School Choice with Control*

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Abstract

Controlled choice over public schools is a common policy of school boards in the United States. It attempts giving choice to parents while maintaining racial and ethnic balance at schools. This paper provides a foundation for controlled school choice programs. We develop a natural notion of fairness and show that assignments, which are fair for same type students and constrained non-wasteful, always exist in controlled choice problems; a "controlled" version of the student proposing deferred acceptance algorithm (CDAA) always finds such an assignment which is also weakly Pareto-optimal. CDAA provides a practical solution for controlled school choice programs.
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1 Introduction

School choice is one of the most widely discussed topics in education. It means giving parents the opportunity to choose the school their child will attend. A central issue is diversity in schools. *Controlled school choice* in the United States attempts to provide parental choice while maintaining the racial, ethnic and socioeconomic balance at schools. Traditionally, children were assigned a public school in their neighborhood. However, neighborhood-based assignment eventually led to economically and racially segregated neighborhoods as wealthy parents used to move to the neighborhoods of schools of their choice. Parents without such means had to send their children to their neighborhood schools, regardless of the quality or appropriateness of those schools for their children. As a result of these concerns, *controlled school choice programs* have become increasingly popular across the United States. Unfortunately, none of the papers in education and in school choice describes *how in practice* to assign students to schools while complying with desegregation guidelines.

The school choice problem, introduced by Abdulkadiroğlu and Sönmez (2003), is closely related to two-sided matching and one-sided matching. Matching theory easily incorporates student preferences as well as school preferences, which may reflect a true preference relation of school principals or an objective priority ordering of students at a school. However, as we will demonstrate in the examples below, control stems from the policies of a third party over assignments of students to schools, be it a school board or a court. In this paper, we develop a theory of matching in order to incorporate such third party policies over matchings. Our theory helps to understand which policies are possible and which ones may lead to incompatibilities.

In some places, control over student assignment is enforced by a court order. For instance, a Racial Imbalance Law that was passed in 1965 in Massachusetts, prohibits racial imbalance and discourages schools from having student enrollments that are more than 50% minority. After a series of legal decisions, the Boston Public Schools (BPS) was ordered to implement a controlled choice plan in 1975.[1] Although BPS has been relieved of legal monitoring, it still continues to try to achieve diversity across ethnic and socioeconomic lines at the city schools (Abdulkadiroğlu, Pathak, Roth, and Sönmez 2005, 2006). Likewise, St. Louis and Kansas City, Missouri, must observe court-ordered racial desegregation guidelines for the placement of students in city schools.[2] In contrast, the White Plains Board of Education employ their nationally recognized Controlled Parents' Choice Program voluntarily.[3]

[1]See http://boston.k12.ma.us/bps/assignmtfacts.pdf for a brief history of student assignment in Boston.

[2]Similarly, Section 228.057 of Florida Statutes requires each school district in the state to design a choice plan. Section 228.057 emphasizes the importance of maintaining socioeconomic, demographic, and racial balance within each school.

[3]The reason behind initiating the choice program was the Board's "belief that balance of the

Other types of control are also present. In New York City, "Educational Option" (EdOpt) schools have to accept students across different ability range. In particular, 16 percent of students that attend an EdOpt school must score above grade level on the standardized English Language Arts test, 68 percent must score at grade level, and the remaining 16 percent must score below grade level (Abdulkadiroglu, Pathak, and Roth, 2005).[4] Miami-Dade County Public Schools control for the socioeconomic status of students in order to diminish concentrations of low-income students at certain schools. Similarly, Chicago Public Schools diversify their student bodies by enrolling students in choice options at schools that are not the students' designated neighborhood schools.[5]

Fairness appears to be the important criterion in student assignment. Quoting from Weaver (1992),

"although controlled-choice districts cannot assign all students to their first-choice schools, districts try to avoid subjective and unfair assignments by establishing clear assignment criteria. This process is often as simple as prioritizing factors such as whether a family has other children in the chosen school, what a student's racial/ethnic background is, where a family lives, and when the application was turned in."[6]

Indeed, a crucial policy of most school choice programs (not only controlled choice programs) is to give some students priority at certain schools. For example, some state and local laws require that students who live in the attendance area of a school must be given priority for that school over students who do not live in the school's attendance area; siblings of students already attending a school must be given priority, and students requiring a bilingual program must be given priority in schools that offer such programs.

Is fairness compatible with controlled choice? That is, given a controlled school choice program, can one guarantee fair assignment of students?

As described by Abdulkadiroğlu and Sönmez (2003), a natural point of departure for school choice is a closely related problem, namely the *college admissions problem* (Gale and

racial and ethnic diversity of the schools' population would promote students' understanding, appreciation, and acceptance of persons of different racial, ethnic, social, and cultural backgrounds. See http://wpcsd.k12.ny.us/1info/index.html". Cambridge has a similar policy of control not only on racial diversity but on socioeconomic diversity as well.

[4]There are similar constraints in other countries as well. For example in England, City Technology Colleges are required to admit a group of students from across the ability range and their student body should be representative of the community in the catchment area (Donald Hirch, 1994, page 120).

[5]We refer the interested reader http://www.buildingchoice.org for an illuminating overview of interdistrict school choice programs including possible desegregation guidelines.

[6]One of the key obstacles identified by the critics of school choice concerns student selection to overdemanded schools (Hirch 1994, p. 14). Because of this reason, the design of a *student assignment mechanism* remains to be an important issue in school choice programs, whether it is controlled or not.

Shapley, 1962; Roth and Sotomayor, 1990).[7] Putting control aside for the time being, the notion of stability in college admissions is equivalent to the following appealing property in the context of school choice: An assignment is fair if there is no unmatched student-school pair (s, c) where student s prefers school c to her assignment and she has higher priority than some other student who is assigned a seat at school c. Therefore, a stable matching in the context of college admissions eliminates *justified envy* in the context of school choice.[8] In particular, the students proposing deferred acceptance algorithm (also known as Gale-Shapley student optimal algorithm) finds the fair assignment which is preferred by every student to any other fair assignment. Moreover, revealing preferences truthfully is a weakly dominant strategy for every student in the preference revelation game in which students submit their preferences over schools first, and then the assignment is determined via the students proposing deferred acceptance algorithm (DAA) using the submitted preferences (Dubins and Freedman, 1981; Roth 1982).[9]

Abdulkadiroğlu and Sönmez (2003) and Abdulkadiroğlu (2005) have considered a relaxed controlled choice problem by employing *type-specific quotas*. Control is imposed on the maximum number of students from each racial/ethnic group which a school can enroll. Their proposed solutions do not capture *controlled choice* to the full extent because they do not exclude segregated schools in fair assignments. For example, consider a school that can enroll 100 students, and at most 50 of these students can be Caucasian. In this case, a student body of 50 Caucasian students would not violate the maximum quota, yet it is fully segregated. Such an assignment would be unacceptable in Minneapolis, White Plains, or St. Louis. Their approach does not provide a complete understanding for the controlled choice problem.

In order to provide a foundation for controlled school choice programs in the United States, a thorough analysis of fairness and controlled choice requires a substantial generalization of the model. Extending the model to fully capture controlled choice brings major difficulties.

The first difficulty concerns the definition of blocking pairs, hence the very definition of stability. By law, every student in the United States is entitled to get enrolled at a

[7]The college admissions problem has been extensively studied and the theory built on this problem has been the basis in designing British and American entry-level labor markets (see Roth (1984,1991,2002) and Roth and Peranson (1999)).

[8]The observation connecting fairness in a one sided matching problem to stability in a corresponding two sided matching problem has previously been made by Balinski and Sönmez (1999) in the context of Turkish college admissions, where they study a college admissions problem with responsive preferences.

[9]Although for schools it is not a weakly dominant strategy to truthfully reveal their preferences in DAA, Kojima and Pathak (2008) have recently shown under some regularity conditions that in DAA the fraction of participants that can gain from misreporting approaches zero as the market becomes large.

public school. The stability and fairness notions above do not incorporate this constraint. In particular, an unmatched student-school pair can cause justified envy (or constitute a blocking pair) even if some other student becomes unassigned after that pair is matched. It is this problem which proves that controlled school choice problems are not equivalent to college admissions problems.

Following the laws of a state or the policies of a school choice program (or of the school board), an assignment is legally *feasible* (or politically acceptable) if both (i) every student is enrolled at a public school and (ii) at each school the desegregation guidelines are respected. We incorporate these constraints in the definition of justified envy, hence in the definition of fairness. The nature of controlled choice imposes that a student-school pair can cause a justified envy (or blocks) only if matching this pair does neither result in any unassigned student nor violate the controlled choice constraints at any school.

This raises the question of existence of fair and legally feasible assignments. We show that feasible student assignments which are fair may not exist. Due to this impossibility, fairness needs to be weakened in order to respect legal constraints. A natural route is to allow envy only among students of the same type. Then only white students can justifiably envy other white students (but not any black students). It turns out that legally feasible assignments, which are fair for same types, may not exist if we require additionally non-wastefulness (Balinski and Sönmez, 1999). In our context, this condition requires that empty seats should not be wasted if students claim them while the legal constraints can be maintained. A positive result emerges if non-wastefulness is constrained: students can claim empty seats only if the resulting assignment does not cause any envy among students of the same type. In particular, a controlled version of the student proposing deferred acceptance algorithm finds for each controlled school choice problem a legally feasible assignment which is both fair for same types and constrained non-wasteful. The assignment found by the controlled student proposing deferred acceptance algorithm (CDAA) has in addition the important welfare property of weak Pareto-optimality: it is impossible to reassign students to schools such that each student is strictly better off with the reassigned school compared to the school chosen for him by CDAA.

Abdulkadiroğlu (2010) considers the same model as here but proposes different stability concepts. In particular, due to the non-existence of feasible and fair student assignments, he relaxes feasibility by not requiring that all students are enrolled at a school and then looks for fair assignments which are not dominated by any other fair assignment.

Students' preferences is the only information which is private. All other components of a controlled school choice problem are commonly known. We show that, unfortunately, it is impossible to elicit true preferences in dominant strategies while maintaining fairness and the legal constraints. More precisely, there does not exist any feasible mechanism which is incentive compatible, fair for same types and constrained non-wasteful. However, giving up constrained non-wastefulness results in a possibility. For instance, when a certain number of seats is reserved for each type at every school and the total number of reserved seats for each race is equal to the number of students of that race, the mechanism that employs DAA is incentive compatible and fair for same types. However, empty seats may be wasted and the assignment may be highly inefficient. In addition, that mechanism is very rigid in the sense that each school reserves for each race a fixed number of students of that race.

Depending on the preferences of a controlled school choice program we make several recommendations of how to assign students to the schools. It is impossible to eliminate envy across different races while respecting controlled school choice constraints. Since these constraints are often legal and/or political, many school choice programs comply with them and fairness across types needs to be abandoned. Incentive compatibility is only guaranteed if the program is ready to accept both highly inefficient assignments of students to schools and the waste of empty seats (although the assignment is based on the true preferences). In real life this cost may be too high and school choice programs may consider giving up incentive compatibility and maintaining some form of efficiency. CDAA achieves this goal: the output assignment always respects the controlled choice constraints, it is weakly Pareto-optimal, fair for students belonging to the same race, and constrained non-wasteful.

Although we focus on controlled school choice, all of our results equally apply to centralized matching programs where diversity constraints are wished to be implemented. Think for instance of college or university admissions where one may want to avoid completely segregated student bodies. Other examples are entry-level labor markets where we may wish to exclude gender segregated worker groups meaning that for each firm there are both female and male workers among its hires. In labor markets we may even desire to control for both race and gender.

The paper is organized as follows. Section 2 formalizes controlled school choice and introduces our desirable criteria, namely fairness for same types and non-wastefulness. Section 3 shows that it is impossible to use results from college admissions problems for controlled school choice. Section 4 shows that there may not exist any feasible assignment which is both fair for same types and non-wasteful. Therefore, we constrain non-wastefulness and show that CDAA always finds a feasible assignment which is both fair for same types and constrained non-wasteful, and which is weakly Pareto-optimal. Section 5 focusses on incentive compatibility and shows that there may not exist any feasible mechanism which is both fair for same types and constrained wasteful. Giving up constrained non-wastefulness allows to divide the school choice problem into several problems, one for each type of students, and for each type DAA is applied. Section 6 summarizes our recommendations for controlled school choice programs. In Appendix A we show that all of our results carry over to controlled school choice with percentage terms. In Appendix B we give an algorithm to check the feasibility of a proposal in CDAA. In Appendix C we allow justified envy across different types and show that the results for fairness across types parallel the corresponding ones for fairness for same types and (constrained) non-wastefulness. Namely, there may not exist any feasible assignment which is fair across types, and there may not exist any feasible mechanism which is both fair across types and incentive compatible.

2 Controlled School Choice

A **controlled school choice problem** or simply a **problem** consists of the following:

1. a finite set of students $S = \{s_1, \ldots, s_n\}$;

2. a finite set of schools $C = \{c_1, \ldots, c_m\}$;

3. a capacity vector $q = (q_{c_1}, \ldots, q_{c_m})$, where q_c is the capacity of school $c \in C$;

4. a students' preference profile $P_S = (P_{s_1}, \ldots, P_{s_n})$, where P_s is the strict preference relation of student $s \in S$ over $C \cup \{s\}$ and each school is acceptable under P_s,[10] i.e. $cP_s s$ for all schools $c \in C$; $cP_s c'$ means that student s strictly prefers school c to school c';

5. a schools' priority profile $\succ_C = (\succ_{c_1}, \ldots, \succ_{c_m})$, where $\succ_c$ is the strict priority ranking of school $c \in C$ over S; $s \succ_c s'$ means that student s has higher priority than student s' to be enrolled at school c;

6. a type space $T = \{t_1, ..., t_k\}$;

7. a type function $\tau : S \to T$, where $\tau(s)$ is the type of student s;

8. for each school c, two vectors of type specific constraints $\underline{q}_c^T = (\underline{q}_c^{t_1}, \ldots, \underline{q}_c^{t_k})$ and $\overline{q}_c^T = (\overline{q}_c^{t_1}, \ldots, \overline{q}_c^{t_k})$ such that $\underline{q}_c^t \leq \overline{q}_c^t \leq q_c$ for all $t \in T$, and $\sum_{t \in T} \underline{q}_c^t \leq q_c \leq \sum_{t \in T} \overline{q}_c^t$.

$\underline{q}_c^t$ is the minimal number of slots that school c must by law allocate to type t students, called **the floor for type t at school** c, whereas $\overline{q}_c^t$ is the maximal number of slots that

[10]This constraint is implicitly given in school choice because students are not allowed to reject schools assigned to them.

school c is allowed by law to allocate to type t students, called **the ceiling for type t at school** c. The same model is studied by Abdulkadiroğlu (2009).

In summary, a controlled school choice problem is given by

$$\left(S, C, q, P_S, \succ_C, T, \tau, (\underline{q}_c^T, \overline{q}_c^T)_{c \in C}\right).$$

When everything except P_S remains fixed, we simply refer to P_S as a controlled school choice problem.

The set of types may represent different students' characteristics: (i) race; (ii) socioeconomic status (determined by free or reduced-price lunch eligibility); or (iii) the district where the student lives.

Controlled choice constraints deserve further discussion. First, these constraints are imposed by law or the policies of a state (via desegregation orders), and the school choice program has to comply with these constraints. Second, they may be more general. For example, our results would apply if these constraints were given in percentage terms (as in the Minneapolis example above). Third, the type space can be a very rich set. When race is controlled, T is typically composed of {white, black, hispanic, asian}. The type space and type specific quotas (i.e. the model) can further be generalized to divide students into categories of several dimensions. For example, consider a controlled choice problem where both race and gender are controlled. Then T can be constructed as {white, black, hispanic, asian}×{female, male} and $\tau(s) = (\tau^r(s), \tau^g(s)) \in$ {white, black, hispanic, asian}×{female, male} denotes student s's race and gender. Type specific racial quotas may be independent of gender, and gender quotas may be independent of racial background. For example, when counting for black students, we do not consider their gender; and when counting for female students, we do not consider their racial background. Appendix A shows that all our results apply to this generalization as well. For expositional convenience the main text will focus on controlled choice where the type space is one-dimensional (like race). Accordingly, each student is associated with a race and the distribution of types induces a natural partition of the set of students: $(S_t)_{t \in T}$ where $S_t = \{s \in S : \tau(s) = t\}$ is the set of all students of type t.

An **assignment** μ is a function from the set $C \cup S$ to the set of all subsets of $C \cup S$ such that

i. $|\mu(s)| = 1$ for every student s, and[11] $\mu(s) = s$ if $\mu(s) \notin C$;

ii. $|\mu(c)| \leq q_c$ and $\mu(c) \subseteq S$ for every school c;

[11]Because each student is assigned to exactly one school or no school, we will omit set brackets and write $\mu(s) = c$ instead of $\mu(s) = \{c\}$ and $\mu(s) = s$ instead of $\mu(s) = \{s\}$.

iii. $\mu(s) = c$ if and only if $s \in \mu(c)$.

Student s is unassigned if $\mu(s) = s$; otherwise $\mu(s)$ denotes the school that student s is assigned; $\mu(c)$ denotes the set of students that are assigned school c; and $\mu^t(c)$ denotes the students of type t that are assigned to school c, i.e. $\mu^t(c) = \mu(c) \cap S_t$. Note that any school is acceptable under P_s for each student s and for any assignment μ we have $\mu(s) = s$ or $\mu(s) P_s s$. This means that any assignment μ is *individually rational*.

A set of students $S' \subseteq S$ **respects (capacity and controlled choice) constraints at school** c if $|S'| \leq q_c$ and for every type $t \in T$, $\underline{q}_c^t \leq |\{s \in S' : \tau(s) = t\}| \leq \overline{q}_c^t$. An assignment μ **respects constraints** if for every school c, $\mu(c)$ respects constraints at c, i.e. for every type t we have

$$\underline{q}_c^t \leq |\mu^t(c)| \leq \overline{q}_c^t.$$

Remark 1 In many school districts, controlled choice constraints are given in percentage terms. For example, in Minneapolis, a district is allowed to go above or below the district-wide average enrollment rates by up to 15 percent points in determining the racial quotas. In White Plains, after 1988, the Board aimed to achieve at each elementary school a mix among the black, Hispanic, and "other" students that is within ±5% points of the district average for each of these groups in each of the grade levels (Yanofsky and Laurette, 1992).

If the controlled school choice constraints are given in percentage terms, then an assignment μ respects constraints if for every school c and every type t we have

$$\underline{q}_c^t |\mu(c)| \leq |\mu^t(c)| \leq \overline{q}_c^t |\mu(c)|.$$

This means, for example, that at a school at least 30 per cent of the admitted students are white ($\underline{q}_c^w = 0.3$) and at most 70 per cent of the admitted students are white ($\overline{q}_c^w = 0.7$). Percentage terms do not cause any difficulties and Appendix A shows that all of our results carry over to controlled school choice with percentage constraints. In the main text constraints are given in quotas for simplicity.

As outlined before the law of many states in the United States requires students to be assigned to schools such that (i) at each school the constraints are respected and (ii) each student is enrolled at a public school. An assignment μ is **(legally or politically) feasible** if μ respects constraints and every student is assigned a school.

Obviously a controlled school choice problem does not have a feasible solution if there are not enough students of a certain type to fill the minimal number of slots required by law for that type at all schools. Therefore, we will assume that the number of students of any type is bigger than the sum of the floors for that type at all schools, i.e. for each $t \in T$,

$|S_t| \geq \sum_{c \in C} \underline{q}_c^t$. Similarly, in order not to leave any student unassigned we need to have enough slots for each type of students, that is $|S_t| \leq \sum_{c \in C} \overline{q}_c^t$.[12]

From now on we will assume that the legal constraints at schools are such that a legally feasible assignment exists. Otherwise the laws are not compatible with each other and they need to be modified. We will not consider this issue here.

What are desirable properties of feasible assignments in controlled school choice problems? The following notions are the natural adaptations of their counterparts in standard two-sided matching (without type constraints).

The first requirement is that whenever a student prefers an empty slot to the school assigned to him, the legal constraints are violated when assigning the empty slot to this student while keeping all other assignments unchanged.[13]

We say that **student s justifiably claims an empty slot at school c under the feasible assignment μ** if

(nw1) $cP_s\mu(s)$ and $|\mu(c)| < q_c$,

(nw2) $\underline{q}_{\mu(s)}^{\tau(s)} < |\mu^{\tau(s)}(\mu(s))|$, and

(nw3) $|\mu^{\tau(s)}(c)| < \overline{q}_c^{\tau(s)}$.

Here (nw1) means student s prefers an empty slot at school c to the school assigned to him; (nw2) means that the floor of student s's type is not binding at school $\mu(s)$; and (nw3) means that the ceiling of student s's type is not binding at school c. Hence, under (nw1)-(nw3) student s can be assigned an empty slot at the better school c without changing the assignments of the other students and violating the constraints at any school. A feasible assignment μ is **non-wasteful** if no student justifiably claims an empty slot at any school.

A well studied requirement of the literature is fairness or no-envy (Foley, 1967)[14]. In school choice student s envies student s' when s prefers the school at which s' is enrolled, say school c, to her school. However, the nature of controlled school choice imposes the following (legal) constraints: Envy is justified only when

(i) student s has higher priority to be enrolled at school c than student s',

[12] Note that these constraints are not sufficient for the existence of a feasible assignment. For example, consider the problem consisting of three schools and three students. Each student has a different type. The capacities are all equal to 1, the floors are all equal to zero, and the ceilings are given by $\overline{q}_{c_1}^{t_1} = \overline{q}_{c_1}^{t_2} = \overline{q}_{c_1}^{t_3} = 1$, $\overline{q}_{c_2}^{t_1} = \overline{q}_{c_2}^{t_2} = 0$ and $\overline{q}_{c_2}^{t_3} = 1$, and $\overline{q}_{c_3}^{t_1} = \overline{q}_{c_3}^{t_2} = 0$ and $\overline{q}_{c_3}^{t_3} = 1$. There does not exist a feasible assignment because student s_1 or student s_2 has to be left unassigned if the constraints at school c_1 are respected.

[13] This requirement is in the spirit of the property "non-wastefulness" introduced by Balinski and Sönmez (1999).

[14] See for example Tadenuma and Thomson (1991), for an excellent survey, also see Thomson (forthcoming), Thomson (2000) and Young (1995).

(ii) student s can be enrolled at school c without violating controlled choice constraints by removing s' from c, and

(iii) student s' can be enrolled at another school without violating constraints by removing s' from c in favor of s.

Throughout the main text we will require that envy is justified only if both the envying student and the envied student are of the same type. If this is the case, then (ii) and (iii) are always true since then the envying student and the envied student can simply exchange schools. We formulate our notion of fairness more precisely below.

We say that **student s justifiably envies student s' at school c under the feasible assignment μ** if

(f1) $\mu(s') = c$, $cP_s\mu(s)$ and $s \succ_c s'$, and

(f2) $\tau(s) = \tau(s')$.

In (f1), student s' is enrolled at school c and both student s prefers school c to his assigned school $\mu(s)$ and student s has higher priority to be enrolled at school c than student s'. By (f2), student s and student s' are of the same type. Then we obtain a feasible assignment when students s and s' exchange their slots, i.e. choose μ' as follows: $\mu'(s) = \mu(s')$, $\mu'(s') = \mu(s)$, and $\mu'(\hat{s}) = \mu(\hat{s})$ for all $\hat{s} \in S \backslash \{s, s'\}$. The assignment μ' is feasible because s and s' are of the same type and μ was feasible.

A feasible assignment μ is **fair for same types** if no student justifiably envies any student who is of the same type.

We discuss also a stronger notion of fairness where envy is allowed across types, i.e. the envying student and the envied student can be of different types. Such an envy is justified only if student s can be enrolled at school c and student s' at another school while keeping all the other assignments unchanged and satisfying the controlled school choice constraints at all schools. A feasible assignment μ is **fair across types** if no student justifiably envies any student. Independently of his own type, a student is allowed to envy any student. In Appendix C we show that the results for this stronger condition parallel the results for fairness for same types and non-wastefulness.

3 (No) Connection with College Admissions

Previous papers on school choice (or "student placement") successfully associated any problem with a college admissions problem and applied well-known results from this literature. In any of these papers the school choice problem can be reduced to a college admissions problem in which (i) the priority ordering of students at a school reflects that school's preferences over individual students, (ii) a set of students that do not respect the type specific quotas of a school is not acceptable for that school, (iii) a school's preferences over acceptable sets of students is *responsive* to the priority ordering of students at that school. Then fairness in the controlled choice problem corresponds to stability in the corresponding college admissions problem. We will show that this approach is not possible here because controlled school choice imposes legal constraints which are absent in the standard two-sided matching.

Every controlled school choice problem $\left(S,C,q,P_S,\succ_C,T,\tau,(\underline{q}_c^T,\overline{q}_c^T)_{c\in C}\right)$ corresponds to a college admissions problem $\left(S,C,q,P_S,\succ_C,\hat{P}_C,T,\tau,(\underline{q}_c^T,\overline{q}_c^T)_{c\in C}\right)$, where $\hat{P}_C$ is the list of colleges' preferences over *sets of students*, whereas $\succ_C$ is the list of colleges' preferences over *individual students*. Parallel to Abdulkadiroğlu (2005) we will impose for each college c a "responsiveness" condition on $\hat{P}_c$ subject to respecting constraints at c.

A set of students is **acceptable for college** c if and only if it respects capacity and controlled choice constraints at college c; furthermore, c's preferences over acceptable sets of students are responsive to $\succ_c$. That is, for every s, $s' \in S$ and $S' \subseteq S\backslash\{s,s'\}$, if $S'\cup\{s\}$ and $S'\cup\{s'\}$ are both acceptable for c, then $S'\cup\{s\}\hat{P}_cS'\cup\{s'\}$ if and only if $s \succ_c s'$. In addition, for all sets $S', S'' \subseteq S$, if both S' and S'' are unacceptable for college c, then $\emptyset\hat{P}_cS'$, $\emptyset\hat{P}_cS''$ and $S'\hat{I}_cS''$. In other words, any unacceptable set of students is ranked below the empty set and any two unacceptable sets are indifferent.

A **matching** is an assignment. We use assignment for school choice and matching for college admissions. A college-student pair (c,s) **blocks a matching** μ if $cP_s\mu(s)$ and

(s1) either $\mu(c)\cup\{s\}$ respects constraints at c, or equivalently $\mu(c)\cup\{s\}\hat{P}_c\mu(c)$;

(s2) or there exists $s' \in \mu(c)$ such that both $\tau(s) = \tau(s')$ and $s \succ_c s'$, or equivalently both $\tau(s) = \tau(s')$ and $(\mu(c)\backslash\{s'\})\cup\{s\}\hat{P}_c\mu(c)$ (by responsiveness).

A matching μ is **stable** if (i) for every c, $\mu(c)$ is acceptable for c and (ii) it is not blocked by any college-student pair. Here (s1) corresponds to the property "non-wastefulness" introduced in Balinski and Sönmez (1999) and (s2) corresponds to the requirement that no student envies another student who is of the same type.

Controlled school choice is fundamentally different from college admissions. In the definition of justified claim and justified envy, the initial assignment μ is assumed to be feasible. Then student s can be assigned (the possibly empty) slot at his more preferred school and the (possibly) envied student s' can be assigned a slot at another school while respecting controlled choice constraints at *all schools*. In contrast, while checking for a blocking pair in a matching, in (s1) and (s2) we only check whether the new set of students respects constraints at *college* c; we do not check whether s' is matched with another college; or whether removing s from $\mu(s)$ violates constraints at $\mu(s)$. Therefore, a fair and non-wasteful assignment is not necessarily stable in the corresponding college admissions problem.

Conversely, if a matching is stable in the college admissions problem and it is feasible in the corresponding school choice problem, then it will be fair for same types and non-wasteful in the corresponding school choice problem. However, the feasibility of the matching in the school choice problem is not implied by its stability in the college admissions problem, since stability requires neither that every student is matched with a college nor that the controlled choice constraints are respected at all schools. It is possible that there is no feasible assignment which is stable in the corresponding college admissions problem. The following example illustrates these points.

Example 1. There are three white students w_1, w_2, w_3; two black students b_1, b_2; and two colleges c_1 and c_2 each with capacity four. State laws require each college to admit at least one student of each type. Each college's preference over acceptable sets of students is responsive to the following ranking of students: $w_1 \succ_c w_2 \succ_c w_3 \succ_c b_1 \succ_c b_2$. All students prefer college c_1 to college c_2. The only stable matching is the following: $\mu(c_1) = \{w_1, w_2, w_3, b_1\}$ and $\mu(c_2) = \emptyset$, i.e. b_2 is not matched with any college.[15] Clearly, μ is not feasible in the corresponding school choice problem because (i) the minimum quotas for white and black students are not satisfied at school c_2 and (ii) student b_2 is assigned no school although the law entitles him a slot at a public school. Since μ is the only matching which is stable, any feasible assignment of the corresponding school choice problem is unstable in the college admissions problem. Furthermore, the unique feasible assignment which is both fair for same types and non-wasteful in the corresponding school choice problem is the following: $\mu'(c_1) = \{w_1, w_2, b_1\}$ and $\mu'(c_2) = \{w_3, b_2\}$.

4 Existence of Fair Assignments

As described before it is impossible to apply results from college admissions problems to controlled school choice. Since stable matchings always exist in college admissions problems, our first result makes this even clearer: the legal constraints, fairness and non-wastefulness may result in an incompatibility.

[15] Note that this is also the matching which Abdulkadiroğlu and Sönmez (2003)'s top trading cycles mechanism with type specific quotas finds. Therefore, Example 1 also shows that it is impossible to use this mechanism for controlled school choice.

THEOREM 1: The set of feasible assignments which are both fair for same types and non-wasteful may be empty in a controlled school choice problem.

Proof: The proof is by means of an example. Consider the following problem consisting of three schools $\{c_1, c_2, c_3\}$ and two students $\{s_1, s_2\}$. Each school has a capacity of two ($q_c = 2$ for all schools c). All students are of the same type t. The ceiling of type t is equal to two at all schools ($\overline{q}_c^t = 2$ for all schools c). School c_1 has a floor for type t of $\underline{q}_{c_1}^t = 1$. All other floors are equal to zero. The schools' priorities are given by $s_2 \succ_{c_1} s_1$, $s_2 \succ_{c_2} s_1$ and $s_1 \succ_{c_3} s_2$. The students' preferences are given by $c_2 P_{s_1} c_3 P_{s_1} c_1 P_{s_1} s_1$ and $c_3 P_{s_2} c_2 P_{s_2} c_1 P_{s_2} s_2$. This information is summarized in Table 1.

TABLE 1.

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	P_{s_1}	P_{s_2}
	s_2	s_2	s_1	c_2	c_3
	s_1	s_1	s_2	c_3	c_2
				c_1	c_1
				s_1	s_2
capacities	$q_{c_1} = 2$	$q_{c_2} = 2$	$q_{c_3} = 2$		
ceiling for t	$\overline{q}_{c_1}^t = 2$	$\overline{q}_{c_2}^t = 2$	$\overline{q}_{c_3}^t = 2$		
floor for t	$\underline{q}_{c_1}^t = 1$	$\underline{q}_{c_2}^t = 0$	$\underline{q}_{c_3}^t = 0$		

Next we determine the set of assignments which are feasible for this problem. Feasibility requires that student s_1 or student s_2 is assigned school c_1 and all students are enrolled at a school. Therefore,

$$\begin{array}{ccc} \mu_1 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & \emptyset \end{pmatrix} & \xrightarrow{s_2 \text{ claims } c_3} & \mu_2 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & \emptyset & s_2 \end{pmatrix}, \\ s_2 \text{ envies } s_1 \uparrow & & \downarrow s_1 \text{ envies } s_2 \\ \mu_4 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & \emptyset \end{pmatrix} & \xleftarrow[s_1 \text{ claims } c_2]{} & \mu_3 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & \emptyset & s_1 \end{pmatrix} \end{array}$$

and $\mu_5 = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_1, s_2\} & \emptyset & \emptyset \end{pmatrix}$ are the only assignments which are feasible. Now (as indicated above)

(i) μ_1 is wasteful because s_2 justifiably claims an empty slot at c_3,

(ii) μ_2 is not fair for same types because s_1 justifiably envies s_2 at c_3,

(iii) μ_3 is wasteful because s_1 justifiably claims an empty slot at c_2,

(iv) μ_4 is not fair for same types because s_2 justifiably envies s_1 at c_2; and

(v) μ_5 is wasteful because s_1 justifiably claims an empty slot at c_2.

Hence there is no feasible assignment which is both fair for same types and non-wasteful. □

Note that, in contrast to the literature on matching, our impossibility result is *not* obtained by violating the responsiveness condition (or "substitutability") of schools' preferences over sets of students, but by controlled choice. In the example used to prove Theorem 1 (except for μ_5) in any politically feasible assignment each school is assigned at most one student and these conditions do not have any bite. Yet we obtain an impossibility. Furthermore, Theorem 1 does not follow from any previous result, because below we show that a slight twist of non-wastefulness gives us back a possibility result.

Remark 2 Similar to Theorem 1, Theorem 1' in Appendix C shows that the set of feasible assignments which are fair across types may be empty in a controlled school choice problem. Hence, it is impossible to allow envy across types while complying with the (legal) choice constraints. This means that any controlled school choice program needs to give up fairness across types and envy can be only allowed among same type students if choice constraints are respected.

We discuss the robustness of Theorem 1 subject to relaxing constraints. We know from Abdulkadiroğlu (2005) that existence of feasible assignments which are fair for same types and non-wasteful is reestablished if we set all floors equal to zero. If we relax the ceilings, then we may also need to increase the number of seats which are available at a school. In the example used to prove Theorem 1 the ceilings and the capacities are such that each school's capacity is equal to the total number of students and the ceiling for each type at each school is equal to the total number of students of that type. Hence, Theorem 1 is robust subject to relaxing ceilings and capacities. Furthermore, note that in the example there is only one school with a floor greater than zero.

Clearly Theorem 1 is a negative result. We will see later that the answer is affirmative to both (i) the existence of feasible assignments which are fair for same types and (ii) the existence of feasible and non-wasteful assignments. Hence, in controlled school choice problem we may retain fairness for same types or non-wastefulness while giving up the other requirement.

Giving up completely the other requirement is not satisfactory for a controlled school choice program. Since in real-life controlled school choice problems typically the total number of slots available is about the same as the number of students, potential violations of non-wastefulness arise less likely than potential envy among students who are of the same type. Therefore, our primary focus should be on fairness for same types and we propose the following: fairness for same types should be always satisfied and among those assignments we chose one which is constrained non-wasteful meaning that if a student justifiably claims an empty slot at a school, then after assigning him this empty slot the new assignment is no longer fair for same types. Then there will be another student of the same type justifiably envying this student at the new school.

We say that a feasible assignment μ is **constrained non-wasteful** if:

student s justifiably claims an empty slot at school c under μ

$\Rightarrow$ the assignment μ' (where $\mu'(s) = c$ and $\mu'(s') = \mu(s')$ for all $s' \in S \backslash \{s\}$) is not fair for same types.

If the feasible assignment μ is fair for same types and constrained non-wasteful, then the above definition is equivalent to whenever a black student s justifiably claims an empty slot at school c under μ, then some other black student s' justifiably envies student s at school c under the assignment μ' (where μ' is defined as above).

The idea of feasible assignments which are both fair for same types and constrained non-wasteful is similar to the one of "bargaining sets": if a black student s has an objection to μ because s claims an empty slot at c, then there will be a counterobjection once s is assigned to c since some other black student will then justifiably envy s at c. Roughly speaking, an outcome belongs to the "bargaining set" if and only if for any objection to the outcome there exists a counterobjection.

THEOREM 2: The set of feasible assignments which are both fair for same types and constrained non-wasteful is non-empty in a controlled school choice problem.

In showing Theorem 2 we propose a controlled version of the student proposing deferred acceptance algorithm (DAA). Recall that in the classical algorithm of Gale and Shapley (1962) students are put tentatively on waiting lists and at any step the students, who do not belong to any waiting list, simultaneously propose to schools to which they did not propose yet. Each school updates its waiting list by accepting the most preferred students from the new proposals and the students who were previously on its waiting list. The other students are rejected. If each student either belongs to a waiting list or has proposed to all schools, then the algorithm ends and the schools are assigned according to the waiting lists.

Our controlled version will have two important differences. First, proposals cannot be simultaneous. When several students propose simultaneously, it may be infeasible to put them on the waiting lists. In Example 1 all white students propose to school c_1 and admitting all of them at school c_1 makes it impossible to assign at least one white student to school c_2.[16] In our controlled student proposing deferred acceptance algorithm proposals are sequential (say according to when the applications were received): similar to McVitie and Wilson (1970) at each step one student, who does not belong to any waiting list, proposes to the most preferred school to which he did not propose yet.

Second, when putting a student on a waiting list we need to make sure that all tentative assignments are feasible. In other words, we check whether there is some feasible assignment such that all students are assigned the school to which's waiting list they belong. In the standard DAA we check only whether the set of most preferred students from the new proposals and the students on the waiting list respects constraints at that school.

Controlled Student Proposing Deferred Acceptance Algorithm (CDAA)

Start: Fix an order of the students, in which they are allowed to make proposals to schools, say $s_1 - s_2 - \cdots - s_n$. We will always define a tentative assignment ν recording the current waiting lists at all schools. The tentative assignment is such that it is possible to allocate the unassigned students to schools such that the resulting assignment is feasible. Let $\mathcal{F}$ denote the set of all feasible assignments and ν_0 be the empty assignment, i.e. $\nu_0(s) = s$ for all $s \in S$. Let P_S be a controlled school choice problem.

1. Let student s_1 apply to the school which is ranked first under P_{s_1}, say c_1. If there is some $\mu \in \mathcal{F}$ such that $\mu(s_1) = c_1$, then set $\nu_1(s_1) = c_1$ and $\nu_1(s) = \nu_0(s) = s$ for all $s \in S \backslash \{s_1\}$; otherwise s_1 is rejected by school c_1 and we set $\nu_1 = \nu_0$.

$\vdots$

k. If there is some student s such that $\nu_{k-1}(s) = s$ (s is unassigned), then student s did not yet apply to all the schools which are acceptable to him. Let s be the student with minimal index among those students. Let c be the school which is most preferred under P_s among the schools to which s did not apply yet.

(i) If there is $\mu \in \mathcal{F}$ such that $\mu(s) = c$ and $\mu(s') = \nu_{k-1}(s')$ for all students s' such that $\nu_{k-1}(s') \neq s'$, then student s justifiably claims an empty slot at school c under ν_{k-1}. Then we set $\nu_k(s) = c$ and $\nu_k(s') = \nu_{k-1}(s')$ for all $s' \in S \backslash \{s\}$ (Appendix

[16] Here one may consider rejecting student w_3 since w_3 has the lowest priority among the white students. Generally (as in the example used to prove Theorem 1), however each white student could propose to a different school and we would not know which students to put on waiting lists.

B provides an algorithm which checks whether student s can justifiably claim an empty slot at school c or not);

(ii) If (i) is not true but there is a student s' of the same type of s and student s justifiably envies student s' at school c under ν_{k-1}, then let s' be the student who has the lowest priority under $\succ_c$ among all the students of type $\tau(s)$ who are tentatively admitted at school c under ν_{k-1}. Then we set $\nu_k(s) = c$, $\nu_k(s') = s'$, and $\nu_k(s'') = \nu_{k-1}(s'')$ for all $s'' \in S \backslash \{s, s'\}$, i.e. school c rejects s' and puts s on its waiting list; and

(iii) Otherwise (if (i) and (ii) are not true) we set $\nu_k = \nu_{k-1}$ and student s is rejected by school c.

End: The algorithm ends at a Step x where $\nu_x(s) \neq s$ for all $s \in S$. Then the tentative assignments become final and ν_x is the controlled student proposing deferred acceptance assignment for profile P_S.

The assignment found by CDAA may be wasteful because in the example used to prove Theorem 1 the algorithm finds μ_1 and student s_2 justifiably claims an empty slot at school c_3 under μ_1.

THEOREM 3: For any controlled school choice problem CDAA yields a feasible assignment which is both fair for same types and constrained non-wasteful.

Proof: Let P_S be a controlled school choice problem and μ be the assignment that CDAA finds for P_S. We show that (a) μ is feasible, (b) μ is fair for same types, and (c) μ is constrained non-wasteful.

For (a) it suffices to show at Step k, any student, who is unassigned under ν_{k-1}, did not yet propose to all schools on his preference. Suppose that $\nu_{k-1}(s) = s$ and student s proposed to all schools before.

Let student s have been on a waiting list of a school, say school c, until Step h. Then at Step h another student s' proposed to c and school c rejected s. But then there were other schools c' which could have given s' an empty slot keeping all the other matches of ν_h unchanged. But s did not apply to any of those empty slots (because otherwise he would have received that slot). Therefore, this is impossible.

If student s was never on a waiting list, then let h be the step where student s applied to his most preferred school. Since s is rejected at Step h, s could not justifiably claim an empty slot at his most preferred school. But then there were no $\mu' \in \mathcal{F}$ such that $\mu'(s') = \nu_{h-1}(s')$ for all $s' \in S \backslash \{s\}$ with $\nu_{h-1}(s') \neq s'$. But then ν_{h-1} is an impossible waiting list at Step $h-1$, which contradicts the definition of CDAA.

For (b), suppose that μ is not fair for same types. Then there is a student s who justifiably envies student s' at school c under μ and both students s and s' are of the same type. Let s' have lowest priority in $\mu(c)$ among the students who are of type $\tau(s)$. Since $cP_s\mu(s)$, student s applied to school c at some step, say Step k.

If $\nu_k(s) = c$, then by $\mu(s) \neq c$, student s was later rejected by school c because some student of type $\tau(s)$ applied to school c and had higher priority than s under $\succ_c$. Now it is impossible that student s' was put on school c's waiting list later because s' must have had higher priority than s and we have $s \succ_c s'$.

If $\nu_k(s) \neq c$, then (i) was not possible at Step k, i.e. s could not justifiably claim an empty slot at school c under ν_{k-1}. Since (ii) was neither possible, all students of type $\tau(s)$ in $\nu_{k-1}(c)$ had higher priority than s. Now it is again impossible that student s' was put on school c's waiting list later because s' must have had higher priority than s and we have $s \succ_c s'$.

It may be that student s' later justifiably claimed an empty slot at school c. This is also impossible because given a waiting list ν_x, for each school c and each type, the students of that type admitted at the school only increases, i.e. it is not possible that s' claims an empty slot later whereas s could not do that earlier.

For (c), suppose that μ is not constrained non-wasteful. Then a student s justifiably claims an empty slot at school c under μ and μ' (where $\mu'(s) = c$ and $\mu'(s') = \mu(s')$ for all $s' \in S\backslash\{s\}$) is fair for same types. Since s justifiably claims an empty slot at school c, we have $cP_s\mu(s)$ and s must have proposed to c, say at Step k, before proposing to $\mu(s)$. The following is true in CDAA: once a student is admitted on a waiting list, then the student can only be removed from the waiting list if another student of the same type is admitted. Therefore, for all types t and all schools c' we have

$$|\nu^t_{k-1}(c')| \leq |\mu^t(c')|. \tag{1}$$

Now by the feasibility of μ and s's justified claim of an empty slot at c under μ, at Step k there was a feasible assignment $\hat{\mu}$ such that $\hat{\mu}(s) = c$ and $\hat{\mu}(\hat{s}) = \nu_{k-1}(\hat{s})$ for all $\hat{s}$ such that $\nu_{k-1}(\hat{s}) \neq \hat{s}$. Hence, $\nu_k(s) = c$ and s was put on the waiting list of c at Step k. Since $\mu(s) \neq c$, at a later step, say Step k', school c rejected student s and admitted a student s'. Then student s' must be of the same type as s and at Step k' (i) was not true, i.e. student s' could not justifiably claim an empty slot at school c at Step k'. But then by the same property (1) for Step k' no student of type $\tau(s)$ can justifiably claim an empty slot at school c under μ, a contradiction to s's justified claim of an empty slot at c under μ. $\square$

In CDAA students with smaller indices are allowed to propose first (and students may be indexed according to when their applications were received by the controlled school choice program). However, it is easy to verify that the order, in which students are allowed to propose, is irrelevant for the conclusion of Theorem 3. Therefore, at each step alternatively we may choose randomly a student from the students who do not belong to any waiting list. This randomization of the CDAA ensures that the algorithm becomes anonymous. Then using Roth and Rothblum (1999) and Ehlers (2008) it can be shown that in a low information environment it is a weakly dominant strategy for each student to submit his true ranking. Unfortunately, in contrast to McVitie and Wilson's sequential version of DAA, CDAA may yield different outcomes for different orders. For instance, in the example used to prove Theorem 1, CDAA finds μ_1 when student s_1 proposes in Step 1 and it finds μ_3 when student s_2 proposes in Step 1.

Remark 3 In Erdil and Ergin (2008) properties parallel to the ones above have appeared: they propose in uncontrolled school choice with possibly equal priorities a constrained efficient solution called the stable improvement cycles algorithm. Loosely speaking, this algorithm breaks ties in any priority ordering, runs DAA for the school choice problem with the resulting strict priority profile, and in case there exists a "stable improvement cycle" for the output assignment of DAA, students exchange their assigned schools along such a cycle, and so on. The algorithm stops when no stable improvement cycle exists. In such a case the resulting assignment is constrained efficient meaning that it is not Pareto dominated by any fair assignment. When students have equal priority at a school, the outcome of the stable improvement cycles algorithm may depend on how ties are broken and which stable improvement cycles are chosen. If both ties are randomly broken and the cycle choice is random, they show similarly to Roth and Rothblum (1999) and Ehlers (2008) that in a low information environment it is a weakly dominant strategy for each student to submit his true ranking in this algorithm.

It turns out that CDAA has another desirable feature: the output assignment is always weakly Pareto-optimal in the sense that there exists no feasible assignment which all students strictly prefer to the output assignment, i.e. if μ is the assignment found by CDAA for the controlled school choice problem P_S, then there exists no feasible assignment $\bar{\mu}$ such that $\bar{\mu}(s) P_s \mu(s)$ for all students s. If this important welfare property is not satisfied by an assignment procedure, then one may seriously criticize the use of that procedure because all students unanimously may strictly prefer another assignment (or another procedure).

THEOREM 4: For any controlled school choice problem CDAA yields a feasible assignment which is weakly Pareto-optimal.

Proof: Let P_S be a controlled school choice problem and μ be the assignment that CDAA finds for P_S.

Suppose that μ is not weakly Pareto-optimal. Then there exists another feasible assignment $\bar{\mu}$ such that $\bar{\mu}(s) P_s \mu(s)$ for all students s. We derive a contradiction as follows: first we show that for any type t, the school, at which the last type-t student is admitted, admits no type-t student under $\bar{\mu}$; second we show that under both μ and $\bar{\mu}$ each school is assigned the same number of students; and third we show that there is a cyclical exchange among all types of the schools at which the last type-t students are admitted in CDAA.

Let t be a type and let CDAA admit the students of type t at their seats specified by μ in the order i_1, i_2, ..., i_l. This means that student i_1 is the first student s of type t who gets assigned to $\mu(s)$ in CDAA and student i_l is the last student s of type t who gets assigned to $\mu(s)$ in CDAA. Because $\bar{\mu}(s) P_s \mu(s)$ for any student s, each student s applies to $\bar{\mu}(s)$ before applying to $\mu(s)$. Since i_l is the last type-t student to be admitted, i_l must justifiably claim an empty slot at $\mu(i_l)$ in the Step k where i_l proposes to $\mu(i_l)$. But then no student s of type t proposed to $\mu(i_l)$ before proposing to $\mu(s)$ because such a student could have claimed an empty slot at school $\mu(i_l)$ (since i_l was able to claim an empty slot at $\mu(i_l)$ at the later Step k). Hence, by $\bar{\mu}(s) P_s \mu(s)$ for all $s \in S_t$, no student of type t is assigned to $\mu(i_l)$ under $\bar{\mu}$ and we have $\bar{\mu}^t(\mu(i_l)) = \emptyset$.

Since under both $\bar{\mu}$ and μ all students are assigned a school, in showing $|\bar{\mu}(c)| = |\mu(c)|$ for all schools c it suffices to show $|\bar{\mu}(c)| \leq |\mu(c)|$ for all schools c. Suppose that this is not the case, i.e. for some school c we have $|\bar{\mu}(c)| > |\mu(c)|$. Then for some type t we have $|\bar{\mu}^t(c)| > |\mu^t(c)|$. Then $\bar{\mu}^t(c) \neq \emptyset$. Let $i \in \bar{\mu}^t(c)$. Since $c P_i \mu(i)$, student i proposed to c before proposing to $\mu(i)$. Since $|\mu(c)| < q_c$ and $|\bar{\mu}^t(c)| > |\mu^t(c)|$, student i claimed an empty slot at school c when proposing to it. Now this claim must have been justified since the last type-t student to be admitted, student i_l, justifiably claimed an empty slot at $\mu(i_l)$ and no student of type t is assigned to $\mu(i_l)$ under $\bar{\mu}$. Thus, student i must have been assigned an empty slot at c when he proposed to c in CDAA. Since our choice $i \in \bar{\mu}^t(c)$ was arbitrary, school c must admit at least $|\mu^t(c)| + 1$ students of type t in CDAA. Hence, we have shown $|\bar{\mu}(c)| = |\mu(c)|$ for all schools c.

For each type t, let i_l^t denote the last type-t student s to be admitted at $\mu(s)$ in CDAA and let $c^t = \mu(i_l^t)$. Since $|\bar{\mu}(c^t)| = |\mu(c^t)|$ and $\bar{\mu}^t(c^t) = \emptyset$, there exists at least one type t' such that $|\bar{\mu}^{t'}(c^t)| > |\mu^{t'}(c^t)|$ or equivalently some students of type t' would like to claim the slot of i_l^t at school c^t. For the moment, let us treat types as agents and say that type t is endowed with an empty slot at c^t and type t' would like to claim a slot at c^t if $|\bar{\mu}^{t'}(c^t)| > |\mu^{t'}(c^t)|$. Now, similarly as above, type t' is also endowed with an empty slot at $c^{t'}$ and some type t'' would like to claim that slot. Because the set of types is finite and each type is endowed with exactly one empty slot, there must exist at least one cyclical exchange from μ to $\bar{\mu}$: there are types $t_1, \ldots, t_m$ such that t_1 claims the slot c^{t_2}, t_2 claims the slot c^{t_3},..., and t_m claims the slot c^{t_1}.

Now choose the type t such that type t is part of a cyclical exchange and among the types, which are part of a cyclical exchange, type t is the first type to admit all students in CDAA. This means that i_l^t is admitted at $\mu(c^t)$ before any other type t', which is part of a cyclical exchange, admits $i_l^{t'}$ at $c^{t'}$. Because t is part of a cyclical exchange, type t claims the "endowment" of another type, say type t'. Because i_l^t is the last type-t student to be admitted, all type-t students, who would like to claim the slot at $c^{t'}$, proposed to $c^{t'}$ before. Because type t is the first type to admit all students among all types which are part of a cyclical exchange, at this step both the slot at $c^{t'}$ was empty and this cyclical exchange was feasible when the type-t students proposed to $c^{t'}$. But then at this step this type-t student justifiably claims an empty slot at school $c^{t'}$ and CDAA would have assigned this type-t student to school $c^{t'}$, which is a contradiction. □

Remark 4 An immediate consequence of Theorem 4 is that it is impossible to make all white students strictly better off by reassigning their seats and seats left empty among white students while keeping all other students enrolled at their schools. More precisely, for any type t the assignment μ^t is weakly Pareto-optimal in the sense that it is impossible to make all students of type t better off by reassigning (in a feasible way) their seats specified by μ and the seats left empty by μ, i.e. there is no feasible assignment $\bar{\mu}$ such that $\bar{\mu}(s)P_s\mu(s)$ for all $s \in S_t$ and $\bar{\mu}(s') = \mu(s')$ for all $s' \in S \backslash S_t$. This is easily seen by applying Theorem 4 to the problem reduced for type-t students where any school c has $q_c + |\mu^t(c)| - |\mu(c)|$ empty seats available and S_t is the set of students. Applying CDAA to P_{S_t} yields μ^t and μ^t needs to be weakly Pareto-optimal by Theorem 4, the desired conclusion.

Remark 5 Another immediate consequence of Theorem 4 is that it is impossible to make all white students weakly better off by fairly reassigning their seats among white students while keeping all other students enrolled at their schools and all empty slots empty. More precisely, if μ is the output of CDAA, then for any type t the assignment μ^t is "best" from the type-t students' point of view in the following sense: there is no other feasible assignment $\bar{\mu}$ such that (i) $|\bar{\mu}^t(c)| = |\mu^t(c)|$ for all schools c, (ii) $\bar{\mu}$ is fair for same types, and (iii) $\bar{\mu}^t$ Pareto dominates for type-t students the assignment μ^t.[17]

[17]This follows from the fact that when allocating the seats of μ^t to type-t students, μ^t is the output of CDAA restricted to type-t students. But CDAA restricted to type-t students and the seats of μ^t is identical with McVitie and Wilson (1970)'s version of DAA. Then μ^t is the output of DAA to type-t students and the seats of μ^t and we know that for type-t students μ^t is most preferred among all assignments which are fair for same types.

5 Incentive Compatibility

Apart from students' preferences all components of a controlled school choice problem are exogenously determined (like the capacities of the schools) or given by law (like the priority profile and the controlled choice constraints). The only information which is private are students' preferences over schools. They need to be stated by the students to the school choice program. Since students must be assigned schools for any possible reported profile, the program has to be based on a mechanism selecting an assignment for each problem. In a controlled school choice program the mechanism should respect the legal constraints imposed by the state. A mechanism is **(legally) feasible** if it selects a feasible assignment for any reported profile.

Any program would like to elicit the true preferences from students. If students would misreport, then the assignment chosen by the program is based on *false* preferences and may be highly unfair for the *true* preferences.

Avoiding this problem means constructing a mechanism where no student has ever an incentive to misrepresent his true preference for any preferences reported by the other agents. Any mechanism which makes truthful revelation of preferences a dominant strategy for each student is called **(dominant strategy) incentive compatible**. A feasible mechanism is **fair for same types** if it selects for any controlled school choice problem a feasible assignment which is fair for same types. Analogously we define **non-wastefulness** and **constrained non-wastefulness**, respectively, for a mechanism.

In contrast to the school choice problems studied in previous papers it is impossible to construct a mechanism which is incentive compatible, fair for same types and constrained non-wasteful while respecting the diversity constraints given by law. Therefore, it is impossible to choose for each profile an order in which students propose in CDAA such that this mechanism becomes incentive compatible.

THEOREM 5: In controlled school choice there is no feasible mechanism which is incentive compatible, fair for same types and constrained non-wasteful.

Proof: The proof is by means of an example. Consider the following problem consisting of three schools $\{c_1, c_2, c_3\}$ and two students $\{s_1, s_2\}$. Each school has a capacity of two ($q_c = 2$ for all schools c). The type space consists of a single type t, i.e. both students are of the same type t. The ceiling for type t is equal to two for each school ($\overline{q}_c^t = 2$ for all schools c). School c_1 has a floor for type t of $\underline{q}_{c_1}^t = 1$ and both other schools have a floor of 0 for type t. Schools c_1 and c_2 give higher priority to student s_2 whereas school c_3 gives higher priority student s_1. The students' preferences are given by $c_2 P_{s_1} c_1 P_{s_1} c_3 P_{s_1} s_1$ and $c_3 P_{s_2} c_1 P_{s_2} c_2 P_{s_2} s_2$.

This information is summarized in Table 2.

TABLE 2.

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	P_{s_1}	P_{s_2}
	s_2	s_2	s_1	c_2	c_3
	s_1	s_1	s_2	c_1	c_1
				c_3	c_2
				s_1	s_2
capacities	$q_{c_1}=2$	$q_{c_2}=2$	$q_{c_3}=2$		
ceiling for t	$\overline{q}^t_{c_1}=2$	$\overline{q}^t_{c_2}=2$	$\overline{q}^t_{c_3}=2$		
floor for t	$\underline{q}^t_{c_1}=1$	$\underline{q}^t_{c_2}=0$	$\underline{q}^t_{c_3}=0$		

Next we determine the set of feasible assignments. Feasibility requires that one of the students is assigned school c_1 and each student is assigned a school. Then it is straightforward to verify that

$$\mu_1 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & \emptyset & s_2 \end{pmatrix}, \mu_2 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & \emptyset \end{pmatrix}$$

$$\mu_3 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & \emptyset & s_1 \end{pmatrix}, \mu_4 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & \emptyset \end{pmatrix}, \mu_5 = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_1, s_2\} & \emptyset & \emptyset \end{pmatrix}$$

is the set of all feasible assignments.

It is easy to check that μ_1 and μ_4 are the only feasible assignments which are both fair for same types and constrained non-wasteful for this controlled school choice problem. Note that under P_S,

(i) μ_2 and μ_5 are not constrained non-wasteful since s_2 justifiably claims an empty slot at c_3 under both μ_2 and μ_5 and the resulting assignment μ_1 is fair for same types, and

(ii) μ_3 is not constrained non-wasteful since s_1 justifiably claims an empty slot at c_2 under μ_3 and the resulting assignment μ_4 is fair for same types.

Any feasible mechanism which is both fair for same types and constrained non-wasteful must select either the assignment μ_1 or the assignment μ_4. We will show that in each case there is a student who profitably manipulates the mechanism.

Case 1: The mechanism selects μ_1.

Under μ_1 student s_1 is assigned school c_1. We will show that student s_1 gains by misreporting his true preference. Suppose that student s_1 states the (false) preference P'_{s_1} given by $c_2 P'_{s_1} c_3 P'_{s_1} c_1 P'_{s_1} s_1$, and student s_2 were to report his true preference P_{s_2}. Keeping all other components of the above problem fixed, in the new problem the students' preferences are $P'_S = (P'_{s_1}, P_{s_2})$.

In the new problem under μ_1 student s_1 justifiably envies student s_2 at school c_3 since (f1) $\mu_1(s_1) = c_1$, $c_3 P'_{s_1} c_1$ and $s_1 \succ_{c_3} s_2$, and (f2) $\tau(s_1) = \tau(s_2)$. Note that under P'_S,

(i) μ_1 and μ_2 are not fair for same types, and

(ii) μ_3 and μ_5 are not constrained non-wasteful since s_1 justifiably claims an empty slot at c_2 under both μ_3 and μ_5 and the resulting assignment μ_4 is fair for same types.

Thus, the unique feasible assignment, which is both fair for same types and non-wasteful for the new problem, is μ_4. Hence, any feasible mechanism, which is both fair for same types and constrained non-wasteful, must select the assignment μ_4 for the new problem. Under μ_4 student s_1 is assigned school c_2 which is strictly preferred to c_1 under the true preference P_{s_1}. Thus student s_1 does better by stating P'_{s_1} than by stating his true preference P_{s_1}, and the mechanism is not incentive compatible.

Case 2: The mechanism selects μ_4.

Under μ_4 student s_2 is assigned school c_1. Similarly as in Case 1 we will show that student s_2 gains by misreporting his preference. Suppose that student s_2 states the (false) preference P'_{s_2} given by $c_3 P'_{s_2} c_2 P'_{s_2} c_1 P'_{s_2} s_2$, and student s_1 were to report his true preference P_{s_1}. Keeping all other components of the above problem fixed, in the new problem the students' preferences are $P'_S = (P_{s_1}, P'_{s_2})$.

In the new problem under μ_4 student s_2 justifiably envies student s_1 at school c_2 since (f1) $\mu_4(s_2) = c_1$, $c_2 P'_{s_2} c_1$ and $s_2 \succ_{c_2} s_1$, and (f2) $\tau(s_2) = \tau(s_1)$. Note that under P'_S,

(i) μ_4 is not fair for same types,

(ii) μ_2 and μ_5 are not constrained non-wasteful since s_2 justifiably claims an empty slot at c_3 under both μ_2 and μ_5 and the resulting assignment μ_1 is fair for same types, and

(iii) μ_3 is not constrained non-wasteful since s_1 justifiably claims an empty slot at c_1 under μ_3 and the resulting assignment μ_5 is fair for same types.

The unique feasible assignment, which is both fair for same types and constrained non-wasteful for the new problem, is μ_1. Hence, any feasible mechanism, which is both fair for same types and constrained non-wasteful, must select the assignment μ_1 for the new problem. Under μ_1 student s_2 is assigned school c_3 which is strictly preferred to c_1 under the true preference P_{s_2}. Thus student s_2 does better by stating P'_{s_2} than by stating his true preference P_{s_2}, and the mechanism is not incentive compatible. $\square$

Remark 6 The conclusion of Theorem 5 remains unchanged when constrained non-wastefulness is replaced by non-wastefulness. Of course, by Theorem 1 we know that the set of feasible assignments which are both fair for same types and non-wasteful may be empty. Therefore, it is meaningful to require a mechanism to satisfy simultaneously both fairness for same types and non-wastefulness only if there exist feasible assignments which are both fair for same types and non-wasteful. Hence we say that a feasible mechanism is **fair for same types and non-wasteful** if it selects for any controlled school choice problem a feasible assignment which is fair for same types and non-wasteful whenever such an assignment exists. Now the proof of Theorem 5 remains true when constrained non-wastefulness is replaced by non-wastefulness. *Hence in controlled school choice there is no feasible mechanism which is incentive compatible, fair for same types and non-wasteful.*

Remark 7 The non-existence of feasible mechanisms, which are incentive compatible, fair for same types and (constrained) non-wasteful, unambiguously shows that controlled school choice is not equivalent to college admission. In all models of school choice studied so far it was possible to connect the school choice problem to the college admissions problem and show that DAA is a mechanism which is non-wasteful, fair, and incentive compatible. This was due to the absence of diversity constraints (the floors) which are present in controlled choice.

In college admissions any mechanism, which is incentive compatible for students, chooses for each problem the extreme of the lattice of stable matchings which students prefer over any other stable matching. In controlled school choice there is not always a unique candidate for a feasible assignment which is fair for same types and (constrained) non-wasteful. This provides additional reason for Theorem 5 and Remark 6, i.e. for the non-existence of feasible mechanisms which are incentive compatible, fair for same types and (constrained) non-wasteful.

In the example used to prove Theorem 5 we know that

$$\mu_1 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & \emptyset & s_2 \end{pmatrix} \text{ and } \mu_4 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & \emptyset \end{pmatrix}$$

are the only feasible assignments which are both fair for same types and (constrained) non-wasteful for this problem.

Student s_1 prefers c_2 to c_1 under P_{s_1} and student s_2 prefers c_3 to c_1 under P_{s_2}. Thus s_1 strictly prefers μ_4 to μ_1 under P_{s_1} and s_2 strictly prefers μ_1 to μ_4 under P_{s_2}. Hence students' preferences are opposed over the (only) two feasible assignments, which are both fair for same types and (constrained) non-wasteful, and there is no feasible, fair for same types and (constrained) non-wasteful assignment which both students prefer to any other feasible assignment which is both fair for same types and non-wasteful.

When computing the "minimum" $\wedge$ of μ_1 and μ_4 (by assigning each student to the school which he least prefers from μ_1 and μ_4) we obtain the assignment

$$\mu_1 \wedge \mu_4 = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_1, s_2\} & \emptyset & \emptyset \end{pmatrix}$$

which is feasible but not (constrained) non-wasteful.

Remark 8 Theorem 5 implies that for any order of the students CDAA is not incentive compatible. Due to this fact students may misrepresent their preferences over schools. Now if the students play a Nash equilibrium (NE), what are the properties of the outcome (or the assignment) of any NE? It is easy to see that the outcome of any NE must be constrained non-wasteful.[18] Unfortunately, the outcome of a NE may not be fair for same types according to students' true preferences.

For instance, consider a problem consisting of three students $\{s_1, s_2, s_3\}$ (all of the same type) and three schools $\{c_1, c_2, c_3\}$. Each school has a capacity of one ($q_c = 1$ for all schools c). Any ceiling of any type t is equal to one and any floor of any type t is equal to zero. The students' preferences and the schools' priorities are given below:

$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	P_{s_1}	P_{s_2}	P_{s_3}
s_2	s_1	s_3	c_1	c_2	c_2
s_1	s_3	s_2	c_2	c_1	c_3
s_3	s_2	s_1	c_3	c_3	c_1

Now if student s_3 reports $\bar{P}_{s_3} : c_3 c_1 c_2$, then independently of the order, in which students propose, CDAA chooses for $\bar{P}_S = (P_{s_1}, P_{s_2}, \bar{P}_{s_3})$ the assignment

$$\bar{\mu} = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & s_3 \end{pmatrix}.$$

Obviously $\bar{\mu}$ is not fair for same types for the true profile since student s_3 justifiably envies student s_2 at school c_2 (and all students are of the same type). It is easy to check that $\bar{P}_S$

[18]Otherwise a student would justifiably claim an empty slot and after assigning him this empty slot the resulting assignment is fair for same types. Then this student profits from changing his preference such that he proposes to this school before proposing to the school to which he is assigned to.

is a NE in CDAA and that $\bar{P}_S$ is even robust to deviations of any group of students, i.e. $\bar{P}_S$ is a strong Nash equilibrium. However, this feature is not peculiar to controlled school choice since the above problem is a college admissions problem and we know that in college admissions the outcome of a NE may not be stable according to the true preferences.[19]

Any controlled school choice program must give up constrained non-wastefulness, fairness for same types or incentive compatibility. It will wonder whether an existence result reemerges if we give up exactly one of our two basic requirements, namely constrained non-wastefulness or fairness for same types.

Since in real life often the number of available seats is approximately the same as the number of students, potential justified claims of empty seats occur less frequently than potential justified envy. Hence, a school choice program may be ready to give up constrained non-wastefulness while retaining fairness for same types and incentive compatibility. We will demonstrate that this weakening results in existence.[20]

Example 3. *A feasible mechanism which is both fair for same types and incentive compatible.*

Fix a feasible assignment, say μ. We relate any controlled school choice problem with a college admissions problem in the following way: break any school c into k schools $\{c(t_1), \ldots, c(t_k)\}$ where $|T| = k$ and $c(t)$ is the part of school c wanting to fill slots with students of type t. The capacity of school $c(t)$ is $q_{c(t)} = |\mu^t(c)|$ and the preference of $c(t)$ ranks only students of type t acceptable, in the same order as $\succ_c$. Note that some slots are wasted at school c if $|\mu(c)| < q_c$. Any student replaces on his preference school c by $|T|$ copies of c in the order $c(t_1)$, $c(t_2)$,..., $c(t_k)$. Then determine the student optimal matching of this related problem. Because (i) all students rank all schools as acceptable, (ii) for any type t there are exactly $\sum_{c\in C} q_{c(t)} = \sum_{c\in C} |\mu^t(c)| = |S_t|$ slots available and (iii) any school $c(t)$ ranks acceptable exactly all students of type t, the student optimal matching $\bar{\mu}$ of the related problem satisfies for all types t and all schools c, $\bar{\mu}^t(c(t)) \subseteq S_t$ and $|\bar{\mu}(c(t))| = q_{c(t)} = |\mu^t(c)|$.

Thus the feasibility of μ implies that the student optimal matching of the related problem is a feasible assignment of the controlled school choice problem. We know that DAA is incentive compatible. Furthermore the stability of the student optimal matching in the related problem implies that there is no student envying justifiably another student of the

[19]In school choice problems without control and legal constraints, Ergin and Sönmez (2006) consider revelation games induced by the Boston school choice mechanism and DAA.

[20]Giving up fairness for same types also results in existence. A serial dictatorship (which is used frequently for the allocation of indivisible objects) is a feasible mechanism which is both non-wasteful and incentive compatible. A serial dictatorship orders the set of students alphabetically, say s_1, $s_2, \ldots$, and s_n. Then for any problem, first student s_1 picks the feasible assignments which he most prefers, second student s_2 picks the assignments, which he most prefers, among the remaining feasible assignments and so on. This mechanism is fair only if each school's priority ranking is identical with the alphabetical order of the students.

same type. Thus the "related" mechanism is a feasible mechanism which is both fair for same types and incentive compatible. The mechanism is constrained non-wasteful only if the initial assignment μ filled all available slots at each school. Furthermore the mechanism is fair (across types) only if all students are of the same type. When all students are of the same type, the floor at a school may be represent the number of students which is necessary not to shut down that school.

Observe that the above mechanism is "rigid": in Example 3 for each type t, the slots, which will be filled with type-t students, are exogenously given by the feasible assignment μ. This inflexibility was the price for incentive compatibility of this mechanism. In general this price includes giving up weak Pareto-optimality because due to the inflexibility all students may be strictly better off with another feasible assignment compared to the assignment chosen by the mechanism in Example 3. Note that this inefficiency stems from the rigidity of the mechanism and not necessarily from the waste of empty seats.

6 Recommendation to School Choice Programs

Although there is a large literature in education evaluating and estimating the effects of segregation across schools on students' achievements (Hanushek, Kain, and Rivkin (2002), Guryan (2004), Card and Rothstein (2005), and others)[21], and on how to measure segregation and how to determine optimal desegregation guidelines[22], none of these papers discusses the problem of *how in practice* to assign students to schools while complying with these desegregation guidelines. This is exactly what our paper does.

Without controlled choice and legal constraints, the student proposing deferred acceptance algorithm eliminates any justified envy and makes truthful revelation of preferences a dominant strategy for students (Abdulkadiroğlu and Sönmez, 2003). Once controlled choice constraints are imposed it may be impossible to eliminate any justified envy. The legal constraints allow to eliminate justified envy only among students of the same type (and not of different types). Any state in the United States needs to decide whether controlled choice and legal constraints are more important or whether elimination of any justified envy is more important. In university admissions it is likely that fairness is regarded more important and in those contexts DAA assigns students to schools in a satisfactory manner. In school choice

[21]We will refer the interested reader to Echenique, Fryer, and Kaufman (2006) for an illuminating account of this literature.

[22]School segregation can be purely racial or, as in Echenique, Fryer and Kaufman (2006), school segregation is measured according to the spectral segregation index of Echenique and Fryer (2006) which uses the intensity of social interactions among the members of a group (see also Cutler and Glaeser (1997)).

it is unlikely that controlled choice and legal constraints are ignored completely and envy can be eliminated only among students of the same race.

Fairness across types and fairness for same types are the extreme in allowing envy among different agents: under fairness across types envy is allowed among all agents and under fairness for same types envy is only allowed among same type students. As we showed even allowing envy only among same type students still results in impossibilities when coupled with non-wastefulness whereas allowing envy among all students alone results in impossibilities. This is a particular feature of controlled school choice problems.

Of course, in the absence of floor constraints, then assignments which are fair across types always exist. In such situations, a straightforward modification of DAA by Abdulkadiroğlu (2005) finds an assignment which is fair across types. However, the assignment found is not necessarily feasible because it may not assign all students to schools. This is achieved by CDAA.

Acceptance of controlled choice and legal constraints means for any school choice program to decide whether incentive compatibility is more important or whether weak Pareto-optimality or/and constrained non-wastefulness is more important. If the program insists on incentive compatibility, the incentive compatible mechanism we propose basically segregates the problem into several problems, one for each race, and applies DAA to each problem separately. Making truth telling a dominant strategy brings many serious flaws with it (even though the assignment is based on true preferences). Any incentive compatible mechanism may implement assignments which are highly inefficient and highly wasteful (of empty seats). Due to these flaws many school choice programs may prefer a mechanism which is weakly Pareto-optimal and constrained non-wasteful: this is achieved in practice by CDAA. Furthermore, in low information environments this mechanism is immune to manipulation. Similar observations have been made by Erdil and Ergin (2008) in (uncontrolled) school choice with indifferences: there exists no incentive compatible and fair mechanism which is constrained efficient; applying DAA may yield assignments which are Pareto-dominated by other fair assignments whereas the stable improvement cycles algorithm is constrained efficient and fair but not incentive compatible. In a low information environment the random version of this mechanism is immune to manipulation.

Controlled choice comes with a price. Any program has to disregard at least one desirable property when following the state's laws or the policies of school boards. More generally, third party policies may override students' preferences and schools' priorities, and "less" segregated assignments may be more preferred among the feasible ones. For instance, a student body of 50% black-50% white may be strictly preferred by a school board to a student body of 40%black-60% white (even if both student bodies comply with the laws).

It is obvious that all our incompatibilities apply to this generalization as well. Our results raise the question whether some state laws or policies need to be modified.

APPENDIX A: PERCENTAGE CONSTRAINTS

Below we generalize our results to school choice with percentage constraints. For all $c \in C$ and all $t \in T$, let $0 \leq \underline{q}_c^t \leq \overline{q}_c^t \leq 1$. Recall that if the constrained school choice constraints are given in percentage terms, then an assignment μ respects percentage constraints if for every school c and every type t we have

$$\underline{q}_c^t |\mu(c)| \leq |\mu^t(c)| \leq \overline{q}_c^t |\mu(c)|.$$

Again we suppose that the set of assignments respecting percentage constraints is non-empty.

Throughout Appendix A we say that that **student s justifiably claims an empty slot at school c under the feasible assignment μ** if

(nw1) $cP_s\mu(s)$ and $|\mu(c)| < q_c$,

(nw2) $\underline{q}_{\mu(s)}^{\tau(s)}(|\mu(\mu(s))| - 1) \leq |\mu^{\tau(s)}(\mu(s))| - 1 \leq q_{\mu(s)}^{\tau(s)}(|\mu(\mu(s))| - 1)$, and

(nw3) $\underline{q}_c^{\tau(s)}(|\mu(c)| + 1) \leq |\mu^{\tau(s)}(c)| + 1 \leq \overline{q}_c^{\tau(s)}(|\mu(c)| + 1)$.

Hence, under (nw1)-(nw3) student s can be assigned an empty slot at the better school c without changing the assignments of the other students and violating the percentage terms at any school. A feasible assignment μ is **non-wasteful** if no student justifiably claims an empty slot at any school.

THEOREM 1$^{\%}$: The set of feasible assignments which are both fair for same types and non-wasteful may be empty in a controlled school choice problem with percentage constraints.

Proof: The proof is by means of modifying the example in the proof of Theorem 1. Consider the following problem consisting of three schools $\{c_1, c_2, c_3\}$ and 31 students $\{s_1, \ldots, s_{31}\}$. Each school has a capacity of 11 ($q_c = 11$ for all schools c). There are two types t and t' and 13 students are of type t, say $S_t = \{s_1, \ldots, s_{13}\}$ and 18 students of type t', say $S_{t'} = \{s_{14}, \ldots, s_{31}\}$. The ceiling of each type is equal to 0.6 at all schools ($\overline{q}_c^t = 0.6 = \overline{q}_c^{t'}$ for all schools c) and the floor of each type is equal to 0.4 at all schools ($\underline{q}_c^t = 0.4 = \underline{q}_c^{t'}$ for all schools c).

We make the following observation. Suppose that the assignment μ respects percentage constraints. Then we must have for all schools c, $|\mu^{t'}(c)| = 6$: if not, then by the fact that there are 18 type-t' students and three schools, for some school c we have $|\mu^{t'}(c)| \geq 7$; since $|\mu(c)| \leq 11$, we must have $\frac{|\mu^{t'}(c)|}{|\mu(c)|} \geq \frac{7}{11} > 0.6 = \overline{q}_c^{t'}$, and μ does not respect percentage constraints, a contradiction.

Now for any feasible μ, we have $|\mu^{t'}(c)| = 6$ for all schools c and hence

$$|\mu^t(c)| \geq 4 \text{ for all schools } c. \tag{2}$$

Now suppose that students s_3, s_4, s_5 all strictly prefer school c_1 to both c_2 and c_3 and all have higher priority than students s_1 and s_2 at c_1; students s_6, s_7, s_8, s_9 all strictly prefer school c_2 to both c_1 and c_3 and all have higher priority than students s_1 and s_2 at c_2; and students $s_{10}, s_{11}, s_{12}, s_{13}$ all strictly prefer school c_3 to both c_2 and c_3 and all have higher priority than students s_1 and s_2 at c_3. Then it is immediate from (2) that for any feasible μ, if μ is fair for same types, then $\{s_3, s_4, s_5\} \subseteq \mu^t(c_1)$, $\{s_6, s_7, s_8, s_9\} \subseteq \mu^t(c_2)$, and $\{s_{10}, s_{11}, s_{12}, s_{13}\} \subseteq \mu^t(c_3)$.

Now by (2), $|\mu^t(c_1)| \geq 4$. Thus, the above implies that s_1 or s_2 must be assigned to c_1. Now if P_{s_1} and P_{s_2} and schools' priorities between s_1 and s_2 are as in the example in the proof of Theorem 1, this problem becomes identical with the proof of Theorem 1. Hence, there is no feasible assignment which is both fair for same types and non-wasteful in a controlled school choice problem with percentage constraints. $\square$

Note that in the example above, the capacities of 11 facilitate the proof but they are not essential. Even if the capacities are set equal to the total number of students we obtain the incompatibility result (possibly by assuring that the 18 type-t' students must be distributed evenly among the three schools in any feasible assignment which is fair for same types and non-wasteful).

Furthermore, unidimensional types is a special case of multi-dimensional types and for controlled school choice with multi-dimensional types the impossibilities of Theorem 1 and Theorem $1^{\%}$ continue to remain true.

It is straightforward to check that the proofs of Theorem 2, Theorem 3 and Theorem 4 carry over unchanged to controlled school choice with percentage constraints (or with multi-dimensional types). Hence, CDAA finds for any problem with percentage constraints a feasible assignment which is both fair for same types and constrained non-wasteful which is weakly Pareto-optimal.

Similarly as in Theorem $1^{\%}$ we adjust the example in the proof of Theorem 5 to controlled school choice with percentage constraints.

THEOREM $5^{\%}$: In controlled school choice with percentage constraints there is no feasible mechanism which is incentive compatible, fair for same types and constrained non-wasteful.

Proof: Consider the same problem as in the proof of Theorem $1^{\%}$. Then for any feasible μ, we have $|\mu^{t'}(c)| = 6$ for all schools c, and, if μ is fair for same types, then $\{s_3, s_4, s_5\} \subseteq \mu^t(c_1)$, $\{s_6, s_7, s_8, s_9\} \subseteq \mu^t(c_2)$, and $\{s_{10}, s_{11}, s_{12}, s_{13}\} \subseteq \mu^t(c_3)$.

Again, by $|\mu^t(c_1)| \geq 4$, the above implies that s_1 or s_2 must be assigned to c_1. Now if P_{s_1} and P_{s_2} and schools' priorities between s_1 and s_2 are as in the proof of Theorem 5, this problem and Cases 1 and 2 become identical with the proof of Theorem 5. Hence, in controlled school choice with percentage constraints there is no feasible mechanism which is incentive compatible, fair for same types and constrained non-wasteful. $\square$

APPENDIX B: FEASIBILITY CHECKING ALGORITHM

Below we provide an algorithm to check in (i) of any step in CDAA whether the proposing student can be assigned an empty slot at the school she proposed to or not.

We will use the following terminology: given $\mu \in \mathcal{F}$ and the tentative assignments ν, we say μ makes ν feasible if for all $t \in T$ and all $c \in C$, $|\nu^t(c)| \leq |\mu^t(c)|$. Obviously, in order to check whether there exists some $\mu' \in \mathcal{F}$ such that $\mu'(s) = \nu(s)$ for all s with $\nu(s) \neq s$ it suffices to find $\mu \in \mathcal{F}$ making ν feasible.

Recall that $\mathcal{F} \neq \emptyset$. Fix $\mu_0 \in \mathcal{F}$. Note that $\nu_0(s) = s$ for all $s \in S$ and μ_0 makes ν_0 feasible. By induction we will simultaneously determine whether the proposing student can be assigned an empty slot at the school she proposed to or not and construct for any Step $k+1$ a matching $\mu_{k+1} \in \mathcal{F}$ making the tentative assignments of ν_{k+1} feasible.

The following notation will be useful. Let $\mu \in \mathcal{F}$. For any sequence of distinct students $s_1, \ldots, s_l \in S$ and $c \in C$, let $(\mu)^{s_1 \to s_2 \to \cdots \to s_l \to c}$ denote the assignment μ' obtained from μ by assigning s_l an empty slot at c and for $h = 1, \ldots, l-1$, s_h takes up the slot left empty by s_{h+1}, i.e. $\mu'(s_l) = c$, and for $h = 1, \ldots, l-1$, $\mu'(s_h) = \mu(s_{h+1})$ (and for all $s \in S \backslash \{s_1, \ldots, s_l\}$, $\mu'(s) = \mu(s)$). Furthermore, for any sequence of distinct students $s_1, \ldots, s_l \in S$, let $(\mu)^{\circlearrowright(s_1, \ldots, s_l)}$ denote the assignment μ'' obtained from μ by a cyclical exchange of the slots of the students $s_1, \ldots, s_l$, i.e. for all $h = 1, \ldots, l-1$, $\mu''(s_h) = \mu(s_{h+1})$ and $\mu''(s_l) = \mu(s_1)$ (and for all $s \in S \backslash \{s_1, \ldots, s_l\}$, $\mu''(s) = \mu(s)$).

Consider any Step $k+1$ ($k \geq 0$) of CDAA. Then the previous tentative assignments are given by ν_k and μ_k makes ν_k feasible (where $\mu_k \in \mathcal{F}$). In Step $k+1$, let student $\hat{s}$ apply to $\hat{c}$ and $\tau(\hat{s}) = \hat{t}$. Let $\nu_{k+1}(\hat{s}) = \hat{c}$ and $\nu_{k+1}(s) = \nu_k(s)$ for all $s \in S \backslash \{\hat{s}\}$.

The following lemma will be instrumental for our feasibility checking algorithm. It shows that in order to check whether ν_{k+1} is feasible it suffices to consider starting from μ_k cyclical exchanges of the form $\circlearrowright (s_1, \ldots, s_l)$ or chains of the form $s_1 \to s_2 \to \cdots \to s_l \to c$ with all students having distinct types.

Lemma 9 *Let μ_k make ν_k feasible and μ_k not make ν_{k+1} feasible. If there exists some $\mu \in \mathcal{F}$ making ν_{k+1} feasible, then there exists a sequence of students $s_1, \ldots, s_l \in S$ having all* *distinct types such that either* $(\mu_k)^{s_1 \to s_2 \to \cdots \to s_l \to c} \in \mathcal{F}$ *(for some school* c*) makes* ν_{k+1} *feasible or* $(\mu_k)^{\circlearrowleft(s_1,\ldots,s_l)} \in \mathcal{F}$ *makes* ν_{k+1} *feasible.*

Proof: The proof is by inductive construction. Starting from μ and μ_k consider the following graph: the set of nodes is equal to all types t such that $|\mu^t(c)| \neq |\mu_k^t(c)|$ for at least one school c and the set of schools such that $|\mu^t(c)| \neq |\mu_k^t(c)|$ for at least one type t; for any type t and any school c, let an arrow point from t to c if $|\mu^t(c)| < |\mu_k^t(c)|$, and let an arrow point from c to t if $|\mu^t(c)| > |\mu_k^t(c)|$. Now the resulting graph may possess a cycle or not. If the resulting graph possesses a cycle, then this cycle must consist of at least three arrows and by construction, any arrow either points from a type to a school or from a school to a type. Choose a cycle, say $t_1 \to c_1 \to t_2 \to c_2 \to \cdots \to c_l \to t_1$, which does not contain any subcycle. Then all the types $t_1, \ldots, t_l$ are distinct and we may choose distinct students $s_1, \ldots, s_l$ such that for $i = 1, \ldots, l$, $\tau(s_i) = t_i$ and $\mu_k(s_i) = c_i$. If an arrow pointing from $\hat{t}$ to $\hat{c}$ is part of the cycle, then $(\mu_k)^{\circlearrowleft(s_l,s_{l-1},\ldots,s_1)} \in \mathcal{F}$ makes ν_{k+1} feasible, the desired conclusion.

Otherwise, let any cycle not contain an arrow pointing from $\hat{t}$ to $\hat{c}$. Then as above, choose a cycle, say $t_1 \to c_1 \to t_2 \to c_2 \to \cdots \to c_l \to t_1$, which does not contain any subcycle. Then all the types $t_1, \ldots, t_l$ are distinct and we may choose distinct students $s_1, \ldots, s_l$ such that for $i = 1, \ldots, l$, $\tau(s_i) = t_i$ and $\mu(s_i) = c_{i-1}$ (with the convention $c_0 = c_l$). Now we set $\mu' = (\mu)^{\circlearrowleft(s_1,\ldots,s_l)}$. Note that by construction, μ' makes ν_{k+1} feasible. Now we repeat the same graph construction for μ' and μ_k. Note that for any type t and any school c, we have either $|\mu^t(c)| \leq |(\mu')^t(c)| \leq |\mu_k^t(c)|$ or $|\mu^t(c)| \geq |(\mu')^t(c)| \geq |\mu_k^t(c)|$. Hence, both the set of nodes and set of arrows shrink and any cycle in the graph resulting from μ' and μ_k was also a cycle in the graph resulting from μ and μ_k. We repeat the above construction until we arrive at an assignment $\hat{\mu}$ such that the graph resulting from $\hat{\mu}$ and μ does not possess any cycle.

Since any cycle did not contain an arrow pointing from $\hat{t}$ to $\hat{c}$, $\hat{\mu}$ makes ν_{k+1} feasible (and μ_k does not make ν_{k+1} feasible). Thus, $|\mu_k^{\hat{t}}(\hat{c})| = |\nu_k^{\hat{t}}(\hat{c})| < |\hat{\mu}^{\hat{t}}(\hat{c})| \leq \overline{q}_{\hat{c}}^{\hat{t}}$. Now there exists $c \in C \backslash \{\hat{c}\}$ and $s \in S_{\hat{t}}$ such that both $s \in \mu_k(c)$ and $|\mu_k^{\hat{t}}(c)| > |\hat{\mu}^{\hat{t}}(c)|$. If $|\mu_k(\hat{c})| < q_{\hat{c}}$, then $(\mu_k)^{s \to \hat{c}} \in \mathcal{F}$ and $(\mu_k)^{s \to \hat{c}}$ makes ν_{k+1} feasible, the desired conclusion.

If $|\mu_k(\hat{c})| = q_{\hat{c}}$, then there exists $t_1 \in T \backslash \{\hat{t}\}$ such that $|\hat{\mu}^{t_1}(\hat{c})| > |\mu_k^{t_1}(\hat{c})|$ and in the graph resulting from $\hat{\mu}$ and μ_k there is an arrow pointing from $\hat{c}$ to t_1. Choose $s_1 \in \hat{\mu}^{t_1}(\hat{c})$. Since $|\hat{\mu}^{t_1}(\hat{c})| > |\mu_k^{t_1}(\hat{c})|$, there exists $c_1 \in C \backslash \{\hat{c}\}$ such that $|\hat{\mu}^{t_1}(c_1)| < |\mu_k^{t_1}(c_1)|$ and there is an arrow pointing from t_1 to c_1. Since the graph resulting from $\hat{\mu}$ and μ does not possess any cycle, we cannot have $\hat{\mu}(s) = c_1$.

Now if $|\mu_k(c_1)| < q_{c_1}$, then $(\mu_k)^{s \to s_1 \to c_1} \in \mathcal{F}$ and $(\mu_k)^{s \to s_1 \to c_1}$ makes ν_{k+1} feasible, the desired conclusion. Otherwise, let $|\mu_k(c_1)| = q_{c_1}$.

In general, let $s_1, \ldots, s_l$ and $t_1, \ldots, t_l$ and $c_1, \ldots, c_l$ be chosen as above, i.e. in the graph resulting from $\hat{\mu}$ and μ_k, we have $t_1 \to c_1 \to t_2 \to \cdots \to t_l \to c_l$, all the types $t_1, \ldots, t_l$ are distinct, and both $\tau(s_i) = t_i$ and $s_i \in \hat{\mu}^{t_i}(c_{i-1})$ for $i = 2, \ldots, l$.

Now if $|\mu_k(c_l)| < q_{c_l}$, then $(\mu_k)^{s \to s_1 \to \cdots \to s_l \to c_l}$ makes ν_{k+1} feasible, the desired conclusion.

If $|\mu_k(c_l)| = q_{c_l}$, then by the fact that the graph resulting from $\hat{\mu}$ and μ_k does not contain any cycle, there exists $t_{l+1} \in T \backslash \{t_1, \ldots, t_l, \hat{t}\}$ and in the graph resulting from $\hat{\mu}$ and μ_k there is an arrow pointing from c_l to t_{l+1}. Choose $s_{l+1} \in \hat{\mu}^{t_{l+1}}(c_l)$. Since both $\hat{\mu}$ and μ are feasible, there exists c_{l+1} such that such that $|\hat{\mu}^{t_{l+1}}(c_{l+1})| < |\mu_k^{t_{l+1}}(c_{l+1})|$ and in the graph resulting from $\hat{\mu}$ and μ there is an arrow pointing from t_{l+1} to c_{l+1}. Since the graph does not contain any cycle, we must have $c_{l+1} \in C \backslash \{c_1, \ldots, c_l, \hat{c}\}$. Now we have chosen $s_1, \ldots, s_l, s_{l+1}$ and $t_1, \ldots, t_l, t_{l+1}$ and $c_1, \ldots, c_l, c_{l+1}$ and we continue as above.

Since both the set of types and the set of schools is finite and the graph resulting from $\hat{\mu}$ and μ_k does not possess any cycle, the above process must terminate. This yields the desired conclusion. $\square$

Before we formulate our algorithm, we introduce some more notation. For any school c, let

$$T(c) = \{t \in T \backslash \{\hat{t}\} : \max\{\underline{q}_c^t, |\nu_k^t(c)|\} < |\mu_k^t(c)|\}$$

denote the set of types (except for $\hat{t}$) which can possibly liberate a slot at c without violating constraints and the tentative assignments made by ν_k. For any type t, let $Y(t) = \{c \in C \backslash \{\hat{c}\} : |\mu_k^t(c)| < \overline{q}_c^t\}$ denote the set of schools (except for $\hat{c}$) which can accept one more type-t student without violating constraints, and $E(t) = \{c \in Y(t) : |\mu_k(c)| < q_c\}$ denote the set of schools (except for $\hat{c}$) where a type-t student can possibly fill an empty slot without violating constraints.

Feasibility Checking Algorithm (FCA)

START: At Step k of CDAA, let $\mu_k \in \mathcal{F}$ make ν_k feasible. At Step $k+1$ let student $\hat{s}$ propose to $\hat{c}$ and $\tau(\hat{s}) = \hat{t}$. Let $\nu_{k+1}(\hat{s}) = \hat{c}$ and $\nu_{k+1}(s) = \nu_k(s)$ for all $s \in S \backslash \{\hat{s}\}$. We check whether (or not) there exists some $\mu \in \mathcal{F}$ making ν_{k+1} feasible.

If $|\nu_k^{\hat{t}}(\hat{c})| < |\mu_k^{\hat{t}}(\hat{c})|$, then μ_k makes ν_{k+1} feasible and we set $\mu_{k+1} = \mu_k$ (Stop). Otherwise, let $|\nu_k^{\hat{t}}(\hat{c})| = |\mu_k^{\hat{t}}(\hat{c})|$.

STEP 1: Let $C(\hat{t} \to \hat{c}) = \{c \in C : \max\{\underline{q}_c^{\hat{t}}, |\nu_k^{\hat{t}}(c)|\} < |\mu_k^{\hat{t}}(c)|\}$ denote the set of schools from where type-$\hat{t}$ students can be moved to $\hat{c}$. If $C(\hat{t} \to \hat{c}) = \emptyset$, then ν_{k+1} is not feasible and we set $\mu_{k+1} = \mu_k$ (Stop). Otherwise, let $C(\hat{t} \to \hat{c}) \neq \emptyset$ and set $C_1 = \{\hat{c}\}$ and $T_1 = \emptyset$; goto STEP 2.

STEP $l+1$: Set $T_{l+1} = \cup_{c \in C_l} T(c)$. If $T_{l+1} \subseteq \cup_{h=1}^{l} T_h$, then ν_{k+1} is not feasible and we set $\mu_{k+1} = \mu_k$ (Stop). Otherwise, we check whether for some $t \in T_{l+1}$, $E(t)$ is empty or not. If for some $t \in T_{l+1}$, $E(t)$ is non-empty, then ν_{k+1} is feasible (via some chain): choose $t_{l+1} \in T_{l+1}$ and c' such that $c' \in E(t_{l+1})$, for $i = l, l-1, \ldots, 2$, choose $t_i \in T_i \setminus \cup_{h=1}^{i-1} T_h$ such that $t_{i+1} \in T(c_i)$ for some $c_i \in Y(t_i)$; then choose for $i = 2, \ldots, l+1$, $s_i \in \mu_k^{t_{i+1}}(c_i)$, and $s_1 \in \mu_k^{\hat{t}}(c)$ for some $c \in C(\hat{t} \to \hat{c})$; now $(\mu_k)^{s_1 \to s_2 \to \cdots \to s_{l+1} \to c'} \in \mathcal{F}$ makes ν_{k+1} feasible and we set $\mu_{k+1} = (\mu_k)^{s_1 \to s_2 \to \cdots \to s_{l+1} \to c'}$ (Stop).

Otherwise, we check whether for some $t \in T_{l+1}$, $Y(t) \cap C(\hat{t} \to \hat{c})$ is non-empty or not. If for some $t \in T_{l+1}$, $Y(t) \cap C(\hat{t} \to \hat{c})$ is non-empty, then ν_{k+1} is feasible (via some cyclical exchange): choose $t_{l+1} \in T_{l+1}$ and c' such that $c' \in Y(t_{l+1}) \cap C(\hat{t} \to \hat{c})$, for $i = l, l-1, \ldots, 2$, choose $t_i \in T_i \setminus \cup_{h=1}^{i-1} T_h$ such that $t_{i+1} \in T(c_i)$ for some $c_i \in Y(t_i)$; then choose for $i = 2, \ldots, l+1$, $s_i \in \mu_k^{t_{i+1}}(c_i)$, and $s_1 \in \mu_k^{\hat{t}}(c')$; now $(\mu_k)^{\circlearrowright(s_1, \ldots, s_l, s_{l+1})} \in \mathcal{F}$ makes ν_{k+1} feasible and we set $\mu_{k+1} = (\mu_k)^{\circlearrowright(s_1, \ldots, s_l, s_{l+1})}$ (Stop). Otherwise, we set $C_{l+1} = \cup_{t \in T_{l+1}} Y(t)$ and goto Step $l+2$.

Note that the FCA terminates in a finite number of steps, because if not, then for any Step $l+1$ we would have $T_{l+1} \nsubseteq \cup_{h=1}^{l} T_h$, which would contradict the finiteness of T. Furthermore, by Lemma 9, if there exists some $\mu \in \mathcal{F}$ making ν_{k+1} feasible, then there exists a sequence of students $s_1, \ldots, s_l \in S$ having all distinct types, say $\tau(s_i) = t_i$ for all $i = 1, \ldots, l$, such that either $(\mu_k)^{s_1 \to s_2 \to \cdots \to s_l \to c}$ (for some school c) makes ν_{k+1} feasible or $(\mu_k)^{\circlearrowright(s_1, \ldots, s_l)}$ makes ν_{k+1} feasible. But then we must have for some $i = 1, \ldots, l$, $t_i = \hat{t}$. Without loss of generality, let $t_1 = \hat{t}$. But then $t_i \in T_i$ for all $i = 1, \ldots, l$ (where T_i is determined in Step i of FCA) and at Step l we have either $E(t_l) \neq \emptyset$ if $(\mu_k)^{s_1 \to s_2 \to \cdots \to s_l \to c}$ (for some school $c \in E(t_l)$) makes ν_{k+1} feasible or $Y(t_l) \cap C(\hat{t} \to \hat{c}) \neq \emptyset$ if $(\mu_k)^{\circlearrowright(s_1, \ldots, s_l)}$ makes ν_{k+1} feasible. Hence, FCA determines whether $\hat{s}$ can justifiably claim an empty slot in (i) of Step k of CDAA.

APPENDIX C: FAIRNESS ACROSS TYPES

We formulate fairness across types precisely below and show that Theorem 1 and Theorem 5 remain unchanged when fairness across types replaces fairness for same types and (constrained) non-wastefulness. Even though these theorems are replicas of their counterparts in the main text, they do not follow from these theorems in the main text: fairness across types does not imply (constrained) non-wastefulness; and fairness for same types together with (constrained) non-wastefulness does not imply fairness across types.

We say that **student s justifiably envies student s' at school c under the feasible assignment μ** if there exists another feasible assignment μ' such that

(f1) $\mu(s') = c$, $cP_s\mu(s)$ and $s \succ_c s'$,

(f2) $\mu'(s) = c$, $\mu'(s') \neq c$, and $\mu'(\hat{s}) = \mu(\hat{s})$ for all $\hat{s} \in S \setminus \{s, s'\}$.

Because μ' is feasible, (f2) simply says that $(\mu(c)\backslash\{s'\}) \cup \{s\}$ respects the controlled choice constraints at school c and student s' can be enrolled at school $c' = \mu'(s')$ such that $(\mu(c')\backslash\{s\}) \cup \{s'\}$ respects the controlled choice constraints at c'; in other words assigning s a slot at c, s' a slot at c', and keeping all the other assignments intact does not violate any controlled choice constraint at any school.

A feasible assignment μ is **fair across types** if no student justifiably envies any student. Independently of his own type, a student is allowed to envy any student.

Since fairness for same types is a weaker requirement than fairness across types, Theorem 1 also shows that there may not exist any feasible assignment which is both fair across types and non-wasteful in a controlled school choice problem. Unfortunately fairness across types alone may be enough for this non-existence result. This can be seen by modifying the example, which is used in the proof of Theorem 1, by introducing a third (dummy) student, who is ranked at the bottom of all priority rankings and whose type is different than t. All the proofs are omitted in Appendix C and they are available in the expanded form (Appendix D) of it.

THEOREM 1': The set of feasible assignments which are fair across types may be empty in a controlled school choice problem.

Similarly as before, a feasible mechanism is **fair across types** if it selects an assignment which is both feasible and fair across types for any controlled school choice problems having a non-empty set of feasible assignments which are fair across types. Parallel to Theorem 5 and Remark 6, fairness across types is incompatible with feasibility and incentive compatibility.

THEOREM 5': In controlled school choice there is no feasible mechanism which is both incentive compatible and fair across types.

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APPENDIX D: FAIRNESS ACROSS TYPES
(expanded form of APPENDIX C)

We formulate fairness across types precisely below and show that Theorem 1 and Theorem 5 remain unchanged when fairness across types replaces fairness for same types and (constrained) non-wastefulness. Even though these theorems are replicas of their counterparts in the main text, they do not follow from these theorems in the main text: fairness across types does not imply (constrained) non-wastefulness; and fairness for same types together with (constrained) non-wastefulness does not imply fairness across types.

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(f1) $\mu(s') = c$, $cP_s\mu(s)$ and $s \succ_c s'$,

(f2) $\mu'(s) = c$, $\mu'(s') \neq c$, and $\mu'(\hat{s}) = \mu(\hat{s})$ for all $\hat{s} \in S \backslash \{s, s'\}$.

Because μ' is feasible, (f2) simply says that $(\mu(c)\backslash\{s'\}) \cup \{s\}$ respects the controlled choice constraints at school c and student s' can be enrolled at school $c' = \mu'(s')$ such that $(\mu(c')\backslash\{s\}) \cup \{s'\}$ respects the controlled choice constraints at c'; in other words assigning s a slot at c, s' a slot at c', and keeping all the other assignments intact does not violate any controlled choice constraint at any school.

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Since fairness for same types is a weaker requirement than fairness across types, Theorem 1 also shows that there may not exist any feasible assignment which is both fair across types and non-wasteful in a controlled school choice problem. Unfortunately fairness across types alone may be enough for this non-existence result. This can be seen by modifying the example, which is used in the proof of Theorem 1, by introducing a third (dummy) student, who is ranked at the bottom of all priority rankings and whose type is different than t.

THEOREM 1': The set of feasible assignments which are fair across types may be empty in a controlled school choice problem.

Proof: The proof is by means of an example. The basic idea is similar to the one used in proving Theorem 1. Consider the following problem consisting of three schools $\{c_1, c_2, c_3\}$ and three students $\{s_1, s_2, s_3\}$. Each school has a capacity of one ($q_c = 1$ for all schools c). The type space consists of two types t_1 and t_2. Students s_1 and s_2 are of type t_1 whereas student s_3 is of type t_2. For all types the ceiling is equal to one at all schools ($\overline{q}_c^t = 1$ for all types t and all schools c). School c_1 has a floor for type t_1 of $\underline{q}_{c_1}^{t_1} = 1$. All other floors are equal to zero. The schools' priorities are given by $s_2 \succ_{c_1} s_1 \succ_{c_1} s_3$, $s_2 \succ_{c_2} s_1 \succ_{c_2} s_3$ and $s_1 \succ_{c_3} s_2 \succ_{c_3} s_3$. The students' preferences are given by $c_2 P_{s_1} c_3 P_{s_1} c_1 P_{s_1} s_1$, $c_3 P_{s_2} c_2 P_{s_2} c_1 P_{s_2} s_2$ and $c_2 P_{s_3} c_3 P_{s_3} c_1 P_{s_3} s_3$. This information is summarized in Table 3.

TABLE 3.

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	P_{s_1}	P_{s_2}	P_{s_3}
	s_2	s_2	s_1	c_2	c_3	c_2
	s_1	s_1	s_2	c_3	c_2	c_3
	s_3	s_3	s_3	c_1	c_1	c_1
				s_1	s_2	s_3
capacities	$q_{c_1} = 1$	$q_{c_2} = 1$	$q_{c_3} = 1$			
ceiling for t_1	$\overline{q}_{c_1}^{t_1} = 1$	$\overline{q}_{c_2}^{t_1} = 1$	$\overline{q}_{c_3}^{t_1} = 1$			
floor for t_1	$\underline{q}_{c_1}^{t_1} = 1$	$\underline{q}_{c_2}^{t_1} = 0$	$\underline{q}_{c_3}^{t_1} = 0$			
ceiling for t_2	$\overline{q}_{c_1}^{t_2} = 1$	$\overline{q}_{c_2}^{t_2} = 1$	$\overline{q}_{c_3}^{t_2} = 1$			
floor for t_2	$\underline{q}_{c_1}^{t_2} = 0$	$\underline{q}_{c_2}^{t_2} = 0$	$\underline{q}_{c_3}^{t_2} = 0$			

Next we determine the set of assignments which are both feasible and fair across types for this problem. Feasibility requires that student s_1 or student s_2 is assigned school c_1 and all students are enrolled at a school. Therefore,

$$\begin{tikzcd}
\mu_1 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & s_3 \end{pmatrix} \arrow[r, "s_2 \text{ envies } s_3"] & \mu_2 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_3 & s_2 \end{pmatrix}, \arrow[d, "s_1 \text{ envies } s_2"] \\
\mu_4 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & s_3 \end{pmatrix} \arrow[u, "s_2 \text{ envies } s_1"] & \mu_3 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_3 & s_1 \end{pmatrix} \arrow[l, "s_1 \text{ envies } s_3"]
\end{tikzcd}$$

are the only assignments which are feasible. Now (as indicated above)

(i) μ_1 is not fair across types because s_2 justifiably envies s_3 at c_3,

(ii) μ_2 is not fair across types because s_1 justifiably envies s_2 at c_3,

(iii) μ_3 is not fair across types because s_1 justifiably envies s_3 at c_2, and

(iv) μ_4 is not fair across types because s_2 justifiably envies s_1 at c_2.

Hence there is no assignment which is both feasible and fair across types. □

Similarly as before, a feasible mechanism is **fair across types** if it selects an assignment which is both feasible and fair across types for any controlled school choice problems having a non-empty set of feasible assignments which are fair across types. Parallel to Theorem 5 and Remark 6, fairness across types is incompatible with feasibility and incentive compatibility.

THEOREM 5': In controlled school choice there is no feasible mechanism which is both incentive compatible and fair across types.

Proof: The proof is by means of modifying the example in the proof of Theorem 4. Consider the following problem consisting of three schools $\{c_1, c_2, c_3\}$ and three students $\{s_1, s_2, s_3\}$. Each school has a capacity of one ($q_c = 1$ for all schools c). The type space consists of two types t_1 and t_2. Students s_1 and s_2 are of type t_1 whereas student s_3 is of type t_2. For all types the ceiling is equal to one at all schools ($\overline{q}_c^t = 1$ for all types t and all schools c). School c_1 has a floor for type t_1 of $\underline{q}_{c_1}^{t_1} = 1$. All other floors are equal to zero. The schools' priorities are given by $s_2 \succ_{c_1} s_1 \succ_{c_1} s_3$, $s_2 \succ_{c_2} s_1 \succ_{c_2} s_3$ and $s_1 \succ_{c_3} s_2 \succ_{c_3} s_3$. The students' preferences are given by $c_2 P_{s_1} c_1 P_{s_1} c_3 P_{s_1} s_1$, $c_3 P_{s_2} c_1 P_{s_2} c_2 P_{s_2} s_2$ and $c_2 P_{s_3} c_3 P_{s_3} c_1 P_{s_3} s_3$. This information is summarized in Table 4.

TABLE 4.

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	P_{s_1}	P_{s_2}	P_{s_3}
	s_2	s_2	s_1	c_2	c_3	c_2
	s_1	s_1	s_2	c_1	c_1	c_3
	s_3	s_3	s_3	c_3	c_2	c_1
				s_1	s_2	s_3
capacities	$q_{c_1} = 1$	$q_{c_2} = 1$	$q_{c_3} = 1$			
ceiling for t_1	$\overline{q}_{c_1}^{t_1} = 1$	$\overline{q}_{c_2}^{t_1} = 1$	$\overline{q}_{c_3}^{t_1} = 1$			
floor for t_1	$\underline{q}_{c_1}^{t_1} = 1$	$\underline{q}_{c_2}^{t_1} = 0$	$\underline{q}_{c_3}^{t_1} = 0$			
ceiling for t_2	$\overline{q}_{c_1}^{t_2} = 1$	$\overline{q}_{c_2}^{t_2} = 1$	$\overline{q}_{c_3}^{t_2} = 1$			
floor for t_2	$\underline{q}_{c_1}^{t_2} = 0$	$\underline{q}_{c_2}^{t_2} = 0$	$\underline{q}_{c_3}^{t_2} = 0$			

Next we determine the set of assignments which are both feasible and fair across types for this problem. Feasibility requires that student s_1 or student s_2 is assigned school c_1 and all students are enrolled at a school. If student s_1 is assigned school c_1, then s_2 needs to be assigned school c_3 since otherwise s_3 is assigned school c_3, s_2 school c_2, and s_2 justifiably envies s_3 at c_3. Similarly, if student s_2 is assigned school c_1, then s_1 needs to be assigned school c_2 since otherwise s_3 is assigned school c_2, s_1 school c_3, and s_1 justifiably envies s_3 at

c_2. Now it is straightforward to verify that

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_3 & s_2 \end{pmatrix} \text{ and } \bar{\mu} = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & s_3 \end{pmatrix}$$

are the only assignments which are both feasible and fair across types for this problem.

Any mechanism which is both feasible and fair across types must select either the assignment μ or the assignment $\bar{\mu}$. We will show that in each case there is a student who profitably manipulates the mechanism.

Case 1: The mechanism selects μ.

Under μ student s_1 is assigned school c_1. We will show that student s_1 gains by misreporting his true preference. Suppose that student s_1 states the (false) preference P'_{s_1} given by $c_2 P'_{s_1} c_3 P'_{s_1} c_1 P'_{s_1} s_1$, and all other students were to state their true preferences. Keeping all other components of the above problem fixed, in the new problem the students' preferences are $P'_S = (P'_{s_1}, P_{s_2}, P_{s_3})$.

In the new problem under μ student s_1 justifiably envies student s_2 at school c_3 through the feasible assignment

$$\mu' = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_3 & s_1 \end{pmatrix}$$

since $\mu(s_1) = c_1$, $c_3 P'_{s_1} c_1$ and $s_1 \succ_{c_3} s_2$. Now it is straightforward to verify that the unique feasible and fair across types assignment of the new problem is $\bar{\mu}$. Thus any mechanism, which is both feasible and fair across types, must select the assignment $\bar{\mu}$ for the new problem. Under $\bar{\mu}$ student s_1 is assigned school c_2 which is strictly preferred to c_1 under the true preference P_{s_1}. Thus student s_1 is better off by stating P'_{s_1} than by stating his true preference P_{s_1}, and the mechanism is not incentive compatible.

Case 2: The mechanism selects $\bar{\mu}$.

Under $\bar{\mu}$ student s_2 is assigned school c_1. Similarly as in Case 1 we will show that student s_2 gains by misreporting his preference. Suppose that student s_2 states the (false) preference P'_{s_2} given by $c_3 P'_{s_2} c_2 P'_{s_2} c_1 P'_{s_2} s_2$, and all other students were to state their true preferences. Keeping all other components of the above problem fixed, in the new problem the students' preferences are $P'_S = (P_{s_1}, P'_{s_2}, P_{s_3})$.

In the new problem under $\bar{\mu}$ student s_2 justifiably envies student s_1 at school c_2 through the feasible assignment

$$\bar{\mu}' = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & s_3 \end{pmatrix}$$

since $\bar{\mu}(s_2) = c_1$, $c_2 P'_{s_2} c_1$ and $s_2 \succ_{c_2} s_1$. Now it is straightforward to verify that μ is the unique feasible assignment which is fair across types for the new problem. Thus any mechanism, which is both feasible and fair across types, must select the assignment μ for the new problem. Under μ student s_2 is assigned school c_3 which is strictly preferred to c_1 under the true preference P_{s_2}. Thus student s_2 does better by stating P'_{s_2} than by stating his true preference P_{s_2}, and the mechanism is not incentive compatible. □

Remark 10 Similar to Remark 7, in controlled school choice there is not always a unique candidate for a feasible assignment which is fair across types. This provides again additional reason for Theorems 5'.

In the example used to prove Theorem 5' let the controlled school choice problem be given by Table 4 except for school c_1's capacity constraints and ceilings: let $q_{c_1} = 2$ and $\overline{q}_{c_1}^{t_1} = \overline{q}_{c_1}^{t_2} = 2$. Then it is straightforward to verify that

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_3 & s_2 \end{pmatrix} \text{ and } \bar{\mu} = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & s_3 \end{pmatrix}$$

are assignments which are both feasible and fair across types for this problem. Since student s_1 prefers c_2 to c_1 under P_{s_1} and student s_2 prefers c_3 to c_1 under P_{s_2}, students' preferences are opposed over μ and $\bar{\mu}$. Obviously, there is no feasible and fair assignment which students s_1 and s_2 prefer to μ and $\bar{\mu}$: if there were such an assignment, then neither s_1 nor s_2 is assigned c_1 and constraints at school c_1 are violated since the floor for type t_1 at school c_1 is equal to one.

When computing the "minimum" $\wedge$ of μ and $\bar{\mu}$ (by assigning each student to the school which he least prefers from μ and $\bar{\mu}$) we obtain the assignment

$$\mu \wedge \bar{\mu} = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_1, s_2\} & \emptyset & s_3 \end{pmatrix}$$

which is feasible but not fair across types since student s_2 justifiably envies student s_3 at school c_3.