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A Test of New Aggregate Demand Curvature Properties

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1. INTRODUCTION

The economic theory of individual consumer behavior is well-known. Under certain assumptions, individual demand functions are differentiable and satisfy the properties of additivity, homogeneity, symmetry, and negativity. On the other hand, even if individual demand functions are well behaved (in this sense), aggregate demand functions need not satisfy symmetry and negativity. However, Hildenbrand (1983) and Grandmont (1984) have shown that under additional assumptions about the distribution of income (Hildenbrand) or preferences (Grandmont) aggregate demand functions may satisfy the "Law of Demand" (as well as negative quasi-definiteness of the matrix of uncompensated price effects) even if they do not satisfy all of the properties of individual demand functions. In particular, the matrix of uncompensated price effects may be negative quasi-definite even if the matrix of compensated price effects is not negative semi-definite.

In this paper we present and implement an econometric test of both the negative semi-definiteness of the matrix of compensated price effects and of the negative quasi-definiteness of the matrix of uncompensated price effects. This test allows us to evaluate two alternative characterizations of aggregate demand systems: the first, that they behave like individual demand functions, and the second, that they respect the properties implied by the assumptions proposed by Hildenbrand (1983) or Grandmont (1984).

Key words: Complete demand systems; curvature properties; econometric tests; aggregate demand.
II. MEAN AGGREGATE DEMAND

We assume for each consumer $i$ a strongly quasi-concave, monotonic, and twice continuously differentiable utility function. Then, there exists a differentiable demand function with total differential of the form

$$
(1) \quad dx_i = K_1 \, dp + K_2 \, dy_1 - x_i^T \, dp
$$

$$
= K_1 \, dp + K_2 \, x_i^T \, dx_i
$$

where, for each consumer $i$, $x_i$ is a vector of demands, $p$ is a vector of corresponding prices, $y_i$ is income, $K_1$ is a matrix of substitution effects, and $K_2$ is a vector of income effects.

It is easily shown (see, for example, Barten and Bohm, 1982) that $K_1$ and $K_2$ satisfy

$$
(2.1) \quad p^T K_1 = 0 \quad \text{and} \quad p^T K_2 = 1 \quad \text{(additivity)}
$$

$$
(2.2) \quad K_1 p = 0 \quad \text{(homogeneity)}
$$

$$
(2.3) \quad K_1 = K_1^T \quad \text{(symmetry)}
$$

$$
(2.4) \quad 0^T K_2 0 < 0 \quad \text{for} \quad 0 \neq p, \quad 0 \in \mathbb{R} \quad \text{(negativity)}
$$

In order to differentiate the mean demand, we rewrite (1) as follows:

$$
(3) \quad I - \left[ K_1 - \bar{x} \right] \bar{p}^T \, dx_1 - \bar{x} \bar{p} \, x_i^T \, dx_i
$$

$$
= K_1 \, dp + \bar{x} \bar{p} \, x_i^T \, dx_i
$$

where $\bar{x} = 1/n \sum_{i=1}^{n} x_i$ is the mean income effect, and $n$ is the total number of consumers in the economy. Sum (3) over the $n$ consumers and divide by $n$

$$
\left( \frac{1}{n} \right) \sum_{i=1}^{n} \left[ I - \left[ K_1 - \bar{x} \right] \bar{p}^T \right] \, dx_i = \left( \frac{1}{n} \right) \sum_{i=1}^{n} K_1 \, dp + \left( \frac{1}{n} \right) \bar{x} \bar{p} \, x_i^T \, dx_i
$$

$$
\text{Denote} \quad \bar{x} = \left( \frac{1}{n} \right) \sum_{i=1}^{n} x_i, \quad \text{the matrix of substitution effects for the mean demand}; \quad \bar{x} = \left( \frac{1}{n} \right) \sum_{i=1}^{n} \, dx_i, \quad \text{the differential of the mean demand}; \quad \text{and we may write} \quad dx_i = \delta_i \, d\bar{x}, \quad \text{where} \quad \delta_i \quad \text{is a diagonal matrix, as long as} \quad d\bar{x} \text{is not equal to zero}. \quad \text{Equation (4) may then be rewritten as}
$$

$$
(4) \quad \left( \frac{1}{n} \right) \sum_{i=1}^{n} \left[ I - \left[ K_1 - \bar{x} \right] \bar{p}^T \right] \, dx_i = \left( \frac{1}{n} \right) \sum_{i=1}^{n} K_1 \, dp + \left( \frac{1}{n} \right) \bar{x} \bar{p} \, x_i^T \, dx_i.
$$

$$
\text{Let} \quad \delta = \left( \frac{1}{n} \right) \sum_{i=1}^{n} \delta_i. \quad \text{Then, since} \quad \frac{n}{\bar{x}} \left[ K_1 - \bar{x} \right] \bar{p}^T \delta = 0,
$$

we can write

$$
\left( \frac{1}{n} \right) \sum_{i=1}^{n} \left[ K_1 - \bar{x} \right] \bar{p}^T \delta_i = \left( \frac{1}{n} \right) \sum_{i=1}^{n} \left[ K_1 - \bar{x} \right] \bar{p}^T \delta_i
$$

$$
= 0.
$$

Where $\Omega$ is, roughly speaking, a covariance matrix of income effects and real income that arises from the distributions of income and tastes across consumers (see Appendix 1 for an alternative interpretation).

Equation (5) can be rewritten as

$$
\left( \frac{1}{n} \right) \sum_{i=1}^{n} \left[ K_1 - \bar{x} \right] \bar{p}^T \delta_i = \left( \frac{1}{n} \right) \sum_{i=1}^{n} \left[ K_1 - \bar{x} \right] \bar{p}^T \delta_i = 0
$$

$$
\Omega
$$

It is easily verified that, by definition of $\left[ 1 - \Omega \right] = \bar{p}^T \left[ 1 - \Omega \right] = \bar{p}^T$, so that $\bar{p} \, \Omega = 0$. Moreover the matrix $\left[ 1 - \Omega \right]$ is regular. To see this, suppose that $\left[ 1 - \Omega \right]$ is singular. Then there is a vector $\delta \neq 0$
such that $\hat{q}^T(I - G) = 0$. This implies $\hat{q}^T K = 0$ by (6) and then $\hat{q}^T = 0$ (\( \bar{p}^T \)). But $\hat{q}^T[I - G] = \hat{p}^T$, so $\hat{q}^T[I - G] = 0$ if and only if $\hat{p} = 0$, or equivalently, if and only if $\hat{q} = 0$, a contradiction.

Then we may rewrite the inverse of $[I - G]$ as

$$[I - G]^{-1} = [I - \rho(G)]^{-1}.$$

(7)

(Remark: $\rho(G) = -[I - G]^{-1}G$). It is also immediate to prove that $\hat{p}^T \rho(G) = 0$. Using (7), we may rewrite the equation (6) as

$$d\hat{x} = [I - \rho(G)] \hat{K} dp + [I - \rho(G)] \hat{K} \bar{p}^T d\hat{x}.$$

$$= \hat{K} dp + \bar{p}^T d\hat{x},$$

(8)

where $\hat{K} = \hat{K} - \rho(G) \hat{K} = \hat{K} - \overline{Z}$; and $\bar{K} = \hat{K} - \rho(G) \hat{K} = \hat{K} - \overline{Z}$.

From the individual characterization (equation 2), one can prove that

$$\hat{p}^T K = 0 \text{ and } \hat{p}^T \bar{K} = 1 \quad \text{(additivity of the mean demand)}$$

(9.1)

$$\bar{p}^T \bar{p} = 0 \quad \text{(homogeneity of the mean demand).}$$

(9.2)

The presence of $\overline{Z}$ doesn't change the properties of the mean demand. This arises from the fact that $\hat{p}^T \rho(G) = 0$. On the other hand, the presence of $\overline{Z}$ implies that the symmetry and the negativity of the mean demand is not assured (see Debreu, 1974; Sonnenschein, 1974; Mantel, 1974).

However, according to the assumptions made about the distribution of income or tastes, additional properties may be obtained that facilitate estimation of the demand system.

II.1 THE "STANDARD" CHARACTERIZATION

For estimation, it is convenient to ignore the matrix $Z$, or equivalently, to assume that its presence does not prevent the individual characterization from pertaining to the aggregate system. In this case, the mean demand functions satisfy symmetry ($\bar{K} = \bar{K}^T$) and negativity ($\bar{p}^T K \bar{p} \leq 0$ for $\bar{p} \neq 0$, where $a$ is a real scalar). When estimating aggregate demand systems, researchers typically impose symmetry, and, more recently, have begun to impose negativity as well, using a Cholesky decomposition (see Dievert and Wales, 1987, and Morey, 1986, for examples).

However, recent theoretical developments imply that we should not conclude that the demand aggregation is not meaningful if the mean demand functions do not satisfy symmetry and negativity. In particular, assumptions about the distribution of income or tastes give rise to alternative characterizations of aggregate demand systems.

II.2 THE HILDENBRAND/GRANDMONT CHARACTERIZATION

Hildenbrand (1983) shows that the matrix of uncompensated price effects $[\bar{K} - \overline{Z} \bar{K}^T]$ is negative quasi-definite under certain assumptions about the distribution of income across consumers. Hence, mean demand functions satisfy the Law of Demand: they are downward sloping in their own prices. Hildenbrand derives this result under the assumption that the distribution of income (for each class of consumers having identical preferences) is everywhere non-increasing, an assumption that is unlikely to be verified by empirical inquiry.

Grandmont (1984) interchanges the roles of income and preferences. By assuming that all consumer's preferences are homothetic transformations of one another, he derives a one-parameter distribution,
f(a), of preferences. If, for each class of consumers having identical income, f is such that

(10) \( f'(a) + 2f(a) > 0 \) for \( a > 0 \),

and, if the support of f is unbounded, then the matrix of uncompensated price effects is negative quasi-definite.

Equation (10) has a useful interpretation. In order to satisfy (10), f(a) must be neither too concentrated nor too diffuse. If the distribution of preferences is too concentrated around the mean, then, in the limit, the aggregate demand system collapses to a single-consumer's demand system, and the matrix of uncompensated price effects need not be negative quasi-definite (and, of course, the Law of Demand need not hold, since individuals may exhibit "perverse", e.g. Giffen, behavior). On the other hand, if the distribution of preferences is too diffuse, then there may not be enough individuals with "normal" preferences such that, through aggregation, the "perverse" (Giffen) individuals are outweighed, and hence, such that the mean demand curves slope down in their own prices.

In short, both the Hildenbrand and Grandmont formulations lead to the same characterization of the mean demand functions: namely, the matrix of uncompensated price effects will be negative quasi-definite. This contrasts with the characterization that results from the standard aggregation rule: that the matrix of compensated price effects will be negative semi-definite.

III. TESTS OF CURVATURE PROPERTIES

Section II highlights the importance of tests of negativity for evaluation of aggregate demand theories. Typically, negativity of the compensated price effects is tested (if at all) by using a Cholesky decomposition of the substitution matrix (Barten and Gayskens, 1975; Lau, 1978; Morey, 1986). The advantage of this method is that one need only test whether the Cholesky values (the elements of the diagonal matrix) are non-positive. A disadvantage is that symmetry must be imposed in the estimation procedure in order to insure that the Cholesky values exist and are real. In the case where the Cholesky values do not exist, or are imaginary, negativity cannot be tested. The test presented here allows the negativity conditions to be tested even if eigenvalues are not real or do not exist. Furthermore, where the matrix is thought to be non-symmetric on theoretical grounds, as in the Hildenbrand/Grandmont characterization, imposition of symmetry cannot be justified.

Another disadvantage of the Cholesky decomposition is that decomposition takes place prior to estimation. Recovery of estimates of the parameters of interest and their variance-covariance matrix, while not impossible, requires additional and possibly arduous calculations.

Here we propose an alternative test for negativity. It has the advantage of being applicable to non-symmetric matrices, and, since decomposition takes place after estimation of the demand system, no additional calculations are required to recover the estimated parameters of the demand system or their variance-covariance matrix.

We develop this test in the context of the Rotterdam model (see Theil, 1976 and 1980; and Barnett, 1979).

(11) \( \tilde{w} \log \tilde{x} = B \log p + \tilde{b} \log y - \tilde{x}^T \log p \);

where \( \tilde{w} \) is the diagonal matrix of expenditure shares; \( \tilde{x} \) is the corresponding column vector; \( \log \tilde{x} \) is a vector of differentials of the log of demand; \( B \) is a matrix of coefficients; \( \log p \) is a vector of differentials of log prices; \( \tilde{b} \) is a vector of coefficients; and \( \log y \) is the differential of the log of mean income.
The Rotterdam model (11) is related to (8) by

\[ B = \left( \frac{1}{T} \right) \Sigma \hat{P} \Sigma; \]

(13) \[ b = \hat{P} \times; \]

where \( P \) is the diagonal matrix of prices.

We can rewrite the properties (2) for the Rotterdam model as

\[ b^T B = 0 \quad \text{(additivity)} \]
\[ b^T B = 1 \quad \text{(homogeneity)} \]
\[ \Sigma = B^T \Sigma \quad \text{(symmetry)} \]
\[ \hat{b}^T B \hat{b} < 0 \quad \text{for} \quad \hat{b} \neq 0, \quad \hat{b} \in \mathbb{R}. \quad \text{(negativity)} \]

where \( \hat{b} \) is a vector of ones.

The first two properties are preserved by definition. If mean demands behave as individual demands, (14.3) and (14.4) will also hold. If the Hildrebrand or Grandmont assumptions are valid, then instead of (14.3) and (14.4) we have

\[ \hat{b}^T C \hat{b} < 0 \quad \text{for all} \quad \hat{b} \neq 0, \]

where \( C = [B - b b^T] \). Note that the matrix of uncompensated price effects \( \Sigma \) is negative quasi-definite if and only if \( C \) is.

We propose a test of negativity for \( B \) and \( C \) in order to test the "standard" and Hildrebrand/Grandmont characterizations of the mean demand systems. First, estimate equation (11) using maximum likelihood methods. To test the negativity of \( B \) (a matrix of estimated coefficients) form the symmetric matrix \( \Sigma = \left( \frac{1}{2} \right) \Sigma + \Sigma^T \). The matrix \( \Sigma \) will be negative definite if and only if \( B \) is. Next, calculate the eigenvalues of \( \Sigma \) (they exist and are real since \( \Sigma \) is symmetric). In Appendix 2, it is shown that the asymptotic variance vector of the eigenvalues is, where (*) indicates estimated variables

\[ \text{avar} \left( \hat{\Sigma} \right) = \frac{\delta \hat{\Sigma}^T}{\delta \hat{b}} \text{avar} \left( \hat{b} \right) \frac{\delta \hat{\Sigma}}{\delta \hat{b}} \]

where \( \lambda \) is the vector of the \( J \) eigenvalues of the estimated matrix \( \hat{\Sigma} \), one for each good \( j=1,...,J \); \( \hat{b} \) is the vector formed by stacking the columns of \( \hat{b} \); and \( \text{avar}(\hat{b}) \) is the asymptotic variance-covariance matrix of \( \hat{b} \).

In order for \( \Sigma \) to be negative semi-definite, all the eigenvalues \( \lambda_j \) must be non-positive. Therefore, one can use a Bonferroni test to evaluate the negativity condition. Since \( \Sigma \) is a matrix of coefficients only, and if the demand system consists of only a few goods, the test is easy to apply. We show in Appendix 2 that \( \lambda \) is asymptotically normally distributed with variance-covariance matrix given by equation (15). Therefore, although asymptotic tests can be conducted using the statistic

\[ \frac{\hat{\lambda}_j}{\text{avar}(\lambda)_j^{1/2}} \]

for all \( j=1,...,J \).

We reject the null hypothesis "\( \Sigma \) is not negative semi-definite" at confidence level at least 1-Ja, only if we reject the hypothesis that each \( \lambda_j \) is positive at confidence level a. An alternative and weaker test of the theory is to reject the null hypothesis of "\( \Sigma \) negative semi-definite" at confidence level at least 1-Ja, if we cannot reject that each \( \lambda_j \) is non-positive at confidence level 1-a.

The test for the uncompensated price effect matrix, \( C \), is similar, although it is more complicated due to the fact that \( C \) is a function of both observed variables (expenditure shares) and estimated coefficients. As with \( B \), form the symmetric matrix \( \Sigma = \left( \frac{1}{2} \right) \Sigma + \Sigma^T \) and compute the eigenvalues. The asymptotic variance is (see Appendix 2):
The demand system consists of four categories of commodities (durables, semi-durables, non-durables, and services) and is completed by including a financial asset (personal savings). To account for expectations, we add a vector of lagged variables including one- and four-period lagged consumer price indexes, one- and four-period lagged real income, and a four-period lagged durable demand. (For theoretical arguments for introducing a financial asset in a demand system, as well as an application of the present framework, see Bronsard, 1983; Bronsard and Salas-Bromard, 1986.)

Since the demand system is complete, additivity is assured. Therefore, we test the homogeneity, negativity and symmetry properties.

Table 1 presents the results from estimation of the discrete version of (11). It is worth noting that only three diagonal elements of the matrix $B$ (the first five columns of the table) both have the expected negative sign, and are statistically significant at the .05 level. As expected, real income is positively related to demand for all commodities and the financial asset (the sixth column of the table), and these coefficients are statistically significant at the .05 level. Using a likelihood ratio test, we cannot reject homogeneity of the mean demand system. The following describes the tests of the assumptions underlying the "standard" and the "Hildenbrand/Grandmont" characterizations.

IV.1. TEST OF THE "STANDARD" CHARACTERIZATION

Using a maximum likelihood test, we reject the symmetry of $B$ (as predicted by the "standard" characterization of aggregate demand) at the .05 level.

For negativity, we apply the test developed in Section III. The computed eigenvalues of $D$ (where $D = (1/2) (\Omega + \Omega^T)$) and their
corresponding asymptotic $t$-statistics are reported in Table 2. (Eigenvalues and eigenvectors are computed using the subroutine EIGRF from the IMSL library.) Individually, two of the five eigenvalues are negative and (asymptotically) statistically significant at the .05 level, while two are negative but not significant. The fifth eigenvalue is positive and statistically significant at the .05 level.

The null hypothesis that $\mathbf{B}$ is not negative semi-definite is rejected at least the .05 level if all the eigenvalues of $\mathbf{D}$ are statistically non-positive at the .01 level. Since only two of five eigenvalues are statistically non-positive, we cannot reject the null hypothesis that $\mathbf{B}$ is not negative semi-definite.

The null hypothesis that $\mathbf{B}$ is negative semi-definite is rejected at least the .05 level if the five eigenvalues of $\mathbf{D}$ are not significantly non-negative at the .01 level. We reject the null hypothesis at least the .05 level because one of the eigenvalues is significantly positive.

In short, the data reject the "standard" characterization of the mean demand system.

IV.2. TEST OF THE HILDENBRAND/GRANDMONT CHARACTERIZATION

Recall that with the Hildenbrand/Grandmont characterization, we need only test for the negativity of the matrix $\mathbf{E}$, where

$$\mathbf{E} = (1/2) \left[ (\mathbf{B} - \mathbf{b} \mathbf{W}^T) + (\mathbf{B} - \mathbf{b} \mathbf{W}^T)^T \right].$$

The computed eigenvalues (evaluated at the period 1985:4) and corresponding asymptotic $t$-statistics are reported in Table 3. Three of the five eigenvalues are negative and individually (asymptotically) statistically significant, while a fourth is negative but not significant. The fifth is statistically significantly positive at about the .001 level (one-tailed asymptotic $t$-test). By the reasoning of Section IV.1, we reject the negativity of $\mathbf{E}$, or alternatively, we cannot reject the non-negativity of $\mathbf{E}$. In sum, the negativity of the matrix of uncompensated price effects is rejected, and hence, so too is the Hildenbrand/Grandmont characterization of mean demand.

V. CONCLUSION

This paper examined the relationship between individual and mean aggregate demand. The properties of the mean demand were shown to depend on the assumptions made about the distribution of income and/or tastes. We presented an "easy" test for two alternative characterizations of the mean demand system: the "standard" and the "Hildenbrand/Grandmont" characterizations.

Overall, while the estimation of the complete demand system does not allow us to reject homogeneity, we do reject both the "standard" and the "Hildenbrand/Grandmont" characterizations.

We wish to point out, however, that in rejecting the two characterizations we do not reject the economic theory of demand aggregation, which implies that aggregate demand systems need only respect the homogeneity and additivity properties.

In addition, while there are more powerful tests available (e.g. Kodde and Palm 1986 and 1987). The test employed here is simple to compute, and, apparently, has enough power to reject the two characterizations.
REFERENCES


APPENDIX I

An alternative interpretation is that this term represents a covariance matrix of income effects and tastes. This is apparent once we recognize, that, given a utility function $U$, $rac{1}{\mu_1} = \mu_1$ for each consumer $i$, where $\mu_1$ is the Lagrange multiplier in the consumer's problem. We have

$$\partial \bar{X} = \frac{1}{n} \sum_{i=1}^{n} \left[ k_i \frac{\partial}{\partial \bar{X}} \left( \frac{\partial}{\partial \bar{X}} + \frac{\partial}{\partial \bar{X}} \right) \right] \bar{X}.$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[ k_i \frac{\partial}{\partial \bar{X}} \left( \frac{\partial}{\partial \bar{X}} + \frac{\partial}{\partial \bar{X}} \right) \right] \bar{X}.$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[ k_i \frac{\partial}{\partial \bar{X}} \left( \frac{\partial}{\partial \bar{X}} + \frac{\partial}{\partial \bar{X}} \right) \right] \bar{X}.$$

But $\frac{\partial}{\partial \bar{X}} \left( \frac{\partial}{\partial \bar{X}} + \frac{\partial}{\partial \bar{X}} \right) = \frac{\partial}{\partial \bar{X}}$, so we have

$$\partial \bar{X} = \frac{1}{n} \sum_{i=1}^{n} \left[ k_i \frac{\partial}{\partial \bar{X}} \left( \frac{\partial}{\partial \bar{X}} + \frac{\partial}{\partial \bar{X}} \right) \right] \bar{X}.$$

This term is a covariance matrix of the income effects and tastes.
APPENDIX 2  Asymptotic variance of eigenvalues

We begin with the following useful theorem, which is proven by Magnus (1985).

THEOREM A.1: Let $A_o$ be a real, symmetric $nxn$ matrix, and let $\lambda_o$ be a simple eigenvalue of $A_o$ with associated standardized eigenvector $u_o$. Then a real-valued function $\lambda$ and a vector-valued function $u$ are defined on a neighborhood $N(A_o) = \mathbb{R}^{nxn}$ of $A_o$ such that $\lambda(A) = \lambda_o$, $u(A) = u_o$, and

$$Au(A) = u(A)\lambda(A), \quad u(A)^T u(A) = 1$$

for all $A \in N(A_o)$. In addition, $\lambda$ is differentiable on $N(A_o)$ with differential

$$d\lambda = u^T(dA)u.$$

This result has an immediate corollary, which is useful in the present context.

COROLLARY A.1: Let $A(\theta)$ be a real $nxn$ matrix function of $\theta \in \mathbb{R}^p$, and denote the $(i,j)$th element of $A$ by $A_{ij}$. For some $\theta_0 \in \mathbb{R}^p$, let $\lambda_o$ be a simple eigenvalue of the real, symmetric matrix $[A(\theta_0)^T + A(\theta_0)]/2$ with associated eigenvector $u_o$. Then the functions $\lambda$, $u$ exist as in Theorem 1, and, if $A$ is differentiable in a neighborhood of $\theta_0$,

$$(a.1) \quad d\lambda \left[ (A^T + A_o)/2 \right] = \lambda_o^{-1} \left[ dA(\theta_0)^T / \theta_0 \right] u_o$$

where values subscripted by "o" are evaluated at $\theta_o$. 

We can apply this result first to the matrix

$$(A_o^T + B_o)/2$$

where $\theta_o = \text{vec} (B_o) = \vec b_o$. Take $A(\theta) = B$ in Corollary A.1 and note that

$$\frac{d\lambda^B}{\theta_{ij}} = e_{ij}$$

where $e_{ij}$ is the $nxn$ matrix with $(i,j)$th element equal to one, and zeros elsewhere. Therefore, if $\hat{\lambda}_k(B)$ is the $k$th eigenvalue of the matrix $(B^T + B)/2$ with associated eigenvector $u_k(B)$, then

$$\frac{d\lambda_k^B}{\theta} = \text{vec} u_k^T (B_o) u_k (B_o)^T.$$

This shows that to compute the $1 \times n^2$ derivative of the $k$th eigenvalue of the matrix $(B^T + B)/2$ evaluated at the point $B_o$, we need only compute the associate eigenvector of $(B_o^T + B_o)/2$.

If we have an estimator of $B_o$ say $\hat B_o$, then the asymptotic variance of the $k$th eigenvalue of $(\hat B_o^T + \hat B_o)/2$ is easily obtained by the "delta method". Let $\hat{\lambda}_{tk}$ the $k$th eigenvalue of $(\hat B_o^T + \hat B_o)/2$ and let $u_{tk}$ be the associated eigenvector. Then

$$\text{avar} \hat{\lambda}_{tk} = \frac{d\lambda^B}{\theta} |_{\theta = \hat B_o} \text{avar} \hat B_o |_{\theta = \hat B_o} \frac{d\lambda^B}{\theta} |_{\theta = \theta_o}$$

and, replacing the quantities on the right hand side by consistent estimators,

$$(a.2) \quad \text{avar} \hat{\lambda}_{tk} = |_{\theta = \hat B_o} \frac{d\lambda^B}{\theta} |_{\theta = \hat B_o} \text{avar} \hat B_o \text{vec} u_k^T (\hat B_o).$$

The asymptotic standard error of $\hat{\lambda}_{tk}$ is the square root of this quantity.
We can also apply Corollary A.1 to the matrix

\[ C_o \equiv B_o - b \omega^T. \]

In this case, \( \omega \equiv (\text{vec}(B)^T, \beta^T)^T \) and

\[ A(\omega) = B - b \omega^T. \]

Letting \( b_{ij} \) be the \((i,j)\)th element of \( B \) and \( b_j \) the \( j \)th element of \( b \), we have

\[ \frac{\partial A(\omega)}{\partial b_{ij}} = e_{ij} \]

\[ \frac{\partial A(\omega)}{\partial b_j} = (0, 0, \ldots, 0, w_i, 0, \ldots, 0)^T \]

where \( \omega \) is an \( n \times 1 \) vector of zeros and \( w \) is in the \( j \)th column. By Corollary A.1, let \( \lambda_k(C) \) denote the \( k \)th eigenvalue of \((C + C)^T/2\), and let \( \omega_k(C) \) be the associated eigenvector. Then

\[ \frac{\partial \lambda_k(C)}{\partial \omega^T} = \text{vec} \left( \omega_k \omega_k^T \right)^T \]

\[ \frac{\partial \lambda_k(C)}{\partial b} = (u_k^T 0, u_k^T 0, \ldots, u_k^T 0) \]

where values superscripted by "o" are evaluated at \( C_o \). Therefore,

\[ V_{\lambda_k}(C) = \text{vec} \left( \omega_k \omega_k^T \right)^T, \quad u_k^T 0, u_k^T 0, \ldots, u_k^T 0 \]

This is consistently estimated by replacing \( \omega_0, \beta_0 \) by their maximum likelihood estimators \( \hat{\omega}_k, \hat{\beta}_j \) denote the resulting estimate by \( V_{\lambda_k}(\hat{C}) \).

Then the asymptotic variance of \( \lambda_k(C) \) is consistently estimated by

\[ \text{avar} \lambda_k(C) = V_{\lambda_k}(C) \text{avar}(0, 0, \ldots, 0) \lambda_k(C)^T \]

where \( \text{avar} \lambda_k(C) = (\hat{C}_k^T \hat{C}_k) \).

---

**APPENDIX 3 Data description**

1. Population: Statistics Canada # 91-001, Table 5.

2. Personal expenditures on consumer goods and services in current dollars: Statistics Canada # 13-001, Table 7
   Statistics Canada # 13-533, Table 7.

3. Personal expenditures on consumers goods and services in 1971 dollars: Statistics Canada # 13-001, Table 8
   Statistics Canada # 13-533, Table 8.

4. Personal Savings: Statistics Canada # 13-001, Table 5
   Statistics Canada # 13-533, Table 5.

5. Consumer Price Index: Statistics Canada # 62-010, Table 1.
### Table 1

| Vector | Vector | Vector | Vector | Vector | Vector | Vector | Vector | Vector | Vector | Vector | Vector | Vector | Vector | Vector | Vector |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
|        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |

### Table 2

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\lambda / \sqrt{\text{var} \lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.484</td>
<td>-12.17</td>
</tr>
<tr>
<td>-0.183</td>
<td>-3.74</td>
</tr>
<tr>
<td>-0.051</td>
<td>-0.90</td>
</tr>
<tr>
<td>-0.013</td>
<td>-0.20</td>
</tr>
<tr>
<td>0.183</td>
<td>7.62</td>
</tr>
</tbody>
</table>

Eigenvalues and asymptotic t-statistics associated to $|\theta + \omega^n|$.
Table 3  
Eigenvalues and asymptotic t-statistics associated to \( \{ (u - \mu)^2 + (u - \bar{u})^2 \} \) for 1985.4

<table>
<thead>
<tr>
<th>( \bar{\lambda} )</th>
<th>( \bar{\lambda} / \text{sd}[\bar{\lambda}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.498</td>
<td>-10.47</td>
</tr>
<tr>
<td>-0.231</td>
<td>-3.82</td>
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<tr>
<td>-0.161</td>
<td>-3.66</td>
</tr>
<tr>
<td>-0.065</td>
<td>-1.02</td>
</tr>
<tr>
<td>0.251</td>
<td>-5.99</td>
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</tbody>
</table>