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Some Robust Exact Results on Sample Autocorrelations and Tests of Randomness*

by

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ABSTRACT

Several exact results on the second moments of sample autocorrelations, for both Gaussian and non-Gaussian series, are presented. General formulae for the mean, variance and covariances of sample autocorrelations are given for the case where the variables in a sequence are exchangeable. Bounds for the variances and covariances of sample autocorrelations from an arbitrary random sequence are derived. The bounds on variances are used to obtain exact upper limits on critical values for tests of randomness based on sample autocorrelations, without any assumption on the form of the distribution. Exact and explicit formulae for the variances and covariances of sample autocorrelations from a Gaussian white noise are given. It is observed that the latter results hold for all spherically symmetric distributions. A simulation experiment, with Gaussian series, indicates that normalizing each sample autocorrelation with its exact mean and variance, instead of the usual approximate moments, can improve considerably the accuracy of the asymptotic N(0,1) distribution to obtain critical values for tests of randomness. The exact second moments of rank autocorrelations are also studied.

RESUME

Ce texte présente plusieurs résultats exacts sur les seconds moments des autocorrélations échantillonnales, pour des séries gaussiennes ou non-gaussiennes. Nous donnons d'abord des formules générales pour la moyenne, la variance et les covariances des autocorrélations échantillonnales, dans le cas où les variables de la série sont interchangeables. Nous déduisons de celles-ci des bornes pour les variances et les covariances des autocorrélations échantillonnales. Ces bornes sont utilisées pour obtenir des limites exactes sur les points critiques lorsqu'on teste le caractère aléatoire d'une série chronologique, sans qu'aucune hypothèse soit nécessaire sur la forme de la distribution sous-jacente. Nous donnons des formules exactes et explicites pour les variances et covariances des autocorrélations dans le cas où la série est un bruit blanc gaussien. Nous montrons que ces résultats sont aussi valides lorsque la distribution de la série est sphériquement symétrique. Nous présentons les résultats d'une simulation qui indiquent clairement qu'on approxime beaucoup mieux la distribution des autocorrélations échantillonnales en normalisant celles-ci avec la moyenne et la variance exactes et en utilisant la loi N(0, 1) asymptotique, plutôt qu'en employant les seconds moments approximatifs couramment en usage. Nous étudions aussi les variances et covariances exactes d'autocorrélations basées sur les rangs des observations.

1. INTRODUCTION

Sample autocorrelations are one of the main instruments of time series analysis. They are especially useful to test the randomness of a time series and to assess dependence at various lags. Several definitions have been proposed. We consider here the most standard one, as it is used for example to identify time series model (Box and Jenkins, 1976, p.32): given n observations $X_1,\ldots,X_n$, the sample autocorrelation at lag k is

$$r_k = \frac{\sum_{i=1}^{n-k} (X_i-\bar{X})(X_{i+k}-\bar{X})}{\sum_{i=1}^{n} (X_i-\bar{X})^2}, \quad 1 \le k \le n-1, \tag{1.1}$$

where $\bar{X} = \sum_{i=1}^{n} X_i/n$ is the sample mean. We find especially important that the data be expressed in deviations from their sample mean because, in most practical situations, the true mean is unknown. This characteristic will play an important role below.

We will be concerned here by some exact distributional properties of sample autocorrelations, under the important null hypothesis of randomness. Both normal and non-normal distributions will be considered. Tests based on sample autocorrelations typically use critical values based on their asymptotic normal distribution (Bartlett, 1946; Anderson and Walker, 1964): both the moments of r_k (mean and variance) and the form of the distribution are approximate. Despite the fact that autocorrelation coefficients are widely applied in empirical research, few exact results have been published on their sampling properties; see the reviews of Anderson (1971, Chap. 6) and Kendall, Stuart and Ord (1983, Chap. 48). Moran (1948) gave the exact mean of r_k, $k \ge 1$, for an arbitrary random series, and the exact variance of the first autocorrelation r_1 for a normal random series; later (1967a), he obtained an upper bound on the variance of r_1, valid for all random series. Using the method of Sawa (1978), De Gooijer (1980) gave formulae that allow to evaluate numerically the first four moments of each sample autocorrelation, when the data come from a general autoregressive moving average Gaussian process: his formulae however are not explicit and require numerical integrations that may be expensive. Actually, no author has given exact and explicit formulae for the variances $\mathrm{Var}(r_k)$, $k \geq 2$, or their covariances, even when the series is a normal white noise. The vast majority of the results available either deal with alternative definitions of autocorrelations (coefficients with known mean, circular definition, etc...) or remain approximate; see, for example, R.L. Anderson (1942), T.W. Anderson (1971, Chap. 6), Anderson and Walker (1964), O.D. Anderson (1979 a,b; 1982), Davies, Triggs and Newbold (1977), Davies and Newbold (1980), Jenkins (1954, 1956), Kendall (1954), Koopmans (1942), Kendall, Stuart and Ord (1983, Chap. 48), Knoke (1977, 1979), Marriott and Pope (1954), Ochi (1983), Quenouille (1949), Phillips (1977, 1978), Ramasubban (1972), Sawa (1978), Shenton and Johnson (1965), Taibah and Kassab (1981), von Neumann (1942), White (1961, 1962). It is important to note, in particular, that exact results that hold for sample autocorrelations of the form $\sum_{t=1}^{n-k} X_t X_{t+k} / \sum_{t=1}^{n} X_t^2$ (where the mean is assumed to be zero), are not generally valid for standard autocorrelations, as defined in (1.1). Concerning non-normal series, evidence on the properties of sample autocorrelations for specific distributions was given by Cox (1966), Moran (1967 a,b; 1970), Quenouille (1948) and Knoke (1977).

In this paper, we present several exact results on the first and second moments of sample autocorrelations, for both normal and non-normal series, and discuss their application in testing the randomness of a time series. We consider in turn four wide classes of series: A) series of exchangeable random variables; B) random series (or random samples), i.e. independent and identically distributed (i.i.d.) random variables with an arbitrary distribution; C) series with a spherically symmetric distribution; D) normal random series. Though we are most interested by the hypothesis of randomness (B or D), we will see that many results that hold for B or D actually hold under the more general assumptions A or C.

In Section 2, we derive general formulae for the mean, variance and covariances of sample autocorrelations, from an arbitrary series of exchangeable random variables, for all lags and sample sizes. Since random series belong to this class, these formulae apply to any sequence of i.i.d. random variables. An important case of variables that are exchangeable without being independent is the sequence of ranks from a sample of i.i.d. random variables. In the sequel, we apply and specialize these formulae. First, we obtain upper bounds on the variances as well as upper and lower bounds for the covariances of autocorrelation coefficients (at all lags) when the variables in the series are exchangeable. Consequently these hold for any sequence of i.i.d. variables, irrespective of the form of the distribution. The bounds are tight in the sense that they are very close to what one gets assuming the variables are i.i.d. normal. Second, we discuss how to apply these inequalities to obtain upper bounds on the critical values of tests of randomness based on sample autocorrelations, and thereby get simple, exact as well as distribution-free tests of randomness. Third, we specialize the general formulae to the case of rank autocorrelations obtained by replacing each observation in (1.1) by its rank. Previous studies of such coefficients gave only approximate expression for $\mathrm{Var}(r_k)$; see Wald and Wolfowitz (1943), Knoke (1977), Bartels (1982), Dufour, Lepage and Zeidan (1982).

In Section 3, we consider series of i.i.d. normal random variables and, more generally, series that obey a spherically symmetric (s.s.) distribution. We first note that the distribution of sample autocorrelations is exactly the same under these two assumptions: accordingly, to study the latter case, we can assume normality. Note, on the other hand, that certain s.s. distributions, e.g. the multivariate Cauchy, differ considerably from that of a normal white noise. We then give exact and explicit formulae for the variances and covariances of sample autocorrelations, applicable to all lags and sample sizes. We see also in numerical comparisons that the variances obtained under the normality assumption are very close to the upper bounds given in Section 2, except possibly when n is small ($n < 20$). Finally, we consider the standard problem of testing the randomness of a normal time series using sample autocorrelations. We suggest that each coefficient r_k can and should be normalized with the exact mean and variance given above, as opposed to the approximate mean (zero) and variance typically used: through a Monte Carlo simulation, we find that exactly normalized sample autocorrelations have distributions that are generally better approximated by the asymptotic N(0,1) distribution and thus yield more accurate critical values; in many cases, the difference is important. We end in Section 4 with a few concluding remarks.

2. RESULTS FOR EXCHANGEABLE VARIABLES

2.1 Definitions and notations

Let $X_1,\ldots,X_n$ be a sequence of exchangeable random variables: i.e. for any permutation $(d_1,\ldots,d_n)$ of the integers $(1,\ldots,n)$, the distribution of $(X_{d_1},\ldots,X_{d_n})$ is the same as the distribution of $(X_1,\ldots,X_n)$. Clearly, independent and identically distributed random variables are exchangeable.

On the other hand, exchangeable variables are not necessarily independent. For example, random variables having a joint symmetric normal distribution (see Rao, 1973, p. 196) are exchangeable even if the correlation ρ between any two of them is large (e.g. $\rho=0.99$). The ranks of independent observations from a common continuous distribution have a uniform distribution and thus form a sequence of exchangeable variables; yet they are not independent. The same results on ranks actually holds if we only assume that the observations are exchangeable and have a continuous distribution, a common hypothesis in nonparametric statistics (see Hájek and Šidák, 1967, p. 37). We will use below the following property of exchangeable variables: if $M = M(X_1,\ldots,X_n)$ is a permutation-symmetric function of the observations, i.e.

$$M(X_{d_1},\ldots,X_{d_n}) = M(X_1,\ldots,X_n)$$

for any permutation $(d_1,\ldots,d_n)$ of $(1,\ldots,n)$, then the variables $X_1-M,\ldots,X_n-M$ are also exchangeable (see Fligner, Hogg and Killeen, 1976). For further details on the notion of exchangeability, see Galambos (1982) and the references therein.

If we define

$$Z_i = X_i - \bar{X}\ ,\ i = 1,\ldots,n,$$

where $\bar{X}$ is the mean of the X_i's, we can write

$$r_k = \frac{\sum_{i=1}^{n-k} Z_i Z_{i+k}}{\sum_{i=1}^{n} Z_i^2}\ ,\ 1 \le k \le n-1. \qquad (2.1)$$

If the X_i's are exchangeable, the Z_i's are also exchangeable since $\bar{X}$ is a permutation-symmetric function of $X_1,\ldots,X_n$.

Assuming $P[X_1 = X_2 = \ldots = X_n] = 0$, we will now derive results on the variances and covariances of the sample autocorrelations that hold under the mere assumption of exchangeability of the variables $X_1,\ldots,X_n$. In particular, they hold whenever $X_1,\ldots,X_n$ are i.i.d. with an arbitrary continuous distribution.

2.2 Variance of r_k

Under the assumption that $X_1,\ldots,X_n$ are i.i.d. (with a continuous distribution), it is possible to show that

$$E[r_k] = -\frac{(n-k)}{n(n-1)}, \quad 1 \le k \le n-1; \tag{2.2}$$

see Moran (1948), Kendall, Stuart and Ord (1983, Vol. 3, p. 551). However one sees easily that the proof of this result depends only on the exchangeability of $Z_1,\ldots,Z_n$ and thus the result holds whenever $X_1,\ldots,X_n$ are exchangeable. We require $P[X_1 = X_2 = \ldots = X_n] = 0$ to ensure that r_k exists with probability 1.

To obtain the variance of r_k, we first observe that the numerator of r_k^2 can be written as

$$\left(\sum_{i=1}^{n} Z_i Z_{i+k}\right)^2 = \sum_{i=1}^{n-k} Z_i^2 Z_{i+k}^2 + 2\sum_{i=1}^{n-2k} Z_i Z_{i+k}^2 Z_{i+2k} + \Sigma_* Z_i Z_{i+k} Z_j Z_{j+k}$$

where Σ_* denotes summation over $i,j = 1,\ldots,n-k$ such that i, $i+k$, j and $j+k$ are all distinct. From the exchangeability of $Z_1,\ldots,Z_n$, we can write

$$E[r_k^2] = E\left[\left(\sum_{i=1}^{n} Z_i^2\right)^{-2}\left\{(n-k)\, Z_1^2 Z_2^2 + 2(n-2k)\, Z_1^2 Z_2 Z_3 + ((n-k)^2 - 2(n-2k) - (n-k))\, Z_1 Z_2 Z_3 Z_4\right\}\right]$$

$$= E\left[\left(\sum_{i=1}^{n} Z_i^2\right)^{-2} \left\{ \frac{(n-k)}{n(n-1)} \Sigma^* Z_i^2 Z_j^2 + \frac{2(n-2k)}{n(n-1)(n-2)} \Sigma^* Z_i^2 Z_j Z_\ell \right.\right.$$

$$\left.\left. + \frac{((n-k)^2 - 2(n-2k) - (n-k))}{n(n-1)(n-2)(n-3)} \Sigma^* Z_i Z_j Z_\ell Z_m \right\}\right]$$

where Σ^* denotes summation over all unequal suffixes varying from 1 to n. Denote the power sums by

$$S_r = \sum_{i=1}^{n} Z_i^r \quad , \quad r \geq 1 .$$

Using the following identities (Kendall, Stuart and Ord, 1983, Vol. 3, p. 708)

$$\Sigma^* Z_i^2 Z_j^2 = S_2^2 - S_4 \quad ,$$

$$\Sigma^* Z_i^2 Z_j Z_\ell = 2S_4 - S_2^2 \ ,$$

$$\Sigma^* Z_i Z_j Z_\ell Z_m = 3S_2^2 - 6S_4 \ ,$$

we get that

$$E\,[r_k^2] = \frac{(n-k)}{n(n-1)} \left(1 - E[S_4/S_2^2]\right) + \frac{\{2n(n-2k) - 3(n-k)(n-k-1)\}}{n(n-1)(n-2)(n-3)} \left(2E[S_4/S_2^2] - 1\right)$$

$$= \frac{1}{n(n-1)(n-2)(n-3)} \left[\{-n^3 + (k+3)n^2 - k(n+6k)\}\, E[S_4/S_2^2] + \{n^2(n-k-4) + 3(n-k) + 3k(n+k)\}\right] . \qquad (2.3)$$

The variance then follows from the familiar formula $Var(r_k) = E[r_k^2] - (E[r_k])^2$, where $E[r_k]$ is given by (2.1). In order to obtain an explicit formula for $Var(r_k)$, all we need is $E[S_4/S_2^2]$. We will consider below two cases: i) the r_k's are computed from the ranks of exchangeable observations; ii) the sample $X_1,\ldots,X_n$ has a spherically symmetric distribution (which include the case of a normal random sample). When $E[S_4/S_2^2]$ cannot be evaluated analytically, the approximation discussed by Moran (1967a,1970) can be useful.

2.3 Covariance between r_k and r_h

Let $k < h$. The numerator of $r_k r_h$ can be written as

$$\sum_{i=1}^{n-k} \sum_{j=1}^{n-h} Z_i \, Z_{i+k} \, Z_j \, Z_{j+h}$$

$$= \sum_{i=1}^{n-h} Z_i^2 \, Z_{i+k} \, Z_{i+h} + \sum_{j=1}^{n-h-k} Z_{j+h}^2 \, Z_{j+h+k} \, Z_j + \sum_{i=1}^{n-h-k} Z_i \, Z_{i+k}^2 \, Z_{i+h+k}$$

$$+ \sum_{j=1}^{n-h} Z_{j+(h-k)} \, Z_{j+h}^2 \, Z_j + \Sigma_* \, Z_i \, Z_{i+k} \, Z_j \, Z_{j+h} \, , \qquad (2.4)$$

where Σ_* denotes summation over $i = 1,\ldots,n-k$ and $j = 1,\ldots,n-h$ such that i, $i+k$, j and $j+h$ are all distinct. By a development similar to the one used to obtain $E[r_k^2]$, we find (for $k < h$)

$$E[r_k r_h] = E\left[S_2^{-2} \{[2(n-h) + 2(n-h-k)] \, Z_1^2 \, Z_2 \, Z_3 \right.$$

$$\left. + [(n-k)(n-h) - 4(n-h) + 2k] \, Z_1 \, Z_2 \, Z_3 \, Z_4\} \right]$$

$$= E\left[S_2^{-2} \left\{ \frac{[4(n-h) - 2k]}{n(n-1)(n-2)} (2S_4 - S_2^2) \right.\right.$$

$$\left.\left. + \frac{[(n-h)(n-k-4) + 2k]}{n(n-1)(n-2)(n-3)} (3S_2^2 - 6S_4) \right\} \right]$$

$$= \frac{\{(n-h)(n+k) - 2kh\}}{n(n-1)(n-2)(n-3)} (2E[S_4/S_2^2] - 1) \, . \qquad (2.5)$$

The covariance follows from the familiar formula

$$\mathrm{Cov}(r_k, r_h) = E[r_k r_h] - E[r_k] \, E[r_h] \, .$$

2.4 Bounds for $Var(r_k)$ and $Cov(r_k, r_h)$

Using the above results, we will now give bounds on the second moments of r_k. For this purpose, we will use the following inequality on S_4/S_2^2 : for any sequence of real numbers $Z_1, \ldots, Z_n$,

$$1/n \le S_4/S_2^2 \le 1 \quad ; \qquad (2.6)$$

if $Z_i = 0$, $i = 1, \ldots, n$, we adopt the convention $S_4/S_2^2 = 1$. The lower bound is obtained by applying Cauchy's inequality (see Moran, 1967a, p. 397). To get the upper bound, set

$$W_i = \frac{Z_i}{\left(\sum_{j=1}^{n} Z_j^2\right)^{1/2}} \quad , \quad i = 1, \ldots, n \ ;$$

it is then immediate that $0 \le W_i^2 \le 1$, $\sum_{i=1}^{n} W_i^2 = 1$ and

$$S_4/S_2^2 = \sum_{i=1}^{n} W_i^4 \le \sum_{i=1}^{n} W_i^2 = 1$$

Thus, for any probability distribution on the Z_i's, we have

$$1/n \le E\,[S_4/S_2^2] \le 1 \ . \qquad (2.7)$$

If we notice that the coefficient of $E\,[S_4/S_2^2]$ in (2.3) is negative for all k (whenever $n > 3$), we get an upper bound for $Var(r_k)$ by replacing $E\,[S_4/S_2^2]$ by $1/n$:

$$Var(r_k) \le \frac{n^4 - (k+7)n^3 + (7k+16)n^2 + 2(k^2-9k-6)n - 4k(k-4)}{n(n-1)^2(n-2)(n-3)} \ , \qquad (2.8)$$

where $k \geq 1$ and $n > 3$. For $k = 1$, we retrieve the result of Moran (1967):

$$\mathrm{Var}(r_1) \leq \frac{n-2}{n(n-1)}$$

If we use polynomial division and retain only the terms up to order n^{-2}, the upper bound for r_k becomes

$$\mathrm{Var}(r_k) \leq \frac{n-k}{n^2} + 0(n^{-3})$$

We could not show that $\mathrm{Var}(r_k)$ is bounded away from zero. Indeed, if we substitute 1 for $E\,[S_4/S_2^2]$ in (2.3), we find

$$E\,[r_k^2] \geq \frac{\{2n(n-2k) - 3(n-k)(n-k-1)\}}{n(n-1)(n-2)(n-3)}$$

which is negative for k small with respect to n.

We get bounds for $E\,[r_k r_h]$ and $\mathrm{Cov}(r_k, r_h)$ from (2.5) and (2.7). If $(n-h)(n+k) - 2kh \geq 0$ (this inequality holds if $k,\ h \leq n/2$), we have (for $k < h$)

$$-\frac{\{(n-h)(n+k) - 2kh\}}{n^2(n-1)(n-3)} \leq E\,[r_k r_h] \leq \frac{\{(n-h)(n+k) - 2kh\}}{n(n-1)(n-2)(n-3)} \quad . \tag{2.9}$$

Bounds for $\mathrm{Cov}(r_k, r_h)$ follow by substracting $E[r_k]E[r_h]$ from each member of (2.9). Up to order n^{-3}, the bounds are (for $k < h$)

$$-\frac{2(n-h+3)}{n^3} + 0(n^{-4}) \leq \mathrm{Cov}(r_k, r_h) \leq \frac{2(k+2)}{n^3} + 0(n^{-4}) \quad . \tag{2.10}$$

For $(n-h)(n+k) - 2kh < 0$, upper and lower bounds in (2.9) and (2.10) are interchanged.

2.5 Rank serial correlation

Let $X_1,\ldots,X_n$ be exchangeable random variables and let $(R_1,\ldots,R_n)$ be the corresponding vector of ranks. Then

$$P\,[(R_1,\ldots,R_n) = (d_1,\ldots,d_n)] = \frac{1}{n!}$$

for any permutation $(d_1,\ldots,d_n)$ of $(1,\ldots,n)$ and thus the ranks are also exchangeable variables. The rank serial correlation at lag k is defined by

$$r_k = \frac{\sum_{i=1}^{n-k} (R_i-\bar{R})\ (R_{i+k}-\bar{R})}{\sum_{i=1}^{n} (R_i-\bar{R})^2}\ , \quad 1 \le k \le n-1 \quad ,$$

where $\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_i = \frac{n+1}{2}$. In this case, the denominator of r_k is constant so that it is equivalent to study the rank serial covariances

$$C_k = \sum_{i=1}^{n-k} (R_i - \bar{R})(R_{i+k} - \bar{R})\ , \quad 1 \le k \le n-1\ .$$

Wald and Wolfowitz (1943) proposed to use a circular version of r_k to test randomness and proved its asymptotic normality. Rank serial correlations, in circular and noncircular form, were studied further or compared with other tests by various authors: e.g. Stuart (1956), Knoke (1977), Dufour (1981), Bartels (1982). In particular, Bartels shows that asymptotically, the rank version of the von Neumann ratio statistic is a linear transformation of r_1.

In order to obtain the exact variance-covariance structure of the rank (non-circular) autocorrelations, we need to evaluate $E\,[S_4/S_2^2]$. In this situation, we see easily that

$$S_2 = \sum_{i=1}^{n} (R_i - \bar{R})^2 = \frac{n(n+1)(n-1)}{12} ,$$

$$S_4 = \frac{n(n+1)}{240} (3n^3 - 3n^2 - 23n + 7) .$$

Consequently,

$$\frac{S_4}{S_2^2} = \frac{3}{5} \frac{(3n^3 - 3n^2 - 23n + 7)}{n(n+1)(n-1)^2} . \tag{2.11}$$

$Var(r_k)$ and $Cov(r_k, r_h)$ can be obtained directly by substituting (2.11) in (2.3) and (2.5). For example, the variance of r_1 is

$$Var(r_1) = \frac{5n^4 - 24n^3 + 29n^2 + 54n - 16}{5n^2(n-1)^3} .$$

2.6 Distribution-free tests based on sample autocorrelations

Sample autocorrelations are frequently used to test the randomness of a time series $X_1, \ldots, X_n$, against serial correlation alternatives. For example, serial dependence at lag k may be represented as

$$H_k : \ Corr(X_t, X_{t+k}) \equiv \rho_k \neq 0, \ t=1, \ldots, n-k.$$

where $1 \leq k \leq n-1$. Under the null hypothesis of randomness (H_0), the expected value of r_k is zero (or very close to it) while, under an alternative of the form H_k, it is close to ρ_k (at least for n-k sufficiently large). Actually, from (2.2), we know that the expected value of r_k is

$$\mu_k = -(n-k)/\{n(n-1)\}$$

under H_0. Tests that use sample autocorrelations are typically based on an asymptotic normal approximation with an approximate standard error for

r_k : $n^{-\frac{1}{2}}$ or $\{(n-k)/n(n+2)\}^{\frac{1}{2}}$ (see Box and Jenkins, 1976, p. 35; Ljung and Box, 1978). Further, even under the standard normality assumption, the distribution of r_k is not well tabulated; the only exact test available in this respect is the van Neumann's test for serial dependence at lag 1 (von Neumann, 1941; Hart, 1942; Hart and von Neumann, 1942).

In view of this, an interesting application of the variance bounds given in Section (2.4) is to provide very general bounds on critical values for tests based on sample autocorrelations: such bounds are valid irrespective of any assumption on the distribution of the X_t's or, more generally, of any further knowledge of the distribution of r_k. The only restriction is $P[X_1 = X_2 = \ldots = X_n] = 0$. We can obtain such bounds easily by using Chebyshev's inequality: under H_0, we have that

$$P[|r_k-\mu_k| \geq \lambda\sigma_k] \leq 1/\lambda^2$$

for any $\lambda > 0$, where $\sigma_k = Var(r_k)$. Further, let σ_{kU}^2 be the bound on $Var(r_k)$ given in (2.8). Since $\sigma_k \leq \sigma_{kU}$, we also have

$$P[|r_k-\mu_k| \geq \lambda\sigma_{kU}] \leq 1/\lambda^2 \; , \; \lambda > 0 \; . \qquad (2.12)$$

Under H_0, $E[r_k] - \mu_k = 0$. Thus to test H_0 against an alternative of serial dependence at lag k, it is natural to consider a critical region of the form $|r_k-\mu_k| \geq c_\alpha\sigma_k$, where c_α is chosen so that

$$P[|r_k-\mu_k| \geq c_\alpha\sigma_k] = \alpha \; ,$$

the level of the test. However, neither σ_k nor c_α are known generally. From (2.12), we know that

$$P[|r_k-\mu_k| \geq \lambda_\alpha\sigma_{kU}] \leq \alpha$$

where $\lambda_\alpha = (1/\alpha)^{\frac{1}{2}}$, so that $\lambda_\alpha \sigma_{kU} \geq c_\alpha \sigma_k$. If $|r_k - \mu_k| \geq \lambda_\alpha \sigma_{kU}$, we know that r_k is significantly different from its expected value under H_0 for a test of level less than or equal to α. Typical values of λ_α are $\lambda_{.05} = 4.47$, $\lambda_{.10} = 3.16$.

The critical region $|r_k - \mu_k| \geq \sigma_{kU}/\alpha^{\frac{1}{2}}$ yields a level-α exact conservative test of H_0 against H_k. Though, the bound is somewhat crude, it has the attractive property of being valid irrespective of the form of the underlying distribution. It provides a threshold beyond which r_k must be viewed significant at level α even if we do not know the exact critical value or the form of the underlying distribution. For example, if $n = 80$, $k = 1$, $\alpha = 0.10$ and $r_1 = 0.5$, we have $\mu_1 = -0.0125$ and $\sigma_{1U} = 0.111$, so that $|r_1 - \mu_1| > (3.16)\sigma_{1U} = 0.351$: a value of r_1 as low as 0.5 can be viewed as significant at the 10% level under very weak assumptions.

3. RESULTS FOR NORMAL AND SPHERICALLY SYMMETRIC DISTRIBUTIONS

We will now specialize the above results to the case of a normal random sample. Since results obtained under the normality assumption remain exactly valid for the more general class of spherically symmetric distributions, we will cast them in this framework.

3.1 Spherically symmetric distributions

Let $\underline{X}$ and $\underline{\mu}$ be $n \times 1$ vectors with $\underline{X}$ random and $\underline{\mu}$ fixed. The vector $\underline{X}$ has a spherically symmetric (s.s.) distribution about $\underline{\mu}$ if and only if $G(\underline{X} - \underline{\mu})$ has the same distribution as $\underline{X} - \underline{\mu}$

for all orthogonal n×n matrices G. This class of distributions is extensively studied in Kelker (1970) and Lord (1954). Various statistical applications are discussed by Kariya and Eaton (1977), Kariya (1977), King (1980, 1981) and Zellner (1976). Chmielewski (1981) provides a bibliography.

The density of a vector $\underline{X}$ with a s.s. distribution, if it exists, is a function of the norm of $\underline{X} - \underline{\mu}$ only and its characteristic function $\phi(\underline{t})$ is of the form $\phi(\underline{t}) = \psi(\underline{t}'\underline{t}) \exp(i\underline{t}'\underline{\mu})$, where $\underline{t} = (t_1, \ldots, t_n)' \in \mathbb{R}^n$. The class of s.s. distributions includes such distributions as the multivariate normal and the multivariate Student-t with covariance matrix $\sigma^2 I_n$, a multivariate Cauchy, a multivariate exponential, etc.

Let $\underline{X} = (X_1, \ldots, X_n)'$ and $\underline{\mu} = \mu \underline{1}$ where $\underline{1} = (1, \ldots, 1)'$ is n×1. Denote $Z_i = X_i - \bar{X}$, i=1,...,n and $\underline{Z} = (Z_1, \ldots, Z_n)'$. We can write

$$\underline{Z} = M\underline{X} \tag{3.1}$$

where $M = I_n - (1/n)\underline{1}\,\underline{1}'$ is a n×n symmetric idempotent matrix of rank n-1. Further we can find a n×n orthogonal matrix P such that

$$P'MP = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix} .$$

Let $P = (P_1, P_2)$ where P_1 is n×(n-1) and P_2 is n×1. Then, if $\underline{X}$ has a s.s. distribution about $\underline{\mu}$, the vector $\underline{W} = \underline{Z}/\|\underline{Z}\|$ has a distribution identical to the one of the vector $P_1(\underline{U}/\|\underline{U}\|)$, where $\underline{U}$ has a multinormal distribution $N(\underline{0}, I_{n-1})$; $\|\cdot\|$ denotes the Euclidean norm. We can see this as follows. Let $\underline{v} = P'\underline{X} = (\underline{v}_1', \underline{v}_2')'$ where $\underline{v}_1 = P_1'\underline{X}$ and $\underline{v}_2 = P_2'\underline{X}$. It is then simple to check that

$$\underline{Z} = P_1\underline{v}_1, \quad \underline{Z}'\underline{Z} = \underline{v}_1'\underline{v}_1, \quad \underline{W} = P_1\ (\underline{v}_1/\|\underline{v}_1\|) \tag{3.2}$$

where $P_1'P_1 = I_{n-1}$ and $P_1'\,\underline{1} = \underline{0}$. Further, by considering the characteristic function of $\underline{v}_1$, we can see easily that $\underline{v}_1$ has a s.s. distribution about zero. The result then follows by applying Theorem 2.1 of Kariya and Eaton (1977).

A useful consequence of this property is the following: any statistic of the form $T(\underline{W})$ has a distribution which is independent of the functional form of the s.s. distribution of $\underline{X}$, provided $\underline{\mu} = \mu\underline{1}$. We can thus study its distribution assuming $\underline{X}$ is $N(\mu\underline{1}, I_n)$. In particular, from the definition of sample autocorrelations, we have

$$r_k = \sum_{i=1}^{n-k} W_i W_{i+k} \quad , \quad 1 \le k \le n-1$$

where $\underline{W} = (W_1, \ldots, W_n)'$. Therefore, the vector of sample autocorrelations has the same distribution whenever $\underline{X}$ has a s.s. distribution with $\underline{\mu} = \mu\underline{1}$: we can study its distribution by assuming $\underline{X}$ is $N(\mu\underline{1}, I_n)$.

3.2 Exact variances and covariances

To obtain explicit formulae for $\mathrm{Var}(r_k)$ and $\mathrm{Cov}(r_k, r_h)$, we need $E[S_4/S_2^2]$. Since

$$S_4/S_2^2 = \sum_{i=1}^{n} W_i^2$$

where $\underline{W} = \underline{Z}/\|\underline{Z}\|$, we know from the previous section that the distribution of S_4/S_2^2 is the same for all s.s. distributions. Assuming normality, Moran (1948) found that

$$E[S_4/S_2^2] = \frac{3(n-1)}{n(n+1)} \quad . \tag{3.3}$$

If we substitute (3.3) into (2.3), we find after some algebra:

$$\mathrm{Var}(r_k) = \frac{n^4 - (k+3)n^3 + 3kn^2 + 2k(k+1)n - 4k^2}{(n+1)n^2\,(n-1)^2} \quad , \tag{3.4}$$

$1 \le k \le n-1$. With $k = 1$, we retrieve the result of Moran (1948)

$$\mathrm{Var}(r_1) = \frac{(n-2)^2}{n^2(n-1)} \, .$$

Up to order n^{-2}, $\mathrm{Var}(r_k)$ is given by

$$\mathrm{Var}(r_k) = \frac{n-(k+2)}{n^2} + 0(n^{-3}).$$

We derive the covariance between r_k and r_h in a similar way from (2.5) and get

$$\mathrm{Cov}(r_k, r_h) = \frac{2\{kh(n-1) - (n-h)(n^2-k)\}}{(n+1)n^2 (n-1)^2} , \qquad (3.5)$$

where $1 \le k < h \le n-1$. Developing up to order n^{-2}, we have

$$\mathrm{Cov}(r_k, r_h) = -\frac{2}{n^2} + 0(n^{-3})$$

which is in agreement with a result of Fuller (1976, p. 242).

In a Monte Carlo study on the asymptotic efficiency of tests for randomness, Knoke (1977) compared various test statistics. One of them is

$$T = \sum_{j=1}^{n-1} \frac{1}{j} r_j$$

and the critical region was determined from a normal approximation with the exact mean obtained from (2.2) and an empirical variance. If we use expressions (3.4) and (3.5) of this paper, it is straightforward to compute the exact variance of T.

3.3 Comparison of the normal variance with the upper bound

For large n, the exact variance for a normal sample, say σ^2_{kN}, is almost identical to the upper bound σ^2_{kU} obtained for exchangeable random variables. Effectively, we have

$$\sigma^2_{kN} = \frac{n-(k+2)}{n^2} + 0(n^{-3})$$

and

$$\sigma^2_{kU} = \frac{n-k}{n^2} + 0(n^{-3}) .$$

It is immediate that $\lim_{n\to\infty} \sigma^2_{kU}/\sigma^2_{kN} = 1$. Table 1 gives the exact ratio

$\sigma^2_{kU}/\sigma^2_{kN}$ for various values of k and n. We see that the upper bound is nearly attained in the normal case even for samples as small as 20.

Table 1: Values of the ratio $\sigma^2_{kU}/\sigma^2_{kN}$ for various values of k and n.

n \ k	1	2	3	4	5	10	15	20	25
5	1.67	1.70	1.96	2.42	--	--	--	--	--
10	1.25	1.25	1.24	1.25	1.26	--	--	--	--
15	1.15	1.15	1.15	1.15	1.15	1.17	--	--	--
20	1.11	1.11	1.11	1.11	1.11	1.11	1.13	--	--
25	1.09	1.09	1.09	1.09	1.09	1.09	1.09	1.10	--
30	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.09
40	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05
50	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04
100	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02
200	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01

3.4 Some numerical results

Tests of randomness that use sample autocorrelations r_k are usually based on an asymptotic normal distribution with mean 0 and approximate standard error $n^{-\frac{1}{2}}$ (Box and Pierce, 1970) or $\{(n-k)/n(n+2)\}^{\frac{1}{2}}$ (Ljung and Box, 1978). It is worthwhile to see what is the gain realized by replacing the approximate mean and variance by the exact mean in (2.2) and the exact variance in (3.4).

To investigate this issue, we conducted the following Monte Carlo experiment. For each of five different series lenghts (n=10,20,30,50,100), 10 000 independent realizations of a normal white noise were generated using the subroutine GGUBS of IMSL(1980), and for each realization, sample autocorrelations r_k at several lags were computed. We then examined the quality of the asymptotic N(0,1) approximation for three different versions of the normalized statistics $R_k = (r_k - \mu_k)/\sigma_k$. The three normalizations S1, S2 and S3 were defined as follows: for S1, $\mu_k = 0$ and $\sigma_k = n^{-\frac{1}{2}}$; for S2,

$\mu_k = 0$ and $\sigma_k = \{(n-k)/n(n+2)\}^{\frac{1}{2}}$; for S3, μ_k is the exact mean in (2.2) and σ_k the exact standard error in (3.4). To appreciate the accuracy of the N(0,1) approximation, we examined the empirical frequencies of rejection of the null hypothesis of randomness by tests with three different nominal levels (5, 10 and 20 percent). Further, for each value of n and k, we considered three types of tests: one-sided tests against positive serial dependence (R), one-sided tests against negative serial dependence (L) and two-sided tests (B).

The results of the experiment are presented in Table 2. We make the following observations. First, for S1, the N(0,1) distribution provides a relatively poor approximation, even for series of 100 observations. Second, the approximation is better for S2, but the empirical significance levels of the one-sided tests remain appreciably different of the theoritical levels (at least for short series of 50 observations or less). Third, the best results are obtained with the normalization S3: the agreement between the empirical and the theoritical levels is very good both for one-sided and two-sided tests and the approximation is satisfactory even for series of 10 observations. These results clearly suggest that the normalization based on the exact mean and variance of r_k is preferable to the approximate normalizations currently in use. Further, it is easy to implement the exact formulae in computer programs.

4. CONCLUDING REMARKS

We described, in this paper, a number of exact and robust properties of sample autocorrelations. By robust properties, we mean properties that hold under wide distributional assumptions, not necessarily properties of robustness to outliers (Martin, 1981). We gave general expressions as well

Table 2: Empirical levels of tests based on sample autocorrelations for different normalizations (in percentage)

Test	%level	Side	n=10, k=1	n=10, k=3	n=10, k=5	n=20, k=1	n=20, k=3	n=20, k=5	n=20, k=10	n=30, k=1	n=30, k=3	n=30, k=5	n=30, k=10	n=30, k=15
S1	5	R	0.8	0.5	0.0	1.9	1.6	1.4	0.1	2.5	2.5	2.0	1.2	0.4
		L	5.5	2.8	0.0	5.8	4.7	3.6	0.8	6.5	5.7	4.5	2.8	0.8
		B	2.2	0.7	0.0	3.0	2.7	1.6	0.0	4.3	3.5	2.6	1.4	0.2
	10	R	3.2	2.6	0.9	5.1	4.6	3.8	1.9	5.9	5.7	5.2	3.9	2.4
		L	13.4	8.8	3.3	12.7	11.1	9.1	4.4	13.0	12.0	10.2	7.8	4.3
		B	6.3	3.3	0.0	7.7	6.3	4.9	0.9	9.0	8.1	6.5	4.1	1.2
	20	R	8.9	8.1	7.9	13.0	11.5	10.8	8.9	14.1	13.5	12.9	11.7	9.5
		L	28.2	23.0	18.0	25.6	23.9	22.0	15.5	25.2	23.9	22.2	18.7	14.7
		B	16.7	11.3	4.1	17.8	15.7	12.9	6.3	18.9	17.7	15.5	11.6	6.7
S2	5	R	1.9	2.7	3.3	2.6	2.9	3.1	3.8	3.1	3.4	3.5	3.8	4.4
		L	9.8	9.4	9.4	7.7	7.8	7.6	7.5	7.6	7.7	7.1	7.4	7.2
		B	5.3	5.8	4.7	4.6	5.1	5.2	5.2	5.4	5.5	5.2	5.6	5.1
	10	R	5.2	6.0	8.3	6.3	6.4	6.6	8.4	6.7	7.2	7.4	7.9	8.7
		L	18.6	17.7	18.6	14.9	15.0	14.7	14.7	14.6	14.5	13.7	13.7	13.7
		B	11.7	12.1	12.7	10.3	10.7	10.7	11.3	10.7	11.1	10.6	11.2	11.6
	20	R	11.4	12.5	16.9	14.5	14.2	14.6	17.2	14.9	15.3	15.5	16.5	17.8
		L	33.2	31.7	32.2	27.6	27.6	27.5	27.2	26.4	25.9	25.5	25.8	25.8
		B	23.8	23.7	26.9	21.1	21.4	21.3	23.1	21.3	21.8	21.0	21.6	22.4
S3	5	R	5.6	5.6	5.9	4.9	5.1	5.0	5.3	4.8	5.0	5.0	5.1	5.5
		L	4.6	4.8	3.3	4.6	4.8	5.0	5.0	5.2	5.2	4.8	5.3	4.9
		B	4.6	4.9	2.6	4.2	4.7	4.5	4.3	4.9	5.1	4.7	5.1	4.5
	10	R	10.6	10.5	12.6	10.4	9.9	10.0	10.9	9.8	10.2	10.0	10.4	10.6
		L	10.6	10.2	10.3	10.1	10.2	10.2	10.2	10.6	10.7	10.1	10.6	10.7
		B	10.2	10.5	9.3	9.5	9.9	10.0	10.3	9.9	10.2	9.9	10.3	10.4
	20	R	20.7	20.3	22.8	20.9	20.2	20.4	21.5	19.9	20.0	20.2	20.4	20.8
		L	21.5	21.4	22.3	20.4	20.6	20.9	21.1	21.0	20.7	20.5	20.5	21.0
		B	21.2	20.7	23.0	20.5	20.1	20.3	21.1	20.4	20.8	20.1	21.1	21.3

Test	%level	Side	n=50, k=1	n=50, k=3	n=50, k=5	n=50, k=10	n=50, k=15	n=50, k=25	n=100, k=1	n=100, k=3	n=100, k=5	n=100, k=10	n=100, k=15	n=100, k=25	n=100, k=50
S1	5	R	3.1	2.8	2.9	2.1	1.7	0.5	3.8	3.6	3.5	3.4	3.0	2.1	1.0
		L	6.1	5.4	5.0	3.8	2.8	1.0	5.6	5.6	5.3	5.1	4.2	3.3	1.2
		B	4.2	3.9	3.3	2.6	1.7	0.3	4.1	4.6	3.9	3.9	3.2	2.3	0.6
	10	R	7.2	6.5	6.4	5.5	5.1	2.3	7.6	7.7	7.8	7.3	6.8	5.6	3.1
		L	12.1	11.3	10.4	9.3	7.2	4.2	11.6	11.0	10.8	10.6	9.6	8.0	3.9
		B	9.1	8.1	7.9	5.8	4.5	1.4	9.4	9.2	8.8	8.4	7.1	5.3	2.1
	20	R	16.1	14.8	15.1	13.2	13.8	9.8	16.7	16.8	16.5	16.3	15.5	14.2	11.2
		L	23.1	23.3	22.0	20.8	18.2	14.3	23.2	22.2	22.5	22.0	20.7	19.0	13.1
		B	19.3	17.8	16.8	14.8	12.3	6.6	19.3	18.7	18.6	17.9	16.4	13.6	7.0
S2	5	R	3.4	3.3	3.8	3.8	4.4	4.1	3.9	3.9	3.9	4.3	4.2	3.9	4.9
		L	6.7	6.4	6.4	6.5	6.1	6.5	6.0	6.0	6.2	6.3	5.9	6.1	5.7
		B	4.9	4.8	4.9	4.8	5.0	5.0	4.5	5.1	4.8	5.2	5.1	4.7	5.3
	10	R	7.8	7.5	7.6	7.7	9.0	8.6	8.0	8.3	8.5	8.5	8.5	8.3	9.8
		L	13.0	12.6	12.4	12.7	11.8	12.8	12.0	11.6	11.7	12.3	12.0	11.7	11.6
		B	10.1	9.7	10.2	10.3	10.4	10.6	9.9	9.9	10.1	10.6	10.1	10.0	10.6
	20	R	16.8	15.8	16.6	15.7	17.8	17.8	17.0	17.4	17.4	17.8	17.6	17.5	19.0
		L	23.8	24.7	23.9	24.0	23.7	23.6	23.7	22.8	23.5	23.5	23.2	22.9	21.9
		B	20.7	20.1	20.0	20.4	20.8	21.5	20.0	19.8	20.2	20.8	20.5	20.0	21.3
S3	5	R	5.0	4.6	4.9	5.0	5.6	5.0	4.8	4.8	4.9	5.3	5.1	4.8	5.5
		L	4.8	4.8	4.8	4.9	4.6	5.1	4.7	5.1	4.9	5.2	4.8	4.9	4.8
		B	4.6	4.7	4.7	4.7	4.9	4.5	4.6	5.0	4.8	5.1	5.0	4.6	5.2
	10	R	10.1	9.7	10.0	9.9	10.7	10.2	9.9	9.9	10.1	10.2	10.0	9.9	10.8
		L	9.9	9.8	9.7	10.0	9.6	10.5	10.2	9.9	9.9	10.3	10.2	10.1	10.0
		B	9.9	9.5	9.8	9.9	10.2	10.0	9.5	9.9	9.8	10.4	9.9	9.7	10.3
	20	R	20.5	19.8	20.3	19.5	21.1	20.4	19.7	20.1	20.4	20.3	20.5	19.8	21.0
		L	19.7	20.2	19.7	20.4	19.9	20.5	20.4	20.2	20.2	20.8	20.3	20.2	19.8
		B	20.0	19.4	19.7	19.9	20.3	20.7	20.1	19.7	20.1	20.5	20.2	20.0	20.9

Tests are based on asymptotic N(0,1) approximation of $R_k=(r_k-\mu_k)/\sigma_k$ where : $\mu_k=0$ for S1 and S2, $\mu_k=-(n-k)/\{n(n-1)\}$ for S3, $\sigma_k=n^{-\frac{1}{2}}$ for S1, $\sigma_k=\{(n-k)/n(n+2)\}^{\frac{1}{2}}$ for S2, and $\sigma_k=\{Var(r_k)\}^{\frac{1}{2}}$ from formula (3.4) for S3. R and L refer to one-sided test against positive and negative dependence respectively. B refers to a two-sided test.

as bounds for the second moments of sample autocorrelations, when the variables in a sequence are exchangeable. The latter thus hold for an arbitrary random sequence. We also gave the exact variances and covariances of autocorrelation coefficients in two important cases: rank autocorrelations from an arbitrary random sequence and standard sample autocorrelations from a normal white noise. The latter also hold for s.s. distributions.

The bounds on the second moments are surprisingly tight: the upper bounds on the variances are of order n^{-1} while those one the absolute value of the covariances are of order n^{-2}. The bounds on the variances also yield exact upper limits on critical values for tests of randomness based on sample autocorrelations, without any assumption on the form of the distribution. In the normal case, we found in a Monte Carlo simulation that normalizing the sample autocorrelations with their exact mean and variance, instead of the usual approximations, can improve considerably the accuracy of the asymptotic N(0,1) distribution. Actually, in the simulations performed, the empirical levels obtained with the exact normalization were practically identical to the theoretical levels (of 5, 10 or 20 percent). We thus recommend strongly to use the exact means and variances when testing randomness with sample autocorrelations. Note here that tail probabilities for sample autocorrelations (in the normal case) can in principle be obtained by using the method of Imhof (1961); see Ramasubban (1972), Sneek (1983). This remains,however, relatively costly and no table of exact critical values for sample autocorrelations is yet available. Clearly simple improvements in the quality of the asymptotic normal approximation, as described above, remain an attractive practical alternative.

Finally, we can note that the robust properties given above also illustrate clearly the fact (known in theory) that the sample autocorrelations are not always a good indicator of independence. For example, $E[r_1] = -n^{-1}$ for both an arbitrary random sample and for a symmetric normal distribution where the pairwise correlation between any two observations is $\rho=0.99$ (since the latter yelds exchangeable random variables). Similarly, the distributions of sample autocorrelations is the same for all spherically symmetric distributions, including that of a normal white noise: except for the latter, variables from a s.s. distribution are not independent (Kelker, 1970). This does not mean that sample autocorrelations cannot be used to test randomness. But appropriate assumptions excluding certain forms of dependence are required.

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