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SOLUTION ALGORITHMS FOR A COMBINED
RESIDENTIAL LOCATION AND
TRANSPORTATION MODEL

by

Marc Los and Sang Nguyen

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ABSTRACT

This article specifies a model for predicting residential location, modal choice and transportation system performance which is formulated as a convex programming problem. Two solution algorithms belonging to the class of feasible descent direction methods are proposed to solve that convex programming problem. Several extensions of the basic model are discussed and the associated solution algorithms are outlined.

RÉSUMÉ

Cet article spécifie un modèle de prévision de la localisation résidentielle, du choix modal et de la performance du système de transport, qui est formulé comme problème de programmation convexe. Deux algorithmes appartenant à la classe des méthodes de direction de descente réalisables sont proposés pour résoudre le programme. Plusieurs extensions du modèle de base sont discutées et les algorithmes associés sont indiqués brièvement.

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I - Introduction

This paper is concerned with the prediction for the medium term of the demand for housing by type and location within an urban area, of the demand for different modes of travel to work, and of the resulting utilization and level of performance of the transportation system. A combined residential location and transportation predictive model was proposed in a previous paper by one of the authors for that purpose.¹ That model can be considered either as an extension of the Senior-Wilson residential location model² or as an extension of the Florian-Nguyen combined trip-distribution modal split and trip-assignment model³. The actual practical utilization of that model relies on the solution of a convex mathematical programming problem of very large scale if there is to be sufficient zonal disaggregation of the urban area, sufficient disaggregation of households by income groups (at least) and sufficient disaggregation of housing between different housing types (renting vs owning; different density types such as single family house, duplex or triplex, tower apartment; different dwelling sizes, etc...) and finally if a distinction is to be made between peak-hour and non peak-hour trips. Such a degree of detail is necessary for the model to be useful to urban and transportation planners.

In this paper we propose efficient algorithms for the combined residential location and transportation model which we believe are capable of handling a realistic level of spatial detail as well as a realistic level of disaggregation for housing types and household types.

¹ Model P3 of Los (1979a).

² See Senior and Wilson (1974) or Senior (1977).

³ See Florian and Nguyen (1978).

The model can be used for determining the impact of transportation decisions (new investments such as a new subway line, or changes in transit fares, in gasoline taxes) on residential location. Reciprocally, it can be used to study the impact of changes in the supply and location of housing on the use (e.g. choice of mode for work trips) and performance (travel times) of the transportation system. In addition the structure of the model allows a determination of the impact on residential location, modal choice and transportation system performance of changes in income distribution, employment location and also of changes in work-schedules. Figure 1 describes the choices open to each household in the model.

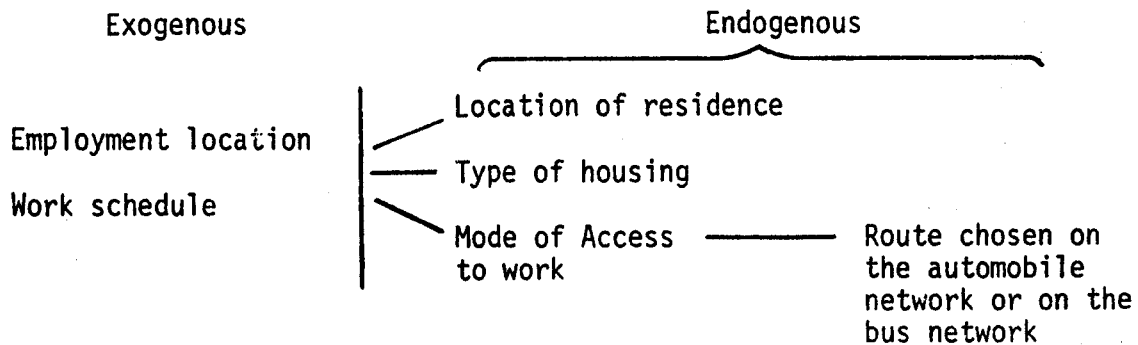


Figure 1: Choices open to each household in the model

The need for a joint model of residential location and transportation comes from the interdependencies that exist between housing choices and modal choice for a given household: for instance a sizable increase in gasoline prices may lead to a simultaneous change in location and modal choice (from living in a suburban location and commuting by car to the city, to living closer to the employment location and commuting to work by bus). In addition, household residential location decisions depend

on accessibility to work, among other things, while accessibility itself depends on mode choice and transportation congestion. Reciprocally transportation congestion depends on travel demand which depends itself on residential location.

The rest of the paper is organized as follows. Section II states the assumptions of the model and its mathematical expression. A methodology for calibrating and solving the model is presented. Section III presents the algorithms to be used for prediction: two cases are considered, one with and another without automobile congestion. For the latter case, two distinct computational approaches belonging to the category of feasible descent direction methods are presented in detail. Section IV discusses some possible extensions of the model and of the associated algorithms. Section V concludes the paper.

II - A combined residential location and transportation model

The spatial distribution and nature of the housing stock, as well as the transportation system (road network, bus network, frequency, etc...) are exogenous. The price of housing and the money cost of travel by different modes are also assumed known. So are the spatial distribution of employment and the distribution of households among income groups. The model is concerned with work trips and neglects other trip purposes. Work schedules are assumed known.

Residential household preferences are translated by means of the bid rent concept.¹ All the households of a given type have a uniform and

¹ See Alonso (1964), Herbert and Stevens (1960), Wheaton (1974) and MacKinnon (1979).

prespecified level of utility. Thus the bid rents of households for different types of housing in different locations are assumed given. There is one worker per household.

Bid rents are obtained as follows. A household of type w has the following *linear* utility function:

$$U^w = M_{ij}^{kw} + \zeta^w \cdot \tilde{h}_i^k - \theta c_{ij} \quad (1)$$

where

M_{ij}^{kw} is the expenditure on nonlocational goods if the household head works in zone j , lives in zone i in a house of type k and has income w ;

\tilde{h}_i^k is a vector of housing characteristics for a type k house in zone i ;

c_{ij} is the time spent travelling to work from zone i to zone j (it depends on the mode and on the level of congestion);

ζ^w and θ are estimated coefficients; in particular θ is the value of time assumed the same for all household types.

For a household (j,w) choosing the (i,k) housing bundle we have the budget constraint

$$w = b_{ij}^{kw,m} + M_{ij}^{kw} + \tilde{c}_{ij}^m \quad m=au, tr \quad (2)$$

where \tilde{c}_{ij}^{au} (resp. \tilde{c}_{ij}^{tr}) is the annual money cost of travel between zones i and j by auto (resp. by transit) and where $b_{ij}^{kw,m}$, the residual budget, is the bid rent of household (j,w) for a dwelling (i,k) when the worker commutes by mode m . From equations (1) and (2) we obtain the bid rent as a function of the level of utility and of the value of time spent travelling to work:

$$b_{ij}^{kw,m} = w - U^W + \xi^W \cdot h_i^k - \tilde{c}_{ij}^m - \theta c_{ij} \quad m=au, tr \quad (3)$$

or
$$b_{ij}^{kw,m} = \beta_{ij}^{kw,m}(U^W) - \theta c_{ij} \quad (4)$$

where
$$\beta_{ij}^{kw,m}(U^W) = w - U^W + \xi^W \cdot h_i^k - \tilde{c}_{ij}^m \quad m=au, tr \quad (5)$$

Utility functions can be estimated either by econometric techniques (Wheaton, 1974) or by household survey techniques (Harsman and Snickars, 1975). The estimated utility functions have to be linearized in order for the bid rents to be linear functions of travel time as in equation (4). The assumption of exogenous utility levels is consistent with that of an *open city*, i.e. of a city which is part of a system of cities among which households can migrate freely. The job location is supposed to be *prior* to the choice of a residence.

The transportation system is composed of two modes: the private automobile and the bus. Every worker owns a car; there is congestion on the road network and there is a uniform car-occupancy factor, so that it is indifferent whether one measures flows on the road network in vehicles or in persons.

A is the set of arcs a of the road network; $s_a(v_a)$ is the congestion function on link a, assumed to be a continuous increasing function of the flow v_a ; the transit network consists of a set, S, of access arcs, transfer arcs and transit-line segments. A transit line is composed of a number of segments. A time, c_s , is associated with each arc and line segment, $s \in S$ of the transit network as follows:

- . a walking and waiting time with an access arc;
- . a walking and a waiting time with a transfer arc;
- . an in-vehicle time with a line segment.

The link times c_s are not functions of the transit-link volumes, v_s and there is no congestion effect assumed on the transit network: travellers on the transit network choose the shortest path between an origin and a destination. It is assumed that the proportion of the total number of work-trips taking place during peak hours is exogenous, that automobile congestion occurs only during peak hours and that bus frequency may vary between peak hours and non-peak hours.

The following additional notation is used:

$E_{j,\pi}^w$ is the number of jobs yielding income w in zone j , whose associated trips occur during peak hours;

$E_{j,\bar{\pi}}^w$ is the number of jobs yielding income w in zone j , whose associated trips take place outside peak hours;

H_i^k is the number of houses of type k in zone i ;

r_i^k is the *actual* annual rent paid for a type k house, in zone i ; ¹

$u_{ij,\pi}^{tr}$ is the travel time from zone i to zone j on the bus network during peak hours;

$c_{ij,\bar{\pi}}^{au}$ is the travel time from zone i to zone j by private automobile outside peak hours.

The quantities $u_{ij,\pi}^{tr}$, $u_{ij,\bar{\pi}}^{tr}$ and $c_{ij,\bar{\pi}}^{au}$ are all constant. This implies that $c_{ij,\bar{\pi}}^{au}$ is obtained from the "free-flow" travel times on the links of the road network.

$$\bar{\beta}_{ij}^{kw,m} = \beta_{ij}^{kw,m} - r_i^k \quad m=au, tr \quad (6)$$

$T_{ij,\pi}^{kw,au}$ is the number of trips made by (j,w) heads of household living in an (i,k) residential bundle, using an automobile during peak hours;

¹ In the case of homeowners an *equivalent* annual rent payment has to be determined for a given price, taking into account also property taxes.

$T_{ij,\pi}^{kw,au}$ is the number of trips made by (j,w) heads of household living in an (i,k) residential bundle, using an automobile outside peak hours;

$T_{ij,\pi}^{kw,tr}$ is the number of trips made by (j,w) heads of household living in an (i,k) residential bundle, using public transit during peak hours;

$T_{ij,\pi}^{kw,tr}$ is the number of trips made by (j,w) heads of household living in an (i,k) residential bundle, using public transit outside peak hours;

$h_{m,ij}^{au}$ is the flow of private automobiles using path m between zones i and j, converted into person trips by the uniform car-occupancy factor.

The total flow of vehicles on link a (converted into persons) is:

$$v_a = \sum_{m,ij} \delta_{am,ij} h_{m,ij}^{au} + v_a^{tr} \quad (7)$$

where v_a^{tr} is the vehicle equivalent of the number of buses that use link a (again converted into person trips by use of the uniform car-occupancy factor). The quantity v_a^{tr} may be computed by determining the number of automobiles equivalent to a bus and by considering all the buses that use the link per time period. The assumptions made on the two modes imply that private automobiles do not have any influence on the speed or frequency of buses, whereas buses do interfere with the traffic of automobiles and increase congestion. This assumes traffic-management decisions giving priority to buses over cars on the road — for instance, special bus lanes on which buses have absolute priority but which are open to cars in the absence of buses.

We require the quantities $T_{ij,\pi}^{kw,au}$, $T_{ij,\bar{\pi}}^{kw,au}$, $T_{ij,\pi}^{kw,tr}$, $T_{ij,\bar{\pi}}^{kw,tr}$ and $h_{m,ij}^{au}$ to provide the solution to the following minimization problem:

$$\begin{aligned}
 P: \quad \text{Min} \quad & \sum_{i,j,k,w} T_{ij,\pi}^{kw,au} \ln T_{ij,\pi}^{kw,au} + \sum_{i,j,k,w} T_{ij,\bar{\pi}}^{kw,au} \ln T_{ij,\bar{\pi}}^{kw,au} \\
 & + \sum_{i,j,k,w} T_{ij,\pi}^{kw,tr} \ln T_{ij,\pi}^{kw,tr} + \sum_{i,j,k,w} T_{ij,\bar{\pi}}^{kw,tr} \ln T_{ij,\bar{\pi}}^{kw,tr} \\
 & - \mu \left[\sum_{i,j,k,w} \hat{\beta}_{ij}^{kw,au} (T_{ij,\bar{\pi}}^{kw,au} + T_{ij,\pi}^{kw,au}) + \sum_{i,j,k,w} \hat{\beta}_{ij}^{kw,tr} (T_{ij,\pi}^{kw,tr} + T_{ij,\bar{\pi}}^{kw,tr}) \right] \\
 & - \theta \left[\sum_{a \in A} \int_0^v s_a(x) dx + \sum_{i,j,k,w} T_{ij,\pi}^{kw,tr} u_{ij,\pi}^{tr} \right. \\
 & \left. + \sum_{i,j,k,w} T_{ij,\bar{\pi}}^{kw,au} c_{ij,\bar{\pi}}^{au} + \sum_{i,j,k,w} T_{ij,\bar{\pi}}^{kw,tr} u_{ij,\bar{\pi}}^{tr} \right] \quad (8)
 \end{aligned}$$

s. t.

$$\sum_{i,k} (T_{ij,\pi}^{kw,au} + T_{ij,\pi}^{kw,tr}) = E_{j,\pi}^w \quad \text{all } (j,w) \quad (9)$$

$$\sum_{i,k} (T_{ij,\bar{\pi}}^{kw,au} + T_{ij,\bar{\pi}}^{kw,tr}) = E_{j,\bar{\pi}}^w \quad \text{all } (j,w) \quad (10)$$

$$\sum_{j,w} (T_{ij,\pi}^{kw,au} + T_{ij,\bar{\pi}}^{kw,au} + T_{ij,\pi}^{kw,tr} + T_{ij,\bar{\pi}}^{kw,tr}) \leq H_i^k \quad \text{all } (i,k) \quad (11)$$

$$\sum_{k,w} T_{ij,\pi}^{kw,au} = \sum_m h_{m,ij}^{au} \quad \text{all } (i,j) \quad (12)$$

$$v_a = \sum_{m,i,j} \delta_{am,ij} h_{m,ij}^{au} + v_a^{tr} \quad \text{all } a \quad (13)$$

$$T_{ij,\pi}^{kw,au} \geq 0, T_{ij,\bar{\pi}}^{kw,au} \geq 0, T_{ij,\pi}^{kw,tr} \geq 0, T_{ij,\bar{\pi}}^{kw,tr} \geq 0 \quad \text{all } i,j,k,w$$

$$h_{m,ij}^{au} \geq 0 \quad \text{all } m,i,j, \quad (14)$$

$$\delta_{am,ij} = \begin{cases} 1 & \text{if } a \text{ belongs to path } m \text{ between } i \text{ and } j \\ 0 & \text{otherwise} \end{cases} \quad (15)$$

Constraint (12) represents the conservation of flows constraint for the automobile trips during peak hours. Constraints (9) and (10) allocate work trips exogenously between peak-hour and non peak-hour trips depending on the employment type or location. Constraint (11) is a housing-supply constraint. The objective function can be written as

$$H + \mu F$$

$$\begin{aligned} \text{with } H \equiv & \sum_{ijkw} T_{ij,\pi}^{kw,au} \ln T_{ij,\pi}^{kw,au} + \sum_{ijkw} T_{ij,\bar{\pi}}^{kw,au} \ln T_{ij,\bar{\pi}}^{kw,au} \\ & + \sum_{ijkw} T_{ij,\pi}^{kw,tr} \ln T_{ij,\pi}^{kw,tr} + \sum_{ijkw} T_{ij,\bar{\pi}}^{kw,tr} \ln T_{ij,\bar{\pi}}^{kw,tr} \end{aligned} \quad (16)$$

$$\begin{aligned} \text{and } F \equiv & \theta \left[\sum_{a \in A} \int_0^v a_s a(x) dx + \sum_{ijkw} T_{ij,\pi}^{kw,tr} u_{ij,\pi}^{tr} \right. \\ & + \sum_{ijkw} T_{ij,\bar{\pi}}^{kw,au} c_{ij,\bar{\pi}}^{au} + \sum_{ijkw} T_{ij,\bar{\pi}}^{kw,tr} u_{ij,\bar{\pi}}^{tr} \left. \right] \\ & - \sum_{ijkw} \hat{\beta}_{ij}^{kw,au} (T_{ij,\bar{\pi}}^{kw,au} + T_{ij,\pi}^{kw,au}) - \sum_{ijkw} \hat{\beta}_{ij}^{kw,tr} (T_{ij,\pi}^{kw,tr} + T_{ij,\bar{\pi}}^{kw,tr}) \end{aligned} \quad (17)$$

H is an entropy term while F coincides with (minus) the net aggregate willingness to pay (or aggregate net bids) if the peak-hour automobile travel times are constant. The coefficient μ , to be estimated, weighs the two parts of the objective function.

The objective function is strictly convex in the variables $T_{ij,\pi}^{kw,au}$, $T_{ij,\bar{\pi}}^{kw,au}$, $T_{ij,\pi}^{kw,tr}$, $T_{ij,\bar{\pi}}^{kw,tr}$ and v_a . There is a unique solution and the Kuhn-Tucker conditions are necessary and sufficient for optimality. They are:

$$T_{ij,\pi}^{kw,au} = \exp(-1-v_{j,\pi}^w - \alpha_i^k) \exp [\mu(\hat{\beta}_{ij}^{kw,au} - \theta u_{ij,\pi}^{au})] \quad (18)$$

$$T_{ij,\bar{\pi}}^{kw,au} = \exp(-1-v_{j,\bar{\pi}}^w - \alpha_i^k) \exp [\mu(\hat{\beta}_{ij}^{kw,au} - \theta c_{ij,\bar{\pi}}^{au})] \quad (19)$$

$$T_{ij,\pi}^{kw,tr} = \exp(-1-v_{j,\pi}^w - \alpha_i^k) \exp [\mu(\hat{\beta}_{ij}^{kw,tr} - \theta u_{ij,\pi}^{tr})] \quad (20)$$

$$T_{ij,\bar{\pi}}^{kw,tr} = \exp(-1-v_{j,\bar{\pi}}^w - \alpha_i^k) \exp [\mu(\hat{\beta}_{ij}^{kw,tr} - \theta u_{ij,\bar{\pi}}^{tr})] \quad (21)$$

$$h_{m,ij}^{au} > 0 \Rightarrow \sum_a s_a(v_a) \delta_{am,ij} = u_{ij,\pi}^{au} \quad (22)$$

$$h_{m,ij}^{au} = 0 \Rightarrow \sum_a s_a(v_a) \delta_{am,ij} \geq u_{ij,\pi}^{au} \quad (23)$$

$$\alpha_i^k > 0 \Rightarrow \sum_{j,w} (T_{ij,\pi}^{kw,au} + T_{ij,\bar{\pi}}^{kw,au} + T_{ij,\pi}^{kw,tr} + T_{ij,\bar{\pi}}^{kw,tr}) = H_i^k \quad (24)$$

and $\sum_{j,w} (T_{ij,\pi}^{kw,au} + T_{ij,\bar{\pi}}^{kw,au} + T_{ij,\pi}^{kw,tr} + T_{ij,\bar{\pi}}^{kw,tr}) < H_i^k \Rightarrow \alpha_i^k = 0 \quad (25)$

where $u_{ij,\pi}^{au} = \mu \theta u_{ij,\pi}^{au}$, α_i^k , $v_{j,\pi}^w$ and $v_{j,\bar{\pi}}^w$ are the dual variables associated respectively with constraints (12), (11), (9) and (10). The conditions (22)-(23) are the Wardrop's user equilibrium conditions on the automobile network during peak hours and $u_{ij,\pi}^{au}$ is the equilibrium travel time by private automobile between zones i and j during peak hours.

The modal share for a quadruplet (i,j,k,w) can be shown from (4), (5) and (6) to be given during peak hours by:

$$\frac{T_{ij,\pi}^{kw,au}}{T_{ij,\pi}^{kw,tr} + T_{ij,\pi}^{kw,au}} = \frac{\exp[-\mu(\tilde{c}_{ij}^{au} + \theta u_{ij,\pi}^{au})]}{\exp[-\mu(\tilde{c}_{ij}^{au} + \theta u_{ij,\pi}^{au})] + \exp[-\mu(\tilde{c}_{ij}^{tr} + \theta u_{ij,\pi}^{tr})]} \quad (26)$$

During non peak hours the modal share for a quadruplet (i,j,k,w) is given by

$$\frac{T_{ij,\bar{\pi}}^{kw,au}}{T_{ij,\bar{\pi}}^{kw,tr} + T_{ij,\bar{\pi}}^{kw,au}} = \frac{\exp[-\mu(\tilde{c}_{ij}^{au} + \theta c_{ij,\bar{\pi}}^{au})]}{\exp[-\mu(\tilde{c}_{ij}^{au} + \theta c_{ij,\bar{\pi}}^{au})] + \exp[-\mu(\tilde{c}_{ij}^{tr} + \theta u_{ij,\bar{\pi}}^{tr})]} \quad (27)$$

Modal choice is independent of k and w and differs between peak hours and non-peak hours. We obtain binary logit models.

From a residential location point of view, conditions (18)-(21) and (24)-(25) express the economic logic of the model: the housing stock in each location tends to be allocated to households with the highest *net* bids, due account being taken of the actual cost of housing and of the offsetting effect of travel time on bids. The quantities $\frac{\alpha_i^k}{\mu}$ are shadow rents accounting for supply limitations when the exogenous rents r_i^k are too low.

A calibration method for μ , extending results by Erlander, Nguyen and Stewart (1979) was proposed in Los (1979a). The method assumes that we know the congestion functions on the links of the road network and that a survey allows us to know (or else estimate) the allocations $\hat{T}_{ij,\pi}^{kw,au}$, $\hat{T}_{ij,\pi}^{kw,au}$, $\hat{T}_{ij,\pi}^{kw,tr}$, $\hat{T}_{ij,\pi}^{kw,tr}$. The link flows \hat{v}_a , consistent with the known $\hat{T}_{ij,\pi}^{kw,au}$ are computed by a traffic equilibrium algorithm.

We set

$$\hat{H}_i^k = \sum_{j,w} (\hat{T}_{ij,\pi}^{kw,au} + \hat{T}_{ij,\pi}^{kw,tr} + \hat{T}_{ij,\pi}^{kr,au} + \hat{T}_{ij,\pi}^{kw,tr})$$

and we replace the inequality constraint (11) by an equality constraint:

$$\sum_{j,w} (T_{ij,\pi}^{kw,au} + T_{ij,\pi}^{kw,au} + T_{ij,\pi}^{kw,tr} + T_{ij,\pi}^{kw,tr}) = \hat{H}_i^k \quad (11')$$

Let $v_a(\mu)$, $T_{ij,\pi}^{kw,au}(\mu)$, $T_{ij,\pi}^{kw,au}(\mu)$, $T_{ij,\pi}^{kw,tr}(\mu)$ and $T_{ij,\pi}^{kw,tr}(\mu)$ be the optimal solution to the minimization problem:

$$\text{Min } H + \mu F$$

s.t. (11'), (9), (10), (12), (14).

Let $F(\mu)$ and $H(\mu)$ be the values of F and H obtained from the optimal solution to the previous minimization problem. The calibrated value of μ , $\hat{\mu}$, is a solution to one of the equations

$$F(\mu) = \hat{F} \quad (26)$$

or $H(\mu) = \hat{H} \quad (27)$

$F(\mu)$ and $H(\mu)$ being *monotone* functions of μ these equations can easily be solved by, for instance, a dichotomic search procedure.

If the model were a perfect specification of reality, solving the two equations should produce the same value of $\hat{\mu}$. In general however we expect the two equations to lead to two distinct values for $\hat{\mu}$. The more similar these two values, the more valid the model. Thus the calibration procedure constitutes an indirect validation technique.

For prediction, μ is replaced by its calibrated value $\hat{\mu}$ and problem P is solved directly, as shown in the next section.

III - Solution algorithms

A. Case without automobile congestion

We consider a version of problem P without automobile congestion for three reasons. Firstly it can be a reasonable approximation of situations where there is very little congestion. Secondly, it can provide a starting feasible solution for the more general case with congestion. Thirdly the algorithm proposed for its solution can be used as part of an algorithm for solving the more general problem.

Let $c_{ij,\pi}^{au}$ be the constant travel time between i and j by private automobile during peak hours. We have to solve the following minimization problem:

$$\begin{aligned}
 \text{Min} \quad & \sum_{ijkw} T_{ij,\pi}^{kw,au} \ln T_{ij,\pi}^{kw,au} + \sum_{ijkw} T_{ij,\bar{\pi}}^{kw,au} \ln T_{ij,\bar{\pi}}^{kw,au} \\
 & + \sum_{ijkw} T_{ij,\pi}^{kw,tr} \ln T_{ij,\pi}^{kw,tr} + \sum_{ijkw} T_{ij,\bar{\pi}}^{kw,tr} \ln T_{ij,\bar{\pi}}^{kw,tr} \\
 & - \mu \left\{ \sum_{ijkw} \hat{\beta}_{ij}^{kw,au} (T_{ij,\pi}^{kw,au} + T_{ij,\bar{\pi}}^{kw,au}) + \sum_{ijkw} \hat{\beta}_{ij}^{kw,tr} (T_{ij,\pi}^{kw,tr} + T_{ij,\bar{\pi}}^{kw,tr}) \right. \\
 & - \theta \left[\sum_{ijkw} T_{ij,\pi}^{kw,au} c_{ij,\bar{\pi}}^{au} + \sum_{ijkw} T_{ij,\pi}^{kw,au} c_{ij,\pi}^{au} \right. \\
 & \left. \left. + \sum_{ijkw} T_{ij,\pi}^{kw,tr} u_{ij,\pi}^{tr} + \sum_{ijkw} T_{ij,\bar{\pi}}^{kw,tr} u_{ij,\bar{\pi}}^{tr} \right] \right\} \quad (28)
 \end{aligned}$$

$$\text{s. t.} \quad \sum_{i,k} (T_{ij,\pi}^{kw,au} + T_{ij,\pi}^{kw,tr}) = E_{j,\pi}^w \quad \text{all } (j,w) \quad (29)$$

$$\sum_{i,k} (T_{ij,\bar{\pi}}^{kw,au} + T_{ij,\bar{\pi}}^{kw,tr}) = E_{j,\bar{\pi}}^w \quad \text{all } (j,w) \quad (30)$$

$$\sum_{j,w} (T_{ij,\pi}^{kw,au} + T_{ij,\bar{\pi}}^{kw,au} + T_{ij,\pi}^{kw,tr} + T_{ij,\bar{\pi}}^{kw,tr}) \leq H_i^k \quad \text{all } (i,k) \quad (31)$$

$$T_{ij,\pi}^{kw,au} \geq 0, T_{ij,\bar{\pi}}^{kw,au} \geq 0, T_{ij,\pi}^{kw,tr} \geq 0, T_{ij,\bar{\pi}}^{kw,tr} \geq 0 \quad (32)$$

The formulation of the problem (28)-(32) can be simplified if we define two new indices as follows:

$$(j,\pi)^w \text{ or } (j,\bar{\pi})^w \rightarrow \delta \quad (33)$$

$$(i)^k \rightarrow \epsilon \quad (34)$$

We define accordingly two sets of coefficients:

$$\tau_{\epsilon\delta}^{au} = \begin{cases} \hat{\beta}_{ij}^{kw,au} - \theta c_{ij,\pi}^{au} \\ \text{or } \hat{\beta}_{ij}^{kw,au} - \theta c_{ij,\bar{\pi}}^{au} \end{cases} \quad (35)$$

$$\tau_{\epsilon\delta}^{tr} = \begin{cases} \hat{\beta}_{ij}^{kw,tr} - \theta u_{ij,\pi}^{tr} \\ \text{or } \hat{\beta}_{ij}^{kw,tr} - \theta u_{ij,\bar{\pi}}^{tr} \end{cases} \quad (36)$$

Problem (28)-(32) can be restated as:

$$\text{Min } \sum_{\epsilon, \delta} T_{\epsilon\delta}^{\text{au}} \ln T_{\epsilon\delta}^{\text{au}} + \sum_{\epsilon, \delta} T_{\epsilon\delta}^{\text{tr}} \ln T_{\epsilon\delta}^{\text{tr}} \quad (37)$$

$$-\mu \left\{ \sum_{\epsilon, \delta} \tau_{\epsilon\delta}^{\text{au}} T_{\epsilon\delta}^{\text{au}} + \sum_{\epsilon, \delta} \tau_{\epsilon\delta}^{\text{tr}} T_{\epsilon\delta}^{\text{tr}} \right\} \quad (38)$$

$$\text{s.t. } \sum_{\epsilon} (T_{\epsilon\delta}^{\text{au}} + T_{\epsilon\delta}^{\text{tr}}) = E_{\delta} \quad \text{all } \delta \quad (39)$$

$$\sum_{\delta} (T_{\epsilon\delta}^{\text{au}} + T_{\epsilon\delta}^{\text{tr}}) \leq H_{\epsilon} \quad \text{all } \epsilon \quad (40)$$

$$T_{\epsilon\delta}^{\text{au}} \geq 0 \quad T_{\epsilon\delta}^{\text{tr}} \geq 0$$

The Kuhn-Tucker conditions of (37)-(40) entail:

$$T_{\epsilon\delta}^{\text{au}} = \exp(-1-\nu_{\delta}) \exp(-\alpha_{\epsilon}) \exp(\mu \tau_{\epsilon\delta}^{\text{au}}) \quad (41)$$

$$T_{\epsilon\delta}^{\text{tr}} = \exp(-1-\nu_{\delta}) \exp(-\alpha_{\epsilon}) \exp(\mu \tau_{\epsilon\delta}^{\text{tr}}) \quad (42)$$

$$\alpha_{\epsilon} \geq 0 \text{ with} \quad (43)$$

$$\alpha_{\epsilon} > 0 \Rightarrow \sum_{\delta} (T_{\epsilon\delta}^{\text{au}} + T_{\epsilon\delta}^{\text{tr}}) = H_{\epsilon} \quad (44)$$

$$\sum_{\delta} (T_{\epsilon\delta}^{\text{au}} + T_{\epsilon\delta}^{\text{tr}}) < H_{\epsilon} \Rightarrow \alpha_{\epsilon} = 0$$

These conditions can be restated as:

$$T_{\epsilon\delta}^{\text{au}} = R_{\delta} S_{\epsilon} Q_{\epsilon\delta}^{\text{au}} \quad (45)$$

$$T_{\epsilon\delta}^{\text{tr}} = R_{\delta} S_{\epsilon} Q_{\epsilon\delta}^{\text{tr}} \quad (46)$$

$$0 < S_{\epsilon} \leq 1 \quad (47)$$

$$S_{\epsilon} < 1 \Rightarrow \sum_{\delta} (T_{\epsilon\delta}^{au} + T_{\epsilon\delta}^{tr}) = H_{\epsilon} \quad (43')$$

$$\sum_{\delta} (T_{\epsilon\delta}^{au} + T_{\epsilon\delta}^{tr}) < H_{\epsilon} \Rightarrow S_{\epsilon} = 1 \quad (44')$$

where

$$R_{\delta} = \exp(-1 - v_{\delta}) \quad (48)$$

$$S_{\epsilon} = \exp(-\alpha_{\epsilon}) \quad (49)$$

$$Q_{\epsilon\delta}^{au} = \exp(\mu\tau_{\epsilon\delta}^{au}) \quad (50)$$

$$Q_{\epsilon\delta}^{tr} = \exp(\mu\tau_{\epsilon\delta}^{tr}) \quad (51)$$

A balancing algorithm can be used to solve exactly problem (37)-(40). It can be interpreted as an iterative technique used to satisfy simultaneously equations (38), (39), (45), (46), (47), (43'), (44'). Its convergence towards the optimum can be proved by using geometric programming (Jefferson and Scott, 1979), by using nonlinear duality theory (Ducharme, 1980). Alternatively it can be proved by appealing to a more general iterative technique proposed by Bregman to solve a class of convex programming problems¹. It works as follows:

Step 1: Choose initial values R_{δ}^0 for R_{δ} , all δ and set $n=0$.

Step 2: Set:

$$S_{\epsilon}^{(n)} = \min \left\{ 1, \frac{H_{\epsilon}}{\sum_{\delta} R_{\delta}^{(n)} (Q_{\epsilon\delta}^{au} + Q_{\epsilon\delta}^{tr})} \right\} \quad \text{all } \epsilon \quad (52)$$

¹ See Bregman (1967), Lamond and Stewart (1980) and Lebeuf, Jörnsten and Stewart (1980).

Step 3: Set

$$R_{\delta}^{(n+1)} = \frac{E_{\delta}}{\sum_{\epsilon} S_{\epsilon}^{(n)} (Q_{\epsilon\delta}^{au} + Q_{\epsilon\delta}^{tr})} \quad \text{all } \delta \quad (53)$$

Step 4: If $|R_{\delta}^{(n+1)} - R_{\delta}^{(n)}| < \epsilon_0$ all δ , stop

Otherwise, go to step 2.

Balancing algorithms are extremely efficient in practice so that one can solve very large size problems at a low computational cost. (See, for instance, Ducharme (1980) for an application of the balancing algorithm with inequality constraints in the context of real park-and-ride data).

B. Case with automobile congestion

Two methods are proposed to solve the general model with congestion. Both are methods of feasible directions.¹ They are analogous and differ in the method for choosing a feasible descent direction. The first algorithm is based on the Frank-and-Wolfe technique.² The second algorithm is a simple extension of an algorithm proposed by Evans (1976) for the solution of the so-called Combined Distribution Assignment problem.

1. The Frank and Wolfe approach

An iteration of this algorithm requires the solution of a linear program that determines the descent direction and the solution of a one-

¹ See Luenberger (1972), Avriel (1976) or Zangwill (1969) for a definition and discussion of that class of techniques.

² See Frank and Wolfe (1956), Zangwill (1969) and Appendix 1 for a presentation of that technique.

dimensional minimization problem that determines the step size that achieves the best improvement in the objective function, given the descent direction. At an intermediate stage of the algorithm, a feasible solution is known: $\bar{T}_{ij,\pi}^{kw,au}$, $\bar{T}_{ij,\bar{\pi}}^{kw,au}$, $\bar{T}_{ij,\pi}^{kw,tr}$, $\bar{T}_{ij,\bar{\pi}}^{kw,tr}$ and path flows $\bar{h}_{m,ij}^{au}$. The descent direction is obtained by solving the problem:

$$\begin{aligned} \text{Min} \quad & \sum_{ijkw} \bar{c}_{ij,\pi}^{kw,au} \gamma_{ij,\pi}^{kw,au} + \sum_{ijkw} \bar{c}_{ij,\bar{\pi}}^{kw,au} \gamma_{ij,\bar{\pi}}^{kw,au} + \sum_{ijkw} \bar{c}_{ij,\pi}^{kw,tr} \gamma_{ij,\pi}^{kw,tr} \\ & + \sum_{ijkw} \bar{c}_{ij,\bar{\pi}}^{kw,tr} \gamma_{ij,\bar{\pi}}^{kw,tr} + \sum_{ijm} \mu \bar{c}_{m,ij}^{au} Z_{m,ij} \end{aligned} \quad (55)$$

$$\text{s.t.} \quad \sum_{kw} \gamma_{ij,\pi}^{kw,au} = \sum_m Z_{m,ij} \quad \text{all } (i,j) \quad (56)$$

$$\sum_{ik} (\gamma_{ij,\pi}^{kw,au} + \gamma_{ij,\pi}^{kw,tr}) = E_{j,\pi}^w \quad \text{all } (j,w) \quad (57)$$

$$\sum_{ik} (\gamma_{ij,\bar{\pi}}^{kw,au} + \gamma_{ij,\bar{\pi}}^{kw,tr}) = E_{j,\bar{\pi}}^w \quad \text{all } (j,w) \quad (58)$$

$$\sum_{jw} [\gamma_{ij,\pi}^{kw,au} + \gamma_{ij,\bar{\pi}}^{kw,au} + \gamma_{ij,\pi}^{kw,tr} + \gamma_{ij,\bar{\pi}}^{kw,tr}] \leq H_i^k \quad \text{all } (i,k) \quad (59)$$

$$\gamma_{ij,\pi}^{kw,au} \geq 0, \gamma_{ij,\bar{\pi}}^{kw,au} \geq 0, \gamma_{ij,\pi}^{kw,tr} \geq 0, \gamma_{ij,\bar{\pi}}^{kw,tr} \geq 0, Z_{m,ij} \geq 0 \quad (60)$$

We have:

$$\bar{c}_{ij,\pi}^{kw,au} = 1 + \lambda \ln \bar{T}_{ij,\pi}^{kw,au} - \mu \hat{\beta}_{ij}^{kw,au} \quad (61)$$

$$\bar{c}_{ij,\bar{\pi}}^{kw,au} = 1 + \lambda \ln \bar{T}_{ij,\bar{\pi}}^{kw,au} - \mu \hat{\beta}_{ij}^{kw,au} + \mu \theta c_{ij,\bar{\pi}}^{au,-} \quad (62)$$

$$\bar{c}_{ij,\pi}^{kw,tr} = 1 + \lambda \ln \bar{T}_{ij,\pi}^{kw,tr} - \mu \hat{\beta}_{ij}^{kw,tr} + \mu \theta u_{ij,\pi}^{tr} \quad (63)$$

$$\bar{c}_{ij,\bar{\pi}}^{kw,tr} = 1 + \lambda \ln \bar{T}_{ij,\bar{\pi}}^{kw,tr} - \mu \hat{\beta}_{ij}^{kw,tr} + \mu \theta u_{ij,\bar{\pi}}^{tr} \quad (64)$$

$\bar{c}_{m,ij}^{au}$ is the travel time on path m for O-D pair (i,j) on the road network when the flows are $\bar{h}_{m,ij}^{au}$. The linear program (55)-(60) simplifies by noting that $\bar{c}_{m,ij}^{au}$ may be replaced by c_{ij}^{*au} , the time on a shortest path between i and j on the network, for the given flows. In an optimal solution, if $\bar{c}_{m,ij}^{au} > c_{ij}^{*au}$ the corresponding $Z_{m,ij}^{au} = 0$, since otherwise the solution could be improved by diminishing a $Z_{m,ij}^{au}$ for which $\bar{c}_{m,ij}^{au} > c_{ij}^{*au}$ and augmenting Z_{ij}^{*au} , the flow on the shortest path. Thus the descent direction is obtained by the following simplified linear program:

$$\text{Min } \sum_{ijkw} \bar{c}_{ij,\pi}^{kw,au} \gamma_{ij,\pi}^{kw,au} + \sum_{ijkw} \bar{c}_{ij,\bar{\pi}}^{kw,au} \gamma_{ij,\bar{\pi}}^{kw,au} + \sum_{ijkw} \bar{c}_{ij,\pi}^{kw,tr} \gamma_{ij,\pi}^{kw,tr} + \sum_{ijkw} \bar{c}_{ij,\bar{\pi}}^{kw,tr} \gamma_{ij,\bar{\pi}}^{kw,tr} \quad (65)$$

$$\text{s.t. } \sum_{ik} (\gamma_{ij,\pi}^{kw,au} + \gamma_{ij,\pi}^{kw,tr}) = E_{j,\pi}^w \quad (66)$$

$$\sum_{ik} (\gamma_{ij,\bar{\pi}}^{kw,au} + \gamma_{ij,\bar{\pi}}^{kw,tr}) = E_{j,\bar{\pi}}^w \quad (67)$$

$$\sum_{jw} (\gamma_{ij,\pi}^{kw,tr} + \gamma_{ij,\bar{\pi}}^{kw,tr} + \gamma_{ij,\pi}^{kw,au} + \gamma_{ij,\bar{\pi}}^{kw,au}) \leq H_i^k \quad (68)$$

$$\gamma_{ij,\pi}^{kw,au} \geq 0, \gamma_{ij,\bar{\pi}}^{kw,au} \geq 0, \gamma_{ij,\pi}^{kw,tr} \geq 0, \gamma_{ij,\bar{\pi}}^{kw,tr} \geq 0 \quad (69)$$

$$\text{where } \bar{c}_{ij,\pi}^{kw,au} = \bar{c}_{ij,\pi}^{kw,au} + \mu \theta c_{ij}^{*au} \quad (70)$$

We define new indices and new coefficients:

$$(j,\pi)^w \quad \text{or} \quad (j,\bar{\pi})^w \rightarrow \delta \quad (71)$$

$$(i)^k \rightarrow \epsilon \quad (72)$$

$$\tau_{\epsilon\delta}^{au} = \begin{cases} =kw, au \\ c_{ij, \bar{\pi}} \end{cases} \text{ or } \begin{cases} -kw, au \\ c_{ij, \pi} \end{cases} \quad (73)$$

$$\tau_{\epsilon\delta}^{tr} = \begin{cases} =kw, tr \\ c_{ij, \pi} \end{cases} \text{ or } \begin{cases} =kw, tr \\ c_{ij, \bar{\pi}} \end{cases} \quad (74)$$

The linear program (65)-(69) can be restated as

$$\text{Min } \sum_{\epsilon\delta} \tau_{\epsilon\delta}^{au} \gamma_{\epsilon\delta}^{au} + \sum_{\epsilon\delta} \tau_{\epsilon\delta}^{tr} \gamma_{\epsilon\delta}^{tr} \quad (75)$$

$$\text{s. t. } \sum_{\epsilon} (\gamma_{\epsilon\delta}^{au} + \gamma_{\epsilon\delta}^{tr}) = E_{\delta} \quad \text{all } \delta \quad (76)$$

$$\sum_{\delta} (\gamma_{\epsilon\delta}^{au} + \gamma_{\epsilon\delta}^{tr}) \leq H_{\epsilon} \quad \text{all } \epsilon \quad (77)$$

$$\gamma_{\epsilon\delta}^{au} \geq 0 \quad \gamma_{\epsilon\delta}^{tr} \geq 0 \quad (78)$$

This appears to be a bimodal version of the transportation problem of linear programming, to be solved in 2 steps:

- by determining the optimal mode for each (ϵ, δ) pair and then
- by solving the remaining transportation problem.¹

The solution elements of problem (65)-(69) give the components of the descent direction vector. We obtain $Z_{m,ij}^{au}$ by assigning $\sum_{kw} \bar{\gamma}_{ij, \pi}^{kw, au}$ for each (i, j) on the shortest route found while computing the c_{ij}^{*au} .

Let $\{\bar{\gamma}_{ij, \pi}^{kw, au}, \bar{\gamma}_{ij, \bar{\pi}}^{kw, au}, \bar{\gamma}_{ij, \pi}^{kw, tr}, \bar{\gamma}_{ij, \bar{\pi}}^{kw, tr}, \bar{Z}_{m,ij}^{au}\}$ be the solution to the linear

¹ Efficient codes exist for solving that problem, such as the code developed by Harris (1976), or the more recent network simplex codes such as RNET. (M.D. Grigoriadis and Hsu, 1979). The Rutgers minimum cost network flow subroutines. Department of Computer Science, Rutgers University, New Brunswick, New Jersey, U.S.A.)

program defining the descent direction. The descent direction for the current iteration is

$$\begin{aligned}
 & (\bar{y}_{ij,\pi}^{kw,au} - \bar{T}_{ij,\pi}^{kw,au}) && \text{all } i,j,k,w \\
 & (\bar{y}_{ij,\pi}^{kw,au} - \bar{T}_{ij,\pi}^{kw,au}) && \text{" " } \\
 & (\bar{y}_{ij,\pi}^{kw,tr} - \bar{T}_{ij,\pi}^{kw,tr}) && \text{" " } \\
 & (\bar{y}_{ij,\pi}^{kr,tr} - \bar{T}_{ij,\pi}^{kw,tr}) && \text{" " } \\
 & (\bar{z}_{m,ij}^{au} - \bar{h}_{m,ij}^{au}) && \text{all } m,i,j
 \end{aligned}$$

$\bar{z}_{m,ij}^{au} = 0$ for all m except for the shortest path, m^* , where

$$\bar{z}_{m^*,ij}^{au} = \sum_{kw} \bar{y}_{ij,\pi}^{kw,au} \tag{79}$$

The optimal step length, λ^* , to find the next feasible solution is given by the solution to the one-dimensional minimization problem:

$$\begin{aligned}
 \text{Min}_{0 \leq \lambda \leq 1} & \sum_{ijkw} x_{ij,\pi}^{kw,au} \ln x_{ij,\pi}^{kw,au} + \sum_{ijkw} x_{ij,\pi}^{kw,au} \ln x_{ij,\pi}^{kw,au} \\
 & \sum_{ijkw} x_{ij,\pi}^{kw,tr} \ln x_{ij,\pi}^{kw,tr} + \sum_{ijkw} x_{ij,\pi}^{kw,tr} \ln x_{ij,\pi}^{kw,tr} \\
 & -\mu \left\{ \sum_{ijkw} \hat{\beta}_{ij}^{kw,au} (x_{ij,\pi}^{kw,au} + x_{ij,\pi}^{kw,au}) + \sum_{ijkw} \hat{\beta}_{ij}^{kw,tr} (x_{ij,\pi}^{kw,tr} + x_{ij,\pi}^{kw,tr}) \right\} \\
 & -\theta \left[\sum_a \int_0^{\phi} s_a(x) dx + \sum_{ijkw} x_{ij,\pi}^{kw,tr} u_{ij,\pi}^{tr} + \sum_{ijkw} x_{ij,\pi}^{kw,tr} u_{ij,\pi}^{tr} \right. \\
 & \left. + \sum_{ijkw} x_{ij,\pi}^{kw,au} c_{ij,\pi}^{au} \right] \tag{80}
 \end{aligned}$$

where

$$x_{ij,\pi}^{kw,au} = \bar{T}_{ij,\pi}^{kw,au} + \lambda (\bar{Y}_{ij,\pi}^{kw,au} - \bar{T}_{ij,\pi}^{kw,au}) \quad (81)$$

$$x_{ij,\bar{\pi}}^{kw,au} = \bar{T}_{ij,\bar{\pi}}^{kw,au} + \lambda (\bar{Y}_{ij,\bar{\pi}}^{kw,au} - \bar{T}_{ij,\bar{\pi}}^{kw,au}) \quad (82)$$

$$x_{ij,\pi}^{kw,tr} = \bar{T}_{ij,\pi}^{kw,tr} + \lambda (\bar{Y}_{ij,\pi}^{kw,tr} - \bar{T}_{ij,\pi}^{kw,tr}) \quad (83)$$

$$x_{ij,\bar{\pi}}^{kw,tr} = \bar{T}_{ij,\bar{\pi}}^{kw,tr} + \lambda (\bar{Y}_{ij,\bar{\pi}}^{kw,tr} - \bar{T}_{ij,\bar{\pi}}^{kw,tr}) \quad (84)$$

$$\phi_a = \bar{v}_a + \lambda(\bar{w}_a - \bar{v}_a) \quad (85)$$

$$\bar{w}_a = \sum_{mij} \delta_{am,ij} \bar{z}_{m,ij}^{au} \quad (86)$$

$$\bar{v}_a = \sum_{mij} \delta_{am,ij} \bar{h}_{m,ij}^{au} \quad (87)$$

The complete solution method can be summarized as follows:

Step 1: Obtain an initial feasible solution $(\bar{T}_{ij,\pi}^{kw,au}, \bar{T}_{ij,\bar{\pi}}^{kw,au}, \bar{T}_{ij,\pi}^{kw,tr}, \bar{T}_{ij,\bar{\pi}}^{kw,tr}, \bar{v}_a)$.

Step 2: For each arc a compute the current cost $s_a(\bar{v}_a)$

Step 3: For each origin-destination pair (i,j) determine the shortest path τ_{ij} , let c_{ij}^{*au} be the travel time on τ_{ij} .

Step 4: Compute $\bar{c}_{ij,\bar{\pi}}^{kw,au}, \bar{c}_{ij,\pi}^{kw,tr}, \bar{c}_{ij,\bar{\pi}}^{kw,tr}, \bar{c}_{ij,\pi}^{kw,au}$ using equations (61)-(64) and (70).

Step 5: Solve the linear program (65)-(69) by transforming it into a bimodal transportation problem (75)-(78). This gives $\bar{Y}_{ij,\pi}^{kw,au}, \bar{Y}_{ij,\bar{\pi}}^{kw,au}, \bar{Y}_{ij,\pi}^{kw,tr}, \bar{Y}_{ij,\bar{\pi}}^{kw,tr}$.

Step 6: Initialize $\bar{w}_a = 0$ for all arcs a ; for all (i,j) set

$$\bar{w}_a = \bar{w}_a + \sum_{kw} \bar{y}_{ij,\pi}^{kw,au} \text{ for } a \in \tau_{ij}.$$

Step 7: Stopping criterion: compute

$$\begin{aligned} \Delta = & \left| \sum_{ijkw} \bar{c}_{ij,\pi}^{kw,au} (\bar{y}_{ij,\pi}^{kw,au} - \bar{T}_{ij,\pi}^{kw,au}) + \sum_{ijkw} \bar{c}_{ij,\pi}^{kw,au} (\bar{T}_{ij,\pi}^{kw,au} - \bar{y}_{ij,\pi}^{kw,au}) \right. \\ & + \sum_{ijkw} \bar{c}_{ij,\pi}^{kw,tr} (\bar{y}_{ij,\pi}^{kw,tr} - \bar{T}_{ij,\pi}^{kw,tr}) + \sum_{ijkw} \bar{c}_{ij,\pi}^{kw,tr} (\bar{y}_{ij,\pi}^{kw,tr} - \bar{T}_{ij,\pi}^{kw,tr}) \\ & \left. + \mu\theta \sum_{a \in A} s_a(\bar{v}_a) (\bar{w}_a - \bar{v}_a) \right| \end{aligned} \quad (88)$$

If $\Delta \leq \epsilon_0$ (a predetermined tolerance) stop, otherwise, go to step 8.

Step 8: Solve the one-dimensional problem (80) to determine the optimal step length λ^* . Revise the allocations and flows as follows:

$$\bar{T}_{ij,\pi}^{kw,au} = \bar{T}_{ij,\pi}^{kw,au} + \lambda^* (\bar{y}_{ij,\pi}^{kw,au} - \bar{T}_{ij,\pi}^{kw,au}) \quad (89)$$

$$\bar{T}_{ij,\bar{\pi}}^{kw,au} = \bar{T}_{ij,\bar{\pi}}^{kw,au} + \lambda^* (\bar{y}_{ij,\bar{\pi}}^{kw,au} - \bar{T}_{ij,\bar{\pi}}^{kw,au}) \quad (90)$$

$$\bar{T}_{ij,\pi}^{kw,tr} = \bar{T}_{ij,\pi}^{kw,tr} + \lambda^* (\bar{y}_{ij,\pi}^{kw,tr} - \bar{T}_{ij,\pi}^{kw,tr}) \quad (91)$$

$$\bar{T}_{ij,\bar{\pi}}^{kw,tr} = \bar{T}_{ij,\bar{\pi}}^{kw,tr} + \lambda^* (\bar{y}_{ij,\bar{\pi}}^{kw,tr} - \bar{T}_{ij,\bar{\pi}}^{kw,tr}) \quad (92)$$

$$\bar{v}_a = \bar{v}_a + \lambda^* (\bar{w}_a - \bar{v}_a) \quad \text{all } a \quad (93)$$

Return to step 2

This solution procedure is extremely similar to the procedure proposed for the combined trip-distribution-trip assignment problem by Florian *et al* (1975).

2. Adaptation of an algorithm proposed by Evans for the combined-distribution assignment problem

The algorithm proposed by Evans for the solution of the combined distribution-assignment problem (Evans, 1976) can be interpreted as a particular case of a method of feasible directions and can be generalized in the following way as mentioned in Stewart (1979).

Generalized "Evans" algorithm

Consider the problem of minimizing a convex and separable function subject to linear constraints:

$$\text{Min } f(X,Y) = f_1(X) + f_2(Y) \quad (94)$$

$$\text{s.t. } A \begin{bmatrix} X \\ Y \end{bmatrix} \leq B \quad (95)$$

$$\text{and } \begin{bmatrix} X \\ Y \end{bmatrix} \geq 0 \quad (96)$$

The detailed steps of the generalized "Evans" algorithm for solving this problem are as follows:

Step 1: Given a feasible solution (X^1, Y^1) , set $\ell=1$.

Step 2: Determine (U^ℓ, V^ℓ) that minimizes

$$\nabla f_1(X^\ell) (U - X^\ell) + f_2(V) \quad (97)$$

$$\text{s.t. } A \begin{bmatrix} U \\ V \end{bmatrix} \leq B \quad (98)$$

$$\begin{bmatrix} U \\ V \end{bmatrix} \geq 0 \quad (99)$$

Step 3: Set the descent direction $\begin{bmatrix} U^l - X^l \\ V^l - Y^l \end{bmatrix}$.

$$\text{If } |\nabla f_1(X^l)(U^l - X^l) + \nabla f_2(Y^l)(V^l - Y^l)| \leq \epsilon_0 \quad (100)$$

terminate, where ϵ_0 is a convergence parameter: (X^l, Y^l) is the optimal solution.

Step 4: Find the optimal step length that minimizes $f_1[X^l + \lambda(U^l - X^l)] + f_2[Y^l + \lambda(V^l - Y^l)]$ for $0 \leq \lambda \leq 1$.

Step 5: Revise the current solution

$$X^{l+1} = X^l + \lambda^l (U^l - X^l) \quad (101)$$

$$Y^{l+1} = Y^l + \lambda^l (V^l - Y^l) \quad (102)$$

Set $l=l+1$ and return to step 2.

Appendix 2 proves that step 2 does define a feasible descent direction and that the algorithm converges to the optimum.

This algorithm can be applied to the general combined residential location and transportation model (problem P) by making the choices:

$$X \leftarrow (h_{m,ij}^{au}) \quad (103)$$

$$Y \leftarrow (T_{ij,\pi}^{kw,au}, T_{ij,\pi}^{kw,au}, T_{ij,\pi}^{kw,tr}, T_{ij,\pi}^{kw,tr}) \quad (104)$$

$$f_1(X) \leftarrow \mu\theta \sum_{a \in A} \int_0^v s_a(x) dx \quad (105)$$

$$\begin{aligned}
 f_2(Y) + & \sum_{ijkw} T_{ij,\pi}^{kw,au} \ln T_{ij,\pi}^{kw,au} + \sum_{ijkw} T_{ij,\pi}^{kw,au} \ln T_{ij,\pi}^{kw,au} \\
 & + \sum_{ijkw} T_{ij,\pi}^{kw,tr} \ln T_{ij,\pi}^{kw,tr} + \sum_{ijkw} T_{ij,\pi}^{kw,tr} \ln T_{ij,\pi}^{kw,tr} \\
 & - \mu \left\{ \sum_{ijkw} \hat{\beta}_{ij}^{kw,au} (T_{ij,\pi}^{kw,au} + T_{ij,\pi}^{kw,au}) + \sum_{ijkw} \hat{\beta}_{ij}^{kw,tr} (T_{ij,\pi}^{kw,tr} + T_{ij,\pi}^{kw,tr}) \right\} \\
 & - \theta \left[\sum_{ijkw} T_{ij,\pi}^{kw,tr} u_{ij,\pi}^{tr} + \sum_{ijkw} T_{ij,\pi}^{kw,au} c_{ij,\pi}^{au} + \sum_{ijkw} T_{ij,\pi}^{kw,tr} u_{ij,\pi}^{tr} \right] \quad (106)
 \end{aligned}$$

Consider an intermediate stage of the application of the algorithm when a feasible solution is known: $\bar{T}_{ij,\pi}^{kw,au}$, $\bar{T}_{ij,\pi}^{kw,au}$, $\bar{T}_{ij,\pi}^{kw,tr}$, $\bar{T}_{ij,\pi}^{kw,tr}$ and path flows $\bar{h}_{m,ij}^{au}$. The descent direction is obtained by solving the problem:

$$\begin{aligned}
 \text{Min } \mu \theta & \sum_{m,ij} \bar{c}_{m,ij}^{au} z_{m,ij}^{au} + \sum_{ijkw} \gamma_{ij,\pi}^{kw,au} \ln \gamma_{ij,\pi}^{kw,au} + \sum_{ijkw} \gamma_{ij,\pi}^{kw,au} \ln \gamma_{ij,\pi}^{kw,au} \\
 & + \sum_{ijkw} \gamma_{ij,\pi}^{kw,tr} \ln \gamma_{ij,\pi}^{kw,tr} + \sum_{ijkw} \gamma_{ij,\pi}^{kw,tr} \ln \gamma_{ij,\pi}^{kw,tr} \\
 & - \mu \left\{ \sum_{ijkw} \hat{\beta}_{ij}^{kw,au} (\gamma_{ij,\pi}^{kw,au} + \gamma_{ij,\pi}^{kw,au}) + \sum_{ijkw} \hat{\beta}_{ij}^{kw,tr} (\gamma_{ij,\pi}^{kw,tr} + \gamma_{ij,\pi}^{kw,tr}) \right\} \\
 & - \theta \left[\sum_{ijkw} \gamma_{ij,\pi}^{kw,tr} u_{ij,\pi}^{tr} + \sum_{ijkw} \gamma_{ij,\pi}^{kw,au} c_{ij,\pi}^{au} + \sum_{ijkw} \gamma_{ij,\pi}^{kw,tr} u_{ij,\pi}^{tr} \right] \quad (107)
 \end{aligned}$$

$$\text{s.t.} \quad \sum_{ik} (\gamma_{ij,\pi}^{kw,au} + \gamma_{ij,\pi}^{kw,tr}) = E_{j,\pi}^w \quad \text{all } (j,w) \quad (108)$$

$$\sum_{ik} (\gamma_{ij,\pi}^{kw,au} + \gamma_{ij,\pi}^{kw,tr}) = E_{j,\pi}^w \quad \text{all } (j,w) \quad (109)$$

$$\sum_{j,w} (\gamma_{ij,\pi}^{kw,au} + \gamma_{ij,\pi}^{kw,au} + \gamma_{ij,\pi}^{kw,tr} + \gamma_{ij,\pi}^{kw,tr}) \leq H_i^k \quad \text{all } (i,k) \quad (110)$$

$$\sum_{k,w} \gamma_{ij,\pi}^{kw,au} = \sum_m z_{m,ij}^{au} \quad \text{all } (i,j) \quad (111)$$

$$\gamma_{ij,\pi}^{kw,au} \geq 0, \gamma_{ij,\pi}^{kw,au} \geq 0, \gamma_{ij,\pi}^{kw,tr} \geq 0, \gamma_{ij,\pi}^{kw,tr} \geq 0, \text{ all } i,j,k,w,$$

$$z_{m,ij}^{au} \geq 0, \text{ all } m,ij \quad (112)$$

As with the Frank and Wolfe descent direction subproblem, problem (107)-(112) simplifies to the following:

$$\begin{aligned} \text{Min } \mu\theta & \sum_{ijkw} c_{ij}^{*au} \gamma_{ij,\pi}^{kw,au} + \sum_{ijkw} \gamma_{ij,\pi}^{kw,au} \ln \gamma_{ij,\pi}^{kw,au} + \sum_{ijkw} \gamma_{ij,\bar{\pi}}^{kw,au} \ln \gamma_{ij,\bar{\pi}}^{kw,au} \\ & + \sum_{ijkw} \gamma_{ij,\pi}^{kw,tr} \ln \gamma_{ij,\pi}^{kw,tr} + \sum_{ijkw} \gamma_{ij,\bar{\pi}}^{kw,tr} \ln \gamma_{ij,\bar{\pi}}^{kw,tr} \\ & - \mu \left\{ \sum_{ijkw} \beta_{ij}^{kw,au} (\gamma_{ij,\pi}^{kw,au} + \gamma_{ij,\bar{\pi}}^{kw,au}) + \sum_{ijkw} \beta_{ij}^{kw,tr} (\gamma_{ij,\pi}^{kw,tr} + \gamma_{ij,\bar{\pi}}^{kw,tr}) \right. \\ & \left. - \theta \left[\sum_{ijkw} \gamma_{ij,\pi}^{kw,tr} u_{ij,\pi}^{kw,tr} + \sum_{ijkw} \gamma_{ij,\bar{\pi}}^{kw,au} c_{ij,\bar{\pi}}^{au} + \sum_{ijkw} \gamma_{ij,\bar{\pi}}^{kw,tr} - u_{ij,\bar{\pi}}^{tr} \right] \right\} \end{aligned} \quad (113)$$

s. t. (108)

..... (109)

..... (110)

$$\gamma_{ij,\pi}^{kw,au} \geq 0, \gamma_{ij,\bar{\pi}}^{kw,au} \geq 0, \gamma_{ij,\pi}^{kw,tr} \geq 0, \gamma_{ij,\bar{\pi}}^{kw,tr} \geq 0 \quad (114)$$

where c_{ij}^{*au} is the travel time on a shortest path between i and j on the road network given the current feasible flows $\bar{h}_{m,ij}^{au}$. It turns out that this subproblem has exactly the same structure as the model without congestion (28)-(32). Thus, after a suitable transformation, it can be solved by a balancing algorithm with inequality constraints, as described in III A.

The solution elements of problem (113), (108), (109), (110), (114) give the components of the descent direction vector. We obtain $Z_{m,ij}^{au}$ by assigning $\sum_{kw} \bar{\gamma}_{ij,\pi}^{kw,au}$ for each (i,j) on the shortest route found while

Step 5: Initialize $\bar{w}_a = 0$ for all arcs a ; for all (i,j) set

$$\bar{w}_a = \bar{w}_a + \sum_{k,w} \bar{y}_{ij,\pi}^{kw,au} \text{ for } a \in \tau_{ij}.$$

Step 6: Stopping criterion: compute Δ as defined by (88), (61)-(64).

If $\Delta \leq \epsilon_0$ stop; otherwise, go to step 7.

Step 7: Solve the one-dimensional problem (80) to determine the optimal step length λ^* . Revise the allocations and flows by (89)-(93). Return to step 2.

IV - Discussion and possible extensions

Previous computational experiments comparing the use of the Frank and Wolfe method containing as subproblem a transportation problem on one hand, with the direct use of a balancing algorithm (Ducharme, 1980) or with the Evans approach (Frank, 1978) on the other hand, indicate clearly the computational superiority of the Evans approach over the Frank and Wolfe approach. Balancing algorithms are extremely efficient and can handle very large size problems since they work by considering one constraint at a time in an iterative fashion. Thus the partial linearization algorithm should be preferred in practice to the Frank and Wolfe algorithm.

Several extensions to model P are possible. It is possible to extend the model to more than one mode of public transportation, one mode being rail rapid transit and adding the possibility of mixed modes such as park'n ride and kiss'n ride. One can also assume limited parking capacities at the rail stations. One would then obtain an extended version of the model proposed by Los (1979a) for modelling the choice between

rail transit and private car, in which the constraints could no longer be reduced to the simple constraint structure of a multimodal transportation problem. An Evans-type algorithm would still be applicable by using generalized balancing algorithms adapted to the specific constraint structure of the model with rail rapid transit. It thus appears that, in addition to being more efficient than the Frank and Wolfe approach for the model discussed in this paper, Evans-type algorithms are sufficiently general in scope to allow straightforward adaptations to a variety of problems seemingly different in structures

The assumption that in-vehicle transit times are constant is in general unrealistic (since special bus lanes are not often used in practice) although it can be a good first approximation if waiting time and access time are more important determinants of behavior than in-vehicle time. In fact, it is more realistic to assume that in-vehicle transit times are functions of automobile travel times. This assumption is made in EMME (see Florian, 1977 and Florian, Chapleau *et al*, 1979), which is a bimodal equilibrium and modal choice model. We have:

$$c_s(v_a) = f_a \{s_a(v_a)\} \quad (115)$$

for all arcs s coinciding with arcs a on the road network.

$$c_s(v_a) = c_s \quad \text{for all arcs } s \quad (116)$$

which do not coincide with a road arc (in particular for access arcs and transfer arcs). With these assumptions, bus travellers are assumed to use the shortest paths on the bus network (Florian, 1980) but these shortest paths and associated travel times are now dependent on the flows of private

vehicles on the road network. The extended model could be solved as follows: Choose initial transit times \bar{c}_s for all links of the bus network and compute the associated travel times by bus, $\bar{u}_{ij,\pi}^{tr}$. Solve the associated mathematical program by the partial linearization method; this produces new flows \bar{v}_a which can be used to recompute new transit times by means of equation (115). The procedure is repeated, with the new transit travel times and stops when transit times have converged to constant values. This algorithm is based on the same relaxation scheme as EMME, the only difference being that the mathematical program P would replace the equilibrium assignment with elastic demand used in EMME.

A rather strong assumption made so far consists in assuming *exogenous* dwelling prices r_i^k . In fact house prices in the home ownership market are highly variable with the forces of supply and demand while rents in the rental market although more sticky, especially if there is some rent control, are nevertheless variable. If one assumes all dwelling prices to be endogenous and if one defines equilibrium prices as prices such that there is no excess demand in the corresponding submarket, a possible approach to the determination of a set of equilibrium prices would work as follows: Start from a hypothesized set of prices \bar{r}_i^k . For each submarket (i,k) for which there is excess demand, i.e. for which the supply constraint is binding, set: $r_i^k = \bar{r}_i^k + \frac{\alpha_i^k}{\mu}$ where $\frac{\alpha_i^k}{\mu}$ is the shadow rent associated with the (i,k) submarket. Rerun the model with these new prices r_i^k and continue to modify the rents r_i^k until no supply constraint is binding. If some prices are exogenous, for instance if there is strict rent control, then the same procedure can be applied for the subset of prices which react to supply and demand, the other prices remaining

fixed, with the possible result that some housing supply constraints may remain binding during the projection period. There is no guarantee that the set of equilibrium prices thus obtained is unique. With this qualification in mind, the model suggested here could be used to study the impact of transportation policies on municipal tax revenues. For instance one could use the model to determine an equitable distribution of the burden of transportation capital expenditures among the different municipalities of a metropolitan area because the impact of the projected investments on property values and thus on property tax revenues (after reassessment) could be predicted by the model (Anas and Lee, 1980).

If the utility levels of the different household groups, U^w , are to be considered endogenous (*closed city hypothesis*), a different methodology than the one suggested in this paper would be needed.¹

Last, but not least importantly, it would be advantageous to find an appropriate calibration method with more than one coefficient μ . For instance, one would like to have as many values of μ as there are income groups w .

V - Conclusion

In this paper, we presented a model for predicting jointly residential location, modal choice and transportation system performance. The proposed advocated solution algorithm based on Evans' partial linearization approach is computationally applicable to large size problems (e.g. 100 zones, 20 housing types, 10 household income groups). However,

¹ See for instance MacKinnon (1979).

the amount of data needed to calibrate the model is enormous. The transportation data needed are the same as those necessary for an application of the EMME model (Florian et al., 1979): a detailed coding of the road network and of the bus network with automobile congestion functions and transit travel time functions. The quantities, $\hat{T}_{ij,\pi}^{kw,au}$ etc..., would have to be obtained by compilation from distinct sources among which would certainly be census data, origin-destination surveys, information on the location and nature of employment etc... Each source of information would provide specific subtotals. For instance an O-D survey would supply the quantities $\hat{T}_{ij,*}^{**},^{au}$, $\hat{T}_{ij,*}^{**},^{tr}$, which represent the summations over the "starred" indices; information on income distribution by municipality could provide subtotals such as $\hat{T}_{I*,*}^{*w*}$, where I is a specific subset of zones; etc... The calibration of μ would have to proceed in two steps:

- 1) Estimate the quantities $\hat{T}_{ij,\pi}^{kw,au}$ and compute the associated equilibrium flows \hat{v}_a on the road network.
- 2) Estimate μ by means of equations (26) or (27).

The information sources should be sufficiently numerous that reliable estimates of the variables $\hat{T}_{ij,\pi}^{kw,au}$ etc... can be obtained. If such were not the case, an appropriately simplified version of the model should be calibrated and used for prediction.

Bid rents are exogenous data in the model. The estimation of bid rent functions for different household types, can be done with household survey data (see for instance Achour and Lapointe, 1980). These functions have to be linearized with respect to travel time to be used in the model proposed in this paper.

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APPENDIX 1

The Frank and Wolfe algorithm (1956)

Consider the problem of minimizing a continuously differentiable convex function subject to linear constraints:

$$\text{Min } Z(X), \text{ subject to } AX \leq b, X \geq 0.$$

The detailed steps of the Frank Wolfe algorithm for solving this problem are:

Step 1 Given a feasible solution X^1 , set $\ell = 1$

Step 2 Determine Y^ℓ that minimizes $\nabla Z(X^\ell) Y$, subject to $AY \leq b, Y \geq 0$.

Step 3 Set the descent direction $d^\ell = Y^\ell - X^\ell$.
If $|\nabla Z(X^\ell) d^\ell| \leq \epsilon$, terminate, where ϵ is a suitable convergence parameter. (X^ℓ is the optimal solution).

Step 4 Find the optimal step length λ^ℓ that minimizes $Z(X^\ell + \lambda d^\ell)$ for $0 \leq \lambda \leq 1$.

Step 5 Revise the current solution $X^{\ell+1} = X^\ell + \lambda^\ell d^\ell$;
set $\ell = \ell + 1$ and return to Step 2.

APPENDIX 2

CONVERGENCE OF THE GENERALIZED EVANS ALGORITHM

Consider the problem

$$(P): \quad \text{Min } f(X,Y) = f_1(X) + f_2(Y)$$

$$\text{subject to} \quad A \begin{pmatrix} X \\ Y \end{pmatrix} \geq b$$

$$X \geq 0$$

$$Y \geq 0$$

where f_1 and f_2 are convex continuously differentiable and the feasible region is compact.

The generalized Evans' algorithm presented in this paper is a feasible direction method which generates from a starting solution (X^1, Y^1) a sequence of points $\{(X^k, Y^k)\}$ where (X^{k+1}, Y^{k+1}) is obtained from (X^k, Y^k) by making a step along a feasible direction d . This algorithm may be decomposed into two separate procedures namely the direction generation mapping and the step length search mapping. If the direction used is a feasible descent direction, the direction generation procedure and the step length search procedure are closed mappings. Then the convergence of the algorithm is insured by a convergence theorem in Zangwill (1969). We need only prove that the generalized Evan's direction is indeed a feasible descent direction, and that the generation procedure is a closed mapping, since the step length search, which is a one-dimensional search on a closed bounded interval, is a closed mapping and composition of closed mappings is a closed mapping (see Zangwill (1969) for example).

Given a feasible solution (\bar{X}, \bar{Y}) the generalized Evans' direction $d = \begin{pmatrix} \bar{U} - \bar{X} \\ \bar{V} - \bar{Y} \end{pmatrix}$ is obtained by solving

$$D(\bar{X}, \bar{Y}): \quad \text{Min } Z(U, V) = \nabla f_1(\bar{X})U + f_2(V)$$

$$\text{s.t.} \quad A \begin{pmatrix} U \\ V \end{pmatrix} \geq B$$

$$U \geq 0$$

$$V \geq 0$$

Clearly direction d is a feasible direction. We now show that d is a descent direction.

Let (X^*, Y^*) be an optimal solution to (P). From the Kuhn-Tucker conditions, one has:

$$\nabla f(X^*, Y^*) + \Pi A \geq 0 \quad (1)$$

$$(\nabla f(X^*, Y^*) + \Pi A) \begin{pmatrix} X^* \\ Y^* \end{pmatrix} = 0 \quad (2)$$

where the Π 's are the Lagrangian multipliers.

Let (U^*, V^*) solve $D(X^*, Y^*)$, then replacing $A \begin{pmatrix} X^* \\ Y^* \end{pmatrix}$ by $A \begin{pmatrix} U^* \\ V^* \end{pmatrix}$ in the second term in (2) gives

$$\nabla f(X^*, Y^*) \begin{pmatrix} X^* \\ Y^* \end{pmatrix} + \Pi A \begin{pmatrix} U^* \\ V^* \end{pmatrix} = 0 \quad (3)$$

From the definition of (U^*, V^*) , one also has

$$\nabla f_1(X^*)U^* + f_2(V^*) - \nabla f_1(X^*)X^* - f_2(Y^*) \leq 0$$

or

$$\nabla f_1(X^*)(U^* - X^*) + (f_2(V^*) - f_2(Y^*)) \leq 0 \quad (4)$$

f_2 convex and differentiable implies then

$$\nabla f_1(x^*)(U^*-x^*) + \nabla f_2(y^*)(V^*-y^*) \leq 0$$

or

$$\nabla f(x^*, y^*) \begin{pmatrix} U^* \\ V^* \end{pmatrix} - \nabla f(x^*, y^*) \begin{pmatrix} x^* \\ y^* \end{pmatrix} \leq 0 \quad (5)$$

Adding (3) to (5) gives

$$(\nabla f(x^*, y^*) + \Pi A) \begin{pmatrix} U^* \\ V^* \end{pmatrix} \leq 0 \quad (6)$$

Taking (1) into account and since $(U^*, V^*) \geq 0$ one obtains that (6) must be satisfied as an equality and so does relation (5), thus

$$\nabla f(x^*, y^*) \begin{pmatrix} U^*-x^* \\ V^*-y^* \end{pmatrix} = 0$$

if and only if (x^*, y^*) is an optimal solution of (P), consequently for a non optimal (\bar{X}, \bar{Y})

$$\nabla f(\bar{X}, \bar{Y}) \begin{pmatrix} \bar{U}-\bar{X} \\ \bar{V}-\bar{Y} \end{pmatrix} < 0.$$

This proves that d is a descent direction at any non optimal point (\bar{X}, \bar{Y})

Given subsequences of $\{(x^k, y^k)\}$ and $\{(U^k, V^k)\}$ which converge respectively to limit points (\bar{X}, \bar{Y}) and (\bar{U}, \bar{V}) , to prove that the direction generation procedure is a closed mapping it is sufficient to establish that (\bar{U}, \bar{V}) solves problem $D(\bar{X}, \bar{Y})$.

Let (U^k, V^k) solve problem $D(x^k, y^k)$, then

$$\nabla f_1(x^k)U^k + f_2(V^k) \leq \nabla f_1(x^k)U + f_2(V)$$

for any feasible (U, V) . Taking the limits in the above inequality, as the

f 's are continuously differentiable one arrives at

$$\nabla f_1(\bar{X})\bar{U} + f_2(\bar{V}) \leq \nabla f_1(\bar{X})U + f_2(V)$$

since all (U^k, V^k) are feasible solutions in a compact set, (\bar{U}, \bar{V}) is also feasible, hence (\bar{X}, \bar{Y}) solves $D(\bar{X}, \bar{Y})$.