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COMBINED RESIDENTIAL LOCATION
AND TRANSPORTATION MODELS

by

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ABSTRACT

This paper presents predictive models which combine residential location with transportation. These models predict residential location and the use and performance of the transportation system for exogenously given total housing stock and employment locations. One can use them to determine the impact on the transportation system of changes in the supply of housing or reciprocally to determine the impact on housing choices of new investments in the transportation system. They are formulated as mathematical programs generalizing the Herbert-Stevens model of residential location to include previously proposed equilibrium models of trip assignment, trip distribution and modal choice. The Kuhn-Tucker conditions corresponding to each of these models are shown to express short run equilibrium in the housing market and user-equilibrium on the transportation network. A solution method is proposed.

I Introduction

Many practical urban planning problems require the prediction of the impact of transportation decisions (new investments, changes in transit fares, etc.) on residential location, and also the prediction of the impact of changes in some determinants of residential location such as new housing construction, or changes in income distribution, on the use and performance of the transportation system. These problems require the development of combined residential location-transportation models. It is the purpose of that paper to present such models and explore their solution methods.

There is by now a long tradition in residential location modelling (Cf. [27] and [28] for an excellent review of the state of the art). At the risk of simplifying, two kinds of modelling approaches have been proposed: microeconomic approaches of which the different versions of the Herbert-Stevens model are well-known examples [17, 18, 20, 21, 22, 31] - and maximum entropy approaches [29, 32]. Syntheses of these two approaches have actually been proposed by Anas [2,3] and Senior and Wilson [30]. The most important reason justifying the popularity of mathematical programming formulations such as the Herbert-Stevens linear programming formulation, is the interpretation of the dual variables. Härsman and Snickars [21] for instance indicate how one could use the scarcity prices associated to the constraints on the housing stock to guide public investments in new housing construction.

Similarly, transportation modelling has made tremendous progress in the recent past both from a theoretical point of view and a computational

point of view. Precise definitions of traffic equilibrium with congestion have been given and used as a basis for efficient mathematical programming methods [9,12]. It has been shown that trip distribution and traffic assignment can be solved simultaneously [7,14] and that modal choice can be introduced as well [13].

So far however residential location models assume that transport costs are known and constant and disregard transportation congestion. Reciprocally transport models assume that residential location, like employment location, is exogenously determined. Clearly, due to congestion on the transport network, residential location and transportation behavior are interdependent: residential choice depends partly on accessibility to work which depends on congestion in the transportation system for peak hour trips, and congestion itself depends on peak hour travel demand which depends itself on residential location. This interdependence has to be taken into account if one is to obtain reliable predictions.

A first attempt at combining a land use model (including service employment and basic employment location in addition to residential location) and a transportation model is due to S. Putman [26]. His approach is to iterate a highly disaggregated Lowry type model [25] with an incremental loading traffic assignment algorithm. There are several difficulties with this approach:

1) The Lowry model is not a good residential location model because its behavioral assumptions are too crude: essentially residential location is assumed to depend on the disutility of work trips and not on housing characteristics for instance. Clearly a model based directly on household preferences would give better predictions. In addition, as stated

above, it does not provide the information given by dual variables in a mathematical programming approach.

2) Incremental loading traffic assignment is a heuristic approach to solve the traffic equilibrium problem. At this point of time, there are efficient and exact traffic assignment methods [9], which are proved to converge towards a "meaningful" equilibrium (Wardrop's user equilibrium). On the other hand, incremental loading techniques converge to different traffic assignments depending on increment size mostly [8].

3) Even if one accepted the Lowry model and if the incremental loading technique was replaced by an "equilibrium" traffic assignment technique, the iterative procedure still would not be guaranteed to converge towards any meaningful equilibrium. It might converge towards different solutions, depending on the starting solution.

The latter point was raised by Boyce who showed in [5] that a combined Lowry-type residential location model and equilibrium traffic assignment problem had the same mathematical structure as the combined trip distribution - trip assignment problem for which S. Evans [7] and Florian, Nguyen and Ferland [14] have proposed two different solution procedures which both converge to the correct equilibrium.

This paper will integrate the Herbert-Stevens residential location model and a traffic equilibrium model in a unified mathematical programming framework and will also consider the case of a bimodal equilibrium. The criterion used to build each relevant mathematical program is that its Kuhn-Tucker conditions express simultaneously equilibrium on the

transportation network (Wardrop's user equilibrium principle) and equilibrium in the housing market with travel costs equal to the "equilibrium" travel costs. The equilibrium conditions on the housing market are the "short-run" equilibrium conditions encompassing household taxes and subsidies, by opposition to "long-run" equilibrium conditions where no taxes or subsidies occur [20, 31].

The rest of the paper is organized as follows: Section II restates one of the possible versions of the Herbert-Stevens model and extends it by relaxing the no-congestion assumption. Then a suboptimal version of the Herbert-Stevens model with congestion is formulated using maximum entropy methodology. Section III presents a possible solution method to calibrate and solve the two models proposed in section II. Section IV presents a new combined model which includes the choice between two modes as well. Section V discusses the possible methods for estimating the bid rents which are exogenously given in the model formulations of sections II and IV. Section VI concludes the paper.

II Combined residential location and transportation models: extensions of the Herbert-Stevens model

The Herbert Stevens model

We will follow closely in this presentation the version of the Herbert-Stevens model stated by Senior and Wilson [30]. We want to allocate households to an exogenously given housing stock so as to maximize aggregate consumer surplus. Employment is given exogenously. The economic rationale for this model has been well reviewed in the urban economics literature and will not be restated here [1, 22, 18, 20, 31, 28].

We assume that we know the bid rents of households for different types of housing in different zones. The Herbert-Stevens model assumes that each household of a given type has the same level of utility, whatever its housing choice, and that the household bid rent is a decreasing function of the level of utility. We will make the additional assumptions here, that the bids are <u>linearly</u> decreasing with the cost of travel between workplace and residence and that the actual rents paid are exogenously given. The former assumption will be discussed in section V. The latter assumption is appropriate when there is rent control. Finally we will assume that there is one worker per household.

T_{ij}^{kw} is the number of working heads of households living in a type k house in zone i earning income w in a job in zone j.

H_i^k is the number of houses of type k in zone i.

E_j^w is the number of jobs yielding income w in zone j.

r_i^k is the actual rent paid for a type k house in zone i.

$$b_{ij}^{kw} = \beta_{ij}^{kw} - \theta c_{ij}$$

where b_{ij}^{kw} is the bid rent by a (j,w) household for a type k house in zone i, and c_{ij} is the cost of travel between i and j. We will assume temporarily that c_{ij} is fixed (this assumption will be relaxed below) and that θ is independent of w. We require T_{ij}^{kw} to be solution to the following problem:

(P1): $\text{Max} \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw} \hat{b}_{ij}^{kw}$ (1)

s.t. $\sum_i \sum_k T_{ij}^{kw} = E_j^w$ all j,w (2)

$\sum_j \sum_w T_{ij}^{kw} \leq H_i^k$ all i,k (3)

$T_{ij}^{kw} \geq 0$ (4)

where $\hat{b}_{ij}^{kw} = b_{ij}^{kw} - r_i^k$ (5)

is the individual consumer surplus.

(P1) is a transportation problem of linear programming in which (1) is the aggregate consumer surplus, (2) states that every household must find a dwelling and (3) states that the supply of the housing stock in a given zone constrains the number of households that can be allocated there.

The dual problem is:

(P1D): $\text{Min} \sum_i \sum_k \alpha_i^k H_i^k + \sum_j \sum_w \nu_j^w E_j^w$ (6)

s.t. $\alpha_i^k + \nu_j^w \geq \hat{b}_{ij}^{kw}$ all i, j, k, w (7)

$\alpha_i^k \geq 0$ (8)

ν_j^w either sign

α_i^k can be interpreted as a scarcity rent, ν_j^w as a tax or subsidy necessary for every household to be located somewhere. For the long run equilibrium the expected levels of utility should be adjusted so that these taxes or subsidies vanish [20, 31]. The rule of complementary slackness entails:

$\alpha_i^k + \nu_j^w > \hat{b}_{ij}^{kw} \Rightarrow T_{ij}^{kw} = 0$ (9)

$T_{ij}^{kw} > 0 \Rightarrow \alpha_i^k + \nu_j^w = \hat{b}_{ij}^{kw}$ (10)

i.e. $\quad T_{ij}^{kw} > 0 \Rightarrow \alpha_i^k + r_i^k = b_{ij}^{kw} - \nu_j^w$ $\qquad\qquad$ (11)

$\qquad \alpha_i^k > 0 \Rightarrow \sum_j \sum_w T_{ij}^{kw} = H_i^k$ $\qquad\qquad$ (12)

$\qquad H_i^k > \sum_j \sum_w T_{ij}^{kw} \Rightarrow \alpha_i^k = 0$ $\qquad\qquad$ (13)

(11) states that the household's ability to pay after tax and subsidy

corrections is equal to the total price of the housing bundle to which

it is allocated. (11) can be restated as

$$T_{ij}^{kw} > 0 \Rightarrow \alpha_i^k + r_i^k = \beta_{ij}^{kw} - \theta c_{ij} - \nu_j^w \qquad\qquad (11')$$

The Herbert-Stevens model with transportation congestion

We now relax the assumption that the travel costs c_{ij} are fixed. We assume

that there is only one mode of transport, the private automobile, that

there is congestion on the road network and that there is a uniform car-

occupancy factor so that it is indifferent to measure flows in vehicles

or in persons.

$A \qquad$ is the set of arcs a of the road network.

$s_a(v_a) \quad$ is the congestion function on link a which we assume to be a convex

$\qquad\qquad$ and increasing function of the flow v_a,

$h_{m,ij} \quad$ is the path flow on path m between zones i and j.

We want to extend the Herbert-Stevens model so as to satisfy (11') with

c_{ij} replaced by u_{ij}, the user-equilibrium travel cost between i and j.

More precisely we want a solution to the extended problem to satisfy the

following relationships:

$$T_{ij}^{kw} > 0 \Rightarrow \alpha_i^k + \nu_j^w = \hat{\beta}_{ij}^{kw} - \theta u_{ij} \qquad\qquad (14)$$

$$\alpha_i^k + \nu_j^w > \hat{\beta}_{ij}^{kw} - \theta u_{ij} \Rightarrow T_{ij}^{kw} = 0 \tag{15}$$

$$\text{with} \qquad \hat{\beta}_{ij}^{kw} = \beta_{ij}^{kw} - r_i^k \tag{16}$$

$$h_{m,ij} > 0 \Rightarrow \sum_a s_a(v_a) \, \delta_{am,ij} = u_{ij} \tag{17}$$

$$h_{m,ij} = 0 \Rightarrow \sum_a s_a(v_a) \, \delta_{am,ij} \geq u_{ij} \tag{18}$$

$$\text{with} \quad \delta_{am,ij} = \begin{cases} 1 & \text{if a belongs to path m between i and j} \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha_i^k > 0 \Rightarrow \sum_j \sum_w T_{ij}^{kw} = H_i^k \tag{19}$$

$$H_i^k > \sum_j \sum_w T_{ij}^{kw} \Rightarrow \alpha_i^k = 0 \tag{20}$$

The required model is the following. We want T_{ij}^{kw} and $h_{m,ij}$ to be solution to:

$$\text{(P2): Max} \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw} \hat{\beta}_{ij}^{kw} - \theta \sum_a \int_0^{v_a} s_a(x) dx \tag{21}$$

$$\text{s.t.} \sum_k \sum_w T_{ij}^{kw} = \sum_m h_{m,ij} \qquad \text{all i,j} \tag{22}$$

$$\sum_i \sum_k T_{ij}^{kw} = E_j^w \qquad \text{all j,w} \tag{23}$$

$$\sum_j \sum_w T_{ij}^{kw} \leq H_i^k \qquad \text{all i,k} \tag{24}$$

$$v_a = \sum_m \sum_i \sum_j \delta_{am,ij} \, h_{m,ij} \qquad \text{all a}$$

$$T_{ij}^{kw} \geq 0 \tag{25}$$

$$h_{m,ij} \geq 0 \tag{26}$$

(P2) is the maximization of a concave function under linear constraints. The Kuhn-Tucker conditions are necessary and sufficient for optimality.

Let α_i^k be the dual variable associated with constraint (24), ν_j^w be the dual variable associated with constraint (23) and $U_{ij} = \theta u_{ij}$ be the dual variable associated with the conservation of flows constraint (22). It can be shown that the Kuhn-Tucker conditions of (P2) are the conditions (14), (15), (17), (18), (19) and (20). (P2) is therefore the required generalization of the Herbert-Stevens model, when there is congestion on the transportation network.

A maximum-entropy formulation of the Herbert Stevens-model with transportation congestion

Model (P2) can be modified to take into account the possibility of some degree of suboptimality in the housing market. Anas [2,3] and Senior and Wilson [30] proposed to use maximum entropy methodology to generate a "suboptimal" model. We will follow this idea here. We require T_{ij}^{kw} and $h_{m,ij}$ to be solution to the following minimization problem.

$$\text{Min} \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw} \ln T_{ij}^{kw} - \mu \left\{ \sum_w \sum_i \sum_j \sum_k \hat{\beta}_{ij}^{kw} T_{ij}^{kw} - \theta \sum_a \int_0^{v_a} s_a(x)dx \right\} \qquad (27)$$

$$\text{s.t.} \qquad \sum_i \sum_k T_{ij}^{kw} = E_j^w \qquad \text{all } j,w \qquad (28)$$

(P3)
$$\sum_j \sum_w T_{ij}^{kw} \leq H_i^k \qquad \text{all } i,k \qquad (29)$$

$$\sum_m h_{m,ij} = \sum_k \sum_w T_{ij}^{kw} \qquad \text{all } i,j \qquad (30)$$

$$v_a = \sum_m \sum_i \sum_j \delta_{am,ij} h_{m,ij}$$

$$h_{m,ij} \geq 0 \qquad (31)$$

$$T_{ij}^{kw} \geq 0 \qquad (32)$$

The parameter μ is to be calibrated from observed data. We require that $T_{ij}^{kw} \ln T_{ij}^{kw} = 0$ at $T_{ij}^{kw} = 0$.

(P3) is the minimization of a convex function subject to linear constraints. The Kuhn-Tucker conditions are necessary and sufficient for optimality. They are:

$$T_{ij}^{kw} = \exp (-1-\nu_j^w-\alpha_i^k) \exp [\mu(\hat{\beta}_{ij}^{kw}-\theta u_{ij})] \tag{33}$$

$$h_{m,ij} > 0 \Rightarrow \sum_a s_a(v_a) \delta_{am,ij} = u_{ij} \tag{34}$$

$$h_{m,ij} = 0 \Rightarrow \sum_a s_a(v_a) \delta_{am,ij} \geq u_{ij} \tag{35}$$

$$\alpha_i^k > 0 \Rightarrow \sum_j \sum_w T_{ij}^{kw} = H_i^k \tag{36}$$

$$\sum_j \sum_w T_{ij}^{kw} < H_i^k \Rightarrow \alpha_i^k = 0 \tag{37}$$

where $U_{ij} = \mu\theta u_{ij}$, α_i^k and ν_j^w are the dual variables associated to (30), (29) and (28).

III Solution method

We will focus on the solution method for the maximum-entropy version of the Herbert-Stevens model with transportation congestion (Problem P3). The same method could be applied to the solution of (P2).

1) Calibration

Let us assume that we know the congestion function on each link of the road network and that a survey allowed us to determine the allocations $\hat{T}_{ij}^{kw}$ and therefore the vacancies in the housing stock for each zone and each housing type as well as the demand for travel $\sum_k \sum_w \hat{T}_{ij}^{kw}$ for each OD pair. We observe an occupation of housing equal to $\hat{H}_i^k$. We can determine the

equilibrium travel times and therefore the travel costs $\hat{c}_{ij}$ between all origin-destination pairs by using a traffic equilibrium technique with fixed demands such as the one developed by S. Nguyen [9,12]. For calibration purposes, we assume the travel costs fixed and equal to their equilibrium values and also that the inequality constraints (29) are replaced by the equality constraints

$$\sum_j \sum_w T_{ij}^{kw} = \hat{H}_i^k \qquad (29')$$

That model is then equivalent to the Senior-Wilson maximum entropy version of the Herbert-Stevens model [30]. μ can be calibrated by the method these authors proposed: Determine μ so that the total observed surplus

$$\hat{Z} = \sum_{wijk} \hat{T}_{ij}^{kw} [\beta_{ij}^{kw} - r_i^k \quad \theta \hat{c}_{ij}] \text{ equals the total surplus} \quad \text{computed by}$$

the model: $\qquad Z = \sum\limits_{wijk} T_{ij}^{kw} [\beta_{ij}^{kw} - r_i^k - \theta \hat{c}_{ij}]$

2) Prediction

We now replace μ by its calibrated value. We solve directly problem (P3) with inequality constraints. (P3) consists in minimizing a convex function subject to linear constraints. This problem can be solved by a technique such as the Frank and Wolfe method[1] [15]. This method has proved successful in solving analogous problems in the past [11, 12, 13, 14]. A typical iteration of this algorithm requires the solution of a linear program that determines the descent direction, and the solution of a one-dimensional minimization problem that determines the step size that achieves the best improvement in the objective function, given the descent direction. Consider an intermediate stage of the application of the

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[1] Cf. the Appendix for a formulation of that method.

algorithm when a feasible solution is known: $\overline{T}_{ij}^{kw}$ and path flows $\overline{h}_{m,ij}$.
The descent direction is obtained by solving the problem:

$$\text{Min} \sum_i \sum_j \sum_k \sum_w \overline{\overline{c}}_{ij}^{kw} \, Y_{ij}^{kw} + \sum_i \sum_j \sum_m \overline{c}_{m,ij} \, Z_{m,ij} \tag{38}$$

$$\text{s.t.} \quad \sum_i \sum_k Y_{ij}^{kw} = E_j^w \qquad\qquad \text{all } j,w \tag{39}$$

$$\sum_j \sum_w Y_{ij}^{kw} \leq H_i^k \qquad\qquad \text{all } i,k \tag{40}$$

$$\sum_m Z_{m,ij} = \sum_{kw} Y_{ij}^{kw} \qquad\qquad \text{all } i, j \tag{41}$$

$$Z_{m,ij} \geq 0, \quad Y_{ij}^{kw} \geq 0 \tag{42}$$

where $\quad \overline{\overline{c}}_{ij}^{kw} = 1 + \ell n \, \overline{T}_{ij}^{kw} - \mu \, \hat{\beta}_{ij}^{kw}$ $\tag{43}$

$\frac{1}{\mu\theta} \, \overline{c}_{m,ij}$ is the length of path m for OD pair (i,j) on the network when
the flows are $\overline{h}_{m,ij}$. The linear program (38)-(42) simplifies by noting
that $\frac{1}{\mu\theta} \, \overline{c}_{m,ij}$ may be replaced by $\frac{1}{\mu\theta} \, c_{ij}^*$ the length of the shortest path
between every pair (i,j) on the network. In an optimal solution, if
$\overline{c}_{m,ij} > c_{ij}^*$, then the corresponding $Z_{m,ij} = 0$, since otherwise the
solution could be improved by diminishing a $Z_{m,ij}$ for which $\overline{c}_{m,ij} > c_{ij}^*$
and augmenting Z_{ij}^* (the flow on the shortest path). Thus the direction
is obtained by the minimization of the objective function

$$\sum_i \sum_j \sum_k \sum_w \overline{\overline{c}}_{ij}^{kw} \, Y_{ij}^{kw} + \sum_i \sum_j c_{ij}^* \left(\sum_m Z_{m,ij} \right) \tag{44}$$

which simplifies to

$$\text{Min} \quad \sum_i \sum_j \sum_k \sum_w \overline{\overline{c}}_{ij}^{kw} \, Y_{ij}^{kw} + \sum_i \sum_j c_{ij}^* \sum_k \sum_w Y_{ij}^{kw} \tag{45}$$

$$\text{or} \quad \text{Min} \quad \sum_i \sum_j \sum_k \sum_w \left(\overline{\overline{c}}_{ij}^{kw} + c_{ij}^* \right) Y_{ij}^{kw} \tag{46}$$

The Linear Program (38)-(42) becomes:

$$\text{Min} \sum_i \sum_j \sum_k \sum_w \overline{c}_{ij}^{kw} \, Y_{ij}^{kw} \tag{47}$$

$$\text{s.t.} \sum_i \sum_j Y_{ij}^{kw} = E_j^w \qquad \text{all } j, w \tag{48}$$

$$\sum_j \sum_w Y_{ij}^{kw} \le H_i^k \qquad \text{all } i, k \tag{49}$$

$$Y_{ij}^{kw} \ge 0 \tag{50}$$

$$\text{where } \overline{c}_{ij}^{kw} = \overline{\overline{c}}_{ij}^{kw} + c_{ij}^* \tag{51}$$

This is a transportation problem. This problem can be solved by an efficient code such as the code developed by Harris [19]. The solution elements are the components of the descent direction vector. We obtain $Z_{m,ij}$ by assigning $\sum_k \sum_w \overline{Y}_{ij}^{kw}$ for each (i,j) on the shortest route that was found while computing the c_{ij}^*. Let $\{\overline{Y}_{ij}, \overline{Z}_{m,ij}\}$ be the solution to the linear program defining the descent direction. The descent direction for the current iteration is:

$$(\overline{Y}_{ij}^{kw} - \overline{T}_{ij}^{kw}) \qquad \text{all } i,j,k,w \tag{52}$$

$$(\overline{Z}_{m,ij} - \overline{h}_{m,ij}) \qquad \text{for all } m, i \text{ and } j \tag{53}$$

Note that $\overline{Z}_{m,ij} = 0$ for all m except for the shortest path m^* where

$$\overline{Z}_{m^*,ij} = \sum_k \sum_w \overline{Y}_{ij}^{kw} \tag{54}$$

The optimal step length λ^* to find the next feasible solution is given by the solution of the one dimensional minimization problem:

$$\text{Min} \sum_i \sum_j \sum_k \sum_w x_{ij}^{kw} \ln x_{ij}^{kw} - \mu \sum_w \sum_i \sum_j \sum_k \hat{\beta}_{ij}^{kw} x_{ij}^{kw} + \mu \, \theta \sum_a F(\phi_a) \tag{55}$$

where

$$x_{ij}^{kw} = \overline{T}_{ij}^{kw} + \lambda(\overline{Y}_{ij}^{kw} - \overline{T}_{ij}^{kw}) \tag{56}$$

$$\phi_a = \overline{v}_a + \lambda(\overline{w}_a - \overline{v}_a) \tag{57}$$

$$\overline{w}_a = \sum_m \sum_i \sum_j \delta_{am,ij} \overline{Z}_{m,ij} \tag{58}$$

$$F(\phi_a) = \int_0^{\phi_a} s_a(x)dx \tag{59}$$

Note that the path flows $h_{m,ij}$ are redundant in the computation of the descent direction and of the step length λ^*. The explicit information of utilized paths between the OD pairs is not necessary. This property has been noted and used in several other cases [9, 13, 14]. The complete solution method to (P3) can be summarized as follows.

Step 1: Obtain an initial feasible solution $\{\overline{T}_{ij}^{kw}, \overline{v}_a\}$ to problem (P3).

Step 2: For each arc a, compute the current cost $\overline{c}_a = s_a(\overline{v}_a)$.

Step 3: For each O-D pair (i,j) determine the shortest path τ_{ij}, let $\frac{1}{\mu\theta} c_{ij}^*$ be the travel cost on τ_{ij}

Step 4: Compute: $\overline{c}_{ij}^{kw} = c_{ij}^* + 1 + \ell n \overline{T}_{ij}^{kw} - \mu \hat{\beta}_{ij}^{kw}$.

Step 5: Solve the transportation problem (47) to (50) to obtain $\overline{Y}_{ij}^{kw}$.

Step 6: Initialize $\overline{w}_a = 0$ for all arcs a. For all (i,j) set $\overline{w}_a = \overline{w}_a + \sum_k \sum_w \overline{Y}_{ij}^{kw}$ for $a \epsilon \tau_{ij}$

Step 7: Stopping criterion:

Compute

$$\Delta = |\sum_i \sum_j \sum_k \sum_w \overline{c}_{ij}^{kw} (\overline{Y}_{ij}^{kw} - \overline{T}_{ij}^{kw}) + \sum_a \overline{c}_a (\overline{w}_a - \overline{v}_a)|$$

If $\Delta \leq \varepsilon$ (a predetermined tolerance) stop; otherwise continue
to the following step.

Step 8: Solve the one-dimensional problem (55) to determine the optimal

step length λ^*. Revise the allocations and flows as follows:

$$\overline{T}_{ij}^{kw} = \overline{T}_{ij}^{kw} + \lambda^* \ (\overline{Y}_{ij}^{kw} - \overline{T}_{ij}^{kw}) \qquad \text{all } i,j,k,w$$

$$\overline{v}_a = \overline{v}_a + \lambda^* \ (\overline{w}_a - \overline{v}_a) \qquad \text{all } a$$

Return to step 2.

This solution procedure is very similar to the procedure proposed
for the combined trip distribution-trip assignment by Ferland,
Florian and Nguyen [14].

IV A combined equilibrium model for residential location, modal split, trip distribution and traffic assignment

Florian and Nguyen developed a combined trip distribution, modal split and
trip assignment model [13]. It is possible to extend it to include the
choice of residential location as well. The model thus obtained can also
be considered as an extension of the Senior-Wilson suboptimal version of
the Herbert-Stevens model [30] or of model (P3) above.

There are two modes: the private automobile and the bus mode. As above
the road network contains a set A of arcs a, each arc having its own
congestion function $s_a(v_a)$. There is also a transit network[1] consisting
of a set S of access arcs, transfer arcs and transit line segments. A
transit line is composed of a number of segments. We associate a time c_s
with each arc and line segment, $s \varepsilon S$, of the transit network as follows:

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[1] Here we follow closely Florian and Nguyen's notation and definitions [13].

- a walking and waiting time with an access arc

- a walking and a waiting time with a transfer arc

- an in-vehicle time with a line segment.

The link times c_s are not functions of the transit link volumes v_s. Therefore there is no congestion effect assumed on the transit network. Travellers on the transit network choose the shortest path between an origin and a destination. Let u_{ij}^{tr} be the constant travel time from i to j on the transit network. The notation is the same as in previous sections with some modifications. Now $\hat{\beta}_{ij}^{kw}$ is a function of mode through the money cost of travel which differs with mode choice. This comes from the fact that $\hat{\beta}_{ij}^{kw}$ is obtained from the definition of the utility function and the budget constraint of the household and takes into account the money cost of travel (Cf. Section V below). We have then, $\hat{\beta}_{ij}^{kw,au}$ and $\hat{\beta}_{ij}^{kw,tr}$ respectively for car and transit. Recall $\hat{\beta}_{ij}^{kw}$ is the consumer surplus that would be obtained by each worker if travel time was zero.

$T_{ij}^{kw,au}$ is the number of heads of households that earn w, work in j, live in a type k house and use the car to travel to work.

$T_{ij}^{kw,tr}$ is the analogous quantity for heads of households that use transit.

$h_{m,ij}^{au}$ corresponds to the flow of private automobiles using path m between i and j, converted into person trips by the uniform car-occupancy factor.

The total flow of vehicles on link a (converted into persons) is:

$$v_a = \sum_i \sum_j \sum_m \delta_{am,ij} \, h_{m,ij}^{au} + v_a^{tr}$$

and v_a^{tr} is the vehicle equivalent of the number of buses that use link a (again converted into person-trips by using the uniform car-occupancy factor). v_a^{tr} may be computed by determining the number of private vehicles

equivalent to a bus and by considering all the buses that use the link per time period. The assumptions made on the two modes imply that private automobiles do not have any influence on the speed or frequency of buses while buses do interfere with the traffic of automobiles and increase congestion. This assumes traffic management decisions giving priority to buses over cars on the road, for instance special bus lanes on which buses have absolute priority but open to cars in the absence of buses. (Such a system actually functions in Paris for instance)

We require $T_{ij}^{kw,au}$, $T_{ij}^{kw,tr}$ and $h_{m,ij}^{au}$ to be solution to the following minimization problem:

$$(P4): \quad \text{Min} \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw,au} \ln T_{ij}^{kw,au} + \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw,tr} \ln T_{ij}^{kw,tr}$$

$$- \mu \left\{ \sum_w \sum_i \sum_j \sum_k \hat{\beta}_{ij}^{kw,au} T_{ij}^{kw,au} + \sum_w \sum_i \sum_j \sum_k \hat{\beta}_{ij}^{kw,tr} T_{ij}^{kw,tr} \right.$$

$$\left. - \theta \left[\sum_{a \in A} \int_0^{v_a} s_a(x) dx + \sum_i \sum_j \sum_k \sum_w T_{ij}^{kw,tr} u_{ij}^{tr} \right] \right\} \tag{60}$$
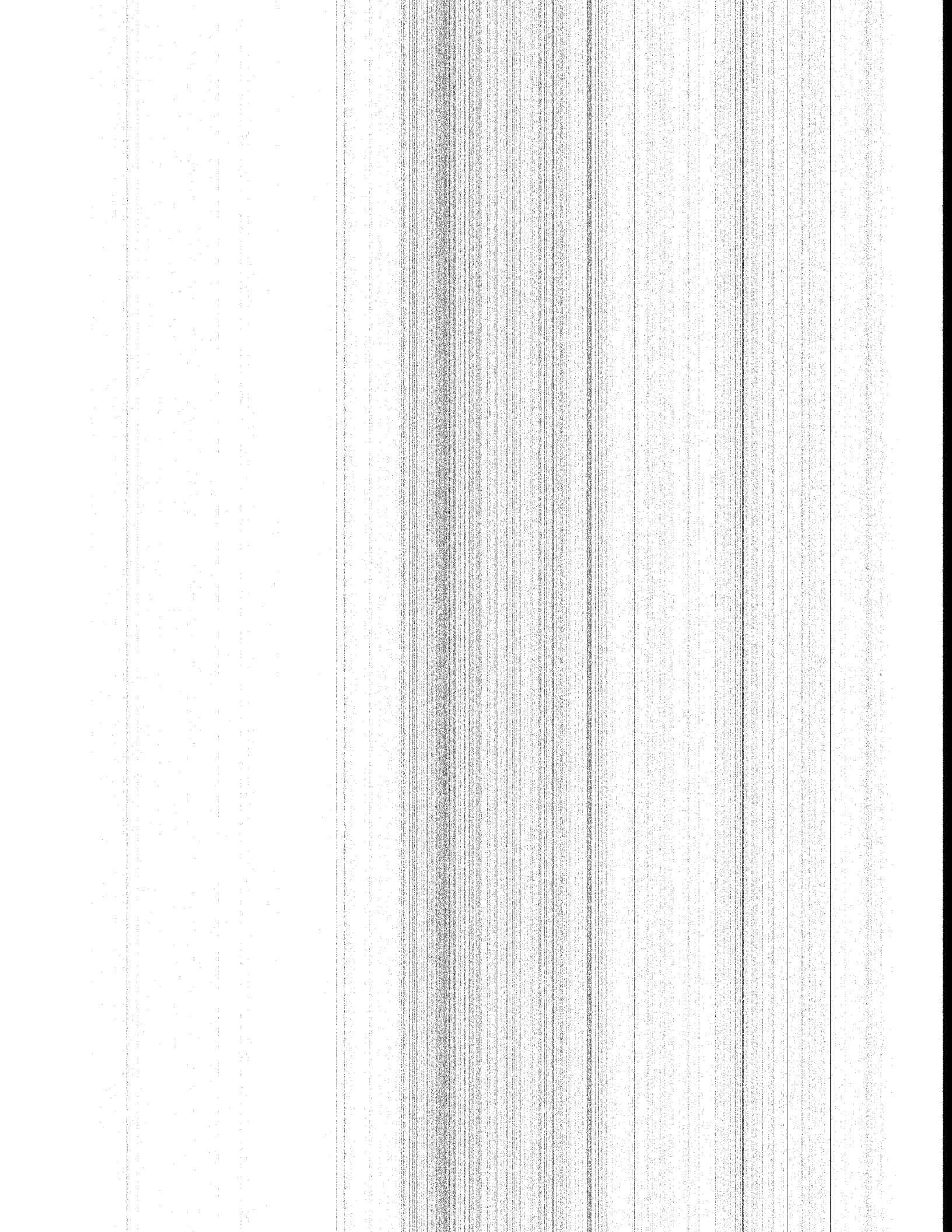
$$\text{s.t.} \quad \sum_j \sum_w (T_{ij}^{kw,au} + T_{ij}^{kw,tr}) \le H_i^k \qquad \text{all } i,k \tag{61}$$

$$\sum_i \sum_j (T_{ij}^{kw,au} + T_{ij}^{kw,tr}) = E_j^w \qquad \text{all } j,w \tag{62}$$

$$\sum_k \sum_w T_{ij}^{kw,au} = \sum_m h_{m,ij}^{au} \qquad \text{all } i,j \tag{63}$$

$$v_a = \sum_i \sum_j \sum_m \delta_{am,ij} h_{m,ij}^{au} + v_a^{tr}$$

$$T_{ij}^{kw,au} \ge 0 \tag{64}$$

$$T_{ij}^{kw,tr} \geq 0 \tag{65}$$

$$h_{m,ij}^{au} \geq 0 \tag{66}$$

We let $T_{ij}^{kw,au} \ln T_{ij}^{kw,au} = 0$ at $T_{ij}^{kw,au} = 0$

and $T_{ij}^{kw,tr} \ln T_{ij}^{kw,tr} = 0$ at $T_{ij}^{kw,tr} = 0$

(60) is then convex in the variables $T_{ij}^{kw,au}$, $T_{ij}^{kw,tr}$ and $h_{m,ij}^{au}$.

The Kuhn-Tucker conditions of (P4) are necessary and sufficient for

optimality and are:

$$T_{ij}^{kw,au} = \exp(-1-\nu_j^w-\alpha_i^k) \exp[\mu(\hat{\beta}_{ij}^{kw,au}-\theta u_{ij}^{au})] \tag{67}$$

$$T_{ij}^{kw,tr} = \exp(-1-\nu_j^w-\alpha_i^k) \exp[\mu(\hat{\beta}_{ij}^{kw,tr}-\theta u_{ij}^{tr})] \tag{68}$$

$$h_{m,ij}^{au} > 0 \Rightarrow \sum_a s_a(v_a)\, \delta_{am,ij} = u_{ij}^{au} \tag{69}$$

$$h_{m,ij}^{au} = 0 \Rightarrow \sum_a s_a(v_a)\, \delta_{am,ij} \geq u_{ij} \tag{70}$$

$$\alpha_i^k > 0 \Rightarrow H_i^k = \sum_j \sum_W (T_{ij}^{kw,au} + T_{ij}^{kw,tr}) \tag{71}$$

$$H_i^k > \sum_j \sum_W (T_{ij}^{kw,au} + T_{ij}^{kw,tr}) \Rightarrow \alpha_i^k = 0 \tag{72}$$

The modal share for a quadruplet (i,j,k,w) is given by:

$$\frac{T_{ij}^{kw,au}}{T_{ij}^{kw,tr}+T_{ij}^{kw,au}} = \frac{\exp[\mu(\hat{\beta}_{ij}^{kw,au}-\theta u_{ij}^{au})]}{\exp[\mu(\hat{\beta}_{ij}^{kw,au}-\theta u_{ij}^{au})] + \exp[\mu(\hat{\beta}_{ij}^{kw,tr}-\theta u_{ij}^{tr})]} \tag{73}$$

$$= \frac{\exp[-\mu(c_{ij}'^{au}+\theta u_{ij}^{au})]}{\exp[-\mu(c_{ij}'^{au}+\theta u_{ij}^{au})] + \exp[-\mu(c_{ij}'^{tr}+\theta u_{ij}^{tr})]} \tag{74}$$

(74) represents a logit model with generalized costs " $c_{ij}'^{au} + \theta u_{ij}^{au}$"

and " $c_{ij}'^{tr} + \theta u_{ij}^{tr}$ ". ($c_{ij}'^{au}$ and $c_{ij}'^{tr}$ are the money

transit). That expression is independent of w and k, and therefore (74) represents the modal share for all work-trips between i and j.

(P4) can be calibrated and solved by the same method as (P3). The only difference is that the linear program that one would have to solve to find a descent direction in the Frank and Wolfe method would be a bimodal version of the Hitchcock problem. However, as shown by Florian and Nguyen [13] this problem can be reduced to an equivalent single mode Hitchcock problem.

If the two alternative modes of travel were rail transit and the private car, one would have to extend to residential location the model developed by Los [24] for modal choice, mode of access to rail, trip distribution and traffic assignment. One would obtain a model analogous to (P4) above. However the linear program one would have to solve to find a descent direction in the Frank and Wolfe algorithm would no longer be reducible to a Hitchcock problem and the method might not be applicable then with a realistic disaggregation of households by income and houses by type, because of size limitations of the linear programs that can be solved by general codes.

V Estimation of bid rents

Assume that a household of type w has the following utility function:

$$u^w = M_{ij}^{kw} + \zeta^w . h_i^k - \theta c_{ij} \tag{75}$$

where M_{ij}^{kw} is the expenditure on non-locational goods if the household head works in j, lives in i in a house of type k and has income W.

h_i^k is a <u>vector</u> of housing characteristics for a type k house in zone i.

c_{ij} is the value of the time spent going to work from i to j (It depends on the mode and possibly on congestion).

ζ^W and θ are coefficients of the utility function to be estimated. Notice that θ is assumed independent of w. If this were not the case we would have to compute an average of the θ^W's, weighted by the relative importance of each group, in order to use the models proposed in the previous sections of this paper, Let us assume that u^W, the level of utility for group w,is known. For instance let it be the current observed value of the utility of each individual household head in group w. For a (j,w) type household choosing the (i,k) housing bundle we have the budget constraint:

$$W = b_{ij}^{kw} + M_{ij}^{kw} + c'_{ij} \tag{76}$$

where c'_{ij} is the money cost of travel between i and j (we could distinguish between c'^{au}_{ij} and c'^{tr}_{ij} if there were 2 modes, car and transit) and where b_{ij}^{kw}, the residual budget, is the bid rent of household (j,w). From (75) and (76) we obtain the bid rent as a function of the level of utility and of travel cost.

$$b_{ij}^{kw} = W - u^W + \zeta^W \cdot h_i^k - c'_{ij} - \theta c_{ij} \tag{77}$$

$$\text{or} \quad b_{ij}^{kw} = \beta_{ij}^{kw}(u^W) - \theta c_{ij} \tag{78}$$

$$\text{where} \quad \beta_{ij}^{kw}(u^W) = W - u^W + \zeta^W \cdot h_i^k - c'_{ij} \tag{79}$$

The bid rents as given by (78) are used as <u>exogenous</u> data for the models of the preceding sections.

Therefore, in order to apply any of the models (P1) to (P4) we have to

estimate the bid rents b_{ij}^{kw}. Two methods have been proposed previously:

1) <u>Estimation of the multiple regression</u> [3, 20, 31]

$$M_{ij}^{kw} = u^w - \zeta^w h_i^k + \theta c_{ij} \qquad (80)$$

with u^w being the constant intercept of the linear regression model, would give the bid rents for various housing types and locations given the utility level u^w. As pointed out by Senior and Wilson [30] this method assumes that "market expenditures" on housing or on nonlocational goods, on which the multiple regression model is fitted, are the same as the "preferred expenditures". This is only the case in the long run equilibrium if no taxes and subsidies are necessary for all households to be located i.e. if their utility level has been properly adjusted. Compounding that problem [16, 21] is the fact that in many countries such as the U.K., Sweden, or Canada, there is rent control, so that rents are predetermined administratively and do not adjust themselves as market rents would in a competitive situation. In fact all tenants tend to enjoy consumer surpluses in such controlled situations. Hence the second approach to bid rent estimation.

2) <u>Direct determination of the utility functions by household surveys</u>
This is the method proposed by Härsman and Snickars [21] and actually applied to Stockholm. Many methods are available today to estimate utility functions by interviews (Cf. [23] for a thorough presentation of the methodology available.)

If the postulated functional form of the utility function is not linear or if the interviews lead to a nonlinear utility function, the utility

function has to be linearized to produce a bid rent that is a linear function of travel time. One way to do this is to choose a representative household in each income group and apply a Taylor series expansion of the first order on the utility function of the representative household around its current housing situation.

VI Conclusion

This paper has shown that it was possible to combine in an integrated model a residential location model of the Herbert-Stevens type (or one of its maximum entropy suboptimal versions) with a transportation model assuming congestion on the road network. The resulting mathematical program (P2) satisfies Kuhn-Tucker conditions which express the required equilibrium conditions on the housing market and the Wardrop's user-equilibrium conditions on the road network. This mathematical program can be solved by an algorithm (Frank and Wolfe) which converges to the equilibrium solution. It is possible to derive a suboptimal version of the basic model and to extend it to modal choice.

In addition the descent directions of the algorithms to solve (P2) or (P3) are obtained by solving transportation problems of the same size as the Herbert-Stevens transportation problem itself (however the size doubles in the bimodal case). Therefore the level of household or housing type disaggregation can be the same for (P2) or (P3) as for (P1), the Herbert-Stevens model itself, and slightly less for (P4), the bimodal version. (In the latter case, we would have to cut the number of income groups by half, for instance). A possible level of disaggregation is that used by

Senior and Wilson in their work [30], i.e. 5 social class types, 28 work-place zones, 6 housing types and 28 possible residence zones. This is well within the capability of the current best codes for the transportation problem [19]. The other computational limit, that on the size of the networks that can be inputs to the current best shortest path codes, is not very constraining [12].

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APPENDIX

The Frank and Wolfe algorithm [15]

Consider the problem of minimizing a convex function subject to linear constraints:

Min $Z(x)$, subject to $AX \leq b$, $X \geq 0$.

The detailed steps of the Frank Wolfe algorithm for solving this problem are:

Step 1   Given a feasible solution X^1, set $\ell = 1$

Step 2   Determine Y^ℓ that minimizes

$\nabla Z(X^\ell) Y$, subject to $AY \leq b$, $Y \geq 0$.

Step 3   Set the descent direction $d^\ell = Y^\ell - X^\ell$.

If $|\nabla Z(X^\ell) d^\ell| \leq \varepsilon$, terminate, where ε is a suitable convergence parameter. (X^ℓ is the optimal solution).

Step 4   Find the optimal step length λ^ℓ that minimizes

$Z(X^\ell + \lambda d^\ell)$ for $0 \leq \lambda \leq 1$.

Step 5   Revise the current solution

$X^{\ell+1} = X^\ell + \lambda^\ell d^\ell$;

set $\ell = \ell + 1$ and return to Step 2.