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A REPRESENTATION THEOREM FOR DOMAINS WITH DISCRETE AND CONTINUOUS VARIABLES

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RÉSUMÉ

Ce papier prouve un nouveau théorème de représentation pour des domaines avec des variables tant discrètes que continues. Le résultat généralise le théorème de représentation bien connu de Debreu dans des domaines connexes. Un renforcement de l'axiome de continuité standard est employé pour garantir l'existence d'une représentation. Une généralisation du théorème principal et une application du résultat plus général sont aussi présentées.

Mots clés : variables continues et discrètes, représentations

ABSTRACT

This paper proves a new representation theorem for domains with both discrete and continuous variables. The result generalizes Debreu's well-known representation theorem on connected domains. A strengthening of the standard continuity axiom is used in order to guarantee the existence of a representation. A generalization of the main theorem and an application of the more general result are also presented.

Key words : continuous and discrete variables, representations

1. Introduction

This paper provides a generalization of Debreu's [1959] classical result on the representability of an ordering. Debreu's [1959, pp. 56–59] theorem shows that any continuous ordering defined on a nonempty, connected subset of a Euclidean space has a continuous representation. In most approaches to the representability problem, the universal set of alternatives is interpreted as a set of consumption bundles or, more generally, a set of vectors of continuous economic variables. Results in more abstract settings make use of continuity properties as well; see, for example, Herden [1995] for a summary of some of these.

We consider 'mixed' domains with both discrete and continuous variables. Our domain can be represented as a Cartesian product of a set of vectors with integer components only and a connected set of vectors in a Euclidean space. We show that a natural generalization of continuity is not sufficient to guarantee the existence of a representation but a suitable strengthening of the continuity condition is.

The domains considered here are of importance in economic models where perfectly divisible goods as well as indivisible goods (or 'bads') are present. Such environments are natural if a comprehensive account of well-being, such as that of Griffin [1986], is employed because many determinants of well-being are, by their very nature, best described as discrete variables.

The main result of the paper—a representation theorem for mixed domains—is a generalization of a result in Blackorby, Bossert and Donaldson [2001]. The current paper uses a more general approach by considering a wider range of applications (Blackorby, Bossert and Donaldson [2001] is concerned exclusively with population ethics) and, unlike the earlier contribution, does not employ any monotonicity conditions.

Section 2 introduces our notation and definitions, along with a brief review of some classical results. Our new representation theorem is stated and proved in Section 3. A generalization of the main theorem is provided in Section 4. It allows the domain to include vectors of different dimensions for both discrete and continuous variables and, furthermore, the possible values of the continuous variables may depend on the values of the discrete variables. This generalization is then used to prove a representation theorem on a domain which is the union of connected subsets of Euclidean spaces, possibly of different dimensions. A straightforward corollary generalizes Debreu's theorem to domains that are unions of connected subsets of the same Euclidean space.

2. Preliminaries

The set of all (positive) integers is denoted by $\mathcal{Z}$ ($\mathcal{Z}_{++}$) and the set of real numbers by $\mathcal{R}$. In addition, for $n \in \mathcal{Z}_{++}$, let $\mathcal{Z}^n$ be the n-fold Cartesian product of $\mathcal{Z}$ and $\mathcal{R}^n$ the n-fold Cartesian product of $\mathcal{R}$.

Let $X \neq \emptyset$ be a universal set of alternatives. An ordering on X is a reflexive, transitive and complete binary relation $\succeq$. The asymmetric and symmetric factors of $\succeq$ are denoted by $\succ$ and $\sim$. A function $U: X \to \mathcal{R}$ is a representation of $\succeq$ if and only if

$$x \succeq y \Leftrightarrow U(x) \geq U(y) \quad \text{for all } x, y \in X.$$

It is obvious that reflexivity, transitivity and completeness are necessary conditions on a relation $\succeq$ on X for the existence of a representation. Though those conditions are, in general, not sufficient, they are in the special case where X is finite or countably infinite. Although the result is well known, we present a proof for the sake of completeness.

Theorem 1: *Let X be a nonempty finite or countably infinite set. $\succeq$ is an ordering on X if and only if there exists a function $U: X \to \mathcal{R}$ that represents it.*

Proof. Clearly, if $\succeq$ is represented by U, it is an ordering.

Suppose that $\succeq$ is an ordering and that X is countably infinite, so that we can write $X = \{x_1, x_2, \ldots\}$. We construct a representation $U: X \to \mathcal{R}$ recursively. Let $U(x_1) = 0$ and suppose that U has been defined for $x_1, \ldots, x_{k-1}$, where $k \in \mathcal{Z}_{++} \setminus \{1\}$. If $\{U(x_j) \mid j \in \{1, \ldots, k-1\} \text{ and } x_j \succ x_k\} \neq \emptyset$, let

$$u_{\min}(x_k) = \min\{U(x_j) \mid j \in \{1, \ldots, k-1\} \text{ and } x_j \succ x_k\}$$

and, if $\{U(x_j) \mid j \in \{1, \ldots, k-1\} \text{ and } x_k \succ x_j\} \neq \emptyset$, let

$$u_{\max}(x_k) = \max\{U_i(x_j) \mid j \in \{1, \ldots, k-1\} \text{ and } x_k \succ x_j\}.$$

Define

$$U(x_k) = \begin{cases} U(x_j) & \text{if } \exists j \in \{1, \ldots, k-1\} \text{ such that } x_j \sim x_k, \\ \max\{U(x_j) \mid j \in \{1, \ldots, k-1\}\} + 1 & \text{if } x_k \succ x_j \ \forall j \in \{1, \ldots, k-1\}, \\ \min\{U(x_j) \mid j \in \{1, \ldots, k-1\}\} - 1 & \text{if } x_j \succ x_k \ \forall j \in \{1, \ldots, k-1\}, \\ \frac{1}{2}[u_{\min}(x_k) + u_{\max}(x_k)] & \text{otherwise.} \end{cases}$$

Clearly, this function is well-defined and represents $\succeq$. The proof for the case where X is finite is a simplified version of the above. ■

If X is uncountable, not every ordering on X has a representation. For example, if $X = \mathcal{R}^c$ for some $c \in \mathcal{Z}_{++}$, lexicographic orderings on X cannot be represented; see Debreu [1959, pp. 72–73]. Thus, in the case of an uncountable universal set X, further restrictions must be imposed on $\succeq$ in order to guarantee its representability. One such condition is continuity, defined as follows. Let $c \in \mathcal{Z}_{++}$, and suppose $X \subseteq \mathcal{R}^c$ is a nonempty and connected set.

Continuity: For all $x \in X$, the sets $\{y \in X \mid y \succeq x\}$ and $\{y \in X \mid x \succeq y\}$ are closed in X.

The following theorem is due to Debreu [1959, pp. 56–59].

Theorem 2: *Let $X \subseteq \mathcal{R}^c$ for some $c \in \mathcal{Z}_{++}$, and suppose X is nonempty and connected. $\succeq$ is an ordering on X satisfying continuity if and only there exists a continuous function $U: X \to \mathcal{R}$ that represents it.*

3. A General Representation Theorem

Suppose that the elements of X are vectors that contain both discrete and continuous components with X equal to the Cartesian product $\mathcal{D} \times \mathcal{C}$, where $\mathcal{D} \subseteq \mathcal{Z}^d$ and $\mathcal{C} \subseteq \mathcal{R}^c$ for some $d, c \in \mathcal{Z}_{++}$. We assume that $\mathcal{D}$ and $\mathcal{C}$ are nonempty and $\mathcal{C}$ is connected. For $x \in X = \mathcal{D} \times \mathcal{C}$, we write $x = (x^D, x^C)$ where $x^D \in \mathcal{D}$ and $x^C \in \mathcal{C}$. For a function $U: \mathcal{D} \times \mathcal{C} \to \mathcal{R}$ with image $U(x^D, x^C)$ for all $x = (x^D, x^C) \in X = \mathcal{D} \times \mathcal{C}$, we refer, for simplicity, to x^D as the first argument of U and to x^C as the second argument of U even though x^D or x^C may be composed of more than one component.

The natural definition of continuity in this setting requires the relevant property with respect to the continuous components to hold conditionally for all fixed values of the discrete components.

Conditional Continuity: For all $x \in X$, the sets $\{y^C \in \mathcal{C} \mid (x^D, y^C) \succeq (x^D, x^C)\}$ and $\{y^C \in \mathcal{C} \mid (x^D, x^C) \succeq (x^D, y^C)\}$ are closed in $\mathcal{C}$.

Conditional continuity is not sufficient to guarantee the existence of a representation. Consider, for example, the relation $\succeq$ on $X = \mathcal{Z}^d \times \mathcal{R}^c$ defined as follows. For all $x, y \in X$,

$$x \succeq y \Leftrightarrow f(x^C) > f(y^C) \text{ or } [f(x^C) = f(y^C) \text{ and } g(x^D) \geq g(y^D)]$$

where $f: \mathcal{R}^c \to \mathcal{R}$ is an arbitrary continuous and increasing function, and $g: \mathcal{Z}^d \to \mathcal{Z}$ is an arbitrary injective function. This relation is an ordering satisfying conditional continuity but it does not have a representation. See Blackorby, Bossert and Donaldson [2001] for a discussion.

The following axiom is a strengthening of conditional continuity.

Unconditional Continuity: For all $x \in X$ and for all $y^D \in \mathcal{D}$, the sets $\{y^C \in \mathcal{C} \mid (y^D, y^C) \succeq (x^D, x^C)\}$ and $\{y^C \in \mathcal{C} \mid (x^D, x^C) \succeq (y^D, y^C)\}$ are closed in $\mathcal{C}$.

Note that, unlike conditional continuity, unconditional continuity applies to all values of the discrete variables. Because the empty set and $\mathcal{C}$ are both closed in $\mathcal{C}$, the axiom is consistent with the possibility that, for some distinct $\bar{x}^D, \bar{y}^D \in \mathcal{D}$, $(\bar{x}^D, x^C) \succ (\bar{y}^D, y^C)$ for all $x^C, y^C \in \mathcal{C}$. In addition, because $\mathcal{C}$ is connected, if there exist $\hat{y}^C, \check{y}^C \in \mathcal{C}$ such that $(y^D, \hat{y}^C) \succ (x^D, x^C)$ and $(x^D, x^C) \succ (y^D, \check{y}^C)$, there must be some $\tilde{y}^C \in \mathcal{C}$ such that $(y^D, \tilde{y}^C) \sim (x^D, x^C)$.

Unconditional continuity is sufficient for the existence of a representation of an ordering on the mixed domain $X = \mathcal{D} \times \mathcal{C}$. Before stating and proving our main theorem, we present some preliminary observations. Note that these lemmas do not require the full force of unconditional continuity—conditional continuity is sufficient. These results generalize similar observations in Blackorby, Bossert and Donaldson [2001]; see also Blackorby and Donaldson [1984].

Lemma 1: *Let $X = \mathcal{D} \times \mathcal{C}$ with $\mathcal{D} \subseteq \mathcal{Z}^d$ and $\mathcal{C} \subseteq \mathcal{R}^c$ for some $d, c \in \mathcal{Z}_{++}$, and suppose $\mathcal{D}$ and $\mathcal{C}$ are nonempty and $\mathcal{C}$ is connected. If $\succeq$ is an ordering on X satisfying conditional continuity, then there exists a family of continuous functions $\{U^{x^D} \mid x^D \in \mathcal{D}\}$, where $U^{x^D}: \mathcal{C} \to \mathcal{R}$ for all $x^D \in \mathcal{D}$, such that, for all $x, y \in X$ such that $x^D = y^D$,*

$$x \succeq y \Leftrightarrow U^{x^D}(x^C) \geq U^{x^D}(y^C).$$

Proof. Lemma 1 is an immediate consequence of applying Theorem 2 for each $x^D \in \mathcal{D}$. ■

Lemma 2: *Let $X = \mathcal{D} \times \mathcal{C}$ with $\mathcal{D} \subseteq \mathcal{Z}^d$ and $\mathcal{C} \subseteq \mathcal{R}^c$ for some $d, c \in \mathcal{Z}_{++}$, and suppose $\mathcal{D}$ and $\mathcal{C}$ are nonempty and $\mathcal{C}$ is connected. If $\succeq$ is an ordering on X satisfying conditional continuity, then there exists an ordering R on $\cup_{x^D \in \mathcal{D}} \big(\{x^D\} \times U^{x^D}(\mathcal{C})\big)$ such that, for all $x^D \in \mathcal{D}$ and for all $\gamma, \beta \in U^{x^D}(\mathcal{C})$,*

$$(x^D, \gamma) R (x^D, \beta) \Leftrightarrow \gamma \geq \beta$$

and, for all $x, y \in X$,

$$x \succeq y \Leftrightarrow \big(x^D, U^{x^D}(x^C)\big) R \big(y^D, U^{y^D}(y^C)\big)$$

where the family of functions $\{U^{x^D} \mid x^D \in \mathcal{D}\}$ is as in the statement of Lemma 1.

Proof. Define R by letting, for all $n, m \in \mathcal{D}$, for all $\gamma \in U^n(\mathcal{C})$ and for all $\beta \in U^m(\mathcal{C})$, $(n, \gamma) R (m, \beta)$ if and only if there exist $x, y \in X$ such that $x^D = n$, $y^D = m$, $U^n(x^C) = \gamma$, $U^m(y^C) = \beta$, and $x \succeq y$. The relation R is well-defined because it does not depend on the choice of x and y with these properties. Furthermore, R is an ordering because $\succeq$ is. By definition, R has the desired properties. ■

We use P and I to denote the asymmetric and symmetric factors of R.

Theorem 3: *Let $X = \mathcal{D} \times \mathcal{C}$ with $\mathcal{D} \subseteq \mathcal{Z}^d$ and $\mathcal{C} \subseteq \mathcal{R}^c$ for some $d, c \in \mathcal{Z}_{++}$, and suppose $\mathcal{D}$ and $\mathcal{C}$ are nonempty and $\mathcal{C}$ is connected. $\succeq$ is an ordering on X satisfying unconditional continuity if and only if there exists a function $U: X \to \mathcal{R}$, continuous in its second argument, that represents it.*

Proof. Suppose, first, that $\succeq$ is an ordering satisfying unconditional continuity. Let $\{U^{x^D} \mid x^D \in \mathcal{D}\}$ be as in the statement of Lemma 1, and let the ordering R be as in the statement of Lemma 2. We first define a function $W\colon \cup_{x^D\in\mathcal{D}}\big(\{x^D\} \times U^{x^D}(\mathcal{C})\big) \to \mathcal{R}$ that represents R. Because $\mathcal{D} \subseteq \mathcal{Z}^d$, the set $\mathcal{D}$ is countable and, therefore, there exists a bijection $\rho\colon \mathcal{D} \to Z$ where $Z = \{1, \ldots, N\}$ for some $N \in \mathcal{Z}_{++}$ if $\mathcal{D}$ is finite and $Z = \mathcal{Z}_{++}$ if $\mathcal{D}$ is countably infinite. Thus, we can without loss of generality assume that $\mathcal{D} = Z$ with $Z = \{1, \ldots, N\}$ or $Z = \mathcal{Z}_{++}$ in order to simplify our exposition.

Clearly, the result follows immediately from Theorem 2 if $\mathcal{D} = Z = \{1\}$. Now suppose $\mathcal{D}$ contains at least two elements. The following sets will be used in the remainder of the proof. For $n, m \in \mathcal{D} = Z$ with $n \neq m$, let

$$\mathcal{I}^n_m = \{\gamma \in U^n(\mathcal{C}) \mid \exists \beta \in U^m(\mathcal{C}) \text{ such that } (n, \gamma)I(m, \beta)\}.$$

Clearly, $\mathcal{I}^n_m \neq \emptyset$ if and only if $\mathcal{I}^m_n \neq \emptyset$ for all $n, m \in \mathcal{D}$ with $n \neq m$. In addition, if $\gamma \in U^n(\mathcal{C}) \setminus \mathcal{I}^n_m$, unconditional continuity implies that either $(n, \gamma)P(m, \beta)$ for all $\beta \in U^m(\mathcal{C})$ or $(m, \beta)P(n, \gamma)$ for all $\beta \in U^m(\mathcal{C})$. We obtain

Lemma 3: *Let $n, m \in \mathcal{D}$ be such that $n \neq m$. If $\mathcal{I}^n_m \neq \emptyset$, then $\mathcal{I}^n_m$ is connected.*

Proof of Lemma 3. By way of contradiction, suppose $\mathcal{I}^n_m \neq \emptyset$ but $\mathcal{I}^n_m$ is not connected. Then there exist $\hat{\gamma}, \check{\gamma} \in \mathcal{I}^n_m$ and $\tilde{\gamma} \in (\check{\gamma}, \hat{\gamma})$ such that $\tilde{\gamma} \notin \mathcal{I}^n_m$. Because U^n is continuous, $U^n(\mathcal{C})$ is an interval and, thus, $\tilde{\gamma} \in U^n(\mathcal{C})$. By definition of $\mathcal{I}^n_m$, there exist $\hat{\beta}, \check{\beta} \in U^m(\mathcal{C})$ such that $(n, \hat{\gamma})I(m, \hat{\beta})$ and $(n, \check{\gamma})I(m, \check{\beta})$. Because $\hat{\gamma} > \tilde{\gamma} > \check{\gamma}$, it follows that $(n, \hat{\gamma})P(n, \tilde{\gamma})P(n, \check{\gamma})$, and the transitivity of R implies $(m, \hat{\beta})P(n, \tilde{\gamma})$ and $(n, \tilde{\gamma})P(m, \check{\beta})$. By unconditional continuity, there exists $\tilde{\beta} \in (\check{\beta}, \hat{\beta})$ such that $(n, \tilde{\gamma})I(m, \tilde{\beta})$, a contradiction. ■

To construct a representation W of the ordering R, we begin by defining a function $W^1\colon \cup_{j\in Z_1}\big(\{n^1_j\} \times U^{n^1_j}(\mathcal{C})\big) \to \mathcal{R}$ where $Z_1 \subseteq Z$ is a nonempty set indexing the components of a vector $\mathbf{n}^1 = (n^1_j)_{j\in Z_1}$. This vector is defined as follows. Let $n^1_1 = 1$. Let $r > 1$, and suppose we have established $r-1$ components of the vector $\mathbf{n}^1$. If there exists no $n \in Z \setminus \{n^1_1, \ldots, n^1_{r-1}\}$ such that $\cup_{j=1}^{r-1}\mathcal{I}^n_{n^1_j} \neq \emptyset$, let $\mathbf{n}^1 = (n^1_1, \ldots, n^1_{r-1})$. If there is such a value of n, let

$$n^1_r = \min\big\{n \in Z \setminus \{n^1_1, \ldots, n^1_{r-1}\} \mid \cup_{j=1}^{r-1}\mathcal{I}^n_{n^1_j} \neq \emptyset\big\}.$$

This procedure generates a vector $\mathbf{n}^1 = (n^1_j)_{j\in Z_1}$ with a finite or countably infinite number of components. We now use this vector to construct the function W^1.

Define a continuous and increasing function $W^1_{n^1_1}\colon U^{n^1_1}(\mathcal{C}) \to \mathcal{R}$ with $W^1_{n^1_1}\big(U^{n^1_1}(\mathcal{C})\big) = \tilde{A}^1_1$, where $\tilde{A}^1_1 \subseteq \mathcal{R}$ is a nonempty and bounded interval. Note that this is possible because $U^{n^1_1}$ is continuous and, thus, $U^{n^1_1}(\mathcal{C})$ is an interval. Let $A^1_1 = \tilde{A}^1_1$.

If $\mathbf{n}^1 = (n_1^1)$, we define W^1 by letting $W^1(n_j^1, \gamma) = W^1_{n_j^1}(\gamma)$ for all $j \in Z_1 = \{1\}$ and for all $\gamma \in U^{n_j^1}(\mathcal{C})$.

If $\mathbf{n}^1 \neq (n_1^1)$, we employ a recursive construction to define W^1. Suppose $\mathbf{n}^1$ has at least $r > 1$ components and a continuous and increasing function $W^1_{n_j^1}\colon U^{n_j^1}(\mathcal{C}) \to \mathcal{R}$ such that $W^1_{n_j^1}\big(U^{n_j^1}(\mathcal{C})\big) = A_j^1$, where A_j^1 is a nonempty and bounded interval, has been defined for every $j \in \{1, \ldots, r-1\}$. Let

$$\tilde{\Gamma}_r^1 = \cup_{j=1}^{r-1} \mathcal{I}_{n_j^1}^{n_r^1},$$

$$\hat{\Gamma}_r^1 = \{\hat{\gamma} \in U^{n_r^1}(\mathcal{C}) \mid \hat{\gamma} > \gamma \text{ for all } \gamma \in \tilde{\Gamma}_r^1\},$$

and

$$\check{\Gamma}_r^1 = \{\check{\gamma} \in U^{n_r^1}(\mathcal{C}) \mid \check{\gamma} < \gamma \text{ for all } \gamma \in \tilde{\Gamma}_r^1\}.$$

By definition of n_r^1, $\tilde{\Gamma}_r^1 \neq \emptyset$. We prove another lemma before continuing with the proof of the theorem.

Lemma 4: *(i) $\tilde{\Gamma}_r^1$ is connected.*

(ii) If $\hat{\Gamma}_r^1 \neq \emptyset$, then $(n_r^1, \hat{\gamma})P(n_j^1, \beta)$ for all $\hat{\gamma} \in \hat{\Gamma}_r^1$, for all $j \in \{1, \ldots, r-1\}$ and for all $\beta \in U^{n_j^1}(\mathcal{C})$.

(iii) If $\hat{\Gamma}_r^1 \neq \emptyset$, then $\inf \hat{\Gamma}_r^1 \in \hat{\Gamma}_r^1$ if and only if there do not exist $\bar{\jmath} \in \{1, \ldots, r-1\}$ and $\bar{\beta} \in U^{n_{\bar{\jmath}}^1}(\mathcal{C})$ such that $(n_{\bar{\jmath}}^1, \bar{\beta})R(n_j^1, \beta)$ for all $j \in \{1, \ldots, r-1\}$ and for all $\beta \in U^{n_j^1}(\mathcal{C})$.

(iv) If $\check{\Gamma}_r^1 \neq \emptyset$, then $(n_j^1, \beta)P(n_r^1, \check{\gamma})$ for all $\check{\gamma} \in \check{\Gamma}_r^1$, for all $j \in \{1, \ldots, r-1\}$ and for all $\beta \in U^{n_j^1}(\mathcal{C})$.

(v) If $\check{\Gamma}_r^1 \neq \emptyset$, then $\sup \check{\Gamma}_r^1 \in \check{\Gamma}_r^1$ if and only if there do not exist $\bar{\jmath} \in \{1, \ldots, r-1\}$ and $\bar{\beta} \in U^{n_{\bar{\jmath}}^1}(\mathcal{C})$ such that $(n_j^1, \beta)R(n_{\bar{\jmath}}^1, \bar{\beta})$ for all $j \in \{1, \ldots, r-1\}$ and for all $\beta \in U^{n_j^1}(\mathcal{C})$.

Proof of Lemma 4. (i) The case $r = 2$ is covered by Lemma 3. Suppose, therefore, that $r > 2$. By definition,

$$\cup_{j=1}^{q-1} \mathcal{I}_{n_j^1}^{n_q^1} \neq \emptyset \text{ for all } q \in \{2, \ldots, r\}. \tag{1}$$

Suppose $\tilde{\Gamma}_r^1$ is not connected. Using Lemma 3, there exist $\hat{\jmath}, \check{\jmath} \in \{1, \ldots, r-1\}$ and $\hat{\gamma}, \tilde{\gamma}, \check{\gamma} \in U^{n_r^1}(\mathcal{C})$ such that $\hat{\gamma} > \tilde{\gamma} > \check{\gamma}$, $\hat{\gamma} \in \mathcal{I}_{n_{\hat{\jmath}}^1}^{n_r^1}$, $\check{\gamma} \in \mathcal{I}_{n_{\check{\jmath}}^1}^{n_r^1}$, and $\tilde{\gamma} \notin \mathcal{I}_{n_j^1}^{n_r^1}$ for all $j \in \{1, \ldots, r-1\}$. Consequently,

$$(n_{\hat{\jmath}}^1, \beta)P(n_r^1, \tilde{\gamma}) \text{ for all } \beta \in U^{n_{\hat{\jmath}}^1}(\mathcal{C}), \tag{2}$$

$$(n_r^1, \tilde{\gamma})P(n_{\check{\jmath}}^1, \beta) \text{ for all } \beta \in U^{n_{\check{\jmath}}^1}(\mathcal{C}), \tag{3}$$

and, by unconditional continuity, for all $j \in \{1, \ldots, r-1\}$,

$$(n_j^1, \beta) P (n_r^1, \tilde{\gamma}) \text{ for all } \beta \in U^{n_j^1}(\mathcal{C}) \tag{4}$$

or

$$(n_r^1, \tilde{\gamma}) P (n_j^1, \beta) \text{ for all } \beta \in U^{n_j^1}(\mathcal{C}). \tag{5}$$

Let $\hat{S}$ be the set of all $j \in \{1, \ldots, r-1\}$ such that (4) is satisfied and $\check{S}$ be the set of all $j \in \{1, \ldots, r-1\}$ such that (5) is satisfied. (2) and (3) imply that both $\hat{S}$ and $\check{S}$ are nonempty, and (4) and (5) imply that $\{\hat{S}, \check{S}\}$ is a partition of $\{1, \ldots, r-1\}$. Furthermore, it follows that, for all $j \in \hat{S}$ and for all $k \in \check{S}$,

$$(n_j^1, \beta) P (n_k^1, \delta) \text{ for all } \beta \in U^{n_j^1}(\mathcal{C}) \text{ and for all } \delta \in U^{n_k^1}(\mathcal{C}). \tag{6}$$

Let $j^0 = \min\{j \in \hat{S}\}$ and $k^0 = \min\{k \in \check{S}\}$. Because $\{\hat{S}, \check{S}\}$ is a partition of $\{1, \ldots, r-1\}$ and $\hat{S}$ and $\check{S}$ are nonempty, one of j^0 and k^0 is greater than one. Letting $q = \max\{j^0, k^0\}$, (6) implies that $\cup_{j=1}^{q-1} \mathcal{I}_{n_j^1}^{n_q^1} = \emptyset$, a contradiction to (1) which completes the proof of (i).

(ii) By way of contradiction, suppose there exist $\tilde{\gamma} \in \hat{\Gamma}_r^1$, $\hat{\jmath} \in \{1, \ldots, r-1\}$ and $\hat{\beta} \in U^{n_{\hat{\jmath}}^1}(\mathcal{C})$ such that $(n_{\hat{\jmath}}^1, \hat{\beta}) R (n_r^1, \tilde{\gamma})$. By definition of $\hat{\Gamma}_r^1$, we must have $(n_{\hat{\jmath}}^1, \hat{\beta}) P (n_r^1, \tilde{\gamma})$ and, using unconditional continuity,

$$(n_{\hat{\jmath}}^1, \beta) P (n_r^1, \tilde{\gamma}) \text{ for all } \beta \in U^{n_{\hat{\jmath}}^1}(\mathcal{C}). \tag{7}$$

Because $\tilde{\Gamma}_r^1 \neq \emptyset$, there exist $\gamma \in \tilde{\Gamma}_r^1$, $\check{\jmath} \in \{1, \ldots, r-1\}$ and $\check{\beta} \in U^{n_{\check{\jmath}}^1}(\mathcal{C})$ such that $(n_r^1, \gamma) I (n_{\check{\jmath}}^1, \check{\beta})$. Because $(n_r^1, \tilde{\gamma}) P (n_r^1, \gamma)$, we have $(n_r^1, \tilde{\gamma}) P (n_{\check{\jmath}}^1, \check{\beta})$ and, again using unconditional continuity,

$$(n_r^1, \tilde{\gamma}) P (n_{\check{\jmath}}^1, \beta) \text{ for all } \beta \in U^{n_{\check{\jmath}}^1}(\mathcal{C}). \tag{8}$$

(7) and (8) are identical to (2) and (3), and a contradiction is obtained using the argument employed in the proof of part (i).

(iii) Let $\bar{\gamma} = \inf \hat{\Gamma}_r^1$. Suppose first that $\bar{\gamma} \in \hat{\Gamma}_r^1$. By part (ii), $(n_r^1, \bar{\gamma}) P (n_j^1, \beta)$ for all $j \in \{1, \ldots, r-1\}$ and for all $\beta \in U^{n_j^1}(\mathcal{C})$. By way of contradiction, suppose there exist $\bar{\jmath} \in \{1, \ldots, r-1\}$ and $\bar{\beta} \in U^{n_{\bar{\jmath}}^1}(\mathcal{C})$ such that $(n_{\bar{\jmath}}^1, \bar{\beta}) R (n_j^1, \beta)$ for all $j \in \{1, \ldots, r-1\}$ and for all $\beta \in U^{n_j^1}(\mathcal{C})$. This implies $(n_{\bar{\jmath}}^1, \bar{\beta}) R (n_r^1, \gamma)$ for all $\gamma \in \tilde{\Gamma}_r^1$. By unconditional continuity, we obtain $(n_{\bar{\jmath}}^1, \bar{\beta}) R (n_r^1, \bar{\gamma})$, a contradiction.

Now suppose $\bar{\gamma} \notin \hat{\Gamma}_r^1$. Therefore, $\bar{\gamma} \in \tilde{\Gamma}_r^1$. By part (ii) and unconditional continuity, $(n_r^1, \bar{\gamma}) R (n_j^1, \beta)$ for all $j \in \{1, \ldots, r-1\}$ and for all $\beta \in U^{n_j^1}(\mathcal{C})$. By definition, there exist $\bar{\jmath} \in \{1, \ldots, r-1\}$ and $\bar{\beta} \in U^{n_{\bar{\jmath}}^1}(\mathcal{C})$ such that $(n_{\bar{\jmath}}^1, \bar{\beta}) I (n_r^1, \bar{\gamma})$ and, thus, $(n_{\bar{\jmath}}^1, \bar{\beta}) R (n_j^1, \beta)$ for all $j \in \{1, \ldots, r-1\}$ and for all $\beta \in U^{n_j^1}(\mathcal{C})$.

The proofs of (iv) and (v) are analogous to the proofs of (ii) and (iii), respectively. ■

Part (i) of Lemma 4 implies that $\{\hat{\Gamma}^1_r, \tilde{\Gamma}^1_r, \check{\Gamma}^1_r\}$ is a partition of $U^{n^1_r}(\mathcal{C})$, where $\hat{\Gamma}^1_r$ or $\check{\Gamma}^1_r$ may be empty. We now define a continuous and increasing function $W^1_{n^1_r}\colon U^{n^1_r}(\mathcal{C}) \to \mathcal{R}$, starting with the points in $\tilde{\Gamma}^1_r$ (which, by construction, is nonempty). For all $\gamma \in \tilde{\Gamma}^1_r$, let $W^1_{n^1_r}(\gamma) = W^1_{n^1_j}(\beta)$, where $j \in \{1, \ldots, r-1\}$ and $\beta \in U^{n^1_j}(\mathcal{C})$ are such that $(n^1_r, \gamma) I (n^1_j, \beta)$. Let $\tilde{A}^1_r = W^1_{n^1_r}(\tilde{\Gamma}^1_r)$.

If $\hat{\Gamma}^1_r \neq \emptyset$, define the values of $W^1_{n^1_r}$ for the points in $\hat{\Gamma}^1_r$ so that $W^1_{n^1_r}(\hat{\Gamma}^1_r) = \hat{A}^1_r$ and $W^1_{n^1_r}$ is continuous and increasing on $\hat{\Gamma}^1_r$, where $\hat{A}^1_r \subseteq \mathcal{R}$ is a nonempty and bounded interval such that $a > b$ for all $a \in \hat{A}^1_r$ and for all $b \in \tilde{A}^1_r$, and $\{h \in \mathcal{R} \mid a > h > b \text{ for all } a \in \hat{A}^1_r \text{ and for all } b \in \tilde{A}^1_r\} = \emptyset$. Part (iii) of Lemma 4 ensures that this construction is possible.

If $\check{\Gamma}^1_r \neq \emptyset$, define the values of $W^1_{n^1_r}$ for the points in $\check{\Gamma}^1_r$ so that $W^1_{n^1_r}(\check{\Gamma}^1_r) = \check{A}^1_r$ and $W^1_{n^1_r}$ is continuous and increasing on $\check{\Gamma}^1_r$, where $\check{A}^1_r \subseteq \mathcal{R}$ is a nonempty and bounded interval such that $a < b$ for all $a \in \check{A}^1_r$ and for all $b \in \tilde{A}^1_r$, and $\{h \in \mathcal{R} \mid a < h < b \text{ for all } a \in \check{A}^1_r \text{ and for all } b \in \tilde{A}^1_r\} = \emptyset$. Part (v) of Lemma 4 ensures that this construction is possible.

Let $A^1_r = \hat{A}^1_r \cup \tilde{A}^1_r \cup \check{A}^1_r$. Clearly, A^1_r is a nonempty and bounded interval, and the function $W^1_{n^1_r}$ is continuous and increasing and maps onto A^1_r. By parts (ii) and (iv) of Lemma 4, we have

$$W^1_{n^1_r}(\gamma) \geq W^1_{n^1_j}(\beta) \Leftrightarrow (n^1_r, \gamma) R (n^1_j, \beta)$$

for all $\gamma \in U^{n^1_r}(\mathcal{C})$, for all $j \in \{1, \ldots, r-1\}$ and for all $\beta \in U^{n^1_j}(\mathcal{C})$.

Because Z_1 contains a finite or countably infinite number of elements, the above recursive construction of the functions $\{W^1_{n^1_j} \mid j \in Z_1\}$ is well-defined. We define the function $W^1\colon \cup_{j \in Z_1} \big(\{n^1_j\} \times U^{n^1_j}(\mathcal{C})\big) \to \mathcal{R}$ by letting

$$W^1(n^1_j, \gamma) = W^1_{n^1_j}(\gamma)$$

for all $(n^1_j, \gamma) \in \cup_{j \in Z_1} \big(\{n^1_j\} \times U^{n^1_j}(\mathcal{C})\big)$.

Now suppose the functions W^i have been constructed in the above manner for all $i \in \{1, \ldots, t-1\}$ for some $t > 1$. If $Z \setminus \big(\cup_{i=1}^{t-1} \cup_{j \in Z_i} \{n^i_j\}\big) \neq \emptyset$, we define $Z_t \subseteq Z$, $\mathbf{n}^t = (n^t_j)_{j \in Z_t}$ and $W^t\colon \cup_{j \in Z_t} \big(\{n^t_j\} \times U^{n^t_j}(\mathcal{C})\big) \to \mathcal{R}$ analogously. Because Z is finite or countably infinite, it follows that either there exists $T \in \mathcal{Z}_{++}$ such that $Z = \cup_{t=1}^{T} \cup_{j \in Z_t} \{n^t_j\}$ or $Z = \cup_{t \in \mathcal{Z}_{++}} \cup_{j \in Z_t} \{n^t_j\}$. In the first case, let $\mathcal{T} = \{1, \ldots, T\}$ and in the second case, let $\mathcal{T} = \mathcal{Z}_{++}$. Letting $A^t = \cup_{j \in Z_t} A^t_j$ for all $t \in \mathcal{T}$, it follows that

$$W^t\Big(\cup_{j \in Z_t} \big(\{n^t_j\} \times U^{n^t_j}(\mathcal{C})\big)\Big) = A^t.$$

Finally, we define the function $W: \cup_{n \in Z}\big(\{n\} \times U^n(\mathcal{C})\big) \to \mathcal{R}$ by letting, for all $t \in \mathcal{T}$, for all $j \in Z_t$, and for all $\gamma \in U^{n_j^t}(\mathcal{C})$,

$$W(n_j^t, \gamma) = H^t\big(W^t(n_j^t, \gamma)\big)$$

where each $H^t: A^t \to \mathcal{R}$ is continuous and increasing, $H^t(A^t)$ is a nonempty and bounded interval, and $H^t(A^t) \cap H^s(A^s) = \emptyset$ for all $s, t \in \mathcal{T}$ such that $s \neq t$. By definition, if $j \in Z_t$, $k \in Z_s$ and $s \neq t$, either $(n_j^t, \gamma)P(n_k^s, \beta)$ for all $\gamma \in U^{n_j^t}(\mathcal{C})$ and for all $\beta \in U^{n_k^s}(\mathcal{C})$ or $(n_k^s, \beta)P(n_j^t, \gamma)$ for all $\gamma \in U^{n_j^t}(\mathcal{C})$ and for all $\beta \in U^{n_k^s}(\mathcal{C})$. Therefore, the functions H^t can be chosen so that all rankings according to R are preserved by W and, thus, W is a representation of R.

Define the function $U: \mathcal{D} \times \mathcal{C} \to \mathcal{R}$ by

$$U(x^D, x^C) = W\Big(\rho(x^D), U^{\rho(x^D)}(x^C)\Big).$$

Because W represents R, Lemma 2 implies that U represents $\succeq$. U is continuous in its second argument by construction.

Now suppose that $\succeq$ is represented by $U: \mathcal{D} \times \mathcal{C} \to \mathcal{R}$ and that U is continuous in its second argument. To show that $\succeq$ satisfies unrestricted continuity, consider any $(x^D, x^C) \in \mathcal{D} \times \mathcal{C}$, $y^D \in \mathcal{D}$, and let $U(x^D, x^C) = \bar{u}$. If $\bar{u} \notin U(y^D, \mathcal{C})$, then $\{y^C \in \mathcal{C} \mid (y^D, y^C) \succeq (x^D, x^C)\}$ is equal to $\mathcal{C}$ or to $\emptyset$ and, in both cases, $\{y^C \in \mathcal{C} \mid (y^D, y^C) \succeq (x^D, x^C)\}$ is closed in $\mathcal{C}$. Similarly, $\{y^C \in \mathcal{C} \mid (x^D, x^C) \succeq (y^D, y^C)\}$ is closed in $\mathcal{C}$.

If $\bar{u} \in U(y^D, \mathcal{C})$, then $\{y^C \in \mathcal{C} \mid (y^D, y^C) \succeq (x^D, x^C)\} = \{y^C \in \mathcal{C} \mid U(y^D, y^C) \geq \bar{u}\}$ and $\{y^C \in \mathcal{C} \mid (x^D, x^C) \succeq (y^D, y^C)\} = \{y^C \in \mathcal{C} \mid U(y^D, y^C) \leq \bar{u}\}$. Both are closed in $\mathcal{C}$ because U is continuous in its second argument. ■

Unconditional continuity is sufficient but not necessary for the existence of a representation. There are orderings that violate unconditional continuity (but satisfy conditional continuity) that are representable, but any representation W of the associated ordering R is necessarily discontinuous in its second argument. See Blackorby, Bossert and Donaldson [2001] for an example and a more detailed discussion.

4. Extensions

A natural generalization of Theorem 3 is possible. Suppose that an individual has preferences over consumption vectors of more than one set of vectors of goods. In such an environment, let $\mathcal{D}$ be the set of vectors of the labels of possible goods. A consumption vector can be described as (x^D, x^C) where $x^D \in \mathcal{D}$ is a vector naming the goods consumed and x^C is a vector of the corresponding quantities. $\mathcal{D}$ may contain vectors of different dimensions (depending on the number of goods) and the set of possible consumption vectors must be of the same dimension as x^D. It follows that the set $\mathcal{C}$ must depend on x^D and we write it as $\bar{\mathcal{C}}(x^D)$ for each $x^D \in \mathcal{D}$.

A second example is one from population ethics. In that case, x^D lists the identities of those alive in an alternative and x^C their lifetime utility levels. Again, because dimensions must match, the set of possible utilities must depend on x^D.

Formally, the domain considered now is defined as follows. Let $X = \{(x^D, x^C) \mid x^D \in \mathcal{D}$ and $x^C \in \bar{\mathcal{C}}(x^D)\}$ such that, for each $x^D \in \mathcal{D}$, there exists $d \in \mathcal{Z}_{++}$ such that $x^D \in \mathcal{Z}^d$ and, for each $x^D \in \mathcal{D}$, there exists $c \in \mathcal{Z}_{++}$ such that $\bar{\mathcal{C}}(x^D) \subseteq \mathcal{R}^c$. Unconditional continuity can easily be rewritten to fit this environment; all that is required is to replace $\mathcal{C}$ with the function $\bar{\mathcal{C}}$.

Unconditional Continuity: For all $x \in X$ and for all $y^D \in \mathcal{D}$, the sets $\{y^C \in \bar{\mathcal{C}}(y^D) \mid (y^D, y^C) \succeq (x^D, x^C)\}$ and $\{y^C \in \bar{\mathcal{C}}(y^D) \mid (x^D, x^C) \succeq (y^D, y^C)\}$ are closed in $\bar{\mathcal{C}}(y^D)$.

A result analogous to that of Theorem 3 can be proved without difficulty in this environment. The only significant change is that the functions $\{U^{x^D}\}$ have different domains for each x^D, that is, they are functions $U^{x^D}: \bar{\mathcal{C}}(x^D) \to \mathcal{R}$. As before, the image of U^{x^D} can vary with x^D.

Theorem 4: *Let $X = \{(x^D, x^C) \mid x^D \in \mathcal{D}$ and $x^C \in \bar{\mathcal{C}}(x^D)\}$ such that, for each $x^D \in \mathcal{D}$, there exists $d \in \mathcal{Z}_{++}$ such that $x^D \in \mathcal{Z}^d$ and, for each $x^D \in \mathcal{D}$, there exists $c \in \mathcal{Z}_{++}$ such that $\bar{\mathcal{C}}(x^D) \subseteq \mathcal{R}^c$, and suppose $\mathcal{D}$ is nonempty and $\bar{\mathcal{C}}(x^D)$ is nonempty and connected for each $x^D \in \mathcal{D}$. $\succeq$ is an ordering on X satisfying unconditional continuity if and only if there exists a function $U: X \to \mathcal{R}$, continuous in its second argument, that represents it.*

The proof of Theorem 4 is almost identical to that of Theorem 3 and is omitted. Theorem 3 is a special case of Theorem 4 in which $\bar{\mathcal{C}}(x^D)$ is the same for all x^D and $\mathcal{D}$ is of fixed dimension.

A related problem arises in population ethics when an anonymity requirement is satisfied. The relevant information for each alternative is the vector of utilities of those who are alive (identities are not needed). The set of possible population sizes is given by $\mathcal{Z}_{++}$, and a social ordering ranks utility vectors in $X = \cup_{j \in \mathcal{Z}_{++}} \mathcal{R}^j$.

A more general domain for this is a union of connected sets of continuous variables only and we write it as $X = \cup_{j \in \mathcal{J}} \mathcal{C}_j$ where $\mathcal{J} = \{1, \ldots, m\}$ with $m \in \mathcal{Z}_{++}$ if the number of sets is finite and $\mathcal{J} = \mathcal{Z}_{++}$ if it is not, and, for all $j \in \mathcal{J}$, there exists $c_j \in \mathcal{Z}_{++}$ such that $\mathcal{C}_j \subseteq \mathcal{R}^{c_j}$ and $\mathcal{C}_j$ is nonempty and connected. Note that it is possible for two or more of these sets to be subsets of the same Euclidean space.

The appropriate continuity axiom takes on a slightly different form in this case.

Extended Continuity: For all $x \in X$ and for all $j \in \mathcal{J}$, the sets $\{y \in \mathcal{C}_j \mid y \succeq x\}$ and $\{y \in \mathcal{C}_j \mid x \succeq y\}$ are closed in $\mathcal{C}_j$.

The ordering $\succeq$ on this domain induces another ordering on a mixed domain. Theorem 4 can be applied to the resulting ordering and used to obtain the following result. For any $c \in \mathcal{Z}_{++}$, we define $\mathcal{C}^c$ to be the set of all the $\mathcal{C}_j$s that are subsets of $\mathcal{R}^c$, that is, $\mathcal{C}^c = \{\mathcal{C}_j \mid \mathcal{C}_j \subseteq \mathcal{R}^c\}$. In addition, let $\mathcal{J}_c = \{j \mid c_j = c\}$ and $\mathcal{Z}_{\mathcal{J}} = \{c \in \mathcal{R}_{++} \mid \mathcal{J}_c \neq \emptyset\}$.

Theorem 5: *Let $X = \cup_{j \in \mathcal{J}} \mathcal{C}_j$ where, for each $j \in \mathcal{J}$, there exists $c_j \in \mathcal{Z}_{++}$ such that $\mathcal{C}_j \subseteq \mathcal{R}^{c_j}$, and suppose that $\mathcal{C}_j$ is nonempty and connected for each $j \in \mathcal{J}$. $\succeq$ is an ordering on X satisfying extended continuity if and only if there exists a function $U\colon X \to \mathcal{R}$, continuous on $\cup_{j \in \mathcal{J}_c} \mathcal{C}_j$ for all $c \in \mathcal{Z}_{\mathcal{J}}$, that represents it.*

Proof. For all $c \in \mathcal{Z}_{\mathcal{J}}$, merge any sets in $\mathcal{J}_c$ whose union is a connected set. Because each $\mathcal{C}_j$ is connected, the merged sets must be disjoint (see Berge [1963, p. 72]). To simplify notation, we assume, without loss of generality, that no such merging is necessary and, thus, $\mathcal{C}_j$ and $\mathcal{C}_k$ are disjoint for all $c \in \mathcal{Z}_{\mathcal{J}}$ and all $\mathcal{C}_j, \mathcal{C}_k \in \mathcal{C}^c$ with $\mathcal{C}_j \neq \mathcal{C}_k$. Hence, each $x \in X$ belongs to a unique $\mathcal{C}_j$.

Let $\tilde{X} = \{(j,x) \mid j \in \mathcal{J} \text{ and } x \in \mathcal{C}_j\}$ and define the ordering $\tilde{R}$ on $\tilde{X}$ by

$$(j,x)\tilde{R}(k,y) \Leftrightarrow x \succeq y$$

for all $(j,x),(k,y) \in \tilde{X}$. Note that $x \in \mathcal{C}_j$ and $y \in \mathcal{C}_k$.

Extended continuity of $\succeq$ is equivalent to unconditional continuity of $\tilde{R}$ and, by Theorem 4, there exists a function $\tilde{U}\colon \tilde{X} \to \mathcal{R}$, continuous in its second argument, that represents $\tilde{R}$. Define the function $U\colon X \to \mathcal{R}$ so that $U(x) = \tilde{U}(j,x)$ where $x \in \mathcal{C}_j$ for all $x \in X$. Because such a j is uniquely determined for each $x \in X$, U is well-defined. For all $x, y \in X$, let $x \in \mathcal{C}_j$ and $y \in \mathcal{C}_k$. Then

$$x \succeq y \Leftrightarrow (j,x)\tilde{R}(k,y) \Leftrightarrow \tilde{U}(j,x) \geq \tilde{U}(k,y) \Leftrightarrow U(x) \geq U(y),$$

so U represents $\succeq$. Consider any $j \in \mathcal{J}$. Because $\tilde{U}$ is continuous in its second argument, U is continuous on $\mathcal{C}_j$.

Now consider any $c \in \mathcal{Z}_{\mathcal{J}}$, and suppose $\mathcal{C}_j, \mathcal{C}_k \in \mathcal{C}^c$ are distinct. $\mathcal{C}_j$ and $\mathcal{C}_k$ are disjoint by assumption. In addition, if a boundary point of $\mathcal{C}_j$ is in $\mathcal{C}_k$, the two would form a connected set. Therefore, all of the boundary points of $\mathcal{C}_j$ in $\cup_{j \in \mathcal{J}^c} \mathcal{C}_j$ are in $\mathcal{C}_j$. It follows that, because U is continuous on each $\mathcal{C}_j \in \mathcal{C}^c$, U is continuous on $\cup_{j \in \mathcal{J}_c} \mathcal{C}_j$.

To establish sufficiency, an argument analogous to the one employed in the proof of Theorem 3 can be used. ■

In Theorem 5, it is possible for each $\mathcal{C}_j$ to be a subset of the same Euclidean space. In that case, extended continuity is equivalent to ordinary continuity and it is sufficient for the existence of a continuous representation. Therefore, we obtain the following result as an immediate corollary of Theorem 5.

Theorem 6: *Let $X = \cup_{j \in \mathcal{J}} \mathcal{C}_j$ where there exists $c \in \mathcal{Z}_{++}$ such that $\mathcal{C}_j \subseteq \mathcal{R}^c$ for each $j \in \mathcal{J}$, and suppose that $\mathcal{C}_j$ is nonempty and connected for each $j \in \mathcal{J}$. $\succeq$ is an ordering on X satisfying continuity if and only if there exists a continuous function $U\colon X \to \mathcal{R}$ that represents it.*

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