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ASYMMETRIC SMILES, LEVERAGE EFFECTS AND STRUCTURAL PARAMETERS

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RÉSUMÉ

Dans cet article, nous caractérisons les asymétries observées dans les courbes de volatilités implicites par la présence d'effets de levier multiples dans un modèle dynamique stochastique d'évaluation des actifs financiers. La dépendance entre les mouvements de prix et la volatilité future est introduite par l'intermédiaire d'un ensemble de variables d'état latentes. Ces variables d'état sont susceptibles de capter non seulement le risque de volatilité et le risque de taux d'intérêt qui peuvent influer sur les prix d'options, mais encore les risques de corrélation et de saut. L'effet de levier financier traditionnel est produit quant à lui par une corrélation instantanée entre les variables d'état qui entrent dans le processus de volatilité stochastique du prix de l'action et le processus du prix de l'action proprement dit. Nous disposons toutefois d'un cadre plus général dans lequel l'asymétrie des courbes de volatilités implicites résulte de toute corrélation instantanée entre les variables d'état et soit le rendement de l'action soit le facteur d'actualisation stochastique. Dans le but de tracer les formes des courbes de volatilités implicites générées par un modèle avec variables latentes, nous spécifions un facteur d'actualisation stochastique fondé sur un modèle d'équilibre avec préférences non séparables dans le temps. Lorsque nous calibrons ce modèle avec des valeurs raisonnables des paramètres, nous reproduisons les diverses formes de courbes de volatilités implicites qui sont produites à partir des données de prix d'options observées sur le marché.

Mots clés : évaluation d'options, facteur d'actualisation stochastique, volatilité stochastique, volatilité implicite de Black-Scholes, effet de sourire, modèle d'équilibre d'évaluation d'options

ABSTRACT

In this paper, we characterize the asymmetries of the smile through multiple leverage effects in a stochastic dynamic asset pricing framework. The dependence between price movements and future volatility is introduced through a set of latent state variables. These latent variables can capture not only the volatility risk and the interest rate risk which potentially affect option prices, but also any kind of correlation risk and jump risk. The standard financial leverage effect is produced by a cross-correlation effect between the state variables which enter into the stochastic volatility process of the stock price and the stock price process itself. However, we provide a more general framework where asymmetric implied volatility curves result from any source of instantaneous correlation between the state variables and either the return on the stock or the stochastic discount factor. In order to draw the shapes of the implied volatility curves generated by a model with latent variables, we specify an equilibrium-based stochastic discount factor with time non-separable preferences. When we calibrate this model to empirically reasonable values of the parameters, we are able to reproduce the various types of implied volatility curves inferred from option market data.

Key words : option pricing, stochastic discount factor, stochastic volatility, Black-Scholes implied volatility, smile effect, equilibrium option pricing

1. Introduction

In the empirical option pricing literature, departures from the Black and Scholes (1973) (BS) model are often characterized by an implied volatility curve, whereby the volatility extracted from the BS option pricing formula given the observed option price is graphed against the moneyness of the option. The empirical biases of the BS model have been dubbed the smile effect in reference to a symmetric implied volatility curve, but numerous distorted smiles in the shape of smirks or frowns are inferred more frequently from market data. In Figure 1, we graph several volatility curves for options on the S&P 500 index on selected dates to reflect the types of shapes that can be observed most frequently.

A stochastic volatility model as in Hull and White (1987) produces a symmetric smile when the returns innovations and the volatility are uncorrelated. With stochastic volatility, the price of the option is expressed as an expectation of the BS price, where the expectation is taken with respect to the distribution of the heterogeneous stochastic volatility factor. The symmetric volatility smile is created by a related Jensen effect (see Renault, 1997 and Renault and Touzi, 1996).

Asymmetric smiles can therefore be potentially explained by an instantaneous correlation between returns and volatility. In Black (1976), an inverse relationship between the level of equity prices and the instantaneous conditional volatility is put forward for individual firms. This inverse relationship is explained by financial leverage. A drop in the price of the stock increases the debt-to-equity ratio and therefore the risk of the firm, which translates into a higher volatility of the stock. Nelson (1991) shows that such a negative correlation exists also for broad market indices. The correlation is still called a leverage effect, but explanations are given in terms of time-varying risk premia and volatility feedback (see Campbell and Hentschel, 1992, among others). If volatility risk is priced, an anticipated increase in volatility raises the discount rate of future expected dividends and lowers the present equity price. From a theoretical perspective, Platen and Schweizer (1998) explain the asymmetric shape of the smile by developing a model in which the diffusion process of the stock price incorporates the technical demand induced by hedging strategies[1]. David and Veronesi (1999) propose an incomplete information

[1] In Grossman and Zhou (1996), an equilibrium model of risk sharing between portfolio insurers and investors generates a negatively skewed smile.

model where investors' uncertainty explains the intertemporal variation in the slope and curvature of implied volatility curves.

To be able to put forward the asymmetric deformations of the smile, we first state necessary and sufficient conditions for symmetry in a general asset pricing setting. These characterizations are given in terms of the option pricing function or, alternatively, through the skewness of the pricing probability measure. We propose a generalized option pricing formula based on a stochastic discount factor containing state variables. As special cases of our formula we obtain the BS formula, the Hull and White (1987) and Bailey and Stulz (1989) stochastic volatility option pricing formulas, as well as the Merton (1973), Turnbull and Milne (1991), and Amin and Jarrow (1992) stochastic interest rate option pricing formulas for equity options.

This extended option pricing framework is motivated by the vast empirical literature aimed at finding option pricing models that will reproduce the cross-sectional patterns and the dynamics of implied volatilities. In the class of deterministic volatility models[2], the local volatility of the underlying asset is a known function of time and of the path and level of the underlying asset price. However, Dumas, Fleming and Whaley (1998) show that deterministic volatility models overfit the smile in sample and loose any predictive power out of sample. Other evidence against the one-dimensional diffusion model can be found in Bakshi, Cao, Chen (2000). They show that call prices often go down when the underlying price goes up and that call prices are not perfectly correlated with each other and the underlying asset. Buraschi and Jackwerth (1997) bring further evidence that deterministic volatility models are not consistent with observed option prices and that stochastic volatility models are more likely to explain the smile.

The main stochastic volatility models that have been submitted to empirical testing are variants of the two-dimensional diffusion models in stock returns and volatility such as in Heston (1993). Although these models produce patterns qualitatively similar to some violations, Bakshi, Cao and Chen (1997), Bates (1996) and Chernov and Ghysels (1999) provide evidence against the stochastic volatility model of Heston (1993). Multi-factor volatility models as in Bates (1997) and

[2]These models include the constant elasticity of variance model of Cox and Ross (1976), the implied binomial tree approach of Rubinstein (1994), the deterministic volatility models of Dupire (1994) and Derman and Kani (1994), and the Kernel approach of Aït-Sahalia and Lo (1998).

Gallant, Hsu, Tauchen (1999) do not improve performance significantly. The main conclusion is that an extremely high volatility of volatility is necessary to generate leptokurtosis of the magnitude consistent with the volatility smirks. Das and Sundaram (1999) confirm that stochastic volatility models are not capable of generating high levels of skewness and kurtosis at short maturities under reasonable parameterizations.

Jump-diffusion models can generate realistic implied volatility smiles at short maturities (see Bates, 1997, Bakshi, Cao and Chen, 1997, Andersen, Benzoni and Lund, 1998), but Das and Sundaram (1999) show that they cannot reproduce the smiles at long horizons. Pan (1999) examines joint time series data on spot and option prices on the S&P 500 and provides evidence of a jump risk premium that responds quickly to market volatility and is important in explaining the volatility smirks. There remains however some misspecifications and some suggestions regarding the inclusion of jumps in the volatility are made.

The conclusion of this empirical literature is that important features for reproducing the cross-sectional patterns and the dynamics of implied volatilities are jumps in both returns and volatility, as well as a correlation between the jumps in returns and volatility. To illustrate how our option pricing formula can incorporate these features, we specialize our latent state variables to a discrete-state Markov process. A discrete change of state will affect simultaneously the mean and variance of equity returns and of the stochastic discount factor, creating jump-like effects also in volatility. In this setting we characterize analytically both the skewness of the returns and the equity leverage effect. We show that the formulas for conditional skewness and leverage effect in the stock are very similar. We establish the conditions for negative skewness or leverage in terms of transition probabilities between states.

To be able to draw the shapes of the implied volatility smiles generated by a model with latent variables, we need to further specify the stochastic discount factor. We choose an example an equilibrium-based stochastic discount factor in the spirit of Rubinstein (1976), Brennan (1979) and Amin and Ng (1993a). We set our equilibrium model in a recursive utility framework with time non-separable preferences (Epstein and Zin [1989])[3]. Our latent variable framework

[3]Two papers have used preferences that disentangle risk aversion from intertemporal substitution in the context of option pricing. Detemple (1990) uses the ordinal certainty equivalence hypothesis in a two-period economy and shows that time preferences play a distinctive and

makes it possible to parameterize parsimoniously the dynamic evolution of the consumption and dividend processes in this equilibrium model. We use a two-state bivariate Markov switching model as in Cecchetti, Lam and Mark (1993) and Bonomo and Garcia (1993, 1996)[4].

When we calibrate this model to empirically reasonable values of the parameters, we are able to reproduce the various types of implied volatility curves inferred from option market data. In particular, we show that a decrease in the persistence of a state accounts for a reversal of the smile skewness. As Bates (1996) emphasized, it is such changing skewness in the smile that poses a challenge to current option pricing models. Moreover, an increase in the persistence of a state can produce a frowning implied volatility curve. In terms of preference parameters, the coefficient governing intertemporal substitution is shown to have a much more pronounced effect on the shape of the volatility curve than the coefficient of relative risk aversion. The source of uncertainty generating the leverage effect is also an important factor. When the leverage effect results from aggregate consumption risk only, the smile is much flatter than when the added leverage effect created by idiosyncratic dividend risk comes into play. We also document various maturity and stochastic volatility effects with and without leverage effects.

The rest of the paper is organized as follows. Section 2 provides general conditions under which the smile is symmetric. Section 3 develops a generalized option pricing formula with latent state variables. Based on this formula, Section 4 characterizes the asymmetric distortions of the smile. A comparison with usual stochastic volatility models with leverage is developed in Section 5 as well as an analytical characterization of leverage when state variables follow a discrete-state Markov process. Section 6 uses an equilibrium-based stochastic discount factor in order to illustrate through simulations the various shapes of the smile that the model can produce. Section 7 concludes and announces further empirical

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significant role in pricing options. For example, option prices change with the expected return on the stock and may decrease when the risk of the stock return increases. Ma (1998) derives a closed-form pricing formula for European call options written on aggregate equity under Kreps-Porteus preferences in an i.i.d. environment and in a Markov setting.

[4] The regime-switching model introduced by Hamilton (1989) has recently enjoyed some popularity in the option pricing literature. See in particular Campbell and Li (1999), Chourdakis and Tzavalis (1999), and David and Veronesi (1999). All these models can be embedded in our framework. A precursor paper in regime-switching option pricing is Naik (1993).

assessment of the proposed option pricing model.

2. The symmetry of the volatility smile

The asymmetry of the implied volatility curves is best characterized with reference to a benchmark model which produces a symmetric curve. When the volatility is stochastic as in the Hull and White (1987) model, Renault (1997) and Renault and Touzi (1996) have shown that the shape of the volatility structure with respect to the moneyness of the option is symmetric when the returns innovations and the volatility are uncorrelated. Moneyness x_t is defined as the logarithm of the ratio of the forward price over the strike price, $x_t = Log\frac{S_t}{KB(t,T)}$ (with S_t the price of the underlying asset, K the strike price and $B(t,T)$ the price of a pure discount bond maturing at T). Since both the BS and the Hull and White option pricing formulas are homogeneous functions of degree one with respect to the pair (S_t, K), we will provide new characterizations of the symmetry of the volatility smile in the context of a general homogeneous option pricing formula, both in terms of the option pricing function and of the pricing probability measure.

The theory for pricing contingent claims in the absence of arbitrage introduces a pricing probability measure $\mathrm{Q}_{t,T}$ under which the price Π_t at time t of any contingent claim maturing at time T is the discounted expectation of its terminal payoff. In the case of a European call option with strike price K, it is given by:

$$\Pi_t = B(t,T)E_t^*(S_T - K)^+, \tag{2.1}$$

where E_t^* denotes the expectation operator with respect to $\mathrm{Q}_{t,T}$ [5]. We will therefore compare a general but homogeneous option pricing formula $\Pi_t(S_t, K)$ as defined in (2.1) with the BS option pricing formula defined itself by a homogeneous function $BS(.,.,\sigma)$, for a given volatility parameter σ, with:

$$\begin{cases} BS(S_t, K, \sigma) = S_t\phi(d_1) - KB(t,T)\phi(d_2), \\ d_1 = \dfrac{1}{\sigma\sqrt{T-t}}\left[x_t + \dfrac{1}{2}\sigma^2(T-t)\right], \\ d_2 = d_1 - \sigma\sqrt{T-t}. \end{cases} \tag{2.2}$$

[5] Existence and unicity of $\mathrm{Q}_{t,T}$ were studied by several authors since the seminal paper of Harrison and Kreps (1979). In this paper, we are only interested in the existence of a well-defined pricing probability measure $Q_{t,T}$, whether it is unique or not.

The BS implied volatility is defined as a function $\sigma_t^*(x_t)$ of the moneyness x_t only, and not of S_t and K separately:

$$\Pi_t(S_t, K) = BS(S_t, K, \sigma_t^*(x_t)), \tag{2.3}$$

since a direct application of the homogeneity of degree one of $\Pi_t(.,.)$ and $BS(.,.,\sigma)$ with respect to the pair (S_t, K) allows one to divide each side of (2.3) by S_t and conclude that $\sigma_t^*(x_t)$ is well-defined as a function of K/S_t or (equivalently) of x_t by :

$$\pi_t(x_t) = bs(x_t, \sigma_t^*(x_t)) \tag{2.4}$$

with the obvious change of notation.

In this setting, we can investigate the slope of the BS implied volatility $\sigma_t^*(x)$ as a function of its distance x to the money, remembering that moneyness x is equal to 0 at the money, that is when the strike price coincides with the forward price $\frac{S_t}{B(t,T)}$. In particular, two strike prices K_1 and K_2 are said to be symmetric with respect to the money if the corresponding x_1 and x_2 are symmetric with respect to zero, since in this case K_1 and K_2 are on each side of the forward price but their geometric average coincides with the forward price. Therefore, the relevant symmetry property of the volatility smile is the following:

$$\sigma_t^*(x) = \sigma_t^*(-x) \quad \text{for any} \quad x. \tag{2.5}$$

In Proposition 2.1 below, we extend a result first stated in Renault and Touzi (1996), which characterizes the symmetry of the smile in terms of the option pricing function[6].

Proposition 2.1. *If option prices are conformable to a homogeneous option pricing formula $x \to \pi(x)$, the volatility smile is symmetric ($\sigma^*(x) = \sigma^*(-x)$ for any x) if and only if, for any x:*

$$\pi(-x) = e^x \pi(x) + 1 - e^x$$

Proof: See Appendix 1.

[6]For sake of notational simplicity, the subscripts t have been dropped.

This characterization of the symmetry of the smile admits an equivalent formulation in terms of the pricing probability measure. While the pricing probability measure is usually characterized through the cumulative distribution function of $\frac{S_T}{S_t}$, it is convenient here to characterize it through either the cumulative distribution function $F_{V_T}(.)$ or the probability density function $f_{V_T}(.)$ of $V_T = Log\frac{S_T B(t,T)}{S_t}$. We are then able to prove (see Appendix 1) the following proposition:

Proposition 2.2. *If the cumulative distribution function $F_{V_T}(.)$ of V_T under a pricing probability measure is absolutely continuous (associated with a density function $f_{V_T}(.) = F'_{V_T}(.)$) and such that* $\exp(V_T)$ *is integrable, the volatility smile is symmetric if and only if one of the following three equivalent properties is fulfilled:*

(i) For any x:

$$\pi(x) = F_{V_T}(x) - e^{-x}[1 - F_{V_T}(-x)]$$

(ii) For any x:

$$F_{V_T}(x) = E_t^*[e^{V_T}\mathbf{1}_{[V_T \geq -x]}]$$

(iii) There exists an even function g(.) such that for any x:

$$f_{V_T}(x) = e^{-x/2}g(x)$$

These characterizations offer various ways to extend the BS formula, while keeping both a homogeneous option pricing function and a symmetric smile. Characterization (i) provides a theoretical support to descriptive approaches which replace the standard normal cumulative distribution function of the BS formula by alternative distribution functions. Characterization (ii) should be interpreted in terms of hedging. Indeed, Garcia and Renault (1998a) have shown that $E_t^*[e^{V_T}\mathbf{1}_{[V_T \geq -x]}]$ is precisely the hedging ratio, in other words the derivative of the option pricing function with respect to the stock price (the so-called delta of the option)[7].

[7] Their proposition 2.1 shows that this characterization of the hedging ratio is a necessary and sufficient condition for homogeneous option pricing. Since hedging is not the primary focus of this paper, we leave to the reader the interpretation of this fairly natural relationship between $F_{V_T}(x)$ and the delta coefficient.

Finally, (iii) characterizes precisely which type of symmetry of the pricing probability measure is required for the symmetry of the smile. In particular, it shows that it is not the density of the log returns that should be symmetric (as it is commonly believed perhaps because of the usual log-normal setting), but the same density rescaled by a suitable exponential function. Indeed, in the log-normal case:

$$V_T = Log\frac{S_T B(t,T)}{S_t} \rightsquigarrow_{(Q_{t,T})} \mathcal{N}(\mu_t, \sigma_t^2),$$

the condition (iii) means that $\mu_t = -\frac{\sigma_t^2}{2}$,which is automatically fulfilled in the absence of arbitrage since, by application of (2.1) with $K = 0$, we have $S_t = B(t,T)E_t^* S_T$.

In the next subsection, we provide sufficient conditions on the pricing probability measure to ensure the homogeneity of the option pricing function in a stochastic framework with state variables. The latter are convenient for capturing the departures from normality in the form of skewness or excess kurtosis usually present in financial time series.

3. A generalized Black-Scholes and Hull-White formula with state variables

Merton (1973) stressed that the desirable homogeneity of option prices will be maintained as soon as asset returns are serially independent. This condition can be generalized by expressing it in conditional terms, given a path of state variables.

3.1. A State-Variable Framework for Homogeneous Option Pricing

In order to specify a dynamic asset pricing model in discrete time, our focus of interest will be the dynamic properties of a positive stochastic discount factor (SDF) denoted[8] by $m_{t,T}$.

In Hansen and Richard (1987), the existence of a positive SDF is shown to be equivalent to the absence of arbitrage in a very general conditional information

[8] The positivity property allows one to compute the probability density function of the pricing probability measure $Q_{t,T}$ as $m_{t,T}/E_t m_{t,T}$. However, in the rest of paper, we will characterize option prices directly in terms of $m_{t,T}$.

setting. The price at time t of a single payoff p_T occurring at time $T > t$ is then characterized by a pricing functional $\pi_t(\cdot) = \pi_t(p_T) = E[m_{t,T} p_T | I_t]$ where I_t is the relevant conditioning information at time t. In particular, the option price will be written as:

$$\pi_t = E_t m_{t,T} (S_T - K)^+ \tag{3.1}$$

Moreover, Hansen and Richard (1987) emphasize that if all the finite variance (given I_t) random variables are feasible payoffs, one and only one among them is a correct SDF. We will therefore refer to *the* SDF[9] and think about it as a payoff. In addition, since agents observe typically more than the econometrician, the information set I_t at time t may contain not only past values of prices and payoffs, but also some latent state variables.

Extending the Hansen and Richard (1987) setting to an intertemporal framework and applying the law of iterated expectations, the log-SDFs necessarily fulfill:

$$\log m_{t,T_2} = \log m_{t,T_1} + \log m_{T_1,T_2} \text{ , for } t < T_1 < T_2. \tag{3.2}$$

and therefore: $m_{t,T} = \prod_{\tau=t}^{T-1} m_\tau$, with: $m_\tau = m_{\tau-1,\tau}$. Following Constantinides (1992), we directly specify the time-series properties of the stochastic process m_t, $t = 1, 2, ..T$ rather than specifying the SDF sequence through a given specification of preferences[10]. The key feature of our asset pricing model is an assumption about the sequence $(m_\tau)_{1\leq\tau\leq T}$ of unit period SDFs which amounts to a factor structure in the longitudinal dimension. A number of state variables summarize the stochastic dependence of the consecutive SDFs, in the sense that, given the state variables, they are mutually conditionally independent. The same assumption is made about the sequence of consecutive returns of the primitive asset of interest on which options are written. Therefore, in terms of the joint distribution of m_t and returns on a given asset price S_t, we maintain the following assumption.

Assumption A1: The variables $(m_{\tau+1}, \frac{S_{\tau+1}}{S_\tau})_{1\leq\tau\leq T-1}$ are conditionally serially independent given the path $U_1^T = (U_t)_{1\leq\tau\leq T}$ of a vector U_t of state variables.

[9]Notice that this unicity property does not refer to any completeness property which would be unrealistic in discrete time.

[10]As stressed by Constantinides (1992), this alternative approach makes it unnecessary to assume an economy with a representative consumer with von Neumann-Morgenstern preferences. Actually, we will consider in section 6 more general non time-separable preferences.

The relevant conditioning information at time t will be: $I_t = \sigma[m_{\tau,} S_\tau, U_\tau, \tau \leq t]$. This model provides two extensions relative to Constantinides (1992). In the latter, since the focus of interest was the term structure of interest rates and options written on bonds, Assumption A1 was only maintained for the SDF sequence (m_t). Resulting bond prices were therefore deterministic functions of the state variables, and Assumption A1 becomes trivial with S_t viewed as a bond price. The second extension relates to the processes considered for the state variables. While Constantinides (1992) considers only AR(1) processes, our setting accommodates any process. In particular, we have in mind Markov switching regime models which can capture any kind of stochastic volatility and jumps in the return process as well as in the volatility.

Generally speaking, regimes are seen as exogenous according to Assumption A2:

Assumption A2: The process $(m_t, \frac{St+1}{S_t})$ does not cause the process (U_t);

The non-causality property may be interpreted equivalently in Granger (1969) or in Sims (1972) terms. Granger causality means that, given the past U_1^t of state variables, the past values $m_\tau, S_\tau, \tau \leq t$ of the return and SDF processes do not bring any relevant information to forecast U_{t+1} (which is in this sense exogenous). Sims causality means that the probability distribution of $(m_{t+1}, \frac{St+1}{S_t})$ given I_t and U_{t+1}^T does not depend upon U_{t+1}^T. Jointly with the conditional independence assumption A1, assumption A2 permits to characterize the joint probability distribution of $(m_{\tau+1}, \frac{S\tau+1}{S_\tau}, U_{\tau+1})_{\tau \geq t}$ given I_t as the following product:

$$\begin{aligned} \ell\left[(m_{\tau+1}, \frac{S_{\tau+1}}{S_\tau}, U_{\tau+1})_{\tau\geq t}|I_t\right] &= l\left[U_{t+1}^T|U_1^t\right].l\left[(m_{\tau+1}, \frac{S_{\tau+1}}{S_\tau})_{\tau\geq t}|U_1^T\right] \qquad (3.3) \\ &= \prod_{h=1}^{T-t}\ell\left[U_{t+h}|U_1^{t+h-1}\right].\prod_{h=1}^{T-t}\ell\left[m_{t+h}, \frac{S_{t+h}}{S_{t+h-1}}|U_1^{t+h}\right]. \end{aligned}$$

Proposition 3.1. *: Under (A1) and (A2) there exists a deterministic function $\Psi_{t,T}$ such that the option price (3.1) can be written as:*

$$\pi_t = \Psi_{t,T}\left[U_1^t, \frac{K}{S_t}\right] S_t.$$

Proposition 3.1 establishes that the option pricing formula is homogeneous of degree one with respect to the pair (S_t, K).

To obtain a generalized BS and Hull-White option pricing formula starting from (3.1), one needs only, in addition to the previous assumptions (A1) and (A2), a joint log-normality assumption of $m_{t,T}$ and $\frac{S_T}{S_t}$ given a path U_1^T of state variables.[11]

Assumption A3: The conditional probability distribution of $(\log m_{t+1}, \log \frac{St+1}{S_t})$ given U_1^{t+1} is, for $t = 1, ..., T-1$, a bivariate normal:

$$\aleph \left[\begin{pmatrix} \mu_{mt+1} \\ \mu_{st+1} \end{pmatrix}, \begin{bmatrix} \sigma^2_{mt+1} & \sigma_{mst+1} \\ \sigma_{mst+1} & \sigma^2_{st+1} \end{bmatrix} \right].$$

Assumption A3 is somehow a consequence of a standard conditional central limit argument which can be applied thanks to Assumption A1 and to the additivity property (3.2) through an arbitrary time scale given a path of state variables. Given these quite standard assumptions[12], one obtains the following generalized Black-Scholes (GBS) option pricing formula:

Proposition 3.2. *Under assumptions A1, A2 and A3:*

$$\frac{\pi_t}{S_t} = \pi_t(x) = E_t \left\{ Q_{ms}(t,T)\Phi(d_1(x)) - \frac{\widetilde{B}(t,T)}{B(t,T)} e^{-x} \Phi(d_2(x)) \right\} \tag{3.4}$$

where $x = Log \dfrac{S_t}{KB(t,T)}$ *and:*

$$\begin{aligned} d_1(x) &= \frac{x}{\overline{\sigma}_{t,T}} + \frac{\overline{\sigma}_{t,T}}{2} + \frac{1}{\overline{\sigma}_{t,T}} Log \left[Q_{ms}(t,T) \frac{B(t,T)}{\widetilde{B}(t,T)} \right] \\ d_2(x) &= d_1(x) - \overline{\sigma}_{t,T} \\ \overline{\sigma}^2_{t,T} &= \sum_{\tau=t}^{T-1} \sigma^2_{s\tau+1}. \end{aligned}$$

[11] In many applications of the state variable concept, Markovianity is usually postulated. Then, the relevant conditioning information is summarized by a few recent lags of the state variable process. Since this Markovianity assumption is not needed at this stage, we maintain in full generality the whole path U_t^T of this process.

[12] Since Clark (1973), there is a long tradition of this approach in financial econometrics. Clark (1973) stressed that non-normality is a puzzle when one has in mind the geometric temporal averaging of the returns and a corresponding central limit theorem argument. In this respect, log normality of returns can be invoked without any significant loss of generality once it is recovered after conditioning on a sufficient number of state variables.

and:

$$
\begin{aligned}
\widetilde{B}(t,T) &= \exp\Big(\sum_{\tau=t}^{T-1} \mu_{m\tau+1} + \frac{1}{2}\sum_{\tau=t}^{T-1} \sigma^2{}_{m\tau+1}\Big), \\
Q_{ms}(t,T) &= \widetilde{B}(t,T) \exp\Big(\sum_{\tau=t}^{T-1} \sigma_{ms\tau+1}\Big) E[\frac{S_T}{S_t} | U_1^T]. \qquad (3.5)
\end{aligned}
$$

To put this general option pricing formula in perspective, we will compare it to pricing formulas based on equilibrium or absence of arbitrage. Concerning the equilibrium approach, our setting is very general since it is based on a stochastic model for the SDF which does not rely on restrictive assumptions about preferences, endowments, or agent heterogeneity. Moreover, our factorization for the SDF is more general than the usual product of intertemporal marginal rates of substitution in time-separable utility models. Indeed, our SDF allows to accommodate non-separable or state-dependent preferences. The non-separable case will be illustrated in the last section by a recursive utility setting. An example of state-dependent preferences could be external habit formation based on state variables[13].

Of course, the benchmark option pricing formulas are based on the absence of arbitrage. Our general formula (3.4) nests a large number of preference-free extensions of the Black-Scholes formula. In particular if:

$$
Q_{ms}(t,T) = 1 \text{ and} \qquad (3.6)
$$

$$
\widetilde{B}(t,T) = \prod_{\tau=t}^{T-1} B(\tau, \tau+1), \qquad (3.7)
$$

one can see that the option price (3.4) is nothing but the conditional expectation of the Black-Scholes price, where the expectation is computed with respect to the joint probability distribution of the rolling-over interest rate $\overline{r}_{t,T} = -\sum_{\tau=t}^{T-1} \log B(\tau, \tau+1)$ and the cumulated volatility $\overline{\sigma}_{t,T}$. This framework nests

[13] These types of preferences have been proposed to explain the equity premium and the risk-free rate puzzles. Campbell and Cochrane (1999) is a recent example which features external habit formation. Their discount factor depends on the state of the consumption surplus. In our setting, the state variables are exogenous and do not depend on the consumption process, which is consistent with the external feature of the habit and allows for more flexibility.

three well-known models. First, the most basic ones, the Black and Scholes (1973) and Merton (1973) formulas, when interest rates and volatility are deterministic. Second, the Hull and White (1987) stochastic volatility extension, since $\overline{\sigma}^2_{t,T} = Var\left[\log \frac{S_T}{S_t} | U_1^T\right]$ corresponds to the integrated volatility $\int_t^T \sigma_u^2 du$ in the Hull and White continuous-time setting. Third, the formula allows for stochastic interest rates as in Turnbull and Milne (1991) and Amin and Jarrow (1992). However, the usefulness of our general formula (3.4) comes above all from the fact that it offers an explicit characterization of instances where the preference-free paradigm cannot be maintained. Usually, preference-free option pricing is underpinned by the absence of arbitrage in a complete market setting. However, our SDF-based option pricing does not preclude incompleteness and points out in which cases this incompleteness will invalidate the preference-free paradigm. The only cases of incompleteness which matter in this respect occur precisely when (3.6) or (3.7) are not fulfilled.

In general, preference parameters appear explicitly in the option pricing formula through $\widetilde{B}(t,T)$ and $Q_{ms}(t,T)$ since these two quantities depend on the characteristics of the SDF:[14] $(\mu_{m\tau+1}, {\sigma^2}_{m\tau+1,}\sigma_{ms\tau+1})_{\tau=t}^{T-1}$. However, in so-called preference-free formulas, it happens that these parameters are eliminated from the option pricing formula through the observation of the bond price and the stock price. Actually, the bond pricing formula and the stock pricing formula provide two dynamic restrictions relating the SDF characteristics to the bond and stock price processes. To avoid cumbersome notation, we will consider for the moment a one-period option price. In this case, the bond pricing equation is given by:

$$B(t,t+1) = E_t\left[\widetilde{B}(t,t+1)\right], \tag{3.8}$$

as shown in Appendix 2. Therefore, observing the bond price will make preference parameters in $\widetilde{B}(t,t+1)$ vanish from the option price as soon as $\widetilde{B}(t,t+1) = B(t,t+1)$, that is if and only if $\widetilde{B}(t,t+1)$, belongs to the information set I_t.

A useful way of writing the stock pricing formula is:

$$E_t\left[Q_{ms}(t,t+1)\right] = 1. \tag{3.9}$$

Therefore, similarly to the bond pricing, the stock pricing will make preference

[14]See Amin and Ng (1993b) for a similar argument in a more specific setting.

parameters vanish from the option price as soon as $Q_{ms}(t, t+1)$ is known at time t and therefore equal to one. From (3.5), we can then express the conditional expected stock return as:

$$E\left[\frac{S_{t+1}}{S_t}|I_t\right] = \frac{1}{B(t, t+1)}\exp[-\sigma_{mst+1}],$$

which is very close to a standard conditional CAPM equation. Therefore, the fact that both $\widetilde{B}(t, t+1)$ and $Q_{ms}(t, t+1)$ are known at time t produces both a preference-free option pricing formula and a CAPM-like stock pricing equation[15].

To conclude, it should be stressed that even in an equilibrium framework with incomplete markets, option pricing is preference-free if and only if there is a kind of predictability property ($\widetilde{B}(t, t+1)$ and $Q_{ms}(t, t+1)$ are known at time t) according to the terminology introduced by Amin and Ng (1993a)[16]. We will see in section 5 that the lack of such predictability corresponds in a general sense to a leverage effect.

4. The asymmetric distortions of the smile

In this section, we focus on the option pricing formula provided by Proposition 3.2 to characterize the cases where the corresponding volatility smiles are symmetric. According to criterion i) of Proposition 2.2 for the symmetry of the smile, it means that the general option pricing formula:

$$\pi_t(x) = E_t\left\{Q_{ms}(t, T)\Phi(d_1(x))\right\} - e^{-x}E_t\left\{\frac{\widetilde{B}(t, T)}{B(t, T)}\Phi(d_2(x))\right\}$$

can be written:

$$\pi(x) = F_{V_T}(x) - e^{-x}[1 - F_{V_T}(-x)].$$

But it can be seen in the derivation of the option pricing formula (3.4) (see Appendix 2) that:

[15] A similar parallel is drawn in an unconditional two-period framework in Breeden and Litzenberger (1978).

[16] Our characterization of preference free option pricing by a predictability property generalizes the one provided by Amin and Ng (1993a) since it does not depend upon a particular equilibrium setting with specific preferences and endowments.

$$[1 - F_{V_T}(-x)] = E_t\left[\frac{\widetilde{B}(t,T)}{B(t,T)}\Phi(d_2(x))\right]. \tag{4.1}$$

Therefore, a necessary and sufficient condition for symmetric smiles of (3.4) is that:

$$E_t\left\{Q_{ms}(t,T)\Phi(d_1(x))\right\} = F_{V_T}(x)$$

or equivalently:

$$E_t\left\{Q_{ms}(t,T)\Phi(d_1(x))\right\} = 1 - E_t\left\{\frac{\widetilde{B}(t,T)}{B(t,T)}\Phi(d_2(-x))\right\} \tag{4.2}$$

But from (3.4):

$$d_2(-x) = -d_1(x) + \frac{2}{\overline{\sigma}_{t,T}}Log\left[Q_{ms}(t,T)\frac{B(t,T)}{\widetilde{B}(t,T)}\right].$$

Thus by taking into account that $E_t[\frac{\widetilde{B}(t,T)}{B(t,T)}] = 1$, the symmetry criterion can be rewritten:

$$E_t\left\{Q_{ms}(t,T)\Phi(d_1(x))\right\} = E_t\left\{\frac{\widetilde{B}(t,T)}{B(t,T)}\Phi(d_1(x) - \frac{2}{\overline{\sigma}_{t,T}}Log\left[Q_{ms}(t,T)\frac{B(t,T)}{\widetilde{B}(t,T)}\right])\right\}$$

We have therefore proven the following proposition:

Proposition 4.1. *A necessary and sufficient condition for a symmetric volatility smile is the following identity :*

$$E_t\left\{Q_{ms}(t,T)\Phi(d_1(x))\right\} = E_t\left\{\frac{\widetilde{B}(t,T)}{B(t,T)}\Phi(d_1(x) - \frac{2}{\overline{\sigma}_{t,T}}Log\left[Q_{ms}(t,T)\frac{B(t,T)}{\widetilde{B}(t,T)}\right])\right\} \tag{4.3}$$

A sufficient condition for a symmetric volatility smile is:

$$Q_{ms}(t,T) = \frac{\widetilde{B}(t,T)}{B(t,T)}, \tag{4.4}$$

It should be stressed that the sufficient condition (4.4) is always fulfilled in expectation because, as shown in Appendix 2, the bond and stock pricing formulas are respectively given by:

$$B(t,T) = E_t\left[\widetilde{B}(t,T)\right], \text{ and} \tag{4.5}$$

$$E_t\left[Q_{ms}(t,T)\right] = 1. \tag{4.6}$$

Moreover, taking into account the highly nonlinear features of the two sides of the necessary and sufficient identity (4.3), Jensen effects are likely to violate it if (4.4) is not fulfilled. In other words, condition (4.4) appears at first sight not too far from being necessary. In order to interpret this symmetry condition, one should first notice that it is likely to be violated at least for long-term options ($T > t+1$) when there is interest rate risk. Actually, the factorization of $\tilde{B}\,(t,T)$ and $Q_{ms}\,(t,T)$ which obviously results from (3.5) gives:

$$\frac{\widetilde{B}(t,T)}{Q_{ms}(t,T)} = \prod_{\tau=t}^{T-1} \frac{\widetilde{B}(\tau,\tau+1)}{Q_{ms}(\tau,\tau+1)}. \tag{4.7}$$

Therefore, the symmetry condition (4.4) maintained for every elementary period $[\tau, \tau+1]$ will imply that:

$$\frac{\widetilde{B}(t,T)}{Q_{ms}(t,T)} = \prod_{\tau=t}^{T-1} B(\tau,\tau+1), \tag{4.8}$$

which cannot coincide with $B(t,T)$ if there is interest rate risk. In the rest of this section, let us assume for sake of simplicity that there is no such interest rate risk. Condition (4.3) can then be expressed as $Q_{ms}(t,T) = 1$, which implies by (3.5) that $E[\frac{S_T}{S_t}|U_1^T]\exp[-\sum_{\tau=t}^{T-1}\sigma_{ms\tau+1}]$ is known at time t as well as a CAPM-like stock pricing equation:

$$E[\frac{S_T}{S_t}|U_1^T] = \frac{1}{B(t,T)}\exp[-\sum_{\tau=t}^{T-1}\sigma_{ms\tau+1}]. \tag{4.9}$$

The expected stock return at time t can then be expressed as:

$$E[\frac{S_T}{S_t}|I_t] = \frac{1}{B(t,T)}E_t\left\{\exp[-\sum_{\tau=t}^{T-1}\sigma_{ms\tau+1}]\right\} \tag{4.10}$$

In other words, the risk-adjusted discounted stock price process appears as a martingale, when the risk adjustment suitably incorporates the market risk, measured by the covariance term σ_{ms}, into the discount factor. This martingale restriction (see Longstaff, 1995) on the stock price at time t, if violated, is likely to introduce skewness in the volatility smile. In the particular setting of an Hull and White model, Renault (1997) provides simulation evidence to show that a very small discrepancy (as low for example as 0.1 per cent) between the stock price S_t and its martingale model-based theoretical value may induce severe skewness in the smile.

In the Hull and White setting, asymmetries of volatility smiles are usually referred to as leverage effects. We will explain in section 5 the relationship between these effects and the violation of our symmetry condition $Q_{ms}(t,T)=1$. However for $T=t+1$, this violation not only means that some of the conditional moments $\mu_{st+1}, {\sigma^2}_{st+1,}$and σ_{mst+1} depend upon the contemporaneous value U_{t+1} of the state variables but that, in addition, this dependence is maintained at the level of the risk-adjusted expected return $\mu_{st+1}+\frac{1}{2}{\sigma^2}_{st+1}+\sigma_{mst+1}$. Moreover, if the condition $Q_{ms}(t,T) = 1$ is fulfilled for $T = t+1$, then it is automatically fulfilled for any value of T since it implies that $\exp[\sum_{\tau=t}^{T-1}\mu_{s\tau+1}+\frac{1}{2}{\sigma^2}_{s\tau+1}+\sigma_{ms\tau+1}]$ is equal to $\frac{1}{\prod_{\tau=t}^{T-1}B(\tau,\tau+1)}$ and is known at time t, even though the conditional moments $\sum_{\tau=t}^{T-1}\mu_{s\tau+1}$, $\sum_{\tau=t}^{T-1}{\sigma^2}_{s\tau+1}$, and $\sum_{\tau=t}^{T-1}\sigma_{ms\tau+1}$ do depend individually on the future path of the state variables. Therefore, apart from interest rate risk distortions, the symmetry of the volatility smile for short-term options implies the same property for longer-term options[17].

5. Leverage Effects and Comparison with Stochastic Volatility Models

The predictability property referred to in the previous sections amounts to the knowledge at time t of the quantities $\tilde{B}\,(t,t+1)$ and $Q_{ms}\,(t,t+1)$ which can be written as:

$$\tilde{B}\,(t,t+1) \quad = \quad E_t\left[m_{t+1}\left|U_1^{t+1}\right.\right]=\exp\left[\mu_{mt+1}+\frac{\sigma^2_{mt+1}}{2}\right]$$

[17]The argument is not as explicit in the Hull and White setting since Renault and Touzi (1996) provide a direct proof that the absence of leverage effect (which is a short-term property) will produce symmetric volatility smiles irrespective of the maturity of the option. However, the proof is based on the maintained hypothesis that the volatility risk premium does not depend on the level of the stock price. If it were not the case, it would introduce a Granger-causality effect in contradiction to assumption A2. But the Hull and White setting does not make explicit the implications in terms of the SDF of such maintained assumptions about risk premia.

$$\begin{aligned} Q_{ms}(t,t+1) &= \tilde{B}(t,t+1)\exp(\sigma_{mst+1}) E_t\left[\frac{S_{t+1}}{S_t}\left|U_1^{t+1}\right.\right] \\ &= \tilde{B}(t,t+1)\exp(\sigma_{mst+1})\exp\left[\mu_{st+1}+\frac{\sigma^2_{st+1}}{2}\right] \end{aligned} \quad (5.1)$$

Therefore, the crucial issue for predictability, which corresponds to preference free option pricing, is to determine if the parameters $\mu_{m,t+1}$, $\mu_{s,t+1}, \sigma^2_{m,t+1}, \sigma^2_{s,t+1}$ and $\sigma_{ms,t+1}$ of the joint conditional probability distribution of $\left(m_{t+1}, \frac{S_{t+1}}{S_t}\right)$ given U_1^{t+1} depend or not upon U_{t+1}, that is if there is or not an instantaneous causality relationship[18] between the state variable process U and the process $\left(m_{t+1}, \frac{S_{t+1}}{S_t}\right)$.

5.1. Returns Volatility and Leverage Effect

We will now check whether this property also eliminates any kind of leverage effect, that is any evidence of negative correlation between stock returns and changes in returns volatility. Volatility tends to rise in response to bad news (excess returns lower than expected) and to fall in response to good news (excess returns higher than expected), as pointed out in Nelson (1991). In our setting, returns volatility can be defined in three ways. The first and most obvious one is the individual stock return volatility process:

$$\begin{aligned} h_t^s &= Var\left[Log\frac{S_{t+1}}{S_t}\left|I_t\right.\right] \\ &= E\left[\sigma^2_{s_{t+1}}\left|U_1^t\right.\right] + Var\left[\mu_{s_{t+1}}\left|U_1^t\right.\right]. \end{aligned}$$

This stock volatility dynamics can give rise to a micro–level leverage effect. However, both on empirical grounds and theoretical aggregation arguments[19], leverage

[18] Our results of equivalence between preference-free option pricing and no instantaneous causality between state variables and asset returns are also consistent with GARCH option pricing. Duan (1995) derived it first in an equilibrium framework, but Kallsen and Taqqu (1998) have shown that it could be obtained with an arbitrage argument. Their idea is to complete the markets by plugging the discrete-time model into a continuous time one, where conditional variance is constant between two integer dates. They show thats such a continuous-time embedding makes possible arbitrage pricing which is per se preference-free. Therefore, preference-free option pricing cannot be recovered in the presence of an instantaneous causality effect, since it is such an effect that prevents the embedding used by Kallsen and Taqqu (1998).

[19] See Bakshi, Karpadia and Madan (2000).

effects may be more important at an aggregate level. One way to capture this aggregate leverage effect is to consider the SDF volatility process:

$$\begin{aligned} h_t^m &= Var\left[\log m_{t+1}\left|I_t\right.\right] \\ &= E\left[\sigma^2_{m_{t+1}}\left|U_1^t\right.\right] + Var\left[\mu_{m_{t+1}}\left|U_1^t\right.\right]. \end{aligned}$$

A third measure of volatility may be defined in terms of non diversifiable risk. Let us consider for instance the simple case where there is no aggregate leverage effect (because there is no instantaneous causality between the state variable process U and the SDF process m) and instantaneous changes in return volatility do not go through the stock variance process (σ^2_{st+1} does not depend upon U_{t+1}). Then, the stock pricing formula (2.19) can still be written in a form close to the standard conditional CAPM equation:

$$E\left[\frac{S_{t+1}}{S_t}\left|I_t\right.\right] = \frac{1}{B(t,t+1)}\frac{1}{E\left[\exp\left(\sigma_{mst+1}\right)\left|U_1^t\right.\right]}$$

$$-\exp\left(\sigma^2_{st+1}\right)\frac{Cov\left[\exp\mu_{st+1},\exp\sigma_{mst+1}\left|U_1^t\right.\right]}{E\left[\exp\left(\sigma_{mst+1}\right)\left|U_1^t\right.\right]}.$$

In other words, $h_t^{ms} = [E[\exp\left(\sigma_{mst+1}\right)\left|U_1^t\right.]]^{-1}$ is the relevant measure of non-diversifiable risk. It defines the part of the stock return volatility which is compensated in equilibrium, up to a correction for "non diversifiable leverage effect", that is a non zero conditional covariance between $\left(\exp\ \mu_{st+1}\right)$ and $\left(\exp\ \sigma_{mst+1}\right)$.

To summarize, we propose to extend the usual definition of leverage effect proposed by Nelson (1991) to any positive or negative conditional covariance, given I_t[20], between the mean returns and the three concepts of volatility just defined, that is $Cov\left[\mu_{st+1},h^s_{t+1}\left|U_1^t\right.\right]$, $Cov\left[\mu_{st+1},h^{ms}_{t+1}\left|U_1^t\right.\right]$, and $Cov\left[\mu_{mt+1},h^{m}_{t+1}\left|U_1^t\right.\right]$. In other words, leverage effect may occur if and only if one of the three following properties is fulfilled: (i) Both μ_{st+1} and σ^2_{st+1} depend upon U_{t+1} (standard leverage effect for the individual stock); (ii) Both μ_{st+1} and σ_{mst+1} depend upon U_{t+1} (leverage effect for the individual stock but only through its non-diversifiable

[20] Of course, some nonlinearity effects may create nontrivial relations between conditional covariances of interest like $Cov\left[\mu_{st+1},h^{ms}_{t+1}\left|U_1^t\right.\right)$ and $Cov\left[\exp\mu_{st+1},\exp\sigma_{mst+1}\left|U_1^t\right.\right]$. These effects are neglected in our informal comments.

risk); (iii) Both μ_{mt+1} and σ^2_{mt+1} depend upon U_{t+1} (standard leverage effect for the portfolio return which mimics the SDF, that is aggregate leverage effect).

Moreover, our state variable setting offers a very flexible framework for parametric models of leverage effect. In standard stochastic volatility models (see Ghysels, Harvey and Renault, 1996, for a survey), the usual leverage effect is generally captured by a (negative) constant linear conditional correlation coefficient between $\log \frac{S_{t+1}}{S_t}$ and h^s_{t+1} (given I_t). In our setting, this correlation coefficient depends on the two functions μ_{st+1} and σ^2_{st+1} of the state variables U_1^{t+1} and upon the exogenous dynamics of these state variables through:

$$Cov_t\left[\log \frac{S_{t+1}}{S_t}, h^s_{t+1}\right] = Cov\left\{\mu_{st+1}, E\left[\sigma^2_{st+2}\left|U_1^{t+1}\right.\right] + Var\left[\mu_{st+2}\left|U_1^{t+1}\right.\right]\left|U_1^t\right.\right\}$$

and the corresponding conditional variances. In other words, the leverage effect of the return process $\log \frac{S_{t+1}}{S_t}$, which features stochastic volatility, can come from two sources[21]. The conditional mean process μ_{st+1} may be a stochastic volatility process which features a leverage effect defined by the negativity of $Cov\left[\mu_{st+1}, Var\left[\mu_{st+2}\left|U_1^{t+1}\right.\right]\left|U_1^t\right.\right]$. Or, the process $\log \frac{S_{t+1}}{S_t}$ itself may be characterized by a leverage effect and then $Cov\left[\mu_{st+1}, E\left[\sigma^2_{st+2}\left|U_1^{t+2}\right.\right]\left|U_1^t\right.\right]$ will be negative, which means that bad news about expected return (when μ_{st+1} is smaller than its unconditional expectation) imply in average a higher expected volatility of $Log\frac{S_{t+1}}{S_t}$, that is a value of $E\left[\sigma^2_{st+2}\left|U_1^{t+2}\right.\right)$ greater than its unconditional mean.

We have seen that Assumption A3 not only allows to capture the standard features of a stochastic volatility model (in terms of heavy tails and leverage effects) but also provides for a richer set of possible dynamics. Moreover, we can certainly extend these ideas to multivariate dynamics either for the joint behavior of market and stock returns or for any portfolio consideration. For instance, the dependence of σ_{mst+1} on the whole set of state variables offers great flexibility to model the stochastic behavior of correlation coefficients, as recently put forward empirically by Andersen et al. (1998). This last feature is clearly highly relevant for asset allocation or conditional beta pricing models. In the next subsection, we will see that a simple Markov switching model offers a versatile framework to capture the exogenous dynamics of the state variables.

[21] This decomposition of the leverage effect in two terms is the exact analogue of the decomposition discussed in Fiorentini and Sentana (1998) and Meddahi (1999) for persistence.

5.2. Comparing Skewness and Leverage Effects with Discrete State Markov Latent Variables

Asymmetric distortions of the volatility smile are often viewed as a signal of some skewness in the underlying probability distribution of returns. Since we have focused our explanation of the volatility smile on various kinds of leverage effects, it is worthwhile to assess how these effects relate, in our state variable setting, to the skewness properties of the conditional probability distribution of returns. To compare analytically the skewness of the returns distribution and the leverage effects, we assume that the latent variables follow a two-state Markov switching process. Therefore, we assume that the log return can be written as:

$$\log \frac{St+1}{S_t} = Y_{t+1} = \mu_0(1-U_{t+1}) + \mu_1 U_{t+1} + [\sigma_0(1-U_{t+1}) + \sigma_1 U_{t+1}]\varepsilon_{st+1}, \quad (5.2)$$

the variable U_{t+1} taking the values 0 or 1 with the probabilities π_0 and π_1. The transition probabilities between states i and j are defined as: $p_{ij} = \Pr[U_{t+1} = j|U_t = i]$, with $p_{ij} = 1 - p_{ii}$ for $i \neq j$,and $\pi_i = \frac{1-p_{jj}}{2-p_{ii}-pjj}$, $i, j = 0, 1$.

The returns are therefore a mixture of normals with two different means and variances. It is well-known that such a mixture model will generate skewness and excess kurtosis in both the conditional and unconditional distributions of returns. We are interested in the conditional distribution of returns Y_{t+1} given I_t, the information available at time t. The moments of $Y_{t+1}|I_t$ are defined as:

$$E[Y_{t+1}|I_t] = m_i^{(1)} \text{ if } U_t = i \quad (5.3)$$

$$E\{[Y_{t+1} - E(Y_{t+1}|I_t)]^n|I_t]\} = m_i^{(n)} \text{ if } U_t = i \quad (5.4)$$

Given the process assumed for Y_{t+1} and U_t, the first three moments are given by:

$$m_i^{(1)} = \begin{cases} p_{00}\mu_0 + (1-p_{00})\mu_1 \text{ if } i = 0 \\ (1-p_{11})\mu_0 + p_{11}\mu_1 \text{ if } i = 1 \end{cases} \quad (5.5)$$

$$m_i^{(2)} = \begin{cases} p_{00}\sigma_0^2 + (1-p_{00})\sigma_1^2 + p_{00}(1-p_{00})(\mu_1-\mu_0)^2 \text{ if } i = 0 \\ (1-p_{11})\sigma_0^2 + p_{11}\sigma_1^2 + p_{11}(1-p_{11})(\mu_1-\mu_0)^2 \text{ if } i = 1 \end{cases} \quad (5.6)$$

$$m_i^{(3)} = \begin{cases} p_{00}(1-p_{00})(\mu_0-\mu_1)[3(\sigma_0^2-\sigma_1^2)+(1-2p_{00})(\mu_1-\mu_0)^2] \text{ if } i=0 \\ p_{11}(1-p_{11})(\mu_1-\mu_0)[3(\sigma_1^2-\sigma_0^2)+(1-2p_{11})(\mu_1-\mu_0)^2] \text{ if } i=1 \end{cases} \tag{5.7}$$

We want to compare skewness, defined as:

$$sk_i = \frac{m_i^{(3)}}{[m_i^{(2)}]^{\frac{3}{2}}}, \tag{5.8}$$

with the leverage effect that we defined as $Cov\left[\mu_{Yt+1}, h_{t+1}^Y \left| U_1^t \right.\right]$. For the Markov case, we have:

$$h_{t+1}^Y = m_i^{(2)} \text{ if } U_t = i \tag{5.9}$$

$$\mu_{t+1}^Y = \mu_0(1-i) + \mu_1 i \text{ if } U_t = i. \tag{5.10}$$

Therefore, we can write:

$$Cov\left[\mu_{Yt+1}, h_{t+1}^Y \left| U_1^t \right.\right] = E_t\left[m_{U_{t+1}}^{(2)}[\mu_0(1-U_{t+1})+\mu_1 U_{t+1}]\right] - m_{U_t}^{(1)} E_t m_{U_{t+1}}^{(2)}. \tag{5.11}$$

After some algebraic manipulations, we obtain the following expressions for the leverage effect:

$$Cov\left[\mu_{Yt+1}, h_{t+1}^Y \left| U_1^t \right.\right] = \begin{cases} p_{00}(1-p_{00})(\mu_0-\mu_1)[(\sigma_0^2-\sigma_1^2)(p_{00}-(1-p_{11})) \\ +(\mu_1-\mu_0)^2(p_{00}(1-p_{00})-p_{11}(1-p_{11}))] \text{ if } i=0 \\ p_{11}(1-p_{11})(\mu_1-\mu_0)[(\sigma_1^2-\sigma_0^2)(p_{11}-(1-p_{00})) \\ +(\mu_1-\mu_0)^2(p_{11}(1-p_{11})-p_{00}(1-p_{00})) \text{ if } i=1 \end{cases} \tag{5.12}$$

First, it appears clearly that the formulas for conditional skewness and leverage effect in the stock are very similar. In both the skewness and leverage expressions, irrespective of the initial state, there is a term in $(\mu_1-\mu_0)(\sigma_1^2-\sigma_0^2)$ and another term in $(\mu_1-\mu_0)^3$. We would like to characterize the bad state as the state where the volatility is high (say $\sigma_1^2 > \sigma_0^2$) and the mean is low ($\mu_1 < \mu_0$). With such a characterization, the skewness in the bad state (state 1) will be negative as long as the bad state is not too persistent ($p_{11} < \frac{1}{2}$). On the contrary, for the good state, the skewness will be negative when the good state is persistent ($p_{00} > \frac{1}{2}$). The conditions for the leverage effect are a little more complex, but it will be negative in both states if $p_{00} + p_{11} > 1$ and $p_{11}(1 - p_{11}) > p_{00}(1 - p_{00})$. The first condition implies some persistence in at least one of the states while the second requires more persistence in the good state. This is consistent with the conditions for skewness. These conditions for negative leverage are consistent with interpretations of the states as business cycle states or bull and bear markets, where typically the good state is more persistent. Of course these are only sufficient conditions. The skewness and the leverage effects can still be negative even if they are not met provided that the means and variances are of the right magnitude.

6. A characterization of the smiles with an equilibrium option pricing model

In this section, we want to illustrate the various types of smiles that can be obtained in a state variable option pricing framework. We propose a setup that leads to a computable formula with an equilibrium version of the SDF. We choose preferences in the recursive utility class (Epstein and Zin, 1989) which are richer than the usual expected utility model. In particular, the elasticity of intertemporal substitution is disentangled from the risk aversion parameter. As we will see, the intertemporal substitution parameter plays an important role for pricing options when preferences matter. In this equilibrium model, the latent state variables affect the fundamentals of the economy and follow a discrete-state Markov process as in the previous section.

6.1. The equilibrium stochastic discount factor

In the recursive utility framework of Epstein and Zin (1989), the stochastic discount factor is given by:

$$m_{t+1} = \beta^{\gamma} (\frac{C_{t+1}}{C_t})^{\gamma(\rho-1)} M_{t+1}^{\gamma-1} \tag{6.1}$$

where $\frac{C_{t+1}}{C_t}$ is the growth rate of consumption in the economy and M_{t+1} represents the return on the market portfolio (computed with the assumption that C_t is the payoff of the market portfolio). The parameters β, γ and ρ are preference parameters. The parameter β is the subjective rate of time preference, while $\alpha = \gamma\rho$ can be interpreted as a relative risk aversion parameter with the degree of risk aversion increasing as α falls ($\alpha \leq 1$). The parameter ρ is associated with intertemporal substitution, since the elasticity of intertemporal substitution is $1/(1-\rho)$[22]. When $\gamma = 1$, we obtain the well-known stochastic discount factor for the expected utility case $m_{t+1} = \beta(\frac{C_{t+1}}{C_t})^{\alpha-1}$.

Given this stochastic discount factor, the price of a European option π_t maturing at time T can be obtained as a particular case of formula (3.4), namely:

$$\frac{\pi_t}{S_t} = E_t\left\{Q_{XY}(t,T)\Phi(d_1) - \frac{K\widetilde{B}(t,T)}{S_t}\Phi(d_2)\right\}, \tag{6.2}$$

where:

$$d_1 = \frac{Log\left[\frac{S_t Q_{XY}(t,T)}{K\widetilde{B}(t,T)}\right]}{(\sum_{\tau=t+1}^{T}\sigma^2_{Y\tau})^{1/2}} + \frac{1}{2}(\sum_{\tau=t+1}^{T}\sigma^2_{Y\tau})^{1/2}, \text{and } d_2 = d_1 - (\sum_{\tau=t+1}^{T}\sigma^2_{Y\tau})^{1/2}.$$

and:

$$\widetilde{B}(t,T) = \beta^{\gamma(T-t)}a_t^T(\gamma)\exp((\alpha-1)\sum_{\tau=t+1}^{T} m_{X\tau} + \frac{1}{2}(\alpha-1)^2\sum_{\tau=t+1}^{T}\sigma^2{}_{X\tau}), \tag{6.3}$$

with: $a_t^T(\gamma) = \prod_{\tau=t}^{T-1}\left[\frac{(1+\lambda(U_1^{\tau+1})}{\lambda(U_1^{\tau})}\right]^{\gamma-1}$, and

$$Q_{XY}(t,T) = \widetilde{B}(t,T)b_t^T\exp((\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau})E[\frac{S_T}{S_t}|U_1^T]. \tag{6.4}$$

with: $b_t^T = \prod_{\tau=t+1}^{T}\frac{(1+\varphi(U_1^{\tau}))}{\varphi(U_1^{\tau})}$.

We define $\lambda_t = \lambda(U_1^t) = \frac{P_t^M}{C_t}$ and $\varphi_t = \varphi(U_1^t) = \frac{S_t}{D_t}$ (with D_t the dividend on the stock) as the solutions to Euler equations for the price of the market portfolio P_t^M and the price of the stock, and $X_t = Log\frac{C_t}{C_{t-1}}$ and $Y_t = Log\frac{D_t}{D_{t-1}}$.

[22] As mentioned in Epstein and Zin (1991), the association of risk aversion with α and intertemporal sustitution with ρ is not fully clear, since at a given level α of risk aversion, changing ρ affects not only the elasticity of intertemporal sustitution but also determines whether the agent will prefer early or late resolution of uncertainty.

6.2. Characterization of the Smiles

We use this equilibrium framework to calibrate the various shapes of the implied volatility curves that can result from the option pricing formula (6.2). In particular, we analyze the sensitivity of the smile skewness to various parameters entering the formula, most notably the parameters of the stochastic process driving the fundamentals and the preference parameters.

6.2.1. A Markov-Chain Setup for the State Variables

The process describing the joint evolution of $X_t = \log C_t/C_{t-1}$ and $Y_t = \log D_t/D_{t-1}$ is parameterized as follows:

$$\begin{aligned} X_t &= m_X(U_t) + \sigma_X(U_t)\varepsilon_{Xt} \\ Y_t &= m_Y(U_t) + \sigma_Y(U_t)\varepsilon_{Yt} \end{aligned}$$

The vector $(\varepsilon_{Xt}, \varepsilon_{Yt})'$ follows a standard bivariate normal distribution with correlation coefficient ρ_{XY}. The time-varying mean and variance parameters are a function of the state variable process $\{U_t\}$, which is assumed to be a discrete first-order Markov chain such that U_t takes values in $\{1, .., N\}$ with $\Pr(U_t = j) = \sum_{i=j}^{N} p_{ij} \Pr(U_{t-1} = i)$ and transition probability $p_{ij} = \Pr(U_t = j|U_{t-1} = i)$ for $i, j = 1, ..., N$.

For given values of the transition probabilities of the Markov chain governing the state variable process and given values of the structural parameters, it is possible to compute the price of an option according to the generalized Black-Scholes and *a fortiori* the Hull-White formula. The steps followed to compute option prices are detailed in Appendix 3.

6.2.2. State variables and the smile

To calibrate the model, we choose parameters for the state variables which mimic roughly business cycle data. We consider the case where the state variable U_t takes values in the set $\{1, 2\}$ and is governed by a first-order Markov chain with a transition probability matrix $[p_{11} = 0.9,\ p_{22} = 0.6]$. The state-contingent parameter values of the consumption and dividend processes are set as $m_{X1} = 0.0015$, $m_{X2} = -0.0009$, $\sigma_{X1} = \sigma_{X2} = 0.003$, $m_{Y1} = m_{Y2} = 0$, $\sigma_{Y1} = 0.02$, $\sigma_{Y2} = 0.12$, and $\rho_{XY} = 0.6$. With this specification, $U_t = 1$ can be interpreted as an expansionary state where consumption growth is positive and stock market volatility is low. On the other hand the recessionary state, $U_t = 2$, is characterized by negative consumption growth and a more volatile stock market. The preference parameters are set as follows: $\gamma = 1$, $\rho = -10$, and hence $\alpha = -10$. This configuration is taken as the benchmark expected utility model for comparison with recursive utility extensions. In the following we explore the various implications that the generalized option pricing model has in terms of the volatility smile.

Stochastic volatility and stochastic interest rates The first implications for the volatility smile that we explore are those arising from the state variable. Figures 2 through 4 illustrate these effects. In order to illustrate the effects due to maturity and stochastic volatility, consider the case where the discount factor $B(t,T)$ is deterministic and where the factor $Q(t,T)$ is unity. The interest rate risk is eliminated by holding constant m_{Xt} and σ_{Xt} which implies constant λ_t and in turn a deterministic discount factor $B(t,T)$. The left panel of Figure 2 illustrates the smile effects arising from stochastic volatility. The three curves show the effect of decreases in the coefficient of variation of the volatility from 0.71 to 0.60 to 0.33 with the flattest representing the least variation. As expected if the coefficient of variation of the volatility goes to zero, Black-Scholes pricing results and the implied volatility curve would be completely flat.

Now holding the coefficient of variation of the volatility constant, the right panel of Figure 2 illustrates the effect of increasing the option's maturity. The most curved smiles in each panel are in fact identical representing a one-period option. The two other curves in the right panel represent options whose maturity is increased by one and two periods with the flattest being associated with the three-period option. We see that as the option's maturity increases, the smile flattens.

Figure 3 illustrates the maturity effect when $Q(t,T) \neq 1$. For comparison, the figure illustrates also the smiles that obtain from preference-free option pricing à la Hull-White. The latter smiles are distinguished by their symmetry with respect to zero. The solid lines are the implied volatility curves for a one-period option whereas the dashed lines are for a two-period option. The left and right panels are associated with states 1 and 2, respectively, as the current state. It is seen that the maturity effect depends on the current state: when state 1 is operative at time t, an increase in maturity results in flatter yet greater implied volatilities, while when in state 2 the flatter implied volatility curves associated with longer maturities are lower. It should be noticed that the smiles are moving to the right in both states. If one considers the expressions developed in section 5.2, this is indicative of a negative skewness or leverage effect due to the assumed configuration of the transition probabilities for the two states.

Next consider the case where $B(t,T)$ is stochastic and $Q(t,T) \neq 1$. The lines and panels of Figure 4, similar to those of the preceding figure, illustrate this case. Comparing the respective lines of figures 3 and 4, reveals that stochastic interest rates imply greater asymmetry in the smile. An important remark is that at longer maturities, the smile is more asymmetric than at shorter maturities. This feature is apparent by noticing that the point of intersection between the symmetric and asymmetric smiles is displaced to the right for the longer maturities. This is consistent with the violation of the symmetry condition of Proposition 3.3 due to the lack of factorization of this condition in a sequence of elementary unit periods.

Average duration and correlation effects Figures 5 and 6 illustrate the effect of changes in the persistence of each state on the implied volatility curves. Figure 5 considers the case where the probability of staying within a given state is greater than its exit probability, $p_{ii} > p_{ij}$. The dashed lines represent an increase in p_{22} from 0.6 to 0.8 and, as previously, the left and right panels are associated with states 1 and 2 respectively as the current states. From each panel it is seen that an increase in the persistence, or average duration $(1 - p_{22})^{-1}$, results in a greater asymmetry of the smile in both states. It is interesting to note from the right panel of Figure 5 that frowns obtain. In Figure 6 we consider the opposite case where $p_{ii} < p_{ij}$. We take as a base case $p_{11} = 0.4$ and $p_{22} = 0.1$. Increasing p_{12} from 0.6 to 0.9 in this case has the same effects in terms of asymmetric distortions of the smile as in Figure 5 but with the roles of current state reversed: frowns in this case arise when state 1 is the time t operative state.

Figure 7 graphs the schedule of option prices across moneyness for two extreme values of ρ_{XY} the correlation between consumption and dividends. The solid line represents the case $\rho_{XY} = 1$ while the dashed line is for $\rho_{XY} = 0$. Regardless of which state is operative at time t, a decrease in the correlation between consumption and dividends results in an upward shift of the entire schedule of option prices. Intuitively when the stock is perfectly correlated with the market portfolio, there is one less source of risk to hedge and hence option prices are lower reflecting the smaller risk premium in this case.

Leverage effects As we have seen, the preference-free option pricing formula à la Hull-White obtains when there are no leverage effects, neither through the market risk nor through the stock risk. Figures 8 and 9 illustrate the implications that these leverage effects separately have in terms of the volatility smile. These implications are explored in the context of a two-period option which is the shortest horizon one can consider here since for a single-period option, absence of leverage through the stock risk implies Black-Scholes pricing.

In Figure 8 the dashed lines show the implications that an absence of leverage through the consumption process has on the volatility smile. It should be mentioned that even in this case, the stochastic feature of interest rates is maintained by including the past value of the state variable in the consumption dynamics. The solid lines are the benchmark case of leverage effects through both the consumption and dividend processes. Notice that the symmetric smiles resulting from preference-free option pricing à la Hull-White are identical whether or not there is leverage through consumption. Absence of leverage through the consumption process leads to a more asymmetric smile as is apparent from both panels of the figure that, as before, are conditional on the current operative state.

Similarly, in Figure 9 the dashed lines show the effect of an absence of leverage through the dividend process. Again we observe an asymmetric smile, except that in this case the asymmetric distortion is far more pronounced than in the previous figure.

6.2.3. Preferences and smile effects

We now proceed to investigate the role played by the preference parameters. We will see that preferences, in particular intertemporal substitution, can have an effect in terms of the asymmetry of the volatility smile but that they play a secondary role compared to that played by the state variable. This is reassuring if one believes that such parameters should stay relatively stable over time in face of the well-documented changing shapes of the volatility smile. The benchmark for comparison here is the expected utility case which obtains when $\gamma = 1$. This was in fact the case up to this point. For example, recall that the preferences underlying Figure 4 are $\gamma = 1$, $\rho = -10$, and hence $\alpha = -10$.

Consider now Figure 10 where the solid lines are the volatility curves for the following configuration of preference parameters: $\gamma = 1$, $\rho = -1$, $\alpha = -1$; whereas the dashed lines are for $\gamma = 1/10$, $\rho = -10$, $\alpha = -1$. Notice that the symmetric preference-free smiles are necessarily identical under both preference configurations. Comparing the solid lines of Figure 10 with those of Figure 4, we see that reducing ρ (and hence α since $\gamma = 1$) from -1 to -10 leads to greater asymmetry in the smile. That this asymmetry is in fact caused by the change in ρ and not by that in α is verified by considering the dashed lines in Figure 10. Since the asymmetric dashed smile of Figure 10 is virtually identical to the asymmetric solid smile of Figure 4, we conclude that it is intertemporal substitution and not risk aversion which explains asymmetric volatility smiles. This is confirmed in figure 11, where increasing α to -1 for one- and two-period options leads to smiles that are practically identical to the ones obtained in Figure 4.

This point can also be made in the following way. The risk aversion parameter α enters the generalized pricing formula through the discounting factor $\widetilde{B}(t,T)$ and through the CAPM-like factor $Q(t,T)$ which accounts for the covariance risk between the stock and market portfolio. If the role of the discounting factor is held fix, then the only difference between the generalized option prices and their preference-free counterparts would be due to risk aversion as this is the only preference parameter that enters the $Q(t,T)$ factor beyond what is already embodied in the $\widetilde{B}(t,T)$ factor. To this end consider Figure 12. The left panel plots $E_t \left\{ S_t(\Phi(d_1^{HW}) - Q_{XY}(t,T)\Phi(d_1^{GBS})) \right\}$; that is, the difference between the first parts of the preference-free and the generalized option pricing formula in which preferences matter. The difference between the second parts of these formulas, $E_t \left\{ K(B(t,T)\Phi(d_2^{HW}) - \widetilde{B}(t,T)\Phi(d_2^{GBS})) \right\}$, is plotted in the right panel. In each panel the solid line represents the case where $B(t,T)$ is stochastic whereas the dashed line represents the case where $B(t,T)$ is deterministic. Comparison of the left and right panels reveals no differences whatsoever.

The conclusion that emerges is that risk aversion plays no role in explaining the departures of generalized option prices in which preferences matter from their preference-free counterpart. In turn this implies that it is intertemporal substitution that explains asymmetric volatility smiles and not risk aversion as has traditionally been thought in a model where these two concepts are entangled.

6.2.4. Errors from preference-free pricing

Given a world in which preferences matter for option pricing, let us consider the relative pricing error that is committed by using the preference-free formula to price options. With the values of the preference parameters and the endowment process given previously as the benchmark for comparison, we generated option prices according to the generalized pricing formula where preferences matter and its preference-free counterpart and plotted the relative difference; i.e. $(\pi_t^{HW} - \pi_t^{GBS})/\pi_t^{GBS}$. Figure 13 shows the relative pricing error across moneyness for options with maturities of one (solid line), two (dashed line) and three periods (short dashes). Again the left and right panels are associated with states 1 and 2, respectively, as the current operative state. In both cases we see that the greatest pricing errors are committed for out-of-the-money options and that the pricing errors are most pronounced at shorter maturities.

Recall from above that when the transition probability matrix was changed from one with persistent states $[p_{ii} > p_{ij}]$ to one where $[p_{ii} < p_{ij}]$, the result was a reversal of the asymmetric bias in the observed volatility smile. Such a change in the transition probabilities has of course a similar effect in terms of relative pricing errors: If $[p_{ii} < p_{ij}]$, Figure 13 gets reversed with in-the-money options being more severely underpriced by the preference-free formula than out-of-the-money options.

This bias reversal is similar to one documented by Hull and White (1987) with respect to the Black-Scholes formula. They found that when there is a positive correlation between the stock price and its volatility, out-of-the-money options are underpriced (by the BS formula), while in-the-money options are overpriced. When the correlation is negative, the effect is reversed. We find something similar with respect to the preference-free option pricing formula à la Hull-White. For the given specification, the covariance between the stock prices and its volatility is positive when $[p_{ii} > p_{ij}]$ and negative when $[p_{ii} < p_{ij}]$. Hence the same pattern emerges: out-of-the-money options are underpriced by the preference-free formula when there is a positive correlation between stock prices and its volatility, and overpriced when this correlation is negative.

This result may well provide an explanation to the empirical mispricings observed by Bakshi, Cao, and Chen (1997) with a pricing model that admits stochastic volatility, stochastic interest rates, and random jumps but which remains preference-free. In Garcia, Luger and Renault (2000), we assess the empirical performance of our model relative to a preference-free formula in terms of pricing errors.

7. Conclusion

In this paper, we have analyzed the symmetry of the so-called implied volatility smiles, which are often used to characterize the European option pricing biases produced by the Black-Scholes formula. We have stated conditions that an option pricing formula must obey to produce a symmetric volatility smile and translated them into conditions on the pricing probability measure. We proposed an option pricing formula with a general stochastic discount factor that generalizes the stochastic volatility option pricing formula. Such a generalization is achieved through a conditioning on state variables. We have shown that two kinds of generalized leverage effects may explain (besides the interest rate risk) asymmetric smiles: either a genuine leverage effect, that is an instantaneous correlation between the return on the stock and its stochastic volatility process, or a stochastic correlation between the return of the stock and the stochastic discount factor. These results provide some theoretical foundations to the observed asymmetric smiles. We have also explained how these leverage effects determine if the option pricing formula is preference-free or not.

Through an equilibrium stochastic discount factor and a Markov regime-switching process for the state variables, we have shown that the model leads itself to a computable formula that can reproduce many of the shapes observed for the implied volatility curves. The remaining task is to show that the parameters estimated from the data in such an extended framework can be used to achieve smaller pricing or hedging errors out of sample. We leave such a task for future research.

Appendix 1

Proof of Proposition 2.1:

We first check that, for any given value of σ,the function $\pi(.) = BS(., \sigma)$ fulfills the announced property:

$$\pi(-x) = e^x \pi(x) + 1 - e^x.$$

Indeed, from (2.2):

$$BS(x, \sigma) = \Phi[d_1(x, \sigma)] - e^{-x}\Phi[d_2(x, \sigma)],$$

with: $d_1(x, \sigma) = \frac{x}{\sigma} + \frac{\sigma}{2}, d_2(x, \sigma) = \frac{x}{\sigma} - \frac{\sigma}{2}$.

But: $\Phi[d_2(-x, \sigma)] = \Phi[-d_1(x, \sigma)] = 1 - \Phi[d_1(x, \sigma)]$,and: $\Phi[d_1(-x, \sigma)] = \Phi[-d_2(x, \sigma)] = 1 - \Phi[d_2(x, \sigma)]$.

Therefore:

$$\begin{aligned} BS(-x, \sigma) &= \Phi[d_1(-x, \sigma)] - e^x\Phi[d_2(-x, \sigma)] \\ &= e^x\Phi[d_1(x, \sigma)] - \Phi[d_2(x, \sigma)] + 1 - e^x \\ &= e^x BS(x, \sigma) + 1 - e^x. \end{aligned}$$

Let us now consider another homogeneous option pricing formula $x \to \pi(x)$. The associated BS implied volatilities are then defined by:

$$\begin{aligned} \pi(x) &= BS[x, \sigma^*(x)], \\ \pi(-x) &= BS[-x, \sigma^*(-x)]. \end{aligned}$$

Therefore, for any x:

$$\begin{aligned} \sigma^*(x) &= \sigma^*(-x) \\ &\iff \pi(-x) = BS[-x, \sigma^*(x)] \\ &\iff \pi(-x) = e^x BS[x, \sigma^*(x)] + 1 - e^x \\ &\iff \pi(-x) = e^x \pi(x) + 1 - e^x. \blacksquare \end{aligned}$$

Proof of Proposition 2.2:

a) First, we prove that the criterion of Proposition 2.1 is equivalent to the property (i) of Proposition 2.2. We can write (2.1) as:

$$\pi_t(S_t, K) = B(t,T)S_t \int_{\frac{K}{S_t}}^{+\infty} \left(\frac{S_T}{S_t} - \frac{K}{S_t}\right) dQ_{t,T}\left(\frac{S_T}{S_t}\right)$$

Therefore, by taking the derivative with respect to K, we obtain the well-known relationship between the option pricing formula and the pricing probability measure.

$$\begin{aligned} \frac{\partial \pi}{\partial K}(S_t, K) &= -B(t,T)Q_t\left[\frac{S_T}{S_t} \geq \frac{K}{S_t}\right] \\ &= -B(t,T)[1 - F_{V_T}(-x)]. \end{aligned}$$

Since:

$$\frac{\partial \pi}{\partial x}(x) = \frac{\partial}{\partial x}[\pi(1, \frac{K}{S_t})] = -\frac{K}{S_t}\frac{\partial}{\partial K}\pi(S_t, K)$$

we have, for any x :

$$\frac{\partial \pi}{\partial x}(x) = e^{-x}[1 - F_{V_T}(-x)].$$

Therefore, the property (i) of Proposition 2.2 may be rewritten as:

$$\pi(x) = 1 - e^{-x}\frac{\partial \pi}{\partial x}(-x) - \frac{\partial \pi}{\partial x}(x)$$

or equivalently:

$$-\frac{\partial \pi}{\partial x}(-x) = e^{x}[\pi(x) + \frac{\partial \pi}{\partial x}(x) - 1].$$

This last equality is obviously a corollary of Proposition 2.2 obtained by taking the derivative with respect to x of the identity in Proposition 2.2. Conversely, this equality implies that for any x :

$$-\int_x^{+\infty} \frac{\partial \pi}{\partial u}(-u)du = \int_x^{+\infty} e^{u}[\pi(u) + \frac{\partial \pi}{\partial u}(u) - 1]du$$

This equation will provide the criterion of Proposition 2.2 if we are able to complete it by the following limit condition:

$$\lim_{x \to +\infty} \pi(-x) = \lim_{x \to +\infty}[e^{x}\pi(x) + 1 - e^{x}].$$

Therefore, the required equivalence will be proved if we show that this limit condition is always guaranteed. But, on the one hand:

$$\begin{aligned}
\lim_{x \to +\infty} \pi(-x) &= \lim_{x \to -\infty} \pi(x) \\
&= \lim_{K \to +\infty} B(t,T)E_t^{*}Max[0, S_T - K] = 0
\end{aligned}$$

by virtue of the Lebesgue dominated convergence theorem since:$Max[0, S_T - K] \longrightarrow_{K \to \infty} 0$ almost surely and $0 \leq Max[0, S_T - K] \leq S_T$, which is by assumption integrable with respect to the pricing probability measure. On the other hand:

$$\begin{aligned}
\lim_{x \to +\infty} e^{x}[\pi(x) - 1] + 1 &= 1 + \lim_{K \to 0^+} \frac{1}{KB(t,T)}\{B(t,T)E_t^{*}Max[0, S_T - K] - B(t,T)E_t^{*}S_T\} \\
&= 1 + \lim_{K \to 0^+} \frac{1}{K}E_t^{*}Max[-S_T, -K] \\
&= 1 - \lim_{K \to 0^+} E_t^{*}Min[\frac{S_T}{K}, 1] \\
&= -\lim_{K \to 0^+} E_t^{*}Min[\frac{S_T}{K} - 1, 0] = 0
\end{aligned}$$

by virtue of the Lebesgue dominated convergence theorem since: $Min[\frac{S_T}{K}-1,0] \longrightarrow_{K\to 0^+} 0$ almost surely and $0 \leq -Min[\frac{S_T}{K}-1,0] \leq 1$. This proves that: $\lim_{x\to+\infty} \pi(-x) = 0 = \lim_{x\to+\infty}[e^x\pi(x)+1-e^x]$ and completes the proof of the required equivalence.

b) We now check that properties (i) and (ii) of Proposition 2.2 are equivalent. The general definition (2.1) of the pricing probability measure implies that:

$$\pi_t(S_t, K) = B(t,T)E_t^*[S_T\mathbf{1}_{[S_T\geq K]}] - B(t,T)KQ_t[S_T \geq K],$$

that is, after dividing by S_t :

$$\pi(x) = E_t^*[e^{V_T}\mathbf{1}_{[V_T\geq -x]}] - e^{-x}[1 - F_{V_T}(-x)]$$

By identification of this formula with condition (i), we see that (i) is equivalent to (ii).

c) Finally, we prove that conditions (i) and (iii) are equivalent. By taking the derivative of (i), we obtain:

$$\frac{\partial \pi}{\partial x}(x) = f_{V_T}(x) - e^{-x}f_{V_T}(-x) + e^{-x}[1 - F_{V_T}(-x)].$$

But, since by part a) of this proof:

$$\frac{\partial \pi}{\partial x}(x) = e^{-x}[1 - F_{V_T}(-x)]$$

we conclude that (i) implies:

$$f_{V_T}(x) = e^{-x}f_{V_T}(-x)$$

or:

$$e^{\frac{x}{2}}f_{V_T}(x) = e^{-\frac{x}{2}}f_{V_T}(-x)$$

which means that the function $x \to e^{\frac{x}{2}}f_{V_T}(x)$ is even, which is exactly condition (iii) of Proposition 2.2. Conversely, if this condition is fulfilled, we have, for any x:

$$\int_x^{+\infty} f_{V_T}(u)du = \int_x^{+\infty} e^{-u}f_{V_T}(-u)du.$$

This equation will provide property (i) of Proposition 2.2 if we complete it by the following limit condition:

$$\lim_{x\to+\infty} \pi(x) = \lim_{x\to+\infty}[F_{V_T}(x) - e^{-x}[1 - F_{V_T}(-x)]].$$

Therefore, the required equivalence will be proved is we show that this limit condition always holds. But it is clear that:

$$\lim_{x\to+\infty}[F_{V_T}(x) - e^{-x}[1 - F_{V_T}(-x)]] = \lim_{x\to+\infty} F_{V_T}(x) = 1$$

and that : $\lim_{x\to+\infty} \pi(x) = 1$, since we have already shown in part a) of this proof that: $\lim_{x\to+\infty} e^x[\pi(x) - 1] = -1$. This completes the proof.■

Appendix 2

In this Appendix, we consider the moneyness of the option as fixed and we denote by k the value of $\frac{K}{S_t}$.

A. Proof of Proposition 3.1:

From (3.1) and the decomposition of $m_{t,T}$ conformable to (A1) and (A2):

$$\frac{\boldsymbol{\pi}_t}{S_t} = E_t \left\{ E \left[(\prod_{\tau=t}^{T-1} m_{\tau+1}) \left[(\prod_{\tau=t}^{T-1} \frac{S_{\tau+1}}{S_\tau}) - k \right]^+ |I_t, U_1^T \right] \right\}$$

But, by (A1), the variables $(m_{\tau+1}, \frac{S_{\tau+1}}{S_\tau})_{\tau \geq t}$ are independent of $(m_{\tau+1}, \frac{S_{\tau+1}}{S_\tau})_{\tau < t}$ given U_1^T. Therefore:

$$\frac{\boldsymbol{\pi}_t}{S_t} = E_t \left\{ E \left[(\prod_{\tau=t}^{T-1} m_{\tau+1}) \left[(\prod_{\tau=t}^{T-1} \frac{S_{\tau+1}}{S_\tau}) - k \right]^+ |U_1^T \right] \right\}$$

is a conditional expectation computed in the conditional probability distribution of U_1^T given I_t. By (A2), this probability distribution depends on I_t only through U_1^t. We are then allowed to denote this expectation by $\Psi_{t,T}(U_1^t, k)$.

B. Proof of Proposition 3.2:

In what follows, we will derive a closed-form formula for $\Psi_{t,T}(U_1^t, k)$ based on the log-normality assumption. We will start from the following decomposition:

$$\Psi_{t,T}(U_1^t, k) = E_t \left\{ \left[G_{t,T}(U_1^T) - k H_{t,T}(U_1^T) \right] \right\}$$

where:

$$G_{t,T}(U_1^T) = E \left[(\prod_{\tau=t}^{T-1} m_{\tau+1} \frac{S_{\tau+1}}{S_\tau}) \mathbf{1}_{[\frac{S_T}{S_t} \geq k]} |U_1^T \right]$$

and:

$$H_{t,T}(U_1^T) = E_t \left[\prod_{\tau=t}^{T-1} m_{\tau+1} \mathbf{1}_{[\frac{S_T}{S_t} \geq k]} |U_1^T \right].$$

Lemma 1 : *If* $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ *is a bivariate Gaussian vector, with:*

$$E \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, Var \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \omega_1^2 & \rho\omega_1\omega_2 \\ \rho\omega_1\omega_2 & \omega_2^2 \end{pmatrix}$$

$E[\exp(Z_1)\mathbf{1}_{[Z_2 \geq 0]}] = \exp[m_1 + \frac{\omega_1^2}{2}]\Phi(\frac{m_2}{\omega_2} + \rho\omega_1)$, with Φ the cumulative normal distribution function.

Let us by Q the probability measure corresponding to the above-specified Gaussian distribution of (Z_1, Z_2) and define the probability $\widetilde{Q}$ by:

$$\frac{d\widetilde{Q}}{dQ}(Z) = \exp[(Z - m_1) - \frac{\omega_1^2}{2}].$$

Then, with obvious notation:

$E[(\exp Z_1)(\mathbf{1}_{[Z_2 \geq 0]})] \quad = \exp(m_1 + \frac{\omega_1^2}{2})\widetilde{Q}[Z_2 \geq 0]$

But by Girsanov theorem, we know that under $\widetilde{Q}$, Z_2 is a Gaussian variable with mean $m_2 + \rho\omega_1\omega_2$ and variance ω_2^2. Therefore:

$$\widetilde{Q}[Z_2 \geq 0] = 1 - \Phi[\frac{-m_2 - \rho\omega_1\omega_2}{\omega_2}] = \Phi[\frac{m_2}{\omega_2} + \rho\omega_1]$$

C. A closed-form formula for $H_{t,T}(U_1^T)$ and bond pricing:

$$H_{t,T}(U_1^T) = E_t\left[\exp[\sum_{\tau=t}^{T-1} \log m_{\tau+1}]\mathbf{1}_{[\sum_{\tau=t}^{T-1} \log \frac{S_{\tau+1}}{S_\tau} \geq \log k]}|U_1^T\right]$$

By virtue of assumption A, this expectation is given by lemma 1 with:

$$Z_1 = \sum_{\tau=t}^{T-1} \log m_{\tau+1} \text{ and } Z_2 = \sum_{\tau=t}^{T-1} \log \frac{S_{\tau+1}}{S_\tau} - \log k$$

so

$$\begin{aligned} m_1 &= \sum_{\tau=t}^{T-1} \mu_{m_{\tau+1}}, m_2 = \sum_{\tau=t}^{T-1} \mu_{S_{\tau+1}} - \log k \\ \omega_1^2 &= \sum_{\tau=t}^{T-1} \sigma^2_{m_{\tau+1}}, \omega_2^2 = \sum_{\tau=t}^{T-1} \sigma^2_{S_{\tau+1}}, \rho\omega_1\omega_2 = \sum_{\tau=t}^{T-1} \sigma_{mS_{\tau+1}}. \end{aligned}$$

Therefore:

$$H_{t,T}(U_1^T) = \exp\left[\sum_{\tau=t}^{T-1} \mu_{m_{\tau+1}} + \frac{1}{2}\sum_{\tau=t}^{T-1} \sigma^2_{m_{\tau+1}}\right]\Phi\left(\frac{1}{\sqrt{\sum_{\tau=t}^{T-1} \sigma^2_{S_{\tau+1}}}}(\sum_{\tau=t}^{T-1} \mu_{S_{\tau+1}} - \log k + \sum_{\tau=t}^{T-1} \sigma_{mS_{\tau+1}})\right)$$

By referring to the notation introduced in proposition 3.2, we first notice that $H_{t,T}(U_1^T)$ can be written as:

$$H_{t,T}(U_1^T) = \widetilde{B}(t,T)\Phi\left(d_2(x_t)\right)$$

with $x_t = \log \frac{S_t}{KB(t,T)}$ and $d_2(x_t)$ defined in proposition 3.2 since:

$$\frac{1}{\sqrt{\sum_{\tau=t}^{T-1} \sigma^2_{S_{\tau+1}}}}(\sum_{\tau=t}^{T-1} \mu_{S_{\tau+1}} - \log \frac{K}{S_t} + \sum_{\tau=t}^{T-1} \sigma_{mS_{\tau+1}}) = \frac{1}{\overline{\sigma}_{t,T}}(x_t + \log B(t,T) + \sum_{\tau=t}^{T-1} \mu_{S_{\tau+1}} + \sum_{\tau=t}^{T-1} \sigma_{mS_{\tau+1}})$$

But:

$$E_t\left[\frac{S_T}{S_t}|U_1^T\right] = \exp(\sum_{\tau=t}^{T-1}\mu_{S_{\tau+1}} + \frac{1}{2}\overline{\sigma}_{t,T}^2)$$

Therefore, the above expression can be rewritten as:

$$\frac{1}{\overline{\sigma}_{t,T}}\left(x_t - \frac{1}{2}\overline{\sigma}_{t,T}^2 + \log\left(E_t\left[\frac{S_T}{S_t}|U_1^T\right]B(t,T)]\right) + \sum_{\tau=t}^{T-1}\sigma_{mS_{\tau+1}}\right)$$

$$= \frac{x_t}{\overline{\sigma}_{t,T}} - \frac{\overline{\sigma}_{t,T}}{2} + \frac{1}{\overline{\sigma}_{t,T}}\log\left(Q_{mS}(t,T)\frac{B(t,T)}{\widetilde{B}(t,T)}\right) = d_1(x_t) - \overline{\sigma}_{t,T} = d_2(x_t)$$

where $d_1(x_t), d_2(x_t)$ and $Q_{mS}(t,T)$ correspond to the expressions given in proposition 3.2.

Finally, it is worth noticing that $\widetilde{B}(t,T)$ can be interpreted in terms of bond pricing. Actually, the general pricing formula (3.1) implies that:

$$B(t,T) = E_t[m_{t,T}] = E_t\left[H_{t,T}(U_1^T)\right]$$

when $H_{t,T}(U_1^T)$ is computed in the limit case $K = +\infty$, that is,

$$H_{t,T}(U_1^T) = \widetilde{B}(t,T), \text{ since } \lim_{K\to+\infty} d_2(x_t) = +\infty$$

therefore the bond pricing equation is given by:

$$B(t,T) = E_t[\widetilde{B}(t,T)]$$

D. A closed-form formula for $G_{t,T}(U_1^T)$ and stock pricing:

$$G_{t,T}(U_1^T) = E\left[\exp(\sum_{\tau=t}^{T-1}\log m_{\tau+1} + \log\frac{S_{\tau+1}}{S_\tau})\mathbf{1}_{[\sum_{\tau=t}^{T-1}\log\frac{S_{\tau+1}}{S_\tau}\geq\log k]}|U_1^T\right]$$

But, by virtue of assumption A, this expectation is given by lemma 1 with:

$$Z_1 = \sum_{\tau=t}^{T-1}\log m_{\tau+1} + \log\frac{S_{\tau+1}}{S_\tau} \text{ and } Z_2 = \sum_{\tau=t}^{T-1}\log\frac{S_{\tau+1}}{S_\tau} - \log k$$

In other words, with respect to part C above, m_2 and ω_2^2 are unchanged while now:

$$m_1 = \sum_{\tau=t}^{T-1}(\mu_{m_{\tau+1}}+\mu_{S_{\tau+1}}), \omega_1^2 = \sum_{\tau=t}^{T-1}(\sigma_{m_{\tau+1}}^2+\sigma_{S_{\tau+1}}^2+2\sigma_{mS_{\tau+1}}), \rho\omega_1\omega_2 = \sum_{\tau=t}^{T-1}(\sigma_{mS_{\tau+1}}+\sigma_{S_{\tau+1}}^2)$$

Therefore:

$$\begin{aligned} G_{t,T}(U_1^T) &= \exp\left[\sum_{\tau=t}^{T-1}(\mu_{m_{\tau+1}} + \mu_{S_{\tau+1}}) + \frac{1}{2}\sum_{\tau=t}^{T-1}(\sigma_{m_{\tau+1}}^2 + \sigma_{S_{\tau+1}}^2 + 2\sigma_{mS_{\tau+1}})\right] \times \\ & \Phi\left(\frac{1}{\sqrt{\sum_{\tau=t}^{T-1}\sigma_{S_{\tau+1}}^2}}\left[\sum_{\tau=t}^{T-1}\mu_{S_{\tau+1}} - \log k + \sum_{\tau=t}^{T-1}(\sigma_{mS_{\tau+1}} + \sigma_{S_{\tau+1}}^2)\right]\right) \end{aligned}$$

But comparison with the above expressions of $H_{t,T}(U_1^T)$ and $E_t\left[\frac{S_T}{S_t}|U_t^T\right]$ we see that:

$$\begin{aligned} G_{t,T}(U_1^T) &= \widetilde{B}(t,T)\exp\left[\sum_{\tau=t}^{T-1}(\mu_{m_{\tau+1}}+\mu_{S_{\tau+1}})+\frac{1}{2}\sum_{\tau=t}^{T-1}(\sigma^2_{S_{\tau+1}}+2\sigma_{mS_{\tau+1}})\right]\Phi\left(d_2(x_t)+\overline{\sigma}_{t,T}\right) \\ &= \widetilde{B}(t,T)E_t\left[\frac{S_T}{S_t}|U_t^T\right]\exp\left[\sum_{\tau=t}^{T-1}\sigma_{mS_{\tau+1}}\right]\Phi\left(d_1(x_t)\right) \end{aligned}$$

that is,

$$G_{t,T}(U_1^T) = Q_{mS}(t,T)\Phi\left(d_1(x_t)\right)$$

Finally, it is worth noticing that $Q_{mS}(t,T)$ can be interpreted in terms of stock pricing. Actually the stock pricing equation corresponds to the general pricing formula (3.1) in the limit case $K=0$, that is:

$$S_t = E_t\left[S_t G_{t,T}(U_1^T)\right]$$

where

$$G_{t,T}(U_1^T) = Q_{mS}(t,T),\ \text{since}\ \lim_{K\to 0} d_1(x_t) = +\infty$$

In other words, the stock pricing equation can be written:

$$1 = E_t[Q_{mS}(t,T)]$$

E. Option pricing formula:

We conclude from parts A, B and C above that the option pricing formula $\boldsymbol{\pi}_t$ is given by:

$$\begin{aligned} \frac{\boldsymbol{\pi}_t}{S_t} &= E_t\left[G_{t,T}(U_1^T)-\frac{K}{S_t}H_{t,T}(U_1^T)\right] \\ &= E_t\left[Q_{mS}(t,T)\Phi\left(d_1(x_t)\right)-\frac{K\widetilde{B}(t,T)}{S_t}\Phi\left(d_2(x_t)\right)\right] \end{aligned}$$

which coincides with the announced formula of proposition 3.2 since:

$$\frac{K\widetilde{B}(t,T)}{S_t} = \frac{\widetilde{B}(t,T)}{B(t,T)}\frac{KB(t,T)}{S_t} = \frac{\widetilde{B}(t,T)}{B(t,T)}\exp(-x_t)$$

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Examples of implied volatility curves inferred from S&P 500 call option Prices

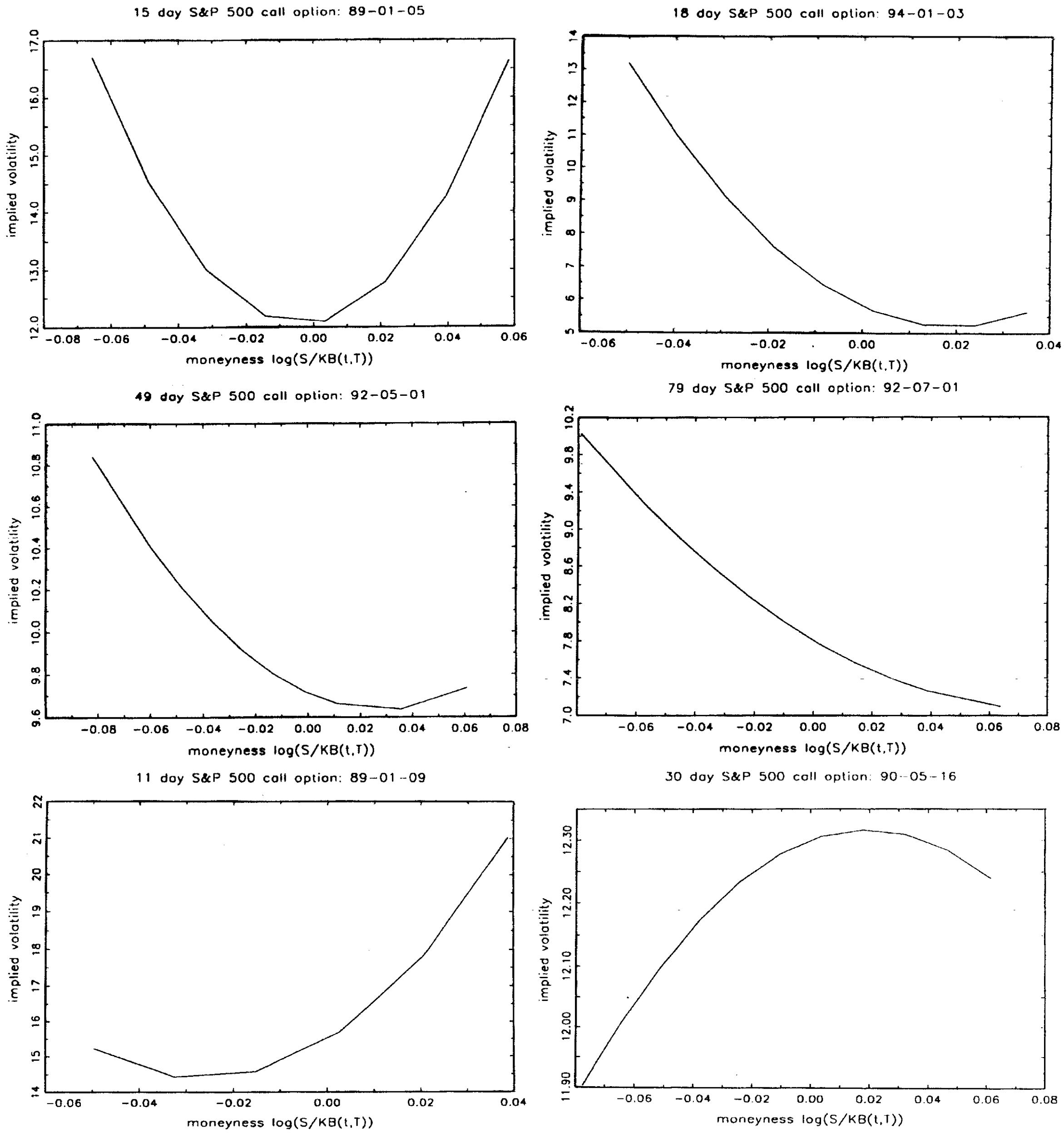

Figure 1

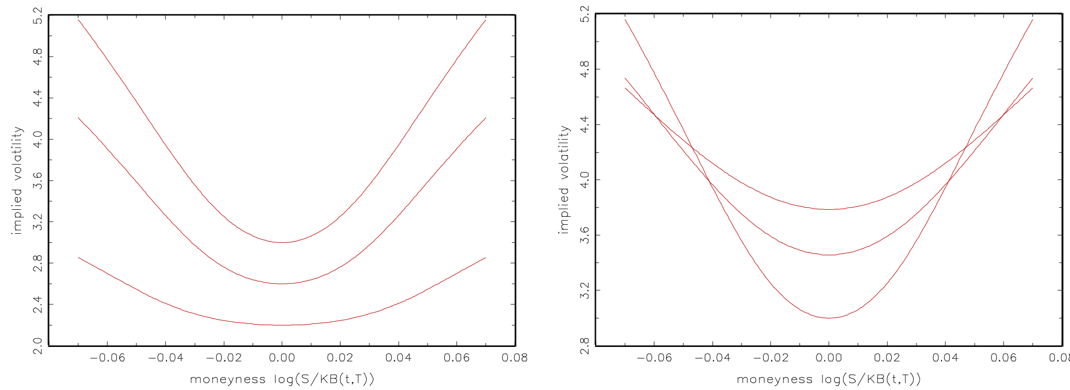

Figure 2: Maturity and stochastic volatility effects when $B(t,T)$ is deterministic and $Q_{XY}(t,T) = 1$. Left panel: The effect of a decrease in the coefficient of variation of the volatility results in flatter smiles. Right panel: As the option's maturity increases, the smile flattens. The most curved smiles in each panel are in fact identical.

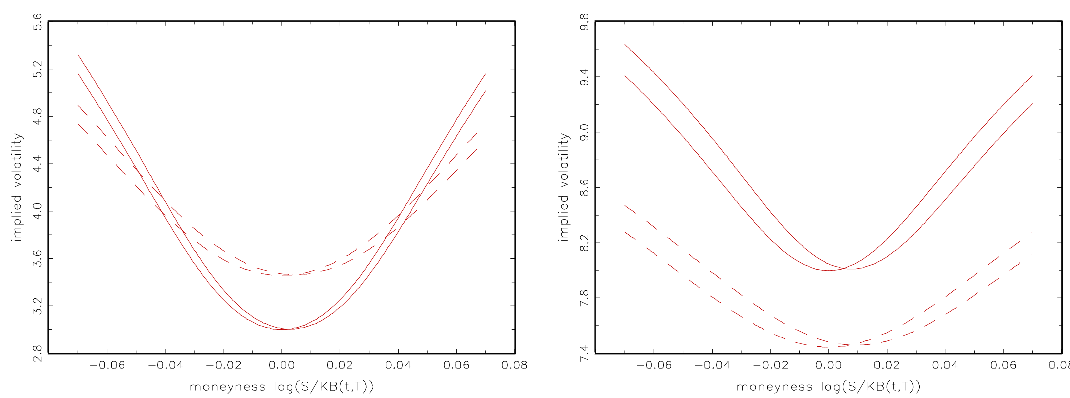

Figure 3: Maturity effects when $B(t,T)$ is deterministic and $Q_{XY}(t,T) \neq 1$. The solid lines are the implied volatility curves for a one-period option whereas the dashed lines are those for an option with a two-period maturity. In each case the symmetric smiles are those resulting from preference-free option pricing à la Hull-White. The preference-free smile is easily recognizable as the one centered on zero. The left and right panels are associated with states 1 and 2, respectively, as the current state.

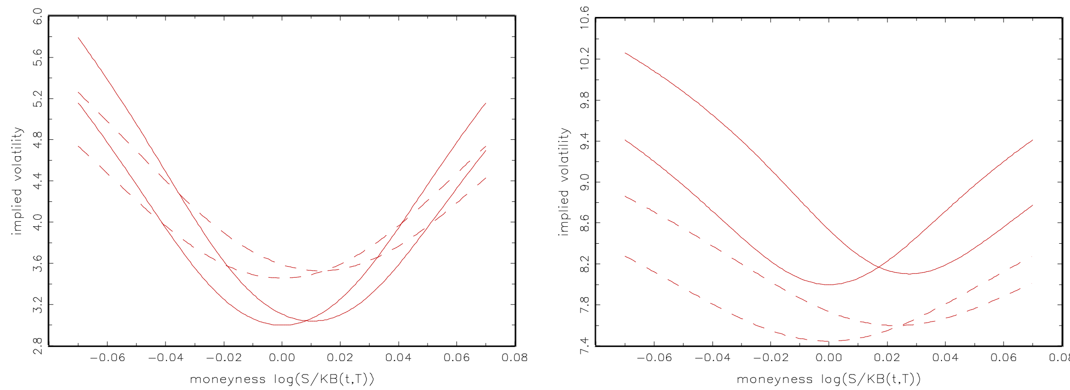

Figure 4: Maturity effects when $B(t,T)$ is stochastic and $Q_{XY}(t,T) \neq 1$. The solid lines are the implied volatility curves for a one-period option whereas the dashed lines are those for an option with a two-period maturity. In each case the symmetric smiles are those resulting from generalized option pricing à la Hull-White. The left and right panels are associated with states 1 and 2, respectively, as the current state.

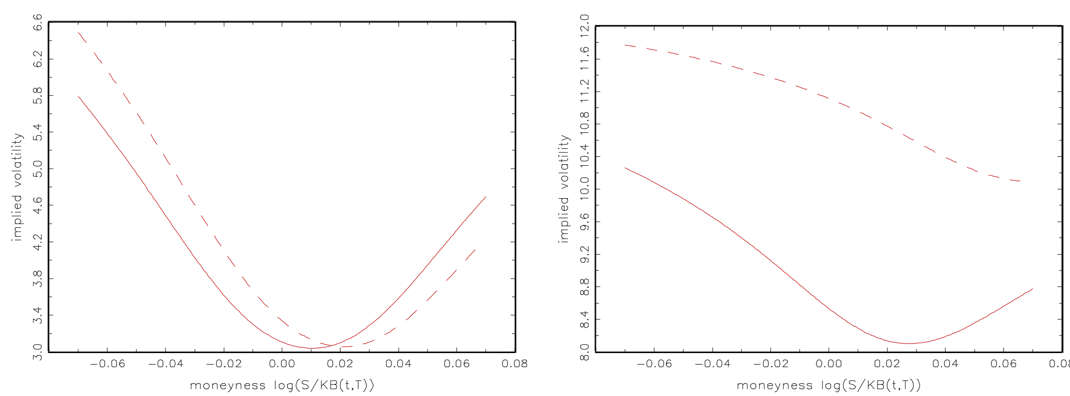

Figure 5: Average duration effects when $p_{ii} > p_{ij}$. The dashed lines represent an increase in p_{ii}. The left and right panels are associated with states 1 and 2, respectively, as the current state.

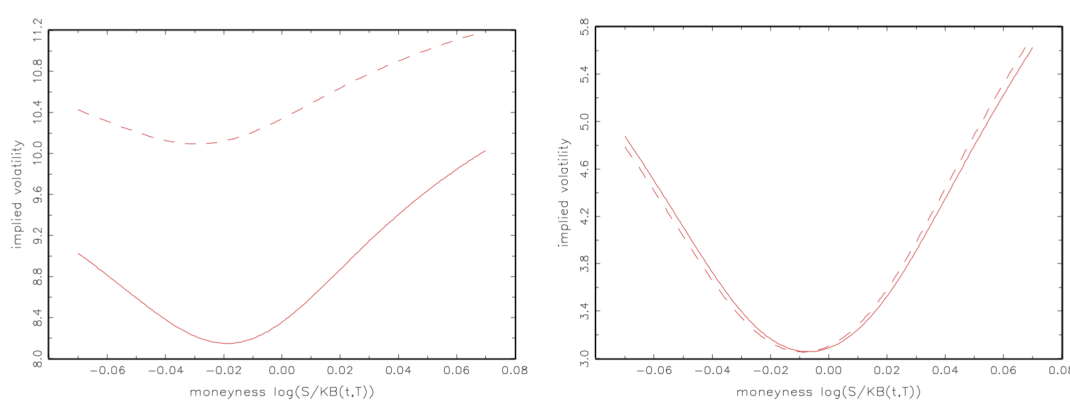

Figure 6: Average duration effects when $p_{ii} < p_{ij}$. The dashed lines represent an increase in p_{ij}. The left and right panels are associated with states 1 and 2, respectively, as the current state.

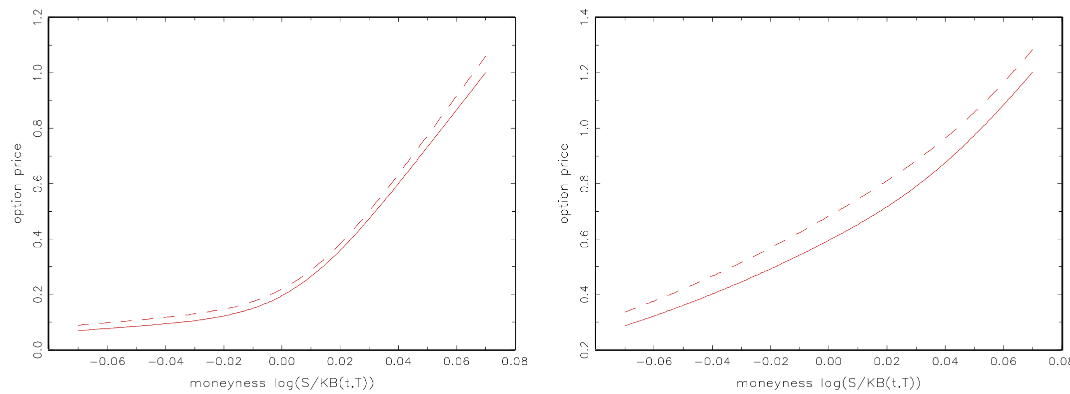

Figure 7: Option prices and the correlation between consumption and dividends. The solid lines represent the schedule of option prices across moneyness when the correlation between consumption and dividends is one. The other extreme of no correlation is represented by the dashed lines. The left and right panels are associated with states 1 and 2, respectively, as the current state.

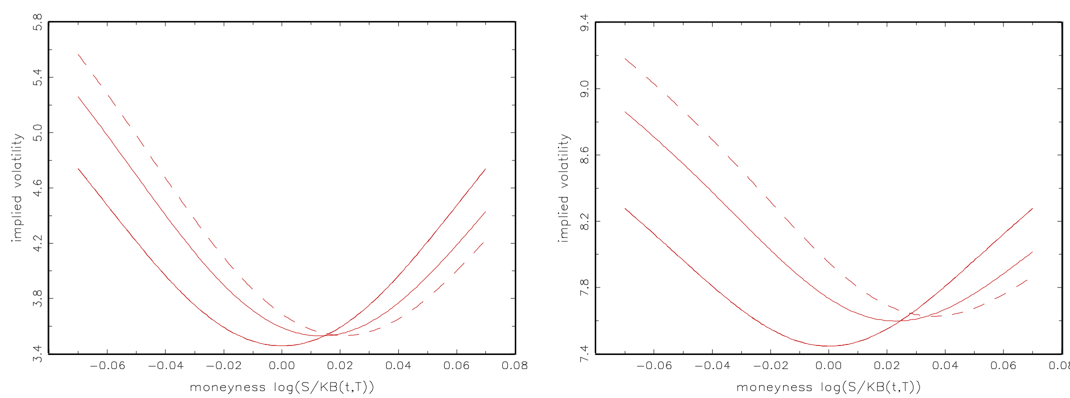

Figure 8: Leverage effects and the volatility smile. The dashed lines show the effect on the volatility smile of an absence of a leverage effect through the consumption process. The solid lines are the benchmark case of leverage effects through both consumption and dividend processes. The left and right panels are associated with states 1 and 2, respectively, as the current state.

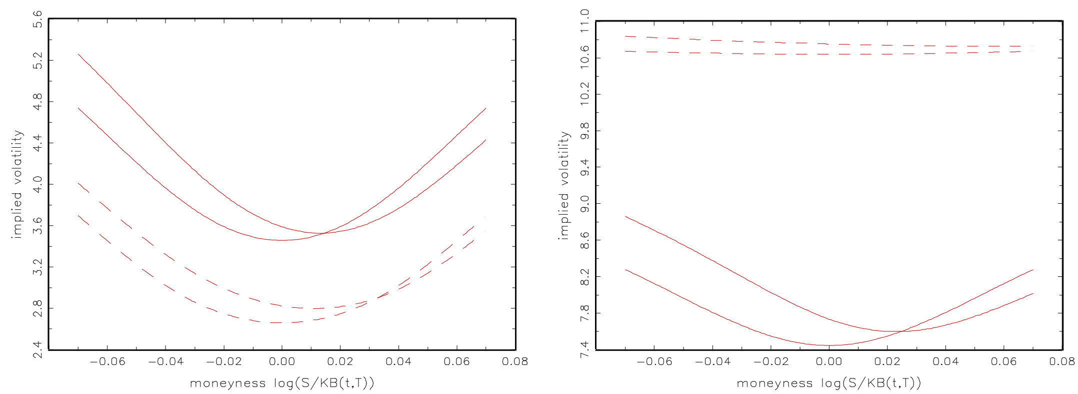

Figure 9: Leverage effects and the volatility smile. The dashed lines show the effect on the volatility smile of an absence of a leverage effect through the dividend process. The solid lines are the benchmark case of leverage effects through both consumption and dividend processes. The left and right panels are associated with states 1 and 2, respectively, as the current state.

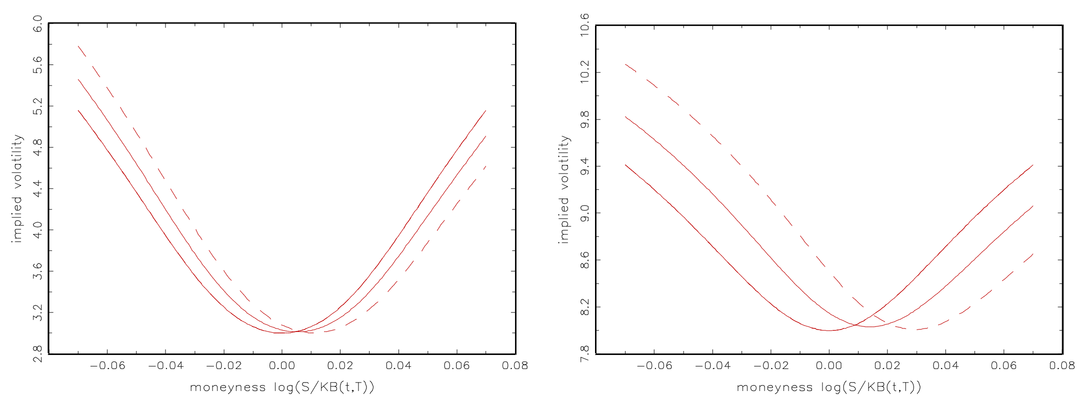

Figure 10: The role of preferences. The solid lines are the volatility smiles for the following configuration of preference parameters: $\gamma = 1$, $\rho = -1$, $\alpha = -1$; whereas the dashed lines are for $\gamma = 1/10$, $\rho = -10$, $\alpha = -1$. The left and right panels are associated with states 1 and 2, respectively, as the current state.

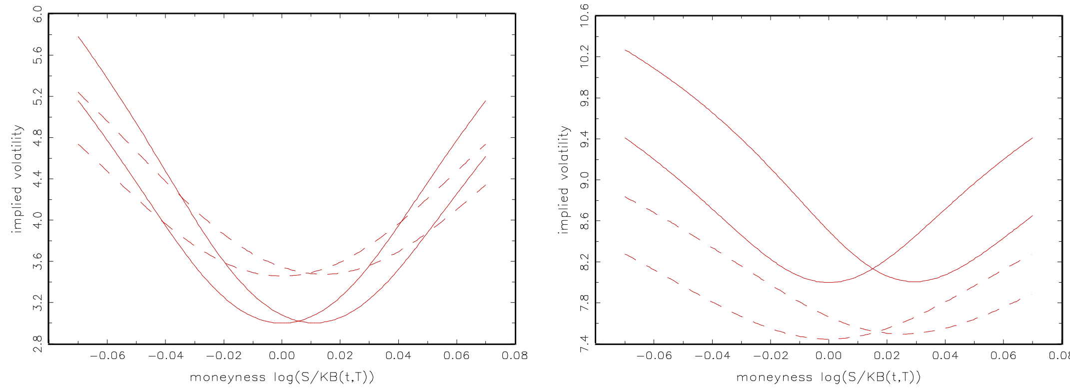

Figure 11: The role of preferences. The solid lines are the volatility smiles for the following configuration of preference parameters: $\gamma = 1/10$, $\rho = -10$, $\alpha = -1$. The dashed lines show the volatility smiles for the same preference parameter configuration but for an option whose maturity is increased by one period. The left and right panels are associated with states 1 and 2, respectively, as the current state.

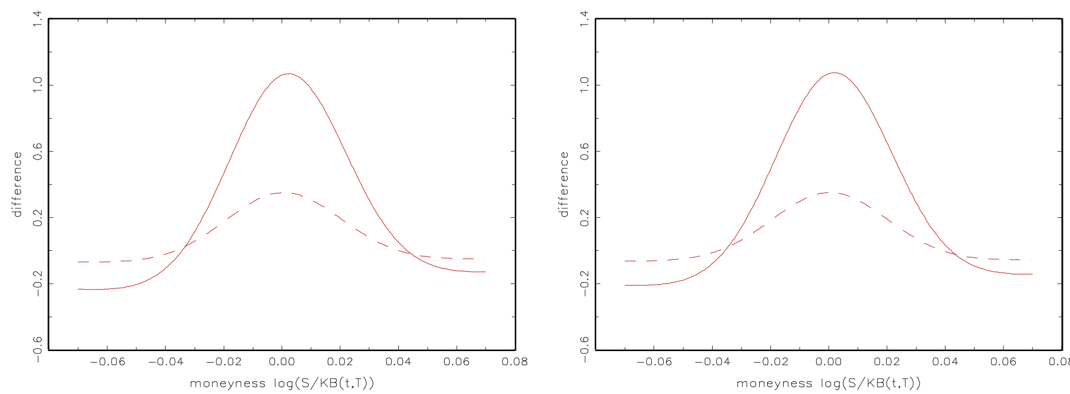

Figure 12: Intertemporal substitution and option pricing. The left panel plots $E_t\left\{S_t(\Phi(d_1^{HW}) - Q_{XY}(t,T)\Phi(d_1^{GBS}))\right\}$; that is, the difference between the first parts of the preference-free and the generalized option pricing formula in which preferences matter. The difference between the second parts of these formulas, $E_t\left\{K(B(t,T)\Phi(d_2^{HW}) - \widetilde{B}(t,T)\Phi(d_2^{GBS}))\right\}$, is plotted in the right panel. The solid lines represent the case where $B(t,T)$ is stochastic whereas the dashed lines represent the case for $B(t,T)$ deterministic.

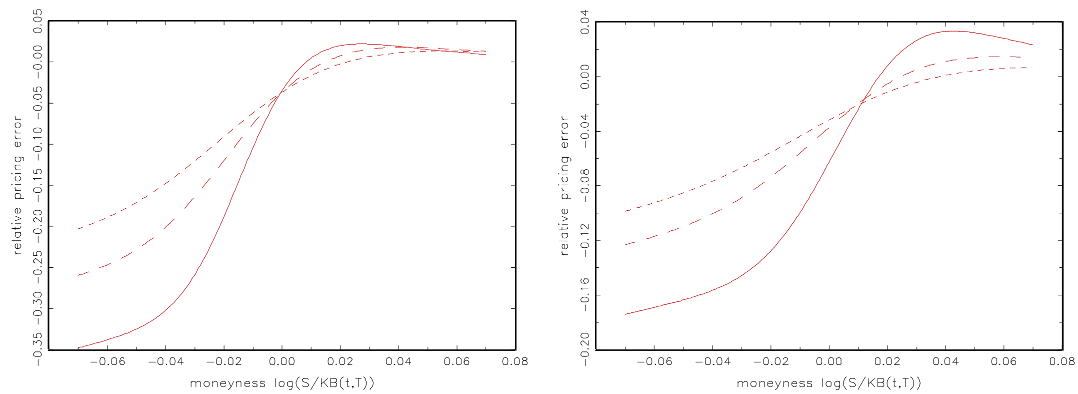

Figure 13: Relative pricing errors. The lines plot the relative difference across moneyness between preference-free option prices and option prices in which preferences matter; that is, the graphs report the relative pricing error that is committed by using a preference-free pricing formula in a world in which preferences matter for option prices. The three lines are for options whose maturity increases by one period with the steepest representing the benchmark one-period option. The left and right panels are associated with states 1 and 2, respectively, as the current state.