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Mahler measure and its generalizations

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Abstract

The (logarithmic) Mahler measure of a non-zero rational function P in n variables is defined as the arithmetic mean of $\log|P|$ restricted to the standard n-torus ($\mathbb{T}^n = \{(x_1, \ldots, x_n) \in (\mathbb{C}^\times)^n : |x_i| = 1, \forall\ 1 \leq i \leq n\}$) with respect to the unique Haar measure (normalized arc measure) on $\mathbb{T}^n$. It has connections to heights, hyperbolic volumes, arithmetic dynamics, and special values of L-functions. Various generalizations of this definition exist in the literature. This thesis is dedicated to exploring two such generalizations: firstly, when the unit torus is substituted by a torus with arbitrary radii $\mathbb{T}^n_{a_1,\ldots,a_n} = \{(x_1, \ldots, x_n) \in (\mathbb{C}^\times)^n : |x_i| = a_i, \forall\ 1 \leq i \leq n\}$ (referred to as the *generalized Mahler measure*), and secondly, when the normalized arc measure on the unit torus is replaced by the normalized area measure on the unit disk (referred to as the *areal Mahler measure*). Our primary objective is to quantify the behavior of the Mahler measure of P under such alterations. This thesis is structured into five projects.

(1) In Chapter 1, we investigate the definition of the generalized Mahler measure for all Laurent polynomials in n-variables when they do not vanish on the integration torus. This work has been published in [**106**].

(2) In Chapter 2, we exhibit some nontrivial evaluations of the areal Mahler measure of multivariable polynomials, defined by Pritsker. This is a joint work with Lalín, and has been published in [**84**].

(3) In Chapter 3, we investigate how the areal Mahler measure changes with the power change of variables. This a joint work with Lalín, and has been published in [**83**].

(4) In Chapter 4, we investigate the Mahler measure of a particular family of rational functions with an arbitrary number of variables and an arbitrary degree in one of the variables. This is a joint work with Lalín and Nair, and will appear in [**81**].

(5) In Chapter 5, we evaluate the areal Mahler measure of a family of polynomials using the areal analogue of the Zeta Mahler measure. This is an ongoing joint work with Lalín, Nair, and Ringeling.

Keywords : Mahler measure; elliptic curve; special values of L-functions; polylogarithm, arbitrary torus, regulator.

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Résumé

La mesure de Mahler (logarithmique) de P, une fonction rationnelle non nulle à n variables, est définie comme la moyenne arithmétique de $\log|P|$ restreinte au tore n-dimensionnel standard ($\mathbb{T}^n = \{(x_1, \dots, x_n) \in (\mathbb{C}^\times)^n : |x_i| = 1, \forall\, 1 \leq i \leq n\}$) par rapport à la mesure de Haar unique (mesure d'arc normalisée) sur $\mathbb{T}^n$. Elle a des liens avec les hauteurs, les volumes hyperboliques, la dynamique arithmétique et les valeurs spéciales des fonctions L. Il existe plusieurs généralisations de cette définition dans la littérature. Cette thèse se consacre à l'exploration de deux de ces généralisations : premièrement, lorsque le tore unité est remplacé par un tore à rayons arbitraires $\mathbb{T}^n_{a_1,\dots,a_n} = \{(x_1, \dots, x_n) \in (\mathbb{C}^\times)^n : |x_i| = a_i, \forall\, 1 \leq i \leq n\}$ (appelée *mesure de Mahler généralisée*), et deuxièmement, lorsque la mesure d'arc normalisée sur le tore unité est remplacée par la mesure d'aire normalisée sur le disque unité (appelée *mesure de Mahler aréale*). Notre objectif principal est de quantifier le comportement de la mesure de Mahler de P sous de telles modifications. Cette thèse est structurée en cinq projets.

(1) Dans le chapitre 1, nous étudions la définition de la mesure de Mahler généralisée pour tous les polynômes de Laurent à n variables lorsqu'ils ne s'annulent pas sur le tore d'intégration. Ce travail est publié dans [**106**].

(2) Le chapitre 2 présente des évaluations non triviales de la mesure de Mahler aréale des polynômes à plusieurs variables, définie par Pritsker. Ce travail est réalisé en collaboration avec Lalín, et publié dans [**84**].

(3) Dans le chapitre 3, nous étudions comment la mesure de Mahler aréale change lorsque l'on effectue un changement de variables par puissance sur les polynômes. Ceci est un travail conjoint avec Lalín, et publié dans [**83**].

(4) Dans le chapitre 4, nous étudions la mesure de Mahler d'une famille particulière de fonctions rationnelles à un nombre arbitraire de variables et à un degré arbitraire dans l'une des variables. Ce travail est réalisé en collaboration avec Lalín et Nair, et sera publié dans [**81**].

(5) Le chapitre 5 est consacré à l'évaluation de la mesure de Mahler aréale d'une famille de polynômes en utilisant l'analogue aréal de la mesure de Mahler zêta. Il s'agit d'un travail collaboratif en cours avec Lalín, Nair et Ringeling.

Mots clés : Mesure de Mahler ; courbe elliptique ; valeurs spéciales des fonctions L ; polylogarithme ; tore arbitraire ; régulateur.

To ma and baba.

To chotomama, in loving memory.

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Introduction

The Mahler measure is a positive real number $M(f)$ that can be associated to a large class of complex-valued functions $f : \mathbb{T}^n \to \mathbb{C}$ defined on the torus $\mathbb{T}^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times : |x_1| = \cdots = |x_n| = 1\}$.[1] The class of non-zero rational functions $P \in \mathbb{C}(x_1, \ldots, x_n)$ is contained in such class of functions. This thesis focuses mainly on certain generalizations of this notion. In this chapter, we provide a broad overview of the theory of Mahler measure and state our main results.

0.1. Mahler measure: a historical introduction

The story starts with Tracy A. Pierce, who developed in **[98]** a method for searching large primes using a generalization of Mersenne's sequence $2^m - 1$. Given a monic polynomial $P \in \mathbb{Z}[x]$, define

$$\Delta_m(P) = \prod_{\alpha \in \mathbb{C}, P(\alpha)=0} (\alpha^m - 1),$$

where $m \in \mathbb{Z}_{\geq 1}$[2]. The integer $\Delta_m(P)$ shares a similar property with Mersenne numbers: it can be factored more easily than a randomly chosen integer. Moreover, one has that

$$\text{if } k = nm, \text{ then } \frac{\Delta_k}{\Delta_m} = \prod_{\alpha \in \mathbb{C}, P(\alpha)=0} \left(\sum_{\ell=0}^{n-1} \alpha^{m\ell} \right) \text{ is an integer.} \tag{0.1.1}$$

Derrick H. Lehmer **[87]** continued the search for large prime numbers with Pierce's method, applying property (0.1.1) to find new large prime numbers of the form $\frac{\Delta_p}{\Delta_1}$, for

1. For a field $\mathbb{S}$, we denote $\mathbb{S}^\times$ as $\mathbb{S} \setminus \{0\}$.
2. Notice that $\Delta_m(x-2) = 2^m - 1$

small primes $p \in \mathbb{N}$. In [**87**], Lehmer further introduced the *Mahler measure for one-variable monic polynomials* $P \in \mathbb{Z}[x]$ as

$$M(P) = \prod_{\alpha \in \mathbb{C}, P(\alpha)=0} \max\{|\alpha|, 1\}, \tag{0.1.2}$$

which, he showed, computes the growth of the sequence of integers $\{\Delta_m(P) : m \geq 1\}$. In particular, he observed that the prime counting function $\left\{p \leq x : \frac{\Delta_p}{\Delta_1} \text{ is prime}\right\}$ seems to grow faster as soon as the Mahler measure $M(P)$ is smaller, when $P(x) \neq x^{\deg P} P(x^{-1})$. Polynomials satisfying $P(x) = x^{\deg P} P(x^{-1})$ are known as *self-reciprocal polynomials*. When P is self-reciprocal, an interesting choice for a prime counting function is $\left\{p \leq x : \sqrt{\frac{\Delta_p}{\Delta_1}} \text{ is prime}\right\}$, since, for $m|k$, the integer $\left|\frac{\Delta_k}{\Delta_m}\right|$ is a square when $k \equiv m (\text{mod } 2)$ (see [**38**, Exercise 1.7]).

Notice that $M(P) \geq 1$ for every non-zero monic polynomial $P \in \mathbb{Z}[x]$. In fact, for a polynomial $P(x) = a_0 \prod_{\alpha \in \mathbb{C}, P(\alpha)=0} (x - \alpha) \in \mathbb{C}[x]$, the definition of the Mahler measure $M(P)$ can be extended to

$$M(P) = |a_0| \prod_{\alpha \in \mathbb{C}, P(\alpha)=0} \max\{|\alpha|, 1\},$$

and, since $|a_0| \geq 1$ when $P \in \mathbb{Z}[x] \setminus \{0\}$, we have $M(P) \geq 1$.

Consider the set $\mathcal{M} = \{M(P) : P \in \mathbb{Z}[x] \setminus \{0\}\}$. It follows from the definition that $M(P)$ is an algebraic integer, and therefore $\mathcal{M} \subseteq \bar{\mathbb{Z}}$. The above discussion also implies that the minimum of the set $\mathcal{M}$ is $\min \mathcal{M} = 1$. In addition, the following theorem due to Kronecker provides necessary and sufficient conditions for a non-zero polynomial $P \in \mathbb{Z}[x]$ such that $M(P) = 1$.

Theorem 0.1.1 (Kronecker, [**71**])**.** *Let α be an algebraic integer. If all the conjugates of α are inside the unit disc, then α is a root of unity.*

Indeed, let $f_\alpha \in \mathbb{Z}[x]$ be the minimal polynomial of α of degree d, and let $\{\alpha_1, \ldots, \alpha_d\}$ be the set of conjugates of α. Since all the conjugates of α are inside the unit disc, we have $|\alpha_j^n| \leq |\alpha_j| \leq 1$ for every j and n. As the coefficients of $f_{n,\alpha}$ are symmetric functions of α_j^n, they are bounded, and therefore, the set $\{f_{n,\alpha}(x) = \prod_{j=1}^d \left(x - \alpha_j^n\right) : n \geq 1\}$ is finite, i.e. $f_{k,\alpha} = f_{\ell,\alpha}$ for some $k \neq \ell$. From this we can infer that there exist i and j such that $i \neq j$

and $\alpha_i^k = \alpha_j^\ell$. If K is the splitting field of α over $\mathbb{Q}$, and $\sigma : K \hookrightarrow \mathbb{C}$ is the embedding such that $\sigma(\alpha_i) = \alpha_j$, then there exists a $t \geq 1$ such that

$$\sigma^t(\alpha_i) = \underbrace{\sigma \circ \sigma \circ \cdots \circ \sigma}_{t \text{ times}}(\alpha_i) = \alpha_i,$$

and therefore

$$\alpha_i^{k^t} = \left(\alpha_j^\ell\right)^{k^{t-1}} = \left(\sigma\left(\alpha_i^\ell\right)\right)^{k^{t-1}} = \left(\sigma\left(\alpha_i^k\right)\right)^{\ell k^{t-2}} = \cdots = \left(\sigma^t(\alpha_i)\right)^{\ell^t} = \alpha_i^{\ell^t},$$

which implies that α_i, as well as all conjugates of α, are roots of unity, which proves Theorem 0.1.1.

Moreover, from (0.1.2), we have $M(f_\alpha) = 1$, since f_α is a *cyclotomic polynomial* (all the roots of f_α are roots of unity, and f_α is irreducible over $\mathbb{Q}$).

On the other hand, if, for some non-zero monic polynomial $P \in \mathbb{Z}[x]$, $M(P) = 1$, then (0.1.2) implies that all the roots of P have absolute values less than or equal to 1. Dividing $P(x)$ by suitable power of x, we can further assume that $P(0) \neq 0$. Then, by Kronecker's Theorem 0.1.1, we have $P(x) = \prod_a \phi_a(x)$, where the ϕ_a are cyclotomic polynomials. In conclusion, we have, for $P \in \mathbb{Z}[x] \setminus \{0\}$,

$$M(P) = 1 \Longleftrightarrow P(x) = x^n \prod_a \phi_a(x). \tag{0.1.3}$$

Given this, Lehmer asked whether the set $\mathcal{M}$ has an isolated minimum at 1. More precisely,

Given $\epsilon > 0$, is there a $P \in \mathbb{Z}[x]$ for which $1 < M(P) < 1 + \epsilon$?

One expects the answer to this question to be negative in the sense that there exists a $\epsilon_0 > 0$, such that $M(P) \geq 1 + \epsilon_0$ for all non-zero $P \in \mathbb{Z}[x]$ which is not cyclotomic. The best guess for $1 + \epsilon_0$ until now is due to Lehmer and Poulet:

$$M(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) = 1.17628081\ldots.$$

Lehmer's question for polynomials with bounded degrees has been extensively studied (see **[20, 114, 118]**). Dobrowski **[49]** showed that if P is not a cyclotomic polynomial and $\deg P = d$, then

$$M(P) \geq 1 + c\left(\frac{\log\log d}{\log d}\right)^3,$$

where c is an explicit constant, which answers Lehmer's question for fixed degree polynomials. Dimitrov improved Dobrowski's bound in his recent proof of the Schinzel–Zassenhauss conjecture [**47**]. In particular, he showed that, for every non-zero monic irreducible polynomial $Q \in \mathbb{Z}[x]$,

$$\max\{|\alpha| : \alpha \in \mathbb{C}, Q(\alpha) = 0\} \geq 2^{\frac{1}{4 \deg Q}},$$

and therefore, $M(Q)$ cannot be too small. However, after extensive study, Lehmer's question remains open.

One of the most interesting attempts towards the resolution of Lehmer's question has been given by Boyd [**27**]. He observed that $\mathcal{M}$ is a countable set of algebraic numbers in the interval $[1, \infty)$, and it is a semigroup under multiplication since $M(PQ) = M(P)M(Q)$. Boyd argued that if one can show that $\mathcal{M}$ is closed, then $\mathcal{M}$ is nowhere dense, and this will answer Lehmer's question in the negative. He further considered an extension of the Mahler measure for n-variable polynomials, and showed that it is improbable that $\mathcal{M}$ is closed and that the closure of $\mathcal{M}$ contains a larger set comprised of algebraic numbers and certain transcendental numbers, namely the set of "Mahler measures of n-variable polynomials".

In order to define such an extension, we first recall Jensen's formula, which states that

$$\frac{1}{2\pi i} \int_{\mathbb{T}^1} \log |Q(x)| \frac{dx}{x} = \log |a_0| + \sum_{j=1}^{d} \log^+ |\alpha_j| = \log M(Q), \tag{0.1.4}$$

where $Q(x) = a_0 \prod_{j=1}^{d} (x - \alpha_j)$ and $\log^+ |\alpha| = \log \max\{|\alpha|, 1\}$. For a non-zero polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$, then the Mahler measure can be defined as

$$M(P) = \exp\left(\frac{1}{(2\pi i)^n} \int_{\mathbb{T}^1} \cdots \int_{\mathbb{T}^1} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \right).$$

In fact, this definition can be extended to a non-zero rational function in $\mathbb{C}(x_1, \ldots, x_n)$ using the multiplicative property of $M(P)$ mentioned above. In this thesis, we are mainly interested in the logarithmic version of $M(P)$. This leads to the following definition of the (logarithmic) Mahler measure of n-variable rational functions, which was first introduced by Mahler [**89**] for n-variable polynomials in his proof of Gelfond's inequality.

Definition 0.1.2. *For a non-zero rational function $P \in \mathbb{C}(x_1, \ldots, x_n)$, the* ***(logarithmic) Mahler measure*** *of P is defined as*

$$\mathrm{m}(P) = \mathrm{m}(P(x_1, \ldots, x_n)) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}, \tag{0.1.5}$$
$$= \int_{[0,1]^n} \log \left|P\left(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}\right)\right| d\theta_1 \cdots d\theta_n,$$

where $\mathbb{T}^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times : |x_1| = \cdots = |x_n| = 1\} = \mathbb{T}^1 \times \cdots \times \mathbb{T}^1$. Then $M(P) = \exp(\mathrm{m}(P))$.

In other words, for $P \in \mathbb{C}(x_1, \ldots, x_n) \setminus \{0\}$, $\mathrm{m}(P)$ is the arithmetic mean of $\log|P|$ over the n-dimensional unit torus $\mathbb{T}^n$ with respect to the unique normalized Haar measure associated to the torus.

Denote, for all $n \geq 1$,

$$\mathcal{M}_n := \{M(P) : P \in \mathbb{Z}[x_1, \ldots, x_n]\}.$$

Then, note that $\mathcal{M}_1 = \mathcal{M}$, $\mathcal{M}_n$ is countable, and $\mathcal{M}_1 \subseteq \mathcal{M}_n \subseteq \mathcal{M}_{n+1}$, for all $n \geq 1$. The following result, due to Boyd and Lawton, shows that $\mathcal{M}_n$ is a subset of the set of limit points of $\mathcal{M}_1$ (we denote the set of limit point as $\mathcal{M}_1^{(1)}$) and hence a subset of the closure of $\mathcal{M}_1$.

Theorem 0.1.3 (Boyd [**27**], Lawton [**86**])**.** *Let $R \in \mathbb{C}(x_1, \ldots, x_n)$ be a non-zero rational function defined by*

$$R(\boldsymbol{x}) = \frac{P(\boldsymbol{x})}{Q(\boldsymbol{x})} = \frac{\sum_{\boldsymbol{j}} a_{\boldsymbol{j}} \boldsymbol{x}^{\boldsymbol{j}}}{\sum_{\boldsymbol{k}} b_{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{k}}},$$

where $P, Q \in \mathbb{C}[x_1, \ldots, x_n]$ are non-zero polynomials, $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{j} = (j_1, \ldots, j_n)$, $\boldsymbol{k} = (k_1, \ldots, k_n)$, $\boldsymbol{x}^{\boldsymbol{j}} = x_1^{j_1} \cdots x_n^{j_n}$, $\boldsymbol{x}^{\boldsymbol{k}} = x_1^{k_1} \cdots x_n^{k_n}$, and the sums are finite. Given a vector $\boldsymbol{r} = (r_1, \ldots, r_n) \in \mathbb{Z}^n \setminus \{0\}$, define

$$\mu(\boldsymbol{r}) := \inf \left\{ \max_{1 \leq j \leq n} |m_j| : \boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n \setminus \{0\}, \sum_{j=1}^n m_j r_j = 0 \right\}.$$

Then

$$\lim_{\mu(\boldsymbol{r}) \to \infty} \mathrm{m}\left(R\left(x^{r_1}, \ldots, x^{r_n}\right)\right) = \mathrm{m}\left(R(x_1, \ldots, x_n)\right). \tag{0.1.6}$$

Remark 0.1.4. *In their recent work* [**48**]*, Dimitrov and Habegger explicitly computed the rate of convergence of* (0.1.6)*, and showed that, given a vector* $\boldsymbol{r} = (r_1, \ldots, r_n) \in \mathbb{Z}^n \setminus \{0\}$ *such that* $\mu(\boldsymbol{r}) > \deg P$,

$$\mathrm{m}\left(P\left(x^{r_1}, \ldots, x^{r_n}\right)\right) - \mathrm{m}(P) \leq C(n,k) \frac{(\deg P)^{16n^2}}{\mu(\boldsymbol{r})^{\frac{1}{16(k-1)}}},$$

where $k \in \mathbb{Z}_{\geq 2}$ *such that* P *has at most* k *non-zero terms, and* $C(n,k)$ *is a positive real constant depending on* n *and* k*.*

Boyd then conjectured that $\bigcup_{n\geq 1} \mathcal{M}_n$ is closed. Since each $\mathcal{M}_n$ is countable, this would simply imply that $\bigcup_{n\geq 1} \mathcal{M}_n$, as well as $\mathcal{M}_1 = \mathcal{M}$, are nowhere dense in $[1, \infty)$, and therefore, it would answer Lehmer's question in the negative.

The theorem of Boyd-Lawton, along with Boyd's conjecture that $\bigcup_{n\geq 1} \mathcal{M}_n$ is closed, helped initiate a thorough systematic study of (logarithmic) Mahler measures of polynomials in several variables. Since this thesis focuses on the (logarithmic) Mahler measure, we will refer to $\mathrm{m}(P)$ as the Mahler measure from this point onward, instead of $M(P)$.

The following theorem, which is due independently to Boyd, Lawton, and Smyth, shows that we can completely characterize the set of polynomials with integer coefficients that achieve the minimal Mahler measure, which is 0. [3]

Theorem 0.1.5 (Boyd [**27**], Lawton [**86**], Smyth [**112**])**.** *For any non-zero Laurent polynomial* $P \in \mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}]$ *such that the greatest common divisor of the coefficients is* 1, $\mathrm{m}(P)$ *is zero if and only if* P *is a product of a monomial and some cylclotomic polynomials evaluated on monomials, i.e.*

$$P(x_1, \ldots . x_n) = x_1^{d_1} \cdots x_n^{d_n} \prod_k \left(x_1^{b_{1,k}} \cdots x_n^{b_{n,k}} \phi_k(x_1^{c_1} \cdots x_n^{c_n}) \right),$$

where $\phi_k(t) \in \mathbb{Z}[t]$ *are cyclotomic polynomials,* $b_{i,k}, c_i, d_i \in \mathbb{Z}$ *for all* i*,* b_i *are chosen minimally such that* $x_1^{b_{1,k}} \cdots x_n^{b_{n,k}} \phi_k(x_1^{c_1} \cdots x_n^{c_n})$ *is a polynomial in* $x_1, \ldots, x_n$*.*

3. Indeed $\log \mathcal{M}_n \subseteq \log \mathcal{M}_1^{(1)} \subseteq [0, \infty)$, and if $P(x_1, \ldots, x_n) = x_1$, then $\mathrm{m}(P) = 0$, i.e. $0 \in \log \mathcal{M}_n$. Here, for $A \subset \mathbb{R}_{>0}$, $\log A := \{\log a : a \in A\}$.

Note that this is a generalization of the one-variable case mentioned in the discussion following Theorem 0.1.1, and, as a result of this and Theorem 0.1.3, a Lehmer-type question in the multivariable case can be reduced to the Lehmer's question in the one-variable case. The following proposition further classifies the changes of variables that keep the Mahler measures of multivariable polynomials invariant.

Theorem 0.1.6 ([**51**, pg. 52])**.** *Let*

$$P(\boldsymbol{x}) = \sum_{\boldsymbol{j}} a_{\boldsymbol{j}} \boldsymbol{x}^{\boldsymbol{j}} \in \mathbb{C}[x_1, \ldots, x_n],$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{j} = (j_1, \ldots, j_n)$, *and* $\boldsymbol{x}^{\boldsymbol{j}} = x_1^{j_1} \cdots x_n^{j_n}$. *Let A be an $n \times n$ integer matrix with non-zero determinant, and define* $P^{(A)}(\boldsymbol{x}) = \sum_{\boldsymbol{j}} a_{\boldsymbol{j}} \boldsymbol{x}^{A\boldsymbol{j}}$.[4] *Then*

$$\mathrm{m}(P(\boldsymbol{x})) = \mathrm{m}(P^{(A)}(\boldsymbol{x})).$$

While the above transformation has been described for polynomials, it is straightforward to generalize it to rational functions. The next step in this search is to evaluate Mahler measures in this multivariable framework.

The computation of the Mahler measure of multivariable polynomials is complicated since we do not have the analogue of Jensen's formula. The pioneering work by Smyth [**112, 27**] showed that some values of $\mathrm{m}(P)$ could be related to special values of L-functions. More precisely, Smyth proved that

$$\begin{aligned} \mathrm{m}(x+y+1) =& L'(\chi_{-3}, -1) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2), && (0.1.7)\\ \mathrm{m}(1+x+y+z) =& \frac{7}{2\pi^2}\zeta(3), && (0.1.8) \end{aligned}$$

where

$$L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \quad \text{with} \quad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \pmod 3, \\ -1 & n \equiv -1 \pmod 3, \\ 0 & n \equiv 0 \pmod 3, \end{cases} \qquad (0.1.9)$$

is a Dirichlet L-function, and ζ is the Riemann ζ-function.

4. For $A = (b_{k\ell})_{1 \le k, \ell \le n}$, $P^{(A)}(x_1, \ldots, x_n) = P\left(x_1^{b_{11}} x_2^{b_{12}} \cdots x_n^{b_{1n}}, \ldots, x_1^{b_{k1}} x_2^{b_{k2}} \cdots x_n^{b_{kn}}, \ldots, x_1^{b_{n1}} x_2^{b_{n2}} \cdots x_n^{b_{nn}}\right)$.

Deninger [**45**] established a link between Mahler measure and the conjectures of Beĭlinson and Bloch–Kato. He proved that the Mahler measure of a suitable class of Laurent polynomials $P \in \bar{\mathbb{Q}}[x_1^{\pm}, \ldots, x_n^{\pm}]$ can be realized as a Deligne period of mixed motives by integrating certain differential forms over a suitable topological chain contained in the smooth part of the zero set of P. Assuming Beĭlinson's conjectures, he found a higher dimensional analogue of (0.1.7) such as

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right) = \frac{r}{(2\pi)^2} L(E, 2)$$

for some $r \in \mathbb{Q}^\times$. Here $L(E, s)$ denotes the L-function associated to the elliptic curve E obtained by taking the projective closure of the zero locus of $x + \frac{1}{x} + y + \frac{1}{y} + 1$ and specifying a suitable origin. It was eventually proved by Rogers and Zudilin [**105**] using modular methods, independent of Beĭlinson's conjectures, and they further showed that $r = 15$.

The identities such as (0.1.7) and (0.1.8), along with Deninger's work, prompted Boyd to start an extensive investigation concerning the relations between Mahler measures of multivariable polynomials and special values of L-functions, which led to the foundational paper [**29**] containing conjectures and numerical calculations relating Mahler measures of two-variable polynomials to special values of L-functions of elliptic curves arising from the polynomials. For example, he conjectured that, for $r \in \mathbb{Z} \setminus \{0, \pm 4\}$,

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + r\right) \stackrel{?}{=} \frac{k_r N(r)}{(2\pi)^2} L(E_{N(r)}, 2) = k_r L'(E_{N(r)}, 0), \tag{0.1.10}$$

where k_r is a non-zero rational number of low height,

$$E_{N(r)} : y^2 = x^3 + \left(\frac{r^2}{4} - 2\right) x^2 + x \tag{0.1.11}$$

is the Weierstraß equation of an elliptic curve (of conductor $N(r)$) which is birationally equivalent to the curve defined by $x + \frac{1}{x} + y + \frac{1}{y} + r = 0$, and the question mark stands for a numerical formula that is true for at least 20 decimal places.

Rodriguez-Villegas also developed a context to explain Boyd conjectures in [**102**], where he focused on the fact that most of the conjectural identities appear in families of polynomials $\{P_\ell \in \mathbb{Z}[\ell][x, y] : \ell \in S, S = \mathbb{Z}, \mathbb{Q}, \mathbb{R} \text{ or } \mathbb{C}\}$. He studied the Mahler measure $\mathrm{m}(P_\ell)$ as a function of ℓ, and, for certain families, related this function to an Eisenstein–Kronecker series (see Definition 0.3.1), which is intimately linked to modular forms. He further proved (0.1.10) for $r = \frac{4}{\sqrt{2}}, 4\sqrt{2}$, where the corresponding elliptic curves have complex multiplication.

Since then, multiple approaches have been made to resolve the conjectural identities, but they are still largely unproven to date. In Section 0.3, we provide more details on the known and proven identities of type (0.1.10), for different families of polynomials.

In this thesis, we consider certain generalizations of the Mahler measure. Indeed, the Mahler measure can be defined in a far more general setting. Let (X, μ_X) be a probability space and let $L^0(X)$ denote the complex vector space of measurable functions $f : X \to \mathbb{C}$ such that f is non-zero almost everywhere. Then, for $r \in \mathbb{R}_{>0}$, the L^r-spaces can be defined as

$$L^r(X) := \left\{ f \in L^0(X) : \|f\|_r := \left(\int_X |f|^r d\mu_X \right)^{1/r} < \infty \right\}.$$

Note that $\|\cdot\|_r$ defines a function from $L^r(X)$ to $\mathbb{R}_{\geq 0}$ for every $r > 0$. Since X is a probability space, we have $L^r(X) \subseteq L^q(X)$ for every $0 \leq q \leq r$. This property can be used to define the *Mahler measure* as the functional

$$\mathrm{m}_X : \bigcup_{r>0} L^r(X) \to \mathbb{R} \cup \{-\infty\}, \qquad f \mapsto \lim_{r \to 0^+} \log \|f\|_r,$$

where $\bigcup_{r>0} L^r(X)$ is a complex vector subspace of $L^0(X)$. Since

$$\lim_{r \to 0^+} \frac{1}{r} \log \left(\int_X |f|^r d\mu_X \right) = \int_X \log |f| d\mu_X,$$

for $X = \mathbb{T}^1$ and $d\mu_X = d\mu_{\mathbb{T}^1} = \frac{1}{2\pi i} \frac{dx}{x}$, we retrieve the one-variable Mahler measure $\mathrm{m}_{\mathbb{T}^1}(\cdot) = \mathrm{m}(\cdot)$ in (0.1.4). This thesis consists of two such generalizations:

(1) when $X = \mathbb{T}^1_a = \{x \in \mathbb{C}^\times : |x| = a\}$ and $d\mu_X = d\mu_{\mathbb{T}^1_a} = d\mu_{\mathbb{T}^1} = \frac{1}{2\pi i} \frac{dx}{x}$ is the normalized arc measure on $\mathbb{T}^1_a$;

(2) when $X = \mathbb{D} = \{x \in \mathbb{C} : |x| \leq 1\}$ and $d\mu_X = d\mu_{\mathbb{D}} = \frac{1}{\pi} dx$ is the normalized area measure on $\mathbb{D}$.

The generalizations (1) and (2) are known as the *generalized Mahler measure* and the *areal Mahler measure*, respectively. They were first introduced by Lalín and Mittal [**80**], and Pritsker [**99**], respectively. We explore these definitions in more detail in Sections 0.5 and 0.6.

The following section examines a different perspective for studying Mahler measures of several variable polynomials.

0.2. Mahler measure as heights and periods

For $\alpha \in K$, with K a number field, the (absolute logarithmic) Weil height is given by

$$h_{\text{Weil}}(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{\substack{v \in M_K \\ v|p}} [K_v : \mathbb{Q}_p] \log \max \{\|\alpha\|_v, 1\},$$

where M_K is an appropriately normalized set of equivalent places (and therefore induced absolute values) on K, so that the product formula is satisfied:

$$\prod_{\substack{v \in M_K \\ v|p}} \|\alpha\|_v^{\frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}} = 1.$$

In the above formulas, p is a rational prime and v is a prime in the ring of integers of K lying above p. Here $\|\alpha\|_v = \left|N_{K_v/\mathbb{Q}_p}(\alpha)\right|_p^{\frac{1}{[K_v:\mathbb{Q}_p]}}$, where $N_{K_v/\mathbb{Q}_p}$ is the norm function on $K_v/\mathbb{Q}_p$ and $|\cdot|_p$ is the normalized p-adic absolute value in $\mathbb{Q}_p$ such that $|p|_p = p^{-1}$. Then the Mahler measure of an algebraic number α, defined as the Mahler measure of its integral minimal polynomial, is the same as the product of $[\mathbb{Q}(\alpha):\mathbb{Q}]$ and $h_{\text{Weil}}(\alpha)$, i.e. if $f_\alpha \in \mathbb{Z}[x]$ is the integral minimal polynomial of $\alpha \in \bar{\mathbb{Q}}^\times$, then

$$\mathrm{m}(f_\alpha) = \log|a_0| + \sum_{\beta \in \mathbb{C}, f_\alpha(\beta)=0} \log^+|\beta| = (\deg f_\alpha)\, h_{\text{Weil}}(\alpha) = [\mathbb{Q}(\alpha):\mathbb{Q}] h_{\text{Weil}}(\alpha).$$

Following the discussion in the previous section, we know that $\mathcal{M}$ is a set of algebraic numbers, i.e. $M(P)$, for $P \in \mathbb{Z}[x]$, is an algebraic number.

The relation between Mahler measure and heights can be extended to several variable polynomials and heights in hypersurfaces. Maillot **[90]** showed that the Mahler measure of a Laurent polynomial $P \in \mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}]$ can be expressed in terms of the canonical height of the hypersurface defined by the polynomial using a toric variety associated to the *Newton polytope* of P. Here the Newton polytope of $P(\mathbf{x}) = \sum_{\mathbf{j}} a_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}$ is defined as the convex hull of the points $\mathbf{j} \in \mathbb{Z}^n$ such that $a_{\mathbf{j}} \neq 0$, where $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{j} = (j_1, \ldots, j_n)$, and $\mathbf{x}^{\mathbf{j}} = x_1^{j_1} \cdots x_n^{j_n}$.

We refer the interested reader to **[62]** and **[60]** for more details on the notions of canonical heights on abelian varieties and toric varieties.

Values of height functions produce computable numbers that measure the complexity of some arithmetic objects, such as algebraic numbers, points on abelian varieties, abelian varieties themselves, etc. Other quantities associated to these arithmetic objects, containing critical information about them, are known as *periods*. Kontsevich and Zagier **[70]** defined them in the following way.

Definition 0.2.1. *A **period** is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients over domains in $\mathbb{R}^n$ given by inequalities with rational coefficients.*

We can replace "rational" with "algebraic" in the above definition to obtain a period, because the algebraic functions occurring in the integrand can be replaced by rational functions by introducing more variables. Note that the countability of $\mathbb{Q}$ implies that the set of all periods $\mathcal{P}$ is countable.

Periods are intended to bridge the gap between algebraic numbers and transcendental numbers. The class of algebraic numbers is too narrow to include many common mathematical constants, while the set of transcendental numbers is not countable, and its members are not generally computable.

For example, the following common constants are periods:

$$\pi = \int_0^1 \frac{4}{x^2+1}dx, \quad \log 2 = \int_1^2 \frac{dx}{x}, \quad \text{and } \zeta(3) = \iiint_{0<x<y<z<1} \frac{dxdydz}{(1-x)yz}.$$

For many purposes, it is convenient to widen our previous definition and consider also elements of the **extended period ring** $\hat{\mathcal{P}} = \mathcal{P}[\frac{1}{2i\pi}]$, which is an algebra. From the definition of Mahler measure (see Definition 0.1.2), it then follows that *the (logarithmic) Mahler measures of rational functions with rational coefficients are elements of* $\hat{\mathcal{P}}$.

Periods can also be seen as values of integrals of algebraically defined differential forms over certain chains in algebraic varieties. For example, let $Y/\bar{\mathbb{Q}}$ be a subvariety of a smooth quasi-projective variety $X/\bar{\mathbb{Q}}$, let ω be a closed algebraic n-form on X vanishing on Y, and let C be a singular n-chain on $X(\mathbb{C})$ with boundary contained in $Y(\mathbb{C})$; then the integral

$$\int_C \omega \quad \text{is a period of the quadruple } (X, Y, \omega, C).$$

Moreover, if these forms and chains depend on parameters (i.e. if $\omega(t)$ or $C(t)$ depend on some parameter t) then the integrals, considered as functions of the parameters, typically satisfy linear differential equations with algebraic coefficients (depending on t). These are called (generalized) *Picard–Fuchs differential equations* or (members of) *Gauss-Manin systems.* Picard-Fuchs differential equations for elliptic surfaces (or curves) are examples of such differential equations, whose solutions describe the periods of the elliptic surfaces (or curves). Special values of the solutions of these differential equations at algebraic arguments produce elements of $\hat{\mathcal{P}}$.

Other examples of periods are special values at algebraic arguments of hypergeometric functions (and their analytic continuations):

$$ {}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; z\right) = \sum_{n \geq 0} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \text{ where } (a)_n = a(a+1)\cdots(a+n-1) \tag{0.2.1}$$

and special values of modular forms and various kinds of L-functions at appropriate arguments.

Kontsevich and Zagier conjectured that any two integral representations of a period should be obtained from each other just by using additivity of integrals, changes of variables, and Stokes' theorem. Examples of this phenomenon include proved identities regarding the relationships between Mahler measure of polynomials and special values of different kinds of L-functions (see Section 0.3 for more examples). We refer the interested reader to [**70**] for more detailed and thorough discussions on periods.

0.3. Mahler measure and special values of L-functions

This section aims to briefly recall the history of the conjectural links between Mahler measure and special values of L-functions. We refer the reader to [**38**, Chapters 1, 3 and 8] for a detailed exposition.

Let $P_k(x) = x^2 + kx + 1 \in \mathbb{Z}[x]$, for $k \in \mathbb{Z}_{\geq 3}$. Using Lehmer's formula (0.1.2), we derive that $\mathrm{m}(P_k) = \log \frac{k+\sqrt{k^2-4}}{2} = \ell \log \epsilon_{D_{\mathbb{Q}\left(\sqrt{k^2-4}\right)}}$, where $\epsilon_{D_{\mathbb{Q}\left(\sqrt{k^2-4}\right)}}$ is the fundamental unit of the real quadratic field $\mathbb{Q}\left(\sqrt{k^2-4}\right)$, $D_{\mathbb{Q}\left(\sqrt{k^2-4}\right)}$ is its fundamental discriminant, and $\ell \in \mathbb{Z}_{\neq 0}$. Recall that the Dirichlet class number formula implies that

$$\sqrt{D_K} L\left(\chi_{D_K}, 1\right) = h(D_K) \log \epsilon_{D_K},$$

where D_K is the fundamental discriminant of a real quadratic field K, ϵ_{D_K} is its fundamental unit, $h(D_K)$ is the class number of K, and

$$L\left(\chi_{D_K}, s\right) = \sum_{n=1}^{\infty} \frac{\chi_{D_K}(n)}{n^s} = \sum_{n \geq 1}^{\infty} \frac{\left(\frac{D_K}{n}\right)}{n^s}. \tag{0.3.1}$$

Combining the above with the functional equation of $L\left(\chi_{D_{\mathbb{Q}\left(\sqrt{k^2-4}\right)}}, s\right)$, we obtain

$$\frac{L'\left(\chi_{D_{\mathbb{Q}\left(\sqrt{k^2-4}\right)}}, 0\right)}{\mathrm{m}(P_k)} \in \mathbb{Q}^{\times}. \tag{0.3.2}$$

Since any real quadratic field K can be represented as $\mathbb{Q}\left(\sqrt{k^2-4}\right)$ for some $k \geq 3$, we have found a suitable polynomial P_k for each K such that (0.3.2) holds.

In the multivariable case, we have already mentioned in (0.1.7) the pioneering result by Smyth:

$$\mathrm{m}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = \frac{3\sqrt{3}}{4\pi} \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^2} = \frac{3\sqrt{3}}{4\pi} \sum_{n=1}^{\infty} \frac{\left(\frac{-3}{n}\right)}{n^2} = L'(\chi_{-3}, -1).$$

Here $F = \mathbb{Q}\left(\sqrt{-3}\right)$, $D_F = -3$, and $\frac{L'\left(\chi_{D_F}, -1\right)}{\mathrm{m}(1+x+y)}$ is a non-zero rational number, namely 1. Smyth's identity is also the first example of identities of the form

$$\frac{L'(\chi_{D_F}, -1)}{\mathrm{m}(Q_F)} \in \mathbb{Q}^{\times},$$

where χ_{D_F} is the quadratic character associated to the imaginary quadratic field F with fundamental discriminant D_F, and Q_F is a polynomial in two variables with integer coefficients. Following this observation, Chinburg conjectured in [**43, 100**] that

For every imaginary quadratic field F, there exists a non-zero polynomial $Q_F \in \mathbb{Z}[x, y]$, such that $\frac{L'(\chi_{D_F}, -1)}{\mathrm{m}(Q_F)}$ is a non-zero rational number.

Identities as conjectured are known in some cases due to Ray [**100**] ($D_F = -4, -7, -8, -20$), Boyd–Rodriguez-Villegas [**31**] ($D_F = -11, -15, -24, -35, -39, -55, -84$), Boyd–Rodriguez-Villegas [**32**] ($D_F = -19, -40, -120$), Liu–Qin [**88**] ($D_F = -23, -303, -755$) and others. Some recent works in this direction are being pursued by Pengo [**97**], Mehrabdollahei [**91**], and others.

Continuing our description of conjectural relations between Mahler measures and special values of L-functions, we now focus on Boyd's extensive numerical computations of Mahler measures of certain families of polynomials $\{G_\ell \in \mathbb{Z}[x,y] : \ell \in \mathbb{C}\}$ whose zero locus is birationally equivalent to elliptic curves $E_{N(\ell)}$ of conductor $N(\ell)$, for almost all $\ell^2 \in \mathbb{Z}$. In particular, Boyd considered the family $x + \frac{1}{x} + y + \frac{1}{y} + r$ for every $r \in \mathbb{Z} \setminus \{0, \pm 4\}$, and conjectured that

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + r\right) \stackrel{?}{=} \frac{k_r N(r)}{(2\pi)^2} L(E_{N(r)}, 2), \qquad k_r \in \mathbb{Q}^\times, \tag{0.3.3}$$

which we have already seen in (0.1.10). Here the curve $C_r : x + \frac{1}{x} + y + \frac{1}{y} + r = 0$ and the elliptic curve $E_{N(r)} : Y^2 = X^3 + \left(\frac{r^2}{4} - 2\right) X^2 + X$ are birationally equivalent via the change of variables

$$\begin{aligned} X &= -\frac{1}{xy}, & x &= \frac{rX - 2Y}{2X(X-1)}, \\ Y &= \frac{(y-x)\left(1 + \frac{1}{xy}\right)}{2xy}, & y &= \frac{rX + 2Y}{2X(X-1)}. \end{aligned} \tag{0.3.4}$$

For $r = \pm 4$, the curve obtained from the zero locus of the polynomial is degenerate in the sense that, for a suitable change of variables, it can be factored into linear polynomials, and then a direct calculation using Jensen's formula yields that

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + 4\right) = \mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} - 4\right) = 2L'(\chi_{-4}, -1),$$

where χ_{-4} is the quadratic character of conductor 4. [5] On the other hand,

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y}\right) = \mathrm{m}\left(\left(x + \frac{1}{y}\right)(x+y)\right) = 0.$$

Prior to stating the next result, we must define Eisenstein–Kronecher series.

5. Note that the curve C_r, as well as $\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + r\right)$, is invariant under the transformation $r \mapsto -r$, and therefore we have the same curve for r and $-r$, as well as the same Mahler measure.

Definition 0.3.1 ([**5**, Section 1.1]). *Let $\Gamma = \omega_1\mathbb{Z} + \omega_2\mathbb{Z} \subset \mathbb{C}$ be a lattice in $\mathbb{C}$ generated by ω_1 and ω_2 with $\operatorname{Im}\left(\frac{\omega_1}{\omega_2}\right) > 0$, and let $A(\Gamma) = \frac{\operatorname{Im}(\omega_1\bar{\omega_2})}{\pi}$. Let a be a non-negative integer. For $z, w \in \mathbb{C} \setminus \Gamma$, we define the **Eisenstein–Kronecker–Lerch series** $K_a(z, w, s; \Gamma)$ by*

$$K_a(z, w, s; \Gamma) := \sum_{\gamma \in \Gamma} \frac{(\bar{z} + \bar{\gamma})^a}{|z + \gamma|^{2s}} \langle \gamma, w \rangle_\Gamma, \qquad \operatorname{Re} s > 1 + \frac{a}{2}$$

*where $\langle \gamma, w \rangle_\Gamma = \exp\left(\frac{(\gamma\bar{w} - w\bar{\gamma})}{A(\Gamma)}\right)$. The **Eisenstein–Kronecker series** $E_1(z, s; \Gamma) : \mathbb{C}^2 \to \mathbb{C}$ is then defined as the analytic continuation of the series $K_1(z, 0, s; \Gamma)$ which converges for $\operatorname{Re} s > \frac{3}{2}$.*

Rodriguez-Villegas [**102**] expressed the Mahler measure of $x + \frac{1}{x} + y + \frac{1}{y} + r$ and other Boyd's families in terms of Eisenstein–Kronecker series. For example, he showed

$$\begin{aligned} \mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + r\right) &= \frac{1}{2} \operatorname{Re}\left[-\log q + 4 \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^2 \frac{q^n}{n}\right] \\ &= \operatorname{Re}\left[\frac{16 \operatorname{Im}(\tau)}{\pi^2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq 0}} \chi(n) \frac{1}{(4m\tau + n)^2 (4m\bar{\tau} + n)}\right], \end{aligned}$$

for $r \in \mathbb{C}$ such that $r^2 = \frac{1}{\mu^2(\tau)}$, where $\tau \in \mathcal{F}$, the fundamental domain formed by the geodesic triangle in $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ with vertices $i\infty, 0, 1/2$ and its reflection along the imaginary axis, and μ is a Hauptmodul for the subgroup of $SL_2(\mathbb{Z})$ associated with $\mathcal{F}$ (i.e., μ induces an isomorphism between $\mathcal{F} \cup \{\text{cusps}\}$ and $\mathbb{P}^1(\mathbb{C})$). Here $\chi(n) = \left(\frac{-4}{n}\right)$ and $q = e^{2\pi i\tau}$. Rodriguez-Villegas further proved Boyd's conjectures for $r = \frac{4}{\sqrt{2}}, 4\sqrt{2}$, where τ is CM-point i.e. a complex quadratic number in the upper half plane $\mathbb{H}$.

Further identities associating Mahler measures of polynomials to special values of L-function of elliptic curves for different families were proved by Bertin and Zudilin [**14**], Brunault [**36, 37, 35**], Lalín [**77**], Rodriguez-Villegas [**102, 103**], Mellit [**92**], Rogers and Zudilin [**104, 105**] et al. Some of these results involving the following families of polynomials are gathered in Table 1:

$$\begin{aligned} \mathcal{Q}_r(x,y) =& x + \frac{1}{x} + y + \frac{1}{y} + r, \\ R_m(x,y) =& (1+x)(1+y)(x+y) - mxy, \end{aligned} \tag{0.3.5}$$

with $r, m \in \mathbb{C}$, where the family $\{\mathcal{Q}_r : r \in \mathbb{C}\}$ comprises of the polynomials considered by Boyd in (0.1.10) (also in (0.3.3)). Here E_N and $\tilde{E}_L$ represent elliptic curves of conductor N and L, respectively.

Identities	Author(s)	year
$\mathrm{m}(\mathcal{Q}_{4\sqrt{2}}) = L'(E_{64}, 0)$	F. Rodriguez-Villegas	1997
$\mathrm{m}(\mathcal{Q}_{4/\sqrt{2}}) = L'(E_{32}, 0)$	F. Rodriguez-Villegas	1997
$\mathrm{m}(\mathcal{Q}_1) = L'(E_{15}, 0)$	M. Rogers and W. Zudilin	2010
$\mathrm{m}(\mathcal{Q}_5) = 6L'(E_{15}, 0)$	M. Lalín	2010
$\mathrm{m}(\mathcal{Q}_{2i}) = L'(E_{40}, 0)$	A. Mellit	2011
$\mathrm{m}(\mathcal{Q}_2) = L'(E_{24}, 0)$	M. Rogers and W. Zudilin	2012
$\mathrm{m}(R_4) = 2L'(\tilde{E}_{20}, 0)$	M. Rogers and W. Zudilin	2012
$\mathrm{m}(\mathcal{Q}_i) = 2L'(E_{17}, 0)$	W. Zudilin	2014
$\mathrm{m}(\mathcal{Q}_3) = 2L'(E_{21}, 0)$	F. Brunault, M. Lalín, D. Samart and W. Zudilin	2015
$\mathrm{m}(\mathcal{Q}_{12}) = 2L'(E_{48}, 0)$	F. Brunault	2015
$\mathrm{m}(R_1) = L'(\tilde{E}_{14}, 0)$	M. J. Bertin and W. Zudilin	2015

Table 1 – Some proven identities of Mahler measure of Boyd's families of polynomials

Remark 0.3.2. *Recall that the change of variables in* (0.3.4) *gives a birational map between the elliptic curves* $E_{N(r)} : Y^2 = X^3 + \left(\frac{r^2}{4} - 2\right) X^2 + X$ *and the curve defined by* $\mathcal{Q}_r(x,y) = 0$. *Similarly, the Deuring form* $\tilde{E}_{N(m)} : Y^2 + (m-2)XY + mY = X^3$ *is birational to the curve* *defined by* $R_m(x,y)=0$ *via the change of variables* [**73**]

$$\begin{aligned} X &= m\frac{x+y+1}{x+y-m}, \qquad & x &= \frac{X-Y}{X-m}, \\ Y &= m\frac{-mx+y+1}{x+y-m}, \qquad & y &= \frac{Y+(m-1)X+m}{X-m}. \end{aligned} \tag{0.3.6}$$

For 3-variable Laurent polynomials, extensive study has been done by Bertin [**9, 10, 12, 11**], Bertin, Feaver, Fuselier, Lalín and Manes [**13**], Samart [**107**], and others. For example, Bertin et al. considered the family of polynomials

$$\left\{L_k(x,y,z) = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + k : k \in \mathbb{C}\right\},$$

and showed that

$$\begin{aligned} \mathrm{m}(L_2) =& 4\frac{|\det T(Y_2)|^{3/2}}{4\pi^3} L(T(Y_2),3), \\ \mathrm{m}(L_3) =& 2\frac{|\det T(Y_3)|^{3/2}}{4\pi^3} L(T(Y_3),3), \end{aligned}$$

and other identities, where Y_k denotes the $K3$-surface (a generalization of elliptic curves in complex dimension 2) associated to the zero locus of L_k, with $T(Y_k)$ being its *transcendental lattice*, a free $\mathbb{Z}$-submodule of the free $\mathbb{Z}$-module $H_2(Y_k,\mathbb{Z})$. Since the results of this thesis do not specifically address $K3$-surfaces, we refer the reader to [**9, 111**] for further expository reading.

0.4. Mahler measure as the integral of a differential form

In this section, we restrict ourselves to 2-variable families of polynomials, and we associate the Mahler measure of them with an integral of certain differential forms over some algebraic chains. This alternative approach was proposed by Deninger [**45**] to express the Mahler measure as a regulator evaluated in a certain K-group, hence establishing a relation between Mahler measure and Beĭlinson conjectures [**38, 45, 97**]. Evaluation of this particular integral representation of the Mahler measure breaks down to two cases: *exact* and *non-exact*. Before going into more details about these cases, we first mention a few facts about polylogarithms, which are related to the *exact* cases. Though the integrals for *non-exact* cases do not appear in our results in the next chapters, we include its discussion for the sake of completion.

0.4.1. Polylogarithms

In this section, we recall some basic properties of polylogarithms.

Definition 0.4.1. *Let $n \in \mathbb{Z}_{\geq 0}$. The* ***polylogarithm*** *is defined as the power series, for $z \in \mathbb{C}$ and $|z| < 1$,*

$$\mathrm{Li}_n(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^n}.$$

The definition and the name come from the analogy with the Taylor series expansion:

$$-\log(1-z) = \sum_{j=1}^{\infty} \frac{z^j}{j} \quad \text{for } |z| < 1.$$

The relation

$$\frac{d}{dz}\mathrm{Li}_n(z) = \frac{\mathrm{Li}_{n-1}(z)}{z}, \quad \text{for } n \geq 2,$$

follows from the definition, and leads to the extension of the domain of definition of Li_n. In particular, the analytic continuation of the dilogarithm is given by

$$\mathrm{Li}_2(z) = -\int_0^z \log(1-u)\frac{du}{u} \quad \text{for } z \in \mathbb{C} \setminus [1, \infty). \tag{0.4.1}$$

In order to extend polylogarithms to the whole complex plane, Zagier [**119**] considered the following version:

$$\mathcal{P}_n(z) := \mathrm{Re}_n\left(\sum_{\ell=0}^{n} \frac{2^\ell B_\ell}{\ell!}(\log|z|^\ell)\mathrm{Li}_{n-\ell}(z)\right), \tag{0.4.2}$$

where B_ℓ is the ℓ-th Bernoulli number, given by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}, \tag{0.4.3}$$

$\mathrm{Li}_0(z)$ is defined as $-\frac{1}{2}$ and Re_n denotes Re or Im depending on whether n is odd or even. The function $\mathcal{P}_n$ is single-valued and continuous in $\mathbb{P}^1(\mathbb{C})$ with $\mathcal{P}_n(\infty) = 0$. Moreover, $\mathcal{P}_n$

is real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, and satisfies certain functional equations. The simplest ones are:

$$\mathcal{P}_n\left(\frac{1}{x}\right) = (-1)^{n-1} P_n(x), \qquad \mathcal{P}_n(\bar{x}) = (-1)^{n-1}\mathcal{P}_n(x). \tag{0.4.4}$$

Some examples of modified polylogarithms are:

— The *Bloch–Wigner dilogarithm* is defined as

$$\mathcal{P}_2(z) = D(z) = \operatorname{Im}(\operatorname{Li}_2(z) + \log(1-z)\log|z|). \tag{0.4.5}$$

It further satisfies the well-known five-term relation [**119**]:

$$D(x) + D(y) + D(1-xy) + D\left(\frac{1-x}{1-xy}\right) + D\left(\frac{1-y}{1-xy}\right) = 0,$$

where $x, y \in \mathbb{P}^1(\mathbb{C})$. For $\theta \in [0, \pi)$, $D(e^{2i\theta})$ admits following representations:

$$-2\int_0^\theta \log|2\sin u|du = D\left(e^{2i\theta}\right) = \sum_{m=1}^{\infty} \frac{\sin(2m\theta)}{m^2}. \tag{0.4.6}$$

Furthermore, some special values of D can also be expressed in terms of Dirichlet L-values, such as

$$D(e^{i\pi/3}) = \frac{3\sqrt{3}}{4} L(\chi_{-3}, 2), \tag{0.4.7}$$

and

$$D(i) = L(\chi_{-4}, 2), \tag{0.4.8}$$

where $L(\chi_{-3}, s)$ and $L(\chi_{-4}, s)$ are the Dirichlet L-functions on the quadratic characters of conductor 3 and 4 respectively, as defined in (0.3.1). These identities are applied in Chapter 2.

— For $n = 3$, the modified trilogarithm is

$$\mathcal{P}_3(z) = \operatorname{Re}\left(\operatorname{Li}_3(z) - \log|z|\operatorname{Li}_2(z) + \frac{1}{3}\log^2|z|\operatorname{Li}_1(z)\right). \tag{0.4.9}$$

$\mathcal{P}_3$ also satisfies functional equations, for example the Spence–Kummer relation [**119**]:

$$\mathcal{P}_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right) + \mathcal{P}_3(xy) + \mathcal{P}_3\left(\frac{x}{y}\right) - \mathcal{P}_3\left(\frac{x(1-y)}{y(1-x)}\right) - 2\mathcal{P}_3\left(\frac{y(1-x)}{y-1}\right)$$
$$- 2\mathcal{P}_3\left(\frac{x(1-y)}{x-1}\right) - 2\mathcal{P}_3\left(\frac{1-y}{1-x}\right) - 2\mathcal{P}_3(x) - 2\mathcal{P}_3(y) + 2\mathcal{P}_3(1) = 0;$$

where $x, y \in \mathbb{P}^1(\mathbb{C})$, and we recall that $\mathcal{P}_3(\infty) = 0$.

We can further generalize the definition of polylogarithms, and define the *multiple polylogarithm* following the notation of Goncharov [**56, 57**].

Definition 0.4.2. *Let $n_1, \dots, n_k \in \mathbb{Z}_{\geq 1}$. The* ***multiple polylogarithm*** *is defined as the power series*

$$\mathrm{Li}_{n_1,\dots,n_k}(z_1,\dots,z_k) := \sum_{0<j_1<j_2<\cdots<j_k} \frac{z_1^{j_1} z_2^{j_2} \dots z_k^{j_k}}{j_1^{n_1} j_2^{n_2} \dots j_k^{n_k}}.$$

This series is convergent for $|z_i| \leq 1$ and $|z_k| < 1$ if $n_k = 1$. The length of the multiple polylogarithm is the number k and its weight is the number $w = n_1 + \cdots + n_k$.

Note that, when $k = 1$, we retrieve the classical polylogarithm $\mathrm{Li}_n(z)$. When $k = 1$ and $n_1 > 1$ the series converges absolutely for $|z| = 1$ and the evaluations at $z = -1, 1$ yield the special values of the Riemann zeta function

$$\mathrm{Li}_n(1) = \zeta(n), \qquad \mathrm{Li}_n(-1) = -\left(1 - \frac{1}{2^{n-1}}\right)\zeta(n). \tag{0.4.10}$$

The evaluations at $z = i$ also give the Riemann zeta function as well as a Dirichlet L-function:

$$\mathrm{Re}(\mathrm{Li}_n(i)) = -\frac{1}{2^n}\left(1 - \frac{1}{2^{n-1}}\right)\zeta(n), \qquad \mathrm{Im}(\mathrm{Li}_n(i)) = L(\chi_{-4}, n). \tag{0.4.11}$$

We also have the following useful identity due to Jonquière [**68**]

$$\mathrm{Li}_n(e^{2\pi i x}) + (-1)^n \mathrm{Li}_n(e^{-2\pi i x}) = -\frac{(2\pi i)^n}{n!} B_n(x),$$

where $B_n(x)$ denotes the Bernoulli polynomial given by

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(x) t^k}{k!}, \tag{0.4.12}$$

and $0 \leq \mathrm{Re}(x) < 1$ if $\mathrm{Im}(x) \geq 0$ and $0 < \mathrm{Re}(x) \leq 1$ if $\mathrm{Im}(x) < 0$. In particular, we have for $0 < \ell < 2r$,

$$\mathrm{Li}_1(\xi_{2r}^{\ell}) - \mathrm{Li}_1(\xi_{2r}^{-\ell}) = \frac{(r-\ell)\pi i}{r}, \tag{0.4.13}$$

where ξ_{2r} is a primitive $2r$-root of the unity.

We can also generalize (0.4.1) to obtain similar integral representations for multiple polylogarithms. To do that, we first need the following definition.

Definition 0.4.3. *Let $n_1, \dots, n_k \in \mathbb{Z}_{\geq 1}$. The* ***hyperlogarithm*** *is defined as the iterated integral*

$$\mathrm{I}_{n_1,\dots,n_m}(a_1 : \dots : a_m : a_{m+1}) :=$$

$$\int_0^{a_{m+1}} \underbrace{\frac{dt}{t-a_1} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n_1} \circ \underbrace{\frac{dt}{t-a_2} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n_2} \circ \cdots \circ \underbrace{\frac{dt}{t-a_m} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n_m}, \tag{0.4.14}$$

where the a_i are complex numbers, and

$$\int_0^{b_{k+1}} \frac{dt}{t-b_1} \circ \cdots \circ \frac{dt}{t-b_k} = \int_{0 \leq t_1 \leq \cdots \leq t_k \leq b_{k+1}} \frac{dt_1}{t_1 - b_1} \cdots \frac{dt_k}{t_k - b_k}.$$

The value of the integral in (0.4.14) depends on the homotopy class of the path connecting 0 and a_{m+1} on $\mathbb{C} \setminus \{a_1, \dots, a_m\}$.

Using hyperlogarithms, we can now integrally represent multiple polylogarithms with the following identities:

$$\mathrm{I}_{n_1,\dots,n_m}(a_1 : \dots : a_m : a_{m+1}) = (-1)^m \mathrm{Li}_{n_1,\dots,n_m}\left(\frac{a_2}{a_1}, \frac{a_3}{a_2}, \dots, \frac{a_m}{a_{m-1}}, \frac{a_{m+1}}{a_m}\right),$$

$$\mathrm{Li}_{n_1,\dots,n_m}(x_1, \dots, x_m) = (-1)^m \mathrm{I}_{n_1,\dots,n_m}((x_1 \dots x_m)^{-1} : \dots : x_m^{-1} : 1),$$

which also give analytic continuation for multiple polylogarithms. We refer the reader to [**55**] for detailed descriptions of multiple polylogarithms and hyperlogarithms.

Some combinations of length 2 polylogarithms can be written in terms of length 1 polylogarithms. To achieve this simplification we will use a certain result due to Nakamura [**94**] and Panzer [**96**]. Here we state the formulation of [**79**].

Theorem 0.4.4. [**79**, Theorem 3] *Let r,s be positive integers, $k = r + s$, and let u, v be complex numbers such that $|u| = |v| = 1$. In addition, we assume that $v \neq 1$ if $s = 1$. Let*

$$\mathrm{Re}_k = \begin{cases} \mathrm{Re} & k \text{ odd,} \\ i\,\mathrm{Im} & k \text{ even.} \end{cases}$$

Then,

$$\begin{aligned}2\mathrm{Re}_k(\mathrm{Li}_{r,s}(u,v)) =&(-1)^k\mathrm{Li}_k(\overline{uv}) + (-1)^{k+1}\mathrm{Li}_r(\overline{u})\mathrm{Li}_s(\overline{v}) + (-1)^{r-1}\mathrm{Li}_r(\overline{u})\mathrm{Li}_s(v)\\
&+ (-1)^{r-1}\left(\binom{k-1}{r-1}\mathrm{Li}_k(\overline{u}) + \binom{k-1}{s-1}\mathrm{Li}_k(v)\right)\\
&+\sum_{m=1}^{k-1}\left(\binom{m-1}{r-1}\mathrm{Li}_m(\overline{u}) + \binom{m-1}{s-1}(-1)^{k+m}\mathrm{Li}_m(v)\right)\\
&\times((-1)^r\mathrm{Li}_{k-m}(uv) + (-1)^{s+m}\mathrm{Li}_{k-m}(\overline{uv})).\end{aligned}$$

Multiple polylogarithms arise in the context of understanding how special values of zeta functions and L-series appear in our results in Section 0.7. The following corollary to Theorem 0.4.4 provides a way to achieve these types of simplifications in Chapters 2 and 4.

Corollary 0.4.5. *Let ξ_{2r} denote a primitive $2r$-root of unity. If h is a nonnegative integer, we have*

$$\begin{aligned}2i\,\mathrm{Im}\left(\mathrm{Li}_{3,2h+1}(i\xi_{2r}^{-\ell}, -i)\right) =&\mathrm{Li}_{2h+4}(\xi_{2r}^{\ell}) - \mathrm{Li}_3(-i\xi_{2r}^{\ell})\mathrm{Li}_{2h+1}(i) + \mathrm{Li}_3(-i\xi_{2r}^{\ell})\mathrm{Li}_{2h+1}(-i)\\
&+\left(\binom{2h+3}{2}\mathrm{Li}_{2h+4}(-i\xi_{2r}^{\ell}) + \binom{2h+3}{2h}\mathrm{Li}_{2h+4}(-i)\right)\\
&+\sum_{t=1}^{2h+3}\left(\binom{t-1}{2}\mathrm{Li}_t(-i\xi_{2r}^{\ell}) + \binom{t-1}{2h}(-1)^t\mathrm{Li}_t(-i)\right)\\
&\times(-\mathrm{Li}_{2h+4-t}(\xi_{2r}^{-\ell}) - (-1)^t\mathrm{Li}_{2h+4-t}(\xi_{2r}^{\ell})).\end{aligned} \tag{0.4.15}$$

If h is a positive integer, we have

$$\begin{aligned}2\,\mathrm{Re}(\mathrm{Li}_{3,2h}\left(\pm\xi_{2r}^{-\ell}, \pm 1\right)) =& -\mathrm{Li}_{2h+3}(\xi_{2r}^{\ell}) + 2\mathrm{Li}_3(\pm\xi_{2r}^{\ell})\mathrm{Li}_{2h}(\pm 1)\\
&+\left(\binom{2h+2}{2}\mathrm{Li}_{2h+3}(\pm\xi_{2r}^{\ell}) + \binom{2h+2}{2h-1}\mathrm{Li}_{2h+3}(\pm 1)\right)\\
&+\sum_{t=1}^{2h+2}\left(\binom{t-1}{2}\mathrm{Li}_t(\pm\xi_{2r}^{\ell}) - \binom{t-1}{2h-1}(-1)^t\mathrm{Li}_t(\pm 1)\right)\\
&\times(-\mathrm{Li}_{2h+3-t}(\xi_{2r}^{-\ell}) + (-1)^t\mathrm{Li}_{2h+3-t}(\xi_{2r}^{\ell})).\end{aligned} \tag{0.4.16}$$

The following technical result enables us to recognize special values of the Riemann zeta function and Dirichlet L-functions from certain integrals involving logarithms.

Lemma 0.4.6. [**75**, Lemma 9] *We have the following length-one identities:*

$$\int_0^1 \log^j x \frac{dx}{x^2-1} = (-1)^{j+1} j! \left(1 - \frac{1}{2^{j+1}}\right) \zeta(j+1), \tag{0.4.17}$$

$$\int_0^1 \log^j x \frac{dx}{x^2+1} = (-1)^j j! L(\chi_{-4}, j+1). \tag{0.4.18}$$

The next lemma simplifies certain sums of polylogarithms at certain roots of unity. This lemma will also allow us to express the sums in terms of special zeta values.

Lemma 0.4.7. *We have*

$$\sum_{\ell=0}^{2r-1} (-1)^\ell \mathrm{Li}_h(\xi_{2r}^\ell) = \frac{2-2^{1-h}}{r^{h-1}} \zeta(h),$$

$$\sum_{\ell=0}^{2r-1} (-1)^\ell \mathrm{Li}_h(-\xi_{2r}^\ell) = (-1)^r \frac{2-2^{1-h}}{r^{h-1}} \zeta(h), \tag{0.4.19}$$

and

$$\sum_{\ell=0}^{2r-1} (-1)^\ell \mathrm{Li}_h(-i\xi_{2r}^\ell) = \frac{2}{r^{h-1}} \left(\mathrm{Li}_h((-i)^r) - 2^{-h} \mathrm{Li}_h((-1)^r)\right).$$

The proofs of Lemmas 0.4.6 and 0.4.7 are included in Section 4.3.

Remark 0.4.8. *Further simplifications of $\zeta(2n)$ and $L(\chi_{-4}, 2n+1)$ are obtained in terms of Bernoulli numbers B_n (see (0.4.3)) and Euler numbers E_n, defined by*

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} \cdot t^n, \tag{0.4.20}$$

as follows

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}, \tag{0.4.21}$$

$$L(\chi_{-4}, 2n+1) = \frac{(-1)^n E_{2n} \pi^{2n+1}}{2^{2n+2}(2n)!}. \tag{0.4.22}$$

The above equalities are widely used in the derivations of our results stated in Sections 0.6.2.2 and 0.7 to obtain expressions in terms of B_n and E_n.

0.4.2. The differential form η in the bivariate case

Our main aim in this section is to establish another integral representation of the Mahler measure of a 2-variable polynomial in terms of a particular differential form.

Let C be a smooth projective curve over $\mathbb{C}$ which defines a compact Riemann surface, and let $\mathbb{C}(C)$ be its field of fractions. For $f, g \in \mathbb{C}(C)^{\times}$, we define

$$\eta(f,g) := \log|f| d\arg g - \log|g| d\arg f, \tag{0.4.23}$$

where $d\arg x$ is defined by $\mathrm{Im}(\frac{dx}{x})$. Let $S_{f,g}$ be a set containing all the zeroes and poles of f and g. Then η is a real C^{∞} differential 1-form on $C \setminus S_{f,g}$. We also note that η evaluates in $\bigwedge^2 \mathbb{C}(C)^{\times}$, the exterior product of $\mathbb{C}(C)^{\times}$. The following lemma lists some useful properties of η, which are extensively used in Section 1.5.

Lemma 0.4.9. *Let $f, g, h, v \in \mathbb{C}(C)^{\times}$ and $a, b \in \mathbb{C}^{\times}$. Then we have*

(1) $\eta(f,g) = -\eta(g,f)$, i.e. η is anti-symmetric,

(2) $\eta(fg, hv) = \eta(f,h) + \eta(g,h) + \eta(f,v) + \eta(g,v)$,

(3) $\eta(a,b) = 0$,

(4) η is a closed differential form.

(5) For $x, 1-x \in \mathbb{C}(C)^{\times}$,

$$\eta(x, 1-x) = dD(x). \tag{0.4.24}$$

where D is the Bloch–Wigner dilogarithm given by (0.4.5).

(1), (2) and (3) of Lemma 0.4.9 follow directly from the definition of η in (0.4.23). Moreover, note that, since the curve C has complex dimension 1, we have $d\eta(f,g) = \mathrm{Im}\left(\frac{df}{f} \wedge \frac{dg}{g}\right) = 0$, and this shows that (4) η is a closed differential form. Assertion (5) follows from the next two implications:

- The Bloch–Wigner dilogarithm function D is a primitive of $\eta(z, 1-z)$ for $z \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, because

$$dD(z) = \log|z| d\arg(1-z) - \log|1-z| d\arg z,$$

- Pulling back to C using $x : C\backslash S_{x,1-x} \to \mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$, we have $\eta(x, 1-x) = dD(x)$, where $D(x) := D \circ x$.

We refer the reader to [**38, 102**] for detailed proofs.

Let $P(x, y) \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ be a non-zero Laurent polynomial in two variables. We may write

$$P(x,y) = \sum_{i=0}^{d} a_i(x) y^i \in \overline{\mathbb{C}(x)}[y],$$

i.e.

$$P(x,y) = P^*(x) \prod_{j=1}^{d} (y - y_j(x)) \in \overline{\mathbb{C}(x)}[y],$$

where $P^*(x) := a_d(x) \in \mathbb{C}[x]$ is the coefficient of the highest power of y, and $y_j := y_j(x)$ are algebraic functions of x for $j \in \{1, 2, \ldots, d\}$.

We apply Jensen's formula with respect to the variable y in the standard Mahler measure formula for $P(x, y)$. Due to the above properties we get

$$\begin{aligned} \mathrm{m}\left(P(x,y)\right) - \mathrm{m}\left(P^*(x)\right) =& \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log \left| P\left(x,y\right) \right| \frac{dx}{x}\frac{dy}{y} - \mathrm{m}\left(P^*(x)\right) \\ =& \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \left(\sum_{j=1}^{d} \log \left| y - y_j\left(x\right) \right| \right) \frac{dx}{x}\frac{dy}{y} \\ =& \frac{1}{2\pi i} \left(\sum_{j=1}^{d} \int_{\{|x|=1, |y_j(x)| \geq 1\}} \log \left| y_j\left(x\right) \right| \frac{dx}{x} \right) \\ =& -\frac{1}{2\pi} \sum_{j=1}^{d} \int_{\gamma_j} \eta\left(x, y_j\right), \end{aligned} \tag{0.4.25}$$

where $\gamma_j = \{|x| = 1, |y_j(x)| \geq 1\}$.[6] When considered over γ_j,

$$\eta(x, y_j) = \log|x| d \arg y_j - \log|y_j| d \arg x = i \log|y_j(x)| \frac{dx}{x}.$$

Here we used the fact that, for $|x| = 1$, $\log|x| = \log 1 = 0$ and $\frac{dx}{x} = d(\log|x| + i \arg x)$, with $\arg(x) \in [-\pi, \pi)$.

Now, to evaluate the integrals in (0.4.25), it is convenient to arrive at one of these two ideal situations:

6. Using multiplicativity of Mahler measure, the argument extends to non-zero rational functions with complex coefficients.

— *Exact Case*: η is exact, and $\partial\gamma_j = \{\text{boundary values of } \gamma_j\} \neq \varnothing$; in this case we can integrate using Stokes' theorem. In other words, if

$$x \wedge y_j = \sum_k s_k x_{j_k} \wedge (1 - x_{j_k}) \in \left(\bigwedge^2 \mathbb{C}(C)^\times\right) \otimes \mathbb{Q},$$

then we can evaluate the integral $\int_{\gamma_j} \eta$ using (5) from Lemma 0.4.9 as

$$\int_{\gamma_j} \eta(x, y_j) = \sum_k s_k \int_{\gamma_j} \eta(x_{j_k}, 1 - x_{j_k}) = \sum_k s_k \int_{\gamma_j} dD(x_{j_k}) = \sum_k s_k \left(D(x_{j_k})\big|_{\partial\gamma_j} \right), \quad (0.4.26)$$

where D is the Bloch–Wigner dilogarithm given by (0.4.5). Here the last equality follows from Stokes' theorem.

— *Non-exact case*: η is not exact and $\partial\gamma_j = \varnothing$, i.e. the integration path γ_j is closed. This case is morally like an evaluation of residues of η. In favorable cases, we obtain special values of L-functions of curves (such as L-functions of elliptic curves).

We should note here that it can happen that η is not exact and γ_j is not closed; in most of these cases, the methods of evaluating the integral are still unknown.

Remark 0.4.10. *As mentioned in* [**117**], *we may have some extra terms of the form* $\eta\,(c, z)$ *in* (0.4.26), *where* c *is a constant complex number and* z *is some algebraic function. In that case, we can still reach a closed formula by integrating* $\eta\,(c, z)$ *directly (i.e. by integrating* $\log|c| d\arg z$*). Also, if* ν *is a constant such that* $|\nu| = 1$, *then* $\eta\,(\nu, z) = \log|\nu| d\arg z = 0$.

If C is a genus 1 non-singular curve, then a favorable integration path in *non-exact* case may belong to the first singular homology group $H_1(C, \mathbb{Z})$, which satisfies $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^2$. We can decompose $H_1(C, \mathbb{Z})$ as

$$H_1\,(C, \mathbb{Z}) = H_1\,(C, \mathbb{Z})^+ \oplus H_1\,(C, \mathbb{Z})^-\,,$$

where the first summand consists of all cycles which are invariant under complex conjugation, and the latter summand consists of the cycles which change signs.

Remark 0.4.11. *The evaluation of the integral* $\eta(x, y)$ *over a path in* $H_1(C, \mathbb{Z})^+$ *is* 0. *Indeed, the path we are considering stays invariant under complex conjugation and* $\overline{\eta(x, y)} = -\eta(x, y)$. *Therefore, we are interested in the cases where the integration path* $\{|x| =$

$1, |y_j(x)| \geq 1\}$ *is closed, and it corresponds to a cycle in the rank* 1 $\mathbb{Z}$*-submodule* $H_1(C,\mathbb{Z})^-$ *rather than just in* $H_1(C,\mathbb{Z})$.

0.4.3. The Elliptic Regulator

We will now recall the definition of the regulator map on the second K-group of an elliptic curve E, given by Bloch and Beĭlinson. Then we will explain its relation with the elliptic dilogarithm, and recover its relationship with Mahler measure. Although the derivations of our results in later chapters do not require the framework of regulators explicitly, we include this section because we believe it provides a valuable perspective to the discussion about Mahler measure of several variable polynomials in general.

Let F be a field. By a theorem of Matsumoto, the second K-group of F can be described as

$$K_2(F) \cong \Lambda^2 F^\times / \{x \otimes (1-x) : x \in F, x \neq 0, 1\}.$$

Recall that, given a two-variable Laurent polynomial $P(x,y) = \sum_{(i,j)\in\mathbb{Z}^2} a_{ij} x^i y^j$, its Newton polytope $N_\Delta(P)$ is the convex hull of the points in $(i,j) \in \mathbb{Z}^2$ such that the coefficient of $x^i y^j$ is non-zero in $P(x,y)$. Let τ denote a side of $N_\Delta(P)$. We parametrize a side clockwise around N_Δ in such a way that $\tau(0), \tau(1), \dots$ are the consecutive lattice points in τ. To every side we then associate a one-variable polynomial

$$P_\tau(U) = \sum_{l\geq 0} a_{\tau(l)} U^l \in \mathbb{C}[U],$$

where

$$a_{\tau(l)} = a_{i_{\tau(l)} j_{\tau(l)}}$$

for $\tau(l) = \left(i_{\tau(l)}, j_{\tau(l)}\right) \in \tau$. Now we have the following definition due to Rodriguez-Villegas [**102**].

Definition 0.4.12. $P(x,y)$ *is called* ***tempered*** *if* $\mathrm{m}\,(P_\tau) = 0$ *for every* τ.

In other words, P is *tempered* is equivalent, by Kronecker's Theorem 0.1.1, to requiring $\mathrm{m}(P_\tau) = 0$ for all τ. In fact, this condition plays a role in understanding the K-theory framework of the regulator. For that, we now need to define regulator maps.

For a field F with discrete valuation ν and maximal ideal $\mathcal{M}$, the tame symbol is given by [**102**]

$$(x,y)_\nu \equiv (-1)^{\nu(x)\nu(y)} \frac{x^{\nu(y)}}{y^{\nu(x)}} \pmod{\mathcal{M}}.$$

Note that in particular, $(x,y)_\nu = 1$ if $\nu(x) = \nu(y) = 0$.

Let C be a smooth projective curve over $\mathbb{C}$ which is a compact Riemann surface, and $\mathbb{C}(C)$ be its field of fractions. A point $P \in C(\mathbb{C})$ defines a valuation ν_P on $\mathbb{C}(C)$, which is determined by the order of the rational functions at the point $P \in C(\mathbb{C})$. We follow the notation in [**102**] to denote the tame symbol given by ν_P as $(\cdot,\cdot)_P$. We also have the residue map, which is a linear form determined by P,

$$\mathrm{Res}_P : H^1(C \setminus \{P\}, \mathbb{R}) \to \mathbb{R},$$

and

$$\mathrm{Res}_P(\eta(x,y)) = \log |(x,y)_P|,$$

where $P \in C(\mathbb{C})$, $x, y \in \mathbb{C}(C)^\times$, $S \subset C(\mathbb{C})$ a finite set containing poles and zeroes of x and y, η is the differential form given by (0.4.23).

Further note that, for a closed path γ in $C \setminus S$, the map

$$\gamma \mapsto \int_\gamma \eta(x,y)$$

only depends on the homology class $[\gamma] \in H_1(C \setminus S, \mathbb{Z})$, and it therefore determines an element in $H^1(C \setminus S, \mathbb{R})$, say $\bar{r}(x,y)$. From (0.4.24) we also have $\eta(x, 1-x) = 0$ in $H^1(C \setminus S, \mathbb{R})$, i.e.

$$\int_\gamma \eta(x, 1-x) = 0 \quad \forall\ [\gamma] \in H_1(C \setminus S, \mathbb{Z}).$$

Given a finite set $S \subset C$, we can define

$$K_{2,S}(C) = \bigcap_{P \notin S} \ker \lambda_P \subset K_2(\mathbb{C}(C)),$$

where $\lambda_P : K_2(\mathbb{C}(C)) \to \mathbb{C}^\times$ is the corresponding map of the tame symbol $(\cdot,\cdot)_P$. Then the following diagram is commutative for every $P \in S$:

$$\begin{array}{ccc} K_{2,S}(C) & \xrightarrow{\bar{r}} & H^1(C \setminus S, \mathbb{R}) \\ \downarrow{\scriptstyle \lambda_P} & & \downarrow{\scriptstyle \mathrm{Res}_P} \\ \mathbb{C}^\times & \xrightarrow{\log|\cdot|} & \mathbb{R} \end{array}$$

Finally, for an elliptic curve E over $\mathbb{Q}$, we can define a tame symbol corresponding to a point $T \in E(\bar{\mathbb{Q}})$ as a map from $K_2(\mathbb{Q}(E))$ to $\mathbb{Q}(T)^\times$. We also have an exact sequence

$$0 \to K_2(E) \otimes \mathbb{Q} \to K_2(\mathbb{Q}(E)) \otimes \mathbb{Q} \to \coprod_{T \in E(\bar{\mathbb{Q}})} \mathbb{Q}(T)^\times \times \mathbb{Q},$$

where the last arrow corresponds to the coproduct of the tame symbols (for more details see [**82**]).

Following the above discussion, we interpret $H^1(E,\mathbb{R})$ as the dual of the first homology group of E with coefficients in $\mathbb{Z}$, namely $H_1(E,\mathbb{Z})$. Let $[\gamma] \in H_1(E,\mathbb{Z})$. Now we can define the regulator map.

Definition 0.4.13. *The regulator map of Bloch* [**22**] *and Beĭlinson* [**15**] *is given by*

$$\begin{aligned} r_E : K_2(E) \otimes \mathbb{Q} &\to H^1(E,\mathbb{R}) \\ \{x,y\} &\to \left\{ [\gamma] \to \int_\gamma \eta(x,y) \right\}. \end{aligned}$$

Remark 0.4.14. *We should note that the regulator is essentially defined over the Néron model $\mathcal{E}$ of E, and $K_2(\mathcal{E}) \otimes \mathbb{Q}$ is a subgroup of $K_2(E) \otimes \mathbb{Q}$ determined by finitely many extra conditions* [**21**].

The condition of $P(x,y)$ being tempered can be seen to be equivalent to the triviality of tame symbols in K-theory [**102**]. Thus, it gives us a way to produce elements in $K_{2,\emptyset}(E)$, where E is an elliptic curve over $\mathbb{Q}$. We can therefore define a map

$$\tilde{r} : K_{2,\emptyset}(E) \to \mathbb{R}, \qquad \varphi \mapsto \frac{1}{2\pi} \bar{r}(\varphi)(c_0),$$

where $c_0 \in H_1(E, \mathbb{Z})$ is the cycle determined by the connected component of $E(\mathbb{R})$. Deninger's [**45**] derivation

$$\mathrm{m}(P) = \frac{1}{2\pi}\bar{r}(\{x, y\})[\gamma] \tag{0.4.27}$$

then establishes a relation between regulators and Mahler measures of polynomials.

The following three sections are dedicated to presenting our results, providing necessary background information, and discussing their applications in the literature.

0.5. Generalized Mahler measure

This section aims to provide an introductory overview for Chapter 1 containing results from [**106**].

Cassaigne and Maillot [**41**] generalized the formula found by Smyth (see (0.1.7)) to $\mathrm{m}(ax+by+c)$ for arbitrary complex constants $a, b,$ and c :

$$\mathrm{m}(ax+by+c) = \begin{cases} \frac{\alpha}{\pi}\log|a| + \frac{\beta}{\pi}\log|b| + \frac{\gamma}{\pi}\log|c| + \frac{D\left(e^{2i\alpha}\right)}{2\pi} + \frac{D\left(e^{2i\beta}\right)}{2\pi} + \frac{D\left(e^{2i\gamma}\right)}{2\pi} & \text{if } \Delta \text{ holds,} \\ \\ \log\max\{|a|, |b|, |c|\} & \text{otherwise,} \end{cases} \tag{0.5.1}$$

where Δ stands for the statement that $|a|, |b|,$ and $|c|$ are the lengths of the sides of a planar triangle. In this case, $\alpha, \beta,$ and γ are the angles opposite to the sides of lengths $|a|, |b|$ and $|c|$ respectively (see Figure 1) and D is the Bloch–Wigner Dilogarithm defined in (0.4.5). We also remark that the condition Δ can also be interpreted as representing the values $(|a|, |b|, |c|)$ such that $ax + by + c$ vanishes on the unit torus.

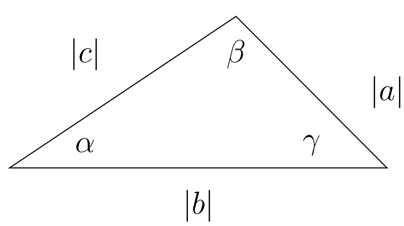

Figure 1 – Condition Δ in Cassaigne and Maillot's formula

Notice that the constant coefficient may be multiplied by a variable without changing the Mahler measure, in the sense that $\mathrm{m}(ax+by+c) = \mathrm{m}(ax+by+cz)$. Additionally, it is immediate to see that Cassaigne and Maillot's result can also be interpreted as

$$\mathrm{m}(ax+by+cz) = \frac{1}{(2\pi i)^3} \int_{\mathbb{T}^3_{|a|,|b|,|c|}} \log|x+y+z| \frac{dx}{x}\frac{dy}{y}\frac{dz}{z},$$

i.e. the standard Mahler measure of $ax+by+cz$ is same as the integral of $\log|x+y+z|$ with respect to the Haar measure $\frac{dx}{x}\frac{dy}{y}\frac{dz}{z}$ over the torus $\mathbb{T}^3_{|a|,|b|,|c|}$, where

$$\mathbb{T}^3_{|a|,|b|,|c|} = \{(x,y,z) \in \mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}^\times : |x| = |a|, |y| = |b|, |z| = |c|\}.$$

This representation of $\mathrm{m}(ax+by+cz)$ makes (0.5.1) a generalization of Smyth's result, and it motivates the following definition.

Definition 0.5.1. *Let $\mathfrak{a} = (a_1, \ldots, a_n) \in (\mathbb{R}_{>0})^n$. The **generalized Mahler measure** of a non-zero rational function $P \in \mathbb{C}(x_1, \ldots, x_n)$ is defined as*

$$\mathrm{m}_{\mathfrak{a}}(P) = \mathrm{m}_{a_1,\ldots,a_n}(P(x_1,\ldots,x_n)) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n_{\mathfrak{a}}} \log|P(x_1,\ldots,x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where

$$\mathbb{T}^n_{\mathfrak{a}} := \{(x_1,\ldots,x_n) \in \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times : |x_1| = a_1, \ldots, |x_n| = a_n\}.$$

Lalín and Mittal [**80**] explored this definition over $\mathbb{T}^2_{a^2,a}$ and $\mathbb{T}^2_{a,a}$ to obtain relations between certain polynomials mentioned in Boyd's paper [**29**], namely

$$\begin{aligned} R_{-2}(x,y) &:= (1+x)(1+y)(x+y) + 2xy, \\ S_{2,-1}(x,y) &:= y^2 + 2xy - x^3 + x, \end{aligned}$$

for some values of $a \in \mathbb{R}_{>0}$. They simultaneously evaluated $\mathrm{m}_{a^2,a}(R_{-2})$ and $\mathrm{m}_{a,a}(S_{2,-1})$ in terms of $\log a$ and special values of L-functions when the polynomials do not vanish on the respective integration torus. In particular, they established a relation between the standard Mahler measure and the generalized Mahler measure. They showed that

$$\mathrm{m}_{a,a}(S_{2,-1}(x,y)) = \begin{cases} 2\log a + 2L'(\tilde{E}_{20}, 0) & \frac{\sqrt{5}-1}{2} \le a \le \frac{\sqrt{5}+1}{2}, \\ 3\log a & a \ge \frac{3+\sqrt{13}}{2}, \\ \log a & 0 < a \le \frac{-3+\sqrt{13}}{2}, \end{cases} \tag{0.5.2}$$

and

$$\mathrm{m}_{a^2,a}(R_{-2}(x,y)) = 3\log a + 3L'(\tilde{E}_{20},0), \tag{0.5.3}$$

when

$$\sqrt{\frac{1+\sqrt{5}-\sqrt{2\sqrt{5}+2}}{2}} \le a \le \sqrt{\frac{1+\sqrt{5}+\sqrt{2\sqrt{5}+2}}{2}}.$$

Here $\tilde{E}_{20} : Y^2 - 4XY - 2Y = X^3$ is the elliptic curve of conductor 20 birationally equivalent to the polynomial R_{-2} via the change of variables in (0.3.6), and

$$L'(\tilde{E}_{20},0) = \frac{20}{4\pi^2}L(\tilde{E}_{20},2).$$

The change of variables

$$X = \frac{(m+1)(x+y)}{x+y-m}, \qquad x = \frac{(m+2)X+2Y}{2(X-m-1}, $$

$$Y = \frac{(m+1)((m-2)x-(m+2)y}{2(x+y-m)}, \qquad y = \frac{(m-2)X-2Y}{2(X-(m+1))},$$

further gives birational maps between R_{-2} and $S_{2,-1}$ when $m=-2$. More generally, it gives a birational transformation

$$\psi : R_m(x,y) \to E_m^{'}(X,Y),$$

where $E_m^{'} : Y^2 + 2XY - \left(X^3 + \left(\frac{m^2}{4} - m - 3\right)X^2 + (m+1)x\right) = 0$, and R_m is given in (0.3.5).

In Chapter 1, we provide a way to obtain relations similar to (0.5.2) and (0.5.3) for a large set of Laurent polynomials. Our search started with the family of Boyd's polynomials mentioned in (0.1.10), namely

$$\left\{\mathcal{Q}_r(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + r : r \in \mathbb{C}\right\}. \tag{0.5.4}$$

An extension of the methods in **[102]** and **[9]** led us to an interesting fact: for an arbitrarily fixed $(a,b) \in \mathbb{R}^2_{>0}$, there exists a large set of $r \in \mathbb{C}$ such that the Mahler measures of these polynomials remain the same irrespective of deforming the integration torus from $\mathbb{T}^2$ $(= \mathbb{T}^2_{1,1})$ to $\mathbb{T}^2_{a,b}$. In fact, we found that this method can be extended to all Laurent polynomials in n variables (where $n \ge 2$) when they do not vanish on the integration torus.

Let $P_k(x_1, \dots, x_n) \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ be a non-zero Laurent polynomial in n variables such that

$$P_k := P_k(x_1, \dots, x_n) = k - P(x_1, \dots, x_n), \tag{0.5.5}$$

where P has no constant term. [7] Let $\mathbb{T}^n_{\mathfrak{a}}$ be the integration torus in the definition of $\mathrm{m}_{\mathfrak{a}}(P_k)$, where $\mathfrak{a} = (a_1, \dots, a_n)$ such that $a_1, \dots, a_n > 0$, and let $\mathcal{K}_{\mathfrak{a}}$ be the image of the map

$$p : \mathbb{T}^n_{\mathfrak{a}} \to \mathbb{C} \quad \text{defined by} \quad (x_1, \dots, x_n) \mapsto P(x_1, \dots, x_n). \tag{0.5.6}$$

Let $\nu^j_{\mathfrak{a},k}$ be the difference between the number of zeroes (counting multiplicities) of $P_k(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_n)$ inside the circle $\mathbb{T}^1_{a_j}$, denoted by $Z^j_{\mathfrak{a},k}$, and the order of the pole of $P_k(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_n)$ at $x_j = 0$, denoted by $P^j_{\mathfrak{a},k}$. In other words,

$$\nu^j_{\mathfrak{a},k} = Z^j_{\mathfrak{a},k} - P^j_{\mathfrak{a},k}.$$

Then, we have the following theorem.

Theorem 0.5.2 ([**106**, Theorem 1.2])**.** *Let $\mathfrak{a} = (a_1, \dots, a_n) \in (\mathbb{R}_{>0})^n$. Let $P_k(x_1, \dots, x_n) = k - P(x_1, \dots, x_n) \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$, such that P has no constant term. Denote $U_{\mathfrak{a}}$ the unbounded open connected component of $\mathbb{C} \setminus \mathcal{K}_{\mathfrak{a}}$ containing a neighbourhood of $k = \infty$. Then, for $k \in U_{\mathfrak{a}} \cap U_{\mathbf{1}}$,*

$$\mathrm{m}_{\mathfrak{a}}(P_k) = \mathrm{m}_{\mathbf{1}}(P_k) + \sum_{j=1}^{n} \nu^j_{\mathfrak{a},k} \log a_j, \tag{0.5.7}$$

where $\nu^j_{\mathfrak{a},k}$ is defined as above, and $\mathrm{m}_{\mathbf{1}}(P_k) = \mathrm{m}(P_k)$. Moreover, for $k \in U_{\mathfrak{a}} \cap U_{\mathbf{1}}$ and $j = 1, \dots, n$, $\nu^j_{\mathfrak{a},k}$ only depends on $\mathfrak{a}$.

Remark 0.5.3. *Notice that any non-zero Laurent polynomial $G_\ell \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ in the family $\{G_\ell : \ell \in \mathbb{C}\}$ is expressible as*

$$G_\ell(x_1, \dots, x_n) = \ell x_1^{\ell_1} \cdots x_n^{\ell_n} - \sum_{\substack{(j_1,\dots,j_n)\in\mathbb{Z}^n \\ (j_1,\dots,j_n)\neq(\ell_1,\dots,\ell_n)}} a_{j_1,\dots,j_n} x_1^{j_1} \cdots x_n^{j_n} = x_1^{\ell_1} \cdots x_n^{\ell_n} \left[\ell - G(x_1, \dots, x_n)\right],$$

7. In fact, any Laurent polynomial can be expressed as the sum of its constant term and a Laurent polynomial with no constant term, as shown in (0.5.5).

where the above sum is finite, $(\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n$*, and* $G(0, \ldots, 0) = 0$*. Then* (0.5.7) *implies that, for* $\ell \in U_{\mathfrak{a},G} \cap U_{\mathbf{1},G}$*,*

$$\mathrm{m}_{\boldsymbol{a}}(G_\ell) = \mathrm{m}_{\boldsymbol{a}}(\tilde{G}_\ell) + \sum_{t=1}^{n} \ell_t \log a_t = \mathrm{m}_{\mathbf{1}}(\tilde{G}_\ell) + \sum_{t=1}^{n} (\nu_{\mathfrak{a},\ell}^t + \ell_t) \log a_t,$$

where $\tilde{G}_\ell(x_1, \ldots, x_n) = \ell - G(x_1, \ldots, x_n)$*. Here* $U_{\mathfrak{a},G}$*,* $U_{\mathbf{1},G}$ *and* $\nu_{\mathfrak{a},\ell}^t$ *are defined similarly as above for* $\tilde{G}_\ell$*. Furthermore,*

$$\bigcup_{(\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n} \{G_\ell \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}] : \ell \in \mathbb{C}\} = \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}],$$

i.e. any Laurent polynomial in n *variables belongs to at least one of the families* $\{G_\ell \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}] : \ell \in \mathbb{C}\}$*. Therefore, we conclude that our result in* (0.5.7) *extends to a larger set of* n*-variable Laurent polynomial.*

For $|k|$ large enough, the relation (0.5.7) between the standard Mahler measure and the generalized Mahler measure of P_k can be obtained by first expanding $\log\left(1 - \frac{P}{k}\right)$ in a convergent series, and then integrating each term individually. We should mention here that, in order to obtain a convergent series expansion of the logarithm, the above procedure is restricted to a smaller subregion contained in the unbounded region of $\mathbb{C} \setminus \mathcal{K}_{\mathfrak{a}}$. Theorem 0.5.2 establishes this equality for a larger set, and since the Mahler measure is the real part of an analytic function **[102]** (in other words, it is harmonic), equality (0.5.7) holds for all k in the unbounded open connected component of $\mathbb{C} \setminus \mathcal{K}_{\mathfrak{a}}$. In particular, we note that the region $U_{\mathfrak{a}} \cap U_{\mathbf{1}}$ contains a neighborhood of $k = \infty$, namely the region

$$\left\{k \in \mathbb{C} : |k| > \max\left\{\max_{(x_1, \ldots, x_n) \in \mathbb{T}_{\mathfrak{a}}^n} |P(x_1, \ldots, x_n)|, \max_{(x_1, \ldots, x_n) \in \mathbb{T}^n} |P(x_1, \ldots, x_n)|\right\}\right\}.$$

Indeed, note that for all those k, $\log\left(1 - \frac{P}{k}\right)$ is well defined and can be expanded in a convergent series, as mentioned above. Also, note that the region is therefore unbounded, and its complement is contained in

$$\left\{k \in \mathbb{C} : |k| \leq \max\left\{\max_{(x_1, \ldots, x_n) \in \mathbb{T}_{\mathfrak{a}}^n} |P(x_1, \ldots, x_n)|, \max_{(x_1, \ldots, x_n) \in \mathbb{T}^n} |P(x_1, \ldots, x_n)|\right\}\right\},$$

which is closed and bounded.

Let $Q(x,y)$ be a non-zero Laurent polynomial in $\mathbb{C}[x^{\pm}, y^{\pm}]$ with no constant term, and define the family of Laurent polynomials $\{Q_r(x,y) : r \in \mathbb{C}\}$ associated to Q as

$$Q_r(x,y) = r - Q(x,y) \in \mathbb{C}[x^{\pm}, y^{\pm}].$$

For $a, b > 0$, let $\mathcal{R}_{a,b}$ be the image of the map

$$q : \mathbb{T}^2_{a,b} \longrightarrow \mathbb{C}, \quad \text{defined by} \quad (x,y) \mapsto Q(x,y). \tag{0.5.8}$$

Then, as a corollary to Theorem 0.5.2, we have the following result in two variables.

Theorem 0.5.4 ([**106**, Theorem 1.3])**.** *Let a and b be positive real numbers, and denote by $U_{a,b}$ the unbounded open connected component of $\mathbb{C} \setminus \mathcal{R}_{a,b}$ containing some neighborhood of $r = \infty$. Then, for $r \in U_{a,b} \cap U_{1,1}$,*

$$\mathrm{m}_{a,b}(Q_r) = \mathrm{m}(Q_r) + \nu^1_{a,b,r} \log a + \nu^2_{a,b,r} \log b,$$

where $\nu^1_{a,b,r}$ is the difference between the number of zeroes (denoted by $Z^1_{a,b,r}$) and the number of poles (denoted by $P^1_{a,b,r}$) of $Q_r(x,b)$ inside the circle $|x| = a$, defined by

$$\nu^1_{a,b,r} = Z^1_{a,b,r} - P^1_{a,b,r}, \tag{0.5.9}$$

$\nu^2_{a,b,r}$ is the difference between the number of zeroes (denoted by $Z^2_{a,b,r}$) and the number of poles (denoted by $P^2_{a,b,r}$) of $Q_r(a,y)$ inside the circle $|y| = b$, defined by

$$\nu^2_{a,b,r} = Z^2_{a,b,r} - P^2_{a,b,r}, \tag{0.5.10}$$

and $\mathrm{m}_{1,1}(Q_r) = \mathrm{m}(Q_r)$. Moreover, for $r \in U_{a,b} \cap U_{1,1}$, $\nu^j_{a,b,r}$ does not depend on r.

A follow-up question can be posed regarding the values of $\mathrm{m}_{\mathfrak{a}}(P_k)$ when k belongs to one of the bounded open connected components of $\mathbb{C} \setminus \mathcal{K}_{\mathfrak{a}}$.[8] The next theorem answers this question when $\nu^j_{\mathfrak{a},k}$ satisfies a particular condition.

We introduce some necessary notation to state the next result. Multiplying P_k with a suitable power of x_j, we can factorise P_k in linear factors with coefficients in $\overline{\mathbb{C}(x_1, \ldots, \widehat{x_j}, \ldots, x_n)}$

8. If $\mathbb{C} \setminus \mathcal{K}_{\mathfrak{a}}$ does not contain any open bounded connected component, then $U_{\mathfrak{a}}$ contains all values of k such that P_k does not vanish on $\mathbb{T}^n_{\mathfrak{a}}$, and therefore, for all such k, $\mathrm{m}_{\mathfrak{a}}(P_k)$ satisfies (0.5.7).

as

$$P_k(x_1,\dots,x_n) = x_j^{-v_j} P^j_{F,k}(x_1,\dots,\widehat{x_j},\dots,x_n) \prod_{l=1}^{d_n} \left(x_j - X_{l,k,j}\left(x_1,\dots,\widehat{x_j},\dots,x_n\right)\right), \quad (0.5.11)$$

where d_j is the degree of P_k as a polynomial in x_j, $X_{l,k,j}$ are algebraic functions of $(x_1,\dots,\widehat{x_j},\dots,x_n)$ for $l=1,\dots,d_n$, $P^j_{F,k}$ is the *leading* coefficient with respect to the variable x_j, and v_j is the largest power of x_j^{-1} in P_k. Let $P^j_{f,k}(x_1,\dots,\widehat{x_j},\dots,x_n)$ denote the *constant* coefficient with respect to the variable x_j. Then

$$P^j_{F,k}(x_1,\dots,\widehat{x_j},\dots,x_n) \prod_{j=1}^{d_n} X_{l,k,j}(x_1,\dots,\widehat{x_j},\dots,x_n) = P^j_{f,k}(x_1,\dots,\widehat{x_j},\dots,x_n). \quad (0.5.12)$$

Suppose $\mathbb{C}\setminus\mathcal{K}_{\mathfrak{a}}$ contains at least one open bounded connected component, then we have the following theorem.

Theorem 0.5.5 ([**106**, Theorem 1.4])**.** *Let* $\mathfrak{a}=(a_1,\dots,a_n)\in\mathbb{R}^n_{>0}$. *Let* $k_0\in\mathbb{C}\setminus\mathcal{K}_{\mathfrak{a}}$ *such that* k_0 *belongs to one of the bounded open connected components of* $\mathbb{C}\setminus\mathcal{K}_{\mathfrak{a}}$*, denoted by* $V_{\mathfrak{a},k_0}$.

(I) For $j=1,\dots,n$*, if all the roots of* $P_{k_0}(a_1,\dots,a_{j-1},x_j,a_{j+1},\dots,a_n)$ *lie entirely inside the circle* $\mathbb{T}^1_{a_j}$*, then, for all* $k\in V_{\mathfrak{a},k_0}$,

$$\mathrm{m}_{\mathfrak{a}}(P_k) = \nu^j_{\mathfrak{a},k}\log a_j + \mathrm{m}_{a_1,\dots,\widehat{a_j},\dots,a_n}\left(P^j_{F,k}\right).$$

(II) For $j=1,\dots,n$*, if all the roots of* $P_{k_0}(a_1,\dots,a_{j-1},x_j,a_{j+1},\dots,a_n)$ *lie entirely outside the circle* $\mathbb{T}^1_{a_j}$*, then, for all* $k\in V_{\mathfrak{a},k_0}$,

$$\mathrm{m}_{\mathfrak{a}}(P_k) = \nu^j_{\mathfrak{a},k}\log a_j + \mathrm{m}_{a_1,\dots,\widehat{a_j},\dots,a_n}\left(P^j_{f,k}\right).$$

Similarly, for the 2-variable case, $Q_r(x,y)$, when considered as a polynomial in y (resp. x) of degree d_y (resp. d_x) with coefficients in $\overline{\mathbb{C}(x)}$ (resp. $\overline{\mathbb{C}(y)}$), can be expressed as

$$\begin{aligned} Q_r(x,y) =& y^{-v_2}\left(Q^y_{F,r}(x)y^{d_y} + Q^y_{f,r}(x) + \sum_{j=1}^{d_y-1} a^y_{j,r}(x)y^j\right) \\ =& x^{-v_1}\left(Q^x_{F,r}(y)x^{d_x} + Q^x_{f,r}(y) + \sum_{j=1}^{d_x-1} a^x_{j,r}(y)x^j\right), \end{aligned}$$

where v_1 and v_2 denote the largest powers of x^{-1} and y^{-1} in $Q_r(x,y)$, respectively, and $Q^u_{F,r}$ and $Q^u_{f,r}$ are the respective *leading* and *constant* coefficient with respect to the variable u, for $u = x$ or y. Suppose $\mathbb{C} \setminus \mathcal{R}_{a,b}$ contains at least one open bounded connected component, then, again as a corollary to Theorem 0.5.5, we have the following result.

Theorem 0.5.6 ([**106**, Theorem 1.2]). *Let a and b be positive real numbers. Let $r_0 \in \mathbb{C}\setminus\mathcal{R}_{a,b}$ such that r_0 belongs to one of the bounded open connected components of $\mathbb{C}\setminus\mathcal{R}_{a,b}$. We denote by V_{a,b,r_0} the bounded open connected component containing r_0.*

(i) If all the roots of $Q_{r_0}(a,y)$ either lie entirely inside the circle $\mathbb{T}^1_b$ or lie entirely outside the circle $\mathbb{T}^1_b$, then, for all $r \in V_{a,b,r_0}$,

$$\mathrm{m}_{a,b}(Q_r) - \nu^2_{a,b,r}\log b = \begin{cases} \mathrm{m}_a(Q^y_{F,r}(x)) & \text{when all roots of } Q_{r_0}(a,y) \text{ lie inside } \mathbb{T}^1_b, \\ \mathrm{m}_a(Q^y_{f,r}(x)) & \text{when all roots of } Q_{r_0}(a,y) \text{ lie outside } \mathbb{T}^1_b. \end{cases}$$

(ii) If all the roots of $Q_{r_0}(x,b)$ either lie entirely inside the circle $\mathbb{T}^1_a$ or lie entirely outside the circle $\mathbb{T}^1_a$, then, for all $r \in V_{a,b,r_0}$,

$$\mathrm{m}_{a,b}(Q_r) - \nu^1_{a,b,r}\log a = \begin{cases} \mathrm{m}_b(Q^x_{F,r}(y)) & \text{when all roots of } Q_{r_0}(x,b) \text{ lie inside } \mathbb{T}^1_a, \\ \mathrm{m}_b(Q^x_{f,r}(y)) & \text{when all roots of } Q_{r_0}(x,b) \text{ lie outside } \mathbb{T}^1_a. \end{cases}$$

Using Theorems 0.5.4 and 0.5.6, Cassaigne and Maillot's result in (0.5.1) follows immediately when the condition Δ does not hold. In this case, let $M_c(x,y) := c - x - y$ for $c \in \mathbb{C}$. For $a, b \in \mathbb{C}^\times$, $\mathcal{R}_{|a|,|b|}$ is now the *closed* annulus $\{z \in \mathbb{C} : |z| \in [||a| - |b||, |a| + |b|]\}$. Note that, when c belongs to the unbounded component of $\mathbb{C} \setminus \mathcal{R}_{|a|,|b|}$, we have $\nu^j_{|a|,|b|,c} = 0$. Then, Theorem 0.5.4 and harmonic properties of Mahler measure imply that, when $|c| > |a| + |b|$,

$$\mathrm{m}_{|a|,|b|}(M_c) = \mathrm{m}(|a|x + |b|y + c) = \log |c|.$$

On the other hand, Theorem 0.5.6 implies that, for $|c| < ||a| - |b||$,

$$\mathrm{m}_{|a|,|b|}(M_c) = \log\max\{|a|, |b|\},$$

since $\nu^1_{|a|,|b|,c} = 1$ (resp. $\nu^2_{|a|,|b|,c} = 1$) when $|a| > |b|$ (resp. $|b| > |a|$). The combination of both equalities leads to a restatement of (0.5.1) when Δ does not hold. We should remark that the condition Δ in (0.5.1) is equivalent to the condition $c \in \mathcal{R}_{|a|,|b|}$, i.e. M_c vanishes on the integration torus. A more involved approach, using Theorems 0.5.4 and 0.5.6 on the family of Laurent polynomials

$$\begin{aligned} R^*_\alpha(x,y) &:= \alpha - x^{-1} - y^{-1} - xy^{-1} - yx^{-1} - x - y, \qquad \alpha \in \mathbb{C}, \\ S^*_{\beta,-1}(x,y) &:= \beta - x^{-1}y + x^2y^{-1} - y^{-1}, \qquad \beta \in \mathbb{C}, \end{aligned}$$

re-establishes the identities obtained in [**80**] for $\alpha = -4$ and $\beta = 2$, since $R_{-2}(x,y) = xy \cdot R^*_{-2}(x,y)$ and $S_{2,-1}(x,y) = xy \cdot S^*_{2,-1}(x,y)$. Note that the aforementioned results involving the generalized Mahler measure of R^*_{-2} (resp. $S^*_{2,-1}$) on the torus $\mathbb{T}^2_{a,b}$ only depend on b since the integration torus is $\mathbb{T}^2_{b^2,b}$ (resp. $\mathbb{T}^2_{b,b}$), i.e. a is a function of b here. Our results, along with the method of the *Lagrange multiplier*, provide a larger set of pairs $(a,b) \in \mathbb{R}^2_{>0}$, such that similar types of identities obtained in [**80**] hold even when a is not a function of b. An analogous statement is exhibited in Section 1.5 in Chapter 1 with a different family of polynomials investigated by Boyd [**29**], namely the family given in (0.5.4).

Given an expression of $\mathrm{m}(P)$ in terms of special values of L-functions, we note that our result establishes a vast amount of identities of the form $\mathrm{m}(Q) = rL'(E,0) + \log|s|$, which were conjectured by Boyd [**29**] for non-tempered polynomials. Here $r \in \mathbb{Q}^\times$, $s \in \bar{\mathbb{Q}}^\times$, $Q(x,y) := P(ax,by)$, where $a,b \in \mathbb{Q}_{>0}$, and E is an elliptic curve associated to P, as well as Q. Indeed, note that, if $a,b \neq 1$, then at least one of the faces of the Newton polytope of Q has non-zero Mahler measure, which makes Q a non-tempered polynomial according to the Definition 0.4.12. Then, for certain non-zero positive rational values of a and b, Theorem 0.5.4 yields such equalities.

Due to the technical difficulties involving the study of the integration path in the definition of Mahler measure, it is challenging to evaluate $\mathrm{m}_{\mathfrak{a}}(P_k)$ explicitly for all $\mathfrak{a} \in (\mathbb{R}_{>0})^n$. In this regard, Theorems 0.5.2 and 0.5.5 have a common feature: *the Laurent polynomial in consideration does not vanish on the integration torus.* Since the methods of proofs are the same for Theorem 0.5.2 (resp. Theorem 0.5.5) and Theorem 0.5.4 (resp. Theorem 0.5.6), we provide proofs of Theorems 0.5.4 and 0.5.6 in Sections 1.2.1 and 1.3, and outline arguments generalizing our methods to derive Theorems 0.5.2 and 0.5.5.

The next statement considers a particular polynomial from our initial family of polynomials in (0.5.4), namely

$$\mathcal{Q}_4(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + 4.$$

It removes the constraint of being non-zero on the integration torus, and evaluates the generalized Mahler measure of $\mathcal{Q}_4(x,y)$ for all $a, b > 0$.

Theorem 0.5.7 ([**106**, Theorem 1.6])**.** *Let $a, b \in \mathbb{R}_{>0}$, and define*

$$c = \sqrt{ab}, \quad d = \sqrt{\frac{b}{a}}, \quad \text{and } \mathcal{A}_{c,d} = \frac{1-d^2}{1+d^2} \cdot \frac{1+c^2}{2c},$$

such that c and d are both positive real numbers. Then,

$$\mathrm{m}_{a,b}(\mathcal{Q}_4(x,y)) = \begin{cases} |\log c| + |\log d| & \text{if } |\mathcal{A}_{c,d}| \geq 1, \\ \frac{2}{\pi}\left[D(ice^{-i\mu}) + D(ice^{i\mu}) - \mu \log d + (\log c)\arctan\left(\frac{c-c^{-1}}{2\cos\mu}\right)\right] & \text{if } |\mathcal{A}_{c,d}| < 1, \end{cases}$$

where $\mu = \sin^{-1}(\mathcal{A}_{c,d}) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and D is the Bloch–Wigner dilogarithm given by (0.4.5).

Under a certain change of variables, the polynomial above can be factored into two linear polynomials [**29**]. This simplification, along with a direct approach involving the differential form η and the Bloch–Wigner dilogarithm (see (0.4.24)), leads us to the explicit formula in the statement of Theorem 0.5.7.

We end this section with a brief description of how the generalized Mahler measure affects the integral (0.4.25) involving the differential form η.

0.5.1. Generalized Mahler measure and the differential form η

We analyze the generalized Mahler measure of a non-zero 2-variable Laurent polynomial P over an arbitrary torus $\mathbb{T}^2_{a,b}$. The following brief description essentially reproduces the analysis in [**80**, Section 3]. For simplicity, we take $d = 2$, where d is the degree of y in $P(x,y)$ once P is multiplied by a suitable power of y to remove any negative power of y.

Let $x = ax'$ and $y = by'$. Then we have, for $P^*(x) \in \mathbb{C}[x]$,

$$\begin{aligned}
\mathrm{m}_{a,b}(P(x,y)) - \mathrm{m}_{a,b}(P^*(x)) =& \frac{1}{(2\pi i)^2} \iint_{|x'|=|y'|=1} \log |P(ax', by')| \frac{dx'}{x'} \frac{dy'}{y'} - \mathrm{m}_{a,b}(P^*(x)) \\
=& 2\log b + \frac{1}{2\pi i} \left(\sum_{j=1}^{2} \int_{|x'|=1, |y_j'|\geq 1} \log |y_j'| \frac{dx'}{x'} \right) \qquad (0.5.13) \\
=& 2\log b - \frac{1}{2\pi} \sum_{j=1}^{2} \int_{|x|=a, |y_j|\geq b} \eta(x/a, y_j/b). \qquad (0.5.14)
\end{aligned}$$

where $y_j = y_j(x) = by_j'$ are algebraic functions of x for $j = 1, 2$, and

$$\eta(x/a, y_j/b) = \eta(x', y_j') = i \log |y_j'| \frac{dx'}{x'},$$

for $j = 1, 2$. The penultimate equality (0.5.13) follows from Jensen's formula. Further simplification of the terms involving the y_j's and the application of (2) of Lemma 0.4.9 imply

$$\mathrm{m}_{a,b}(P(x,y)) - \mathrm{m}_{a,b}(P^*(x)) = 2\log b - \frac{1}{2\pi} \sum_{j=1}^{2} \int_{|x|=a, |y_j|\geq b} [\eta(x, y_j) - \eta(a, y_i) - \eta(x, b)].$$

If $\{|x| = a, |y_j| \geq b\}$ is a closed path, then the integral

$$\frac{1}{2\pi} \sum_{j=1}^{2} \int_{|x|=a, |y_j|\geq b} \eta(x/a, y_j/b)$$

can be evaluated using Stokes' theorem (see **[45]**). In addition, if $\{|x| = a, |y_i| \geq b\}$ is a closed path, the term

$$\frac{1}{2\pi} \int_{|x|=a, |y_j|\geq b} \eta(a, y_i) = \frac{\log a}{2\pi} \int_{|x|=a, |y_j|\geq b} d\arg y_j$$

becomes a multiple of $\log a$. As mentioned in the paragraph preceding this section, note that if we have a genus 0 curve (such as $C_4 : \mathcal{Q}_4(x, y) = 0$) then, instead of proceeding in the direction above, we may be able to use (0.4.24) to relate the Bloch–Wigner dilogarithm and η to evaluate the Mahler measure. The evaluation is much simpler in this case as we will see in the proof of Theorem 1.1.6.

0.6. Areal Mahler measure

In this section, we describe the necessary introductory material needed for Chapter 2 and Chapter 3, which consist of results from two joint works with Lalín [**84, 83**].

A natural counterpart of the Mahler measure is obtained by replacing the normalized arclength measure (unique Haar measure) on the unit torus $\mathbb{T}^1$ by the normalized area measure on the (open) unit disk. Using continuity, we can extend this measure to the closed unit disk $\mathbb{D}$. Namely, we consider the **(logarithmic) areal Mahler measure** defined by Pritsker [**99**] for a non-zero rational function $P \in \mathbb{C}(x_1, \dots, x_n)$.

Definition 0.6.1. *The* ***(logarithmic) areal Mahler measure*** *of a non-zero rational function $P \in \mathbb{C}(x_1, \dots, x_n)$ is defined as*

$$\mathrm{m}_{\mathbb{D}}(P) = \frac{1}{\pi^n} \int_{\mathbb{D}^n} \log |P(x_1, \dots, x_n)| dA(x_1) \dots dA(x_n),$$

where

$$\mathbb{D}^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : |x_1|, \dots, |x_n| \leq 1\}$$

is the product of n unit disks, and $\frac{1}{\pi} dA(x) = \frac{1}{\pi} dx$ is the normalized area measure on $\mathbb{D}^1 = \mathbb{D}$.

Remark 0.6.2. *In the discussion at the end of Section 0.1 about defining Mahler measure on function spaces, we saw that Mahler measure can appear as the logarithm of the "0-th norm" in a suitable function space. For $0 < r < \infty$, the Bergman spaces $A^r(\mathbb{D})$ are the function spaces comprising holomorphic functions f on $\mathbb{D}$ that are absolutely integrable, and such that, for $0 < r < \infty$,*

$$\|f\|_r = \left(\frac{1}{\pi} \int_{\mathbb{D}} |f|^r dA(x) \right)^{\frac{1}{r}} < \infty.$$

Then, the exponential of the (logarithmic) areal Mahler measure of a non-zero rational function $f \in \mathbb{C}(x)$ is the limiting norm of $f \in \bigcup_{r>0} A^r(\mathbb{D})$, i.e.,

$$\|f\|_0 := \exp\left(\mathrm{m}_{\mathbb{D}}(f)\right) = \lim_{r \to 0^+} \|f\|_r.$$

The norm $\|\cdot\|_0$ naturally arises in certain extremal problems for the function space associated to the classical Mahler measure, namely the Hardy space $H^r(\mathbb{D})$. (See **[61, 116]** *for more details on these function spaces.)*

Since, for a non-zero polynomial $P \in \mathbb{C}[x]$, the integral arithmetic means of $\log|P(z)|$ over $\mathbb{T}_r^1$ are increasing with r, we have

$$\mathrm{m}_{\mathbb{D}}(P) \leq \mathrm{m}(P).$$

Choi and Samuels [**44**, Theorem 1.2] showed that, if we further have $|P(0)| = 1$, then

$$\mathrm{m}_{\mathbb{D}}(P) \leq (\mathrm{m}(P))^2,$$

which gives a better bound than above when $\mathrm{m}(P) < 1$. Pritsker further showed that $\mathrm{m}_{\mathbb{D}}(P)$ cannot be arbitrarily small, as

$$\mathrm{m}(P) - \frac{\deg P}{2} \leq \mathrm{m}_{\mathbb{D}}(P) \leq \mathrm{m}(P),$$

where the equality holds in the lower estimate if and only if $P(x) = ax^n$, and the upper estimate holds when the polynomial does not vanish of the disk. The latter condition follows directly from the areal counterpart to Jensen's formula (0.1.4) due to Pritsker [**99**, Theorem 1.1]: for $P(x) = a\prod_{j=1}^{d}(x - \alpha_j) \in \mathbb{C}[x]$,

$$\mathrm{m}_{\mathbb{D}}(P) = \mathrm{m}(P) + \frac{1}{2}\sum_{|\alpha_j|<1}\left(|\alpha_j|^2 - 1\right) = \log|a| + \sum_{j=1}^{d}\log^+|\alpha_j| + \frac{1}{2}\sum_{|\alpha_j|<1}\left(|\alpha_j|^2 - 1\right). \quad (0.6.1)$$

Taking exponential on both sides we get

$$\|P\|_0 = M(P)\exp\left(\frac{1}{2}\sum_{|\alpha_j|<1}\left(|\alpha_j|^2 - 1\right)\right). \quad (0.6.2)$$

Note that, if $P \in \mathbb{Z}[x]$ and $P(0) \neq 0$, then $\mathrm{m}_{\mathbb{D}}(P) \geq 0$, which we have already seen $\mathrm{m}(P)$ to satisfy. The equivalency in (0.1.3) extends to the areal case, in the sense that, for $P \in \mathbb{Z}[x]$ and $P(0) \neq 0$,

$$\|P\|_0 = 1 \Longleftrightarrow P(x) = \prod_a \phi_a(x),$$

where ϕ_a are cyclotomic polynomials. This provides an analogous statement of Kronecker's Theorem. Indeed, if P is a product of cyclotomic polynomials, then by (0.6.1) we have $\|P\|_0 = 1$. On the other hand, if $\|P\|_0 = 1$, then we have

$$M(P) = \exp\left(\frac{1}{2}\sum_{|\alpha_j|<1}\left(1-|\alpha_j|^2\right)\right),$$

and, since $M(P)$ is algebraic and exponential of an algebraic number is transcendental (Lindemann–Weierstraß Theorem), the above equality only holds when $|\alpha_j| = 1$ for all j, i.e., when P is a product of cyclotomic polynomials.

Another problem to consider is the areal analogue of Lehmer's question, which asks whether the point 1 is an isolated limit point in $\mathcal{M}_{\mathbb{D}} = \{\|P\|_0 : P \in \mathbb{Z}[x]\} \subseteq (-\infty, \infty)$. Consider the polynomial family $P_n(x) = x^{3n} - nx^{2n} - nx^n + 1$. Then

$$M(P_n) = \frac{n+1+\sqrt{(n+1)^2-4}}{2},$$

$$\|P_n\|_0 = M(P_n)\exp\left(\frac{n}{2}\left(\left(\frac{n+1-\sqrt{(n+1)^2-4}}{2}\right)^{2/n} - 1\right)\right),$$

and

$$\|P_n\|_0 \longrightarrow 1 \quad \text{as } n \to \infty,$$

which shows that indeed 1 is a limit-point of $\mathcal{M}_{\mathbb{D}}$, and is *not isolated*. This is in contrast to the classical case, which is still open, where it is expected that 1 is indeed an isolated limit-point of $\mathcal{M}$.

Recall that, for $P \in \mathbb{Z}[x]$, $M(P)$ is always algebraic, regardless of whether the roots of P lie inside or outside the unit circle $\mathbb{T}^1$. Since the exponential of an algebraic number is transcendental, the following proposition, which highlights another key difference from the classical case, follows from (0.6.1).

Proposition 0.6.3 ([**99**, Proposition 1.5])**.** *If $P(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{Z}[x]$ has at least one zero in the interior of $\mathbb{D}$, then $\|P\|_0$ is a transcendental number. Otherwise, $\|P\|_0 = M(P) = |a_0|$ is an integer.*

Let E be a closed subset of $\mathbb{C}$ such that the open set $\mathbb{C} \setminus E$ contains a neighborhood of a point of absolute value 1, i.e., $\mathbb{T}^1 \not\subset E$. Consider an arbitrary sequence of integer coefficient polynomials, $\{P_d\}_{d\geq 1}$, such that $\deg P_d = d$, P_d has only simple zeros, and the set of all zeros $\bigcup_{d\geq 1}\{\alpha : P_d(\alpha) = 0\}$ is contained in E. Then Pritsker [**99**, Corollary 2.2] showed that there exists a constant $C(E) > 0$, depending on E, such that

$$\liminf_{d\to\infty} \frac{\mathrm{m}_{\mathbb{D}}(P_d)}{d} \geq C(E) > 0. \tag{0.6.3}$$

This is an areal analogue of the results due to Langevin [**85**] and Dubickas–Smyth [**50**] which show that, for a non-zero non-cyclotomic algebraic number α, if α and its conjugates are contained in a closed subset $\tilde{E}$ of $\mathbb{C}$ which does not contain a neighborhood of a single point on $\mathbb{T}^1$, then, for some constant $c(\tilde{E}) > 0$, we have $\mathrm{m}(f_\alpha) \geq c(\tilde{E}) \deg f_\alpha$, where $f_\alpha \in \mathbb{Z}[x]$ is the integral minimal polynomial of α.

The lower bound in (0.6.3) exhibits the growth of the areal Mahler measure for many families of polynomials such as polynomials with real zeros, etc. Additional results about the areal Mahler measure of one-variable polynomials can be found in the works of Pritsker [**99**], Choi and Samuels [**44**] and Flammang [**52**].

Our main aim in Chapter 2 is to evaluate the areal Mahler measure of some nontrivial multivariable polynomials and rational functions. Pritsker showed that, for a non-zero polynomial

$$P(x_1, \ldots, x_n) = \sum_{k_1+\cdots+k_n\leq d} a_{k_1\cdots k_n} x_1^{k_1} \cdots x_n^{k_n} \in \mathbb{C}[x_1, \ldots, x_n] \tag{0.6.4}$$

of degree at most d,

$$\mathrm{m}(P) - \frac{d}{2} \leq \mathrm{m}_{\mathbb{D}}(P) \leq \mathrm{m}(P),$$

and the equality on the left and right hold when $P(x_1, \ldots, x_n) = a_{k_1\cdots k_n} x_1^{k_1} \cdots x_n^{k_n}$ with $k_1 + \cdots + k_n = d$, and when the polynomial does not vanish in $\mathbb{D}^n$, respectively.

Some simple evaluations in multivariable cases are included in [**99**]. In particular, Pritsker proved that

$$\mathrm{m}_{\mathbb{D}}(x_1 + x_2) = -\frac{1}{4}, \quad \mathrm{m}_{\mathbb{D}}\left(1 + x_1^{k_1} \cdots x_n^{k_n}\right) = 0, \quad \text{for } k_1, \ldots, k_d \geq 0,$$

and if the polynomial P of the form (0.6.4) satisfies

$$|a_{0\cdots 0}| \geq \sum_{k_1+\cdots+k_n \leq d} |a_{k_1\cdots k_n}|, \tag{0.6.5}$$

then $\mathrm{m}_{\mathbb{D}}(P) = \mathrm{m}(P) = \log |a_{0\cdots 0}|$.

In Chapter 2, we provide many more formulas for multivariable areal Mahler measures, most of which involve special values of L-functions and other special functions. For example, we prove the following result.

Theorem 0.6.4 ([**84**, Theorem 1])**.** *We have*

$$\mathrm{m}_{\mathbb{D}}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3},2) + \frac{1}{6} - \frac{11\sqrt{3}}{16\pi}. \tag{0.6.6}$$

Comparing this formula with (0.1.7), we see the same term involving the L-function/dilogarithm and some extra terms, namely

$$\mathrm{m}_{\mathbb{D}}(1+x+y) = \mathrm{m}(1+x+y) - \left(\frac{11\sqrt{3}}{16\pi} - \frac{1}{6}\right) < \mathrm{m}(1+x+y).$$

It is natural to wonder if one can obtain an elegant areal Mahler measure formula for polynomials of the type $ax+by+c$, analogous to the Cassaigne–Maillot's result (0.5.1) in the classical case, where a, b, c are fixed coefficients. It seems to be quite difficult to obtain such a formula for the areal Mahler measure in full generality. To illustrate this, we have the following nontrivial statement.

Theorem 0.6.5 ([**84**, Theorem 2])**.** *We have*

$$\mathrm{m}_{\mathbb{D}}\left(\sqrt{2}+x+y\right) = \frac{L(\chi_{-4},2)}{\pi} + \mathcal{C}_{\sqrt{2}} + \frac{3}{8} - \frac{3}{2\pi},$$

where

$$\mathcal{C}_{\sqrt{2}} = \frac{\Gamma\left(\frac{3}{4}\right)^2}{\sqrt{2\pi^3}} {}_4F_3\left(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}; \frac{1}{2}, \frac{5}{4}, \frac{5}{4}; 1\right) - \frac{\Gamma\left(\frac{1}{4}\right)^2}{72\sqrt{2\pi^3}} {}_4F_3\left(\frac{3}{4}, \frac{3}{4}, \frac{5}{4}, \frac{5}{4}; \frac{3}{2}, \frac{7}{4}, \frac{7}{4}; 1\right),$$

is expressed in terms of generalized hypergeometric functions, as defined in (0.2.1)*.*

We notice that

$$\mathrm{m}\left(\sqrt{2}+x+y\right) = \frac{L(\chi_{-4},2)}{\pi} + \frac{\log 2}{4}.$$

This formula can be obtained by specializing the more general expression for $\mathrm{m}(ax+by+c)$ in (0.5.1).[9]

Remark 0.6.6. *In recent ongoing work with Lalín, Nair, and Ringeling, we obtain a general expression of* $\mathrm{m}_{\mathbb{D}}(k+x+y)$ *in terms of* k*, for* $k \in \mathbb{C}$*, using an areal analogue of the zeta Mahler measure (see Section 0.6.3 and Chapter 5 for more details). Furthermore, evaluating* (0.6.22) *in Theorem 0.6.25 at* $k=\sqrt{2}$ *and comparing it with the results of Cassaigne–Maillot in* (0.5.1)*, we have* $\mathcal{C}_{\sqrt{2}} = \frac{\log 2}{4}$*, which provides a nontrivial hypergeometric identity:*

$$\frac{\Gamma\left(\frac{3}{4}\right)^2}{\sqrt{2\pi^3}}\,{}_4F_3\left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4};\frac{1}{2},\frac{5}{4},\frac{5}{4};1\right) - \frac{\Gamma\left(\frac{1}{4}\right)^2}{72\sqrt{2\pi^3}}\,{}_4F_3\left(\frac{3}{4},\frac{3}{4},\frac{5}{4},\frac{5}{4};\frac{3}{2},\frac{7}{4},\frac{7}{4};1\right) = \frac{\log 2}{4}.$$

Moreover, we have

$$\mathrm{m}_{\mathbb{D}}\left(\sqrt{2}+x+y\right) = \mathrm{m}\left(\sqrt{2}+x+y\right) - \left(\frac{3}{2\pi}-\frac{3}{8}\right) < \mathrm{m}\left(\sqrt{2}+x+y\right).$$

A motivation to study this particular polynomial lies in the fact that it is relatively easy to understand the boundaries of integration upon application of Jensen's formula, due to the particular properties of the constant $\sqrt{2}$.

We also prove the following statement involving a rational function.

Theorem 0.6.7 ([**84**, Theorem 4])**.** *We have*

$$\mathrm{m}_{\mathbb{D}}\left(y+\left(\frac{1-x}{1+x}\right)\right) = \frac{6}{\pi}L\left(\chi_{-4},2\right) - \log 2 - \frac{1}{2} - \frac{1}{\pi}. \tag{0.6.7}$$

The above formula can be compared to the evaluation due to Boyd [**28**]

$$\mathrm{m}\left(y+\left(\frac{1-x}{1+x}\right)\right) = \frac{2}{\pi}L(\chi_{-4},2), \tag{0.6.8}$$

as

$$\mathrm{m}_{\mathbb{D}}\left(y+\left(\frac{1-x}{1+x}\right)\right) = 3\mathrm{m}\left(y+\left(\frac{1-x}{1+x}\right)\right) - \log 2 - \frac{1}{2} - \frac{1}{\pi}.$$

In this case, the term $L(\chi_{-4},2)$ involving the Dirichlet L-function in the character of conductor 4 comes from evaluating the dilogarithm at $\pm i$. However, unlike the situation of equations (0.1.7) and (0.6.6), the dilogarithmic terms in (0.6.7) and (0.6.8) do not have the same coefficients.

9. Take $a=b=1$ and $c=\sqrt{2}$.

For the moment we lack the connection to regulators that could potentially allow us to perform these evaluations more systematically. Nevertheless, our results provide evidence that the areal Mahler measure is also an interesting object deserving of attention and open the door to future considerations of the areal Mahler measure and suggest the search for deeper connections to regulators that could potentially explain such formulas.

0.6.1. Areal Mahler measure under a power change of variables

Recall that, for $A \in GL_n(\mathbb{Z})$, $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{j} = (j_1, \ldots, j_n)$, Theorem 0.1.6 states that Mahler measure of a non-zero polynomial $P(\mathbf{x}) = \sum_{\mathbf{j}} c_{\mathbf{j}} \mathbf{x}^{\mathbf{j}} \in \mathbb{C}[\mathbf{x}]$ is invariant under the transformation defined by $P^{(A)}(\mathbf{x}) := \sum_{\mathbf{j}} c_{\mathbf{j}} \mathbf{x}^{A\mathbf{j}}$, i.e.

$$\mathrm{m}(P) = \mathrm{m}\left(P^{(A)}\right). \tag{0.6.9}$$

In Chapter 3, we investigate the simplest possible case of the above transformation, namely, when one of the variables, x, is replaced by a power of itself, x^r, where r is a positive integer, in the areal Mahler measure case. To illustrate this, we compute the areal Mahler measures of $1 + x^r + y^s$, where r and s are positive integers and we obtain results that are different from (0.6.6), which corresponds to the case $r = s = 1$. More precisely, we prove the following statement.

Theorem 0.6.8 ([**83**, Theorem 1])**.** *Let r,s be positive integers. We have*

$$\begin{aligned}
&\mathrm{m}_{\mathbb{D}}\left(1 + x^r + y^s\right) \\
&= \frac{3\sqrt{3}}{4\pi} L(\chi_{-3},2) - \frac{r}{6} + \frac{\sqrt{3}r}{12\pi}\left[\zeta\left(1,\frac{r+2}{3r}\right) - \zeta\left(1,\frac{2r+2}{3r}\right) + \zeta\left(1,\frac{r+1}{3r}\right) - \zeta\left(1,\frac{2r+1}{3r}\right)\right] \\
&- \frac{2}{\pi}\sum_{1\leq k}\sum_{h=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2h}\frac{(-1)^{h-1}{}_2F_1\left(\frac{1}{2}-h, k-h+\frac{1}{r}+\frac{1}{2}; k-h+\frac{1}{r}+\frac{3}{2}; \frac{1}{4}\right)}{2^{k-2h+1}k(kr+2)\left(2k+\frac{2}{r}-2h+1\right)} + \frac{s}{6}\sum_{1\leq k}\binom{\frac{1}{s}}{k}^2\frac{1}{kr+1} \\
&- \frac{s\sqrt{3}}{\pi}\sum_{0\leq j<k}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{\chi_{-3}(k-j)}{((k+j)r+2)(k-j)} + \frac{s}{4\pi}\sum_{1\leq k}\binom{\frac{1}{s}}{k}^2\frac{{}_2F_1\left(\frac{1}{2}, k+\frac{1}{r}+\frac{1}{2}; k+\frac{1}{r}+\frac{3}{2}; \frac{1}{4}\right)}{(kr+1)\left(2k+1+\frac{2}{r}\right)} \\
&+ \frac{s}{\pi}\sum_{0\leq j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left(\frac{1}{2}-h, k-h+\frac{1}{r}+\frac{1}{2}; k-h+\frac{1}{r}+\frac{3}{2}; \frac{1}{4}\right)}{2^{k-j-2h}\left((k+j)r+2\right)\left(2k+\frac{2}{r}-2h+1\right)},
\end{aligned}$$

where $\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$ *is the Hurwitz zeta-function and* ${}_2F_1(a,b;c;z)$ *is the hypergeometric function given by* (0.2.1)*.*

Remark 0.6.9. *For the case* $r = 1$*, the formula from Theorem 0.6.8 should be interpreted as a regularization, namely the divergent terms* $\zeta(1,1)$ *with opposite signs cancel each other. More precisely, for* $r = 1$*, the line*

$$\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2) - \frac{r}{6} + \frac{\sqrt{3}r}{12\pi}\left[\zeta\left(1,\frac{r+2}{3r}\right) - \zeta\left(1,\frac{2r+2}{3r}\right) + \zeta\left(1,\frac{r+1}{3r}\right) - \zeta\left(1,\frac{2r+1}{3r}\right)\right]$$

should be replaced by

$$\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2) - \frac{1}{4} + \frac{\sqrt{3}}{4\pi}.$$

Remark 0.6.10. *The result of Theorem 0.6.8 should be symmetric with respect to* r *and* s*, which is certainly not obvious to guess from the formula itself! This phenomenon is observed numerically, but we do not have a direct proof of it.*

We also compute the areal Mahler measure of a similar family, namely, $(1+x)^r + y^s$ and obtain interesting results depending on s.

Theorem 0.6.11 ([**83**, Theorem 4])**.** *Let* r, s *be positive integers. We have*

$$\begin{aligned}
&\mathrm{m}_{\mathbb{D}}((1+x)^r + y^s)\\
&= r\left(\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2) + \frac{1}{6} - \frac{\sqrt{3}}{2\pi}\right) - \frac{s}{6} + \frac{s}{6}\frac{\Gamma\left(\frac{2r}{s}+2\right)}{\Gamma\left(\frac{r}{s}+2\right)^2}\\
&- \frac{s\sqrt{3}}{\pi}\sum_{0\le j<k}\binom{\frac{r}{s}}{k}\binom{\frac{r}{s}}{j}\frac{\chi_{-3}(k-j)}{(k+j+2)(k-j)} + \frac{s}{4\pi}\sum_{1\le k}\binom{\frac{r}{s}}{k}^2\frac{{}_2F_1\left(\frac{1}{2},k+\frac{3}{2};k+\frac{5}{2};\frac{1}{4}\right)}{(k+1)(2k+3)}\\
&+ \frac{s}{\pi}\sum_{0\le j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\frac{r}{s}}{k}\binom{\frac{r}{s}}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left(\frac{1}{2}-h,k-h+\frac{3}{2};k-h+\frac{5}{2};\frac{1}{4}\right)}{2^{k-j-2h}(k+j+2)(2k-2h+3)}.
\end{aligned}$$

Remark 0.6.12. *We remark that Theorem 0.6.8 and Theorem 0.6.11 should coincide in the case of* $r = 1$*. This results in the identities*

$$\sum_{1\le k}\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2j}\frac{(-1)^{j-1}{}_2F_1\left(\frac{1}{2}-j,k-j+\frac{3}{2};k-j+\frac{5}{2};\frac{1}{4}\right)}{2^{k-2j}k(k+2)\left(2k-2j+3\right)} = \frac{3\sqrt{3}}{4} - \frac{5\pi}{12} \qquad (0.6.10)$$

and

$$\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{k+1}=-1+\frac{\Gamma\left(\frac{2}{s}+2\right)}{\Gamma\left(\frac{1}{s}+2\right)^2}. \tag{0.6.11}$$

While equation (0.6.11) *is proven in Corollary 3.3.2 in Chapter 3, we do not know how to prove equation* (0.6.10) *independently, which can be seen to be numerically true.*

If, in addition, we set $s=1$, we recover formula (0.6.6) *by employing the evaluation of* ${}_2F_1\left(\frac{1}{2},\frac{5}{2};\frac{7}{2};\frac{1}{4}\right)$, *given by*

$$ {}_2F_1\left(\frac{1}{2},\frac{5}{2};\frac{7}{2};\frac{1}{4}\right)=10\pi-\frac{35\sqrt{3}}{2}.$$

.

We also prove the following result in Chapter 3, which explains the effect of the change $x\mapsto x^r$ in general, as $r\to\infty$.

Theorem 0.6.13 ([**83**, Theorem 6])**.** *Let $P(x_1,\ldots,x_n)\in\mathbb{C}(x_1,\ldots,x_n)$ be a non-zero rational function and let $P(0,x_2,\ldots,x_n)\in\mathbb{C}(x_2,\ldots,x_n)$ be the non-zero rational function resulting from P by setting $x_1=0$. Let r be a positive integer. Then we have*

$$\lim_{r\to\infty}\mathrm{m}_{\mathbb{D}}(P(x_1^r,x_2,\ldots,x_n))=\mathrm{m}_{\mathbb{D}}(P(0,x_2,\ldots,x_n)).$$

0.6.2. Generalized (maximal) areal Mahler measure, multiple areal Mahler measure, and higher areal Mahler measure

Variations of the Mahler measure such as generalized (maximal) Mahler measures [**54**], multiple and higher Mahler measures [**72**], and zeta Mahler measures [**8, 3**] can also be adapted to the areal Mahler measure setting. In the following sections, we will define these analogues and present our findings for each type.

0.6.2.1. Generalized (maximal) areal Mahler measure. Generalized (maximal) Mahler measures were introduced by Gon and Oyanagi [**54**], who studied their basic properties, computed some examples, and related them to multiple sine functions and special values of Dirichlet L-functions. They were also studied in [**67, 76**].

For non-zero rational functions $P_1, \dots, P_r \in \mathbb{C}(x_1, \dots, x_n)$, the *generalized (maximal) Mahler measure* of $P_1, \dots, P_r$ is defined by

$$\mathrm{m}_{\max}(P_1, \dots, P_r) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \max\{\log|P_1|, \dots, \log|P_r|\} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

Observe that, if $P_j(x_1, \dots, x_n) = x_j$ for all $j = 1, \dots, n$, then $\mathrm{m}_{\max}(P_1, \dots, P_n) = 0$, since $\log|x_j| = \log 1 = 0$. In other words, we have

$$\mathrm{m}_{\max}(x_1, \dots, x_n) = 0. \tag{0.6.12}$$

When $P_j(x_1, \dots, x_n) = 1 - x_j$ or $\frac{1-x_j}{1+x_j}$, we have more involved evaluations.

$\mathbf{P = 1 - x}$: Gon–Oyanagi [**54**] showed that, for $k \geq 1$,

$$\begin{aligned} \mathrm{m}(1 - x_1, \dots, 1 - x_{2k}) =& \frac{(-1)^{k+1}(2k)!}{\pi^{2k}} \zeta(2k+1) \\ &+ (2k)! \sum_{j=1}^{k} (-1)^j \frac{1 - 2^{2j}}{(2k-2j)!(2\pi)^{2j}} \zeta(2j+1), \\ \mathrm{m}(1 - x_1, \dots, 1 - x_{2k-1}) =& (2k-1)! \sum_{j=1}^{k-1} (-1)^j \frac{1 - 2^{2j}}{(2k-2j-1)!(2\pi)^{2j}} \zeta(2j+1); \end{aligned}$$

$\mathbf{P = \frac{1-x}{1+x}}$: Lalín [**76**] showed that, for $k \geq 1$,

$$\begin{aligned} \mathrm{m}\left(\frac{1 - x_1}{1 + x_1}, \dots, \frac{1 - x_{2k}}{1 + x_{2k}}\right) =& \frac{(-1)^k (2k)!(1 - 2^{2k+1})}{(2\pi)^{2k}} \zeta(2k+1) \\ &+ (2k)! \sum_{j=1}^{k} (-1)^j \frac{1 - 2^{2j+1}}{(2k-2j)!(2\pi)^{2j}} \zeta(2j+1), \\ \mathrm{m}\left(\frac{1 - x_1}{1 + x_1}, \dots, \frac{1 - x_{2k+1}}{1 + x_{2k+1}}\right) =& (2k-1)! \sum_{j=1}^{k-1} (-1)^j \frac{1 - 2^{2j+1}}{(2k-2j-1)!(2\pi)^{2j}} \zeta(2j+1). \end{aligned}$$

Next, it is natural to consider the areal analogue following the discussions preceding this section.

Definition 0.6.14. *Let $P_1, \dots, P_r \in \mathbb{C}(x_1, \dots, x_n)$ be non-zero rational functions. Then the* ***generalized (maximal) areal Mahler measure*** *of $P_1, \dots, P_r$ is defined by*

$$\mathrm{m}_{\mathbb{D}, \max}(P_1, \dots, P_r) = \frac{1}{\pi^n} \int_{\mathbb{D}^n} \max\{\log|P_1|, \dots, \log|P_r|\} dA(x_1) \dots dA(x_n).$$

For the areal analogue of (0.6.12), we have the following result.

Theorem 0.6.15 ([**84**, Proposition 9])**.** *We have*

$$\mathrm{m}_{\mathbb{D},\max}(x_1,\ldots,x_n) = -\frac{1}{2n}.$$

0.6.2.2. Multiple and higher areal Mahler measures. The *multiple Mahler measure* was defined by Kurokawa, Lalín, and Ochiai in [**72**] as, for non-zero rational functions $P_1,\ldots,P_r \in \mathbb{C}(x_1,\ldots,x_n)$,

$$\mathrm{m}(P_1,\ldots,P_r) := \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n} \log|P_1(x_1,\ldots,x_n)|\cdots\log|P_r(x_1,\ldots,x_n)|\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.$$

For the particular case in which $P_1 = \cdots = P_r = P$, the multiple Mahler measure is called the *r-th higher Mahler measure* and is given by

$$\mathrm{m}_r(P) := \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n} \log^r|P(x_1,\ldots,x_n)|\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}. \tag{0.6.13}$$

The multiple Mahler measure and the higher Mahler measure were considered by various authors who computed specific formulas and proved various limiting properties [**17, 18, 23, 24, 67, 78, 108, 109**]. For example, in [**72**], Kurokawa, Lalín and Ochiai considered the polynomial $P_1(x) = 1-x$ and $P_2(x) = 1+x$, and obtained

$$\mathrm{m}(P_1,P_1,P_2) = \frac{\zeta(3)}{4} = \mathrm{m}(P_1,P_2,P_2),$$

and

$$\mathrm{m}_r(P_1) = \sum_{k_1+\cdots+k_h=r,k_j\ge 2} \frac{(-1)^r r!}{2^{2h}}\zeta(k_1,\ldots,k_h), \tag{0.6.14}$$

where $\zeta(k_1,\ldots,k_h)$ denotes a multizeta value, i.e.,

$$\zeta(k_1,\ldots,k_h) = \sum_{\ell_1<\cdots<\ell_h} \frac{1}{\ell_1^{k_1}\cdots\ell_h^{k_h}}.$$

The multizeta values can be further simplified and expressed in terms of classical zeta values using the next proposition.

Proposition 0.6.16. [**72**, Proposition 4]

$$\sum_{\sigma\in S_h}\zeta(k_{\sigma(1)},\ldots,k_{\sigma(h)}) = \sum_{e_1+\cdots+e_g=h}(-1)^{h-g}\prod_{s=1}^{g}(e_g-1)!\sum\zeta\left(\sum_{b\in\pi_1}k_b\right)\cdots\zeta\left(\sum_{b\in\pi_g}k_b\right),$$

where the sum in the right is taken over all the possible unordered partitions of the set $\{1, \ldots, h\}$ *into* g *subsets* $\pi_1, \ldots, \pi_g$ *with* $e_1, \ldots, e_g$ *elements respectively.*

A more elaborate example is due to Sasaki [**110**], and Lalín and Lechasseur [**78**]:

$$\mathrm{m}_h\left(\frac{1-x}{1+x}\right) = \begin{cases} \dfrac{|E_h|}{2^h}\pi^h & h \text{ even}, \\ \\ 0 & h \text{ odd}, \end{cases} \tag{0.6.15}$$

where E_n denotes the n-th Euler number defined in (0.4.20).

Following the essence of the previous section, it is again natural to consider the areal versions of these constructions.

Definition 0.6.17. *Let* $P_1, \ldots, P_r \in \mathbb{C}(x_1, \ldots, x_n)$ *be non-zero rational functions. Then the* ***multiple areal Mahler measure*** *of* $P_1, \ldots, P_r$ *is defined by*

$$\mathrm{m}_{\mathbb{D},h_1,\ldots,h_r}(P_1, \ldots, P_r) := \frac{1}{\pi^n}\int_{\mathbb{D}^n} \log^{h_1}|P_1(x_1, \ldots, x_n)| \cdots \log^{h_r}|P_r(x_1, \ldots, x_n)| dA(x_1) \ldots dA(x_n).$$

Moreover, the r***-th higher areal Mahler measure*** *is defined by taking* $P_1 = P_2 = \cdots = P_r$.

We remark that in the classical case, we have $\mathrm{m}_{h_1,\ldots,h_n}(x_1, \ldots, x_n) = 0$, which again follows from the fact that $\log|x_j| = 0$ on the n-torus. The areal analogue is non-zero.

Theorem 0.6.18 ([**84**, Proposition 10])**.** *We have*

$$\mathrm{m}_{\mathbb{D},h_1,\ldots,h_n}(x_1, \ldots, x_n) = \frac{(-1)^{h_1+\cdots+h_n} h_1! \cdots h_n!}{2^{h_1+\cdots+h_n+n}}.$$

The following theorem evaluates the areal version of the result in (0.6.15).

Theorem 0.6.19 ([**84**, Theorem 11])**.** *For* $h \in \mathbb{Z}_{>0}$ *even, we have,*

$$\begin{aligned}\mathrm{m}_{\mathbb{D},h}\left(\frac{1-x}{1+x}\right) =& \frac{E_h(\pi i)^h}{2^h} - \frac{E_{h-2}(\pi i)^{h-2}h(h-1)}{2^{h-2}}\log 2 \\ &- \frac{4h!}{2^h}\sum_{m=2}^{h-1}(1-2^{1-m})\zeta(m)\frac{E_{h-m-1}(\pi i)^{h-m-1}}{(h-m-1)!},\end{aligned}$$

where B_n and E_n denote the nth Bernoulli number and the nth Euler number defined in (0.4.3) *and* (0.4.20) *respectively, and the first sum for $h = 2$ should be interpreted as equal to zero.*

For h odd, we have

$$\mathrm{m}_{\mathbb{D},h}\left(\frac{1-x}{1+x}\right) = 0.$$

Section 2.4 in Chapter 2 contains the derivations of the above-mentioned results. Next, we consider certain Zeta functions associated to the Mahler measure, as well as the areal Mahler measure, which collect all r-th higher Mahler measures and r-th higher areal Mahler measures, for $r \geq 1$, respectively.

0.6.3. Areal zeta Mahler measure

The zeta Mahler measure was defined by Akatsuka [**3**] for a nonzero rational function $P \in \mathbb{C}(x_1, \ldots, x_n)$ as

$$Z(s,P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} |P(x_1, \ldots, x_n)|^s \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where s is a complex variable. The integral converges absolutely in $\mathrm{Re}(s) > \sigma_0(P)$, where

$$\sigma_0(P) := \inf\left\{\sigma \in \mathbb{R} : \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} |P(x_1, \ldots, x_n)|^\sigma dA(x_1) \ldots dA(x_n) < \infty\right\} \in \mathbb{R} \cup \{-\infty\}.$$

Since $Z(0, P) = 1 < \infty$, $Z(s, P)$ converges absolutely when $\mathrm{Re}(s) = 0$, and therefore we have $\sigma_0(P) \leq 0$. Akatsuka further showed that $Z(s, P)$ is holomorphic when $\mathrm{Re}(s) > \sigma_0(P)$, and

$$\frac{d^k Z(s, P)}{ds^k} = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} |P(x_1, \ldots, x_n)|^s (\log|P|)^k \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

for all $k \geq 1$. In particular, $\mathrm{m}(P) = \left.\frac{dZ(s,P)}{ds}\right|_{s=0}$.

The zeta Mahler measure was considered in [**17, 19, 26, 72, 101, 110, 115**]. Its Taylor expansion is the exponential generating series of the higher Mahler measure given in (0.6.13):

$$Z(s,P) = \sum_{k=0}^{\infty} \frac{\mathrm{m}_k(P) s^k}{k!}.$$

Next, we include some evaluations of zeta Mahler measure for certain linear polynomials. The first example is due to Akatsuka [**3**], and Kurokawa, Lalín and Ochiai [**72**], where they showed that, for $\mathrm{Re}(s) > -1$,

$$Z(s, x+1) = \exp\left(\sum_{j=2}^{\infty} \frac{(-1)^j}{j}(1-2^{1-j})\zeta(j)s^j\right) = 2^s \pi^{-\frac{1}{2}} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} = \frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2}+1\right)^2}. \quad (0.6.16)$$

Akatsuka further extended (0.6.16) in [**3**] to all $a \in \mathbb{C}$. He showed that, for $a \in \mathbb{C}$ and $\mathrm{Re}(s) > -1$,

$$Z(s, x+a) = \begin{cases} |a|^s {}_2F_1\left(-\frac{s}{2}, -\frac{s}{2}; 1; |a|^{-2}\right) & \text{if } |a| > 1 \\ {}_2F_1\left(-\frac{s}{2}, -\frac{s}{2}; 1; |a|^2\right) & \text{if } |a| < 1 \\ \frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2}+1\right)^2} & \text{if } |a| = 1. \end{cases} \quad (0.6.17)$$

A more involved example is due to Borwein, Straub, Wan and Zudilin [**26**]. They showed that, for s not an odd integer,

$$\begin{aligned} Z(s, x+y+1) =& \frac{1}{2^{2s+1}} \tan\left(\frac{\pi s}{2}\right) \binom{s}{\frac{s-1}{2}}^2 {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{s+3}{2}, \frac{s+3}{2}; \frac{1}{4}\right) \\ &+ \binom{s}{\frac{s}{4}} {}_3F_2\left(-\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2}; 1, -\frac{s-1}{2}; \frac{1}{4}\right). \end{aligned}$$

We can again extend the definition of the zeta Mahler measure to the areal case in a similar sense as in previous sections.

Definition 0.6.20. *Let $P \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$ be a Laurent polynomial. Then the* ***areal zeta Mahler measure*** *of P is defined as*

$$Z_{\mathbb{D}}(s,P) := \frac{1}{\pi^n} \int_{\mathbb{D}^n} |P(x_1, \ldots, x_n)|^s \, dA(x_1) \ldots dA(x_n). \quad (0.6.18)$$

From the discussion at the end of Section 0.1, we know that the normalized area measure $\mu_{\mathbb{D}}$ is finite (see (2)), which implies that the integral expression of $Z_{\mathbb{D}}(s, P)$ absolutely converges when $\mathrm{Re}(s) = 0$. Further, if $P \in \mathbb{C}[x_1, \ldots, x_n]$, then the integral converges absolutely for $\mathrm{Re}(s) > \sigma_1(P)$, where

$$\sigma_1(P) := \inf\left\{\sigma \in \mathbb{R} : \frac{1}{\pi^n} \int_{\mathbb{D}^n} |P(x_1, \ldots, x_n)|^\sigma \, dA(x_1) \ldots dA(x_n) < \infty\right\} \in \mathbb{R} \cup \{-\infty\}.$$

Since $Z_{\mathbb{D}}(0, P) = \int_{\mathbb{D}^n} d\mu_{\mathbb{D}^n} = \mu_{\mathbb{D}^n}(\mathbb{D}^n) = 1 < \infty$, we have $\sigma_1(P) \leq 0$, when $P \in \mathbb{C}[x_1, \ldots, x_n]$. Here $d\mu_{\mathbb{D}^n} = d\mu_{\mathbb{D}} \cdots d\mu_{\mathbb{D}} = \frac{1}{\pi^n} dA(x_1) \cdots dA(x_n)$.

In fact, we will show in Chapter 5 that, for $\mathrm{Re}(s) > \sigma_1(P)$, the integral converges locally uniformly, $Z(s,P)$ is holomorphic, and

$$\frac{d^k Z_{\mathbb{D}}(s,P)}{ds^k} = \frac{1}{\pi^n}\int_{\mathbb{D}^n} |P(x_1,\dots,x_n)|^s (\log|P|)^k dA(x_1)\dots dA(x_n),$$

for all $k \geq 1$. In particular,

$$\left.\frac{dZ_{\mathbb{D}}(s,P)}{ds}\right|_{s=0} = \mathrm{m}_{\mathbb{D}}(P).$$

Remark 0.6.21. *If $P \in \mathbb{C}[x_1^{\pm},\dots,x_n^{\pm}] \setminus \mathbb{C}[x_1,\dots,x_n]$, then there exist non-negative integers $k_1,\dots,k_n$ such that $\tilde{P}(x_1,\dots,x_n) := x_1^{k_1}\cdots x_n^{k_n}P(x_1,\dots,x_n) \in \mathbb{C}[x_1,\dots,x_n]$. Since, the exponential generating function of higher areal Mahler measure of P is the areal zeta Mahler measure of P, we can formally define and evaluate $Z_{\mathbb{D}}(s,P)$ as*

$$\begin{aligned}Z_{\mathbb{D}}(s,P) =& \sum_{k=0}^{\infty}\frac{\mathrm{m}_{\mathbb{D},k}(P)}{k!}s^k = \sum_{k=0}^{\infty}\frac{\mathrm{m}_{\mathbb{D},k}(x_1^{-k_1}\cdots x_n^{-k_n}\tilde{P})}{k!}s^k \\ =& Z_{\mathbb{D}}(s,\tilde{P}) + \sum_{k=1}^{\infty}\sum_{j=1}^{k}(-1)^j\frac{\mathrm{m}_{\mathbb{D},k-j}(\tilde{P})\mathrm{m}_{\mathbb{D},j}(x_1^{k_1}\cdots x_n^{k_n})}{j!(k-j)!}s^k.\end{aligned}$$

In Chapter 2, we follow some arguments from [**72**, Theorem 14] to compute the areal Mahler measure of $x+1$ and prove the following result.

Theorem 0.6.22 ([**84**, Theorem 13])**.** *We have*

$$Z_{\mathbb{D}}(s,x+1) = \exp\left(\sum_{j=2}^{\infty}\frac{(-1)^j}{j}(1-2^{1-j})(\zeta(j)-1)s^j\right) = \frac{s+1}{\left(\frac{s}{2}+1\right)^2}\frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2}+1\right)^2}. \qquad (0.6.19)$$

Note that (0.6.19) can be compared to the classical case (0.6.16):

$$Z_{\mathbb{D}}(s,x+1) = \frac{s+1}{(s/2+1)^2}Z(s,x+1).$$

As the zeta Mahler measure is the exponential generating function of the higher Mahler measure, it leads to formulas for the latter by taking successive derivatives of the former and evaluating at $s=0$. For example, we have the first few examples of $\mathrm{m}_k(x+1)$, for $k = 1,2,3,4,5$, which follows from (0.6.16) (the following examples can also be obtained from the general formula (0.6.14) of higher Mahler measure $\mathrm{m}_k(1+x)$ in [**72**, Example 5]):

$$\begin{aligned}
\mathrm{m}_1(x+1) =&0,\\
\mathrm{m}_2(x+1) =&\frac{\zeta(2)}{2},\\
\mathrm{m}_3(x+1) =&-\frac{3\zeta(3)}{2},\\
\mathrm{m}_4(x+1) =&\frac{3\zeta(2)^2+21\zeta(4)}{4}=\frac{57\zeta(4)}{8},\\
\mathrm{m}_5(x+1) =&-\frac{15\zeta(3)\zeta(2)+45\zeta(5)}{2}.
\end{aligned}$$

Using Theorem 0.6.22, we can further compare their results with their areal counterparts:

$$\begin{aligned}
\mathrm{m}_{\mathbb{D},1}(x+1) =&0,\\
\mathrm{m}_{\mathbb{D},2}(x+1) =&\frac{\zeta(2)-1}{2},\\
\mathrm{m}_{\mathbb{D},3}(x+1) =&-\frac{3\zeta(3)-3}{2},\\
\mathrm{m}_{\mathbb{D},4}(x+1) =&\frac{3(7\zeta(4)+\zeta(2)^2-2\zeta(2)-6)}{4}=\frac{57\zeta(4)-12\zeta(2)-36}{8},\\
\mathrm{m}_{\mathbb{D},5}(x+1) =&-\frac{15\zeta(3)\zeta(2)+45\zeta(5)-15\zeta(3)-15\zeta(2)-30}{2}.
\end{aligned}$$

In Chapter 5, we investigate some fundamental properties of the areal zeta Mahler measure including the convergent domain of the integral in (0.6.18) and certain transformation formulas, and compare them with the classical case. Our methods are influenced by the work of Akatsuka in [**3**].

Our main aim in Chapter 5 is to study the areal Zeta Mahler measure of $|k|+x+y$ for $k\in\mathbb{C}$, for which we have the following result. This is a joint work in progress with Lalín, Nair, and Ringeling.

Theorem 0.6.23. *For* $\mathrm{Re}(s)>-\frac{7}{2}$ *and* $k\in\mathbb{C}$, *we have*

$$Z_{\mathbb{D}}(s,|k|+x+y)=c_0(s)\left(\frac{|k|}{2}\right)^{s+3}F_0\left(\frac{|k|^2}{4};s\right)+c_1(s)F_1\left(\frac{|k|^2}{4};s\right), \tag{0.6.20}$$

where

$$F_0(z;s) = {}_3F_2\left(-\frac{1}{2},\frac{1}{2},\frac{3}{2};\frac{5+s}{2},\frac{5+s}{2};z\right),$$

$$F_1(z;s) = {}_3F_2\left(-2-\frac{s}{2},-1-\frac{s}{2},-\frac{s}{2};1,-\frac{1}{2}-\frac{s}{2};z\right),$$

$$c_0(s) = \frac{2^s\cdot {}_3F_2\left(-\frac{s}{2},-\frac{s}{2},\frac{3}{2};2,3;1\right) - \frac{4}{s+4}\frac{\Gamma(s+2)}{\Gamma\left(\frac{s}{2}+2\right)^2}\cdot {}_3F_2\left(-2-\frac{s}{2},-1-\frac{s}{2},-\frac{s}{2};1,-\frac{1}{2}-\frac{s}{2};1\right)}{{}_3F_2\left(-\frac{1}{2},\frac{1}{2},\frac{3}{2};\frac{5+s}{2},\frac{5+s}{2};1\right)},$$

and $c_1(s) = \frac{4}{s+4}\frac{\Gamma(s+2)}{\Gamma\left(\frac{s}{2}+2\right)^2}$.

Since $\left.\frac{dZ_{\mathbb{D}}(s,|k|+x+y)}{ds}\right|_{s=0} = \mathrm{m}_{\mathbb{D}}(|k|+x+y)$, we have the following corollary.

Corollary 0.6.24. *For* $k\in\mathbb{C}$,

$$\mathrm{m}_{\mathbb{D}}(|k|+x+y) = -\frac{4|k|^3}{9\pi}{}_3F_2\left(-\frac{1}{2},\frac{1}{2},\frac{3}{2};\frac{5}{2},\frac{5}{2};\frac{|k|^2}{4}\right)+\frac{|k|^2}{2}-\frac{1}{4}.$$

From Cassaigne–Maillot's result in (0.5.1) (see also **[7]** for the hypergeometric expression of Cassaigne–Maillot's formula), we obtain, for $k>0$,

$$\mathrm{m}(k+x+y) = \frac{k}{\pi}{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2};\frac{k^2}{4}\right). \tag{0.6.21}$$

Corollary 0.6.24 then yields the following result.

Theorem 0.6.25. *For* $k>0$,

$$\mathrm{m}(k+x+y)-\mathrm{m}_{\mathbb{D}}(k+x+y) = \frac{k\sqrt{4-k^2}(10+k^2)+(8-16k^2)\arccos\left(\frac{k}{2}\right)}{16\pi}. \tag{0.6.22}$$

Remark 0.6.26. *Note that, for any* $\xi\in\mathbb{C}^\times$ *satisfying* $|\xi|=1$, *neither* $\mathrm{m}(k+x+y)$ *nor* $\mathrm{m}_{\mathbb{D}}(k+x+y)$ *changes under the transformation* $k\mapsto k\xi$. *Therefore, we can rewrite* (0.6.22) *as*

$$\mathrm{m}(|k|+x+y)-\mathrm{m}_{\mathbb{D}}(|k|+x+y) = \frac{|k|\sqrt{4-|k|^2}(10+|k|^2)+(8-16|k|^2)\arccos\left(\frac{|k|}{2}\right)}{16\pi}, \tag{0.6.23}$$

which further shows that (0.6.23) *holds for all* $k\in\mathbb{C}$.

When $k=\sqrt{2}$, (0.6.23) yields a simpler expression of the hypergeometric representation of $\mathcal{C}_{\sqrt{2}}$ in Theorem 0.6.5 (see Remark 0.6.6). Furthermore, combining Cassaigne–Maillot's result in (0.5.1) with (0.6.23), we can express $\mathrm{m}_{\mathbb{D}}(|k|+x+y)$ in a more simplified manner involving special values of Bloch–Wigner dilogarithm D.

0.7. Mahler measure of an n-variable family

This section gives an introductory overview of the results from a collaborative project with Lalín and Nair [**81**] in Chapter 4.

Very few examples of Mahler measure of multivariable polynomials are known with more than three variables. Such examples represent important evidence for understanding the relationship between Mahler measure and regulators. In [**74, 75**] Lalín considered the Mahler measures of the following families of rational functions:

$$\begin{aligned} R_n(x_1,\dots,x_n,z) :=& z+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right),\\ S_n(x_1,\dots,x_n,x,y,z) :=& (1+x)z+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)(1+y),\\ T_n(x_1,\dots,x_n,x,y) :=& 1+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)x+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)\right)y. \end{aligned}$$

Notice that multiplication by $(1+x_1)\cdots(1+x_n)$ turns the above functions into polynomials, without changing the Mahler measure. They are written as rational functions for convenience.

For $a_1,\dots a_n\in\mathbb{C}$, define the symmetric functions as the coefficients of the polynomial $(x+a_1)\cdots(x+a_n)$, namely,

$$s_\ell(a_1,\dots,a_n)=\begin{cases}1 & \text{if } \ell=0,\\ \sum_{i_1<\cdots<i_\ell} a_{i_1}\cdots a_{i_\ell} & \text{if } 0<\ell\le n,\\ 0 & \text{if } n<\ell.\end{cases} \tag{0.7.1}$$

We also set $s_0=1$ when $n=0$.

The Mahler measures of the polynomials R_n, S_n, T_n are then given by the following formulas [**75, 78**]. For $k\ge 1$,

$$\mathrm{m}\left(R_{2k}\right)=\sum_{h=1}^{k}\frac{s_{k-h}(2^2,4^2,\dots,(2k-2)^2)}{(2k-1)!}\left(\frac{2}{\pi}\right)^{2h}\mathcal{A}(h),$$

where

$$\mathcal{A}(h) := (2h)!\left(1 - \frac{1}{2^{2h+1}}\right)\zeta(2h+1).$$

For $k \geq 0$,

$$\mathrm{m}\left(R_{2k+1}\right) = \sum_{h=0}^{k} \frac{s_{k-h}(1^2,3^2,\ldots,(2k-1)^2)}{(2k)!}\left(\frac{2}{\pi}\right)^{2h+1}\mathcal{B}(h),$$

where

$$\mathcal{B}(h) := (2h+1)!L(\chi_{-4}, 2h+2).$$

For $k \geq 1$,

$$\mathrm{m}\left(S_{2k}\right) = \sum_{h=1}^{k} \frac{s_{k-h}(2^2,4^2,\ldots,(2k-2)^2)}{(2k-1)!}\left(\frac{2}{\pi}\right)^{2h+2}\mathcal{C}(h), \tag{0.7.2}$$

where

$$\mathcal{C}(h) := \sum_{\ell=1}^{h}\binom{2h}{2\ell}\frac{(-1)^{h-\ell}}{4h}B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell+2)!\left(1-\frac{1}{2^{2\ell+3}}\right)\zeta(2\ell+3),$$

and the Bernoulli numbers B_k are given by (0.4.3).

For $k \geq 0$,

$$\mathrm{m}\left(S_{2k+1}\right) = \sum_{h=0}^{k} \frac{s_{k-h}(1^2,3^2\ldots,(2k-1)^2)}{(2k)!}\left(\frac{2}{\pi}\right)^{2h+3}\mathcal{D}(h), \tag{0.7.3}$$

where

$$\mathcal{D}(h) := \sum_{\ell=0}^{h}\binom{2h+1}{2\ell+1}\frac{(-1)^{h-\ell}}{2(2h+1)}B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell+3)!L(\chi_{-4},2\ell+4).$$

For $k \geq 1$,

$$\mathrm{m}\left(T_{2k}\right) = \frac{\log 2}{2} + \sum_{h=1}^{k} \frac{s_{k-h}(2^2,4^2,\ldots,(2k-2)^2)}{(2k-1)!}\left(\frac{2}{\pi}\right)^{2h}\mathcal{E}(h),$$

where

$$\mathcal{E}(h) := \frac{(2h)!}{2}\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1) + \sum_{\ell=1}^{h}(2^{2(h-\ell)-1}-1)\binom{2h}{2\ell}\frac{(-1)^{h-\ell+1}}{2h}$$
$$\times B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell)!\left(1-\frac{1}{2^{2\ell+1}}\right)\zeta(2\ell+1).$$

For $k \geq 0$,

$$\mathrm{m}\left(T_{2k+1}\right) = \frac{\log 2}{2} + \sum_{h=1}^{k} \frac{s_{k-h}(2^2,4^2,\ldots,(2k-2)^2)}{(2k+1)!} \left(\frac{2}{\pi}\right)^{2h+2} \mathcal{F}(h),$$

where

$$\begin{aligned}\mathcal{F}(h) :=& \frac{(2h+2)!}{2}\left(1-\frac{1}{2^{2h+3}}\right)\zeta(2h+3) + \frac{\pi^2 k^2}{2}(2h)!\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1)\\ &+ k(2k+1)\sum_{\ell=1}^{h}\left[(2^{2(h-\ell)-1}-1)\binom{2h}{2\ell}\frac{(-1)^{h-\ell+1}}{4h}B_{2(h-\ell)}\pi^{2h+2-2\ell}(2\ell)!\right.\\ &\left.\times\left(1-\frac{1}{2^{2\ell+1}}\right)\zeta(2\ell+1)\right].\end{aligned}$$

The above formulas are quite miraculous. Their computations are possible because the Möbius transformation $\frac{1-x}{1+x}$ has a particularly elegant effect mapping the unit circle to the imaginary axis. The resulting differential in the change of variables also has very special properties, allowing for certain recurrences relating the case $n+2$ to the case n, which explains why the above formulas depend on the parity of n.

Nair **[93]** recently explored a similar phenomenon by considering the family

$$Q_n(x_1,\ldots,x_n,z) := z + \left(\frac{\xi_3 + x_1}{1+x_1}\right)\cdots\left(\frac{\xi_3+x_n}{1+x_n}\right),$$

where

$$\xi_3 = \frac{-1+\sqrt{3}i}{2},$$

and he proved similar formulas involving linear combinations of values of $\frac{\zeta(k)}{\pi^{k-1}}$ and $\frac{L(\chi_{-3},k)}{\pi^{k-1}}$ with certain rational coefficients.

In **[30]**, Boyd proposed the study of polynomials of the form $a(x)+b(x)y+c(x)z$, where $a(x), b(x), c(x)$ are products of cyclotomic polynomials. The reason for studying this particular class of polynomials comes from the Cassaigne–Maillot formula for the Mahler measure of $ax+by+c$ in (0.5.1), which has an expression that is particularly convenient for numerical integration. The investigation of such polynomials led to the discovery of several interesting numerical formulas involving L-functions of elliptic curves. Recently Brunault further pursued these computations with higher degree cyclotomic polynomials. This led to the discovery of certain formulas with arbitrary degrees such as

$$\mathrm{m}\left(1+(x^2-x+1)y+(x+1)^r z\right)=r\frac{3\sqrt{3}}{4\pi}L\left(\chi_{-3},2\right), \tag{0.7.4}$$

where r is an arbitrary positive integer.

In Chapter 4, we combine ideas in [**74, 75, 93**], and extend the above study by replacing the coefficients $a(x)$, $b(x)$, and $c(x)$ with $\prod_{j=1}^{n}(1-x_j)^r$, $\prod_{j=1}^{n}(1-x_j)^r$, and $(1+x)\prod_{j=1}^{n}(1+x_j)^r$, respectively, where r is an arbitrary positive integer. Note that the new multivariable *coefficient* polynomials are again products of cyclotomic polynomials. This is a joint work with Lalín and Nair [**81**].

More precisely, we generalize the family S_n to

$$S_{n,r}(x_1,\dots,x_n,x,y,z) := (1+x)z+\left[\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)\right]^r(1+y),$$

and we prove the following result.

Theorem 0.7.1 ([**81**, Theorem 1])**.** *Let $r\geq 1$. For $k\geq 1$, we have*

$$\mathrm{m}(S_{2k,r})=\sum_{h=1}^{k}\frac{s_{k-h}(2^2,4^2,\dots,(2k-2)^2)}{(2k-1)!}\left(\frac{2}{\pi}\right)^{2h}\mathcal{C}_r(h),$$

where

$$\begin{aligned}\mathcal{C}_r(h):=&r(2h)!\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1)\\&+\frac{r^2(2h-1)!}{\pi^2}\left\{\frac{(-1)^{h+1}7B_{2h}\pi^{2h}}{2r^2(2h)!}\zeta(3)\left(2^{2h-1}+(-1)^r2^{2h-1}+(-1)^{r+1}\right)\right.\\&+(2h+2)(2h+1)\frac{1-2^{-2h-3}}{r^{2h+2}}(1-(-1)^r)\zeta(2h+3)\\&-\sum_{\ell=0}^{2r-1}(-1)^\ell\left[\sum_{t=2}^{2h+2}\left(\frac{(t-1)(t-2)}{2}(-1)^t\left(\mathrm{Li}_t(\xi_{2r}^\ell)-\mathrm{Li}_t(-\xi_{2r}^\ell)\right)\right.\right.\\&\left.\left.\left.-\binom{t-1}{2h-1}(2-2^{1-t})\zeta(t)\right)\frac{(2\pi i)^{2h+3-t}}{(2h+3-t)!}B_{2h+3-t}\left(\frac{\ell}{2r}\right)\right]\right\}.\end{aligned}$$

For $k\geq 0$, we have

$$\mathrm{m}(S_{2k+1,r})=\sum_{h=0}^{k}\frac{s_{k-h}(1^2,3^2,\dots,(2k-1)^2)}{(2k)!}\left(\frac{2}{\pi}\right)^{2h+1}\mathcal{D}_r(h),$$

where

$$\begin{aligned}
\mathcal{D}_r(h) :=& r(2h+1)!L(\chi_{-4},2h+2)\\
&+\frac{2ir^2(2h)!}{\pi^2}\Bigg\{\frac{(-1)^{h+1}(2^{2h+4}-1)B_{2h+4}\pi^{2h+4}}{r^{2h+3}(2h+4)!}\\
&-i\frac{(-1)^hE_{2h}\pi^{2h+1}}{r^2 2^{2h}(2h)!}\left(\mathrm{Li}_3((-i)^r)-\frac{1}{8}\mathrm{Li}_3((-1)^r)\right)\\
&+(2h+3)(2h+2)\frac{1}{r^{2h+3}}\left(\mathrm{Li}_{2h+4}((-i)^r)-\frac{1}{2^{2h+4}}\mathrm{Li}_{2h+4}((-1)^r)\right)\\
&+\sum_{\ell=0}^{2r-1}(-1)^\ell\left[\sum_{t=1}^{2h+3}\left(\frac{(t-1)(t-2)}{2}(-1)^t\mathrm{Li}_t(-i\xi_{2r}^\ell)+\binom{t-1}{2h}\mathrm{Li}_t(-i)\right)\right.\\
&\left.\times\frac{(2\pi i)^{2h+4-t}}{(2h+4-t)!}B_{2h+4-t}\left(\frac{\ell}{2r}\right)\right]\Bigg\}.
\end{aligned}$$

In the above formulas, ξ_{2r} denotes a primitive $2r$-root of unity, $\mathrm{Li}_\ell(z)$ denotes the ℓ-th polylogarithm (see Section 0.4.1), and $B_n(t)$ denotes the Bernoulli polynomial given by (0.4.12).

The importance of Theorem 0.7.1 lies in providing formulas for the Mahler measure of families characterized by arbitrarily many variables and arbitrarily large degrees. This stands in stark contrast to previous results which primarily dealt with families such as R_n, S_n, and T_n, having arbitrarily many variables but remaining linear in them. Moreover, the degree r plays a non-crucial role in the Mahler measure of $S_{n,r}$, as varying r fundamentally changes $\mathrm{m}(S_{n,r})$, as opposed to formula (0.7.4), where r is merely a factor in the final formula.

We remark that in the case $r=1$, Theorem 0.7.1 reduces to the cases previously known for S_n, namely,

$$\mathcal{C}_1(h)=\frac{4}{\pi^2}\mathcal{C}(h) \qquad \text{and} \qquad \mathcal{D}_1(h)=\frac{4}{\pi^2}\mathcal{D}(h).$$

The case $r = 2$ also admits an interesting simplification as follows.

$$\begin{aligned}\mathcal{C}_2(h) =&(-1)^{h+1}\frac{7}{4h}B_{2h}\pi^{2h-2}\left(2^{2h}-1\right)\zeta(3)\\&+4\sum_{\ell=0}^{h-1}\binom{2h}{2\ell}\frac{(-1)^{h-\ell}}{h}\left(2^{2h-2\ell}-1\right)B_{2(h-\ell)}\pi^{2h-2\ell-2}(2\ell+2)!\left(1-\frac{1}{2^{2\ell+3}}\right)\zeta(2\ell+3)\\&+\sum_{\ell=1}^{h}\binom{2h-1}{2\ell-1}\frac{(-1)^{h-\ell}}{2^{2h-2\ell-2}}E_{2(h-\ell)}\pi^{2h-2\ell-1}(2\ell+1)!L(\chi_{-4},2\ell+2)\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_2(h) =&(-1)^{h}\frac{21}{2^{2h+2}}E_{2h}\pi^{2h-1}\zeta(3)\\&+8\sum_{\ell=0}^{h-1}\binom{2h+1}{2\ell+1}\frac{(-1)^{h-\ell}}{2h+1}B_{2(h-\ell)}\pi^{2h-2\ell-2}(2\ell+3)!\left(2^{2h-2\ell}-1\right)L(\chi_{-4},2\ell+4)\\&+\sum_{\ell=1}^{h}\binom{2h}{2\ell}\frac{(-1)^{h-\ell}}{2^{2h+1}}E_{2(h-\ell)}\pi^{2h-2\ell-1}(2\ell+2)!\left(2^{2\ell+3}-1\right)\zeta(2\ell+3),\end{aligned}$$

where the E_k are the Euler numbers, defined in (0.4.20).

Tables 2 and 3 record the formulas for the Mahler measures of $S_{n,1}$ and $S_{n,2}$ respectively for the first few values of n. We have included the case $n = 0$, not covered in Theorem 0.7.1, for comparison purposes. We see that, although there is a clear distinction between the cases n even and odd for $\mathrm{m}(S_{n,1})$ in the sense that the formulas for n even only contain special values of the Riemann zeta function, and the formulas for n odd only contain special values of the Dirichlet L-function, for $\mathrm{m}(S_{n,2})$ the formulas are mixed.

When $r > 2$, it is more difficult to evaluate $\mathcal{C}_r(h)$ and $\mathcal{D}_r(h)$ in terms of special values of the Riemann zeta function and Dirichlet L-functions, due to the difficulty relating polylogarithms evaluated at roots of unity of higher order to special values of L-functions. We illustrate the formulas for the Mahler measures of $S_{1,r}$ for the first few values of r in Table 4. We remark the appearance of Dirichlet L-functions in the characters $\chi_{12}(11,\cdot) := \left(\frac{12}{\cdot}\right)$ of conductor 12 and $\chi_8(5,\cdot) := \left(\frac{8}{\cdot}\right)$ of conductor 8. This is a key distinction from the previous results for the families R_n, S_n and T_n. Chapter 4 includes the proof of Theorem 0.7.1 and further continues the above discussion.

$\pi^2 \mathrm{m}\left(1+x+(1+y)z\right)$	$\frac{7}{2}\zeta(3)$
$\pi^4 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)$	$93\zeta(5)$
$\pi^6 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)(1+y)z\right)$	$\frac{1905}{2}\zeta(7)+31\pi^2\zeta(5)$
$\pi^8 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_6}{1+x_6}\right)(1+y)z\right)$	$7154\zeta(9)+635\pi^2\zeta(7)+\frac{248\pi^4}{15}\zeta(5)$
$\pi^3 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right)$	$24L(\chi_{-4},4)$
$\pi^5 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_3}{1+x_3}\right)(1+y)z\right)$	$320L(\chi_{-4},6)+4\pi^2L(\chi_{-4},4)$
$\pi^7 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_5}{1+x_5}\right)(1+y)z\right)$	$2688L(\chi_{-4},8)+160\pi^2L(\chi_{-4},6)+\frac{9\pi^4}{5}L(\chi_{-4},4)$

Table 2 – Mahler measure of $S_{n,1}$ for $n \leq 6$.

$\pi^2 \mathrm{m}\left(1+x+(1+y)z\right)$	$\frac{7}{2}\zeta(3)$
$\pi^4 \mathrm{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\right]^2(1+y)z\right)$	$96\pi L(\chi_{-4},4) - \frac{21\pi^2}{2}\zeta(3)$
$\pi^6 \mathrm{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)\right]^2(1+y)z\right)$	$1280\pi L(\chi_{-4},6) - 372\pi^2\zeta(5) + 112\pi^3 L(\chi_{-4},4) - \frac{21\pi^4}{2}\zeta(3)$
$\pi^8 \mathrm{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_6}{1+x_6}\right)\right]^2(1+y)z\right)$	$10752\pi L(\chi_{-4},8) - 3810\pi^2\zeta(7) + 1920\pi^3 L(\chi_{-4},6) - 496\pi^4\zeta(5) + \frac{596\pi^5}{5} L(\chi_{-4},4) - \frac{21\pi^6}{2}\zeta(3)$
$\pi^3 \mathrm{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\right]^2(1+y)z\right)$	$\frac{21\pi}{2}\zeta(3)$
$\pi^5 \mathrm{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_3}{1+x_3}\right)\right]^2(1+y)z\right)$	$\frac{31\pi}{2}\zeta(5) - 96\pi^2 L(\chi_{-4},4) + \frac{21\pi^3}{2}\zeta(3)$
$\pi^7 \mathrm{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_5}{1+x_5}\right)\right]^2(1+y)z\right)$	$\frac{127\pi}{24}\zeta(7) - 1280\pi^2 L(\chi_{-4},6) + \frac{62\pi^3}{3}\zeta(5) - 112\pi^4 L(\chi_{-4},4) + \frac{21\pi^5}{2}\zeta(3)$

Table 3 – Mahler measure of $S_{n,2}$ for $n \leq 6$.

$\pi^3 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right)$	$24L(\chi_{-4},4)$
$\pi^3 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^2(1+y)z\right)$	$\frac{21\pi}{2}\zeta(3)$
$\pi^3 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^3(1+y)z\right)$	$-8L(\chi_{-4},4)+12\sqrt{3}\pi L(\chi_{12}(11,\cdot),3)$
$\pi^3 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^4(1+y)z\right)$	$-\frac{105\pi}{2}\zeta(3)+64\sqrt{2}\pi L(\chi_8(5,\cdot),3)$

Table 4 – Mahler measure of $S_{1,r}$ for $r \leq 4$.

Chapter 1

Generalized Mahler measures of Laurent polynomials

Building on Lalín and Mittal's work (see (0.5.2) and (0.5.3)) on a generalization of the Mahler measure of two particular Boyd's polynomials by considering the integration torus as arbitrary (see Definition 0.5.1), we extend this definition to all Laurent polynomials that do not vanish on the integration torus in this chapter. This work will appear in [**106**].

1.1. A brief description of the results

This section includes a restatement of Theorems 0.5.2, 0.5.4, 0.5.5, 0.5.6, and 0.5.7 for the reader's convenience, along with a brief discussion on the strategies of the proofs.

Recall that the *generalized Mahler measure* of a non-zero rational function $P \in \mathbb{C}(x_1, \dots, x_n)$, denoted by $\mathrm{m}_{\mathbf{a}}(P)$, is defined as the arithmetic mean of $\log|P|$ over the torus $\mathbb{T}^n_{\mathfrak{a}} = \{(x_1, \dots, x_n) \in \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times : |x_1| = a_1, \dots, |x_n| = a_n\}$ with respect to the unique Haar measure (see Definition 0.6.14). As mentioned in Section 0.5, the notion of the generalized Mahler measure was first introduced by Lalín and Mittal in [**80**] following Cassaigne–Maillot's result in (0.5.1), which explicitly expresses $\mathrm{m}(ax + by + c)$, for $a, b, c \in \mathbb{C}^\times$, and can be reinterpreted as the generalized Mahler measure $\mathrm{m}_{|a|,|b|,|c|}(1+x+y)$. Lalín and Mittal investigated this definition to find $\mathrm{m}_{a,a}(y^2+2xy-x^3+x)$

and $\mathrm{m}_{a^2,a}((1+x)(1+y)(x+y)+2xy)$ for certain positive values of a such that the polynomials do not vanish on the integration torii $\mathbb{T}^2_{a,a}$ and $\mathbb{T}^2_{a^2,a}$, respectively (see (0.5.2) and (0.5.3)).

This chapter provides a generalization of their results for a large set of Laurent polynomials with complex coefficients. We recall some notation from Section 0.5 before (re)stating our results.

Given a non-zero Laurent polynomial $P_k(x_1,\dots,x_n) = k - P(x_1,\dots,x_n) \in \mathbb{C}[x_1^{\pm},\dots,x_n^{\pm}]$ such that $P(0,\dots,0) = 0$, we recall that $\mathcal{K}_{\mathfrak{a}}$ denotes the image of the map from $\mathbb{T}^n_{\mathfrak{a}}$ to $\mathbb{C}$ defined in (0.5.6) by $(x_1,\dots,x_n) \mapsto P(x_1,\dots,x_n)$, which is compact in $\mathbb{C}$. The unbounded open connected component of $\mathbb{C}\setminus\mathcal{K}_{\mathfrak{a}}$ containing a neighbourhood of $k=\infty$ is denoted by $U_{\mathfrak{a}}$. Let $\nu^j_{\mathfrak{a},r}$ be the difference between the number of zeroes (denoted by $Z^j_{\mathfrak{a},r}$) and the number of poles (denoted by $P^j_{\mathfrak{a},r}$) of $P_k(a_1,\dots,a_{j-1},x_j,a_{j+1},\dots,a_n)$ inside the circle $|x_j| = a_j$, namely

$$\nu^j_{\mathfrak{a},r} := Z^j_{\mathfrak{a},r} - P^j_{\mathfrak{a},r}.$$

We further denote by $P^j_{F,k}$ (resp. $P^j_{f,k}$) the *leading* (resp. *constant*) coefficient of P_k when considered as a polynomial in x_j with coefficients in $\mathbb{C}\left[x_1^{\pm},\dots,\widehat{x_j},\dots,x_n^{\pm}\right]$, where $\widehat{\ }$ indicates that the term is omitted from the expression (see (0.5.11) and (0.5.12) for more details). When $n=2$, we redefine the above notation as follows: $x := x_1$, $y := x_2$, $r := k$, $Q_r(x,y) := P_k(x_1,x_2)$, $Q(x,y) := P(x_1,x_2)$, $(a,b) := \mathbf{a} = (a_1,a_2)$, $\mathcal{R}_{a,b} := \mathcal{K}_{\mathbf{a}}$, $Q^x_{F,r} := P^1_{F,k}$, $Q^y_{F,r} := P^2_{F,k}$, $Q^x_{f,r} := P^1_{f,k}$, and $Q^y_{f,r} := P^2_{f,k}$. Then, we have the following theorem.

Theorem 1.1.1 (see Theorem 0.5.2)**.** *Let $\mathfrak{a} = (a_1,\dots,a_n) \in (\mathbb{R}_{>0})^n$. Let $P_k(x_1,\dots,x_n) = k - P(x_1,\dots,x_n) \in \mathbb{C}[x_1^{\pm},\dots,x_n^{\pm}]$, such that P has no constant term. Denote by $U_{\mathfrak{a}}$ the unbounded open connected component of $\mathbb{C}\setminus\mathcal{K}_{\mathfrak{a}}$ containing a neighbourhood of $k=\infty$. Then, for $k \in U_{\mathfrak{a}} \cap U_{\mathbb{1}}$,*

$$\mathrm{m}_{\mathfrak{a}}(P_k) = \mathrm{m}_{\mathbf{1}}(P_k) + \sum_{j=1}^{n} \nu^j_{\mathfrak{a},k} \log a_j,$$

where $\nu^j_{\mathfrak{a},k}$ is defined as above, and $\mathrm{m}_{\mathbb{1}}(P_k) = \mathrm{m}(P_k)$. Moreover, for $k \in U_{\mathfrak{a}} \cap U_{\mathbb{1}}$ and $j = 1,\dots,n$, $\nu^j_{\mathfrak{a},k}$ only depends on $\mathfrak{a}$.

As a corollary to Theorem 1.1.1, we have the following result in two variables.

Theorem 1.1.2 (see Theorem 0.5.4). *Let a and b be positive real numbers, and denote by $U_{a,b}$ the unbounded open connected component of $\mathbb{C} \setminus \mathcal{R}_{a,b}$ containing some neighborhood of $r = \infty$. Then, for $r \in U_{a,b} \cap U_{1,1}$,*

$$\mathrm{m}_{a,b}(Q_r) = \mathrm{m}(Q_r) + \nu^1_{a,b,r} \log a + \nu^2_{a,b,r} \log b,$$

where $\nu^1_{a,b,r}$ is the difference between the number of zeroes (denoted by $Z^1_{a,b,r}$) and the number of poles (denoted by $P^1_{a,b,r}$) of $Q_r(x, b)$ inside the circle $|x| = a$, defined by

$$\nu^1_{a,b,r} = Z^1_{a,b,r} - P^1_{a,b,r}, \tag{1.1.1}$$

$\nu^2_{a,b,r}$ is the difference between the number of zeroes (denoted by $Z^2_{a,b,r}$) and the number of poles (denoted by $P^2_{a,b,r}$) of $Q_r(a, Y)$ inside the circle $|y| = b$, defined by

$$\nu^2_{a,b,r} = Z^2_{a,b,r} - P^2_{a,b,r}, \tag{1.1.2}$$

and $\mathrm{m}_{1,1}(Q_r) = \mathrm{m}(Q_r)$. Moreover, for $r \in U_{a,b} \cap U_{1,1}$, $\nu^j_{a,b,r}$ does not depend on r.

Suppose that $\mathbb{C} \setminus \mathcal{K}_{\mathfrak{a}}$ contains at least one open bounded connected component, then the following result aims to provide a way to evaluate $\mathrm{m}_{\mathfrak{a}}(P_k)$ conditionally when k belongs to one of the bounded connected components of $\mathbb{C} \setminus \mathcal{K}_{\mathfrak{a}}$.

Theorem 1.1.3 (see Theorem 0.5.5). *Let $\mathfrak{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_{>0}$. Let $k_0 \in \mathbb{C} \setminus \mathcal{K}_{\mathfrak{a}}$ such that k_0 belongs to one of the bounded open connected components of $\mathbb{C} \setminus \mathcal{K}_{\mathfrak{a}}$, denoted by $V_{\mathfrak{a},k_0}$.*

(I) For $j = 1, \ldots, n$, if all the roots of $P_{k_0}(a_1, \ldots, a_{j-1}, x_j, a_{j+1}, \ldots, a_n)$ lie entirely inside the circle $\mathbb{T}^1_{a_j}$, then, for all $k \in V_{\mathfrak{a},k_0}$,

$$\mathrm{m}_{\mathfrak{a}}(P_k) = \nu^j_{\mathfrak{a},k} \log a_j + \mathrm{m}_{a_1,\ldots,\widehat{a_j},\ldots,a_n}\left(P^j_{F,k}\right).$$

(II) For $j = 1, \ldots, n$, if all the roots of $P_{k_0}(a_1, \ldots, a_{j-1}, x_j, a_{j+1}, \ldots, a_n)$ lie entirely outside the circle $\mathbb{T}^1_{a_j}$, then, for all $k \in V_{\mathfrak{a},k_0}$,

$$\mathrm{m}_{\mathfrak{a}}(P_k) = \nu^j_{\mathfrak{a},k} \log a_j + \mathrm{m}_{a_1,\ldots,\widehat{a_j},\ldots,a_n}\left(P^j_{f,k}\right).$$

Similarly, suppose that $\mathbb{C}\setminus\mathcal{R}_{a,b}$ contains at least one open bounded connected component, then we have the following theorem for the two-variable case as a corollary.

Theorem 1.1.4 (see Theorem 0.5.6)**.** *Let a and b be positive real numbers. Let $r_0 \in \mathbb{C} \setminus \mathcal{R}_{a,b}$ such that r_0 belongs to one of the bounded open connected components of $\mathbb{C} \setminus \mathcal{R}_{a,b}$. We denote by V_{a,b,r_0} the bounded open connected component containing r_0.*

(i) *If all the roots of $Q_{r_0}(a, y)$ either lie entirely inside the circle $\mathbb{T}_b^1$ or lie entirely outside the circle $\mathbb{T}_b^1$, then, for all $r \in V_{a,b,r_0}$,*

$$\mathrm{m}_{a,b}(Q_r) - \nu_{a,b,r}^2 \log b = \begin{cases} \mathrm{m}_a(Q_{F,r}^y(x)) & \text{when all roots of } Q_{r_0}(a,y) \text{ lie inside } \mathbb{T}_b^1, \\ \mathrm{m}_a(Q_{f,r}^y(x)) & \text{when all roots of } Q_{r_0}(a,y) \text{ lie outside } \mathbb{T}_b^1. \end{cases}$$

(ii) *If all the roots of $Q_{r_0}(x, b)$ either lie entirely inside the circle $\mathbb{T}_a^1$ or lie entirely outside the circle $\mathbb{T}_a^1$, then, for all $r \in V_{a,b,r_0}$,*

$$\mathrm{m}_{a,b}(Q_r) - \nu_{a,b,r}^1 \log a = \begin{cases} \mathrm{m}_b(Q_{F,r}^x(y)) & \text{when all roots of } Q_{r_0}(x,b) \text{ lie inside } \mathbb{T}_a^1, \\ \mathrm{m}_b(Q_{f,r}^x(y)) & \text{when all roots of } Q_{r_0}(x,b) \text{ lie outside } \mathbb{T}_a^1. \end{cases}$$

To prove Theorem 1.1.2, we follow the methods of Rodriguez-Villegas [**102**] and Bertin [**9**]. We consider $\frac{d\mathrm{m}_{a,b}(Q_r)}{dr}$ as a series in r. Since continuous deformation of the integration torus does not change $\frac{d\mathrm{m}_{a,b}(Q_r)}{dr}$ as long as the deformation happens outside the zero-set of Q_r, we show that $\frac{d\mathrm{m}_{a,b}(Q_r)}{dr} = \frac{d\mathrm{m}(Q_r)}{dr}$ for $r \in U_{a,b} \cap U_{1,1}$. Integrating both sides yields a constant term $f(a, b)$ depending only on a and b. Then, suitably varying a and b, we further consider $a\frac{\partial f(a,b)}{\partial a}$ and $b\frac{\partial f(a,b)}{\partial b}$, and express $f(a, b)$ in terms of $\log a$ and $\log b$ to complete the proof. Extending this idea to n variables yields Theorem 1.1.1.

In order to prove Theorem 1.1.4, we apply Rouché's theorem.

Theorem 1.1.5 ([**2**, Corollary to Theorem 18])**.** *Let $U \subset \mathbb{C}$ be an open bounded region with piecewise smooth boundary ∂U. Let f, g be meromorphic functions on $U \cup \partial U$ which have finitely many zeroes, no removable singularities, and no poles on ∂U. Suppose also that $|f(z) - g(z)| < |f(z)|$ for all $z \in \partial U$. Then f and g have the same number of zeroes enclosed by ∂U.*

We show that if for some $r_0 \in V_{a,b,r_0}$, $Q_{r_0}(a, y)$ (resp. $Q_{r_0}(x, b)$) has all the roots inside or outside of the circle $|y| = b$ (resp. $|x| = a$), then the same is true for all the roots of $Q_r(a, y)$ (resp. $Q_r(x, b)$) for all $r \in V_{a,b,r_0}$. This, combined with an application of Jensen's formula

(0.1.4), yields the required results. Similarly, the n-variable case in Theorem 1.1.3 follows from an extension of this idea.

In Section 1.4, we consider Boyd's family of polynomials given in (0.1.10) by

$$\left\{\mathcal{Q}_r(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + r : r \in \mathbb{C}\right\},$$

(see also (0.5.4)) and apply Theorems 1.1.2 and 1.1.4 to evaluate $\mathrm{m}_{a,b}\left(\mathcal{Q}_r\right)$ for all $r \in \mathbb{C}\backslash\mathcal{R}_{a,b}$, i.e. when $\mathcal{Q}_r$ does not vanish on $\mathbb{T}^2_{a,b}$. We should note that our evaluation of $\mathrm{m}_{a,b}(\mathcal{Q}_r)$ for $r \in U_{a,b}$ is expressed in terms of $\mathrm{m}(\mathcal{Q}_r)$. From the known evaluations of $\mathrm{m}(\mathcal{Q}_r)$ in terms of special values of L-functions of certain elliptic curves (see Table 1), we can further obtain explicit expressions of $\mathrm{m}_{a,b}(\mathcal{Q}_r)$ in terms of those special L-values and certain $\mathbb{Z}$-linear combinations of $\log a$ and $\log b$; two of such results are mentioned in Examples 1.4.1 and 1.4.2.

We will end this chapter with a derivation of $\mathrm{m}_{a,b}(\mathcal{Q}_4)$ for all $a, b > 0$, irrespective of whether $\mathcal{Q}_4$ vanishes on $\mathbb{T}^2_{a,b}$ or not. This follows from iterative applications of properties of η in Lemma 0.4.9 (especially the *exactness* property in (0.4.24)), complemented with a change of variables due to Boyd [**29**], which factors $\mathcal{Q}_4$ in linear polynomials. In fact, we have the following result.

Theorem 1.1.6 (see Theorem 0.5.7)**.** *Let $a, b \in \mathbb{R}_{>0}$, and define*

$$c = \sqrt{ab}, \quad d = \sqrt{\frac{b}{a}}, \quad \textit{and } \mathcal{A}_{c,d} = \frac{1-d^2}{1+d^2}\cdot\frac{1+c^2}{2c},$$

such that c and d are both positive real numbers. Then,

$$\mathrm{m}_{a,b}(\mathcal{Q}_4(x,y)) = \begin{cases} |\log c| + |\log d| & \textit{if } |\mathcal{A}_{c,d}| \geq 1, \\ \\ \frac{2}{\pi}\left[D(ice^{-i\mu}) + D(ice^{i\mu}) - \mu\log d + (\log c)\arctan\left(\frac{c-c^{-1}}{2\cos\mu}\right)\right] & \textit{if } |\mathcal{A}_{c,d}| < 1, \end{cases}$$

where $\mu = \arcsin\left(\mathcal{A}_{c,d}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and D is the Bloch–Wigner dilogarithm defined in (0.4.5)*.*

In summary, Chapter 1 is organized as follows. In Section 1.2, we discuss the proof of Theorem 1.1.2 and some auxiliary results required to complete the proof. We conclude the section with a brief argument generalizing our method to the several variables setting and proving Theorem 1.1.1. Section 1.3 is completely dedicated to the proof of Theorem

1.1.4 and subsequently to the proof of Theorem 1.1.3 by a similar generalization. In Section 1.4, we discuss some applications of Theorems 1.1.2 and 1.1.4 to the family of polynomials in (0.5.4). We then prove Theorem 1.1.6 in Section 1.5, where we use properties of the differential form and the Bloch–Wigner dilogarithm mentioned in Section 0.4. We end the chapter with concluding remarks on possible directions to pursue going forward.

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1.2. Proof of Theorem 1.1.2

Recall that any 2-variable Laurent polynomial $Q_r(x,y) \in \mathbb{C}[x^{\pm}, y^{\pm}]$ can be written as $Q_r(x,y) = r - Q(x,y)$, where $Q(x,y) \in \mathbb{C}[x^{\pm}, y^{\pm}]$ has no constant term. In this section, we use the notation

$$\mathrm{m}_{a,b}(Q_r(x,y)) = \mathrm{m}_{a,b}(Q_r) = \mathrm{m}_{a,b}(r) \qquad \text{for } r \in \mathbb{C},$$

for simplicity. Our approach is inspired by the methods of Rodriguez-Villegas [**102**] and Bertin [**9**]. We first show that the required equality between Mahler measures holds for a smaller unbounded region of $\mathbb{C} \setminus \mathcal{R}_{a,b}$, and then, using properties of harmonic functions, we argue that it can be extended to the desired region stated in Theorem 1.1.2.

The following lemma formulates the invariance of $\mathrm{m}_{a,b}(r)$ under certain changes of variables.

Lemma 1.2.1. *Let a, b be positive real numbers. Define $f_r(a,b) := \mathrm{m}_{a,b}(r)$. Then f_r satisfies the following identities:*

$$f_r(a,b) = f_r(b,a) = f_r\left(\frac{1}{a}, b\right) = f_r\left(\frac{1}{a}, \frac{1}{b}\right).$$

Proof. Let $a, b > 0$. For $\tilde{Q}_r(x,y) = Q_r(ax, by)$, the generalized Mahler measure of Q_r satisfies the identity

$$\mathrm{m}_{a,b}(Q_r(x,y)) = \mathrm{m}(\tilde{Q}_r(x,y)) = \mathrm{m}(\tilde{Q}_r).$$

The changes of variables

$$(x,y) \to (y,x), \quad (x,y) \to (x^{-1}, y), \quad (x,y) \to (x^{-1}, y^{-1}),$$

fix $\mathrm{m}(\tilde{Q}_r)$. Since $\mathrm{m}_{a,b}(r) = \mathrm{m}(\tilde{Q}_r)$, we have the required identities involving $f_r(a,b) = \mathrm{m}_{a,b}(r)$. □

In view of Lemma 1.2.1, we may assume without loss of generality that $a > b > 1$.

Our main aim is to study $\mathrm{m}_{a,b}(r)$ in terms of the complex parameter r. Recall that $\mathcal{R}_{a,b}$ is the set of all $r \in \mathbb{C}$ such that $Q_r(x,y)$ vanishes on $\mathbb{T}^2_{a,b}$. Before proceeding to prove Theorem 1.1.2, we state a proposition explaining the following:

— the behaviour of the roots of $Q_r(x,y)$ for each $x \in \mathbb{T}^1_a$; in particular, the number of roots inside the unit circle $\mathbb{T}^1_b$,

— the behaviour of the roots of $Q_r(x,y)$ for each $y \in \mathbb{T}^1_b$; in particular, the number of roots inside the unit circle $\mathbb{T}^1_a$.

This proposition, in particular, gives a formula for the quantities $\nu^2_{a,b,r}$ and $\nu^1_{a,b,r}$ in the statement of Theorem 1.1.2. Since the above two cases are analogous, it suffices to consider the first case.

For $w \in \mathbb{T}^1_a$, let $\varrho^2_{a,b,r}(w)$ denote the number of roots of $Q_r(w,y)$ lying inside the circle $\mathbb{T}^1_b$. In particular, following the discussion preceding the definition in (1.1.2), we have, for $w \in \mathbb{T}^1_a$,

$$\varrho^2_{a,b,r}(w) = Z^2_{w,b,r} \quad \text{and} \quad \varrho^2_{a,b,r}(a) = Z^2_{a,b,r} = \nu^2_{a,b,r} + P^2_{a,b,r} = \nu^2_{a,b,r} + v_2, \tag{1.2.1}$$

where $Z^2_{w,b,r}$ is the number of zeros (counting multiplicities) of $Q_r(w, y)$ inside the circle $\mathbb{T}^1_b$, $P^2_{w,b,r}$ is the order of the pole of $Q_r(w, y)$ at $y = 0$, and v_2 is the largest power of y^{-1} in $Q_r(x, y)$. Then we have the following statement.

Proposition 1.2.2. *Let $r \in \mathbb{C} \setminus \mathcal{R}_{a,b}$. Then $\varrho^2_{a,b,r}(x)$ is constant for all $x \in \mathbb{T}^1_a$.*

Before proceeding with the proof, we first consider the *resultant* of the polynomial Q_r with respect to y.

Recall that

$$Q_r(x, y) = y^{-v_2} Q^y_{F,r}(x) \prod_{j=1}^{d_y} (y - y_{j,r}(x)),$$

where $y_{j,r}(x)$ are algebraic functions in x, and v_2 is as defined above.

Here and in what follows for the rest of this section, we denote $Q_{F,r}(x) := Q^y_{F,r}(x), d := d_y$.

Let $D_r(x)$ denote the *resultant* of $Q_r(x, y)$ and $\frac{\partial}{\partial y} Q_r(x, y)$ with respect to y.

Then the algebraic solutions $y_{j,r}$ are holomorphic in some neighbourhood of x for any $x \in \mathbb{C} \setminus S_r$, where

$$S_r = \{z \in \mathbb{C} : Q_{F,r}(z) D_r(z) = 0\} \tag{1.2.2}$$

is a finite subset of $\mathbb{C}$.

Let $\mathbf{y}_r(x)$ be the d-valued global analytic function, with d-branches $y_{1,r}, \dots y_{d,r}$, such that $Q_r(x, \mathbf{y}_r(x)) = 0$. Then S_r is called the set of *critical points* of $\mathbf{y}_r(x)$. If x' is a critical point of $\mathbf{y}_r(x)$, then x' is either an algebraic branch point or a pole (for more details see [**2**]).

(**1**) If $x' \in S_r$ is an algebraic branch point, i.e. when $D_r(x') = 0$, then, in a sufficiently small neighbourhood $U_{x'}$ of x' (which does not contain any other critical points), the multi-set $\{y_{1,r}, \dots y_{d,r}\}$ can be decomposed into a number of non-intersecting cycles

$$\{f_1(x), \dots, f_{k_1}(x)\}, \dots, \{f_{k_1+\dots+k_{t-1}+1}(x), \dots, f_{k_1+\dots+k_t}(x)\},$$

such that $\sum_{n=1}^{t} k_n = d$, and $f_j(x) = y_{l,r}(x)$ for some $j, l \in \{1, \dots, d\}$. The elements of the first cycle can be represented as convergent Puiseux series of the local parameter $\tau = (x - x')^{1/k_1}$ in a small enough neighbourhood of $\tau = 0$. The elements of the rest of the cycles follow analogous convergent series representations. Therefore, a single turn around x' in a circle $C' \subset U_{x'}$ converts the Puiseux series of elements in one cycle into each other in a cyclic order, i.e. $f_1 \to f_2 \to \cdots \to f_{k_1} \to f_1$ etc.

(**2**) If $x' \in S_r$ is a pole, that is when $Q_{F,r}(x') = 0$, then, substituting y with $yQ_{F,r}(x)$, we return to the first case where the local parameter of the convergent series is $\tau = 1/x$.

Recall that, for $w \in \mathbb{T}_a^1$, $\varrho^2_{a,b,r}(w)$ denote the number of roots of $Q_r(w, y)$ lying inside the circle $\mathbb{T}_b^1$. We are now ready to prove Proposition 1.2.2.

Proof of Proposition 1.2.2. First fix an arbitrary $r \in \mathbb{C} \setminus \mathcal{R}_{a,b}$. Note that $\varrho^2_{a,b,r}$ defines a function from $\mathbb{T}_a^1$ to $\mathbb{Z}$ via the map $x \mapsto \varrho^2_{a,b,r}(x)$, where $\mathbb{Z}$ is equipped with discrete topology.

If $x_0 \in \mathbb{T}_a^1$ is not a critical point of $\mathbf{y}_r$, i.e. $x_0 \notin S_r$, where S_r is given in (1.2.2), then, for all $j = 1, \dots, d$, $y_{j,r}$ is holomorphic in a sufficiently small neighbourhood U_{x_0} of x_0 which does not contain any critical point. Therefore, $|y_{j,r}(x)|$ is continuous in U_{x_0}. Since Q_r does not vanish on $\mathbb{T}^2_{a,b}$, we have $|y_{j,r}(x)| \neq b$ for all $x \in U_{x_0} \cap \mathbb{T}_a^1$, j. Therefore, if, for any $l = 1, \dots, d$, $|y_{l,r}(x_0)| < 1$ (resp. $|y_{l,r}(x_0)| > 1$), then, for all $x \in U_{x_0 \cap \mathbb{T}_a^1}$, $|y_{l,r}(x)| < 1$ (resp. $|y_{l,r}(x_0)| > 1$). In other words, $\varrho^2_{a,b,r}(x)$ is constant for all $x \in U_{x_0} \cap \mathbb{T}_a^1$. In particular, $\varrho^2_{a,b,r}$ is continuous at x_0.

If $x_1 \in \mathbb{T}_a^1 \cap S_r$, then there exists a sufficiently small neighbourhood U_{x_1} of x_1 which does not contain any critical point except x_1. Then the convergent Puiseux series expansions of $y_{1,r}, \dots, y_{d,r}$ in U_{x_1} imply that, for all j, $|y_{j,r}|$ is continuous in U_{x_1}, and this brings us to the previous case. From properties (**1**), (**2**) and the above discussion, we conclude that, in the neighbourhood U_{x_1} of x_1, $\varrho^2_{a,b,r}$ is constant. This implies that $\varrho^2_{a,b,r}$ is continuous at x_1.

We now have a continuous function $\varrho^2_{a,b,r}$ from a connected set $\mathbb{T}_a^1$ to a discrete set $\mathbb{Z}$. Since only connected subsets of $\mathbb{Z}$ are singletons, we derive that $\varrho^2_{a,b,r}$ is constant in $\mathbb{T}_a^1$, and thus completing the proof of the statement. □

1.2.1. Proof of Theorem 1.1.2

Proposition 1.2.2 implies that, for all $x \in \mathbb{T}_a^1$, $\varrho^2_{a,b,r}(x) = \nu^2_{a,b,r} + v_2$. Moreover, (1.2.1) implies that the constant is $\nu^2_{a,b,r} + v_2$, where v_2 is the largest power of y^{-1} in $Q_r(x, y)$. In particular, we have $\nu^2_{w,b,r} = \nu^2_{a,b,r}$ for all $w \in \mathbb{T}_a^1$, where $\nu^2_{w,b,r}$ is given in (1.1.2). Next, we derive Theorem 1.1.2 using Proposition 1.2.2.

Proof of Theorem 1.1.2. For a and b positive real numbers, the torus $\mathbb{T}^2_{a,b}$ is defined as the set $\{(x,y) \in (\mathbb{C}^\times)^2 : |x| = a, |y| = b\}$. By construction, $\mathbb{T}^2_{a,b}$ is compact. Since the map in (0.5.6), namely

$$q : \mathbb{T}^2_{a,b} \longrightarrow \mathbb{C}, \quad \text{defined by} \quad (x,y) \mapsto Q(x,y),$$

is continuous, the image of q is compact. That is, $q(\mathbb{T}^2_{a,b}) = \mathcal{R}_{a,b}$ is compact, and therefore closed and bounded in $\mathbb{C}$. In other words, $\max_{r\in\mathcal{R}_{a,b}} |r|$ exists. We denote

$$R_{a,b} := \max_{r\in\mathcal{R}_{a,b}} |r|, \quad \text{and} \quad R_{a,b,1,1} := \max\{R_{a,b}, R_{1,1}\}.$$

Following a construction in **[102]**, we define

$$\tilde{\mathrm{m}}_{a,b}(r) = \log r - \sum_{n\geq 0} \frac{a_{n,a,b}}{n} r^{-n}, \quad |r| > R_{a,b,1,1}, r \notin (-\infty, 0],$$

where log denotes the principal branch of the logarithm, and $a_{n,a,b}$ is defined as follows:

$$a_{n,a,b} = \left[\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2_{a,b}} \frac{dxdy}{xy(1 - r^{-1}Q(x,y))}\right]_n = \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2_{a,b}} Q(x,y)^n \frac{dx}{x}\frac{dy}{y},$$

Here $[T(s)]_n$ denotes the coefficient of s^{-n} in the series $T(s)$. It is immediate to see that $\tilde{\mathrm{m}}_{a,b}$ is holomorphic in the region defined by $|r| > R_{a,b,1,1}$ and $r \notin (-\infty, 0]$. Also,

$$\mathrm{Re}(\tilde{\mathrm{m}}_{a,b}(r)) = \mathrm{m}_{a,b}(r), \quad |r| > R_{a,b,1,1}.$$

We now claim that, for $|r| > R_{a,b,1,1}$,

$$\frac{d\tilde{\mathrm{m}}_{a,b}}{dr} = \frac{d\tilde{\mathrm{m}}_{1,1}}{dr}.$$

In order to prove our claim, it is enough to show that $a_{n,a,b} = a_{n,1,1}$ for all n. The above construction of the coefficients and the integral expression of these terms in **[102]** yield that

$$\begin{aligned} a_{n,a,b} &= \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2_{a,b}} Q(x,y)^n \frac{dx}{x}\frac{dy}{y} = \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2} Q(ax',by')^n \frac{dx'}{x'}\frac{dy'}{y'} \\ &= [Q(ax',by')^n]_0 = [Q(x',y')^n]_0 \\ &= \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2} Q(x',y')^n \frac{dx'}{x'}\frac{dy'}{y'} \\ &= a_{n,1,1}. \end{aligned}$$

The equality $[Q(ax', by')^n]_0 = [Q(x', y')^n]_0$ follows from the fact that the constant term gathers the terms with degree 0, which are invariant under the multiplications of x and y by a and b, respectively. This concludes the proof.

Due to the above identity, we can denote the coefficients as $a_n := a_{n,a,b} = a_{n,1,1}$ for the rest of the argument. From the definition of $\tilde{\mathrm{m}}_{a,b}$, it follows that

$$\frac{d\tilde{\mathrm{m}}_{a,b}}{dr} = \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2_{a,b}} \frac{1}{r - Q(x,y)}\frac{dx}{x}\frac{dy}{y}, \qquad |r| > R_{a,b,1,1}, \tag{1.2.3}$$

where we include the region $r \in (-\infty, 0] \cap \{|r| > R_{a,b,1,1}\}$ by continuity. We need to show that $\frac{d\tilde{\mathrm{m}}_{a,b}}{dr}$ is in fact holomorphic in $|r| > R_{a,b,1,1}$.

For $r \in \mathbb{C}$, define

$$\mathcal{F}_{a,b}(r) := \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2_{a,b}} \frac{1}{r - Q(x,y)}\frac{dx}{x}\frac{dy}{y}. \tag{1.2.4}$$

Note that the integrand

$$\left.\frac{1}{r - Q(x,y)}\right|_{(x,y)\in\mathbb{T}^2_{a,b}}$$

is holomorphic in r when $|r| > R_{a,b,1,1}$. In fact, we will now show that $\mathcal{F}_{a,b}(r)$ is holomorphic as well on $|r| > R_{a,b,1,1}$. The integrand, as well as the integral in (1.2.3), are bounded on $\mathbb{T}^2_{a,b}$. This implies that $\frac{d^j\mathcal{F}_{a,b}}{dr^j}$ exists and is holomorphic for $j = 1$ (and therefore for all $j \geq 1$). Hence, $\mathcal{F}_{a,b}(r)$ is holomorphic in $|r| > R_{a,b,1,1}$.

Recall that we have, for $|r| > R_{a,b,1,1}$,

$$\frac{d\tilde{\mathrm{m}}_{a,b}(r)}{dr} = \frac{d\tilde{\mathrm{m}}_{1,1}(r)}{dr},$$

and all the quantities are holomorphic in the mentioned region. Integrating both sides with respect to r, we get

$$\tilde{\mathrm{m}}_{a,b}(r) = \tilde{\mathrm{m}}_{1,1}(r) + \tilde{f}(a,b), \qquad \text{for } |r| > R_{a,b,1,1},$$

where $\tilde{f}(a,b)$ is the integration constant which only depends on a and b. Taking the real part of both sides yields

$$\mathrm{m}_{a,b}(r) = \mathrm{m}_{1,1}(r) + f(a,b), \qquad \text{for } |r| > R_{a,b,1,1}, \tag{1.2.5}$$

where $\mathrm{Re}(\tilde{f}(a,b)) = f(a,b)$.

Notice that $\mathrm{m}_{a,b}(r)$ is harmonic on $U_{a,b}$, the unbounded component of $\mathbb{C} \setminus \mathcal{R}_{a,b}$ which contains $\{|r| > R_{a,b}\}$, and $\mathrm{m}_{1,1}(r) + f(a,b)$ is also harmonic on $U_{1,1}$, since $f(a,b)$ is constant for a, b fixed. The equality (1.2.5) implies that $\mathrm{m}_{a,b}(r)$ and $\mathrm{m}(r) + f(a,b)$ coincide in the open neighbourhood $|r| > R_{a,b,1,1}$. Therefore, they must be equal in $U_{a,b} \cap U_{1,1}$, that is

$$\mathrm{Re}(\tilde{\mathrm{m}}_{a,b}(r)) = \mathrm{m}_{a,b}(r) = \mathrm{m}(r) + f(a,b), \qquad \text{for } r \in O_{a,b} := U_{a,b} \cap U_{1,1} \tag{1.2.6}$$

We now proceed to evaluate $f(a,b)$ in terms of a, b. Since $\mathcal{R}_{a,b}$ is compact for $a, b > 0$, it is bounded for such a, b. Let $0 < \delta < 1$ such that $a, b > \delta$. Let $\mathcal{M}_{a,b}$ be the subset of $\mathbb{R}^2_{>0}$ defined by

$$\mathcal{M}_{a,b} = [a-\delta, a+\delta] \times [b-\delta, b+\delta].$$

Note that $(a,b) \in \mathcal{M}_{a,b}$. Since $\mathcal{M}_{a,b}$ is compact, and the map $(\alpha, \beta) \mapsto R_{\alpha,\beta}$ is continuous for all (α,β) in $\mathcal{M}_{a,b}$, we conclude that the subset $\{R_{\alpha,\beta} : (\alpha,\beta) \in \mathcal{M}_{a,b}\}$ is compact in $\mathbb{R}_{>0}$. Then $\tilde{R}_{a,b} := \max_{(\alpha,\beta)\in\mathcal{M}_{a,b}} R_{\alpha,\beta}$ exists, and is finite. Now choose an $R \in \mathbb{R}_{>0}$ such that

$$R > \tilde{R}_{a,b} + R_{1,1}.$$

The choice of R implies that, for $(\alpha,\beta) \in \mathcal{M}_{a,b}$, $\tilde{\mathrm{m}}_{\alpha,\beta}(R)$ is holomorphic, and (1.2.5) yields

$$\mathrm{m}_{\alpha,\beta}(R) = \mathrm{m}_{1,1}(R) + f(\alpha,\beta). \tag{1.2.7}$$

Let $A_{a,b,\delta} \subset \mathbb{C}^2$ be the poly-annulus $A_{a,b,\delta} = A_{a,\delta} \times A_{b,\delta}$, where $A_{a,\delta} = \{z \in \mathbb{C} : a - \delta < |z| < a + \delta\}$ and $A_{b,\delta} = \{z \in \mathbb{C} : b - \delta < |z| < b + \delta\}$. Note that $\mathbb{T}^2_{a,b} \subset A_{a,b,\delta}$. Since

$$Q_R(x,y) \in \mathbb{C} \setminus (-\infty, 0] \qquad \text{for } (x,y) \in A_{a,b,\delta},$$

$\log(Q_R(x,y))$ is holomorphic in $A_{a,b,\delta}$, where log is the principal branch of logarithm. Let $\tilde{W}_{a,b}$ denote the set of all $(\alpha,\beta) \in \mathcal{M}_{a,b}$ such that

$$\tilde{\mathrm{m}}_{\alpha,\beta}(R) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2_{\alpha,\beta}} \log(Q_R(x,y)) \frac{dx}{x} \frac{dy}{y}.$$

Note that $\tilde{W}_{a,b}$ is an open subset of $\mathcal{M}_{a,b}$, and it also contains (a,b).

Next we compute the functions $\alpha \frac{\partial \tilde{\mathrm{m}}_{\alpha,\beta}(R)}{\partial \alpha}$ and $\beta \frac{\partial \tilde{\mathrm{m}}_{\alpha,\beta}(R)}{\partial \beta}$. We only show here the computation of $\alpha \frac{\partial \tilde{\mathrm{m}}_{\alpha,\beta}(R)}{\partial \alpha}$ when α belongs to an open subinterval of $(a - \delta, a + \delta)$ containing a, since the other case is analogous.

Note that $\tilde{\mathrm{m}}_{\alpha,\beta}(R)$ and $\log(Q_R)$ are well-defined and finite-valued on $\tilde{W}_{a,b}$ and $A_{a,b,\delta}$, respectively. Therefore, we can consider their partial derivatives with respect to α, and obtain

$$\begin{aligned}
\alpha\frac{\partial\tilde{\mathrm{m}}_{\alpha,\beta}(R)}{\partial\alpha} &= \alpha\frac{\partial}{\partial\alpha}\left(\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2_{\alpha,\beta}}\log(Q_R(x,y))\frac{dx}{x}\frac{dy}{y}\right)\\
&= \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2_{\alpha,\beta}}\alpha\frac{\partial\log(Q_R(x,y))}{\partial x}\frac{\partial x}{\partial\alpha}\frac{dx}{x}\frac{dy}{y}\\
&= \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2_{\alpha,\beta}}x\frac{\partial_x Q_R(x,y)}{Q_R(x,y)}\frac{dx}{x}\frac{dy}{y}\\
&= \frac{1}{(2\pi i)^2}\int_{|y|=\beta}\left(\int_{|x|=\alpha}\frac{\partial_x Q_R(x,y)}{Q_R(x,y)}dx\right)\frac{dy}{y}, \qquad (1.2.8)
\end{aligned}$$

where $\partial_x = \frac{\partial}{\partial x}$, and the penultimate equality follows from the facts that $x = \alpha e^{i\theta}$ and θ does not depend on α. For a fixed y_0 such that $|y_0| = \beta$, the integrand

$$\int_{|x|=\alpha}\frac{\partial_x Q_R(x,y_0)}{Q_R(x,y_0)}dx = Z^1_{\alpha,y_0,R} - P^1_{\alpha,y_0,R}$$

is an integer, where $Z^1_{\alpha,y_0,R}$ denotes the number of zeros (counting multiplicity) of the Laurent polynomial $Q_R(x,y_0)$ inside the circle $\mathbb{T}^1_\alpha$, and $P^1_{\alpha,y_0,R}$ denote the order of pole of $Q_R(x,y_0)$ at $x=0$. Let $\nu^1_{\alpha,R}(y_0) := Z^1_{\alpha,y_0,R} - P^1_{\alpha,y_0,R}$. From Proposition 1.2.2 (when applied to the torus $\mathbb{T}^2_{\alpha,\beta}$), it follows that $\nu^1_{\alpha,R}(y)$ is constant for all y in $\mathbb{T}^1_\beta$. We define $\nu^1_{\alpha,\beta,R} := \nu^1_{\alpha,R}(y) \in \mathbb{Z}$, for all $y \in \mathbb{T}^1_\beta$. Therefore, (1.2.8) can be simplified to

$$\frac{\partial\tilde{\mathrm{m}}_{\alpha,\beta}(R)}{\partial\alpha} = \frac{\nu^1_{\alpha,\beta,R}}{\alpha}. \qquad (1.2.9)$$

Similarly,

$$\frac{\partial\tilde{\mathrm{m}}_{\alpha,\beta}(R)}{\partial\beta} = \frac{\nu^2_{\alpha,\beta,R}}{\beta}, \qquad (1.2.10)$$

where $\nu^2_{\alpha,\beta,R} = Z^2_{\alpha,\beta,R} - P^2_{\alpha,\beta,R}$. Here $Z^2_{\alpha,\beta,R}$ and $P^2_{\alpha,\beta,R}$ are similarly defined.

Since the integer-valued functions $\nu^1_{\alpha,\beta,R}$ and $\nu^2_{\alpha,\beta,R}$ depend on α and β continuously, they are constant on $\tilde{W}_{a,b} \subset \operatorname{int}(\mathcal{M}_{a,b})$. In other words,

$$\nu^1_{a,b,R} = \nu^1_{\alpha,\beta,R}, \quad \text{and} \quad \nu^2_{a,b,R} = \nu^2_{\alpha,\beta,R}, \qquad \text{for all } (\alpha,\beta) \in \tilde{W}_{a,b}.$$

Integrating (1.2.9) with respect to α and then taking the real part yields

$$\mathrm{m}_{\alpha,\beta}(R) = \mathrm{m}_{1,1}(R) + \nu^1_{a,b,R} \log \alpha + F(\beta),$$

where F is a function of β which does not depend on α and R. A similar process when applied to (1.2.10) implies that

$$\mathrm{m}_{\alpha,\beta}(R) = \mathrm{m}_{1,1}(R) + \nu^2_{a,b,R} \log \beta + G(\alpha),$$

where G is independent of β and R. From the above equalities and (1.2.7), we conclude that

$$\mathrm{m}_{\alpha,\beta}(R) = \mathrm{m}_{1,1}(R) + \nu^1_{a,b,R} \log \alpha + \nu^2_{a,b,R} \log \beta + c, \tag{1.2.11}$$

for all $(\alpha, \beta) \in \mathcal{M}_{a,b}$, and some constant c independent of α, β, R. As $|R| > R_{1,1}$, evaluating (1.2.11) at $\alpha = 1, \beta = 1$ we obtain $c = 0$. Then, combining (1.2.6) and (1.2.11) together, we derive that

$$f(a,b) = \nu^1_{a,b,R} \log a + \nu^2_{a,b,R} \log b, \qquad \text{for } r \in O_{a,b}. \tag{1.2.12}$$

Since $f(a,b)$ in (1.2.6) is independent of r, comparing (1.2.12) with (1.2.6) we obtain that, for $j = 1, 2$, $\nu^j_{a,b,R}$ is constant in $O_{a,b}$, i.e.

$$\nu^j_{a,b,R} = \nu^j_{a,b,r}, \qquad \text{when } r \in O_{a,b}, j \in \{1,2\}.$$

This concludes the proof of Theorem 1.1.2, namely

$$\mathrm{m}_{a,b}(r) = \mathrm{m}_{1,1}(r) + \nu^1_{a,b,r} \log a + \nu^2_{a,b,r} \log b, \qquad \text{for } r \in O_{a,b} = U_{a,b} \cap U_{1,1}.$$

□

Remark 1.2.3. *Theorem 1.1.2 can be explained in terms of in terms of periods when the curve defined by the polynomial has non-zero genus. Following the investigations by Rodriguez-Villegas* [**102**], *Deninger* [**45**] *et al., we conclude that*

$$\frac{d\tilde{\mathrm{m}}(Q_r)}{dr} = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \frac{1}{r - Q(x,y)} \frac{dx}{x} \frac{dy}{y}$$

is a period of the non-singular curve C_r associated to Q_r (when the genus of C_r is non-zero for generic r), and in particular, from Proposition 1.2 *in* [**45**], *we can argue that $\frac{d\tilde{\mathrm{m}}(Q_r)}{dr}$ remains invariant under the continuous deformation of the integration torus $\mathbb{T}^2$ as long as* *the deformation process does not reach any points of* C_r *(see* **[59]***). Therefore, in our case,* $\frac{d\tilde{m}(Q_r)}{dr} = \frac{d\tilde{m}_{a,b}(Q_r)}{dr}$ *as long as* $\mathbb{T}^2$ *is continuously deformed into* $\mathbb{T}^2_{a,b}$ *without approaching a point of* C_r*. Further, the coefficients of* $\log a$ *and* $\log b$ *can also be derived combining our method with Proposition* 1.2 *in* **[45]**.

Theorem 1.1.1 follows from generalizing the argument in the above proof. Indeed, let $K_{\mathfrak{a}} := \max_{k \in \mathcal{K}_{\mathfrak{a}}} |k|$, and let $K_{\mathfrak{a},\imath} = \max\{K_{\mathfrak{a}}, K_{\imath}\}$, where $\imath = (1, \ldots, 1)$. Then, generalizing the steps in **[102]** and **[9]**, we define

$$\tilde{m}_{\mathfrak{a}}(P_k) = \log k - \sum_{m \geq 0} \frac{a_{m,\mathfrak{a}}}{m} k^{-m}, \quad |k| > K_{\mathfrak{a},\imath}, k \notin (-\infty, 0], \tag{1.2.13}$$

where log denotes the principal branch of the logarithm, and $a_{m,\mathfrak{a}}$ is defined as follows:

$$a_{m,\mathfrak{a}} = \left[\frac{1}{(2\pi i)^n} \int_{\mathbb{T}^2_{\mathfrak{a}}} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n (1 - r^{-1} P(x_1, \ldots, x_n))}\right]_m = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n_{\mathfrak{a}}} P(x_1, \ldots, x_n)^n \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where $[T(s)]_m$ denotes the coefficient of s^{-m} in the series $T(s)$.

It is immediate to see that $\tilde{m}_{\mathfrak{a}}(P_k)$ is holomorphic in the region defined by the intersection of $\{|k| > K_{\mathfrak{a},\imath}\}$ and $\mathbb{C} \setminus (-\infty, 0]$. Also,

$$\mathrm{Re}(\tilde{m}_{\mathfrak{a}}(P_k)) = m_{\mathfrak{a}}(P_k), \quad |k| > K_{\mathfrak{a},\imath}.$$

A similar argument as in the 2-variable case shows that $a_{m,\mathfrak{a}} = a_{m,\imath}$ for all $m \geq 0$, and therefore, we have the following equality:

$$\frac{d\tilde{m}_{\mathfrak{a}}(P_k)}{dk} = \frac{d\tilde{m}_{\imath}(P_k)}{dk}, \qquad \text{for } |k| > K_{\mathfrak{a},\imath}. \tag{1.2.14}$$

A similar argument as in the 2-variable case also shows that $\frac{d\tilde{m}_{\mathfrak{a}}(P_k)}{dk}$ is holomorphic in $\{|k| > K_{\mathfrak{a},\imath}\}$.

Now integrating both sides of (1.2.14) with respect to k and then taking real parts on both sides yield that, for $\{|k| > K_{\mathfrak{a},\imath}\}$,

$$m_{\mathfrak{a}}(P_k) = m_{\imath}(P_k) + g(\mathfrak{a}). \tag{1.2.15}$$

Let $U_{\mathfrak{a}}$ be the unbounded open connected component of $\mathbb{C} \setminus \mathcal{K}_{\mathfrak{a}}$ which contains the region $\{|k| > K_{\mathfrak{a},\imath}\}$. As both sides of (1.2.15) are harmonic on $O_{\mathfrak{a}} := U_{\mathfrak{a}} \cap U_{\imath}$, the equality can be extended to $O_{\mathfrak{a}}$. In other words, for $k \in O_{\mathfrak{a}}$, we have $m_{\mathfrak{a}}(P_k) = m_{\imath}(P_k) + g(\mathfrak{a})$.

It only remains to express g explicitly in terms of $\mathfrak{a}$. Consider the functions

$$a_j \frac{\partial \tilde{\mathrm{m}}_{\mathfrak{a}}(P_k)}{\partial a_j} = \frac{1}{(2\pi i)^n} \int_{|x_1|=a_1,\dots,\widehat{|x_j|=a_j},\dots,|x_n|=a_n} \left(\int_{|x_j|=a_j} \frac{\partial_{x_j} P_k}{P_k} dx_j \right) \frac{dx_1}{x_1} \cdots \frac{\widehat{dx_j}}{x_j} \cdots \frac{dx_n}{x_n}$$

for all $j = 1, \dots, n$. Here $\widehat{\ }$ denotes that the term is omitted from the expression. Now, again following the steps for 2-variable case we conclude that $a_j \frac{\partial \tilde{\mathrm{m}}_{\mathfrak{a}}(P_k)}{\partial a_j}$ is constant depending only on a_j. More precisely, we find that

$$a_j \frac{\partial \tilde{\mathrm{m}}_{\mathfrak{a}}(P_k)}{\partial a_j} = \nu^j_{\mathfrak{a},k}, \tag{1.2.16}$$

where $\nu^j_{\mathfrak{a},k}$ is the difference between the number of zeroes (counting multiplicities) of $P_k(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_n)$ inside the circle $\mathbb{T}^1_{a_j}$, denoted by $Z^j_{\mathfrak{a},k}$, and the order of the pole of $P_k(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_n)$ at $x_j = 0$, denoted by $P^j_{\mathfrak{a},k}$. In other words,

$$\nu^j_{\mathfrak{a},k} = Z^j_{\mathfrak{a},k} - P^j_{\mathfrak{a},k}.$$

We also note that $\nu^j_{\mathfrak{a},k}$ is independent of k when $k \in O_{\mathfrak{a}}$, and only depends on $\mathfrak{a}$ and the polynomial $P = k - P_k$. Integrating (1.2.16) with respect to a_j for $j = 1, \dots, n$, derives Theorem 1.1.1.

1.3. Proof of Theorem 1.1.4

In this section, our goal is to provide the proof of Theorem 1.1.4, and eventually evaluate $\mathrm{m}_{a,b}(Q_r)$ when $r \in \mathbb{C} \setminus (\mathcal{R}_{a,b} \cup U_{a,b})$. Our proof uses Proposition 1.2.2 to conclude that, for all r from a small enough neighbourhood in one of the bounded regions under consideration, certain properties of the roots of $Q_r(a, y)$ or $Q_r(x, b)$ remain invariant. This, combined with the properties of harmonic functions along with Rouché's theorem, gives us the desired results.

Recall that $Q_r(x, y)$, considered as a polynomial in y of degree d_y with coefficients in $\overline{\mathbb{C}(x)}$, can be factored in $\overline{\mathbb{C}(x)}[y]$ as

$$Q_r(x, y) =(y)^{-v_2} \left(Q^y_{F,r}(x)(y)^{d_y} + Q^y_{f,r}(x) + \sum_{j=1}^{d_y-1} a^y_{j,r}(x)(y)^j \right) \tag{1.3.1}$$

$$=(y)^{-v_2} Q^y_{F,r}(x) \prod_{j=1}^{d_y} (y - y_{j,r}(x)), \tag{1.3.2}$$

where the $y_{j,r}(x)$ are algebraic functions in x, v_2 is the order of the pole of $Q_r(a, y)$ at $y = 0$, and $Q^y_{F,r}(x)$ and $Q^y_{f,r}(x)$ are the respective *leading* and *constant* coefficient with respect to the variable y. Similarly, we can factor Q_r, considered as a polynomial in x of degree d_x with coefficients in $\overline{\mathbb{C}(y)}$, as

$$Q_r(x, y) =(x)^{-v_1} \left(Q^x_{F,r}(y)(x)^{d_x} + Q^x_{f,r}(y) + \sum_{j=1}^{d_x-1} a^x_{j,r}(y)(x)^j \right)$$

$$=(x)^{-v_1} Q^x_{F,r}(y) \prod_{j=1}^{d_x} (x - x_{j,r}(y)),$$

where the $x_{j,r}(y)$ are algebraic functions in y, v_1 is the order of the pole of $Q_r(x, b)$ at $x = 0$, and $Q^x_{F,r}(y)$ and $Q^x_{f,r}(y)$ are the respective leading and "constant" coefficient with respect to the variable x.

Let $Z^u_{F,r} = \{z \in \mathbb{C} : Q^u_{F,r}(z) = 0\}$, $Z^u_{f,r} = \{z \in \mathbb{C} : Q^u_{f,r}(z) = 0\}$, where $u = x$ or y, and V_{a,b,r_0} denotes the bounded open connected component of $\mathbb{C} \setminus \mathcal{R}_{a,b}$ containing r_0.

Since the proofs of the statements in (**i**) and (**ii**) of Theorem 1.1.4 are similar, here we restrict ourselves in proving the statement (**i**)

Proof of Theorem 1.1.4. In (1.3.1), we see that the polynomial $Q_r(x, y)$ can be expressed in terms of $y_{j,r}(x)$ (algebraic functions in x), $v_2, Q^y_{F,r}(x)$ and $Q^y_{f,r}(x)$. For simplicity we denote

$$Q_{F,r}(x) := Q^y_{F,r}(x), Q_{f,r}(x) := Q^y_{f,r}(x), \text{ and } d := d_y.$$

Proposition 1.2.2 and the assumption in (**i**) in the statement of Theorem 1.1.4 yield that $\varrho^2_{a,b,r_0}(x) = d$ or 0 for all $x \in \mathbb{T}^1_a$. In particular, $\varrho^2_{a,b,r_0}(a) = d$ or 0, depending on whether all the roots of $Q_r(a, y)$ lie entirely inside or entirely outside the circle $\mathbb{T}^1_b$.

The following three cases can occur when $\varrho^2_{a,b,r_0}(a) = d$ or 0.

Case 1: For all $x \in \mathbb{T}_a^1$,

$$Q_{F,r_0}(x) \cdot Q_{f,r_0}(x) \neq 0.$$

Case 2: Q^y_{F,r_0} vanishes on $\mathbb{T}_a^1$, but Q^y_{f,r_0} does not, i.e.

$$Z^y_{F,r_0} \cap \mathbb{T}_a^1 \neq \varnothing, \quad \text{and} \quad Z^y_{f,r_0} \cap \mathbb{T}_a^1 = \varnothing,$$

Case 3: Q^y_{f,r_0} vanishes on $\mathbb{T}_a^1$, but Q^y_{F,r_0} does not, i.e.

$$Z^y_{f,r_0} \cap \mathbb{T}_a^1 \neq \varnothing, \quad \text{and} \quad Z^y_{F,r_0} \cap \mathbb{T}_a^1 = \varnothing.$$

Case 1: Since

$$Q_{F,r_0}(x) \cdot Q_{f,r_0}(x) \neq 0 \qquad \text{for all } x \in \mathbb{T}_a^1,$$

the discussion preceding the proof of Proposition 1.2.2 implies that the algebraic functions $y_{j,r_0}(x)$ may have only an algebraic branch point at $x = a \in \mathbb{T}_a^1$. From Proposition 1.2.2 we know that ν^2_{a,b,r_0} is constant in $\mathbb{T}_a^1$. Therefore, we can in fact assume that $x = a$ is not a branch point of $y_{j,r_0}(x)$ for all j. Indeed, if $x = a$ is branch point, then there exists an $x_0 \in \mathbb{T}_a^1$ close enough to a such that $x_0 \notin S_r$, where S_r is given in (1.2.2). We replace a with x_0 in the statement, and proceed. Here we provide a proof of **Case 1** when $\varrho^2_{a,b,r_0}(a) = d$, since the case when $\varrho^2_{a,b,r_0}(a) = 0$ is similar. Recall that the condition $\varrho^2_{a,b,r_0}(a) = d$ (resp. $\varrho^2_{a,b,r_0}(a) = 0$) is equivalent to the condition that all the roots of $Q_{r_0}(a,y)$ lie inside (resp. outside) the circle $\mathbb{T}_b^1$.

The polynomial $Q_r(x,y)$ has additional structure: $Q_r(x,y) = r - Q(x,y)$ where Q does not contain any constant term, and r is the constant coefficient in Q_r. Therefore, after multiplying Q_r by y^{v_2}, we find from (1.3.1) that one, and only one, among the set of the coefficients

$$\mathrm{Coeff}_{Q_r,x} := \{Q_{F,r}(x), Q_{f,r}(x), a_{1,r}(x), \ldots, a_{d-1,r}(x)\} \subset \overline{\mathbb{C}(x)}$$

contains r as its constant term, namely the coefficient of y^{v_2} in $y^{v_2}Q_r(x,y)$. Let $b_{v_2,r}(x)$ denotes the said coefficient. Then $b_{v_2,r}(x) \in \mathrm{Coeff}_{Q_r,x}$, and $b_{v_2,r}(x) - b_{v_2,r_0}(x) = r - r_0$. Since all the coefficients, except $b_{v_2,r}$, do not depend on r by construction, the above discussion further implies that

$$\{|Q_{F,r}(x) - Q_{F,r_0}(x)|, |Q_{f,r}(x) - Q_{f,r_0}(x)|\} \cup \{|a_{j,r}(x) - a_{j,r_0}(x)| : 1 \leq j \leq d-1\} \tag{1.3.3}$$
$$= \{0, |b_{v_2,r}(x) - b_{v_2,r_0}(x)|\} = \{0, |r - r_0|\}.$$

In other words, if, for example, $Q_{F,r}(x) = b_{v_2,r}(x)$, then

$$Q_{F,r}(x) - Q_{F,r_0}(x) = r - r_0, \quad Q_{f,r}(x) = Q_{f,r_0}(x), \quad \text{and, for all } j,\ a_{j,r}(x) = a_{j,r_0}(x).$$

Next we investigate the relation between $|Q_r(a,y) - Q_{r_0}(a,y)|$ and $|Q_{r_0}(a,y)|$ when y takes values in certain sufficiently small circles.

Let

$$\epsilon_{ij} = \frac{1}{b}|y_{i,r_0}(a) - y_{j,r_0}(a)| \qquad \text{and} \qquad \epsilon_k = \frac{1}{b}\min_{t \in \mathbb{T}_b^1} |y_{k,r_0}(a) - t|.$$

Since all the roots of $Q_{r_0}(a,y)$ are distinct and lie inside the circle $\mathbb{T}_b^1$, the quantities $\epsilon_{ij}, \epsilon_k$ are non-zero for any $i, j, k \in \{1, \ldots, d\}$ such that $i \neq j$.

We denote

$$\Upsilon = \min_{\substack{1 \leq i < j \leq d \\ 1 \leq k \leq d}} \{\epsilon_{ij}, \epsilon_k\}.$$

Note that $\Upsilon > 0$. Let $\epsilon \in (0, \Upsilon) \cap (0,1)$. We define the closed discs

$$D_j = \{z : |z - y_{j,r_0}(a)| \leq \epsilon\}, \quad \text{for } j = 1, \ldots, d.$$

Let $C_j = \partial D_j$ be the boundary of D_j. The choice of ϵ then confirms that the discs D_j are disjoint and $Q_{r_0}(a,y)$ does not vanish on C_j. This implies $\psi_{j,\epsilon,r_0} := \min_{y \in C_j} |Q_{r_0}(a,y)|$ is positive for each j.

Let $\delta_{j,r_0,\epsilon} := \frac{\psi_{j,\epsilon,r_0}}{d+1}$. Then, for $y \in C_j$, and $r \in V_{a,b,r_0}$ such that

$$|r - r_0| < \delta_{j,r_0,\epsilon},$$

we have

$$\begin{aligned}
&|Q_r(a,y) - Q_{r_0}(a,y)| \\
=&\left|(Q_{F,r}(a) - Q_{F,r_0}(a))\,(y)^d + (Q_{f,r}(a) - Q_{f,r_0}(a)) + \sum_{j=1}^{d-1}(a_{j,r}(a) - a_{j,r_0}(a))\,(y)^j\right| \\
\leq&|r-r_0|\left(\sum_{j=0}^{d}|\epsilon|^j\right) \leq (d+1)|r-r_0| < \psi_{j,\epsilon,r_0} \leq |Q_{r_0}(a,y)|\,,
\end{aligned}$$

where the first inequality follows from (1.3.3).

This implies that, for $j = 1, \dots, d$,

$$|Q_r(a,y) - Q_{r_0}(a,y)| < |Q_{r_0}(a,y)|$$

on C_j. Therefore, it follows from Rouché's Theorem (see Theorem 1.1.5) that $Q_r(a,y)$ and $Q_{r_0}(a,y)$ have the same number of root(s) in the interior of D_j when $|r - r_0| < \delta_{j,r_0,\epsilon}$. Moreover, for

$$\delta(\epsilon, r_0) = \min_{1\leq j\leq d} \delta_{j,r_0,\epsilon} > 0,$$

the choice of ϵ implies that, when $|r-r_0| < \delta(\epsilon, r_0)$, all the roots of $Q_r(a,y)$ lie entirely inside the circle $\mathbb{T}^1_b$.

When $|r-r_0| < \delta(\epsilon, r_0)$, another application of Proposition 1.2.2 yields that all the roots of $Q_r(x,y)$ lie inside $\mathbb{T}^1_b$ for every $x \in \mathbb{T}^1_a$. Following the discussion in Section 0.5.1 regarding the Mahler measure over arbitrary tori, we conclude that, for $r \in \{z : |z - r_0| < \delta_{\epsilon,r_0}\} \subset V_{a,b,r_0}$,

$$\begin{aligned}
\mathrm{m}_{a,b}(Q_r(x,y)) &= \mathrm{m}_{a,b}\left((y)^{-v_2}Q_{F,r}(x)\prod_{j=1}^{d}(y - y_{j,r}(x))\right) \\
&= \mathrm{m}_a(Q_{F,r}(x)) - v_2\log b + \mathrm{m}_{a,b}\left(\prod_{j=1}^{d}(y - y_{j,r}(x))\right) \\
&= \mathrm{m}_a(Q_{F,r}(x)) - v_2\log b + d\log b. \qquad (1.3.4)
\end{aligned}$$

Similarly, when all roots of $Q_{r_0}(a,y)$ lie outside the circle $\mathbb{T}_b^1$, we have for $r \in \{z : |z-r_0| < \delta_{\epsilon,r_0}\} \subset V_{a,b,r_0}$,

$$\begin{aligned}\mathrm{m}_{a,b}(Q_r(x,y)) =& \mathrm{m}_a(Q_{F,r}(x)) - v_2 \log b + \mathrm{m}_a(Q_{f,r}(x)) - \mathrm{m}_a(Q_{F,r}(x))\\ =& \mathrm{m}_a(Q_{f,r}(x)) - v_2 \log b. \end{aligned} \tag{1.3.5}$$

Recall that, $\nu^2_{a,b,r}$ denotes the difference between the number of zeros (counting multiplicity) of $Q_r(a,y)$ inside $\mathbb{T}_b^1$ and the order of pole of $Q_r(a,y)$ at $y=0$. Then the above discussion implies that, for $r \in \{z : |z-r_0| < \delta_{\epsilon,r_0}\} \subset V_{a,b}$,

$$\nu^2_{a,b,r} = \nu^2_{a,b,r_0} = \varrho^2_{a,b,r_0}(x) - v_2 = d - v_2 \text{ or } -v_2.$$

Since $\mathrm{m}_a(Q_{F,r}(x))$ and $\mathrm{m}_a(Q_{f,r}(x))$ are harmonic, and $\mathrm{m}_{a,b}(Q_r(x,y))$ is harmonic for all $r \in V_{a,b,r_0} \setminus \mathcal{S}_{a,b,r_0}$ (where $\mathcal{S}_{a,b,r_0}$ is a finite set containing all the $r \in V_{a,b,r_0}$ such that $Q_r(x,y)$ is singular), the equalities in (1.3.4) and (1.3.5) can be extended to a larger set $V_{a,b,r_0} \setminus \mathcal{S}_{a,b,r_0}$. using the harmonicity of Mahler measure. In other words, for $r, r_0 \in V_{a,b,r_0} \setminus \mathcal{S}_{a,b,r_0}$,

$$\mathrm{m}_{a,b}(Q_r) - \nu^2_{a,b,r_0} \log b = \begin{cases} \mathrm{m}_a(Q_{F,r}(x)) & \text{all roots of } Q_{r_0}(a,y) \text{ lie inside } \mathbb{T}_b^1, \\ \mathrm{m}_a(Q_{f,r}(x)) & \text{all roots of } Q_{r_0}(a,y) \text{ lie outside } \mathbb{T}_b^1. \end{cases} \tag{1.3.6}$$

By continuity, (1.3.6) holds for all $r \in V_{a,b,r_0}$, and this concludes the proof of the **Case 1**.

Recall that $Z^y_{F,r} = \{z \in \mathbb{C} : Q^y_{F,r}(z) = 0\}$, and $Z^y_{f,r} = \{z \in \mathbb{C} : Q^y_{f,r}(z) = 0\}$.
Case 2: If

$$Z^y_{F,r_0} \cap \mathbb{T}_a^1 \neq \varnothing, \quad \text{and} \quad Z^y_{f,r_0} \cap \mathbb{T}_a^1 = \varnothing,$$

then there exists $x' \in Z^y_{F,r_0} \cap \mathbb{T}_a^1$, and a $l \in \{1, \ldots, d_y\}$ such that $y_{l,r_0}(x)$ has a pole at x'. Then Proposition 1.2.2 and the conditions in the statement of Theorem 1.1.4 imply that all the roots of $Q_{r_0}(a,y)$ lie outside the circle $\mathbb{T}_b^1$, and we can choose an $x_0 \in \mathbb{T}_a^1$ in a sufficiently small neighbourhood of x', such that x_0 is not a pole of y_{j,r_0} for all j. Such choice is possible since the set of critical points S_{r_0} of the global analytic function $\mathbf{y}_{r_0}$ is a finite set. Then a similar argument as in **Case** 1 implies that, for all $r \in V_{a,b,r_0}$,

$$\mathrm{m}_{a,b}(Q_r) - \nu^2_{a,b,r} \log b = \mathrm{m}_a(Q^y_{f,r}(x)).$$

Case 3: If

$$Z^y_{f,r_0} \cap \mathbb{T}^1_a \neq \varnothing, \quad \text{and} \quad Z^y_{F,r_0} \cap \mathbb{T}^1_a = \varnothing,$$

then there exists $x'' \in Z^y_{f,r_0} \cap \mathbb{T}^1_a$, and a $p \in \{1, \ldots, d_y\}$ such that $y_{p,r_0}(x)$ has a zero at x''. Again, Proposition 1.2.2 and the conditions in the statement of Theorem 1.1.4 imply that all the roots of $Q_{r_0}(a, y)$ lie inside the circle $\mathbb{T}^1_b$, and we can choose an $x_1 \in \mathbb{T}^1_a$, such that $x_1 \notin S_{r_0} \cup Z^y_{f,r_0}$, and $Q_{r_0}(x_1, y)$ has all the roots inside $\mathbb{T}^1_b$. With these conditions, we have, for all $r \in V_{a,b,r_0}$,

$$\mathrm{m}_{a,b}(Q_r) - \nu^2_{a,b,r} \log b = \mathrm{m}_a(Q^y_{F,r}(x)).$$

This concludes the proof of the statement (**i**). Statement (**ii**) follows from an analogous argument. □

Next, we sketch a proof of Theorem 1.1.3. Recall that multiplying P_k with a suitable power of x_j, we can factorise P_k in linear factors with coefficients in $\overline{\mathbb{C}(x_1, \ldots, \widehat{x_j}, \ldots, x_n)}$ as

$$P_k(x_1, \ldots, x_n) = x_j^{-v_j} P^j_{F,k}(x_1, \ldots, \widehat{x_j}, \ldots, x_n) \prod_{l=1}^{d_n} \left(x_j - X_{l,k,j}\left(x_1, \ldots, \widehat{x_j}, \ldots, x_n\right)\right),$$

where d_j is the degree of P_k as a polynomial in x_j, $X_{l,k,j}$ are algebraic functions of $(x_1, \ldots, \widehat{x_j}, \ldots, x_n)$ for $l = 1, \ldots, d_n$, $P^j_{F,k}$ is the *leading* coefficient with respect to the variable x_j, and v_j is the largest power of x_j^{-1} in P_k. Let $P^j_{f,k}(x_1, \ldots, \widehat{x_j}, \ldots, x_n)$ denote the *constant* coefficient with respect to the variable x_j. Then

$$P^j_{F,k}(x_1, \ldots, \widehat{x_j}, \ldots, x_n) \prod_{j=1}^{d_n} X_{l,k,j}(x_1, \ldots, \widehat{x_j}, \ldots, x_n) = P^j_{f,k}(x_1, \ldots, \widehat{x_j}, \ldots, x_n).$$

For $(u_1, \ldots, \widehat{u_j}, \ldots, u_n) \in \mathbb{T}^{n-1}_{a_1, \ldots, \widehat{a_j}, \ldots, a_n}$, let $\varrho^j_{\mathfrak{a},k}\left(u_1, \ldots, \widehat{u_j}, \ldots, u_n\right)$ be the number of zeroes (counting multiplicities) of $P_k(u_1, \ldots, u_{j-1}, x_j, u_{j+1}, \ldots, u_n)$ inside the circle $\mathbb{T}^1_{a_j}$. Then, from the above discussion, we have $P^j_{\mathfrak{a},k} = v_j$, and

$$\varrho^j_{\mathfrak{a},k}\left(a_1, \ldots, \widehat{a_j}, \ldots, a_n\right) = Z^j_{\mathfrak{a},k} = \nu^j_{\mathfrak{a},k} + P^j_{\mathfrak{a},k} = \nu^j_{\mathfrak{a},k} + v_j.$$

An analogous argument as in the proof of Proposition 1.2.2 yields the following proposition.

Proposition 1.3.1. *Let $k \notin \mathcal{K}_{\mathfrak{a}}$. Then $\varrho^j_{\mathfrak{a},k}(x_1, \ldots, \widehat{x_j}, \ldots, x_n)$ is constant for all $(x_1, \ldots, \widehat{x_j}, \ldots, x_n) \in \mathbb{T}^{n-1}_{a_1, \ldots, \widehat{a_j}, \ldots, a_n}$.*

We omit the proof of the proposition here since it is an immediate extension of Proposition 1.2.2, which follows from an induction argument on $n \geq 2$.

Then Proposition 1.3.1, along with a similar argument as in the proof of Theorem 1.1.4, establishes Theorem 1.1.3.

1.4. Generalized Mahler measure of a family of polynomials

In this section, our aim is to apply Theorems 1.1.2 and 1.1.4 to evaluate the generalized Mahler measure of the family of polynomials in (0.5.4), namely $\mathcal{Q}_r = x + \frac{1}{x} + y + \frac{1}{y} + r$, where $r \in \mathbb{C}$.

Before proceeding with this evaluation, we recall some notation associated to the considered family of polynomials for the reader's convenience.

1. The map in (0.5.6) is defined in this case as
$$q : \mathbb{T}^2_{a,b} \mapsto \mathbb{C}, \qquad (x, y) \mapsto x + \frac{1}{x} + y + \frac{1}{y}.$$
2. Any element r of the image of q, denoted by $\mathcal{R}_{a,b}$, are of the form
$$r = \left(a + a^{-1}\right) \cos \alpha + \left(b + b^{-1}\right) \cos \beta + i \left[\left(a - a^{-1}\right) \sin \alpha + \left(b - b^{-1}\right) \sin \beta\right],$$
where $\alpha, \beta \in [-\pi, \pi)$.
3. Since $\mathcal{R}_{a,b}$ is compact, $R_{a,b} = \max_{r \in \mathcal{R}_{a,b}} |r|$ exists.
4. $U_{a,b}$ denotes the unbounded open connected component of $\mathbb{C} \setminus \mathcal{R}_{a,b}$. It contains the region $\{|r| > R_{a,b}\}$; since $R_{1,1} = 4$, we have $U_{a,b} \subseteq U_{1,1}$.

Now we are ready to apply our theorems to evaluate the generalized Mahler measure of $\mathcal{Q}_r$.

1.4.1. Generalized Mahler measure on the unbounded component of $\mathbb{C} \setminus \mathcal{R}_{a,b}$

In **[102]** Rodriguez-Villegas expressed the (standard) Mahler measure of $\mathcal{Q}_r$ in terms of Eisenstein-Kronecker series for any $r \in \mathbb{C}$. Combining his proof and Theorem 1.1.2, we will show that, for fixed $a, b > 0$, there exists a large open subset of $\mathbb{C}$, namely $O_{a,b} = U_{a,b} \cap U_{1,1}$, such that if $r \in O_{a,b}$, then the Mahler measure remains unchanged irrespective of the dependence of the integration torus on (a, b). We will in fact go further and show that $O_{a,b}$ is the unbounded component of $\mathbb{C} \setminus \mathcal{R}_{a,b}$, namely $U_{a,b}$. Later in this section, we will give an explicit expression of the region $O_{a,b}$, as well as of the region $\mathcal{R}_{a,b}$.

Recall that, for fixed $a, b > 0$, $\mathcal{Q}_r$ does not vanish on $\mathbb{T}^2_{a,b}$ if and only if $r \notin \mathcal{R}_{a,b}$. In order to show that, for a fixed $a, b > 0$,

$$\mathrm{m}_{a,b}(\mathcal{Q}_r) = \mathrm{m}(\mathcal{Q}_r), \qquad \text{for all } r \in O_{a,b},$$

it suffices to evaluate $\nu^j_{a,b,r}$ for $j = 1, 2$. Since these quantities are constant in the region $O_{a,b}$, we can choose a suitable r and apply Theorem 1.1.2 to evaluate them. Let

$$R = R_{a,b} + R_{1,1} = a + \frac{1}{a} + b + \frac{1}{b} + 4.$$

Note that $R \in O_{a,b}$ and $R \notin (-\infty, 0]$.

Recall that $\nu^1_{a,b,r}$ denotes the difference between the number of zeros (counting multiplicity), namely $Z^1_{a,b,r}$ and the number of poles (counting multiplicity), namely $P^1_{a,b,r}$, of $\mathcal{Q}_r(x, b)$ inside the circle $\mathbb{T}^1_a$, i.e.

$$\nu^1_{a,b,r} = Z^1_{a,b,r} - P^1_{a,b,r},$$

and that $\nu^2_{a,b,r}$ is also defined in a similar way.

Since $\mathcal{Q}_R(x, b)$ is holomorphic everywhere except for a simple pole at $x = 0$, we have $P^1_{a,b,R} = 1$. Therefore, $x\mathcal{Q}_R(x, b)$ has no pole in $\mathbb{C}$. Now $x\mathcal{Q}_R(x, b)$ can be factored in $\mathbb{C}[x]$ as

$$x\mathcal{Q}_R(x, b) = (x - x_+)\,(x - x_-)\,,$$

where

$$x_\pm = \frac{-\left(R + b + \frac{1}{b}\right) \pm \sqrt{\left(R + b + \frac{1}{b}\right)^2 - 4}}{2}.$$

Notice that $x_+ \cdot x_- = 1$, and since $R + b + \frac{1}{b} > a + \frac{1}{a}$, we also have

$$
\begin{aligned}
|x_-| = \left| \frac{R + b + \frac{1}{b} + \sqrt{\left(R + b + \frac{1}{b}\right)^2 - 4}}{2} \right| &= \frac{R + b + \frac{1}{b} + \sqrt{\left(R + b + \frac{1}{b}\right)^2 - 4}}{2} \\
&= \frac{R + b + \frac{1}{b} + \sqrt{\left(a + \frac{1}{a} + b + \frac{1}{b} + b + \frac{1}{b} + 6\right)\left(a + \frac{1}{a} + b + \frac{1}{b} + b + \frac{1}{b} + 2\right)}}{2} \\
&\geq a + \frac{1}{a}.
\end{aligned}
$$

Since $a + \frac{1}{a} > \max\left\{a, \frac{1}{a}\right\}$, we have $|x_+| \leq \frac{1}{a + \frac{1}{a}} < a$, and therefore, $Z^1_{a,b,R} = 1$. By the definition of $\nu^1_{a,b,R}$, it follows that $\nu^1_{a,b,R} = 0$. A similar argument shows that $\nu^2_{a,b,R} = 0$. Combining Theorem 1.1.2 and the values obtained above, we derive that, for $r \in U_{a,b} \subset U_{1,1}$,

$$\mathrm{m}_{a,b}(\mathcal{Q}_r) = \mathrm{m}(\mathcal{Q}_r),$$

and the required $O_{a,b}$ is in fact the region $U_{a,b}$.

Until now we have been fixing $a, b > 0$ in our discussion. Next, we want to show that our theorem can even be applied to a fixed suitable r in order to obtain certain values of (a, b) such that the equality $\mathrm{m}_{a,b}(\mathcal{Q}_r) = \mathrm{m}(\mathcal{Q}_r)$ still holds.

For some particular values of $r \in \mathbb{R} \cup i\mathbb{R}$, the standard Mahler measure of $\mathcal{Q}_r$ has been proven to be the same as (up to a rational multiple) a special value of L-function of the elliptic curve corresponding to $\mathcal{Q}_r$ due to Boyd [**29**], Rodriguez-Villegas [**102**], Deninger [**45**], Rogers and Zudilin [**105**], Lalín and Rogers [**82**] et al. Therefore, an interesting direction would be to search for values of (a, b) such that changing the integration torus from $\mathbb{T}^2$ $(= \mathbb{T}^2_{1,1})$ to $\mathbb{T}^2_{a,b}$ keeps the Mahler measure fixed. In order to do so, first notice that, for all $r > R_{a,b}$, Theorem 1.1.2 implies that

$$\mathrm{m}_{a,b}(\mathcal{Q}_r) = \mathrm{m}(\mathcal{Q}_r).$$

Since a and b are fixed arbitrarily, we can fix $r = r_0 > 4$, and conclude that, for all 2-tuples (a, b) satisfying

$$a + \frac{1}{a} + b + \frac{1}{b} < r_0,$$

we have $\mathrm{m}_{a,b}(\mathcal{Q}_{r_0}) = \mathrm{m}(\mathcal{Q}_{r_0})$. Since the change of variables $r \mapsto -r$ covers the case when $r < -4$, it is sufficient to consider the $r > 4$ case here.

For $r \in i\mathbb{R}$, it suffices to investigate the imaginary part of $r \in \mathcal{R}_{a,b}$. Indeed, once we calculate the $\max_{r\in\mathcal{R}_{a,b}} \mathrm{Im}(r)$, we can conclude that all $r' \in \mathbb{C}$, such that $\mathrm{Im}(r') > \max_{r\in\mathcal{R}_{a,b}} \mathrm{Im}(r)$, belong to the unbounded component of $\mathbb{C}\backslash\mathcal{R}_{a,b}$, namely $U_{a,b}$. The following discussion results in gathering the required 2-tuples (a, b) such that

$$\mathrm{m}_{a,b}(\mathcal{Q}_{r_0}) = \mathrm{m}(\mathcal{Q}_{r_0})$$

for a fixed $r' = r_0$.

Recall that, any element in $\mathcal{R}_{a,b}$ can be written as

$$r = \left(a + a^{-1}\right)\cos\alpha + \left(b + b^{-1}\right)\cos\beta + i\left[\left(a - a^{-1}\right)\sin\alpha + \left(b - b^{-1}\right)\sin\beta\right],$$

where $\alpha, \beta \in [-\pi, \pi)$. Notice that,

$$|\mathrm{Im}(r)| = \left|\left(a - a^{-1}\right)\sin\alpha + \left(b - b^{-1}\right)\sin\beta\right| \le \left|a - a^{-1}\right| + \left|b - b^{-1}\right|,$$

and, for $\alpha = \beta \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}$, we have

$$r_{\max, i\mathbb{R}} = i\left[\left|a - a^{-1}\right| + \left|b - b^{-1}\right|\right].$$

Therefore, when a and b are fixed, we have $\mathrm{m}_{a,b}(\mathcal{Q}_r) = \mathrm{m}(\mathcal{Q}_r)$ for all $r \in \left\{z \in i\mathbb{R} : |z| > |a - a^{-1}| + |b - b^{-1}|\right\}$. Then a similar argument as in the real case shows that, for a fixed $r_0 \in i\mathbb{R}_{>0}$, the Mahler measure of $\mathcal{Q}_{r_0}$ over the integration torus $\mathbb{T}^2_{a,b}$ is same as the standard Mahler measure, i.e.

$$\mathrm{m}_{a,b}(\mathcal{Q}_{r_0}) = \mathrm{m}(\mathcal{Q}_{r_0}),$$

for all the 2-tuples (a, b) satisfying

$$\left|a - a^{-1}\right| + \left|b - b^{-1}\right| < |r_0|.$$

Here we mention two such examples for $r = 8$ and $r = 2i$.

Example 1.4.1 ($\mathbf{r = 8}$)**.** *We provide two cases: (I) when $b = a$, and (II) when $b = \sqrt{a}$. Notice that, case (I) keeps the symmetry of the polynomial*

$$\mathcal{Q}_8(x, y) = x + \frac{1}{x} + y + \frac{1}{y} + 8$$

in the variables x and y. In other words, under the change of variables $x \mapsto y$ and $y \mapsto x$, the polynomial $\mathcal{Q}_8$ remains unchanged, and so does the integration torus $\mathbb{T}^2_{a,a}$. On the other hand, case (II) breaks the symmetry as then the above changes of variables change the integration torus from $\mathbb{T}^2_{a,\sqrt{a}}$ to $\mathbb{T}^2_{\sqrt{a},a}$. Despite the differences between these two cases, there are certain values of a such that

$$\mathrm{m}_{a,\sqrt{a}}(\mathcal{Q}_8) = \mathrm{m}_{a,a}(\mathcal{Q}_8) = \mathrm{m}(\mathcal{Q}_8) = 4L'(E_{24}, 0),$$

where E_{24} is an elliptic curve of conductor 24 associated to $\mathcal{Q}_8$. Here the last equality follows from combining the results due to Rogers and Zudilin [**105**]*, and Lalín and Rogers* [**82**]*, where they showed*

$$\mathrm{m}(\mathcal{Q}_8(x,y)) = \mathrm{m}(\mathcal{Q}_2(x,y)) = L'(E_{24}, 0).$$

From the above discussion, we find that, when (I) $a = b$, the equality

$$\mathrm{m}_{a,a}(\mathcal{Q}_8) = \mathrm{m}(\mathcal{Q}_8)$$

holds for all a satisfying

$$a + \frac{1}{a} < 4 \Longleftrightarrow 2 - \sqrt{3} < a < 2 + \sqrt{3}.$$

Similarly, when (II) $b = \sqrt{a}$, we find that, for

$$\begin{aligned} & a + \frac{1}{a} + \sqrt{a} + \frac{1}{\sqrt{a}} < 8 \\ \Longleftrightarrow & \frac{17 - \sqrt{41} - \sqrt{2\left(157 - 17\sqrt{41}\right)}}{4} < a < \frac{17 - \sqrt{41} + \sqrt{2\left(157 - 17\sqrt{41}\right)}}{4}, \end{aligned}$$

the equality

$$\mathrm{m}_{a,\sqrt{a}}(\mathcal{Q}_8) = \mathrm{m}(\mathcal{Q}_8)$$

holds. Since,

$$\frac{17 - \sqrt{41} + \sqrt{2\left(157 - 17\sqrt{41}\right)}}{4} > 2 + \sqrt{3}$$

and

$$\frac{17 - \sqrt{41} - \sqrt{2\left(157 - 17\sqrt{41}\right)}}{4} = \left[\frac{17 - \sqrt{41} + \sqrt{2\left(157 - 17\sqrt{41}\right)}}{4}\right]^{-1},$$

we obtain

$$\mathrm{m}_{a,\sqrt{a}}(\mathcal{Q}_8) = \mathrm{m}_{a,a}(\mathcal{Q}_8) = \mathrm{m}(\mathcal{Q}_8) = 4L'(E_{24},0) \qquad \text{for all } a \in \left(2-\sqrt{3}, 2+\sqrt{3}\right).$$

Example 1.4.2 (r = 2i). *In 2011, Mellit* **[120]** *showed that*

$$\mathrm{m}(\mathcal{Q}_{2i}) = L'(E_{40},0),$$

where E_{40} is an elliptic curve of conductor 40, associated to $\mathcal{Q}_{2i}$.

When $b = a$, Theorem 1.1.2 implies that $\mathrm{m}_{a,a}(\mathcal{Q}_{2i}) = \mathrm{m}(\mathcal{Q}_{2i})$ is true for

$$\left|a - a^{-1}\right| < 1 \Longleftrightarrow \frac{\sqrt{5}-1}{2} < a < \frac{\sqrt{5}+1}{2}.$$

Similarly, when $b = \sqrt{a}$, the equality

$$\mathrm{m}_{a,\sqrt{a}}(\mathcal{Q}_{2i}) = \mathrm{m}(\mathcal{Q}_{2i})$$

holds for all a satisfying

$$\left|a - \frac{1}{a}\right| + \left|\sqrt{a} - \frac{1}{\sqrt{a}}\right| < 2 \Leftrightarrow a_0 < a < a_1,$$

where $a_0 \approx 0.530365\ldots$ and $a_1 \approx 1.88549\ldots$ satisfy $T - 1/T + \sqrt{T} - 1/\sqrt{T} - 2 = 0$ and $T - 1/T + \sqrt{T} - 1/\sqrt{T} + 2 = 0$, respectively. Since $a_0 < \frac{\sqrt{5}-1}{2}$ and $a_1 > \frac{\sqrt{5}+1}{2}$, we obtain

$$\mathrm{m}_{a,a}(\mathcal{Q}_{2i}) = \mathrm{m}_{a,\sqrt{a}}(\mathcal{Q}_{2i}) = \mathrm{m}(\mathcal{Q}_{2i}) = L'(E_{40},0)$$

for all $a \in \left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}\right)$.

1.4.2. Generalized Mahler measure on the bounded components of $\mathbb{C} \setminus \mathcal{R}_{a,b}$

In this section, our goal is to evaluate $\mathrm{m}_{a,b}(\mathcal{Q}_r)$ when r belongs to the bounded connected component(s) of $\mathbb{C} \setminus \mathcal{R}_{a,b}$. In particular, we show there can be at most one such component of $\mathbb{C} \setminus \mathcal{R}_{a,b}$. Later, we apply Theorem 1.1.4 to calculate $\mathrm{m}_{a,b}(\mathcal{Q}_r)$ for all r in said component.

1.4.2.1. Existence of at most one open connected bounded component of $\mathbb{C}\setminus\mathcal{R}_{a,b}$. Our aim here is to show that there exists at most one bounded open connected component of $\mathbb{C}\setminus\mathcal{R}_{a,b}$ for any $a,b>0$.

Recall that the elements of $\mathcal{R}_{a,b}$ are of the form

$$r=\left(a+a^{-1}\right)\cos\alpha+\left(b+b^{-1}\right)\cos\beta+i\left[\left(a-a^{-1}\right)\sin\alpha+\left(b-b^{-1}\right)\sin\beta\right], \qquad (1.4.1)$$

where $\alpha,\beta\in[-\pi,\pi)$. We have

$$R_{a,b}=\max_{r\in\mathcal{R}_{a,b}}|r|=a+a^{-1}+b+b^{-1},\ \text{and } r_{a,b}:=\min_{r\in\mathcal{R}_{a,b}}|r|=\left(a+a^{-1}\right)-\left(b+b^{-1}\right).$$

Then

$$a=b\Longleftrightarrow r_{a,b}=0\Longleftrightarrow 0\in\mathcal{R}_{a,b}.$$

From this point onwards we assume that $a\geq b\geq 1$. The other cases follow analogously using Lemma 1.2.1.

Proposition 1.4.3. *There exists at most one bounded open connected component of* $\mathbb{C}\setminus\mathcal{R}_{a,b}$, *and if it exists, then it contains* 0.

Note that, if $0\in\mathcal{R}_{a,b}$, then Proposition 1.4.3 implies that there is no bounded open connected component in $\mathbb{C}\setminus\mathcal{R}_{a,b}$. Before we proceed to prove the proposition, we note some useful properties of $\mathcal{R}_{a,b}$.

(**A**) The points $r\in\mathcal{R}_{a,b}$ can be interpreted as points on the ellipses

$$E_{b,z}:|r-(z+2)|+|r-(z-2)|=2\left(b+b^{-1}\right), \qquad (1.4.2)$$

where $z\in\mathbb{C}$ lies on the ellipse

$$e_a:|z-2|+|z+2|=2\left(a+a^{-1}\right). \qquad (1.4.3)$$

In other words, elements of $\mathcal{R}_{a,b}$ can be identified with points on ellipses $E_{b,z}$ defined by (1.4.2) with centres on the ellipse e_a in (1.4.3). Note that, the centre (c) and the foci (c_1 and c_2) of the ellipse e_a are the points $c=0, c_1=-2$ and $c_2=2$. The centre (C_z) and the foci ($C_{1,z}$ and $C_{2,z}$) of the ellipse $E_{b,z}$ are

$$C_z=z, C_{1,z}=z-2,\ \text{and } C_{2,z}=z+2.$$

Any point $p \in \mathbb{C}$ lying inside (resp. outside) the ellipse $E_{b,z}$ satisfies $|p - C_{1,z}| + |p - C_{2,z}| < 2\left(b + b^{-1}\right)$ (resp. $|p - C_{1,z}| + |p - C_{2,z}| > 2\left(b + b^{-1}\right)$). Since the length of the minor axis of e_a is $2\left(a - a^{-1}\right)$, we derive that, for $z \in e_a$,

$$|\mathrm{Im}\left(C_z\right)| \leq \left(a - a^{-1}\right),$$

and, for all $h \in \left(-a + a^{-1}, a - a^{-1}\right)$, there exists a $\tilde{z} \in e_a$ such that

$$\mathrm{Im}\left(C_{\tilde{z}}\right) = h. \tag{1.4.4}$$

(**B**) The region $\mathcal{R}_{a,b}$ is symmetric with respect to the imaginary and real axes, i.e. if $r \in \mathcal{R}_{a,b}$, then $-\bar{r}, \bar{r} \in \mathcal{R}_{a,b}$. Let $S_{\pm,\pm}$ denote the four quadrants of $\mathbb{C}$, namely $S_{+,+} = \{s \in C : \mathrm{Re}(s) \geq 0, \mathrm{Im}(s) \geq 0\}, S_{+,-} = \{s \in C : \mathrm{Re}(s) \geq 0, \mathrm{Im}(s) \leq 0\}$ and so on. Then, the changes of variable $\{\alpha \mapsto \pi - \alpha, \beta \mapsto \pi - \beta\}$ and $\{\alpha \mapsto -\alpha, \beta \mapsto -\beta\}$ applied to (1.4.1) take points $r \in \mathcal{R}_{a,b} \cap S_{+,\pm}$ to $-\bar{r} \in \mathcal{R}_{a,b} \cap S_{-,\pm}$, and $r \in \mathcal{R}_{a,b} \cap S_{\pm,+}$ to $\bar{r} \in \mathcal{R}_{a,b} \cap S_{\pm,-}$.

For the rest of this section, $\mathcal{R}^c_{a,b}$ denotes the region defined by $\mathbb{C} \setminus \mathcal{R}_{a,b}$.

Proof of Proposition 1.4.3. Recall that $U_{a,b}$ denotes the connected unbounded component of $\mathcal{R}^c_{a,b}$. First we consider the case $a > b$.

From the discussion at the beginning of this section, we have $0 \in \mathcal{R}^c_{a,b}$, and moreover the open disc $\{u \in \mathbb{C} : |u| < r_{a,b}\}$ is contained in one of the bounded open connected components of $\mathcal{R}^c_{a,b}$. Let $V_{a,b}$ denote this component. Note that $0 \in V_{a,b}$.

In order to prove the statement, we note that the property (**B**) above implies that we can restrict ourselves to the quadrant $S_{+,+}$.

Since $a > b$, the ellipse E_{b,z_0} lies completely in the interior of $S_{+,-}$, where $z_0 = -i\left(a - a^{-1}\right)$. This implies that, for $P \in S_{+,+}$, we have $|P - C_{1,z_0}| + |P - C_{2,z_0}| > 2\left(b + b^{-1}\right)$, where C_{1,z_0} and C_{2,z_0} are the foci of E_{b,z_0}. In other words, the point P lies completely outside the ellipse E_{b,z_0} with centre at $z_0 = -i\left(a - a^{-1}\right) \in e_a$. Let z_1 and z_2 denote the points $a + a^{-1}$ and $i\left(a - a^{-1}\right)$, respectively. We now consider the following cases:

(I) $\mathrm{Im}(P) > \mathrm{Im}(z_2) = \left(a - a^{-1}\right)$,

(II) $0 \leq \mathrm{Im}(P) \leq \mathrm{Im}(z_2) = \left(a - a^{-1}\right)$.

Let $E^{a,b}$ and $e^{a,b}$ denote the boundaries of $U_{a,b}$ and $V_{a,b}$ respectively, i.e.

$$E^{a,b} = \overline{U}_{a,b} \setminus U_{a,b}, \quad \text{and} \quad e^{a,b} = \overline{V}_{a,b} \setminus V_{a,b}.$$

Therefore, $E^{a,b} \cup e^{a,b} \subset \mathcal{R}_{a,b}$. If $P \notin U_{a,b} \cup V_{a,b}$, then P lies in the region bounded by $E^{a,b}$ and $e^{a,b}$. Then the line $L_{\mathrm{Re}(P)} : t = \mathrm{Re}(P)$ intersects $E^{a,b}$ at some point $P' \in Q_{+,+}$. By construction, $\mathrm{Re}(P) = \mathrm{Re}(P')$, and $\mathrm{Im}(P) \leq \mathrm{Im}(P')$, where the equality holds iff $P = P'$. Note that $P = P'$ is the trivial case. Therefore, we assume that $P \neq P'$.

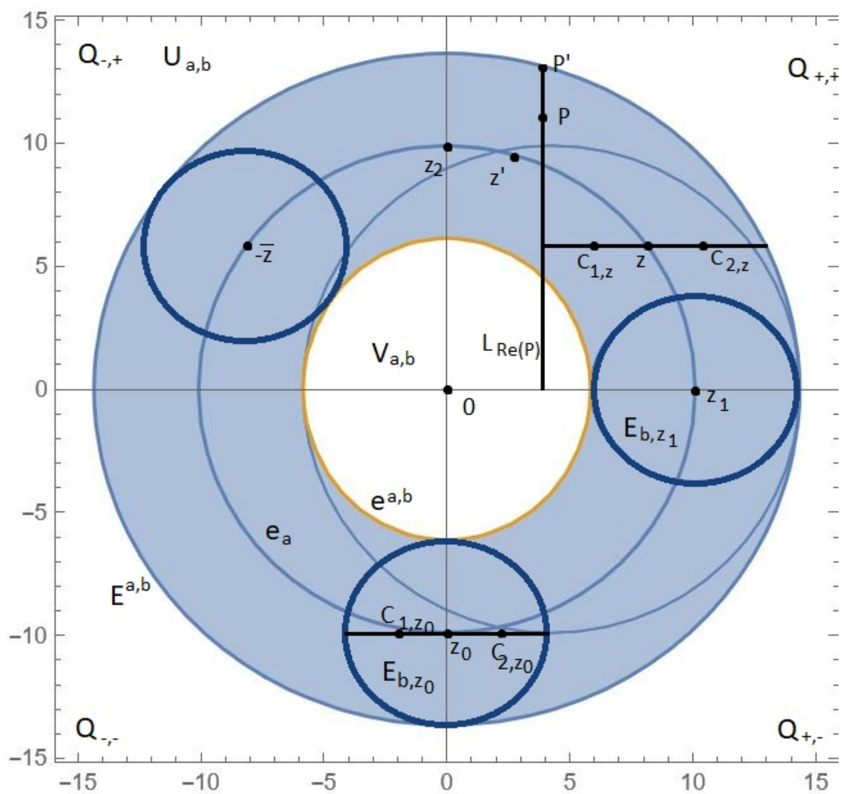


Figure 1 – $\mathcal{R}_{a,b}, U_{a,b}, V_{a,b}, e_a, C_{j,z}, P, P', L_{\mathrm{Re}(P)}$ when $a > b$ and (I) holds.

Claim: *For $P \in S_{+,+}$, if (I) holds, and $P \notin V_{a,b} \cup U_{a,b}$, then there exists an ellipse $E_{b,z'}$, with centre at $z' \in e_a$, such that*

$$|P - C_{1,z'}| + |P - C_{2,z'}| < 2\left(b + b^{-1}\right).$$

Since $\operatorname{Im}(C_{j,z}) \le a - a^{-1}$ for all $z \in e_a$ and $j = 1, 2$, case (I) implies that $|P' - C_{j,z}|^2 - |P - C_{j,z}|^2 > |P' - P|^2 > 0$ (see Figure 1), and then we have

$$|P - C_{1,z}| + |P - C_{2,z}| < |P' - C_{1,z}| + |P' - C_{2,z}|.$$

On the other hand, $P' \in E_{a,b} \subset \mathcal{R}_{a,b}$, i.e. there exists a $z' \in e_a$ such that $|P' - C_{1,z'}| + |P' - C_{2,z'}| = 2\left(b + b^{-1}\right)$. This concludes the proof of the claim.

Now note that $z \in e_a$ can also be written as $z = (a + a^{-1})\cos\alpha_z + i\,(a - a^{-1})\sin\alpha_z$, for $\alpha_z \in [-\pi, \pi)$. Recall that $C_{1,z} = z - 2$ and $C_{2,z} = z + 2$. Therefore, when a is fixed, $|P - C_{j,z}|$ is a continuous function of α_z for $j = 1, 2$. Let

$$\Theta(\alpha_z) := |P - C_{1,z}| + |P - C_{2,z}| - 2\left(b + b^{-1}\right)$$

define a function from $[-\pi, \pi)$ to $\mathbb{R}$. From the claim, we have already concluded that, for the case (I), either $P \in U_{a,b} \cup V_{a,b}$, or there are $z_0, z' \in e_b$ such that

$$|P - C_{1,z_0}| + |P - C_{2,z_0}| > 2\left(b + b^{-1}\right), \quad \text{and } |P - C_{1,z'}| + |P - C_{2,z'}| < 2\left(b + b^{-1}\right).$$

This implies that Θ is a continuous function which takes both negative and positive values, and, using Mean Value Theorem (MVT) on Θ, we derive that there exists $z_1 \in e_a$ such that $\Theta(\alpha_{z_1}) = 0$. In other words, $P \in \mathcal{R}_{a,b}$, which completes the proof of the proposition for $a > b$ when (I) holds.

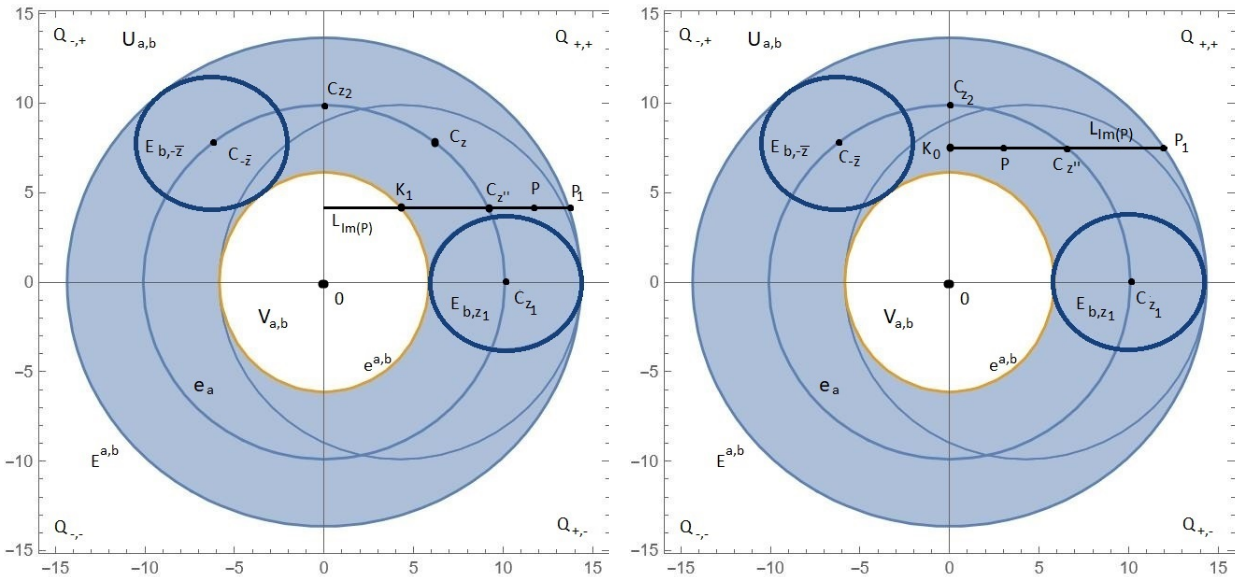
For case (II), note that

$$0 \le \operatorname{Im}(P), \operatorname{Im}(C_z) \le a - a^{-1}, \qquad \text{for all } z \in e_a,$$

where C_z $(= z)$ is the centre of $E_{b,z}$ given in property (**A**). Then, the *continuous property* of $\operatorname{Im}(C_z)$ mentioned in (1.4.4) implies that there exists a $z'' \in e_a$ such that $\operatorname{Im}(P) = \operatorname{Im}(C_{z''})$ (see Figure 2). Let $L_{\operatorname{Im}(P)} : t = \operatorname{Im}(P)$ denote the line joining P and $C_{z''}$, and let $L_{\operatorname{Im}(P)}$ intersect $E^{a,b}$ at P_1 in $Q_{+,+}$ such that any $t \in L_{\operatorname{Im}(P)}$ satisfying $\operatorname{Re}(t) > \operatorname{Re}(P_1)$ lies in $U_{a,b}$. Then P_1 has a representation as in (1.4.1), namely

$$P_1 = \left(a + a^{-1}\right)\cos\alpha + \left(b + b^{-1}\right)\cos\beta + i\left[\left(a - a^{-1}\right)\sin\alpha + \left(b - b^{-1}\right)\sin\beta\right],$$

for some $\alpha, \beta \in [0, \pi/2)$. Moreover, there exist $\gamma \in [0, \pi/2)$ such that $\operatorname{Im}(P) = \operatorname{Im}(P') = \operatorname{Im}(C_{z''}) = (a - a^{-1})\sin\gamma$. The case $\operatorname{Re}(P) = \operatorname{Re}(P_1)$ is trivial as the above discussion implies that $P = P_1$. If $\text{Re}(P) > \text{Re}(P_1)$, then from the definition of P_1 it follows that $P \in U_{a,b}$. Therefore, it only remains to investigate the case when $0 \leq \text{Re}(P) < \text{Re}(P_1)$.



(a) (b)

Figure 2 – $\mathcal{R}_{a,b}, U_{a,b}, V_{a,b}, e_a, C_z, P, P_1, L_{\text{Im}(P)}$ when $a > b$ and (II) holds; left (resp. right) figure shows the case when $L_{\text{Im}(P)} \cap e^{a,b} \neq \varnothing$ (resp. $L_{\text{Im}(P)} \cap e^{a,b} = \varnothing$).

Define the function $f : [\alpha, \gamma] \to [0, \beta]$, which sends ψ to $f(\psi)$ such that

$$\left(a - a^{-1}\right) \sin\gamma = \left(a - a^{-1}\right) \sin\alpha + \left(b - b^{-1}\right) \sin\beta = \left(a - a^{-1}\right) \sin\psi + \left(b - b^{-1}\right) \sin f(\psi).$$

Note that this is a continuous onto function from a connected set. Then the graph of this function, namely $\Gamma_f := \{(\psi, f(\psi)) : \psi \in [\alpha, \gamma]\}$, is also a connected set. Consider another function $g : \Gamma_f \to \mathbb{R}$, defined by

$$(\psi, f(\psi)) \mapsto \left(a + a^{-1}\right) \cos\psi + \left(b + b^{-1}\right) \cos f(\psi).$$

Note that this function is a continuous function, and there exist $\chi, \xi \in \Gamma_f$ such that $g(\chi) = \text{Re}(P_1) = (a + a^{-1}) \cos\alpha + (b + b^{-1}) \cos\beta$, and $g(\xi) = \text{Re}\left(C_{z''}\right) = (a + a^{-1}) \cos\gamma$.

Now if $\text{Re}(P) \in \left(\text{Re}\left(C_{z''}\right), \text{Re}(P_1)\right)$, we claim that there exists a $\psi_0 \in [\alpha, \gamma]$ such that

$$P = \left(a + a^{-1}\right) \cos\psi_0 + \left(b + b^{-1}\right) \cos f(\psi_0) + i \left[\left(a - a^{-1}\right) \sin\psi_0 + \left(b - b^{-1}\right) \sin f(\psi_0)\right],$$

which will imply that $P \in \mathcal{R}_{a,b}$. Indeed, g is a continuous function on a connected set, and $g(\xi) < g(\chi)$. Therefore, all the values of the interval $(g(\xi), g(\chi))$ are attained by g. In particular, such ψ_0 exists. This proves the statement of the proposition when $\mathrm{Im}(P) \in [0, a - a^{-1}]$ and $\mathrm{Re}(P) \geq \mathrm{Re}\left(C_{z''}\right)$.

Therefore, it remains to consider the case when (II) holds along with $\mathrm{Re}(P) \in [0, \mathrm{Re}\left(C_{z''}\right))$.

If the line $L_{\mathrm{Im}(P)}$ intersect $e^{a,b}$ in $S_{+,+}$, then we consider the intersection point with the smallest non-negative real part. In other words, if $K_1, \ldots, K_l \in S_{+,+} \cap e^{a,b} \cap L_{\mathrm{Im}(P)}$ are distinct with $0 \leq \mathrm{Re}(K_1) < \cdots < \mathrm{Re}(K_l)$, then consider $\mathrm{Re}(K_1)$. We want to show that, in fact there can be at most one intersection point of $L_{\mathrm{Im}(P)}$ and $e^{a,b}$ in $S_{+,+}$. That is, if $\mathrm{Re}(P) \in (\mathrm{Re}(K_1), \mathrm{Re}(C_{z''}))$, then $P \in \mathcal{R}_{a,b}$; but this follows from a similar argument as above. Therefore, it only remains to investigate the case when $L_{\mathrm{Im}(P)}$ does not intersect $e^{a,b}$ in $S_{+,+}$. Let K_0 be the intersection point of $L_{\mathrm{Im}(P)}$ and the imaginary axis. Then we have $\mathrm{Re}(K_0) = 0$, and $\mathrm{Re}(P) \in (\mathrm{Re}(K_0), \mathrm{Re}(C_{z''})) \subset L_{\mathrm{Im}(P)}$. Again, an analogous argument as above implies that $P \in \mathcal{R}_{a,b}$.

We collect all the results above, and then using the symmetry of $\mathcal{R}_{a,b}$ (see property (**B**)) we conclude that if $P \notin U_{a,b} \cup V_{a,b}$, then $P \in \mathcal{R}_{a,b}$, which completes the proof of the proposition for $a > b$. The case $a = b$ follows from a similar argument. □

1.4.2.2. Application of Theorem 1.1.4 to the bounded component. Now we are ready to apply Theorem 1.1.4 to $V_{a,b}$, and evaluate $\mathrm{m}_{a,b}(\mathcal{Q}_r)$. Firstly, we need to investigate the roots of $x\mathcal{Q}_{r_0}(x, b)$. Since, $0 \in V_{a,b}$, we can choose $r_0 = 0$ in our theorem. In particular, we need to count the number of roots of $x\mathcal{Q}_0(x, b)$ lying inside the circle $|x| = a$. By Lemma 1.2.1, we can also assume $a > b > 1$.

Factoring $x\mathcal{Q}_0(x, b)$ in $\mathbb{C}[x]$, we obtain that

$$x\mathcal{Q}_0(x, b) = x^2 + \left(b + \frac{1}{b}\right) x + 1 = (x + b)\left(x + \frac{1}{b}\right).$$

Since $a > b > 1$, both roots of $x\mathcal{Q}_0(x, b))$ lies inside the circle $|x| = a$. Also note that $\mathcal{Q}^x_{F,0}(y)$ and $\mathcal{Q}^x_{f,0}(y)$ in (1.3.1) are equal to the constant function **1**. Applying Theorem 1.1.4, we have, for $a > b > 1$ and $r \in V_{a,b}$,

$$\mathrm{m}_{a,b}(\mathcal{Q}_r) = \nu^1_{a,b,0} \log a = \log a,$$

where the last equality follows from the fact that

$$\nu^1_{a,b,0} = Z^1_{a,b,0} - P^1_{a,b,0} = 2 - 1 = 1.$$

Other cases, such as $b > a > 1$, $a > 1 > b$ etc, follow from a combination Lemma 1.2.1 and a similar arguments as above.

1.5. Generalized Mahler measure of $x + \frac{1}{x} + y + \frac{1}{y} + 4$

In this section, our goal is to provide a proof of Theorem 1.1.6, and evaluate

$$\mathrm{m}_{a,b}(\mathcal{Q}_4) := \mathrm{m}_{a,b}(\mathcal{Q}_4(x,y)) = \mathrm{m}_{a,b}\left(x + \frac{1}{x} + y + \frac{1}{y} + 4\right)$$

for all $a, b > 0$.

Our method of proof is mostly inspired from the proof of Theorem 12 in [**42**]. We apply the change of variables considered by Boyd (see Section 2A in [**29**]), namely

$$x \mapsto \frac{w}{z} \quad \text{and} \quad y \mapsto wz,$$

to $\mathcal{Q}_4(x,y)$, and this yields that

$$P(w,z) = \mathcal{Q}_4\left(\frac{w}{z}, wz\right) = \frac{1}{wz}(1 + iw + iz + wz)(1 - iw - iz + wz). \tag{1.5.1}$$

Since $\mathrm{m}_{a,b}(S(x,y)T(x,y)) = \mathrm{m}_{a,b}(S(x,y)) + \mathrm{m}_{a,b}(T(x,y))$, it is sufficient to evaluate the Mahler measures of the linear polynomials $(1 \pm iw \pm iz + wz)$ over $\mathbb{T}^2_{c,d} = \{(w,z) \in \mathbb{C}^\times \times \mathbb{C}^\times : |w| = c, |z| = d\}$, where

$$c = \sqrt{ab},\ d = \sqrt{\frac{b}{a}}.$$

Afterwards, using the changes of variables, we can evaluate $\mathrm{m}_{a,b}(\mathcal{Q}_4)$. The changes of variables

$$w \mapsto -w \quad \text{and} \quad z \mapsto -z$$

transform $(1+iw+iz+wz)$ to $(1-iw-iz+wz)$. As these changes of variables preserve the Mahler measure, we find that

$$\begin{aligned}
\mathrm{m}_{a,b}(\mathcal{Q}_4) = \mathrm{m}_{c,d}(P(w,z)) =& \mathrm{m}_{c,d}\left(\frac{1}{wz}\right) + \mathrm{m}_{c,d}\left(1+iw+iz+wz\right) + \mathrm{m}_{c,d}\left(1-iw-iz+wz\right) \\
=& -\log cd + 2\mathrm{m}_{c,d}\left(1+iw+iz+wz\right) \\
=& -\log b + 2\mathrm{m}_{c,d}\left(1+iw+iz+wz\right), \qquad (1.5.2)
\end{aligned}$$

where the last equality follows from the fact that $cd = \sqrt{ab}\cdot\sqrt{\frac{b}{a}} = b$.

Among the terms in (1.5.2), it remains to evaluate

$$\frac{1}{2}\left(\mathrm{m}_{c,d}(P) + \log cd\right) = \frac{1}{2}\left(\mathrm{m}_{a,b}(\mathcal{Q}_4) + \log b\right) = \mathrm{m}_{c,d}(1+iw+iz+wz).$$

Note that $z(w) = -\frac{1+iw}{i+w}$ is the only root of $R(w,z) = 1+iw+iz+wz$, when considered as a polynomial in z. Therefore,

$$\begin{aligned}
\mathrm{m}_{c,d}(R(w,z)) &= \mathrm{m}_{c,d}(w+i) + \mathrm{m}_{c,d}\left(z + \frac{1+iw}{i+w}\right) \\
&= \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2_{c,d}} \log|w+i|\frac{dw}{w}\frac{dz}{z} + \mathrm{m}_{c,d}\left(z + \frac{1+iw}{i+w}\right) \\
&= \frac{1}{2\pi i}\int_{|w|=c} \log|w+i|\frac{dw}{w} + \mathrm{m}_{c,d}\left(z + \frac{1+iw}{i+w}\right). \qquad (1.5.3)
\end{aligned}$$

To evaluate the first integral, we apply the change of variables $w = cw'$ and Jensen's formula (see (0.1.4)) to obtain

$$\frac{1}{2\pi i}\int_{|w|=c} \log|w+i|\frac{dw}{w} = \log c + \frac{1}{2\pi i}\int_{|w'|=1} \log\left|w' + \frac{i}{c}\right|\frac{dw'}{w'} = \begin{cases} \log c & \text{if } c > 1, \\ 0 & \text{if } c \leq 1. \end{cases} \qquad (1.5.4)$$

It now suffices to evaluate

$$\begin{aligned}
\mathrm{m}_{c,d}\left(z + \frac{1+iw}{i+w}\right) &= \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2_{c,d}} \log\left|z + \frac{1+iw}{i+w}\right|\frac{dw}{w}\frac{dz}{z} \qquad (1.5.5) \\
&= \frac{1}{2\pi i}\int_{|w|=c}\left(\frac{1}{2\pi i}\int_{|z|=d} \log\left|z + \frac{1+iw}{i+w}\right|\frac{dz}{z}\right)\frac{dw}{w}
\end{aligned}$$

to complete the proof. Note that $\frac{1}{2\pi i}\int_{|z|=d}\log\left|z+\frac{1+iw}{i+w}\right|\frac{dz}{z}$ can be simplified to

$$\frac{1}{2\pi i}\int_{|z|=d}\log\left|z+\frac{1+iw}{i+w}\right|\frac{dz}{z}=\begin{cases}\log\left|\frac{1+iw}{i+w}\right| & \text{if } \left|\frac{1+iw}{i+w}\right|>d,\\ \\ \log d & \text{if } \left|\frac{1+iw}{i+w}\right|\le d\end{cases}\tag{1.5.6}$$

following an application of Jensen's formula.

Let $\gamma_{>d}$ and $\gamma_{\le d}$ be the two collections of arcs defined by

$$\gamma_{>d}=\{w:|w|=c,|z(w)|>d\},\qquad \gamma_{\le d}=\{w:|w|=c,|z(w)|\le d\}.$$

Then, applying Jensen's formula with respect to the variable z, (1.5.5) can be expressed as

$$\begin{aligned}\mathrm{m}_{c,d}\left(z+\frac{1+iw}{i+w}\right)&=\frac{1}{2\pi i}\int_{|w|=c}\left(\frac{1}{2\pi i}\int_{|z|=d}\log\left|z+\frac{1+iw}{i+w}\right|\frac{dz}{z}\right)\frac{dw}{w}\\&=\frac{1}{2\pi i}\int_{\gamma_{>d}}\log\left|\frac{1+iw}{i+w}\right|\frac{dw}{w}+\frac{1}{2\pi i}\int_{\gamma_{\le d}}\log d\frac{dw}{w}.\end{aligned}\tag{1.5.7}$$

Since $\operatorname{Im}\left(\frac{dw}{w}\right)=d\arg w$, the differential form can be represented in terms of η as

$$\log\left|\frac{1+iw}{i+w}\right|\frac{dw}{w}=\log|z(w)|\frac{dw}{w}=-i\left(\eta(w,z(w))-\eta(c,z(w))\right).$$

The second term above can be further simplified to

$$\eta(c,z(w))=\eta(c,iz(w))-\eta(c,i)=\eta(c,iz(w))=(\log c)d\arg\left(\frac{1+iw}{1-iw}\right),$$

where $iz(w)=i\frac{1+iw}{i+w}=\frac{1+iw}{1-iw}$. Therefore, once we have determined $\gamma_{>d}$ and $\gamma_{\ge d}$ explicitly, the integrals in (1.5.7) can be evaluated individually using the properties of η and the following two lemmas.

Lemma 1.5.1. *For $w,z(w)$ mentioned above, $\eta\left(w,z(w)\right)$ decomposes as*

$$\eta(w,z(w))=\eta\left(-iw,1+iw\right)-\eta\left(iw,1-iw\right).$$

Lemma 1.5.2 (Lemma 16, **[42]**)**.** *For $c\in\mathbb{R}_{>0}$ and $\theta\in[-\pi,\pi)$, let $w=ce^{i\theta}$ and $\psi=\theta+\frac{\pi}{2}$. Then*

$$d\arg\left(\frac{1+iw}{1-iw}\right)=\frac{2\left(c^{-1}-c\right)\cos\psi}{\left(c^{-1}-c\right)^2+4\sin^2\psi}d\psi.$$

Using property (0.4.24) of η we can rewrite $\eta(w, z(w))$ in Lemma 1.5.1 as

$$\eta(w, z(w)) = dD\left(-iw\right) - dD\left(iw\right), \tag{1.5.8}$$

where D is the Bloch–Wigner dilogarithm given in (0.4.5).

The evaluation of the remaining integral involving $\eta(c, z(w))$ $(= \log c \; d\arg\left(\frac{1+iw}{1-iw}\right))$ over the integration path $\gamma_{>d}$ follows from the lemma below.

Lemma 1.5.3. *For $c \in \mathbb{R}_{>0}$ and $\theta \in [-\pi, \pi)$, let $w = ce^{i\theta}$. Let $\alpha, \beta \in [-\pi, \pi)$. Then*

$$\int_{w(\alpha)}^{w(\beta)} d\arg\left(\frac{1+iw}{1-iw}\right) = \arctan\left(\frac{2\cos\alpha}{c-c^{-1}}\right) - \arctan\left(\frac{2\cos\beta}{c-c^{-1}}\right),$$

where $w(\alpha) = ce^{i\alpha}$ and $w(\beta) = ce^{i\beta}$.

We omit the proof of Lemma 1.5.2 since it is an intermediate step in Lemma 16 of [**42**]. We should also remark that Lemma 1.5.3 is a generalized version of Lemma 16 in [**42**], which states the above result for the case $\alpha = -\pi$ and $\beta = 0$. We will see later that the proof of Lemma 1.5.3 also follows from an argument similar to the proof in [**42**]. We now provide the proofs of Lemma 1.5.1 and 1.5.3.

Proof of Lemma 1.5.1. Using properties of η in Lemma 0.4.9, $\eta(w, z(w))$ decomposes as

$$\begin{aligned}
\eta(w, z(w)) &= \eta\left(w, \frac{1+iw}{i+w}\right) \\
&= \eta(w, 1+iw) - \eta(w, i+w) \\
&= \eta(-iw, 1+iw) - \eta(-i, 1+iw) - \eta(iw, i+w) + \eta(i, i+w) \\
&= \eta(-iw, 1+iw) - \eta(iw, 1-iw) - \eta(iw, i) \\
&= \eta(-iw, 1+iw) - \eta(iw, 1-iw),
\end{aligned}$$

where we applied Remark 0.4.10, which implies that $\eta(\zeta, f(w)) = 0 = \eta(f(w), \zeta)$ for any root of unity ζ and any function $f(w)$ of w. □

Proof of Lemma 1.5.3: We first assume that $c \in \mathbb{R}_{>0} \setminus \{1\}$, and the case $c = 1$ follows from a continuity argument.

For $\alpha, \beta \in [-\pi, \pi)$ and $w = ce^{i\theta}$, Lemma 1.5.2 yields that

$$\int_{w(\alpha)}^{w(\beta)} d\arg\left(\frac{1+iw}{1-iw}\right) = \int_{\alpha+\frac{\pi}{2}}^{\beta+\frac{\pi}{2}} \frac{2\left(c^{-1}-c\right)\cos\psi}{\left(c^{-1}-c\right)^2+4\sin^2\psi}d\psi = -\int_{\cos\alpha}^{\cos\beta} \frac{2(c-c^{-1})}{(c-c^{-1})^2+4t^2}dt,$$

where $\psi = \theta + \frac{\pi}{2}$, $w(\phi) = ce^{i\phi}$, and the last equality follows from the change of variables $\sin\psi \mapsto t$.

Further, the change of variables $\frac{2t}{c-c^{-1}} \mapsto u$ gives that

$$\int_{w(\alpha)}^{w(\beta)} d\arg\left(\frac{1+iw}{1-iw}\right) = -\int_{\frac{2\cos\alpha}{c-c^{-1}}}^{\frac{2\cos\beta}{c-c^{-1}}} \frac{du}{1+u^2} = \arctan\left(\frac{2\cos\alpha}{c-c^{-1}}\right) - \arctan\left(\frac{2\cos\beta}{c-c^{-1}}\right),$$

which proves the lemma. □

Now we have everything to complete the proof of Theorem 1.1.6.

Proof of Theorem 1.1.6. In order to apply Lemma 1.5.1 and Lemma 1.5.2 to (1.5.7), it is necessary to explicitly express $\gamma_{\le d}$ and $\gamma_{>d}$. Since $\gamma_{>d}$ and $\gamma_{\le d}$ are disjoint, and

$$\{w : |w| = c\} = \gamma_{>d} \cup \gamma_{\le d},$$

it suffices to understand $\gamma_{>d}$.

Recall that $z(w) = -\frac{1+iw}{i+w}$. Then, $|z(w)| > d \Leftrightarrow \left|\frac{1+iw}{i+w}\right| > d \Leftrightarrow |1+iw| > d\,|i+w|$. Since both sides of the inequality are non-negative, we can square them and get

$$\begin{aligned} |1+iw| > d\,|i+w| &\Leftrightarrow |1+iw|^2 > d^2\,|i+w|^2 \\ &\Leftrightarrow 2(1+d^2)\operatorname{Re}(iw) > (d^2-1)(1+|w|^2) \\ &\Leftrightarrow \operatorname{Re}(ie^{i\theta}) > \frac{d^2-1}{1+d^2}\cdot\frac{1+c^2}{2c}, \end{aligned}$$

where the last inequality follows from the fact that $w = ce^{i\theta}$ for $\theta \in [-\pi, \pi)$. In other words, the condition $|z(w)| > d$ is equivalent to the condition

$$-1 \le -\operatorname{Re}(ie^{i\theta}) = \sin\theta < \frac{1-d^2}{1+d^2}\cdot\frac{1+c^2}{2c}, \tag{1.5.9}$$

with $\theta \in [-\pi, \pi)$. We recall that

$$\mathcal{A}_{c,d} = \frac{1-d^2}{1+d^2}\cdot\frac{1+c^2}{2c}.$$

As $|\sin\theta| \le 1$, there are three cases to consider.

Case 1: If $\mathcal{A}_{c,d} \leq -1$, then $\gamma_{>d} = \varnothing \quad \text{and} \quad \gamma_{\leq d} = \{w : |w| = c\}$.

Case 2: If $\mathcal{A}_{c,d} \geq 1$, then $\gamma_{>d} = \{w : |w| = c\} \quad \text{and} \quad \gamma_{\leq d} = \varnothing$.

Case 3: If $|\mathcal{A}_{c,d}| < 1$, then, for $w = ce^{i\theta}$ with $\theta \in [-\pi, \pi)$,

$$\gamma_{>d} = \{w : |w| = c, -1 \leq \sin\theta < \mathcal{A}_{c,d}\} \text{ and } \gamma_{\leq d} = \{w : |w| = c, \mathcal{A}_{c,d} \leq \sin\theta \leq 1\}.$$

Now we have everything needed to evaluate (1.5.7), namely

$$\mathrm{m}_{c,d}\left(z + \frac{1+iw}{i+w}\right) = \frac{1}{2\pi i}\int_{\gamma_{>d}} \log\left|\frac{1+iw}{i+w}\right| \frac{dw}{w} + \frac{1}{2\pi i}\int_{\gamma_{\leq d}} \log d \frac{dw}{w}.$$

Case 1: Since $\gamma_{>d} = \varnothing$, the integrals in (1.5.7) can be evaluated individually to obtain

$$\frac{1}{2\pi i}\int_{\gamma_{>d}} \log\left|\frac{1+iw}{i+w}\right| \frac{dw}{w} = 0, \quad \text{and} \quad \frac{1}{2\pi i}\int_{\gamma_{\leq d}} \log d \frac{dw}{w} = \frac{1}{2\pi i}\int_{|w|=c} \log d \frac{dw}{w} = \log d.$$

Therefore, in this case we have

$$\mathrm{m}_{c,d}\left(z + \frac{1+iw}{i+w}\right) = \log d. \tag{1.5.10}$$

Case 2: Since $\gamma_{\leq d} = \varnothing$, the second integral in (1.5.7) contributes nothing. However the first integral can be decomposed into simpler integrals, i.e.

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_{>d}} \log\left|\frac{1+iw}{i+w}\right| \frac{dw}{w} &= \frac{1}{2\pi i}\int_{|w|=c} \log\left|\frac{1+iw}{i+w}\right| \frac{dw}{w} \\ &= \frac{1}{2\pi i}\int_{|w|=c} \log|1+iw| \frac{dw}{w} - \frac{1}{2\pi i}\int_{|w|=c} \log|i+w| \frac{dw}{w} \quad = 0.\end{aligned}$$

Therefore, when $\gamma_{>d} = \{|w| = c\}$ (and $\gamma_{\leq d} = \varnothing$), then

$$\mathrm{m}_{c,d}\left(z + \frac{1+iw}{i+w}\right) = 0. \tag{1.5.11}$$

Case 3: Since

$$|\mathcal{A}_{c,d}| < 1,$$

we have two sub-cases to consider.

3A: When

$$-1 < \mathcal{A}_{c,d} < 0,$$

then $\arcsin(\mathcal{A}_{c,d}) \in [-\pi, 0)$. For simplicity, we denote $\tau = \arcsin(\mathcal{A}_{c,d})$ such that $\tau \in \left(-\frac{\pi}{2}, 0\right)$. Note that $\sin\tau = \sin(-\pi-\tau)$. Then the boundary values of $\gamma_{>d}$ are

$$\partial\gamma_{>d} = \{w(-\pi-\tau), w(\tau)\} = \{ce^{i(-\pi-\tau)}, ce^{i\tau}\} = \{-ce^{-i\tau}, ce^{i\tau}\},$$

where $w(\theta) = ce^{i\theta}$. The integration path $\gamma_{\le d}$ is then the union of the arcs joining $w(-\pi)$ and $w(-\pi-\tau)$, and joining $w(\tau)$ and $w(\pi)$. Therefore

$$\partial\gamma_{\le d} = \{w(-\pi), w(-\pi-\tau), w(\tau), w(\pi)\}$$

are the boundary values of $\gamma_{\le d}$. All the paths are assumed to be traversed counter-clockwise. Now, we have all the tools to calculate (1.5.7) in this case. Combining Lemma 1.5.1 and (1.5.8) with the above discussion we obtain

$$\begin{aligned}
\mathrm{m}_{c,d}\left(z + \frac{1+iw}{i+w}\right) =& \frac{1}{2\pi i}\int_{\gamma_{>d}} \log\left|\frac{1+iw}{i+w}\right|\frac{dw}{w} + \frac{1}{2\pi i}\int_{\gamma_{\le d}} \log d\frac{dw}{w} \\
=& -\frac{1}{2\pi}\int_{\gamma_{>d}} \eta(w, z(w)) + \frac{1}{2\pi}\int_{\gamma_{>d}} \eta(c, z(w)) + \frac{1}{2\pi i}\int_{\gamma_{\le d}} \log d\frac{dw}{w} \\
=& -\frac{1}{2\pi}\int_{\gamma_{>d}} (dD(-iw) - dD(iw)) \\
&+ \frac{\log c}{2\pi}\int_{w(-\pi-\tau)}^{w(\tau)} d\arg\left(\frac{1+iw}{1-iw}\right) + \frac{\log d}{2\pi}\left(\int_{-\pi}^{-\pi-\tau} + \int_{\tau}^{\pi}\right) d\theta, \quad (1.5.12)
\end{aligned}$$

where the simplification of the last integral follows from the above discussion regarding $\partial\gamma_{\le d}$ and substituting w with $w(\theta) = ce^{i\theta}$. The first integral in (1.5.12) can be evaluated using Stokes' theorem as

$$\begin{aligned}
\frac{1}{2\pi}\int_{\gamma_{>d}} (dD(-iw) - dD(iw)) =& \frac{1}{2\pi}\left[D(-iw) - D(iw)\right]_{\partial\gamma_{>d}} \\
=& \frac{1}{2\pi}\left[D(-iw) - D(iw)\right]_{w(-\pi-\tau)}^{w(\tau)} \\
=& -\frac{1}{\pi}\left(D(ice^{-i\tau}) + D(ice^{i\tau})\right), \quad (1.5.13)
\end{aligned}$$

where the last equality follows from the property (0.4.4) of the Bloch-Wigner dilogarithm.

Substituting $\alpha = -\pi - \tau$ and $\beta = \tau$ in the statement of Lemma 1.5.3, we evaluate the second integral in (1.5.12):

$$\frac{\log c}{2\pi} \int_{w(-\pi-\tau)}^{w(\tau)} d \arg \left(\frac{1+iw}{1-iw} \right) = -\frac{\log c}{\pi} \arctan \left(\frac{2\cos\tau}{c-c^{-1}} \right). \tag{1.5.14}$$

The remaining integral's contribution is

$$\frac{\log d}{2\pi} \left(\int_{-\pi}^{-\pi-\tau} + \int_{\tau}^{\pi} \right) d\theta = \frac{\log d}{2\pi} \left[-\pi - \tau + \pi + \pi - \tau \right] = \frac{\pi - 2\tau}{2\pi} \log d. \tag{1.5.15}$$

Then (1.5.13), (1.5.14) and (1.5.15) together yield that

$$\mathrm{m}_{c,d} \left(z + \frac{1+iw}{i+w} \right) = \frac{1}{\pi} \left[D(ice^{-i\tau}) + D(ice^{i\tau}) - (\log c) \arctan \left(\frac{2\cos\tau}{c-c^{-1}} \right) + \left(\frac{\pi}{2} - \tau \right) \log d \right].$$

3B: It remains to evaluate the case when

$$0 < \mathcal{A}_{c,d} < 1.$$

This condition is equivalent to

$$\arcsin(\mathcal{A}_{c,d}) \in (0, \pi).$$

Again, for simplicity, we denote $\kappa = \arcsin(\mathcal{A}_{c,d})$ such that $\kappa \in \left(0, \frac{\pi}{2}\right)$. Since $\sin\kappa = \sin(\pi - \kappa)$ and $\sin\pi = 0$, the boundary values in this case are

$$\partial\gamma_{>d} = \{w(-\pi), w(\kappa), w(\pi-\kappa), w(\pi)\},$$

and

$$\partial\gamma_{\leq d} = \{w(\kappa), w(\pi-\kappa)\}.$$

The arcs are considered to be oriented in a counter-clockwise direction. From a similar argument as before, we deduce that

$$\mathrm{m}_{c,d} \left(z + \frac{1+iw}{i+w} \right) = \frac{1}{\pi} \left[D(ice^{-i\kappa}) + D(ice^{i\kappa}) - (\log c) \arctan \left(\frac{2\cos\kappa}{c-c^{-1}} \right) + \left(\frac{\pi}{2} - \kappa \right) \log d \right].$$

We combine the results obtained in **3A** and **3B** to obtain

$$\mathrm{m}_{c,d} \left(z + \frac{1+iw}{i+w} \right) = \frac{1}{\pi} \left[D(ice^{-i\mu}) + D(ice^{i\mu}) - (\log c) \arctan \left(\frac{2\cos\mu}{c-c^{-1}} \right) + \left(\frac{\pi}{2} - \mu \right) \log d \right], \tag{1.5.16}$$

where $\mu = \arcsin(\mathcal{A}_{c,d}) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This concludes the evaluation of $\mathrm{m}_{c,d}\left(z + \frac{1+iw}{i+w}\right)$ for the different cases.

Recall that $R(w,z) = 1 + iw + iz + wz$. In order to evaluate $\mathrm{m}_{c,d}(R(w,z))$, it suffices to collect the equalities in (1.5.4), (1.5.10), (1.5.11) and (1.5.16). We deduce

$$\mathrm{m}_{c,d}(R(w,z)) = \max\{\log c, 0\} + \begin{cases} \max\{\log d, 0\} & \text{if } |\mathcal{A}_{c,d}| \geq 1, \\ \\ \frac{1}{\pi}\left[D(ice^{-i\mu}) + D(ice^{i\mu}) \right. \\ \left. -(\log c)\arctan\left(\frac{2\cos\mu}{c-c^{-1}}\right) + \left(\frac{\pi}{2} - \mu\right)\log d\right] & \text{if } |\mathcal{A}_{c,d}| < 1, \end{cases}$$

where $\mu = \arcsin\left(\mathcal{A}_{c,d}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Notice that $\mathrm{m}_{c,d}(R(w,z)) = \frac{1}{2}\left(\mathrm{m}_{a,b}(\mathcal{Q}_4) + \log b\right)$, where $a = \frac{c}{d}$ and $b = cd$. This implies that when $|\mathcal{A}_{c,d}| \geq 1$, we have

$$\mathrm{m}_{a,b}(\mathcal{Q}_4) = \max\{\log c, -\log c\} + \max\{\log d, -\log d\}. \tag{1.5.17}$$

On the other hand, when $|\mathcal{A}_{c,d}| < 1$,

$$\begin{aligned}
\mathrm{m}_{a,b}(\mathcal{Q}_4) =& \max\{\log c, -\log c\} - \log d + \frac{2}{\pi}\left[D(ice^{-i\mu}) + D(ice^{i\mu})\right] \\
& - \frac{2\log c}{\pi}\arctan\left(\frac{2\cos\mu}{c - c^{-1}}\right) + \left(1 - \frac{2\mu}{\pi}\right)\log d \\
=& \frac{2}{\pi}\left[D(ice^{-i\mu}) + D(ice^{i\mu}) - \mu\log d\right] \\
& + \left[\max\left\{\frac{\pi}{2}\cdot\frac{2\log c}{\pi}, -\frac{\pi}{2}\cdot\frac{2\log c}{\pi}\right\} - \frac{2\log c}{\pi}\arctan\left(\frac{2\cos\mu}{c - c^{-1}}\right)\right] \\
=& \frac{2}{\pi}\left[D(ice^{-i\mu}) + D(ice^{i\mu}) - \mu\log d\right] + \frac{2\log c}{\pi}\arctan\left(\frac{c - c^{-1}}{2\cos\mu}\right),
\end{aligned}$$

where $\mu = \arcsin\left(\mathcal{A}_{c,d}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and the simplification of the last term follows from the fact that

$$\text{if } x > 0, \quad \pi/2 - \arctan(x) = \arctan\left(x^{-1}\right),$$

and

$$\text{if } x < 0, \quad -\pi/2 - \arctan(x) = \arctan\left(x^{-1}\right).$$

Therefore, (1.5.17) along with the above discussion implies that

$$\mathrm{m}_{a,b}(\mathcal{Q}_4) = \begin{cases} |\log c| + |\log d| & \text{if } |\mathcal{A}_{c,d}| \geq 1, \\ \\ \frac{2}{\pi}\left[D(ice^{-i\mu}) + D(ice^{i\mu}) - \mu\log d + (\log c)\arctan\left(\frac{c-c^{-1}}{2\cos\mu}\right)\right] & \text{if } |\mathcal{A}_{c,d}| < 1, \end{cases}$$

where $\mu = \arcsin(\mathcal{A}_{c,d}) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which completes the proof of Theorem 1.1.6. $\square$

1.6. Conclusion

There are several directions for further exploration. The most immediate question one can ask is, for any two variable Laurent polynomial $Q_r(x, y)$, how to evaluate $\mathrm{m}_{a,b}(Q_r)$ when $r \in \mathcal{R}_{a,b}$. A primary observation in this case is that the integration path is not necessarily closed (and in most cases it is not). This turns out to be a challenging problem since the integration path in this case cannot be easily identified as a cycle in the homology group. We face a similar obstacle while evaluating the Mahler measure of $Q_r(x, y)$ on the bounded connected components when the number of roots of $Q_r(a, y)$ (counting multiplicity) (or $Q_r(x, b)$) inside $\mathbb{T}^1_b$ (or $\mathbb{T}^1_a$) is strictly less than the degree of the polynomials. In this situation it is frequently required to integrate the algebraic functions coming from the factorisation of $Q_r(x, y)$ (when considered as a polynomial in either x or y), on paths which are not closed. These similar challenges also extend to the n-variable cases when $n \geq 3$.

A different direction would be to consider the family of rational polynomials

$$P_k(x_1, \ldots, x_n) = k - \frac{P(x_1, \ldots, x_n)}{Q(x_1, \ldots, x_n)} \in \mathbb{C}(x_1, \ldots, x_n), \qquad \text{for } k \in \mathbb{C}.$$

Our method of proof for Theorems 1.1.2 and 1.1.4 extends to this type of rational polynomials when $Q(x_1, \ldots, x_n)$ is a monomial, which essentially recovers Theorems 1.1.1 and 1.1.3.

The expression of $\nu^j_{\mathfrak{a},k}$ in (1.2.16) appears in the work of Forsberg, Passare, and Tsikh [**53**], where it is denoted as the *order* of an element in the complement of the *Amoeba* associated to the respected polynomial. Our theorems also re-establish certain properties of the *Ronkin function* associated to amoebas mentioned in [**53**]. Therefore, it would be also natural to explore the generalized Mahler measure in terms of the Ronkin function associated to amoebas in more depth.

Chapter 2

Evaluations of the areal Mahler measure of multivariable polynomials

In this chapter, we derive some nontrivial evaluations of the areal Mahler measure of multivariable polynomials, defined by Pritsker [**99**] (see Definition 0.6.1). As in the case of the classical Mahler measure, we find examples yielding special values of L-functions. This is a joint work with Lalín, and was published in [**84**].

2.1. A brief description of the results

In [**99**], Pritsker provided some evaluations of areal Mahler measure in the multivariable cases, namely $\mathrm{m}_{\mathbb{D}}(x+y) = -\frac{1}{4}$ and $\mathrm{m}_{\mathbb{D}}(1+x_1^{k_1}\cdots x_n^{k_n}) = 0$ for $k_1,\ldots,k_d \geq 0$. We continue his study in the following sections by providing explicit formulas of the areal Mahler measure of some nontrivial multivariable polynomials and rational functions, most of which involve special values of L-functions and other special functions. In particular, we prove the following result.

Theorem 2.1.1 (see Theorem 0.6.4)**.** *We have*

$$\mathrm{m}_{\mathbb{D}}(1+x+y) = \frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2) + \frac{1}{6} - \frac{11\sqrt{3}}{16\pi}. \tag{2.1.1}$$

Comparing this formula with (0.1.7), we see that $\mathrm{m}_{\mathbb{D}}(1+x+y) < \mathrm{m}(1+x+y)$, as $\frac{1}{6} - \frac{11\sqrt{3}}{16\pi} < 0$.

We further investigate the polynomials $ax+by+c$, since such a formula does exist in the classical case due to Cassaigne and Maillot (see (0.5.1)). Due to some technical difficulties in obtaining such a formula for the areal Mahler measure in full generality, we restrict ourselves to $a=1=b$ and $c=\sqrt{2}$. In Chapter 5, we overcome a few such difficulties with different techniques and obtain hypergeometric expressions for the areal Mahler measure when $a=1=b$ and $c\in\mathbb{C}$.

Theorem 2.1.2 (see Theorem 0.6.5)**.** *We have*

$$\mathrm{m}_{\mathbb{D}}\left(\sqrt{2}+x+y\right)=\frac{L(\chi_{-4},2)}{\pi}+\mathcal{C}_{\sqrt{2}}+\frac{3}{8}-\frac{3}{2\pi}, \tag{2.1.2}$$

where

$$\mathcal{C}_{\sqrt{2}}=\frac{\Gamma\left(\frac{3}{4}\right)^2}{\sqrt{2\pi^3}}{}_4F_3\left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4};\frac{1}{2},\frac{5}{4},\frac{5}{4};1\right)-\frac{\Gamma\left(\frac{1}{4}\right)^2}{72\sqrt{2\pi^3}}{}_4F_3\left(\frac{3}{4},\frac{3}{4},\frac{5}{4},\frac{5}{4};\frac{3}{2},\frac{7}{4},\frac{7}{4};1\right),$$

is expressed in terms of generalized hypergeometric functions, as defined in (0.2.1)*.*

As suggested in Remark 0.6.6, the calculation showing $\mathcal{C}_{\sqrt{2}}=\frac{\log 2}{4}$ is included in Chapter 5, where we consider the family $\{x+y+k : k\in\mathbb{C}\}$. Although our proof of Theorem 0.6.25 in Chapter 5 holds for general k, we include our proof of Theorem 2.1.2 from [**84**] in this chapter because we believe it contains a different essence and methodology to achieve the result.

We also prove the following statement involving a rational function.

Theorem 2.1.3 (see Theorem 0.6.7)**.** *We have*

$$\mathrm{m}_{\mathbb{D}}\left(y+\left(\frac{1-x}{1+x}\right)\right)=\frac{6}{\pi}L\left(\chi_{-4},2\right)-\log 2-\frac{1}{2}-\frac{1}{\pi}. \tag{2.1.3}$$

We further adapt variations of the classical Mahler measure, such as generalized (maximal) Mahler measures, multiple and higher Mahler measures [**72**], and zeta Mahler measures [**3**], to the areal setting (see Sections 0.6.2.2 and 0.6.18). We evaluate the higher areal Mahler of measure $\frac{1-x}{1+x}$.

Theorem 2.1.4 (see Theorem 0.6.19). *For $h \in \mathbb{Z}_{>0}$ even, we have,*

$$\mathrm{m}_{\mathbb{D},h}\left(\frac{1-x}{1+x}\right) = \frac{E_h(\pi i)^h}{2^h} - \frac{E_{h-2}(\pi i)^{h-2}h(h-1)}{2^{h-2}}\log 2 - \frac{4h!}{2^h}\sum_{m=2}^{h-1}(1-2^{1-m})\zeta(m)\frac{E_{h-m-1}(\pi i)^{h-m-1}}{(h-m-1)!},$$

where E_n denote the nth Euler number defined in (0.4.20) *respectively, and the first sum for $h=2$ should be interpreted as equal to zero.*

For h odd, we have

$$\mathrm{m}_{\mathbb{D},h}\left(\frac{1-x}{1+x}\right) = 0.$$

We further compute the areal zeta Mahler measure of $x+1$.

Theorem 2.1.5 (See Theorem 0.6.22). *We have*

$$Z_{\mathbb{D}}(s,x+1) = \exp\left(\sum_{j=2}^{\infty}\frac{(-1)^j}{j}(1-2^{1-j})(\zeta(j)-1)s^j\right). \tag{2.1.4}$$

The techniques for proving formulas for the areal Mahler measure and its variants are not unlike the techniques employed in the classical Mahler measure case and include inventive changes of variables in integrals, as well as connections to polylogarithms and other special functions such as generalized hypergeometric series, and their properties.

Chapter 2 is structured as follows. In Section 2.2 we present the areal Mahler measure of $x+y$ and more generally $x_1\cdots x_m + y_1\cdots y_n$ as a prelude to more involved arguments that follow in subsequent sections. We prove the main areal Mahler measure formulas given by Theorems 2.1.1, 2.1.2, and 2.1.3 in Section 2.3. Finally, in Section 2.4 we recall the areal analogue of the generalized, multiple, and higher Mahler measure and the zeta Mahler measure, and give examples of evaluations in each of these cases, including the derivations of Theorems 2.1.4 and 2.1.5.

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2.2. A basic result

The simplest possibly non-trivial polynomial that we can consider in this context is a linear polynomial in one variable. Equation (0.6.1) gives us

$$\mathrm{m}_{\mathbb{D}}(x-\alpha) = \begin{cases} \log^+|\alpha| & |\alpha| \geq 1, \\ \frac{|\alpha|^2-1}{2} & |\alpha| \leq 1. \end{cases} \tag{2.2.1}$$

Given the above formula, it is natural to pose the question about the areal Mahler measure of $x+y$. The following result is due to Pritsker, but we reprove it here for completeness.

Proposition 2.2.1. [**99**, Example 5.2] *We have*

$$\mathrm{m}_{\mathbb{D}}(x+y) = -\frac{1}{4}.$$

Proof. We first consider the integral over the variable y by exploiting formula (2.2.1). This gives

$$\mathrm{m}_{\mathbb{D}}(x+y) = \frac{1}{\pi^2}\int_{\mathbb{D}^2} \log|x+y|dA(y)dA(x) = \frac{1}{2\pi}\int_{\mathbb{D}}(|x|^2-1)dA(x).$$

Parametrizing $x=\rho e^{i\theta}$ with $0 \leq \rho \leq 1$ and $-\pi \leq \theta \leq \pi$, the above integral becomes

$$\mathrm{m}_{\mathbb{D}}(x+y) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1(\rho^2-1)\rho d\rho d\theta = \int_0^1(\rho^2-1)\rho d\rho = \left(\frac{\rho^4}{4}-\frac{\rho^2}{2}\right)\Big|_0^1 = -\frac{1}{4}.$$

□

We remark that Proposition 2.2.1 exhibits a point of difference between the classical case m and the areal case $\mathrm{m}_{\mathbb{D}}$. Indeed, it is known that the classical Mahler measure of an homogeneous polynomial is the same as the Mahler measure of any dehomogenization, and in particular, $\mathrm{m}(x+y) = \mathrm{m}(x+1) = 0$. However, equation (2.2.1) shows that $\mathrm{m}_{\mathbb{D}}(x+1) = 0$, while Proposition 2.2.1 gives $\mathrm{m}_{\mathbb{D}}(x+y) = -\frac{1}{4}$.

This discrepancy is even more general. Indeed, while $\mathrm{m}(x_1 \cdots x_m + y_1 \cdots y_n) = 0$, the areal Mahler measure $\mathrm{m}_{\mathbb{D}}(x_1 \cdots x_m + y_1 \cdots y_n)$ is a rational number depending on m and n as shown in the following result.

Theorem 2.2.2. *We have for* $m \geq 1$,

$$\mathrm{m}_{\mathbb{D}}(x_1 \cdots x_m + y) = \frac{1}{2^{m+1}} - \frac{1}{2}.$$

For $m,n \geq 2$,

$$\mathrm{m}_{\mathbb{D}}(x_1 \cdots x_m + y_1 \cdots y_n) = -\frac{1}{2^{m+n+1}} \left[m \sum_{r=0}^{n-1} \binom{m+n-1-r}{m} 2^r + n \sum_{r=0}^{m-1} \binom{m+n-1-r}{n} 2^r \right].$$

Before proving Theorem 2.2.2, we need an auxiliary result.

Lemma 2.2.3. *For* $a, b \in \mathbb{Z}_{\geq 1}$,

$$\sum_{r=0}^{a} \binom{a+b-r}{b} 2^r + \sum_{r=0}^{b} \binom{a+b-r}{a} 2^r = 2^{a+b+1}.$$

Proof. A version of Chu–Vandermonde identity states that, for $m,n,r \in \mathbb{Z}_{\geq 0}$ such that $m + n \leq r$, we have

$$\sum_{k=n}^{r-m} \binom{r-k}{m} \binom{k}{n} = \binom{r+1}{m+n+1}. \tag{2.2.2}$$

(See for example Equation (25) in [**69**, 1.2.6].)

Since $2^r = \sum_{p=0}^{r} \binom{r}{p}$, we obtain

$$\begin{aligned}
\sum_{r=0}^{a} \binom{a+b-r}{b} 2^r &= \sum_{r=0}^{a} \binom{a+b-r}{b} \sum_{p=0}^{r} \binom{r}{p} \\
&= \sum_{p=0}^{a} \sum_{r=p}^{a} \binom{r}{p} \binom{a+b-r}{b} \\
&= \sum_{p=0}^{a} \binom{a+b+1}{b+p+1},
\end{aligned}$$

where the last equality follows from (2.2.2). Similarly,

$$\sum_{r=0}^{b} \binom{a+b-r}{a} 2^r = \sum_{q=0}^{b} \binom{a+b+1}{a+q+1} = \sum_{q=0}^{b} \binom{a+b+1}{b-q}.$$

Therefore, we have

$$\begin{aligned}\sum_{r=0}^{a}\binom{a+b-r}{b}2^r+\sum_{r=0}^{b}\binom{a+b-r}{a}2^r=&\sum_{p=0}^{a}\binom{a+b+1}{b+p+1}+\sum_{q=0}^{b}\binom{a+b+1}{b-q}\\=&\sum_{m=0}^{a+b+1}\binom{a+b+1}{m}\\=&2^{a+b+1},\end{aligned}$$

where the penultimate inequality follows from rearranging the terms. □

Proof of Theorem 2.2.2. Without loss of generality we can assume that $m \geq 2$ and $n \geq 1$. By symmetry, this only excludes the case $m = n = 1$, which was already treated in Proposition 2.2.1. By equation (2.2.1) applied to $\mathrm{m}_{\mathbb{D}}(x) = -\frac{1}{2}$, we have $\mathrm{m}_{\mathbb{D}}(x_1 \cdots x_{m-1}) = -\frac{m-1}{2}$. By multiplicity, we get

$$\mathrm{m}_{\mathbb{D}}(x_1 \cdots x_m + y_1 \cdots y_n) = \mathrm{m}_{\mathbb{D}}\left(x_m + \frac{y_1 \cdots y_n}{x_1 \cdots x_{m-1}}\right) - \frac{m-1}{2}.$$

Applying the definition of the areal Mahler measure and integrating respect to x_m by means of (2.2.1), we obtain

$$\begin{aligned}&\mathrm{m}_{\mathbb{D}}(x_1 \cdots x_m + y_1 \cdots y_n) + \frac{m-1}{2}\\=&\frac{1}{\pi^{m+n}}\int_{\mathbb{D}^{m+n}}\log\left|x_m+\frac{y_1\cdots y_n}{x_1\cdots x_{m-1}}\right|dA(x_1)\dots dA(x_m)dA(y_1)\dots dA(y_n)\\=&\frac{1}{2\pi^{m+n-1}}\int_{\mathbb{D}^{m+n-1}\cap\{|y_1\cdots y_n|\leq|x_1\cdots x_{m-1}|\}}\left(\left|\frac{y_1\cdots y_n}{x_1\cdots x_{m-1}}\right|^2-1\right)dA(x_1)\dots dA(x_{m-1})dA(y_1)\dots dA(y_n)\\&+\frac{1}{\pi^{m+n-1}}\int_{\mathbb{D}^{m+n-1}\cap\{|y_1\cdots y_n|\geq|x_1\cdots x_{m-1}|\}}\log\left|\frac{y_1\cdots y_n}{x_1\cdots x_{m-1}}\right|dA(x_1)\dots dA(x_{m-1})dA(y_1)\dots dA(y_n).\end{aligned}$$

We now consider the change of variables to polar coordinates $x_j = \rho_j e^{i\theta_j}$ and $y_k = \sigma_k e^{i\tau_k}$, for $j = 1, \dots, m-1$ and $k = 1, \dots, n$, where $0 \leq \theta_j, \tau_k \leq 2\pi$, and $0 \leq \rho_j, \sigma_k \leq 1$. Since the functions under consideration are independent of θ_j, τ_k, we can directly integrate respect to those variables. We have

$$\begin{aligned}
&\mathrm{m}_{\mathbb{D}}(x_1\cdots x_m+y_1\cdots y_n)+\frac{m-1}{2}\\
=&\frac{2^{m+n-1}}{2}\int_0^1\cdots\int_0^1\int_{\sigma_1\cdots\sigma_n\leq\rho_1\cdots\rho_{m-1}}\left(\left(\frac{\sigma_1\cdots\sigma_n}{\rho_1\cdots\rho_{m-1}}\right)^2-1\right)\rho_1\cdots\rho_{m-1}\sigma_1\cdots\sigma_n d\rho_1\dots d\rho_{m-1}d\sigma_1\dots d\sigma_n\\
&+2^{m+n-1}\int_0^1\cdots\int_0^1\int_{\sigma_1\cdots\sigma_n\geq\rho_1\cdots\rho_{m-1}}\log\left(\frac{\sigma_1\cdots\sigma_n}{\rho_1\cdots\rho_{m-1}}\right)\rho_1\cdots\rho_{m-1}\sigma_1\cdots\sigma_n d\rho_1\dots d\rho_{m-1}d\sigma_1\dots d\sigma_n.
\end{aligned}$$

We further consider the change of variables $\alpha_j=\rho_1\cdots\rho_j$, and $\beta_k=\sigma_1\cdots\sigma_k$ for $j=1,\dots,m-1$ and $k=1,\dots,n$. This transformation leads to

$$\begin{aligned}
&\mathrm{m}_{\mathbb{D}}(x_1\cdots x_m+y_1\cdots y_n)+\frac{m-1}{2}\\
=&2^{m+n-2}\int_{\substack{0\leq\alpha_{m-1}\leq\cdots\leq\alpha_1\leq1\\0\leq\beta_n\leq\cdots\leq\beta_1\leq1\\\beta_n\leq\alpha_{m-1}}}\left(\frac{\beta_n^3}{\alpha_{m-1}}-\alpha_{m-1}\beta_n\right)\frac{d\alpha_1\dots d\alpha_{m-1}}{\alpha_1\cdots\alpha_{m-2}}\frac{d\beta_1\dots d\beta_n}{\beta_1\cdots\beta_{n-1}}\\
&+2^{m+n-1}\int_{\substack{0\leq\alpha_{m-1}\leq\cdots\leq\alpha_1\leq1\\0\leq\beta_n\leq\cdots\leq\beta_1\leq1\\\beta_n\geq\alpha_{m-1}}}(\log\beta_n-\log\alpha_{m-1})\alpha_{m-1}\beta_n\frac{d\alpha_1\dots d\alpha_{m-1}}{\alpha_1\cdots\alpha_{m-2}}\frac{d\beta_1\dots d\beta_n}{\beta_1\cdots\beta_{n-1}}.
\end{aligned}$$

Integrating respect to $\alpha_1,\dots,\alpha_{m-2}$ as well as $\beta_1,\dots,\beta_{n-1}$ leads to

$$\begin{aligned}
&2^{m+n-2}\Bigg[\int_{0\leq\beta_n\leq\alpha_{m-1}\leq1}\left(\frac{\beta_n^3}{\alpha_{m-1}}-\alpha_{m-1}\beta_n\right)\frac{(-1)^{m-2}}{(m-2)!}\log^{m-2}\alpha_{m-1}\frac{(-1)^{n-1}}{(n-1)!}\log^{n-1}\beta_n d\alpha_{m-1}d\beta_n\\
&+2\int_{0\leq\alpha_{m-1}\leq\beta_n\leq1}(\log\beta_n-\log\alpha_{m-1})\alpha_{m-1}\beta_n\frac{(-1)^{m-2}}{(m-2)!}\log^{m-2}\alpha_{m-1}\frac{(-1)^{n-1}}{(n-1)!}\log^{n-1}\beta_n d\alpha_{m-1}d\beta_n\Bigg]\\
=&\frac{(-1)^{m+n-1}2^{m+n-2}}{(m-2)!(n-1)!}\int_{0\leq\beta\leq\alpha\leq1}\left(\frac{\beta^3}{\alpha}-\alpha\beta\right)\log^{m-2}\alpha\log^{n-1}\beta d\alpha d\beta\\
&+\frac{(-1)^{m+n-1}2^{m+n-1}}{(m-2)!(n-1)!}\int_{0\leq\alpha\leq\beta\leq1}(\log\beta-\log\alpha)\alpha\beta\log^{m-2}\alpha\log^{n-1}\beta d\alpha d\beta.
\end{aligned}$$

The above integral can be decomposed into a sum of similar terms, which can be evaluated from the following general formula (see formula 2.722 in [**58**]):

$$\int x^j\log^k x dx=x^{j+1}\sum_{r=0}^{k}\frac{(-1)^r r!}{(j+1)^{r+1}}\binom{k}{r}\log^{k-r}x+C. \tag{2.2.3}$$

Formula (2.2.3) allows us to compute

$$\int_{0\leq\beta\leq\alpha\leq1}\alpha\beta\log^a\alpha\log^b\beta=\frac{(-1)^{a+b}a!b!}{2^{2a+2b+3}}\sum_{r=0}^{b}\binom{a+b-r}{a}2^r. \tag{2.2.4}$$

We also have

$$\int_{0\leq\beta\leq\alpha\leq 1}\frac{\beta^3}{\alpha}\log^{m-2}\alpha\log^{n-1}\beta d\alpha d\beta = -\frac{1}{m-1}\int_0^1 \beta^3\log^{m+n-2}\beta d\beta$$
$$=\frac{(-1)^{m+n-1}(m+n-2)!}{(m-1)4^{m+n-1}}. \tag{2.2.5}$$

Combining (2.2.4), (2.2.5), as well as Lemma 2.2.3, we obtain

$$\begin{aligned}
&\mathrm{m}_{\mathbb{D}}(x_1\cdots x_m+y_1\cdots y_n)\\
=&-\frac{m-1}{2}+\binom{m+n-2}{m-1}\frac{1}{2^{m+n}}-\frac{1}{2^{m+n-1}}\sum_{r=0}^{n-1}\binom{m+n-3-r}{m-2}2^r\\
&-\frac{n}{2^{m+n}}\sum_{r=0}^{m-2}\binom{m+n-2-r}{n}2^r+\frac{m-1}{2^{m+n}}\sum_{r=0}^{m-1}\binom{m+n-2-r}{n-1}2^r\\
=&\binom{m+n-2}{m-1}\frac{1}{2^{m+n}}-\frac{1}{2^{m+n-1}}\sum_{r=0}^{n-1}\binom{m+n-3-r}{m-2}2^r\\
&-\frac{n}{2^{m+n}}\sum_{r=0}^{m-2}\binom{m+n-2-r}{n}2^r-\frac{m-1}{2^{m+n}}\sum_{r=0}^{n-1}\binom{m+n-2-r}{m-1}2^r.
\end{aligned}\tag{2.2.6}$$

Specializing the above for $n=1$ and $m>1$, we obtain

$$\mathrm{m}_{\mathbb{D}}(x_1\cdots x_m+y)=\frac{1}{2^{m+1}}-\frac{1}{2}.$$

Moreover, by comparing with Proposition 2.2.1, this formula is also true for $m=1$.

When $n>1$, expression (2.2.6) can be made symmetric by exchanging m and n and taking the average. Applying Lemma 2.2.3 again (2.2.4), this gives the expression in the case where both m,$n>1$:

$$\begin{aligned}
\mathrm{m}_{\mathbb{D}}(x_1\cdots x_m+y_1\cdots y_n)=&\frac{1}{4}+\binom{m+n-2}{m-1}\frac{1}{2^{m+n}}\\
&-\frac{1}{2^{m+n}}\left[\sum_{r=0}^{n-1}\binom{m+n-3-r}{m-2}2^r+\sum_{r=0}^{m-1}\binom{m+n-3-r}{n-2}2^r\right]\\
&-\frac{1}{2^{m+n+1}}\left[m\sum_{r=0}^{n-1}\binom{m+n-1-r}{m}2^r+n\sum_{r=0}^{m-1}\binom{m+n-1-r}{n}2^r\right].
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
&-\frac{1}{2^{m+n}}\left[\sum_{r=0}^{n-1}\binom{m+n-3-r}{m-2}2^r+\sum_{r=0}^{m-1}\binom{m+n-3-r}{n-2}2^r\right]\\
=&-\frac{1}{2^{m+n}}\left[\binom{m+n-3}{m-2}+\binom{m+n-3}{n-2}+\sum_{r=1}^{n-1}\binom{m+n-3-r}{m-2}2^r+\sum_{r=1}^{m-1}\binom{m+n-3-r}{n-2}2^r\right]\\
=&-\frac{1}{2^{m+n}}\left[\binom{m+n-3}{m-2}+\binom{m+n-3}{m-1}+\sum_{r=1}^{n-1}\binom{m+n-3-r}{m-2}2^r+\sum_{r=1}^{m-1}\binom{m+n-3-r}{n-2}2^r\right]\\
=&-\frac{1}{2^{m+n}}\left[\binom{m+n-2}{m-1}+2\sum_{r'=0}^{n-2}\binom{m+n-4-r'}{m-2}2^{r'}+2\sum_{r'=0}^{m-2}\binom{m+n-4-r'}{n-2}2^{r'}\right]\\
=&-\binom{m+n-2}{m-1}\frac{1}{2^{m+n}}-\frac{1}{2^{m+n}}\cdot 2^{m+n-3+1}=-\binom{m+n-2}{m-1}\frac{1}{2^{m+n}}-\frac{1}{4},
\end{aligned}
$$

where $r'=r-1$ and the penultimate equality follows from Lemma 2.2.3. Therefore, we have, for $m,n>1$,

$$
\mathrm{m}_{\mathbb{D}}(x_1\cdots x_m+y_1\cdots y_n)=-\frac{1}{2^{m+n+1}}\left[m\sum_{r=0}^{n-1}\binom{m+n-1-r}{m}2^r+n\sum_{r=0}^{m-1}\binom{m+n-1-r}{n}2^r\right].
$$

□

2.3. Evaluations of the areal Mahler measure

In this section we consider evaluations of the areal Mahler measures of some particular polynomials and rational functions.

2.3.1. The areal Mahler measure of $1+x+y$

We now consider Smyth's polynomial $1+x+y$ and prove Theorem 2.1.1.

Proof of Theorem 2.1.1. By definition and application of (2.2.1), we have that

$$
\begin{aligned}
\mathrm{m}_{\mathbb{D}}(1+x+y)=&\frac{1}{\pi^2}\int_{\mathbb{D}^2}\log|1+x+y|dA(y)dA(x)\\
=&\frac{1}{2\pi}\int_{\mathbb{D}\cap\{|1+x|\le 1\}}(|1+x|^2-1)dA(x)+\frac{1}{\pi}\int_{\mathbb{D}\cap\{|1+x|\ge 1\}}\log|1+x|dA(x).
\end{aligned}
\tag{2.3.1}
$$

We treat the first integral above. Write $x = \rho e^{i\theta}$ with $0 \leq \rho \leq 1$ and $-\pi \leq \theta \leq \pi$. We have that $|1+x|^2 = |1+\rho e^{i\theta}|^2 = \rho^2 + 2\rho\cos\theta + 1$, and $|1+x| \leq 1$ if and only if $0 \leq \rho \leq -2\cos\theta$ (provided that $\cos\theta \leq 0$). Therefore, when $\frac{2\pi}{3} \leq |\theta| \leq \pi$, we need to integrate in $0 \leq \rho \leq 1$, while for $\frac{\pi}{2} \leq |\theta| \leq \frac{2\pi}{3}$, we need to integrate in $0 \leq \rho \leq -2\cos\theta$. Separating these two cases, we obtain,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{D}\cap\{|1+x|\leq 1\}} (|1+x|^2 - 1)dA(x)\\
=&\frac{1}{\pi}\int_{\frac{2\pi}{3}}^{\pi}\int_0^1 (\rho^2 + 2\rho\cos\theta)\rho d\rho d\theta + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\int_0^{-2\cos\theta}(\rho^2 + 2\rho\cos\theta)\rho d\rho d\theta\\
=&\frac{1}{\pi}\int_{\frac{2\pi}{3}}^{\pi}\left(\frac{1}{4} + \frac{2}{3}\cos\theta\right)d\theta + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\left(-\frac{4}{3}\cos^4\theta\right)d\theta. \qquad (2.3.2)
\end{aligned}$$

Notice that

$$\int_{\frac{2\pi}{3}}^{\pi}\cos\theta d\theta = -\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2},$$

and

$$\begin{aligned}
\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\cos^4\theta d\theta &= \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\left(\frac{1}{4} + \frac{\cos(2\theta)}{2} + \frac{\cos^2(2\theta)}{4}\right)d\theta\\
&= \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\left(\frac{1}{4} + \frac{\cos(2\theta)}{2} + \frac{1+\cos(4\theta)}{8}\right)d\theta = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\left(\frac{3}{8} + \frac{\cos(2\theta)}{2} + \frac{\cos(4\theta)}{8}\right)d\theta\\
&=\frac{\pi}{16} + \frac{1}{4}\sin\left(\frac{4\pi}{3}\right) + \frac{1}{32}\sin\left(\frac{8\pi}{3}\right) = \frac{\pi}{16} - \frac{7\sqrt{3}}{64}.
\end{aligned}$$

Thus, equation (2.3.2) equals

$$\frac{1}{12} - \frac{1}{\sqrt{3}\pi} - \frac{1}{12} + \frac{7}{16\sqrt{3}\pi} = -\frac{3\sqrt{3}}{16\pi}. \qquad (2.3.3)$$

We now consider the second integral in (2.3.1). We make the change of variables $y = 1+x$ and set $y = \rho e^{i\theta}$ with $1 \leq \rho$ and $-\pi \leq \theta \leq \pi$. We have that $|y-1|^2 = |\rho e^{i\theta} - 1|^2 = \rho^2 - 2\rho\cos\theta + 1$, and $|y-1| \leq 1$ if and only if $0 \leq \rho \leq 2\cos\theta$ (provided that $2\cos\theta \geq 0$). Putting these conditions together, we integrate when $0 \leq |\theta| \leq \frac{\pi}{3}$ and $1 \leq \rho \leq 2\cos\theta$. This leads to

$$\frac{1}{\pi}\int_{\mathbb{D}\cap\{|1+x|\geq 1\}}\log|1+x|dA(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{3}}\int_1^{2\cos\theta}(\log\rho)\rho d\rho d\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{3}}\frac{1}{2}\left(\rho^2\log\rho-\frac{\rho^2}{2}\right)\Big|_1^{2\cos\theta}d\theta$$
$$= \frac{1}{\pi}\int_0^{\frac{\pi}{3}}\left(4\cos^2\theta\log(2\cos\theta)-2\cos^2\theta+\frac{1}{2}\right)d\theta. \tag{2.3.4}$$

Note that

$$\int_0^{\frac{\pi}{3}}\cos^2\theta d\theta = \int_0^{\frac{\pi}{3}}\left(\frac{1+\cos(2\theta)}{2}\right)d\theta = \frac{\pi}{6}+\frac{1}{4}\sin\left(\frac{2\pi}{3}\right) = \frac{\pi}{6}+\frac{\sqrt{3}}{8}. \tag{2.3.5}$$

It remains to compute the integral of $\cos^2\theta\log(2\cos\theta)$. We start by making the change of variables $\tau = \frac{\pi}{2}-\theta$ and use (0.4.6) to obtain

$$\begin{aligned}\int_0^{\frac{\pi}{3}}\cos^2\theta\log(2\cos\theta)d\theta &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sin^2\tau\log(2\sin\tau)d\tau\\ &= -\frac{1}{2}\sin^2\tau D\left(e^{2i\tau}\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}+\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}2\sin\tau\cos\tau D\left(e^{2i\tau}\right)d\tau\\ &= \frac{1}{8}D(e^{i\pi/3})+\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sin(2\tau)\sum_{n=1}^{\infty}\frac{\sin(2n\tau)}{n^2}d\tau\\ &= \frac{1}{8}D(e^{i\pi/3})+\frac{1}{4}\int_{\frac{\pi}{3}}^{\pi}\sin(t)\sum_{n=1}^{\infty}\frac{\sin(nt)}{n^2}dt,\end{aligned} \tag{2.3.6}$$

where we have set $t = 2\tau$.

Note that

$$\int_{\frac{\pi}{3}}^{\pi}\sin^2(t)dt = \frac{\pi}{3}+\frac{\sqrt{3}}{8}, \tag{2.3.7}$$

and, for $n \neq 1$,

$$\begin{aligned}\int_{\frac{\pi}{3}}^{\pi}\sin(t)\sin(nt)dt &= \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}\left(\cos((n-1)t)-\cos((n+1)t)\right)dt\\ &= \frac{1}{2}\left(\frac{\sin((n-1)t)}{n-1}-\frac{\sin((n+1)t)}{n+1}\right)\Big|_{\frac{\pi}{3}}^{\pi}\\ &= -\frac{1}{2}\left(\frac{e^{i(n-1)\pi/3}-e^{-i(n-1)\pi/3}}{2(n-1)i}-\frac{e^{i(n+1)\pi/3}-e^{-i(n+1)\pi/3}}{2(n+1)i}\right).\end{aligned}$$

Incorporating the sum for $n \geq 2$ gives

$$\begin{aligned}
&-\frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{e^{i(n-1)\pi/3}-e^{-i(n-1)\pi/3}}{2(n-1)i}-\frac{e^{i(n+1)\pi/3}-e^{-i(n+1)\pi/3}}{2(n+1)i}\right)\frac{1}{n^2}\\
=&-\frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{e^{i(n-1)\pi/3}-e^{-i(n-1)\pi/3}}{2i}\left(\frac{1}{n-1}-\frac{1}{n}-\frac{1}{n^2}\right)\right.\\
&\left.-\frac{e^{i(n+1)\pi/3}-e^{-i(n+1)\pi/3}}{2i}\left(\frac{1}{n^2}-\frac{1}{n}+\frac{1}{n+1}\right)\right)\\
=&-\frac{1}{2}\operatorname{Im}(\mathrm{Li}_1(e^{i\pi/3}))+\frac{1}{2}\operatorname{Im}\left(e^{-i\pi/3}\left(\mathrm{Li}_1(e^{i\pi/3})-e^{i\pi/3}\right)+e^{-i\pi/3}\left(\mathrm{Li}_2(e^{i\pi/3})-e^{i\pi/3}\right)\right)\\
&+\frac{1}{2}\operatorname{Im}\left(e^{i\pi/3}\left(\mathrm{Li}_2(e^{i\pi/3})-e^{i\pi/3}\right)-e^{i\pi/3}\left(\mathrm{Li}_1(e^{i\pi/3})-e^{i\pi/3}\right)\right)\\
&+\frac{1}{2}\operatorname{Im}\left(\mathrm{Li}_1(e^{i\pi/3})-e^{i\pi/3}-\frac{e^{i2\pi/3}}{2}\right)\\
=&\frac{1}{2}\operatorname{Im}\left(-\sqrt{3}i\mathrm{Li}_1(e^{i\pi/3})+\mathrm{Li}_2(e^{i\pi/3})\right)-\frac{3\sqrt{3}}{8}\\
=&\frac{1}{2}D(e^{i\pi/3})-\frac{3\sqrt{3}}{8}.
\end{aligned}$$

By combining this with (2.3.7), and incorporating it in (2.3.6), we obtain

$$\int_0^{\frac{\pi}{3}}\cos^2\theta\log(2\cos\theta)d\theta=\frac{1}{4}D(e^{i\pi/3})+\frac{\pi}{12}-\frac{\sqrt{3}}{16}.$$

Applying this, as well as (2.3.5) and (0.4.7), we obtain that (2.3.4) equals

$$\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)+\frac{1}{6}-\frac{\sqrt{3}}{2\pi}.$$

Combining the above with (2.3.3) yields the desired result.

□

2.3.2. The areal Mahler measure of $\sqrt{2}+x+y$

We proceed to consider the polynomial $\sqrt{2}+x+y$ and prove Theorem 2.1.2.

Proof of Theorem 2.1.2. By definition, we have

$$\begin{aligned} \mathrm{m}_{\mathbb{D}}(\sqrt{2}+x+y) =& \frac{1}{\pi^2}\int_{\mathbb{D}^2}\log|\sqrt{2}+x+y|dA(y)dA(x)\\ =& \frac{1}{2\pi}\int_{\mathbb{D}\cap\{|\sqrt{2}+x|<1\}}(|\sqrt{2}+x|^2-1)dA(x)+\frac{1}{\pi}\int_{\mathbb{D}\cap\{|\sqrt{2}+x|\geq 1\}}\log|\sqrt{2}+x|dA(x). \end{aligned} \tag{2.3.8}$$

We treat the first integral above. Write $x=\rho e^{i\theta}$ with $0\leq\rho\leq 1$ and $0\leq\theta\leq 2\pi$. We have that $|\sqrt{2}+x|^2=|\sqrt{2}+\rho e^{i\theta}|^2=\rho^2+2\sqrt{2}\rho\cos\theta+2$, and $|\sqrt{2}+x|\leq 1$ if and only if $\sin^2\theta\leq\frac{1}{2}$ and $\max\{0,-\sqrt{2}\cos\theta-\sqrt{1-2\sin^2\theta}\}\leq\rho\leq\min\{1,-\sqrt{2}\cos\theta+\sqrt{1-2\sin^2\theta}\}$. Since

$$\left|\sqrt{2}\cos\theta\right|\geq\left|\sqrt{1-2\sin^2\theta}\right|,\quad\text{and}\quad\cos\theta\in\left[-1,-\frac{1}{\sqrt{2}}\right)\text{ when }|\pi-\theta|\leq\frac{\pi}{4},$$

we need to integrate the first integral in $|\pi-\theta|\leq\frac{\pi}{4}$ and $-\sqrt{2}\cos\theta-\sqrt{1-2\sin^2\theta}\leq\rho\leq 1$. Thus, we have

$$\begin{aligned} &\frac{1}{2\pi}\int_{\mathbb{D}\cap\{|\sqrt{2}+x|<1\}}(|\sqrt{2}+x|^2-1)dA(x)\\ &=\frac{1}{2\pi}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}}\int_{-\sqrt{2}\cos\theta-\sqrt{1-2\sin^2\theta}}^{1}(\rho^2+2\sqrt{2}\rho\cos\theta+1)\rho d\rho d\theta\\ &=\frac{1}{2\pi}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}}\left(\frac{8}{3}\cos^4\theta-2\cos^2\theta+\frac{2\sqrt{2}}{3}\cos\theta+1+\frac{2\sqrt{2}}{3}\cos\theta(1-2\sin^2\theta)^{3/2}\right)d\theta. \end{aligned} \tag{2.3.9}$$

We remark that

$$\begin{aligned} \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}}\cos\theta(1-2\sin^2\theta)^{3/2}d\theta =& \frac{1}{16}\left(3\sqrt{2}\arcsin\left(\sqrt{2}\sin\theta\right)+2(2\sin\theta+\sin(3\theta))\sqrt{\cos(2\theta)}\right)\Bigg|_{\frac{3\pi}{4}}^{\frac{5\pi}{4}}\\ =&-\frac{3\sqrt{2}\pi}{16}. \end{aligned}$$

By proceeding as in the evaluation of (2.3.2), we have that (2.3.9) becomes

$$\frac{1}{4}-\frac{1}{2\pi}+\frac{\sqrt{2}}{3\pi}\left(-\frac{3\sqrt{2}\pi}{16}\right)=\frac{1}{8}-\frac{1}{2\pi}.$$

For the second integral in (2.3.8), we write $y=\sqrt{2}+x$ and $y=\rho e^{i\theta}$ with $1\leq\rho$. We have $|y-\sqrt{2}|^2=|\rho e^{i\theta}-\sqrt{2}|^2=\rho^2-2\sqrt{2}\rho\cos\theta+2$ and $|y-\sqrt{2}|\leq 1$ iff $-\frac{\pi}{4}\leq\theta\leq\frac{\pi}{4}$ and

$1 \leq \rho \leq \sqrt{2}\cos\theta + \sqrt{1-2\sin^2\theta}$. Therefore,

$$
\begin{aligned}
&\frac{1}{\pi}\int_{\mathbb{D}\cap\{|\sqrt{2}+x|\geq 1\}} \log|\sqrt{2}+x|dx \\
=&\frac{2}{\pi}\int_0^{\frac{\pi}{4}}\int_1^{\sqrt{2}\cos\theta+\sqrt{1-2\sin^2\theta}}(\log\rho)\rho d\rho d\theta \\
=&\frac{2}{\pi}\int_0^{\frac{\pi}{4}}\frac{1}{2}\left(\rho^2\log\rho-\frac{\rho^2}{2}\right)\Big|_1^{\sqrt{2}\cos\theta+\sqrt{1-2\sin^2\theta}}d\theta \\
=&\frac{1}{\pi}\int_0^{\frac{\pi}{4}}\left(\left(\sqrt{2}\cos\theta+\sqrt{1-2\sin^2\theta}\right)^2\log\left(\sqrt{2}\cos\theta+\sqrt{1-2\sin^2\theta}\right)\right. \\
&\left.-\frac{\left(\sqrt{2}\cos\theta+\sqrt{1-2\sin^2\theta}\right)^2}{2}+\frac{1}{2}\right)d\theta \\
=&\frac{1}{8}+\frac{1}{\pi}\int_0^{\frac{\pi}{4}}\left(\sqrt{2}\cos\theta+\sqrt{1-2\sin^2\theta}\right)^2\log\left(\sqrt{2}\cos\theta+\sqrt{1-2\sin^2\theta}\right)d\theta \\
&-\frac{1}{2\pi}\int_0^{\frac{\pi}{4}}\left(\sqrt{2}\cos\theta+\sqrt{1-2\sin^2\theta}\right)^2 d\theta. \qquad (2.3.10)
\end{aligned}
$$

Substituting $x = \sqrt{2}\cos\theta + \sqrt{1-2\sin^2\theta}$, we have $x^{-1} = \sqrt{2}\cos\theta - \sqrt{1-2\sin^2\theta}$, and

$$d\theta = -\frac{x-x^{-1}}{x\sqrt{4-(x-x^{-1})^2}}dx.$$

This gives

$$\int_0^{\frac{\pi}{4}}\left(\sqrt{2}\cos\theta+\sqrt{1-2\sin^2\theta}\right)^2\log\left(\sqrt{2}\cos\theta+\sqrt{1-2\sin^2\theta}\right)d\theta = \int_1^{\sqrt{2}+1}\frac{x(x-x^{-1})\log x}{\sqrt{4-(x-x^{-1})^2}}dx, \qquad (2.3.11)$$

and

$$\int_0^{\frac{\pi}{4}}\left(\sqrt{2}\cos\theta+\sqrt{1-2\sin^2\theta}\right)^2 d\theta = \int_1^{\sqrt{2}+1}\frac{x\left(x-x^{-1}\right)}{\sqrt{4-(x-x^{-1})^2}}dx. \qquad (2.3.12)$$

Applying integration by parts to (2.3.11) gives

$$
\begin{aligned}
&\int_1^{\sqrt{2}+1}\frac{x(x-x^{-1})\log x}{\sqrt{4-(x-x^{-1})^2}}dx \\
=&\left[\log x\int^x\frac{u(u-u^{-1})}{\sqrt{4-(u-u^{-1})^2}}du-\int\frac{1}{x}\left(\int^x\frac{u(u-u^{-1})}{\sqrt{4-(u-u^{-1})^2}}du\right)dx\right]_1^{\sqrt{2}+1}. \qquad (2.3.13)
\end{aligned}
$$

We now apply the change of variables $z = u - u^{-1}$ and find that

$$\begin{aligned}\int \frac{u(u-u^{-1})}{\sqrt{4-(u-u^{-1})^2}}du =& \frac{1}{4}\int \frac{z\left(z+\sqrt{z^2+4}\right)^2}{\sqrt{4-z^2}\sqrt{4+z^2}}dz \\ =& \frac{1}{4}\left(2\int \frac{z^3dz}{\sqrt{16-z^4}} + 4\frac{zdz}{\sqrt{16-z^4}} + 2\int \frac{z^2dz}{\sqrt{4-z^2}}\right) \\ =& \frac{1}{4}\left(-\sqrt{16-z^4} + 2\arcsin\left(\frac{z^2}{4}\right) - z\sqrt{4-z^2} + 4\arcsin\left(\frac{z}{2}\right)\right),\end{aligned} \tag{2.3.14}$$

where the domain under consideration for z is $0 \le z \le 2$. Therefore, the first integral in (2.3.13) evaluates to

$$\begin{aligned}&\frac{1}{4}\log\left(\frac{z+\sqrt{z^2+4}}{2}\right)\left(-\sqrt{16-z^4} + 2\arcsin\left(\frac{z^2}{4}\right) - z\sqrt{4-z^2} + 4\arcsin\left(\frac{z}{2}\right)\right)\Bigg|_0^2 \\ =& \frac{3\pi}{4}\log\left(\sqrt{2}+1\right).\end{aligned}$$

Equation (2.3.14) also allows us to evaluate the integral in (2.3.12) as

$$\int_0^{\frac{\pi}{4}} \left(\sqrt{2}\cos\theta + \sqrt{1-2\sin^2\theta}\right)^2 d\theta = 1 + \frac{3\pi}{4}.$$

Using the same changes of variables (namely $u \mapsto u - u^{-1}$ and $x \mapsto x - x^{-1} = z$), the second integral in (2.3.13) can be written as

$$\begin{aligned}&\int_1^{\sqrt{2}+1} \frac{1}{x}\left(\int^x \frac{u(u-u^{-1})}{\sqrt{4-(u-u^{-1})^2}}du\right)dx \\ =& \frac{1}{4}\int_0^2 \left(-\sqrt{16-z^4} + 2\arcsin\left(\frac{z^2}{4}\right) - z\sqrt{4-z^2} + 4\arcsin\left(\frac{z}{2}\right)\right)\frac{dz}{\sqrt{4+z^2}} \\ =& -\frac{1}{4}\int_0^2 \sqrt{4-z^2}dz - \frac{1}{4}\int_0^2 \frac{z\sqrt{4-z^2}}{\sqrt{4+z^2}}dz + \frac{1}{2}\int_0^2 \frac{\arcsin\left(\frac{z^2}{4}\right)}{\sqrt{4+z^2}}dz + \int_0^2 \frac{\arcsin\left(\frac{z}{2}\right)}{\sqrt{4+z^2}}dz \\ =& -\frac{1}{4}\left[\frac{1}{2}z\sqrt{4-z^2} + 2\arcsin\left(\frac{z}{2}\right)\right]_0^2 - \frac{1}{8}\left[\sqrt{16-z^4} + 4\arcsin\left(\frac{z^2}{2}\right)\right]_0^2 \\ &+ \frac{1}{2}\int_0^2 \frac{\arcsin\left(\frac{z^2}{4}\right)}{\sqrt{4+z^2}}dz + \int_0^2 \frac{\arcsin\left(\frac{z}{2}\right)}{\sqrt{4+z^2}}dz \\ =& -\frac{\pi}{2} + \frac{1}{2} + \frac{1}{2}\int_0^2 \frac{\arcsin\left(\frac{z^2}{4}\right)}{\sqrt{4+z^2}}dz + \int_0^2 \frac{\arcsin\left(\frac{z}{2}\right)}{\sqrt{4+z^2}}dz.\end{aligned}$$

Thus, to evaluate (2.3.10), it only remains to evaluate the last two integrals above. The change of variable $v = z/2$ yields

$$\int_0^2 \frac{\arcsin\left(\frac{z}{2}\right)}{\sqrt{4+z^2}}dz = \int_0^1 \frac{\arcsin v}{\sqrt{1+v^2}}dv, \quad \int_0^2 \frac{\arcsin\left(\frac{z^2}{4}\right)}{\sqrt{4+z^2}}dz = \int_0^1 \frac{\arcsin v^2}{\sqrt{1+v^2}}dv. \tag{2.3.15}$$

The first integral in (2.3.15) equals

$$\int_0^1 \frac{\arcsin v}{\sqrt{1+v^2}}dv = \left[\arcsin v \int^v \frac{dw}{\sqrt{1+w^2}} - \int \frac{1}{\sqrt{1-v^2}}\left(\int^v \frac{dw}{\sqrt{1+w^2}}\right)dv\right]_0^1$$
$$= \arcsin v \log\left(v+\sqrt{1+v^2}\right)\Big|_0^1 - \int_0^1 \frac{\log\left(v+\sqrt{1+v^2}\right)}{\sqrt{1-v^2}}dv.$$

We will use

$$\int_0^1 \frac{\log\left(v+\sqrt{1+v^2}\right)}{\sqrt{1-v^2}}dv = \text{ Catalan's constant } = D(i),$$

which is equation (26) in [**33**], after a suitable change of variables. We have that the first integral in (2.3.15) is

$$\int_0^1 \frac{\arcsin v}{\sqrt{1+v^2}}dv = \arcsin v \log\left(v+\sqrt{1+v^2}\right)\Big|_0^1 - D(i) = \frac{\pi}{2}\log\left(\sqrt{2}+1\right) - D(i).$$

In order to compute the second integral in (2.3.15), we make the change of variables $u = v^2$ and notice that

$$\int_0^1 \frac{\arcsin(v^2)}{\sqrt{1+v^2}}dv = \frac{1}{2}\int_0^1 \frac{\arcsin(u)}{\sqrt{u(1+u)}}du = \arcsin(u)\operatorname{arcsinh}(\sqrt{u})\big|_0^1 - \int_0^1 \frac{\operatorname{arcsinh}(\sqrt{u})}{\sqrt{1-u^2}}du$$
$$= \frac{\pi}{2}\log\left(\sqrt{2}+1\right) - \int_0^1 \frac{\operatorname{arcsinh}(\sqrt{u})}{\sqrt{1-u^2}}du.$$

We recall that

$$\operatorname{arcsinh}(\sqrt{u}) = \sum_{j=0}^{\infty} \frac{(-1)^j u^{j+\frac{1}{2}}}{4^j(2j+1)}\binom{2j}{j}.$$

(See formula 4.6.31 in [**1**].) Thus, we have to compute

$$\int_0^1 \frac{\operatorname{arcsinh}(\sqrt{u})}{\sqrt{1-u^2}}du = \sum_{j=0}^{\infty} \frac{(-1)^j}{4^j(2j+1)}\binom{2j}{j}\int_0^1 \frac{u^{j+\frac{1}{2}}}{\sqrt{1-u^2}}du.$$

The change of variables $v = u^2$ allows us to express the previous integral in terms of the beta function. (See formulas 6.2.1 and 6.2.2 in [**1**].) This gives

$$\int_0^1 \frac{u^{j+\frac{1}{2}}}{\sqrt{1-u^2}}du = \frac{1}{2}\int_0^1 v^{\frac{2j-1}{4}}(1-v)^{-\frac{1}{2}}dv = \frac{\Gamma\left(\frac{2j+3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2j+5}{4}\right)} = \frac{2^{j-\frac{1}{2}}\Gamma\left(\frac{2j+3}{4}\right)^2}{\Gamma\left(\frac{2j+3}{2}\right)},$$

where the last equality follows from the Lagrange's duplication formula for the Gamma function (Equation 6.1.18 in [**1**])

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z), \tag{2.3.16}$$

and the identity $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Therefore, we have

$$\int_0^1 \frac{\operatorname{arcsinh}(\sqrt{u})}{\sqrt{1-u^2}}du = \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{j+\frac{1}{2}}(2j+1)}\binom{2j}{j}\frac{\Gamma\left(\frac{2j+3}{4}\right)^2}{\Gamma\left(\frac{2j+3}{2}\right)}.$$

Using (2.3.16) again, we obtain

$$\Gamma\left(\frac{2j+3}{2}\right) = \frac{2^{-2j-1}\sqrt{\pi}\Gamma(2j+2)}{\Gamma(j+1)} = \frac{2^{-2j-1}\sqrt{\pi}(2j+1)!}{j!}.$$

Since

$$\Gamma\left(\frac{2j+3}{4}\right) = \begin{cases} \left(\frac{2j-1}{4}\right)\left(\frac{2j-5}{4}\right)\cdots\frac{3}{4}\left(-\frac{1}{4}\right)\Gamma\left(\frac{-1}{4}\right) & j \text{ even,} \\ \left(\frac{2j-1}{4}\right)\left(\frac{2j-5}{4}\right)\cdots\frac{1}{4}\Gamma\left(\frac{1}{4}\right) & j \text{ odd,} \end{cases}$$

this finally gives

$$\begin{aligned}\int_0^1 \frac{\operatorname{arcsinh}(\sqrt{u})}{\sqrt{1-u^2}}du =& \frac{1}{\sqrt{\pi}}\sum_{\ell=0}^{\infty}\frac{2^{2\ell+\frac{1}{2}}}{(4\ell+1)^2(2\ell)!}\Gamma\left(\frac{4\ell+3}{4}\right)^2 - \frac{1}{\sqrt{\pi}}\sum_{\ell=0}^{\infty}\frac{2^{2\ell+\frac{3}{2}}}{(4\ell+3)^2(2\ell+1)!}\Gamma\left(\frac{4\ell+5}{4}\right)^2 \\ =& \frac{\Gamma\left(\frac{-1}{4}\right)^2}{\sqrt{\pi}}\sum_{\ell=0}^{\infty}\frac{\prod_{k=0}^{\ell}(4k-1)^2}{2^{2\ell+\frac{7}{2}}(4\ell+1)^2(2\ell)!} - \frac{\Gamma\left(\frac{1}{4}\right)^2}{\sqrt{\pi}}\sum_{\ell=0}^{\infty}\frac{\prod_{k=0}^{\ell}(4k+1)^2}{2^{2\ell+\frac{5}{2}}(4\ell+3)^2(2\ell+1)!} \\ =& \frac{2\Gamma\left(\frac{3}{4}\right)^2}{\sqrt{2\pi}}{}_4F_3\left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4};\frac{1}{2},\frac{5}{4},\frac{5}{4};1\right) - \frac{\Gamma\left(\frac{1}{4}\right)^2}{36\sqrt{2\pi}}{}_4F_3\left(\frac{3}{4},\frac{3}{4},\frac{5}{4},\frac{5}{4};\frac{3}{2},\frac{7}{4},\frac{7}{4};1\right).\end{aligned}$$

The last equality follows from comparing the sums with the corresponding expressions for ${}_4F_3$ as follows

$$\begin{aligned}\frac{2\Gamma\left(\frac{3}{4}\right)^2}{\sqrt{2\pi}}{}_4F_3\left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4};\frac{1}{2},\frac{5}{4},\frac{5}{4};1\right) =&\frac{2\Gamma\left(\frac{3}{4}\right)^2}{\sqrt{2\pi}}\sum_{\ell=0}^{\infty}\frac{\left(\frac{1}{4}\right)_\ell^2\left(\frac{3}{4}\right)_\ell^2}{\left(\frac{1}{2}\right)_\ell\left(\frac{5}{4}\right)_\ell^2}\frac{1}{\ell!}\\
=&\frac{\Gamma\left(\frac{-1}{4}\right)^2}{2^3\sqrt{2\pi}}\sum_{\ell=0}^{\infty}\frac{1}{2^{2\ell}(4\ell+1)^2(2\ell)!}\prod_{k=0}^{\ell}(4k-1)^2\\
=&\frac{\Gamma\left(\frac{-1}{4}\right)^2}{\sqrt{\pi}}\sum_{\ell=0}^{\infty}\frac{\prod_{k=0}^{\ell}(4k-1)^2}{2^{2\ell+\frac{7}{2}}(4\ell+1)^2(2\ell)!},\end{aligned}$$

and similarly,

$$\frac{\Gamma\left(\frac{1}{4}\right)^2}{36\sqrt{2\pi}}{}_4F_3\left(\frac{3}{4},\frac{3}{4},\frac{5}{4},\frac{5}{4};\frac{3}{2},\frac{7}{4},\frac{7}{4};1\right) = \frac{\Gamma\left(\frac{1}{4}\right)^2}{\sqrt{\pi}}\sum_{\ell=0}^{\infty}\frac{\prod_{k=0}^{\ell}(4k+1)^2}{2^{2\ell+\frac{5}{2}}(4\ell+3)^2(2\ell+1)!}.$$

Therefore we have

$$\begin{aligned}\int_1^{\sqrt{2}+1}\frac{x(x-x^{-1})\log x}{\sqrt{4-(x-x^{-1})^2}}dx =&\frac{\pi}{2}-\frac{1}{2}+D(i)+\frac{\Gamma\left(\frac{3}{4}\right)^2}{\sqrt{2\pi}}{}_4F_3\left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4};\frac{1}{2},\frac{5}{4},\frac{5}{4};1\right)\\
&-\frac{\Gamma\left(\frac{1}{4}\right)^2}{72\sqrt{2\pi}}{}_4F_3\left(\frac{3}{4},\frac{3}{4},\frac{5}{4},\frac{5}{4};\frac{3}{2},\frac{7}{4},\frac{7}{4};1\right).\end{aligned}$$

This concludes the evaluation of (2.3.11), and therefore of (2.3.10). Combining with (0.4.8), this concludes the proof. □

2.3.3. The areal Mahler measure of $y+\left(\frac{1-x}{1+x}\right)$

In this section, we consider the rational function $y+\left(\frac{1-x}{1+x}\right)$. More precisely, we prove Theorem 2.1.3.

Proof of Theorem 2.1.3. As in previous cases, we have, by definition,

$$\begin{aligned}\mathrm{m}_{\mathbb{D}}\left(y+\left(\frac{1-x}{1+x}\right)\right) =&\frac{1}{\pi^2}\int_{\mathbb{D}^2}\log\left|y+\left(\frac{1-x}{1+x}\right)\right|dA(y)dA(x)\\
=&\frac{1}{2\pi}\int_{\mathbb{D}\cap\left\{\left|\frac{1-x}{1+x}\right|<1\right\}}\left(\left|\frac{1-x}{1+x}\right|^2-1\right)dA(x)+\frac{1}{\pi}\int_{\mathbb{D}\cap\left\{\left|\frac{1-x}{1+x}\right|\geq 1\right\}}\log\left|\frac{1-x}{1+x}\right|dA(x).\end{aligned} \tag{2.3.17}$$

We consider the first integral above. Note that, for $\theta \in [-\pi, \pi)$ and $x = \rho e^{i\theta}$,

$$\left|\frac{1-x}{1+x}\right| \le 1 \Leftrightarrow \frac{1-2\rho\cos\theta+\rho^2}{1+2\rho\cos\theta+\rho^2} \le 1 \Leftrightarrow \cos\theta \ge 0 \Leftrightarrow \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{2.3.18}$$

Therefore, we have

$$\frac{1}{2\pi}\int_{\mathbb{D}\cap\left\{\left|\frac{1-x}{1+x}\right|\le 1\right\}} \left(\left|\frac{1-x}{1+x}\right|^2 - 1\right) dA(x) = -\frac{1}{\pi}\int_0^1 \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\rho\cos\theta}{1+2\rho\cos\theta+\rho^2} d\theta d\rho. \tag{2.3.19}$$

The integral with respect to θ in (2.3.19) is evaluated to be

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\rho\cos\theta}{1+2\rho\cos\theta+\rho^2} d\theta =& 2\int_0^{\frac{\pi}{2}} \frac{2\rho\cos\theta}{1+2\rho\cos\theta+\rho^2} d\theta \\
=& 2\left[\int_0^{\frac{\pi}{2}} d\theta - (1+\rho^2)\int_0^{\frac{\pi}{2}} \frac{1}{1+2\rho\cos\theta+\rho^2} d\theta\right] \\
=& \pi - 2(1+\rho^2)\int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{\theta}{2}\right)}{(1+\rho)^2 + (1-\rho)^2\tan^2\left(\frac{\theta}{2}\right)} d\theta.
\end{aligned}$$

By applying the change of variables $u = \tan\left(\frac{\theta}{2}\right)$, we find that

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\rho\cos\theta}{1+2\rho\cos\theta+\rho^2} d\theta =& \pi - 4(1+\rho^2)\int_0^1 \frac{du}{(1+\rho)^2 + (1-\rho)^2 u^2} \\
=& \pi - \frac{4(1+\rho^2)}{1-\rho^2}\arctan\left(\frac{1-\rho}{1+\rho}\right).
\end{aligned} \tag{2.3.20}$$

Incorporating (2.3.20) in (2.3.19), we obtain

$$\begin{aligned}
\int_0^1 \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\rho\cos\theta}{1+2\rho\cos\theta+\rho^2} d\theta d\rho =& \int_0^1 \rho\left[\pi - \frac{4(1+\rho^2)}{1-\rho^2}\arctan\left(\frac{1-\rho}{1+\rho}\right)\right] d\rho \\
=& \frac{\pi}{2} - \int_0^1 \frac{4\rho(1+\rho^2)}{1-\rho^2}\arctan\left(\frac{1-\rho}{1+\rho}\right) d\rho \\
=& \frac{\pi}{2} - \left[\arctan\left(\frac{1-\rho}{1+\rho}\right)\int^{\rho} \frac{4r(1+r^2)}{1-r^2} dr\Big|_0^1\right. \\
&\left. + \int_0^1 \frac{1}{1+\rho^2}\int^{\rho} \frac{4r(1+r^2)}{1-r^2} dr d\rho\right].
\end{aligned} \tag{2.3.21}$$

Applying the change of variables $v = 1-\rho^2$, we have

$$\int \frac{4\rho(1+\rho^2)}{1-\rho^2} d\rho = -2\int \frac{2-v}{v} dv = -4\log v + 2v + C = -4\log\left(1-\rho^2\right) + 2(1-\rho^2) + C.$$

Then, from (2.3.21), we derive that

$$
\begin{aligned}
\int_0^1 \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\rho\cos\theta}{1+2\rho\cos\theta+\rho^2} d\theta d\rho =& \frac{\pi}{2} - \left[-\frac{\pi}{2} - \int_0^1 \frac{1}{1+\rho^2} \left(4\log\left(1-\rho^2\right) - 2(1-\rho^2) \right) d\rho \right] \\
=& \pi + 4 \int_0^1 \frac{\log(1-\rho^2)}{1+\rho^2} d\rho - 2 \int_0^1 \frac{1-\rho^2}{1+\rho^2} d\rho \\
=& \pi \log 2 + 2 + 4 \int_0^1 \frac{\log\left(\frac{1-\rho^2}{2}\right)}{1+\rho^2} d\rho \\
=& 2 + \pi \log 2 - 4D(i), \qquad (2.3.22)
\end{aligned}
$$

where the last equality from the integral representation of the Catalan's constant $D(i)$ (equation (19) in **[33]**).

By incorporating the result of (2.3.22) into (2.3.19), we obtain

$$
\frac{1}{2\pi} \int_{\mathbb{D}\cap\left\{\left|\frac{1-x}{1+x}\right|\leq 1\right\}} \left(\left|\frac{1-x}{1+x}\right|^2 - 1 \right) dA(x) = \frac{4D(i)}{\pi} - \log 2 - \frac{2}{\pi}. \qquad (2.3.23)
$$

It remains to evaluate the second integral in (2.3.17). Recall from (2.3.18) that we have

$$
\left|\frac{1-x}{1+x}\right| \geq 1 \Leftrightarrow \cos\theta \leq 0 \Leftrightarrow \theta \in \left[-\pi, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right),
$$

where $x = \rho e^{i\theta}$ and $-\pi \leq \theta < \pi$.

Therefore, (2.3.17) can be written as

$$
\begin{aligned}
\int_{\mathbb{D}\cap\left\{\left|\frac{1-x}{1+x}\right|\geq 1\right\}} \log\left|\frac{1-x}{1+x}\right| dA(x) =& \frac{1}{2} \int_0^1 \rho \left[\int_{-\pi}^{-\frac{\pi}{2}} \log\left(\frac{1-2\rho\cos\theta+\rho^2}{1+2\rho\cos\theta+\rho^2} \right) d\theta \right. \\
& \left. + \int_{\frac{\pi}{2}}^{\pi} \log\left(\frac{1-2\rho\cos\theta+\rho^2}{1+2\rho\cos\theta+\rho^2} \right) d\theta \right] d\rho \\
=& \int_0^1 \rho \left[\int_{\frac{\pi}{2}}^{\pi} \log\left(\frac{1-2\rho\cos\theta+\rho^2}{1+2\rho\cos\theta+\rho^2} \right) d\theta \right] d\rho. \qquad (2.3.24)
\end{aligned}
$$

We develop the power series of $\log\left(\frac{1-2\rho\cos\theta+\rho^2}{1+2\rho\cos\theta+\rho^2}\right)$ to get

$$\begin{aligned}\log\left(\frac{1-2\rho\cos\theta+\rho^2}{1+2\rho\cos\theta+\rho^2}\right) =&\log\left(1-\frac{2\rho\cos\theta}{1+\rho^2}\right)-\log\left(1+\frac{2\rho\cos\theta}{1+\rho^2}\right)\\ =&\sum_{k=1}^{\infty}\frac{(-1)^k-1}{k}\left(\frac{2\rho\cos\theta}{1+\rho^2}\right)^k\\ =&-2\sum_{j=0}^{\infty}\frac{1}{2j+1}\left(\frac{2\rho\cos\theta}{1+\rho^2}\right)^{2j+1}.\end{aligned}$$

Now, for $n\geq 2$, a repetitive use of the fact

$$\int\cos^n\theta d\theta=\frac{1}{n}\cos^{n-1}\theta\sin\theta+\frac{n-1}{n}\int\cos^{n-2}\theta d\theta$$

yields

$$\begin{aligned}\int\cos^n\theta d\theta =&\frac{1}{n}\cos^{n-1}\theta\sin\theta+\frac{1}{n}\sum_{k=1}^{\left\lfloor\frac{n}{2}\right\rfloor-1}\left(\prod_{\ell=1}^{k}\frac{n-2\ell+1}{n-2\ell}\right)\cos^{n-2k-1}\theta\sin\theta\\ &+\frac{(n-1)(n-3)\cdots\left(n-2\left\lfloor\frac{n}{2}\right\rfloor+1\right)}{n(n-2)\cdots\left(n-2\left\lfloor\frac{n}{2}\right\rfloor+2\right)}\int\cos^{n-2\left\lfloor\frac{n}{2}\right\rfloor}\theta d\theta,\end{aligned}$$

where $\lfloor x\rfloor$ denotes the largest integer less than or equal to x.

Therefore, when $n=2j+1$, we have, for $j=0$ and $j\geq 1$,

$$\int_{\frac{\pi}{2}}^{\pi}\cos\theta d\theta=-1,\quad \int_{\frac{\pi}{2}}^{\pi}\cos^{2j+1}\theta d\theta=\frac{(2j)(2j-2)\cdots 2}{(2j+1)(2j-1)\cdots 3}\int_{\frac{\pi}{2}}^{\pi}\cos\theta d\theta=-\frac{4^j(j!)^2}{(2j+1)!},$$

respectively. This implies that

$$\int_{\frac{\pi}{2}}^{\pi}\log\left(\frac{1-2\rho\cos\theta+\rho^2}{1+2\rho\cos\theta+\rho^2}\right)d\theta =2\left(\frac{2\rho}{1+\rho^2}+\sum_{j=1}^{\infty}\frac{1}{2j+1}\frac{4^j(j!)^2}{(2j+1)!^2}\left(\frac{2\rho}{1+\rho^2}\right)^{2j+1}\right). \tag{2.3.25}$$

Therefore, in order to compute the integral over ρ in (2.3.24), we need to first consider the individual integrals

$$\int_0^1\left(\frac{2\rho}{1+\rho^2}\right)^{2j+1}\rho d\rho,\qquad \text{for all } j\geq 0. \tag{2.3.26}$$

When $j=0$, we have

$$\int_0^1\frac{2\rho^2}{1+\rho^2}d\rho=2\int_0^1\left(1-\frac{1}{1+\rho^2}\right)d\rho=2\left[\rho-\arctan\rho\right]_0^1=2-\frac{\pi}{2}.$$

For $j \geq 1$, using integration by parts (where $\frac{\rho}{(1+\rho^2)^{2j+1}}$ is integrated and the rest is differentiated), the integrals in (2.3.26) give

$$\begin{aligned}
&\int_0^1 \left(\frac{2\rho}{1+\rho^2}\right)^{2j+1} \rho d\rho \\
=&4^j \left(\frac{-\rho^{2j+1}}{2j\left(1+\rho^2\right)^{2j}}\Bigg|_0^1 + \frac{2j+1}{2j}\int_0^1 \frac{\rho^{2j}}{\left(1+\rho^2\right)^{2j}} d\rho \right) \\
=&4^j \left[\frac{-\rho^{2j+1}}{2j\left(1+\rho^2\right)^{2j}} + \frac{2j+1}{2(2j)}\left(-\frac{\rho^{2j-1}}{(2j-1)\left(1+\rho^2\right)^{2j-1}} + \int \frac{\rho^{2j-2}}{\left(1+\rho^2\right)^{2j-1}} d\rho \right) \right]_0^1 \\
=&-\frac{2}{2j-1} + \frac{4^j(2j+1)}{4j}\int_0^1 \frac{\rho^{2j-2}}{\left(1+\rho^2\right)^{2j-1}} d\rho. \qquad (2.3.27)
\end{aligned}$$

The change of variables $u = \rho^2$ yields

$$\int_0^1 \left(\frac{2\rho}{1+\rho^2}\right)^{2j+1} \rho d\rho = -\frac{2}{2j-1} + \frac{4^{j-1}(2j+1)}{2j}\int_0^1 \frac{u^{j-\frac{3}{2}}}{(1+u)^{2j-1}} du.$$

Making the change of variables $v = \frac{u}{1+u}$, we have

$$\int_0^1 \frac{u^{j-\frac{3}{2}}}{(1+u)^{2j-1}} du = \int_0^{\frac{1}{2}} v^{j-\frac{3}{2}}(1-v)^{j-\frac{3}{2}} dv$$

By the change of variables $w = 1 - v$, we have that

$$\int_0^{\frac{1}{2}} v^{j-\frac{3}{2}}(1-v)^{j-\frac{3}{2}} dv = \int_{\frac{1}{2}}^1 w^{j-\frac{3}{2}}(1-w)^{j-\frac{3}{2}} dw.$$

Therefore, we obtain the beta function

$$\begin{aligned}
\int_0^1 \frac{u^{j-\frac{3}{2}}}{(1+u)^{2j-1}} du =& \frac{1}{2}\int_0^1 v^{j-\frac{3}{2}}(1-v)^{j-\frac{3}{2}} dv = \frac{\Gamma\left(j-\frac{1}{2}\right)^2}{2\Gamma\left(2j-1\right)} = \frac{4^{3-2j}\pi\Gamma\left(2j-2\right)^2}{2\Gamma(j-1)^2\Gamma\left(2j-1\right)} \\
=& \frac{\pi}{2^{4j-3}}\binom{2j-2}{j-1}. \qquad (2.3.28)
\end{aligned}$$

Incorporating (2.3.28) into (2.3.27), we obtain

$$\int_0^1 \left(\frac{2\rho}{1+\rho^2}\right)^{2j+1} \rho d\rho = -\frac{2}{2j-1} + \frac{(2j+1)\pi}{4^j j}\binom{2j-2}{j-1}. \qquad (2.3.29)$$

Next, combining (2.3.24) and (2.3.25) along with (2.3.29), we derive that

$$\begin{aligned}
&\int_{\mathbb{D}\cap\left\{\left|\frac{1-x}{1+x}\right|\geq 1\right\}} \log\left|\frac{1-x}{1+x}\right| dA(x)\\
=&2\int_0^1\left(\frac{2\rho}{1+\rho^2}+\sum_{j=1}^{\infty}\frac{1}{2j+1}\frac{4^j(j!)^2}{(2j+1)!}\left(\frac{2\rho}{1+\rho^2}\right)^{2j+1}\right)\rho d\rho\\
=&2\left(2-\frac{\pi}{2}+\sum_{j=1}^{\infty}\frac{1}{2j+1}\frac{4^j(j!)^2}{(2j+1)!}\left(-\frac{2}{2j-1}+\frac{(2j+1)\pi}{4^j j}\binom{2j-2}{j-1}\right)\right)\\
=&4-\pi-2\sum_{j=1}^{\infty}\frac{2}{4j^2-1}\frac{4^j(j!)^2}{(2j+1)!}+\pi\sum_{j=1}^{\infty}\frac{1}{j^2(2j+1)}\frac{(j!)^2}{(2j-1)!}\binom{2j-2}{j-1}.
\end{aligned}$$

Simplifying the above sums individually, we obtain

$$\pi\sum_{j=1}^{\infty}\frac{1}{j^2(2j+1)}\frac{(j!)^2}{(2j-1)!}\binom{2j-2}{j-1}=\pi\sum_{j=1}^{\infty}\frac{1}{4j^2-1}=\frac{\pi}{2}\sum_{j=1}^{\infty}\left(\frac{1}{2j-1}-\frac{1}{2j+1}\right)=\frac{\pi}{2},$$

and

$$\begin{aligned}
\sum_{j=1}^{\infty}\frac{2}{4j^2-1}\frac{4^j(j!)^2}{(2j+1)!}=&\sum_{j=1}^{\infty}\frac{1}{2j-1}\frac{4^j(j!)^2}{(2j+1)!}-\sum_{j=1}^{\infty}\frac{1}{2j+1}\frac{4^j(j!)^2}{(2j+1)!}\\
=&\frac{1}{2}\sum_{j=1}^{\infty}\left(\frac{1}{2j-1}-\frac{1}{2j+1}\right)\frac{4^j(j!)^2}{(2j)!}-\sum_{j=1}^{\infty}\frac{1}{2j+1}\frac{4^j(j!)^2}{(2j+1)!}\\
=&1+\frac{1}{2}\sum_{j=2}^{\infty}\frac{2j}{(2j-1)^2}\frac{4^{j-1}((j-1)!)^2}{(2(j-1))!}-\frac{1}{2}\sum_{j=2}^{\infty}\frac{1}{2j-1}\frac{4^{j-1}((j-1)!)^2}{(2(j-1))!}\\
&-\sum_{j=1}^{\infty}\frac{1}{2j+1}\frac{4^j(j!)^2}{(2j+1)!}\\
=&1+\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{2k+1}\frac{4^k(k!)^2}{(2k+1)!}-\sum_{j=1}^{\infty}\frac{1}{2j+1}\frac{4^j(j!)^2}{(2j+1)!}\\
=&1-\frac{1}{2}(2D(i)-1)\\
=&\frac{3}{2}-D(i),
\end{aligned}$$

where the evaluation of the sum follows from [**34**, Theorem 2]. (See also equation (61) in [**33**].) Therefore, the integral over the domain $\mathbb{D} \cap \left\{ \left| \frac{1-x}{1+x} \right| \geq 1 \right\}$ yields

$$\frac{1}{\pi} \int_{\mathbb{D} \cap \left\{ \left| \frac{1-x}{1+x} \right| \geq 1 \right\}} \log \left| \frac{1-x}{1+x} \right| dA(x) = \frac{1}{\pi} \left[4 - \pi - 2 \left(\frac{3}{2} - D(i) \right) + \frac{\pi}{2} \right] = \frac{1}{\pi} - \frac{1}{2} + \frac{2}{\pi} D(i).$$

By combining this with (2.3.23) and (0.4.8), the result follows. □

2.4. The areal versions of generalized, higher and zeta Mahler measures

In this section, we recall the areal versions of some variants of the Mahler measure, namely generalized, higher and zeta Mahler measure from Sections 0.6.2 and 0.6.3, give examples in each cases, and derive Theorems 0.6.15, 0.6.18, 0.6.19, and 0.6.22.

2.4.1. Generalized areal Mahler measure

Recall that the generalized areal Mahler measure of non-zero rational functions $P_1, \dots, P_r \in \mathbb{C}(x_1, \dots, x_n)$ is defined by (see Definition 0.6.14)

$$\mathrm{m}_{\mathbb{D}, \max}(P_1, \dots, P_r) = \frac{1}{\pi^n} \int_{\mathbb{D}^n} \max\{\log |P_1|, \dots, \log |P_r|\} dA(x_1) \dots dA(x_n).$$

Then we have the following result.

Theorem 2.4.1 (see Theorem 0.6.15)**.** *We have*

$$\mathrm{m}_{\mathbb{D},\max}(x_1, \dots, x_n) = -\frac{1}{2n}.$$

Proof. As usual, we proceed to apply the definition together with the change of variables $x_j = \rho_j e^{i\theta_j}$. This gives

$$\begin{aligned} \mathrm{m}_{\mathbb{D},\max}(x_1, \dots, x_n) =& \frac{1}{\pi^n} \int_{\mathbb{D}^n} \max\{\log |x_1|, \dots, \log |x_n|\} dA(x_1) \dots dA(x_n) \\ =& 2^n \int_0^1 \cdots \int_0^1 \max\{\log \rho_1, \dots, \log \rho_n\} \rho_1 \cdots \rho_n d\rho_1 \dots d\rho_n. \end{aligned}$$

Notice that the above integral can be written as a sum of $n!$ integrals, where each term corresponds to a certain order of the variables ρ_j. The advantage of considering the ordered variables lies in the fact that the maximum is then easy to describe. Thus, the above becomes

$$\begin{aligned}
&2^n n! \int_{0\le\rho_1\le\cdots\le\rho_n\le 1} \log\rho_n \rho_1\cdots\rho_n d\rho_1\dots d\rho_n\\
=&2^n n! \int_{0\le\rho_2\le\cdots\le\rho_n\le 1} \log\rho_n \frac{\rho_2^3}{2}\cdots\rho_n d\rho_2\dots d\rho_n\\
=&2^n n! \int_{0\le\rho_3\le\cdots\le\rho_n\le 1} \log\rho_n \frac{\rho_3^5}{2\cdot 4}\cdots\rho_n d\rho_3\dots d\rho_n\\
=&\cdots\\
=&2^n n! \int_{0\le\rho_n\le 1} \log\rho_n \frac{\rho_n^{2n-1}}{2\cdots(2n-2)} d\rho_n\\
=&2n\,\frac{\rho_n^{2n}}{(2n)^2}(2n\log\rho_n - 1)\Big|_0^1\\
=&-\frac{1}{2n}.
\end{aligned}$$

□

2.4.2. Multiple and higher areal Mahler measures

Recall that, the *multiple areal Mahler measure* is defined by, for non-zero rational functions $P_1,\dots,P_r \in \mathbb{C}(x_1,\dots,x_n)$,

$$\mathrm{m}_{\mathbb{D},h_1,\dots,h_r}(P_1,\dots,P_r) := \frac{1}{\pi^n}\int_{\mathbb{D}^n} \log^{h_1}|P_1(x_1,\dots,x_n)|\cdots\log^{h_r}|P_r(x_1,\dots,x_n)|dA(x_1)\dots dA(x_n),$$

and the *r-th higher areal Mahler measure* is the the *multiple areal Mahler measure* when $P_1 = \cdots = P_r$ (see Definition 0.6.17). Before proceeding to the proof of Theorem 0.6.18, we restate it below for the readers' convenience.

Theorem 2.4.2 (see Theorem 0.6.18)**.** *We have*

$$\mathrm{m}_{\mathbb{D},h_1,\dots,h_n}(x_1,\dots,x_n) = \frac{(-1)^{h_1+\cdots+h_n}h_1!\cdots h_n!}{2^{h_1+\cdots+h_n+n}}.$$

Proof. First we recall equation (2.2.3), which in particular gives

$$\int_0^1 x \log^k x dx = \frac{(-1)^k k!}{2^{k+1}}.$$

Proceeding as usual by setting $x_j = \rho_j e^{i\theta_j}$, we have

$$\begin{aligned}
\mathrm{m}_{\mathbb{D},h_1,\ldots,h_n}(x_1,\ldots,x_n) =& \frac{1}{\pi^n}\int_{\mathbb{D}^n} \log^{h_1}|x_1|\cdots\log^{h_n}|x_n| dA(x_1)\ldots dA(x_n) \\
=& 2^n \int_0^1 \cdots \int_0^1 \log^{h_1}\rho_1\cdots\log^{h_n}\rho_n \rho_1\cdots\rho_n d\rho_1\ldots d\rho_n \\
=& \frac{(-1)^{h_1+\cdots+h_n} h_1!\cdots h_n!}{2^{h_1+\cdots+h_n+n}}.
\end{aligned}$$

□

In Theorem 2.1.4, we evaluate the h-th higher areal Mahler measure of the rational functions $\frac{1-x}{1+x}$. Before proceeding to the proof, we need the following lemma.

Lemma 2.4.3. *Let $h \in \mathbb{Z}_{>2}$ $|\alpha|, |\beta| \le 1$. We have*

$$\sum_{b>1} \frac{\beta^b}{b^{h-1}} \sum_{a=1}^{b-1} \frac{\alpha^a}{a} = \mathrm{Li}_{1,h-1}(\alpha,\beta), \tag{2.4.1}$$

and for $\alpha \neq 1$,

$$\sum_{b>1} \frac{\beta^b}{b^{h+1}} \sum_{a=1}^{b-1} a\alpha^a = \frac{1}{(\alpha-1)^2}(\alpha \mathrm{Li}_h(\alpha\beta) - \alpha\mathrm{Li}_{h+1}(\alpha\beta) - \mathrm{Li}_h(\alpha\beta) + \alpha \mathrm{Li}_{h+1}(\beta)). \tag{2.4.2}$$

Proof. Identity (2.4.1) follows directly from the definition of multiple polylogarithms. For identity (2.4.2) we have

$$\begin{aligned}
\sum_{b>1} \frac{\beta^b}{b^{h+1}} \sum_{a=1}^{b-1} a\alpha^a =& \sum_{b\ge1} \frac{\beta^b}{b^{h+1}} \frac{\alpha((b-1)\alpha^b - b\alpha^{b-1}+1)}{(\alpha-1)^2} \\
=& \frac{1}{(\alpha-1)^2}(\alpha \mathrm{Li}_h(\alpha\beta) - \alpha\mathrm{Li}_{h+1}(\alpha\beta) - \mathrm{Li}_h(\alpha\beta) + \alpha \mathrm{Li}_{h+1}(\beta)).
\end{aligned}$$

□

Proof of Theorem 2.1.4. By definition, we have

$$\mathrm{m}_{\mathbb{D},h}\left(\frac{1-x}{1+x}\right) = \frac{1}{\pi}\int_{\mathbb{D}} \log^h \left|\frac{1-x}{1+x}\right| dA(x).$$

We remark that the change of variables $z = \frac{1-x}{1+x}$ takes the unit disk to the right half plane $\mathbb{H}_+$ defined by $\mathrm{Re}(z) \geq 0$. The areal measure of the unit disk is $dA(x) = dx_1 dx_2$, and under the map $f : x \mapsto \frac{1-x}{1+x} = z$, by definition, the areal measure of $\mathbb{H}_+$ is

$$dA(z) = |x_{1,z_1}x_{2,z_2} - x_{1,z_2}x_{2,z_1}|\, dz_1 dz_2, \quad \text{where} \quad \frac{\partial z_j}{\partial x_k} = z_{j,x_k} \text{ for } j = 1,2 \text{ and } k = 1,2.$$

Since the map f is conformal on $\mathbb{D}$, it satisfies the Cauchy–Riemann relations, and therefore

$$|x_{1,z_1}x_{2,z_2} - x_{1,z_2}x_{2,z_1}|\, dz_1 dz_2 = \left|\frac{dx}{dz}\right|^2 dz_1 dz_2 = \left|\frac{d}{dz}\left(\frac{1-z}{1+z}\right)\right|^2 dz_1 dz_2 = \frac{4 dz_1 dz_2}{|z+1|^4}.$$

In sum, we have that

$$\int_{\mathbb{D}} \log^h \left|\frac{1-x}{1+x}\right| dA(x) = 4 \int_{\mathbb{H}_+} \log^h |z| \frac{dz_1 dz_2}{|z+1|^4}. \tag{2.4.3}$$

We further consider the change to polar coordinates $z = \rho e^{i\theta}$ with $\rho \geq 0$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. The integral in (2.4.3) can be written as

$$\begin{aligned}
\int_{\mathbb{H}_+} \log^h |z| \frac{dz_1 dz_2}{|z+1|^4} &= \int_{\mathrm{Re}(z)\geq 0} \log^h |z| \frac{dz_1 dz_2}{|z+1|^4} \\
&= \int_0^\infty \rho \log^h \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{|\rho e^{i\theta}+1|^4} d\rho \\
&= \int_0^1 \rho \log^h \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{|\rho e^{i\theta}+1|^4} d\rho + (-1)^h \int_0^1 \rho \log^h \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{|\rho e^{i\theta}+1|^4} d\rho,
\end{aligned}$$

where the last equality is obtained by separating the integral over ρ into $0 \leq \rho \leq 1$ and $1 \leq \rho$, and then applying the change of variables $\rho \mapsto \rho^{-1}$ in the $1 \leq \rho$ term. The above derivation implies that

$$\int_{\mathbb{H}_+} \log^h |z| \frac{dz_1 dz_2}{|z+1|^4} = \begin{cases} 2 \displaystyle\int_0^1 \rho \log^h \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{|\rho e^{i\theta}+1|^4} d\rho & h \text{ is even,} \\ 0 & h \text{ is odd.} \end{cases}$$

For what follows we will assume that h is even. First we will also assume that $h \neq 2$, as this will guarantee that certain series converge. The case $h = 2$ will be treated later.

Evaluating the individual terms in the above formula leads to

$$\begin{aligned}
&\int_0^1 \rho \log^h \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\left|\rho e^{i\theta}+1\right|^4} d\rho \\
=&\int_0^1 \rho \log^h \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\left(\rho e^{i\theta}+1\right)^2\left(\rho e^{-i\theta}+1\right)^2} d\rho \\
=&\int_0^1 \rho \log^h \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sum_{k\geq 0}(k+1)(-\rho)^k e^{ik\theta}\right)\left(\sum_{\ell\geq 0}(\ell+1)(-\rho)^\ell e^{-i\ell\theta}\right) d\theta d\rho \\
=&\pi \sum_{k\geq 0}\int_0^1 (k+1)^2\rho^{2k+1}\log^h \rho d\rho \qquad (2.4.4)\\
&+\frac{2}{i}\sum_{k>\ell\geq 0}\int_0^1 \frac{(k+1)(\ell+1)}{k-\ell}(-1)^{k+\ell}\left(i^{k-\ell}-(-i)^{k-\ell}\right)\rho^{k+\ell+1}\log^h \rho d\rho. \qquad (2.4.5)
\end{aligned}$$

From (2.2.3) we have

$$\int_0^1 x^j \log^k x dx = \frac{(-1)^k k!}{(j+1)^{k+1}}.$$

Since h is even, we can calculate the integral in (2.4.4) and obtain, for $h>2$,

$$\sum_{k\geq 0}\int_0^1 (k+1)^2\rho^{2k+1}\log^h \rho d\rho = \sum_{k\geq 0}\frac{h!(k+1)^2}{(2k+2)^{h+1}} = \frac{h!}{2^{h+1}}\sum_{k\geq 0}\frac{1}{(k+1)^{h-1}} = \frac{h!}{2^{h+1}}\zeta(h-1).$$

For the integral in (2.4.5), first notice that, since $k+\ell$ and $k-\ell$ have the same parity, it suffices to consider

$$\sum_{k>\ell\geq 0}\int_0^1 \frac{(k+1)(\ell+1)}{k-\ell} I_{k-\ell}\rho^{k+\ell+1}\log^h \rho d\rho, \quad \text{where} \quad I_j = (-i)^j - i^j.$$

Setting $k^* = k+1$, $\ell^* = \ell+1$ first, and then $a = k^* - \ell^*$, $b = k^* + \ell^*$, we find that

$$\begin{aligned}
&\sum_{k>\ell\geq 0}\int_0^1 \frac{(k+1)(\ell+1)}{k-\ell} I_{k-\ell}\rho^{k+\ell+1}\log^h \rho d\rho \\
=&h!\sum_{k>\ell\geq 0}\frac{(k+1)(\ell+1)}{(k-\ell)(k+\ell+2)^{h+1}} I_{k-\ell} = h!\sum_{k^*>\ell^*\geq 1}\frac{k^*\ell^*}{(k^*-\ell^*)(k^*+\ell^*)^{h+1}} I_{k^*-\ell^*} \\
=&\frac{h!}{4}\left[\sum_{k^*>\ell^*\geq 1}\frac{I_{k^*-\ell^*}}{(k^*-\ell^*)(k^*+\ell^*)^{h-1}} - \sum_{k^*>\ell^*\geq 1}\frac{(k^*-\ell^*)I_{k^*-\ell^*}}{(k^*+\ell^*)^{h+1}}\right] \\
=&\frac{h!}{4}\left[\sum_{\substack{b>a\geq 1\\ a\equiv b \bmod 2}}\frac{I_a}{ab^{h-1}} - \sum_{\substack{b>a\geq 1\\ a\equiv b \bmod 2}}\frac{aI_a}{b^{h+1}}\right] \\
=&\frac{h!}{8}\left[\sum_{b>a\geq 1}\frac{(1+(-1)^{a+b})I_a}{ab^{h-1}} - \sum_{b>a\geq 1}\frac{a(1+(-1)^{a+b})I_a}{b^{h+1}}\right] \\
=&\frac{h!}{8}\left[\sum_{b>a\geq 1}\frac{(-i)^a - i^a + i^a(-1)^b - (-i)^a(-1)^b}{ab^{h-1}}\right. \\
&\left. - \sum_{b>a\geq 1}\frac{a((-i)^a - i^a + i^a(-1)^b - (-i)^a(-1)^b)}{b^{h+1}}\right].
\end{aligned}$$

Applying Lemma 2.4.3 gives

$$\begin{aligned}
&\sum_{k>\ell\geq 0}\int_0^1 \frac{(k+1)(\ell+1)}{k-\ell} I_{k-\ell}\rho^{k+\ell+1}\log^h \rho d\rho \\
=&\frac{h!}{8}\left[\sum_{b>a\geq 1}\frac{(-i)^a - i^a + i^a(-1)^b - (-i)^a(-1)^b}{ab^{h-1}} - \sum_{b>a\geq 1}\frac{a((-i)^a - i^a + i^a(-1)^b - (-i)^a(-1)^b)}{b^{h+1}}\right] \\
=&\frac{h!}{8}[\mathrm{Li}_{1,h-1}(-i,1) - \mathrm{Li}_{1,h-1}(i,1) + \mathrm{Li}_{1,h-1}(i,-1) - \mathrm{Li}_{1,h-1}(-i,-1) \\
&- \frac{1}{(-i-1)^2}\left(-i\mathrm{Li}_h(-i) + i\mathrm{Li}_{h+1}(-i) - \mathrm{Li}_h(-i) - i\mathrm{Li}_{h+1}(1)\right) \\
&+ \frac{1}{(i-1)^2}\left(i\mathrm{Li}_h(i) - i\mathrm{Li}_{h+1}(i) - \mathrm{Li}_h(i) + i\mathrm{Li}_{h+1}(1)\right) \\
&- \frac{1}{(i-1)^2}\left(i\mathrm{Li}_h(-i) - i\mathrm{Li}_{h+1}(-i) - \mathrm{Li}_h(-i) + i\mathrm{Li}_{h+1}(-1)\right) \\
&+ \frac{1}{(-i-1)^2}\left(-i\mathrm{Li}_h(i) + i\mathrm{Li}_{h+1}(i) - \mathrm{Li}_h(i) - i\mathrm{Li}_{h+1}(-1)\right)].
\end{aligned}$$

The length 2 polylogarithms above can be written in terms of length 1 polylogarithms by means of Theorem 0.4.4 as follows (recall that $h > 2$ is even),

$$\begin{aligned}
&\mathrm{Li}_{1,h-1}(-i,1) - \mathrm{Li}_{1,h-1}(i,1) + \mathrm{Li}_{1,h-1}(i,-1) - \mathrm{Li}_{1,h-1}(-i,-1)\\
=&2\mathrm{Re}_h(\mathrm{Li}_{1,h-1}(-i,1)) + 2\mathrm{Re}_h(\mathrm{Li}_{1,h-1}(i,-1))\\
=&3\mathrm{Li}_h(i) + \mathrm{Li}_h(-i) + (h-1)(\mathrm{Li}_h(1) + \mathrm{Li}_h(-1))\\
&-(\mathrm{Li}_{h-1}(1) + \mathrm{Li}_{h-1}(-1))(-\mathrm{Li}_1(-i) + \mathrm{Li}_1(i))\\
&+\sum_{m=1}^{h-1}(\mathrm{Li}_m(i) + \mathrm{Li}_m(-i))(-\mathrm{Li}_{h-m}(-i) + (-1)^{m-1}\mathrm{Li}_{h-m}(i)).
\end{aligned}$$

For the length 1 polylogarithms in the expression of $\sum_{k>\ell\geq 0}\int_0^1 \frac{(k+1)(\ell+1)}{k-\ell} I_{k-\ell}\rho^{k+\ell+1}\log^h \rho d\rho$, we have

$$\begin{aligned}
&-\frac{1}{(-i-1)^2}\left(-i\mathrm{Li}_h(-i) + i\mathrm{Li}_{h+1}(-i) - \mathrm{Li}_h(-i) - i\mathrm{Li}_{h+1}(1)\right)\\
&+\frac{1}{(i-1)^2}\left(i\mathrm{Li}_h(i) - i\mathrm{Li}_{h+1}(i) - \mathrm{Li}_h(i) + i\mathrm{Li}_{h+1}(1)\right)\\
&-\frac{1}{(i-1)^2}\left(i\mathrm{Li}_h(-i) - i\mathrm{Li}_{h+1}(-i) - \mathrm{Li}_h(-i) + i\mathrm{Li}_{h+1}(-1)\right)\\
&+\frac{1}{(-i-1)^2}\left(-i\mathrm{Li}_h(i) + i\mathrm{Li}_{h+1}(i) - \mathrm{Li}_h(i) - i\mathrm{Li}_{h+1}(-1)\right)\\
=&\frac{i}{2}\left(-i\mathrm{Li}_h(-i) + i\mathrm{Li}_{h+1}(-i) - i\mathrm{Li}_{h+1}(1)\right) + \frac{i}{2}\left(i\mathrm{Li}_h(i) - i\mathrm{Li}_{h+1}(i) + i\mathrm{Li}_{h+1}(1)\right)\\
&-\frac{i}{2}\left(i\mathrm{Li}_h(-i) - i\mathrm{Li}_{h+1}(-i) + i\mathrm{Li}_{h+1}(-1)\right) - \frac{i}{2}\left(-i\mathrm{Li}_h(i) + i\mathrm{Li}_{h+1}(i) - i\mathrm{Li}_{h+1}(-1)\right)\\
=&-(\mathrm{Li}_h(i) - \mathrm{Li}_h(-i)) + (\mathrm{Li}_{h+1}(i) - \mathrm{Li}_{h+1}(-i)).
\end{aligned}$$

Putting everything together, we have

$$\begin{aligned}
&\frac{8}{h!}\sum_{k>\ell\geq 0}\int_0^1 \frac{(k+1)(\ell+1)}{k-\ell} I_{k-\ell}\rho^{k+\ell+1}\log^h \rho d\rho \\
=&2(\mathrm{Li}_h(i)+\mathrm{Li}_h(-i))+(h-1)(\mathrm{Li}_h(1)+\mathrm{Li}_h(-1))-(\mathrm{Li}_{h-1}(1)+\mathrm{Li}_{h-1}(-1))(-\mathrm{Li}_1(-i)+\mathrm{Li}_1(i)) \\
&+\sum_{m=1}^{h-1}(\mathrm{Li}_m(i)+\mathrm{Li}_m(-i))(-\mathrm{Li}_{h-m}(-i)+(-1)^{m-1}\mathrm{Li}_{h-m}(i))+(\mathrm{Li}_{h+1}(i)-\mathrm{Li}_{h+1}(-i)) \\
=&2^{2-h}\mathrm{Li}_h(-1)+(h-1)2^{1-h}\zeta(h)-2^{2-h}\zeta(h-1)\frac{\pi i}{2}+2iL(h+1,\chi_{-4}) \\
&+\sum_{\substack{m=1\\ m\,\mathrm{odd}}}^{h-1} 2^{1-m}\mathrm{Li}_m(-1)2iL(h-m,\chi_{-4})-\sum_{\substack{m=1\\ m\,\mathrm{even}}}^{h-1} 2^{1-m}\mathrm{Li}_m(-1)2^{1-h+m}\mathrm{Li}_{h-m}(-1) \\
=&2^{2-h}(2^{1-h}-1)\zeta(h)+(h-1)2^{1-h}\zeta(h)-i\pi 2^{1-h}\zeta(h-1)+2iL(h+1,\chi_{-4}) \\
&-2i\log(2)L(h-1,\chi_{-4})+i\sum_{\substack{m=2\\ m\,\mathrm{odd}}}^{h-1} 2^{2-m}(2^{1-m}-1)\zeta(m)L(h-m,\chi_{-4}) \\
&-\sum_{\substack{m=2\\ m\,\mathrm{even}}}^{h-1} 2^{2-h}(2^{1-m}-1)(2^{1-h+m}-1)\zeta(m)\zeta(h-m).
\end{aligned}$$

Using the expressions of $\zeta(2n)$ and $L(\chi_{-4}, 2n+1)$ in terms of Bernoulli numbers B_n (see (0.4.3)) and Euler numbers E_n (see (0.4.20)) in (0.4.21) and (0.4.22) (see Remark 0.4.8), we can further simplify the above expression to obtain

$$\begin{aligned}
&2(1-2^{1-h})\frac{i^h B_h \pi^h}{h!}-(h-1)\frac{i^h B_h\pi^h}{h!}-i\pi 2^{1-h}\zeta(h-1)+\frac{i^{h+1}E_h\pi^{h+1}}{2^{h+1}h!} \\
&+\log(2)\frac{i^{h+1}E_{h-2}\pi^{h-1}}{2^{h-1}(h-2)!}+\sum_{\substack{m=2\\ m\,\mathrm{odd}}}^{h-1} 2(2^{1-m}-1)\zeta(m)\frac{i^{h-m}E_{h-m-1}\pi^{h-m}}{2^h(h-m-1)!} \\
&-\frac{i^h\pi^h}{h!}\sum_{\substack{m=2\\ m\,\mathrm{even}}}^{h-1}\binom{h}{m}(2^{1-m}-1)(2^{1-h+m}-1)B_m B_{h-m}.
\end{aligned}$$

Following the above simplification, we get, for $h > 2$ even,

$$
\begin{aligned}
&\frac{1}{\pi}\int_{\mathbb{D}} \log^h \left|\frac{1-x}{1+x}\right| dA(x) \\
=&\frac{h!}{2^{h-2}}\zeta(h-1) + \frac{2h!}{\pi i}\Bigg[2(1-2^{1-h})\frac{i^h B_h \pi^h}{h!} - (h-1)\frac{i^h B_h \pi^h}{h!} - i\pi 2^{1-h}\zeta(h-1) \\
&+\frac{i^{h+1}E_h \pi^{h+1}}{2^{h+1}h!} + \log(2)\frac{i^{h+1}E_{h-2}\pi^{h-1}}{2^{h-1}(h-2)!} + \sum_{\substack{m=2\\ m\,\text{odd}}}^{h-1} 2(2^{1-m}-1)\zeta(m)\frac{i^{h-m}E_{h-m-1}\pi^{h-m}}{2^h(h-m-1)!} \\
&-\frac{i^h\pi^h}{h!}\sum_{\substack{m=2\\ m\,\text{even}}}^{h-1}\binom{h}{m}(2^{1-m}-1)(2^{1-h+m}-1)B_m B_{h-m}\Bigg] \\
=&-2(h-1)B_h(\pi i)^{h-1} + \frac{E_h(\pi i)^h}{2^h} - \log(2)\frac{E_{h-2}(\pi i)^{h-2}h(h-1)}{2^{h-2}} \\
&-\frac{4h!}{2^h}\sum_{m=2}^{h-1}(1-2^{1-m})\zeta(m)\frac{E_{h-m-1}(\pi i)^{h-m-1}}{(h-m-1)!} \\
&-2(\pi i)^{h-1}\sum_{m=0}^{h}\binom{h}{m}(1-2^{1-m})(1-2^{1-h+m})B_m B_{h-m},
\end{aligned}
\tag{2.4.6}
$$

where we have used that $B_n = E_n = 0$ when n is odd (with the exception of B_1). Further observe that, from the generating series of B_n in (0.4.3), we have the following identities

$$\frac{x}{e^{\frac{x}{2}}-1} = \sum_{n=0}^{\infty}\frac{2^{1-n}B_n x^n}{n!}, \tag{2.4.7}$$

$$\frac{x^2 e^x}{(e^x-1)^2} = 1 - \sum_{n=1}^{\infty}\frac{(n-1)B_n x^n}{n!}, \tag{2.4.8}$$

$$\frac{x}{e^x-1} - \frac{x}{e^{\frac{x}{2}}-1} = \sum_{n=0}^{\infty}\left(1-2^{1-n}\right)\frac{B_n x^n}{n!}, \tag{2.4.9}$$

where (2.4.7) follows from replacing x with $\frac{x}{2}$ in (0.4.3), (2.4.8) follows from differentiating (0.4.3) with respect to x, and (2.4.9) is the difference between (0.4.3) and (2.4.7). Squaring both sides of (2.4.9) and comparing them with (2.4.8), we have

$$\begin{aligned}
\left(\frac{x}{e^x-1}-\frac{x}{e^{\frac{x}{2}}-1}\right)^2 &= \left(\sum_{n=0}^{\infty}\left(1-2^{1-n}\right)\frac{B_nx^n}{n!}\right)\left(\sum_{\ell=0}^{\infty}\left(1-2^{1-\ell}\right)\frac{B_\ell x^\ell}{\ell!}\right),\\
\Leftrightarrow \frac{x^2e^x}{\left(e^x-1\right)^2} &= \sum_{t=0}^{\infty}\left(\sum_{\ell=0}^{t}\binom{t}{\ell}\left(1-2^{1-\ell}\right)\left(1-2^{1-t+\ell}\right)B_\ell B_{t-\ell}\right)\frac{x^t}{t!},\\
\Leftrightarrow 1-\sum_{t=1}^{\infty}\frac{(t-1)B_tx^t}{t!} &= \sum_{t=0}^{\infty}\left(\sum_{\ell=0}^{t}\binom{t}{\ell}\left(1-2^{1-\ell}\right)\left(1-2^{1-t+\ell}\right)B_\ell B_{t-\ell}\right)\frac{x^t}{t!}.
\end{aligned}$$

For $t\geq 2$, comparing the coefficients of $\frac{x^t}{t!}$, we have

$$-(t-1)B_t=\sum_{\ell=0}^{t}\binom{t}{\ell}\left(1-2^{1-\ell}\right)\left(1-2^{1-t+\ell}\right)B_\ell B_{t-\ell}.$$

This further simplifies (2.4.6), and we finally obtain, for $h\geq 2$ even,

$$\begin{aligned}
\frac{1}{\pi}\int_{\mathbb{D}}\log^h\left|\frac{1-x}{1+x}\right|dA(x)=&\frac{E_h(\pi i)^h}{2^h}-\log(2)\frac{E_{h-2}(\pi i)^{h-2}h(h-1)}{2^{h-2}}\\
&-\frac{4h!}{2^h}\sum_{m=2}^{h-1}(1-2^{1-m})\zeta(m)\frac{E_{h-m-1}(\pi i)^{h-m-1}}{(h-m-1)!}. \qquad (2.4.10)
\end{aligned}$$

Note that the above computation fails to converge when $h=2$. Therefore, we need to evaluate the $h=2$ case in a different way. Since $\log^2\left|\frac{1-x}{1+x}\right| = \log^2|1-x| - 2\log|1-x|\log|1+x| + \log^2|1+x|$, and $\mathrm{m}_{\mathbb{D},2}(1-x) = \mathrm{m}_{\mathbb{D},2}(1+x)$, we have

$$\mathrm{m}_{\mathbb{D},2}\left(\frac{1-x}{1+x}\right)=2\mathrm{m}_{\mathbb{D},2}(1+x)-\frac{2}{\pi}\int_{\mathbb{D}}\log|1-x|\log|1+x|\,dx.$$

It only remains to compute the second integral. Following the method in the proof of [**72**, Theorem 7], we derive

$$\begin{aligned}
\int_{\mathbb{D}}\log|1-x|\log|1+x|\,dx &= \int_0^1\rho\left[\int_0^{2\pi}\mathrm{Re}\left(\log\left(1-\rho e^{i\theta}\right)\right)\mathrm{Re}\left(\log\left(1+\rho e^{i\theta}\right)\right)d\theta\right]d\rho\\
&= \int_0^1\rho\left[\int_0^{2\pi}\left(-\sum_{k\geq 1}\frac{\rho^k}{k}\cos(k\theta)\right)\left(-\sum_{\ell\geq 1}\frac{(-1)^\ell\rho^\ell}{\ell}\cos(\ell\theta)\right)d\theta\right]d\rho\\
&= \int_0^1\sum_{k,\ell\geq 1}\frac{(-1)^\ell\rho^{k+\ell+1}}{k\ell}\left[\int_0^{2\pi}\cos(k\theta)\cos(\ell\theta)d\theta\right]d\rho.
\end{aligned}$$

On the other hand, we have

$$\int_0^{2\pi} \cos(k\theta)\cos(\ell\theta)d\theta = \begin{cases} \pi & \text{if } k = \ell, \\ 0 & \text{if } k \neq \ell. \end{cases}$$

This implies that

$$\begin{aligned}
\int_{\mathbb{D}} \log|1-x| \log|1+x|\, dx =& \int_0^1 \sum_{k,\ell\geq 1} \frac{(-1)^\ell \rho^{k+\ell+1}}{k\ell} \left[\int_0^{2\pi} \cos(k\theta)\cos(\ell\theta)d\theta\right] d\rho \\
=&\pi \sum_{k\geq 1} \frac{(-1)^k}{k^2} \int_0^1 \rho^{2k+1} d\rho \\
=&\frac{\pi}{2} \sum_{k\geq 1} \frac{(-1)^k}{k^2(k+1)} \\
=&\frac{\pi}{2} \sum_{k\geq 1} (-1)^k \left[\frac{1}{k^2} - \frac{1}{k} + \frac{1}{k+1}\right] \\
=&\frac{\pi}{2} \left[\mathrm{Li}_2(-1) - 2\mathrm{Li}_1(-1) - 1\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{m}_{\mathbb{D},2}\left(\frac{1-x}{1+x}\right) =& 2\mathrm{m}_{\mathbb{D},2}(1+x) - \mathrm{Li}_2(-1) + 2\mathrm{Li}_1(-1) + 1 \\
=&\frac{\pi^2}{6} - 1 + \frac{\pi^2}{12} - 2\log 2 + 1 \\
=&\frac{\pi^2}{4} - 2\log 2.
\end{aligned}$$

Finally we remark that this value is also obtained by replacing $h = 2$ in (2.4.10). □

2.4.3. Areal zeta Mahler measure

In Section 0.6.3, we encountered how the Taylor expansion of the zeta Mahler measure (first defined by Akatsuka [**3**]) collects all h-th higher Mahler measure for $h \geq 0$, namely

$$Z(s,P) = \sum_{k=0}^{\infty} \frac{\mathrm{m}_k(P)s^k}{k!}.$$

This relationship in fact extends to the areal case. Indeed, from Definition 0.6.20, we have that the areal zeta Mahler measure of $P \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$, denoted by

$$Z_{\mathbb{D}}(s,P) := \frac{1}{\pi^n} \int_{\mathbb{D}^n} |P(x_1, \dots, x_n)|^s \, dA(x_1) \dots dA(x_n),$$

is the exponential generating function of the higher areal Mahler measures of P.

Next, we follow some arguments from [**72**, Theorem 14] to evaluate $\mathbb{Z}_{\mathbb{D}}(s, x+1)$ and derive Theorem 2.1.5.

Before proceeding to the proof, we recall an equality due to Akatsuka, which appeared in his derivation of the zeta Mahler measure $Z(s,x-c)$ for a constant c. In particular, his result in [**3**, Theorem 2] implies the following formula for $\rho < 1$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\rho e^{i\theta} + 1|^s d\theta = {}_2F_1\left(-\frac{s}{2}, -\frac{s}{2}; 1; \rho^2\right). \tag{2.4.11}$$

Proof of Theorem 2.1.5. By definition and the usual change of variables, we have

$$Z_{\mathbb{D}}(s,x+1) = \frac{1}{\pi} \int_{\mathbb{D}} |x+1|^s dA(x) = \frac{1}{\pi} \int_0^1 \int_{-\pi}^{\pi} |\rho e^{i\theta} + 1|^s \rho d\theta d\rho.$$

By applying Akatsuka's result (2.4.11), we then have to compute

$$\begin{aligned} Z_{\mathbb{D}}(s,x+1) =& 2\int_0^1 {}_2F_1\left(-\frac{s}{2}, -\frac{s}{2}; 1; \rho^2\right) \rho d\rho = 2\int_0^1 \sum_{n=0}^{\infty} \frac{(-s/2)_n^2}{(n!)^2} \rho^{2n+1} d\rho \\ =& \sum_{n=0}^{\infty} \binom{s/2}{n}^2 \frac{1}{(n+1)} = \frac{1}{s/2+1} \sum_{n=0}^{\infty} \binom{s/2}{n} \binom{s/2+1}{s/2-n} = \frac{1}{s/2+1} \binom{s+1}{s/2} \\ =& \frac{s+1}{(s/2+1)^2} \frac{\Gamma(s+1)}{\Gamma(s/2+1)^2}. \end{aligned}$$

We now apply the Weierstrass product of the Gamma function

$$\Gamma(s+1)^{-1} = e^{\gamma s} \prod_{k=1}^{\infty} \left(1 + \frac{s}{k}\right) e^{-s/k},$$

to obtain

$$\begin{aligned}
Z_{\mathbb{D}}(s,x+1) &= \prod_{k=2}^{\infty} \frac{\left(1+\frac{s}{2k}\right)^2}{1+\frac{s}{k}}\\
&= \exp\left(\sum_{k=2}^{\infty}\left(2\log\left(1+\frac{s}{2k}\right)-\log\left(1+\frac{s}{k}\right)\right)\right)\\
&= \exp\left(\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\sum_{k=2}^{\infty}\left(\frac{2}{(2k)^j}-\frac{1}{k^j}\right)s^j\right)\\
&= \exp\left(\sum_{j=1}^{\infty}\frac{(-1)^{j}}{j}(1-2^{1-j})\sum_{k=2}^{\infty}\frac{s^j}{k^j}\right)\\
&= \exp\left(\sum_{j=2}^{\infty}\frac{(-1)^{j}}{j}(1-2^{1-j})(\zeta(j)-1)s^j\right).
\end{aligned}$$

□

2.5. Conclusion

We have obtained formulas for the areal Mahler and its generalizations for various rational functions. In most cases, the results are connected to evaluations of polylogarithms leading to special values of functions with arithmetic significance such as the Riemann zeta function, and Dirichlet L-functions. In this sense, the results are analogous to what is obtained in the case of the standard Mahler measure.

However, there is a crucial difference between the areal case and the standard case. In the standard case, there is a way to assign a weight to the terms in the formula, that typically results in formulas of homogeneous weight 1. For this, we recall that the length one n-th polylogarithm has weight n. The logarithm is associated to $\mathrm{Li}_1(z)$ and therefore has weight 1. The constant π arises as an evaluation of the logarithm and therefore has weight 1. Finally, we assign weight one to the Mahler measure itself. Taking the weight multiplicatively, we have, for example, in Smyth's formula (0.1.7) that $\mathrm{m}(1+x+y)$ has weight 1, while $L(\chi_{-3},2)$ has weight 2 (as it is the evaluation of a dilogarithm), while π has weight 1, giving a total weight of 1 on the right-hand side. In contrast, the terms on the right-hand side of the areal Mahler measure (2.1.1) do not have a homogeneous weight. While the term $\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)$ has weight 1, the term $\frac{1}{6}$ has weight 0 and the term $-\frac{11\sqrt{3}}{16\pi}$ has weight -1. This suggests that if there is a connection between $\mathrm{m}_{\mathbb{D}}$ and the regulator, it will be more difficult to describe than in the standard case. It would be nevertheless interesting to explore the possibility of such a connection.

None of the formulas considered in this chapter correspond to rational functions whose Mahler measure is related to special values of other L-functions, such as L-functions attached to elliptic curves. A natural question and direction of future research would be to evaluate the areal version of the formulas in Table 1 involving the family $\mathcal{Q}_r$ in (0.1.10).

Chapter 3

The areal Mahler measure under a power change of variables

In this chapter, we investigate how the areal Mahler measure changes with the transformation $x \mapsto x^r$, where r is an integer, and provide some specific examples. This is based on a joint work with Lalín, and will appear in [**83**].

3.1. A brief description of the results

Recall that Theorem 0.1.6 implies that the change of variables induced by a matrix $A \in GL_n(\mathbb{Z})$ (see (0.6.9)) on a non-zero n-variable rational function P does not affect the Mahler measure $\mathrm{m}(P)$.

In this chapter, we investigate the transformation $\{x \mapsto x^r,\ y \mapsto y^s\}$ for the polynomials $1 + x + y$ in the areal Mahler measure case, where $r, s \geq 1$. Observe that the matrix representing this transformation is

$$A = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in GL_2(\mathbb{Z}).$$

More precisely, we derive the following result.

Theorem 3.1.1 (see Theorem 0.6.8)**.** *Let* r,s *be positive integers. We have*

$$
\begin{aligned}
&\mathrm{m}_{\mathbb{D}}\left(1+x^r+y^s\right)\\
&=\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)-\frac{r}{6}+\frac{\sqrt{3}r}{12\pi}\left[\zeta\left(1,\frac{r+2}{3r}\right)-\zeta\left(1,\frac{2r+2}{3r}\right)+\zeta\left(1,\frac{r+1}{3r}\right)-\zeta\left(1,\frac{2r+1}{3r}\right)\right]\\
&-\frac{2}{\pi}\sum_{1\le k}\sum_{h=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2h}\frac{(-1)^{h-1}{}_2F_1\left(\frac12-h,k-h+\frac1r+\frac12;k-h+\frac1r+\frac32;\frac14\right)}{2^{k-2h+1}k(kr+2)\left(2k+\frac2r-2h+1\right)}+\frac{s}{6}\sum_{1\le k}\binom{\frac1s}{k}^2\frac{1}{kr+1}\\
&-\frac{s\sqrt{3}}{\pi}\sum_{0\le j<k}\binom{\frac1s}{k}\binom{\frac1s}{j}\frac{\chi_{-3}(k-j)}{((k+j)r+2)(k-j)}+\frac{s}{4\pi}\sum_{1\le k}\binom{\frac1s}{k}^2\frac{{}_2F_1\left(\frac12,k+\frac1r+\frac12;k+\frac1r+\frac32;\frac14\right)}{(kr+1)\left(2k+1+\frac2r\right)}\\
&+\frac{s}{\pi}\sum_{0\le j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\frac1s}{k}\binom{\frac1s}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left(\frac12-h,k-h+\frac1r+\frac12;k-h+\frac1r+\frac32;\frac14\right)}{2^{k-j-2h}\left((k+j)r+2\right)\left(2k+\frac2r-2h+1\right)},
\end{aligned}
$$

where $\zeta(s,x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^s}$ *is the Hurwitz zeta-function and* ${}_2F_1(a,b;c;z)$ *is the hypergeometric function given in* (3.3.1)*.*

Note that the above result is different from (0.6.6), which corresponds to the case $r=s=1$.

We extend our analysis to a similar family, namely $(1+x)^r+y^s$. Here, the areal Mahler measure exhibits an interesting dependence on the parameter s.

Theorem 3.1.2 (see Theorem 0.6.11)**.** *Let* r,s *be positive integers. We have*

$$
\begin{aligned}
&\mathrm{m}_{\mathbb{D}}((1+x)^r+y^s)\\
&=r\left(\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)+\frac16-\frac{\sqrt{3}}{2\pi}\right)-\frac{s}{6}+\frac{s}{6}\frac{\Gamma\left(\frac{2r}{s}+2\right)}{\Gamma\left(\frac{r}{s}+2\right)^2}\\
&-\frac{s\sqrt{3}}{\pi}\sum_{0\le j<k}\binom{\frac rs}{k}\binom{\frac rs}{j}\frac{\chi_{-3}(k-j)}{(k+j+2)(k-j)}+\frac{s}{4\pi}\sum_{1\le k}\binom{\frac rs}{k}^2\frac{{}_2F_1\left(\frac12,k+\frac32;k+\frac52;\frac14\right)}{(k+1)(2k+3)}\\
&+\frac{s}{\pi}\sum_{0\le j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\frac rs}{k}\binom{\frac rs}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left(\frac12-h,k-h+\frac32;k-h+\frac52;\frac14\right)}{2^{k-j-2h}(k+j+2)(2k-2h+3)}.
\end{aligned}
$$

We also prove the following result, which investigates the limiting behavior of the areal Mahler measure under the transformation $x\mapsto x^r$ in general, as r tends to infinity.

Theorem 3.1.3 (see Theorem 0.6.13)**.** *Let $P(x_1,\dots,x_n)\in\mathbb{C}(x_1,\dots,x_n)$ be a non-zero rational function and let $P(0,x_2,\dots,x_n)\in\mathbb{C}(x_2,\dots,x_n)$ be the non-zero rational function resulting from P by setting $x_1=0$. Let r be a positive integer. Then we have*

$$\lim_{r\to\infty}\mathrm{m}_{\mathbb{D}}(P(x_1^r,x_2,\dots,x_n))=\mathrm{m}_{\mathbb{D}}(P(0,x_2,\dots,x_n)).$$

Here the elements of the transformation matrix $A=(b_{k\ell})_{1\le k,\ell\le n}$ are $b_{11}=r$, $b_{kk}=1$, and $b_{k\ell}=0$, for $2\le k\le n$, $1\le \ell\le n$, and $k\ne\ell$.

The content of this chapter is organized as follows. We start with Section 3.2, where we compute the areal Mahler measures of polynomials with two terms, namely $\mathrm{m}_{\mathbb{D}}(x^r-a)$ and $\mathrm{m}_{\mathbb{D}}(x^r+y^s)$. We review some necessary background on hypergeometric functions in Section 3.3. Theorem 3.1.2 is proven in Section 3.4, while Theorem 3.1.1 is proven in Section 3.5. The order is reversed because the proof of Theorem 3.1.1 is considerably more involved. Finally, we close this chapter with the proof of Theorem 3.1.3, which is a result of a different flavour than the others, in Section 3.6.

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3.2. The case of two terms

In this section, we consider the effect of the transformation $x\mapsto x^r$ in the simplest possible polynomials, namely those with only two monomials. For the linear case of one variable, we have the following result.

Proposition 3.2.1. *Let r be a positive integer. We have*

$$\mathrm{m}_{\mathbb{D}}(x^r - a) = \begin{cases} \log^+ |a| & |a| \geq 1, \\ \frac{r}{2}\left(|a|^{\frac{2}{r}} - 1\right) & |a| \leq 1. \end{cases}$$

Proof. Let ξ_r denote a primitive rth root of unity and let $a^{\frac{1}{r}}$ denote any rth root of a. By multiplicativity we have

$$\mathrm{m}_{\mathbb{D}}(x^r - a) = \mathrm{m}_{\mathbb{D}}\left(\prod_{j=0}^{r-1}\left(x - a^{\frac{1}{r}}\xi_r^j\right)\right) = \sum_{j=0}^{r-1} \mathrm{m}_{\mathbb{D}}\left(x - a^{\frac{1}{r}}\xi_r^j\right).$$

The conclusion follows immediately from (2.2.1), which shows that

$$\mathrm{m}_{\mathbb{D}}(x - a^{\frac{1}{r}}\xi_r^j) = \begin{cases} \frac{1}{r}\log^+ |a| & |a| \geq 1, \\ \frac{|\alpha|^{\frac{2}{r}} - 1}{2} & |a| \leq 1. \end{cases}$$

□

Now we consider the case of $x^r + y^s$.

Theorem 3.2.2. *Let r,s be positive integers. We have*

$$\mathrm{m}_{\mathbb{D}}(x^r + y^s) = -\frac{rs}{2(r+s)}.$$

Proof. By definition and by Proposition 3.2.1,

$$\begin{aligned} \mathrm{m}_{\mathbb{D}}(x^r + y^s) =& \frac{1}{\pi^2}\int_{\mathbb{D}^2} \log|x^r + y^s| dA(x) dA(y) \\ =& \frac{r}{2\pi}\int_{\mathbb{D}}\left(|y|^{\frac{2s}{r}} - 1\right) dA(y) \\ =& r\int_0^1 \left(\rho^{\frac{2s}{r}} - 1\right)\rho d\rho \\ =& -\frac{rs}{2(r+s)}. \end{aligned}$$

□

3.3. Background on hypergeometric functions

In this section, we recall some standard results of hypergeometric functions, defined in (0.2.1) by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{3.3.1}$$

(where $(a)_n$ denotes the Pochhammer symbol), which will be needed for the proofs of Theorems 3.1.1 and 3.1.2.

Theorem 3.3.1. *[Gauss Hypergeometric Theorem, Eq. 15.1.20 in* [**1**]*] Let a,b,c $\in \mathbb{C}$ such that $c \notin \mathbb{Z}_{\leq 0}$ and* $\mathrm{Re}(c-a-b) > 0$. *Then*

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Corollary 3.3.2. *Let $t > 0$. Then*

$$\sum_{0\leq k} \binom{t}{k}^2 \frac{1}{k+1} = \frac{\Gamma\left(2(t+1)\right)}{\Gamma\left(t+2\right)^2}.$$

Proof. We apply Theorem 3.3.1 with $a = b = -t$ and $c = 2$ together with the fact that $\Gamma(2) = 1$ to obtain

$$\frac{\Gamma\left(2(t+1)\right)}{\Gamma\left(t+2\right)^2} = {}_2F_1(-t,-t;2;1) = \sum_{0\leq k}\frac{(-t)_k^2}{(2)_k k!} = \sum_{0\leq k}\frac{[t(t-1)\cdots(t-k+1)]^2}{(k+1)!k!} = \sum_{0\leq k}\binom{t}{k}^2\frac{1}{k+1}.$$

□

The next theorem shows that, given certain conditions on a, b, c and z, ${}_2F_1(a,b;c;z)$ admits an integral representation.

Theorem 3.3.3. [**4**, Theorem 2.2.1] *If $|z| < 1$, $a, b, c \in \mathbb{C}^*$ with $c \notin \mathbb{Z}_{\leq 0}$ and* $\min\{\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(c-a)\} > 0$, *then we can express ${}_2F_1(a,b;c;z)$ as*

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 y^{a-1}(1-y)^{c-a-1}(1-zy)^{-b}dy, \tag{3.3.2}$$

where $\Gamma(\cdot)$ denotes the Gamma function. Here it is understood that $\arg y = \arg(1-y) = 0$, *and $(1-zy)^{-b}$ has its principal value.*

Lemma 3.3.4. *Let $\beta > -1$ be a real number and n be a non negative integer (possibly 0). Then*

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (-2\cos\theta)^{\beta}\cos(n\theta)d\theta =& (-1)^n \int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}} (2\cos\tau)^{\beta}\cos(n\tau)d\tau \\ =& \sum_{h=0}^{\lfloor \frac{n}{2}\rfloor} \binom{n}{2h} \frac{(-1)^{h+n} {}_2F_1\left(\frac{\beta+n+1}{2}-h, \frac{1}{2}-h; \frac{\beta+n+3}{2}-h; \frac{1}{4}\right)}{2^{n-2h+1}(\beta+n+1-2h)}.\end{aligned}$$

Proof. We first notice that the equality between the integrals follows from the change of variables $\theta + \pi = \tau$. We remark that $\cos(n\theta) = T_n(\cos\theta)$, where $T_n(x)$ is the Chebyshev polynomial of the first kind. By using this, together with the change of variables $t = -2\cos\theta$, we have

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (-2\cos\theta)^{\beta}\cos(n\theta)d\theta = \int_0^1 t^{\beta} T_n\left(-\frac{t}{2}\right)\frac{dt}{\sqrt{4-t^2}}. \tag{3.3.3}$$

The Chebyshev polynomials can be explicitly expressed as

$$T_n(x) = \frac{1}{2}\left[\left(x-\sqrt{x^2-1}\right)^n + \left(x+\sqrt{x^2-1}\right)^n\right] = \sum_{h=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2h}\left(x^2-1\right)^h x^{n-2h}.$$

Then, using this, we can evaluate the integral in (3.3.3) as

$$\begin{aligned}\int_0^1 t^{\beta} T_n\left(-\frac{t}{2}\right)\frac{dt}{(4-t^2)^{\frac{1}{2}}} =& \sum_{h=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2h}\int_0^1 t^{\beta}\left(\frac{t^2}{4}-1\right)^h\left(-\frac{t}{2}\right)^{n-2h}\frac{dt}{(4-t^2)^{\frac{1}{2}}} \\ =& \sum_{h=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2h}\frac{(-1)^{n+h}}{2^{n-2h+1}}\int_0^1 t^{\beta+n-2h}\left(1-\frac{t^2}{4}\right)^{h-\frac{1}{2}}dt \\ =& \sum_{h=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2h}\frac{(-1)^{n+h}{}_2F_1\left(\frac{\beta+n+1}{2}-h, \frac{1}{2}-h; \frac{\beta+n+3}{2}-h; \frac{1}{4}\right)}{2^{n-2h+1}(\beta+n+1-2h)},\end{aligned}$$

where the last identity follows from making the change $u = t^2$ and then applying equation 15.3.1 in [**1**]. □

Applications of Lemma 3.3.4 will naturally lead to evaluations of the hypergeometric function at $z = \frac{1}{4}$. Here we record two identities that will be useful for simplifying some formulas:

$${}_2F_1\left(\frac{1}{2}, \frac{3}{2}; \frac{5}{2}; \frac{1}{4}\right) = 2\pi - 3\sqrt{3} \tag{3.3.4}$$

and

$$
{}_2F_1\left(\frac{1}{2},\frac{5}{2};\frac{7}{2};\frac{1}{4}\right) = 10\pi - \frac{35\sqrt{3}}{2}. \tag{3.3.5}
$$

Equation (3.3.4) follows from the more general formula 07.23.03.2888.01 in [**63**] :

$$
{}_2F_1\left(\frac{1}{2},\frac{3}{2};\frac{5}{2};z\right) = \frac{3}{2z^{3/2}}\left(\arcsin\left(\sqrt{z}\right) - \sqrt{z(1-z)}\right),
$$

by setting $z=\frac{1}{4}$, while equation (3.3.5) follows from formula 07.23.03.2933.01 in [**64**]:

$$
{}_2F_1\left(\frac{1}{2},\frac{5}{2};\frac{7}{2};z\right) = \frac{5}{8z^{5/2}}\left(3\arcsin\left(\sqrt{z}\right) - \sqrt{z(1-z)}(3+2z)\right),
$$

by setting $z=\frac{1}{4}$.

3.4. The areal Mahler measure of $(1+x)^r + y^s$

In this section we prove Theorem 3.1.2, which is simpler than Theorem 3.1.1. To place the result in perspective, we first consider the classical case.

Lemma 3.4.1. *Let r, s be positive integers. We have*

$$
\mathrm{m}((1+x)^r + y^s) = r\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2).
$$

Proof. First notice that the left-hand side is completely independent of s, since a particular case of equation (0.6.9) implies

$$
\mathrm{m}((1+x)^r + y^s) = \mathrm{m}((1+x)^r + y).
$$

Let ξ_r denote a primitive rth root of unity. We have

$$
\mathrm{m}((1+x)^r+y) = \mathrm{m}((1+x)^r-y^r) = \mathrm{m}\left(\prod_{j=0}^{r-1}(1+x-\xi_r^j y)\right) = \sum_{j=0}^{r-1}\mathrm{m}(1+x-\xi_r^j y) = r\,\mathrm{m}(1+x+y),
$$

since $\mathrm{m}(1+x-\xi_r^j y) = \mathrm{m}(1+x+y)$ for any j. The result follows from equation (0.1.7). □

Proof of Theorem 3.1.2. Let ξ_s be a primitive s root of unity and let $\alpha = \frac{r}{s}$. When $x \in \mathbb{D}$, $(1+x)^\alpha$ is well-defined, and we can consider the principal branch of the α-th power. By multiplicativity we have

$$\mathrm{m}_{\mathbb{D}}((1+x)^r + y^s) = \mathrm{m}_{\mathbb{D}}((1+x)^r - y^s) = \sum_{j=0}^{s-1} \mathrm{m}_{\mathbb{D}}\left((1+x)^\alpha - \xi_s^j y\right),$$

where we have extended the definition of $\mathrm{m}_{\mathbb{D}}$ to the algebraic functions $(1+x)^\alpha - \xi_s^j y$ in the natural way using the integral.

By definition and by application of equation (2.2.1),

$$\begin{aligned}\mathrm{m}_{\mathbb{D}}((1+x)^\alpha - \xi_s y) =& \frac{1}{\pi^2}\int_{\mathbb{D}^2} \log|(1+x)^\alpha - \xi_s y|\, dA(y) dA(x)\\ =& \frac{\alpha}{\pi}\int_{\mathbb{D}\cap\{|1+x|\geq 1\}} \log|1+x| dA(x) + \frac{1}{2\pi}\int_{\mathbb{D}\cap\{|1+x|\leq 1\}} \left(|1+x|^{2\alpha} - 1\right) dA(x).\end{aligned} \tag{3.4.1}$$

The first integral was already computed in the proof of the case of $1+x+y$ (see Section 2.3.1) and is given by

$$\frac{\alpha}{\pi}\int_{\mathbb{D}\cap\{|1+x|\geq 1\}} \log|1+x| dA(x) = \alpha\left(\frac{3\sqrt{3}}{4\pi} L(\chi_{-3},2) + \frac{1}{6} - \frac{\sqrt{3}}{2\pi}\right).$$

We now treat the second integral in (3.4.1). Writing $x = \rho e^{i\theta}$, we have

$$\begin{aligned}&\frac{1}{2\pi}\int_{\mathbb{D}\cap\{|1+x|\leq 1\}} \left(|1+x|^{2\alpha} - 1\right) dA(x)\\ =& \frac{1}{\pi}\left[\int_{\frac{2\pi}{3}}^{\pi}\int_0^1 ((1+\rho e^{i\theta})^\alpha (1+\rho e^{-i\theta})^\alpha - 1)\rho d\rho d\theta + \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\int_0^{-2\cos\theta} ((1+\rho e^{i\theta})^\alpha (1+\rho e^{-i\theta})^\alpha - 1)\rho d\rho d\theta\right]\\ =& \frac{1}{\pi}\sum_{0\leq j,k}\binom{\alpha}{k}\binom{\alpha}{j}\left[\int_{\frac{2\pi}{3}}^{\pi}\int_0^1 \rho^{k+j+1} e^{i(k-j)\theta} d\rho d\theta + \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\int_0^{-2\cos\theta} \rho^{k+j+1} e^{i(k-j)\theta} d\rho d\theta\right]\\ &- \frac{1}{\pi}\left[\int_{\frac{2\pi}{3}}^{\pi}\int_0^1 \rho d\rho d\theta + \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\int_0^{-2\cos\theta} \rho d\rho d\theta\right]\\ =& \frac{1}{\pi}\sum_{0\leq j,k}\binom{\alpha}{k}\binom{\alpha}{j}\left[\int_{\frac{2\pi}{3}}^{\pi}\int_0^1 \rho^{k+j+1} e^{i(k-j)\theta} d\rho d\theta + \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\int_0^{-2\cos\theta} \rho^{k+j+1} e^{i(k-j)\theta} d\rho d\theta\right] \qquad (3.4.2)\\ &- \frac{1}{6} + \frac{1}{\pi}\left(\frac{\sqrt{3}}{4} - \frac{\pi}{6}\right).\end{aligned}$$

We treat the integrals in (3.4.2) separately for the cases $k=j$ and $k\neq j$. When $k=j$, we have

$$
\begin{aligned}
&\frac{1}{\pi}\sum_{0\le k}\binom{\alpha}{k}^2\left[\int_{\frac{2\pi}{3}}^{\pi}\int_0^1 \rho^{2k+1}d\rho d\theta+\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\int_0^{-2\cos\theta}\rho^{2k+1}d\rho d\theta\right]\\
=&\frac{1}{\pi}\sum_{0\le k}\binom{\alpha}{k}^2\frac{1}{2(k+1)}\left[\frac{\pi}{3}+\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}(-2\cos\theta)^{2(k+1)}d\theta\right]\\
=&\frac{\Gamma(2\alpha+2)}{6\Gamma(\alpha+2)^2}+\frac{1}{4\pi}\sum_{0\le k}\binom{\alpha}{k}^2\frac{{}_2F_1\left(\frac{1}{2},k+\frac{3}{2};k+\frac{5}{2};\frac{1}{4}\right)}{(k+1)(2k+3)}\\
=&\frac{\Gamma(2\alpha+2)}{6\Gamma(\alpha+2)^2}+\frac{1}{6}-\frac{\sqrt{3}}{4\pi}+\frac{1}{4\pi}\sum_{1\le k}\binom{\alpha}{k}^2\frac{{}_2F_1\left(\frac{1}{2},k+\frac{3}{2};k+\frac{5}{2};\frac{1}{4}\right)}{(k+1)(2k+3)} \qquad (3.4.3)
\end{aligned}
$$

where we have applied Corollary 3.3.2, Lemma 3.3.4, and equation (3.3.4).

Note that the expression in (3.4.2) is conjugated under the change $(k,j)\to(j,k)$, when $k\neq j$. Therefore, when $k\neq j$, we derive that

$$
\begin{aligned}
&\frac{1}{\pi}\sum_{\substack{0\le j,k\\ k\neq j}}\binom{\alpha}{k}\binom{\alpha}{j}\int_{\frac{2\pi}{3}}^{\pi}\int_0^1\rho^{k+j+1}e^{i(k-j)\theta}d\rho d\theta\\
=&\frac{1}{\pi}\sum_{\substack{0\le j,k\\ k\neq j}}\binom{\alpha}{k}\binom{\alpha}{j}\frac{1}{k+j+2}\int_{\frac{2\pi}{3}}^{\pi}e^{i(k-j)\theta}d\theta\\
=&\frac{2}{\pi}\sum_{0\le j<k}\binom{\alpha}{k}\binom{\alpha}{j}\frac{1}{k+j+2}\int_{\frac{2\pi}{3}}^{\pi}\cos((k-j)\theta)d\theta\\
=&-\frac{2}{\pi}\sum_{0\le j<k}\binom{\alpha}{k}\binom{\alpha}{j}\frac{1}{(k+j+2)(k-j)}\sin\left(\frac{2(k-j)\pi}{3}\right)\\
=&-\frac{\sqrt{3}}{\pi}\sum_{0\le j<k}\binom{\alpha}{k}\binom{\alpha}{j}\frac{\chi_{-3}(k-j)}{(k+j+2)(k-j)}, \qquad (3.4.4)
\end{aligned}
$$

and

$$\frac{1}{\pi}\sum_{\substack{0\leq j,k\\ k\neq j}}\binom{\alpha}{k}\binom{\alpha}{j}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\int_{0}^{-2\cos\theta}\rho^{k+j+1}e^{i(k-j)\theta}d\rho d\theta$$

$$=\frac{2}{\pi}\sum_{0\leq j<k}\binom{\alpha}{k}\binom{\alpha}{j}\frac{1}{k+j+2}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}(-2\cos\theta)^{k+j+2}\cos((k-j)\theta)d\theta$$

$$=\frac{2}{\pi}\sum_{0\leq j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\alpha}{k}\binom{\alpha}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left(\frac{1}{2}-h,k-h+\frac{3}{2};k-h+\frac{5}{2};\frac{1}{4}\right)}{2^{k-j-2h+1}(k+j+2)(2k-2h+3)},\quad(3.4.5)$$

by Lemma 3.3.4.

Combining (3.4.2), (3.4.3), (3.4.4), and (3.4.5), we obtain

$$\frac{1}{2\pi}\int_{\mathbb{D}\cap\{|1+x|\leq 1\}}(|1+x|^{2\alpha}-1)dA(x)$$

$$=\frac{\Gamma(2\alpha+2)}{6\Gamma(\alpha+2)^2}-\frac{1}{6}+\frac{1}{4\pi}\sum_{1\leq k}\binom{\alpha}{k}^2\frac{{}_2F_1\left(\frac{1}{2},k+\frac{3}{2};k+\frac{5}{2};\frac{1}{4}\right)}{(k+1)(2k+3)}-\frac{\sqrt{3}}{\pi}\sum_{0\leq j<k}\binom{\alpha}{k}\binom{\alpha}{j}\frac{\chi_{-3}(k-j)}{(k+j+2)(k-j)}$$

$$+\frac{1}{\pi}\sum_{0\leq j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\alpha}{k}\binom{\alpha}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left(\frac{1}{2}-h,k-h+\frac{3}{2};k-h+\frac{5}{2};\frac{1}{4}\right)}{2^{k-j-2h}(k+j+2)(2k-2h+3)}.$$

By recalling that $\alpha=\frac{r}{s}$, we obtain the result.

□

3.5. The areal Mahler measure of $1+x^r+y^s$

In this section we prove Theorem 3.1.1, our main result. Before proceeding to its proof, we show the following auxiliary statement.

Lemma 3.5.1. *For $r>1$, we have*

$$\sum_{1\leq k}\frac{(-1)^{k-1}\chi_{-3}(k)}{k^2(kr+2)}=\frac{3}{4}L(\chi_{-3},2)-\frac{\pi r}{6\sqrt{3}}$$
$$+\frac{r}{12}\left[\zeta\left(1,\frac{r+2}{3r}\right)-\zeta\left(1,\frac{2r+2}{3r}\right)+\zeta\left(1,\frac{r+1}{3r}\right)-\zeta\left(1,\frac{2r+1}{3r}\right)\right],$$

and for $r = 1$,

$$\sum_{1\le k} \frac{(-1)^{k-1}\chi_{-3}(k)}{k^2(k+2)} = \frac{3}{4}L(\chi_{-3},2) - \frac{\pi}{4\sqrt{3}} + \frac{1}{4}.$$

Proof. From the identity

$$\frac{1}{k^2(kr+2)} = \frac{1}{2k^2} - \frac{r}{4k} + \frac{r^2}{4(kr+2)},$$

we have that

$$\sum_{1\le k} \frac{(-1)^{k-1}\chi_{-3}(k)}{k^2(kr+2)} = \frac{1}{2}\sum_{1\le k} \frac{(-1)^{k-1}\chi_{-3}(k)}{k^2} - \frac{r}{4}\sum_{1\le k} \frac{(-1)^{k-1}\chi_{-3}(k)}{k} + \frac{r^2}{4}\sum_{1\le k} \frac{(-1)^{k-1}\chi_{-3}(k)}{kr+2}. \tag{3.5.1}$$

We consider the different terms on the right-hand side of (3.5.1). They are

$$\sum_{1\le k} \frac{(-1)^{k-1}\chi_{-3}(k)}{k^2} = \sum_{1\le k} \frac{\chi_{-3}(k)}{k^2} - 2\sum_{1\le k} \frac{\chi_{-3}(2k)}{(2k)^2} = \frac{3}{2}\sum_{1\le k} \frac{\chi_{-3}(k)}{k^2} = \frac{3}{2}L(\chi_{-3},2), \tag{3.5.2}$$

$$\sum_{1\le k} \frac{(-1)^{k-1}\chi_{-3}(k)}{k} = \sum_{1\le k} \frac{\chi_{-3}(k)}{k} - 2\sum_{1\le k} \frac{\chi_{-3}(2k)}{2k} = 2\sum_{1\le k} \frac{\chi_{-3}(k)}{k} = \frac{2\pi}{3\sqrt{3}}, \tag{3.5.3}$$

and finally

$$\begin{aligned}
\sum_{1\le k} \frac{(-1)^{k-1}\chi_{-3}(k)}{kr+2} &= \sum_{1\le k} \frac{\chi_{-3}(k)}{kr+2} - 2\sum_{1\le k} \frac{\chi_{-3}(2k)}{2kr+2} = \sum_{1\le k} \frac{\chi_{-3}(k)}{kr+2} + \sum_{1\le k} \frac{\chi_{-3}(k)}{kr+1} \\
&= \sum_{0\le j} \frac{1}{3jr+r+2} - \sum_{0\le j} \frac{1}{3jr+2r+2} + \sum_{0\le j} \frac{1}{3jr+r+1} - \sum_{0\le j} \frac{1}{3jr+2r+1} \\
&= \frac{1}{3r}\left[\zeta\left(1,\frac{r+2}{3r}\right) - \zeta\left(1,\frac{2r+2}{3r}\right) + \zeta\left(1,\frac{r+1}{3r}\right) - \zeta\left(1,\frac{2r+1}{3r}\right)\right].
\end{aligned} \tag{3.5.4}$$

By combining (3.5.2), (3.5.3), and (3.5.4) with (3.5.1), we get the result for $r > 1$.

When $r = 1$, (3.5.4) becomes

$$\frac{1}{3}\left[\zeta\left(1,\frac{2}{3}\right) - \zeta\left(1,\frac{4}{3}\right)\right] = 1 - \frac{\pi}{3\sqrt{3}}.$$

□

Proof of Theorem 3.1.1. Our goal is to calculate $\mathrm{m}_{\mathbb{D}}\left(1+x^r+y^s\right)$. Since $1+x^r+y^s = \prod_{j=0}^{s-1}\left(\sqrt[s]{1+x^r}+\xi_s^j y\right)$ and for any $k \neq \ell$ we have $\mathrm{m}_{\mathbb{D}}\left(\sqrt[s]{1+x^r}+\xi_s^k y\right) = \mathrm{m}_{\mathbb{D}}\left(\sqrt[s]{1+x^r}+\xi_s^\ell y\right)$, we can write

$$\mathrm{m}_{\mathbb{D}}\left(1+x^r+y^s\right)=\sum_{j=0}^{s-1}\mathrm{m}_{\mathbb{D}}\left(\sqrt[s]{1+x^r}+\xi_s^j y\right)=s\mathrm{m}_{\mathbb{D}}\left(\sqrt[s]{1+x^r}+y\right).$$

Here we note that the function $\sqrt[s]{1+x^r}$ is well-defined when $x\in\mathbb{D}$, and, from now on, we consider the principal branch of the s-th root.

By definition and by application of equation (2.2.1), we obtain

$$\begin{aligned}
&\mathrm{m}_{\mathbb{D}}\left(\sqrt[s]{1+x^r}+y\right)\\
=&\frac{1}{\pi^2}\int_{\mathbb{D}^2}\log\left|\sqrt[s]{1+x^r}+y\right|dA(x)dA(y)\\
=&\frac{1}{s\pi}\int_{\mathbb{D}\cap\{|1+x^r|\geq 1\}}\log|1+x^r|\,dA(x)+\frac{1}{2\pi}\int_{\mathbb{D}\cap\{|1+x^r|\leq 1\}}\left(|1+x^r|^{\frac{2}{s}}-1\right)dA(x).
\end{aligned}\tag{3.5.5}$$

For $x=\rho e^{i\theta}$ with $0\leq\rho\leq 1$ and θ defined modulo 2π, the condition $|1+x^r|\geq 1$ is equivalent to

$$1+\rho^{2r}+2\rho^r\cos(r\theta)\geq 1\Longleftrightarrow\rho^r+2\cos(r\theta)\geq 0.$$

Therefore, for $\ell\in\mathbb{Z}\cap[0,r-1]$, the condition $|1+x^r|\geq 1$ holds when $\frac{(4\ell-1)\pi}{2r}\leq\theta\leq\frac{(4\ell+1)\pi}{2r}$ and $0\leq\rho\leq 1$, and, when $\frac{(4\ell+1)\pi}{2r}\leq\theta\leq\frac{(6\ell+2)\pi}{3r}$ as well as $\frac{(6\ell-2)\pi}{3r}\leq\theta\leq\frac{(4\ell-1)\pi}{2r}$ and $\sqrt[r]{-2\cos(r\theta)}\leq\rho\leq 1$.

Similarly, for $\ell\in\mathbb{Z}\cap[0,r-1]$, the condition $|1+x^r|\leq 1$ implies that that the second integral needs to be evaluated when $\frac{(4\ell+1)\pi}{2r}\leq\theta\leq\frac{(6\ell+2)\pi}{3r}$ as well as $\frac{(6\ell+4)\pi}{3r}\leq\theta\leq\frac{(4\ell+3)\pi}{2r}$ and $0\leq\rho\leq\sqrt[r]{-2\cos(r\theta)}$, and when $\frac{(6\ell+2)\pi}{3r}\leq\theta\leq\frac{(6\ell+4)\pi}{3r}$ and $0\leq\rho\leq 1$.

We start by evaluating the first integral in (3.5.5). Following the above discussion, we have

$$\begin{aligned}
\int_{\mathbb{D}\cap\{|1+x^r|\geq 1\}}\log|1+x^r|\,dA(x)=&\sum_{\ell=0}^{r-1}\mathrm{Re}\left[\int_{\frac{(4\ell-1)\pi}{2r}}^{\frac{(4\ell+1)\pi}{2r}}\int_0^1\log\left(1+\rho^re^{ir\theta}\right)\rho d\rho d\theta\right.\\
&+\left.\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_{\sqrt[r]{-2\cos(r\theta)}}^1\log\left(1+\rho^re^{ir\theta}\right)\rho d\rho d\theta+\int_{\frac{(6\ell-2)\pi}{3r}}^{\frac{(4\ell-1)\pi}{2r}}\int_{\sqrt[r]{-2\cos(r\theta)}}^1\log\left(1+\rho^re^{ir\theta}\right)\rho d\rho d\theta\right].
\end{aligned}$$

Since $\log\left(1+\rho^r e^{ir\theta}\right) = \sum_{1\le k}(-1)^{k-1}\frac{\rho^{kr}e^{ikr\theta}}{k}$, we have

$$\begin{aligned}
\mathrm{Re}\left[\int_{\frac{(4\ell-1)\pi}{2r}}^{\frac{(4\ell+1)\pi}{2r}}\int_0^1 \log\left(1+\rho^r e^{ir\theta}\right)\rho d\rho d\theta\right] =& \mathrm{Re}\left[\sum_{1\le k}\frac{(-1)^{k-1}}{k}\int_{\frac{(4\ell-1)\pi}{2r}}^{\frac{(4\ell+1)\pi}{2r}}\int_0^1 \rho^{kr+1}e^{ikr\theta}d\rho d\theta\right]\\
=& \mathrm{Re}\left[\sum_{1\le k}\frac{(-1)^{k-1}}{k(kr+2)}\int_{\frac{(4\ell-1)\pi}{2r}}^{\frac{(4\ell+1)\pi}{2r}} e^{ikr\theta}d\theta\right]\\
=& \mathrm{Re}\left[\sum_{1\le k}\frac{(-1)^k i}{k^2r(kr+2)}\left(e^{\frac{ik(4\ell+1)\pi}{2}} - e^{\frac{ik(4\ell-1)\pi}{2}}\right)\right]\\
=& \frac{2}{r}\sum_{1\le k}\frac{1}{k^2(kr+2)}\sin\left(\frac{k\pi}{2}\right).
\end{aligned}$$

Now, using the fact that $-2\cos(r\theta) = 2\cos(r\theta+\pi)$, we have

$$\begin{aligned}
&\mathrm{Re}\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_{\sqrt[r]{2\cos(r\theta+\pi)}}^1 \log\left(1+\rho^r e^{ir\theta}\right)\rho d\rho d\theta\right]\\
=& \mathrm{Re}\left[\sum_{1\le k}\frac{(-1)^{k-1}}{k}\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_{\sqrt[r]{2\cos(r\theta+\pi)}}^1 \rho^{kr+1}e^{ikr\theta}d\rho d\theta\right]\\
=& \mathrm{Re}\left[\sum_{1\le k}\frac{(-1)^{k-1}}{k(kr+2)}\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\left(1-(2\cos(r\theta+\pi))^{k+\frac{2}{r}}\right)e^{ikr\theta}d\theta\right]\\
=& \frac{1}{r}\,\mathrm{Re}\left[\sum_{1\le k}\frac{(-1)^{k-1}}{k(kr+2)}\int_{\frac{(4\ell+3)\pi}{2}}^{\frac{(6\ell+5)\pi}{3}}\left(1-(2\cos(\tau))^{k+\frac{2}{r}}\right)e^{ik(\tau-\pi)}d\tau\right]\\
=& \frac{1}{r}\sum_{1\le k}\frac{(-1)^{k-1}}{k(kr+2)}\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}}\left(1-(2\cos(\tau))^{k+\frac{2}{r}}\right)\cos(k\tau-k\pi)d\tau\\
=& \frac{1}{r}\sum_{1\le k}\frac{(-1)^{k-1}\cos(k\pi)}{k(kr+2)}\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}}\left(1-(2\cos(\tau))^{k+\frac{2}{r}}\right)\cos(k\tau)d\tau\\
=& \frac{1}{r}\sum_{1\le k}\frac{(-1)^{k-1}\cos(k\pi)}{k^2(kr+2)}\left(\sin\left(\frac{5k\pi}{3}\right)-\sin\left(\frac{3k\pi}{2}\right)\right)\\
&-\frac{1}{r}\sum_{1\le k}\frac{(-1)^{k-1}\cos(k\pi)}{k(kr+2)}\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}}(2\cos(\tau))^{k+\frac{2}{r}}\cos(k\tau)d\tau,
\end{aligned}$$

where we have set $\tau = r\theta + \pi$. By Lemma 3.3.4, we have

$$\begin{aligned}
&\mathrm{Re}\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}} \int_{\sqrt[r]{2\cos(r\theta+\pi)}}^{1} \log\left(1+\rho^r e^{ir\theta}\right)\rho d\rho d\theta\right] \\
=&\frac{1}{r}\sum_{1\le k}\frac{1}{k^2(kr+2)}\left(\sin\left(\frac{k\pi}{3}\right)-\sin\left(\frac{k\pi}{2}\right)\right) \\
&-\frac{1}{r}\sum_{1\le k}\sum_{h=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2h}\frac{(-1)^{h-1}{}_2F_1\left(\frac{1}{2}-h,k-h+\frac{1}{r}+\frac{1}{2};k-h+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right)}{2^{k-2h+1}k(kr+2)\left(2k+\frac{2}{r}-2h+1\right)}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\mathrm{Re}\left[\int_{\frac{(6\ell-2)\pi}{3r}}^{\frac{(4\ell-1)\pi}{2r}} \int_{\sqrt[r]{-2\cos(r\theta)}}^{1} \log\left(1+\rho^r e^{ir\theta}\right)\rho d\rho d\theta\right] \\
=&\frac{1}{r}\sum_{1\le k}\frac{1}{k^2(kr+2)}\left(\sin\left(\frac{k\pi}{3}\right)-\sin\left(\frac{k\pi}{2}\right)\right) \\
&-\frac{1}{r}\sum_{1\le k}\sum_{h=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2h}\frac{(-1)^{h-1}{}_2F_1\left(\frac{1}{2}-h,k-h+\frac{1}{r}+\frac{1}{2};k-h+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right)}{2^{k-2h+1}k(kr+2)\left(2k+\frac{2}{r}-2h+1\right)}.
\end{aligned}$$

Therefore, combining the above results we obtain

$$\begin{aligned}
&\int_{\mathbb{D}\cap\{|1+x^r|\ge 1\}} \log|1+x^r|\,dA(x) \\
=&2\sum_{1\le k}\frac{\sin\left(\frac{k\pi}{3}\right)}{k^2(kr+2)}-2\sum_{1\le k}\sum_{h=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2h}\frac{(-1)^{h-1}{}_2F_1\left(\frac{1}{2}-h,k-h+\frac{1}{r}+\frac{1}{2};k-h+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right)}{2^{k-2h+1}k(kr+2)\left(2k+\frac{2}{r}-2h+1\right)} \\
=&\sqrt{3}\sum_{1\le k}\frac{(-1)^{k-1}\chi_{-3}(k)}{k^2(kr+2)}-2\sum_{1\le k}\sum_{h=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2h}\frac{(-1)^{h-1}{}_2F_1\left(\frac{1}{2}-h,k-h+\frac{1}{r}+\frac{1}{2};k-h+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right)}{2^{k-2h+1}k(kr+2)\left(2k+\frac{2}{r}-2h+1\right)}.
\end{aligned} \tag{3.5.6}$$

The second integral in (3.5.5) yields

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{D}\cap\{|1+x^r|\le 1\}} \left(|1+x^r|^{\frac{2}{s}}-1\right) dA(x)\\
=&\frac{1}{2}\sum_{\ell=0}^{r-1}\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}} \int_0^{\sqrt[r]{-2\cos(r\theta)}} \left(\left(1+\rho^r e^{ir\theta}\right)^{\frac{1}{s}}\left(1+\rho^r e^{-ir\theta}\right)^{\frac{1}{s}}-1\right)\rho d\rho d\theta\right.\\
&+\int_{\frac{(6\ell+4)\pi}{3r}}^{\frac{(4\ell+3)\pi}{2r}} \int_0^{\sqrt[r]{-2\cos(r\theta)}} \left(\left(1+\rho^r e^{ir\theta}\right)^{\frac{1}{s}}\left(1+\rho^r e^{-ir\theta}\right)^{\frac{1}{s}}-1\right)\rho d\rho d\theta\\
&\left.+\int_{\frac{(6\ell+2)\pi}{3r}}^{\frac{(6\ell+4)\pi}{3r}} \int_0^{1} \left(\left(1+\rho^r e^{ir\theta}\right)^{\frac{1}{s}}\left(1+\rho^r e^{-ir\theta}\right)^{\frac{1}{s}}-1\right)\rho d\rho d\theta\right]\\
=&\frac{1}{2}\sum_{\ell=0}^{r-1}\sum_{\substack{0\le k,j\\(k,j)\ne(0,0)}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}} \int_0^{\sqrt[r]{-2\cos(r\theta)}} \rho^{(k+j)r+1}e^{ir(k-j)\theta}d\rho d\theta\right. && (3.5.7)\\
&+\int_{\frac{(6\ell+4)\pi}{3r}}^{\frac{(4\ell+3)\pi}{2r}} \int_0^{\sqrt[r]{-2\cos(r\theta)}} \rho^{(k+j)r+1}e^{ir(k-j)\theta}d\rho d\theta && (3.5.8)\\
&\left.+\int_{\frac{(6\ell+2)\pi}{3r}}^{\frac{(6\ell+4)\pi}{3r}} \int_0^{1} \rho^{(k+j)r+1}e^{ir(k-j)\theta}d\rho d\theta\right]. && (3.5.9)
\end{aligned}
$$

For $k=j$, the inner sum for integral (3.5.9) gives

$$
\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\int_{\frac{(6\ell+2)\pi}{3r}}^{\frac{(6\ell+4)\pi}{3r}}\int_0^1 \rho^{2kr+1}d\rho d\theta = \frac{\pi}{3r}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{kr+1}. \tag{3.5.10}
$$

The combination of the inner sums of integrals (3.5.7) and (3.5.8) yields, when $k=j$,

$$\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_0^{\sqrt[r]{2\cos(r\theta+\pi)}}\rho^{2kr+1}d\rho d\theta+\int_{\frac{(6\ell+4)\pi}{3r}}^{\frac{(4\ell+3)\pi}{2r}}\int_0^{\sqrt[r]{2\cos(r\theta-\pi)}}\rho^{2kr+1}d\rho d\theta\right]$$
$$=\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{2(kr+1)}\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}(2\cos(r\theta+\pi))^{2k+\frac{2}{r}}\,d\theta+\int_{\frac{(6\ell+4)\pi}{3r}}^{\frac{(4\ell+3)\pi}{2r}}(2\cos(r\theta-\pi))^{2k+\frac{2}{r}}\,d\theta\right]$$
$$=\frac{1}{r}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{2(kr+1)}\left[\int_{\frac{(4\ell+3)\pi}{2}}^{\frac{(6\ell+5)\pi}{3}}(2\cos(\tau))^{2k+\frac{2}{r}}\,d\tau+\int_{\frac{(6\ell+1)\pi}{3}}^{\frac{(4\ell+1)\pi}{2}}(2\cos(\tau))^{2k+\frac{2}{r}}\,d\tau\right]$$
$$=\frac{1}{r}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{2(kr+1)}\left[\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}}(2\cos(\tau))^{2k+\frac{2}{r}}\,d\tau+\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(2\cos(\tau))^{2k+\frac{2}{r}}\,d\tau\right]$$
$$=\frac{1}{r}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{kr+1}\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}}(2\cos(\tau))^{2k+\frac{2}{r}}\,d\tau.$$

Applying Lemma 3.3.4, we get

$$\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_0^{\sqrt[r]{2\cos(r\theta+\pi)}}\rho^{2kr+1}d\rho d\theta+\int_{\frac{(6\ell+4)\pi}{3r}}^{\frac{(4\ell+3)\pi}{2r}}\int_0^{\sqrt[r]{2\cos(r\theta+\pi)}}\rho^{2kr+1}d\rho d\theta\right]$$
$$=\frac{1}{r}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{{}_2F_1\left(\frac{1}{2},k+\frac{1}{r}+\frac{1}{2};k+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right)}{2(kr+1)\left(2k+1+\frac{2}{r}\right)}. \tag{3.5.11}$$

We now treat the case when $k\neq j$. For the inner sum of integral (3.5.9), the $k\neq j$ case yields

$$\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\int_{\frac{(6\ell+2)\pi}{3r}}^{\frac{(6\ell+4)\pi}{3r}}\int_0^1\rho^{(k+j)r+1}e^{ir(k-j)\theta}d\rho d\theta$$
$$=\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{1}{(k+j)r+2}\int_{\frac{(6\ell+2)\pi}{3r}}^{\frac{(6\ell+4)\pi}{3r}}e^{ir(k-j)\theta}d\theta$$
$$=\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{2(-1)^{k-j}}{r((k+j)r+2)(k-j)}\sin\left(\frac{(k-j)\pi}{3}\right)$$
$$=-\frac{\sqrt{3}}{r}\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{\chi_{-3}(k-j)}{((k+j)r+2)(k-j)}. \tag{3.5.12}$$

Now, the inner sum for integral (3.5.7) in the $k \neq j$ case gives

$$\sum_{\substack{0\leq k,j\\ k\neq j}} \binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j} \int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}} \int_0^{\sqrt[r]{2\cos(r\theta+\pi)}} \rho^{r(k+j)+1} e^{ir(k-j)\theta} d\rho d\theta$$

$$= \sum_{\substack{0\leq k,j\\ k\neq j}} \binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j} \frac{1}{(k+j)r+2} \int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}} (2\cos(r\theta+\pi))^{k+j+\frac{2}{r}} e^{ir(k-j)\theta} d\theta$$

$$= \frac{1}{r} \sum_{\substack{0\leq k,j\\ k\neq j}} \binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j} \frac{1}{(k+j)r+2} \int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}} (2\cos(\tau))^{k+j+\frac{2}{r}} e^{i(k-j)(\tau-\pi)} d\tau$$

$$= \frac{1}{r} \sum_{\substack{0\leq k,j\\ k\neq j}} \binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j} \frac{(-1)^{k-j}}{(k+j)r+2} \int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}} (2\cos(\tau))^{k+j+\frac{2}{r}} e^{i(k-j)\tau} d\tau.$$

The above expression gets conjugated under the change $(k,j) \to (j,k)$. That means that it suffices to take the real part, and therefore it suffices to find

$$\frac{2}{r} \sum_{0\leq j<k} \binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j} \frac{(-1)^{k-j}}{(k+j)r+2} \int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}} (2\cos(\tau))^{k+j+\frac{2}{r}} \cos((k-j)\tau) d\tau.$$

By Lemma 3.3.4,

$$\sum_{\substack{0\leq k,j\\ k\neq j}} \binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j} \int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}} \int_0^{\sqrt[r]{2\cos(r\theta+\pi)}} \rho^{r(k+j)+1} e^{ir(k-j)\theta} d\rho d\theta$$

$$= \frac{1}{r} \sum_{0\leq j<k} \sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor} \binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\binom{k-j}{2h} \frac{(-1)^{k-j+h} {}_2F_1\left(\frac{1}{2}-h, k-h+\frac{1}{r}+\frac{1}{2}; k-h+\frac{1}{r}+\frac{3}{2}; \frac{1}{4}\right)}{2^{k-j-2h}\left((k+j)r+2\right)\left(2k+\frac{2}{r}-2h+1\right)}. \tag{3.5.13}$$

Similarly, we have

$$\sum_{\substack{0\leq k,j\\ k\neq j}} \binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j} \int_{\frac{(6\ell+4)\pi}{3r}}^{\frac{(4\ell+3)\pi}{2r}} \int_0^{\sqrt[r]{2\cos(r\theta+\pi)}} \rho^{r(k+j)+1} e^{ir(k-j)\theta} d\rho d\theta$$

$$= \frac{1}{r} \sum_{0\leq j<k} \sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor} \binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\binom{k-j}{2h} \frac{(-1)^{k-j+h} {}_2F_1\left(\frac{1}{2}-h, k-h+\frac{1}{r}+\frac{1}{2}; k-h+\frac{1}{r}+\frac{3}{2}; \frac{1}{4}\right)}{2^{k-j-2h}\left((k+j)r+2\right)\left(2k+\frac{2}{r}-2h+1\right)}. \tag{3.5.14}$$

Collecting (3.5.10) and (3.5.12), we have that the integral in (3.5.9) yields

$$\frac{1}{2}\sum_{\ell=0}^{r-1}\sum_{\substack{0\le k,j\\(k,j)\neq(0,0)}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\int_{\frac{(6\ell+2)\pi}{3r}}^{\frac{(6\ell+4)\pi}{3r}}\int_0^1\rho^{(k+j)r+1}e^{ir(k-j)\theta}d\rho d\theta$$

$$=\frac{\pi}{6}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{kr+1}-\frac{\sqrt{3}}{2}\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{\chi_{-3}(k-j)}{((k+j)r+2)(k-j)}.$$

Combining (3.5.11), (3.5.13) and (3.5.14), we derive that the integrals in (3.5.7) and (3.5.8) yield

$$\frac{1}{2}\sum_{\ell=0}^{r-1}\sum_{\substack{0\le k,j\\(k,j)\neq(0,0)}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_0^{\sqrt[r]{-2\cos(r\theta)}}\rho^{(k+j)r+1}e^{ir(k-j)\theta}d\rho d\theta\right.$$

$$\left.+\int_{\frac{(6\ell+4)\pi}{3r}}^{\frac{(4\ell+3)\pi}{2r}}\int_0^{\sqrt[r]{-2\cos(r\theta)}}\rho^{(k+j)r+1}e^{ir(k-j)\theta}d\rho d\theta\right]$$

$$=\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{{}_2F_1\left(\frac{1}{2},k+\frac{1}{r}+\frac{1}{2};k+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right)}{4(kr+1)\left(2k+1+\frac{2}{r}\right)}$$

$$+\sum_{0\le j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left(\frac{1}{2}-h,k-h+\frac{1}{r}+\frac{1}{2};k-h+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right)}{2^{k-j-2h}\left((k+j)r+2\right)\left(2k+\frac{2}{r}-2h+1\right)}.$$

Therefore,

$$\frac{1}{2}\int_{\mathbb{D}\cap\{|1+x^r|\le 1\}}\left(|1+x^r|^{\frac{2}{s}}-1\right)dA(x)$$

$$=\frac{\pi}{6}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{kr+1}-\frac{\sqrt{3}}{2}\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{\chi_{-3}(k-j)}{((k+j)r+2)(k-j)}$$

$$+\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{{}_2F_1\left(\frac{1}{2},k+\frac{1}{r}+\frac{1}{2};k+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right)}{4(kr+1)\left(2k+1+\frac{2}{r}\right)}$$

$$+\sum_{0\le j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left(\frac{1}{2}-h,\frac{1}{2}-h+k+\frac{1}{r};\frac{3}{2}-h+k+\frac{1}{r};\frac{1}{4}\right)}{2^{k-j-2h}\left((k+j)r+2\right)\left(2k+\frac{2}{r}-2h+1\right)}.$$

By combining the above with the result of (3.5.6) and Lemma 3.5.1 in equation (3.5.5) we conclude the proof of the statement. □

3.6. A limiting property for the areal Mahler measure

In this section we prove Theorem 3.1.3, which sheds light on how the change of variables $x \mapsto x^r$ interacts with the areal Mahler measure as $r \to \infty$ in generality.

Proof of Theorem 3.1.3. Without loss of generality, we can consider the polynomial case. Let

$$P(x_1, \dots ,x_n) = \sum_{m_1,\dots,m_n \geq 0} c_{m_1,\dots,m_n} x_1^{m_1} \cdots x_n^{m_n} \in \mathbb{C}[x_1, \dots ,x_n]$$

be a non-zero polynomial and recall that $P(0,x_2, \dots ,x_n)$ denotes the polynomial resulting from P by setting $x_1 = 0$. Given $0 \leq R < 1$, let $\mathbb{D}_R$ denote the disk at the origin of radius R. We have

$$\begin{aligned}
&\lim_{r\to\infty} \int_{\mathbb{D}^{n-1}} \int_{\mathbb{D}_R} \log |P(x_1^r,x_2, \dots , x_n)|\, dA(x_1) \cdots dA(x_n)\\
&= \lim_{r\to\infty} \int_{\mathbb{D}^{n-1}} \int_{\mathbb{D}_R} \log \left| \sum_{m_1,\dots,m_n \geq 0} c_{m_1,\dots,m_n} x_1^{rm_1} \cdots x_n^{m_n} \right| dA(x_1) \cdots dA(x_n)\\
&= \int_{\mathbb{D}^{n-1}} \int_0^{2\pi} \int_0^R \lim_{r\to\infty} \log \left| \sum_{m_1,\dots,m_n \geq 0} c_{m_1,\dots,m_n} \rho^{rm_1} e^{irm_1\theta} x_2^{m_2} \cdots x_n^{m_n} \right| \rho d\rho d\theta dA(x_2) \cdots dA(x_n)\\
&=\pi R^2 \int_{\mathbb{D}^{n-1}} \log \left| \sum_{m_1=0,m_2,\dots,m_n \geq 0} c_{0,m_2,\dots,m_n} x_2^{m_2} \cdots x_n^{m_n} \right| dA(x_2) \cdots dA(x_n)\\
&=\pi R^2 \int_{\mathbb{D}^{n-1}} \log |P(0,x_2, \dots , x_n)|\, dA(x_2) \cdots dA(x_n) = \pi^n R^2 \mathrm{m}_{\mathbb{D}} \left(P(0,x_2, \dots , x_n)\right),
\end{aligned}$$

where the exchanges between integrals and limits follow from the fact that the integrand is bounded above by $\log \left(\sum_{m_1,\dots,m_n \geq 0} |c_{m_1,\dots,m_n}| \right)$. Then

$$\lim_{r\to\infty} \mathrm{m}_{\mathbb{D}} \left(P\left(x_1^r,x_2, \dots , x_n\right)\right) = \lim_{R\to 1^-} R^2 \mathrm{m}_{\mathbb{D}} \left(P\left(0,x_2, \dots , x_n\right)\right) = \mathrm{m}_{\mathbb{D}} \left(P\left(0,x_2, \dots , x_n\right)\right).$$

This concludes the proof of Theorem 3.1.3. □

3.7. Conclusion

In this chapter, we have explored how the areal Mahler measure varies under the change of variables $x \mapsto x^r$, where r is a positive integer. This change of variables does not affect the

r	s	$m_{\mathbb{D}}$	r	s	$m_{\mathbb{D}}$	r	s	$m_{\mathbb{D}}$	r	s	$m_{\mathbb{D}}$
1	1	0.111	2	1	0.074	5	1	0.037	10	1	0.020
1	2	0.074	2	2	0.049	5	2	0.024	10	2	0.013
1	3	0.056	2	3	0.036	5	3	0.018	10	3	0.010
1	4	0.045	2	4	0.029	5	4	0.014	10	4	0.008
1	5	0.037	2	5	0.024	5	5	0.011	10	5	0.006
1	10	0.020	2	10	0.013	5	10	0.006	10	10	0.003
1	20	0.011	2	20	0.006	5	20	0.003	10	20	0.002

Table 1 – Values of $m_{\mathbb{D}}(1 + x^r + y^s)$ given by Theorem 3.1.1.

standard Mahler measure and therefore represents a clear distinction between the standard definition and the areal version.

While it would be difficult to explore the result of these limits directly from the formulas given in Theorems 3.1.1 and 3.1.2, one can see Theorem 3.1.3 in action by doing some numerical experiments. This is illustrated in Table 1, where the values $m_{\mathbb{D}}(1 + x^r + y^r)$ are listed for some choices of r and s. We see, first of all, the symmetry resulting from exchanging r and s, and we also see that the value of $m_{\mathbb{D}}(1 + x^r + y^r)$ approaches zero when r or s grow, as they approach $m_{\mathbb{D}}(1 + y^s) = 0$ or $m_{\mathbb{D}}(1 + x^r) = 0$ respectively.

Similarly Table 2 illustrates the values of $m_{\mathbb{D}}((1 + x)^r + y^s)$ for some choices of r and s. We see again that as s grows, the value of $m_{\mathbb{D}}((1 + x)^r + y^s)$ approaches zero, the value of $m_{\mathbb{D}}((1 + x)^r)$. The table also shows that the value of $m_{\mathbb{D}}((1 + x)^r + y^s)$ grows when r grows. Presumably, the areal Mahler measure is multiplied by r.

It would be interesting to understand these phenomena in full generality, including characterizing the difference between $m_{\mathbb{D}}(P)$ and $m_{\mathbb{D}}(P^{(A)})$ for A an $n \times n$ integer matrix with non-zero discriminant as in (0.6.9).

r	s	$\mathrm{m}_{\mathbb{D}}$	r	s	$\mathrm{m}_{\mathbb{D}}$	r	s	$\mathrm{m}_{\mathbb{D}}$
1	1	0.11069	2	1	0.29242	10	1	1.96069
1	2	0.07440	2	2	0.22139	10	2	1.80754
1	3	0.05600	2	3	0.17800	10	3	1.67597
1	4	0.04490	2	4	0.14880	10	4	1.56188
1	5	0.03746	2	5	0.12781	10	5	1.46209
1	10	0.02050	2	10	0.07493	10	10	1.10694
1	10^2	0.00224	2	10^2	0.00886	10	10^2	0.20495
1	10^3	0.00023	2	10^3	0.00090	10	10^3	0.02239

Table 2 – Values of $\mathrm{m}_{\mathbb{D}}((1+x)^r + y^s)$ given by Theorem 3.1.2.

Chapter 4

The Mahler measure of an n-variable family with non-linear degree

In this chapter, we investigate the Mahler measure of a particular family of rational functions with an arbitrary number of variables and an arbitrary degree in one of the variables, generalizing previous results for families of an arbitrary number of variables but linear dependence in each variable obtained in [**75**]. The results are based on a joint work with Lalín and Nair [**81**].

4.1. A brief description of the results

In [**74, 75**], among other families of rational functions, Lalín considered the Mahler measures of the following family:

$$S_n(x_1,\ldots,x_n,x,y,z) := (1+x)z + \left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)(1+y),$$

and evaluated their Mahler measures in terms of special ζ-values and Dirichlet L-values (see (0.7.2) and (0.7.3) for the explicit expressions). These types of examples represent important evidence towards understanding the relationship between Mahler measure and regulators

In this chapter, our aim is to evaluate the Mahler measure of the following generalization of the family S_n:

$$S_{n,r}(x_1,\ldots,x_n,x,y,z) := (1+x)z + \left[\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)\right]^r(1+y).$$

We prove the following result.

Theorem 4.1.1 (see Theorem 0.7.1)**.** *Let $r \geq 1$. For $k \geq 1$, we have*

$$\mathrm{m}(S_{2k,r}) = \sum_{h=1}^{k} \frac{s_{k-h}(2^2,4^2,\ldots,(2k-2)^2)}{(2k-1)!} \left(\frac{2}{\pi}\right)^{2h} \mathcal{C}_r(h),$$

where

$$\begin{aligned}\mathcal{C}_r(h) :=& r(2h)!\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1)\\ &+\frac{r^2(2h-1)!}{\pi^2}\Bigg\{\frac{(-1)^{h+1}7B_{2h}\pi^{2h}}{2r^2(2h)!}\zeta(3)\left(2^{2h-1}+(-1)^r2^{2h-1}+(-1)^{r+1}\right)\\ &+(2h+2)(2h+1)\frac{1-2^{-2h-3}}{r^{2h+2}}(1-(-1)^r)\zeta(2h+3)\\ &-\sum_{\ell=0}^{2r-1}(-1)^\ell\Bigg[\sum_{t=2}^{2h+2}\Bigg(\frac{(t-1)(t-2)}{2}(-1)^t\left(\mathrm{Li}_t(\xi_{2r}^\ell)-\mathrm{Li}_t(-\xi_{2r}^\ell)\right)\\ &-\binom{t-1}{2h-1}(2-2^{1-t})\zeta(t)\Bigg)\frac{(2\pi i)^{2h+3-t}}{(2h+3-t)!}B_{2h+3-t}\left(\frac{\ell}{2r}\right)\Bigg]\Bigg\}.\end{aligned}$$

For $k \geq 0$, we have

$$\mathrm{m}(S_{2k+1,r}) = \sum_{h=0}^{k} \frac{s_{k-h}(1^2,3^2,\ldots,(2k-1)^2)}{(2k)!} \left(\frac{2}{\pi}\right)^{2h+1} \mathcal{D}_r(h),$$

where

$$
\begin{aligned}
\mathcal{D}_r(h) :=& r(2h+1)!L(\chi_{-4},2h+2)\\
&+\frac{2ir^2(2h)!}{\pi^2}\Bigg\{\frac{(-1)^{h+1}(2^{2h+4}-1)B_{2h+4}\pi^{2h+4}}{r^{2h+3}(2h+4)!}\\
&-i\frac{(-1)^hE_{2h}\pi^{2h+1}}{r^2 2^{2h}(2h)!}\left(\mathrm{Li}_3((-i)^r)-\frac{1}{8}\mathrm{Li}_3((-1)^r)\right)\\
&+(2h+3)(2h+2)\frac{1}{r^{2h+3}}\left(\mathrm{Li}_{2h+4}((-i)^r)-\frac{1}{2^{2h+4}}\mathrm{Li}_{2h+4}((-1)^r)\right)\\
&+\sum_{\ell=0}^{2r-1}(-1)^\ell\left[\sum_{t=1}^{2h+3}\left(\frac{(t-1)(t-2)}{2}(-1)^t\mathrm{Li}_t(-i\xi_{2r}^\ell)+\binom{t-1}{2h}\mathrm{Li}_t(-i)\right)\right.\\
&\left.\times\frac{(2\pi i)^{2h+4-t}}{(2h+4-t)!}B_{2h+4-t}\left(\frac{\ell}{2r}\right)\right]\Bigg\}.
\end{aligned}
$$

In the above formulas, ξ_{2r} denotes a primitive $2r$-root of unity, $\mathrm{Li}_\ell(z)$ denotes the ℓ-th polylogarithm (see Section 0.4.1), and $B_n(t)$ denotes the Bernoulli polynomial, as defined in (0.4.12).

The proof of Theorem 4.1.1 relies on similar recursive strategies as used in the proofs of the previous results from [**75, 93**] discussed in Section 0.7. For Theorem 4.1.1 we introduce a clever application of partial fractions that allows us to write the Mahler measure in terms of hyperlogarithms evaluated at the roots of unity. This new idea allows us to make the important transition from the previous results at $r = 1$ to the more general case of arbitrary r. These hyperlogarithms give rise to multiple polylogarithms that can then be reduced to length-one polylogarithms.

Chapter 4 is organized as follows. Section 4.2 presents some preliminary results on evaluating certain necessary integrals that where proven in previous work ([**74, 75, 78**]). The derivations of Lemmas 0.4.6 and 0.4.7 are given in Section 4.3. The proof of Theorem 4.1.1 is given in Sections 4.4 and 4.5. More precisely, Section 4.4 describes the iterative process that leads to the Mahler measure being expressed in terms of integrals that can be related to hyperlogarithms, while these integrals are evaluated in Section 4.5. Discussions of the case $r = 2$ and of the cases $n = 1$ and $r = 3,4$ are included in Section 4.6.

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4.2. Some preliminary results

The goal of this section is to state some results concerning the evaluation of certain integrals that were proven in [**74, 75, 78**] and that are necessary for the proof of Theorem 4.1.1.

Let $P_\alpha(y,w,z) = 1 + y + \alpha(1+w)z$. The Mahler measure of this polynomial was initially computed by Smyth [**27, 113**]. We state here a version given in [**74**, Theorem 17].

Theorem 4.2.1.

$$\pi^2 \mathrm{m}(1+y+\alpha(1+w)z) = \begin{cases} 2\mathcal{L}_3(|\alpha|) & \text{for} \quad |\alpha| \leq 1, \\ \\ \pi^2 \log|\alpha| + 2\mathcal{L}_3(|\alpha|^{-1}) & \text{for} \quad |\alpha| > 1, \end{cases}$$

where, for $\beta > 0$*,*

$$\mathcal{L}_3(\beta) = -\frac{2}{\beta}\int_0^1 \frac{dt}{t^2 - \frac{1}{\beta^2}} \circ \frac{dt}{t} \circ \frac{dt}{t} := -\frac{2}{\beta}\int_{0 \leq t_1 \leq t_2 \leq t_3 \leq 1} \frac{dt_1}{t_1^2 - \frac{1}{\beta^2}} \frac{dt_2}{t_2} \frac{dt_3}{t_3}.$$

The following proposition allows us to compute an integral that will be key for the iterative process leading to Theorem 4.1.1.

Proposition 4.2.2. [**75**, Proposition 5]*,* [**78**, Proposition 5.5] *Let* $a, b > 0$ *and* $k \in \mathbb{Z}_{\geq 0}$*. We have*

$$\int_0^\infty \frac{x \log^k x dx}{(x^2+a^2)(x^2+b^2)} = \left(\frac{\pi}{2}\right)^{k+1} \frac{A_k\left(\frac{2\log a}{\pi}\right) - A_k\left(\frac{2\log b}{\pi}\right)}{a^2 - b^2},$$

where the $A_k(x)$ are polynomials in $\mathbb{Q}[x]$ given by

$$R(T;x) = \frac{e^{xT}-1}{\sin T} = \sum_{k\geq 0} A_k(x)\frac{T^k}{k!}.$$

Remark 4.2.3. *The polynomials $A_k(x)$ satisfy the following recurrence.*

$$A_k(x) = \frac{x^{k+1}}{k+1} + \frac{1}{k+1}\sum_{\substack{j>1\\ \text{odd}}}^{k+1}(-1)^{\frac{j+1}{2}}\binom{k+1}{j}A_{k+1-j}(x),$$

and can be explicitly given by

$$A_k(x) = -\frac{2}{k+1}\sum_{h=0}^{k} B_h\binom{k+1}{h}(2^{h-1}-1)i^h x^{k+1-h},$$

where the B_n are the Bernoulli numbers. (See the Appendix to [**75**] *and* [**78**, Lemma 5.2]*.)*

4.3. Integrals and polylogarithms

To understand how special values of zeta functions and L-series arise in our formulas, we derive Lemmas 0.4.6 and 0.4.7 in this section.

A proof of Lemma 0.4.6 can be found in [**75**, Lemma 9]. We include it here to familiarize the reader with the proof strategy, which we adapt frequently throughout this chapter to establish our results.

Proof of Lemma 0.4.6. We derive the equality in (0.4.18) below. A similar argument applies to obtain (0.4.17), which is provided in the proof of Lemma 9 in [**75**].

Our goal is to translate the integral into hyperlogarithms.

$$\int_0^1 \log^j x \frac{dx}{x^2+1} = \frac{(-1)^j j!}{2i}\int_0^1 \left(\frac{1}{x-i} - \frac{1}{x+i}\right) dx \circ \underbrace{\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{j\ \text{times}},$$

where we use the fact that $\int_x^1 \frac{dt}{t} = -\log x$. Since $\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}$ are ordered in the above hyperlogarithm integral, we have the factor $j!$ in the left-hand side as the number of possible permutation of the variable t. Therefore, we have

$$\frac{(-1)^j j!}{2i}\int_0^1\left(\frac{1}{x-i}-\frac{1}{x+i}\right)dx\circ\underbrace{\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{j\ \text{times}}=\frac{(-1)^j j!}{2i}\left[\mathrm{Li}_{j+1}(i)-\mathrm{Li}_{j+1}(-i)\right]$$

$$=(-1)^j j! L\left(\chi_{-4}, j+1\right),$$

where the last equality follows from (0.4.11). □

We now prove Lemma 0.4.7, which allows us to express the sums polylogarithms at roots of unity in terms of special zeta values.

Proof of Lemma 0.4.7. Indeed, we have

$$\begin{aligned}\sum_{\ell=0}^{2r-1}(-1)^\ell \mathrm{Li}_h(\xi_{2r}^\ell) =&\sum_{n=1}^\infty\sum_{\ell=0}^{2r-1}\frac{(-1)^\ell\xi_{2r}^{\ell n}}{n^h}=\sum_{n=1}^\infty\sum_{\ell=0}^{2r-1}\frac{(\xi_{2r}^{n+r})^\ell}{n^h}\\ =&2r\sum_{\substack{n=1\\ n\equiv r \bmod 2r}}^\infty\frac{1}{n^h}=\frac{2r}{r^h}\sum_{j=0}^\infty\frac{1}{(2j+1)^h}\\ =&\frac{2(1-2^{-h})}{r^{h-1}}\zeta(h).\end{aligned}$$

The proof of (0.4.19) is similar. We also have

$$\begin{aligned}\sum_{\ell=0}^{2r-1}(-1)^\ell \mathrm{Li}_h(-i\xi_{2r}^\ell) =&\sum_{n=1}^\infty\sum_{\ell=0}^{2r-1}\frac{(-1)^\ell(-i\xi_{2r}^{\ell})^n}{n^h}=\sum_{n=1}^\infty\frac{(-i)^n}{n^h}\sum_{\ell=0}^{2r-1}(\xi_{2r}^{n+r})^\ell\\ =&2r\sum_{\substack{n=1\\ n\equiv r \bmod 2r}}^\infty\frac{(-i)^n}{n^h}=\frac{2r}{r^h}\sum_{j=0}^\infty\frac{(-i)^{(2j+1)r}}{(2j+1)^h}\\ =&\frac{2}{r^{h-1}}\left(\mathrm{Li}_h((-i)^r)-2^{-h}\mathrm{Li}_h((-1)^r)\right).\end{aligned}$$

□

4.4. General set-up

We start by first describing a general setting that could be applied to various rational functions. Then we will specialize this setting in the particular polynomial from the statement.

Let $P_\alpha \in \mathbb{C}(\mathbf{x})$ be a non-zero rational function such that its coefficients depend (as rational functions) on a parameter $\alpha \in \mathbb{C}$. We replace α by $\left[\left(\frac{x_1-1}{x_1+1}\right)\cdots\left(\frac{x_n-1}{x_n+1}\right)\right]^r$ and obtain a new rational function $\tilde{P} \in \mathbb{C}(\mathbf{x}, x_1, \ldots, x_n)$. By definition of the Mahler measure, one can see that

$$\mathrm{m}(\tilde{P}) = \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n} \mathrm{m}\left(P_{\left[\left(\frac{x_1-1}{x_1+1}\right)\cdots\left(\frac{x_n-1}{x_n+1}\right)\right]^r}\right)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.$$

We first apply a change of variables to polar coordinates, $x_j = e^{i\theta_j}$:

$$= \frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \mathrm{m}\left(P_{\left[i^n \tan\left(\frac{\theta_1}{2}\right)\cdots\tan\left(\frac{\theta_n}{2}\right)\right]^r}\right) d\theta_1\ldots d\theta_n.$$

Now let $y_i = \tan\left(\frac{\theta_i}{2}\right)$. We get,

$$\begin{aligned}
=&\frac{1}{\pi^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathrm{m}\left(P_{(i^n y_1\cdots y_n)^r}\right)\frac{dy_1}{y_1^2+1}\cdots\frac{dy_n}{y_n^2+1}\\
=&\frac{2^{n-1}}{\pi^n}\int_{0}^{\infty}\cdots\int_{0}^{\infty} \mathrm{m}\left(P_{(i^n y_1\cdots y_n)^r}\right)\frac{dy_1}{y_1^2+1}\cdots\frac{dy_n}{y_n^2+1}\\
&+\frac{2^{n-1}}{\pi^n}\int_{0}^{\infty}\cdots\int_{0}^{\infty} \mathrm{m}\left(P_{(-i^n y_1\cdots y_n)^r}\right)\frac{dy_1}{y_1^2+1}\cdots\frac{dy_n}{y_n^2+1}.
\end{aligned}$$

By making one more change of variables, $\hat{x}_1 = y_1, \ldots, \hat{x}_{n-1} = y_1\cdots y_{n-1}, \hat{x}_n = y_1\cdots y_n$, we finally obtain

$$\begin{aligned}
=&\frac{2^{n-1}}{\pi^n}\int_{0}^{\infty}\cdots\int_{0}^{\infty} \mathrm{m}\left(P_{(i^n\hat{x}_n)^r}\right)\frac{\hat{x}_1 d\hat{x}_1}{\hat{x}_1^2+1}\frac{\hat{x}_2 d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2}\cdots\frac{\hat{x}_{n-1}d\hat{x}_{n-1}}{\hat{x}_{n-1}^2+\hat{x}_{n-2}^2}\frac{d\hat{x}_n}{\hat{x}_n^2+\hat{x}_{n-1}^2}\\
&+\frac{2^{n-1}}{\pi^n}\int_{0}^{\infty}\cdots\int_{0}^{\infty} \mathrm{m}\left(P_{(-i^n\hat{x}_n)^r}\right)\frac{\hat{x}_1 d\hat{x}_1}{\hat{x}_1^2+1}\frac{\hat{x}_2 d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2}\cdots\frac{\hat{x}_{n-1}d\hat{x}_{n-1}}{\hat{x}_{n-1}^2+\hat{x}_{n-2}^2}\frac{d\hat{x}_n}{\hat{x}_n^2+\hat{x}_{n-1}^2}.
\end{aligned}$$

Thus, to obtain our final formula, we need to compute this integral.

By iterating Proposition 4.2.2, the above integral can be written as a linear combination, with coefficients that are rational numbers and powers of π in such a way that the weights are homogeneous, of integrals of the form

$$\int_0^{\infty} \mathrm{m}\left(P_{(i^n x)^r}\right)\log^j x\frac{dx}{x^2\pm 1} + \int_0^{\infty} \mathrm{m}\left(P_{(-i^n x)^r}\right)\log^j x\frac{dx}{x^2\pm 1}. \qquad (4.4.1)$$

One can see that j is even if and only if n is odd and the corresponding sign in that case is "+". This leads to the following construction.

Definition 4.4.1. [**75**, Definition15] *Let $a_{k,j} \in \mathbb{Q}$ be defined for $k \geq 1$, $n = 2k$ and $j = 0, \dots, k-1$ by*

$$\int_0^\infty \cdots \int_0^\infty \mathrm{m}\left(P_{(\pm i^n \hat{x}_n)^r}\right) \frac{\hat{x}_1 d\hat{x}_1}{\hat{x}_1^2+1} \frac{\hat{x}_2 d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2} \cdots \frac{\hat{x}_{n-1} d\hat{x}_{n-1}}{\hat{x}_{n-1}^2+\hat{x}_{n-2}^2} \frac{d\hat{x}_n}{\hat{x}_n^2+\hat{x}_{n-1}^2}$$
$$= \sum_{h=1}^{k} a_{k,h-1} \left(\frac{\pi}{2}\right)^{2k-2h} \int_0^\infty \mathrm{m}\left(P_{(\pm i^n x)^r}\right) \log^{2h-1} x \frac{dx}{x^2-1}.$$

Let $b_{k,j} \in \mathbb{Q}$ be defined for $k \geq 0$, $n = 2k+1$ and $j = 0, \dots, k$ by

$$\int_0^\infty \cdots \int_0^\infty \mathrm{m}\left(P_{(\pm i^n \hat{x}_n)^r}\right) \frac{\hat{x}_1 d\hat{x}_1}{\hat{x}_1^2+1} \frac{\hat{x}_2 d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2} \cdots \frac{\hat{x}_{n-1} d\hat{x}_{n-1}}{\hat{x}_{n-1}^2+\hat{x}_{n-2}^2} \frac{d\hat{x}_n}{\hat{x}_n^2+\hat{x}_{n-1}^2}$$
$$= \sum_{h=0}^{k} b_{k,h} \left(\frac{\pi}{2}\right)^{2k-2h} \int_0^\infty \mathrm{m}\left(P_{(\pm i^n x)^r}\right) \log^{2h} x \frac{dx}{x^2+1}.$$

The following result is proven in [**75**].

Theorem 4.4.2. [**75**, Theorem 17] *For $k \geq 1$ and $h = 0, \dots, k-1$, we have*

$$a_{k,h} = \frac{s_{k-1-h}(2^2, \dots, (2k-2)^2)}{(2k-1)!}.$$

For $k \geq 0$ and $h = 0, \dots, k$, we have

$$b_{k,h} = \frac{s_{k-h}(1^2, \dots, (2k-1)^2)}{(2k)!},$$

where we recall that the symmetric polynomials are given by (0.7.1)*.*

It remains to evaluate the integrals of the type (4.4.1).

4.5. Integral reduction

In this section, we focus on evaluating the integral

$$\mathcal{I}_{r,j} := \int_0^\infty \mathrm{m}\left(P_{(i^n x)^r}\right) \log^j x \frac{dx}{x^2+(-1)^j}$$

for the polynomial $P_\alpha = 1 + y + \alpha(1+w)z$ and we deduce our main result. Notice that in this case the Mahler measure is independent of the complex argument of α, and it therefore suffices to evaluate $\mathrm{m}(P_{x^r})$. We have the following result.

Proposition 4.5.1. *Let $P_\alpha = 1 + y + \alpha(1+w)z$. When $h \geq 0$ we have*

$$\begin{aligned}
\mathcal{I}_{r,2h} =& \frac{2ir^2(2h)!}{\pi^2}\Bigg\{\frac{(-1)^{h+1}(2^{2h+4}-1)B_{2h+4}\pi^{2h+4}}{r^{2h+3}(2h+4)!} - i\frac{(-1)^h E_{2h}\pi^{2h+1}}{r^2 2^{2h}(2h)!}\left(\mathrm{Li}_3((-i)^r) - \frac{1}{8}\mathrm{Li}_3((-1)^r)\right)\\
&+ (2h+3)(2h+2)\frac{1}{r^{2h+3}}\left(\mathrm{Li}_{2h+4}((-i)^r) - \frac{1}{2^{2h+4}}\mathrm{Li}_{2h+4}((-1)^r)\right)\\
&+ \sum_{\ell=0}^{2r-1}(-1)^\ell\Bigg[\sum_{t=1}^{2h+3}\left(\frac{(t-1)(t-2)}{2}(-1)^t\mathrm{Li}_t(-i\xi_{2r}^\ell) + \binom{t-1}{2h}\mathrm{Li}_t(-i)\right)\\
&\times \frac{(2\pi i)^{2h+4-t}}{(2h+4-t)!}B_{2h+4-t}\left(\frac{\ell}{2r}\right)\Bigg]\Bigg\} + r(2h+1)!L(\chi_{-4}, 2h+2).
\end{aligned}$$

When $h \geq 1$ we have

$$\begin{aligned}
\mathcal{I}_{r,2h-1} =& \frac{r^2(2h-1)!}{\pi^2}\Bigg\{\frac{(-1)^{h+1}7B_{2h}\pi^{2h}}{2r^2(2h)!}\zeta(3)\left(2^{2h-1} + (-1)^r 2^{2h-1} + (-1)^{r+1}\right)\\
&+ (2h+2)(2h+1)\frac{1-2^{-2h-3}}{r^{2h+2}}(1-(-1)^r)\zeta(2h+3)\\
&- \sum_{\ell=0}^{2r-1}(-1)^\ell\Bigg[\sum_{t=2}^{2h+2}\Bigg(\frac{(t-1)(t-2)}{2}(-1)^t\left(\mathrm{Li}_t(\xi_{2r}^\ell) - \mathrm{Li}_t(-\xi_{2r}^\ell)\right)\\
&- \binom{t-1}{2h-1}(2-2^{1-t})\zeta(t)\Bigg)\frac{(2\pi i)^{2h+3-t}}{(2h+3-t)!}B_{2h+3-t}\left(\frac{\ell}{2r}\right)\Bigg]\Bigg\}\\
&+ r(2h)!\left(1 - \frac{1}{2^{2h+1}}\right)\zeta(2h+1).
\end{aligned}$$

Proof. We start by splitting the integral according to $0 \leq x \leq 1$ and $1 \leq x$.

$$\mathcal{I}_{r,j} = \int_0^1 \mathrm{m}\left(P_{x^r}\right)\log^j x \frac{dx}{x^2 + (-1)^j} + \int_1^\infty \mathrm{m}\left(P_{x^r}\right)\log^j x \frac{dx}{x^2+(-1)^j}.$$

By applying Theorem 4.2.1, we obtain

$$\begin{aligned}
\mathcal{I}_{r,j} =& \int_0^1\left(-\frac{4}{x^r\pi^2}\right)\int_0^1 \frac{dt}{t^2 - \frac{1}{x^{2r}}}\circ\frac{dt}{t}\circ\frac{dt}{t}\frac{\log^j x dx}{x^2+(-1)^j}\\
&+ \int_1^\infty\left(\log\left(x^r\right) + \left(-\frac{4x^r}{\pi^2}\right)\int_0^1\frac{dt}{t^2-x^{2r}}\circ\frac{dt}{t}\circ\frac{dt}{t}\right)\frac{\log^j x dx}{x^2+(-1)^j}.
\end{aligned}$$

Denoting the t-variables by $0 \leq t_1 \leq t_2 \leq t_3 \leq 1$, we consider the following changes of variables. For the first term above, we let

$$t_1 = \frac{s_1^r}{x^r}, \quad t_2 = \frac{s_2^r}{x^r}, \quad t_3 = \frac{s_3^r}{x^r},$$

and for the second term we let

$$t_1 = \frac{x^r}{s_1^r}, \quad t_2 = \frac{x^r}{s_2^r}, \quad t_3 = \frac{x^r}{s_3^r}.$$

This leads to

$$\begin{aligned}\mathcal{I}_{r,j} = &-\frac{4}{\pi^2}\int_0^1 \frac{rs^{r-1}ds}{s^{2r}-1} \circ \frac{rds}{s} \circ \frac{rds}{s} \circ \frac{\log^j x dx}{x^2+(-1)^j} \\ &+ r\int_1^\infty \frac{\log^{j+1} x dx}{x^2+(-1)^j} - \frac{4}{\pi^2}\int_1^\infty \frac{\log^j x dx}{x^2+(-1)^j} \circ \frac{(-r)ds}{s} \circ \frac{(-r)ds}{s} \circ \frac{(-r)s^{r-1}ds}{1-s^{2r}}.\end{aligned}$$

In the last two integrals, we reverse $s \to \frac{1}{s}$ and $x \to \frac{1}{x}$ to get

$$\begin{aligned}\mathcal{I}_{r,j} &= -\frac{4}{\pi^2}\int_0^1 \frac{rs^{r-1}ds}{s^{2r}-1} \circ \frac{rds}{s} \circ \frac{rds}{s} \circ \frac{\log^j x dx}{x^2+(-1)^j} \\ &\quad - r\int_0^1 \frac{\log^{j+1} x dx}{x^2+(-1)^j} - \frac{4}{\pi^2}\int_0^1 \frac{rs^{r-1}ds}{s^{2r}-1} \circ \frac{rds}{s} \circ \frac{rds}{s} \circ \frac{\log^j x dx}{x^2+(-1)^j} \\ &= -\frac{8r^2}{\pi^2}\int_0^1 \frac{rs^{r-1}ds}{s^{2r}-1} \circ \frac{ds}{s} \circ \frac{ds}{s} \circ \frac{\log^j x dx}{x^2+(-1)^j} - r\int_0^1 \frac{\log^{j+1} x dx}{x^2+(-1)^j} \\ &= -\frac{4r^2 j!(-1)^j}{\pi^2 i^{j+1}}\int_0^1 \frac{rs^{r-1}ds}{s^{2r}-1} \circ \frac{ds}{s} \circ \frac{ds}{s} \circ \left(\frac{1}{x-i^{j+1}} - \frac{1}{x+i^{j+1}}\right) dx \circ \underbrace{\frac{du}{u} \circ \cdots \circ \frac{du}{u}}_{j \text{ times}} \\ &\quad - r\int_0^1 \frac{\log^{j+1} x dx}{x^2+(-1)^j}.\end{aligned}$$

Let ξ_{2r} be a primitive $(2r)^{\text{th}}$ root of unity. We can then write

$$s^r - 1 = \prod_{\ell=0}^{r-1}(s - \xi_{2r}^{2\ell}) \quad \text{and} \quad s^r + 1 = \prod_{\ell=0}^{r-1}(s - \xi_{2r}^{2\ell+1}).$$

By applying the logarithmic derivatives above, we get

$$\frac{rs^{r-1}}{s^r-1} = \sum_{\ell=0}^{r-1}\frac{1}{s-\xi_{2r}^{2\ell}} \quad \text{and} \quad \frac{rs^{r-1}}{s^r+1} = \sum_{\ell=0}^{r-1}\frac{1}{s-\xi_{2r}^{2\ell+1}}.$$

This gives

$$\frac{rs^{r-1}}{s^{2r}-1} = \frac{rs^{r-1}}{2(s^r-1)} - \frac{rs^{r-1}}{2(s^r+1)} = \frac{1}{2}\sum_{\ell=0}^{2r-1} \frac{(-1)^\ell}{s-\xi_{2r}^\ell}.$$

Finally we have

$$\begin{aligned}
\mathcal{I}_{r,j} =& -\frac{2r^2 j!(-1)^j}{\pi^2 i^{j+1}} \sum_{\ell=0}^{2r-1} (-1)^\ell I_{3,j+1}\left(\xi_{2r}^\ell : i^{j+1} : 1\right) \\
&+ \frac{2r^2 j!(-1)^j}{\pi^2 i^{j+1}} \sum_{\ell=0}^{2r-1} (-1)^\ell I_{3,j+1}\left(\xi_{2r}^\ell : -i^{j+1} : 1\right) - r\int_0^1 \frac{\log^{j+1} x dx}{x^2+(-1)^j} \\
=& -\frac{2r^2 j!(-1)^j}{\pi^2 i^{j+1}} \sum_{\ell=0}^{2r-1} (-1)^\ell \mathrm{Li}_{3,j+1}(i^{j+1}\xi_{2r}^{-\ell}, i^{-j-1}) \\
&+ \frac{2r^2 j!(-1)^j}{\pi^2 i^{j+1}} \sum_{\ell=0}^{2r-1} (-1)^\ell \mathrm{Li}_{3,j+1}(-i^{j+1}\xi_{2r}^{-\ell}, -i^{-j-1}) - r\int_0^1 \frac{\log^{j+1} x dx}{x^2+(-1)^j}.
\end{aligned} \tag{4.5.1}$$

By Lemma 0.4.6, we have

$$-r\int_0^1 \frac{\log^{j+1} x dx}{x^2+(-1)^j} = \begin{cases} r(j+1)!\left(1-\dfrac{1}{2^{j+2}}\right)\zeta(j+2) & j \text{ odd}, \\ \\ r(j+1)! L(\chi_{-4}, j+2) & j \text{ even}. \end{cases} \tag{4.5.2}$$

When $j = 2h$ is even, we have that

$$\begin{aligned}
&-\frac{2r^2 j!(-1)^j}{\pi^2 i^{j+1}} \sum_{\ell=0}^{2r-1} (-1)^\ell \left(\mathrm{Li}_{3,j+1}(i^{j+1}\xi_{2r}^{-\ell}, i^{-j-1}) - \mathrm{Li}_{3,j+1}(-i^{j+1}\xi_{2r}^{-\ell}, -i^{-j-1})\right) \\
&= \frac{2r^2 i(-1)^h (2h)!}{\pi^2} \sum_{\ell=0}^{2r-1} (-1)^\ell \left(\mathrm{Li}_{3,2h+1}(i(-1)^h\xi_{2r}^{-\ell}, i(-1)^{h+1}) - \mathrm{Li}_{3,2h+1}(i(-1)^{h+1}\xi_{2r}^{-\ell}, i(-1)^h)\right) \\
&= \frac{2r^2 i(2h)!}{\pi^2} \sum_{\ell=0}^{2r-1} (-1)^\ell \left(\mathrm{Li}_{3,2h+1}(i\xi_{2r}^{-\ell}, -i) - \mathrm{Li}_{3,2h+1}(-i\xi_{2r}^{-\ell}, i)\right) \\
&= \frac{2r^2 i(2h)!}{\pi^2} \sum_{\ell=0}^{2r-1} \left((-1)^\ell \mathrm{Li}_{3,2h+1}(i\xi_{2r}^{-\ell}, -i) - (-1)^{2r-\ell}\mathrm{Li}_{3,2h+1}(-i\xi_{2r}^{2r-\ell}, i)\right) \\
&= -\frac{4r^2 (2h)!}{\pi^2} \sum_{\ell=0}^{2r-1} (-1)^\ell \operatorname{Im}\left(\mathrm{Li}_{3,2h+1}(i\xi_{2r}^{-\ell}, -i)\right).
\end{aligned} \tag{4.5.3}$$

When $j = 2h-1$ is odd, we have that

$$
\begin{aligned}
&-\frac{2r^2 j!(-1)^j}{\pi^2 i^{j+1}}\sum_{\ell=0}^{2r-1}(-1)^\ell\left(\mathrm{Li}_{3,j+1}(i^{j+1}\xi_{2r}^{-\ell}, i^{-j-1}) - \mathrm{Li}_{3,j+1}(-i^{j+1}\xi_{2r}^{-\ell}, -i^{-j-1})\right)\\
&=\frac{2r^2(-1)^h(2h-1)!}{\pi^2}\sum_{\ell=0}^{2r-1}(-1)^\ell\left(\mathrm{Li}_{3,2h}((-1)^h\xi_{2r}^{-\ell}, (-1)^h) - \mathrm{Li}_{3,2h}(-(-1)^h\xi_{2r}^{-\ell}, -(-1)^h)\right)\\
&=\frac{2r^2(2h-1)!}{\pi^2}\sum_{\ell=0}^{2r-1}(-1)^\ell\left(\mathrm{Li}_{3,2h}(\xi_{2r}^{-\ell}, 1) - \mathrm{Li}_{3,2h}(-\xi_{2r}^{-\ell}, -1)\right). \qquad (4.5.4)
\end{aligned}
$$

Notice that one can combine

$$
\begin{aligned}
(-1)^\ell\mathrm{Li}_{3,2h}\left(\xi_{2r}^{-\ell},1\right) + (-1)^{2r-\ell}\mathrm{Li}_{3,2h}\left(\xi_{2r}^{-2r+\ell},1\right) =&(-1)^\ell\mathrm{Li}_{3,2h}\left(\xi_{2r}^{-\ell},1\right) + (-1)^\ell\mathrm{Li}_{3,2h}\left(\xi_{2r}^{\ell},1\right)\\
=&(-1)^\ell 2\,\mathrm{Re}(\mathrm{Li}_{3,2h}\left(\xi_{2r}^{-\ell},1\right))
\end{aligned}
$$

and similarly with

$$
(-1)^\ell\mathrm{Li}_{3,2h}\left(-\xi_{2r}^{-\ell},-1\right) + (-1)^{2r-\ell}\mathrm{Li}_{3,2h}\left(-\xi_{2r}^{-2r+\ell},-1\right) =(-1)^\ell 2\,\mathrm{Re}(\mathrm{Li}_{3,2h}\left(-\xi_{2r}^{-\ell},-1\right)).
$$

By combining the above with (4.5.4), we finally have that, when $j = 2h-1$ is odd,

$$
\begin{aligned}
&-\frac{2r^2 j!(-1)^j}{\pi^2 i^{j+1}}\sum_{\ell=0}^{2r-1}(-1)^\ell\left(\mathrm{Li}_{3,j+1}(i^{j+1}\xi_{2r}^{-\ell}, i^{-j-1}) - \mathrm{Li}_{3,j+1}(-i^{j+1}\xi_{2r}^{-\ell}, -i^{-j-1})\right)\\
&=\frac{2r^2(2h-1)!}{\pi^2}\sum_{\ell=0}^{2r-1}(-1)^\ell\left(\mathrm{Re}\left(\mathrm{Li}_{3,j+1}(\xi_{2r}^{-\ell}, 1)\right) - \mathrm{Re}\left(\mathrm{Li}_{3,j+1}(-\xi_{2r}^{-\ell}, -1)\right)\right). \qquad (4.5.5)
\end{aligned}
$$

In order to continue the simplification, we apply Corollary 0.4.5. Equation (0.4.15) gives, for $j = 2h$,

$$
\begin{aligned}
&2i\sum_{\ell=0}^{2r-1}(-1)^\ell\,\mathrm{Im}\left(\mathrm{Li}_{3,2h+1}(i\xi_{2r}^{-\ell}, -i)\right)\\
&=\sum_{\ell=0}^{2r-1}(-1)^\ell\Bigg[\mathrm{Li}_{2h+4}(\xi_{2r}^{\ell}) - \mathrm{Li}_3(-i\xi_{2r}^{\ell})\left(\mathrm{Li}_{2h+1}(i) - \mathrm{Li}_{2h+1}(-i)\right) + \binom{2h+3}{2}\mathrm{Li}_{2h+4}(-i\xi_{2r}^{\ell})\\
&+\sum_{t=1}^{2h+3}\left(\binom{t-1}{2}\mathrm{Li}_t(-i\xi_{2r}^{\ell}) + \binom{t-1}{2h}(-1)^t\mathrm{Li}_t(-i)\right)\left(-\mathrm{Li}_{2h+4-t}(\xi_{2r}^{-\ell}) - (-1)^t\mathrm{Li}_{2h+4-t}(\xi_{2r}^{\ell})\right)\Bigg].
\end{aligned}
$$

We now apply part of Lemma 0.4.7 and other identities from Section 0.4.1, such as (0.4.10), (0.4.11), (0.4.13), (0.4.21), and (0.4.22), to find that the above equals

$$\frac{(-1)^{h+1}(2^{2h+4}-1)B_{2h+4}\pi^{2h+4}}{r^{2h+3}(2h+4)!} - i\frac{(-1)^h E_{2h}\pi^{2h+1}}{r^2 2^{2h}(2h)!}\left(\mathrm{Li}_3((-i)^r) - \frac{1}{8}\mathrm{Li}_3((-1)^r)\right)$$
$$+ (2h+3)(2h+2)\frac{1}{r^{2h+3}}\left(\mathrm{Li}_{2h+4}((-i)^r) - \frac{1}{2^{2h+4}}\mathrm{Li}_{2h+4}((-1)^r)\right)$$
$$+ \sum_{\ell=0}^{2r-1}(-1)^\ell\left[\sum_{t=1}^{2h+3}\left(\frac{(t-1)(t-2)}{2}(-1)^t\mathrm{Li}_t(-i\xi_{2r}^\ell) + \binom{t-1}{2h}\mathrm{Li}_t(-i)\right)\right.$$
$$\left.\times\frac{(2\pi i)^{2h+4-t}}{(2h+4-t)!}B_{2h+4-t}\left(\frac{\ell}{2r}\right)\right]. \tag{4.5.6}$$

Equation (0.4.16) gives for $j = 2h-1$,

$$2\sum_{\ell=0}^{2r-1}(-1)^\ell\left(\mathrm{Re}\left(\mathrm{Li}_{3,j+1}(\xi_{2r}^{-\ell},1)\right) - \mathrm{Re}\left(\mathrm{Li}_{3,j+1}(-\xi_{2r}^{-\ell},-1)\right)\right)$$
$$= \sum_{\ell=0}^{2r-1}(-1)^\ell\left[2\mathrm{Li}_3(\xi_{2r}^\ell)\mathrm{Li}_{2h}(1) - 2\mathrm{Li}_3(-\xi_{2r}^\ell)\mathrm{Li}_{2h}(-1) + \binom{2h+2}{2}\left(\mathrm{Li}_{2h+3}(\xi_{2r}^\ell) - \mathrm{Li}_{2h+3}(-\xi_{2r}^\ell)\right)\right.$$
$$+ \sum_{t=1}^{2h+2}\left(\binom{t-1}{2}\left(\mathrm{Li}_t(\xi_{2r}^\ell) - \mathrm{Li}_t(-\xi_{2r}^\ell)\right) - \binom{t-1}{2h-1}(-1)^t\left(\mathrm{Li}_t(1) - \mathrm{Li}_t(-1)\right)\right)$$
$$\left.\times\left(-\mathrm{Li}_{2h+3-t}(\xi_{2r}^{-\ell}) + (-1)^t\mathrm{Li}_{2h+3-t}(\xi_{2r}^\ell)\right)\right].$$

Again, we apply part of Lemma 0.4.7 and identities from Section 0.4.1 to see that the above equals

$$\frac{(-1)^{h+1}7B_{2h}\pi^{2h}}{2r^2(2h)!}\zeta(3)\left(2^{2h-1} + (-1)^r 2^{2h-1} + (-1)^{r+1}\right)$$
$$+ (2h+2)(2h+1)\frac{1-2^{-2h-3}}{r^{2h+2}}(1-(-1)^r)\zeta(2h+3)$$
$$- \sum_{\ell=0}^{2r-1}(-1)^\ell\left[\sum_{t=2}^{2h+2}\left(\frac{(t-1)(t-2)}{2}(-1)^t\left(\mathrm{Li}_t(\xi_{2r}^\ell) - \mathrm{Li}_t(-\xi_{2r}^\ell)\right) - \binom{t-1}{2h-1}(2-2^{1-t})\zeta(t)\right)\right.$$
$$\left.\times\frac{(2\pi i)^{2h+3-t}}{(2h+3-t)!}B_{2h+3-t}\left(\frac{\ell}{2r}\right)\right]. \tag{4.5.7}$$

Combining equations (4.5.6) and (4.5.7) with (4.5.3), (4.5.5), and (4.5.2) in (4.5.1) concludes the proof of the statement. □

Proof of Theorem 4.1.1. By Definition 4.4.1, we have that

$$\mathrm{m}(S_{2k,r}) = \sum_{h=1}^{k} a_{k,h-1} \left(\frac{2}{\pi}\right)^{2h} \mathcal{I}_{r,2h-1}$$

and

$$\mathrm{m}(S_{2k+1,r}) = \sum_{h=0}^{k} b_{k,h} \left(\frac{2}{\pi}\right)^{2h+1} \mathcal{I}_{r,2h}.$$

The result the follows from Theorem 4.4.2 and Proposition 4.5.1, by setting $\mathcal{C}_r(h) := \mathcal{I}_{r,2h-1}$ and $\mathcal{D}_r(h) := \mathcal{I}_{r,2h}$. □

4.6. Some particular cases

In this section we focus on the simplest cases, for low values of r or n.

For the case $r = 1$, and $j = 2h$, we have, from (4.5.6),

$$\begin{aligned}
&\sum_{\ell=0}^{1} (-1)^{\ell} \operatorname{Im}\left(\mathrm{Li}_{3,2h+1}(i(-1)^{-\ell}, -i)\right) \\
&= -(2h+3)(h+1)L(\chi_{-4}, 2h+4) + (2h+1)L(\chi_{-4}, 2h+2)\frac{\pi^2}{4} \\
&+ \sum_{s=2}^{h+1} (2s-1)(s-1)L(\chi_{-4}, 2s)\frac{(-1)^{h+1-s}B_{2h+4-2s}\pi^{2h+4-2s}}{(2h+4-2s)!}.
\end{aligned}$$

This gives, for $r = 1$,

$$\mathcal{I}_{1,2h} = \frac{2}{\pi^2}\sum_{\ell=0}^{h} \binom{2h+1}{2\ell+1} \frac{(-1)^{h-\ell}}{2h+1} B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell+3)!L(\chi_{-4}, 2\ell+4) = \frac{4}{\pi^2}\mathcal{D}(h),$$

where we have set $s = \ell + 2$.

For the case $r = 1$, and $j = 2h - 1$, we have, from (4.5.7),

$$\begin{aligned}
&\sum_{\ell=0}^{1}(-1)^{\ell}\left(\mathrm{Re}\left(\mathrm{Li}_{3,j+1}((-1)^{-\ell},1)\right)-\mathrm{Re}\left(\mathrm{Li}_{3,j+1}(-(-1)^{-\ell},-1)\right)\right)\\
=&(h+1)(2h+1)\left(2-\frac{1}{2^{2h+2}}\right)\zeta(2h+3)-h\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1)\pi^2\\
&-\sum_{s=2}^{h}s(2s-1)\left(2-\frac{1}{2^{2s}}\right)\zeta(2s+1)\frac{(-1)^{h-s}B_{2h+2-2s}\pi^{2h+2-2s}}{(2h+2-2s)!}.
\end{aligned}$$

This gives, for $r = 1$,

$$\mathcal{I}_{1,2h-1}=\frac{1}{\pi^2}\sum_{\ell=1}^{h}\binom{2h}{2\ell}\frac{(-1)^{h-\ell}}{h}B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell+2)!\left(1-\frac{1}{2^{2\ell+3}}\right)\zeta(2\ell+3)=\frac{4}{\pi^2}\mathcal{C}(h),$$

where we have set $s = \ell + 1$.

For the case $r = 2$ and $j = 2h$, we have, from (4.5.6),

$$\begin{aligned}
&\sum_{\ell=0}^{3}(-1)^{\ell}\,\mathrm{Im}\left(\mathrm{Li}_{3,2h+1}(i^{1-\ell},-i)\right)\\
=&\frac{(-1)^{h+1}21\pi^{2h+1}}{2^{2h+6}(2h)!}E_{2h}\zeta(3)+(2h+1)L(\chi_{-4},2h+2)\frac{\pi^2}{8}\\
&-\sum_{s=2}^{h+1}\binom{2s-1}{2}(-1)^{h-s}L(\chi_{-4},2s)\left(2^{2h+4-2s}-1\right)\frac{\pi^{2h+4-2s}}{(2h+4-2s)!}B_{2h+4-2s}\\
&+\sum_{s=2}^{h+1}\binom{2s}{2}(-1)^{h-s}\left(2^{2s+1}-1\right)\zeta(2s+1)\frac{\pi^{2h+3-2s}}{2^{2h+4}(2h+2-2s)!}E_{2h+2-2s}.
\end{aligned}$$

This gives, for $r = 2$,

$$\begin{aligned}
\mathcal{I}_{2,2h}=&\frac{(-1)^{h}21}{2^{2h+2}}E_{2h}\pi^{2h-1}\zeta(3)\\
&+8\sum_{\ell=0}^{h-1}\binom{2h+1}{2\ell+1}\frac{(-1)^{h-\ell}}{2h+1}B_{2(h-\ell)}\pi^{2h-2\ell-2}(2\ell+3)!\left(2^{2h-2\ell}-1\right)L(\chi_{-4},2\ell+4)\\
&+\sum_{\ell=1}^{h}\binom{2h}{2\ell}\frac{(-1)^{h-\ell}}{2^{2h+1}}E_{2(h-\ell)}\pi^{2h-2\ell-1}(2\ell+2)!\left(2^{2\ell+3}-1\right)\zeta(2\ell+3)\\
=&\mathcal{D}_2(h).
\end{aligned}$$

For the case $r=2$, and $j=2h-1$, we have, from (4.5.7),

$$\begin{aligned}
&\sum_{\ell=0}^{3}(-1)^\ell \left(\mathrm{Re}\left(\mathrm{Li}_{3,2h}(i^{-\ell},1)\right) - \mathrm{Re}\left(\mathrm{Li}_{3,2h}(-i^{-\ell},-1)\right)\right)\\
=&\frac{(-1)^{h+1}7\pi^{2h}}{16(2h)!}B_{2h}\left(2^{2h}-1\right)\zeta(3) - h\left(2-\frac{1}{2^{2h}}\right)\zeta(2h+1)\frac{\pi^2}{4}\\
&-\sum_{s=1}^{h}\binom{2s}{2}(-1)^{h-s}\left(2-\frac{1}{2^{2s}}\right)\zeta(2s+1)\left(2^{2h+2-2s}-1\right)\frac{\pi^{2h+2-2s}}{(2h+2-2s)!}B_{2h+2-2s}\\
&-\sum_{s=1}^{h+1}\binom{2s-1}{2}(-1)^{h-s}L(\chi_{-4},2s)\frac{\pi^{2h+3-2s}}{2^{2h+2-2s}(2h+2-2s)!}E_{2h+2-2s}.
\end{aligned}$$

This gives, for $r=2$,

$$\begin{aligned}
\mathcal{I}_{2,2h-1} =&\frac{(-1)^{h+1}7}{4h}B_{2h}\pi^{2h-2}\left(2^{2h}-1\right)\zeta(3)\\
&+4\sum_{\ell=0}^{h-1}\binom{2h}{2\ell}\frac{(-1)^{h-\ell}}{h}\left(2^{2h-2\ell}-1\right)B_{2(h-\ell)}\pi^{2h-2\ell-2}(2\ell+2)!\left(1-\frac{1}{2^{2\ell+3}}\right)\zeta(2\ell+3)\\
&+\sum_{\ell=1}^{h}\binom{2h-1}{2\ell-1}\frac{(-1)^{h-\ell}}{2^{2h-2\ell-2}}E_{2(h-\ell)}\pi^{2h-2\ell-1}(2\ell+1)!L(\chi_{-4},2\ell+2)\\
=&\mathcal{C}_2(h).
\end{aligned}$$

The evaluation of $\mathcal{I}_{r,j}$ and $\mathrm{m}(S_{n,r})$ for $r>2$ quickly becomes computationally involved. We will focus on the case $n=1$. This corresponds to the case $k=h=0$ and $\mathcal{I}_{r,0}$. We remark that for $j=0$ we have

$$\begin{aligned}
\mathcal{I}_{r,0} =&\mathrm{Re}\left[\frac{12i}{\pi^2 r}\mathrm{Li}_4((-i)^r) - \frac{2}{\pi}\mathrm{Li}_3((-i)^r) + \frac{2r}{\pi}\sum_{\ell=0}^{2r-1}(-1)^\ell \mathrm{Li}_3(-i\xi_{2r}^\ell)\ell\right]\\
&-\frac{3r^2}{16\pi}\zeta(3) + \frac{1}{4\pi}\mathrm{Li}_3((-1)^r)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{m}(S_{1,r}) =&\mathrm{Re}\left[\frac{24i}{\pi^3 r}\mathrm{Li}_4((-i)^r) - \frac{4}{\pi^2}\mathrm{Li}_3((-i)^r) + \frac{4r}{\pi^2}\sum_{\ell=0}^{2r-1}(-1)^\ell \mathrm{Li}_3(-i\xi_{2r}^\ell)\ell\right]\\
&-\frac{3r^2}{8\pi^2}\zeta(3) + \frac{1}{2\pi^2}\mathrm{Li}_3((-1)^r).
\end{aligned}$$

We get different cases according to the class of $r \bmod 4$.

For $r = 2s+1$, we have

$$\mathrm{m}(S_{1,2s+1}) = \frac{24(-1)^s}{(2s+1)\pi^3} L(\chi_{-4},4) - \frac{3(2s+1)^2}{8\pi^2}\zeta(3) + \frac{4(2s+1)}{\pi^2}\sum_{\ell=0}^{4s+1}(-1)^\ell \operatorname{Re}(\mathrm{Li}_3(-i\xi_{4s+2}^\ell))\ell.$$

For $r = 4s$, we have

$$\mathrm{m}(S_{1,4s}) = -\frac{12s^2+7}{2\pi^2}\zeta(3) + \frac{16s}{\pi^2}\sum_{\ell=0}^{8s-1}(-1)^\ell \operatorname{Re}(\mathrm{Li}_3(-i\xi_{8s}^\ell))\ell.$$

For $r = 4s+2$, we have

$$\mathrm{m}(S_{1,4s+2}) = -\frac{6s^2+6s-2}{\pi^2}\zeta(3) + \frac{16s+8}{\pi^2}\sum_{\ell=0}^{8s+3}(-1)^\ell \operatorname{Re}(\mathrm{Li}_3(-i\xi_{8s+4}^\ell))\ell.$$

Specializing in $r = 1{,}2$ we recover the formulas for the Mahler measures of $S_{1,1}$ and $S_{1,2}$. We now provide additional details for the cases $r = 3{,}4$.

For $r = 3$, we must find

$$\begin{aligned}\sum_{\ell=0}^{5}(-1)^\ell \operatorname{Re}(\mathrm{Li}_3(-i\xi_6^\ell))\ell =& -\operatorname{Re}(\mathrm{Li}_3(e^{\frac{11\pi i}{6}})) + 2\operatorname{Re}(\mathrm{Li}_3(e^{\frac{\pi i}{6}})) - 3\operatorname{Re}(\mathrm{Li}_3(i))\\ &+ 4\operatorname{Re}(\mathrm{Li}_3(e^{\frac{5\pi i}{6}})) - 5\operatorname{Re}(\mathrm{Li}_3(e^{\frac{7\pi i}{6}}))\\ =& \operatorname{Re}(\mathrm{Li}_3(e^{\frac{\pi i}{6}})) - \operatorname{Re}(\mathrm{Li}_3(e^{\frac{5\pi i}{6}})) - 3\operatorname{Re}(\mathrm{Li}_3(i)),\end{aligned}$$

since $\mathrm{Li}(\bar{z}) = \overline{\mathrm{Li}(z)}$. Now consider

$$\begin{aligned}\operatorname{Re}(\mathrm{Li}_3(e^{\frac{\pi i}{6}})) &= \sum_{k=1}^{\infty}\frac{\cos\frac{k\pi}{6}}{k^3}\\ &= \frac{\sqrt{3}}{2}\left(\frac{1}{1^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{11^3} + \cdots\right) + \frac{1}{2}\left(\frac{1}{2^3} - \frac{1}{4^3} - \frac{2}{6^3} - \frac{1}{8^3} + \frac{1}{10^3} + \frac{2}{12^3} + \cdots\right).\end{aligned}$$

This sum is absolutely convergent and we may rearrange the terms as desired. Let $\chi_{12}(11,n)$ be the Dirichlet character of conductor 12 given by $\left(\frac{12}{n}\right)$. This corresponds to the character $\chi_{12,4}$ according to Mathematica. Its values are given by

n	1	5	7	11
$\chi_{12}(11{,}n)$	1	-1	-1	1

so that

$$\frac{\sqrt{3}}{2}\left(\frac{1}{1^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{11^3} + \cdots\right) = \frac{\sqrt{3}}{2}\cdot L(\chi_{12}(11,\cdot),3).$$

We can also write

$$\frac{1}{2}\left(\frac{1}{2^3}-\frac{1}{4^3}-\frac{2}{6^3}-\frac{1}{8^3}+\frac{1}{10^3}+\frac{2}{12^3}+\cdots\right)=\frac{1}{2}\left(\frac{1}{2^3}-\frac{1}{4^3}+\frac{1}{6^3}-\frac{1}{8^3}+\frac{1}{10^3}-\frac{1}{12^3}+\cdots\right)$$
$$-\frac{3}{2}\left(\frac{1}{6^3}-\frac{1}{12^3}+\frac{1}{18^3}+\cdots\right)$$
$$=-\left(\frac{1}{2\cdot 2^3}-\frac{3}{2\cdot 6^3}\right)\mathrm{Li}_3(-1)$$
$$=\frac{\zeta(3)}{24}.$$

Therefore,

$$\mathrm{Re}(\mathrm{Li}_3(e^{\frac{\pi i}{6}}))=\frac{\zeta(3)}{24}+\frac{\sqrt{3}}{2}L(\chi_{12}(11,\cdot),3).$$

Similarly, we can show that

$$\mathrm{Re}(\mathrm{Li}_3(e^{\frac{5\pi i}{6}}))=\frac{\zeta(3)}{24}-\frac{\sqrt{3}}{2}L(\chi_{12}(11,\cdot),3),$$

and using that $\mathrm{Re}(\mathrm{Li}_3(i))=-\frac{3}{32}\zeta(3)$, we obtain

$$\sum_{\ell=0}^{5}(-1)^{\ell}\,\mathrm{Re}(\mathrm{Li}_3(-i\xi_6^{\ell}))\ell=\frac{9}{32}\zeta(3)+\sqrt{3}L(\chi_{12}(11,\cdot),3),$$

which gives

$$\mathrm{m}(S_{1,3})=\frac{12\sqrt{3}}{\pi^2}L(\chi_{12}(11,\cdot),3)-\frac{8}{\pi^3}L(\chi_{-4},4).$$

When $m=4$, using similar manipulations, we can also show

$$\mathrm{Re}(\mathrm{Li}_3(e^{\frac{\pi i}{4}}))=-\frac{3}{4^4}\zeta(3)+\frac{1}{\sqrt{2}}L(\chi_8(5,\cdot),3),$$
$$\mathrm{Re}(\mathrm{Li}_3(e^{\frac{3\pi i}{4}}))=-\frac{3}{4^4}\zeta(3)-\frac{1}{\sqrt{2}}L(\chi_8(5,\cdot),3),$$

where $\chi_8(5,n)$ is the Dirichlet character of conductor 8 given by $\left(\frac{8}{n}\right)$. This corresponds to the character $\chi_{8,2}$ according to Mathematica. Its values are given by

n	1	3	5	7
$\chi_8(5,n)$	1	-1	-1	1

.

Thus,

$$\sum_{\ell=0}^{7}(-1)^\ell \operatorname{Re}(\mathrm{Li}_3(-i\xi_8^\ell))\ell = -4\operatorname{Re}(\mathrm{Li}_3(e^{\frac{\pi i}{4}})) - 12\operatorname{Re}(\mathrm{Li}_3(e^{\frac{3\pi i}{4}})) - \frac{23}{8}\zeta(3)$$
$$= -\frac{43}{16}\zeta(3) + 4\sqrt{2}L(\chi_8(5,\cdot),3),$$

and

$$\mathrm{m}(S_{1,4}) = -\frac{105}{2\pi^2}\zeta(3) + \frac{64\sqrt{2}}{\pi^2}L(\chi_8(5,\cdot),3).$$

4.7. Conclusion

Our results show that the Mahler measure of the family $S_{n,r}$ is even richer and more interesting than the previously known Mahler measure of $S_{n,1}$. It is clear from the case $n = 1$ that we can not expect a formula of the form (0.7.4). Such a formula is certainly true if we consider an analogous construction for the R_n family, namely, if we let

$$R_{n,r}(x_1,\dots,x_n,z) := z + \left[\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)\right]^r,$$

Then, we trivially have that

$$\mathrm{m}(R_{n,r}(x_1,\dots,x_n,z)) = \mathrm{m}(R_{n,r}(x_1,\dots,x_n,-z^r)) = \sum_{j=0}^{r}\mathrm{m}\left(z - \xi_r^j\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)\right)$$
$$= r\,\mathrm{m}(R_{n,1}(x_1,\dots,x_n,z)).$$

Thus, the case of $R_{n,r}$ is trivial. Similar considerations apply to the family $Q_{n,r}$ given by

$$Q_{n,r}(x_1,\dots,x_n,z) := z + \left[\left(\frac{\xi_3+x_1}{1+x_1}\right)\cdots\left(\frac{\xi_3+x_n}{1+x_n}\right)\right]^r.$$

An interesting project would be to consider the construction of this chapter for the family T_n:

$$T_{n,r}(x_1,\dots,x_n,x,y) := 1+\left[\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)\right]^r x+\left(1-\left[\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)\right]^r\right)y.$$

As we remarked in the introduction, there is a clear distinction between the cases n even and odd for $\mathrm{m}(S_n)$, namely, the formulas for n even only contain special values of the Riemann zeta function, and the formulas for n odd only contain special values of the Dirichlet L-function at χ_{-4}. However, for $\mathrm{m}(S_{n,2})$, the formulas are mixed. The case of $\mathrm{m}(R_n)$ also shows an alternation of formulas involving special values of the Riemann zeta function or special values of the Dirichlet L-function, and by the discussion above, since $\mathrm{m}(R_{n,r}) = r\mathrm{m}(R_n)$, the same is true for $\mathrm{m}(R_{n,r})$ independently of r. Finally, all the formulas involving $\mathrm{m}(T_n)$ are given in terms of $\log 2$ and special values of the Riemann zeta function. It would be interesting to see how this extends to $\mathrm{m}(T_{n,r})$.

Chapter 5

The areal zeta Mahler measure of a family of polynomials

In this chapter, we derive some fundamental properties of the areal zeta Mahler measure and we evaluate the areal Mahler measure of the polynomial family $\{k+x+y : k \in \mathbb{C}\}$ using the associated areal zeta Mahler measure. This is an ongoing joint work with Lalín, Nair, and Ringeling.

5.1. A brief description of the results

For a non-zero polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$, the areal zeta Mahler measure is given in (0.6.18) by

$$Z_{\mathbb{D}}(s,P) := \frac{1}{\pi^n} \int_{\mathbb{D}^n} |P(x_1, \dots, x_n)|^s \, dA(x_1) \dots dA(x_n). \tag{5.1.1}$$

The goal of this chapter is to evaluate $\mathrm{m}_{\mathbb{D}}(k+x+y)$ explicitly in terms of hypergeometric series, as well as in terms of special values of the Bloch–Wigner dilogarithm, for all $k \in \mathbb{C}$. Following the discussion in Remark 0.6.26, it suffices to compute $\mathrm{m}_{\mathbb{D}}(|k|+x+y)$, since

$$\mathrm{m}_{\mathbb{D}}(k+x+y) = \mathrm{m}_{\mathbb{D}}(k\xi+x+y) = \mathrm{m}_{\mathbb{D}}(|k|+x+y)$$

for any $\xi \in \mathbb{C}^\times$ such that $|\xi| = 1$. We achieve the evaluation in two steps, given by the next two theorems.

Theorem 5.1.1 (see Theorem 0.6.23)**.** *For* $\mathrm{Re}(s) > -2$ *and* $k \in \mathbb{C}$*, we have*

$$Z_{\mathbb{D}}(s, |k| + x + y) = c_0(s) \left(\frac{|k|}{2}\right)^{s+3} F_0\left(\frac{|k|^2}{4}; s\right) + c_1(s) F_1\left(\frac{|k|^2}{4}; s\right), \tag{5.1.2}$$

where

$$F_0(z; s) = {}_3F_2\left(-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{5+s}{2}, \frac{5+s}{2}; z\right),$$

$$F_1(z; s) = {}_3F_2\left(-2 - \frac{s}{2}, -1 - \frac{s}{2}, -\frac{s}{2}; 1, -\frac{1}{2} - \frac{s}{2}; z\right),$$

$$c_0(s) = \frac{2^s \cdot {}_3F_2\left(-\frac{s}{2}, -\frac{s}{2}, \frac{3}{2}; 2, 3; 1\right) - \frac{4}{s+4}\frac{\Gamma(s+2)}{\Gamma\left(\frac{s}{2}+2\right)^2} \cdot {}_3F_2\left(-2 - \frac{s}{2}, -1 - \frac{s}{2}, -\frac{s}{2}; 1, -\frac{1}{2} - \frac{s}{2}; 1\right)}{{}_3F_2\left(-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{5+s}{2}, \frac{5+s}{2}; 1\right)},$$

and $c_1(s) = \frac{4}{s+4}\frac{\Gamma(s+2)}{\Gamma\left(\frac{s}{2}+2\right)^2}$.

Using Theorem 5.1.1, we compute

$$\left.\frac{dZ_{\mathbb{D}}(s, |k| + x + y)}{ds}\right|_{s=0} = \mathrm{m}_{\mathbb{D}}(|k| + x + y), \tag{5.1.3}$$

which leads to the statement of Corollary 0.6.24. Comparing this with (0.6.21), which represents $\mathrm{m}(|k| + x + y)$ in terms of a hypergeometric series, namely,

$$\mathrm{m}(|k| + x + y) = \frac{|k|}{\pi} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{|k|^2}{4}\right),$$

we obtain the following result.

Theorem 5.1.2 (see Theorem 0.6.25)**.** *For* $k > 0$,

$$\mathrm{m}(k + x + y) - \mathrm{m}_{\mathbb{D}}(k + x + y) = \frac{k\sqrt{4 - k^2}(10 + k^2) + (8 - 16k^2)\arccos\left(\frac{k}{2}\right)}{16\pi}. \tag{5.1.4}$$

Before proceeding with the proofs of the above statements, we will first discuss some fundamental properties of the areal zeta Mahler measure, which is in itself a very interesting object to study.

5.2. Some fundamental properties of the areal zeta Mahler measure

In order to compute (5.1.3), we need to verify that $Z_{\mathbb{D}}$ is differentiable at $s=0$. To see this, we first consider the holomorphic regions of $Z_{\mathbb{D}}(s,P)$ for any polynomial $P \in \mathbb{C}[x_1,\dots,x_n]$.

We first specify some notation that we use frequently in our arguments below. Let $\lambda_n(A)$ denote *the Lebesgue measure of a set* $A \in \mathbb{R}^n$, and let S_n denote *the Symmetric group of* n *elements.*

Recall that, for any $P \in \mathbb{C}[x_1^{\pm},\dots,x_n^{\pm}]$, we denote

$$\sigma_1(P) := \inf\left\{\sigma \in \mathbb{R} : \frac{1}{\pi^n}\int_{\mathbb{D}^n} |P(x_1,\dots,x_n)|^{\sigma}\, dA(x_1)\dots dA(x_n) < \infty\right\} \in \mathbb{R}\cup\{-\infty\}.$$

For $P \in \mathbb{C}[x_1,\dots,x_n]$, since $\frac{1}{\pi^n}\int_{\mathbb{D}^n} |P(x_1,\dots,x_n)|^0\, dA(x_1)\dots dA(x_n) = 1 < \infty$, we have $\sigma_1(P) \le 0$. Following an argument analogous to the classical case in [**3**, Proposition 2.1], we have the following result.

Proposition 5.2.1. *Let* $P \in \mathbb{C}[x_1,\dots,x_n]$ *be a non-zero polynomial. Then the integral in* (5.1.1) *converges absolutely and locally uniformly in* $\mathrm{Re}(s) > \sigma_1(P)$*. Furthermore, for* $\mathrm{Re}(s) > \sigma_1(P)$*,* $Z_{\mathbb{D}}(s,P)$ *is holomorphic and we have*

$$\frac{d^k Z_{\mathbb{D}}(s,P)}{ds^k} = \frac{1}{\pi^n}\int_{\mathbb{D}^n} |P(x_1,\dots,x_n)|^{s}\,(\log|P|)^k dA(x_1)\dots dA(x_n).$$

Proof. We assume $n=2$ for simplicity of notation, as the same idea extends to the general case. First observe that

$$\frac{1}{\pi^2}\int_{\mathbb{D}^2} |P(x,y)|^s\, dA(x)dA(y) = 2^2\int_{[0,1]^4} \left|P\left(\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i\theta_2}\right)\right|^s \rho_1\rho_2 d\rho_1 d\rho_2 d\theta_1 d\theta_2.$$

Let $\epsilon > 0$ be arbitrary and $T > \max\{\sigma_1(P)+\epsilon, 0\}$. Then, from the definition of $\sigma_1(P)$, it follows that there exists at least one $\sigma \in [\sigma_1(P), \sigma_1(P)+\epsilon)$ such that

$$2^2\int_{[0,1]^4} \left|P\left(\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i\theta_2}\right)\right|^{\sigma} \rho_1\rho_2 d\rho_1 d\rho_2 d\theta_1 d\theta_2 < \infty. \tag{5.2.1}$$

Note that, assuming the integral in (5.1.1) converges absolutely for all $\sigma_1(P)+\epsilon \le \mathrm{Re}(s) \le T$, we can extend this region of absolute convergence to $(\sigma_1(P), \infty)$ by taking $\epsilon \to 0$

and $T \to \infty$. Therefore, it suffices to prove the absolute convergence of the integral in (5.1.1) for all $\mathrm{Re}(s) \in [\sigma_1(P) + \epsilon, T]$.

For $t \geq 0$, we divide $[0,1]^4$ into two parts

$$A_2^{\geq t}(P) := \{(\rho_1, \rho_2, \theta_1, \theta_2) \in [0,1]^4 : \left|P\left(\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i\theta_2}\right)\right| \geq t\}, \tag{5.2.2}$$

$$A_2^{<t}(P) := \{(\rho_1, \rho_2, \theta_1, \theta_2) \in [0,1]^4 : \left|P\left(\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i\theta_2}\right)\right| < t\}. \tag{5.2.3}$$

Let $R_P := \max_{(x,y)\in\mathbb{D}^2} |P(x,y)|$, which is well-defined since $\mathbb{D}^2$ is compact and $|P(x,y)|$ is continuous. Then, on the one hand, for all $(\rho_1, \rho_2, \theta_1, \theta_2) \in A_2^{\geq 1}(P)$ and $\mathrm{Re}(s) \in [\sigma_1(P)+\epsilon, T]$, we have

$$||P(\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i\theta_2})|^s| = |P(\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i\theta_2})|^{\mathrm{Re}(s)} \leq R_P^T.$$

On the other hand, for $(\rho_1, \rho_2, \theta_1, \theta_2) \in A_2^{<1}(P)$, we have

$$||P(\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i\theta_2})|^s| = |P(\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i\theta_2})|^{\mathrm{Re}(s)} \leq |P(\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i\theta_2})|^{\sigma},$$

since $\sigma < \sigma_1(P) + \epsilon \leq \mathrm{Re}(s)$ and $|P(\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i\theta_2})| < 1$.

Observe that $\max\left\{\lambda_4\left(A_2^{<1}(P)\right), \lambda_4\left(A_2^{\geq 1}(P)\right)\right\} \leq \lambda_4([0,1]^4) = 1$. Then, from (5.1.1), we have

$$\begin{aligned}
&\left|\frac{1}{\pi^2}\int_{\mathbb{D}^2} |P(x,y)|^s \, dA(x)dA(y)\right| \\
=&\left|2^2\int_{[0,1]^4} \left|P\left(\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i\theta_2}\right)\right|^s \rho_1\rho_2 d\rho_1 d\rho_2 d\theta_1 d\theta_2\right| \\
\leq& \; 2^2\int_{A_2^{\geq 1}(P)} \left|P\left(\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i\theta_2}\right)\right|^{\mathrm{Re}(s)} \rho_1\rho_2 d\rho_1 d\rho_2 d\theta_1 d\theta_2 \\
&+ 2^2\int_{A_2^{<1}(P)} \left|P\left(\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i\theta_2}\right)\right|^{\mathrm{Re}(s)} \rho_1\rho_2 d\rho_1 d\rho_2 d\theta_1 d\theta_2 \\
\leq& \; 2^2 R_P^T \lambda_4\left(A_2^{\geq 1}(P)\right) + 2^2\int_{A_2^{<1}(P)} \left|P\left(\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i\theta_2}\right)\right|^{\sigma} \rho_1\rho_2 d\rho_1 d\rho_2 d\theta_1 d\theta_2 \\
<& \; \infty,
\end{aligned}$$

where the finiteness of the second integral in the penultimate line follows from (5.2.1). This concludes the proof that the integral representation of $Z_{\mathbb{D}}(s,P)$ is absolutely convergent in $\sigma_1(P) + \epsilon \leq \mathrm{Re}(s) \leq T$, as well as in $\mathrm{Re}(s) > \sigma_1(P)$.

Let $\delta > 0$. For $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > \sigma_1(P)$, we denote $D(s,\delta) = \{z \in \mathbb{C} : |z-s| < \delta\}$. Observe that R_P^s is locally uniformly continuous on $\mathrm{Re}(s) > \sigma_1(P)$, [1] i.e. for every $\epsilon_1 > 0$ there exists a $\delta_{s,\epsilon_1} := \min\left\{\frac{\epsilon_1}{2(e-1)R_p^{\mathrm{Re}(s)}|\log R_P|}, \frac{1}{|\log R_P|}\right\} > 0$ such that, for all $z, w \in D(s, \delta_{s,\epsilon_1})$, we have

$$\begin{aligned}|R_P^z - R_P^w| \leq & \ |R_P^z - R_P^s| + |R_P^s - R_P^w| \\ &= R_P^{\mathrm{Re}(s)}\left(\left|e^{(z-s)\log R_P} - 1\right| + \left|e^{(s-w)\log R_P} - 1\right|\right) \\ &\leq (2(e-1)R_P^{\mathrm{Re}(s)}|\log R_p|)\delta_{s,\epsilon_1} = \epsilon_1,\end{aligned}$$

where last inequality follows from the following simplification

$$\begin{aligned}\left|e^{(z-s)\log R_P} - 1\right| &= \left|\sum_{j=1}^{\infty} \frac{(z-s)^j \log^j R_P}{j!}\right| \leq |z-s||\log R_P| \left(\sum_{j=1}^{\infty} \frac{|z-s|^{j-1}\log^{j-1} R_P}{j!}\right) \\ &\leq \delta_{s,\epsilon_1}|\log R_P| \left(\sum_{j=1}^{\infty} \frac{1}{j!}\right) = (e-1)\delta_{s,\epsilon_1}|\log R_P|.\end{aligned}$$

First we consider all s satisfying $\mathrm{Re}(s) > \max\{0, \sigma_1(P)\}$. In this region we have $|P(x,y)|^{\mathrm{Re}(s)} \leq R_P^{\mathrm{Re}(s)}$ for all $(x,y) \in \mathbb{D}^2$. Notice that, for every $\epsilon_2 > 0$, there exists a $\delta'_{s,\epsilon_2} := \min\left\{\frac{\epsilon_2}{2(e-1)R_p^{\mathrm{Re}(s)}|\log R_P|}, \frac{1}{|\log R_P|}\right\} > 0$ such that, for all $z, w \in D(s, \delta'_{s,\epsilon_2})$, we have

$$\begin{aligned}|Z_{\mathbb{D}}(z,P) - Z_{\mathbb{D}}(w,P)| &= \left|\frac{1}{\pi^2}\int_{\mathbb{D}^2} \left(|P(x,y)|^z - |P(x,y)|^w\right) dA(x)dA(y)\right| \\ &\leq \frac{1}{\pi^2}\int_{\mathbb{D}^2} \left||P(x,y)|^z - |P(x,y)|^w\right| dA(x)dA(y) \\ &\leq \frac{2(e-1)}{\pi^2}\int_{\mathbb{D}^2} |P(x,y)|^{\mathrm{Re}(s)}\, \delta'_{s,\epsilon_2}|\log|P(x,y)||dA(x)dA(y) \\ &\leq 2(e-1)R_P^{\mathrm{Re}(s)}|\log R_P|\delta'_{s,\epsilon_2} = \epsilon_2,\end{aligned}$$

which implies that the integral representation of $Z_{\mathbb{D}}(s,P)$ in (5.1.1) is locally uniformly convergent in $\mathrm{Re}(s) > \max\{0, \sigma_1(P)\}$ for $s \in \mathbb{C}$. If $\sigma_1(P) < 0$, then, for the region $\sigma_1(P) < \mathrm{Re}(s) < 0$, a similar approach on $A_2^{\leq 1}(P)$ and $A_2^{\geq 1}(P)$ extend the region of locally uniform convergence of the integral in (5.1.1) from $\{\mathrm{Re}(s) > 0\}$ to $\{\mathrm{Re}(s) > \sigma_1(P)\}$. This further implies that $Z_{\mathbb{D}}(s,P)$ is holomorphic in $\mathrm{Re}(s) > \sigma_1(P)$, which concludes our proof. □

1. It follows from the fact that e^z is locally uniformly continuous in $\mathbb{C}$.

The next proposition gives an estimate for $\sigma_1(P)$. First note that $\sigma_1(P^\tau) = \sigma_1(P)$ for any $\tau \in S_n$, where $P^\tau(x_1, \ldots, x_n) = P(x_{\tau(1)}, \ldots, x_{\tau(n)})$. Following the notation of [**3**], we define

$$\begin{aligned} d_1(P) &:= \deg P & &\text{if } n = 1, \\ d_n(P) &:= \deg_{x_n}(P) + d_{n-1}(Q) & &\text{if } n > 1, \end{aligned}$$

where Q is the coefficient of $x_n^{\deg_{x_n}(P)}$ in P. We also define

$$d_n^{\min}(P) := \min_{\tau \in S_n} d(P^\tau).$$

Then $d_n^{\min}(P) \le d_n(P) \le \deg P$. We have the following result.

Proposition 5.2.2. *Let $P \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-zero polynomial. Then*

(1) $\sigma_1(P) \le -1/d_n^{\min}(P)$.

(2) If P does not vanish on $\mathbb{D}^n$, then $\sigma_1(P) = -\infty$.

Our argument to prove Proposition 5.2.2 is again inspired by the derivation of [**3**, Theorem 8]. Before proceeding with the proof, we define $A_r^{<t}(S)$ and $A_r^{\ge t}(S)$ as the generalizations of (5.2.2) and (5.2.3) by

$$A_r^{\ge t}(S) := \{(\rho_1, \ldots, \rho_r, \theta_1, \ldots, \theta_r) \in [0,1]^{2r} : \left|S\left(\rho_1 e^{2\pi i\theta_1}, \ldots, \rho_r e^{2\pi i\theta_r}\right)\right| \ge t\}, \tag{5.2.4}$$

$$A_r^{<t}(S) := \{(\rho_1, \ldots, \rho_r, \theta_1, \ldots, \theta_r) \in [0,1]^{2r} : \left|S\left(\rho_1 e^{2\pi i\theta_1}, \ldots, \rho_r e^{2\pi i\theta_r}\right)\right| < t\}. \tag{5.2.5}$$

Our proof relies on the following extension of Lemma 2.5 in [**3**] to the areal case.

Lemma 5.2.3. *For a non-zero polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$, there exists a constant $C = C_n(P) > 0$, such that*

$$\lambda_{2n}\left(A_n^{<t}(P)\right) \le C t^{\frac{1}{d_n^{\min}(P)}}. \tag{5.2.6}$$

Proof. We again mimic the argument given in [**3**], and proceed by induction on n.

For $n = 1$, we start with $P(x) = a\prod_{j=1}^d (x - \alpha_j) \in \mathbb{C}[x]$, where $d = \deg P$ and $\{\alpha_j : 1 \le j \le d\}$ is the set of roots of P. Then the condition $|P(x)| < t$ for some $x \in \mathbb{C}$, implies that at least one of the α_j satisfies that $|x - \alpha_j| < \left(\frac{t}{|a|}\right)^{1/d}$. For convenience, let $B_{a,d,t} := \left(\frac{t}{|a|}\right)^{1/d}$.

Then

$$\lambda_2\left(A_1^{<t}(P)\right) \leq \lambda_2\left(\bigcup_{j=1}^{d} A_1^{<B_{a,d,t}}(x-\alpha_j)\right)$$

$$\leq \sum_{j=1}^{d} \lambda_2\left(A_1^{<B_{a,d,t}}(x-\alpha_j)\right) = \sum_{j=1}^{d} \lambda_2\left(\{(\rho,\theta)\in[0,1]^2 : |\rho e^{2\pi i\theta}-\alpha_j| < B_{a,d,t}\}\right)$$

$$\leq \sum_{j=1}^{d} \lambda_2\left(\{(\rho,\theta)\in[0,1]^2 : |\rho e^{2\pi i\theta}-|\alpha_j|| < B_{a,d,t}\}\right), \tag{5.2.7}$$

where the last inequality follows from the periodicity of $e^{2\pi i\theta}$. Suppose that there exists a $(\rho,\theta)\in[0,1]^2$ such that $|\rho e^{2\pi i\theta}-\alpha_j| < B_{a,d,t}$. Then, by the triangle inequality, we have $|\rho-|\alpha_j|| \leq |\rho e^{2\pi i\theta}-\alpha_j| < B_{a,d,t}$ and

$$|\rho e^{2\pi i\theta}-\rho| \leq |\rho e^{2\pi i\theta}-|\alpha_j|| + |\rho-|\alpha_j|| \leq 2B_{a,d,t}.$$

Hence (5.2.7) is

$$\leq \sum_{j=1}^{d} \lambda_2\left(\{(\rho,\theta)\in[0,1]^2 : |\rho e^{2\pi i\theta}-\rho| < 2B_{a,d,t}\}\right)$$

$$= \sum_{j=1}^{d} \lambda_2\left(\{(\rho,\theta)\in[0,1]^2 : \rho\sin(\pi\theta) < B_{a,d,t}\}\right)$$

$$\leq 2\sum_{j=1}^{d} \lambda_2\left(\{(\rho,\theta)\in[0,1]\times[0,1/2] : \rho\sin(\pi\theta) < B_{a,d,t}\}\right)$$

$$\leq 2\sum_{j=1}^{d} \lambda_2\left(\left\{(\rho,\theta)\in[0,1]\times[0,1/2] : \rho\theta < \frac{B_{a,d,t}}{2}\right\}\right), \tag{5.2.8}$$

where the last equality follows from the fact that $\sin(\pi\theta)\geq 2\theta$ for any $\theta\in[0,1/2]$. It only remains to compute the area under the curve $\rho\theta = \frac{B_{a,d,t}}{2}$ enclosed by the lines $\rho=0$, $\rho=1$, $\theta=0$, and $\theta=\frac{1}{2}$. In fact, if $B_{a,d,t}\leq 1$, we have

$$\lambda_2\left(\left\{(\rho,\theta)\in[0,1]\times[0,1/2] : \rho\theta < \frac{B_{a,d,t}}{2}\right\}\right) = \int_0^1 \min\left\{\frac{B_{a,d,t}}{2\rho}, \frac{1}{2}\right\} d\rho$$

$$= \int_0^{B_{a,d,t}} \frac{1}{2} d\rho + \int_{B_{a,d,t}}^1 \frac{B_{a,d,t}}{2\rho} d\rho = \frac{B_{a,d,t}}{2} + \frac{B_{a,d,t}}{2}\log B_{a,d,t} \leq \frac{B_{a,d,t}}{2}.$$

If $B_{a,d,t} > 1$, then $\lambda_2\left(\left\{(\rho,\theta)\in[0,1]\times[0,1/2] : \rho\theta < \frac{B_{a,d,t}}{2}\right\}\right) = \frac{1}{2} < \frac{B_{a,d,t}}{2}$. Therefore, from the above discussion, we can derive an upper bound for equation (5.2.8) as

$$2\sum_{j=1}^{d}\lambda_2\left(\left\{(\rho,\theta)\in[0,1]\times[0,1/2] : \rho\theta < \frac{B_{a,d,t}}{2}\right\}\right) \le 2\sum_{j=1}^{d}\frac{B_{a,d,t}}{2} = dB_{a,d,t}, \qquad (5.2.9)$$

which implies that

$$\lambda_2\left(A_1^{<t}(P)\right) \le dB_{a,d,t} = d|a|^{-\frac{1}{d}}t^{\frac{1}{d}}.$$

We conclude that $C_1(P) = d|a|^{-\frac{1}{d}}$.

Let $n \ge 2$, and suppose the statement holds up to $n-1$. Let $t' > 0$, which we will later choose in terms of t.

We now factor $P(x_1,\ldots,x_n)$ as

$$P(x_1,\ldots,x_n) = a(x_1,\ldots,x_{n-1})\prod_{j=1}^{h}(x_n - \alpha_j(x_1,\ldots,x_n)),$$

where $h = \deg_{x_n}(P)$, $a(x_1,\ldots,x_{n-1})$ is the *leading* coefficient of P as a polynomial in x_n, and $\alpha_j(x_1,\ldots,x_{n-1})$ are suitable branches of algebraic functions in $x_1,\ldots,x_n$. We divide $A_n^{<t}(P)$ in two parts: $A_n^{<t}(P)\cap A_{n-1}^{\ge t'}(a)$ and $A_n^{<t}(P)\cap A_{n-1}^{<t'}(a)$, where $A_{n-1}^{\ge t'}(a)$ and $A_{n-1}^{<t'}(a)$ are defined following (5.2.4) and (5.2.5), where $r = n-1$ and $S = a$. Our aim is to estimate Lebesgue measure of each set.

By induction, we have

$$\lambda_{2n}\left(A_n^{<t}(P)\cap A_{n-1}^{<t'}(a)\right) \le \lambda_{2(n-1)}\left(A_{n-1}^{<t'}(a)\right) \le C_{n-1}(a)(t')^{\frac{1}{d_{n-1}(a)}}.$$

For the other component of $A_n^{<t}(P)$, we have the following upper bound:

$$\lambda_{2n}\left(A_n^{<t}(P)\cap A_{n-1}^{\ge t'}(a)\right) \le \sum_{j=1}^{h}\lambda_{2n}\left(A_n^{<t/t'}\left(\prod_{j=1}^{h}(x_n - \alpha_j(x_1,\ldots,x_n))\right)\right) \le h\left(\frac{t}{t'}\right)^{1/h},$$

where the last inequality follows from a similar argument as in the $n = 1$ case. Choosing

$$t' = t^{\left(\deg_{x_n}(P)^{-1}\left(\frac{1}{d_{n-1}(a)}+\frac{1}{\deg_{x_n}(P)}\right)^{-1}\right)} = t^{\frac{d_{n-1}(a)}{d_n(P)}},$$

we have

$$\lambda_{2n}\left(A_n^{<t}(P)\right) \le C_n(P)t^{1/d_n(P)} \le C_n(P)t^{1/d_n^{\min}(P)},$$

which concludes the proof. □

Now we are ready to prove Proposition 5.2.2.

Proof of Proposition 5.2.2. Note that, for any $\sigma \in \left(-\frac{1}{d_n(P)}, 0\right)$, we have

$$\begin{aligned}
&2^n \int_{[0,1]^{2n}} \left|P\left(\rho_1 e^{2\pi i\theta_1}, \ldots, \rho_n e^{2\pi i\theta_n}\right)\right|^\sigma \rho_1 \cdots \rho_n d\rho_1 \cdots d\rho_n d\theta_1 \cdots d\theta_n \\
&\le 2^n \lambda_{2n}\left(A_n^{\ge 1}(P)\right) + 2^n \int_{A_n^{<1}(P)} \left|P\left(\rho_1 e^{2\pi i\theta_1}, \ldots, \rho_n e^{2\pi i\theta_n}\right)\right|^\sigma \rho_1 \cdots \rho_n d\rho_1 \cdots d\rho_n d\theta_1 \cdots d\theta_n \\
&= 2^n \lambda_{2n}\left(A_n^{\ge 1}(P)\right) \\
&\quad + 2^n \sum_{\ell=0}^{\infty} \int_{A_n^{\ge 2^{-\ell-1}}(P) \cap A_n^{<2^{-\ell}}(P)} \left|P\left(\rho_1 e^{2\pi i\theta_1}, \ldots, \rho_n e^{2\pi i\theta_n}\right)\right|^\sigma \rho_1 \cdots \rho_n d\rho_1 \cdots d\rho_n d\theta_1 \cdots d\theta_n \\
&\le 2^n \lambda_{2n}\left(A_n^{\ge 1}(P)\right) + 2^n \sum_{\ell=0}^{\infty} 2^{-\sigma(\ell+1)} \lambda_{2n}\left(A_n^{<2^{-\ell}}(P)\right) \\
&\le 2^n \left(\lambda_{2n}\left(A_n^{\ge 1}(P)\right) + C_n(P) \sum_{\ell=0}^{\infty} 2^{-\sigma(\ell+1) - \frac{\ell}{d_n^{\min}(P)}}\right) < \infty. \qquad (5.2.10)
\end{aligned}$$

Based on (5.2.10), we can infer that $\sigma_1(P) \le -\frac{1}{d_n^{\min}(P)}$. This proves statement (1).

If P does not vanish on $\mathbb{D}^n$, then there exists $r_P > 0$, such that $r_P \le |P(x_1, \ldots, x_n)| \le R_P$ for any $(x_1, \ldots, x_n) \in \mathbb{D}^n$. Then, the integral in (5.1.1) converges absolutely for all $s \in \mathbb{C}$. In other words, $\sigma_1(P) = -\infty$, which implies statement (2). $\square$

The next proposition lists other fundamental properties of the zeta function $Z_{\mathbb{D}}$ that follow directly from its definition and Proposition 5.2.1.

Proposition 5.2.4. *Let $P \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-zero polynomial. Then*

(1) for any $s \in \mathbb{C}$, $Z_{\mathbb{D}}(s, 1) = 1$;

(2) for any $a \in \mathbb{C}^\times$ and $\mathrm{Re}(s) > \sigma_1(P)$, $Z_{\mathbb{D}}(s, aP) = |a|^s Z_{\mathbb{D}}(s, P)$;

(3) for any $k \in \mathbb{Z}_{\ge 1}$ and $\mathrm{Re}(s) > \sigma_1(P)/k$, $Z_{\mathbb{D}}(s, P^k) = Z_{\mathbb{D}}(ks, P)$.

In the next sections, our goal is to investigate $Z_{\mathbb{D}}$ for the family of polynomials $\{k+x+y : k \in \mathbb{C}\}$, and eventually derive the results of Theorems 5.1.1 and 5.1.2, which, along with Cassaigne–Maillot's formula (0.5.1), will help us explicitly compute $\mathrm{m}_{\mathbb{D}}(k + x + y)$ in terms of special values of dilogarithms, for any $k \in \mathbb{C}$.

5.3. The areal zeta Mahler measure of $k + x + y$

Following the works of Borwein and Straub [**24**], Borwein, Straub, Wan, and Zudilin [**26**], Ringeling [**101**], and others, we compute $Z_{\mathbb{D}}(k + x + y)$ as a solution of a certain hypergeometric differential equation, and derive the result of Theorem 5.1.1.

We start with the simple case when $k = 0$.

Theorem 5.3.1. *For $s \in \mathbb{C}$,*

$$Z_{\mathbb{D}}(s, x + y) = \frac{4}{s+4} \frac{\Gamma(s+2)}{\Gamma\left(\frac{s}{2}+2\right)^2}. \tag{5.3.1}$$

Proof. By Definition 0.6.20, we have

$$\begin{aligned}
Z_{\mathbb{D}}(s, x + y) =& \frac{1}{\pi^2} \int_{\mathbb{D}} \int_{\mathbb{D}} |x + y|^s dA(x) dA(y) \\
=& \frac{1}{\pi^2} \int_0^1 \int_0^1 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\rho_1 e^{i\theta_1} + \rho_2 e^{i\theta_2}|^s \rho_1 \rho_2 d\theta_1 d\theta_2 d\rho_1 d\rho_2 \\
=& \frac{1}{\pi^2} \int_0^1 \int_0^1 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \rho_2^s |\rho_1 \rho_2^{-1} e^{i(\theta_1 - \theta_2)} + 1|^s \rho_1 \rho_2 d\theta_1 d\theta_2 d\rho_1 d\rho_2 \\
=& \frac{2}{\pi} \int_{0 \le \rho_2 \le \rho_1 \le 1} \rho_2 \rho_1^{s+1} \int_{-\pi}^{\pi} |\rho_2 \rho_1^{-1} e^{i\tau} + 1|^s d\tau d\rho_1 d\rho_2 \\
&+ \frac{2}{\pi} \int_{0 \le \rho_1 \le \rho_2 \le 1} \rho_1 \rho_2^{s+1} \int_{-\pi}^{\pi} |\rho_1 \rho_2^{-1} e^{i\tau} + 1|^s d\tau d\rho_1 d\rho_2,
\end{aligned}$$

where we have set $\tau = \theta_1 - \theta_2$. Using (0.6.17),[2] we have

$$\begin{aligned}
Z_{\mathbb{D}}(s, x + y) =& 4 \int_{0 \le \rho_2 \le \rho_1 \le 1} \rho_1^{s+1} \rho_2 \, {}_2F_1 \left(-\frac{s}{2}, -\frac{s}{2}; 1; \rho_2^2 \rho_1^{-2} \right) d\rho_1 d\rho_2 \\
&+ 4 \int_{0 \le \rho_1 \le \rho_2 \le 1} \rho_1 \rho_2^{s+1} \, {}_2F_1 \left(-\frac{s}{2}, -\frac{s}{2}; 1; \rho_1^2 \rho_2^{-2} \right) d\rho_1 d\rho_2.
\end{aligned} \tag{5.3.2}$$

2. We use the identity in (0.6.17) when $|a| = \rho_2 \rho_1^{-1} < 1$

Under the change of variables $\rho_1 \mapsto \rho_2$ and $\rho_2 \mapsto \rho_1$, the first integral transforms into the second integral. Hence (5.3.2) becomes

$$\begin{aligned}
Z_{\mathbb{D}}(s, x+y) =& 8\int_{0\le\rho_2\le\rho_1\le 1} \rho_1^{s+1}\rho_2\, {}_2F_1\left(-\frac{s}{2}, -\frac{s}{2}; 1; \rho_2^2\rho_1^{-2}\right) d\rho_1 d\rho_2 \\
=& 8\int_0^1 \rho_1^{s+3}\int_0^1 \sigma\, {}_2F_1\left(-\frac{s}{2}, -\frac{s}{2}; 1; \sigma^2\right) d\sigma d\rho_1 \\
=& \frac{8}{s+4}\cdot\frac{1}{2}\int_0^1 {}_2F_1\left(-\frac{s}{2}, -\frac{s}{2}; 1; \sigma^2\right) d\sigma^2 \\
=& \frac{4}{s+4}{}_2F_1\left(-\frac{s}{2}, -\frac{s}{2}; 2; 1\right) \\
=& \frac{4}{s+4}\frac{\Gamma(s+2)}{\Gamma\left(\frac{s}{2}+2\right)^2},
\end{aligned}$$

where the last equality follows from Theorem 3.3.1 (by taking $a = b = -\frac{s}{2}$ and $c = 2$). □

The argument for general k starts with the case when $|k| > 2$. Recall that, for $|k| > 2$, we have $\mathrm{m}_{\mathbb{D}}(k + x + y) = \log |k|$ from (0.6.5). We will express the $Z_{\mathbb{D}}(s, k + x + y)$ in terms of hypergeometric functions depending on $|k|$ and s when $|k| > 2$, compute the differential equation satisfied by the obtained expression of $Z_{\mathbb{D}}(s, k + x + y)$ and argue that, for general k, $Z_{\mathbb{D}}(s, k + x + y)$ is, in fact, a particular solution of that differential equation. This leads to the identity in (5.1.2). A similar argument as the one given in Remark 0.6.26 implies that $Z_{\mathbb{D}}(s, k+x+y)$ is invariant under $k \mapsto |k|$. Therefore, it suffices to compute $Z_{\mathbb{D}}(s, |k|+x+y)$ instead.

Let X and Y be the random variables defined by walks of lengths ρ_1 and ρ_2 along the directions θ_1 and θ_2, uniformly distributed on $[0, 1)$, respectively. In particular, X takes values $x = \rho_1 e^{2\pi i\theta_1}$ and Y takes values $y = \rho_2 e^{2\pi i\theta_2}$. Let Z be another random walk of unit length and direction θ.

We define a new random variable T_1 as the absolute value $|X + Y|$. Let p_{T_1} denote the probability density function of T_1, which has support on $[0, 2]$. We further define the random variable T_2 as the absolute value $||k|Z + X + Y|$. Let p_{T_2} denote the probability density function of T_2, which has support on $[|k| - 2, |k| + 2]$, when $|k| > 2$.

Observe that

$$
\begin{aligned}
Z_{\mathbb{D}}(s,|k|+x+y) =&\frac{1}{\pi^2}\int_{\mathbb{D}^2} ||k|+x+y|^s dA(x)dA(y)\\
=&2^2\int_{[0,1]^4} ||k|+\rho_1 e^{2\pi i\theta_1}+\rho_2 e^{2\pi i\theta_2}|^s \rho_1\rho_2 d\rho_1 d\rho_2 d\theta_1 d\theta_2\\
=&2^2\int_{[0,1]^5} ||k|e^{2\pi i\theta}+\rho_1 e^{2\pi i\theta_1}+\rho_2 e^{2\pi i\theta_2}|^s \rho_1\rho_2 d\rho_1 d\rho_2 d\theta_1 d\theta_2 d\theta. \qquad (5.3.3)
\end{aligned}
$$

We now apply the change of variables $||k|e^{2\pi i\theta}+x+y| = u$ and $|x+y| = v$ to obtain, for $|k| > 2$,

$$
\begin{aligned}
Z_{\mathbb{D}}(s,|k|+x+y) =&\frac{1}{\pi^2}\int_0^1\int_{\mathbb{D}}\int_{\mathbb{D}} ||k|e^{2\pi i\theta}+x+y|^s dA(x)dA(y)d\theta\\
=&\int_0^2\int_{|k|-v}^{|k|+v} u^s\,\mathbb{P}(T_2=u\,|\,T_1=v)\,\mathbb{P}(T_1=v)dudv. \qquad (5.3.4)
\end{aligned}
$$

where the normalized area measures of the variables x and y in the first integral are substituted by the new probability measures obtained from functions of the variables x and y, namely u and v. In (5.3.4), we have

$$
\mathbb{P}(T_2=u\,|\,T_1=v) = p_{T_2|T_1}(u|v), \text{ and } \mathbb{P}(T_1=v) = p_{T_1}(v),
$$

where

$$
\mathbb{P}(T_2=u\,|\,T_1=v) = \frac{\mathbb{P}(T_2=u,T_1=v)}{\mathbb{P}(T_1=v)} = \frac{p_{T_2,T_1}(u,v)}{p_{T_1}(v)} = p_{T_2|T_1}(u|v). \qquad (5.3.5)
$$

Here p_{T_2,T_1} is the joint probability density function of the random variable (T_2, T_1).

Remark 5.3.2. *Note that, since T_2 is dependent on T_1, the moment generating function of $\log T_2$ can be expressed as follows:*

$$\begin{aligned}
\mathbb{E}[e^{s\log T_2}] = \mathbb{E}[T_2^s] &= \int_{||k|-2|}^{|k|+2} u^s \mathbb{P}(T_2 = u) du \\
&= \int_{||k|-2|}^{|k|+2} u^s \left(\int_0^2 \mathbb{P}(T_2 = u, T_1 = v) dv \right) du \qquad (5.3.6) \\
&= \int_0^2 \int_{||k|-2|}^{|k|+2} u^s \mathbb{P}(T_2 = u, T_1 = v) du dv \\
&= \int_0^2 \int_{||k|-v|}^{|k|+v} u^s \mathbb{P}(T_2 = u | T_1 = v) \mathbb{P}(T_1 = v) du dv, \qquad (5.3.7)
\end{aligned}$$

where the last equality follows from (5.3.5). *After integrating with respect to u,* (5.3.7) *becomes*

$$\begin{aligned}
&= \int_0^2 \mathbb{E}[T_2^s | T_1 = v] \mathbb{P}(T_1 = v) dv \\
&= \mathbb{E}[\mathbb{E}[T_2^s | T_1]], \qquad (5.3.8)
\end{aligned}$$

where the equality in (5.3.8) *follows from considering $\mathbb{E}[T_2|T_1 = v]$ as a function of v (and therefore as a function of T_1), and then evaluating the expectation of the function with respect to the variable T_1. This also shows that the assertions* (5.3.3) *and* (5.3.7) *together imply that the areal zeta Mahler measure of $|k| + x + y$ coincides with the moment generating function of $\log T_2$, namely $\mathbb{E}\left[e^{s\log T_2}\right]$.*

It remains to compute (5.3.4), for which we need the next two lemmas.

Lemma 5.3.3. *For $0 \le v \le 2$, we have*

$$\mathbb{P}(T_1 = v) = p_{T_1}(v) = \frac{v}{\pi}\left(2\pi - v\sqrt{4 - v^2} - 4\arcsin\left(\frac{v}{2}\right)\right). \qquad (5.3.9)$$

Proof. Following the discussion in Remark 5.3.2, we observe that the areal zeta Mahler measure of $x + y$ coincides with the moment generating function of T_1, namely (see Theorem 5.3.1)

$$Z_{\mathbb{D}}(s, x + y) = \frac{4}{s + 4} \frac{\Gamma(s + 2)}{\Gamma\left(\frac{s}{2} + 2\right)^2} = \mathbb{E}[T_1^s] = \int_0^\infty v^s p_{T_1}(v) dv = \int_0^2 v^s p_{T_1}(v) dv.$$

Further, the above equality implies that $Z_{\mathbb{D}}(s-1,x+y)$ is the Mellin transform of $p_{T_1}(v)$. Then, we can retrieve $p_{T_1}(v)$ by considering the inverse Mellin transform of

$$Z_{\mathbb{D}}(s-1,x+y)=\frac{4}{s+3}\frac{\Gamma(s+1)}{\Gamma\left(\frac{s-1}{2}+2\right)^2}.$$

In order to obtain the expression of $p_{T_1}(v)$ given in (5.3.9), we first note that the inverse Mellin transform of $Z_{\mathbb{D}}(s-1,x+y)$ is

$$p_{T_1}(v)=\sum_{k=1}^{\infty}\mathrm{Res}_{s=-k}\left[\frac{4}{s+3}\frac{\Gamma(s+1)}{\Gamma\left(\frac{s-1}{2}+2\right)^2}v^{-s}\right]=2v-\frac{4}{\pi}v^2+\sum_{k=4}^{\infty}\frac{4}{(3-k)}\frac{(-v)^k}{\Gamma(k+1)\,\Gamma\left(2-\frac{k+1}{2}\right)^2}. \tag{5.3.10}$$

To show that the series in (5.3.10) is indeed $\frac{v}{\pi}\left(2\pi-v\sqrt{4-v^2}-4\arcsin\left(\frac{v}{2}\right)\right)$ for $0\le v\le 2$, we consider the coefficients of v^j of the Taylor series expansion of the function

$$g(v):=2v-\frac{4}{\pi}v^2+\sum_{k=4}^{\infty}\frac{4}{(3-k)}\frac{(-v)^k}{\Gamma(k+1)\,\Gamma\left(2-\frac{k+1}{2}\right)^2}-\frac{v}{\pi}\left(2\pi-v\sqrt{4-v^2}-4\arcsin\left(\frac{v}{2}\right)\right)$$

at $v=0$. These coefficients are given by $\frac{1}{j!}\left.\frac{d^jg(v)}{dv^j}\right|_{v=0}$, and a simple evaluation shows that $\left.\frac{d^jg(v)}{dv^j}\right|_{v=0}=0$ for all $j\ge 0$. This implies that $g(v)$ is identically 0, which further shows that, for $0\le v\le 2$,

$$p_{T_1(v)}=2v-\frac{4}{\pi}v^2+\sum_{k=4}^{\infty}\frac{4}{(3-k)}\frac{(-v)^k}{\Gamma(k+1)\,\Gamma\left(2-\frac{k+1}{2}\right)^2}=\frac{v}{\pi}\left(2\pi-v\sqrt{4-v^2}-4\arcsin\left(\frac{v}{2}\right)\right).$$

□

Lemma 5.3.4. *Given $0\le v\le 2$, for $u\in(||k|-v|,||k|+v|)$, we have*

$$\mathbb{P}(T_2=u\,|\,T_1=v)=p_{T_2|T_1}(u|v)=\frac{2u}{\pi\sqrt{4v^2|k|^2-(u^2-v^2-|k|^2)^2}}. \tag{5.3.11}$$

Proof. We start by computing the cumulative distribution function of T_2 at $T_1=v$ given by

$$f_v(u)=\mathbb{P}(T_2\le u\,|\,T_1=v).$$

Since $p_{T_2|T_1}(u|v)$ is the partial derivative of $f_v(u)$ with respect to u, it only remains to compute f_v.

Writing $Z = e^{2\pi i\theta}$, i.e. θ is the direction of Z, where θ is uniformly distributed in $[-\pi, \pi)$, we have

$$f_v(u) = \mathbb{P}(T_2 \le u \,|\, T_1 = v) = \mathbb{P}(||k|e^{2i\pi\theta} + v| \le u \,|\, T_1 = v) = \frac{1}{2\pi}\int_{[-\pi,\pi)\cap I_{u,v}} d\theta,$$

where $I_{u,v}$ is the set of values of θ such that

$$T_2^2 = ||k|e^{2i\pi\theta} + v|^2 = v^2 + |k|^2 + 2v|k|\cos\theta \le u^2 \iff \cos\theta \le \frac{u^2 - v^2 - |k|^2}{2v|k|}.$$

Since $v \in [0, 2]$ and $u \in (||k|-v|, ||k|+v|)$, $I_{u,v}$ is a non-empty set. Let $\beta = \arccos\frac{u^2-v^2-|k|^2}{2v|k|} \in [0, \pi)$. Then

$$f_v(u) = \frac{1}{2\pi}\int_{[-\pi,\pi)\cap I_{u,v}} d\theta = 1 - \frac{1}{2\pi}\int_{-\beta}^{\beta} d\theta = 1 - \frac{\beta}{\pi} = 1 - \frac{\arccos\frac{u^2-v^2-|k|^2}{2v|k|}}{\pi}.$$

Taking the derivative with respect to u, we obtain the required expression in (5.3.11). ☐

We remark that Lemmas 5.3.3 and 5.3.4 are derived independently of the assumption $|k| > 2$ or $|k| \le 2$.

We are now ready to compute the integral in (5.3.4). When $|k| > 2$, evaluation of the integral follows from the following two crucial lemmas.

Lemma 5.3.5. *For* $\mathrm{Re}(s) > 0$ *and* $|k| > 2$, *we have*

$$\frac{2}{\pi}\int_{|k|-v}^{|k|+v} u^s \cdot \frac{u}{\sqrt{4v^2|k|^2 - (v^2 - u^2 - |k|^2)^2}} du = |k|^s \cdot {}_2F_1\left(-\frac{s}{2}, -\frac{s}{2}; 1; \frac{v^2}{|k|^2}\right). \qquad (5.3.12)$$

Proof. Since $|k| > 2$, we have $|k| > v$. Evaluating the inner integral in (5.3.4) with respect to u and using (5.3.11), we have

$$\frac{2}{\pi}\int_{|k|-v}^{|k|+v} u^s \frac{u}{\sqrt{4v^2|k|^2 - (u^2 - v^2 - |k|^2)^2}} du.$$

Setting $w = u^2$, the above integral equals

$$\frac{1}{\pi}\int_{(|k|-v)^2}^{(|k|+v)^2} \frac{w^{s/2}}{\sqrt{4v^2|k|^2 - (w - v^2 - |k|^2)^2}} dw. \qquad (5.3.13)$$

After the change of variables

$$w' = \frac{w - (|k| - v)^2}{(|k| + v)^2 - (|k| - v)^2} = \frac{w - (|k| - v)^2}{4v|k|},$$

(5.3.13) becomes

$$\frac{1}{\pi}\int_0^1 \frac{(4|k|vw' + (|k|-v)^2)^{s/2}}{\sqrt{w'(1-w')}}dw' = (|k|-v)^s \ {}_2F_1\left(-\frac{s}{2}, \frac{1}{2}; 1; \frac{-4kv}{(|k|-v)^2}\right).$$

The last equality follows from the integral representation of the hypergeometric function given in (3.3.2). We conclude that the above integral converges when $\mathrm{Re}(s) > 0$ from the convergence condition mentioned in Theorem 3.3.3.

Using a quadratic transformation for the hypergeometric function **[65]** [3], we find, for $|k| > 2 \geq v$,

$$(|k|-v)^s \ {}_2F_1\left(-\frac{s}{2}, \frac{1}{2}; 1; \frac{-4|k|v}{(|k|-v)^2}\right) = |k|^s \cdot {}_2F_1\left(-\frac{s}{2}, -\frac{s}{2}; 1; \frac{v^2}{|k|^2}\right).$$

This concludes the proof. □

We need the next lemma to completely evaluate the integral in (5.3.4) for $|k| > 2$.

Lemma 5.3.6. *For all* $\mathrm{Re}(s) > 0$, *we have the following integral evaluation*

$$\int_0^2 p_{T_1}(v) \cdot {}_2F_1\left(-\frac{s}{2}, -\frac{s}{2}; 1; \frac{v^2}{|k|^2}\right) dv = {}_3F_2\left(-\frac{s}{2}, -\frac{s}{2}, \frac{3}{2}; 2, 3; \frac{4}{|k|^2}\right).$$

Proof. We expand the hypergeometric function into its series and interchange the integral and the sum. In other words, we have

$$\begin{aligned}\int_0^2 p_{T_1}(v) \cdot {}_2F_1\left(-\frac{s}{2}, -\frac{s}{2}; 1; \frac{v^2}{|k|^2}\right) dv &= \sum_{n=0}^{\infty} \frac{\left(-\frac{s}{2}\right)_n^2}{n!^2} \frac{1}{|k|^{2n}} \int_0^2 v^{2n} p_{T_1}(v)dv \\ &= \sum_{n=0}^{\infty} \frac{\left(-\frac{s}{2}\right)_n^2}{n!^2} \frac{1}{|k|^{2n}} \frac{4^n \left(\frac{3}{2}\right)_n}{(n+1)\,(3)_n},\end{aligned}$$

where the last equality follows from (5.3.1) and the discussion in Remark 5.3.2, since

$$\int_0^2 v^{2n} p_{T_1}(v)dv = Z_{\mathbb{D}}(2n, x+y) = \frac{4}{2n+4}\frac{\Gamma(2n+2)}{\Gamma(n+2)^2} = \frac{2\cdot 2^n}{n+1}\frac{3\cdot 5\cdots 2n+1}{2\cdot (3)_n} = \frac{4^n\left(\frac{3}{2}\right)_n}{(n+1)\,(3)_n}.$$

Furthermore,

$$\sum_{n=0}^{\infty} \frac{\left(-\frac{s}{2}\right)_n^2}{n!^2} \frac{1}{|k|^{2n}} \frac{4^n \left(\frac{3}{2}\right)_n}{(n+1)\,(3)_n} = \sum_{n=0}^{\infty} \frac{\left(-\frac{s}{2}\right)_n^2 \left(\frac{3}{2}\right)_n}{(3)_n (2)_n} \frac{\left(\frac{4}{|k|^2}\right)^n}{n!}$$

3. For $|z| < 1$, we have ${}_2F_1(a, b; a-b+1; z) = (1+\sqrt{z})^{-2a} {}_2F_1\left(a, a-b+\frac{1}{2}; 2a-2b+1; \frac{4\sqrt{z}}{(1+\sqrt{z})^2}\right)$.

coincides with the series representation of ${}_3F_2\left(-\frac{s}{2}, -\frac{s}{2}, \frac{3}{2}; 2, 3; \frac{4}{|k|^2}\right)$, and this concludes the proof. ∎

Combining Lemmas 5.3.5 and 5.3.6, we have, for $|k| > 2$ and $\mathrm{Re}(s) > 0$,

$$Z_{\mathbb{D}}(s, |k| + x + y) = |k|^s {}_3F_2\left(-\frac{s}{2}, -\frac{s}{2}, \frac{3}{2}; 2, 3; \frac{4}{|k|^2}\right). \tag{5.3.14}$$

We now extend our method for $|k| \leq 2$. Notice that in this case the boundary points of the inner integral in (5.3.4) will be $||k| - v|$ and $|k| + v$, since for $0 \leq v \leq 2$, there exists at least one v such that $|k| \leq v$. Denote

$$F(|k|) := |k|^s \cdot {}_2F_1\left(-\frac{s}{2}, -\frac{s}{2}; 1; \frac{v^2}{|k|^2}\right). \tag{5.3.15}$$

Now, to evaluate the integral in (5.3.4) with boundary points $||k| - v|$ and $|k| + v$, we have the following lemma.

Lemma 5.3.7. *Let s be a real positive number that is not an odd integer. Then, for $|k| \leq 2$, we have*

$$\int_{||k|-v|}^{|k|+v} u^s \cdot \mathbb{P}(T_2 = u \,|\, T_1 = v)\, du = \mathrm{Re}\left(F(|k|)\right) - \cot\left(\frac{\pi s}{2}\right) \mathrm{Im}\left(F(|k|)\right). \tag{5.3.16}$$

Proof. For $|k| \geq v$, the result follows from Lemma 5.3.5, since the integral is real, i.e. $\mathrm{Im}(F(|k|)) = 0$. Assume $|k| < v$. We can split the integral in (5.3.12) as

$$\begin{aligned} F(|k|) &= \int_{|k|-v}^{|k|+v} u^s \cdot \frac{u du}{\sqrt{4v^2|k|^2 - (v^2 - u^2 - |k|^2)^2}} \\ &= \int_{v-|k|}^{|k|+v} u^s \cdot \frac{u du}{\sqrt{4v^2|k|^2 - (v^2 - u^2 - |k|^2)^2}} + \int_0^{v-|k|} u^s \cdot \frac{u du}{\sqrt{4v^2|k|^2 - (v^2 - u^2 - |k|^2)^2}} \\ &\quad + \int_{|k|-v}^{0} u^s \cdot \frac{u du}{\sqrt{4v^2|k|^2 - (v^2 - u^2 - |k|^2)^2}} && (5.3.17) \\ &= \int_{v-|k|}^{|k|+v} \cdot \frac{u^{s+1} du}{\sqrt{4v^2|k|^2 - (v^2 - u^2 - |k|^2)^2}} + (1 - e^{\pi i s}) \int_0^{v-|k|} \cdot \frac{u^{s+1} du}{\sqrt{4v^2|k|^2 - (v^2 - u^2 - |k|^2)^2}} && (5.3.18) \\ &= \int_{v-|k|}^{|k|+v} \cdot \frac{u^{s+1} du}{\sqrt{4v^2|k|^2 - (v^2 - u^2 - |k|^2)^2}} + (1 - e^{\pi i s}) \int_0^{v-|k|} \cdot \frac{u^{s+1} du}{\sqrt{4v^2|k|^2 - (v^2 - u^2 - |k|^2)^2}}. && (5.3.19) \end{aligned}$$

The penultimate equality follows from applying the change of variables $u \to -u$ to the third integral in (5.3.17). Observe that in (5.3.18), the integrand is real on the interval $[v-|k|,|k|+v]$, while it is purely imaginary on the intervals $[0, v-|k|]$ and $[|k|-v, 0]$. Note that we need s to be real to express $(-1)^s$ as $e^{\pi i s}$ in (5.3.18). Next, taking the complex conjugate of (5.3.19), we have

$$\overline{F(|k|)} = \int_{v-|k|}^{|k|+v} \frac{u^{s+1}du}{\sqrt{4v^2|k|^2-(v^2-u^2-|k|^2)^2}} - (1-e^{-\pi i s}) \int_0^{v-|k|} \frac{u^{s+1}du}{\sqrt{4v^2|k|^2-(v^2-u^2-|k|^2)^2}}.$$

Combining the above equality with (5.3.19), we finally have

$$\begin{aligned}\int_{v-|k|}^{|k|+v} \frac{u^{s+1}}{\sqrt{4v^2|k|^2-(v^2-u^2-|k|^2)^2}}du &= \frac{F(|k|)-e^{\pi i s}\overline{F(|k|)}}{1-e^{\pi i s}} \\ &= \mathrm{Re}\,(F(|k|)) - \cot\left(\frac{\pi s}{2}\right)\mathrm{Im}\,(F(|k|)).\end{aligned}$$

□

Now integrating $F(|k|)\cdot p_{T_1}(v)$ as in Lemma 5.3.6 and considering the real and imaginary parts, we have the following proposition.

Proposition 5.3.8. *For real $s > 0$, not an odd integer,*

$$Z_{\mathbb{D}}(s, |k|+x+y) = \mathrm{Re}\,(G(|k|)) - \cot\left(\frac{\pi s}{2}\right)\mathrm{Im}\,(G(|k|)), \tag{5.3.20}$$

where

$$G(|k|) = |k|^s \cdot {}_3F_2\left(-\frac{s}{2}, -\frac{s}{2}, \frac{3}{2}; 2, 3; \frac{4}{|k|^2}\right).$$

A consequence of Proposition 5.3.8 is that $Z_{\mathbb{D}}(s, |k|+x+y)$ satisfies the same differential equation for both $|k| \le 2$ and $|k| > 2$, as both contain the same hypergeometric function $G(|k|)$ in their expressions (compare (5.3.14) with (5.3.20)). For $z = \frac{|k|^2}{4}$, denote

$$\mathcal{G}(z) := G(\sqrt{4z}) = (4z)^{-\frac{s}{2}} {}_3F_2\left(-\frac{s}{2}, -\frac{s}{2}, \frac{3}{2}; 2, 3; \frac{1}{z}\right). \tag{5.3.21}$$

The goal here is to find a differential equation satisfied by $\mathcal{G}(z)$ and a particular solution $H(z) = H\left(\frac{|k|^2}{4}\right)$ to the differential equation which coincides with $G(|k|)$ when $|k| > 2$ and $s > 0$. Then, by the analytic properties of $H\left(\frac{|k|^2}{4}\right)$, we will conclude that $Z_{\mathbb{D}}(s, |k|+x+y) = H\left(\frac{|k|^2}{4}\right)$ for all $k \in \mathbb{C}$ and for a larger region of $s \in \mathbb{C}$.

In order to obtain a differential equation satisfied by $\mathcal{G}$, we start with a hypergeometric differential equation satisfied by ${}_3F_2\left(-\frac{s}{2}, -\frac{s}{2}, \frac{3}{2}; 2, 3; z\right)$. From [**95**], we have that the hypergeometric function ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ satisfies the differential equation

$$z\prod_{n=1}^{p}\left(z\frac{d}{dz}+a_n\right)\mathcal{W}(z) = z\frac{d}{dz}\prod_{\ell=1}^{q}\left(z\frac{d}{dz}+b_\ell-1\right)\mathcal{W}(z), \tag{5.3.22}$$

where $\left(z\frac{d}{dz}\right)^2 f(z) = z\frac{d}{dz}zf'(z) = zf'(z) + z^2 f''(z)$ for a twice differentiable function f. Let $\Theta := z\frac{d}{dz}$. Then, replacing $p = 3, q = 2, a_1 = a_2 = -\frac{s}{2}, a_3 = \frac{3}{2}, b_1 = 2$, and $b_3 = 3$ in (5.3.22), we obtain the third order differential equation

$$\begin{aligned} \left[z\left(\Theta-\frac{s}{2}\right)\left(\Theta-\frac{s}{2}\right)\left(\Theta+\frac{3}{2}\right)-\Theta\left(\Theta+1\right)\left(\Theta+2\right)\right]\mathcal{W}(z) =&0\\ \Longleftrightarrow \left[(z-1)\,\Theta^3-\left(3-\frac{3z}{2}+zs\right)\Theta^2-\left(2+\frac{3zs}{2}-\frac{zs^2}{4}\right)\Theta+\frac{3zs^2}{8}\right]\mathcal{W}(z) =&0, \end{aligned}$$

which is satisfied by

$$\frac{\mathcal{G}(z^{-1})}{(4z)^{\frac{s}{2}}} = {}_3F_2\left(-\frac{s}{2}, -\frac{s}{2}, \frac{3}{2}; 2, 3; z\right).$$

Now, to find the differential equation satisfied by $\mathcal{G}$, we substitute $\mathcal{W}(z)$ with $\frac{\mathcal{G}(z^{-1})}{(4z)^{\frac{s}{2}}}$ above. Next, using the change of variables $z \mapsto \frac{1}{z}$ and further simplifying, we obtain a third order differential equation

$$s(8+6s+s^2)\mathcal{V}(z)-2(2+3s^2z+s(2+6z))\mathcal{V}'(z)-4z(-3+s-3sz)\mathcal{V}''(z)-8(z-1)z^2\mathcal{V}'''(z) = 0, \tag{5.3.23}$$

which is satisfied by $\mathcal{G}(z)$. Dividing both sides of (5.3.23) by the coefficient of $\mathcal{V}'''(z)$, we find that this differential equation has a regular singularity at $z = 0$ (see [**16**]).

It remains to obtain a fundamental set of solutions of (5.3.23) around $z = 0$. Using the method of Frobenius to find power-series solutions to differential equations [**95**],[4] we show that the local exponents at $z = 0$ are $0, 0$ and $(3+s)/2$, and we further obtain that the differential equation has a basis of solutions around $z = 0$ of the form

$$z^{(3+s)/2}F_0(z;s), \qquad F_1(z;s) \qquad \text{and} \qquad F_2(z;s)+\log(z)F_1(z;s),$$

4. See [**66**, §16.3] for more details.

where F_0, F_1 and F_2 are holomorphic, non-zero at $z=0$, and defined by

$$F_0(z;s) = {}_3F_2\left(-\frac{1}{2},\frac{1}{2},\frac{3}{2};\frac{5+s}{2},\frac{5+s}{2};z\right), \tag{5.3.24}$$

$$F_1(z;s) = {}_3F_2\left(-2-\frac{s}{2},-1-\frac{s}{2},-\frac{s}{2};1,-\frac{1}{2}-\frac{s}{2};z\right), \tag{5.3.25}$$

$$F_2(z;s) = G^{3\,3}_{4\,3}\left(\frac{2+s}{2},\frac{4+s}{2},\frac{6+s}{2};0,0,\frac{3+s}{2};z\right), \tag{5.3.26}$$

where the Meijer G-functions $G^{m\,n}_{p\,q}$ are generalization of hypergeometric functions:

$$\begin{aligned} &G^{m\,n}_{p\,q}(a_1,\ldots,a_p;b_1,\ldots,b_q;z) \\ =&\sum_{h=1}^{m}\frac{\prod_{\substack{j=1\\ j\neq h}}^{m}\Gamma(b_j-b_h)\ \prod_{j=1}^{n}\Gamma(1+b_h-a_j)\ z^{b_h}}{\prod_{j=m+1}^{q}\Gamma(1+b_h-b_j)\ \prod_{j=n+1}^{p}\Gamma(a_j-b_h)} \\ &\times {}_pF_{q-1}\left(1+b_h-a_1,\ldots,1+b_h-a_p;1+b_h-b_1,\ldots,\widehat{1+b_h-b_h},\ldots,1+b_h-b_n;z\right), \end{aligned}$$

where $a_\ell, b_j \in \mathbb{C}^\times$ and $\widehat{\ }$ indicates that the term corresponding to $j=h$ is omitted.

Now we have all the elements to prove Theorem 5.1.1.

Proof of Theorem 5.1.1. From Theorem 5.3.1, we have, for almost all[5] s, that $Z_{\mathbb{D}}(s,|k|+x+y)$ converges as $k\to 0$. This eliminates $F_2(z;s)+\log(z)F_1(z;s)$ as a possible contributor to the expression of $Z_{\mathbb{D}}(s,|k|+x+y)$, since it does not converge as $z\to 0$ and $z=\frac{|k|^2}{4}$. Therefore, $Z_{\mathbb{D}}(s,|k|+x+y)$ is a linear combination of $z^{(3+s)/2}F_0(z;s)$ and $F_1(z;s)$, where $F_0(z;s)$ and $F_1(z;s)$ are as defined in (5.3.24) and (5.3.25), respectively.

Further, using the initial conditions $Z_{\mathbb{D}}(s,2+x+y)=G(2)$ and $Z_{\mathbb{D}}(s,x+y)=\frac{4}{s+4}\frac{\Gamma(s+2)}{\Gamma\left(\frac{s}{2}+2\right)^2}=G(0)$ we conclude that

$$Z_{\mathbb{D}}(s,|k|+x+y)=c_0(s)\left(\frac{|k|}{2}\right)^{s+3}F_0\left(\frac{|k|^2}{4};s\right)+c_1(s)F_1\left(\frac{|k|^2}{4};s\right), \tag{5.3.27}$$

5. The only problematic s are coming from the zeros of $\Gamma\left(\frac{s}{2}+2\right)$ and $s+4$, and the poles of $\Gamma(s+2)$.

where $c_1(s)=\frac{4}{s+4}\frac{\Gamma(s+2)}{\Gamma\left(\frac{s}{2}+2\right)^2}$ and

$$\begin{aligned} c_0(s) &= \frac{G(2)-c_1(s)F_1(1;s)}{F_0(1;s)} \\ &= \frac{2^s\cdot {}_3F_2\left(-\frac{s}{2},-\frac{s}{2},\frac{3}{2};2,3;1\right)-\frac{4}{s+4}\frac{\Gamma(s+2)}{\Gamma\left(\frac{s}{2}+2\right)^2}\cdot {}_3F_2\left(-2-\frac{s}{2},-1-\frac{s}{2},-\frac{s}{2};1,-\frac{1}{2}-\frac{s}{2};1\right)}{{}_3F_2\left(-\frac{1}{2},\frac{1}{2},\frac{3}{2};\frac{5+s}{2},\frac{5+s}{2};1\right)}. \end{aligned}$$

Since a hypergeometric series ${}_{q+1}F_q(a_1,\dots,a_{q+1};b_1,\dots,b_q;z)$ absolutely converges at $z=1$ when $\mathrm{Re}\left(\sum_{j=1}^q b_j-\sum_{\ell=1}^{q+1}a_\ell\right)>0$, we have that $F_0(z;s)$ and $F_1(z;s)$ converge absolutely for $\mathrm{Re}(s)>-\frac{7}{2}$. Since $\Gamma(s+2)$ has simple poles at $s=-2$, the expression in (5.3.27) converges absolutely in $\mathrm{Re}(s)>-2$, and this concludes the proof.

□

5.4. The areal Mahler measure of $|k|+x+y$

The areal Mahler measure of $|k|+x+y$ can be obtained by computing $\frac{d}{ds}Z_{\mathbb{D}}(s,|k|+x+y)|_{s=0}$. For $|k|\geq 2$, we find $\mathrm{m}_{\mathbb{D}}(|k|+x+y)=\mathrm{m}_{\mathbb{D}}(k+x+y)=\log|k|$. For $|k|<2$, we differentiate the expression (5.3.27) with respect to s. By expanding the hypergeometric series in the numerator of $c_0(s)$, we find that

$$c_0(s)=\frac{\log 2-\frac{7}{4}}{{}_3F_2\left(-\frac{1}{2},\frac{1}{2},\frac{3}{2};\frac{5}{2},\frac{5}{2};1\right)}s+\mathcal{O}(s^2)=-\frac{32}{9\pi}s+\mathcal{O}(s^2).$$

Moreover, we also have that

$$\begin{aligned} c_1(s) &= 1-\frac{1}{4}s+\mathcal{O}(s^2), \\ F_0\left(\frac{|k|^2}{4};s\right) &= {}_3F_2\left(-\frac{1}{2},\frac{1}{2},\frac{3}{2};\frac{5}{2},\frac{5}{2};\frac{|k|^2}{4}\right)+\mathcal{O}(s), \\ F_1\left(\frac{|k|^2}{4};s\right) &= 1+\frac{|k|^2}{2}s+\mathcal{O}(s^2). \end{aligned}$$

Therefore, computing $\frac{d}{ds}Z_{\mathbb{D}}(s,|k|+x+y)|_{s=0}$, we conclude that

$$\mathrm{m}_{\mathbb{D}}(k+x+y)=\mathrm{m}_{\mathbb{D}}(|k|+x+y)=-\frac{4|k|^3}{9\pi}{}_3F_2\left(-\frac{1}{2},\frac{1}{2},\frac{3}{2};\frac{5}{2},\frac{5}{2};\frac{|k|^2}{4}\right)+\frac{|k|^2}{2}-\frac{1}{4}, \quad (5.4.1)$$

which is the identity in Corollary 0.6.24. Combining (5.4.1) and (0.6.21), we derive the required relation between $\mathrm{m}(k+x+y)$ and $\mathrm{m}_{\mathbb{D}}(k+x+y)$ in the statement of Theorem 5.1.2.

5.5. Comparison with $\mathrm{m}(k+x+y)$

By Cassaigne–Maillot's formula (0.5.1), the Mahler measure of $k+x+y$ for $0<|k|<2$ is given by

$$\mathrm{m}(k+x+y)=\frac{1}{\pi}\left(2\log|k|\arcsin\frac{|k|}{2}+\frac{1}{2}D\left(e^{4i\arcsin\frac{|k|}{2}}\right)+D\left(e^{2i\arccos\frac{|k|}{2}}\right)\right),\qquad(5.5.1)$$

where $\arcsin\frac{|k|}{2},\arccos\frac{|k|}{2}\in[0,\pi)$. Since (5.3.27) is written in terms of hypergeometric series in $|k|$, we need a hypergeometric formula for $\mathrm{m}(k+x+y)=\mathrm{m}(|k|+x+y)$. The following result provides such a formula.

Proposition 5.5.1 ([**7**])**.** *For $k\in\mathbb{C}$, we have*

$$\mathrm{m}(k+x+y)=\frac{|k|}{\pi}{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2};\frac{|k|^2}{4}\right).\qquad(5.5.2)$$

Since the argument to derive (5.5.2) using the zeta Mahler measure is analogous to the ones in Sections 5.3 and 5.4, we do not include the proof here. Instead we provide a brief sketch of the derivation of (5.5.2) in Appendix A.

Comparing (5.4.1) and (5.5.2), we conclude that

$$\mathrm{m}(k+x+y)-\mathrm{m}_{\mathbb{D}}(k+x+y)=\frac{|k|\sqrt{4-|k|^2}(10+|k|^2)+(8-16|k|^2)\arccos\left(\frac{|k|}{2}\right)}{16\pi},$$

which proves Theorem 5.1.2. Evaluating the above equality at $k=\sqrt{2}$, we have

$$m\left(\sqrt{2}+x+y\right)-\mathrm{m}_{\mathbb{D}}\left(\sqrt{2}+x+y\right)=\frac{3}{2\pi}-\frac{3}{8}.$$

Now comparing Theorem 0.6.5 together with the fact that

$$\mathrm{m}\left(\sqrt{2}+x+y\right)=\frac{L(\chi_{-4},2)}{\pi}+\frac{\log 2}{4},$$

we complete the derivation of the result in Remark 0.6.6, namely,

$$\mathcal{C}_{\sqrt{2}}=\frac{\Gamma\left(\frac{3}{4}\right)^2}{\sqrt{2\pi^3}}{}_4F_3\left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4};\frac{1}{2},\frac{5}{4},\frac{5}{4};1\right)-\frac{\Gamma\left(\frac{1}{4}\right)^2}{72\sqrt{2\pi^3}}{}_4F_3\left(\frac{3}{4},\frac{3}{4},\frac{5}{4},\frac{5}{4};\frac{3}{2},\frac{7}{4},\frac{7}{4};1\right)=\frac{\log 2}{4}.$$

Further combining the identities (5.1.4) and (5.5.1) together yields a much simpler expression for $\mathrm{m}_{\mathbb{D}}(k+x+y)$ in terms of special values of Bloch–Wigner dilogarithm for all

$k \in \mathbb{C}$, namely,

$$\mathrm{m}_{\mathbb{D}}(k+x+y) = \frac{1}{\pi}\left(2\log|k|\arcsin\frac{|k|}{2} + \frac{1}{2}D\left(e^{4i\arcsin\frac{|k|}{2}}\right) + D\left(e^{2i\arccos\frac{|k|}{2}}\right)\right) - \frac{|k|\sqrt{4-|k|^2}(10+|k|^2) + (8-16|k|^2)\arccos\left(\frac{|k|}{2}\right)}{16\pi}.$$

5.6. Conclusion

In this chapter (as in Chapters 2 and 3), we only considered polynomials which have standard Mahler measures expressed in terms of Dirichlet L-values and ζ-values. An interesting direction for the future is to consider certain polynomial families $\{Q_r : r \in \mathbb{C}\}$ (such as $\{(1+x)(1+y)+rz : r \in \mathbb{C}\}$, $\{x + \frac{1}{x} + y + \frac{1}{y} + r : r \in \mathbb{C}\}$, etc.) where, for some values of $r \in \mathbb{C}$, $\mathrm{m}(Q_r)$ can be expressed in terms of special values of L-functions of elliptic curves (see Table 1). Due to difficulties in obtaining hypergeometric expressions of their zeta areal Mahler measures, we propose an *implicit* approach to calculate their areal Mahler measures. The idea is to consider the integral representation of the zeta areal Mahler measure of Q_r as the moment generating function of the random variable $\log T_r = \log|Q_r|$ (as given in Remark 5.3.2), and to express the areal Mahler measure as a single integral by taking the derivative of $Z_{\mathbb{D}}(s, Q_r)$ at $s = 0$. In particular, we have $\mathrm{m}_{\mathbb{D}}(Q_r) = \mathbb{E}[\log T_r]$. The next step in this approach would be to investigate the integral representation of $\mathbb{E}[\log T_r]$ in order to obtain an expression of $\mathrm{m}(Q_r) - \mathrm{m}_{\mathbb{D}}(Q_r)$ in terms of $|r|$, ultimately yielding examples of the areal Mahler measure involving special values of L-functions of elliptic curves for certain values of r.

Another direction to pursue is to obtain an expression of $Z_{\mathbb{D}}(s, x+y+z)$ from the general expression of $Z_{\mathbb{D}}(s, |k|+x+y)$ in equation (5.3.27), and to apply an analogous approach as the one discussed in Section 5.3 in order to evaluate $Z_{\mathbb{D}}(s, r+x+y+z)$ in terms of hypergeometric series in $|r|$ and s, where $r \in \mathbb{C}$.

An active area of research is computing $Z(s, x_1 + \cdots + x_n)$ in terms of hypergeometric series and Meijer G-functions using random walks and an iterative process involving $Z(s, x_1 + \cdots + x_\ell)$, where $1 \le \ell < n$. Hypergeometric expressions of $Z(s, x_1 + x_2 + x_3)$ and $Z(s, x_1 + x_2 + x_3 + x_4)$ can be found in **[26]**. This method can be extended to the areal case. In other words, an iterative process then could yield an evaluation of $Z_{\mathbb{D}}(s, r + x_1 + \cdots + x_n)$ in terms of hypergeometric series, for $n \geq 3$ and $r \in \mathbb{C}$. Finally, taking the derivative of $Z_{\mathbb{D}}(s, r + x_1 + \cdots + x_n)$ with respect to s at $s = 0$ would likely lead to a hypergeometric expression of $\mathrm{m}_{\mathbb{D}}(r + x_1 + \cdots + x_n)$.

Chapter 6

Conclusion

The Mahler measure bridges diverse areas of mathematics, connecting topics such as hyperbolic volumes, heights, Beilinson conjectures, and random walks. This thesis aims to provide a pathway into the vast realm of Mahler measure and its various generalizations. We refer readers to the end of each chapter for outlines of future directions that can be pursued in each generalization.

We would like to conclude this thesis by proposing some further questions that might be interesting to explore in the future. It is already known that the Mahler measure of a polynomial can be computed as the entropy of a dynamical system. Recent works by Deninger [**46**], Carter, Lalín, Manes, and Miller [**40**], among others, have opened the door to the search for suitable dynamical systems giving rise to Mahler measures. Similar to the classical case, if one can associate a notion of height (such as the canonical height) with certain dynamical systems, will there exist a Mahler measure arising from that system? Exploring this direction would be fascinating, and could further enrich the already diverse world of Mahler measures. The dynamical approach established in Benedetto's work [**6**] involving canonical heights (first considered by Call and Silverman [**39**]) on specific Julia sets contains a potential to express the Mahler measure of a one-variable polynomial over function fields in terms of such heights, and pursue a similar analysis to the one described in [**40**]. An ideal goal would be to evaluate this version of Mahler measure on non-trivial multivariable polynomials in order to express it in terms of special values of L-functions of the associated varieties over $\mathbb{F}_q$. A more ambitious approach would be to seek a universal theory encompassing different generalizations of Mahler measures, as the known ones seem to be connected in various ways.

Bibliography

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. National Bureau of Standards Applied Mathematics Series, No. 55. U. S. Government Printing Office, Washington, D.C., 1964.

[2] L. Ahlfors. *Complex analysis—an introduction to the theory of analytic functions of one complex variable*. AMS Chelsea Publishing, Providence, RI, third edition, [2021] ©2021. Reprint of the 1978 original.

[3] H. Akatsuka. Zeta Mahler measures. *J. Number Theory*, 129(11):2713–2734, 2009.

[4] G. E. Andrews, R. Askey, and R. Roy. *The Hypergeometric Functions*, pages 61–123. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1999.

[5] K. Bannai and S. Kobayashi. Algebraic theta functions and the p-adic interpolation of Eisenstein-Kronecker numbers. *Duke Math. J.*, 153(2):229–295, 2010.

[6] R. L. Benedetto. Heights and preperiodic points of polynomials over function fields. *Int. Math. Res. Not.*, (62):3855–3866, 2005.

[7] B. C. Berndt and A. Straub. On a secant Dirichlet series and Eichler integrals of Eisenstein series. *Math. Z.*, 284(3-4):827–852, 2016.

[8] I. N. Bernšteĭn and S. I. Gel'fand. Meromorphy of the function P^{λ}. *Funkcional. Anal. i Priložen.*, 3(1):84–85, 1969.

[9] M. J. Bertin. Mesure de Mahler d'une famille de polynômes. *J. Reine Angew. Math.*, 569:175–188, 2004.

[10] M. J. Bertin. Mesure de Mahler et régulateur elliptique: preuve de deux relations "exotiques". In *Number theory*, volume 36 of *CRM Proc. Lecture Notes*, pages 1–12. Amer. Math. Soc., Providence, RI, 2004.

[11] M. J. Bertin. Mahler's measure and L-series of $K3$ hypersurfaces. In *Mirror symmetry V*, volume 38 of *AMS/IP Stud. Adv. Math.*, pages 3–18. Amer. Math. Soc., Providence, RI, 2006.

[12] M. J. Bertin. Mesure de Mahler d'hypersurfaces $K3$. *J. Number Theory*, 128(11):2890–2913, 2008.

[13] M.-J. Bertin, A. Feaver, J. Fuselier, M. Lalín, and M. Manes. Mahler measure of some singular $K3$-surfaces. In *Women in numbers 2: research directions in number theory*, volume 606 of *Contemp. Math.*, pages 149–169. Amer. Math. Soc., Providence, RI, 2013.

[14] M. J. Bertin and W. Zudilin. On the Mahler measure of a family of genus 2 curves. *Math. Z.*, 283(3-4):1185–1193, 2016.

[15] A. A. Beĭlinson. Higher regulators and values of L-functions. In *Current problems in mathematics, Vol. 24*, Itogi Nauki i Tekhniki, pages 181–238. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.

[16] F. Beukers. Notes on differential equations and hypergeometric functions. *Unpublished notes, https://pages.uoregon.edu/njp/beukers.pdf.*

[17] A. Biswas. Asymptotic nature of higher Mahler measure. *Acta Arith.*, 166(1):15–22, 2014.

[18] A. Biswas and C. Monico. Limiting value of higher Mahler measure. *J. Number Theory*, 143:357–362, 2014.

[19] A. Biswas and M. R. Murty. The zeta Mahler measure of $(z^n-1)/(z-1)$. *Hardy-Ramanujan J.*, 41:77–84, 2018.

[20] P. E. Blanksby and H. L. Montgomery. Algebraic integers near the unit circle. *Acta Arith.*, 18:355–369, 1971.

[21] S. Bloch and D. Grayson. K_2 and L-functions of elliptic curves: computer calculations. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, volume 55 of *Contemp. Math.*, pages 79–88. Amer. Math. Soc., Providence, RI, 1986.

[22] S. J. Bloch. *Higher regulators, algebraic K-theory, and zeta functions of elliptic curves*, volume 11 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2000.

[23] D. Borwein, J. M. Borwein, A. Straub, and J. Wan. Log-sine evaluations of Mahler measures, II. *Integers*, 12(6):1179–1212, 2012.

[24] J. M. Borwein and A. Straub. Log-sine evaluations of Mahler measures. *J. Aust. Math. Soc.*, 92(1):15–36, 2012.

[25] J. M. Borwein, A. Straub, and J. Wan. Three-step and four-step random walk integrals. *Exp. Math.*, 22(1):1–14, 2013.

[26] J. M. Borwein, A. Straub, J. Wan, and W. Zudilin. Densities of short uniform random walks. *Canad. J. Math.*, 64(5):961–990, 2012. With an appendix by Don Zagier.

[27] D. W. Boyd. Speculations concerning the range of Mahler's measure. *Canad. Math. Bull.*, 24(4):453–469, 1981.

[28] D. W. Boyd. Two sharp inequalities for the norm of a factor of a polynomial. *Mathematika*, 39(2):341–349, 1992.

[29] D. W. Boyd. Mahler's measure and special values of L-functions. *Experiment. Math.*, 7(1):37–82, 1998.

[30] D. W. Boyd. Conjectural explicit formulas for the Mahler measure of some three variable polynomials. Personal communication to M. Lalín, 2006.

[31] D. W. Boyd and F. Rodriguez-Villegas. Mahler's measure and the dilogarithm. I. *Canad. J. Math.*, 54(3):468–492, 2002.

[32] D. W. Boyd, F. Rodriguez-Villegas, and N. M. Dunfield. Mahler's Measure and the Dilogarithm (II). *arXiv Mathematics e-prints*, Aug. 2003.

[33] D. Bradley. Representations of Catalan's constant. https://www.researchgate.net/publication/2325473, 2001.

[34] D. M. Bradley. A class of series acceleration formulae for Catalan's constant. *Ramanujan J.*, 3(2):159–173, 1999.

[35] F. Brunault. *Value at 2 of the L-function of an elliptic curve*. Theses, Université Paris-Diderot - Paris VII, Dec. 2005.

[36] F. Brunault. Version explicite du théorème de Beilinson pour la courbe modulaire $X_1(N)$. *C. R. Math. Acad. Sci. Paris*, 343(8):505–510, 2006.

[37] F. Brunault. Regulators of Siegel units and applications. *J. Number Theory*, 163:542–569, 2016.

[38] F. Brunault and W. Zudilin. *Many variations of Mahler measures—a lasting symphony*, volume 28 of *Australian Mathematical Society Lecture Series*. Cambridge University Press, Cambridge, 2020.

[39] G. S. Call and J. H. Silverman. Canonical heights on varieties with morphisms. *Compositio Math.*, 89(2):163–205, 1993.

[40] A. Carter, M. Lalín, M. Manes, A. B. Miller, and L. Mocz. Two-variable polynomials with dynamical Mahler measure zero. *Res. Number Theory*, 8(2):Paper No. 25, 22, 2022.

[41] J. Cassaigne and V. Maillot. Hauteur des hypersurfaces et fonctions zêta d'Igusa. *J. Number Theory*, 83(2):226–255, 2000.

[42] A. Champanerkar, I. Kofman, and M. N. Lalín. Mahler measure and the vol-det conjecture. *J. London Math Soc.*, 99(3):872–900, 2019.

[43] T. Chinburg. Mahler measures and derivatives of l-functions at non-positive integers. *Unpublished manuscript*, 1984.

[44] K.-K. S. Choi and C. L. Samuels. Two inequalities on the areal Mahler measure. *Illinois J. Math.*, 56(3):825–834, 2012.

[45] C. Deninger. Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbf{Z}^n$-actions. *J. Amer. Math. Soc.*, 10(2):259–281, 1997.

[46] C. Deninger. Dynamical systems for arithmetic schemes. *arXiv e-prints*, page arXiv:1807.06400, July 2018.

[47] V. Dimitrov. A proof of the Schinzel-Zassenhaus conjecture on polynomials. *arXiv e-prints*, page arXiv:1912.12545, Dec. 2019.

[48] V. Dimitrov and P. Habegger. Galois orbits of torsion points near atoral sets. *arXiv e-prints*, page arXiv:1909.06051, Sept. 2019.

[49] E. Dobrowolski. On a question of Lehmer and the number of irreducible factors of a polynomial. *Acta Arith.*, 34(4):391–401, 1979.

[50] A. Dubickas and C. J. Smyth. The Lehmer constants of an annulus. *J. Théor. Nombres Bordeaux*, 13(2):413–420, 2001.

[51] G. Everest and T. Ward. *Heights of polynomials and entropy in algebraic dynamics*. Universitext. Springer-Verlag London, Ltd., London, 1999.

[52] V. Flammang. The Mahler measure and its areal analog for totally positive algebraic integers. *J. Number Theory*, 151:211–222, 2015.

[53] M. Forsberg, M. Passare, and A. Tsikh. Laurent determinants and arrangements of hyperplane amoebas. *Adv. Math.*, 151(1):45–70, 2000.

[54] Y. Gon and H. Oyanagi. Generalized Mahler measures and multiple sine functions. *Internat. J. Math.*, 15(5):425–442, 2004.

[55] A. B. Goncharov. Deninger's conjecture of L-functions of elliptic curves at $s = 3$. *J. Math. Sci.*, 81(3):2631–2656, 1996. Algebraic geometry 4.

[56] A. B. Goncharov. Explicit regulator maps on polylogarithmic motivic complexes. In *Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998)*, volume 3 of *Int. Press Lect. Ser.*, pages 245–276. Int. Press, Somerville, MA, 2002.

[57] A. B. Goncharov. Regulators. In *Handbook of K-theory. Vol. 1, 2*, pages 295–349. Springer, Berlin, 2005.

[58] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, seventh edition, 2007. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX).

[59] P. A. Griffiths. On the periods of certain rational integrals. I, II. *Ann. of Math. (2)*, 90:460–495; 90 (1969), 496–541, 1969.

[60] R. Gualdi. Heights of hypersurfaces in toric varieties. *Algebra Number Theory*, 12(10):2403–2443, 2018.

[61] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge, at the University Press,, 1952. 2d ed.

[62] M. Hindry and J. H. Silverman. *Diophantine geometry*, volume 201 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[63] W. R. Inc. Hypergeometric ${}_2F_1$. *http://functions.wolfram.com/07.23.03.2888.01*.

[64] W. R. Inc. Hypergeometric ${}_2F_1$. *http://functions.wolfram.com/07.23.03.2933.01*.

[65] W. R. Inc. Hypergeometric ${}_2F_1$. *http://functions.wolfram.com/07.23.17.0094.01*.

[66] E. L. Ince. *Ordinary Differential Equations*. Dover Publications, New York, 1944.

[67] Z. Issa and M. Lalín. A generalization of a theorem of Boyd and Lawton. *Canad. Math. Bull.*, 56(4):759–768, 2013.

[68] A. Jonquière. Note sur la série $\sum_{n=1}^{\infty} \frac{x^n}{n^s}$. *Bull. Soc. Math. France*, 17:142–152, 1889.

[69] D. E. Knuth. *The art of computer programming. Vol. 1*. Addison-Wesley, Reading, MA, 1997. Fundamental algorithms, Third edition.

[70] M. Kontsevich and D. Zagier. Periods. In *Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.

[71] L. Kronecker. Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten. *J. Reine Angew. Math.*, 53:173–175, 1857.

[72] N. Kurokawa, M. Lalín, and H. Ochiai. Higher Mahler measures and zeta functions. *Acta Arith.*, 135(3):269–297, 2008.

[73] M. Lalín and G. Wu. Regulator proofs for Boyd's identities on genus 2 curves. *Int. J. Number Theory*, 15(5):945–967, 2019.

[74] M. N. Lalín. Some examples of Mahler measures as multiple polylogarithms. *J. Number Theory*, 103(1):85–108, 2003.

[75] M. N. Lalín. Mahler measure of some n-variable polynomial families. *J. Number Theory*, 116(1):102–139, 2006.

[76] M. N. Lalín. Mahler measures and computations with regulators. *J. Number Theory*, 128(5):1231–1271, 2008.

[77] M. N. Lalín. On a conjecture by Boyd. *Int. J. Number Theory*, 6(3):705–711, 2010.

[78] M. N. Lalín and J.-S. Lechasseur. Higher Mahler measure of an n-variable family. *Acta Arith.*, 174(1):1–30, 2016.

[79] M. N. Lalín and J.-S. Lechasseur. A reduction formula for length-two polylogarithms and some applications. *Rev. Un. Mat. Argentina*, 59(2):285–309, 2018.

[80] M. N. Lalín and T. Mittal. The Mahler measure for arbitrary tori. *Res. Number Theory*, 4(2):Art. 16, 23, 2018.

[81] M. N. Lalín, S. S. Nair, and S. Roy. The Mahler measure of a multivariate family with non-linear degree. *Acta Arith.*, to appear.

[82] M. N. Lalín and M. D. Rogers. Functional equations for Mahler measures of genus-one curves. *Algebra Number Theory*, 1(1):87–117, 2007.

[83] M. N. Lalin and S. Roy. The areal Mahler measure under a power change of variables. *Illinois J. Math.*, 68(2):309–330, 2024.

[84] M. N. Lalin and S. Roy. Evaluations of the areal Mahler measure of multivariable polynomials. *J. Number Theory*, 254:103–145, 2024.

[85] M. Langevin. Méthode de Fekete–Szegő et problème de Lehmer. *C. R. Acad. Sci. Paris Sér. I Math.*, 301(10):463–466, 1985.

[86] W. M. Lawton. A problem of Boyd concerning geometric means of polynomials. *J. Number Theory*, 16(3):356–362, 1983.

[87] D. H. Lehmer. Factorization of certain cyclotomic functions. *Ann. of Math. (2)*, 34(3):461–479, 1933.

[88] H. Liu and H. Qin. Mahler measure of families of polynomials defining genus 2 and 3 curves. *Exp. Math.*, 32(2):321–336, 2023.

[89] K. Mahler. On some inequalities for polynomials in several variables. *J. London Math. Soc.*, 37:341–344, 1962.

[90] V. Maillot. Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables. *Mém. Soc. Math. Fr. (N.S.)*, (80):vi+129, 2000.

[91] M. Mehrabdollahei. *The Mahler measure of a family of exact polynomials*. PhD thesis, Sorbonne Université, 07 2022.

[92] A. Mellit. Elliptic dilogarithms and parallel lines. *J. Number Theory*, 204:1–24, 2019.

[93] S. S. Nair. The Mahler measure of a family of polynomials with arbitrarily many variables. *arXiv:2309.16966*, 2023.

[94] T. Nakamura. A simple proof of the functional relation for the Lerch type Tornheim double zeta function. *Tokyo J. Math.*, 35(2):333–337, 2012.

[95] N. E. Nørlund. Hypergeometric functions. *Acta Math.*, 94:289–349, 1955.

[96] E. Panzer. The parity theorem for multiple polylogarithms. *J. Number Theory*, 172:93–113, 2017.

[97] R. Pengo. *Mahler measures, special values of L-functions and complex multiplication*. PhD thesis, University of Copenhagen, 10 2020.

[98] T. A. Pierce. The numerical factors of the arithmetic forms $\prod_{i=1}^{n}(1 \pm \alpha_i^m)$. *Ann. of Math. (2)*, 18(2):53–64, 1916.

[99] I. E. Pritsker. An areal analog of Mahler's measure. *Illinois J. Math.*, 52(2):347–363, 2008.

[100] G. A. Ray. Relations between Mahler's measure and values of L-series. *Canad. J. Math.*, 39(3):694–732, 1987.

[101] B. Ringeling. Special zeta Mahler functions. *Res. Number Theory*, 8(2):Paper No. 29, 27, 2022.

[102] F. Rodriguez-Villegas. Modular Mahler measures. I. In *Topics in number theory (University Park, PA, 1997)*, volume 467 of *Math. Appl.*, pages 17–48. Kluwer Acad. Publ., Dordrecht, 1999.

[103] F. Rodriguez-Villegas. Identities between Mahler measures. In *Number theory for the millennium, III (Urbana, IL, 2000)*, pages 223–229. A K Peters, Natick, MA, 2002.

[104] M. Rogers and W. Zudilin. From L-series of elliptic curves to Mahler measures. *Compos. Math.*, 148(2):385–414, 2012.

[105] M. Rogers and W. Zudilin. On the Mahler measure of $1 + X + 1/X + Y + 1/Y$. *Int. Math. Res. Not. IMRN*, (9):2305–2326, 2014.

[106] S. Roy. Generalized Mahler measures of Laurent polynomials. *Ramanujan J.*, 64(3):581–627, 2024.

[107] D. Samart. The elliptic trilogarithm and Mahler measures of $K3$ surfaces. *Forum Math.*, 28(3):405–423, 2016.

[108] Y. Sasaki. On multiple higher Mahler measures and multiple L values. *Acta Arith.*, 144(2):159–165, 2010.

[109] Y. Sasaki. On multiple higher Mahler measures and Witten zeta values associated with semisimple Lie algebras. *Commun. Number Theory Phys.*, 6(4):771–784, 2012.

[110] Y. Sasaki. Zeta Mahler measures, multiple zeta values and L-values. *Int. J. Number Theory*, 11(7):2239–2246, 2015.

[111] M. Schütt and T. Shioda. Elliptic surfaces. In *Algebraic geometry in East Asia—Seoul 2008*, volume 60 of *Adv. Stud. Pure Math.*, pages 51–160. Math. Soc. Japan, Tokyo, 2010.

[112] C. J. Smyth. Mahler measure and the vol-det conjecture. *Canad. Math. Bull.*, 24(4):447–452, 1981.

[113] C. J. Smyth. An explicit formula for the Mahler measure of a family of 3-variable polynomials. *J. Théor. Nombres Bordeaux*, 14(2):683–700, 2002.

[114] C. L. Stewart. Algebraic integers whose conjugates lie near the unit circle. *Bull. Soc. Math. France*, 106(2):169–176, 1978.

[115] A. Straub and W. Zudilin. Short walk adventures. In *From analysis to visualization*, volume 313 of *Springer Proc. Math. Stat.*, pages 423–439. Springer, Cham, [2020].

[116] G. Szegő. Beiträge zur Theorie der Toeplitzschen Formen. *Math. Z.*, 9(3-4):167–190, 1921.

[117] S. Vandervelde. The Mahler measure of parametrizable polynomials. *J. Number Theory*, 128(8):2231–2250, 2008.

[118] M. Waldschmidt. Sur le produit des conjugués extérieurs au cercle unité d'un entier algébrique. *Enseign. Math. (2)*, 26(3-4):201–209 (1981), 1980.

[119] D. Zagier. The dilogarithm function. In *Frontiers in number theory, physics, and geometry. II*, pages 3–65. Springer, Berlin, 2007.

[120] W. Zudilin. Regulator of modular units and Mahler measures. *Math. Proc. Cambridge Philos. Soc.*, 156(2):313–326, 2014.

Appendix A

The hypergeometric expression of $\mathrm{m}(k+x+y)$

A.1. Derivation of Proposition 5.5.1

Here we provide a sketch of the proof of Proposition 5.5.1. We assume that $k>0$ below, since

$$\mathrm{m}(k+x+y)=\mathrm{m}(|k|+x+y).$$

Sketch of the proof of Proposition 5.5.1. Our method of proof is similar to the proofs of Theorems 5.1.1 and 5.1.2 in Sections 5.3 and 5.4. We follow an analogous argument to derive the hypergeometric expression of $\mathrm{m}(k+x+y)$.

We define random variables $U=|X+Y|$ and $V=|k+X+Y|$, where X and Y are random walks of unit length and directions θ_1 and θ_2, uniformly distributed in $[-\pi,\pi)$. Following the discussion in Remark 5.3.2, we then have, for $k>2$,

$$\begin{aligned} Z(s,k+x+y) &= \mathbb{E}(V^s)=\mathbb{E}\left[\mathbb{E}\left[V^s|U\right]\right] \\ &= \int_0^2\int_{k-u}^{k+u} v^s\,\mathbb{P}(V=v\,|\,U=u)\,\mathbb{P}(U=u)dvdu. \end{aligned} \tag{A.1.1}$$

Since the zeta Mahler measure of $x+y$ is (see (0.6.16))

$$Z(s,x+y)=Z(s,x+1)=\frac{2^s}{\sqrt{\pi}}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)}=\int_0^\infty u^s\mathbb{P}(U=u)du,$$

the Mellin inverse of $Z(s-1, x+y)$ yields, for $0 \leq u \leq 2$,

$$\mathbb{P}(U = u) = \frac{2}{\pi\sqrt{4-u^2}}.$$

A similar approach as in Section 5.3 gives

$$\mathbb{P}(V = v \,|\, U = u) = \frac{2v}{\pi\sqrt{4k^2u^2 - (v^2 - u^2 - k^2)^2}}.$$

Then, from Lemmas 5.3.5 and 5.3.7 we have, for s real and positive,

$$\begin{aligned}\int_{|k-u|}^{k+u} v^s \cdot \mathbb{P}(V = v \,|\, U = u)\, dv &= \int_{|k-u|}^{k+u} v^s \cdot \frac{2v}{\pi\sqrt{4u^2k^2 - (v^2 - u^2 - k^2)^2}} dv \\ &= \mathrm{Re}(F(k)) - \cot\left(\frac{\pi s}{2}\right) \mathrm{Im}(F(k)),\end{aligned}$$

where $F(k)$ is given in (5.3.15) as

$$F(k) = k^s \cdot {}_2F_1\left(-\frac{s}{2}, -\frac{s}{2}; 1; \frac{u^2}{k^2}\right).$$

Therefore we have, for all $k > 2$,

$$k^s \int_0^2 {}_2F_1\left(-\frac{s}{2}, -\frac{s}{2}; 1; \frac{u^2}{k^2}\right) \mathbb{P}(U = u) du = k^s {}_3F_2\left(-\frac{s}{2}, -\frac{s}{2}, \frac{1}{2}; 1, 1; \frac{4}{k^2}\right).$$

An analogous approach as in the areal case further shows that, for all real $s > 0$ which is not an odd integer and $k > 0$,

$$Z(s, k + x + y) = \mathrm{Re}(J(k)) - \cot\left(\frac{\pi s}{2}\right) \mathrm{Im}(J(k)),$$

where

$$J(k) := k^s {}_3F_2\left(-\frac{s}{2}, -\frac{s}{2}, \frac{1}{2}; 1, 1; \frac{4}{k^2}\right).$$

Now, to analytically extend the above inequality to all $k \in \mathbb{C}$ and to a larger region of $s \in \mathbb{C}$, we need to find a differential equation for

$$\mathcal{H}(z) = (4z)^{-\frac{s}{2}} {}_3F_2\left(-\frac{s}{2}, -\frac{s}{2}, \frac{1}{2}; 1,1; \frac{1}{z}\right),$$

where $z = \frac{k^2}{4}$. Again, a procedure similar to the areal case yields the required differential equation:

$$s^2\mathcal{U}(z) - 2z(2 + 2s + s^2 - 4z)\mathcal{U}'(z) - 4z^2(5 + 2s - 6z)\mathcal{U}''(z) + 4z^2(1-z)\mathcal{U}'''(z) = 0.$$

The above differential equation has a regular singularity at $z = 0$ with local exponents $0, 0$, and $\frac{1+s}{2}$ (see [**16**] for more details on local exponents). Then, using Frobenius' method to solve the differential equation [**95**], we find the following set of solutions:

$$G_0(z; s), \quad z^{\frac{1+s}{2}} G_1(z; s), \quad G_2(z; s) + \log z \, G_0(z; s)$$

where

$$G_1(z; s) = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{3+s}{2}, \frac{3+s}{2}; z\right), \qquad G_0(z; s) = {}_3F_2\left(-\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2}; 1, \frac{1-s}{2}; z\right),$$

and G_2 is a Meijer G-function. Since, for $z \to 0$ (i.e. $k \to 0$), $Z(s, k + x + y)$ exists, namely $Z(s, x + y) = Z(s, x + 1) = \frac{2^s}{\sqrt{\pi}} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)}$, the quantity $G_2(z; s) + \log z \, G_0(z; s)$ does not appear in the expression of $Z(s, k+x+y)$. This implies that $Z(s, k+x+y)$ is a linear combination of $G_0(z; s)$ and $z^{\frac{1+s}{2}} G_1(z; s)$. In particular, substituting $z = \frac{k^2}{4}$, we have

$$Z(s, k + x + y) = c_0(s) \left(\frac{k}{2}\right)^{1+s} G_1\left(\frac{k^2}{4}; s\right) + c_1(s) G_0\left(\frac{k^2}{4}; s\right), \tag{A.1.2}$$

where

$$c_1(s) = Z(s, x + y) = \frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2}+1\right)^2},$$

and

$$c_0(s) = \frac{1}{2^s} \tan\left(\frac{\pi s}{2}\right) \binom{s}{\frac{s-1}{2}}^2.$$

The expression $c_0(s)$ follows from the case $k = 1$ in [**25**, Corollary 1].

Similar to the areal case, we investigate the order of s in each of c_0, c_1 and G_0, and we have

$$c_0(s) = \frac{2s}{\pi} + \mathcal{O}(s^2).$$

$$c_1(s) = 1 + \mathcal{O}(s^2).$$

$$G_0 = 1 + \mathcal{O}(s^3).$$

Therefore, differentiating the expression of $Z(s, k+x+y)$ in (A.1.2), and evaluating at $s = 0$, we finally have

$$\mathrm{m}(k + x + y) = \frac{k}{\pi} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{k^2}{4}\right),$$

which completes the proof of the statement. ☐