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Universal Numerical Series

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Universal Numerical Series

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Résumé

Dans ce mémoire, nous allons nous concentrer sur le sujet de l'universalité en analyse complexe. Tout d'abord, nous allons énumérer de nombreux résultats découverts dans ce domaine, tout en soulignant que, dans la plupart des cas, les preuves d'existence des éléments universels sont implicites et non pas constructives. Nous examinerons en détail une preuve spécifique de l'existence des séries universelles de Taylor qui se voulait constructive et nous déterminerons si tel est le cas ou non. Pour atteindre cet objectif, nous introduirons un nouvel élément universel que nous appellerons les séries numériques universelles. Ce sont des séries complexes telles que leurs sommes partielles sont denses dans le plan complexe. Nous donnerons une preuve constructive de l'existence de ces éléments et, afin de déterminer pleinement si la preuve susmentionnée de l'existence des séries universelles de Taylor est constructive, nous allons la comparer avec notre preuve de l'existence des séries numériques universelles. Enfin, nous examinerons les propriétés topologiques et algébriques des séries numériques universelles, en montrant sous quelles conditions elles sont topologiquement génériques et algébriquement génériques dans l'ensemble de toutes séries formelles à termes complexes.

Keywords: Universalité, Constructibilité, Séries, Analyse Complexe, Généricité Algébrique, Généricité Topologique

Abstract

This master's thesis will be centered around the subject of universality in complex analysis. First, we will provide a summary of many of the results that have been discovered in the field of universality. We will show that, in most cases, the proofs of existence of the universal elements are not constructive, but, rather, implicit. We will perform an in-depth analysis of a specific proof of the existence of Universal Taylor series which was intended to be constructive and we will determine whether or not this goal was achieved. To do this, we will introduce a new universal element, which we will call Universal numerical series. These are complex numerical series such that the partial sums of the series are dense in the complex plane. We will give a constructive proof of the existence of these elements and, in order to fully determine whether the aforementioned proof of the existence of the Universal Taylor series is constructive, we will compare it to our proof of the existence of the Universal numerical series. Finally, we will examine the topological and algebraic properties of the Universal numerical series, showing under which conditions they are topologically generic and algebraically generic in the set of all complex numerical series.

Keywords: Universality, Constructibility, Series, Complex Analysis, Algebraic Genericity, Topological Genericity

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List of acronyms and abbreviations

$\mathbb{C}^\infty$	The set of all sequences of complex numbers
$\mathbb{Q}+i\mathbb{Q}$	The set of rational complex numbers
NS($\mathbb{C}$)	The set of complex numerical series
UNS($\mathbb{C}$)	The set of Universal complex numerical series

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Introduction

0.1. Universality

The subject of universality has stimulated a great deal of research over the past century. Paraphrasing a remark Grosse-Erdmann once made [**10**], one can use the term *universal* to refer to some sequence which diverges in the worst way possible because there is some maximal set M such that, for every point $m \in M$, there is some subsequence of the overall sequence which converges to the point m. More formally, let X and Y be topological spaces. Let $\{T_i\}_{i\in I}$, where I is some index set, be a set of transformations such that, for all $i \in I$, we have that $T_i : X \longrightarrow Y$. We say that some element $x \in X$ is *universal* for the set of transformations $\{T_i\}_{i\in I}$ if

$$\overline{\{T_i(x) \mid i \in I\}} = Y.$$

In other words, universality is a property where a chosen set of transformations act on a single element to produce a set of results which are dense in the range. Generally, this produces a countable set of results which is ordered in a sequence, which is the origin of Grosse-Erdmann's description.

This phenomenon obviously depends greatly on the chosen set of transformations and on the chosen spaces X and Y. In this thesis, we will avoid obvious, overly simple transformations in favor of the more interesting cases. We begin with a general summary of the historical developments in the field of universality, although we note that this summary is in no way comprehensive and that the literature on the subject is vast and intricate. We will discuss important results and general trends in this field of research.

0.2. Summary of Universality

Before starting our summary, we introduce the following notation: first, we will let $\mathbb{N} = \{1, 2, 3, \dots\}$. Second we will let $\mathbb{D} = B(0{,}1)$, which is the unit disk in the complex plane.

Next, we cover some theory which we will need. The Baire Category Theorem states that, in a complete metric space, the countable intersection of open dense sets must be a dense set. A G_δ set is a countable intersection of open sets. A *nowhere dense* set is a set N such that $\mathrm{Int}(\overline{N}) = \emptyset$ and, more significantly, a set is called *residual* if its complement can be written as a countable union of nowhere dense sets.

We now begin our resume of the work done on this subject. We will discuss several important early results in this field. The earliest result regarding universal series was published in 1914, when Fekete [**20**] proved the existence of a formal real power series $\sum_{t=1}^{\infty} a_t z^t$ such that, for any function $h \in C([-1,1])$ for which $h(0) = 0$, there exists some subsequence $(n_j)_{j=1}^{\infty}$ of $\mathbb{N}$ such that

$$\lim_{j\to\infty} \sup_{z\in[-1,1]} \left| \sum_{t=1}^{n_j} a_t z^t - h(z) \right| = 0.$$

More forms of universality began to crop up as time went on. In 1929, Birkhoff [**2**] discovered that there exists an entire function f such that, given any entire function g, there exists a sequence of natural numbers $(n_j)_{j=1}^{\infty}$ for which the sequence $(f(z+n_j))_{j=1}^{\infty}$ converges uniformly to g on any compact subset of $\mathbb{C}$.

Next, in 1935, Marcinkiewicz [**14**] discovered what was termed a universal primitive. This was, in fact, the first time that the word *universal* was used to describe a set of transformations acting on a single element to produce a dense set. It was also the first time that a set of universal elements was shown to be residual. Marcinkiewicz showed that if $(h_j)_{j=1}^{\infty}$ is a set of real numbers for which $\lim_{j\to\infty} h_j = 0$, then there exists some function $f \in C([0,1])$ such that, for every measurable function g on $[0,1]$, there exists some subsequence $(n_j)_{j=1}^{\infty}$ of $\mathbb{N}$ such that

$$\lim_{j\to\infty} \frac{f(x+h_{n_j}) - f(x)}{h_{n_j}} = g \text{ a.e. in } [0,1].$$

In 1945, Menchoff [**16**] demonstrated the existence of many trigonometric series, denoted $\sum_{k=-\infty}^{\infty} a_k e^{ikt}$, which have a universality property. For all complex, measurable, 2π-periodic functions, denoted f, there exists some subsequence $(n_j)_{j=1}^{\infty}$ of $\mathbb{N}$ such that

$$\lim_{j\to\infty} \sum_{k=-n_j}^{n_j} a_k e^{ikt} = f \text{ a.e.}$$

These series were called universal trigonometric series. Significantly, Menchoff demonstrated that the coefficients of these series can be made to tend towards 0.

In 1952, G. R. MacLane [**13**] showed that there exists an entire function f such that the derivatives of f had universal behaviours. MacLane showed that if $G \subset \mathbb{C}$ is a simply connected domain and if $g \in H(G)$, then there exists a subsequence $(n_j)_{j=1}^{\infty}$ of $\mathbb{N}$ such that, for any compact subset $K \subset G$, we have that

$$\lim_{j\to\infty} \sup_{z\in K} |f^{(n_j)}(z) - g(z)| = 0.$$

In 1970, Wolfgang Luh [**12**] proved the existence of a Taylor series with a universality behaviour. Let $A = (a_{n,k})_{n,k=0}^{\infty,\infty}$ be a matrix such that the following properties hold:

1) If $n,k \in \mathbb{N}$, then $a_{n,k} \in \mathbb{C}$.

2) If $k > n$, then $a_{n,k} = 0$.

3) We have the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{n,k} = 1.$$

4) For any fixed $k \in \mathbb{N}$, we get that

$$\lim_{n\to\infty} a_{n,k} = 0.$$

Luh found that there exists some power series $f = \sum_{t=0}^{\infty} a_t z^t$ with radius of convergence equal to 1 with the following property: for any bounded simply connected domain G such that $G \cap \{z \in \mathbb{C} \mid |z| \leq 1\} = \emptyset$ and for every function $f \in H(G)$, there exists some subsequence $(n_j)_{j=1}^{\infty}$ of $\mathbb{N}$ such that, for any compact set $K \subset G$, we have that

$$\lim_{j\to\infty} \sup_{z\in K} \left| \sum_{v=0}^{n_j} a_{n_j,v} \sum_{t=0}^{v} a_t z^t - f(z) \right| = 0.$$

In 1971, Chui and Parnes [**3**] proved the existence of a power series, denoted $\sum_{t=0}^{\infty} a_t z^t$, which is centered at 0 and which converges on the unit disk $\mathbb{D}$, such that if K is a compact set, K^c is connected and $K \cap \overline{\mathbb{D}} = \emptyset$ and if $f \in A(K)$, then there exists some subsequence $(n_j)_{j=1}^{\infty}$ of $\mathbb{N}$ such that

$$\lim_{j\to\infty} \sup_{z\in K} \left| \sum_{t=0}^{n_j} a_t z^t - h(z) \right| = 0.$$

These are some the main early results discovered in this field. It is quite clear that there is a great variety of forms of universality. Now that we have seen several specific examples of objects with a universality property, we can begin studying general trends in the research in this field. We begin doing so by examining some recent developments which enormously sped up the discovery of new results.

Vassili Nestoridis has done a great deal of work in the field of universality. Both working alone and with other authors, Nestoridis has written many papers examining several different types of universal elements. In 1996, [**19**] he proved the existence of Universal Taylor series. He defined these series to be Taylor series $\sum_{t=0}^{\infty} a_t z^t$ which converge on the unit disk $\mathbb{D}$ and such that, for any compact set K such that K^c is connected and $K \cap \mathbb{D} = \emptyset$ and for any $f \in A(K)$, there exists some subsequence $(n_j)_{j=1}^{\infty}$ of $\mathbb{N}$ such that

$$\lim_{j\to\infty} \sup_{z\in K} \left| \sum_{t=0}^{n_j} a_t z^t - f(z) \right| = 0.$$

This result is very similar to that of Chui and Parnes, which they described in their previously mentioned 1971 paper. However, there are two important differences between theses results. Firstly, in their paper, Chui and Parnes required that $K \cap \overline{\mathbb{D}} = \emptyset$. Nestoridis improved on this by requiring only that $K \cap \mathbb{D} = \emptyset$, demonstrating that K can, in fact, intersect the boundary of the unit disk.

Secondly, and more importantly. whereas the work of Chui and Parnes was constructive in nature, the proof of Nestoridis is implicit. This is because Nestoridis demonstrated that the set of Universal Taylor series is equal to a countable intersection of open, dense sets in the set $H(\mathbb{D})$, which is a complete metric space. Thus, the Baire Category Theorem implies that the set of Universal Taylor series must be a dense G_δ set in $H(\mathbb{D})$. This shows that the set of Universal Taylor series cannot be empty, meaning that they must exist.

This proof is an important milestone because this was one of the first instances in which the Baire Category Theorem was used to demonstrate the existence of a universal object. In fact, not only does this technique show that the chosen universal objects must exist, but it also shows that the set of all such elements are dense in the overall set. We will call this the Baire Category Theorem strategy. This strategy has the advantage of simplifying the work needed to prove the existence of a great variety of universal elements. As a result, this proof strategy has become ubiquitous in the field of universality, having been used by a multitude of authors to demonstrate the existence of many other universal elements. For further examples of such elements, some suggested texts include [**5**],[**6**],[**7**],[**8**]and [**15**].

In fact, it has become increasingly uncommon to find proofs which do not depend on the Baire Category Theorem strategy. A rare example of such a proof, one which will be of great importance in this thesis, was written in a 2012 paper by Nestoridis, Mouze, Papadoperakis and Tsirivas [**17**]. In this paper, they collaborated in creating a second proof of the existence of the Universal Taylor series which Nestoridis defined in 1996. This time, they worked to produce a proof which did not depend on the Baire Category Theorem. They were successful in doing so and the work they did will be covered in detail in Chapter 2.

Having finished our description of the first trend in the research, we will now look at the second. We will say that a set X is *lineable* if there exists a vector space V such that all the vectors in V, with the possible exception of the 0 vector, are contained in the set X. A set $E \subset X$ is called a *dense-lineable* subset of X if E contains, with the possible exception of the 0 vector, a vector space which is dense in X. Once it was discovered that it is possible for a chosen set of universal elements to be lineable, it opened up an entire new avenue for research in the field of universality. Naturally, many researchers went beyond this point and attempted to determine if their chosen set of universal elements was dense-lineable. This leads to the second trend present in the research, which is is that, very frequently, the set of chosen universal elements can be shown to be dense-lineable in their overall parent space.

To illustrate both of these trends in the research, we now mention a 2008 paper by Bayart, Nestoridis, Grosse-Erdmann and Papadimitropoulos [**1**]. In this paper, they demonstrated the existence of a great deal of different universal elements, some of which had already been demonstrated in the papers discussed above. Significantly, in most cases, they used the Baire Category Theorem strategy to show that the set of universal elements in question are a dense G_δ set in the overall space in which they reside. Secondly, in most cases, they showed that the set of the chosen universal elements is dense-lineable in the overall space. Among the objects that they studied were Universal Taylor series on simply connected domains, Universal Dirichlet series, Universal Faber series, Universal Laurent series and Universal Laurent-Faber series, Universal expansions of harmonic functions, and Universal Taylor series on doubly connected domains.

We now conclude our discussion of the work done in the field of universality by examining a notable example of a class of explicitly defined elements with a universality property. This is the set of Dirichlet L-functions. In 1975, S. M. Voronin wrote a paper [**22**] concerning a universality property which was exhibited by the Riemann Zeta-function. Let $\zeta(z)$ denote the Riemann Zeta-function. Let $0 < r < \frac{1}{4}$ and let f be a function which is analytic inside the disc $|z| \leq r$ and continuous up to the boundary of the disc, such that f has no zeroes inside the disc $|z| \leq r$. Voronin proved that for all $\epsilon > 0$, there exists some real number $T(\epsilon)$ such that

$$\max_{|z| \leq r} \left| f(z) - \zeta\left(z + iT(\epsilon) + \frac{3}{4}\right) \right| < \epsilon.$$

Voronin stated that analogous results hold for all other Dirichlet L-functions.

0.3. Our Objectives

In short, by examining the totality of the literature on the subject of universality, it is clear that there are two patterns in much of the research which has been done. Very often, the proofs demonstrate the existence of the universal element by using the Baire Category Theorem strategy, which specifically shows that the set of universal elements is a dense G_δ set in the parent space. Also, many of the papers show that the chosen set of universal objects is dense-lineable in the parent space.

Thus, because the set of universal elements is quite often a dense G_δ set and a dense-lineable set, the overall pattern in the research is that, for the majority of the different types of universality, almost all of the objects in the parent space will have the desired universality properties. That being said, it is clear that there is another pair of less desirable patterns occurring in the research as well.

The first pattern is that the proofs of existence are rarely constructive in nature. Some of the earliest proofs, such as that which was written by Chui and Parnes, were constructive. However, constructive proofs became rarer once the Baire Category Theorem strategy was created. This strategy can only offer an implicit proof of the existence of the chosen universal object, not a constructive one. Once this strategy was popularized, the majority of subsequent proofs relied upon it almost exclusively, resulting in fewer constructive proofs being discovered.

Despite this, some researchers have continued to search for constructive approaches. One such proof was mentioned earlier in our summary of the field of universality, where we explained that we would dissect it in greater detail later on. This was the proof written by Mouze et al. in 2012 [**17**] and it outlined a sequence of steps which, if followed, would build a Universal Taylor series. In short, Mouze et al. claim to have found a constructive proof which demonstrates that Universal Taylor series must exist. While their proof does succeed in demonstrating the existence of these series, there is some question over whether their approach was actually constructive.

The second pattern, which is closely related to the first, is that in most of the different cases of universality discussed above, while the chosen universal object was shown to exist and, even, to be generic in the parent space, the authors do not provide an actual, explicitly defined example of their chosen universal object. In other words, in nearly all the papers summarized above, the research did not pinpoint a specific element of the space which was universal.

There are several reasons for this. Firstly, the proofs relying on the Baire Category Theorem strategy can offer only an implicit proof of the existence of the chosen universal element. Thus, they cannot obtain any specific examples of an object with the universality properties in question. In addition, the constructive proofs of existence, such as the proof written by Chui and Parnes, do not describe the universal elements in explicit detail. Instead, they show that one could theoretically be constructed. Nearly always, such proofs are very complicated and involve an infinite number of steps, meaning that they cannot be used to create a fully defined example of the desired universal element. They can only be used to partially describe the object.

As a result of this, Dirichlet L-functions are one of the few concrete examples of an explicit object with universality properties. It is important to note that the Dirichlet L-functions were already defined when Voronin discovered that they had universality properties. He did not find a constructive proof which both demonstrated the existence of an element with universality properties and provided a specific example of one. Voronin simply found that an object which was already fully defined had certain universal behaviors.

In this thesis we will attempt to find a universal element which can help us overcome the challenges outlined above. In short, we will offer a constructive approach which creates an explicit universal element such that, topologically and algebraically, the set of these new universal elements behaves analogously to other, previously discovered universal elements. We will also use our constructive approach to answer the lingering questions surrounding the proof of Mouze et al.

More precisely, the purpose of our work in this thesis will be threefold. Our first goal is to define a novel form of universality, one which, as far as we know, has not yet been studied by other mathematicians. After describing the desired properties of this universal object, we will demonstrate its existence. However, unlike the majority of the results discussed in the introduction, we will offer a proof of the element's existence which is constructive rather than implicit in nature and we will fully describe this element in explicit detail.

Our second goal is to determine whether the approach of Mouze et al. is, in fact, constructive. To do so, we will study the similarities and differences between the construction of our universal element and the proof of Mouze et al. These first two goals will be achieved in Chapter 1 of this thesis.

Our third goal will be to demonstrate that the set of these new universal elements has topological and algebraic properties identical to those of the majority of the universal elements referred to in the introduction. Specifically, we will demonstrate that these new universal elements are a dense G_δ set by using the Baire Category Theorem strategy and we will demonstrate that these elements are dense-lineable. Accomplishing this goal will include the largest part of the original research done in this thesis and will take up Chapter 2 and Chapter 3.

Chapter 1

Constructibility

1.1. The Proof of Mouze et al.

We will now examine the 2012 paper written by Mouze, Nestoridis, Papadoperakis and Tsirivas [**17**]. Throughout the rest of this thesis, we will refer to this paper numerous times. Instead of using a citation each time, we will simply say that we are discussing the proof of Mouze et al. Mouze et al begin their proof by stating that the following is a well-know Theorem.

Theorem 1.1.1. *Let Ω be a simply connected domain in $\mathbb{C}$ and let $z_0 \in \Omega$. There exists some holomorphic function $F \in H(\Omega)$ such that the Taylor series of F centered at z_0, denoted $\sum_{t=0}^{\infty} a_t(z-z_0)^t$, has the following property: if $K \subset \mathbb{C}$ is a compact set such that $K \cap \Omega = \emptyset$ and K^c is connected, then, for any $h \in A(K)$, there exists some subsequence $(n_j)_{j=1}^{\infty}$ of $\mathbb{N}$ such that*

$$\lim_{j\to\infty} \sup_{z\in K} \left| \sum_{t=0}^{n_j} a_t(z-z_0)^t - h(z) \right| = 0.$$

Mouze et al. noted that, so far, no examples of such a function has been discovered, although, as stated in the introduction, their existence has been proven using the Baire Category Theorem strategy. In their paper, Mouze et al. claimed to have found a constructive approach which builds a Universal Taylor series. While it is undisputed that their method succeeds in proving that Universal Taylor series must exist, there is some contention regarding whether their approach is, in fact, constructive. To determine whether or not this is so, we will reproduce the relevant aspects of their proof and subsequently analyze it.

Their proof requires a certain amount of setup before it can be written out. Mouze et al. declared that they would concentrate on the case where the Universal Taylor series converges on the set $\Omega = \mathbb{D}$ and $z_0 = 0$, meaning that the Taylor series of F centered at z_0 will be of the form $\sum_{t=0}^{\infty} a_t z^t$. The other cases are all very similar.

They let $(L_n)_{n=1}^{\infty}$ be an exhausting sequence of compact sets inside of $\mathbb{D}$ such that, for all $n \in \mathbb{N}$, we have the inclusion $L_n \subset L_{n+1}$. This sequence can be easily created in the following manner:

$$L_n = \left\{ z \in \mathbb{D} \text{ such that } |z| \leq 1 - \frac{1}{n+1} \right\}.$$

It is clear that, for all $n \in \mathbb{N}$, we will have that $0 \in L_n$.

Next, it is also clear that $\Omega = \mathbb{D}$ is a bounded simply connected domain. Mouze et al. demonstrated the following lemma regarding simply connected domains. Next, it is also clear that Ω is a bounded simply connected domain. Mouze et al. demonstrated the following lemma regarding simply connected domains.

Lemma 1.1.2. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain and let $z_0 \in \Omega$. Then there exists a set of compact sets $\{K_m\}_{m=1}^{\infty}$ such that, for all $m \in \mathbb{N}$, we have that $K_m \cap \Omega = \emptyset$ and K_m^c is connected, and such that, for any compact set K such that $K \cap \Omega = \emptyset$ and K^c is connected, there exists some $m_K \in \mathbb{N}$ such that $K \subseteq K_{m_K}$.*

Proof. We will briefly sketch the proof of this lemma. First, we say that, for any two sets $A, B \subset \mathbb{C}$, we define $\text{dist}(A,B) = \inf\{|x-y| \text{ such that } x \in A,\ y \in B\}$ as the distance between A and B.

Mouze et al. consider the set Γ of all polygonal lines of finite length with vertices in $\mathbb{Q}+i\mathbb{Q}$ and which start at z_0 and end at some natural number n_0+1, for some $n_0 \in \mathbb{N}$. They show this set of polygonal lines is countable. Then, for any such polygonal line $\alpha \in \Gamma$ ending at some $n_0 + 1 \in \mathbb{N}$ and, for any $n \in \mathbb{N}$, they take the set of all points $z \in \Omega^c$ such that $\text{dist}(\{x\},\alpha) \geq \frac{1}{n}$ and such that $|z| < n_0$. Mouze et al. do this for all $\alpha \in \Gamma$ and for all $n \in \mathbb{N}$. They show that this creates a countable set of compact sets with connected complements. These are the sets $\{K_m\}_{m=1}^{\infty}$.

Mouze et al. demonstrate that, if K is a compact set such that $K \cap \Omega = \emptyset$ and K^c is connected, then there exists some polygonal line from z_0 to some $(n_0+1) \in \mathbb{N}$ such that the line is contained entirely in the set K^c. They use this to show that there is some $m_K \in \mathbb{N}$ such that $K \subseteq K_{m_K}$. □

Next, they let $\{f_j\}_{j=1}^{\infty}$ be the set of all polynomials such that all their coefficients are elements of $\mathbb{Q} + i\mathbb{Q}$. We will call this the set of rational polynomials. It is easy to see that the set of these polynomials is countable.

They then use the sets $\{f_j\}_{j=1}^{\infty}$ and $\{K_m\}_{m=1}^{\infty}$ to create the set $\{(K_m, f_j)\}_{j,m\in\mathbb{N}}$, which is the set of all possible pairings of a polynomial in the set $\{f_j\}_{j=1}^{\infty}$ with a compact set in the set

$\{K_m\}_{m=1}^{\infty}$. Clearly, the set $\{(K_m, f_j)\}_{j,m\in\mathbb{N}}$ is countable and there exists some enumeration $(K_{m_i}, f_{j_i})_{i=1}^{\infty}$ of its elements.

Finally, with all this in place, we can describe the necessary portions of the proof of Mouze et al. The objective of the proof is to build a function F that satisfies the conditions of Theorem 1.1.1. This will be done by adding up a countably infinite set of polynomials. There are two parts to this proof. Part 1 will involve discussing how adding certain polynomials will create the requisite Universal Taylor series. We will study this part very closely. Part 2 explains how those specific polynomials are chosen. We will not examine this part in detail as it is not necessary for the work done in this thesis.

1.1.1. Part 1 of the Proof of Mouze et al.

Before we begin, we will need to discuss some notation. For any set $S \subseteq \mathbb{C}$, the set $C(S)$ is the set of all continuous $\mathbb{C}$-valued functions on the set S. For any set $S \subset \mathbb{C}$, we will use the symbol $A(S)$ to denote the set of all $\mathbb{C}$-valued functions which are continuous on S and holomorphic in the interior of S. If S has no interior then $A(S) = C(S)$. We always equip the set $A(S)$ with the topology of uniform convergence on compact sets, making $A(S)$ a complete metric space [**11**]. If S is a compact set then the topology of uniform convergence on compact sets is equivalent to the topology induced by the supremum norm. Similarly, we let $H(S)$ be the set of all functions which are holomorphic on S. We also equip this with the topology of uniform convergence on compact sets, which is also equivalent to the topology induced by the supremum norm if S is compact. As mentioned in the introduction, the set $H(\mathbb{D})$ is also a complete metric space.

Next we prove the following Proposition, which we will use in the demonstration of the proof of Mouze et al.

Proposition 1.1.3. *We consider a power series centered at* 0*, which we denote* $\sum_{j=0}^{\infty} a_j z^j$*. If a subsequence* $(\sum_{j=0}^{k_m} a_j z^j)_{m=1}^{\infty}$ *of the partial sums of the power series converges locally uniformly on the unit disk* $\mathbb{D}$ *to a function* F*, then the power series* $\sum_{j=0}^{\infty} a_j z^j$ *converges to* F *on the unit disk as well.*

Proof. Let us discuss some notation. We will let

$$P_1(z) = \sum_{j=0}^{k_1} a_j z^j \text{ and, for all } t > 1, \text{ we will let } P_t(z) = \sum_{j=1+k_{t-1}}^{k_t} a_j z^j.$$

Because the sequence of functions $(\sum_{t=0}^{k_m} a_t z^t)_{m=1}^{\infty}$ converges locally uniformly to F on $\mathbb{D}$, the sequence $(\sum_{t=1}^{m} P_t(z))_{m=1}^{\infty}$ also converges locally uniformly to F on $\mathbb{D}$. Thus, if $z \in \mathbb{D}$, then

$$F(z) = \lim_{m\to\infty} \sum_{j=0}^{k_m} a_j z^j = \lim_{m\to\infty} \sum_{j=1}^{m} P_t(z) = \sum_{t=1}^{\infty} P_t(z).$$

It follows that if $z \in \mathbb{D}$, then

$$F(z) = \sum_{t=1}^{\infty} P_t(z).$$

Next, the sequence of partial sums $(\sum_{t=0}^{k_m} a_t z^t)_{m=1}^{\infty}$ is obviously a sequence of polynomials and, thus, a sequence of holomorphic functions. Because this sequence converges locally uniformly to F on $\mathbb{D}$, we know that F is holomorphic on $\mathbb{D}$. Thus, the Taylor series of the function F centered at 0 converges uniformly on the disk $B(0,r)$, for every r such that $0 < r < 1$. Of course, the Taylor series of the function F centered at 0 is written as $\sum_{j=0}^{\infty} b_j z^j$, where, for all $n \in \{0\} \cup \mathbb{N}$, we know that

$$b_n = \frac{F^{(n)}(0)}{n!}.$$

Now we show that, for all $n \in \mathbb{N}$, the coefficient $b_n = \frac{F^{(n)}(0)}{n!}$ is equal to a_n, which is the n-th coefficient of the power series $\sum_{j=0}^{\infty} a_j z^j$. This will imply that the Taylor series centered at 0 of F is the series $\sum_{j=0}^{\infty} a_j z^j$, meaning that the series $\sum_{j=0}^{\infty} a_j z^j$ converges to F on the set $\mathbb{D}$.

By definition, we know that $0 \leq k_1 < k_2 < k_3 < \ldots$. This means that, for all $n \in (\mathbb{N} \cup \{0\})$, either $0 \leq n \leq k_1$ or there exists some $t_n \in \mathbb{N}$ such that $k_{t_n} < n \leq k_{(1+t_n)}$. We write $k_{(1+t_n)}$ instead of k_{1+t_n} to make the notation clearer to read. We will do the case where $n > k_1$ and there exists some $t_n \in \mathbb{N}$ such that $k_{t_n} < n \leq k_{(1+t_n)}$. The case where $0 \leq n \leq k_1$ is very similar.

As described above, we denote

$$P_{(1+t_n)}(z) = \sum_{j=1+k_{t_n}}^{k_{(1+t_n)}} a_j z^j.$$

So we need to find the following:

$$F^{(n)}(z) = \frac{d^n}{dz^n}\left[\sum_{t=1}^{\infty} P_t(z)\right] = \frac{d^n}{dz^n}\left[\sum_{t=1}^{t_n} P_t(z) + P_{(1+t_n)}(z) + \sum_{t=2+t_n}^{\infty} P_t(z)\right].$$

To do this, we recall that the sequence $(\sum_{t=1}^{m} P_t(z))_{m=1}^{\infty}$ converges locally uniformly to F on $\mathbb{D}$. Because the convergence is locally uniform, the Cauchy integral formula implies that the convergence of the derivatives will also be locally uniform and the series can be differentiated term by term. This implies that

$$\frac{d^n}{dz^n}\left[\sum_{t=1}^{t_n} P_t(z) + P_{(1+t_n)}(z) + \sum_{t=2+t_n}^{\infty} P_t(z)\right] = \sum_{t=1}^{t_n}\left(\frac{d^n}{dz^n}P_t\right)(z) + \left(\frac{d^n}{dz^n}P_{(1+t_n)}\right)(z) + \sum_{t=2+t_n}^{\infty}\left(\frac{d^n}{dz^n}P_t\right)(z),$$

which, in turn, implies that

$$F^{(n)}(z) = \sum_{t=1}^{t_n}\left(\frac{d^n}{dz^n}P_t\right)(z) + \left(\frac{d^n}{dz^n}P_{(1+t_n)}\right)(z) + \sum_{t=2+t_n}^{\infty}\left(\frac{d^n}{dz^n}P_t\right)(z).$$

For an arbitrary polynomial P, it is clear that $P^{(n)}(0)$ is $n!$ times the coefficient of z^n. We know that the term $a_n z^n$ is only in the polynomial $P_{1+t_n}(z)$. Altogether, therefore, we know that

$$F^{(n)}(0) = \sum_{t=1}^{t_n}\left(\frac{d^n}{dz^n}P_t\right)(0) + \left(\frac{d^n}{dz^n}P_{(1+t_n)}\right)(0) + \sum_{t=2+t_n}^{\infty}\left(\frac{d^n}{dz^n}P_t\right)(0) = 0 + a_n(n!) + 0,$$

which is equivalent to saying that

$$F^{(n)}(0) = a_n(n!).$$

This allows us to finally conclude that, for all $n \in (\mathbb{N} \cup \{0\})$,

$$b_n = \frac{F^{(n)}(0)}{n!} = \frac{a_n(n!)}{n!} = a_n$$

and, therefore, we know that

$$F(z) = \sum_{j=0}^{\infty} b_j z^j = \sum_{j=0}^{\infty} a_j z^j.$$

$\square$

We can now begin the construction of the Universal Taylor series.

We consider (K_{m_1}, f_{j_1}). Let $F_1(z) = \begin{cases} 0 & \text{if } z \in L_1 \\ f_{j_1}(z) & \text{if } z \in K_{m_1} \end{cases}$.

We then use the methods which Mouze et al. describe in Part 2 of their proof to obtain a polynomial

$$P_1(z) = \sum_{t=0}^{k_1} a_t z^t$$

for which $0 \le k_1$ and

$$\sup_{z \in (L_1 \bigcup K_{m_1})} |F_1(z) - P_1(z)| < \frac{1}{2}.$$

We consider (K_{m_2}, f_{j_2}). Let $F_2(z) = \begin{cases} 0 & \text{if } z \in L_2 \\ f_{j_2}(z) - P_1(z) & \text{if } z \in K_{m_2} \end{cases}$.

Again, we use the methods of Part 2 to find a polynomial

$$P_2(z) = \sum_{t=1+k_1}^{k_2} a_t z^t$$

such that $k_1 < k_2$ and

$$\sup_{z \in (L_2 \bigcup K_{m_2})} |F_2(z) - P_2(z)| < \frac{1}{2^2}.$$

We consider (K_{m_3}, f_{j_3}). Let $F_3(z) = \begin{cases} 0 & \text{if } z \in L_3 \\ f_{j_3}(z) - P_1(z) - P_2(z) & \text{if } z \in K_{m_3} \end{cases}$.

Again, we use Part 2 to find a polynomial

$$P_3(z) = \sum_{t=1+k_2}^{k_3} a_t z^t$$

such that $k_2 < k_3$ and

$$\sup_{z \in (L_3 \bigcup K_{m_3})} |F_3(z) - P_3(z)| < \frac{1}{2^3}.$$

We continue this process inductively. In the induction step, we chose some $n \in \mathbb{N}$ such that $n > 2$ and we consider (K_{m_n}, f_{j_n}) and we let

$$F_n(z) = \begin{cases} 0 & \text{if } z \in L_n \\ f_{j_n}(z) - \sum_{i=1}^{n-1} P_i(z) & \text{if } z \in K_{m_n} \end{cases}$$

and, using Part 2, we create a polynomial

$$P_n(z) = \sum_{t=1+k_{(n-1)}}^{k_n} a_t z^t,$$

(where we write $k_{(n-1)}$ instead of k_{n-1} in the index to make the notation clearer to read,) such that $k_{(n-1)} < k_n$ and

$$\sup_{z \in (L_n \bigcup K_{m_n})} |F_n(z) - P_n(z)| < \frac{1}{2^n}.$$

This infinite process gives us a set of polynomials $\{P_n\}_{n=1}^{\infty}$.

We now define a new set of functions using the set of polynomials $\{P_n\}_{n=1}^{\infty}$. For all $n \in \mathbb{N}$, we let

$$R_n(z) = \sum_{t=1}^{n} P_t(z).$$

Now we must discuss some of our notation. We recall that we denoted

$$P_1(z) = \sum_{t=0}^{k_1} a_t z^t \text{ and that, if } n > 1, \text{ then } P_n(z) = \sum_{t=1+k_{(n-1)}}^{k_n} a_t z^t.$$

However, because we write $R_n(z) = \sum_{t=1}^{n} P_t(z)$, we will now change our notation for each $P_n(z)$ slightly. First, we write $P_t(z)$ instead of $P_n(z)$. Second, the index of each sum will now be written using the letter j instead of the letter t. Therefore, for the rest of the duration of Chapter 1, we write

$$P_1(z) = \sum_{j=0}^{k_1} a_j z^j \text{ and, if } t > 1, \text{ then we write } P_t(z) = \sum_{j=1+k_{(t-1)}}^{k_t} a_j z^j,$$

Altogether, for all $n \in \mathbb{N}$, we let

$$R_n(z) = \sum_{t=1}^{n} P_t(z) = \sum_{j=0}^{k_n} a_j z^j.$$

It is clear that the sequence $(R_n)_{n=1}^{\infty}$ is a subsequence of the partial sums of the series $\sum_{j=0}^{\infty} a_j z^j$. It is also clear that, for all $n \in \mathbb{N}$, we have that $R_n \in H(\mathbb{D})$. We now demonstrate a lemma regarding the sequence $(R_n)_{n=1}^{\infty}$.

Lemma 1.1.4. *The above defined sequence of functions $(R_n)_{n=1}^{\infty}$ will converge locally uniformly to some function $F \in H(\mathbb{D})$.*

Proof. We will demonstrate that the sequence $(R_n)_{n=1}^{\infty}$ is a Cauchy sequence in the space $H(\mathbb{D})$. Because $H(\mathbb{D})$ is a complete metric space, this will prove that there exists some $F \in H(\mathbb{D})$ to which $(R_n)_{n=1}^{\infty}$ must converge.

It suffices to show that, for any compact set $L \subset \mathbb{D}$, the sequence $(R_n)_{n=1}^{\infty}$ converges uniformly to F on L [**4**, p 142-152]. To do this, we show that the sequence $(R_n)_{n=1}^{\infty}$ is a Cauchy sequence in the space $A(L)$. This was stated to be a complete metric space at the beginning of Section 1.1.1. In other words, we must show that for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that if $n,m > N$, then we get that

$$\sup_{z\in L} |R_n(z) - R_m(z)| < \epsilon.$$

We choose some $\epsilon > 0$. Clearly, there exists some $n_\epsilon \in \mathbb{N}$ such that

$$\sum_{t=n_\epsilon}^{\infty} \frac{1}{2^t} < \epsilon.$$

We recall that, earlier, we created a set of exhausting compact sets $(L_n)_{n=1}^{\infty}$. Clearly, there exists some n_L such that, for all $t \geq n_L$, we have the inclusion $L \subset L_t$. We choose some $N \in \mathbb{N}$ such that $N > \max\{n_\epsilon, n_L\}$.

We let $n,m \in \mathbb{N}$, such that $n,m > N$ and $n \neq m$. Without loss of generality, let us assume that $n > m$. We get that

$$\sup_{z\in L} |R_n(z) - R_m(z)| = \sup_{z\in L} \left| \sum_{t=0}^{n} P_t(z) - \sum_{t=0}^{m} P_t(z) \right| = \sup_{z\in L} \left| \sum_{t=m+1}^{n} P_t(z) \right|$$

$$\leq \sup_{z\in L} \sum_{t=m+1}^{n} |P_t(z)| \leq \sup_{z\in L} \sum_{t=m+1}^{\infty} |P_t(z)| \leq \sum_{t=m+1}^{\infty} \sup_{z\in L} |P_t(z)|.$$

Therefore,

$$\sup_{z\in L} |R_n(z) - R_m(z)| \leq \sum_{t=m+1}^{\infty} \sup_{z\in L} |P_t(z)|.$$

Because $t \geq m+1 > m > N > n_L$, we know that $t > n_L$ and $L \subset L_t$. This implies that, for any $t \geq m+1$, we have that $\sup_{z\in L} |P_t(z)| \leq \sup_{z\in L_t} |P_t(z)|$. This gives us that

$$\sup_{z\in L} |R_n(z) - R_m(z)| \leq \sum_{t=m+1}^{\infty} \sup_{z\in L} |P_t(z)| \leq \sum_{t=m+1}^{\infty} \sup_{z\in L_t} |P_t(z)|.$$

This means that

$$\sup_{z\in L}|R_n(z)-R_m(z)| \leq \sum_{t=m+1}^{\infty}\sup_{z\in L_t}|P_t(z)|.$$

We recall that, for all $t \in \mathbb{N}$, if $z \in L_t$ then $F_t(z) = 0$. We also recall that, for all $t \in \mathbb{N}$,

$$\sup_{z\in(L_t\bigcup K_t)}|F_t(z)-P_t(z)| < \frac{1}{2^t}.$$

So we have that

$$\sup_{z\in L}|R_n(z)-R_m(z)| \leq \sum_{t=m+1}^{\infty}\sup_{z\in L_t}|P_t(z)| = \sum_{t=m+1}^{\infty}\sup_{z\in L_t}|P_t(z)-0|$$

$$= \sum_{t=m+1}^{\infty}\sup_{z\in L_t}|P_t(z)-F_t(z)| \leq \sum_{t=m+1}^{\infty}\sup_{z\in(L_t\bigcup K_t)}|P_t(z)-F_t(z)| \leq \sum_{t=m+1}^{\infty}\frac{1}{2^t}.$$

We deduce that

$$\sup_{z\in L}|R_n(z)-R_m(z)| \leq \sum_{t=m+1}^{\infty}\frac{1}{2^t}.$$

Here, we recall that $m+1 > m > N > n_\epsilon$. This means that

$$\sup_{z\in L}|R_n(z)-R_m(z)| \leq \sum_{t=m+1}^{\infty}\frac{1}{2^t} < \sum_{t=n_\epsilon}^{\infty}\frac{1}{2^t} < \epsilon.$$

Hence,

$$\sup_{z\in L}|R_n(z)-R_m(z)| < \epsilon.$$

Thus, $(R_n)_{n=1}^{\infty}$ is a Cauchy sequence in $A(\mathbb{D})$ which converges to some $F \in A(\mathbb{D})$ and, if $z \in \mathbb{D}$, then

$$F(z) = \lim_{n\to\infty} R_n(z).$$

□

By Proposition 1.1.3 and Lemma 1.1.4, the series $\sum_{j=0}^{\infty} a_j z^j$ is the Taylor series expansion of F centered at 0. We now show that this Taylor series has the desired universality properties, which is to say that $\sum_{j=0}^{\infty} a_j z^j$ is a Universal Taylor series and that it satisfies the requirements of Theorem 1.1.1.. We do so in the following Proposition.

Proposition 1.1.5. *Let K be some compact set such that $K \cap \mathbb{D} = \emptyset$ and such that K^c is connected. Let $h \in A(K)$. There exists a subsequence of the partial sums of the Taylor series $\sum_{j=0}^{\infty} a_j z^j$ which converges uniformly to h on K.*

Proof. For any $n \in \mathbb{N}$, we will denote the n-th partial sum of the Taylor series centered at 0 of the function F as $S_n(F,0)$. By construction, we know that

$$P_1(z) = \sum_{j=0}^{k_1} a_j z^j \text{ and, for all } t \in \mathbb{N} \text{ such that } t > 1,\ P_t(z) = \sum_{j=1+k_{(t-1)}}^{k_t} a_j z^j.$$

Also, that $\deg(P_1) = k_1 < \deg(P_2) = k_2 < \deg(P_3) = k_3 < \ldots$ We consider the sequence of partial sums $(S_{k_i}(F,0)(z))_{i=1}^{\infty}$. It is clear that, for any $i \in \mathbb{N}$, we have the equality

$$S_{k_i}(F,0)(z) = \sum_{j=0}^{k_i} a_j z^j = \sum_{t=1}^{i} P_t(z).$$

Thus,

$$S_{k_i}(F,0)(z) = \sum_{t=1}^{i} P_t(z).$$

We will construct a sequence $i_1, i_2, i_3, \ldots$ such that the subsequence $(S_{k_{i_s}}(F,0)(z))_{s=1}^{\infty}$ of $(S_{k_i}(F,0)(z))_{i=1}^{\infty}$ converges uniformly to h on K. It is clear that $(S_{k_{i_s}}(F,0)(z))_{s=1}^{\infty}$ will be a subsequence of the overall sequence of the partial sums of the series $\sum_{j=0}^{\infty} a_j z^j$, and, thus, the proof will be done.

To begin, clearly there exists some subsequence $(g_j)_{j=1}^{\infty}$ of $\mathbb{N}$ such that, for all $j \in \mathbb{N}$, the number g_j is large enough that

$$\frac{2}{2^{g_j}} < \frac{1}{j}.$$

Obviously, if $i \in \mathbb{N}$ and $i > g_j$, then

$$\frac{1}{2^i} < \frac{1}{2^{g_j}}.$$

Now we will begin to select the numbers in the sequence $(i_s)_{s=1}^{\infty}$. We recall the collection $\{K_m\}_{m=1}^{\infty}$, which was previously defined above in Lemma 1.1.2. Let K_{m^*} be one of the sets in $\{K_m\}_{m=1}^{\infty}$ such that $K \subseteq K_{m^*}$. By Mergelyan's Theorem, because K_{m^*} is compact sets with connected complement, there is an infinite set of polynomials which approximate the holomorphic function h within $\frac{1}{2^{g_1}}$ on K_{m^*} [**21**, page 390]. Because any polynomial can be approximated by rational polynomials as closely as desired on a compact set using the sup norm, it follows that there exists an infinite number of rational polynomials which approximate h within $\frac{1}{2^{g_1}}$ on K_{m^*}.

By construction, the enumeration $(K_{m_i}, f_{j_i})_{i=1}^{\infty}$ contains all possible pairings of the compact set K_{m^*} with a rational polynomial. Thus, there is an infinite subsequence $(i_v^*)_{v=1}^{\infty}$ of $\mathbb{N}$ such that the subsequence $(K_{m_{i_v^*}}, f_{j_{i_v^*}})_{v=1}^{\infty}$ of the enumeration $(K_{m_i}, f_{j_i})_{i=1}^{\infty}$ has the following properties: for all $v \in \mathbb{N}$, we have that $K_{m_{i_v^*}} = K_{m^*}$ and that

$$\sup_{z \in K_{m^*}} |h(z) - f_{j_{i_v^*}}(z)| < \frac{1}{2^{g_1}}.$$

Essentially, the terms of the subsequence are pairings of the set K_{m^*} with rational polynomials which approximate h within $\frac{1}{2^{g_1}}$ on K_{m^*}.

Because this subsequence is infinitely long, the indices of the terms of this subsequence grow infinitely large. Therefore, we can choose some $v \in \mathbb{N}$ such that $i_v^* > g_1$ and we let $i_v^* = i_1$. This means that, for the pair $(K_{m_{i_1}}, f_{j_{i_1}})$ of the enumeration $(K_{m_i}, f_{j_i})_{i=1}^{\infty}$, we have that $K \subseteq K_{m_{i_1}} = K_{m^*}$ and

$$\sup_{z \in K_{m_{i_1}}} |h(z) - f_{j_{i_1}}(z)| < \frac{1}{2^{g_1}}.$$

Because we chose $i_1 > g_1$, we get that

$$\frac{1}{2^{i_1}} < \frac{1}{2^{g_1}}.$$

Now we chose i_2. Using the same logic described above, we can chose some $i_2 \in \mathbb{N}$ such that $i_2 > \max\{g_2, i_1\}$ and such that, for the pair $(K_{m_{i_2}}, f_{j_{i_2}})$ of the enumeration $(K_{m_i}, f_{j_i})_{i=1}^{\infty}$, we have that $K \subseteq K_{m_{i_2}} = K_{m^*}$ and

$$\sup_{z \in K_{m_{i_2}}} |h(z) - f_{j_{i_2}}(z)| < \frac{1}{2^{g_2}}.$$

Because we chose $i_2 > g_2$, we get that

$$\frac{1}{2^{i_2}} < \frac{1}{2^{g_2}}.$$

We continue this process inductively. In the induction step, for some $s > 2$, we use the same logic described above to pick i_s such that $i_s > \max\{g_s, i_{s-1}\}$ and such that, for the pair $(K_{m_{i_s}}, f_{j_{i_s}})$ of the enumeration $(K_{m_i}, f_{j_i})_{i=1}^{\infty}$, we have that $K \subseteq K_{m_{i_s}} = K_{m^*}$ and

$$\sup_{z\in K_{m_{i_s}}} |h(z) - f_{j_{i_s}}(z)| < \frac{1}{2^{g_s}}.$$

Because we chose $i_s > g_s$, we get that

$$\frac{1}{2^{i_s}} < \frac{1}{2^{g_s}}.$$

Altogether, the construction implies the following pair of statements: first, it is obvious that $i_1 < i_2 < i_3 < \dots$. Second, for all $s \in \mathbb{N}$,

$$\sup_{z\in K} |h(z) - S_{k_{i_s}}(F,0)(z)| = \sup_{z\in K} \left| h(z) - \sum_{t=1}^{i_s} P_t(z) \right| \leq \sup_{z\in K_{m_{i_s}}} \left| h(z) - \sum_{t=1}^{i_s} P_t(z) \right|$$

$$\leq \sup_{z\in K_{m_{i_s}}} \left(|h(z) - f_{j_{i_s}}(z)| + \left| f_{j_{i_s}}(z) - \sum_{t=1}^{i_s} P_t(z) \right| \right) \leq \sup_{z\in K_{m_{i_s}}} |h(z) - f_{j_{i_s}}(z)| + \sup_{z\in K_{m_{i_s}}} \left| f_{j_{i_s}}(z) - \sum_{t=1}^{i_s} P_t(z) \right|$$

$$< \frac{1}{2^{g_s}} + \sup_{z\in K_{m_{i_s}}} \left| f_{j_{i_s}}(z) - \sum_{t=1}^{i_s} P_t(z) \right| = \frac{1}{2^{g_s}} + \sup_{z\in K_{m_{i_s}}} \left| f_{j_{i_s}}(z) - \sum_{t=1}^{i_s-1} P_t(z) - P_{i_s}(z) \right|.$$

We can summarize this by saying

$$\sup_{z\in K} |h(z) - S_{k_{i_s}}(F,0)(z)| \leq \frac{1}{2^{g_s}} + \sup_{z\in K_{m_{i_s}}} \left| f_{j_{i_s}}(z) - \sum_{t=1}^{i_s-1} P_t(z) - P_{i_s}(z) \right|.$$

Here we remember that, at the beginning of section 1.1.1, we defined the function $F_{i_s}(z)$ such that

$$F_{i_s}(z) = \begin{cases} 0 & \text{if } z \in L_{i_s} \\ f_{j_{i_s}}(z) - \sum_{t=1}^{i_s-1} P_t(z) & \text{if } z \in K_{m_{i_s}} \end{cases}$$

and that

$$\sup_{z\in (L_{i_s} \bigcup K_{m_{i_s}})} |F_{i_s}(z) - P_{i_s}(z)| < \frac{1}{2^{i_s}}.$$

This means that

$$\sup_{z\in K} |h(z) - S_{k_{i_s}}(F,0)(z)| \leq \frac{1}{2^{g_s}} + \sup_{z\in K_{m_{i_s}}} \left| f_{j_{i_s}}(z) - \sum_{t=1}^{i_s-1} P_t(z) - P_{i_s}(z) \right| = \frac{1}{2^{g_s}} + \sup_{z\in K_{m_{i_s}}} |F_{i_s}(z) - P_{i_s}(z)|$$

$$\leq \frac{1}{2^{g_s}} + \sup_{z\in(L_{i_s}\bigcup K_{m_{i_s}})} |F_{i_s}(z) - P_{i_s}(z)| < \frac{1}{2^{g_s}} + \frac{1}{2^{i_s}} < \frac{1}{2^{g_s}} + \frac{1}{2^{g_s}} = \frac{2}{2^{g_s}} < \frac{1}{s}.$$

Hence,

$$\sup_{z\in K} |h(z) - S_{k_{i_s}}(F,0)(z)| < \frac{1}{s}.$$

It is easy to see that the subsequence of partial sums $(S_{k_{i_s}}(F,0)(z))_{s=1}^{\infty}$ converges uniformly to $h(z)$ on K. □

1.1.2. Part 2 of the Proof of Mouze et al.

Here, Mouze et al. describe how the polynomials in the set $\{P_n\}_{n=1}^{\infty}$ are chosen. They must be selected so that each P_n not only approximates F_n, but also so that, for some strictly increasing sequence $k_1, k_2, k_3, \ldots$, for any $n \in \mathbb{N}$, we have that $\deg(P_n) = k_n$ and that $P_n = \sum_{t=1+k_{(n-1)}}^{k_n} a_t(z - z_0)^t$. Essentially, the polynomials must be chosen such that, when the polynomials are summed up, their coefficients line up to create one long Taylor series. Most crucially, this selection method cannot simply state the existence of some polynomial which will provide the desired approximation of F_n, which is what is done by Theorems such as Mergelyan's Theorem and Runge's Theorem. To be constructive, this selection method must produce a single, fully defined polynomial each time we choose a polynomial P_n.

While their work in this section is interesting and worth reading, as mentioned earlier, it does not have sufficient relevance to the topic discussed in this thesis and it will not be studied in detail.

1.2. Universal Numerical Series

We have now finished relating the necessary aspects of the proof written by Mouze et al. As we promised in the introduction, we now present the new type of universality which we discovered. We initially proceed in this endeavor by examining a question asked by Fournier: Is there a complex numerical series for which the partial sums of the series are dense in $\mathbb{C}$, assuming the Euclidean metric on $\mathbb{C}$?

Before we begin, we must lay some groundwork. Let us denote the set of (formal) complex numerical series as NS($\mathbb{C}$), where NS stands for Numerical Series. Let us now discuss our notation. In this thesis if we write that $R \in$ NS($\mathbb{C}$), then the symbol R represents some Numerical series $\sum_{t=1}^{\infty} r_t \in$ NS($\mathbb{C}$). It is important to note that R does not represent a number to which the series $\sum_{t=1}^{\infty} r_t$ converges. Instead, R represents the (formal) complex numerical series itself and we denote $R = \sum_{t=1}^{\infty} r_t$.

An important mathematical property of any (formal) complex numerical series is that it has a sequence of partial sums. Let $R \in \mathrm{NS}(\mathbb{C})$, where we denote $R = \sum_{t=1}^{\infty} r_t$. If $(x_t)_{t=1}^{\infty}$ is the sequence of the partial sums of R, then we will say that R is *associated* to the sequence of partial sums $(x_t)_{t=1}^{\infty}$ or that $(x_t)_{t=1}^{\infty}$ is the sequence of partial sums *associated* to the series R. During most of this thesis, for the sake of brevity, we will shorten this terminology by simply saying that that R is *associated* to the sequence $(x_t)_{t=1}^{\infty}$ or that $(x_t)_{t=1}^{\infty}$ is the sequence *associated* to the series R.

If R is *associated* to the sequence $(x_t)_{t=1}^{\infty}$, then by definition, for all $n \in \mathbb{N}$,

$$x_n = \sum_{t=1}^{n} r_t.$$

We will now prove a pair of lemmas regarding this association.

Lemma 1.2.1. *A sequence of complex numbers $(x_t)_{t=1}^{\infty}$ is the associated sequence of partial sums of some Numerical series $R = \sum_{t=1}^{\infty} r_t$ if and only if $x_1 = r_1$ and, if $j > 1$, then $r_j = x_j - x_{j-1}$.*

Proof. We first show the right implication. Let the series R be associated to the sequence $(x_t)_{t=1}^{\infty}$. By definition, we know that

$$x_1 = \sum_{t=1}^{1} r_t = r_1.$$

We also know that, if $j \in \mathbb{N}$ and $j > 1$, then

$$x_j = \sum_{t=1}^{j} r_t \text{ and } x_{j-1} = \sum_{t=1}^{j-1} r_t.$$

This means that, if $j \in \mathbb{N}$ and $j > 1$, then we find the following:

$$r_j + x_{j-1} = r_j + \sum_{t=1}^{j-1} r_t = \sum_{t=1}^{j} r_t = x_j.$$

Hence,

$$r_j = x_j - x_{j-1}.$$

We now show the left implication. Because $x_1 = r_1$, we know that

$$\sum_{t=1}^{1} r_t = r_1 = x_1.$$

We know that, if $j > 1$, then $r_t = x_j - x_{j-1}$. If $n > 1$, then

$$\sum_{t=1}^{n} r_t = r_1 + \sum_{t=2}^{n} r_t = x_1 + \sum_{t=2}^{n} (x_t - x_{t-1}).$$

The series on the right is obviously telescopic and so we find that

$$x_1 + \sum_{t=2}^{n} (x_t - x_{t-1}) = x_1 + (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_t - x_{t-1}) = x_n$$

and so,

$$\sum_{t=1}^{n} r_t = r_1 + \sum_{t=2}^{n} r_t = x_1 + \sum_{t=2}^{n} (x_t - x_{t-1}) = x_n.$$

Hence,

$$\sum_{t=1}^{n} r_t = x_n.$$

This means that for all $n \in \mathbb{N}$, we have that

$$x_n = \sum_{t=1}^{n} r_t$$

and so the sequence $(x_t)_{t=1}^{\infty}$ is the sequence of the partial sums of the series $R = \sum_{t=1}^{\infty} r_t$. □

In the following lemma and throughout the rest of this thesis, we will denote the set of sequences of complex numbers as $\mathbb{C}^{\infty}$.

Lemma 1.2.2. *The above association between Numerical series and their respective sequence of partial sums is a bijective map from* $\mathrm{NS}(\mathbb{C})$ *to* $\mathbb{C}^{\infty}$.

Proof. We demonstrate injectivity. Let us choose two different sequences of complex numbers, denoted $(x_t)_{t=1}^{\infty}$ and $(y_t)_{t=1}^{\infty}$. Because they are different sequences, there exists at least one $t \in \mathbb{N}$ such that $x_t \neq y_t$. Let $t^* = \min\{t \in \mathbb{N} \mid x_t \neq y_t\}$. Let $x_1 = r_1$ and, for any $t \in \mathbb{N}$ such that $t \geq 2$, let $r_t = x_t - x_{t-1}$. Let $y_1 = s_1$ and, if $t \in \mathbb{N}$ such that $t \geq 2$, let $s_t = y_t - y_{t-1}$. Let $R = \sum_{t=1}^{\infty} r_t$ and $S = \sum_{t=1}^{\infty} s_t$. Clearly, by Lemma 1.2.1, the series R is associated to the sequence $(x_t)_{t=1}^{\infty}$ and the series S is associated to the sequence $(y_t)_{t=1}^{\infty}$.

If $t^* = 1$ then $s_1 = y_1 \neq x_1 = r_1$ and so $s_1 \neq r_1$. This means that $R \neq S$ because their first terms are not equal. If $t^* > 1$, then, because we have that $t^* = \min\{t \in \mathbb{N} \mid x_t \neq y_t\}$, it is clear that $x_{(t^*-1)} = y_{(t^*-1)}$. Thus, we find that

$$x_{t^*} \neq y_{t^*}$$

$$x_{t^*} - x_{(t^*-1)} \neq y_{t^*} - y_{(t^*-1)}$$

$$r_{t^*} = x_{t^*} - x_{(t^*-1)} \neq y_{t^*} - y_{(t^*-1)} = s_{t^*}$$

$$r_{t^*} \neq s_{t^*}.$$

This similarly means $R \neq S$. This shows that the association is injective.

Now we demonstrate surjectivity. Let $R = \sum_{t=1}^{\infty} r_t$ be a complex numerical series, meaning that $R \in \mathrm{NS}(\mathbb{C})$. We must find a point in $\mathbb{C}^\infty$ which is mapped to R by the association. We create a sequence by letting $x_1 = r_1$ and, for any $t \in \mathbb{N}$ such that $t > 1$, letting $x_t = r_t + x_{t-1}$. This is equivalent to writing that, for all $t > 1$, we have that $r_t = x_t - x_{t-1}$. Then $(x_t)_{t=1}^{\infty}$ is a sequence of complex numbers and, thus, it is a point in $\mathbb{C}^\infty$. By Lemma 1.2.1, it is also clear that the sequence $(x_t)_{t=1}^{\infty}$ is the sequence associated to the series R. This shows that the association is surjective.

□

When discussing the characteristics of the sequence $(x_t)_{t=1}^{\infty}$ associated to some series $R \in \mathrm{NS}(\mathbb{C})$, we will use the following terminology: we say that $(x_t)_{t=1}^{\infty}$ is a *dense sequence* or a *sequence which is dense* if the set of its terms is dense in $\mathbb{C}$. In other words, if $\overline{\{x_t\}_{t=1}^{\infty}} = \mathbb{C}$. Sequences of this form clearly exist; we can pick any sequence that enumerates the set $(\mathbb{Q} + i\mathbb{Q})$, which is countable and dense in $\mathbb{C}$.

With this information, we can return to the question. Fournier found that the answer to his question is yes and, not only can these series be shown to exist, but explicit examples can be created. These series are called the Universal complex numerical series, however, for the sake of brevity, we shall simply call them Universal numerical series. We will denote the set of such series as $\mathrm{UNS}(\mathbb{C})$, where UNS stands for Universal numerical series.

We can write a much more formal definition of the set $\mathrm{UNS}(\mathbb{C})$. For all $k \in \mathbb{N}$, if $\sum_{t=1}^{\infty} r_t = R \in \mathrm{NS}(\mathbb{C})$, then let us denote the partial sums of R as $h_k(R) = \sum_{t=1}^{k} r_t$. We will use this notation often throughout this thesis.

Definition 1.2.3. *A Universal Numerical series is any series $R \in \mathrm{NS}(\mathbb{C})$ such that*

$$\overline{\{h_k(R)\}_{k=1}^{\infty}} = \mathbb{C}.$$

This is equivalent to saying that a Universal Numerical series is any series $R \in \mathrm{NS}(\mathbb{C})$ *which is associated to a dense sequence. Let the set* $\mathrm{UNS}(\mathbb{C})$ *be the set of all such series.*

Thus, to create an explicit example of a Universal numerical series, one can simply consider any explicit enumeration, denoted $(x_t)_{t=1}^{\infty}$, of a dense countable set in $\mathbb{C}$, such as the set $\mathbb{Q} + i\mathbb{Q}$. This will clearly be a dense sequence. By Lemma 1.2.2, there must exists a single Numerical series, denoted $R = \sum_t^{\infty} r_t$, which is associated to the dense sequence $(x_t)_{t=1}^{\infty}$. Thus, by Definition 1.2.3, we will have that $R \in \mathrm{UNS}(\mathbb{C})$. For all $t \in \mathbb{N}$, Lemma 1.2.1 can be used to find the term r_t of the series R. This will give a full and explicit description of a Universal numerical series.

1.3. Is the Proof of Mouze et al. constructive?

We have now covered how Universal numerical series are constructed. We can now finally examine the question of whether or not the procedure described by the paper of Mouze et al. is actually constructive. To answer this question, we will compare and contrast the work of Mouze et al. with our construction of the set of Universal numerical series. In doing so, we will uncover the limitations of the work done by Mouze et al. We will finish by discussing the admirable aspects of their paper.

In order to examine this topic, we must ask what it means for a process to be constructive. We can provide explicit examples of geometric series or continuous functions. However, if a proof is constructive, does this automatically imply that an explicitly defined example must be provided?

In general, this is not true. For example: in Cantor's famous diagonal argument, the proof assumes that the real numbers are countable and that they can be written as a sequence. Then it finds a specific example of a real number that is not included in the sequence. This demonstrates that the assumption that the set of real numbers is countable must, in fact, be false.

While this proof does demonstrate how the specific real number is assembled, it does not state which exact number is chosen. Despite this, it is a constructive argument. It proves that a specific example of the desired element does exist and shows how it is assembled, even though it does not fully describe the element. Similarly, the proof of Mouze et al. indicates how to construct a Universal Taylor series without providing an exact example of one. Therefore, it is constructive. However, the proof of Mouze et al. is limited in that it is not explicit.

Let us suppose that the series $R = \sum_{t=1}^{\infty} r_t$ is associated to some dense sequence $(x_t)_{t=1}^{\infty}$. Let us further suppose that we choose to determine the identity of the term r_n, for some $n \in \mathbb{N}$. We notice that the term r_n only depends on, at most, two terms of the dense sequence. It does not depend on any previous terms in the series. This is because $r_1 = x_1$ and, if $n > 1$, then $r_n = x_n - x_{n-1}$. Essentially, for all $n \in \mathbb{N}$, the term r_n can be determined immediately from the sequence $(x_n)_{n=1}^{\infty}$, even if r_1,$r_2, \ldots$,r_{n-1} are unknown. Thus, the Universal numerical series is encoded in the structure of the dense sequence $(x_n)_{n=1}^{\infty}$.

However, this is not possible with the arguments of Mouze et al. For any $n \in \mathbb{N}$, the construction of P_n depends on the polynomials $P_1, P_2, \ldots, P_{n-1}$ and, therefore, if $n > 1$, then an induction on $P_1, P_2, \ldots, P_{n-1}$ is necessary to construct the polynomial P_n. This means that there is no structure which can encode the full identity of a Universal Taylor series that is being built using the proof of Mouze et al.

We conclude the following: in our work, we know more about the universal object being constructed. The method of Mouze et al. is blinkered in that it cannot jump ahead, so to speak. It can only ever describe, for some $N \in \mathbb{N}$, the first N polynomials in the construction of the Universal Taylor series and nothing beyond that. This means that it can never be used to obtain a complete Universal Taylor series, only part of one. On the other hand, our construction allows us to find any term in the Universal numerical series, regardless of whether the other terms are known. This is almost, if not exactly, the same thing as writing out a complete, explicit example. In this respect, our work provides a more explicit construction than that of Mouze et al. To summarize, whereas the method of Mouze et al. require an induction to build each polynomial P_n when $n > 1$, for the construction of the Universal numerical series, the sequence $(x_t)_{t=1}^{\infty}$ encodes all of the information regarding the summands of the series and no induction is needed.

We conclude with the worthwhile parts of the work done by Mouze et al. Firstly, there is no question that they have created a mathematically valid program for constructing a Universal Taylor series. Given infinite time, this series can be built, polynomial by polynomial. Also, quite impressively, they managed to create a proof of the existence of the Universal Taylor series which did not depend, as nearly all other proofs on the subject did, on the use of the Baire Category Theorem. Part 2 of their proof is especially praiseworthy in how it constructively builds a specific polynomial which approximates a chosen function on a specific compact set and such that, for some chosen number $m \in \mathbb{N}$, the first m coefficients of the polynomial are equal to zero.

Chapter 2

Topological Properties of the set $\mathrm{UNS}(\mathbb{C})$

2.1. Topology

We now examine the topological and algebraic aspects of the set of Universal numerical series. The present chapter focuses on the topological properties and the following chapter combines these topological properties with algebraic properties. In order to study this topic we must first discuss some elements of topology.

To begin with, we denote the Euclidean metric distance between any $z_1,z_2 \in \mathbb{C}$ as $d(z_1,z_2) = |z_1 - z_2|$, where $|\cdot|$ is the modulus symbol. A property of the modulus, which we will use frequently in what follows, is that for any complex number $z \in \mathbb{C}$, we know that $|\operatorname{Re}(z)| \leq |z|$. This is easily proven to be the case:

$$|\operatorname{Re}(z)| = \sqrt{(\operatorname{Re}(z))^2} \leq \sqrt{(\operatorname{Re}(z))^2 + (\operatorname{Im}(z))^2} = |z|$$

$$|\operatorname{Re}(z)| \leq |z|.$$

Also, to denote the open Euclidean ball of radius ϵ around a point $z_0 \in \mathbb{C}$, we will use the notation $B(z_0,\epsilon)$.

Let us consider any subset V of $\mathbb{C}^\infty$ of the form $V = \prod_{t=1}^{\infty} V_t$ such that, for all $t \in \mathbb{N}$, the set V_t is a non-empty subset of $\mathbb{C}$. Let $p = (p_1, p_2, p_3, p_4, \dots) \in \mathbb{C}^\infty$. Clearly, we declare that $p \in V$ if and only if, for all $t \in \mathbb{N}$, we have that $p_t \in V_t$.

There are several topologies which can be used on the set $\mathbb{C}^\infty$. We will be using two of them, including the most well-known, which is the product topology. The other will be the uniform topology.

Let us examine these topologies. The product topology has a basis, [**18**, p 231] which we will denote

$$\mathbf{B}_\pi = \{\prod_{t=1}^{\infty} O_t \mid \forall t \in \mathbb{N},\, O_t \text{ is open and non-empty in } \mathbb{C} \\ \text{and } \exists N \in \mathbb{N} \text{ such that, } \forall t > N,\, O_t = \mathbb{C}\}.$$

By the definition of a topological basis, the set of all possible non-empty unions of the basis elements equals the set of all non-empty open sets of the product topology.

While we are on the topic of the product topology, we will prove a lemma, which will be useful in many of the following calculations.

Lemma 2.1.1. *Let E be a dense set in $\mathbb{C}$. Let $D \subset \mathbb{C}^\infty$ be such that, for any $N \in \mathbb{N}$, for any finite, ordered sequence of complex numbers $(z_1, z_2, \dots, z_N)$ such that $z_t \in E$ for all $t \in \{1, 2, 3, \dots, N\}$, there exists $p \in D$ such that $p = (p_1, p_2, p_3, p_4, p_5, \dots)$ and such that $p_1 = z_1$, $p_2 = z_2$, $p_3 = z_3$, $\dots$ and $p_N = z_N$. Then D is dense in $\mathbb{C}^\infty$ equipped with the product topology.*

Proof. We will prove that any open set of the product topology contains at least one point of D. To do this we recall that all non-empty open sets of the product topology are unions of basis elements. This means that it suffices to show that any basis element in $\mathbf{B}_\pi$ contains some point of D.

We pick some arbitrary basis element $O \in \mathbf{B}_\pi$ and denote it $O = \prod_{t=1}^{\infty} O_t$. We must show that $O \cap D \neq \emptyset$. We know that there exists some $N \in \mathbb{N}$ such that, if $t > N$, then $O_t = \mathbb{C}$. We also know that if $t \in \{1, 2, \dots, N\}$, then O_t is open and non-empty. Because the set E is dense in $\mathbb{C}$, we can select some $z_1 \in O_1 \cap E$, some $z_2 \in O_2 \cap E$, some $z_3 \in O_3 \cap E$, $\dots$ and some $z_N \in O_N \cap E$. In short, for any $t \in \{1, 2, 3, \dots, N\}$, we have that $z_t \in O_t \cap E$ and, thus, that $z_t \in O_t$.

By hypothesis, there exists some $p \in D$ such that $p = (p_1, p_2, p_3, p_4, p_5, \dots)$ and such that $p_1 = z_1$, $p_2 = z_2$, $p_3 = z_3$, $\dots$ and $p_N = z_N$. Thus, $p_1 \in O_1$, $p_2 \in O_2$, $p_3 \in O_3$, $\dots$ and $p_N \in O_N$. Next, if $t \in \mathbb{N}$ such that $t > N$, then $O_t = \mathbb{C}$ and, therefore, it is automatically true that $p_t \in O_t$. Thus, for all $t \in \mathbb{N}$, we have shown that $p_t \in O_t$. This implies that $p \in O$, which, in turn means that $p \in O \cap D$. We conclude that $O \cap D \neq \emptyset$. □

We now discuss the second topology, which is the uniform topology. This topology is induced by a metric which we will use at various times in this thesis. Let $p = (p_1, p_2, p_3, p_4, p_5, \dots)$ and $q = (q_1, q_2, q_3, q_4, q_5, \dots)$ be points of the set $\mathbb{C}^\infty$. The uniform metric is defined and denoted as $d_u(p,q) = \sup_{t \in \mathbb{N}}\{\min\{1, |p_t - q_t|\}\}$. We denote the open uniform ball of radius ϵ around a point $p \in \mathbb{C}^\infty$ by writing $B_{\text{unif}}(p,\epsilon)$ [**18**, pp 121-125].

Often, we will not use $\epsilon > 1$ in this topology, because the uniform metric gives that

$$d_u(p,q) = \sup_{t\in\mathbb{N}}\{\min\{1,|p_t - q_t|\}\} \leq \sup_{t\in\mathbb{N}}\{1\} = 1.$$

This means that, if $\epsilon > 1$, then $d_u(p,q) < \epsilon$ for all $p,q \in \mathbb{C}^\infty$. Thus, for any chosen $p \in \mathbb{C}^\infty$, for all $q \in \mathbb{C}^\infty$ we would have that $q \in B_{\text{unif}}(p,\epsilon)$. This would mean that $B_{\text{unif}}(p,\epsilon) = \mathbb{C}^\infty$, which is not very useful in most cases. For this reason, in the uniform topology, we will generally choose ϵ such that $1 \geq \epsilon > 0$.

There is a calculation which is frequently necessary to perform when we choose some ϵ such that $1 \geq \epsilon > 0$. Instead of writing it several times in this thesis, we will simply write out this calculation once as a lemma.

Lemma 2.1.2. *Let $p,q \in \mathbb{C}^\infty$ and let $0 < \epsilon \leq 1$. We denote $p = (p_1,p_2,p_3,p_4,\dots)$ and $q = (q_1,q_2,q_3,q_4,\dots)$. If $q \in B_{\text{unif}}(p,\epsilon)$ or $p \in B_{\text{unif}}(q,\epsilon)$, then for all $t \in \mathbb{N}$, we have that*

$$|p_t - q_t| < \epsilon.$$

Proof. We will do the case where $q \in B_{\text{unif}}(p,\epsilon)$, as the other case is similar. We consider the metric ball $B_{\text{unif}}(p,\epsilon)$. We know that $q \in B_{\text{unif}}(p,\epsilon)$ if and only if

$$d_u(p,q) = \sup_{t\in\mathbb{N}}\{\min\{1,|p_t - q_t|\}\} < \epsilon,$$

which implies that

$$\forall t \in \mathbb{N},\ \min\{1,|p_t - q_t|\} < \epsilon.$$

If for some $t \in \mathbb{N}$, we have that $1 = \min\{1,|p_t - q_t|\}$, then it would mean that $1 < \epsilon \leq 1$. This would imply that $1 < 1$, which is impossible. Thus, for all $t \in \mathbb{N}$, we find that $\min\{1,|p_t - q_t|\} = |p_t - q_t|$ and that $|p_t - q_t| < \epsilon$.

□

We now compare the product and uniform topologies. These two topologies interact in the following way: for countably infinite products of a metric space, the uniform topology is finer than the product topology. The product topology is favored for most work done in the sets $\mathbb{C}^\infty$ or $\mathbb{R}^\infty$. While it may not seem that the product topology is an intuitive notion of what a topology should be on a countably infinite product of metric spaces, it has an important advantage. Most of the results which will hold in a finite product of metric spaces will hold in a countably infinite product of metric spaces equipped with the product topology. Conversely, in general, they will not hold if the countably infinite product of metric spaces is equipped with the uniform topology or any other, finer topology. For this reason, the product topology is used more frequently.

The main use of the uniform topology is that many results which hold in the uniform topology also hold in finer topologies. For example, if we show that a set is not dense in the uniform topology, then this will automatically imply that the set is not dense in any finer topology. Since the uniform topology is metrizable, it is often more convenient to prove results in the uniform topology and then show that the results carry over into finer topologies. As it so happens, in this thesis, all the results involving the uniform topology will also hold in any finer topology as well. This can be easily verified by the reader.

We recall that we denote the set of complex numerical series as NS($\mathbb{C}$). In order to induce a topology on this set, we *identify* each series $\sum_{t=1}^{\infty} r_t = R \in$ NS($\mathbb{C}$) to the sequence of its summands, which is the point $(r_1, r_2, r_3, r_4, r_5, \dots) \in \mathbb{C}^\infty$.

Thus, the set NS($\mathbb{C}$) is bijectively mapped to $\mathbb{C}^\infty$, which we use to confer a topology on the set NS($\mathbb{C}$). We say that a set $E \subseteq$ NS($\mathbb{C}$) is identified to the set $F \subseteq \mathbb{C}^\infty$ if the set E is bijectively mapped to F by the identification. We select one of the two previously discussed topologies and use it on $\mathbb{C}^\infty$. Then a set $E \subseteq$ NS($\mathbb{C}$) is considered to be open if the set E is identified to a set $F \subseteq \mathbb{C}^\infty$ such that F is open. Throughout this thesis, if we confer some topology on the set NS($\mathbb{C}$) using the above process, we will simply say that we equip NS($\mathbb{C}$) with the chosen topology.

Also, if we equip NS($\mathbb{C}$) with the uniform topology, then, by the same process described above, we obtain a metric on the set NS($\mathbb{C}$). We consider two series $R,S \in$ NS($\mathbb{C}$) and we denote them as $R = \sum_{t=1}^{\infty} r_t$ and $S = \sum_{t=1}^{\infty} s_t$. The distance between R and S using the uniform metric will, therefore, be $d_u(R,S) = \sup_{t\in\mathbb{N}}\{\min\{1,|r_t - s_t|\}\}$.

2.2. Density of UNS($\mathbb{C}$) in NS($\mathbb{C}$)

The set UNS($\mathbb{C}$) can be shown to be dense in the space NS($\mathbb{C}$) equipped with the product topology. It can also be shown that this is false when NS($\mathbb{C}$) is equipped with the uniform topology.

We begin with a pair of lemmas and a pair of corollaries.

Lemma 2.2.1. *If $(x_t)_{t=1}^{\infty}$ is a dense sequence in $\mathbb{C}$, then for all $N \in \mathbb{N}$, so is $(x_t)_{t=N}^{\infty}$.*

The proof is omitted.

Corollary 2.2.2. *Let the partial sums of the series $\sum_{t=1}^{\infty} r_t$ be dense in $\mathbb{C}$. For all $N \in \mathbb{N}$, the partial sums $\sum_{t=1}^{n} r_t$ such that $n > N$ are also dense in $\mathbb{C}$.*

Proof. For any $n \in \mathbb{N}$, let $\sum_{t=1}^{n} r_t = h_n$. We know that the sequence $(h_t)_{t=1}^{\infty}$ is dense in $\mathbb{C}$. Our proof is equivalent to showing that, for all $N \in \mathbb{N}$, the sequence $(h_t)_{t=N+1}^{\infty}$ is dense in $\mathbb{C}$. By Lemma 2.2.1, we are done. □

Corollary 2.2.3. *Let $(x_t)_{t=1}^{\infty}$ be dense in $\mathbb{C}$ and let $N \in \mathbb{N}$. If for all $t \in \{1, 2, \ldots, N\}$, we choose some arbitrary complex number $y_t \in \mathbb{C}$, then the sequence*

$$(y_1, y_2 \ldots, y_{N-1}, y_N, x_{N+1}, x_{N+2}, x_{N+3}, \ldots)$$

is dense in $\mathbb{C}$.

In other words, if we change first N terms of the sequence such that x_1 becomes y_1, x_2 becomes y_2, ... and x_N becomes y_N, then the resulting sequence would be dense in $\mathbb{C}$.

Proof. Because $(x_t)_{t=1}^{\infty}$ is a dense set in $\mathbb{C}$, by Lemma 2.2.1, for all $N \in \mathbb{N}$ we have that the sequence $(x_t)_{t=N+1}^{\infty}$ is also dense in $\mathbb{C}$. By definition, this means that $\overline{\{x_t\}_{t=N+1}^{\infty}} = \mathbb{C}$. In turn, this implies the following:

$$\{x_t\}_{t=N+1}^{\infty} \subset \{y_t\}_{t=N+1}^{N} \cup \{x_t\}_{t=N+1}^{\infty} \subset \mathbb{C}$$

$$\mathbb{C} = \overline{\{x_t\}_{t=N+1}^{\infty}} \subset \overline{\{y_t\}_{t=N+1}^{N} \cup \{x_t\}_{t=N+1}^{\infty}} \subset \mathbb{C}$$

$$\mathbb{C} \subset \overline{\{y_t\}_{t=N+1}^{N} \cup \{x_t\}_{t=N+1}^{\infty}} \subset \mathbb{C}$$

Hence,

$$\mathbb{C} = \overline{\{y_t\}_{t=N+1}^{N} \cup \{x_t\}_{t=N+1}^{\infty}}.$$

It is clear that the set of terms of the sequence $(y_1, y_2 \ldots, y_{N-1}, y_N, x_{N+1}, x_{N+2}, \ldots)$ is a dense set. Thus, by definition, $(y_1, y_2 \ldots, y_{N-1}, y_N, x_{N+1}, x_{N+2}, \ldots)$ is a dense sequence. □

Lemma 2.2.4. *Let $R \in \mathrm{UNS}(\mathbb{C})$, where we denote $R = \sum_{t=1}^{\infty} r_t$. For any $n \in \mathbb{N}$, we replace r_1 with some $s_1 \in \mathbb{C}$, r_2 with some $s_2 \in \mathbb{C}$, r_3 with some $s_3 \in \mathbb{C} \ldots$ and r_N with some $s_N \in \mathbb{C}$, and if we define the sequence*

$$w_t = \begin{cases} s_t & \text{if } t \in \{1, 2, 3, \ldots, N\} \\ r_t & \text{if } t > N \end{cases},$$

then the resulting series $\sum_{t=1}^{\infty} w_t$ is also a Universal numerical series.

Proof. Let $R \in \text{UNS}(\mathbb{C})$, where we denote $R = \sum_{t=1}^{\infty} r_t$. We choose some $N \in \mathbb{N}$ and choose some subset $\{s_1, s_2, s_3, \ldots, s_N\} \subset \mathbb{C}$. Let

$$w_t = \begin{cases} s_t & \text{if } t \in \{1, 2, 3, \ldots, N\} \\ r_t & \text{if } t > N \end{cases}$$

and, for all $t \in \{1, 2, 3, \ldots, N\}$, let $y_t = r_t - s_t$. For convenience, we will let $A = \sum_{t=1}^{N} y_t$.

It is obvious that $A \in \mathbb{C}$. Clearly, for all $z \in \mathbb{C}$, we get that $(z + A) \in \mathbb{C}$. Therefore, since $R \in \text{UNS}(\mathbb{C})$, for any $z \in \mathbb{C}$ and any $\epsilon > 0$, by Corollary 2.2.2, there exists $N_z \in \mathbb{N}$ such that $N_z > N$ and such that $|(z + A) - \sum_{t=1}^{N_z} r_t| < \epsilon$. We also have that

$$\left|(z + A) - \sum_{t=1}^{N_z} r_t\right| = \left|z + \sum_{t=1}^{N} y_t - \sum_{t=1}^{N_z} r_t\right| = \left|z + \sum_{t=1}^{N} (r_t - s_t) - \sum_{t=1}^{N_z} r_t\right|$$

$$= \left|z + \sum_{t=1}^{N} (r_t - s_t) - \sum_{t=1}^{N} r_t - \sum_{t=N+1}^{N_z} r_t\right| = \left|z + \sum_{t=1}^{N} (r_t - s_t - r_t) - \sum_{t=N+1}^{N_z} r_t\right|$$

$$= \left|z - \sum_{t=1}^{N} s_t - \sum_{t=N+1}^{N_z} r_t\right| = \left|z - \sum_{t=1}^{N_z} w_t\right|.$$

Hence,

$$\left|(z + A) - \sum_{t=1}^{N_z} r_t\right| = \left|z - \sum_{t=1}^{N_z} w_t\right|.$$

This implies that

$$\left|z - \sum_{t=1}^{N_z} w_t\right| = \left|(z + A) - \sum_{t=1}^{N_z} r_t\right| < \epsilon.$$

This demonstrates that the series $\sum_{t=1}^{\infty} w_t$ is a Universal numerical series. □

While these may have been seemingly minor points, they will prove useful in the following work. Another way to think of Lemma 2.2.1, Lemma 2.2.4 and Corollary 2.2.2 and Corollary 2.2.3 is to state that the universality of a sequence or a series is in the tail of said sequence or series.

Now we show that these series are dense in $\text{NS}(\mathbb{C})$ equipped with the product topology. We also show that, conversely, these series are not dense in $\text{NS}(\mathbb{C})$ equipped with the uniform topology.

Proposition 2.2.5. *The set* $\text{UNS}(\mathbb{C})$ *is dense in* $\text{NS}(\mathbb{C})$ *equipped with the product topology.*

Proof. Let $D \subset \mathbb{C}^\infty$ be the set of sequences to which the set UNS($\mathbb{C}$) is identified. By the definition of the product topology on NS($\mathbb{C}$), it is sufficient to show that D is dense in $\mathbb{C}^\infty$ equipped with the product topology.

We consider any $R \in \text{UNS}(\mathbb{C})$, where we denote $R = \sum_{t=1}^{\infty} r_t$. By definition, we know that $(r_t)_{t=1}^{\infty} \in D$. Clearly, $\mathbb{C}$ is dense in $\mathbb{C}$. For any arbitrary $N \in \mathbb{N}$, we choose some arbitrary finite sequence $(z_1, z_2, \ldots, z_N)$ such that $z_t \in \mathbb{C}$ for all $t \in \{1, 2, 3, \ldots, N\}$. Then we consider the series $R^* = \sum_{t=1}^{\infty} r_t^*$ such that

$$r_t^* = \begin{cases} z_t & \text{if } t \in \{1, 2, 3, \ldots, N\} \\ r_t & \text{if } t > N \end{cases}.$$

By Lemma 2.2.4, we know that $R^* \in \text{UNS}(\mathbb{C})$. This implies that $(r_t^*)_{t=1}^{\infty} \in D$. Thus, by Lemma 2.1.1, the set D is dense in $\mathbb{C}^\infty$ and so UNS($\mathbb{C}$) is dense in NS($\mathbb{C}$). ☐

To show that UNS($\mathbb{C}$) is not dense in NS($\mathbb{C}$) when it is equipped with the uniform topology, we produce a lemma.

Lemma 2.2.6. *For some series $R \in \text{NS}(\mathbb{C})$, we denote $R = \sum_{t=1}^{\infty} r_t$. Let R be such that at least one of the following are true:*

a) $\exists N \in \mathbb{N}$ such that, if $t \in \mathbb{N}$ and $t > N$, then $\text{Re}(r_t) \geq 0$;

b) $\exists N \in \mathbb{N}$ such that, if $t \in \mathbb{N}$ and $t > N$, then $\text{Re}(r_t) \leq 0$;

c) $\exists N \in \mathbb{N}$ such that, if $t \in \mathbb{N}$ and $t > N$, then $\text{Im}(r_t) \geq 0$;

d) $\exists N \in \mathbb{N}$ such that, if $t \in \mathbb{N}$ and $t > N$, then $\text{Im}(r_t) \leq 0$.

Then $R \notin \text{UNS}(\mathbb{C})$.

Proof. Let us assume that a.) holds. The other cases are similar.

Let $\epsilon > 0$. Then we choose some $C \in \mathbb{R}$ such that $C > 0$ and such that, if $n \in \mathbb{N}$ and $n \leq N$, then $\text{Re}(\sum_{t=1}^{n} r_t) > (-C + \epsilon)$. Hence, if $n \leq N$, this gives us the following:

$$\text{Re}(\sum_{t=1}^{n} r_t) + C > -C + \epsilon + C = \epsilon > 0$$

$$\text{Re}(\sum_{t=1}^{n} r_t) + C > \epsilon > 0.$$

This means that

$$\mathrm{Re}(\sum_{t=1}^{n} r_t) + C > 0 \text{ and } \mathrm{Re}(\sum_{t=1}^{n} r_t) + C > \epsilon,$$

which implies that

$$\left|\mathrm{Re}(\sum_{t=1}^{n} r_t) + C\right| = \mathrm{Re}(\sum_{t=1}^{n} r_t) + C > \epsilon.$$

Consequently,

$$\left|\mathrm{Re}(\sum_{t=1}^{n} r_t) + C\right| > \epsilon.$$

So if $n \leq N$, then

$$d(\sum_{t=1}^{n} r_t, -C)) = \left|\sum_{t=1}^{n} r_t - (-C)\right| = \left|\sum_{t=1}^{n} r_t + C\right| \geq \left|\mathrm{Re}(\sum_{t=1}^{n} r_t + C)\right|$$
$$= \left|\mathrm{Re}(\sum_{t=1}^{n} r_t) + \mathrm{Re}(C)\right| = \left|\mathrm{Re}(\sum_{t=1}^{n} r_t) + C\right| > \epsilon.$$

Hence,

$$d(\sum_{t=1}^{n} r_t, -C)) > \epsilon.$$

Next, using steps similar to those above, for all $n > N$ we find that

$$d(\sum_{t=1}^{n} r_t, -C)) \geq \left|\mathrm{Re}(\sum_{t=1}^{n} r_t) + C\right| = \left|\mathrm{Re}(\sum_{t=1}^{N} r_t + \sum_{t=N+1}^{n} r_t) + C\right|$$
$$= \left|\mathrm{Re}(\sum_{t=1}^{N} r_t) + C + \mathrm{Re}(\sum_{t=N+1}^{n} r_t)\right| = \left|\mathrm{Re}(\sum_{t=1}^{N} r_t) + C + \sum_{t=N+1}^{n} \mathrm{Re}(r_t)\right|.$$

Thus,

$$d(\sum_{t=1}^{n} r_t, -C)) \geq \left|\mathrm{Re}(\sum_{t=1}^{N} r_t) + C + \sum_{t=N+1}^{n} \mathrm{Re}(r_t)\right|.$$

We can now remove the absolute value sign because

$$\sum_{t=N+1}^{n} \mathrm{Re}(r_t) \geq \sum_{t=N+1}^{n} 0 = 0$$

and

$$\operatorname{Re}(\sum_{t=1}^{N} r_t) + C > \epsilon > 0.$$

This means that

$$\operatorname{Re}(\sum_{t=1}^{N} r_t) + C + \sum_{t=N+1}^{n} \operatorname{Re}(r_t) > 0$$

and, thus,

$$d(\sum_{t=1}^{n} r_t, -C)) \geq \left|\operatorname{Re}(\sum_{t=1}^{N} r_t) + C + \sum_{t=N+1}^{n} \operatorname{Re}(r_t)\right| = \operatorname{Re}(\sum_{t=1}^{N} r_t) + C + \sum_{t=N+1}^{n} \operatorname{Re}(r_t) > \epsilon + \sum_{t=N+1}^{n} 0 > \epsilon + 0 = \epsilon.$$

Altogether, we find that

$$d(\sum_{t=1}^{n} r_t, -C)) > \epsilon.$$

Thus, for all $n \in \mathbb{N}$, we get that $d(\sum_{t=1}^{n} r_t, -C)) > \epsilon$, which demonstrates that the partial sums of the series are not a dense set in $\mathbb{C}$. By Definition 1.2.3, we know that $R \notin \operatorname{UNS}(\mathbb{C})$.

In case b. we similarly use a number $C \in \mathbb{R}$ such that $C > 0$ and such that, if $n \in \mathbb{N}$ and $n \leq N$, then $(C - \epsilon) > \operatorname{Re}(\sum_{t=1}^{n} r_t)$.

In case c. we similarly use a number $C \in \mathbb{R}$ such that $C > 0$ and such that, if $n \in \mathbb{N}$ and $n \leq N$, then $i(-C + \epsilon) < \operatorname{Im}(\sum_{t=1}^{n} r_t)$.

In case d. we similarly use a number $C \in \mathbb{R}$ such that $C > 0$ and such that, if $n \in \mathbb{N}$ and $n \leq N$, then $i(C - \epsilon) > \operatorname{Im}(\sum_{t=1}^{n} r_t)$. □

With this lemma we can proceed.

Proposition 2.2.7. *The set* $\operatorname{UNS}(\mathbb{C})$ *is not dense in* $\operatorname{NS}(\mathbb{C})$ *when it is equipped with the uniform topology.*

Proof. We consider the series $S \in \operatorname{NS}(\mathbb{C})$, which we denote $S = \sum_{t=1}^{\infty} s_t$, such that, for all $t \in \mathbb{N}$, we have that $s_t = 2$.

Let us then take the ball $B_{\text{unif}}(S, \frac{1}{2})$ in the uniform topology. For the sake of contradiction, suppose that the set $\operatorname{UNS}(\mathbb{C})$ is dense in $\operatorname{NS}(\mathbb{C})$ equipped with the uniform topology. This implies that there exists some series $R \in \operatorname{NS}(\mathbb{C})$, which we will denote $R = \sum_{t=1}^{\infty} r_t$, for which we will have that $R \in (\mathrm{UNS}(\mathbb{C}) \cap B_{\mathrm{unif}}(S,\frac{1}{2}))$. Clearly, this will mean that $R \in B_{\mathrm{unif}}(S,\frac{1}{2})$. Because $0 < \frac{1}{2} < 1$, by Lemma 2.1.2, for all $t \in \mathbb{N}$ we must have that $|r_t - s_t| < \frac{1}{2}$.

For any $t \in \mathbb{N}$, we defined $s_t = 2$. Therefore, for all $t \in \mathbb{N}$, we see that

$$\frac{1}{2} > |r_t - s_t| = |r_t - 2| \geq |\operatorname{Re}(r_t - 2)| = |\operatorname{Re}(r_t) - \operatorname{Re}(2)| = |\operatorname{Re}(r_t) - 2|.$$

Consequently,

$$\frac{1}{2} > |\operatorname{Re}(r_t) - 2|.$$

Thus, for all $t \in \mathbb{N}$, we know that

$$|\operatorname{Re}(r_t) - 2| < \frac{1}{2}$$

$$-\frac{1}{2} < \operatorname{Re}(r_t) - 2 < \frac{1}{2}$$

$$2 - \frac{1}{2} < \operatorname{Re}(r_t) < 2 + \frac{1}{2}$$

$$\frac{3}{2} < \operatorname{Re}(r_t) < \frac{5}{2}.$$

This means that

$$\frac{3}{2} < \operatorname{Re}(r_t).$$

Therefore, for all $t \in \mathbb{N}$, we have found that $\frac{3}{2} < \operatorname{Re}(r_t)$. By Lemma 2.2.6, we know that $R \notin \mathrm{UNS}(\mathbb{C})$. This is a contradiction. □

2.3. UNS($\mathbb{C}$) is a G_δ Set in the Product Topology

As demonstrated in the introduction, many proofs of the existence of a chosen universal element first show that the selected universal elements are a countable intersection of dense, open sets in some complete metric space. The Baire Category Theorem states that a countable intersection of open dense sets in a complete metric space is a dense set, which implies that the universal elements form a dense set. If the universal elements do not exist they cannot be a dense set in the overall space, which implies that they must, in fact, exist. In the introduction, we named this the Baire Category Theorem strategy.

This leads to another question asked by Fournier: Is it possible to prove the existence of the Universal numerical series by using the Baire Category Theorem strategy? We will show that the answer is "yes" in the space NS($\mathbb{C}$) equipped with the product topology, but that the answer is "no" if NS($\mathbb{C}$) is equipped with the uniform topology. It is an established fact that the uniform topology on NS($\mathbb{C}$) is completely metrizable [**18**, p 267], therefore, the Baire Category Theorem can be used in that space. However, if the set UNS($\mathbb{C}$) were a countable intersection of open, dense sets in the uniform topology, then the Baire Category Theorem would imply that the set UNS($\mathbb{C}$) should be a dense set in NS($\mathbb{C}$). Proposition 2.2.7 states that this is impossible if NS($\mathbb{C}$) is equipped with the uniform topology. Thus, UNS($\mathbb{C}$) cannot be proven to be non-empty using the Baire Category Theorem strategy if NS($\mathbb{C}$) is equipped with the uniform topology.

A formal statement which answers Fournier's question is the following:

Theorem 2.3.1. *The set* UNS($\mathbb{C}$) *is a countable intersection of open, dense sets in* NS($\mathbb{C}$) *equipped with the product topology. By the Baire Category Theorem,* UNS($\mathbb{C}$) *is a dense set in* NS($\mathbb{C}$)*, which implies that it is non-empty.*

The proof of this Theorem has several parts. We begin with the following proposition, where we show that the Universal numerical series are a countable intersection of open sets, or, in other words, a G_δ set.

Proposition 2.3.2. *The set* UNS($\mathbb{C}$) *is a G_δ set in the product topology and the uniform topology.*

We will prove this result in the product topology. This will imply that the result holds in the uniform topology as well, since it is finer than the product topology.

Before starting, we must prepare some groundwork for this proof. First, let $\{x_t\}_{t=1}^{\infty}$ be a set such that $\overline{\{x_t\}_{t=1}^{\infty}} = \mathbb{C}$. Next, for all $R \in$ NS($\mathbb{C}$) and for all $k \in \mathbb{N}$, if we write $R = \sum_{t=1}^{\infty} r_t$, then we recall that we denote the k-th partial sum of R as $h_k(R) = \sum_{t=1}^{k} r_t$.

Next, we define a collection of sets $\{E_{n,s,k}\}_{n,s,k\in\mathbb{N}}$ such that

$$\forall n,s,k \in \mathbb{N},\ E_{n,s,k} = \left\{ R \in NS(\mathbb{C}) \mid |h_k(R) - x_n| < \frac{1}{s} \right\}.$$

Now we begin the proof.

Lemma 2.3.3. *If for all $n,s,k \in \mathbb{N}$, the sets $E_{n,s,k}$ are defined as above, then we have that* UNS($\mathbb{C}$) $= \cap_{s=1}^{\infty}\cap_{n=1}^{\infty}\cup_{k=1}^{\infty} E_{n,s,k}$.

Proof. For all $n,s,k \in \mathbb{N}$, we know that

$$E_{n,s,k} = \left\{ R \in \mathrm{NS}(\mathbb{C}) \mid |h_k(R) - x_n| < \frac{1}{s} \right\}$$

$$\cup_{k=1}^{\infty} E_{n,s,k} = \left\{ R \in \mathrm{NS}(\mathbb{C}) \mid \exists k \in \mathbb{N} \text{ such that } |h_k(R) - x_n| < \frac{1}{s} \right\}$$

$$\cap_{n=1}^{\infty} \cup_{k=1}^{\infty} E_{n,s,k} = \left\{ R \in \mathrm{NS}(\mathbb{C}) \mid \forall n \in \mathbb{N}, \exists k_n \in \mathbb{N} \text{ such that } |h_{k_n}(R) - x_n| < \frac{1}{s} \right\}.$$

So we finally get that

$$\cap_{s=1}^{\infty} \cap_{n=1}^{\infty} \cup_{k=1}^{\infty} E_{n,s,k} = \left\{ R \in \mathrm{NS}(\mathbb{C}) \mid \forall n,s \in \mathbb{N}, \exists k_{n,s} \in \mathbb{N} \text{ such that } |h_{k_{n,s}}(R) - x_n| < \frac{1}{s} \right\}.$$

This will soon be used in our proof.

We first show that $\mathrm{UNS}(\mathbb{C}) \subset \cap_{s=1}^{\infty} \cap_{n=1}^{\infty} \cup_{k=1}^{\infty} E_{n,s,k}$. By the definition of the series in $\mathrm{UNS}(\mathbb{C})$, if $R \in \mathrm{UNS}(\mathbb{C})$, then for any $s,n \in \mathbb{N}$, there exists some $k_{n,s} \in \mathbb{N}$ such that $|h_{k_{n,s}}(R) - x_n| < \frac{1}{s}$. Thus, for all $R \in \mathrm{UNS}(\mathbb{C})$, we have that $R \in \cap_{s=1}^{\infty} \cap_{n=1}^{\infty} \cup_{k=1}^{\infty} E_{n,s,k}$. This implies that $\mathrm{UNS}(\mathbb{C}) \subset \cap_{s=1}^{\infty} \cap_{n=1}^{\infty} \cup_{k=1}^{\infty} E_{n,s,k}$.

Next, we show that $\mathrm{UNS}(\mathbb{C}) \supset \cap_{s=1}^{\infty} \cap_{n=1}^{\infty} \cup_{k=1}^{\infty} E_{n,s,k}$. Let $R \in \cap_{s=1}^{\infty} \cap_{n=1}^{\infty} \cup_{k=1}^{\infty} E_{n,s,k}$. This immediately implies that $R \in \mathrm{NS}(\mathbb{C})$ and we denote $R = \sum_{t=1}^{\infty} r_t$. We let $y_1 = r_1$ and, for all $t \in \mathbb{N}$ such that $t > 1$, we choose y_t such that $y_t = r_t + y_{t-1}$, which is equivalent to saying that $r_t = y_t - y_{t-1}$. By Lemma 1.2.1, this gives us that R is associated to the sequence $(y_t)_{t=1}^{\infty}$. By definition, for all $k \in \mathbb{N}$, we must have that $h_k(R) = y_k$.

Because R is associated to the sequence $(y_t)_{t=1}^{\infty}$, by Definition 1.2.3, we can show that $R \in \mathrm{UNS}(\mathbb{C})$ by showing that $(y_t)_{t=1}^{\infty}$ is a dense sequence. By definition, this means we must prove that $\overline{\{y_t\}_{t=1}^{\infty}} = \mathbb{C}$. Let us choose any $n,s \in \mathbb{N}$. Because $R \in \cap_{s=1}^{\infty} \cap_{n=1}^{\infty} \cup_{k=1}^{\infty} E_{n,s,k}$, there exists some $k_{n,s} \in \mathbb{N}$ such that

$$|h_{k_{n,s}}(R) - x_n| < \frac{1}{s}.$$

Because for all $k \in \mathbb{N}$, we know that $y_k = h_k(R)$, we find that

$$|y_{k_{n,s}} - x_n| < \frac{1}{s}.$$

Because the n and s are arbitrary, this implies that for all $n \in \mathbb{N}$, we get that $x_n \in \overline{\{y_t\}_{t=1}^{\infty}}$. This means that

$$\{x_n\}_{n=1}^{\infty} \subset \overline{\{y_t\}_{t=1}^{\infty}} \subset \mathbb{C}$$

$$\mathbb{C} = \overline{\{x_n\}_{n=1}^{\infty}} \subset \overline{\overline{\{y_t\}_{t=1}^{\infty}}} = \overline{\{y_t\}_{t=1}^{\infty}} \subset \mathbb{C}$$

$$\mathbb{C} \subset \overline{\{y_t\}_{t=1}^{\infty}} \subset \mathbb{C}.$$

Hence,

$$\mathbb{C} = \overline{\{y_t\}_{t=1}^{\infty}}.$$

This implies that, for all $R \in \cap_{s=1}^{\infty}\cap_{n=1}^{\infty}\cup_{k=1}^{\infty}E_{n,s,k}$, we have that $R \in \text{UNS}(\mathbb{C})$. Therefore, we have established that $\cap_{s=1}^{\infty}\cap_{n=1}^{\infty}\cup_{k=1}^{\infty}E_{n,s,k} \subset \text{UNS}(\mathbb{C})$.

So the claim $\text{UNS}(\mathbb{C}) = \cap_{s=1}^{\infty}\cap_{n=1}^{\infty}\cup_{k=1}^{\infty}E_{n,s,k}$ has been proven. □

We finish the proof that the set UNS($\mathbb{C}$) is a G_δ set by showing that $\cap_{s=1}^{\infty}\cap_{n=1}^{\infty}\cup_{k=1}^{\infty}E_{n,s,k}$ is a countable intersection of open sets. To do this, it suffices to show that, for all $n,k,s \in \mathbb{N}$, the set $E_{n,s,k}$ is open in the product topology. We recall that the union of open sets is open. Thus, for all $n,s \in \mathbb{N}$, the union $\cup_{k=1}^{\infty}E_{n,s,k}$ is an open set and $\cap_{s=1}^{\infty}\cap_{n=1}^{\infty}\cup_{k=1}^{\infty}E_{n,s,k}$ is a countable intersection of open sets.

Lemma 2.3.4. *For all $n,k,s \in \mathbb{N}$, the set $E_{n,s,k}$ is open in the product topology.*

Proof. Fix $n,s,k \in \mathbb{N}$. Let $R \in E_{n,s,k}$, where we denote $R = \sum_{t=1}^{\infty} r_t$.

Our strategy will be the following: we will show that there is some open set O_R such that $R \in O_R \subseteq E_{n,s,k}$. Because R is arbitrary, this will imply that

$$E_{n,s,k} = \bigcup_{R \in E_{n,s,k}} O_R$$

and, thus, that $E_{n,s,k}$ is an open set. As before, we will let $h_k(R) = \sum_{t=1}^{k} r_t$.

First, because $R \in E_{n,s,k}$, we get that $|h_k(R) - x_n| < \frac{1}{s}$, which implies that

$$(\frac{1}{s} - |h_k(R) - x_n|) > 0.$$

Next, let us consider any series $S \in \text{NS}(\mathbb{C})$, where we denote $S = \sum_{t=1}^{\infty} s_t$. We get that

$$|h_k(S) - x_n| = |h_k(S) - h_k(R) + h_k(R) - x_n| \leq |h_k(S) - h_k(R)| + |h_k(R) - x_n|$$

$$= |h_k(S-R)| + |h_k(R)-x_n| = \left|\sum_{t=1}^{k}(s_t - r_t)\right| + |h_k(R)-x_n| \leq \sum_{t=1}^{k}|s_t - r_t| + |h_k(R)-x_n|.$$

This means that

$$|h_k(S)-x_n| \leq \sum_{t=1}^{k}|r_t - s_t| + |h_k(R)-x_n|.$$

This is true for any $S \in \mathrm{NS}(\mathbb{C})$. We will use this in a moment.

We now build the open set O_R. Let $C_R = (\frac{1}{s} - |h_k(R)-x_n|)$. Therefore, $C_R > 0$. We consider the open set $O_R = \prod_{t=1}^{\infty} O_t$ such that

$$O_t = \begin{cases} B(r_t, \frac{C_r}{k}) & \text{if } t \in \{1,2,3,....,k\} \\ \mathbb{C} & \text{if } t > k \end{cases}.$$

This is a basis element of the product topology and so it is clearly an open set in the product topology. Also, it clearly contains R.

To finish the proof, we must show now that $O_R \subset E_{n,s,k}$. To do this, we show that if $S \in O_R$, where we again denote $S = \sum_{t=1}^{\infty} s_t$, then $S \in E_{n,s,k}$.

If $S \in O_R$, then for any $t \in \mathbb{N}$ such that $1 \leq t \leq k$, we must have that $s_t \in O_t = B(r_t, \frac{C_R}{k})$. In other words, if $1 \leq t \leq k$, then $|s_t - r_t| < \frac{C_R}{k}$. This gives us that

$$\sum_{t=1}^{k}|r_t - s_t| < \sum_{t=1}^{k}\frac{C_R}{k} = k(\frac{C_R}{k}) = C_R.$$

Hence,

$$\sum_{t=1}^{k}|r_t - s_t| < C_R.$$

Hence, we deduce that

$$|h_k(S)-x_n| \leq \sum_{t=1}^{k}|r_t - s_t| + |h_k(R)-x_n| < C_R + |h_k(R)-x_n| = \frac{1}{s} - |h_k(R)-x_n| + |h_k(R)-x_n| = \frac{1}{s}.$$

Therefore,

$$|h_k(S)-x_n| < \frac{1}{s},$$

which means that $S \in E_{n,s,k}$. $\square$

By Lemmas 2.3.3 and 2.3.4, Proposition 2.3.2 is proven.

2.4. $\mathrm{UNS}(\mathbb{C})$ is a Dense G_δ Set in the Product Topology

We have now proven that $\mathrm{UNS}(\mathbb{C})$ is a countable intersection of open sets in the two topologies considered in this thesis. Our goal was to show that $\mathrm{UNS}(\mathbb{C})$ is a countable intersection of dense open sets in the product topology. An intuitive first attempt to achieve this goal would be to prove that, for all $n,s,k \in \mathbb{N}$, the set $E_{n,s,k}$ is a dense set. This would imply that the union $\cup_{k=1}^{\infty} E_{n,s,k}$ is an open dense set and, because $\mathrm{UNS}(\mathbb{C}) = \cap_{s=1}^{\infty} \cap_{n=1}^{\infty} \cup_{k=1}^{\infty} E_{n,s,k}$, we would have that $\mathrm{UNS}(\mathbb{C})$ is a countable intersection of open dense sets.

While this may seem like an excellent solution, we will soon demonstrate that this will not work because, for all $n,s,k \in \mathbb{N}$, the set $E_{n,s,k}$ is *not* a dense set in $\mathrm{UNS}(\mathbb{C})$. So we will instead show that, for all $n,s \in \mathbb{N}$, the union $\cup_{k=1}^{\infty} E_{n,s,k}$ is a dense set. By the same logic as the preceding paragraph, this will imply that the set $\mathrm{UNS}(\mathbb{C})$ is an intersection of open dense sets in the product topology. By the Baire Category Theorem, this will show that the set $\mathrm{UNS}(\mathbb{C})$ is dense in $\mathrm{NS}(\mathbb{C})$ and, thus, that $\mathrm{UNS}(\mathbb{C}) \neq \emptyset$.

We will do the work outlined above in just a moment. Before doing so, we recall that, because of Proposition 2.2.7, it is not possible to show that the set $\mathrm{UNS}(\mathbb{C})$ is a countable intersection of open dense sets in $\mathrm{NS}(\mathbb{C})$ equipped with the product topology. Therefore, the strategy outlined above for the product topology will not work in the uniform topology. Our first step is to examine why this strategy will fail.

Lemma 2.4.1. *For all $n,s \in \mathbb{N}$, the set $\cup_{k=1}^{\infty} E_{n,s,k}$ is not dense in* $\mathrm{UNS}(\mathbb{C})$ *equipped with the uniform topology.*

Proof. We recall that

$$\forall n,s,k \in \mathbb{N},\ E_{n,s,k} = \left\{ R \in \mathrm{NS}(\mathbb{C}) \mid |h_k(R) - x_n| < \frac{1}{s} \right\}.$$

Let $n,s \in \mathbb{N}$. This implies that $\frac{1}{s} \leq 1$. We choose the series $Q = \sum_{t=1}^{\infty} q_t$ such that, for all $t \in \mathbb{N}$,

$$q_t = |x_n| + \frac{3}{s}.$$

It is clear that, for all $t \in \mathbb{N}$, we have that $(|x_n| + \frac{3}{s}) \in \mathbb{R}$ and, therefore, we have that $\mathrm{Re}(|x_n| + \frac{3}{s}) = |x_n| + \frac{3}{s}$. We take the ball $B_{\mathrm{unif}}(Q, \frac{1}{s})$. If $\cup_{k=1}^{\infty} E_{n,s,k}$ is dense in $\mathrm{NS}(\mathbb{C})$ equipped with the uniform topology, then there exists some series R, denoted $R = \sum_{t=1}^{\infty} r_t$, such that

$R \in B_{\text{unif}}(Q, \frac{1}{s}) \cap (\cup_{k=1}^{\infty} E_{n,s,k})$. As a consequence, $R \in B_{\text{unif}}(Q, \frac{1}{s})$. Because $0 < \frac{1}{s} \leq 1$, by Lemma 2.1.2, for all $t \in \mathbb{N}$ we must have that $|r_t - s_t| < \frac{1}{s}$.

Additionally, we see that

$$|r_t - q_t| = |r_t - (|x_n| + \frac{3}{s})| \geq |\operatorname{Re}(r_t - (|x_t| + \frac{3}{s}))| = |\operatorname{Re}(r_t) - \operatorname{Re}(|x_t| + \frac{3}{s})| = |\operatorname{Re}(r_t) - (|x_t| + \frac{3}{s})|$$

$$|r_t - q_t| \geq |\operatorname{Re}(r_t) - (|x_t| + \frac{3}{s})| = |\operatorname{Re}(r_t) - |x_t| - \frac{3}{s}|.$$

Hence,

$$|r_t - q_t| \geq |\operatorname{Re}(r_t) - |x_t| - \frac{3}{s}|.$$

This means that, for all $t \in \mathbb{N}$, we have that

$$\frac{1}{s} > |r_t - q_t| \geq |\operatorname{Re}(r_t) - |x_t| - \frac{3}{s}|$$

$$\frac{1}{s} > |\operatorname{Re}(r_t) - |x_t| - \frac{3}{s}|$$

$$-\frac{1}{s} < \operatorname{Re}(r_t) - |x_n| - \frac{3}{s} < \frac{1}{s}.$$

Consequently,

$$|x_n| + \frac{2}{s} < \operatorname{Re}(r_t) < |x_n| + \frac{4}{s}.$$

This shows that, if $t \in \mathbb{N}$, then $\operatorname{Re}(r_t) > |x_n| + \frac{2}{s}$. We will use this to demonstrate that $R \notin \cup_{k=1}^{\infty} E_{n,s,k}$, which will contradict the assumption that $R \in B_{\text{unif}}(Q, \frac{1}{s}) \cap (\cup_{k=1}^{\infty} E_{n,s,k})$.

We recall that

$$\cup_{k=1}^{\infty} E_{n,s,k} = \left\{ R \in \operatorname{NS}(\mathbb{C}) \mid \exists k \in \mathbb{N} \text{ such that } |h_k(R) - x_n| < \frac{1}{s} \right\}.$$

We assume, by contradiction, that $R \in \cup_{k=1}^{\infty} E_{n,s,k}$. This implies that, for some $k \in \mathbb{N}$, we have that $\frac{1}{s} > |h_k(R) - x_n|$. This mean that

$$\frac{1}{s} > |h_k(R) - x_n| \geq |\operatorname{Re}(h_k(R) - x_n)| = |\operatorname{Re}(h_k(R)) - \operatorname{Re}(x_n)|$$

$$\geq |\operatorname{Re}(h_k(R))| - |\operatorname{Re}(x_n)| = \left|\operatorname{Re}(\sum_{t=1}^{k} r_t)\right| - |\operatorname{Re}(x_n)| \geq \operatorname{Re}(\sum_{t=1}^{k} r_t) - |\operatorname{Re}(x_n)|$$

$$= \sum_{t=1}^{k} \operatorname{Re}(r_t) - |\operatorname{Re}(x_n)| > \sum_{t=1}^{k} (|x_n| + \frac{2}{s}) - |\operatorname{Re}(x_n)| = k(|x_n| + \frac{2}{s}) - |\operatorname{Re}(x_n)|.$$

Thus,

$$\frac{1}{s} > k(|x_n| + \frac{2}{s}) - |\operatorname{Re}(x_n)|.$$

Here, we recognize that $|x_n| \geq |\operatorname{Re}(x_n)| \geq 0$, meaning that $|x_n| - |\operatorname{Re}(x_n)| \geq 0$. Also, since $k \in \mathbb{N}$, we know that $k \geq 1$. Therefore, $k - 1 \geq 0$, which implies that $(k-1)|x_n| \geq 0$. Thus,

$$\frac{1}{s} > k(|x_n| + \frac{2}{s}) - |\operatorname{Re}(x_n)| = k|x_n| + \frac{2k}{s} - |\operatorname{Re}(x_n)| = \frac{2k}{s} + (k - 1 + 1)|x_n| - |\operatorname{Re}(x_n)|$$

$$= \frac{2k}{s} + (k-1)|x_n| + |x_n| - |\operatorname{Re}(x_n)| \geq \frac{2k}{s} + (k-1)|x_n| \geq \frac{2k}{s}.$$

Therefore,

$$\frac{1}{s} > \frac{2k}{s},$$

which implies that $1 > 2k$. Because $k \geq 1$, this is a contradiction.

□

Thus, Lemma 2.4.1 show us that the equality $\mathrm{UNS}(\mathbb{C}) = \cap_{s=1}^{\infty} \cap_{n=1}^{\infty} \cup_{k=1}^{\infty} E_{n,s,k}$ cannot be used to imply that UNS($\mathbb{C}$) is an intersection of open dense sets. This equality can only be used to show that UNS($\mathbb{C}$) is an intersection of open sets. Thus, the Baire Category Theorem cannot be applied in this situation.

Our next step is to show why the first strategy we outlined, in which we attempt to show that the sets $E_{n,s,k}$ are dense, will not work.

Lemma 2.4.2. *For all* n,s,$k \in \mathbb{N}$, *the set* $E_{n,s,k}$ *is not dense in* UNS($\mathbb{C}$) *equipped with either the product topology or the uniform topology.*

Proof. It is sufficient to prove that this holds in the product topology, because the uniform topology is finer. First, we recall that, by definition, for all $n,s,k \in \mathbb{N}$, we know that $E_{n,s,k} = \left\{R \in \mathrm{NS}(\mathbb{C}) \mid |h_k(R) - x_n| < \frac{1}{s}\right\}$. We choose some $n \in \mathbb{N}$. We show that there are open sets of the product topology that do not intersect $E_{n,s,k}$, regardless of which k,s are selected.

We select any desired $k \in \mathbb{N}$ and consider the set $O = \prod_{t=1}^{\infty} O_t$ such that, for any $t \in \mathbb{N}$,

$$O_t = \begin{cases} B(3 + |x_n|,1) & \text{if } t \in \{1, 2, 3, \ldots, k\} \\ \mathbb{C} & \text{if } t > k \end{cases}.$$

This set O is a basis element of the product topology and, therefore, it is an open set.

We assume, by contradiction, that $E_{n,s,k}$ is dense in the product topology. This would mean that there exists some $R = \sum_{t=1}^{\infty} r_t$ such that $R \in (E_{n,s,k} \cap O)$. Therefore, since $R \in O$, we get that if $1 \leq t \leq k$, then $r_t \in O_t = B(3+|x_n|,1)$, which implies that $1 > |r_t - (|x_n|+3)|$. Also, it is clear that $\mathrm{Re}(|x_n| + 3) = |x_n| + 3$. Therefore, if $1 \leq t \leq k$, then

$$1 > |r_t - (|x_n|+3)| \geq |\,\mathrm{Re}(r_t - (|x_n|+3))| = |\,\mathrm{Re}(r_t) - \mathrm{Re}(|x_n|+3)| = |\,\mathrm{Re}(r_t) - (|x_n|+3)| = |\,\mathrm{Re}(r_t) - |x_n| - 3|$$

$$1 > |\,\mathrm{Re}(r_t) - |x_n| - 3|$$

$$-1 < \mathrm{Re}(r_t) - |x_n| - 3 < 1.$$

As a result,

$$2 + |x_n| < \mathrm{Re}(r_t) < 4 + |x_n|.$$

Thus, if $1 \leq t \leq k$, then we get that $2 + |x_n| < \mathrm{Re}(r_t)$. We will use this to obtain a contradiction. Similarly to what was done in the proof of Lemma 2.4.1, we know that $|x_n| \geq |\,\mathrm{Re}(x_n)| \geq 0$, meaning that $|x_n| - |\,\mathrm{Re}(x_n)| \geq 0$. Also, since $k \in \mathbb{N}$, we know that $k \geq 1$. Therefore, $k - 1 \geq 0$, which implies that $(k-1)|x_n| \geq 0$. This means that

$$|x_n - h_k(R)| \geq |\,\mathrm{Re}(x_n - h_k(R))| = |\,\mathrm{Re}(x_n) - \mathrm{Re}(h_k(R))| = \left|\mathrm{Re}(x_n) - \mathrm{Re}(\sum_{t=1}^{k} r_t)\right|$$

$$\geq \left|\mathrm{Re}(\sum_{t=1}^{k} r_t)\right| - |\,\mathrm{Re}(x_n)| \geq \mathrm{Re}(\sum_{t=1}^{k} r_t) - |\,\mathrm{Re}(x_n)| = \sum_{t=1}^{k} \mathrm{Re}(r_t) - |\,\mathrm{Re}(x_n)|$$

$$> \sum_{t=1}^{k} (2 + |x_n|) - |\operatorname{Re}(x_n)| = 2k + k|x_n| - |\operatorname{Re}(x_n)| = 2k + (k - 1 + 1)|x_n| - |\operatorname{Re}(x_n)|$$

$$= 2k + (k-1)|x_n| + |x_n| - |\operatorname{Re}(x_n)| \geq 2k + (k-1)|x_n| \geq 2k.$$

Thus,

$$|x_n - h_k(R)| > 2k.$$

Since $R \in E_{n,s,k}$, we have that $\frac{1}{s} > |x_n - h_k(R)| > 2k$, and, since $s \in \mathbb{N}$, we have that $\frac{1}{s} \leq 1$. This gives us that

$$1 \geq \frac{1}{s} > |x_n - h_k(R)| > 2k,$$

which implies that

$$1 > 2k.$$

Because $k \geq 1$, this is a contradiction. $\square$

Now we return to our goal and finish demonstrating that the set UNS($\mathbb{C}$) is a countable intersection of open dense sets in the product topology.

Lemma 2.4.3. *For all n,$s \in \mathbb{N}$, the set $\cup_{k=1}^{\infty} E_{n,s,k}$ is dense in* NS($\mathbb{C}$) *equipped with the product topology.*

Proof. Let us consider some $N \in \mathbb{N}$. Let $(z_1,z_2,\ldots,z_N)$ be a finite, ordered sequence such that, if $1 \leq t \leq N$, then $z_t \in \mathbb{C}$. The number N and the finite ordered sequence $(z_1,z_2,\ldots,z_N)$ are arbitrary and $\mathbb{C}$ is obviously dense in $\mathbb{C}$. We will show that there exists some series $R \in \cup_{k=1}^{\infty} E_{n,s,k}$, where we denote $R = \sum_{t=1}^{\infty} r_t$, such that if $1 \leq t \leq N$, then $r_t = z_t$. By Lemma 2.1.1, this will prove that the set $\cup_{k=1}^{\infty} E_{n,s,k}$ is dense in the set NS($\mathbb{C}$).

To do this, we will define the summands of the series R. For any $t \in \mathbb{N}$ such that $1 \leq t \leq N$, let $r_t = z_t$. Let $r_{N+1} = x_n - \sum_{t=1}^{N} r_t$. If $t > N+1$, then let r_t be any complex number.

This means that, if we show that $R \in \cup_{k=1}^{\infty} E_{n,s,k}$, then we will be done. We have that

$$|x_n - h_{N+1}(R)| = \left| x_n - \sum_{t=1}^{N+1} r_t \right| = \left| x_n - \sum_{t=1}^{N} r_t - r_{N+1} \right| = \left| (x_n - \sum_{t=1}^{N} r_t) - (x_n - \sum_{t=1}^{N} r_t) \right| = 0.$$

Hence,

$$|x_n - h_{N+1}(R)| = 0.$$

Since $s \in \mathbb{N}$, we know that $0 < \frac{1}{s}$, and so

$$|x_n - h_{N+1}(R)| = 0 < \frac{1}{s}.$$

We therefore know that

$$R \in E_{n,s,N+1} \subset \cup_{k=1}^{\infty} E_{n,s,k},$$

which implies that $R \in \cup_{k=1}^{\infty} E_{n,s,k}$. As previously stated, this concludes the proof. □

Taken together, Proposition 2.3.2, Lemma 2.4.3 and the Baire Category Theorem prove Theorem 2.3.1.

2.5. Separability

We now consider the separability of the set UNS($\mathbb{C}$) in different topologies.

Proposition 2.5.1. *The set* UNS($\mathbb{C}$) *is separable when it is equipped with the product topology but not if it is equipped with the uniform topology.*

Proof. It is very simple to show that the set UNS($\mathbb{C}$) is separable in the product topology. We pick any $R \in$ UNS($\mathbb{C}$) and we denote it as $R = \sum_{t=1}^{\infty} r_t$. We can take its first term r_1 and change it to any rational complex number. We denote D_1 the set of all such series, meaning that

$$D_1 = \left\{ \sum_{t=1}^{\infty} c_t \mid c_1 \in (\mathbb{Q} + i\mathbb{Q}) \text{ and } \forall t > 1,\, c_t = r_t \right\}.$$

This set is countable because the rational complex numbers are countable. By Lemma 2.2.4, for all $S \in D_1$, we have that $S \in$ UNS($\mathbb{C}$), which implies that $D_1 \subset$ UNS($\mathbb{C}$).

We can similarly create the set D_2, where we change the first two summands of R to any rational complex number. So we can write that

$$D_2 = \left\{ \sum_{t=1}^{\infty} c_t \mid c_1, c_2 \in (\mathbb{Q} + i\mathbb{Q}) \text{ and } \forall t > 2,\, c_t = r_t \right\}.$$

Just as before, this is a countable because $(\mathbb{Q} + i\mathbb{Q}) \times (\mathbb{Q} + i\mathbb{Q})$ is a countable set. Also, just as before, we know that $D_2 \subset$ UNS($\mathbb{C}$).

For all $n \in \mathbb{N}$ such that $n > 2$, we similarly create the set

$$D_n = \left\{ \sum_{t=1}^{\infty} c_t \mid c_1, c_2, \ldots, c_n \in (\mathbb{Q} + i\mathbb{Q}) \text{ and } \forall t > n,\, c_t = r_t \right\}.$$

By the same logic which we outlined above, each of these sets is countable and each is a subset of UNS($\mathbb{C}$). Thus, the union $D = \cup_{n=1}^{\infty} D_n$ is countable and $D \subset$ UNS($\mathbb{C}$). Because UNS($\mathbb{C}$) $\subset$ NS($\mathbb{C}$), if we show that D is dense in NS($\mathbb{C}$), then the proof is done.

We know that $(\mathbb{Q} + i\mathbb{Q})$ is dense in $\mathbb{C}$. By construction, for any $N \in \mathbb{N}$, for any finite sequence $(z_1, z_2, z_3, \ldots, z_N)$ such that $z_t \in (\mathbb{Q} + i\mathbb{Q})$ for all $t \in \{1, 2, 3, \ldots, N\}$, there is some series $S^* \in D_N \subset D$ such that $\sum_{t=1}^{\infty} s_t = S^*$ and $s_t = z_t$ for $t \in \{1, 2, 3, \ldots, N\}$. Therefore, by Lemma 2.1.1, the set D is dense in UNS($\mathbb{C}$).

However, the set UNS($\mathbb{C}$) is not separable in the uniform topology. We will use a variation of Cantor's diagonal argument to do this proof.

Let us assume, to obtain a contradiction, that there is some dense, countable set $\{R_n\}_{n=1}^{\infty} \subset$ UNS($\mathbb{C}$). Let each $R_n = \sum_{t=1}^{\infty} r_{n,t}$. We choose some arbitrary ϵ such that $1 \geq \epsilon > 0$ and consider the set of balls $\{B_{\text{unif}}(R_n,\epsilon)\}_{n=1}^{\infty}$. Because the set $\{R_n\}_{n=1}^{\infty}$ is dense in UNS($\mathbb{C}$), we should have that UNS($\mathbb{C}$) $\subset \cup_{n=1}^{\infty} B_{\text{unif}}(R_n,\epsilon)$.

We will now develop a Universal numerical series that is not contained in any of the balls in the set $\{B_{\text{unif}}(R_n,\epsilon)\}_{n=1}^{\infty}$ and, therefore, not contained by the union $\cup_{n=1}^{\infty} B_{\text{unif}}(R_n,\epsilon)$. This will create a contradiction and the proof will be done.

We consider an enumeration of the rational complex numbers, which we will denote $\{q_m\}_{m=1}^{\infty}$. We will build a series $S = \sum_{t=1}^{\infty} s_t$ such that, for any $m \in \mathbb{N}$, we have that $\sum_{t=1}^{2m} s_t = q_m$. Thus, the even-numbered partial sums will be the dense set $(\mathbb{Q} + i\mathbb{Q})$. This will imply that the partial sums of S are dense in $\mathbb{C}$. Thus, by Definition 1.2.3, we will have that $S \in$ UNS($\mathbb{C}$). Additionally, we will choose the terms of S in such a way that $S \notin \cup_{n=1}^{\infty} B_{\text{unif}}(R_n,\epsilon)$.

We examine the metric balls $\{B_{\text{unif}}(R_n,\epsilon)\}_{n=1}^{\infty}$ to see how we can design the series S so that it is not contained in any of the metric balls. Let us assume that for some $n \in \mathbb{N}$ and some $S \in$ UNS($\mathbb{C}$), we have that $S \in B_{\text{unif}}(R_n,\epsilon)$. Because $0 < \epsilon \leq 1$, by Lemma 2.1.2, for all $t \in \mathbb{N}$ we must have that $|r_{n,t} - s_t| < \epsilon$.

In other words, if $S \in B_{\text{unif}}(R_n,\epsilon)$, then it implies that for all $t \in \mathbb{N}$, we must have that $s_t \in B(r_{n,t},\epsilon)$. From this, we can deduce that, if $n \in \mathbb{N}$ and if there exists some $t^* \in \mathbb{N}$ such that $s_{t^*} \notin B(r_{n,t^*},\epsilon)$, then $S \notin B_{\text{unif}}(R_n,\epsilon)$.

With this information we can proceed. We construct a sequence of numbers $\{K_n\}_{n=1}^{\infty}$ such that, for all $n \in \mathbb{N}$,

$$K_n = \sup_{z \in B(r_{n,n},\epsilon)} |z|.$$

Essentially, each K_n is the supremum of the norms of the set of points in $B(r_{n,n},\epsilon)$. Obviously, if we consider any $z \in \mathbb{C}$ such that $|z| > K_n$, then $z \notin B(r_{n,n}, \epsilon)$.

Next, we create another set of numbers, denoted $\{C_m\}_{m=1}^{\infty}$. Let $C_1 = 2\max\{|q_1|, K_1, K_2\}$ and, for all $m \in \mathbb{N}$ such that $m > 1$, let $C_m = 3\max\{|q_{m-1}|, |q_m|, K_{2m-1}, K_{2m}\}$. We will use the numbers in the set $\{C_m\}_{m=1}^{\infty}$ to construct the series $S = \sum_{t=1}^{\infty} s_t$.

Let $s_1 \in \mathbb{C}$ be such that $|s_1| > C_1 = 2\max\{|q_1|, K_1, K_2\}$. This implies that

$$|s_1| > C_1 > K_1.$$

This means that $s_1 \notin B(r_{1,1},\epsilon)$, and, thus, that

$$S \notin B_{\text{unif}}(R_1,\epsilon).$$

Then, we let $s_2 = q_1 - s_1$. Just as we did several times before in this thesis, we denote the k-th partial sum of S as $h_k(S)$. Thus,

$$h_2(S) = \sum_{t=1}^{2} s_t = s_1 + q_1 - s_1 = q_1.$$

Hence,

$$h_2(S) = q_1.$$

Now we must show that $s_2 \notin B(r_{2,2}, \epsilon)$, which will imply that $S \notin B_{\text{unif}}(R_2,\epsilon)$. We have that $|s_1| > C_1 = 2\max\{|q_1|, K_1, K_2\}$ and, therefore,

$$|s_2| = |q_1 - s_1| \geq |s_1| - |q_1| > C_1 - q_1$$

$$= 2\max\{|q_1|, K_1, K_2\} - |q_1| = \max\{|q_1|, K_1, K_2\} + (\max\{|q_1|, K_1, K_2\} - |q_1|)$$

$$\geq \max\{|q_1|, K_1, K_2\} \geq K_2.$$

Consequently,

$$|s_2| > K_2,$$

which means that $s_2 \notin B(r_{2,2}, \epsilon)$.

We proceed similarly with each following pairs of terms, with a slight difference.

Let $s_3 \in \mathbb{C}$ be such that $|s_3| > C_2 = 3\max\{|q_1|, |q_2|, K_3, K_4\}$. This implies that

$$|s_3| > C_2 > K_3.$$

This means that $s_3 \notin B(r_{3,3}, \epsilon)$, and, thus, that

$$S \notin B_{\text{unif}}(R_3,\epsilon).$$

Then we let $s_4 = q_2 - s_3 - q_1$. Because of this, the 4-th partial sum of S is

$$h_4(S) = h_2(S) + s_3 + s_4 = q_1 + s_3 + q_2 - s_3 - q_1 = q_2.$$

Hence,

$$h_4(S) = q_2.$$

Now we must show that $s_4 \notin B(r_{4,4}, \epsilon)$, which would imply that $S \notin B_{\text{unif}}(R_4,\epsilon)$. We have that $|s_3| > C_2 = 3\max\{|q_1|, |q_2|, K_3, K_4\}$ and, therefore,

$$|s_4| = |q_2 - s_3 - q_1| \geq |s_3| - |q_2| - |q_1| > C_3 - |q_2| - |q_1|$$

$$= 3\max\{|q_1|, |q_2|, K_3, K_4\} - |q_1| - |q_2| = \max\{|q_1|, |q_2|, K_3, K_4\} + (2\max\{|q_1|, |q_2|, K_3, K_4\} - |q_1| - |q_2|)$$

$$\geq \max\{|q_1|, |q_2|, K_3, K_4\} \geq K_4.$$

This gives us that

$$|s_4| > K_4,$$

which means that $s_4 \notin B(r_{4,4}, \epsilon)$.

We continue this process inductively. In the induction step, for some $m \in \mathbb{N}$ such that $m > 2$, we do the m-th case. We know that $h_{2m-2}(S) = q_{m-1}$. Let $s_{2m-1} \in \mathbb{C}$ such that $|s_{2m-1}| > C_m = 3\max\{|q_{m-1}|, |q_m|, K_{2m-1}, K_{2m}\}$. This implies that

$$|s_{2m-1}| > C_{2m-1} > K_{2m-1}.$$

This means that $s_{2m-1} \notin B(r_{2m-1,2m-1}, \epsilon)$, and, hence, that

$$S \notin B_{\text{unif}}(R_{2m-1}, \epsilon).$$

Then, we let $s_{2m} = q_m - s_{2m-1} - q_{m-1}$. The $2m$-th partial sum of S is

$$h_{2m}(S) = h_{2m-2}(S) + s_{2m-1} + s_{2m} = q_{m-1} + s_{2m-1} + q_m - s_{2m-1} - q_{m-1} = q_m.$$

As a result,

$$h_{2m}(S) = q_m.$$

Now we must show that $s_{2m} \notin B(r_{2m,2m}, \epsilon)$, which will imply that $S \notin B_{\text{unif}}(R_{2m}, \epsilon)$. We have that $|s_{2m-1}| > C_m = 3\max\{|q_{m-1}|, |q_m|, K_{2m-1}, K_{2m}\}$ and, therefore,

$$|s_{2m}| = |q_m - s_{2m-1} - q_{m-1}| \geq |s_{2m-1}| - |q_m| - |q_{m-1}| > C_m - |q_m| - |q_{m-1}|$$

$$= 3\max\{|q_{m-1}|, |q_m|, K_{2m-1}, K_{2m}\} - |q_{m-1}| - |q_m|$$

$$= \max\{|q_{m-1}|, |q_m|, K_{2m-1}, K_{2m}\} + (2\max\{|q_{m-1}|, |q_m|, K_{2m-1}, K_{2m}\} - |q_{m-1}| - |q_m|)$$

$$\geq \max\{|q_{m-1}|, |q_m|, K_{2m-1}, K_{2m}\} \geq K_{2m}.$$

Therefore, we know that

$$|s_{2m}| > K_{2m},$$

which means that $s_{2m} \notin B(r_{2m,2m}, \epsilon)$.

It is clear that, for any $n \in \mathbb{N}$, the series $S = \sum_{t=1}^{\infty} s_t$ is not in $B_{\text{unif}}(R_n,\epsilon)$ and that $S \in \text{UNS}(\mathbb{C})$ because, for all $m \in \mathbb{N}$, we have that $h_{2m}(S) = q_m$. Thus, we have found the contradiction we needed. □

Chapter 3

The Universal Numerical Series are Dense-Lineable

3.1. Arithmetic with Series in NS($\mathbb{C}$)

Another question asked by Fournier was whether the set UNS($\mathbb{C}$) is dense-lineable. The answer is “yes” if UNS($\mathbb{C}$) is equipped with the product topology. The answer is “no” if it is equipped with the uniform topology. It is clear that the 0 vector will be the series $\sum_{t=1}^{\infty} 0$ and that $\sum_{t=1}^{\infty} 0 \notin$ UNS($\mathbb{C}$). It is also clear that, if there are two series $\sum_{t=1}^{\infty} r_t = R$ and $\sum_{t=1}^{\infty} s_t = S$ such that $S = R$, then for all $t \in \mathbb{N}$, we know that $s_t = r_t$. We will need some definitions before we can continue.

Definition 3.1.1. *A subset M of a vector space X is lineable if $M \cup \{0\}$ contains an infinite dimensional vector subspace.*

Definition 3.1.2. *A subset M of a topological vector space X is dense-lineable in X if $M \cup \{0\}$ contains an infinite dimensional vector space which is dense in X.*

The set NS($\mathbb{C}$) is a vector space. Let us define vector addition and scalar multiplication of vectors in the set NS($\mathbb{C}$). Let us consider any two series R,$S \in$ NS($\mathbb{C}$), where we denote $\sum_{t=1}^{\infty} r_t = R$ and $\sum_{t=1}^{\infty} s_t = S$. We define the addition of the two series as

$$R + S = \sum_{t=1}^{\infty} (r_t + s_t).$$

Let us define the multiplication of a series $\sum_{t=1}^{\infty} r_t = R$ by the constant $c \in \mathbb{C}$ as

$$cR = \sum_{t=1}^{\infty} cr_t.$$

We now produce a Lemma regarding how the vector space operations affect the associated sequences of series in NS($\mathbb{C}$).

Lemma 3.1.3. *Let* R,$S \in$ NS($\mathbb{C}$)*, where we denote* $\sum_{t=1}^{\infty} r_t = R$ *and* $\sum_{t=1}^{\infty} s_t = S$*, such that* R *and* S *are associated to* $(x_t)_{t=1}^{\infty}$ *and* $(y_t)_{t=1}^{\infty}$*, respectively. Then we have that the series* $R+S = \sum_{t=1}^{\infty}(r_t + s_t)$ *is associated to the sequence* $(x_t+y_t)_{t=1}^{\infty}$ *and the series* $cR = \sum_{t=1}^{\infty} cr_t$ *is associated to the sequence* $(cx_t)_{t=1}^{\infty}$.

The proof is omitted.

In what will follow, we will say that we have added two sequences $X = (x_t)_{t=1}^{\infty}$ and $Y = (y_t)_{t=1}^{\infty}$ to mean that we have added the sequences term by term to produce the sequence $(x_t + y_t)_{t=1}^{\infty}$. We will denote this new sequence as $X + Y$. Also, we will say that we have multiplied a sequence $X = (x_t)_{t=1}^{\infty}$ by the constant $c \in \mathbb{C}$ to mean that we have multiplied each term of the sequence by the constant c to produce the sequence $(cx_t)_{t=1}^{\infty}$. We will denote this new sequence as cX.

With addition and scalar multiplication so-defined, NS($\mathbb{C}$) and $\mathbb{C}^\infty$ are complex vector spaces and the bijective association NS($\mathbb{C}$) $\longleftrightarrow \mathbb{C}^\infty$, $\sum_{t=1}^{\infty} r_t \longleftrightarrow (x_t)_{t=1}^{\infty}$, which we defined in Chapter 1, is linear.

We now formally answer Fournier's question.

Theorem 3.1.4. *The set* $UNS(\mathbb{C}) \cup \{\sum_{t=1}^{\infty} 0\}$ *contains an uncountably infinite dimensional vector space which is dense in* NS($\mathbb{C}$) *equipped with the product topology. Thus, the set* UNS($\mathbb{C}$) *is dense-lineable.*

The proof of this requires a good deal of work and will be the focus of most of the rest of Chapter 3. From here on we will assume that, until otherwise specified, we are dealing with the product topology.

3.2. Examples and Overview.

Because the work involved in completing this objective is long and potentially confusing, we begin by examining several examples. They will essentially provide a heuristic sketch of the proofs which will be presented over the remainder of this chapter. This is intended to make it easier to follow the upcoming arguments.

3.2.1. Examples

Example 3.2.1. *Let* $(x_t)_{t=1}^{\infty}$ *be a dense sequence in* $\mathbb{C}$ *such that, for all* $t \in \mathbb{N}$, $x_t \neq 0$. *Let us consider the following sequences:* $S_1 = (s_{1,t})_{t=1}^{\infty}$ *and* $S_2 = (s_{2,t})_{t=1}^{\infty}$. *For all* $t \in \mathbb{N}$, *we let*

$$s_{1,t} = \begin{cases} x_{\left(\frac{t}{2}\right)} & \textit{if } t \textit{ is even} \\ 0 & \textit{otherwise} \end{cases}$$

and

$$s_{2,t} = \begin{cases} x_{\left(\frac{t+1}{2}\right)} & \textit{if } t \textit{ is odd} \\ 0 & \textit{otherwise} \end{cases}.$$

This gives us that

$S_1 = (0, x_1, 0, x_2, 0, x_3, 0, x_4, 0, x_5, 0, x_6, \dots),$

$S_2 = (x_1, 0, x_2, 0, x_3, 0, x_4, 0, x_5, 0, x_6, 0 \dots).$

It is clear that these sequences are dense. Let us examine the pair of Universal numerical series, denoted R_1 and R_2, such that R_1 is associated to the sequence S_1 and such that R_2 is associated to the sequence S_2. For any non-zero $c, d \in \mathbb{C}$, we consider the addition $R = cR_1 + dR_2$. From Lemma 3.1.3, we know that the series R is associated to the sequence $cS_1 + dS_2 = S = (s_t)_{t=1}^{\infty}$, where, for all $t \in \mathbb{N}$, we have that

$$s_t = cs_{1,t} + ds_{2,t}.$$

This means that $S = (cs_{1,t} + ds_{2,t})_{t=1}^{\infty}$. Because, for all $t \in \mathbb{N}$, we have already defined the values of $s_{1,t}$ and $s_{2,t}$, this sequence can be re-written as

$$S = (0 + dx_1, cx_1 + 0, 0 + dx_2, cx_2 + 0, 0 + dx_3, cx_3 + 0, 0 + dx_4, cx_4 + 0, \dots),$$

which is to say that

$$S = (dx_1, cx_1, dx_2, cx_2, dx_3, cx_3, dx_4, cx_4, \dots).$$

It is obvious that this new sequence will be a dense sequence because one of its subsequences is the sequence $(cx_t)_{t=1}^{\infty}$. It is equally valid to say that it is dense because one of its subsequences is the sequence $(dx_t)_{t=1}^{\infty}$. We will offer more rigorous proofs of these statements later on. Ultimately, we have that the series R is associated to the dense sequence S and, thus, by Definition 1.2.3, we know that $R \in \mathrm{UNS}(\mathbb{C})$.

We can also see that the sequences S_1 and S_2 are not multiples of each other. We consider any non-zero $c \in \mathbb{C}$ and the term $s_{2,1}$ and we find that

$$s_{2,1} = x_1 \neq 0 = c \cdot 0 = cs_{1,1}$$

$$s_{2,1} \neq cs_{1,1}.$$

Thus, we have that $S_1 \neq cS_2$. Later, we will rigorously prove that this implies that R_1 and R_2 must be linearly independent.

Clearly, this generates a 2-dimensional vector space which, except 0, is contained in the set UNS($\mathbb{C}$). We can extend this into a 3-dimensional vector space using very much the same process. Let R_1, R_2 and R_3 be series associated to the sequences

$S_1 = (s_{1,t})_{t=1}^{\infty} = (x_1, 0, 0, x_2, 0, 0, x_3, 0, 0, x_4, 0, 0, x_5, 0, 0, \dots)$,

$S_2 = (s_{2,t})_{t=1}^{\infty} = (0, x_1, 0, 0, x_2, 0, 0, x_3, 0, 0, x_4, 0, 0, x_5, 0, \dots)$,

$S_3 = (s_{3,t})_{t=1}^{\infty} = (0, 0, x_1, 0, 0, x_2, 0, 0, x_3, 0, 0, x_4, 0, 0, x_5, \dots)$,

respectively. It is obvious that S_1, S_2, S_3 are dense sequences and, therefore, we get that $R_1, R_2, R_3 \in$ UNS($\mathbb{C}$). It is also possible to see that the same arguments used above regarding linear independence will hold here as well. If we take the term $s_{1,1}$, it is clear that, for all non-zero constants $c, d \in \mathbb{C}$, we have that

$$s_{1,1} = x_1 \neq 0 = 0 + 0 = c \cdot 0 + d \cdot 0 = cs_{2,1} + ds_{3,1}$$

$$s_{1,1} \neq cs_{2,1} + ds_{3,1}.$$

This means that $S_1 \neq cS_2 + dS_3$ and, thus, the series R_1 is not a linear combination of the other series R_2 and R_3. A similar statement holds regarding R_2 and is proven using $s_{2,2}$. A third similar statement holds for R_3 and is proven using $s_{3,3}$. This demonstrates that none of these series are linearly dependent. Also, using a strategy similar to what was done above in the 2-dimensional case, we can show that the span of these series is contained, except 0, in the set of Universal numerical series. Thus, the series R_1, R_2 and R_3 are the basis elements of a 3-dimensional vector space contained, except 0, in the set UNS($\mathbb{C}$).

In fact, it should be clear that, for any $n \in \mathbb{N}$, we can modify the construction above to create an n-dimensional vector space which, except 0, is contained in UNS($\mathbb{C}$).

We can push this even further with the following example. In the previous cases, the sequences were built so that, for any $t \in \mathbb{N}$, if the t-th term of one of the sequences is not equal to 0, then for all the other sequences, the t-th term is equal to 0. If we look at the 3-dimensional example, when $t = 1$ we see that $s_{1,1} = x_1 \neq 0$ and $s_{2,1} = s_{3,1} = 0$. It is easy to see that, in the 3-dimensional example, similar results holds for any $t \in \mathbb{N}$ such that $t > 1$.

We can now design a situation where this is not the case. In other words, there is some subset $M \subset \mathbb{N}$ such that, if $t \in M$, then for more than one sequence, the t-th term is non-zero. We mention here that, when we later on begin the actual proofs of these concepts, we will consider only the cases where M is a finite set.

Example 3.2.2. *Let us consider the following sequences:* $S_1 = (s_{1,t})_{t=1}^{\infty}$ *and* $S_2 = (s_{2,t})_{t=1}^{\infty}$. *We let*

$S_1 = (x_1, x_2, 0, x_3, x_4, 0, x_5, 0, x_6, 0, x_7, 0, x_8, \dots)$ *and*

$S_2 = (x_1, x_2, x_3, 0, x_4, x_5, 0, x_6, 0, x_7, 0, x_8, 0, \dots)$.

We consider the series R_1 *and* R_2*, which are associated to* S_1 *and* S_2*, respectively.*

We see that, for all $t \in \mathbb{N}$ such that $t > 5$, if $s_{1,t} \neq 0$ then $s_{2,t} = 0$ and vice versa. This means that R_1 and R_2 are linear independent. We see that for any non-zero $c, d \in \mathbb{C}$, we have the following:

$$S = cS_1 + dS_2 = (cs_{1,t} + ds_{2,t})_{t=1}^{\infty}$$

$$S = (cs_{1,t} + ds_{2,t})_{t=1}^{\infty}.$$

In other words,

$$S = (cx_1 + dx_1, cx_2 + dx_2, dx_3, cx_3, cx_4 + dx_4, dx_5, cx_5, dx_6, cx_6, dx_7, cx_7, \dots).$$

We see that S has two dense subsequences: $(cx_t)_{t=5}^{\infty}$ and $(dx_t)_{t=5}^{\infty}$. Either of them can be used to prove that S is a dense sequence in $\mathbb{C}$. In turn, by Definition 1.2.3, this demonstrates that, if the series R is associated to S, then $R \in \text{UNS}(\mathbb{C})$. This shows that it is not necessary to impose the condition that, for all $t \in \mathbb{N}$, if the t-th term of one of the sequences is not equal to 0, then the t-th terms in all the other sequences must be 0. It suffices to impose this condition on all but a finite number of $t \in \mathbb{N}$.

The overall point is straightforward. By creating sequences with dense subsequences and by arranging those subsequences so that, in layman's terms, they do not overlap too much and eventually separate, we can create basis elements for a vector space that is contained in the set UNS($\mathbb{C}$) $\cup \{\sum_{t=1}^{\infty} 0\}$. We will work all of these details out with full proofs in what follows.

3.2.2. Overview

Finally, we include an overview of the upcoming proofs, in order to make our work clearer to read.

In Step 1, we construct an uncountable number of sequences of natural numbers which have specific properties. We will denote this set as $W_1 \cup W_2 = \{\varphi_n\}_{n=1}^{\infty} \cup \{\Psi_\alpha\}_{\alpha \in I}$, where it will be the case that $|I| > \mathbb{N}$.

In Step 2, we use the sequences in the set $W_1 \cup W_2$ to construct another uncountable set of sequences, denoted $\{B_n\}_{n=1}^{\infty} \cup \{\xi_\alpha\}_{\alpha \in I}$. Then we use the sequences in $\{B_n\}_{n=1}^{\infty} \cup \{\xi_\alpha\}_{\alpha \in I}$ to build an uncountable set of series, which we denote $\{R_n\}_{n=1}^{\infty} \cup \{R_\alpha\}_{\alpha \in I}$. While this may sound like two steps in one, the second part is trivial.

In Step 3, we prove several lemmas which we will use in Step 4.

Finally, in Step 4, we prove that the set of series $\{R_n\}_{n=1}^{\infty} \cup \{R_\alpha\}_{\alpha \in I}$ is the basis of an uncountably infinite dimensional vector space which is contained, except for the 0-vector, in the set UNS($\mathbb{C}$) and which is dense in the set NS($\mathbb{C}$) equipped with the product topology. After each step is complete, it will be announced.

3.3. Uncountably many Almost Disjoint Sequences

We now start with Step 1, building the set of sequences $W_1 \cup W_2$. We begin with some terminology. Let $A = (a_t)_{t=1}^{\infty}$ and $B = (b_t)_{t=1}^{\infty}$ be a pair subsequences of $\mathbb{N}$. We say that $A = B$ if, for all $t \in \mathbb{N}$, we have that $a_t = b_t$.

Let A^* be the set of all the terms in A, meaning that $A^* = \{a_t\}_{t=1}^{\infty}$. Let $B^* = \{b_t\}_{t=1}^{\infty}$ represent the same set for the sequence B. We can say that these two sequences are *disjoint* if $|A^* \cap B^*| = 0$ and we can say that these two sequences are *almost disjoint* if we find that $|A^* \cap B^*| < |\mathbb{N}|$. It is clear that, if two sequences A and B are *disjoint* or *almost disjoint*, then $A \neq B$. Also, clearly, *almost disjoint* does not imply *disjoint*, but *disjoint* implies *almost disjoint*. Furthermore, we can say that a set of sequences is a set of *disjoint* sequences if any pair of sequences in the set are *disjoint*. We define a set of *almost disjoint* sequences in the same way.

We consider the set of natural numbers $\mathbb{N} = \{1, 2, 3, 4, 5, 6 \dots\}$. It is possible to create a countably infinite number of subsets of the natural numbers, which we can denote $\{\varphi_n^*\}_{n=1}^{\infty}$, such that, for all $n \in \mathbb{N}$, we have that $|\varphi_n^*| = |\mathbb{N}|$ and such that, for all $n,m \in \mathbb{N}$ for which $n \neq m$, we have that $\varphi_n^* \cap \varphi_m^* = \emptyset$. Basically, there exists a countably infinite number of countably infinite subsets of $\mathbb{N}$ such that any two of them are disjoint.

Next, for all $n \in \mathbb{N}$, we let φ_n be the sequence of all the numbers of φ_n^*, arranged in increasing order. It is clear that, for all $n \in \mathbb{N}$, we have that φ_n is a subsequence of the natural numbers. Let us denote, for all $n \in \mathbb{N}$, that $\varphi_n = (\varphi_{n,1}, \varphi_{n,2}, \varphi_{n,3}, \varphi_{n,4}, \varphi_{n,5}, \dots)$.

We denote

$$W_1 = \{\varphi_n\}_{n=1}^{\infty}.$$

By construction, for all $n,m \in \mathbb{N}$ such that $n \neq m$, we have that $|\varphi_n^* \cap \varphi_m^*| = |\emptyset| = 0$. This gives that W_1 is a countably infinite set of disjoint sequences. This implies that they are an almost disjoint set of sequences as well.

Next, we will create an uncountably infinite set of sequences, which we will call W_2, such that $W_1 \cap W_2 = \emptyset$. We specify that these new sequences will be of the following form: they will be sequences $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \dots)$ such that, for all $t \in \mathbb{N}$, we have that $\Psi_t \in \varphi_t^*$ and $\Psi_t < \Psi_{t+1}$. It is clear that any sequence of this form must be a subsequence of $\mathbb{N}$. For convenience, we will call the form described above the *Sequence Jumping Form*. We chose the name *Sequence Jumping Form* because any sequence Ψ which has the properties described above "jumps" from φ_1^* to φ_2^* to φ_3^* and so on.

Note: It is possible to specify that $\Psi_t < \Psi_{t+1}$ because, for all $n \in \mathbb{N}$, the set φ_n^* was chosen such that $|\varphi_n^*| = |\mathbb{N}|$. So regardless of what Ψ_t is equal to, there must exist some $\Psi_{t+1} \in \varphi_{t+1}^*$ such that $\Psi_t < \Psi_{t+1}$. Also, it is clear that these Sequence Jumping sequences will depend greatly on the set $\{\varphi_n^*\}_{n=1}^{\infty}$. For the rest of the thesis, if we discuss sequences of the sequence Jumping Form, we will assume the set $\{\varphi_n^*\}_{n=1}^{\infty}$ has already been chosen.

We now produce a lemma.

Lemma 3.3.1. *Let $\{\varphi_n\}_{n=1}^{\infty}$ be a set of disjoint sequences. If Ψ is of the Sequence Jumping Form, then for all $n \in \mathbb{N}$, the sequences Ψ and φ_n are almost disjoint.*

Proof. Let $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \dots)$ be a sequence of the Sequence Jumping Form. Let Ψ^* be the set of all the terms of Ψ. We must prove that, for all $n \in \mathbb{N}$,

$$|\varphi_n^* \cap \Psi^*| < |\mathbb{N}|.$$

By the construction of the Sequence Jumping Form, for all $n \in \mathbb{N}$, we know that $\Psi_n \in \varphi_n^*$. We consider any $n_1,n_2 \in \mathbb{N}$ such that $n_1 \neq n_2$. It is clear that $\Psi_{n_1} \in \varphi_{n_1}^*$ and $\Psi_{n_2} \in \varphi_{n_2}^*$. Therefore, if we had that $\Psi_{n_2} \in \varphi_{n_1}^*$, it would imply that

$$\Psi_{n_2} \in (\varphi^*_{n_1} \cap \varphi^*_{n_2}) \implies (\varphi^*_{n_1} \cap \varphi^*_{n_2}) \neq \emptyset.$$

However, because the set $\{\varphi_n\}_{n=1}^{\infty}$ is a set of disjoint sequence, we must also have that

$$(\varphi^*_{n_1} \cap \varphi^*_{n_2}) = \emptyset.$$

This would be a contradiction. A similar contradiction would arise if $\Psi_{n_1} \in \varphi^*_{n_2}$.

This implies that, if n_1,$n_2 \in \mathbb{N}$ such that $n_1 \neq n_2$, then

$$\Psi_{n_1} \notin \varphi^*_{n_2}.$$

Thus, for all $n \in \mathbb{N}$, we find the following:

$$\varphi^*_n \cap \Psi^* = \{\Psi_n\}$$

$$|\varphi^*_n \cap \Psi^*| = |\{\Psi_n\}| = 1 < |\mathbb{N}|.$$

□

Now we produce a lemma regarding countable sets of almost disjoint sequences that are of the Sequence Jumping Form.

Lemma 3.3.2. *Let E be a countable set of almost disjoint sequences which are all of the Sequence Jumping Form. Then there exists a subsequence of $\mathbb{N}$, denoted Φ, such that $\Phi \notin E$ and $E \cup \{\Phi\}$ is a set of almost disjoint sequences of the Sequence Jumping Form.*

Proof. We show this using a modified version of Cantor's diagonal argument. Because E is countable, either $E = \{\Psi_n\}_{n=1}^{\infty}$ or $E = \{\Psi_n\}_{n=1}^{N}$, for some $N \in \mathbb{N}$. We will do the case where $E = \{\Psi_n\}_{n=1}^{\infty}$ as, with minor modifications, the same proof is used in the case where $E = \{\Psi_n\}_{n=1}^{N}$. For all $n \in \mathbb{N}$, we will denote that $\Psi_n = (\Psi_{n,1}, \Psi_{n,2}, \Psi_{n,3}, \Psi_{n,4}, \dots)$ and $\Psi^*_n = \{\Psi_{n,t}\}_{t=1}^{\infty}$.

We know that $|\varphi^*_1| = |\mathbb{N}|$, so there exists some $\Phi_1 \in \varphi^*_1$ such that $\Phi_1 > \Psi_{1,1}$. Let this be the first term of the sequence Φ.

Next, we know that $|\varphi^*_2| = |\mathbb{N}|$, so there exists some $\Phi_2 \in \varphi^*_2$ such that the inequality $\Phi_2 > \max\{\Psi_{1,2}, \Psi_{2,2}, \Phi_1\}$ will hold. Let this be the second term of the sequence Φ.

Next, we know that $|\varphi^*_3| = |\mathbb{N}|$, so there exists some $\Phi_3 \in \varphi^*_3$ such that the inequality $\Phi_3 > \max\{\Psi_{1,3}, \Psi_{2,3}, \Psi_{3,3}, \Phi_2\}$ will hold. Let this be the third term of the sequence Φ.

We continue this process inductively. In the induction step, we chose some $t \in \mathbb{N}$ such that $t > 3$. We know that $|\varphi_t^*| = |\mathbb{N}|$, and so there exists some $\Phi_t \in \varphi_t^*$ such that the inequality $\Phi_t > \max\{\Psi_{1,t}, \Psi_{2,t}, \Psi_{3,t}, \ldots, \Psi_{t,t}, \Phi_{t-1}\}$ will hold. Let this be the t-th term of the sequence Φ.

By construction, it is clear that, for all $t \in \mathbb{N}$, we have that $\Phi_t \in \varphi_t^*$ and $\Phi_t < \Phi_{t+1}$. So $\Phi = (\Phi_1, \Phi_2, \Phi_3, \ldots)$ is of the Sequence Jumping Form. Obviously, all elements of the set $E \cup \{\Phi\}$ are of the Sequence Jumping Form.

We now demonstrate that $E \cup \{\Phi\}$ is an almost disjoint set of sequences. We fix some pair of natural numbers $j,t \in \mathbb{N}$ such that $j \neq t$. We will first show that, for all $n \in \mathbb{N}$, we have that $\Phi_t \neq \Psi_{n,j}$. By construction, for any $n,t,j \in \mathbb{N}$, we have that $\Phi_t \in \varphi_t^*$ and $\Psi_{n,j} \in \varphi_j^*$. So, if we show that $\Phi_t \notin \varphi_j^*$, then we immediately have that $\Phi_t \neq \Psi_{n,j}$.

We know that the set $\{\varphi_n\}_{n=1}^{\infty}$ is a set of disjoint sequences and that $j \neq t$, which implies that $(\varphi_t^* \cap \varphi_j^*) = \emptyset$. Because $\Phi_t \in \varphi_t^*$, if $\Phi_t \in \varphi_j^*$, then we would have that $\Phi_t \in (\varphi_t^* \cap \varphi_j^*)$. This would imply that $(\varphi_t^* \cap \varphi_j^*) \neq \emptyset$, which is a contradiction.

So we have shown that for all $n \in \mathbb{N}$, if $t,j \in \mathbb{N}$ such that $j \neq t$, then $\Phi_t \neq \Psi_{n,j}$. We know that if $\Phi_t \in \Psi_n^* = \{\Psi_{n,t}\}_{t=1}^{\infty}$, then there exists some $j \in \mathbb{N}$ such that $\Phi_t = \Psi_{n,j}$. Together, these facts imply that if $n,t \in \mathbb{N}$ and if $\Phi_t \in \Psi_n^*$, then $\Phi_t = \Psi_{n,t}$. Therefore, if $n,t \in \mathbb{N}$ and $\Phi_t \neq \Psi_{n,t}$, then $\Phi_t \notin \Psi_n^*$.

By construction, for all $n,t \in \mathbb{N}$ such that $t \geq n$, we have that $\Phi_t > \Psi_{n,t}$, because

$$\Phi_t > \max\{\Psi_{1,t}, \Psi_{2,t}, \Psi_{3,t}, \ldots, \Psi_{n,t}, \ldots, \Psi_{t,t}, \Phi_{t-1}\} \geq \Psi_{n,t}.$$

Thus, for any $t \geq n$, we know that $\Phi_t \neq \Psi_{n,t}$, which means that $\Phi_t \notin \Psi_n^*$.

This means that, if $\Phi^* = \{\Phi_t\}_{t=1}^{\infty}$, then $(\Phi^* \cap \Psi_n^*) \subset \{\Phi_t\}_{t=1}^{n-1}$ and we deduce that

$$|\Phi^* \cap \Psi_n^*| \leq |\{\Phi_t\}_{t=1}^{n-1}| = n - 1 < |\mathbb{N}|,$$

and, therefore,

$$|\Phi^* \cap \Psi_n^*| < |\mathbb{N}|.$$

Thus, Φ and any sequence in E are almost disjoint. Because E is already a set of almost disjoint sequences, this means that $E \cup \{\Phi\}$ must also be a set of almost disjoint sequences. □

In Lemma 3.3.2, we have shown that there exists countably infinite sets of almost disjoint sequences of the Sequence Jumping form. Now we can demonstrate the existence of an uncountable set W_2 of almost disjoint sequences of the Sequence Jumping Form. However, we cannot give an explicit example of such a set, because the proof of its existence relies on Zorn's Lemma.

Proposition 3.3.3. *For any countably infinite set of disjoint sequences, which we denote* $W_1 = \{\varphi_n\}_{n=1}^{\infty}$*, there exists an uncountable set of sequences* $W_2 = \{\Psi_\alpha\}_{\alpha \in I}$*, where* $|I| > |\mathbb{N}|$*, such that the sequences in* W_2 *are of the Sequence Jumping Form and such that* $W_1 \cup W_2$ *is a set of almost disjoint sequences.*

Proof. We know there exists sets of almost disjoint sequences which are all of the Sequence Jumping Form. Because such sets exist, let us consider the collection of all such sets and call it Λ.

We can impose a partial order on Λ in the form of $\subset$, which denotes set inclusion. We take any chain of sets in Λ. We know the sets of the chain are nested due to the partial order we imposed. Let us denote the sets in the chain as $\{E_\alpha\}_{\alpha \in I} \subset \Lambda$, where I is an index set, either countable or uncountable. We denote the union of these sets in the chain as

$$U = \bigcup_{\alpha \in I} E_\alpha.$$

Clearly, U is an upper bound on the chain. Also, clearly U is a set of sequences of the Sequence Jumping Form.

Since U is a union of nested sets, for any two sequences Ψ_1,$\Psi_2 \in U$, there must exist some $\alpha \in I$ such that Ψ_1,$\Psi_2 \in E_\alpha \in \Lambda$. Because E_α is an almost disjoint set of sequences, this means that Ψ_1 and Ψ_2 are almost disjoint sequences. Because Ψ_1 and Ψ_2 are arbitrary, we know that U is a set of almost disjoint sequences.

Altogether, we know that the upper bound U is a set of almost disjoint sequences of the Sequence Jumping Form and, thus, $U \in \Lambda$. This means that Λ is a set equipped with a partial order such that every chain of elements in Λ has some upper bound which is an element of Λ. Therefore, we can apply Zorn's Lemma to demonstrate that there must be some maximal element in the set Λ. Call it M, for maximal.

If $|M| \leq |\mathbb{N}|$, which is to say that $|M|$ is countable, then we apply Lemma 3.3.2 to show that there exists some subsequence Φ of $\mathbb{N}$ such that $\Phi \notin M$ and for which $(M \cup \{\Phi\})$ is a set of almost disjoint sequences of the Sequence Jumping Form. This implies that $(M \cup \{\Phi\}) \in \Lambda$. However, this would mean that M is not maximal since $M \subset (M \cup \{\Phi\})$.

This is a contradiction and so we know that $|M| > |\mathbb{N}|$, which is to say that M is uncountable. Therefore, we let $W_2 = M$.

All the sequences of W_2 are almost disjoint by construction. The same holds for the sequences in $W_1 = \{\varphi_n\}_{n=1}^{\infty}$, since they are a set of disjoint sequences. Since all sequences in W_2 are of the Sequence Jumping Form, by Lemma 3.3.1, for all $\Psi \in W_2$ and for all $n \in \mathbb{N}$, the sequences Ψ and φ_n are almost disjoint. Thus, any pair of sequences in $W_1 \cup W_2$ are almost disjoint and the set $W_1 \cup W_2$ is an almost disjoint set of sequences. □

We now add a final detail.

Lemma 3.3.4. *If W_1 and W_2 are constructed as described above, then $W_1 \cap W_2 = \emptyset$.*

Proof. We consider some $\Psi \in W_2$ and we show that $\Psi \notin W_1$. We chose some $n \in \mathbb{N}$. Because Ψ is of the Sequence Jumping Form, we know that the sequences Ψ and φ_n are almost disjoint and, thus, they are not equal. So, because n is arbitrary, we find that $\Psi \notin \{\varphi_n\}_{n=1}^{\infty} = W_1$. Because Ψ was arbitrary, this proves that $W_1 \cap W_2 = \emptyset$. □

Thus, we have created the set $W_1 \cup W_2$, which is an uncountable, almost disjoint set of sequences of natural numbers. They are all subsequences of $\mathbb{N}$. This finishes Step 1.

3.4. The Basis Elements of the Vector Space

For Step 2, we will use the sequences in the set $W_1 \cup W_2$ to create another set of sequences, which we will denote $\{B_n\}_{n=1}^{\infty} \cup \{\xi_\alpha\}_{\alpha \in I}$. We start by adding two lemmas here.

Lemma 3.4.1. *If some sequence $A = (A_t)_{t=1}^{\infty}$ has a dense subsequence $(A_{t_j})_{j=1}^{\infty}$, then A is also dense.*

The proof is omitted.

Lemma 3.4.2. *If $(x_t)_{t=1}^{\infty}$ is dense in $\mathbb{C}$, then for all $c \in \mathbb{C}$ such that $c \neq 0$, the sequence $(cx_t)_{t=1}^{\infty}$ is also dense.*

The proof is omitted.

We now introduce some more new terminology. We say that a sequence of complex numbers $A = (A_1, A_2, A_3, A_4, A_5, \dots)$ is *made* by some increasing sequence of natural numbers $S = (s_1, s_2, s_3, \dots)$ and some dense sequence $(x_t)_{t=1}^{\infty}$ if, for all $t \in \mathbb{N}$,

$$A_t = \begin{cases} x_k & \text{if t} = s_k \\ 0 & \text{otherwise} \end{cases}.$$

In other words, for the terms in the subsequence $(A_{s_k})_{k=1}^{\infty}$ of A, if $k \in \mathbb{N}$, then $A_{s_k} = x_k$. All the other the terms of A are equal to 0.

We show an example to make this concept clearer.

Example 3.4.3. *Let* $\varphi = (1, 3, 5, 7, 9 \dots)$ *and let* $(x_t)_{t=1}^{\infty}$ *be a dense sequence. In this case, the sequence* A *made by* $\varphi = (1, 3, 5, 7, 9 \dots)$ *and* $(x_t)_{t=1}^{\infty}$ *would be the sequence*

$$A = (x_1, 0, x_2, 0, x_3, 0, x_4, 0, x_5, 0 \dots).$$

In short, the sequence φ informs us of where each x_t must be placed the sequence A.

We can now start creating the set $\{B_n\}_{n=1}^{\infty} \cup \{\xi_\alpha\}_{\alpha \in I}$. Let $(x_t)_{t=1}^{\infty}$ be a dense sequence such that, for all $t \in \mathbb{N}$, we have that $x_t \neq 0$. This will be an important point later on in Lemma 3.6.2. We begin with the set of sequences $W_1 = \{\varphi_n\}_{n=1}^{\infty}$. We recall that, for all $n \in \mathbb{N}$, we denote $\varphi_n = (\varphi_{n,1}, \varphi_{n,2}, \varphi_{n,3}, \varphi_{n,4}, \varphi_{n,5}, \dots)$. We construct a set of sequences $\{A_n\}_{n=1}^{\infty}$ such that, for all $n \in \mathbb{N}$, the sequence A_n was made by the sequence φ_n and the dense sequence $(x_t)_{t=1}^{\infty}$.

This means that, if $n \in \mathbb{N}$ and if we denote $A_n = (A_{n,1}, A_{n,2}, A_{n,3}, A_{n,4}, A_{n,5}, \dots)$, then for all $t \in \mathbb{N}$, we will have that

$$A_{n,t} = \begin{cases} x_k & \text{if t} = \varphi_{n,k} \\ 0 & \text{otherwise} \end{cases}.$$

It also means that, for any $n \in \mathbb{N}$, the sequence $(x_t)_{t=1}^{\infty}$ is a subsequence of A_n. By Lemma 3.4.1, this implies that A_n is a dense sequence.

We now modify a finite number of terms of each sequence A_n. For all $n \in \mathbb{N}$, let us construct the sequence

$$B_n = (B_{n,1}, B_{n,2}, B_{n,3}, B_{n,4}, B_{n,5}, B_{n,6} \dots)$$

such that

$$B_{n,t} = \begin{cases} 0 & \text{if } t \in \{1, 2, 3, \dots, n-1\} \\ 1 & \text{if } t = n \\ A_{n,t} & \text{if } t > n \end{cases},$$

which gives us that

$$B_n = (0, 0, 0, \dots 0, 1, A_{n,n+1}, A_{n,n+2}, A_{n,n+3} \dots),$$

where the 1 is the n-th term. By Corollary 2.2.3, for all $n \in \mathbb{N}$ we get that B_n is a dense sequence because, aside from the first n terms, B_n identical to the dense sequence A_n.

Next, we do something similar with the sequences in the the set $W_2 = \{\Psi_\alpha\}_{\alpha\in I}$, where we recall that $|I| > |\mathbb{N}|$. Let us denote, for all $\alpha \in I$, that ξ_α is the sequence made by the sequence Ψ_α and the dense sequence $(x_t)_{t=1}^\infty$. In other words, if $\alpha \in I$ and if we denote $\xi_\alpha = (\xi_{\alpha,1}, \xi_{\alpha,2}, \xi_{\alpha,3}, \xi_{\alpha,4}, \xi_{\alpha,5}, \dots)$, then for all $t \in \mathbb{N}$, we will have that

$$\xi_{\alpha,t} = \begin{cases} x_k & \text{if t} = \Psi_{\alpha,k} \\ 0 & \text{otherwise} \end{cases}.$$

Just as before, for all $\alpha \in I$, the sequence $(x_t)_{t=1}^\infty$ is a subsequence of ξ_α. By Lemma 3.4.1, the sequence ξ_α is a dense sequence. This gives us the set of sequences $\{B_n\}_{n=1}^\infty \cup \{\xi_\alpha\}_{\alpha\in I}$.

For all $n \in \mathbb{N}$, let R_n be the complex numerical series associated to the sequence B_n. Because each B_n is a dense sequence, by Definition 1.2.3, we know that $\{R_n\}_{n=1}^\infty \subset \text{UNS}(\mathbb{C})$. For all $n \in \mathbb{N}$, we will denote $R_n = \sum_{t=1}^\infty r_{n,t}$. Next, for all $\alpha \in I$, let R_α be the complex numerical series associated to the sequence ξ_α. As we said before, because each ξ_α is a dense sequence, by Definition 1.2.3, we know that $\{R_\alpha\}_{\alpha\in I} \subset \text{UNS}(\mathbb{C})$. For all $\alpha \in I$, we denote $R_\alpha = \sum_{t=1}^\infty r_{\alpha,t}$. This finishes the construction of the set $\{R_n\}_{n=1}^\infty \cup \{R_\alpha\}_{\alpha\in I}$. This completes Step 2.

3.5. Several Preliminary Lemmas

Now we begin Step 3, which is writing the lemmas that are needed in Step 4.

We first prove a lemma regarding a specific method for constructing a dense vector space in NS($\mathbb{C}$).

Note: This vector space need not be a subset of UNS($\mathbb{C}$).

Lemma 3.5.1. *Let $\{S_n\}_{n=1}^\infty \subset \text{NS}(\mathbb{C})$ where, for all $n \in \mathbb{N}$, we denote $S_n = \sum_{t=1}^\infty s_{n,t}$. Suppose that, for all $n \in \mathbb{N}$, we have $s_{n,n} = 1$ and if $t < n$, then we have that $s_{n,t} = 0$. Then the set $\text{span}\{S_n\}_{n=1}^\infty$ is dense in NS($\mathbb{C}$).*

In other words, we take some set of series $\{S_n\}_{n=1}^\infty$, such that, for all $n \in \mathbb{N}$, we have that $s_{n,1} = s_{n,2} = s_{n,3} = \cdots = s_{n,n-1} = 0$ and $s_{n,n} = 1$. Also, there is no restriction on what the rest of the terms of S_n must be. The span of such a set of series will be dense in NS($\mathbb{C}$).

Proof. We choose some $N \in \mathbb{N}$ and we choose an arbitrary finite sequence $(z_1, z_2, z_3, \dots z_N)$ such that $\{z_1, z_2, z_3, \dots z_N\} \subset \mathbb{C}$. We will prove that there is some series $S^* \in \text{span}\{S_n\}_{n=1}^\infty$ such that, if we denote $S^* = \sum_{t=1}^\infty s_t^*$ and if $t \in \{1,2,\dots,N\}$, then $z_t = s_t^*$. By Lemma 2.1.1, because the finite sequence $(z_1, z_2, z_3, \ldots, z_N)$ is arbitrary and because the set $\mathbb{C}$ is obviously dense in $\mathbb{C}$, this will prove that the set $\text{span}\{S_n\}_{n=1}^{\infty}$ is dense in NS($\mathbb{C}$).

So we take a sum

$$S^* = c_1S_1 + c_2S_2 + \cdots + c_NS_N$$

where $c_1,\ c_2,\ldots, c_N$ will be specified momentarily. Due to the definition of addition and scalar multiplication of vectors in the vector space NS($\mathbb{C}$), we know that

$$\sum_{t=1}^{\infty} s_t^* = S^* = c_1S_1 + c_2S_2 + \cdots + c_NS_N = \sum_{t=1}^{\infty} (c_1s_{1,t} + c_2s_{2,t} + \cdots + c_Ns_{N,t})$$

which implies that, for all $t \in \mathbb{N}$, the t-th term of S^* must be

$$s_t^* = c_1s_{1,t} + c_2s_{2,t} + \cdots + c_Ns_{N,t}.$$

We will consider this sum term by term. Let $t \in \mathbb{N}$ such that $t \leq N$. By hypothesis, we have that $s_{t,t} = 1$. Also by hypothesis, if $n \in \mathbb{N}$ such that $t < n$, then $s_{n,t} = 0$. Thus, because $t < t + 1$, we know that $s_{t+1,t} = 0$. Similarly, because $t < t + 2 < \cdots < N$, we find that $s_{t+2,t} = \cdots = s_{N,t} = 0$.

Altogether, for any $t \leq N$,

$$s_t^* = c_1s_{1,t} + c_2s_{2,t} + \cdots + c_{t-1}s_{t-1,t} + c_ts_{t,t} + c_{t+1}s_{t+1,t} + \cdots + c_Ns_{N,t}$$

$$= (c_1s_{1,t} + c_2s_{2,t} + \cdots + c_{t-1}s_{t-1,t}) + (c_t \cdot 1) + (c_{t+1} \cdot 0) + \cdots + (c_N \cdot 0)$$

$$= (c_1s_{1,t} + c_2s_{2,t} + \cdots + c_{t-1}s_{t-1,t}) + c_t.$$

Hence,

$$s_t^* = (c_1s_{1,t} + c_2s_{2,t} + \cdots + c_{t-1}s_{t-1,t}) + c_t.$$

For all $t \in \{1,2,\ldots,N\}$, we let

$$c_t = z_t - (c_1s_{1,t} + c_2s_{2,t} + \cdots + c_{t-1}s_{t-1,t}),$$

which implies that

$$s_t^* = (c_1s_{1,t} + c_2s_{2,t} + \cdots + c_{t-1}s_{t-1,t}) + c_t$$

$$= (c_1 s_{1,t} + c_2 s_{2,t} + \cdots + c_{t-1} s_{t-1,t}) + z_t - (c_1 s_{1,t} + c_2 s_{2,t} + \cdots + c_{t-1} s_{t-1,t}) = z_t.$$

This means

$$s_t^* = z_t.$$

This means that, if $t \in \{1,2\ldots,N\}$, then $s_t^* = z_t$. □

The next three lemmas will be used in proving that the set of series $\{R_n\}_{n=1}^{\infty} \cup \{R_\alpha\}_{\alpha \in I}$ is the basis of a vector space contained, except 0, in UNS($\mathbb{C}$).

Lemma 3.5.2. *Let $E = \{S_n\}_{n=1}^{N}$, for some $N \in \mathbb{N}$, be a finite set of almost disjoint subsequences of $\mathbb{N}$. Then there exists some $t^* \in \mathbb{N}$ such that, if $n \in \{1, 2, \ldots, N\}$ and t is a term in the sequence S_n such that $t > t^*$, then t is not a term in any other sequence in the set $E \setminus \{S_n\}$.*

Proof. Let $E = \{S_n\}_{n=1}^{N}$, for some $N \in \mathbb{N}$, and if $1 \leq n \leq N$, then let S_n^* denote the set of all the terms in the sequence S_n. For all $n \in \{1, 2, \ldots, N\}$, because the sequence S_n is a subsequence of $\mathbb{N}$, we know that $S_n^* \subset \mathbb{N}$ and that $|S_n^*| = |\mathbb{N}|$.

We consider the set,

$$U = \bigcup_{1 \leq n < m \leq N} (S_n^* \cap S_m^*).$$

Clearly, it is the case that $U \subset \mathbb{N}$. Because $E = \{S_n\}_{n=1}^{N}$ is a set of almost disjoint sequences, if $1 \leq n < m \leq N$, then $|S_n^* \cap S_m^*| < |\mathbb{N}|$. This means that

$$|U| = \left| \bigcup_{1 \leq n < m \leq N} (S_n^* \cap S_m^*) \right| \leq \left(\sum_{1 \leq n < m \leq N} |S_n^* \cap S_m^*| \right) < |\mathbb{N}|.$$

Therefore, we will let $t^* = \max\{t \mid t \in U\}$.

We let $n \in \mathbb{N}$ such that $n \in \{1, 2, \ldots, N\}$. We consider any $t \in S_n^*$ such that $t > t^*$. Such a t exists because $|S_n^*| = |\mathbb{N}|$. By contradiction, let us assume that there exists some $m \in \{1, 2, \ldots, N\}$ such that $n \neq m$ and such that t is a term in the sequence S_m. This would imply that $t \in S_m^*$ and, thus,

$$t \in (S_n^* \cap S_m^*) \subset U.$$

Since t^* is the maximum of the set U, we would find that

$$t \leq t^* < t.$$

This is a contradiction.

Thus, if $n \in \{1, 2, \ldots, N\}$ and t is a term in the sequence S_n such that $t > t^*$, then t is not term in any other sequence in the set $E \setminus \{S_n\}$. $\square$

For Lemma 3.5.3, Lemma 3.5.4, Lemma 3.6.1 and Lemma 3.6.2, because the indices become relatively complicated, we will change our notation to make it easier to read. If $S_n = (s_{n,1}, s_{n,2}, s_{n,3}, s_{n,4}, \ldots)$ is a sequence and $t \in \mathbb{N}$, then we will write $s_{(n,t)}$ instead of writing $s_{n,t}$ to denote the t-th term of the sequence S_n. Similarly, when describing a series $R_n = \sum_{t=1}^{\infty} r_{n,t}$, we will denote the t-th term of the series as $r_{(n,t)}$ instead of $r_{n,t}$.

Additionally, while studying Lemma 3.5.3 and Lemma 3.5.4, the reader is encouraged to refer back to Example 3.2.1 and Example 3.2.2. These examples help to illustrate what is proven by the two lemmas.

Lemma 3.5.3. *Let $N \in \mathbb{N}$. Let $\{S_n\}_{n=1}^{N}$ be a finite set of sequences such that, for all $n \in \{1, 2, \ldots, N\}$, we denote $S_n = \left(s_{(n,1)}, s_{(n,2)}, s_{(n,3)}, s_{(n,4)}, s_{(n,5)}, \ldots\right)$. Let R_1 be the series associated to S_1, R_2 be the series associated to S_2 and so on.*

Suppose that there exists some $k \in \{1, 2, \ldots, N\}$ such that there exists some $t_p \in \mathbb{N}$ for which $s_{(k,t_p)} \neq 0$ and for which, if $n \in \{1, 2, 3, \ldots, N\}$ and $n \neq k$, then $s_{(n,t_p)} = 0$. Then R_k is not a linear combination of the series in the set $(\{R_n\}_{n=1}^{N} \setminus \{R_k\})$.

Proof. By contradiction, we assume that R_k is a linear combination of the series in the set

$$(\{R_n\}_{n=1}^{N}) \setminus \{R_k\}.$$

This implies that, for some set of coefficients $c_1, c_2, \ldots, c_N \in \mathbb{C}$ such that $c_k = 0$, we have that

$$R_k = \sum_{n=1}^{N} c_n R_n.$$

By Lemma 3.1.3, for all $n \in \{1, 2, \ldots, N\}$, we have that $c_n R_n$ is associated to the sequence

$$\left(c_n s_{(n,1)}, c_n s_{(n,2)}, c_n s_{(n,3)}, c_n s_{(n,4)}, c_n s_{(n,5)}, \ldots\right).$$

Also by Lemma 3.1.3, we have that $\sum_{n=1}^{N} c_n R_n$ is a series associated to the sequence

$$\left(\sum_{n=1}^{N} c_n s_{(n,t)}\right)_{t=1}^{\infty} = \left(\sum_{n=1}^{N} c_n s_{(n,1)}, \sum_{n=1}^{N} c_n s_{(n,2)}, \sum_{n=1}^{N} c_n s_{(n,3)} \ldots\right).$$

We know that R_k is already associated to the sequence $S_k = \left(s_{(k,1)}, s_{(k,2)}, s_{(k,3)}, s_{(k,4)}, \ldots\right)$, and our assumption implies that it is also associated to the sequence $(\sum_{n=1}^{N} c_n s_{(n,t)})_{t=1}^{\infty}$.

Therefore, by Lemma 1.2.2, we must have that $S_k = (\sum_{n=1}^{N} c_n s_{(n,t)})_{t=1}^{\infty}$. Therefore, for all $t \in \mathbb{N}$,

$$s_{(k,t)} = \sum_{n=1}^{N} c_n s_{(n,t)}.$$

Because $t_p \in \mathbb{N}$, this implies that

$$s_{(k,t_p)} = \sum_{n=1}^{N} c_n s_{(n,t_p)}.$$

However, by hypothesis, this would imply that

$$0 \neq s_{(k,t_p)} = \sum_{n=1}^{N} c_n s_{(n,t_p)} = c_k s_{(k,t_p)} + \sum_{\substack{n=1 \\ n\neq k}}^{N} c_n s_{(n,t_p)} = (0 \cdot s_{(k,t_p)}) + \sum_{\substack{n=1 \\ n\neq k}}^{N} (c_n \cdot 0) = 0 + 0 = 0.$$

This implies that

$$0 \neq 0,$$

which is a contradiction. □

Lemma 3.5.4. *Let $(x_t)_{t=1}^{\infty}$ be a set that is dense in $\mathbb{C}$. Let $N \in \mathbb{N}$ and $\{S_n\}_{n=1}^{N}$ be a finite set of sequences where, for all $n \in \{1, 2, \ldots, N\}$, we denote $S_n = \big(s_{(n,1)}, s_{(n,2)}, s_{(n,3)}, s_{(n,4)}, \ldots\big)$. Let R_1 be the series that is associated to the sequence S_1, let R_2 be the series that is associated to the sequence S_2, and so on.*

Let $k \in \{1, 2, \ldots, N\}$. Suppose there exists some $T \in \mathbb{N}$ and some subsequence of $\mathbb{N}$, denoted $(t_p)_{p=T+1}^{\infty}$, such that if $p > T$, then the following are true: first, that $s_{(k,t_p)} = x_p$ and, second, that for every $n \in \{1, 2, \ldots, N\}$ for which $n \neq k$, we have that $s_{(n,t_p)} = 0$.

Then, for all $c_1, c_2, c_3, \ldots, c_N \in \mathbb{C}$ such that $c_k \neq 0$, we get that $(\sum_{n=1}^{N} c_n R_n) \in \mathrm{UNS}(\mathbb{C})$.

Proof. We assume that we have selected the complex numbers $c_1, c_2, c_3, \ldots, c_N \in \mathbb{C}$ such that $c_k \neq 0$. By Lemma 3.1.3, for all $n \in \{1, 2, \ldots, N\}$, we have that $c_n R_n$ is associated to the sequence

$$\big(c_n s_{(n,1)}, c_n s_{(n,2)}, c_n s_{(n,3)}, c_n s_{(n,4)}, \ldots\big),$$

which, for convenience, we will denote $c_n S_n$. Also by Lemma 3.1.3, we know that $\sum_{n=1}^{N} c_n R_n$ is a series associated to the sequence

$$\left(\sum_{n=1}^{N} c_n s_{(n,t)}\right)_{t=1}^{\infty} = \left(\sum_{n=1}^{N} c_n s_{(n,1)}, \sum_{n=1}^{N} c_n s_{(n,2)}, \sum_{n=1}^{N} c_n s_{(n,3)} \dots\right),$$

which, for convenience, we can denote $\sum_{n=1}^{N} c_n S_n$.

We recall the subsequence $(t_p)_{p=T+1}^{\infty}$ of $\mathbb{N}$ from the hypothesis of the lemma and consider the subsequence $(\sum_{n=1}^{N} c_n s_{(n,t_p)})_{p=1}^{\infty}$ of $\sum_{n=1}^{N} c_n S_n$. Again by hypothesis, for all $p > T$, we know that $s_{(k,t_p)} = x_p$. Furthermore, by hypothesis, for all $n \in \{1, 2, \dots, N\}$ for which $n \neq k$, we know that $s_{(n,t_p)} = 0$. Therefore, for all $p > T$,

$$\sum_{n=1}^{N} c_n s_{(n,t_p)} = c_k s_{(k,t_p)} + \sum_{\substack{n=1 \\ n\neq k}}^{N} c_n s_{(n,t_p)} = c_k x_p + \sum_{\substack{n=1 \\ n\neq k}}^{N} (c_n \cdot 0) = c_k x_p + \sum_{\substack{n=1 \\ n\neq k}}^{N} 0 = c_k x_p + 0 = c_k x_p.$$

Hence,

$$\sum_{n=1}^{N} c_n s_{(n,t_p)} = c_k x_p.$$

This means that $(c_k x_p)_{p=T+1}^{\infty}$ is a subsequence of $\sum_{n=1}^{N} c_n S_n$. Because $c_k \neq 0$, by Lemma 3.4.2, the sequence $(c_k x_p)_{p=1}^{\infty}$ is dense and, thus, by Lemma 2.2.1, the subsequence $(c_k x_p)_{p=T+1}^{\infty}$ is dense. This implies that the sequence $\sum_{n=1}^{N} c_n S_n$ contains a dense subsequence. By Lemma 3.4.1, the sequence $\sum_{n=1}^{N} c_n S_n$ is dense and, by Definition 1.2.3, this means that $(\sum_{n=1}^{N} c_n R_n) \in \mathrm{UNS}(\mathbb{C})$. □

This is the last of the Lemmas we need for Step 4. This completes Step 3.

Note: It may seem odd that in notation of Lemma 3.5.3 and Lemma 3.5.4, we used the symbol t_p without explaining what is represented by the subscript p. Our reasons for doing so will make much greater sense once the reader has seen how Lemma 3.5.3 and Lemma 3.5.4 are used in the proof of Lemma 3.6.2.

3.6. A Dense Vector Space in the set UNS($\mathbb{C}$)

We now combine all of our previous work to finally do Step 4.

We will show that the set $\{R_n\}_{n=1}^{\infty} \cup \{R_\alpha\}_{\alpha\in I}$, which we defined at the end of Step 2, is the basis of a dense, uncountably infinite dimensional vector space in $(\mathrm{UNS}(\mathbb{C}) \cup \{\sum_{t=1}^{\infty} 0\})$. We will do this by proving the following 3 properties about the set $\{R_n\}_{n=1}^{\infty} \cup \{R_\alpha\}_{\alpha\in I}$:

1) The set $\mathrm{span}(\{R_n\}_{n=1}^{\infty} \cup \{R_\alpha\}_{\alpha\in I})$ is dense in $\mathrm{NS}(\mathbb{C})$.

2) We have the inclusion

$$\operatorname{span}(\{R_n\}_{n=1}^{\infty} \cup \{R_\alpha\}_{\alpha\in I}) \subset \left(\mathrm{UNS}(\mathbb{C}) \cup \left\{\sum_{t=1}^{\infty} 0\right\}\right).$$

3) The set $\{R_n\}_{n=1}^{\infty} \cup \{R_\alpha\}_{\alpha\in I}$ is a linearly independent set of series.

We begin by proving 1).

Lemma 3.6.1. *The vector space* $\operatorname{span}(\{R_n\}_{n=1}^{\infty} \cup \{R_\alpha\}_{\alpha\in I})$ *is a dense set in* $\mathrm{NS}(\mathbb{C})$.

Proof. We begin by considering $R_1 = \sum_{t=1}^{\infty} r_{(1,t)}$, which we know is associated to the sequence $B_1 = \left(B_{(1,1)}, B_{(1,2)}, B_{(1,3)}, B_{(1,4)}, B_{(1,5)}, \dots\right)$ such that $B_{(1,1)} = 1$. By Lemma 1.2.1, this means that the term $r_{(1,1)} = B_{(1,1)} = 1$. For all $n \in \mathbb{N}$ such that $n > 1$, we know that the series $R_n = \sum_{t=1}^{\infty} r_{(n,t)}$ is associated to the sequence B_n. This means that $r_{(n,1)} = B_{(n,1)}$ and, if $t \in \mathbb{N}$ such that $t > 1$, then $r_{(n,t)} = B_{(n,t)} - B_{(n,t-1)}$. We also understand that $B_n = \left(B_{(n,1)}, B_{(n,2)}, B_{(n,3)}, B_{(n,4)}, B_{(n,5)}, \dots\right)$ such that

$$B_{(n,t)} = \begin{cases} 0 & \text{if } t \in \{1, 2, 3, \dots, n-1\} \\ 1 & \text{if } t = n \\ A_{(n,t)} & \text{if } t > n \end{cases},$$

where we recall from Step 2 that, aside from the first n terms, the sequences B_n and A_n are identical.

So if $n = 1$, then we conclude that

$$r_{(n,1)} = B_{(n,1)} = 0$$

and, if $t \in \{2, 3, \dots, n-1\}$, then

$$r_{(n,t)} = B_{(n,t)} - B_{(n,t-1)} = 0 - 0 = 0$$

$$r_{(n,t)} = 0.$$

For $t = n$ we get

$$r_{(n,n)} = B_{(n,n)} - B_{(n,n-1)} = 1 - 0 = 1$$

$$r_{(n,n)} = 1.$$

Thus, for all $n \in \mathbb{N}$, we have that $r_{(n,1)} = r_{(n,2)} = \cdots = r_{(n,n-1)} = 0$ and $r_{(n,n)} = 1$. By Lemma 3.5.1, the set $\text{span}(\{R_n\}_{n=1}^{\infty})$ is dense in $\text{NS}(\mathbb{C})$, which immediately implies that $\text{span}(\{R_n\}_{n=1}^{\infty} \cup \{R_\alpha\}_{\alpha \in I})$ is dense in $\text{NS}(\mathbb{C})$. $\square$

Now we move on to demonstrating 1) and 2). We will do both in the following lemma.

Lemma 3.6.2. *The set of series* $(\{R_n\}_{n=1}^{\infty} \cup \{R_\alpha\}_{\alpha \in I})$ *is the basis of an uncountably infinite dimensional vector space which is a subset of* $(\text{UNS}(\mathbb{C}) \cup \{\sum_{t=1}^{\infty} 0\})$.

Proof. Let $N, M \in (\mathbb{N} \cup \{0\})$. We choose some arbitrary set $\{n_h\}_{h=1}^{N} \subset \mathbb{N}$ and another arbitrary set $\{\alpha_g\}_{g=1}^{M} \subset I$. If $N = 0$, then we let $\{n_h\}_{h=1}^{N} = \emptyset$. The same applies to $M = 0$. Also, we will not consider the case where $M = N = 0$. This gives us two sets of series:

$$\{R_{n_h}\}_{h=1}^{N} \subset \{R_n\}_{n=1}^{\infty} \text{ and } \{R_{\alpha_g}\}_{g=1}^{M} \subset \{R_\alpha\}_{\alpha \in I}.$$

We consider any $\{c_h\}_{h=1}^{N} \subset \mathbb{C}$ and $\{d_g\}_{g=1}^{M} \subset \mathbb{C}$ such that, if $h \in \{1, 2, \ldots, N\}$ and if $g \in \{1, 2, \ldots, M\}$, then $c_h \neq 0$ and $d_g \neq 0$. We must show that

$$(\{R_{n_h}\}_{h=1}^{N} \cup \{R_{\alpha_g}\}_{g=1}^{M})$$

is a linearly independent set of series, which we will call Objective 1, and we must show that

$$\left(\sum_{h=1}^{N} c_h R_{n_h} + \sum_{g=1}^{M} d_g R_{\alpha_g} \right) \in \text{UNS}(\mathbb{C}),$$

which we will call Objective 2. This proof is relatively long and, to better organize our work, we split this proof into 4 parts.

Part 1.

We begin by describing the notation which we will use in this proof.

We recall that if $h \in \{1, 2, \ldots, N\}$, then the series R_{n_h} is associated to the sequence $B_{n_h} = \big(B_{(n_h,1)}, B_{(n_h,2)}, B_{(n_h,3)} \ldots\big)$ and that, aside from the first n_h terms, the sequence B_{n_h} is identical to the sequence $A_{n_h} = \big(A_{(n_h,1)}, A_{(n_h,2)}, A_{(n_h,3)} \ldots\big)$. Each A_{n_h} was made by the sequence $\varphi_{n_h} = \big(\varphi_{(n_h,1)}, \varphi_{(n_h,2)}, \varphi_{(n_h,3)} \ldots\big)$ and the dense sequence $(x_p)_{p=1}^{\infty}$. Also, we recall that if $g \in \{1, 2, \ldots, M\}$, then R_{α_g} is associated to the sequence $\xi_{\alpha_g} = \big(\xi_{(\alpha_g,1)}, \xi_{(\alpha_g,2)}, \xi_{(\alpha_g,3)} \ldots\big)$. Each ξ_{α_g} was made by the sequence $\Psi_{\alpha_g} = \big(\Psi_{(\alpha_g,1)}, \Psi_{(\alpha_g,2)}, \Psi_{(\alpha_g,3)} \ldots\big)$ and the dense sequence $(x_p)_{p=1}^{\infty}$.

Now, as we can see, the indices of the sets above are slightly complicated. To make the following easier to read we will abuse the notation and say that if $g \in \{1, 2, \ldots, M\}$, then

$R_{\alpha_g} = R_g$. This give us the set $\{R_g\}_{g=1}^{M} \subset \{R_\alpha\}_{\alpha \in I}$. Also, we say that each R_g is associated to $\xi_g = (\xi_{(g,t)})_{t=1}^{\infty}$, where each ξ_g was made by the sequence $\Psi_g = (\Psi_{(g,p)})_{p=1}^{\infty}$ and the dense sequence $(x_p)_{p=1}^{\infty}$.

Similarly, if $h \in \{1, 2, \ldots, N\}$, then we will let $R_{n_h} = R_h$ and say that each R_h is associated to $B_h = (B_{(h,t)})_{t=1}^{\infty}$, where each B_h was constructed by altering the first n_h terms of the sequence $A_h = (A_{(h,t)})_{t=1}^{\infty}$. We will also say that each sequence A_h was made by $\varphi_h = (\varphi_{(h,p)})_{p=1}^{\infty}$ and the dense sequence $(x_p)_{p=1}^{\infty}$. We let $t^{**} = \max\{n_1, n_2, \ldots, n_N\}$, which will be used later.

In the above paragraphs, we used t as the index for the sequences $B_h = (B_{(h,t)})_{t=1}^{\infty}$ and $\xi_g = (\xi_{(g,t)})_{t=1}^{\infty}$. We used p as the index for the sequences $\varphi_h = (\varphi_{(h,p)})_{p=1}^{\infty}$ and $\Psi_g = (\Psi_{(g,p)})_{p=1}^{\infty}$ and for the dense set $(x_p)_{p=1}^{\infty}$. We do this to be able to tell them apart.

Finally, we denote $(\{R_h\}_{h=1}^{N} \cup \{R_g\}_{g=1}^{M}) = \widehat{R}$ and $(\{\varphi_h\}_{h=1}^{N} \cup \{\Psi_g\}_{g=1}^{M}) = \Theta$ for the sake of convenience.

Part 2.

With our notation fully explained, we will now examine an important property of the terms of the sequences in the set Θ.

Because $\Theta \subset (W_1 \cup W_2)$, we know that Θ is a set of almost disjoint sequences. So, by Lemma 3.5.2, there exists some $t^* \in \mathbb{N}$ such that, if t is a term in some sequence $S \in \Theta$ and $t > t^*$, then t is not a term in any of the sequences in the set $\Theta \setminus \{S\}$.

So if $g^* \in \{1, 2, \ldots, M\}$, if $t > t^*$ and if t is a term in the sequence Ψ_{g^*}, then t is not a term of any of the other sequences in the set $\Theta \setminus \{\Psi_{g^*}\}$. Similarly, if $h^* \in \{1, 2, \ldots, N\}$, if $t > t^*$ and if t is a term in the sequence φ_{h^*}, then t is not a term of any of the other sequences in the set $\Theta \setminus \{\varphi_{h^*}\}$.

Part 3.

Next, we will consider several important aspects of the terms of the sequences in the set $\{B_h\}_{h=1}^{N} \cup \{\xi_g\}_{g=1}^{M}$.

First, if $g \in \{1, 2, \ldots, M\}$ and if $t \in \mathbb{N}$, then we know that

$$\xi_{(g,t)} = \begin{cases} x_p & \text{if t} = \Psi_{(g,p)} \\ 0 & \text{otherwise} \end{cases} .$$

This implies that if some $t \in \mathbb{N}$ is not a term of the sequence Ψ_g, then $\xi_{(g,t)} = 0$. This also implies that, for all $p \in \mathbb{N}$, if $t = \Psi_{(g,p)}$, then

$$\xi_{(g,t)} = x_p.$$

Second, if $h \in \{1, 2, \ldots, N\}$ and if $t \in \mathbb{N}$, then we know that

$$A_{(h,t)} = \begin{cases} x_p & \text{if t} = \varphi_{(h,p)} \\ 0 & \text{otherwise} \end{cases} .$$

The sequence B_h was constructed from A_h by changing, at most, the first t^{**} terms of A_h. So if $h \in \{1,2, \ldots ,N\}$ and $t > t^{**}$, then $B_{(h,t)} = A_{(h,t)}$ and

$$B_{(h,t)} = \begin{cases} A_{(h,t)} = x_p & \text{if t} = \varphi_{(h,p)} \\ A_{(h,t)} = 0 & \text{otherwise} \end{cases} .$$

This implies that if $t \in \mathbb{N}$ such that $t > t^{**}$ and t is not a term of the sequence φ_h, then $B_{(h,t)} = 0$. This also implies that, for all $p \in \mathbb{N}$ such that $\varphi_{(h,p)} > t^{**}$, if $t = \varphi_{(h,p)}$, then $t > t^{**}$ and

$$B_{(h,t)} = x_p.$$

Part 4.

With all of the above information regarding the sequences in the sets $\{B_h\}_{h=1}^{N} \cup \{\xi_g\}_{g=1}^{M}$ and Θ fully laid out, we will now do the proof by examining two cases. For both cases we will find two results which will be used to prove the overall result. For Case 1, the two results will be denoted A.1 and B.1. For Case 2, the two results will be denoted A.2 and B.2.

For both cases, we let $T = \max\{t^*, t^{**}\}$. We crucially recall that we chose the dense sequence $(x_p)_{p=1}^{\infty}$ such that $x_p \neq 0$ for all $p \in \mathbb{N}$. Also, in both cases, we will use the well known fact that if $(t_p)_{p=1}^{\infty}$ is a subsequence of $\mathbb{N}$ and if $p, T \in \mathbb{N}$ such that $p > T$, then $t_p > T$.

Case 1. We assume that $M > 0$. We choose some $g^* \in \{1, 2, \ldots, M\}$ and we consider $\Psi_{g^*} = (\Psi_{(g^*,p)})_{p=1}^{\infty}$, which is a subsequence of $\mathbb{N}$. To simplify our notation, for all $p \in \mathbb{N}$, we let $t_p = \Psi_{(g^*,p)}$. Thus, $\Psi_{g^*} = (t_p)_{p=1}^{\infty}$, which means that $(t_p)_{p=1}^{\infty}$ is a subsequence of $\mathbb{N}$.

We now show that the sequence $(t_p)_{p=1}^{\infty}$ has an important property: if $p \in \mathbb{N}$ such that $p > T$, if $h \in \{1, 2, \ldots, N\}$ and if $g \in \{1, 2, \ldots, M\}$ such that $g \neq g^*$, then

$$\xi_{(g,t_p)} = B_{(h,t_p)} = 0 \text{ and } \xi_{(g^*,t_p)} = x_p \neq 0.$$

First, from what was stated in Part 3, if $g \in \{1, 2, \ldots, M\}$ and $t_p = \Psi_{(g,p)}$, then we get that $\xi_{(g,t_p)} = x_p$. For all $p \in \mathbb{N}$, we know that $t_p = \Psi_{(g^*,p)}$. This implies that, for all $p \in \mathbb{N}$,

$$\xi_{(g^*,t_p)} = x_p \neq 0.$$

Second, from what was stated in Part 2, we know that if $t_p = \Psi_{(g^*,p)} > T$, then t_p is not a term in any of the sequences in the set $\Theta \setminus \{\Psi_{g^*}\}$. Thus, if $h \in \{1, 2, \ldots, N\}$ and $t_p > T$, then t_p is not a term in the sequence φ_h and, as stated in Part 3, we find that $B_{(h,t_p)} = 0$. Likewise, if $g \in \{1, 2, \ldots, M\}$ such that $g \neq g^*$ and if $t_p > T$, then t_p is not a term in the sequence Ψ_g and, as stated in Part 3, we find that $\xi_{(g,t_p)} = 0$.

Thus, if $p \in \mathbb{N}$ such that $p > T$, then $t_p > T$ and

$$\xi_{(g,t_p)} = 0 \text{ and } B_{(h,t_p)} = 0.$$

Thus, we have demonstrated the desired property.

For Result A.1 and Result B.1, we recall that if $h \in \{1, 2, \ldots, N\}$, then the series R_h is associated to the sequence B_h and if $g \in \{1, 2, \ldots, M\}$, then the series R_g is associated to the sequence ξ_g.

Result A.1. We choose some $p > T$. The property states that, if $p \in \mathbb{N}$ such that $p > T$, if $h \in \{1, 2, \ldots, N\}$ and if $g \in \{1, 2, \ldots, M\}$ such that $g \neq g^*$, then $\xi_{(g,t_p)} = B_{(h,t_p)} = 0$ and $\xi_{(g^*,t_p)} \neq 0$. By Lemma 3.5.3, we know that R_{g^*} is linearly independent of the other series in the set $\widehat{R} \setminus \{R_{g^*}\}$.

It is important to note that the above work proving Result A.1 can be done with any $g^* \in \{1, 2, \ldots, M\}$. So, for any $g^* \in \{1, 2, \ldots, M\}$, the series R_{g^*} is linearly independent from the other series in the set $\widehat{R} \setminus \{R_{g^*}\}$.

Result B.1. The property also states that, for all $p \in \mathbb{N}$ such that $p > T$, if we have $h \in \{1, 2, \ldots, N\}$ and if $g \in \{1, 2, \ldots, M\}$ such that $g \neq g^*$, then $\xi_{(g,t_p)} = B_{(h,t_p)} = 0$ and $\xi_{(g^*,t_p)} = x_p$. By Lemma 3.5.4, we have that

$$\left(\sum_{h=1}^{N} c_h R_h + \sum_{g=1}^{M} d_g R_g \right) \in \mathrm{UNS}(\mathbb{C}).$$

Case 2. We assume that $N > 0$. We choose some $h^* \in \{1,2,\ldots,N\}$ and we consider $\varphi_{h^*} = (\varphi_{(h^*,p)})_{p=1}^{\infty}$, which is a subsequence of $\mathbb{N}$. To simplify our notation, for all $p \in \mathbb{N}$, we let $t_p = \varphi_{(h^*,p)}$. Thus, $\varphi_{h^*} = (t_p)_{p=1}^{\infty}$, which means that $(t_p)_{p=1}^{\infty}$ is a subsequence of $\mathbb{N}$.

We now show that the sequence $(t_p)_{p=1}^{\infty}$ has an important property: if $p \in \mathbb{N}$ such that $p > T$, if $g \in \{1, 2, \ldots, M\}$ and if $h \in \{1, 2, \ldots, N\}$ such that $h \neq h^*$, then

$$\xi_{(g,t_p)} = B_{(h,t_p)} = 0 \text{ and } B_{(h^*,t_p)} = x_p \neq 0.$$

First, from what was stated in Part 3, if $h \in \{1, 2, \ldots, N\}$ and $t_p = \varphi_{(h,p)} > T$, then we have that $B_{(h,t_p)} = x_p$. For all $p \in \mathbb{N}$, we know that $t_p = \varphi_{(h^*,p)}$ and that, if $p > T$, then $\varphi_{(h,p)} = t_p > T$. This implies that, for all $p \in \mathbb{N}$ such that $p > T$, we have that

$$B_{(h^*,t_p)} = x_p \neq 0.$$

Second, from what was stated in Part 2, we know that if $t_p = \varphi_{(h^*,p)} > T$, then t_p is not a term in any of the sequences in the set $\Theta \setminus \{\varphi_{h^*}\}$. Thus, if $h \in \{1, 2, \ldots, N\}$ such that $h \neq h^*$ and if $t_p > T$, then t_p is not a term in the sequence φ_h and, as stated in Part 3, we find that $B_{(h,t_p)} = 0$. Likewise, if $g \in \{1, 2, \ldots, M\}$ and $t_p > T$, then t_p is not a term in the sequence Ψ_g and, as stated in Part 3, we find that $\xi_{(g,t_p)} = 0$.

Thus, if $p \in \mathbb{N}$ such that $p > T$, then $t_p > T$ and

$$\xi_{(g,t_p)} = 0 \text{ and } B_{(h,t_p)} = 0.$$

Thus, we have demonstrated the desired property.

For Result A.2 and Result B.2, we recall that if $h \in \{1, 2, \ldots, N\}$, then the series R_h is associated to the sequence B_h and if $g \in \{1, 2, \ldots, M\}$, then the series R_g is associated to the sequence ξ_g.

Result A.2. We choose some $p > T$. The property states that if $p \in \mathbb{N}$ such that $p > T$, if $g \in \{1, 2, \ldots, M\}$ and if $h \in \{1, 2, \ldots, N\}$ for which $h \neq h^*$, then $\xi_{(g,t_p)} = B_{(h,t_p)} = 0$ and $B_{(h^*,t_p)} \neq 0$. By Lemma 3.5.3 we have that R_{h^*} is linearly independent of the other series in the set $\widehat{R} \setminus \{R_{h^*}\}$.

It is important to note that the above work proving Result A.2 can be done with any h^* such that $h^* \in \{1, 2, \ldots, N\}$. So, for any h^* such that $h^* \in \{1, 2, \ldots, N\}$, the series R_{h^*} is linearly independent from the other series in the set $\widehat{R} \setminus \{R_{h^*}\}$.

Result B.2. The property also states that, for all $p \in \mathbb{N}$ such that $p > T$, if we have that $g \in \{1, 2, \ldots, M\}$ and if $h \in \{1, 2, \ldots, N\}$ for which $h \neq h^*$, then $\xi_{(g,t_p)} = B_{(h,t_p)} = 0$ and $B_{(h^*,t_p)} = x_p$. By Lemma 3.5.4, we have that

$$\left(\sum_{h=1}^{N} c_h R_h + \sum_{g=1}^{M} d_g R_g \right) \in \text{UNS}(\mathbb{C}).$$

We can now use the results $A.1$, $A.2$, $B.1$ and $B.2$ to finally prove the statements of the lemma. If we are in a situation where $M > 0$ and $N = 0$, then we use $A.1$ to prove Objective 1 and $B.1$ to prove Objective 2. If we are in a situation where $N > 0$ and $M = 0$, then we use $A.2$ to prove Objective 1 and $B.2$ to prove Objective 2. If both $M > 0$ and $N > 0$, then we use Results $A.1$ and $A.2$ to prove Objective 1 and, for Objective 2, we can use either Result $B.1$ or $B.2$.

Overall, this shows that the whole set $\widehat{R} = (\{R_h\}_{h=1}^{N} \cup \{R_g\}_{g=1}^{M})$ is linearly independent and that any linear combination of these series is a Universal numerical series, which was the overall goal of the proof. □

Together, Lemma 3.6.1 and Lemma 3.6.2 finally prove Theorem 3.1.4.

3.7. Vector Spaces in UNS($\mathbb{C}$) with Finer Topologies

If the set NS($\mathbb{C}$) is equipped with the uniform topology and if UNS($\mathbb{C}$) were dense-lineable, it would imply that UNS($\mathbb{C}$) is a dense set in NS($\mathbb{C}$). By Lemma 2.2.7, we know that this is impossible. This means that if the set NS($\mathbb{C}$) is equipped with the uniform topology, then UNS($\mathbb{C}$) is not dense-lineable in NS($\mathbb{C}$). We can go further by proving the stronger result that no subset of UNS($\mathbb{C}$) is dense-lineable in UNS($\mathbb{C}$) equipped with the uniform topology.

Proposition 3.7.1. *The set* UNS($\mathbb{C}$) *is not dense-lineable in* NS($\mathbb{C}$) *equipped with the uniform topology.*

Proof. We will show that UNS($\mathbb{C}$) does not contain a vector space that is dense in the set UNS($\mathbb{C}$) $\cup$ $(\{\sum_{t=1}^{\infty} 0\})$ equipped with the uniform topology. Because UNS($\mathbb{C}$) $\subset$ NS($\mathbb{C}$), this will also demonstrate that UNS($\mathbb{C}$) does not contain a vector space that is dense in the set NS($\mathbb{C}$) equipped with the uniform topology.

We assume, by contradiction, that there exists a vector space, denoted V, which is contained in and which is dense in the set UNS($\mathbb{C}$) $\cup$ $(\{\sum_{t=1}^{\infty} 0\})$ equipped with the uniform topology. We will show that the density of V implies the existence of two series, which we will denote R_1,$R_2 \in V$, such that $(R_1 - R_2) \notin$ (UNS($\mathbb{C}$) $\cup$ $\{\sum_{t=1}^{\infty} 0\}$). This will contradict the assumption that $V \subset$ (UNS($\mathbb{C}$) $\cup$ $\{\sum_{t=1}^{\infty} 0\}$).

We select some sequence $(x_t)_{t=1}^{\infty}$ which is dense. We build two sequences, which we will denote $A = (a_1, a_2, a_3, a_4, a_5, \dots)$ and $B = (b_1, b_2, b_3, b_4, b_5, \dots)$, such that, for all $t \in \mathbb{N}$, we select $a_{2t-1} = x_t$ and $a_{2t} = x_t - 2t$ and we select $b_t = a_t + t$. Thus, we have that

$$A = (a_1, a_2, a_3, a_4, a_5, \dots)$$

$$\text{and } B = (a_1 + 1, a_2 + 2, a_3 + 3, a_4 + 4, a_5 + 5, \dots).$$

We now show that both A and B are sequences with dense subsequences. In the case of A, the subsequence $(a_{2t-1})_{t=1}^{\infty}$ is dense because, for all $t \in \mathbb{N}$, $a_{2t-1} = x_t$. In the case of B, the subsequence $(b_{2t})_{t=1}^{\infty}$ is dense because, for all $t \in \mathbb{N}$,

$$b_{2t} = a_{2t} + 2t = x_t - 2t + 2t = x_t$$

$$b_{2t} = x_t.$$

By Lemma 3.4.1, the sequences A and B are dense in $\mathbb{C}$. We associate the sequence A to the series $S_1 = \sum_{t=1}^{\infty} s_{1,t}$ and the sequence B to the series $S_2 = \sum_{t=1}^{\infty} s_{2,t}$. Next, because A and B are dense sequences, by Definition 1.2.3, we know that $S_1, S_2 \in \mathrm{UNS}(\mathbb{C})$. It is not relevant to our proof whether either series is in V.

By Lemma 3.1.3, we have that $S_2 - S_1$ is associated to the sequence

$$B - A = (a_1 + 1 - a_1, a_2 + 2 - a_2, a_3 + 3 - a_3, a_4 + 4 - a_4, a_5 + 5 - a_5, \dots)$$

$$= (1, 2, 3, 4, 5, \dots).$$

Thus, the sequence $(1, 2, 3, 4, 5, \dots)$ is associated to the series $S^* = S_2 - S_1$, which we will denote $S^* = \sum_{t=1}^{\infty} s_t^*$. By Lemma 1.2.1, this implies that

$$s_1^* = 1,$$

and that

$$s_2^* = 2 - 1 = 1,$$

$$s_3^* = 3 - 2 = 1,$$

$$s_4^* = 4 - 3 = 1,$$

and so on.

In other words, it is clear that, for all $t \in \mathbb{N}$, we have that $s_t^* = 1$. By the definition of addition and scalar multiplication of vectors in the vector space $\mathrm{NS}(\mathbb{C})$, we know that

$$\sum_{t=1}^{\infty} s_t^* = S^* = S_2 - S_1 = \sum_{t=1}^{\infty} s_{2,t} - \sum_{t=1}^{\infty} s_{1,t} = \sum_{t=1}^{\infty} (s_{2,t} - s_{1,t}).$$

From this, we deduce that

$$\sum_{t=1}^{\infty} s_t^* = \sum_{t=1}^{\infty} (s_{2,t} - s_{1,t}).$$

Thus, for all $t \in \mathbb{N}$, we know that $s_{2,t} - s_{1,t} = s_t^*$. This gives us the crucial fact that, for all $t \in \mathbb{N}$,

$$s_{2,t} - s_{1,t} = 1.$$

We take the metric balls $B_{\text{unif}}(S_1, \frac{1}{10})$ and $B_{\text{unif}}(S_2, \frac{1}{10})$. We assumed that V is a dense vector space. So there must exist some $R_1, R_2 \in V$, which are denoted $R_1 = \sum_{t=1}^{\infty} r_{1,t}$ and $R_2 = \sum_{t=1}^{\infty} r_{2,t}$, such that $R_1 \in B_{\text{unif}}(S_1, \frac{1}{10})$ and $R_2 \in B_{\text{unif}}(S_2, \frac{1}{10})$.

So now let

$$R_2 - R_1 = R^* = \sum_{t=1}^{\infty} r_t^*.$$

We will show that $R^* \notin (\text{UNS}(\mathbb{C}) \cup \{\sum_{t=1}^{\infty} 0\})$, which is the desired contradiction. We will demonstrate this by showing that, for all $t \in \mathbb{N}$, we have that $\text{Re}(r_t^*) > 0$. First, this will imply that, for all $t \in \mathbb{N}$, the term $r_t^* \neq 0$. This will show that $R^* \neq \sum_{t=1}^{\infty} 0$. Second, by Lemma 2.2.6, it will also imply that $R^* \notin \text{UNS}(\mathbb{C})$. Together, these results will show that $R^* \notin (\text{UNS}(\mathbb{C}) \cup \{\sum_{t=1}^{\infty} 0\})$.

First, we examine the metric balls $B_{\text{unif}}(S_1, \frac{1}{10})$ and $B_{\text{unif}}(S_2, \frac{1}{10})$. Let $i \in \{1,2\}$. Because $R_i \in B_{\text{unif}}(S_i, \frac{1}{10})$ and $0 < \frac{1}{10} < 1$, by Lemma 2.1.2, for all $t \in \mathbb{N}$ we must have that $|r_{i,t} - s_{i,t}| < \frac{1}{10}$. Thus, for all $t \in \mathbb{N}$, we know that,

$$|r_{1,t} - s_{1,t}| < \frac{1}{10} \text{ and } |r_{2,t} - s_{2,t}| < \frac{1}{10}.$$

Second, using identical logic to what was done above for the series $S^* = S_2 - S_1$, because we know that $R^* = R_2 - R_1$, we can conclude that for all $t \in \mathbb{N}$,

$$r_t^* = r_{2,t} - r_{1,t}.$$

We recall that, if $t \in \mathbb{N}$, then $s_{2,t} - s_{1,t} = 1$. Also, we can write that $r_{1,t} = s_{1,t} + (r_{1,t} - s_{1,t})$ and $r_{2,t} = s_{2,t} + (r_{2,t} - s_{2,t})$. Therefore, for all $t \in \mathbb{N}$, we find that

$$r_t^* = r_{2,t} - r_{1,t} = (s_{2,t} + (r_{2,t} - s_{2,t})) - (s_{1,t} + (r_{1,t} - s_{1,t}))$$

$$= (s_{2,t} - s_{1,t}) + (r_{2,t} - s_{2,t}) - (r_{1,t} - s_{1,t}) = 1 + (r_{2,t} - s_{2,t}) - (r_{1,t} - s_{1,t}).$$

Therefore,

$$r_t^* = 1 + (r_{2,t} - s_{2,t}) - (r_{1,t} - s_{1,t}).$$

We have shown that $|r_{1,t} - s_{1,t}| < \frac{1}{10}$ and $|r_{2,t} - s_{2,t}| < \frac{1}{10}$. From this, we deduce the following:

$$|\operatorname{Re}(r_{1,t} - s_{1,t})| \leq |r_{1,t} - s_{1,t}| < \frac{1}{10}$$

$$|\operatorname{Re}(r_{1,t} - s_{1,t})| < \frac{1}{10}$$

$$-\frac{1}{10} < \operatorname{Re}(r_{1,t} - s_{1,t}) < \frac{1}{10}.$$

We can rearrange this as

$$\frac{1}{10} > -\operatorname{Re}(r_{1,t} - s_{1,t}) > -\frac{1}{10}.$$

Therefore, $-\operatorname{Re}(r_{1,t} - s_{1,t}) > -\frac{1}{10}$. Similarly, we find that

$$-\frac{1}{10} < \operatorname{Re}(r_{2,t} - s_{2,t}) < \frac{1}{10},$$

and, therefore, we also get that $\operatorname{Re}(r_{2,t} - s_{2,t}) > -\frac{1}{10}$.

This implies that, for all $t \in \mathbb{N}$,

$$\operatorname{Re}(r_t^*) = \operatorname{Re}(1 + (r_{2,t} - s_{2,t}) - (r_{1,t} - s_{1,t})) = \operatorname{Re}(1) + \operatorname{Re}(r_{2,t} - s_{2,t}) - \operatorname{Re}(r_{1,t} - s_{1,t})$$

$$> 1 - \frac{1}{10} - \frac{1}{10} = \frac{8}{10} > 0.$$

This gives us that

$$\operatorname{Re}(r_t^*) > 0.$$

This finishes the proof. □

Chapter 4

Conclusion

4.1. Our work

In this Master's thesis we have shown that there are universal objects which can be proven to exist using constructive methods which yield a specific, fully defined element. This goes deeper than demonstrating implicitly that the universal elements must exist or finding a partially but not fully detailed example of such an element, which was the form of the proofs of existence of many other universal objects. We examined the proof of Mouze et al. in detail to determine whether it was constructive, as they claimed. We found that, while it was constructive, their method could not fully describe any specific Universal Taylor series, whereas the construction in our proof provides completely detailed examples of Universal numerical series.

In our original work in this thesis, we have shown the set UNS($\mathbb{C}$) has the following properties: in Theorem 2.3.1, we proved that, in the product topology, the set of Universal numerical series is topologically generic. More specifically, using the Baire Category Theorem strategy, we showed that it is a dense G_δ set. This result provides a second proof of the existence of the Universal numerical series. In Theorem 3.1.4, we demonstrated that, in the product topology, the set of Universal numerical series is densely-lineable. More specifically, we showed that UNS($\mathbb{C}$) contains, except 0, an uncountably infinite-dimensional vector space which is dense in the set NS($\mathbb{C}$). We additionally showed, in Proposition 2.5.1, that the set UNS($\mathbb{C}$) is separable in the product topology.

Considering that, aside from being separable, most previously discovered sets of universal elements have these properties, it was expected that the set of Universal numerical series should have them as well. Thus, in Theorem 2.3.1 and Theorem 3.1.4, we verified that the

Universal numerical series fit into this overarching pattern which governs most universal objects.

Also, rather than simply working with the usual topologies, we dug deeper to determine if the properties discussed above would hold in finer topologies. We showed, in Proposition 2.2.7 and Proposition 3.7.1, that the results of Theorem 2.3.1 and Theorem 3.1.4 do not hold if finer topologies are used. In Proposition 2.5.1, we similarly proved that UNS($\mathbb{C}$) is not separable in finer topologies. Thus, we have shown that the topological and algebraic properties of universal elements are not necessarily invariant under refinement.

4.2. Future Work

While we did discuss the dense-lineability of the set UNS($\mathbb{C}$) in this thesis, we did not discuss a closely related and similar topic, namely, whether the set UNS($\mathbb{C}$) is spaceable. A subset E of an infinite dimensional linearly-topological space X is spaceable if $E \cup \{0\}$ contains an infinite dimensional and topologically closed vector space [**9**]. Considering that dense-lineability and spaceability are often mentioned together in the literature, now that it has been proven that UNS($\mathbb{C}$) is densely-lineable, it would be natural to also uncover the conditions for which UNS($\mathbb{C}$) is spaceable.

Further research could also concern itself with whether there are subsets of the Universal numerical series which are algebraically generic, topologically generic and spaceable. For example, Paul Gauthier suggested that it might be possible to show that a modified version of Theorems 2.3.1 and 3.1.4 would hold for the set of all Universal numerical series which have only non-zero summands. It is easy to show that such series exist. There are possibly other subsets which could be similarly investigated, such as the set of Universal numerical series $\sum_{t=1}^{\infty} a_t$ such that $\lim_{t\to\infty} a_t = 0$ and the set of all Universal numerical series such that the terms a_t are all complex rational numbers. It is also easy to show that these series must exist.

A positive development is that it is potentially possible to use some of the techniques created in this thesis to achieve some of the goals listed above. We recall the uncountably infinite set of almost disjoint sequences developed in this thesis and we can speculate that, with some modifications, they may additionally prove useful in demonstrating that the set UNS($\mathbb{C}$) is spaceable or that subsets of UNS($\mathbb{C}$) are dense-lineable and so on. In fact, this technique might even be useful in creating vector spaces for other types of universal series, such as Universal Taylor series. While it has already been proven that Universal Taylor series are densely-lineable and spaceable, we may be able to find alternative proofs for these properties.

Having mentioned the Universal Taylor series, it is worthwhile to discuss the links between the set of Universal Taylor series and Universal numerical series. It has been noted by Fournier that a Universal Taylor series produces a Universal numerical series when it is evaluated at any point outside of the region of convergence of the Universal Taylor series. Thus, through sufficient study of the Universal numerical series, it may be possible to use these links to find the full list of the coefficients of a Universal Taylor series.

Meanwhile, on a slightly unrelated note, an analogous form of universality suggests itself when one studies the set UNS($\mathbb{C}$). Instead of studying the set of complex series $\sum_{t=1}^{\infty} a_t$ such that the set $\{\sum_{t=1}^{n} a_t \mid n \in \mathbb{N}\}$ is dense in the complex plane, it would be interesting to study the set of sequences $(a_t)_{t=1}^{\infty} \in \mathbb{C}^{\infty}$ such that the set $\{\prod_{t=1}^{n} a_t \mid n \in \mathbb{N}\}$ is dense in $\mathbb{C}$. These sequences can easily be shown to exist. We choose some dense sequence $(x_t)_{t=1}^{\infty}$ such that, for all $t \in \mathbb{N}$, $x_t \neq 0$. Then we let $a_1 = x_1$ and, for all $t \in \mathbb{N}$ such that $t > 1$, we let $a_t = \frac{x_t}{x_{t-1}}$. It is clear that, for all $n \in \mathbb{N}$, we would get that $\prod_{t=1}^{n} a_t = x_n$ and, thus, the set $\{\prod_{t=1}^{n} a_t \mid n \in \mathbb{N}\}$ is dense in $\mathbb{C}$. It would certainly be a worthwhile problem to determine whether the set of such sequences is topologically and algebraically generic.

Finally, this thesis shows that there is more work that can be done with universal objects which have already been discovered. Firstly, the results in this thesis made it clear that the topological and algebraic genericity of a universal element could depend on the chosen topology. Thus, one could check the topological and algebraic properties of known universal elements in different topologies to see which properties are preserved and which are not. Secondly, it would be beneficial to find constructive proofs for the many other forms of universality which have previously been shown to exist using only implicit methods. Not only is it a worthwhile endeavour in and of itself, it may lead to the discovery of other proof techniques which can be used to demonstrate further results.

References

[1] F. Bayart, K.-G. Grosse-Erdmann, V. Nestoridis et C. Papadimitropoulos : Abstract theory of universal series and applications. *Proc. Lond. Math. Soc. (3)*, 96(2):417–463, 2008.

[2] G. D. Birkhoff : Démonstration d'un théorème élémentaire sur les fonctions entières. *C. R. Acad. Sci. Paris*, 189:473–475, 1929.

[3] C. K. Chui et M. N. Parnes : Approximation by overconvergence of a power series. *J. Math. Anal. Appl.*, 36:693–696, 1971.

[4] J. B. Conway : *Functions of one complex variable*, volume 11 de *Graduate Texts in Mathematics*, pages xiii+317. Springer-Verlag, New York-Berlin, second édition, 1978.

[5] G. Costakis : Some remarks on universal functions and Taylor series. *Math. Proc. Cambridge Philos. Soc.*, 128(1):157–175, 2000.

[6] G. Costakis et V. Vlachou : A generic result concerning univalent universal functions. *Arch. Math. (Basel)*, 82(4):344–351, 2004.

[7] O. Demanze et A. Mouze : On universal formal power series. *J. Math. Anal. Appl.*, 338(1):662–674, 2008.

[8] R. Fournier et V. Nestoridis : Non-normal sequences of holomorphic functions and universality. *Comput. Methods Funct. Theory*, 11(1):309–316, 2011.

[9] S. Głąb et F. Strobin : Spaceability of sets in $L^p \times L^q$ and $C_0 \times C_0$. *J. Math. Anal. Appl.*, 440(2):451–465, 2016.

[10] K.-G. Grosse-Erdmann : Universal families and hypercyclic operators. *Bull. Amer. Math. Soc. (N.S.)*, 36(3):345–381, 1999.

[11] J. L. Kelley : *General topology.* Graduate Texts in Mathematics, No. 27. Springer-Verlag, New York-Berlin, 1975. Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.].

[12] W. Luh : Approximation analytischer Funktionen durch überkonvergente Potenzreihen und deren Matrix-Transformierten. *Mitt. Math. Sem. Giessen*, 88:i+56, 1970.

[13] G. R. MacLane : Sequences of derivatives and normal families. *J. Analyse Math.*, 2:72–87, 1952.

[14] J. Marcinkiewicz : Sur les nombres dérivés. *Fund. Math*, 24:305–308, 1935.

[15] K. Maronikolakis et V. Nestoridis : An extension of the universal power series of Seleznev. *Results Math.*, 75(4):Paper No. 131, 8, 2020.

[16] D. Menchoff : Sur les séries trigonométriques universelles. *C. R. (Doklady) Acad. Sci. URSS (N.S.)*, 49:79–82, 1945.

[17] A. Mouze, V. Nestoridis, I. Papadoperakis et N. Tsirivas : Determination of a universal series. *Comput. Methods Funct. Theory*, 12(1):173–199, 2012.

[18] J. Munkres : *Topology.* Prentice Hall, Upper Saddle River, 2nd édition, 2000.

[19] V. Nestoridis : Universal taylor series. *Ann. Inst. Fourier (Grenoble)*, 46(5):1293–1306, 1996.

[20] J. Pal : Zwei kleine Bemerkungen. *Tohôku Math J*, (6):42–43, 1914/1915.

[21] Walter Rudin : *Real and complex analysis*. McGraw-Hill Book Co., New York, third édition, 1987.

[22] S. M. Voronin : A theorem on the "universality" of the Riemann zeta-function. *Izv. Akad. Nauk SSSR Ser. Mat.*, 39(3):475–486, 1975.