Title: Eikonal Equation: Computation

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# Eikonal Equation: Computation

## Synonyms

Eikonal equation; eikonal-diffusion equation

## Mathematics Subject Classification

35J60; 65N30

## Short Definition

The eikonal equation is a nonlinear partial differential equation that describes wave propagation in terms of arrival times and wave front velocity. Applications include modeling seismic waves, combustion, computational geometry, image processing and cardiac electrophysiology.

# Description

## Problem Statement

A wave propagation process may be meaningfully represented by its arrival time $\tau(\mathbf{x})$ at every point $\mathbf{x}$ in space (e.g., shock wave, seismic wave, sound propagation). The local propagation velocity, which can be computed as $\|\nabla\tau\|^{-1}$, is often determined by the physical properties of the medium and therefore may be assumed to be a known positive scalar field $c(\mathbf{x})$. This relation leads to the so-called eikonal equation (Sethian, 1999; Keener and Sneyd, 2001)

$$c\,\|\nabla\tau\| = 1 \tag{1}$$

whose purpose is to compute arrival times from local propagation velocity. The zero of arrival time is defined on a curve $\Gamma_0$ as Dirichlet boundary condition $\tau = 0$ (the wave front originates from the source $\Gamma_0$). The eikonal equation may also be derived from the (hyperbolic) wave equation (Landau and Lifshitz, 1975).

The eikonal-diffusion equation is a generalization that involves an additional diffusive term (Tomlinson et al, 2002):

$$\|\mathbf{c}\nabla\tau\| = 1 + \nabla\cdot(\mathbf{D}\nabla\tau)\ . \tag{2}$$

To account for possible anisotropic material properties, the propagation velocity $\mathbf{c}$ and the diffusion coefficient $\mathbf{D}$ are symmetric positive definite tensor fields. The boundary condition is $\tau = 0$ on $\Gamma_0$ and $\mathbf{n}\cdot\mathbf{D}\nabla\tau = 0$ on other boundaries ($\mathbf{n}$ is normal to the boundary). The diffusive term creates wave front curvature-dependent propagation velocity, smoothens the solution and enforces numerical stability. In the context of wave propagation in nonlinear reaction-diffusion systems (e.g. electrical impulse propagation in the heart), the eikonal-diffusion equation may also be derived from the reaction-diffusion equation using singular perturbation theory (Franzone et al, 1990).

The objective is to compute the arrival time field ($\tau$) knowing the material properties ($\mathbf{c}$ and $\mathbf{D}$) and the location of the source ($\Gamma_0$).

## Fast Marching Method for the Eikonal Equation

The fast marching method (Sethian, 1999) is an efficient algorithm to solve Eq. (1) in a single pass. Its principle, based on Dijkstra's shortest path algorithm, exploits the causality of wave front propagation. In a structured grid with space steps $\Delta x$ and $\Delta y$ (the value of a field $F$ at coordinate $(i\Delta x, j\Delta y)$ is denoted by $F_{i,j}$), Eq. (1) is discretized as (Sethian and Vladimirsky, 2011)

$$\max\left(D^{-x}_{i,j}\tau,\ -D^{+x}_{i,j}\tau,\ 0\right)^2 + \max(D^{-y}_{i,j}\tau,\ -D^{+y}_{i,j}\tau,\ 0)^2 = 1/c^2_{i,j} \tag{3}$$

where the finite difference operators are defined as $D^{-x}_{i,j}\tau = (\tau_{i,j} - \tau_{i-1,j})/\Delta x$, $D^{+x}_{i,j}\tau = (\tau_{i+1,j} - \tau_{i,j})/\Delta x$, and similarly for $D^{-y}_{i,j}$ and $D^{+y}_{i,j}$. The algorithm maintains three lists of nodes: *accepted* nodes (for which $\tau$ has been determined), *considered* nodes (for which $\tau$ is being computed, one grid point away from an accepted node) and *far* nodes (for which $\tau$ is set to $+\infty$). The quadratic equation (3) is used to determine the values of $\tau$ in increasing order. At each step, $\tau$ is computed at *considered* nodes from known values at *accepted* nodes and the smallest value of $\tau$ among those *considered* becomes *accepted*. The lists are then updated and another step is performed until all nodes are *accepted*. The efficiency of the method relies on the implementation of list data structures and sorting algorithms. The fast marching method can be extended to triangulated surface (Sethian and Vladimirsky, 2011; Qian et al, 2007) by adapting Eq. (3). A Matlab/C implementation (used here) has been made available by Gabriel Peyr for both structured and unstructured meshes.

## Newton-based Method for the Eikonal-Diffusion Equation

When physically relevant (e.g. for cardiac propagation, see Pernod et al (2011)), the eikonal-diffusion equation may be used to refine the solution provided by the fast marching algorithm. In this case, if $\tau$ is an approximate solution to (2) satisfying the boundary conditions, a better approximation $\tau + \theta$ can be obtained by substituting $\tau + \theta$ into (2) and finding a solution $\theta$ up to second order in $\theta$. This is equivalent to Newton iterations for nonlinear system solving. Taylor expansion of (2) leads to the following steady-state convection-diffusion equation for the correction $\theta$:

$$\|\mathbf{c}\nabla\tau\| - \nabla\cdot(\mathbf{D}\nabla\tau) - 1 = \nabla\cdot(\mathbf{D}\nabla\theta) - \|\mathbf{c}\nabla\tau\|^{-1}\ \nabla\tau\ \mathbf{c}^*\mathbf{c}\nabla\theta \tag{4}$$

with boundary condition $\mathbf{n}\cdot\mathbf{D}\nabla\theta = 0$ and $\theta = 0$ in $\mathit{\Gamma}_0$.

This linearized equation can be solved using finite elements. The procedure is given here for a triangular mesh composed of a set of nodes $i \in \mathcal{V}$ and a set of triangles $(ijk) \in \mathcal{T}$, of area $\mathit{\Omega}_{ijk}$, with $i, j, k \in \mathcal{V}$. Linear shape functions, denoted by $N_i$ for $i \in \mathcal{V}$, are used to reconstruct the scalar fields $\tau = \sum_{i\in\mathcal{V}} \tau_i N_i$ and $\theta = \sum_{i\in\mathcal{V}} \theta_i N_i$. These functions are linear in each triangle; the gradient operator evaluated in triangle $(ijk)$ is noted $\nabla_{ijk}$. Similarly, the parameters $\mathbf{c}_{ijk}$ and $\mathbf{D}_{ijk}$ denote the values of $\mathbf{c}$ and $\mathbf{D}$ at the center of gravity of the triangle $(ijk)$. The application of the Galerkin finite element approach (Huebner et al, 2001) to (4) leads to the linear system $\mathbf{A}(\tau)\boldsymbol{\theta} = \mathbf{f}(\tau)$, where the matrix and the right hand side are computed as:

$$\begin{aligned} A_{mn}(\tau) = &- \sum_{(ijk)\in\mathcal{T}} \mathit{\Omega}_{ijk}\nabla_{ijk}N_m \cdot \mathbf{D}_{ijk}\nabla_{ijk}N_n \\ &- \sum_{(ijk)\in\mathcal{T}} \frac{\mathit{\Omega}_{ijk}}{3}\|\mathbf{c}_{ijk}\nabla_{ijk}\tau\|^{-1}\ (\mathbf{c}_{ijk}\nabla_{ijk}\tau)^* \cdot (\mathbf{c}_{ijk}\nabla_{ijk}N_n) \end{aligned} \tag{5}$$

$$f_m(\tau) = \sum_{\substack{(ijk)\in\mathcal{T}\\ m\in\{ijk\}}} \frac{\mathit{\Omega}_{ijk}}{3}\left(\|\mathbf{c}_{ijk}\nabla_{ijk}\tau\| - 1 + 3\,\nabla_{ijk}N_m \cdot \mathbf{D}_{ijk}\nabla_{ijk}\tau\right)\ . \tag{6}$$

For vertices $m \in \mathit{\Gamma}_0$, the boundary condition $\theta = 0$ is applied by setting $A_{mn} = \delta_{mn}$ and $f_m = 0$, which ensures that $\mathbf{A}$ is not singular. An easy and efficient implementation in Matlab based on sparse matrix manipulation is possible after reformulation (Jacquemet, 2011).

Practically, the first estimate $\tau^0$ is given by the fast marching method (neglecting diffusion). At iteration $n+1$, the correction $\theta^{n+1}$ is obtained by solving the linear system $\mathbf{A}(\tau^n)\,\boldsymbol{\theta}^{n+1} = \mathbf{f}(\tau^n)$. Then $\tau^{n+1} = \tau^n + \theta^{n+1}$ is updated until the norm of the correction falls below a given tolerance $\|\theta^{n+1}\| < tol$.

## Extension to Reentrant Waves

The eikonal-diffusion equation, due to its local nature, is also valid for reentrant wave propagation. In this case, the wave does not originate from a focal source but instead is self-maintained by following a closed circuit. To account for the periodicity of the propagation pattern and avoid phase unwrapping issue, a phase transformation $\phi = \exp(i\tau)$ is applied, where $\tau$ is now normalized between 0 and $2\pi$. The transformed eikonal-diffusion equation reads (Jacquemet, 2010):

$$\|\mathbf{c}\nabla\phi\| = 1 + \mathrm{Im}\ \nabla \cdot (\phi^* \mathbf{D}\nabla\phi) \tag{7}$$

The boundary condition $\mathbf{n} \cdot \mathbf{D}\nabla\phi = 0$ still holds and the constraint $|\phi| = 1$ must be preserved. The star (*) denotes the complex conjugate and 'Im' the imaginary part.

The same Newton-based method as in the previous subsection can be used. If $\phi$ approximates the solution to (7), $\phi \exp(i\theta)$ is substituted in (7) and the linearized equation is solved for $\theta$:

$$\|\mathbf{c}\nabla\phi\| - \mathrm{Im}\ \nabla \cdot (\phi^* \mathbf{D}\nabla\phi) - 1 = \nabla \cdot (\mathbf{D}\nabla\theta) + \|\mathbf{c}\nabla\phi\|^{-1}\ \mathrm{Im}\ (\phi\nabla\phi^* \mathbf{c}^* \mathbf{c}\nabla\theta) \tag{8}$$

This equation has the same steady-state convection-diffusion form as (4). The same Galerkin approach can therefore be applied, leading to a linear system $\mathbf{A}(\phi)\,\boldsymbol{\theta} = \mathbf{f}(\phi)$

whose matrix elements are defined as (Jacquemet, 2010):

$$
\begin{aligned}
A_{mn}(\phi) = & - \sum_{(ijk)\in\mathcal{T}} \varOmega_{ijk} \nabla_{ijk} N_m \cdot \mathbf{D}_{ijk} \nabla_{ijk} N_n \\
& + \sum_{(ijk)\in\mathcal{T}} \frac{\varOmega_{ijk}}{3} \|\mathbf{c}_{ijk}\nabla_{ijk}\phi\|^{-1} \operatorname{Im} \frac{\phi_i+\phi_j+\phi_k+\phi_m}{4} \left(\mathbf{c}_{ijk}\nabla_{ijk}\phi\right)^* \cdot \left(\mathbf{c}_{ijk}\nabla_{ijk}N_n\right) \quad (9)
\end{aligned}
$$

$$
f_m(\phi) = \sum_{\substack{(ijk)\in\mathcal{T}\\ m\in\{ijk\}}} \frac{\varOmega_{ijk}}{3} \left( \|\mathbf{c}_{ijk}\nabla_{ijk}\phi\| - 1 + 3\,\nabla_{ijk}N_m \cdot \operatorname{Im} \frac{\phi_i^*+\phi_j^*+\phi_k^*}{3} \mathbf{D}_{ijk}\nabla_{ijk}\phi \right) . \quad (10)
$$

The matrix $\mathbf{A}$ is singular due to the symmetry $\phi \mapsto \phi\exp(i\theta_0)$ with spatially-uniform $\theta_0$. This issue is solved by replacing the system $\mathbf{A}(\phi)\boldsymbol{\theta} = \mathbf{f}(\phi)$ by the non-singular (deflated) system $\left(\mathbf{A}(\phi) - n_v^{-1}\mathbf{e}\,\mathbf{e}^*\right)\boldsymbol{\theta} = \mathbf{f}(\phi)$ where $\mathbf{e}$ is a $n_v$-vector whose all elements are 1 and $n_v$ is the dimension of $\mathbf{A}$.

Practically, the first estimate $\phi^0$ is obtained from a supposedly known circuit $\varGamma$ (closed curve) by interpolating $\phi^0$ in the entire surface from the values $\exp(2\pi i\ell/L)$ along the curve $\varGamma$ using Laplacian interpolation ($\ell$ is the curvilinear coordinate and $L$ the length of the curve). At iteration $n+1$, the correction $\theta^{n+1}$ is computed by solving the linear system $\left(\mathbf{A}(\phi^n) - n_v^{-1}\mathbf{e}\,\mathbf{e}^*\right)\boldsymbol{\theta}^{n+1} = \mathbf{f}(\phi^n)$. To avoid the use of a fully-populated matrix, the system can be solved iteratively (e.g. BiCGstab with incomplete LU preconditioner) with matrix-vector multiply being implemented as $\mathbf{A}\boldsymbol{\theta} - \mathrm{mean}(\boldsymbol{\theta})\,\mathbf{e}$. The uniform phase shift is suppressed by subtracting the mean: $\tilde{\theta}^{n+1} = \theta^{n+1} - \mathrm{mean}(\theta^{n+1})$. Then $\tau^{n+1} = \tau^n + \tilde{\theta}^{n+1}$ is updated until the norm of the correction falls below a given tolerance $\|\tilde{\theta}^{n+1}\| < tol$. Jacquemet (2011) showed that the period of reentry compatible with the propagation velocity specified in the eikonal-diffusion equation is approximately $T = 2\pi/(1 + \mathrm{mean}(\theta^{n+1}))$.

## Examples in cardiac electrophysiology

The eikonal approach is illustrated here in a triangular mesh (about 5,000 nodes) representing the atrial epicardium derived from magnetic resonance images of a patient.

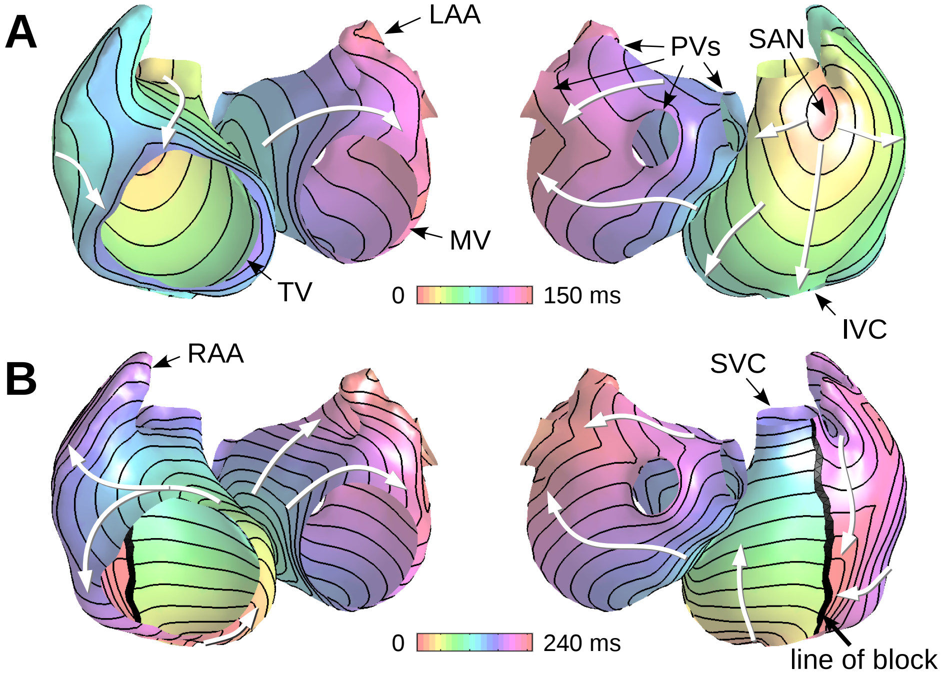


**Fig. 1.** Propagation of the electrical impulse in an anisotropic surface model of the human atria computed using the eikonal-diffusion equation. (A) Normal propagation from the sino-atrial node region. Activation time is color-coded. Isochrones are displayed every 10 ms. White arrows illustrate propagation pathways. (B) Reentrant propagation similar to typical atrial flutter in a model with slower propagation velocity. The line of block is represented as a thick black line. TV: tricuspid valve; MV: mitral valve; LAA: left atrial appendage; RAA: right atrial appendage; PVs: pulmonary veins; IVC: inferior vena cava; SVC: superior vena cava; SAN: sino-atrial node.

Fiber orientation (anisotropic properties) was obtained from anatomical and histological data. Propagation velocity was set to 100 cm/s (along fiber) and 50 cm/s (across fiber). $\Gamma_0$ was placed near the anatomical location of the sino-atrial node. The diffusion coefficient $D$ was set to 10 cm$^2$. The activation map (arrival times) computed using the eikonal-diffusion solver (iteration from the solution provided by the fast marching algorithm) is displayed in Fig. 1A. With $tol = 10^{-10}$, 16 iterations were needed.

Figure 1B shows a reentrant activation map corresponding to an arrhythmia called typical atrial flutter, simulated using the eikonal-diffusion solver extended for reentrant propagation. The reentrant pathway $\Gamma$ was formed by a closed circuit connecting the two vena cava. Propagation velocity was reduced by 40%. With $tol = 10^{-10}$, 50 iterations were needed. The resulting period of reentry was 240 ms, a value within physiological range.

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