Université de Montréal

Diamètre spectral et cohomologie symplectique

par

Pierre-Alexandre Mailhot

Département de mathématiques et de statistique
Faculté des arts et des sciences

Thèse présentée en vue de l'obtention du grade de
Philosophiæ Doctor (Ph.D.)
en mathématiques pures

31 août 2023

Université de Montréal

Faculté des arts et des sciences

Cette thèse intitulée

Diamètre spectral et cohomologie symplectique

présentée par

Pierre-Alexandre Mailhot

a été évaluée par un jury composé des personnes suivantes :

Octav Cornea

(président-rapporteur)

Egor Shelukhin

(directeur de recherche)

Iosif Polterovich

(membre du jury)

Jungsoo Kang

(examinateur externe)

Octav Cornea

(représentant du doyen de la FESP)

RÉSUMÉ

Le groupe de difféomorphismes hamiltoniens à support compact d'une variété symplectique admet une distance naturelle bi-invariante, d'après les travaux de Viterbo, Schwarz, Oh, Frauenfelder et Schlenk, construite à partir des invariants spectraux en homologie de Floer Hamiltonienne. Cette distance, appelée la norme spectrale, s'est révélée être un outil fort utile en topologie symplectique. Par contre, son diamètre reste inconnu en général. En fait, pour les variétés symplectiques fermées, il n'existe même pas de critère pour déterminer si la norme spectrale a un diamètre fini ou infini. Il a été conjecturé que, pour les variétés symplectiquement asphériques, le diamètre de la norme spectrale est infini.

Dans cette thèse, nous démontrons que pour tout domaine de Liouville, la norme spectrale a un diamètre infini si et seulement si la cohomologie symplectique du domaine de Liouville en question est non nulle. Ceci généralise un résultat de Monzner-Vichery-Zapolsky et admet plusieurs applications dans le cadre des variétés symplectiques fermées. En particulier, nous démontrons que le produit de deux variétés symplectiquement asphériques a un diamètre spectral infini. Plus généralement, nous démontrons que toute variété symplectiquement asphérique contenant un domaine de Liouville incompressible de codimension zéro avec cohomologie symplectique non nulle doit avoir un diamètre spectral infini.

Mots clés : Topologie symplectique, topologie de contact, domaines de Liouville, groupe de difféomorphismes hamiltoniens, cohomologie symplectique, norme spectrale, topologie symplectique C^0.

ABSTRACT

The group of compactly supported Hamiltonian diffeomorphisms of a symplectic manifold is endowed with a natural bi-invariant distance, due to Viterbo, Schwarz, Oh, Frauenfelder and Schlenk, coming from spectral invariants in Hamiltonian Floer homology. This distance, called the spectral norm, has found numerous applications in symplectic topology. However, its diameter is still unknown in general. In fact, for closed symplectic manifolds there is no unifying criterion for the diameter to be finite or infinite. It has been conjectured that for closed symplectically aspherical manifolds, the spectral norm has infinite diameter.

In this thesis, we prove that for any Liouville domain the spectral norm has infinite diameter if and only if its symplectic cohomology does not vanish. This generalizes a result of Monzner-Vichery-Zapolsky and has applications in the setting of closed symplectic manifolds. For instance, we show that the product of two closed symplectically aspherical manifold has an infinite spectral diameter . More generally, we prove that any symplectically aspherical manifold which contains an incompressible Liouville domain of codimension zero with non-vanishing symplectic cohomology must have infinite spectral diameter.

Key words : Symplectic topology, contact topology, Liouville domains, group of Hamiltonian diffeomorphisms, symplectic cohomology, spectral norm, C^0 symplectic topology.

CONTENTS

LIST OF FIGURES

REMERCIEMENTS

Je tiens tout d'abord à remercier Egor Shelukhin, mon directeur, pour m'avoir proposé un sujet de recherche fascinant qui regorge de questions intrigantes. Les multiples pistes de réflexion sur lesquelles il m'a lancé m'ont non seulement donné l'opportunité de répondre à plusieurs desdites questions, mais elles m'ont aussi permises de développer mes compétences de recherche en mathématiques. Grâce à son support et ses conseils, j'ai pu véritablement m'épanouir dans mes études au doctorat.

Je remercie Octav Cornea pour ses commentaires et recommandations à l'égard de la première version des résultats qui figurent dans cette thèse. Avec ses conseils et encouragements, il a su me guider lorsque je me questionnais encore sur mon parcours académique.

Les questions intéressantes de Leonid Polterovich et Shira Tanny ont poussées ma recherche dans des directions fort intéressantes. Je les remercie de m'avoir invité à donner mes toutes premières présentations qui portaient sur ma recherche et de m'avoir mis à l'aise lors de ces dernières.

Les innombrables discussions mathématiques que j'ai eu la chance d'entretenir avec mes collègues et ami.e.s du DMS et d'ailleurs ont étés essentielles à l'écriture de cette thèse. Je tiens donc à remercier Jean-Philippe, Dominique, Jonathan, Élise, Xu Rui, Alexis, Julien, Dylan, Marcello, Habib, Jakob, John, Filip, Jordan, Dustin, et François.

Je veux aussi remercier mes ami.e.s hors du monde des mathématiques qui m'ont toujours encouragé et qui ont remplies de joie et de rires mon parcours universitaire. Pour cela, je remercie mes amis et camarades du bac, François, Adrien et Maxime, mes ami.e.s de Folie Technique, Alexandre (Turbo), Bénédicte (Béryllium) et Étienne (Ampère), et mes amis de longue date, Charles-Étienne, Olivier et Steve.

Au cours de mon parcours aux études supérieures, j'ai eu la chance de développer ma passion pour l'enseignement et la communication scientifique. Cela n'aurait pas été possible sans la générosité des organisateurs et organisatrices du Club Math. Je remercie aussi le DMS de m'avoir fait confiance avec les charges de cours que j'ai eu la chance d'enseigner. Je dois également remercier Julie Doucet-Lamoureux de m'avoir donné l'opportunité de transmettre mon amour des sciences aux plus jeunes.

Un merci tout spécial à ma famille. Sans leur amour et leur soutien, je n'aurai pas pu imaginer entreprendre un pareil projet. Merci aux nombreux membres des familles Mailhot, Monette, Duong, Tan et Gosselin d'être de si bonne compagnie.

Je tiens à remercier du fond du cœur ma meilleure amie et fiancée Leslie. Au cours des dix dernières années, elle a toujours été là pour moi et continue de l'être. Je la remercie d'avoir passé avec moi les plus beaux moments. Son aide lors de la conception des figures dans cette thèse fut aussi indispensable.

Finalement, je ne pourrai jamais assez remercier ma mère Murielle de m'avoir apporté réconfort dans les moments les plus difficiles, mon père Luc de m'avoir toujours poussé à aller plus loin, ma sœur Annabelle sur qui je peux toujours compter, mes grands-parents Connie et Robert qui savent toujours comment me remonter le moral et mon grand-père Bernard qui a su éveiller en moi ma curiosité scientifique.

Chapter 1

INTRODUCTION

1.1. Symplectic geometry

Symplectic geometry is concerned with studying manifolds M equipped with an area form, i.e. a non-degenerate closed 2-form $\omega_M \in \Omega^2(M)$. The natural symmetries, called *symplectomorphisms*, which one studies on these spaces are those that preserve the *symplectic form* ω_M. It follows from an elementary cohomological argument that symplectic manifolds can only be of even dimension. More surprisingly perhaps, is the fact that symplectic geometry is inherently a global theory. Indeed, in virtue of Darboux's theorem, all symplectic manifolds of a given dimension are locally symplectomorphic.

Examples of symplectic manifolds appear naturally in classical mechanics [**Arn89**]. To entirely describe the behaviour of a free particle moving inside a closed n-manifold Q, one needs to specify an initial state for the particle and an energy function. A state corresponds to a point (position) $q \in Q$ and a covector p (momentum) in the fibre $\mathrm{T}^*_q Q \subset \mathrm{T}^*Q$ over q. The energy of the particle, which might depend on time, is given by a function $H : \mathbb{R} \times \mathrm{T}^*Q \to \mathbb{R}$, which we call a *Hamiltonian*. On T^*Q there exists a canonical symplectic form ω_{std} which, in local coordinates, can be written as

$$\omega_{\mathrm{std}} = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i.$$

Globally, $\omega_{\text{std}} = \mathrm{d}\lambda_{\text{std}}$ where the form λ_{std}, called the *Liouville 1-form*, can be locally written as $\lambda_{\text{std}} = \sum_{i=1}^{n} p_i \mathrm{d}q_i$. A particular class of symplectomorphisms also appears naturally in classical mechanics. According to *Hamilton's equations*, the evolution of the free particle through time is given by the flow $\varphi^t_{X^H} : \mathrm{T}^*Q \to \mathrm{T}^*Q$ of the *Hamiltonian vector field* X_H associated to H defined by

$$X_{H_t} \lrcorner\, \omega_{\text{std}} = -\mathrm{d}H_t.$$

Liouville's theorem then assures us that $\varphi^t_{X^H}$ preserves ω_{std}. Given a function $H : \mathbb{R} \times M \to M$ on a general symplectic manifold, such a construction can be carried out verbatim and the time 1 map of the resulting family of symplectomorphisms is called a *Hamiltonian diffeomorphism*. The set of all such tansformations forms a group which is denoted by $\mathrm{Ham}(M,\omega_M)$.

1.1.1. Contact manifolds

There exists an odd-dimensional analogue of symplectic manifolds which are called *contact manifolds*. Such a $(2k-1)$-manifold N is, by definition, endowed with a *contact 1-form* $\alpha_N \in \Omega^1(N)$ defined by the fact that it is non-degenerate. Namely, its associated volume form $\alpha_N \wedge (\mathrm{d}\alpha_N)^{\wedge(k-1)}$ does not vanish. The natural symmetries of contact manifolds are called *contactomorphisms* and they preserve, by definition, the contact form.

In the context of classical mechanics addressed above, contact manifolds arise naturally as energy levels

$$N_H^E = \{(q,p) \in \mathrm{T}^*Q \mid H(p,q) = E\}$$

of Hamiltonians $H : \mathrm{T}^*Q \to \mathbb{R}$ of the form $H(q,p) = \frac{1}{2}\|p\|^2 + V(q)$ [HZ94]. Another example is given by the space consisting of states $(q,p) = \mathrm{T}^*Q$ with fixed momentum $\|p\| = r \in \mathbb{R}$. This space corresponds to the cotangent sphere bundle S_r^*Q of radius r over Q. These two examples, are special cases of an important class of co-dimension 1 submanifolds of symplectic manifolds. An hypersurface $S \subset M$ is said to be of *contact type* if there exists a vector field Y transverse to S and defined in a neighbourhood of S

such that $L_Y\omega = \omega$. As the name suggests, contact type surfaces are contact manifolds with contact 1-form given by $\alpha_S = Y \lrcorner\, \omega$.

There is always a way to construct a symplectic manifold from a given contact manifold N. The manifold $\mathrm{Symp}(N) = N \times (0,\infty)$, when equipped with the exact 2-form $\mathrm{d}(e^t\alpha_N)$, where t is the coordinate on $(0,\infty)$, is symplectic. We call $(\mathrm{Symp}(N), \mathrm{d}(e^t\alpha_N))$ the *symplectization of* N. Looking at the vector field $Y = \partial_t$, we see that any contact manifold can be realized as a surface of contact type in its symplectization.

1.1.2. Orbits

One of the simplest dynamical question one may ask is if a given system admits periodic orbits and if so, how many. On a symplectic manifold M, and in physics in general, an important class of periodic orbits are those of Hamiltonian diffeomorphisms of period 1: curves $\gamma : S^1 \to M$ that can be written as $\gamma(t) = \varphi_H^t(x_0)$ for some fixed point $x_0 \in M$ of φ_H^1 and Hamiltonian H. One of the main conjecture regarding these fixed points was posed by V.I. Arnol'd in the 60's:

> **Conjecture 1.1 • [Arn65]**
>
> For a compact symplectic manifold M the time 1 map of a non-degenerate Hamiltonian diffeomorphism possesses at least as many fixed points as a Morse function on M.

To this day, Conjecture 1.1 is still open. It was proven to hold for the $2n$-torus by Conley and Zehnder [**CZ83**] and for surfaces of genus at least 2 and for certain classes of Kähler manifolds by Floer [**Flo86**]. A weaker version of Conjecture 1.1, called the Homological Arnol'd Conjecture, is known to hold in general:

Theorem 1.2

Let (M,ω) be a closed symplectic manifold and H a nondegenerate time periodic Hamiltonian on M. Then,

$$\#\left|\mathrm{Fix}(\varphi_H^1)\right| \geq \sum_{i=0}^{\dim M} \dim H_i(M;\mathbb{Q}).$$

Some of the first steps made towards a proof Theorem 1.2 were initiated by Floer in his revolutionary paper [**Flo89**]. There, he combined the variational approach of the symplectic action functional developed by Conley-Zehnder [**CZ83**] with Gromov's pseudo holomorphic curves [**Gro85**] to construct a chain complex $(\mathrm{CF}_\bullet(H),\partial)$, now called the *Floer complex*, generated by $\mathrm{Fix}(\varphi_H^1)$. The differential ∂ counts pseudo holomorphic cylinders between the one periodic orbits of $\varphi_H^\bullet$. These cylinders are interpreted as negative gradient flow lines of the symplectic action functional thus yielding an infinite dimensional Morse-type theory. In particular, Floer showed that for a C^2-small Hamiltonian h, the *Floer homology* $\mathrm{HF}_\bullet(h) = \mathrm{H}_\bullet(CF_\bullet(H),\partial)$ is isomorphic to the Morse homology of h. Using this approach, Floer gave a proof Theorem 1.2 for closed monotone symplectic manifolds ($[\omega_M] = \lambda c_1(M)$ for some $\lambda \in \mathbb{R}$). Theorem 1.2 was then established in full generality by later work of Fukaya-Ono [**FO99a, FO99b**], Ruan [**Rua99**], Liu-Tian [**LT98**] and Pardon [**Par16**]. Floer theory is now a cornerstone of contemporary symplectic geometry and has far reaching applications in other fields such as the theory of 3-dimensional manifolds [**Man15**].

From a physical perspective, one might be interested in the existence of orbits that sit in a given energy level. Using a symplectic language, we ask , given a Hamiltonian H, if any regular level set $N \subset M$ of contact type of H contains at least one periodic orbit of $\varphi_H^\bullet$. This question can be reformulated entirely in terms of the contact structure on N. Indeed, it can be shown that Hamiltonian orbits on N are determined by N. Only their parametrization depends on H. In fact, Hamiltonian trajectories correspond to the Reeb trajectories of α_N. By definition, the latter are trajectories of the Reeb vector field R_α which is defined by

$$R_{\alpha_N} \lrcorner\, \alpha_N = 1, \quad R_{\alpha_N} \lrcorner\, \mathrm{d}\alpha_N = 0$$

and only depends on α_N. Recall that any contact manifold can be realized as a contact type hypersurface. This leads us to the Weinstein Conjecture.

Conjecture 1.3 • [Wei79]

Any compact contact manifold carries at least one periodic Reeb orbit.

Conjecture 1.3 is still open in general. Many of the settings where it holds depend on which class of symplectic manifolds M the contact manifold N can be embedded in as a contact hypersurface. The Weinstein conjecture was first established for $M = \mathbb{R}^{2n}$ by Viterbo [**Vit87**]. It was then extended to $M = \mathrm{T}^*Q$ by Hofer-Viterbo [**HV88**] and for products $M = P \times \mathbb{C}$, where P is symplectically aspherical, by Floer-Hofer-Viterbo [**FHV90**]. In dimension 3 one does not need to rely on an embedding of N in a symplectic manifold to prove the conjecture. Taubes [**Tau07**] proved that any closed contact 3-manifold admits a Reeb orbit using a variant of Seiberg-Witten Floer Homology.

1.1.3. Liouville domains

Liouville domains, a special class of symplectic manifolds with boundary, offer a more general setting than symplectizations where one can study properties of contact manifolds from the symplectic point of view. They are characterized by their exact symplectic form $\omega = \mathrm{d}\lambda$ and their boundary of contact type. Liouville domains regroup under a common theoretical framework many important classes of symplectic manifolds. Examples of such manifolds include cotangent disk bundles over closed manifolds, complements of Donaldson divisors [**Gir17**], preimages of some intervals under exhausting functions of Stein manifolds [**CE12**], positive regions of convex hypersurfaces in contact manifolds [**Gir91**] and total spaces of Lefschetz fibrations.

To better control the behaviour of objects defined near the boundary of ∂D, one considers the *extension* $(\hat{D}, \mathrm{d}\hat{\lambda})$ of $\hat{D}$. To construct this extension, first glue the cylinder $[1,\infty) \times \partial D$ to ∂D using the flow of the Liouville vector Y near ∂D. Then, equip the resulting manifold $\hat{D}$ with the Liouville 1-form $\hat{\lambda}$ which coincides with λ on D and is given by $t\lambda|_{\partial D}$ on the cylindrical part of $\hat{D}$.

A key invariant of a Liouville domain D is its symplectic cohomology $\mathrm{SH}^\bullet(D)$ which beautifully combines the dynamics of Reeb orbits on ∂D with the Floer cohomology of D. It was first defined by Cielieback, Floer and Hofer [FH94, CFH95] and later developed by Viterbo [Vit99]. Symplectic cohomology is built from a class of Hamiltonians $H : \hat{D} \to \mathbb{R}$, called *admissible*, which are affine in the radial coordinate on the cylindrical part of $\hat{D}$. Floer cohomology groups $\mathrm{HF}^\bullet(H)$ of such Hamiltonians are well defined and only depend on the *slope* τ_H of H on $[1,\infty) \times \partial D$. Contrary to the closed case, we do not have, in general, an isomorphism between the singular cohomology of D and $\mathrm{HF}^\bullet(H)$. However, when the slope τ_F of an admissible Hamiltonian F is less than the minimal period of a Reeb orbit on ∂D, we have $\mathrm{H}^\bullet(D) \cong \mathrm{HF}^\bullet(F)$. Taking an increasing sequence of admissible Hamiltonians $\{H_i\}_i$ with corresponding slopes $\{\tau_i\}_i$ satisfying $\tau_i \to +\infty$, one can define the symplectic cohomology $\mathrm{SH}^\bullet(D)$ of D as

$$\mathrm{SH}^\bullet(D) = \varinjlim_{H_i} \mathrm{HF}^\bullet(H_i).$$

In [Vit99], Viterbo showed that the symplectic cohomology of D comes equipped with a map

$$v^\bullet : \mathrm{H}^\bullet(D) \longrightarrow \mathrm{SH}^\bullet(D).$$

The failure of $v^\bullet$ to be an isomorphism signals the presence of Reeb orbits on the boundary of D. Thus, $\mathrm{SH}^\bullet(D)$ is a useful tool to study the Weinstein conjecture. Using this strategy, Viterbo [Vit99] proved the Weinstein conjecture for the boundary of subcritical Stein manifolds.

One reason to work with symplectic cohomology instead of its homological counterpart is that $\mathrm{SH}^\bullet(D)$ inherits a product with unit 1_D from the pair of pants product on the Floer cohomology groups used to define it [**Rit13**]. The algebraic convenience of this approach will be made clear in section 1.2.

1.1.4. Topology of $\mathrm{Ham}(M,\omega_M)$

Another central topic in symplectic geometry, which is strongly related to periodic orbits, is the topology of the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M,\omega_M)$. This group admits many different metrics two of which are of interest here: the Hofer metric and the spectral metric.

The *Hofer norm* $\|H\|$ of a compactly supported Hamiltonian H is defined as the integral of its oscillation over time :

$$\|H\| = E_+(H) - E_-(H)$$
$$E_- = -\int_0^1 \max_{p\in M} H(t,p)\,\mathrm{d}t, \quad E_+ = -\int_0^1 \min_{p\in M} H(t,p)\,\mathrm{d}t$$

Following the work of Hofer [**Hof90**] and Lalonde-McDuff [**LM95**], a bi-invariant metric d_H on $\mathrm{Ham}(M,\omega_M)$, called the *Hofer metric*, can be defined by

$$d_H(\varphi,\psi) = d_H(\varphi\psi^{-1},\mathrm{id}), \quad d_H(\varphi,\mathrm{id}) = \inf\{\|H\| \mid \varphi = \varphi_H^1\}.$$

To illustrate the link between the Hofer geometry of $\mathrm{Ham}(M,\omega_M)$ and periodic orbits of Hamiltonian diffeomorphisms, it is worthwhile to take a look at the action functional which appears, among other things, in the construction of Floer homology.

Denote by $\mathrm{Fix}^0(\varphi_H^1)$ the set of fixed point x_0 of φ_H^1 for which the associated 1-periodic orbit $\varphi_H^t(x_0)$ is contractible. For any $x_0 \in \mathrm{Fix}^0(\varphi_H^1)$, choose a 2-disk Σ bounded by the image of $\varphi_H^t(x_0)$. The *symplectic action* $\mathcal{A}_H(x_0)$ of H at x_0 is given by

$$\mathcal{A}_H(x_0) = \int_\Sigma \omega_M - \int_0^1 H(t, \varphi_H^t(x_0))\,\mathrm{d}t.$$

Note that when ω_M is exact or $\pi_2(M) = 0$, $\mathcal{A}_H(x_0)$ is independent of the choice of Σ. We call the image $\mathrm{Spec}(H)$ of the symplectic action functional applied to $\mathrm{Fix}^0(\varphi^1_H)$ the *action spectrum*. When two Hamiltonians are equivalent, i.e. there exists an Hamiltonian homotopy $\{G_s\}_{s\in[0,1]}$ such that each G_s generates the same Hamiltonian diffeomorphism $\varphi_H = \varphi^1_{G_s} = \varphi_K$, their action spectrum are equal [**Oh15**, 18.3]. This allows one to define the action spectrum on the level of $\mathrm{Ham}(M,\omega_M)$ as

$$\mathrm{Spec}(\varphi) = \mathrm{Spec}(H) \text{ for any } H \text{ such that } \varphi = \varphi_H.$$

The action spectrum can be studied from the point of view of Floer theory using *spectral invariants* which are functions that associate to any pair $(\alpha,H) \in \mathrm{H}^\bullet(M) \times C^\infty_c(S^1 \times M)$ a real number $c(\alpha,H)$, that belongs to $\mathrm{Spec}(H)$[1]. These incredibly useful functions were first defined on $\mathbb{R}^{2n}$ from the point of view of generating functions by Viterbo in [**Vit92**]. They were then constructed on general closed symplectic manifolds by Oh in [**Oh05**] (see also [**Ush13**]). Following the work of Schwartz [**Sch00**] on symplectically aspherical manifolds and the work of Frauenfelder-Schlenk [**FS07**] on Liouville domains, spectral invariants can be defined in both these settings. To construct $c(\cdot, H)$, first notice that the Hamiltonian action functional $\mathcal{A}_H$ induces a filtration on Floer cohomology thus allowing us to work with the filtered Floer cohomology groups $\mathrm{HF}^\bullet_{(a,b)}(H)$. Then we define $c(\alpha,H)$ as the minimal action at which the cohomology class α appears in $\mathrm{HF}^\bullet(H) := \mathrm{HF}^\bullet_{(-\infty,+\infty)}(H)$. More precisely,

$$c(\alpha,H) = \inf\left\{c \in \mathbb{R} \mid \pi_{>c}(\Phi_H(\alpha)) = 0\right\}.$$

Here, $\Phi_H : \mathrm{H}^\bullet(M) \to \mathrm{HF}^\bullet(M)$ is the PSS isomorphism and $\pi_{>c} : \mathrm{HF}^\bullet(H) \to \mathrm{HF}^*_{(c,+\infty)}(H)$ is the natural map induced by the projection of subcomplexes. Some of the most basic properties of spectral invariants are the following

- **Spectrality.** $c(\alpha, H) \in \mathrm{Spec}(H)$

[1] at least if the Hamiltonian satisfies certain technical conditions.

- **Continuity.**

$$E_-(H-K) \leq c(\alpha,H) - c(\alpha,K) \leq E_+(H-K)$$

- **Homotopy invariance.** If H and K are equivalent,

$$c(\alpha,H) = c(\alpha,K).$$

In particular, homotopy invariance allows us to define spectral invariance on the level of $\mathrm{Ham}(M,\omega_M)$ as

$$c(\alpha,\varphi) = c(\alpha,H) \text{ for any } H \text{ such that } \varphi = \varphi_H.$$

In general, these three properties of spectral invariants can be used to study the relationship between the topology of $\mathrm{Ham}(M,\omega_M)$ and the action spectrum all under the setting of Floer theory. A direct application is a lower bound for d_H proved by Schwarz [**Sch00**]. By spectrality, we have $\min \mathrm{Spec}(\varphi) \leq |c(\alpha,H)|$ for any H such that $\varphi_H^1 = \varphi$. Moreover, continuity implies that $|c(\alpha,H)| \leq \|H\|$. Picking a sequence of Hamiltonians with Hofer norm converging to $d_H(\varphi,\mathrm{id})$ and using homotopy invariance yields the following result.

> **Theorem 1.4 • [Sch00]**
>
> Let (M,ω_M) be a closed symplectic manifold with $\pi_2(M) = 0$. Then, for every $\varphi \in \mathrm{Ham}(M,\omega_M)$
>
> $$\min \mathrm{Spec}(\varphi) \leq d_H(\mathrm{id},\varphi)$$

Using Theorem 1.4 Ostrover constructed a path $\{\varphi_t\}_{t\in\mathbb{R}_{\geq 0}} \subset \mathrm{Ham}(M,\omega_M)$ with infinite Hofer norm while its graph in $(M \times M, -\omega_M \oplus \omega_M)$ stays at a fixed Hofer distance $\rho(\mathrm{graph}(\mathrm{id}), \mathrm{graph}(\varphi_t))$ (see [**Che00**]) from the graph of the identity.

Theorem 1.5 • [Ost03]

Suppose M is closed and $\pi_2(M) = 0$. Then there exist a family $\{\varphi_t\}_{t\in\mathbb{R}_{\geq 0}} \subset \mathrm{Ham}(M,\omega_M)$ and a constant c such that:

- $d_{\mathrm{H}}(\mathrm{id}, \varphi_t) \to +\infty$ as $t \to +\infty$,
- $\rho(\mathrm{graph}(\mathrm{id}), \mathrm{graph}(\varphi_t)) = c$ for all $t \in [0, +\infty)$.

The second metric of interest announced at the beginning of this section section can be constructed from spectral invariants taken with respect to the unit 1 in $\mathrm{H}^\bullet(M)$. The *spectral norm* of $\varphi \in \mathrm{Ham}(M,\omega_M)$ is defined [FS07, Section 7] as

$$\gamma(\varphi) = c(1,H) + c(1,\overline{H})$$

where H and $\overline{H}$ generate φ and φ^{-1} respectively. The *spectral metric* d_γ is then given by

$$d_\gamma(\varphi,\psi) = \gamma(\varphi \circ \psi^{-1}).$$

A key property, in the spirit of Theorem 1.4, for the spectral norm is the fact that it bounds the Hofer norm from below. The boundedness (or unboundedness) of these norms have far reaching implications on the topology of $\mathrm{Ham}_c(M,\omega_M)$. For instance Kawamoto [Kaw22b] proved that, on rational symplectic manifolds ($\langle\omega_M, \pi_2(M)\rangle = c_0\mathbb{Z}$ for some constant $c_0 > 0$), when the values of spectral norms are bounded by a number strictly smaller than the rationality constant c_0, then γ is C^0-continuous on $\mathrm{Ham}(M,\omega_M)$. Alternatively, the unboundedness of the spectral norm has applications to the study of open balls in $\mathrm{Ham}(M,\omega_M)$. It can be used to study the following question posed by Le Roux:

Question 1.6 • [LR10]

For any $A > 0$, let

$$E_A(M,\omega) := \{\varphi \in \mathrm{Ham}(M,\omega) \mid d_H(\mathrm{Id},\varphi) > A\}$$

be the complement of the closed ball of radius A in Hofer's metric. For all $A > 0$, does $E_A(M,\omega)$ have non-empty C^0-interior?

In the case of closed symplectically aspherical manifolds with infinite spectral diameter, a positive answer to Question 1.6 was given by Buhovsky, Humilière and Seyfaddini (see also [**Kaw22a**, **Kaw22b**] for the positive and negative monotone cases).

Theorem 1.7 • [BHS21]

Let (M,ω) be a closed, connected and symplectically aspherical manifold. If the spectral norm is unbounded, then $E_A(M,\omega)$ has non-empty C^0-interior for all $A > 0$.

In view of these results, it is therefore natural to study the finiteness of the *spectral diameter*

$$\mathrm{diam}_\gamma(M) = \sup\left\{\gamma(\varphi) \mid \varphi \in \mathrm{Ham}_c(M)\right\}.$$

1.1.5. Finiteness of the spectral diameter

It has been known for some time now [**EP03**] that for $(\mathbb{C}P^n,\omega_{\mathrm{FS}})$,

$$\mathrm{diam}_\gamma(\mathbb{C}P^n) \le \int_{\mathbb{C}P^1} \omega_{\mathrm{FS}}.$$

The above upper bound was latter optimized by Kislev and Shelukhin in [**KS21**, Theorem G] to

$$\mathrm{diam}_\gamma(\mathbb{C}P^n) = \frac{n}{n+1}\int_{\mathbb{C}P^1} \omega_{\mathrm{FS}}.$$

However, for a surfaces of strictly positive genus, the spectral diameter is infinite.

Proposition 1.8

Let Σ^g be a surface of genus g. Then,

$$g \ge 1 \implies \mathrm{diam}_\gamma(\Sigma^g) = +\infty.$$

There exists multiple proofs of this property which stem from much more general results. In particular, Proposition 1.8 is covered by the following theorem of Kislev and Shelukhin [KS21, Theorem D] which is a sharpening of a result of Usher [Ush13, Theorem 1.1].

> **Theorem 1.9**
>
> Let (M,ω) be a closed symplectic manifold that admits an autonomous Hamiltonian $H \in C^\infty(M,\mathbb{R})$ such that
>
> **[U1]** all the contractible periodic orbits of X_H are constant.
>
> Then $\mathrm{diam}_\gamma(M) = +\infty$.

Theorem 1.9 allows one to prove that the spectral diameter is infinite in many more cases. A list of examples in which condition **[U1]** holds can be found in [Ush13, Section 1]. As mentioned above, surfaces of positive genus satisfy **[U1]**. Also, if (N,ω_N) satisfies **[U1]** then so does $(M \times N,\omega_M \oplus \omega_N)$ for any other closed symplectic manifold (M,ω_M). In [Kaw22b], Kawamoto proves that the spectral diameter of the quadrics Q^2 and Q^4 (of real dimension 4 and 8 respectively) and certain stabilizations of them is infinite.

1.2. Main results

Let us now give a brief overview of the main results of this Thesis. They concern the links that exist between the topology of a Liouville domain, in particular its symplectic cohomology, and the metric properties of its group of Hamiltonian diffeomorphisms. Application are found to the spectral diameter of symplectically aspherical manifolds.

1.2.1. Boundedness of γ on Liouville domains

The main result of this thesis gives a characterization of the finiteness of $\mathrm{diam}_\gamma(D)$ for a Liouville domain $(D,\mathrm{d}\lambda)$ in terms of its symplectic cohomology.

> **Theorem A**
>
> Let $(D,\mathrm{d}\lambda)$ be a Liouville domain. Then,
>
> $$\mathrm{diam}_\gamma(D) = +\infty \quad \Longleftrightarrow \quad \mathrm{SH}^\bullet(D) \neq 0.$$

It was already shown by Benedetti-Kang [BK22], that the spectral diameter of a Liouville domain is finite whenever its symplectic cohomology vanishes. All that was missing to prove Theorem A was to show that if $\mathrm{SH}^\bullet(D) \neq 0$ then $\mathrm{diam}_\gamma(D) = +\infty$. To prove this we work on the level of the spectral invariants from which γ is build. In particular, we compute, for a suitable class of Hamiltonians, spectral invariants that can be made arbitrarily large.

Let us delve a little deeper into the proof of Theorem A to see how the non-vanishing of $\mathrm{SH}^\bullet(D)$ comes into play. We first consider, following an approach of Cieliebak-Frauenfelder-Oancea [CFO10] used to compute filtered symplectic cohomology, the family of Hamiltonians $H_{\delta,A}$, illustrated in Figure 1, where $\delta \in (0,1)$ and $A > 0$ is not a period of a Reeb orbit on ∂D.

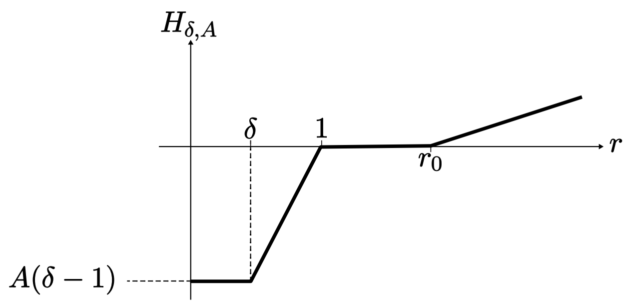


Fig. 1. The special admissible Hamiltonian $H_{\delta,A}$ used in the proof of Theorem A. The constant $r_0 > 0$ is fixed.

We distinguish 4 types of 1-periodic orbits for $H_{\delta,A}$:

I. critical points in $\hat{D} \setminus [\delta,\infty) \times \partial D$;

II. non-constant 1-periodic orbits near $\{\delta\} \times \partial D$;

III. non-constant 1-periodic orbits near $\{1\} \times \partial D$;

IV. critical points in $D^{r_0} \setminus D$;

The trajectories of orbits of type II and III correspond to trajectories of Reeb orbits on ∂D. This classification of orbits is motived by the fact that, their image under the symplectic action functional, form disjoint intervals.

Indeed, we can choose $\delta \in (0,1)$ and $\varepsilon > 0$ small enough so that, in terms of action, we have the series of inequalities

$$IV < \mathrm{III} < A - \varepsilon < \mathrm{I} < \mathrm{II}.$$

In the case where $\mathrm{SH}^\bullet(D) \neq 0$, the natural map $\Psi : \mathrm{HF}^\bullet(H_{\delta,A}) \to \mathrm{SH}^\bullet(D)$, given to us by the direct limit construction of symplectic cohomology, carries the unit $1_{H_{\delta,A}} = \Phi_H(1)$ in $\mathrm{HF}^\bullet(H_{\delta,A})$ to a unit 1_D in $\mathrm{SH}^\bullet(D)$. We then construct a map $\Psi_{\mathrm{I,II}}$ from the cohomology $H^\bullet(\mathrm{C}^*_{\mathrm{I,II}})$ of the quotient complex generated by orbits of type I and II. Such that Diagram (1.2.1) commutes.

$$\begin{tikzcd}
\mathrm{HF}^\bullet(H_{\delta,A}) \arrow[r, "{[\pi_{>A-\varepsilon}]}"] \arrow[dr, "\Psi"'] & \mathrm{H}^\bullet(\mathrm{C}^*_{\mathrm{I,II}}) \arrow[d, "\Psi_{\mathrm{I,II}}"] \\
& \mathrm{SH}^\bullet(D)
\end{tikzcd} \tag{1.2.1}$$

Therefore, since $\Psi(1_{H_{\delta,A}}) = 1_D \in \mathrm{SH}^\bullet(D)$, $[\pi_{>A-\varepsilon}](1_{H_{\delta,A}}) \neq 0$ and the definition of spectral invariant yields the inequality

$$c(1, H_{\delta,A}) \geq A - \varepsilon$$

as desired. Notice that we still need to deal with the term $c(1,\overline{H_{\delta,A}})$ in $\gamma(H_{\delta,A})$. This is done by adapting a result of Ganor-Tanny [**GT23**] to the setting of Floer cohomology on Liouville domains.

Lemma B

Let H be a compactly supported Hamiltonian on a Liouville domain $(D,\mathrm{d}\lambda)$. Then,

$$c(1, H) \geq 0.$$

Theorem A thus follows from the computation of $c(1,H_{\delta,A})$ and the application of Lemma B to $c(1,H_{\delta,A})$.

1.2.2. An isometric embedding result

From a continuity argument involving the symplectic contraction principle of Polterovich [**Pol14**], we can *pull* $H_{\delta,A}$ towards D. This allows us, when the symplectic cohomology of D does not vanish, to exactly compute the spectral invariant of negative compactly supported Hamiltonians which are constant on the skeleton $\mathrm{Sk}(D)$ of D. This special subset can be seen as the result of pulling ∂D towards the interior of D with the flow of the Liouville vector field Y.

> **Lemma C**
>
> Suppose (D,λ) is a Liouville domain such that $\mathrm{SH}^\bullet(D) \neq 0$. Let H be a compactly supported autonomous Hamiltonian on D such that
>
> $$H\Big|_{\mathrm{Sk}(D)} = -A \quad \text{and} \quad -A \leq H\Big|_D \leq 0$$
>
> for a constant $A > 0$. Then
>
> $$c(1,H) = A.$$

Lemma C allows us to construct an explicit isometric group embedding of $\mathbb{R}$ equipped with the standard Euclidian metric d_{std} in $(\mathrm{Ham}_c(D),d_\gamma)$.

> **Theorem D**
>
> Let (D,λ) be a Liouville domain such that $\mathrm{SH}^\bullet(D) \neq 0$. Consider a compactly supported Hamiltonian H on D such that
>
> $$H\Big|_{\mathrm{Sk}(D)} = -1 \quad \text{and} \quad -1 \leq H\Big|_D \leq 0$$
>
> Then, the map $\iota : (\mathbb{R}, d_{\mathrm{std}}) \to (\mathrm{Ham}_c(D),d_\gamma)$ defined by
>
> $$\iota(s) = \varphi^1_{sH}$$
>
> is an isometric group embedding.

Notice that Theorem D is a refinement of Theorem A. The proof of the former can be seen as an extension of the proof of the latter.

1.2.3. Symplectically aspherical manifolds

We can use Theorem A to compute the spectral diameter of certain symplectically aspherical manifolds. This approach relies on finding a certain type of embedded Liouville D with $\mathrm{SH}^\bullet(D) \neq 0$ inside a symplectically aspherical M. Namely, we need the map $\pi_1(D) \to \pi_1(M)$ induced by the inclusion of D inside M to be injective. These kind of submanifolds are called *incompressible*.

Proposition E

Let (M,ω) be a closed symplectically aspherical manifold of dimension $2n$. Suppose there exists an incompressible Liouville domain D of codimension 0 embedded inside M with $SH^\bullet(D) \neq 0$. Then, $\mathrm{diam}_\gamma(M) = +\infty$.

The proof of Proposition E is an almost direct application of Theorem A.

The first ingredient is a translation of a result of Ganor-Tanny [**GT23**] to cohomology. If M is a symplectically aspherical manifold M which contains an embedded incompressible Liouville of codimension 0, we can express spectral invariants of compactly supported Hamiltonian in D in terms of spectral invariants on M as a whole. Namely, we get

$$c_D(\beta,H) = \max_{\substack{\alpha\in \mathrm{H}^\bullet(M) \\ \iota^\bullet(\alpha)=\beta}} c_M(\alpha,H).$$

For this formula to hold, incompressibility of D is indispensable. The result still holds even if $\mathrm{SH}^\bullet(D) = 0$.

The second ingredient is an implicit definition of spectral invariants taken with respect to the unit. For every compactly supported Hamiltonian H in M, $c(1,H)$ can be expressed, using properties of spectral invariants of cup products in cohomology, as the maximum over all spectral invariants of H with respect to any cohomology class. Using this computation in conjunction with the equality of the previous step, we get

$$c_D(1_D,H) = c_M(1_M,H).$$

In particular $\gamma_D(H) = \gamma_M(H)$.

The third and final ingredient simply consists in using the non-vanishing of $\mathrm{SH}^\bullet(D)$ and Theorem A on $\gamma_D(H)$.

1.2.4. Products of symplectically aspherical manifolds

In general, not a lot of examples of symplectically aspherical manifolds satisfying the conditions of Proposition E are known. The simplest example we can give of such a manifold is the symplectic product $(M',\omega') = (M \times M,\omega_M \oplus -\omega_M)$ of a symplectically aspherical manifold M with itself. By a classical result of Weinstein, the diagonal $\Delta \subset M'$ admits a tubular neighbourhood W symplectomorphic to the cotangent unit disk bundle $\mathrm{D}^*\Delta$. Thus, W is a codimension 0 Liouville submanifoild of M'. Also, W is incompressible in M' since Δ is a deformation retract of W and the diagonal itself is incompressible in M'. From the work of Monzner-Vichery-Zapolsky [**MVZ12**], we know that D^*N has infinite spectral diameter for closed N. Combining this with Proposition E we obtain the following corollary.

Corollary F

Let (M,ω) be a closed symplectically aspherical manifold. Then,

$$\mathrm{diam}_\gamma(\mathrm{Ham}(M \times M, \omega_M \oplus -\omega_M)) = +\infty.$$

This gives a brand new, and quite general, example of symplectically aspherical manifold with infinite spectral diameter.

1.2.5. Hofer geometry

Using Theorem 1.7 in conjunction with Corollary F, we directly obtain the following answer to Question 1 of Section 1.1.4 in the specific setting of Corollary F.

Corollary G

Let (M,ω) be a closed symplectically aspherical manifold. Then, $E_A(M \times M, \omega \oplus -\omega)$ has a non-empty C^0-interior for all $A > 0$.

1.3. Organisation of the thesis

In Chapter 2 we recall the definition of Liouville domains. We then discuss the classes Hamiltonians and almost complex structures, called admissible, that we will consider on these manifolds. The chapter ends with a *no-escape* Lemma which guarantees that the Floer trajectories for admissible pairs (H,J) do not escape at infinity.

Chapter 3 contains a review of Floer cohomology of admissible Hamiltonians. We define the τ-extension of time-dependent Hamiltonians $\mathcal{C}(D)$ which are supported inside a Liouville domain D. This allows us to define the Floer cohomology of any $H \in \mathcal{C}(D)$ by proving that the resulting cohomology group is independent of the chosen extension. We define symplectic cohomology using the class of admissible Hamiltonians defined in the previous chapter.

Chapter 4 has two main objectives. First, we adapt a method of Ritter [**Rit13**] to Hamiltonians in $\mathcal{C}(D)$ in order to build a filtered product between the Floer cohomologies of such Hamiltonians. Second, we explain, following the work of Ritter, how the operations on Floer cohomology induce a unital ring strcuture on symplectic cohomology.

In Chapter 5 we construct spectral invariants for Hamiltonians compactly supported in Liouville domains. In particular, we prove the triangle inequality property using the product from the previous chapter. This allows us to define the spectral norm.

Chapter 6 contains an adaptation of the Ganor-Tanny barricade construction [**GT23**] to the Floer cohomology of certain admissible Hamiltonians on Liouville domains.

Finally, in Chapter 7 we give complete proofs of the main results.

Chapter 2

LIOUVILLE DOMAINS

In this chapter we recall the definition of Liouville domains, specify the class of Hamiltonians we will restrict our attention to and describe how their Floer trajectories behave at infinity.

2.1. Completion of Liouville domains

A Liouville domain $(D,\mathrm{d}\lambda,Y)$ is an exact symplectic manifold with boundary on which the vector field Y, defined by $Y \lrcorner\, \mathrm{d}\lambda = \lambda$ and called the Liouville vector field, points outwards along ∂D. Denote by $\hat{D} = D \cup [1,\infty) \times \partial D$ the completion of D and (r,x) the coordinates on $[1,\infty) \times \partial D$. Here, we glue ∂D and $\{1\} \times \partial D$ with respect to the reparametrization $\psi_Y^{\ln r}$ of the Liouville flow generated by Y. Given $\delta > 0$, let

$$D^{\delta} = \psi_Y^{\ln \delta}(D) = \hat{D} \setminus (\delta,\infty) \times \partial D.$$

We extend the Liouville form λ to $\hat{D}$ by defining $\hat{\lambda} : T\hat{D} \to \mathbb{R}$ as

$$\hat{\lambda}\,|_D = \lambda \quad \text{and} \quad \hat{\lambda}\,|_{\hat{D}\setminus D} = r\alpha$$

where $\alpha = \lambda|_{\partial D}$. The cylindrical portion $[1,\infty) \times \partial D$ of $\hat{D}$ is thus equipped with the symplectic form $\omega = \mathrm{d}(r\alpha)$.

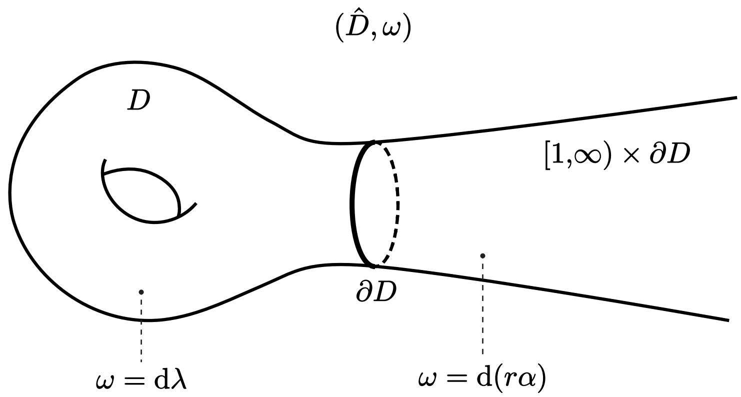


Fig. 1. A Liouville domain with its completion.

The skeleton $\mathrm{Sk}(D)$ of $(D,\mathrm{d}\lambda,Y)$ is defined by

$$\mathrm{Sk}(D) = \bigcap_{0<r<1} \psi_Y^{\ln r}(D).$$

Denote by R_α the Reeb vector field on ∂D associated to α, meaning

$$R_\alpha \lrcorner \, \mathrm{d}\alpha = 0, \quad \alpha(R) = 1.$$

We define $\mathrm{Spec}(\partial D,\alpha)$ to be the set of periods of closed characteristics, the periodic orbits generated by R_α, on ∂D and put

$$T_0 = \min \mathrm{Spec}(\partial D,\lambda).$$

As a subset of $\mathbb{R}$, $\mathrm{Spec}(\partial D,\alpha)$ is known to be closed and nowhere dense. For any $A \in \mathbb{R}$, let η_A denote the distance between A and $\mathrm{Spec}(\partial D,\lambda)$.

Cotangent disk bundles. A class of Liouville domains which will be of interest here are unit cotangent disk bundles over closed manifolds. Let (N,g) be a closed manifold of dimension n equipped with a Riemannian metric g. The unit cotangent disk bundle D^*N over N is defined as

$$\mathrm{D}^*N = \left\{(q,p) \in N \times T_q^*N \mid \|p\|_g < 1\right\} \subset \mathrm{T}^*N.$$

We endow D^*N with the standard symplectic structure ω_{std} which it inherits from T^*N. Notice that $\omega_{\mathrm{std}} = \mathrm{d}\lambda_{\mathrm{std}}$ for $\lambda_{\mathrm{std}} = \sum_{i=1}^n p_i \mathrm{d}q_i$ Consider the vector field Y defined by

$$Y = \sum_{i=1}^n p_i \partial_{p_i}.$$

Then, we compute

$$Y \lrcorner\, \omega_{\mathrm{std}} = \sum_{i=1}^n \mathrm{d}p_i(Y) \cdot \mathrm{d}q_i = \sum_{i=1}^n p_i \mathrm{d}q_i = \lambda_{\mathrm{std}}.$$

Therefore, Y is a Liouville vector field. This implies, by definition, that D^*N is a Liouville domain.

2.2. Admissible Hamiltonians and almost complex structures

2.2.1. Periodic orbits and action functional

Given a Hamiltonian $H : S^1 \times \hat{D} \to \mathbb{R}$, one defines its time-dependent Hamiltonian vector field $X_H^t : \hat{D} \to T\hat{D}$ by

$$X_H^t \lrcorner\, \omega = -dH_t$$

where $H_t(p) = H(t,p)$. We denote by $\varphi_H^t : \hat{D} \to \hat{D}$ the flow generated by X_H^t. The set of all contractible 1-periodic orbits of φ_H^t is denoted by $\mathcal{P}(H)$. An orbit $x \in \mathcal{P}(H)$ is said to be *non-degenerate* if

$$\det\Big(\mathrm{id} - d_{x(0)}\varphi_H^1\Big) \neq 0$$

and *transversally non-degenerate* if the eigenspace associated to the eigenvalue 1 of the map $d_{x(0)}\varphi^1$ is of dimension 1.

Let $\mathcal{L}\hat{D}$ be the space of contractible loops in $\hat{D}$. For a Hamiltonian $H : S^1 \times \hat{D} \to \mathbb{R}$, the *Hamiltonian action functional* $\mathcal{A}_H : \mathcal{L}\hat{D} \to \mathbb{R}$ associated to H is defined as

$$\mathcal{A}_H(x) = \int_0^1 x^*\hat{\lambda} - \int_0^1 H_t(x(t))\, \mathrm{d}t.$$

It is well known that the elements of $\mathcal{P}(H)$ correspond to the critical points of $\mathcal{A}_H$, see [**AD14**, section 6]. The image of $\mathcal{P}(H)$ under the Hamiltonian action functional is called the *action spectrum of* H and is denoted by $\mathrm{Spec}(H)$. For an open set $U \subset \hat{D}$ we define

$$\mathcal{P}_U(H) = \{x \in \mathcal{P}(H) \mid \operatorname{im} x \subset U\}\,.$$

2.2.2. Admissible Hamiltonians

The completion of a Liouville domain is obviously non-compact. We thus need to control the behavior at infinity of Hamiltonians we use in order for them to have finitely many 1-periodic contractible orbits.

Definition 2.1 • Admissible Hamiltonians

Let $r_0 > 1$. A Hamiltonian H is r_0-*admissible* if $\exists \rho_0 \in (0,r_0)$ such that

- $H(t,x,r) = h(r)$ on $\hat{D} \setminus D^{\rho_0}$,
- $h(r) = \tau_H r + \eta_H$ on $(r_0, +\infty)$ for $\tau_H \in (0,\infty) \setminus \mathrm{Spec}(\partial D, \alpha)$,
- H is regular: every element of $\mathcal{P}_{D^\rho}(H)$ is non-degenerate and every element of $\mathcal{P}_{\hat{D}\setminus D^\rho}(H)$ is transversally non-degenerate.

We denote the set of such Hamiltonians $\mathcal{H}_{r_0}$. The function h is called the *profile function* of H. We will also consider the set $\mathcal{H}^0_{r_0} \subset \mathcal{H}_{r_0}$ of r_0-admissible Hamiltonians which are negative on D. In some cases, it is not necessary to specify r_0 as long as it is greater than 1. For that purpose, we define

$$\mathcal{H} = \bigcup_{r_0>1} \mathcal{H}_{r_0}, \quad \mathcal{H}^0 = \bigcup_{r_0>1} \mathcal{H}^0_{r_0}.$$

Remark 2.2: Suppose $H \in \mathcal{H}$. If $x \in \mathcal{P}_{\hat{D}\setminus D^{\rho_0}}(H)$ is non constant, then it is necessarily transversally non-degenerate. Indeed, since H is time-independent there by definition, for any $c \in \mathbb{R}$, $x(t-c)$ is also a 1-periodic orbit of H.

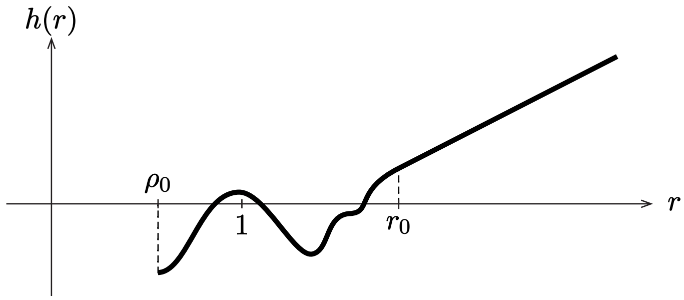


Fig. 2. the profile function h of an r_0-admissible Hamiltonian H.

Lemma 2.3

If $H \in \mathcal{H}$, then $|\mathcal{P}_{D^{\rho_0}}(H)|$ is finite and $\mathcal{P}_{\hat{D}\setminus D^{\rho_0}}(H)$ consists of a finite number S^1 families of periodic orbits.

Proof. Since $\overline{D^{\rho_0}}$ is compact and elements of $\mathcal{P}_{D^{\rho_0}}(H)$ are non-degenerate, there is a finite number of 1-periodic orbits of H inside it.

Next, we look at the elements of $\mathcal{P}_{\hat{D}\setminus D^{\rho_0}}(H)$. On $\hat{D} \setminus D^{\rho_0}$, we know that $H = h(r)$ and $\omega = \mathrm{d}\hat{\lambda}$. Therefore, on $\hat{D} \setminus D^{\rho_0}$

$$
\begin{aligned}
X_H \lrcorner\, \omega &= X_H \lrcorner\, (\mathrm{d}r \wedge \alpha + r\mathrm{d}\alpha) \\
&= \mathrm{d}r(X_H)\alpha - \alpha(X_H)\mathrm{d}r + rX_H \lrcorner\, \mathrm{d}\alpha
\end{aligned}
$$

and $\mathrm{d}H = h'(r)\mathrm{d}r$. Hamilton's equation thus yields

$$
\mathrm{d}r(X_H) = 0 = X_H \lrcorner\, \mathrm{d}\alpha, \quad \alpha(X_H) = h'(r).
$$

The three equations above imply the following two facts,

- on $\hat{D} \setminus D^{\rho_0}$, $X_H = h'(r)R_\alpha$;
- if $x \in \mathcal{P}(H)$ is such that $x \cap \hat{D} \setminus D^{\rho_0} \neq \varnothing$, then $\operatorname{im} x \subset \{r\} \times \partial D$ for some $r > \rho_0$.

We conclude that a 1-periodic orbit x of H which lies inside $\{r\} \times \partial D$ corresponds to a Reeb orbit of period $h'(r)$. Notice that since $\tau_H \notin (0, +\infty) \cap \operatorname{Spec}(\partial D, \alpha)$, $\mathcal{P}_{\hat{D}\setminus D^{\rho_0}}(H) = \mathcal{P}_{D^{r_0}\setminus D^{\rho_0}}(H)$. Therefore, since $\overline{D^{r_0} \setminus D^{\rho_0}}$ is compact and every element of $\mathcal{P}_{D^{r_0}\setminus D^{\rho_0}}(H)$ is transversally non-degenerate by definition, $\mathcal{P}_{D^{r_0}\setminus D^{\rho_0}}(H)$ is finite. $\square$

Remark 2.4: The fact that admissible Hamiltonians are radial on the cylindrical part of $\hat{D}$ allows us to express the action of the 1-periodic orbits inside $\hat{D}\setminus D$ in terms of that radial function. To see this, we fix $H \in \mathcal{H}$ and compute the action of a non constant orbit $x \in \mathcal{P}(H) \cap (\hat{D}\setminus D)$ which we suppose lies inside $\{r\} \times \partial D$ for $r > 1$:

$$\begin{aligned}\mathcal{A}_H(x) &= \int_0^1 x^*\hat{\lambda} - \int_0^1 H \circ x \,\mathrm{d}t \\ &= \int_0^1 r\alpha(X_H)\mathrm{d}t - \int_0^1 h(r)\mathrm{d}t = rh'(r) - h(r).\end{aligned}$$

The function $A_H(r) = rh'(r) - h(r)$ on the right hand side of the above equation has a nice geometric interpretation. Looking at the graph of h , we notice that $A_H(r')$ corresponds to minus the y-coordinate of the intersection of the tangent at the point $(r',h(r'))$ and the y-axis.

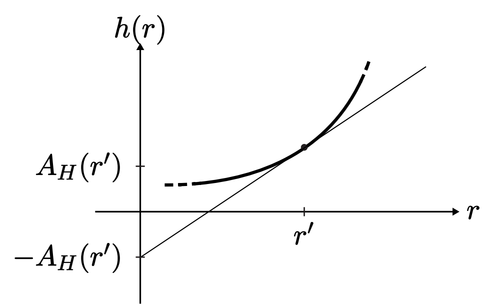


Fig. 3. Action value of a periodic orbit contained in $\{r'\} \times \partial D$.

2.2.3. Monotone homotopies

We will need to also restrict the types of Hamiltonian homotopies we consider to the following class.

Definition 2.5

Let $H_s = \{H_s\}_{s\in\mathbb{R}}$ be a smooth homotopy from $H_+ \in \mathcal{H}_{r_0}$ to $H_- \in \mathcal{H}_{r'_0}$ We say that H_s is a *monotone homotopy* if the following conditions hold

- $\exists S > 0$ such that $H_{s'} = H_-$ for $s' < -S$ and $H_{s'} = H_+$ for $s' > S$,
- $H_s = h_s(r)$ on $\hat{D} \setminus D^\rho$ for $\rho = \max\{\rho_0,\rho'_0\}$,
- for $R = \max\{r_0,r'_0\}$, $h_s(r) = \tau_s r + \eta_s$ on $(R,+\infty)$ for smooth functions τ_s, η_s of s,
- $\partial_s H_s(t,p) \leq 0$ for $(t,p,s) \in S^1 \times \hat{D} \times \mathbb{R}$

For $H_+ \in \mathcal{H}_{r_0}$ and $H_- \in \mathcal{H}_{r'_0}$ such that $H_+ \leq H_-$ pointwise everywhere on $\hat{D}$, we can explicitly construct a monotone homotopy in the following way. Fix a positive constant $S > 0$. Let $b : \mathbb{R} \to [0,1]$ be a smooth function such that $b(s) = 0$ for $s \leq -S$, $b(s) = 1$ for $s \geq S$ and $b'(s) > 0$ for all $s \in (-S,S)$. Define

$$H_s = H_- + b(s)(H_+ - H_-).$$

Notice that, since $b'(s) \geq 0$ and $H_+ \leq H_-$, we have

$$\partial_s H_s = b'(s)(H_+ - H_-) \leq 0$$

For $R = \max\{r_0,r'_0\}$ we have, on $\hat{D} \setminus D^R$,

$$H_s(t,r,p) = (b(s)(\tau_+ - \tau_-) + \tau_-)r + b(s)(\eta_+ - \eta_-) + \eta_- = h_s(r)$$

as desired and

$$\partial_s \partial_r h_s(r) = b'(s)(\tau_+ - \tau_-) \leq 0. \tag{2.2.1}$$

This inequality will be needed for the maximum principle of Section 2.3.2.

2.2.4. Admissible almost complex structures

Let J be an almost complex structure on $\hat{D}$. Recall that J is ω-compatible if the map $g_J : TM \otimes TM \to \mathbb{R}$ defined by

$$g_J(v,w) = \omega(v,Jw)$$

is a Riemannian metric. To control the behavior of ω-compatible almost complex structures at infinity, we make the following definition.

Definition 2.6

Let J be an ω-compatible almost complex structure on $\hat{D}$. We say that J is *admissible* if $J_1 = J|_{\hat{D}\setminus D}$ is of *contact type*. Namely, we ask that

$$J_1^* \hat{\lambda} = \mathrm{d}r.$$

We denote the set of such almost complex structures by $\mathcal{J}$. A pair (H,J) where $H \in \mathcal{H}_{r_0}$ and $J \in \mathcal{J}$ is called an r_0-*admissible pair.*

2.3. Floer trajectories and maximum principle.

In this subsection, we recall some analytical aspects of Floer theory on Liouville domains. Issues regarding transversality will be dealt with in the next section.

2.3.1. Floer trajectories

Consider a Hamiltonian $H : S^1 \times \hat{D} \to \mathbb{R}$ and two 1-periodic orbits $x_\pm \in \mathcal{P}(H)$. Let J be an ω-compatible almost complex structure on $\hat{D}$. A *Floer trajectory* between x_- and x_+ is a solution $u : \mathbb{R} \times S^1 \to \hat{D}$ to the *Floer equation*

$$\partial_s u + J(\partial_t u - X_H) = 0$$

that converges uniformly in t to x_- and x_+ as $s \to \pm\infty$:

$$\lim_{s \to \pm\infty} u(s,t) = x_\pm(t).$$

We denote the moduli space of such trajectories $\mathcal{M}'(x_-,x_+;H)$. We may reparametrize a solution $u \in \mathcal{M}'(x_-,x_+;H)$ in the $\mathbb{R}$-coordinate by adding a constant. Thus, Floer trajectories occur in $\mathbb{R}$-families. The space of unparametrized solutions is denoted by $\mathcal{M}(x_-,x_+;H) = \mathcal{M}'(x_-,x_+;H)/\mathbb{R}$. When the context is clear, we will drop H from the notation and simply write $\mathcal{M}(x_-,x_+)$.

If we replace H with a monotone homotopy $H_\bullet = \{H_s\}_{s\in\mathbb{R}}$, we can instead consider solutions $u : \mathbb{R} \times S^1 \to \hat{D}$ to the *s-dependent Floer equation*

$$\partial_s u + J(\partial_t u - X_{H_s}) = 0 \tag{2.3.1}$$

that converge uniformly in t to $x_\pm \in \mathcal{P}(H_\pm)$ as $s \to \pm\infty$. The moduli space of such trajectories is denoted by $\mathcal{M}(x_-,x_+;H_\bullet)$. Unlike the s-independent case, $\mathcal{M}(x_-,x_+;H_\bullet)$ does not admit a free $\mathbb{R}$-action by which we can quotient.

2.3.2. Maximum principle

To define Floer cohomology of $\hat{D}$, we need to control the behavior of the Floer trajectories. In particular, we have to make sure they do not escape to infinity. Admissible Hamiltonians and admissible complex structures allow us to achieve that requirement. The first result in that direction is the maximum principle for Floer trajectories. In what follows we say that v is a local Floer solution of (H,J) in $\hat{D} \setminus D$ if

$$v = u\Big|_{u^{-1}(\operatorname{im} u \cap \hat{D}\setminus D)} : u^{-1}(\operatorname{im} u \cap \hat{D} \setminus D) \longrightarrow \hat{D} \setminus D$$

for some $u \in \mathcal{M}(x_-,x_+;H)$.

Lemma 2.7 • Generalized maximum principle [Vit99]

Let (H,J) be an r_0-admissible pair on $\hat{D}$. Suppose v is a local Floer solution of (H,J) in $\hat{D} \setminus D^{r_0}$. Then, the r-coordinate $r \circ v$ of v does not admit an interior maximum unless $r \circ v$ is constant.

Remark 2.8: The generalized maximum principle still holds if we replace $H \in \mathcal{H}$ by a monotone homotopy H_s between $H_+ \in \mathcal{H}_{r_0}$ and $H_- \in \mathcal{H}_{r_0'}$ and if v is a local solution of the s-dependent Floer equation

$$\partial_s v + J(\partial_t v - X_{H_s}) = 0$$

inside $\hat{D} \setminus D^{r_0}$. Here it is crucial that $\partial_s \partial_r h_s(r) \leq 0$ for large enough r. This always holds for monotone homotopies (see Equation (2.2.1)).

From the maximum principle above, we immediately obtain the following corollary which guarantees that Floer trajectories do not escape to infinity.

Corollary 2.9 • No escape

Let (H,J) be an r_0-admissible pair on $\hat{D}$ and let $x_\pm \in \mathcal{P}(H)$. If $u \in \mathcal{M}(x_-,x_+)$, then

$$\operatorname{im} u \subset D^R, \quad \text{for } R = \max\{r \circ x_-, r \circ x_+, r_0\}.$$

If H_s is a monotone homotopy between $H_- \in \mathcal{H}_{r_0}$ and $H_+ \in \mathcal{H}_{r_0'}$ and u is a solution to the s-dependent Floer equation between $x_- \in \mathcal{P}(H_-)$ and $x_+ \in \mathcal{P}(H_+)$, then

$$\operatorname{im} u \subset D^R, \quad \text{for } R = \max\{r \circ x_-, r \circ x_+, r_0, r_0'\}.$$

2.3.3. Energy

An important quantity which is associated to a Floer trajectory is its *energy*. It is defined as

$$E(u) = \frac{1}{2} \int_{\mathbb{R} \times S^1} \left(|\partial_s u|_J^2 + |\partial_t u - X_H|_J^2 \right) \mathrm{d}s \wedge \mathrm{d}t$$

where $|\cdot|_J$ is the norm corresponding to g_J. Using the Floer equation, we can write

$$|\partial_t u - X_H|_J^2 = \omega(J\partial_s u, -\partial_s u) = \omega(\partial_s u, J\partial_s u) = |\partial_s u|_J^2.$$

Thus, the energy can be written more compactly as

$$E(u) = \int_{\mathbb{R}\times S^1} |\partial_s u|_J^2 \, \mathrm{d}s \wedge \mathrm{d}t.$$

It is often useful to estimate the difference in Hamiltonian action on the ends of a Floer trajectory in terms of the energy of that trajectory. This can be achieved using the maximum principle and Stokes Theorem.

Lemma 2.10 • Energy estimate for Floer trajectories

Let (H,J) be an r_0-admissible pair and let $u \in \mathcal{M}'(x_-,x_+;H)$ for $x_\pm \in \mathcal{P}(H)$. Then,

$$0 \leq E(u) = \mathcal{A}_H(x_+) - \mathcal{A}_H(x_-).$$

If H_s is a monotone homotopy between $H_+ \in \mathcal{H}_{r_0}$ and $H_- \in \mathcal{H}_{r_0'}$ that is constant in the s-coordinate for $s > |S|$ then

$$0 \leq E(u) \leq \mathcal{A}_{H_+}(x_+) - \mathcal{A}_{H_-}(x_-) + \sup_{\substack{s\in[-S,S],\\ t\in S^1, p\in D^\delta}} \partial_s H_s(t,p)$$

where $\delta = \max\{r \circ x_-, r \circ x_+, r_0, r_0'\}$.

Chapter 3

FLOER COHOMOLOGY

We present in this subsection a brief overview of Floer cohomology for completions of Liouville domains and their symplectic cohomology. For more details we refer the reader to [**CFH95**], [**CFHW96**], [**Vit99**], [**Web06**], [**CFO10**] and [**Rit13**].

3.1. Filtered Floer Cohomology

3.1.1. The Floer cochain complex

Let (H,J) be an admissible pair. As mentioned in Remark 2.2, the 1-periodic orbits of H on $\hat{D}\backslash D^{\rho_0}$ come in a finite number of S^1-families which we denote by $\hat{x}_i$. To break each $\hat{x}_i$ in a finite number of isolated periodic orbits, we first choose an open neighborhoods U_i of each $\hat{x}_i$ such that $U_i \cap U_j = \varnothing$ for $i \neq j$. Then, we define on each $\hat{x}_i$ a Morse function f_i having exactly two critical points : one of index 0 and another of index 1. We extend each f_i to its corresponding U_i. When added to H, these perturbations, which can be chosen as small as we want, break each of the S^1-families into two critical points. In virtue of the action formula derived in Remark 2.4, the actions of the new critical points are as close as we want to the action of their original S^1-family. We denote by H_1 the Hamiltonian resulting from this procedure. By abuse of notation we will write $\mathcal{P}(H)$ for the set of 1-periodic orbits of H_1.

We define the *Floer cochain group of H* as the $\mathbb{Z}_2$-vector space [1]

$$\mathrm{CF}^\bullet(H) = \bigoplus_{x \in \mathcal{P}(H)} \mathbb{Z}_2 \langle x \rangle .$$

As the notation above suggests, $\mathrm{CF}^\bullet(H)$ is in fact a *graded* $\mathbb{Z}_2$-vector space. Assuming that the first Chern class $c_1(\omega) \in \mathrm{H}^2(\hat{D};\mathbb{Z})$ of $(T\hat{D},J)$ vanishes on $\pi_2(\hat{D})$, the Conley-Zehnder index $\mathrm{CZ}(x) \in \mathbb{Z}$ of a 1-periodic orbit $x \in \mathcal{P}(H)$ is well defined [**SZ92**]. We can therefore equip $\mathrm{CF}^\bullet(H)$ with the degree

$$|x| = \frac{\dim \hat{D}}{2} - \mathrm{CZ}(x)$$

and define

$$\mathrm{CF}^k(H) = \bigoplus_{\substack{x \in \mathcal{P}(H) \\ |x|=k}} \mathbb{Z}_2 \langle x \rangle .$$

Here, CZ is normalized such that for a C^2-small time-independent admissible Hamiltonian F,

$$\mathrm{CZ}(x) = \frac{\dim \hat{D}}{2} - \mathrm{ind}(x)$$

where $\mathrm{ind}(x)$ corresponds to the Morse index of $x \in \mathrm{Crit}(F) = \mathcal{P}(F)$. In particular, if x is a local minimum of F, then $|x| = 0$. This convention therefore ensures that the cohomological unit has degree zero.

For a generic perturbation of J, the space $\mathcal{M}(x_-,x_+;H)$ is a smooth manifold of dimension

$$\dim \mathcal{M}(x_-,x_+;H) = \mathrm{CZ}(x_+) - \mathrm{CZ}(x_-) - 1.$$

In the case where $|x_-| = |x_+| + 1$, Corollary 2.9 and Lemma 2.10 allow us to use the standard compactness arguments, as in [**AD14**, Chapter 8] to show that $\mathcal{M}(x_-,x_+;H)$ is a compact manifold of dimension 0. Knowing that, we

[1] We use $\mathbb{Z}_2$ coefficients here for simplicity but the cohomological construction that follows can be carried out with any coefficient ring.

define the co-boundary operator $\partial : \mathrm{CF}^k(H) \to \mathrm{CF}^{k+1}(H)$ by

$$\partial x_+ = \sum_{|x_-|=k+1} \#_2\mathcal{M}(x_-,x_+;H)x_-$$

where $\#_2\mathcal{M}(x_-,x_+;H)$ is the count modulo 2 of components in $\mathcal{M}(x_-,x_+,H)$.

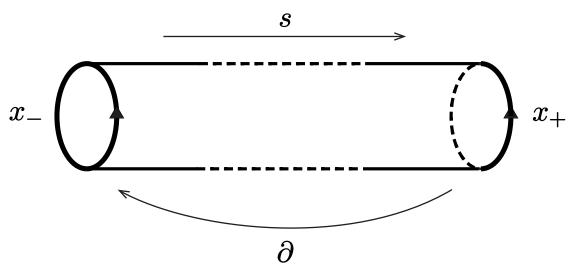


Fig. 1. The differential in Floer cohomology goes from right to left.

Using once again Corollary Corollary 2.9, $\partial \circ \partial = 0$ holds by standard arguments which appear in [**AD14**, Chapter 9]. The pair $(\mathrm{CF}^\bullet(H),\partial)$ is thus a graded cochain complex that we call the *Floer cochain complex of H*.

3.1.2. Filtered Floer cochain complex

The Hamiltonian action functional induces a filtration on the Floer cochain complex. For $a \in (\mathbb{R} \cup \{\pm\infty\}) \setminus \mathrm{Spec}(H)$, we define

$$\mathrm{CF}^k_{<a}(H) = \bigoplus_{\substack{x\in\mathcal{P}(H) \\ |x|=k,\ \mathcal{A}_H(x)<a}} \mathbb{Z}_2 \langle x \rangle .$$

By definition, we have $\mathrm{CF}^\bullet(H) = \mathrm{CF}^\bullet_{<+\infty}(H)$. Lemma 2.10 ensures that ∂ decreases the action. Thus, the restriction $\partial_{<a} : \mathrm{CF}^k_{<a}(H) \to \mathrm{CF}^{k+1}_{<a}(H)$ of the co-boundary operator is well defined and $(\mathrm{CF}^\bullet_{<a}(H),\partial_{<a})$ is a sub-complex of $(\mathrm{CF}^\bullet(H),\partial)$. Now, for $a,b \in (\mathbb{R} \cup \{\pm\infty\}) \setminus \mathrm{Spec}(H)$ such that $a < b$, we can define the Floer cochain complex in the action window (a,b)

as the quotient

$$\mathrm{CF}^\bullet_{(a,b)}(H) = \frac{\mathrm{CF}^\bullet_{<b}(H)}{\mathrm{CF}^\bullet_{<a}(H)},$$

on which we denote the projection of the co-boundary operator by

$$\partial_{(a,b)} : \mathrm{CF}^k_{(a,b)}(H) \longrightarrow \mathrm{CF}^{k+1}_{(a,b)}(H).$$

Therefore, for $a,b,c \in (\mathbb{R} \cup \{\pm\infty\}) \setminus \mathrm{Spec}(H)$ such that $a < b < c$, we have an inclusion and a projection

$$\iota_a^{b,c} : \mathrm{CF}^\bullet_{(a,b)}(H) \longrightarrow \mathrm{CF}^\bullet_{(a,c)}(H), \quad \pi^c_{a,b} : \mathrm{CF}^\bullet_{(a,c)}(H) \longrightarrow \mathrm{CF}^\bullet_{(b,c)}(H)$$

that produce the short exact sequence

$$0 \longrightarrow \mathrm{CF}^\bullet_{(a,b)}(H) \xrightarrow{\iota_a^{b,c}} \mathrm{CF}^\bullet_{(a,c)}(H) \xrightarrow{\pi^c_{a,b}} \mathrm{CF}^\bullet_{(b,c)}(H) \longrightarrow 0.$$

For simplicity, we define $\iota^{<c} = \iota_{-\infty}^{+\infty,c}$ and $\pi_{>b} = \pi^{+\infty}_{-\infty,b}$.

3.1.3. Filtered Floer cohomology

Let $a,b \in (\mathbb{R}\cup\{\pm\infty\})\setminus\mathrm{Spec}(H)$ such that $a < b$. The above filtered cochain complexes allow us to define the *Floer cohomology group of H* in the action window (a,b) *as*

$$\mathrm{HF}^\bullet_{(a,b)}(H) = \frac{\ker \partial_{(a,b)}}{\mathrm{im}\, \partial_{(a,b)}}.$$

The full Floer cohomology group of H is defined as $\mathrm{HF}^\bullet(H) = \mathrm{HF}^\bullet_{(-\infty,+\infty)}(H)$. For $a,b,c \in (\mathbb{R} \cup \{\pm\infty\}) \setminus \mathrm{Spec}(H)$ such that $a < b < c$, the short exact sequence on the cochain level induces a long exact sequence in cohomology:

$$\begin{tikzcd}
 & \mathrm{HF}^\bullet_{(a,c)}(H) \arrow[dr, "{[\pi^c_{a,b}]}"] & \\
\mathrm{HF}^\bullet_{(a,b)}(H) \arrow[ur, "{[\iota_a^{b,c}]}"] & & \mathrm{HF}^\bullet_{(b,c)}(H) \arrow[ll, "{[+1]}"]
\end{tikzcd} \tag{3.1.1}$$

For C^2-small admissible Hamiltonians on D with small slope at infinity, the Floer cohomology recovers the standard cohomology of D.

Lemma 3.1 • [Rit13, Section 15.2]

Let $H \in \mathcal{H}$ be C^2-small on D with $\tau_H < T_0$ for $T_0 = \min \operatorname{Spec}(\partial D, \lambda)$. Then, we have an isomorphism

$$\Phi_H : \mathrm{H}^\bullet(D) \longrightarrow \mathrm{HF}^\bullet(H)$$

Proof. Since H is C^2-small on D, all the element of $\mathcal{P}(H)$ which sit in D are critical points. The condition $\tau_H < T_0$ assures us that H does not admit any 1-periodic orbits outside D. Multiplying H by a small enough constant and perturbing J so that H is Morse-Smale for the metric g_J, every Floer trajectory for (H,J) can be made t-independent. Floer's equation then becomes

$$\partial_s u = JX_H = -\nabla H$$

and thus, the Floer differential for (H,J) counts negative g_J-gradient trajectories between critical points of H. From the grading convention introduced in Section 3.1.1, the index of $x \in \mathcal{P}(H)$ is equal to its Morse index. We thus obtain an identification between $\mathrm{H}^\bullet(D)$ and $\mathrm{HF}^\bullet(H)$. □

3.1.4. Continuation maps

Let $K \in \mathcal{H}_{r_0}$ and $F \in \mathcal{H}_{r_0'}$ such that $F \leq K$. Consider a monotone homotopy $H_\bullet$ from F to K. Then from Corollary 2.9 and Lemma 2.10 in the case of homotopies, we can apply the techniques shown in [**AD14**, Chapter 11] to show that, for $x_- \in \mathcal{P}(K)$ and $x_+ \in \mathcal{P}(F)$ with $|x_-| = |x_+|$, $\mathcal{M}(x_-,x_+;H_\bullet)$ is a smooth compact manifold of dimension zero. The *continuation map* $\Phi^{H_\bullet} : \mathrm{CF}^k(F) \to \mathrm{CF}^k(K)$ *induced by* H_s on the cochain level is defined as

$$\Phi^{H_\bullet}(x_+) = \sum_{|x_-|=k} \#_2\mathcal{M}(x_-,x_+;H_\bullet)x_-$$

where $\#_2\mathcal{M}(x_-,x_+;H_\bullet)$ counts modulo 2 the number of components in $\mathcal{M}(x_-,x_+;H_\bullet)$. The map

$$[\Phi^{H_\bullet}] : \mathrm{HF}^\bullet(F) \longrightarrow \mathrm{HF}^\bullet(K)$$

is independent of the chosen monotone homotopy and we can denote it by $[\Phi^{K,F}]$. Consider the monotone homotopy

$$H_s = K + b(s)(F-K)$$

described in Section 2.2.3. We note that $\partial_s H_s \leq 0$ since $F \leq K$ and $b' \geq 0$. Thus the action estimate given by Lemma 2.10 for homotopies yields

$$\mathcal{A}_K(x_-) \leq \mathcal{A}_H(x_+) + \sup_{\substack{s\in[-S,S],\\ t\in S^1,p\in D^R}} \partial_s H_s(t,p) \leq \mathcal{A}_H(x_+)$$

for $x_- \in \mathcal{P}(K)$ and $x_+ \in \mathcal{P}(F)$. Therefore, the continuation map decreases the action and hence induces maps

$$[\Phi^{K,F}_{(a,b)}] : \mathrm{HF}^\bullet_{(a,b)}(F) \longrightarrow \mathrm{HF}^\bullet_{(a,b)}(K)$$

that commute with the inclusion and restriction maps as follows [**Rit13**, Section 8]:

$$\begin{tikzcd}
\mathrm{HF}^*_{(a,b)}(F) \arrow[r, "{[\iota^{b,c}_a]}"] \arrow[d, "{[\Phi^{K,F}_{(a,b)}]}"] & \mathrm{HF}^*_{(a,c)}(F) \arrow[r, "{[\pi^c_{a,b}]}"] \arrow[d, "{[\Phi^{K,F}_{(a,c)}]}"] & \mathrm{HF}^*_{(b,c)}(F) \arrow[d, "{[\Phi^{K,F}_{(b,c)}]}"] \arrow[ll, bend right, "{[+1]}"'] \\
\mathrm{HF}^*_{(a,b)}(K) \arrow[r, "{[\iota^{b,c}_a]}"] & \mathrm{HF}^*_{(a,c)}(K) \arrow[r, "{[\pi^c_{a,b}]}"] & \mathrm{HF}^*_{(b,c)}(K) \arrow[ll, bend left, "{[+1]}"]
\end{tikzcd} \tag{3.1.2}$$

Suppose we are given another Hamiltonian $H \geq K$, then we have the commutative diagram

$$\begin{tikzcd} \mathrm{HF}^*_{(a,b)}(F) \arrow[r, "{[\Phi^{K,F}_{(a,b)}]}"] \arrow[rr, bend right, "{[\Phi^{H,F}_{a,b}]}"'] & \mathrm{HF}^*_{(a,b)}(K) \arrow[r, "{[\Phi^{H,K}_{(a,b)}]}"] & \mathrm{HF}^*_{(a,b)}(H) \end{tikzcd}$$

As opposed to the closed case, for completion of Liouville domains, continuation maps do not necessarily yield isomorphisms. One case in which they do is when both Hamiltonians have the same slope.

Lemma 3.2 • [Rit09, Section 2.12]

Let $F,K \in \mathcal{H}$ and suppose τ_F and τ_K are both contained in an open interval that does not intersect $\mathrm{Spec}(\partial D,\alpha)$. Then, if $\tau_F \leq \tau_K$

$$[\Phi^{K,F}] : \mathrm{HF}^\bullet(F) \longrightarrow \mathrm{HF}^\bullet(K)$$

is an isomorphism.

In action windows, we have the following isomorphisms.

Lemma 3.3 • [Vit99, Proposition 1.1]

Let $H_\bullet$ be a monotone homotopy between $H_\pm \in \mathcal{H}$ that is constant in the s-coordinate for $|s| > S > 0$. Suppose $a_s,b_s : \mathbb{R} \to \mathbb{R}$ are functions which are constant outside $[-S,S]$ and $a_s,b_s \notin \mathrm{Spec}(H_s)$ for all s. Then,

$$[\Phi^{H_-,H_+}] : \mathrm{HF}^\bullet_{(a_+,b_+)}(H_+) \xrightarrow{\cong} \mathrm{HF}^\bullet_{(a_-,b_-)}(H_-)$$

for $a_\pm = \lim_{s\to\pm\infty} a_s$ and $b_\pm = \lim_{s\to\pm\infty} b_s$.

3.1.5. Compactly supported Hamiltonians

We can define the Floer cohomology of compactly supported Hamiltonians on Liouville domains by first extending to affine functions on the cylindrical portion of $\hat{D}$. This will allow us to define, in Chapter 5, the spectral invariants and spectral norm of such Hamiltonians.

For $0 < \varepsilon < 1$ and $r_0 \geq 1$, define the Hamiltonian $K_{\varepsilon,r_0,\tau}$ as follows

- $K_{\varepsilon,r_0,\tau}$ is the constant zero function on D^{r_0},
- $K_{\varepsilon,r_0,\tau} = k_{\varepsilon,r_0,\tau}(r)$ on $\hat{D} \setminus D^{r_0-\varepsilon}$,
- $k_{\varepsilon,r_0,\tau}(r)$ is convex for $r \in [r_0 - \varepsilon, r_0]$ with $k^{(\ell)}_{\varepsilon,r_0,\tau}(1) = 0$ for all $\ell \geq 0$, $k'_{\varepsilon,r_0,\tau}(1+\varepsilon) = \tau$ and $k^{(\ell)}_{\varepsilon,r_0,\tau}(1+\varepsilon) = 0$ for all $\ell > 1$,
- $k_{\varepsilon,r_0,\tau}(r) = \tau(r - (r_0 - \varepsilon/2))$ for $r \in [r_0, +\infty)$.

This allows us to define the τ-extension.

Definition 3.4 • Extension of a compactly supported Hamiltonian

Denote by $\mathcal{C}(D)$ the set of Hamiltonians with support in $S^1 \times (D \setminus \partial D)$. Let $H \in \mathcal{C}(D)$. Choose ε small enough so that the support of H is contained inside $S^1 \times D^{r_0-\varepsilon}$. We define the τ-extension $H^\tau \in \mathcal{H}_1$ of H as

$$H^\tau = H + K_{\varepsilon,r_0,\tau}$$

We perturb H^τ so that it is r_0-admissible. The Floer cohomology of H is defined as

$$\mathrm{HF}^\bullet_{(a,b)}(H) = \mathrm{HF}^\bullet_{(a,b)}(H^\tau)$$

where $0 < \tau < T_0$.

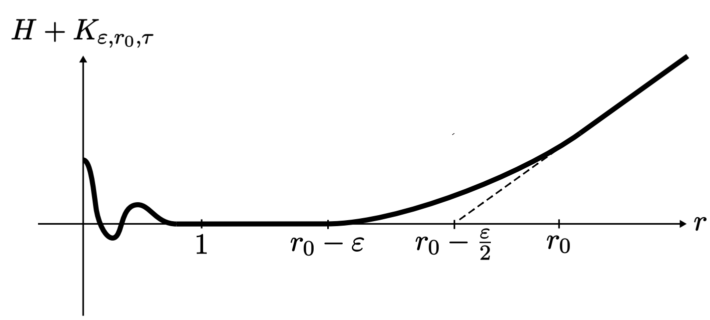


Fig. 2. The τ-extension of a compactly supported Hamiltonian.

The following Lemma assures us that the filtered Floer cohomology groups of a compactly supported Hamiltonian does not depend on the chosen τ-extension. This guarantees that $\mathrm{HF}^{\bullet}_{(a,b)}(H)$ from Definition 3.4 is well defined.

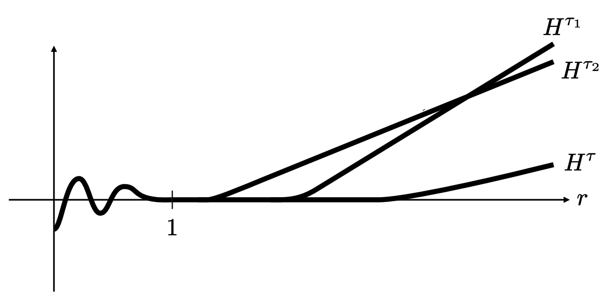
Lemma 3.5 • Independence of extension

Let $H \in \mathcal{C}(D)$ be a compactly supported Hamiltonian. Consider two extensions H^{τ_1} and H^{τ_2} of H. Then,

$$\mathrm{HF}^{\bullet}_{(a,b)}(H^{\tau_1}) \cong \mathrm{HF}^{\bullet}_{(a,b)}(H^{\tau_2})$$

Proof. The profile function of two extensions H^{τ_1} and H^{τ_2} of H might intersect at some point. However, we can always find a third extension H^{τ} with $\tau < \tau_1, \tau_2$ such that

$$H^{\tau} \leq H^{\tau_1}, H^{\tau_2}.$$



Thus, there exists monotone homotopies $F^1_{\bullet}$ from H^{τ} to H^{τ_1} and $F^2_{\bullet}$ from H^{τ} to H^{τ_2}. Note that, by definition of the extension, τ,τ_1,$\tau_2 < T_0$. Moreover, the action windows stay fixed and their limits do not cross the spectrum of any of the Hamiltonians under consideration. Lemma 3.3 then implies that we have two isomorphisms

$$\mathrm{HF}^{\bullet}_{(a,b)}(H^{\tau_1}) \xleftarrow{[\Phi^{H^{\tau_1},H^{\tau}}]} \mathrm{HF}^{\bullet}_{(a,b)}(H^{\tau}) \xrightarrow{[\Phi^{H^{\tau_2},H^{\tau}}]} \mathrm{HF}^{\bullet}_{(a,b)}(H^{\tau_2}).$$

Taking $[\Phi^{H^{\tau_1},H^{\tau}}]^{-1} \circ [\Phi^{H^{\tau_2},H^{\tau}}]$ gives us the desired isomorphism. □

3.1.6. Time reparametrization of Hamiltonians

Let G be a compactly supported Hamiltonian and G^{τ} a τ-extensions of G. The following procedure is adapted from [**Pol14**, Section 1.4], [**MVZ12**, Remark 2.5] and [**Oh15**, Section 21.4]. It allows one to replace any compactly supported Hamiltonian on D with one that vanishes for t close to 0 and 1.

Definition 3.6 • Smoothing of Hamiltonians

Let $\sigma : [0,1] \longrightarrow [0,1]$ be a smooth function with $\sigma'(t) \geq 0$ such that $\sigma(s) = 0$ for s near 0 and $\sigma(s) = 1$ for s near 1. The *σ-smoothing of* $G \in \mathcal{C}(D)$ is defined as

$$\widetilde{G}(t,p) = \sigma'(t)G(\sigma(t),p).$$

The τ-extension of the smoothed Hamiltonian $\widetilde{G}$ is denoted by $\widetilde{G}^{\tau}$.

Remark 3.7: It is important to note here that the smoothing is always applied to G before the τ-extension. Otherwise, we would end up with an Hamiltonian which depends on t at infinity and is therefore not admissible.

Since the slope of the τ-extension is less than the minimal Reeb period T_0, $\widetilde{G}^{\tau}$ only admits 1-periodic orbits inside D. A direct computation shows that, inside D, the Hamiltonian flow of $\widetilde{G}^{\tau}$ is given by

$$\varphi^t_{\widetilde{G}} = \varphi^{\sigma(t)}_G$$

which just corresponds to the σ time reparametrization of the Hamiltonian flow of G. In particular, G and $\widetilde{G}$ both have the same time-1 map. Thus, since σ is monotone increasing, there is a bijection between the 1-periodic orbits $\mathcal{P}_D(\widetilde{G}^\tau)$ of $\widetilde{G}^\tau$ inside D and the 1-periodic orbits $\mathcal{P}(G)$ of G given by

$$\begin{aligned} \mathcal{P}_D(G) &\longrightarrow \mathcal{P}_D(\widetilde{G}^\tau) \\ x &\longmapsto x \circ \sigma \end{aligned}$$

This bijection preserves the action. Indeed, from the change of variable $t \longmapsto \sigma(t)$ we have, for $x \in \mathcal{P}(G)$,

$$\begin{aligned} \mathcal{A}_{\widetilde{G}^\tau}(x \circ \sigma) &= \int_0^1 (x \circ \sigma)^* \lambda - \int_0^1 \widetilde{G}^\tau(t, x \circ \sigma)\, \mathrm{d}t \\ &= \int_0^1 (x \circ \sigma(t))^* \lambda - \int_0^1 \sigma'(t) G(\sigma(t), x \circ \sigma(t))\, \mathrm{d}t \\ &= \int_0^1 x^* \lambda - \int_0^1 G(t, x(t))\, \mathrm{d}t = \mathcal{A}_G(x). \end{aligned} \tag{3.1.3}$$

From now on, we consider that every compactly supported Hamiltonian is smoothed. This assumption is useful to compute action estimates for the pair of pants product of Section 4.2.3.

Another reason why the *smoothed* assumption is useful concerns the composition of Hamiltonian flows. Given two smoothed $H, K \in \mathcal{C}(D)$, the *concatenation* $H \# K : [0,1] \times D \longrightarrow \mathbb{R}$ *of* H *and* K is defined by

$$H \# K(t,p) = \begin{cases} 2H(2t, p) & \text{for } t \in [0,1/2], \\ 2K(2t-1, p) & \text{for } t \in [1/2,1]. \end{cases} \tag{3.1.4}$$

The *inverse* of a smoothed Hamiltonian is defined by

$$\overline{H}(t,p) = -H(t-1,p). \tag{3.1.5}$$

The Hamiltonian isotopy generated by the inverse Hamiltonian $\overline{H}(t,p)$ is given by

$$\varphi_{\overline{H}}^t = \varphi_H^{1-t} \circ (\varphi_H^1)^{-1}.$$

As the notation suggest $H \# \overline{H}$ is equivalent to the zero function on D in the sense that there exists an Homotopy $F_\bullet$ from 0 to $H \# \overline{H}$ so that $\varphi_{F_s}^1 = \mathrm{id}_D$ for all s. In particular, $H \# \overline{H}$ generates the identity on D as its time-1 map.

3.2. Filtered Symplectic cohomology

Equip the set of admissible Hamiltonians $\mathcal{H}^0$ negative on D with the partial order

$$H \preceq K \iff H(t,p) \leq K(t,p) \quad \forall (t,p) \in S^1 \times \hat{D}.$$

Let $\{H_i\}_{i\in I} \subset \mathcal{H}^0$ be a cofinal sequence with respect to $\preceq$. We define the *symplectic cohomology of* D as the direct limit

$$\mathrm{SH}^\bullet_{(a,b)}(D) = \varinjlim_{H_i} \mathrm{HF}^\bullet_{(a,b)}(H_i)$$

taken with respect to the continuation maps

$$[\Phi^{H_j,H_i}_{(a,b)}] : \mathrm{HF}^\bullet_{(a,b)}(H_i) \longrightarrow \mathrm{HF}^\bullet_{(a,b)}(H_j)$$

for $i < j$. We denote $\mathrm{SH}^\bullet(D) = \mathrm{SH}^\bullet_{(-\infty,+\infty)}(D)$. The long exact sequence on Floer cohomology carries through the direct limit and we also have a long exact sequence on symplectic cohomology

$$\begin{tikzcd}[column sep=small] & \mathrm{SH}^*_{(a,c)}(D) \arrow[dr, "{[\iota^{b,c}_{a,a}]}"] & \\ \mathrm{SH}^*_{(a,b)}(D) \arrow[ur, "{[\iota^{b,c}_{a}]}"] & & \mathrm{SH}^*_{(b,c)}(D) \arrow[ll, "{[+1]}"] \end{tikzcd}$$

3.2.1. The Viterbo map

Let $F \in \mathcal{H}$ and consider $H \in \mathcal{H}^0$ with $\tau_H = \tau_F$. Then, by Lemma 3.2, we have $\mathrm{HF}^\bullet(F) \cong \mathrm{HF}^\bullet(H)$ and there exist, by definition of symplectic cohomology, a map

$$j_F : \mathrm{HF}^\bullet(F) \cong \mathrm{HF}^\bullet(H) \longrightarrow \mathrm{SH}^\bullet(D) \tag{3.2.1}$$

sending each element of $\mathrm{HF}^\bullet(H)$ to its equivalence class. Now, for $H \in \mathcal{H}^0$ with slope $\tau_H < T_0$ we can define, by Lemma 3.1, the map $v^\bullet : \mathrm{H}^\bullet(D) \to$

$\mathrm{SH}^\bullet(D)$ first introduced in [Vit99] by

$$\begin{tikzcd} \mathrm{H}^\bullet(D) \arrow[r, "\Phi_H"] \arrow[rr, bend right, "v^\bullet"'] & \mathrm{HF}^\bullet(H) \arrow[r, "j_H"] & \mathrm{SH}^\bullet(D) \end{tikzcd}$$

Chapter 4

OPERATIONS ON FLOER COHOMOLOGY

The Floer cohomology of an admissible Hamiltonian H is equipped with a product structure

$$[\Psi_{\mathrm{P}}^{H}] : \mathrm{HF}^{\bullet}(H) \otimes \mathrm{HF}^{\bullet}(H) \longrightarrow \mathrm{HF}^{\bullet}(2H)$$

called the *pair of pants product*, hence the P in the notation. On the chain level, this product is computed by counting punctured spheres with two positive ends assymptotic to two periodic orbits in $\mathcal{P}(H)$ and a negative end assymptotic to a periodic orbit in $\mathcal{P}(2H)$. Composing with continuation maps, which corresponds to gluing continuation cylinders to the ends of the pair of pants, we can extend $[\Psi_{\mathrm{P}}]$ to a product

$$[\Psi_{\mathrm{P}}^{(H,K)}] : \mathrm{HF}^{\bullet}(H) \otimes \mathrm{HF}^{\bullet}(K) \longrightarrow \mathrm{HF}^{\bullet}(H\#K)$$

for pairs (K,F) of compactly supported Hamiltonians. An action estimate for the periodic orbits in the output in terms of the action for the periodic orbits in the input will allow us to restrict the product $[\Psi_{P}^{F,K}]$ on action windows. This product is compatible with the cup product on singular cohomology. This can be viewed through the scope of Morse theory.

4.1. Operations for one Hamiltonian

The methods used to construct the product Ψ_{P}^{H} arise from a more general framework developed by Schwarz in the mid 90's for closed symplectically aspherical manifolds (M,ω). Indeed, in [Sch95], Schwarz constructs a functor Z which associates to any surface $\Sigma_{p,q}^{g}$ of genus g with p negative ends and q positive ends an operation on Floer cohomology.

We give a brief overview of the construction of Z. Negative and positive ends correspond to connected components Z_k^- and Z_ℓ^+ of the complement of a compact subset of $\Sigma_{p,q}^{g}$ which are parameterized by $(-\infty,0)\times S^1$ and $(0,+\infty)\times S^1$ respectively. On $\Sigma_{p,q}^{g}$ a complex structure j is chosen so that, on the ends, it agrees with the standard complex structure on $\mathbb{R}\times S^1$. To each negative end Z_k^-, associate a $Z_k^-\times M$ dependent regular pair (H_k^-, J_k^-) which is independent of s for large enough s and such that H_k^- vanishes for small enough s. To each positive end Z_ℓ^+, associate a regular pair (H_ℓ^+, J_ℓ^+) with the same properties. On $\Sigma_{p,q}^{g}\times M$ fix an almost complex structure J that agrees with the ones chosen on the ends. Then, given 1-periodic contractible orbits $x_k^-\in\mathcal{P}(H_k^-)$ and $x_\ell^+\in\mathcal{P}(H_\ell^+)$, one considers the space of maps $u:\Sigma_{p,q}^{g}\to M$ which are J-holomorphic away from the ends and restrict, on the ends, to maps $u_i^\pm: Z_i^\pm\to M$ satisfying the Floer equation

$$\partial_s u_i^\pm + J_i^\pm(s,t,u_i^\pm)(\partial_t u_i^\pm - X_{H_i^\pm}(s,t,u_i^\pm)) = 0$$

and the asymptotics

$$\lim_{s\to\pm\infty} u_i^\pm(s) = x_i^\pm.$$

Schwarz showed that these spaces of solutions are compact smooth manifolds and that counting the number of components of the zero dimensional ones yields an operation in Floer cohomology

$$Z(\Sigma_g^{p,q}) : \bigotimes_{\ell=1}^{q} \mathrm{HF}^\bullet(H_\ell^+) \longrightarrow \bigotimes_{k=1}^{p} \mathrm{HF}^\bullet(H_k^-).$$

This operation is independent of the choices we made for J, j, the cylindrical coordinates and the order in which we labelled the ends.

In his influential survey [**Sei08**, Section 8a], Seidel proposed an extension of the construction of Schwarz to symplectic cohomology. The product was explored further by McLean in [**McL09**, Section 10]. The full details of that extension were latter carried by Ritter in [**Rit13**]. We now delve into that more recent work highlighting the main differences and similarities with the case of closed manifolds along the way.

4.1.1. The setting

Start with a single admissible pair (H,J) where H is autonomous and C^2 small on D. The goal here is to associate to any surface $\Sigma^g_{p,q}$ an operation $[\Psi^H_{p,q,g}]$ between the Floer cohomologies of multiples of H

$$[\Psi^H_{p,q,g}] : \bigotimes_{\ell=1}^{q} \mathrm{HF}^\bullet(B_\ell H) \longrightarrow \bigotimes_{k=1}^{p} \mathrm{HF}^\bullet(A_k H).$$

and not arbitrary admissible Hamiltonians akin to the product $Z(\Sigma^g_{p,q})$ described above. Here is all the data we need to define $[\Psi^H_{p,q,g}]$ in addition to the pair (H,J).

- Let $(\Sigma^g_{p,q}, j)$ be a surface of genus g with q positive ends (inputs) and p negative ends (outputs). Denote by U^-_k cylindrical neighbourhoods of the negative ends and by U^+_ℓ cylindrical neighbourhoods of the positive ends. We choose these neighbourhoods so that they do not intersect pairwise. Fix parametrizations

  $$(-\infty,0) \times S^1 \xrightarrow{\varphi^-_k} U^-_k, \quad U^+_\ell \xleftarrow{\varphi^+_\ell} (0,+\infty) \times S^1$$

  Chose the complex structure j so that, near the punctures $j\partial_s = \partial_t$.
- Fix a set of *weights* $\{A_1, \ldots, A_p, B_1, \ldots, B_q\}$ with $A_k, B_\ell > 0$ so that

  $$\sum_{k=1}^{p} A_k \geq \sum_{\ell=1}^{q} B_\ell. \tag{4.1.1}$$

- Fix a 1-form $\beta \in \Omega^1(\Sigma^g_{p,q})$ such that $\mathrm{d}\beta \leq 0$ and it's restrictions to the negative and positive ends are respectively given by

$$(\varphi_k^-)^*\beta = A_k \mathrm{d}t, \quad (\varphi_\ell^+)^*\beta = B_\ell \mathrm{d}t. \tag{4.1.2}$$

for large enough s.

The triple $(\Sigma^g_{p,q},\beta,j)$ is called a *model surface*. We consider disjoint unions of model surfaces to be model surfaces themselves. Depending on the form β we choose, we need to make assumptions on the sign of H (see Section 4.1.4). The assumptions are as follows:

- If $\mathrm{d}\beta \leq 0$, we impose $H \geq 0$.
- If $\mathrm{d}\beta = 0$, there are no further restrictions on H.

The form β, which always exists by Lemma 4.1, is needed here because a global form $\mathrm{d}t$ does not exist on $\Sigma^g_{p,q}$ for $p,q > 1$. We will use β to interpolate between the Floer equations for the Hamiltonians $A_k H$ and $B_\ell H$ at the ends. Integrating the differential of β, which is negative, on $\Sigma^g_{p,q}$ using Stokes' Theorem yields

$$\int_{\Sigma^g_{p,q}} \mathrm{d}\beta = \sum_{\ell=1}^{q} \left(\int_{\{+\infty\}\times S^1} (\varphi_\ell^+)^*\beta \right) - \sum_{k=1}^{p} \left(\int_{\{-\infty\}\times S^1} (\varphi_k^-)^*\beta \right) = \sum_{\ell=1}^{q} B_\ell - \sum_{k=1}^{p} A_k.$$

Since $\mathrm{d}\beta \leq 0$, this motivates condition (4.1.1) imposed on the weights. In particular, it forces $\Sigma^g_{p,q}$ to have at least one negative end ($p > 0$).

> **Lemma 4.1 • Existence of β [Rit13]**
>
> Let $C = \sum_{k=1}^{p} A_k - \sum_{\ell=1}^{q} B_\ell$. There exists a 1-form $\beta \in \Omega^1(\Sigma^g_{p,q})$ satisfying Equation 4.1.2 such that
> - if $C = 0$, $\mathrm{d}\beta = 0$,
> - if $C > 0$, $\mathrm{d}\beta \leq 0$.

Proof. To make the proof readable, let $\Sigma = \Sigma^g_{p,q}$. First, we relate the De Rham cohomology of Σ with the one for its boundary. We have the exact

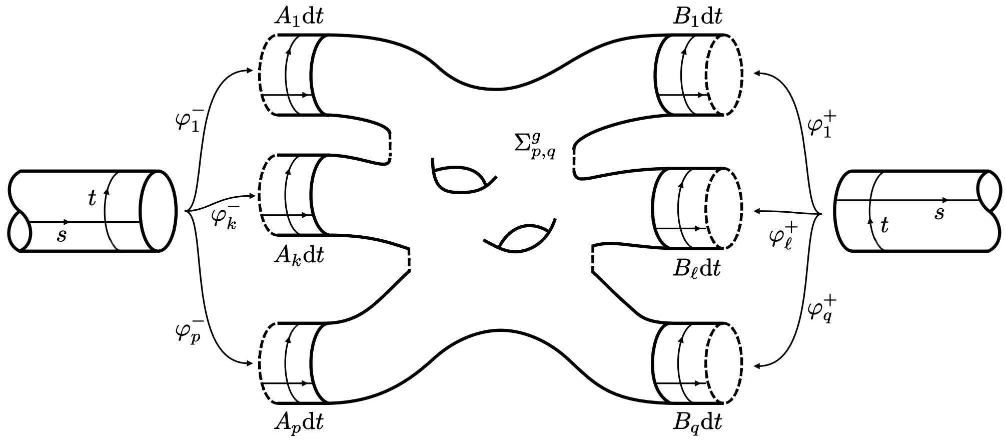


Fig. 1. The surface $\Sigma^g_{p,q}$ with its negative and positive ends parameterized by $(-\infty,0)\times S^1$ and $(0,+\infty)\times S^1$ respectively. Near each end, the value of β is written.

sequence in cohomology

$$H^1(\Sigma) \xrightarrow{\ f\ } H^1(\partial\Sigma) \xrightarrow{\ g\ } H^2(\Sigma,\partial\Sigma) \longrightarrow 0.$$

The map f corresponds to the pullback by the inclusion of $\partial\Sigma$ in Σ: $f[\lambda]=[\lambda|_{\partial\Sigma}]$. To construct g, take a representative of a class $[\eta]\in H^1(\partial\Sigma)$, extend it to a 1-form $\tilde\eta$ on Σ and consider the class of its differential which sits naturally in $H^2(\Sigma,\partial\Sigma)$: $g[\eta]=[\mathrm{d}\tilde\eta]$. Recall that $H^2(\Sigma,\partial\Sigma)$ is identified with $\mathbb{R}$ under integration:

$$\begin{aligned} H^2(\Sigma,\partial\Sigma) &\longrightarrow \mathbb{R} \\ [\nu] &\longmapsto \int_S \nu \end{aligned} \tag{4.1.3}$$

We construct β in the case where $\sum_{k=1}^p A_k=\sum_{\ell=1}^q B_\ell$. Let $[\eta]\in H^1(\partial\Sigma)$ be such that, on the negative ends

$$\left(\varphi_k^-|_{\{-\infty\}\times S^1}\right)^*\eta = A_k\mathrm{d}t$$

and on the positive ends

$$\left(\varphi_\ell^+|_{\{+\infty\}\times S^1}\right)^*\eta = B_\ell \mathrm{d}t.$$

It follows from the identification $H^2(\Sigma, \partial\Sigma) \cong \mathbb{R}$ and from Stokes' Theorem that

$$\begin{aligned} g[\eta] = \int_\Sigma \mathrm{d}\tilde{\eta} &= \sum_{\ell=1}^{q}\left(\int_{\{+\infty\}\times S^1}(\varphi_\ell^+)^*\eta\right) - \sum_{k=1}^{p}\left(\int_{\{-\infty\}\times S^1}(\varphi_k^-)^*\eta\right) \\ &= \sum_{\ell=1}^{q}\left(\int_{S^1} B_\ell \mathrm{d}t\right) - \sum_{k=1}^{p}\left(\int_{S^1} A_k \mathrm{d}t\right) = 0. \end{aligned}$$

By exactness of the sequence 4.1.3, there exists a closed $\beta \in \Omega^1(P)$ such that $[\beta|_{\partial S}] = [\eta]$. By construction of η, β has the desired behaviour on the positive and negative ends of Σ modulo an exact 1-form supported near the ends.

Now we construct β in the case where $\sum_{k=1}^{p} A_k \geq \sum_{\ell=1}^{q} B_\ell$. Define

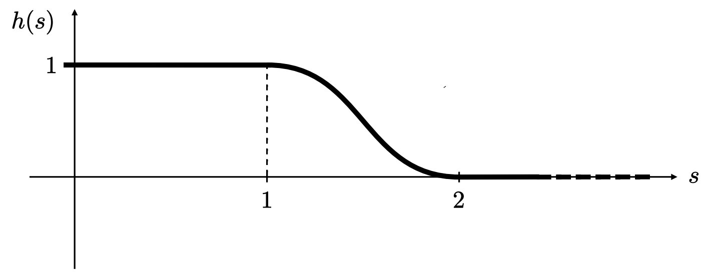
$$C = \sum_{k=1}^{p} A_k - \sum_{\ell=1}^{q} B_\ell.$$

From our hypothesis on the weights, $C > 0$. Consider the surface Σ^+ obtained from Σ by adding one additional positive end labelled $q+1$ with weight C. Applying the previous method, we can build an exact 1-form $\beta^+ \in \Omega^1(\Sigma^+)$ with

$$(\varphi_k^-)^*\beta^+ = A_k \mathrm{d}t, \quad (\varphi_\ell^+)^*\beta^+ = B_\ell \mathrm{d}t, \quad (\varphi_{q+1}^+)^*\beta^+ = C\mathrm{d}t$$

for large enough s. Extend β^+ to a one form β on Σ by progressively changing the value of β^+ on the $q+1$-th positive end to 0:

$$(\varphi^+_{q+1})^*\beta = h(s)(\varphi^+_{q+1})^*\beta^+ = h(s)C\mathrm{d}t.$$



Then, $\mathrm{d}\beta = 0$ except on $\varphi^+_{q+1}((1,2)\times S^1)$ where $h'(s)\neq 0$. There,

$$(\varphi^+_{q+1})^*\mathrm{d}\beta = h'(s)\mathrm{d}s\wedge \mathrm{d}t \leq 0.$$

This completes the construction of β. $\square$

4.1.2. The moduli spaces

Fix $p+q$ contractible 1-periodic orbits $x_k \in \mathcal{P}(A_k H)$, $y_\ell \in \mathcal{P}(B_\ell H)$ for all $k\in\{1,\dots,p\}$ and $\ell\in\{1,\dots,p\}$. Consider the moduli pace $\mathcal{M}(x_k, y_\ell; \Sigma^g_{p,q}, \beta)$ of curves $u:\Sigma^g_{p,q}\to \hat{D}$ satisfying the β-purturbed Floer's equation

$$(\mathrm{d}u - X_H\otimes\beta)^{0,1} = 0 \tag{4.1.4}$$

and which converge uniformely in t on the positive ends to *inputs* y_ℓ and on the negative ends to *outputs* x_ℓ:

$$\lim_{s\to+\infty} u\circ\varphi^+_\ell(s,t) = y_\ell(t), \quad \lim_{s\to-\infty} u\circ\varphi^-_k(s,t) = x_k(t). \tag{4.1.5}$$

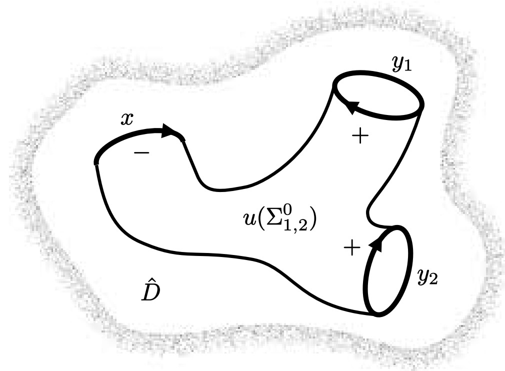


Fig. 2. The image of an element $u \in \mathcal{M}(x,y_1,y_2;\Sigma^0_{1,2},\beta)$ in $\hat{D}$. We indicate the image of both positive ends with $+$ and the image of the negative end with $-$.

When working with multiple Hamiltonians, we will indicate for which one the moduli space is built by writing $\mathcal{M}(x_k, y_\ell; \Sigma^g_{p,q}, \beta, H)$.

It is shown in [**Rit13**, Appendix A] that, after taking small generic $\Sigma^g_{p,q}$-dependent perturbation of J, $\mathcal{M}(x_k, y_\ell; \Sigma^g_{p,q}, \beta)$ is a smooth manifold of dimension

$$\dim \mathcal{M}(x_k, y_\ell; \Sigma^g_{p,q}, \beta) = \sum_{k=1}^{p} |x_k| - \sum_{\ell=1}^{q} |y_\ell| + 2n(1-g-p).$$

To show that $\mathcal{M}(x_k, y_\ell; \Sigma^g_{p,q}, \beta)$ admits a compactification, one can use the same methods employed in [**Sch95**] (see [**Rit13**, Appendix A]). The boundary of such a compactification consists of curves that break at only one end to a new 1-periodic orbit z for the Hamiltonian on that end. See Figure 3. As in the closed case, we need to show an a priori upper bound on the energy of any $u \in \mathcal{M}(x_k, y_\ell; \Sigma^g_{p,q}, \beta)$. This bound should depend on the action of the orbits at the ends of u. Moreover, since we are working on a Liouville domain and hence not a closed symplectic manifold, another technical detail needs to be addressed. The image of any curve u must stay within a compact set inside $\hat{D}$. We show that these two conditions hold in Section

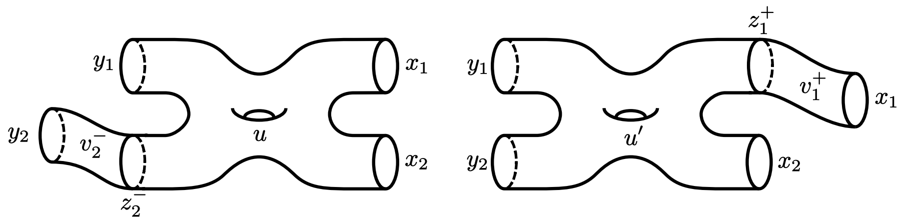


Fig. 3. Two curves in the boundary of the compactification of $\mathcal{M}(x_k, y_\ell; \Sigma^1_{2,2}, \beta)$. The curve on the left is given by $v_2^- \# u \in \mathcal{M}(y_2, z_2^-; A_2 H) \times \mathcal{M}(y_1, z_2^-, x_1, x_2; \Sigma^1_{2,2}, \beta)$ and the one on the right by $u' \# v_1^+ \in \mathcal{M}(y_1, y_2, z_1^+, x_2; \Sigma^1_{2,2}, \beta) \times \mathcal{M}(z_1^+, x_1; B_1 H)$

4.1.4. Note that, since $\omega = \mathrm{d}\lambda$ is exact, no bubbling can occur. Therefore, both conditions enumerated above are sufficient to prove compactification.

4.1.3. Defining the operations

Restricting ourselves to $p+q$ tuples of orbits $\{x_1, \ldots, x_p, y_1, \ldots, y_q\}$ satisfying

$$\sum_{k=1}^{p} |x_k| = \sum_{\ell=1}^{q} |y_\ell| - 2n(1-g-p),$$

$\mathcal{M}(x_k, y_\ell; \Sigma^g_{p,q}, \beta)$ will be of dimension 0 and, since it is compact, will consist of a finite number of points. Counting these points modulo 2 yields an operation on the chain level

$$\begin{aligned} \Psi^H_{p,q,g} : \left(\bigotimes_{\ell=1}^{q} \mathrm{CF}(B_\ell H) \right)^d &\longrightarrow \left(\bigotimes_{k=1}^{p} \mathrm{CF}(A_k H) \right)^{d-2n(1-g-p)} \\ \boldsymbol{y}_\ell &\longmapsto \sum_{|\boldsymbol{x}_k| = d-2n(1-g-p)} \#_2 \mathcal{M}(x_k, y_\ell; \Sigma^g_{p,q}, \beta) \boldsymbol{x}_k \end{aligned} \tag{4.1.6}$$

and extend it linearly. Here, $\boldsymbol{y}_\ell$ and $\boldsymbol{x}_k$ denote elementary tensors $y_1 \otimes \cdots \otimes y_q$ and $x_1 \otimes \cdots \otimes x_p$ respectively. The degree $|\cdot|$ of an elementary tensor is given by the sum of the degrees of its elements.

For $\Psi^H_{p,q,g}$ to descend on cohomology, it needs to be a chain map. Let us see what differentials we are wroking with here. Denote by ∂^+_ℓ the Floer differential on $\mathrm{CF}^\bullet(B_\ell H)$ and by ∂_k^- the differential on $\mathrm{CF}^\bullet(A_k H)$. The tensor products

$$\left(\bigotimes_{k=1}^{p} \mathrm{CF}(A_k H)\right)^d = \bigoplus_{d=d_1+\cdots+d_p} \left(\bigotimes_{k=1}^{p} \mathrm{CF}^{d_k}(A_k H)\right)$$
$$\left(\bigotimes_{\ell=1}^{q} \mathrm{CF}(B_\ell H)\right)^d = \bigoplus_{d=d_1+\cdots+d_p} \left(\bigotimes_{\ell=1}^{q} \mathrm{CF}^{d_\ell}(B_\ell H)\right)$$

are respectively equipped with the differentials ∂^- and ∂^+ defined by

$$\begin{aligned}\partial^+(\boldsymbol{y}_\ell) = \partial_1^+ y_1 \otimes y_2 \otimes \cdots \otimes y_q \\ + (-1)^{|y_1|}\, y_1 \otimes \partial_2^+ y_2 \otimes y_3 \otimes \cdots \otimes y_q + \cdots \\ \cdots + (-1)^{|y_1|+\cdots+|y_{q-1}|}\, y_1 \otimes \cdots \otimes y_{q-1} \otimes \partial_q^+ y_q \\ \partial^-(\boldsymbol{x}_k) = \partial_1^- x_1 \otimes x_2 \otimes \cdots \otimes x_p \\ + (-1)^{|x_1|}\, x_1 \otimes \partial_2^- x_2 \otimes x_3 \otimes \cdots \otimes x_p + \cdots \\ \cdots + (-1)^{|x_1|+\cdots+|x_{p-1}|}\, x_1 \otimes \cdots \otimes x_{p-1} \otimes \partial_p^- y_p.\end{aligned}$$

The sign corrections, derived from the Koszul sign convention, are needed to ensure $\partial^+ \circ \partial^+ = 0 = \partial^- \circ \partial^-$. With respect to these differentials, $\Psi^H_{\Sigma^g_{p,q}}$ is a chain map, i.e. the following diagram commutes

$$\begin{tikzcd} \left(\bigotimes_{k=1}^{p} \mathrm{CF}(A_k H)\right)^{d-2n(1-g-p)+1} & \left(\bigotimes_{\ell=1}^{q} \mathrm{CF}(B_\ell H)\right)^{d+1} \arrow[l, "\Psi^H_{p,q,g}"'] \\ \left(\bigotimes_{k=1}^{p} \mathrm{CF}(A_k H)\right)^{d-2n(1-g-p)} \arrow[u, "\partial^-"] & \left(\bigotimes_{\ell=1}^{q} \mathrm{CF}(B_\ell H)\right)^{d} \arrow[l, "\Psi^H_{p,q,g}"'] \arrow[u, "\partial^+"'] \end{tikzcd}$$

Since all chain complexes here are over $\mathbb{Z}_2$, a field, the Künneth formula yields the isomorphisms

$$\mathrm{H}^d\left(\bigotimes_{k=1}^{p}\mathrm{CF}(A_kH)\right)\cong\bigoplus_{d=d_1+\cdots+d_p}\left(\bigotimes_{k=1}^{p}\mathrm{HF}^{d_k}(A_kH)\right),$$
$$\mathrm{H}^d\left(\bigotimes_{\ell=1}^{q}\mathrm{CF}(B_\ell H)\right)\cong\bigoplus_{d=d_1+\cdots+d_q}\left(\bigotimes_{\ell=1}^{q}\mathrm{HF}^{d_\ell}(B_\ell H)\right).$$

Using these equivalences and the fact that $\Psi^H_{\Sigma^g_{p,q}}=:\Psi^H_{p,q,g}$ is a chain map, the latter descends to cohomology:

$$[\Psi^H_{p,q,g}]:\bigotimes_{\ell=1}^{q}\mathrm{HF}^{d_\ell}(B_\ell H)\longrightarrow\left(\bigotimes_{k=1}^{p}\mathrm{HF}(A_kH)\right)^{\sum_\ell d_\ell-2n(1-g-p)}$$

Moreover, $[\Psi^H_{p,q,g}]$ does not depend on the choice of data (β,j,J) relative to ends.

4.1.4. Energy estimate and No-escape Lemma

As mentioned in Section 4.1, to ensure compactness of the moduli spaces $\mathcal{M}(x_k,y_\ell;\Sigma^g_{p,q})$, we need to make sure that, just as in the closed case, the energy[1]

$$E(u)=\frac{1}{2}\int_\Sigma\|\mathrm{d}u-X_H\otimes\beta\|^2\,\mathrm{vol}_\Sigma$$

of curves u in these spaces of solutions have bounded energy. Another key technical detail which is particular to the case of Liouville domains is that such curves can't escape to infinity.

The fact that any curve $u\in\mathcal{M}(x_k,y_\ell;\Sigma^g_{p,q})$ stays within a compact subset of $\hat{D}$ is ensured by the integrated maximum principle of Abouzaid and Seidel in [**AS10**, Section 7]. Note that, when β is exact, we can achieve a better bound on the size of the compact set in which u is contained.

[1]Also called *geometric energy*.

Lemma 4.2 • Integrated maximum principle

Suppose the images of the orbits x_k and y_ℓ all lie inside D^δ for some $\rho_0 < \delta < r_0$. Let $u \in \mathcal{M}(x_k,y_\ell;\Sigma,\beta)$.

- If $\mathrm{d}\beta = 0$, then
$$\operatorname{im} u \subset D^\delta.$$
- If $\mathrm{d}\beta \leq 0$, suppose $H \geq 0$. Then,
$$\operatorname{im} u \subset D^{r_0}.$$

The a priori energy estimate on u also gives us an action estimate. That estimate relates the action of the orbits in the output of u to the action of the orbits in the input.

Lemma 4.3 • Energy estimates for operations

Let $u \in \mathcal{M}(x_k, y_\ell; \Sigma, \beta)$.

- If $\mathrm{d}\beta = 0$, then
$$E(u) = \sum_{\ell=1}^{q} \mathcal{A}_{B_\ell H}(y_\ell) - \sum_{k=1}^{p} \mathcal{A}_{A_k H}(x_k).$$
- If $\mathrm{d}\beta \leq 0$, suppose $H \geq 0$. Then,
$$E(u) \leq \sum_{\ell=1}^{q} \mathcal{A}_{B_\ell H}(y_\ell) - \sum_{k=1}^{p} \mathcal{A}_{A_k H}(x_k).$$

In both cases, since $E(u) \geq 0$, we have the action estimate
$$\sum_{k=1}^{p} \mathcal{A}_{A_k H}(x_k) \leq \sum_{\ell=1}^{q} \mathcal{A}_{B_\ell H}(y_\ell).$$

Before proving Lemma 4.2 and Lemma 4.3, we need some preliminary computations.

Suppose H is an r_0-admissible Hamiltonian. We denote by $\rho_0 > 0$ the radius at which H become radial (see Definition 2.1). Let $\Sigma = \Sigma^g_{p,q}$. For local holomorphic coordinates $s+it$ on (Σ,j), the volume form can be written as $\mathrm{vol}_\Sigma = \mathrm{d}s \wedge \mathrm{d}t$. In these coordinates, any one 1-form ζ on $T\Sigma$ with values in $u^*T\hat{D}$ can be written as $\zeta = \zeta_s \mathrm{d}s + \zeta_t \mathrm{d}t$ where $\zeta_s = \zeta(\partial_s)$ and $\zeta_t = \zeta(\partial_t)$.

We then define

$$\|\zeta\|^2 = |\zeta_s|_J^2 + |\zeta_t|_J^2$$

for $|\cdot|_J$ the norm corresponding to the Riemannian metric g_J (see Section 2.3.3).

It is useful to rewrite the energy so that H appears explicitly in the case where u is a Floer solution in $\mathcal{M}(x_k, y_\ell; \Sigma, \beta)$. Consider the $u^*T\hat{D}$-valued 1-form on $T\Sigma$ given by $\zeta = \mathrm{d}u - X_H \otimes \beta$. Locally, $\mathrm{d}u = \partial_s u\,\mathrm{d}s + \partial_t u\,\mathrm{d}t$, $\beta = \beta_s\,\mathrm{d}s + \beta_t\,\mathrm{d}t$ and

$$\zeta = (\partial_s u - X_H(u)\beta_s)\mathrm{d}s + (\partial_t u - X_H(u)\beta_t)\mathrm{d}t = \zeta_s\,\mathrm{d}s + \zeta_t\,\mathrm{d}t. \tag{4.1.7}$$

From Floer's equation (4.1.4), $0 = \zeta^{0,1} = \frac{1}{2}(\zeta + J \circ \zeta \circ j)$ which implies $\zeta \circ j = J \circ \zeta$. Locally, since $j\partial_s = \partial_t$, the previous equality can be written as $\zeta_t = J\zeta_s$. Therefore,

$$\|\zeta\|^2 = \omega(\zeta_s, J\zeta_s) + \omega(\zeta_t, J\zeta_t) = \omega(\zeta_s, \zeta_t) - \omega(\zeta_t, \zeta_s) = 2\omega(\zeta_s, \zeta_t). \tag{4.1.8}$$

Combining Equations (4.1.7) and (4.1.8), we can write the 2-form appearing in the definition of the energy as

$$\begin{aligned}
\frac{1}{2}\|\zeta\|^2 \mathrm{vol}_\Sigma &= \omega(\zeta_s, \zeta_t)\,\mathrm{d}s \wedge \mathrm{d}t \\
&= (\omega(\partial_s u, \partial_t u) + \beta_t \omega(X_H, \partial_s u) - \beta_s \omega(X_H, \partial_t u))\ \mathrm{d}s \wedge \mathrm{d}t \\
&= \omega(\partial_s u, \partial_t u)\,\mathrm{d}s \wedge \mathrm{d}t + (\beta_s \mathrm{d}H(\partial_t u) - \beta_t \mathrm{d}H(\partial_s u))\,\mathrm{d}s \wedge \mathrm{d}t \\
&= u^*\omega - u^*(\mathrm{d}H) \wedge \beta.
\end{aligned}$$

where the third equality is a consequence of Hamilton's equation. This allows us to make the following definitions.

Definition 4.4 • The geometric and topological energy

Let $u : (\Sigma, \beta, j) \to \hat{D}$. The *geometric energy of* u is given by

$$E(u) = \int_\Sigma (u^*\omega - u^*\mathrm{d}H \wedge \beta). \tag{4.1.9}$$

The *topological energy of* u is defined as

$$E_{\text{top}}(u) = \int_{\Sigma} (u^*\omega - \mathrm{d}(u^*H \cdot \beta)) = E(u) - \int_{\Sigma} u^*H \cdot \mathrm{d}\beta. \tag{4.1.10}$$

We denote by

$$K_H(u) = E(u) - E_{\text{top}}(u) = \int_{\Sigma} u^*H \cdot \mathrm{d}\beta$$

the H-*curvature* of u.

The topological energy only depends on the ends of Σ and is therefore an homotopy invariant, hence the name. In fact, if $u \in \mathcal{M}(x_k, y_\ell; \Sigma, \beta)$, the topological energy only depends on the action of the orbits to which u is asymptotic to. Indeed, since $\omega = \mathrm{d}\lambda$, Stokes's theorem guarantees that

$$\begin{aligned} E_{\text{top}}(u) &= \int_{\partial\Sigma} (u^*\lambda - u^*H \cdot \beta) &(4.1.11)\\ &= \int_{S_1} \left(\sum_{\ell=1}^{q} y_\ell^* \lambda - \sum_{k=1}^{p} x_k^* \lambda \right) - \int_{S_1} \left(\sum_{\ell=1}^{q} B_\ell H \circ y_\ell - \sum_{k=1}^{p} A_k H \circ x_k \right) \mathrm{d}t \\ &= \sum_{\ell=1}^{q} \mathcal{A}_{B_\ell H}(y_\ell) - \sum_{k=1}^{p} \mathcal{A}_{A_k H}(x_k). &(4.1.12) \end{aligned}$$

Therefore, if we want to have a chance to compute energy estimates for curves in $\mathcal{M}(x_k, y_\ell; \Sigma, \beta)$, we need to control the curvature term K_H. It's here that the two sets of conditions on $\mathrm{d}\beta$ and H introduced in Section 4.1.1 become important. Let us see what can be done in both case.

$(\mathrm{d}\beta = 0)$ **:** The H-curvature $K_H(u)$ vanishes and we have

$$0 \leq E(u) = E_{\text{top}}(u). \tag{4.1.13}$$

$(\mathrm{d}\beta \leq 0$ **and** $H \geq 0)$ **:** Here, $K_H(u) \leq 0$. Therefore,

$$0 \leq E(u) \leq E_{\text{top}}(u). \tag{4.1.14}$$

With these preliminary computations in hand, we are ready to prove Lemma 4.2 and Lemma 4.3.

Proof of Lemma 4.3. Suppose $\mathrm{d}\beta = 0$. Combining Equation (4.1.12) and Equation (4.1.13), we get

$$E(u) = \sum_{\ell=1}^{q} \mathcal{A}_{B_\ell H}(y_\ell) - \sum_{k=1}^{p} \mathcal{A}_{A_k H}(x_k)$$

as desired.

Now, suppose $\mathrm{d}\beta \leq 0$ and $H \geq 0$. From Equation (4.1.12) and Equation (4.1.14) we have,

$$E(u) \leq \sum_{\ell=1}^{q} \mathcal{A}_{B_\ell H}(y_\ell) - \sum_{k=1}^{p} \mathcal{A}_{A_k H}(x_k).$$

This completes the proof. □

Proof of Lemma 4.2. Choose $\delta > 0$ so that

$$\delta = \max\left\{r \circ x_1, \ldots, x_p, r \circ y_1, \ldots, r \circ y_q, \rho_0\right\}.$$

Let $u \in \mathcal{M}(x_k, y_\ell; \Sigma, \beta)$ and suppose, in view of a contradiction, that u leaves D^δ. If needed, slightly increase δ while keeping it under r_0 to ensure that u intersects ∂D^δ transversely. Denote by

$$S = u^{-1}(\hat{D} \setminus \mathrm{int} D^\delta) \subset \Sigma$$

the subsurface of Σ which is sent outside of D^δ (see Figure 4). Note that S is nor necessarily connected and write it, and its boundary, as disjoint union of connected components

$$S = \bigsqcup_{i=1}^{m} S_i, \quad \partial S = \bigsqcup_{i=1}^{m} \partial S_i.$$

Following the integrated maximum method introduced by Abouzaid and Seidel in [**AS10**, Section 7], we prove a more precise statement than what appears in Lemma 4.2. First, we show that the energy

$$E(u|_S) = \int_S \|\mathrm{d}u - X_H(u) \otimes \beta\|^2 \ \mathrm{vol}_S$$

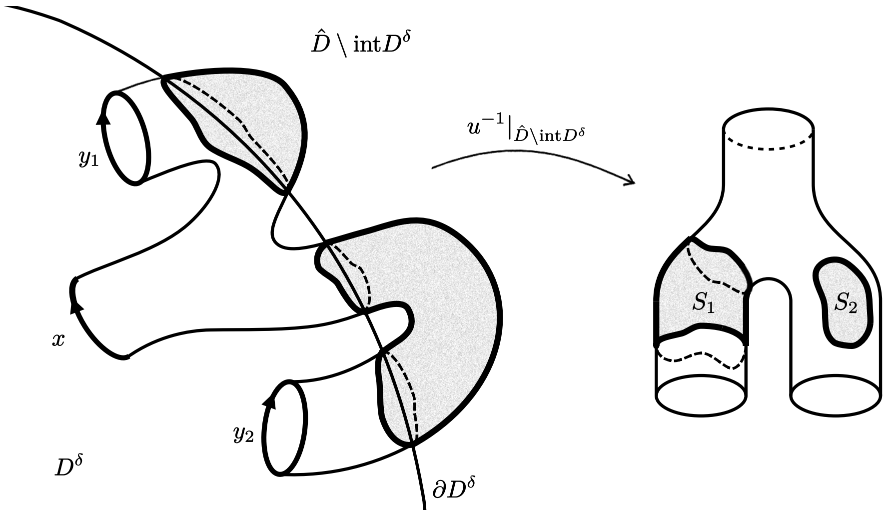


Fig. 4. The preimage S of the part of $u(\Sigma)$ which escapes D^δ. To simplify the figure, we took $\Sigma = \Sigma^0_{1,2}$.

of u restricted to S is negative and thus $E(u|_S) = 0$. By definition of the energy, this implies that

$$\mathrm{d}u = X_H(u) \otimes \beta$$

and thus each connected component of $u|_S$ is contained in an X_H-orbit. From Section 2.2.2 we know that X_H-orbits in $(\rho_0, r_0) \times \partial D$ lie entirely in subsets of the form $\{r\} \times \partial D$, thus $u(S) \subset \partial D^\delta$. This contradicts the fact that u leaves D^δ.

Now, suppose $\mathrm{d}\beta = 0$. We show that $E(u|_S)$ is negative. From Equation (4.1.13) and Equation (4.1.11) the energy of $u|_S$ is given by its topological energy

$$E(u|_S) = E_{\mathrm{top}}(u|_S) = \int_{\partial S} (\lambda \circ \mathrm{d}u - H(u) \cdot \beta) \tag{4.1.15}$$

We want to factor λ in the previous equation. To do that, we prove $\int_{\partial S} H(u)\cdot\beta = \int_{\partial S}\lambda(X_H(u))\cdot\beta$. Indeed, since $u(\partial S)\subset\partial D^\delta$, $\lambda(X_H(u)) = \delta h'(\delta)$ and

$$\begin{aligned}\int_{\partial S}(\lambda(X_H(u)) - H(u))\cdot\beta &= \int_{\partial S}(\delta h'(\delta) - h(\delta))\beta \\ &= \int_{\partial S} A_h(\delta)\beta \\ &= \int_S A_h(\delta)\mathrm{d}\beta = 0.\end{aligned} \tag{4.1.16}$$

The third equation follows from Stokes' theorem, the fourth equation holds because $A_h(\delta)$ is constant and the last equation follows from the fact β is closed. Equation 4.1.15 can now be written as

$$\begin{aligned}E(u|_S) &= \int_{\partial S}(\lambda\circ\mathrm{d}u - \lambda(X_H(u))\cdot\beta) \\ &= \int_{\partial S}\lambda(\mathrm{d}u - X_H(u)\otimes\beta) = \int_{\partial S}\lambda(J\circ(\mathrm{d}u - X_H(u)\otimes\beta)\circ(-j))\end{aligned}$$

since $(\mathrm{d}u - X_H(u)\otimes\beta)^{0,1} = 0$. Recall that, since J is cylindrical near ∂D^δ, $\lambda\circ J = \mathrm{d}r$ along ∂D^δ. Moreover, since $X_H(u)\subset T\partial D^\delta$, $\mathrm{d}r(X_H(u)) = 0$ along ∂S. Thus,

$$\begin{aligned}E(u|_S) &= \int_{\partial S}\lambda(J\circ(\mathrm{d}u - X_H(u)\otimes\beta)\circ(-j)) \\ &= \int_{\partial S}(\mathrm{d}r\circ\mathrm{d}u - \mathrm{d}r(X_H(u))\otimes\beta)\circ(-j)) \\ &= \int_{\partial S}\mathrm{d}r\circ\mathrm{d}u\circ(-j).\end{aligned}$$

It suffices now to prove that the expression on the right of the last equality is negative. On each ∂S_i choose a normal vector $\boldsymbol{n}_i$ pointing outwards. Then, ∂S_i is oriented by $j\boldsymbol{n}$. Let Y_i be a tangent vector to ∂S_i pointing in the direction induced by $j\boldsymbol{n}_i$. Then, jY_i points inside S_i. See Figure 5. Therefore, since $u(S_i)$ is outside D^δ by definition, $\mathrm{d}u(jY_i)$ points outwards along ∂D^δ and $\mathrm{d}r(\mathrm{d}u(jY))\geq 0$. We conclude that

$$E(u|_S) = \int_{\partial S}\mathrm{d}r\circ\mathrm{d}u\circ(-j)\leq 0.$$

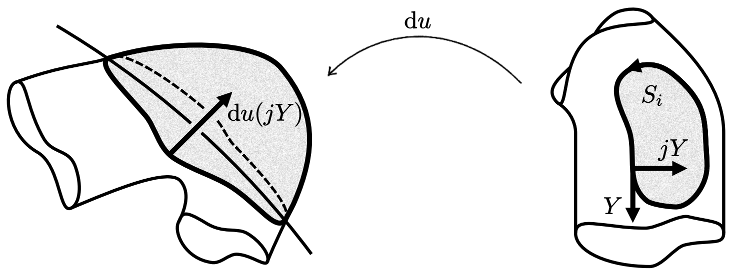


Fig. 5. The image under du of tangent vector to ∂S_i points inwards along ∂D^δ

Now let us take care of the case where $\mathrm{d}\beta \le 0$ and $H \ge 0$. Since H is r_0-admissible, $H|_{\hat{D}\setminus D^{r_0}} = h(r_0)$. Further suppose that

$$A_h(r_0) = r_0 h'(r_0) - h(r_0) \ge 0. \tag{4.1.17}$$

This is true for r_0-admissible Hamiltonians with $\eta_H \le 0$. We mimic the proof for the previous case with $\delta = r_0$. From Equation (4.1.14) and Equation (4.1.11) the energy of $u|_S$ is bounded from above by its topological energy and thus

$$E(u|_S) \le E_{\mathrm{top}}(u|_S) = \int_{\partial S} (\lambda \circ \mathrm{d}u - H(u) \cdot \beta)$$

Just like in the case $\mathrm{d}\beta = 0$, we want to factor λ. To do this, we first prove that $-\int_{\partial S} H(u)\beta \le -\int_{\partial S} \lambda(X_H(u))\beta$. Indeed, from Equation (4.1.16), we have

$$\int_{\partial S} (\lambda(X_H(u)) - H(u)) \cdot \beta = \int_{\partial S} A_h(r_0)\beta = A_h(r_0) \int_S \mathrm{d}\beta \le 0$$

since we assumed $\mathrm{d}\beta \le 0$ and $A_h(r_0) \ge 0$. Then,

$$E(u|_S) \le \int_{\partial S} \lambda(\mathrm{d}u - X_H(u) \otimes \beta)$$

The rest of the proof is the same as the case $\mathrm{d}\beta = 0$. □

4.2. Operations, action and compactly supported Hamiltonians

We now study how the operations induced by model surfaces act on the filtered Floer cohomology groups. We do this for the three operations of interest to us: the *capping* $C = \Sigma^0_{1,0}$, the *continuation cylinder* $Z = \Sigma^0_{1,1}$ and the *pair of pants product* $P = \Sigma^0_{1,2}$.

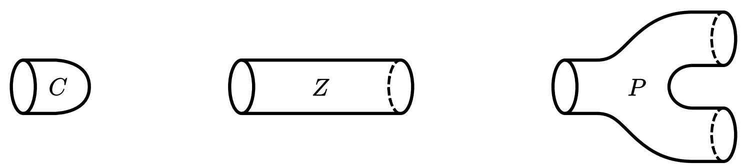


Fig. 6. Form left to right: the capping operation, the continuation cylinder and the pair of pants product.

Throughout this section, we consider compactly supported Hamiltonians and their τ-extension (see Definition 3.4) for τ strictly smaller than the minimum Reeb period T_0 and the same $K_{\varepsilon,r_0,\tau}$. Note that, since the Hamiltonians under consideration here depend on time and are not necessarily C^2-small on D, we need to adapt the construction from the previous section. We will also see how to relax the condition $H \geq 0$ when β is not closed.

Hamiltonian 1-forms. Let (Σ,β,j) be a model surface of genus 0 with $p = 1$. We adapt the methods of [**KS21**, Section 2] to Ritter's approach to deal with the arbitrary nature of the Hamiltonians on D. We consider Hamiltonian 1-forms $\boldsymbol{F} \in \Omega^1(\Sigma, C^\infty(\hat{D}, \mathbb{R}))$ that are written as

$$\boldsymbol{F} = \boldsymbol{F}^{r_0} + K_{\varepsilon,r_0,\tau} \otimes \beta$$

where $\boldsymbol{F}^{r_0} \in \Omega^1(\Sigma, C^\infty_c(D,\mathbb{R}))$. For such Hamiltonian 1-forms, Floer's equation takes the more general form

$$(\mathrm{d}u - X_{\boldsymbol{F}}(u))^{(0,1)} = 0 \tag{4.2.1}$$

where $X_{\boldsymbol{F}}$ is defined such that for any $\xi \in T_z\Sigma$, $X_{\boldsymbol{F}}(\xi)$ corresponds to the Hamiltonian vector field of $\boldsymbol{F}(\xi)$. On the ends, we require $\boldsymbol{F}$ to be of the form $H^{\pm} \otimes \mathrm{d}t$ on D^{r_0} for $H \in \mathcal{C}(D)$ and $K_{\varepsilon,r_0,\tau} \otimes W\mathrm{d}t$ on $\hat{D} \setminus D^{r_0}$ for a weight $W > 0$. The operations are then build by counting the zero dimensional components of the moduli space $\mathcal{M}(x_1,y_\ell;\Sigma^g_{p,q},X_{\boldsymbol{F}})$ of curves $u : \Sigma \to \hat{D}$ satisfying Equation (4.2.1) and converging, for $s \to \pm\infty$, to orbits of the Hamiltonians on the ends. Once again, to prove that these moduli spaces are smooth compact manifolds of finite dimension, we need to compute a priori energy estimates and show that their curves do not escape at infinity. The energy of a curve $u \in \mathcal{M}(x_1,y_\ell;\Sigma^g_{p,q},X_{\boldsymbol{F}})$ can be written as (see [**KS21**, Section 2.5])

$$E(u) = \underbrace{\sum_{\ell=1}^{q} \mathcal{A}_{(H_\ell^+)^\tau}(y_\ell) - \mathcal{A}_{(H^-)^\tau}(x_1)}_{E_{\mathrm{top}}(u)} + K_{\boldsymbol{F}}(u). \tag{4.2.2}$$

The curvature term is given by

$$K_{\boldsymbol{F}}(u) = \int_\Sigma (\mathrm{d}\boldsymbol{F}(u) - \{\boldsymbol{F},\boldsymbol{F}\}(u))$$

where, in local coordinates (s,t) on Σ, the curvature two-form can be written as

$$(\mathrm{d}\boldsymbol{F} - \{\boldsymbol{F},\boldsymbol{F}\})(\partial_s,\partial_t) = \partial_s\boldsymbol{F}(\partial_t) - \partial_t\boldsymbol{F}(\partial_s) - \{\boldsymbol{F}(\partial_s),\boldsymbol{F}(\partial_t)\}.$$

4.2.1. The capping operation

For C we choose $\Sigma^0_{1,0} = \mathbb{C}$. The negative end, which is chosen to be a neighbourhood of the point at ∞, is parameterized by

$$\begin{aligned} \varphi_1^- : (-\infty,0] \times S^1 &\longrightarrow \mathbb{C} \\ (s,t) &\longmapsto e^{-2\pi(s+it)} \end{aligned}$$

To construct β, first pick a function $b(s) : (-\infty,0] \to [0,1]$ such that $b'(s) \leq 0$, $b|_{(-\infty,-2]} \equiv 1$ and $b|_{[-1,0]} \equiv 0$. Then, on the negative end define β so that

$$(\varphi_1^-)^*\beta = b(s)\mathrm{d}t$$

and extend β to 0 everywhere else on C. We have only one positive weight here given by $A_1 = 1$. Note that $\mathrm{d}\beta \leq 0$. Let H be compactly supported and consider its τ-extension H^τ by the Hamiltonian $K_{\varepsilon,r_0,\tau}$. We define our Hamiltonian 1-form $\boldsymbol{F}$ such that

$$\boldsymbol{F} = \boldsymbol{F}^{r_0} + K_{\varepsilon,r_0,\tau} \otimes \beta.$$

where

$$(\varphi_1^-)^* \boldsymbol{F}^{r_0} = H_t \otimes b(s)\mathrm{d}t$$

and is extended to zero everywhere else on C. We will see that we do not need to only consider positive Hamiltonians for Lemma 4.2 to work. Doing so, we will be able to compute an a priori energy bound for u which only depends on H. This allows us to define an operation

$$[\Psi_C^{H^\tau}] : \mathbb{Z}_2 \longrightarrow \mathrm{HF}^0(H^\tau) = \mathrm{HF}^0(H)$$

which is independent of the chosen τ-extension.

No escape. Let $u \in \mathcal{M}(x_1; C, \boldsymbol{F})$ and suppose that $\operatorname{im} u$ leaves D^{r_0}. Denote $S = u^{-1}(\hat{D} \setminus D^{r_0}) \subset C$. From Equation (4.2.2) we have

$$E(u|_S) = E_{\mathrm{top}}(u|_S) + K_{\boldsymbol{F}}(u|_S).$$

The $\boldsymbol{F}$-curvature is given here by

$$K_{\boldsymbol{F}}(u) = \int_{(-\infty,0]\times S^1} (\varphi_1^-)^*(\mathrm{d}\boldsymbol{F}(u) - \{\boldsymbol{F} - \boldsymbol{F}\}(u)) \tag{4.2.3}$$

$$= \int_{(-\infty,0]\times S^1} b'(s)(\varphi_1^-)^*(H_t^\tau(u))\mathrm{d}s \wedge \mathrm{d}t. \tag{4.2.4}$$

Since $u(S) \subset \hat{D} \setminus D^{r_0}$, $(u|_S)^* H_t^\tau = k_{\varepsilon,r_0,\tau}(r) \geq 0$ by definition of the τ-extension. Thus, taking into account that $b'(s) \leq 0$ by construction, $E(u|_S) \leq E_{\mathrm{top}}(u|_S)$. The proof of Lemma 4.2 in the case $\mathrm{d}\beta \leq 0$ can then be carried out. Therefore, $\operatorname{im} u \subset D^{r_0}$.

Energy estimate. Let us now compute the energy of u. Equation (4.2.2) allows us to write

$$0 \leq E(u) = -\mathcal{A}_{H^\tau}(x_1) + K_{\boldsymbol{F}}(u)$$

Since the image of u stays within D^{r_0} by the previous paragraph and since H_t has support in D, $(\varphi_1^-)^* u^* H_t^\tau \geq \min_D H_t$. Therefore, taking into account that $b'(s) \leq 0$,

$$K_{\boldsymbol{F}}(u) \leq \int_{S^1} \int_{(-\infty,0]} \min_D(H) \cdot b'(s) \, \mathrm{d}s\mathrm{d}t \leq -\int_{S^1} \min_D(H) \, \mathrm{d}t = E_+(H).$$

This yields the energy estimate

$$0 \leq E(u) \leq -\mathcal{A}_{H^\tau}(x_1) + E_+(H)$$

which only depends on H and possibly its τ-extension H^τ.

We gather the results we obtained in this the section in the following Lemma.

Lemma 4.5 • Filtered capping operation

Let $H \in \mathcal{C}(D)$ and let H^τ be a τ-extension of H. For any $u \in \mathcal{M}(x_1; C, \boldsymbol{F})$, $\operatorname{im} u \subset D^{r_0}$. Moreover, by the energy estimate of such curves,

$$\mathcal{A}_{H^\tau}(x_1) \leq E_+(H).$$

In action windows, we thus have an operation

$$[\Psi_C^H] : \mathbb{Z}_2 \longrightarrow \mathrm{HF}^*_{<E_+(H)}(H).$$

Remark 4.6: Even though the energy estimate seems to depend on the extension of H, the image of $[\Psi_C^H]$ in the previous Lemma does not. Indeed, from Lemma 3.5, we know that, for any two extensions H^{τ_1} and H^{τ_2} of H there is an isomorphism $\mathrm{HF}^*_{(-\infty,E_+(H)]}(H^{\tau^1}) \cong \mathrm{HF}^*_{(-\infty,E_+(H)]}(H^{\tau^2})$ which preserves the action windows.

4.2.2. Continuation cylinders revisited

For Z we choose $\Sigma^0_{1,1} = \mathbb{R} \times S^1$. On Z we can pick the global $\beta = \mathrm{d}t$ 1-form. Let $F_\pm \in \mathcal{C}(D)$ and consider their τ-extension F_-^τ and F_+^τ with the same $K_{\varepsilon,r_0,\tau}$. Let $f : \mathbb{R} \to [0,1]$ be a smooth function with $f'(s) \leq 0$

interpolating between 1 and 0 which is constant outside $(-1,1) \subset \mathbb{R}$. Let $F_\bullet$ be the homotopy between $(F_-^\tau)_t$ and $(F_+^\tau)_t$:

$$(F_s)_t = (F_+^\tau)_t + f(s)((F_-^\tau)_t - (F_+^{\tau'})_t). \tag{4.2.5}$$

Consider the Hamiltonian 1-form

$$\boldsymbol{F} = (F_s)_t \otimes \mathrm{d}t = ((F_+)_t + f(s)((F_-)_t - (F_+)_t)) \otimes \mathrm{d}t + K_{\varepsilon,r_0,\tau} \otimes \mathrm{d}t$$

No escape. Let $u \in \mathcal{M}(x_1, y_1; Z, \boldsymbol{F})$ and suppose that $\operatorname{im} u$ leaves D^{r_0}. Denote $S = u^{-1}(\hat{D} \setminus D^{r_0}) \subset Z$. The curvature term here is given by

$$K_{\boldsymbol{F}}(u) = \int_{S^1} \int_{\mathbb{R}} f'(s)((F_-)_t(u) - (F_+)_t(u))\mathrm{d}s\mathrm{d}t \tag{4.2.6}$$

Restricting u to $S \subset Z$, we have $u|_S^*((F_-)_t - (F_+)_t) = 0$ since $(F_\pm)_t$ have support in $S^1 \times D^{r_0}$, and Equation (4.2.6) yields

$$K_{\boldsymbol{F}}(u|_S) = 0.$$

and thus $E(u|_S) = E_{\mathrm{top}}(u|_S)$. The proof of the no escape lemma 4.2 in the case $\mathrm{d}\beta = 0$ can then be carried out with very little modifications. Therefore, $\operatorname{im} u \subset D^{r_0'}$.

Energy estimate. Let us now compute the energy of u. Equation (4.1.11) and Equation (4.2.6) allows us to write

$$\mathcal{A}_{F_-^\tau}(x_1) \leq \mathcal{A}_{F_+^\tau}(y_1) + \int_Z f'(s)u^*((F_-)_t - (F_+)_t)\,\mathrm{d}s \wedge \mathrm{d}t$$

From the previous paragraph, we know the curve u does not escape $D^{r_0'}$. Therefore,

$$u^*((F_-^\tau)_t - (F_+^{\tau'})_t) \geq \min_D((F_-)_t - (F_+)_t).$$

and since $f'(s) \leq 0$,

$$\begin{aligned}
\int_Z f'(s)u^*((F_-)_t - (F_+)_t)\,\mathrm{d}s \wedge \mathrm{d}t &\leq \int_C f'(s)\min_D((F_-)_t - (F_+)_t)\,\mathrm{d}s \wedge \mathrm{d}t \\
&= -\int_{S^1} \min_D((F_-)_t - (F_+)_t)\,\mathrm{d}t \\
&= E_+(F_- - F_+).
\end{aligned}$$

The two previous paragraphs are summarized in the following lemma.

Lemma 4.7 • Filtered continuation cylinder

Let $F_-,F_+ \in \mathcal{C}(D)$ and let F_-^τ and F_+^τ be τ-extensions. For any $u \in \mathcal{M}(x_1, y_1; Z, \boldsymbol{F})$, $\operatorname{im} u \subset D^{r_0'}$. Moreover, by the energy estimate of such curves,

$$\mathcal{A}_{F_-^\tau}(x_1) \leq \mathcal{A}_{F_+^{\tau'}}(y_1) + E_+(F_- - F_+).$$

In action windows, we thus have an operation

$$[\Psi_Z^{F_-,F_+}] = [\Phi^{F_-,F_+}] : \mathrm{HF}^\bullet_{<a}(F_+) \longrightarrow \mathrm{HF}^\bullet_{<a+E_+(F_- - F_+)}(F_-).$$

4.2.3. Pair of pants product for two different Hamiltonians

To prove some properties of spectral invariants, like the triangle inequality (see Proposition 5.1), we need a product defined between the extension of two compactly supported Hamiltonians H and K. This operation should have as output classes in the Floer cohomology of the concatenation $H\#K$ of H and K:

$$\mathrm{HF}^\bullet(H) \otimes \mathrm{HF}^\bullet(K) \longrightarrow \mathrm{HF}^\bullet(H\#K)$$

To construct this operation, we use a P dependent Hamiltonian 1-form $\boldsymbol{F} \in \Omega^1(P, C^\infty(S^1 \times \hat{D}, \mathbb{R}))$ which interpolates between H, K and $H\#K$ on disjoint strips in P with disjoint image. This technique can be generalized, for instance, to any number of inputs (see [**KS21**, Section 2.5] for a detailed treatment). To carry out this approach, it is crucial that H and K are smoothed in the sense of Definition 3.6.

In this section, we suppose for simplicity that all τ-extensions have $r_0 = 1$.

The model surface and Hamiltonians We consider the pair of pants model surface (P,β,j) with weights $A_1 = 2$, $B_1 = 1 = B_2$. By our choice of weights, Section 4.1.1 assures us that β can be chosen so that $\mathrm{d}\beta = 0$. We are given smoothed Hamiltonians H and K with compact support inside D^δ for some $0 < \delta < 1$. We consider their τ-extensions with respect to the Hamiltonian

$K_{\varepsilon,1,\tau}$ so that $2\tau < T_0$. The concatenation $H\#K$ defined by

$$H\#K(t,p) = \begin{cases} 2H(2t,p) & \text{for } t \in [0,1/2], \\ 2K(2t-1,p) & \text{for } t \in [1/2,1]. \end{cases}$$

is also smoothed and has compact support in D^δ.

Strips and Hamiltonian 1-form. Denote by (ρ,t) the natural coordinates on $\mathbb{R} \times (0,1)$. We consider two strips $\phi_1^{-,+}, \phi_2^{-,+} : \mathbb{R} \times (0,1) \longrightarrow P$ which are smooth proper embeddings that satisfy the following compatibility conditions with respect to the cylindrical ends on P. Let $\rho_\pm \in \mathbb{R}_{>0}$ be two sufficiently large constants.

- For all $\rho \geq \rho_+$,

$$\phi_\ell^{-,+}(\rho,t) = \varphi_\ell^+(\rho - \rho_+, t)\,. \tag{4.2.7}$$

  where $\ell \in \{1,2\}$.

- For all $\rho \leq -\rho_-$,

$$\begin{aligned} \phi_1^{-,+}(\rho,t) &= \varphi_1^-\left(\rho + \rho_-, \frac{t}{2}\right), \\ \phi_2^{-,+}(\rho,t) &= \varphi_1^-\left(\rho + \rho_-, \frac{t}{2} + \frac{1}{2}\right). \end{aligned} \tag{4.2.8}$$

We now build our Hamiltonian 1-form $\boldsymbol{F} \in \Omega^1(P, C^\infty(S^1 \times \hat{D}, \mathbb{R}))$. On the strips, $\boldsymbol{F}$ is given by

$$\begin{aligned} (\phi_1^{-,+})^*\boldsymbol{F} &= H_t \otimes \mathrm{d}t + K_{\varepsilon,1,\tau} \otimes (\phi_1^{-,+})^*\beta \\ (\phi_2^{-,+})^*\boldsymbol{F} &= K_t \otimes \mathrm{d}t + K_{\varepsilon,1,\tau} \otimes (\phi_2^{-,+})^*\beta \end{aligned}$$

and we extend $\boldsymbol{F}$ to $K_{\varepsilon,1,\tau} \otimes \beta$ everywhere else on P. From Equation (4.2.7) and Equation (4.2.8), the definition of $\boldsymbol{F}$ on the strips and the value of β on the cylindrical ends of P, we can recover the value of $\boldsymbol{F}$ on these ends. On the positive ends, we have

$$(\varphi_1^+)^*(\boldsymbol{F}) = H_t^\tau \otimes \mathrm{d}t, \quad (\varphi_2^+)^*(\boldsymbol{F}) = K_t^\tau \otimes \mathrm{d}t,$$

and on the negative end, we have, for $t \in [0,1/2]$

$$
\begin{aligned}
(\varphi_1^-)^* \boldsymbol{F} &= (\phi_1^{-,+}(\rho - \rho_-,2t))^* \boldsymbol{F}^{r_0} + (\varphi_1^-)^*(K_{\varepsilon,1,\tau} \otimes \beta) \\
&= H_{2t} \otimes 2\mathrm{d}t + K_{\varepsilon,1,\tau} \otimes 2\mathrm{d}t
\end{aligned}
$$

and for $t \in [1/2,1]$ we obtain

$$
\begin{aligned}
(\varphi_1^-)^* \boldsymbol{F} &= (\phi_2^{-,+}(\rho - \rho_-,2t-1))^* \boldsymbol{F}^{r_0} + (\varphi_1^-)^*(K_{\varepsilon,1,\tau} \otimes \beta) \\
&= K_{2t-1} \otimes 2\mathrm{d}t + K_{\varepsilon,1,\tau} \otimes 2\mathrm{d}t.
\end{aligned}
$$

Thus, by definition of $H\#K$, we have

$$
(\varphi_1^-)^* \boldsymbol{F} = (H\#K)^{2\tau} \otimes \mathrm{d}t.
$$

Therefore, on the strips, $\boldsymbol{F}$ interpolates from H and K on the positive ends, to $H\#K$ on the negative end.

No-escape. Consider a curve $u \in \mathcal{M}(x_1, y_1, y_2; P, \boldsymbol{F})$. The $\boldsymbol{F}$-curvature is given by

$$
K_{\boldsymbol{F}}(u) = \sum_{\ell=1}^{2} \int_{\mathbb{R}\times(0,1)} (\varphi_\ell^+)^*(\mathrm{d}\boldsymbol{F}(u) - \{\boldsymbol{F},\boldsymbol{F}\}(u)).
$$

Since the Hamiltonian terms in $\boldsymbol{F}$ depend only on t on the strips and since $\mathrm{d}\beta = 0$ on P, a direct computation shows that $K_{\boldsymbol{F}}(u) = 0$. Thus, $E(u) = E_{\mathrm{top}}(u)$ and the proof of Lemma 4.2 in the case $\mathrm{d}\beta = 0$ can be carried out. Therefore by our choice of $r_0 = 1$, the image of u stays inside D.

Energy estimate. Since $E(u) = E_{\mathrm{top}}(u)$ for any $u \in \mathcal{M}(x_1, y_1, y_2; Z, \beta, F_P)$, we have a sharp energy estimate

$$
\mathcal{A}_{H\#K}(x_1) \leq \mathcal{A}_H(y_1) + \mathcal{A}_K(y_2).
$$

The two previous paragraphs prove the following Lemma.

Lemma 4.8

Let $H,K \in \mathcal{C}(D)$. For a slope $\tau > 0$ such that $2\tau < T_0$ choose extensions H^τ, K^τ and $H\#K^{2\tau}$ with $r_0 = 1$. Then, we have the pair of pants product

$$[\Psi_P^{H,K}] : \mathrm{HF}^\bullet_{<a}(H) \otimes \mathrm{HF}^\bullet_{<b}(K) \longrightarrow \mathrm{HF}^\bullet_{<a+b}(H\#K).$$

4.3. Gluing, composition and naturality of the operations

Consider two model surfaces $\Sigma_1 = (\Sigma^g_{p_1,q_1}, \beta_1, j_1)$ and $\Sigma_2 = (\Sigma^g_{p_2,q_2}, \beta_2, j_2)$. We say that Σ_1 and Σ_2 *match* if

- Σ_1 has as many positive ends as Σ_2 has negative ends: $q_1 = p_2$,
- the positive weights of Σ_1 are the same as the negative weights of Σ_2: $B_{1,i} = A_{2,i}$ for all $i \in \{1, \ldots, q_1\}$.
- the restriction of j_2 on the i-th negative end of Σ_2 agrees with the restriction of j_1 on the i-th positive end of Σ_1 for large enough s.

If Σ_1 and Σ_2 match, then we can obtain a third model surface $\Sigma = \Sigma_1\#\Sigma_2$ with

$$p = p_1, \quad q = q_2, \quad g = g_1 + g_2 + (q_1 - 1) = g_1 + g_2 + (p_2 - 1)$$[2]

by gluing, along the positive ends of Σ_1 and negative ends of Σ_2, Σ_1 to Σ_2, β_1 to β_2 and j_1 to j_2. We call $\Sigma_1\#\Sigma_2$ the *gluing of* Σ_1 *to* Σ_2.

If two surfaces do not match, we can add constant continuation cylinders Z_A (see Section 4.2.2) to either one of the surfaces:

- If $q_1 = p_2 + q'$ with extra weights $B_{1,q_1}, B_{1,q_1+1}, \ldots, B_{1,q_1+q'}$ then

$$\Sigma_1 \quad \text{and} \quad \Sigma_2 \bigsqcup_{m=q_1+1}^{q_1+q'} Z_{B_{1,q_1}} \quad \text{match.}$$

[2] This formula for the glued model surface only holds if Σ_1 and Σ_2 are connected.

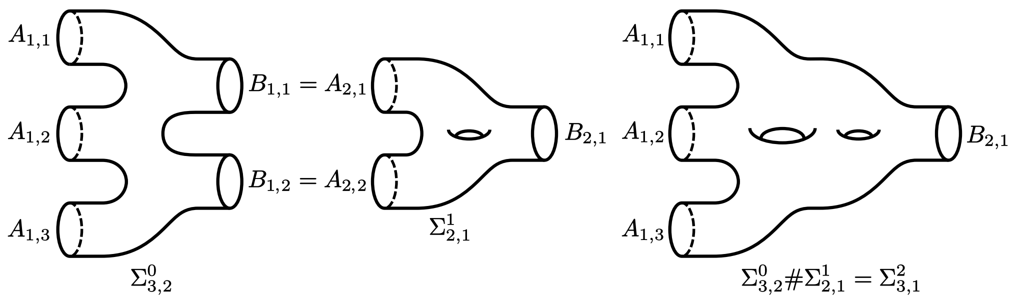


Fig. 7. The gluing $\Sigma^0_{3,2} \# \Sigma^1_{2,1}$ of two matching model surfaces $\Sigma_1 = \Sigma^0_{3,2}$ and $\Sigma_2 = \Sigma^1_{2,1}$.

- If $p_2 = q_1 + p'$ with extra weights $A_{2,p_2}, A_{2,p_2+1}, \dots A_{2,p_2+p'}$ then

$$\Sigma_1 \bigsqcup_{m=p_2+2}^{p_1+p'} Z_{A_{2,p_2}} \quad \text{and} \quad \Sigma_2 \quad \text{match.}$$

This procedures is well defined because the operation induced by each constant continuation cylinder is the identity.

Any model surface can be expressed as the gluing of elements in a finite set of model surfaces. This finite set is composed of the model surfaces studied in Section 4.2 and the inversed pair of pants $P' \cong \Sigma^0_{2,1}$.

Lemma 4.9

Let (Σ,β,j) be a connected model surface. Then, there exists model surfaces $\{(\Sigma_i,\beta_i,j)\}_{i=1}^m$ such that

$$\Sigma = \Sigma_1 \# \cdots \# \Sigma_m$$

and $\Sigma_i \in \{C,Z,P,P'\}$ for all $i \in \{1,\dots,m\}$.

The gluing of two matching model surfaces Σ_1 and Σ_2 is compatible with the composition of operations in the following sense.

Lemma 4.10 • Gluing = composition [Rit13, Appendix A.10]

Let Σ_1 and Σ_2 be two matching model surfaces. Then, the following diagram commutes

$$\begin{tikzcd}
 & \bigotimes_{m=1}^{q_1} \mathrm{HF}^\bullet(B_{1,m}H) \arrow[d, equal] \arrow[dr, "\Psi^H_{\Sigma_1}"] & \\
\bigotimes_{\ell=1}^{q_2} \mathrm{HF}^\bullet(B_{2,\ell}H) \arrow[r, "\Psi^H_{\Sigma_2}"] \arrow[rr, bend right, "\Psi^H_{\Sigma_1 \# \Sigma_2}"'] & \bigotimes_{m=1}^{p_2} \mathrm{HF}^\bullet(A_{2,m}H) & \bigotimes_{k=1}^{p_1} \mathrm{HF}^\bullet(A_{1,k}H)
\end{tikzcd}$$

Lemma 4.10 is the key ingredient to prove that the operations Ψ^H_Σ are natural with respect to continuation maps. This property, which basically states that the operations induced by model surfaces commute with continuation maps, is the object of the following theorem.

Theorem 4.11 • [Rit13, Theorem A.14]

Let (Σ_1, β_1, j) and (Σ_2, β_2, j) be model surfaces with sets of weights $\{A_{1,k}, B_{1,\ell}\}$ and $\{A_{2,k}, B_{2,\ell}\}$ such that $\Sigma_1 \cong \Sigma^g_{p,q} \cong \Sigma_2$. Let H_1 and H_2 be admissible Hamiltonians satisfying the conditions of Section 4.1.1 such that for every k and ℓ there exists monotone homotopies which induce the continuation maps

$$\mathrm{HF}^\bullet(A_{1,k}H_1) \xleftarrow{[\Phi^{1,2}_{A_k}]} \mathrm{HF}^\bullet(A_{2,k}H_2),$$

$$\mathrm{HF}^\bullet(B_{1,\ell}H_1) \xleftarrow{[\Phi^{1,2}_{B_\ell}]} \mathrm{HF}^\bullet(B_{2,\ell}H_2).$$

Then, the following diagram commutes.

$$\begin{tikzcd}
\bigotimes \mathrm{HF}^\bullet(A_{1,k}H_1) & & \bigotimes \mathrm{HF}^\bullet(B_{1,\ell}H_1) \arrow[ll, "{[\Psi^{H_1}_{\Sigma_1}]}"'] \\
\bigotimes \mathrm{HF}^\bullet(A_{2,k}H_2) \arrow[u, "{[\Phi^{1,2}_{A_1}]\otimes\cdots\otimes[\Phi^{1,2}_{A_p}]}"] & & \bigotimes \mathrm{HF}^\bullet(B_{2,\ell}H_2) \arrow[ll, "{[\Psi^{H_2}_{\Sigma_2}]}"] \arrow[u, "{[\Phi^{1,2}_{B_1}]\otimes\cdots\otimes[\Phi^{1,2}_{B_q}]}"']
\end{tikzcd}$$

4.4. Operations and Morse cohomology

In Lemma 3.1, we saw that when an admissible Hamiltonian F with small slope $\tau_F < T_0$ is C^2-small on D, then there is an isomorphism $[\Phi_F] : \mathrm{H}^*(D) \to \mathrm{HF}^*(F)$. One could then use Lemma 3.2 to show that any admissible Hamiltonian H with small slope has Floer cohomology isomorphic to $H^*(D)$. However, when dealing with operations in Morse and Floer cohomology, it is useful to have a direct isomorphism between $\mathrm{H}^*(D)$ and $\mathrm{HF}^*(H)$ that does not rely on an auxiliary small Hamiltonian. Such a map, called the *PSS isomorphism*, was developed by Piunikhin, Salamon and Schwarz in [**PSS96**] for closed symplectic manifolds. In this section we review operations in Morse cohomology [**BC94**, **Fuk97**] and the extension of the PSS isomorphism to Liouville domains carried out by Ritter in [**Rit13**]. This allows us to relate operations in Morse cohomology to the ones in Floer cohomology.

4.4.1. Morse cohomology

Let us recall the definition of Morse cohomology. Start with a Morse function $f : \overline{D} \longrightarrow \mathbb{R}$ on the closure $\overline{D}$ of D and suppose its negative gradient $-\nabla f$ points **inwards** along a neighbourhood of ∂D.

Let $z_\pm \in \mathrm{Crit}(f)$ be critical points. Denote by $\widetilde{\mathcal{N}}(z_-,z_+;f)$ the moduli space negative gradient lines $u : \mathbb{R} \longrightarrow \overline{D}$ joining z to w, i.e.

$$\frac{\mathrm{d}u}{\mathrm{d}s} = -\nabla f \tag{4.4.1}$$

$$\lim_{s\to-\infty} u = z_-, \qquad \lim_{s\to+\infty} u = z_+.$$

There is a natural $\mathbb{R}$-action on $\widetilde{\mathcal{N}}$ by which we quotient to obtain $\mathcal{N}(z,w;f) = \widetilde{\mathcal{N}}/\sim$, where $u(s) \sim v(s)$ if and only if $u(s) = v(s+s')$ for some $s' \in \mathbb{R}$. For a generic choice of f, $\mathcal{N}(z,w;f)$ is a smooth compact manifold of dimension $\mathrm{ind}_f(z_-) - \mathrm{ind}_f(z_+) - 1$ for ind_f the Morse index of f. Moreover, $\mathcal{N}(z_-,z_+;f)$ admits a compactification by broken trajectories. The *d-th Morse-Witten complex of f* correspond to the $\mathbb{Z}_2$ vector space

$$\mathrm{CM}^d(f) = \bigoplus_{\mathrm{ind}_f(z)=d} \mathbb{Z}_2 \langle z \rangle .$$

The differential on $\mathrm{CM}^d(f)$ is defined as

$$\begin{aligned} \partial_f : \mathrm{CM}^k(f) &\longrightarrow \mathrm{CM}^{k+1}(f) \\ z_+ &\longmapsto \sum_{\mathrm{ind}_f(z_-)=\mathrm{ind}_f(z_+)+1} \#_2\mathcal{N}(z_-,z_+;f) z_- \end{aligned}$$

For a generic choice of f, $\partial_f \circ \partial_f = 0$, making $(\mathrm{CM}^\bullet(f),\partial_f)$ a co-chain complex. We denote the cohomology of that complex $\mathrm{HM}^\bullet(f)$ and call it the *Morse cohomology* of f. It is now well known that Morse cohomology computes the singular cohomology $H^\bullet(D)$ of D.

Theorem 4.12

Let f be a Morse function on $\overline{D}$ such that $-\nabla f$ points inwards along ∂D. Then,

$$[\varphi_f] : H^\bullet(D) \cong \mathrm{HM}^\bullet(f).$$

4.4.2. Operations in Morse cohomology

Operations can be defined in Morse cohomology in a similar fashion as to how we defined operations in Floer cohomology. In the Morse framework however, operations are built from graphs with positive and negative ends instead of surfaces [**BC94**, **Fuk97**]. Since our goal is to relate operations in both cohomology theories, we will build graph from the surfaces $\Sigma^g_{p,q}$.

Associated graphs. An *oriented parameterized graph* Γ *with* p *negative ends and* q *positive ends* is a finite oriented graph with

- $p \geq 0$ edges e_k^- parameterized by $(-\infty,0]$,
- $q > 0$ edges e_k^+ parameterized by $[0,+\infty)$,
- and all other edges e_i parameterized by $[0,1]$.

These edges are called *negative*, *positive* and *internal* respectively. When we draw such graphs, the vertices at the infinite ends will be denoted by empty circles $\circ$ and the vertices at the ends of internal edges will be represented by a circle bullet $\bullet$.

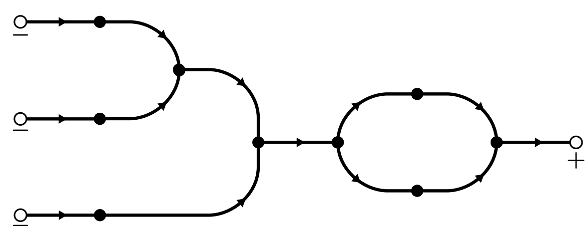


Fig. 8. An oriented parameterized graph with 3 negative edges and 1 positive edge.

From any surface $\Sigma^g_{p,q}$ we build a oriented parameterized graph $\Gamma(\Sigma^g_{p,q})$, called the *associated graph* of $\Sigma^g_{p,q}$, in the following way. For the basic model surfaces C, Z, P and P', we have one negative edge per negative end and one positive edge per positive end. We then connect the finite vertex of these edges together. Now, by Lemma 4.9, we can express $\Sigma^g_{p,q}$ as a gluing

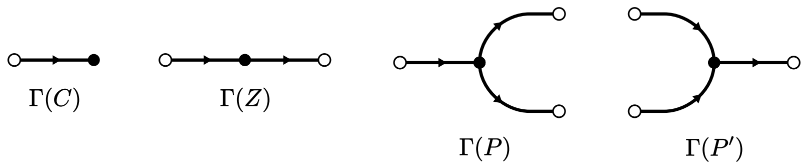


Fig. 9. Associated graphs to the basic model surfaces C, Z, P and P'.

of $\Sigma_i \in \{C,Z,P,P'\}$:

$$\Sigma^g_{p,q} = \Sigma_1 \# \cdots \# \Sigma_m.$$

We replace each Σ_i in the decomposition of $\Sigma^g_{p,p}$ by their associated graph $\Gamma(\Sigma_i)$ and glue their ends together to obtain the associated graph $\Gamma(\Sigma^g_{p,q})$ of $\Sigma^g_{p,q}$:

$$\Gamma(\Sigma^g_{p,q}) = \Gamma(\Sigma_1) \# \cdots \# \Gamma(\Sigma_m).$$

Model graphs. A *model graph* is a quintuple $(\Gamma^g_{p,q}, L_i, f_k, f_i, f_\ell)$ consisting of

- the associated graph $\Gamma^g_{p,q}$ of a surface $\Sigma^g_{p,q}$,
- a set of lengths $\{L_i\}$ for each internal edge,
- Morse functions f_k on $\overline{D}$ associated to each of the p negative edges of $\Gamma^g_{p,q}$,
- Morse functions f_i on $\overline{D}$ associated to each of the internal edges $\Gamma^g_{p,q}$,
- and Morse functions f_ℓ on $\overline{D}$ associated to each of the positive edges $\Gamma^g_{p,q}$.

We will often abbreviate the notation for a model graph to $(\Gamma^q_{p,q}, f_e)$ where $\{f_e\}$ is the list of all Morse functions associated to the edges of $\Gamma^q_{p,q}$.

The moduli spaces. Let $(\Gamma^g_{p,q}, L_i, f_e)$ be a model graph. Let $w_k \in \mathrm{Crit}(f_{e^-_k})$ and $z_\ell \in \mathrm{Crit}(f_{e^+_\ell})$. A continuous curve $u : \Gamma^g_{p,q} \longrightarrow \overline{D}$ lies in the moduli space $\mathcal{M}(w_k, z_\ell; \Gamma^g_{p,q}, L_i, f_e)$ associated to the given model graph if and only if

$$\frac{\mathrm{d}}{\mathrm{d}s} u|_e = -\nabla f_e$$

for any edge e of $\Gamma^g_{p,q}$ and

$$\lim_{s\to-\infty} u|_{e_k^-} = w_k, \quad \lim_{s\to+\infty} u|_{e_\ell^+} = z_\ell$$

for all k and ℓ. The value of an interior vertex under u corresponds to the intersection of the stable and unstable manifolds for the Morse functions associated to the edges meeting at that vertex. The ensure that $\mathcal{M}(w_k,z_\ell;\Gamma^g_{p,q},L_i,f_e)$ is a smooth manifolds, these intersections need to be transverse. This can be achieved by choosing the Morse functions f_e generically [**BC94**, Theorem 1]. Moreover, the dimension of $\mathcal{M}(w_k,z_\ell;\Gamma^g_{p,q},L_i,f_e)$ is given by

$$\begin{aligned}\dim \mathcal{M}(w_k,z_\ell;\Gamma^g_{p,q},L_i,f_e) = \sum_{k=1}^{p} \operatorname{ind}_{f_k}(w_k) - \sum_{\ell=1}^{q} \operatorname{ind}_{f_\ell}(z_\ell)& \\ - 2n(1-p-b_1(\Gamma^g_{p,q}))& \qquad (4.4.2)\end{aligned}$$

where $b_1(\Gamma^g_{p,q})$ is the first Betti number of $\Gamma^g_{p,q}$. By construction of the associated graph,

$$b_1(\Gamma^g_{p,q}) = g.$$

The moduli space $\mathcal{M}(w_k,z_\ell;\Gamma^g_{p,q},L_i,f_e)$ also admits a compactification by broken trajectories [**BC94**, Theorem 3].

The operations. Recall the notation introduced in Section 4.1.3 regarding tensor product of co-chain complexes. Using the properties of the moduli space $\mathcal{M}(w_k,z_\ell;\Gamma^g_{p,q},L_i,f_e)$ defined above we can define operations

$$\begin{aligned}\psi_{\Gamma^g_{p,q}} : \left(\bigotimes_{\ell=1}^{q} \mathrm{CM}^\bullet(f_\ell)\right)^d &\longrightarrow \left(\bigotimes_{k=1}^{p} \mathrm{CM}^\bullet(f_k)\right)^{d-2n(1-p-g)} \\ \boldsymbol{w} &\longmapsto \sum_{\operatorname{ind}(\boldsymbol{z})=d-2n(1-p-g)} \#_2\mathcal{M}(w_k,z_\ell;\Gamma^g_{p,q},L_i,f_e)\boldsymbol{z}\end{aligned}$$

where $\operatorname{ind}(\boldsymbol{z}) = \sum_{\ell=1}^{q} \operatorname{ind}_{f_\ell}(w_\ell)$. The map $\psi_{\Gamma^g_{p,q}}$ is a chain map and does not depend on the choice of graph associated to $\Sigma^g_{p,q}$ and the choices of L_i, f_k and f_ℓ. This can be proven using gluing and continuation results as in

Section 4.3. Thus $\psi^g_{p,q}$ descends on cohomology to a map

$$[\psi_{\Gamma^g_{p,q}}] : \bigotimes_{\ell=1}^{q} \mathrm{HM}^{d_\ell}(f_\ell) \longrightarrow \left(\bigotimes_{k=1}^{p} \mathrm{HM}(f_k)\right)^{\sum_\ell d_\ell - 2n(1-g-p)}$$

Composing with the isomorphisms of Theorem 4.12 for each of the Morse functions, we get operations on the singular cohomology $\mathrm{H}^\bullet(D)$:

$$\bigotimes_{k=1}^{p} [\varphi_{f_k}]^{-1} \circ [\psi_{\Gamma^g_{p,q}}] \circ \bigotimes_{\ell=1}^{q} [\varphi_{f_\ell}] : \bigotimes_{\ell=1}^{q} \mathrm{H}^\bullet(D) \longrightarrow \bigotimes_{k=1}^{p} \mathrm{H}^\bullet(D).$$

A particular operation of interest to us in the Morse setting is the product $[\psi^0_{\Gamma_{1,2}}]$. As one expects, it coincides with the cup product $\smile$ in singular cohomology.

Lemma 4.13 • [BC94, Example 3], [Fuk97, Proposition 1.11]

Let $\beta_1,\beta_2 \in \mathrm{H}^\bullet(D)$. Then, for a model graph $(\Gamma^0_{1,2}, L_i, f_1^-, f_1^+, f_2^+)$ the following diagram commutes

$$\begin{tikzcd}
\mathrm{H}^\bullet(D) \otimes \mathrm{H}^\bullet(D) \arrow[r, "{[\varphi_{f_1^+}] \otimes [\varphi_{f_2^+}]}"] \arrow[d, "\cdot \smile \cdot"'] & \mathrm{HM}^\bullet(f_1^+) \otimes \mathrm{HM}^\bullet(f_2^+) \arrow[d, "{[\psi_{\Gamma^0_{1,2}}]}"] \\
\mathrm{H}^\bullet(D) \arrow[r, "{[\varphi_{f_1^-}]}"'] & \mathrm{HM}^\bullet(f_1^+)
\end{tikzcd}$$

4.4.3. The PSS isomorphism

Let H be an admissible Hamiltonian and f a generic Morse function on $\overline{D}$. Consider an homotopy H_s interpolating between H and 0. On the chain level, the *PSS map* is defined as

$$\begin{aligned}
\Phi^{\mathrm{PSS}}_H : \mathrm{CM}^d(f) &\longrightarrow \mathrm{CF}^d(H) \\
z &\longmapsto \sum_{|x| = \mathrm{ind}_f(z)} \#_2 \mathcal{N}(x, z; H, f)\, x
\end{aligned}$$

The moduli space $\mathcal{N}(x,z;H,f)$ consists of *spiked disk* between x and z. These curves consist of a map $u:\mathbb{C}\to\hat{D}$ such that $u(e^{2\pi(s+it)})$ satisfies Floer's continuation equation for H_s. For the asymptotics, we impose that $\lim_{s\to-\infty}u=x$ and that $\lim_{s\to+\infty}u=u(0)$ sits inside the unstable manifold of z. The *spike* refers to the negative gradient flow line connecting z to $u(0)$. The PSS map is a chain map and descends on cohomology

$$[\Phi_H^{\mathrm{PSS}}]:\mathrm{HM}^k(f)\longrightarrow\mathrm{HF}^d(H)$$

When considering admissible Hamiltonians with small slopes, the PSS map induces an isomorphism on Floer cohomology just as in the closed case.

Theorem 4.14 • [Rit13, Section 15.2]

Let H be an admissible Hamiltonian and suppose $\tau_H<T_0$. Then, we have an isomorphism

$$[\Phi_H^{\mathrm{PSS}}]:\mathrm{H}^d(D)\xrightarrow{\cong}\mathrm{HF}^d(H)$$

Moreover, when H is C^2 small on D,

$$[\Phi_H^{\mathrm{PSS}}]=[\Phi_H]$$

where $[\Phi_H]$ is the isomorphism defined in Lemma 3.1.

We now have all the tools to express how operations in Floer cohomology are compatible with operations in Morse cohomology.

Theorem 4.15 • [Rit13, Section 15.5]

Let H be as in Section 4.1.3 and suppose $\tau_H<T_0$. Consider a model surface $(\Sigma^g_{p,q},\beta,j)$ such that $A_k\tau_H,B_\ell\tau_H<T_0$ and a model graph

$(\Gamma^g_{p,q}, f_k^-, f_\ell^+)$. Then, the following diagram commutes

$$\begin{tikzcd}
\bigotimes_{k=1}^{p} \mathrm{HF}^\bullet(A_k H) & & \bigotimes_{\ell=1}^{q} \mathrm{HF}^\bullet(B_\ell H) \arrow[ll, "{[\Psi^H_{p,q,g}]}"'] \\
\bigotimes_{k=1}^{p} \mathrm{HM}^\bullet(f_k^-) \arrow[u, "{[\Phi^{\mathrm{PSS}}_{A_1 H}]\otimes\cdots\otimes[\Phi^{\mathrm{PSS}}_{A_\ell H}]}"] & & \bigotimes_{\ell=1}^{q} \mathrm{HM}^\bullet(f_\ell^+) \arrow[ll, "{[\psi_{\Gamma^g_{p,q}}]}"] \arrow[u, "{[\Phi^{\mathrm{PSS}}_{B_1 H}]\otimes\cdots\otimes[\Phi^{\mathrm{PSS}}_{B_k H}]}"']
\end{tikzcd}$$

We can adapt Theorem 4.15 for the pair of pants product $[\Psi_P^{H,K}]$ between two Hamiltonians $H,K \in \mathcal{C}(D)$. For a model graph $(\Gamma^0_{1,2}, f_1^-, f_1^+, f_2^+)$, this adaptation yields the following diagram:

$$\begin{tikzcd}
\mathrm{HF}^\bullet((H\#K)^{2\tau}) & & \mathrm{HF}^\bullet(H^\tau)\otimes\mathrm{HF}^\bullet(K^\tau) \arrow[ll, "{[\Psi_P^{H,K}]}"'] \\
\mathrm{HM}^\bullet(f_1^-) \arrow[u, "{[\Phi^{\mathrm{PSS}}_{(H\#K)^{2\tau}}]}"] & & \mathrm{HM}^\bullet(f_1^+)\otimes\mathrm{HM}^\bullet(f_2^+) \arrow[ll, "{[\psi_{\Gamma^0_{1,2}}]}"] \arrow[u, "{[\Phi^{\mathrm{PSS}}_{H^\tau}]\otimes[\Phi^{\mathrm{PSS}}_{K^\tau}]}"']
\end{tikzcd}$$

4.5. The symplectic cohomology TQFT

From Theorem 4.11, we know that operations in Floer cohomology commute with continuation maps. These operations thus carry through the direct limit used to define symplectic cohomology. Therefore, for a model surface $\Sigma^g_{p,q}$, we have operations in symplectic cohomology

$$[\Psi_{p,q,g}] : \bigotimes_{\ell=1}^{q} \mathrm{SH}^\bullet(D) \longrightarrow \bigotimes_{k=1}^{p} \mathrm{SH}^\bullet(D)$$

Of course, passing through the direct limit removes any dependence on H. The induced operation is also independent on the choice of weights. Thus, $[\Psi_{p,q,g}]$ only depends on the topology (completely described by p, q and g) of the model surface used to define it. In fact the operations just defined satisfy the topological quantum field theory (TQFT) axioms defined by Atiyah in [**Ati88**]. In particular, the pair of pants product $[\Psi_{1,2,0}]$ induces a unital ring structure on $\mathrm{SH}^\bullet(M)$. The unit 1_D for this product is given by the image of $1 \in \mathbb{Z}_2$ under the map $[\Psi_{1,0,0}] : \mathbb{Z}_2 \longrightarrow \mathrm{SH}^0(D)$ induced by the capping operation.The Viterbo map of Section 3.2.1 is a unital ring map.

Theorem 4.16 • [Rit13]

The Viterbo map $v^\bullet : \mathrm{H}^\bullet(D) \to SH^\bullet(D)$ is a unital ring map with respect to the cup product $\smile$ on $\mathrm{H}^\bullet(D)$ and the pair of pants product $[\Psi_{1,2,0}]$ on $\mathrm{SH}^\bullet(D)$. In particular, the unit on $\mathrm{SH}^\bullet(D)$ is given by the image of the unit $e_D \in \mathrm{H}^\bullet(D)$. In terms of action,

$$v^\bullet(e_D) \in \mathrm{im}\left([\iota^{\varepsilon,\infty}_{-\infty,\infty}] : \mathrm{SH}^*_{(-\infty,\varepsilon)}(D) \longrightarrow \mathrm{SH}^\bullet(D) \right).$$

Chapter 5

SPECTRAL INVARIANTS AND THE SPECTRAL NORM

5.1. Spectral invariants

Denote by $\mathrm{Ham}_c(D,\mathrm{d}\lambda)$ the group of compactly supported Hamiltonian diffeomorphisms of $(D,\mathrm{d}\lambda)$ and by $\mathrm{Symp}_c(D,\mathrm{d}\lambda)$ the group of compactly supported symplectomorphisms of $(D,\mathrm{d}\lambda)$. The Hofer norm of a compactly supported Hamiltonian $H \in \mathcal{C}(D)$ is defined as

$$\|H\| = E_+(H) - E_-(H)$$

$$E_- = -\int_0^1 \max_{p\in D} H(t,p)\,\mathrm{d}t, \quad E_+ = -\int_0^1 \min_{p\in D} H(t,p)\,\mathrm{d}t \tag{5.1.1}$$

Using the Hofer norm, we can define a bi-invariant metric [**Hof90**, **LM95**] on $\mathrm{Ham}_c(D,\mathrm{d}\lambda)$ by

$$d_H(\varphi,\psi) = d_H(\varphi\psi^{-1},\mathrm{id}), \quad d_H(\varphi,\mathrm{id}) = \inf\{\|H\| \mid \varphi = \varphi_H\}.$$

Recall that, since we assumed every element of $\mathcal{C}(D)$ to be smoothed (see Section 3.1.6), the concatenation of $H,K \in \mathcal{C}(D)$ is given by

$$H\#K(t,p) = \begin{cases} 2H(2t,p) & \text{for } t \in [0,1/2], \\ 2K(2t-1,p) & \text{for } t \in [1/2,1]. \end{cases} \tag{5.1.2}$$

with the inverse of some $H \in \mathcal{C}(D)$ given by $\overline{H}(t,p) = -H(1-t,p)$.

From Lemma 3.1 and by definition of $\mathrm{HF}^\bullet(H)$ for $H \in \mathcal{C}(D)$, we know that $\mathrm{HF}^\bullet(H) \cong \mathrm{H}^\bullet(D)$. For $\alpha \in \mathrm{H}^\bullet(D)$, we define, following [**Sch00**], the *spectral invariant of H relative to α as*

$$c(\alpha,H) = \inf\left\{\ell \in \mathbb{R} \mid \Phi_H(\alpha) \in \operatorname{im}\left([\iota^{\ell,\infty}_{-\infty,-\infty}] : \mathrm{HF}^*_{(-\infty,\ell)}(H) \to \mathrm{HF}^\bullet(H)\right)\right\}$$

which is, by exactness of the long exact sequence (3.1.1), equivalent to

$$c(\alpha,H) = \inf\left\{\ell \in \mathbb{R} \mid [\pi^{\infty,\infty}_{-\infty,\ell}] \circ \Phi_H(\alpha) = 0\right\}.$$

The following proposition gathers all the properties of spectral invariants we need for the rest of the text. Proofs of these properties can be found[1] in [**FS07**, Section 5].

Proposition 5.1 • Properties of spectral invariants

Let $\alpha,\eta \in \mathrm{H}^\bullet(D)$ and let $H,K \in \mathcal{C}(D)$. Then,

- **Continuity.**

$$E_-(H-K) \leq c(\alpha,H) - c(\alpha,K) \leq E_+(H-K)$$

- **Spectrality.** $c(\alpha,H) \in \mathrm{Spec}(H)$.
- **Triangle inequality.** $c(\alpha \smile \eta, H\#K) \leq c(\alpha,H) + c(\eta,K)$.
- **Monotonicity.** If $H(t,x) \leq K(t,x)$ for all $(t,x) \in [0,1] \times D$, then $c(\alpha,H) \geq c(\alpha,K)$.

Remark 5.2: The continuity property of Proposition 5.1 allows us to define spectral invariants of compactly supported continuous Hamiltonians $H \in C^0_c([0,1] \times D)$. They satisfy continuity, the triangle inequality and monotonicity.

[1]Note that the signs for continuity and monoticity differ from [**FS07**, Section 5] because of differences in sign conventions.

5.1.1. Proving the properties of spectral invariants

In this section we prove all the properties appearing in Proposition 5.1 except for spectrality which follows from exactly the same arguments as in [**Sch00**, Section 2.2]. We adapt the proofs of [**Sch00**] and [**FS07**] for the setting of compactly supported Hamiltonians on Liouville domains. In what follows, $\alpha \in \mathrm{H}^d(D)$, $\eta \in \mathrm{H}^{d'}(D)$ and $H,K \in \mathcal{C}(D)$.

5.1.1.1. Continuity. Consider two τ-extensions H^τ and K^τ. Choose an homotopy $F_\bullet$ from H^τ to K^τ which is monotone at infinity. Define $e_+ = E_+(H-K)$ and $c = c(\alpha,K)$. In order to relate $c(\alpha,H)$ and $c(\alpha,K)$, consider the following diagram.

$$\begin{tikzcd}
 & \mathrm{HF}^d(K) \arrow[r, "{[\pi^K_{>c}]}"] \arrow[d, "{[\Phi^{H,K}]}", "\cong"'] & \mathrm{HF}^d_{>c}(K) \arrow[d, "{[\Phi^{H,K}]}"] \\
\mathrm{H}^d(D) \arrow[ur, "{[\Phi^{\mathrm{PSS}}_K]}"] \arrow[dr, "{[\Phi^{\mathrm{PSS}}_H]}"'] & & \\
 & \mathrm{HF}^d(H) \arrow[r, "{[\pi^H_{>c+e_+}]}"'] & \mathrm{HF}^d_{>c+e_+}(H)
\end{tikzcd} \tag{5.1.3}$$

We notice the following:

- Theorem 4.14 assures us that the PSS isomorphisms $[\Phi^{\mathrm{PSS}}_K]$ and $[\Phi^{\mathrm{PSS}}_H]$ preserve the grading. Therefore, we only consider the Floer and singular cohomologies in degree d which corresponds to the degree of the class α. Moreover, the triangle on the left of diagram 5.1.3 commutes.

- Since the homotopy $F_\bullet$ is not monotone on D, the restriction of its associated rectriction map $[\Phi^{H,K}]$ on action windows shifts the upper bound by e_+. This is a direct application of Lemma 4.7. The vertical arrow on the far left of the diagram take this into account.

- By Lemma 3.2, the vertical arrow given by the continuation map between the full Floer cohomologies of H and K is an isomorphism.
- Recall that continuation maps commute with the projection $\pi_{>c}$. Thus, the square in 5.1.3 commutes.

By definition of $c(\alpha,K)$, we have

$$[\pi^K_{>c}] \circ [\Phi^{\mathrm{PSS}}_K](\alpha) = 0 \in \mathrm{HF}^d_{>c}(K) \tag{5.1.4}$$

By the above points, we can compute an upper bound for $c(\alpha,H)$. Indeed, form Equation (5.1.4), we have

$$[\pi^H_{>c+e_+}] \circ [\Phi^{\mathrm{PSS}}_K](\alpha) = [\Phi^{\mathrm{PSS}}_K] \circ [\pi^K_{>c}] \circ [\Phi^{\mathrm{PSS}}_K](\alpha) = [\Phi^{\mathrm{PSS}}_K](0) = 0.$$

Thus, from the definition of spectral invariants

$$c(\alpha,H) \leq c + e_+ = c(\alpha,K) + E_+(H-K)$$

from which we deduce the right hand side of the continuity property.

To prove the left hand side, repeat the same argument with an homotopy from K to H. Doing so yields

$$-E_+(K-H) \leq c(\alpha,H) - c(\alpha,K). \tag{5.1.5}$$

Notice that, by definition of E_+ and E_-,

$$-E_+(K-H) = \int_0^1 \min_D (K-H)\mathrm{d}t = -\int_0^1 \max_D (H-K)\mathrm{d}t = E_-(H-K).$$

Combining the two previous equations yields the left hand side of the continuity property.

5.1.1.2. Monotonicity. The proof of monotonicity follows from the proof of continuity. Suppose $H \leq K$ on $S^1 \times D$. Then,

$$-E_+(K-H) = \int_0^1 \min_D (K-H)\mathrm{d}t \geq 0.$$

Therefore, Equation (5.1.5) implies

$$c(\alpha,H) \geq c(\alpha,K) - E_+(K-H) \geq c(\alpha,K).$$

5.1.1.3. Triangle inequality. Let H,K be two smoothed Hamiltonians compactly supported on D. Suppose $c(\alpha,H) = a$ and $c(\eta,K) = b$. Consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}^\bullet(D) \otimes \mathrm{H}^\bullet(D) \arrow[r, "\cdot\smile\cdot"] \arrow[d, "{[\Phi_H^{\mathrm{PSS}}]\otimes[\Phi_K^{\mathrm{PSS}}]}"] & \mathrm{H}^\bullet(D) \arrow[d, "{[\Phi_{H\#K}^{\mathrm{PSS}}]}"] \\
\mathrm{HF}^\bullet(H) \otimes \mathrm{HF}^\bullet(K) \arrow[r, "{[\Psi_P^{H,K}]}"] & \mathrm{HF}^\bullet(H\#K) \\
\mathrm{HF}^\bullet_{<a}(H) \otimes \mathrm{HF}^\bullet_{<b}(K) \arrow[r, "{[\Psi_P^{H,K}]}"] \arrow[u, "{[\iota^{<a}]\otimes[\iota^{<b}]}"] & \mathrm{HF}^\bullet_{<a+b}(H\#K) \arrow[u, "{[\iota^{<a+b}]}"']
\end{tikzcd} \tag{5.1.6}$$

In virtue of Lemma 4.8, the bottom rectangle in Diagram (5.1.6) commutes. Moreover, by Theorem 4.15 and Lemma 4.13 the top rectangle in Diagram Equation (5.1.6) also commutes.

By definition of the constants a,b and of spectral invariants, there exists $[y_1] \in \mathrm{HF}^\bullet(H)$ and $[y_2] \in \mathrm{HF}^\bullet(K)$ such that

$$[\Phi_H^{\mathrm{PSS}}] \otimes [\Phi_K^{\mathrm{PSS}}](\alpha \otimes \eta) = [\iota^{<a}] \otimes [\iota^{<b}]([y_1] \otimes [y_2]).$$

Then, by commutativity of Diagram (5.1.6), we have

$$[\Phi_{H\#K}^{\mathrm{PSS}}](\alpha \smile \eta) = [\iota^{<a+b}] \circ [\Psi_P^{H,K}]([y_1] \otimes [y_2]).$$

Therefore, by definition of spectral invariants

$$c(\alpha \smile \eta, H\#K) \leq a + b = c(\alpha,H) + c(\eta,K).$$

This completes the proof of the triangle inequality.

5.1.2. Additional properties of spectral invariants

The following lemma assures us that spectral invariants are well defined on $\mathrm{Ham}_c(D,\mathrm{d}\lambda)$. The proof relies on the spectrality and the triangle inequality.

Lemma 5.3

Let $H,K \in \mathcal{C}(D)$ such that $\varphi_H = \varphi_K$ and let $\alpha \in \mathrm{H}^\bullet(D)$. Then,

$$c(\alpha,H) = c(\alpha,K)$$

Proof. We have $\varphi_{H\#\overline{K}} = \varphi_0 = \mathrm{id}$ and in that case $\mathrm{Spec}(H\#\overline{K}) = \{0\}$. Now, by spectrality of spectral invariants, $c(\alpha,H\#\overline{K}) = 0$. Thus, the triangle inequality yields

$$c(\alpha,H) = c(\alpha,H\#\overline{K}\#K) \leq c(\alpha,H\#\overline{K}) + c(\alpha,K) = c(\alpha,K).$$

Repeating the same argument with $K\#\overline{H}$ instead of $H\#\overline{K}$, we obtain $c(\alpha,K) \leq c(\alpha,H)$ which concludes the proof. □

The spectral invariant with respect to the cohomological unit admits an implicit definition which depends on the spectral invariants with respect to all other cohomology classes in $\mathrm{H}^\bullet(D)$. This follows directly from the triangle inequality.

Lemma 5.4

Let $H \in \mathcal{C}(D)$. Then,

$$c(1,H) = \max_{\alpha \in \mathrm{H}^\bullet(D)} c(\alpha,H).$$

Proof. Let $\alpha \in H^\bullet(D)$. By definition of the unit and the concatenation of Hamiltonians, we have

$$c(\alpha,H) = c(\alpha \smile 1,H) = c(\alpha \smile 1,0\#H).$$

Then, since $c(\alpha,0) = 0$, the triangle inequality guaranties that

$$c(\alpha,H) = c(\alpha \smile 1,0\#H) \leq c(\alpha,0) + c(1,H) = c(1,H).$$

The choice of α being arbitrary, this concludes the proof. □

5.1.3. The symplectic contraction principle

We conclude this section by recalling the symplectic contraction technique introduced by Polterovich [Pol14, Section 5.4]. This principle allows one to describe the effect of the Liouville flow $\{\psi_Y^{\log r}\}_{0<r<1}$ on spectral invariants.

First, we need to describe how the Liouville flow acts on the symplectic form ω of D and on compactly supported Hamiltonians on D. Since $L_Y\omega = \omega$, we have that the Liouville flow contracts the symplectic form :

$$\left(\psi_Y^{\log r}\right)^* \omega = r\omega$$

Now, consider a Hamiltonian $H \in \mathcal{C}(D)$ supported in $U \subset D$. For fixed $0 < r < 1$ define the Hamiltonian

$$H_r(t,x) = \begin{cases} rH\left(t, \left(\psi_Y^{\log r}\right)^{-1}(x)\right) & \text{if } x \in \psi_Y^{\log r}(U), \\ 0 & \text{if } x \notin \psi_Y^{\log r}(U). \end{cases} \tag{5.1.7}$$

It then follows from the two previous equations that $\mathrm{Spec}(H_r) = r\,\mathrm{Spec}(H)$. This allows one to prove

Lemma 5.5 • Symplectic contraction principle [Pol14]

Suppose $H \in \mathcal{C}(D)$ and let $H_r \in \mathcal{C}(D)$ be as in Equation 5.1.7. Then,

$$c(1,H_r) = rc(1,H).$$

5.2. Spectral norm

We define the *spectral norm* $\gamma(H)$ *of* $H \in \mathcal{C}(D)$ as

$$\gamma(H) = c(1,H) + c(1,\overline{H}).$$

For $\varphi \in \mathrm{Ham}_c(D,\mathrm{d}\lambda)$ such that $\varphi = \varphi_H$, define

$$\gamma(\varphi) = \gamma(H)$$

In virtue of Lemma 5.3, this is well defined.

From [**FS07**, Section 7], we have the following theorem which justifies calling γ a norm.

Theorem 5.6

Let $\varphi,\psi \in \mathrm{Ham}_c(D,\mathrm{d}\lambda)$ and let $\chi \in \mathrm{Symp}_c(D,\mathrm{d}\lambda)$. Then,

- **Non-degeneracy.** $\gamma(\mathrm{id}) = 0$ and $\gamma(\varphi) > 0$ if $\gamma \neq \mathrm{id}$,
- **Triangle inequality.** $\gamma(\varphi\psi) \leq \gamma(\varphi) + \gamma(\psi)$,
- **Symplectic invariance.** $\gamma(\chi \circ \varphi \circ \chi^{-1}) = \gamma(\varphi)$,
- **Symmetry.** $\gamma(\varphi) = \gamma(\varphi^{-1})$,
- **Hofer bound.** $\gamma(\varphi) \leq d_H(\varphi,\mathrm{id})$.

Chapter 6

COHOMOLOGICAL GANOR-TANNY BARRICADES

In [GT23] Ganor and Tanny introduced a particular perturbation of Hamiltonians compactly supported inside *contact incompressible boundary* domains (CIB) of closed aspherical symplectic manifolds. For instance, if $U \subset M$ is an incompressible open set which is a Liouville domain, then U is a CIB. In Floer homology, the aforementioned Hamiltonian perturbation, which is called a barricade, prohibits the existence of Floer trajectories exiting and entering the CIB. We consider barricades in the particular case of Liouville domains and adapt them to Floer cohomology.

In the present setting, we define barricades for a special class of admissible Hamiltonians.

Definition 6.1 • Barricade admissible Hamiltonians

A Hamiltonian H is said to be r_0-*barricade-admissible* if $H \in \mathcal{H}_{r_0}$ and the following conditions hold:

- $H(t,x,r) = h(r)$ on $\hat{D} \setminus D^{\rho_0}$ for some $\rho_0 \in (0,1)$,
- $h(r)$ is C^2-small on $(1,r_0 - \varepsilon)$,
- $h(r)$ is strictly convex on $(r_0 - \varepsilon,r_0)$.

Here $\varepsilon > 0$ is small enough so that $1 < r_0 - \varepsilon$. We denote the set of r_0-barricade-admissible Hamiltonians by $\overline{\mathcal{H}}_{r_0}$.

We say that $(F_\bullet, J)$ is an r_0-*barricade-admissible pair* if $F_\bullet$ is a monotone homotopy such that $F_s \in \overline{\mathcal{H}}_{r_0}$ for all s and J is an admissible almost complex structure.

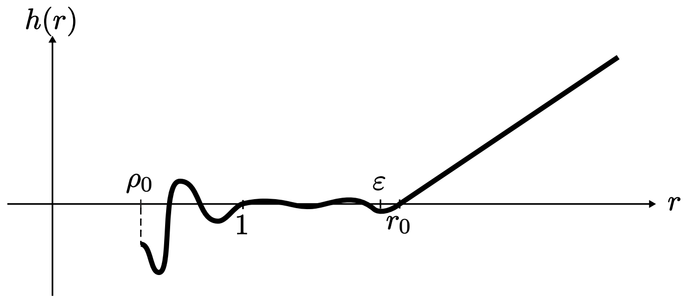


Fig. 1. An r_0-barricade-admissible Hamiltonian.

Remark 6.2: By Definition 3.4, the τ-extension H^τ of any Hamiltonian H compactly supported in D can be chosen so that it is r_0-barricade-admissible.

Definition 6.3 • Barricade

Let $r_0 > 1$ and $0 < \varepsilon < r_0 - 1$. Define $B_{r_0,\varepsilon} = D^{r_0-\varepsilon} \setminus D$ where, for $\rho > 0$, $D^\rho = \Psi_Y^{\log \rho}(D)$. Suppose $(F_\bullet,J)$ is an r_0-barricade-admissible pair from F_+ to F_-. We say that $(F_\bullet,J)$ admits a barricade on $B_{r_0,\varepsilon}$ if for every $x_\pm \in \mathcal{P}(F_\pm)$ and every Floer trajectory $u : \mathbb{R} \times S^1 \to \hat{D}$ connecting $x_\pm$, we have, for $D_{\mathrm{b}} := D^{r_0-\varepsilon} = D \cup B_{r_0,\varepsilon}$

(1) If $x_- \in D$, then $\mathrm{im}(u) \subset D$,
(2) If $x_+ \in D_b$, then $\mathrm{im}(u) \subset D_{\mathrm{b}}$.

Remark 6.4: In the language of [**GT23**], a barricade on $B_{r_0,\varepsilon}$ as described above would be called a barricade in $D^{r_0-\varepsilon}$ around D.

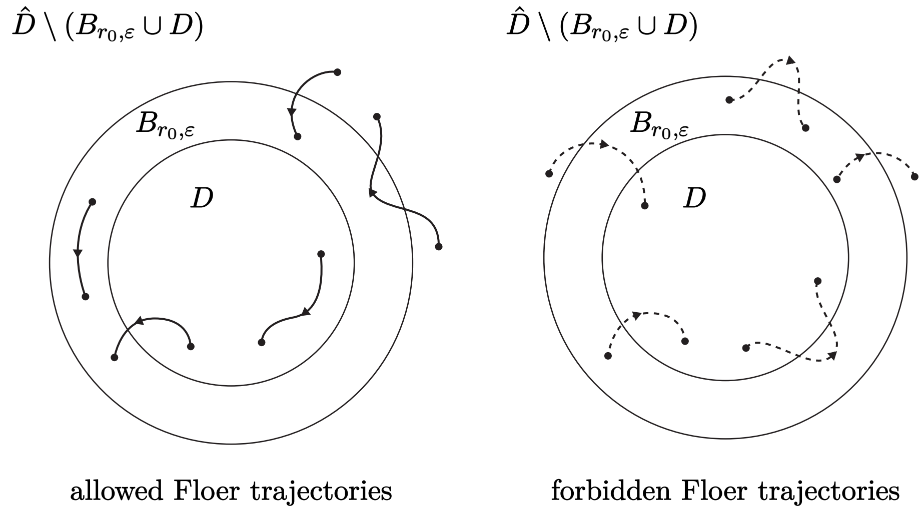


Fig. 2. Floer cylinders in a barricade. The arrows follow the direction of the Floer differential and the continuation map: from x_+ to x_-.

6.1. How to construct barricades

To construct barricades, we need to consider special classes of pairs of Hamiltonians and almost complex structures. These are defined using a refinement of Definition 3.5 in [**GT23**].

> **Definition 6.5 • Cylindrical bumps**
>
> Let $r_0 > 1$, $\sigma \in (0, +\infty) \setminus \mathrm{Spec}(\partial D, \lambda)$ and $0 < \varepsilon < r_0 - 1$. An r_0-barricade-admissible pair $(F_\bullet, J)$ admits a cylindrical bump of slope σ on $B_{r_0,\varepsilon}$ if
>
> (1) $F = 0$ on $\partial B_{r_0,\varepsilon} \times S^1 \times \mathbb{R}$,
>
> (2) $JY = R_\alpha$, for Y the Liouville vector field on D, on a neighborhood of $\partial B_{r_0,\varepsilon}$, i.e. J is cylindrical near $\partial B_{r_0,\varepsilon} = \partial D \sqcup (\{r_0 - \varepsilon\} \times \partial D)$.
>
> (3) $\nabla_J F = \sigma Y$ near $(\{1\} \times \partial D) \times S^1 \times \mathbb{R}$ and $\nabla_J F = -\sigma Y$ near $(\{r_0 - \varepsilon\} \times \partial D) \times S^1 \times \mathbb{R}$. Here, ∇_J denotes the gradient induced by the metric g_J.

(4) All 1-periodic orbits of $F_\pm$ contained in $B_{r_0,\varepsilon}$ are critical points with values in the interval $(-\sigma,\sigma)$. (In particular, $\sigma < T_0$.)

A cohomological adaptation of Lemma 3.3 in [**GT23**] yields the following action estimates for pairs with cylindrical bumps.

Lemma 6.6

Suppose that the r_0-barricade-admissible pair (F,J) admits a cylindrical bump of slope σ on $B_{r_0,\varepsilon}$. For every finite energy solution u connecting $x_\pm \in \mathcal{P}(F_\pm)$, then

(1) $\operatorname{im} x_- \subset D$ and $\operatorname{im} x_+ \subset \hat{D} \setminus D \implies \mathcal{A}_{F_+}(x_+) > \sigma$,

(2) $\operatorname{im} x_+ \subset D_\mathrm{b}$ and $\operatorname{im} x_- \subset \hat{D} \setminus D_\mathrm{b} \implies \mathcal{A}_{F_-}(x_-) < -\sigma$.

Lemma 6.6 and the maximum principle are all we need to prove that every pair with a cylindrical bump admits a barricade. More precisely, we have the following proposition.

Proposition 6.7

Let (F,J) be a r_0-barricade-admissible pair with a cylindrical bump of slope σ on $B_{r_0,\varepsilon}$. Then, (F,J) admits a barricade on $B_{r_0,\varepsilon}$.

Proof. Suppose $u : \mathbb{R} \times S^1 \to \hat{D}$ is a Floer trajectory between $x_\pm \in \mathcal{P}(F_\pm)$. We only need to study the case where $\operatorname{im} x_- \subset D$ and the case where $\operatorname{im} x_+ \subset D_\mathrm{b}$.

Suppose that $\operatorname{im} x_- \subset D$. We first establish that x_+ must lie inside D. Indeed, if $\operatorname{im} x_+ \subset \hat{D}\setminus D$, part (1) of Lemma 6.6 assures us that $\mathcal{A}_{F_+}(x_+) > \sigma$ which contradicts the fact that orbits on $\hat{D} \setminus D$ must have action in the interval $(-\sigma,\sigma)$ by the construction of the cylindrical bump. Therefore, $\operatorname{im} x_+ \subset D$ as desired. Now, since $\operatorname{im} x_\pm \subset D$, the maximum principle guarantees that $\operatorname{im} u \subset D$.

To finish the proof, we look at the case where $\operatorname{im} x_+ \subset D_\mathrm{b}$. Similarly to the previous case, we prove that x_- also lies inside D_b. If $\operatorname{im} x_- \subset \hat{D} \setminus D_\mathrm{b}$, part 2 of Lemma 6.6 imposes $\mathcal{A}_{F_-}(x_-) < -\sigma$, which is again impossible by construction of the cylindrical bump. Therefore, $\operatorname{im} x_- \subset D_{\mathrm{b}}$ and the maximum principle implies $\operatorname{im} u \subset D_{\mathrm{b}}$. □

Given a pair (F,J) and $\sigma > 0$ small, we can add to F a $\mathcal{C}^\infty$-small radial bump function χ with support inside $B_{r_0,\varepsilon}$ such that $(F+\chi,J)$ has a cylindrical bump of slope σ on $B_{r_0,\varepsilon}$. By Proposition 6.7, the perturbed pair will also admit a barricade on $B_{r_0,\varepsilon}$. A second perturbation of the Hamiltonian term at its ends, under which the barricade survives, allows us to achieve Floer regularity for the pair. This procedure is carried out carefully in [**GT23**, section 9] and yields the following.

Theorem 6.8 • [GT23]

Let $F_\bullet$ be a monotone homotopy. Then, there exists a $\mathcal{C}^\infty$-small perturbation $F^{\mathrm{b}}_\bullet$ of $F_\bullet$ and an almost complex structure J such that the pairs $(F^{\mathrm{b}}_\bullet,J)$ and $(F^{\mathrm{b}}_\pm,J)$ are Floer-regular and have a barricade on $B_{r_0,\varepsilon}$.

6.2. Decomposition of the Floer cochain complex

Let us investigate what structure barricades impose on the Floer co-chain complex. Let $H \in \overline{\mathcal{H}}_{r_0}$ and suppose the pair (H,J) admits a barricade on $B_{r_0,\varepsilon}$. For an open subset $U \subset \hat{D}$, denote by $\mathrm{C}^\bullet(U,H)$ the set of 1-periodic orbits of H in U. By definition of the differential ∂ on Floer cohomology, $\mathrm{C}^\bullet(D_{\mathrm{b}},H)$ is closed under ∂ and it therefore forms a sub-complex of $\mathrm{CF}^\bullet(H)$. Moreover, for $D_{\mathrm{c}} = \hat{D} \setminus D_{\mathrm{b}}$, we also have that

$$\mathrm{C}^\bullet(D_{\mathrm{c}},H) = \frac{\mathrm{CF}^\bullet(H)}{\mathrm{C}^\bullet(D_{\mathrm{b}},H)}$$

is a well defined cochain complex. In terms of vector spaces, we have the decomposition

$$\mathrm{CF}^\bullet(H) \cong \mathrm{C}^\bullet(D_{\mathrm{b}},H) \oplus \mathrm{C}^\bullet(D_{\mathrm{c}},H).$$

The direct product gives us injections ι_{b}^H, ι_{c}^H and projections π_{b}^H, π_{c}^H for which the diagram

$$\begin{tikzcd}
 & \mathrm{C}^\bullet(D_{\mathrm{b}}, H) & \\
\mathrm{C}^\bullet(D_{\mathrm{b}}, H) \arrow[ur, "\mathrm{id}"] \arrow[dr, "0"'] \arrow[r, "\iota_{\mathrm{b}}^H"] & \mathrm{CF}^\bullet(H) \arrow[u, "\pi_{\mathrm{b}}^H"] \arrow[d, "\pi_{\mathrm{c}}^H"] & \mathrm{C}^\bullet(D_{\mathrm{c}}, H) \arrow[ul, "0"'] \arrow[dl, "\mathrm{id}"] \arrow[l, "\iota_{\mathrm{c}}^H"] \\
 & \mathrm{C}^\bullet(D_{\mathrm{c}}, H) &
\end{tikzcd}$$

commutes and the equation

$$\iota_{\mathrm{b}}^H \circ \pi_{\mathrm{b}}^H(q) + \iota_{\mathrm{c}}^H \circ \pi_{\mathrm{c}}^H(q) = q$$

holds for any $q \in \mathrm{CF}^\bullet(H)$. Here, the projection π_{c}^H coincides with the canonical projection

$$\mathrm{CF}^\bullet(H) \longrightarrow \frac{\mathrm{CF}^\bullet(H)}{\mathrm{C}^\bullet(D_{\mathrm{b}}, H)}$$

The differential ∂_{b} on $\mathrm{C}^\bullet(D_{\mathrm{b}})$ is simply the restriction of the differential ∂ of $\mathrm{CF}^\bullet(H)$ on $\mathrm{C}^\bullet(D_{\mathrm{b}})$. The differential ∂_{c} on $\mathrm{C}^\bullet(D_{\mathrm{c}})$ is the quotient complex differential defined by

$$\partial_{\mathrm{c}} \pi_{\mathrm{c}}^H(p) = \pi_{\mathrm{c}}^H(\partial p).$$

6.2.1. Continuation maps

Let $(F_\bullet, J)$ be an r_0-barricade-admissible pair that admits a barricade on $B_{r_0,\varepsilon}$. Then, since the continuation map $\Phi_{F_\bullet} : \mathrm{CF}^\bullet(F_+) \to \mathrm{CF}^\bullet(F_-)$ counts Floer trajectories of F connecting 1-periodic orbits of F_+ to 1-periodic orbits of F_-, it restricts, due to the barricade, to a chain map

$$\Phi_F^{\mathrm{b}} : \mathrm{C}^\bullet(D_{\mathrm{b}}, F_+) \longrightarrow \mathrm{C}^\bullet(D_{\mathrm{b}}, F_-)$$

Moreover, in virtue of Lemma 6.10 below, Φ_F projects to a chain map

$$\Phi_F^{\mathrm{c}} : \mathrm{C}^\bullet(D_{\mathrm{c}},F_+) \longrightarrow \mathrm{C}^\bullet(D_{\mathrm{c}},F_-)$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{CF}^\bullet(F_+) \arrow[r, "\Phi_{F_\bullet}"] \arrow[d, "\pi_{\mathrm{b}}^+"'] & \mathrm{CF}^\bullet(F_-) \arrow[d, "\pi_{\mathrm{b}}^-"] \\
\mathrm{C}^\bullet(D_{\mathrm{c}}, F_+) \arrow[r, "\Phi_{F_\bullet}^{\mathrm{c}}"'] & \mathrm{C}^\bullet(D_{\mathrm{c}}, F_-)
\end{tikzcd}$$

where we write $\pi_{\mathrm{b}}^+ = \pi_{\mathrm{b}}^{F_+}$ and $\pi_{\mathrm{b}}^- = \pi_{\mathrm{b}}^{F_-}$.

6.2.2. Chain homotopies

Let $(F_\pm,J)$ be r_0-barricade-admissible pairs that admit cylindrical bumps of slope σ on $B_{r_0,\varepsilon}$ such that F_+ and F_- have the same slope $\tau_+ = \tau_-$ at infinity. Consider the linear homotopy

$$F_s = F_- + b(s)(F_+ - F_-)$$

where $b : \mathbb{R} \to [0,1]$ is a smooth function such that $b(s) = 0$ for $s \le -1$, $b(s) = 1$ for $s \ge 1$ and $b'(s) > 0$ for all $s \in (-1,1)$. Denote by $\overline{F}_\bullet$ the inverse homotopy defined by $\overline{F}_s = F_{-s}$. For $\rho > 1$ large, we define the concatenation $F\#\overline{F}_\bullet$ as

$$(F\#\overline{F})_s = \begin{cases} F_{s+\rho} & \text{for } s \le 0 \\ \overline{F}_{s-\rho} & \text{for } s \ge 0 \end{cases}.$$

Using the definition of $F_\bullet$ and $\overline{F}_\bullet$, we can simply write

$$(F\#\overline{F})_s = F_- + b_\rho(s)(F_+ - F_-)$$

for $b_\rho(s) = b(-|s| + \rho)$. The homotopy $F\#\overline{F}_\bullet$ generates the composition of continuation homomorphisms $\Phi_F \circ \Phi_{\overline{F}} : \mathrm{CF}^\bullet(F_-) \to \mathrm{CF}^\bullet(F_-)$ which is chain homotopic to the identity on $\mathrm{CF}^\bullet(F_-)$,

$$\Phi_{F_\bullet} \circ \Phi_{\overline{F}_\bullet} - \mathrm{id}_- = \partial_- \circ \Psi_- - \Psi_- \circ \partial_-$$

for $\Psi_- : \mathrm{CF}^\bullet(F_-) \to \mathrm{CF}^{\bullet-1}(F_-)$ and ∂_- the differential on $\mathrm{CF}^\bullet(F_-)$. The chain homotopy Ψ_- is built by counting Floer solutions of the homotopy $\{\Gamma^\kappa\}_{\kappa\in[0,1]}$ between $F\#\overline{F}_\bullet$ and the constant homotopy F_- which is defined by

$$\Gamma^\kappa_s = F_- + \kappa b_\rho(s)(F_+ - F_-).$$

For $x \in \mathcal{P}(F_-)$ and $y \in \mathcal{P}(F_+)$, define

$$\mathcal{M}^\Gamma(x,y) = \left\{(\kappa,u) \mid \kappa \in [0,1],\, u \in \mathcal{M}(x,y;\Gamma^k_\bullet)\right\}.$$

We can perturb Γ with a C^∞-small function in order to make it regular [**AD14**, Chapter 11]. Now, since the pairs $(F_\pm,J)$ admit cylindrical bumps of slope σ on $B_{r_0,\varepsilon}$, and thus have barricades on $B_{r_0,\varepsilon}$, solutions to the parametric Floer equation for Γ^κ also admit cylindrical bumps of slope σ on $B_{r_0,\varepsilon}$ and have barricades on $B_{r_0,\varepsilon}$. To see this, first fix $\kappa \in [0,1]$, We need to show that Γ^κ satisfies conditions (1) through (4) of Definition 6.5. For (1), we have, on $\partial B_{r_0,\varepsilon} \times S^1 \times \mathbb{R}$,

$$\Gamma^\kappa = F_- + \kappa\alpha_\rho(s)(F_+ - F_-) = 0 + \kappa b_\rho(s)(0-0) = 0.$$

Condition (2) is automatically satisfied since J is fixed. For condition (3), we have on $(\{1\}\times\partial D)\times S^1\times\mathbb{R}$,

$$\nabla_J\Gamma^\kappa = \nabla_J F_- + \kappa\alpha_\rho(s)(\nabla_J F_+ - \nabla_J F_-) = \sigma Y + \kappa\alpha_\rho(s)(\sigma Y - \sigma Y) = \sigma Y$$

and, by the same computation, $\nabla_J\Gamma^\kappa = -\sigma Y$ on $(\{r_0-\varepsilon\}\times\partial D)\times S^1\times\mathbb{R}$. Condition (4) is also satisfied since $\Gamma^\kappa_{\pm\infty} = F_-$. All of this still holds with regular perturbations of Γ.

Lemma 6.9

Let $F_-,F_+ \in \overline{\mathcal{H}}_{r_0}$ with same slope at infinity and suppose they both admit barricades on $B_{r_0,\varepsilon}$. Furthermore, suppose that solutions to the parametric Floer equation for Γ^κ also admit barricades on $B_{r_0,\varepsilon}$. Then, for any C^∞-small perturbation Γ' of Γ which satisfies $\mathcal{P}(F'_\pm) = \mathcal{P}(F_\pm)$, Floer trajectories in $\mathcal{M}^{\Gamma'}$ follow the rules of the barricade on $B_{r_0,\varepsilon}$.

Proof. The proof follows the same ideas as the proof of Proposition 9.21 in [**GT23**]. By Gromov compactness, any sequence $(\kappa_n,u_n) \in \mathcal{M}^{\Gamma}(x_-,y_+)$ of solutions to the parametric Floer equation converges, up to taking a subsequence, to a broken trajectory $(\kappa,\bar{v})$ where $\bar{v} = (v_1,\dots,v_k,w,v'_1,\dots,v'_\ell)$ connects two orbits $x_\pm \in \mathcal{P}(F_\pm)$. The fact that $F_\pm$ both admit a barricade on $B_{r_0,\varepsilon}$ assures us that

- $x_- \in D \implies \bar{v} \subset D$
- $x_+ \in D \implies \bar{v} \subset D_{\mathrm{b}}$.

Now, consider a sequence of regular homotopies $\{\Gamma_n\}_n$ with ends $\lim_{s\to\pm\infty}\Gamma_{s,n} = F_{n\pm}$ converging to Γ such that $\mathcal{P}(F_{n\pm}) = \mathcal{P}(F_\pm)$ for all n. Then, the above two implications regarding broken trajectories imply that every trajectory $(\kappa_n,u'_n) \in \mathcal{M}^{\Gamma'}(x_-,x_+)$, for $x_\pm \in \mathcal{P}(F_\pm)$, obey the rules of the barricade. $\square$

Thus, Ψ_- restricts to a map $\Psi_-^b : \mathrm{C}^\bullet(D_{\mathrm{b}},F_-) \to \mathrm{C}^{\bullet-1}(D_{\mathrm{b}},F_-)$ and by Lemma 6.11 below, we can define its projection $\Psi_-^c : \mathrm{C}^\bullet(D_{\mathrm{c}},F_-) \to \mathrm{C}^{\bullet-1}(D_{\mathrm{c}},F_-)$.

6.2.3. Some technical lemmas

When adapting computations from homology to cohomology, we often have to rely on quotient complexes instead of sub-complexes. Here are a few simple results from homological algebra which will be useful in that regard. Let (A,d_A) and (C,d_C) be cochain complexes and let $B \subset A$ and $D \subset C$ be sub-complexes.

Lemma 6.10

Suppose $f : (A,B) \to (C,D)$ is a chain map. Then, there exists a unique chain map $\bar{f} : A/B \to C/D$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\pi_B"'] & C \arrow[d, "\pi_D"] \\ A/B \arrow[r, "\bar{f}"'] & C/D \end{tikzcd}$$

for π_B and π_D the canonical projections. It follows that, on cohomology, we have the following commutative diagram.

$$\begin{tikzcd} \mathrm{H}^\bullet(A) \arrow[r, "{[f]}"] \arrow[d, "{[\pi_B]}"'] & \mathrm{H}^\bullet(C) \arrow[d, "{[\pi_D]}"] \\ \mathrm{H}^\bullet(A/B) \arrow[r, "{[\bar{f}]}"'] & \mathrm{H}^\bullet(C/D) \end{tikzcd}$$

Proof. Define, for all $x \in A$,

$$\bar{f}(\pi_B(x)) = \pi_D(f(x)).$$

We first need to show that $\bar{f}$ is well defined. Suppose $x' = x + b$ for $x \in A$ and $b \in B$. Then, since f restricts to a map from B to D, there exists $d \in D$ such that $f(b) = d$ and we have

$$\bar{f}(\pi_B(x')) = \pi_D(f(x+b)) = \pi_D(f(x) + d) = \pi_D(f(x)).$$

Thus, $\bar{f}$ is well defined.

To prove uniqueness, we simply use the definition of $\bar{f}$. Suppose we have another map $\bar{g} : A/B \to C/D$ which makes the above diagram commute as well. Then, for all $x \in A$,

$$\bar{f}(\pi_B(x)) - \bar{g}(\pi_B(x)) = \pi_D(f(x)) - \pi_D(f(x)) = 0.$$

$\square$

Lemma 6.11

Suppose $f:(A,B)\to(C,D)$ and $g:(C,D)\to(A,B)$ are chain maps such that $f\circ g$ is chain homotopic to the identity

$$f\circ g-\mathrm{id}_C=d_C\circ\psi-\psi\circ d_C$$

where the chain homotopy is a map $\psi:(C,D)\to(C,D)$. Then, $\bar{f}\circ\bar{g}:C/D\to C/D$ is also chain homotopic to the identity.

Proof. Since the chain homotopy $\psi:(C,D)\to(C,D)$ is a chain map of pairs, Lemma 6.10 allows us to define its projection $\bar{\psi}:C/D\to C/D$. Thus, for all $y\in C$,

$$\begin{aligned}
\bar{f}\circ\bar{g}(\pi_D(y))-\mathrm{id}_{C/D}(\pi_D(y)) &= \bar{f}\circ\pi_B(g(y))-\pi_D(\mathrm{id}_C(y))\\
&=\pi_D(f\circ g(y))-\pi_D(\mathrm{id}_C(y))\\
&=\pi_D((d_C\circ\psi-\psi\circ d_C)(y))\\
&=(d_{C/D}\circ\pi_D\circ\psi-\pi_D\circ\psi\circ d_C)(y)\\
&=d_{C/D}\circ\bar{\psi}(\pi_D(y))-\bar{\psi}\circ d_{C/D}(\pi_D(y))
\end{aligned}$$

which proves that $\bar{f}\circ\bar{g}$ is chain homotopic to the identity on C/D since any $z\in C/D$ is of the form $z=\pi_D(y)$. $\square$

Chapter 7

SPECTRAL DIAMETER AND SYMPLECTIC COHOMOLOGY

7.1. Infinite spectral diameter

In this section we give a proof of Theorem A announced in the introduction.

Theorem A

Let $(D,\mathrm{d}\lambda)$ be a Liouville domain. Then,

$$\mathrm{diam}_\gamma(D) = +\infty \quad \Longleftrightarrow \quad \mathrm{SH}^\bullet(D) \neq 0.$$

Fix $A \in (0,\infty) \setminus \mathrm{Spec}(\partial D,\lambda)$. The idea of the proof is to construct a special admissible Hamiltonian $H_{\delta,A}$ for which $c(1,\cdot)$ is bounded from below by $A-\varepsilon$ for ε a small constant which depends on A. The construction of $H_{\delta,A}$ is inspired by [CFO10, Proposition 2.5]. Then, we use the fact that $c(1,\cdot) \geq 0$ to conclude.

7.1.1. Construction of the Hamiltonian

Fix some $r_0 > 1$. For any $\delta \in (0,1)$ and $\sigma \in (0,T_0)$, we define the Hamiltonian $H_{\delta,A}$ as follows :

- $H_{\delta,A}$ is the constant function $A(\delta-1)$ on D^δ,
- $H_{\delta,A}(r,x) = A(r-1)$ on $D \setminus D^\delta$,

- ∘ $H_{\delta,A}(r,x) = 0$ on $D^{r_0} \setminus D$
- ∘ $H_{\delta,A}(r,x) = \sigma(r - r_0)$ on $\hat{D} \setminus D^{r_0}$.

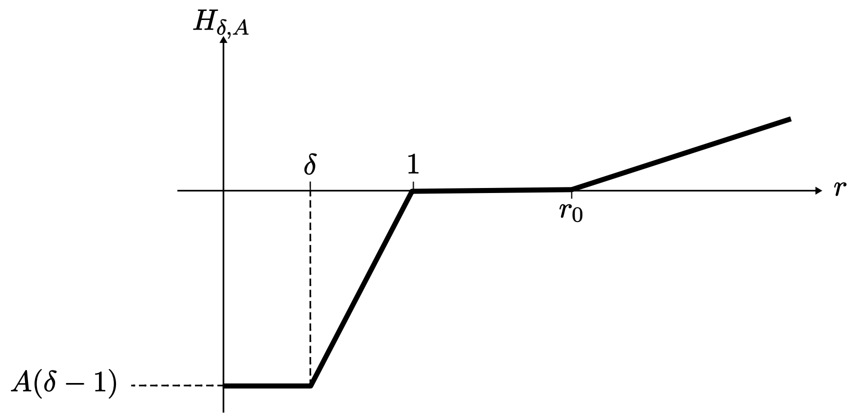


Fig. 1. Radial portion of the Hamiltonian $H_{\delta,A}$.

We add a small perturbation to $H_{\delta,A}$ so that it lies in $\overline{\mathcal{H}}_{r_0}$. Denote by $h_{\delta,A}$ the function of one variable for which $H_{\delta,A} = h_{\delta,A} \circ r$ on D^c. If γ is a 1-periodic orbit of $h_{\delta,A}$ inside the level set $\{r\} \times \partial D$, its action can be written as

$$\mathcal{A}_{H_{\delta,A}}(\gamma) = \mathcal{A}_{H_{\delta,A}}(r) = rh'_{\delta,A}(r) - h_{\delta,A}(r).$$

The 1-periodic orbits of $H_{\delta,A}$ can be classified in three different categories. Recall that η_A denotes the distance between A and $\mathrm{Spec}(\partial D,\alpha)$.

(I) Critical points in D^δ with action close to $r_{\mathrm{I}} := (1 - \delta)A$

(II) Non-constant 1-periodic orbits near $\{\delta\} \times \partial D$ with action in a small neighborhood of the interval

$$I_{\mathrm{II}} = [\delta T_0 + (1 - \delta)A, A - \delta\eta_A].$$

(III) Non-constant 1-periodic orbits near $\{1\} \times \partial D$ with action in a small neighborhood of the interval

$$I_{\mathrm{III}} = [T_0, A - \eta_A].$$

(IV) Critical points in $D^{r_0} \setminus D$ with action close to $r_{\mathrm{IV}} := 0$.

Note that there are no non-constant 1-periodic orbits near $\{r_0\} \times \partial D$, since the slope of the Hamiltonian there ranges from 0 to σ which is less than T_0 by assumption.

We now want to construct a Floer complex $\mathrm{C}^\bullet_{\mathrm{I,II}}$ which will contain the orbits of type (I) and (II) and another complex $\mathrm{C}^\bullet_{\mathrm{III,IV}}$ containing orbits of type (III) and (IV). To that end, pick $0 < \delta < 1$ small enough so that $\delta A < \eta_A$. Now choose $\varepsilon > 0$ such that

$$\delta A < \varepsilon < \eta_A.$$

Then, we have the following inequalities :

$$r_{\mathrm{IV}} < I_{\mathrm{III}} < A - \varepsilon < r_{\mathrm{I}} < I_{\mathrm{II}}.$$

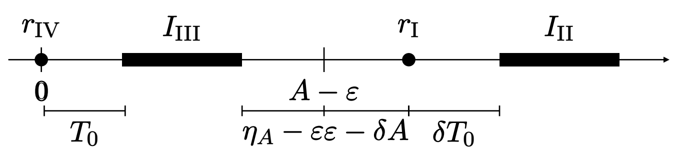


Fig. 2. Distances that separate the action windows under consideration.

As shown in Figure 2, r_{I}, I_{II}, I_{III} and r_{IV} are all separated by distances which depend only on T_0, A, η_A, δ and ε. Thus, we can choose the perturbation we add to $H_{\delta,A}$ to be small enough so that, in terms of action, we have

$$\text{(IV)} < \text{(III)} < A - \varepsilon < \text{(I)} < \text{(II)}.$$

Therefore, since the Floer differential decreases the action, we can define the Floer co-chain complexes as

$$\mathrm{C}^\bullet_{\mathrm{III,IV}} = \mathrm{CF}^\bullet_{(-\infty,A-\varepsilon)}(H_{\delta,A}), \quad \mathrm{C}^\bullet_{\mathrm{I,II}} = \frac{\mathrm{CF}^\bullet(H_{\delta,A})}{\mathrm{C}^\bullet_{\mathrm{III,IV}}} = \mathrm{CF}^\bullet_{(A-\varepsilon,\infty)}(H_{\delta,A})$$

and they yield the Floer cohomology groups

$$\mathrm{H}^\bullet(\mathrm{C}^\bullet_{\mathrm{III,IV}}) = \mathrm{HF}^\bullet_{(-\infty,A-\varepsilon)}(H_{\delta,A}), \quad \mathrm{H}^\bullet(\mathrm{C}^\bullet_{\mathrm{I,II}}) = \mathrm{HF}^\bullet_{(A-\varepsilon,\infty)}(H_{\delta,A}).$$

A quick look at the action windows under consideration informs us that the above complexes fit into the following short exact sequence

$$0 \longrightarrow \mathrm{C}^\bullet_{\mathrm{III,IV}} \xrightarrow{\iota^{A-\varepsilon,+\infty}_{-\infty}} \mathrm{CF}^\bullet(H_{\delta,A}) \xrightarrow{\pi^{+\infty}_{-\infty,A-\varepsilon}} \mathrm{C}^\bullet_{\mathrm{I,\,II}} \longrightarrow 0.$$

which in turn yields an exact triangle in cohomology

$$\begin{tikzcd}
 & \mathrm{HF}^\bullet(H_{\delta,A}) \arrow[dr, "{[\pi^{+\infty}_{-\infty,A-\varepsilon}]}"] & \\
\mathrm{H}^\bullet(\mathrm{C}^\bullet_{\mathrm{III,IV}}) \arrow[ur, "{[\iota^{A-\varepsilon,+\infty}_{-\infty}]}"] & & \mathrm{H}^\bullet(\mathrm{C}^\bullet_{\mathrm{I,II}}) \arrow[ll, "{[+1]}"]
\end{tikzcd}$$

7.1.2. Factoring a map to $\mathrm{SH}^\bullet(D)$

We now build maps Ψ and $\Psi_{\mathrm{I,II}}$ such that the diagram

$$\begin{tikzcd}
\mathrm{HF}^\bullet(H_{\delta,A}) \arrow[r, "{[\pi^{+\infty}_{-\infty,A-\varepsilon}]}"] \arrow[dr, "\Psi"'] & \mathrm{H}^\bullet(\mathrm{C}^*_{\mathrm{I,II}}) \arrow[d, "\Psi_{\mathrm{I,II}}"] \\
 & \mathrm{SH}^\bullet(D)
\end{tikzcd} \tag{7.1.1}$$

commutes. We need to construct Ψ so that it coincides with the map $j_{H_{\delta,A}}$: $\mathrm{HF}^\bullet(H_{\delta,A}) \to \mathrm{SH}^\bullet(D)$ (see Equation 3.2.1). In virtue of Theorem 4.16, this assures us that Ψ is a map of unital algebras.

First, we construct $\Psi_{\mathrm{I,II}}$ in three steps.

STEP 1. $[\Phi_1] : \mathrm{H}^\bullet(\mathrm{C}^\bullet_{\mathrm{I,II}}) \cong \mathrm{HF}^\bullet_{(\delta A-\varepsilon,\infty)}(H_{\delta,A} + A(1-\delta))$. This isomorphism follows from a simple shift of $A(1-\delta)$ in the Hamiltonian term which translates to a shift of $A(\delta-1)$ in action (see Figure 3). In what follows, we denote $\hat{H}_{\delta,A} := H_{\delta,A} + A(1-\delta)$.

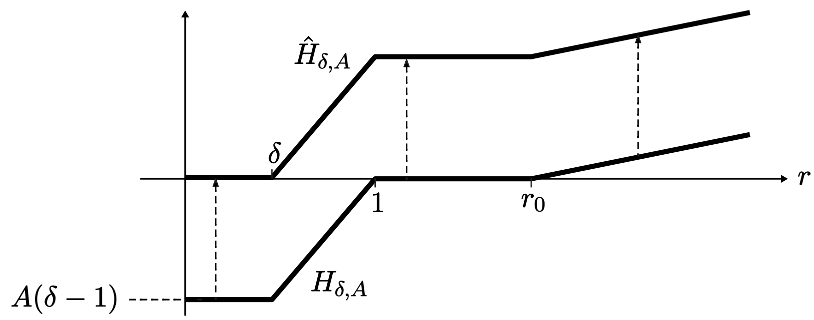


Fig. 3. Homotopy from $H_{\delta,A}$ to $\hat{H}_{\delta,A}$.

For the next steps, we need to define another special family of Hamiltonians. Given $r_1 \in (0,+\infty)$ and $\tau \in (0,\infty) \setminus \mathrm{Spec}(\partial D,\lambda)$, recall that we define the Hamiltonian $K_{r_1,\tau}$ as follows (see Figure 4).

- $K_{r_1,\tau}$ is the constant zero function on D^{r_1},
- $K_{r_1,\tau}(x,r) = \tau(r-r_1)$ on $\hat{D} \setminus D^{r_1}$.

We add a small perturbation to $K_{r_1,\tau}$ so that it r_1-admissible. The 1-periodic orbits of $K_{r_1,\tau}$ fall in two categories.

(I') Critical points in D^{r_1} with action near zero,

(II') Non-constant 1-periodic orbits near $\{r_1\} \times \partial D$ with action in a small neighborhood of the interval

$$[r_1 T_0, r_1\tau - r_1\eta_\tau].$$

By the same argument used for $H_{\delta,A}$, the action windows (I$'$) and (II$'$) are separated if we choose a small enough perturbation.

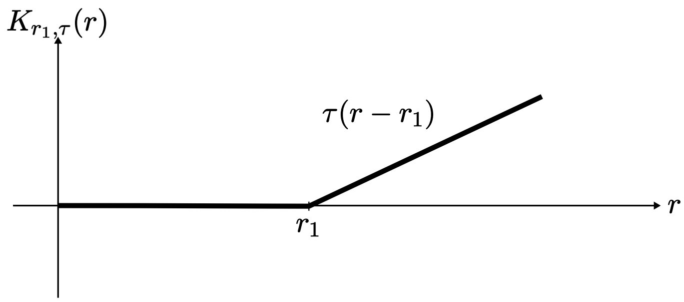


Fig. 4. Radial portion of the Hamiltonian $K_{r_1,\tau}$.

STEP 2. $[\Phi_2] : \mathrm{HF}^\bullet_{(\delta A-\varepsilon,\infty)}(\hat{H}_{\delta,A}) \cong \mathrm{HF}^\bullet(K_{\delta,A})$. Consider the homotopy

$$F_s = (1-b(s))K_{\delta,A} + b(s)\hat{H}_{\delta,A},$$

where $b : \mathbb{R} \to [0,1]$ is a smooth function such that $b(s) = 0$ for $s \leq -1$, $b(s) = 1$ for $s \geq 1$ and $b'(s) > 0$ for all $s \in (-1,1)$ (see Figure 5). Denote by

$$\Phi_{F_\bullet} : \mathrm{CF}^\bullet(\hat{H}_{\delta,A}) \longrightarrow \mathrm{CF}^\bullet(K_{\delta,A})$$

the continuation map generated by $F_\bullet$.

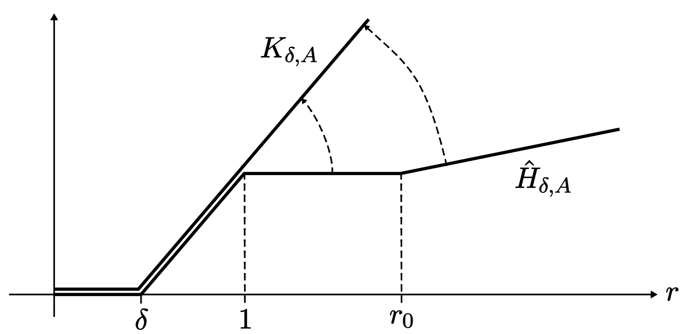


Fig. 5. Homotopy from $\hat{H}_{\delta,A}$ to $K_{\delta,A}$.

Notice that since $H_{\delta,A} \preceq K_{\delta,A}$ we can restrict the continuation map on the action window $(\delta A-\varepsilon,\infty)$. Thus,

$$[\Phi_{F_\bullet}] : \mathrm{HF}^\bullet_{(\delta A-\varepsilon,\infty)}(\hat{H}_{\delta,A}) \longrightarrow \mathrm{HF}^\bullet_{(\delta A-\varepsilon,\infty)}(K_{\delta,A})$$

is well defined. Moreover, since $\delta A - \varepsilon < 0$, $K_{\delta,A}$ has no orbits outside the action window $(\delta A - \varepsilon, \infty)$ and thus

$$[\iota^{-\infty,\infty}_{\delta A-\varepsilon,\infty}] : \mathrm{HF}^{\bullet}_{(\delta A-\varepsilon,\infty)}(K_{\delta,A}) \longrightarrow \mathrm{HF}^{\bullet}(K_{\delta,A})$$

is an isomorphism. We define $[\Phi_2]$ to be the composition $[\iota^{-\infty,\infty}_{\delta A-\varepsilon,\infty}] \circ [\Phi_{F_\bullet}]$.

STEP 3. Recall from Equation 3.2.1, that we have a natural map

$$j_{K_{\delta,A}} : \mathrm{HF}^{\bullet}(K_{\delta,A}) \longrightarrow \mathrm{SH}^{\bullet}(D)$$

We define $\Psi_{\mathrm{I,II}} : \mathrm{H}^{\bullet}(\mathrm{C}^{\bullet}_{\mathrm{I,II}}) \to \mathrm{SH}^{\bullet}(D)$ to be the composition

$$\Psi_{\mathrm{I,II}} = j_{K_{\delta,A}} \circ [\Phi_2] \circ [\Phi_1].$$

The morphism Ψ is built in a similar fashion. We define it as the composition of the maps

$$\begin{tikzcd} \mathrm{HF}^{\bullet}(H_{\delta,A}) \arrow[r, "\cong_{[\Phi'_1]}"] & \mathrm{HF}^{\bullet}(\hat{H}_{\delta,A}) \arrow[d, "{[\Phi'_2]}"] & \\ & \mathrm{HF}^{\bullet}(K_{\delta,A}) \arrow[r, "j_{K_{\delta,A}}"] & \mathrm{SH}^{\bullet}(D) \end{tikzcd}$$

Here, the isomorphism $[\Phi'_1]$ follows from the fact that both $H_{\delta,A}$ and $\hat{H}_{\delta,A}$ have the same slope at infinity. We defined $[\Phi'_2]$ to be the continuation map $[\Phi^{K_{\delta,A}\hat{H}_{\delta,A}}]$. The last map is given, just as in STEP 3, by $j_{K_{\delta,A}}$: $\mathrm{HF}^{\bullet}(K_{\delta,A}) \to \mathrm{SH}^{\bullet}(D)$. By construction, we therefore have

$$\Psi = j_{K_{\delta,A}} \circ [\Phi'_2] \circ [\Phi'_1] = j_{K_{\delta,A}} \circ [\Phi_{K_{\delta,A}\hat{H}_{\delta,A}}] \circ [\Phi'_1] = j_{H_{\delta,A}}$$

as desired.

Now, we need to prove that Diagram (7.1.1) commutes. Writing the maps Ψ and $\Psi_{\mathrm{I,II}}$ explicitly, we have the following diagram:

$$
\begin{tikzcd}
\mathrm{HF}^\bullet(H_{\delta,A}) \arrow[r, "{[\pi^{+\infty,+\infty}_{-\infty,A-\varepsilon}]}"] \arrow[d, "{[\Phi_1']}"] & \mathrm{H}^\bullet(\mathrm{C}^\bullet_{\mathrm{I,II}}) \arrow[d, "{[\Phi_1]}"'] \\
\mathrm{HF}^\bullet(\hat{H}_{\delta,A}) \arrow[r, "{[\pi^{+\infty,+\infty}_{-\infty,\delta A-\varepsilon}]}"] \arrow[d, "{[\Phi_{K_{\delta,A}\hat{H}_{\delta,A}}]}"] & \mathrm{HF}^*_{(\delta A-\varepsilon,+\infty)}(\hat{H}_{\delta,A}) \arrow[d, "{[\Phi_2]}"'] \\
\mathrm{HF}^\bullet(K_{\delta,A}) \arrow[r, "\mathrm{id}"] & \mathrm{HF}^\bullet(K_{\delta,A}) \arrow[d, "j_{K_{\delta,A}}"'] \\
 & \mathrm{SH}^\bullet(D)
\end{tikzcd}
\tag{7.1.2}
$$

The top square in Diagram (7.1.2) commutes because, given that $\hat{H}_{\delta,A} \geq H_{\delta,A}$, there exists a continuation map from $\mathrm{HF}^\bullet(H_{\delta,A}) \cong \mathrm{HF}^*_{(\delta A-\varepsilon,\infty)}(H_{\delta,A})$ to $\mathrm{HF}^\bullet_{(\delta A-\varepsilon,+\infty)}(\hat{H}_{\delta,A})$ where the isomorphism follows from the fact that $H_{\delta,A}$ has no orbits outside the action window $(\delta A-\varepsilon,\infty)$. Now, since the projection $[\pi^{+\infty,+\infty}_{-\infty,\delta A-\varepsilon}]$ commutes with continuation maps (see Diagram (3.1.2)), the bottom square in Diagram (7.1.2) also commutes. Therefore, we can conclude that Diagram (7.1.1) commutes.

7.1.3. Spectral invariant and spectral norm of $H_{\delta,A}$

Recall that, by definition,

$$c(1,H_{\delta,A}) = \inf\{\ell \in \mathbb{R} \mid [\pi^{+\infty,+\infty}_{-\infty,\ell}] \circ [\iota^{\ell,+\infty}_{-\infty,-\infty}](1_{H_{\delta,A}}) = 0\}$$

Since Ψ is a morphism of unital algebras, the commutative diagram (7.1.1) assures us that $[\pi^{+\infty,+\infty}_{-\infty,A-\varepsilon}](1_{H_{\delta,A}}) \neq 0$ since we assume that $\mathrm{SH}^\bullet(D) \neq 0$. Thus, from the exact triangle in cohomology induced by $[\iota^{A-\varepsilon,+\infty}_{-\infty,-\infty}]$ and

$[\pi^{+\infty,+\infty}_{-\infty,A-\varepsilon}]$, we have $1_{H_{\delta,A}} \notin \mathrm{im}[\iota^{A-\varepsilon,+\infty}_{-\infty,-\infty}]$ and therefore,

$$c(1,H_{\delta,A}) \geq A - \varepsilon.$$

Now, we turn our attention to the spectral norm $\gamma(H_{\delta,A})$. We know from Lemma B that $c(1,H_{\delta,A}),c(1,\overline{H}_{\delta,A}) \geq 0$. It thus follows from the previous inequality that

$$\gamma(H_{\delta,A}) = c(1,H_{\delta,A}) + c(1,\overline{H}_{\delta,A}) \geq A - \varepsilon$$

as desired. This completes the proof.

7.2. Positive spectral invariants

We give a proof of Lemma B.

> **Lemma B**
>
> Let H be a compactly supported Hamiltonian on a Liouville domain $(D,\mathrm{d}\lambda)$. Then,
>
> $$c(1, H) \geq 0.$$

This result relies on the decomposition of the Floer complex induced by the Ganor-Tanny barricade discussed in Chapter 6. We expect that Lemma B could also be proven using Poincaré duality between filtered Floer cohomology and filtered Floer homology (as in [**CO18**, Section 3]) and Lemma 4.1 of [**GT23**].

Let $H \in \mathcal{H}_{r_0}$ with slope $0 < \tau_H < T_0$. Consider an homotopy $F_\bullet$ from $F_+ = K_{r_0,\tau_H}$ (see Figure 4) to $F_- = H$. There exists a small perturbation $f_\bullet$ of $F_\bullet$ and an almost complex structure J such that the pairs $(f_\bullet,J)$ and $(f_\pm,J)$ admit a barricade on $B_{r_0,\varepsilon}$ for $\varepsilon > 0$ small enough. Fix $\delta > 0$. The construction of Theorem 6.1 allows us to choose J time independent [**GT23**, Remark 3.7] and f such that

$$-\delta \leq \int_0^1 \min_{\hat{D}\setminus(r_0,+\infty)\times\partial D} (f_- - H)\mathrm{d}t \leq \delta.$$

We may assume further that f_+ has a local minimum point $p \in D_\mathrm{c} = \hat{D} \setminus D_\mathrm{b}$, since f_+ is C^2-small there. It follows from Lemma 3.1 that $1_{f_+} = [p] \in \mathrm{HF}^\bullet(f_+)$ is the image of the unit $e_D \in \mathrm{H}^\bullet(D)$ under the isomorphism $\Phi_{f_+} : \mathrm{H}^\bullet(D) \to \mathrm{HF}^\bullet(f_+)$. Moreover, since f_+ and f_- have the same slope at infinity, Lemma 3.2 assures us that the isomorphism $[\Phi_{f_\bullet}] : \mathrm{HF}^\bullet(f_+) \to \mathrm{HF}^\bullet(f_-)$ induced by the continuation morphism $\Phi_{f_\bullet} : \mathrm{CF}^\bullet(f_+) \to \mathrm{CF}^\bullet(f_-)$ preserves the unit. To summarize, we have

$$\Phi_{f_+}(e_D) = [p] = 1_{f_+} \quad \text{and} \quad [\Phi_{f_\bullet}(p)] = [\Phi_{f_\bullet}](1_{f_+}) = 1_{f_-}.$$

By the continuity of spectral invariants, we know that

$$c(1,H) - c(1,f_-) \geq \int_0^1 \min_{D^{r_0}} (f_- - H)\mathrm{d}t.$$

Therefore, since f_- is chosen to be arbitrarily close to H, we have

$$c(1,H) \geq -\delta + c(1,f_-).$$

To complete the proof, it suffices to show that $c(1,f_-) \geq -k\delta$ for $k>0$ independent of f_-. However, the definition of spectral invariants guarantees the existence of $q \in \mathrm{CF}^\bullet(f_-)$ cohomologous to 1 for which $c(1,f_-) \geq \mathcal{A}_{f_-}(q) - \delta$. We thus only need to prove that $\mathcal{A}_{f_-}(q) \geq -\delta$. In the case where q is a combination $q_1 + \cdots + q_k$ of orbits, the action of q is defined as

$$\mathcal{A}_{f_-}(q) = \max_i \mathcal{A}_{f_-}(q_i).$$

Recall from Section 6.2 that the barricade construction assures that we have, in terms of vector spaces, the decomposition

$$\mathrm{CF}^\bullet(f_\pm) \cong \mathrm{C}^\bullet(D_\mathrm{b},f_\pm) \oplus \mathrm{C}^\bullet(D_\mathrm{c}, f_\pm)$$

with inclusions and projections respectively given by

$$\iota_\heartsuit^\pm : \mathrm{C}^\bullet(D_\heartsuit,f_\pm) \to \mathrm{CF}^\bullet(f_\pm) \quad \text{and} \quad \pi_\heartsuit^\pm : \mathrm{CF}^\bullet(f_\pm) \to \mathrm{C}^\bullet(D_\heartsuit,f_\pm)$$

for $\heartsuit \in \{\mathrm{b}, \mathrm{c}\}$. Moreover, Floer trajectories starting in D_b must have ends in D_b and Floer trajectories starting in D_c can have ends in D_b and D_c.

Thus,

$$\Phi_{f_\bullet}(p) = p_{\mathrm{b}} + p_{\mathrm{c}} \text{ and } q = p_{\mathrm{b}} + p_{\mathrm{c}} + \partial(r_{\mathrm{b}} + r_{\mathrm{c}})$$

for $p_{\mathrm{b}}, r_{\mathrm{b}} \in \operatorname{im} i_{\mathrm{b}}^-$ and $p_{\mathrm{c}}, r_{\mathrm{c}} \in \operatorname{im} \iota_{\mathrm{c}}^-$. Furthermore,

$$\partial(r_{\mathrm{b}}) = r_{\mathrm{bb}} \quad \text{and} \quad \partial(r_{\mathrm{c}}) = r_{\mathrm{cb}} + r_{\mathrm{cc}}$$

where $r_{\mathrm{bb}}, r_{\mathrm{cb}} \in \operatorname{im} \iota_{\mathrm{b}}^-$ and $r_{\mathrm{cc}} \in \operatorname{im} \iota_{\mathrm{c}}^-$. See Figure 6 for an illustration of the Floer trajectories under consideration here.

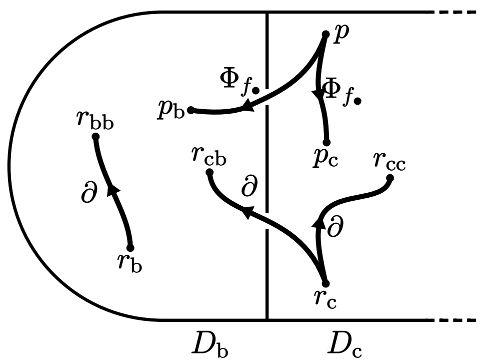


Fig. 6. The possible trajectories for the differential of r_{b}, r_{c} and the continuation map applied to p according to the rules of the barricade.

Notice that since f_- is C^2-small on D_{c}, $\mathcal{A}_{f_-}(p_{\mathrm{c}}+r_{\mathrm{cc}}) \geq -\delta$. Thus, if $p_{\mathrm{c}}+r_{\mathrm{cc}} \neq 0$, we have

$$\mathcal{A}_{f_-}(q) = \mathcal{A}_{f_-}(p_{\mathrm{b}} + p_{\mathrm{c}} + r_{\mathrm{bb}} + r_{\mathrm{cb}} + r_{\mathrm{cc}}) \geq \mathcal{A}_{f_-}(p_{\mathrm{c}} + r_{\mathrm{cc}}) \geq -\delta.$$

We now prove that $p_{\mathrm{c}} + r_{\mathrm{cc}} \neq 0$. This is equivalent to showing that the class $[\pi_{\mathrm{c}}^-(p_{\mathrm{c}})]$ in $\mathrm{H}^\bullet(D_{\mathrm{c}}, f_-)$ is nonzero. Indeed, if $p_{\mathrm{c}} + r_{\mathrm{cc}} = 0$, we have by definition of r_{cc}, $p_{\mathrm{c}} = -\partial r_{\mathrm{c}}$ and thus

$$[\pi_{\mathrm{c}}^-(p_{\mathrm{c}})] = [\pi_{\mathrm{c}}^-(-\partial r_{\mathrm{c}})] = [-\partial_{\mathrm{c}} \pi_{\mathrm{c}}^-(r_{\mathrm{c}})] = 0.$$

Denote by $\Phi_{\bar{f}_\bullet} : \mathrm{CF}^\bullet(f_-) \to \mathrm{CF}^\bullet(f_+)$ the continuation map generated by the inverse homotopy $\overline{f}_s = f_{-s}$. We know that both $\Phi_{\bar{f}_\bullet} \circ \Phi_{f_\bullet}$ and $\Phi_{f_\bullet} \circ \Phi_{\bar{f}_\bullet}$

are chain homotopic to the identity :

$$\Phi_{\bar{f}_\bullet} \circ \Phi_{f_\bullet} - \mathrm{id}_+ = \partial_+ \circ \Psi_+ - \Psi_+ \circ \partial_+$$
$$\Phi_{f_\bullet} \circ \Phi_{\bar{f}_\bullet} - \mathrm{id}_- = \partial_- \circ \Psi_- - \Psi_- \circ \partial_-$$

for the differentials $\partial_\pm : \mathrm{CF}^\bullet(f_\pm) \to \mathrm{CF}^{\bullet+1}(f_\pm)$ and chain homotopies $\Psi_\pm : \mathrm{CF}^\bullet(f_\pm) \to \mathrm{CF}^{\bullet-1}(f_\pm)$. (In fact, for our purpose here, we only need the first homotopy relation.) Since $\Psi_\pm$ also obey the rules of the barricade by Lemma 6.9, the composition of the projections $\Phi^{\mathrm{c}}_{f_\bullet} : \mathrm{C}^\bullet(D_{\mathrm{c}}, f_+) \to \mathrm{C}^\bullet(D_{\mathrm{c}}, f_-)$ and $\Phi^{\mathrm{c}}_{\bar{f}_\bullet} : \mathrm{C}^\bullet(D_{\mathrm{c}}, f_-) \to \mathrm{C}^\bullet(D_{\mathrm{c}}, f_+)$ are chain homotopic to the identity on $\mathrm{C}^\bullet(D_{\mathrm{c}}, f_+)$ by Lemma 6.11. Therefore, on cohomology, the morphism

$$[\Phi^{\mathrm{c}}_{\bar{f}_\bullet} \circ \Phi^{\mathrm{c}}_{f_\bullet}] : \mathrm{H}^\bullet(D_{\mathrm{c}}, f_+) \to \mathrm{H}^\bullet(D_{\mathrm{c}}, f_+)$$

is given by the identity. Moreover, recall that by definition, $p \in D_{\mathrm{c}}$ which guarantees that, as a cycle, $p \in \operatorname{im} \iota^+_{\mathrm{c}}$ and since $[p] = 1_{f_+}$, we have $[\pi^+_{\mathrm{c}}(p)] \neq 0$. Therefore,

$$[\pi^-_{\mathrm{c}}(p_{\mathrm{c}})] = [\Phi^{\mathrm{c}}_{f_\bullet} \circ \pi^+_{\mathrm{c}}(p)] = [\Phi^{\mathrm{c}}_{f_\bullet}]([\pi^+_{\mathrm{c}}(p)]) \neq 0.$$

This concludes the proof.

7.3. Computing spectral invariants.

Lemma C allows one to compute spectral invariants of negative Hamiltonians which are constant on the skeleton of a Liouville domain with non-vanishing symplectic cohomology.

Lemma C

Suppose (D, λ) is a Liouville domain such that $\mathrm{SH}^\bullet(D) \neq 0$. Let H be a compactly supported autonomous Hamiltonian on D such that

$$H\Big|_{\mathrm{Sk}(D)} = -A \quad \text{and} \quad -A \leq H\Big|_D \leq 0$$

for a constant $A > 0$. Then

$$c(1, H) = A.$$

The proof of Lemma C has three main steps.

- **STEP 1:** We show, using the continuity of spectral invariants, that when $\delta \to 0$, the spectral invariant with respect to the unit of the Hamiltonian $H_{\delta,A}$ is given the value A of $H_{\delta,A}$ on $\mathrm{Sk}(D)$.
- **STEP 2:** Using step 1, the monotonicity of spectral invariants and the symplectic contraction principle of Lemma 5.5, we prove Lemma C for negative Hamiltonians H which are constant *in a neighborhood of* $\mathrm{Sk}(D)$.
- **STEP 3:** We construct an homotopy from H to $H_{0,\delta}$ and prove Lemma 5.5 in full generality using the continuity of spectral invariants.

STEP 1. Let $0 < \delta < 1$ be small enough so that

$$\delta A < \delta A + \delta\eta_A < \eta_A.$$

Then, following the proof of Theorem A with $\varepsilon = \delta(A + \eta_A)$, we have that

$$c(1,H_{\delta,A}) \geq A - \delta(A + \eta_A).$$

Notice that $H_{\delta,A}$ converges uniformly as $\delta \to 0$ to the continuous function $H_{0,A}$ (see Figure 7). Then, by continuity of spectral invariants and the previous equation, we have

$$c(1,H_{0,A}) = \lim_{\delta\to 0} c(1,H_{\delta,A}) \geq \lim_{\delta\to 0} (A - \delta(A + \eta_A)) = A.$$

Moreover, since $H_{0,A} \geq -A$, continuity of spectral invariants yields

$$c(1,H_{0,A}) \leq \max_{x\in D} -H_{0,A} = A$$

which allows us to conclude that $c(1,H_{0,A}) = A$.

STEP 2. We prove the Lemma for Hamiltonians which are constant on an open neighborhood of the Skeleton of D. Consider an autonomous Hamiltonian $H \in \mathcal{C}(D)$ such that $H\big|_V = -A$ and $-A \leq H \leq 0$ for an open

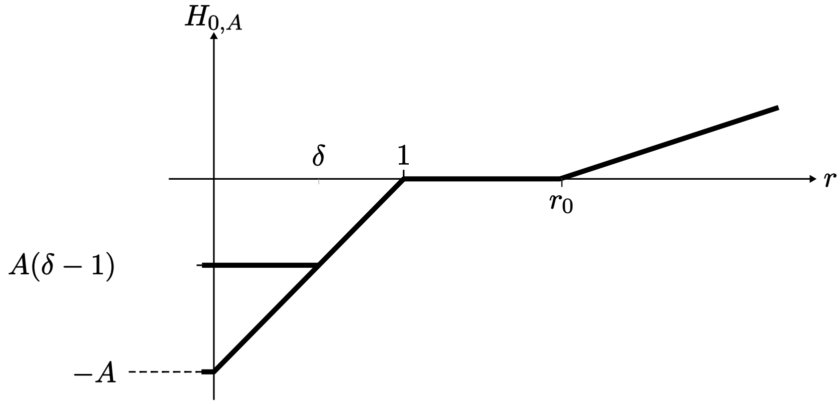


Fig. 7. The continuous Hamiltonian $H_{0,A}$.

neighborhood V of $\mathrm{Sk}(D)$ and a constant $A > 0$. The last condition on H allows us to use continuity of spectral invariance to conclude that

$$c(1,H) \leq A. \tag{7.3.1}$$

All we need to do now is prove that A bounds $c(1,H)$ from below.

Define $F \in \mathcal{C}(D)$ to be the continuous autonomous Hamiltonian that agrees with $H_{0,A/r'}$ on D for some $0 < r' < 1$. Since $H\big|_V = -A$, we can choose r' so that the r'-contraction $F_{r'}$ of F under the Liouville flow (see Equation 5.1.7 and Figure 8), has support in V and $-A \leq F_{r'} \leq 0$. Therefore,

$$F_{r'}(x) \geq H(x), \quad \forall x \in D. \tag{7.3.2}$$

From the contraction principle stated in Lemma 5.5 and the computation of $c(1,H_{0,A})$ above, we have

$$c(1,F_{r'}) = r'c(1,F) = r'c(1,H_{0,A/r'}) = A.$$

This computation and Equation 7.3.2 yield, by virtue of the monoticity of spectral invariants, the lower bound $A = c(1,F_{r'}) \leq c(1,H)$ as desired. In conjunction with Equation 7.3.1, we conclude that $c(1,H) = A$.

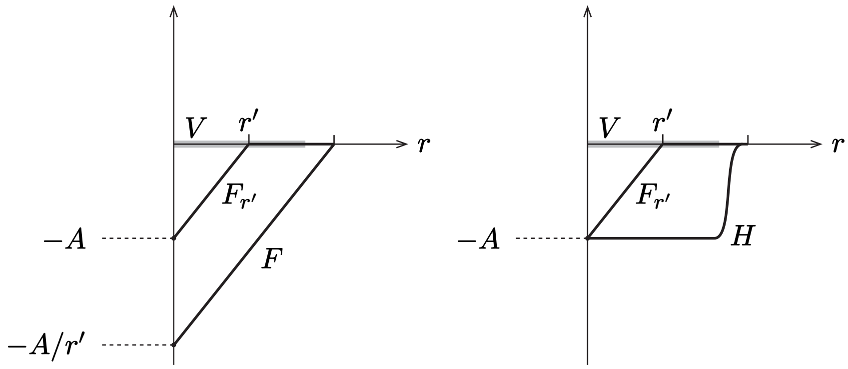


Fig. 8. The Hamiltonians F, $F_{r'}$ and H.

STEP 3. Now, we prove the Lemma in general. Suppose $H\big|_{\mathrm{Sk}(D)} = -A$ and $-A \leq H \leq 0$. For any $\varepsilon \in (0,1)$, there exists a compactly supported Hamiltonian H_ε such that $H_\varepsilon\big|_{V_\varepsilon} = -A$ for an open neighborhood V_ε of $\mathrm{Sk}(D)$ and $H_\varepsilon \leq H$ everywhere. Indeed, define H_ε as follows : $H_\varepsilon\big|_{\mathrm{Sk}(D)} = -A$,

$$H_\varepsilon\Big|_{D^\varepsilon \setminus \mathrm{Sk}(D)} = b_\varepsilon(r)H + (1 - b_\varepsilon(r))(-A)$$

where $b_\varepsilon : (0,1) \to \mathbb{R}$ is such that

- $b_\varepsilon\big|_{(0,\varepsilon]} \equiv 0$,
- $b'_\varepsilon\big|_{(\varepsilon,2\varepsilon/3)} > 0$,
- $b_\varepsilon\big|_{(2\varepsilon/3,1)} \equiv 1$.

Then, H_ε satisfies the required conditions and converges uniformly to H as $\varepsilon \to 0$. We have $c(1,H_\varepsilon) = A$ by the previous computation and by continuity of spectral invariants, we can conclude that

$$c(1,H) = c(1,H_\varepsilon) = A.$$

This completes the proof.

7.4. An isometric group embedding

The computation of spectral invariants carried out in Section 7.3 can be used to construct an explicit isometric group embedding of $\mathbb{R}$ into $\mathrm{Ham}_c(D)$.

> **Theorem D**
>
> Suppose $SH^\bullet(D) \neq 0$. There exists an isometric group embedding of $\mathbb{R}$ equipped with the standard Euclidean metric d_{std} into $(\mathrm{Ham}_c(D),d_\gamma)$.

Let $H \in \mathcal{C}(D)$ be an autonomous Hamiltonian such that $H\big|_V = -1$ and $-1 \leq H \leq 0$ everywhere for an open neighborhood V of $\mathrm{Sk}(D)$.

Define $\iota : \mathbb{R} \to \mathrm{Ham}_c(D)$ as

$$\iota(s) = \varphi_{sH},$$

where $\varphi_{sH} \in \mathrm{Ham}_c(D)$ is the time-one map associated to sH. We claim that ι is the desired embedding.

We first bound $d_\gamma(\iota(s),\iota(s'))$ from above. If $F \in \mathcal{C}(D)$, then $\gamma(\varphi_F) \leq \|F\|$. Moreover, since H is autonomous, $sH\#\overline{s'H} = (s - s')H$. Therefore,

$$d_\gamma(\iota(s),\iota(s')) = \gamma(\iota(s)\iota(s')^{-1}) \leq \|(s - s')H\| = |s - s'|.$$

Now, we bound $d_\gamma(\iota(s),\iota(s'))$ from below. Since d_γ is symmetric, we can assume that $s \geq s'$. Then, by Lemma B and Lemma C, we have

$$d_\gamma(\iota(s),\iota(s')) \geq c(1,(s - s')H) = s - s',$$

which completes the proof.

REFERENCES

[AD14] M. Audin and M. Damian. *Morse theory and Floer homology*. Universitext. Springer, London; EDP Sciences, Les Ulis, 2014. Translated from the 2010 French original by Reinie Erné.

[Arn65] V. I. Arnol'd. Sur une propriété topologique des applications globalement canoniques de la mécanique classique. *C. R. Acad. Sci. Paris*, 261:3719–3722, 1965.

[Arn89] V. I. Arnol'd. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, [1989]. Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein, Corrected reprint of the second (1989) edition.

[AS10] M. Abouzaid and P. Seidel. An open string analogue of Viterbo functoriality. *Geom. Topol.*, 14(2):627–718, 2010.

[Ati88] M. Atiyah. Topological quantum field theories. *Inst. Hautes Études Sci. Publ. Math.*, (68):175–186 (1989), 1988.

[BC94] M. Betz and R. L. Cohen. Graph moduli spaces and cohomology operations. *Turkish J. Math.*, 18(1):23–41, 1994.

[BHS21] L. Buhovsky, V. Humilière, and S. Seyfaddini. The action spectrum and C^0 symplectic topology. *Math. Ann.*, 380(1-2):293–316, 2021.

[BK22] G. Benedetti and J. Kang. Relative hofer–zehnder capacity and positive symplectic homology. *Journal of Fixed Point Theory and Applications*, 24(2):44, 2022.

[CE12] K. Cieliebak and Y. Eliashberg. *From Stein to Weinstein and back*, volume 59 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2012. Symplectic geometry of affine complex manifolds.

[CFH95] K. Cieliebak, A. Floer, and H. Hofer. Symplectic homology. II. A general construction. *Math. Z.*, 218(1):103–122, 1995.

[CFHW96] K. Cieliebak, A. Floer, H. Hofer, and K. Wysocki. Applications of symplectic homology. II. Stability of the action spectrum. *Math. Z.*, 223(1):27–45, 1996.

[CFO10] K. Cieliebak, U. Frauenfelder, and A. Oancea. Rabinowitz Floer homology and symplectic homology. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(6):957–1015, 2010.

[Che00] Yu. V. Chekanov. Invariant Finsler metrics on the space of Lagrangian embeddings. *Math. Z.*, 234(3):605–619, 2000.

[CO18] K. Cieliebak and A. Oancea. Symplectic homology and the Eilenberg–Steenrod axioms. *Algebraic and Geometric Topology*, 18(4):1953–2130, 2018. v3: corrected Lemma 7.11. Various other minor modifications and reformatting. Final version to be published in Algebraic and Geometric Topology.

[CZ83] C. C. Conley and E. Zehnder. The Birkhoff-Lewis fixed point theorem and a conjecture of V. I. Arnol'd. *Invent. Math.*, 73(1):33–49, 1983.

[EP03] M. Entov and L. Polterovich. Calabi quasimorphism and quantum homology. *Int. Math. Res. Not.*, (30):1635–1676, 2003.

[FH94] A. Floer and H. Hofer. Symplectic homology. I. Open sets in $\mathbf{C}^n$. *Math. Z.*, 215(1):37–88, 1994.

[FHV90] A. Floer, H. Hofer, and C. Viterbo. The Weinstein conjecture in $P \times \mathbf{C}^l$. *Math. Z.*, 203(3):469–482, 1990.

[Flo86] A. Floer. Proof of the Arnol'd conjecture for surfaces and generalizations to certain Kähler manifolds. *Duke Math. J.*, 53(1):1–32, 1986.

[Flo89] A. Floer. Symplectic fixed points and holomorphic spheres. *Comm. Math. Phys.*, 120(4):575–611, 1989.

[FO99a] K. Fukaya and K. Ono. Arnold conjecture and Gromov-Witten invariant. *Topology*, 38(5):933–1048, 1999.

[FO99b] K. Fukaya and K. Ono. Arnold conjecture and Gromov-Witten invariant for general symplectic manifolds. In *The Arnoldfest (Toronto, ON, 1997)*, volume 24 of *Fields Inst. Commun.*, pages 173–190. Amer. Math. Soc., Providence, RI, 1999.

[FS07] U. Frauenfelder and F. Schlenk. Hamiltonian dynamics on convex symplectic manifolds. *Israel J. Math.*, 159:1–56, 2007.

[Fuk97] K. Fukaya. Morse homotopy and its quantization. In *Geometric topology (Athens, GA, 1993)*, volume 2 of *AMS/IP Stud. Adv. Math.*, pages 409–440. Amer. Math. Soc., Providence, RI, 1997.

[Gir91] E. Giroux. Convexité en topologie de contact. *Comment. Math. Helv.*, 66(4):637–677, 1991.

[Gir17] E. Giroux. Remarks on Donaldson's symplectic submanifolds. *Pure Appl. Math. Q.*, 13(3):369–388, 2017.

[Gro85] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[GT23] Y. Ganor and S. Tanny. Floer theory of disjointly supported Hamiltonians on symplectically aspherical manifolds. *Algebr. Geom. Topol.*, 23(2):645–732, 2023.

[Hof90] H. Hofer. On the topological properties of symplectic maps. *Proc. Roy. Soc. Edinburgh Sect. A*, 115(1-2):25–38, 1990.

[HV88] H. Hofer and C. Viterbo. The Weinstein conjecture in cotangent bundles and related results. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 15(3):411–445 (1989), 1988.

[HZ94] H. Hofer and E. Zehnder. *Symplectic invariants and Hamiltonian dynamics.* Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 1994.

[Kaw22a] Y. Kawamoto. Homogeneous quasimorphisms, C^0-topology and Lagrangian intersection. *Comment. Math. Helv.*, 97(2):209–254, 2022.

[Kaw22b] Y. Kawamoto. On C^0-continuity of the spectral norm for symplectically non-aspherical manifolds. *Int. Math. Res. Not. IMRN*, (21):17187–17230, 2022.

[KS21] A. Kislev and E. Shelukhin. Bounds on spectral norms and barcodes. *Geom. Topol.*, 25(7):3257–3350, 2021.

[LM95] F. Lalonde and D. McDuff. The geometry of symplectic energy. *Ann. of Math. (2)*, 141(2):349–371, 1995.

[LR10] F. Le Roux. Six questions, a proposition and two pictures on Hofer distance for Hamiltonian diffeomorphisms on surfaces. In *Symplectic topology and measure preserving dynamical systems*, volume 512 of *Contemp. Math.*, pages 33–40. Amer. Math. Soc., Providence, RI, 2010.

[LT98] G. Liu and G. Tian. Floer homology and Arnold conjecture. *J. Differential Geom.*, 49(1):1–74, 1998.

[Man15] C. Manolescu. Floer theory and its topological applications. *Jpn. J. Math.*, 10(2):105–133, 2015.

[McL09] M. McLean. Lefschetz fibrations and symplectic homology. *Geom. Topol.*, 13(4):1877–1944, 2009.

[MVZ12] A. Monzner, N. Vichery, and F. Zapolsky. Partial quasimorphisms and quasi-states on cotangent bundles, and symplectic homogenization. *J. Mod. Dyn.*, 6(2):205–249, 2012.

[Oh05] Y.-G. Oh. Construction of spectral invariants of Hamiltonian paths on closed symplectic manifolds. In *The breadth of symplectic and Poisson geometry*, volume 232 of *Progr. Math.*, pages 525–570. Birkhäuser Boston, Boston, MA, 2005.

[Oh15] Y.-G. Oh. *Symplectic topology and Floer homology. Vol. 2*, volume 29 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2015. Floer homology and its applications.

[Ost03] Y. Ostrover. A comparison of Hofer's metrics on Hamiltonian diffeomorphisms and Lagrangian submanifolds. *Commun. Contemp. Math.*, 5(5):803–811, 2003.

[Par16] John Pardon. An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves. *Geom. Topol.*, 20(2):779–1034, 2016.

[Pol14] L. Polterovich. Symplectic geometry of quantum noise. *Comm. Math. Phys.*, 327(2):481–519, 2014.

[PSS96] S. Piunikhin, D. Salamon, and M. Schwarz. Symplectic Floer-Donaldson theory and quantum cohomology. In *Contact and symplectic geometry (Cambridge, 1994)*, volume 8 of *Publ. Newton Inst.*, pages 171–200. Cambridge Univ. Press, Cambridge, 1996.

[Rit09] A. F. Ritter. Novikov-symplectic cohomology and exact Lagrangian embeddings. *Geometry and Topology*, 13(2):943 – 978, 2009.

[Rit13] A. F. Ritter. Topological quantum field theory structure on symplectic cohomology. *J. Topol.*, 6(2):391–489, 2013.

[Rua99] Y. Ruan. Virtual neighborhoods and pseudo-holomorphic curves. In *Proceedings of 6th Gökova Geometry-Topology Conference*, volume 23, pages 161–231, 1999.

[Sch95] M. Schwarz. *Cohomology operations from S^1-cobordisms in Floer homology.* Doctoral thesis, ETH Zurich, Zürich, 1995. Diss. Math. Wiss. ETH Zürich, Nr. 11182, 1995. Ref.: Helmut Hofer ; Korref.: Eduard Zehnder.

[Sch00] M. Schwarz. On the action spectrum for closed symplectically aspherical manifolds. *Pacific J. Math.*, 193(2):419–461, 2000.

[Sei08] P. Seidel. A biased view of symplectic cohomology. In *Current developments in mathematics, 2006*, pages 211–253. Int. Press, Somerville, MA, 2008.

[SZ92] D. Salamon and E. Zehnder. Morse theory for periodic solutions of Hamiltonian systems and the Maslov index. *Comm. Pure Appl. Math.*, 45(10):1303–1360, 1992.

[Tau07] C. H. Taubes. The Seiberg-Witten equations and the Weinstein conjecture. *Geom. Topol.*, 11:2117–2202, 2007.

[Ush13] M. Usher. Hofer's metrics and boundary depth. *Ann. Sci. Éc. Norm. Supér. (4)*, 46(1):57–128 (2013), 2013.

[Vit87] C. Viterbo. A proof of Weinstein's conjecture in $\mathbf{R}^{2n}$. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 4(4):337–356, 1987.

[Vit92] C. Viterbo. Symplectic topology as the geometry of generating functions. *Math. Ann.*, 292(4):685–710, 1992.

[Vit99] C. Viterbo. Functors and computations in Floer homology with applications. I. *Geom. Funct. Anal.*, 9(5):985–1033, 1999.

[Web06] J. Weber. Noncontractible periodic orbits in cotangent bundles and Floer homology. *Duke Math. J.*, 133(3):527–568, 2006.

[Wei79] A. Weinstein. On the hypotheses of Rabinowitz' periodic orbit theorems. *Journal of Differential Equations*, 33(3):353–358, 1979.