Département de sciences économiques

Cahier 2000-01

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Ce cahier a également été publié par le Centre interuniversitaire de recherche en économie quantitative (CIREQ) sous le numéro 2000-01.

This working paper was also published by the Center for Interuniversity Research in Quantitative Economics (CIREQ), under number 2000-01.

ISSN 0709-9231

CAHIER 2000-01

LATENT VARIABLE MODELS FOR STOCHASTIC DISCOUNT FACTORS

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January 2000

The first author gratefully acknowledges financial support from the Fonds pour la Formation de chercheurs et l'aide à la recherche (FCAR) du Québec, the Social Sciences and Humanities Research Council (SSHRC) of Canada and the Network of Centres of Excellence MITACS. The second author thanks CIRANO and C.R.D.E. for financial support.

RÉSUMÉ

En finance, les modèles à variables latentes apparaissent à la fois dans les théories d'évaluation des actifs financiers et dans l'analyse de séries chronologiques. Ces deux courants de littérature font appel à deux concepts différents de structures latentes qui servent tous deux à réduire la dimension d'un modèle statistique de séries temporelles sur les prix ou les rendements de plusieurs actifs. Dans les modèles CAPM ou APT, où l'évaluation est fonction de coefficients bêtas, la réduction de dimension est de nature transversale, tandis que dans les modèles de séries chronologiques espace-état, la dimension est réduite longitudinalement en supposant l'indépendance conditionnelle entre les rendements consécutifs, étant donné un petit nombre de variables d'état. Dans ce texte, nous utilisons le concept de facteur d'actualisation stochastique (SDF) ou noyau de valorisation comme principe unificateur en vue d'intégrer ces deux concepts de variables latentes. Les relations de valorisation avec coefficients bêtas reviennent à caractériser les facteurs comme une base d'un espace vectoriel pour le SDF. Les coefficients du SDF par rapport aux facteurs sont spécifiés comme des fonctions déterministes de certaines variables d'état qui résument leur évolution dynamique. Dans ces modèles d'évaluation à coefficients bêtas, on dit souvent que seul le risque factoriel est compensé puisque le risque résiduel idiosyncratique est diversifiable. Implicitement, cet argument peut être interprété comme une structure factorielle transversale conditionnelle, c'est-à-dire une indépendance conditionnelle entre les rendements contemporains d'un grand nombre d'actifs, étant donné un petit nombre de facteurs, comme dans l'analyse factorielle standard. Nous établissons cette analyse unificatrice dans le contexte des modèles conditionnels d'équilibre à coefficients bêtas de même que dans des modèles d'évaluation des actifs financiers avec volatilité stochastique, taux d'intérêt stochastiques et autres variables d'état. Nous adressons la question générale de la spécification économétrique des modèles dynamiques d'évaluation des actifs financiers qui regroupent la littérature moderne des modèles à facteurs conditionnellement hétéroscédastiques, ainsi que les modèles d'équilibre d'évaluation des actifs financiers avec une spécification intertemporelle des préférences et des processus fondamentaux du marché. Nous interprétons diverses relations de causalité instantanées entre les variables d'état et les processus fondamentaux du marché comme des effets de levier et discutons le rôle central qu'elles jouent dans la validité des modèles de référence tels que le CAPM pour les actions ou les modèles d'évaluation sans paramètres de préférence pour les options.

Mots clés : facteurs d'actualisation stochastiques, variables latentes, évaluation des actifs avec coefficients bêtas, modèles à facteurs conditionnels, modèles d'équilibre d'évaluation des actifs financiers avec variables latentes

ABSTRACT

Latent variable models in finance originate both from asset pricing theory and time series analysis. These two strands of literature appeal to two different concepts of latent structures, which are both useful to reduce the dimension of a statistical model specified for a multivariate time series of asset prices. In the CAPM or APT beta pricing models, the dimension reduction is cross-sectional in nature, while in time-series state-space models, dimension is reduced longitudinally by assuming conditional independence between consecutive returns, given a small number of state variables. In this paper, we use the concept of Stochastic Discount Factor (SDF) or pricing kernel as a unifying principle to integrate these two concepts of latent variables. Beta pricing relations amount to characterize the factors as a basis of a vectorial space for the SDF. The coefficients of the SDF with respect to the factors are specified as deterministic functions of some state variables which summarize their dynamics. In beta pricing models, it is often said that only the factorial risk is compensated since the remaining idiosyncratic risk is diversifiable. Implicitly, this argument can be interpreted as a conditional cross-sectional factor structure, that is, a conditional independence between contemporaneous returns of a large number of assets, given a small number of factors, like in standard Factor Analysis. We provide this unifying analysis in the context of conditional equilibrium beta pricing as well as asset pricing with stochastic volatility, stochastic interest rates and other state variables. We address the general issue of econometric specifications of dynamic asset pricing models, which cover the modern literature on conditionally heteroskedastic factor models as well as equilibrium-based asset pricing models with an intertemporal specification of preferences and market fundamentals. We interpret various instantaneous causality relationships between state variables and market fundamentals as leverage effects and discuss their central role relative to the validity of standard CAPM-like stock pricing and preference-free option pricing.

Keywords : stochastic discount factors, latent variables, conditional beta pricing, conditional factor models, equilibrium asset pricing models with latent variables

1. Introduction

Latent variable models in finance have traditionally been used in asset pricing theory and in time series analysis. In asset pricing models, a factor structure is imposed to a collection of asset returns to describe their joint distribution at a point in time, while in time series, the dynamic behavior of a series of multivariate returns depends on common factors for which a time series process is assumed. In both cases, the fundamental role of factors is to reduce the number of correlations between a large set of variables. In the first case, the dimension reduction is cross-sectional, in the second longitudinal. Factor analysis postulates that there exists a number of unobserved common factors or latent variables which explain observed correlations. To reduce dimension, a conditional independence is assumed between the observed variables given the common factors.

Arbitrage pricing theory (APT) is the standard financial model where returns of an infinite sequence of risky assets with a positive definite variance-covariance matrix are assumed to depend linearly on a set of common factors and on idiosyncratic residuals. Statistically, the returns are mutually independent given the factors. Economically, the idiosyncratic risk can be diversified away to arrive at an approximate linear beta pricing: the expected return of a risky asset in excess of a risk-free asset is equal to the scalar product of the vector of asset risks, as measured by the factor betas, with the corresponding vector of prices for the risk factors.

The latent GARCH factor model of Diebold and Nerlove (1989) best illustrates the type of time series models used to characterize the dynamic behavior of a set of financial returns. All returns are assumed to depend on a common latent factor and on noise. A longitudinal dimension reduction is achieved by assuming that the factor captures and subsumes the dynamic behavior of returns[1]. The imposed statistical structure is a conditional absence of correlation between the factor and the noise terms given the whole past of the factor and the noise, while the conditional variance of the factor follows a GARCH structure. This autoregressive conditional variance structure is important for financial applications such as portfolio allocations or value-at-risk calculations.

In this paper, we aim at providing a unifying analysis of these two strands of literature through the concept of stochastic discount factor (SDF). The SDF (m_{t+1}), also called pricing kernel, discounts future payoffs p_{t+1} to determine the current price π_t of assets:

$$\pi_t = E[m_{t+1}p_{t+1}|J_t], \tag{1.1}$$

conditionally to the information set at time t, J_t. We summarize in Section 2 the mathematics

[1] A cross-sectional dimension reduction is also achieved if the variance-covariance matrix of residuals is assumed to be diagonal.

of the SDF in a conditional setting according to Hansen and Richard (1987). Practical implementation of an asset pricing formula like (1.1) requires a statistical model to characterize the joint probability distribution of (m_{t+1}, p_{t+1}) given J_t. We specify in Section 3 a dynamic statistical framework to condition the discounted payoffs on a vector of state variables. Assumptions are made on the joint probability distribution of the SDF, asset payoffs and state variables to provide a state-space modeling framework which extends standard models.

Beta pricing relations amount to characterize a vector space basis for the SDF through a limited number of factors. The coefficients of the SDF with respect to the factors are specified as deterministic functions of the state variables. Factor Analysis and beta pricing with conditioning on state variables are reviewed in Section 4.

In dynamic asset pricing models, one can distinguish between reduced-form time-series models such as conditionally heteroskedastic factor models and asset pricing models based on equilibrium. We propose in Section 5 an intertemporal asset pricing model based on a conditioning on state variables which includes as a particular case stochastic volatility models. In this respect, we stress the importance of timing in conditioning to generate instantaneous correlation effects called leverage effects and show how it affects the pricing of stocks, bonds and European options. We make precise how this general model with latent variables relates to standard models such as CAPM for stocks and Black and Scholes (1973) or Hull and White (1987) for options.

2. Stochastic discount factors and conditioning information

Since Harrison and Kreps (1979) and Chamberlain and Rothschild (1983), it is well-known that, when asset markets are frictionless, portfolio prices can be characterized as a linear valuation functional that assigns prices to the portfolio payoffs. Hansen and Richard (1987) analyze asset pricing functions in the presence of conditioning information. Their main contribution is to show that these pricing functions can be represented using random variables included in the collection of payoffs from portfolios. In this section we summarize the mathematics of a stochastic discount factor in a conditional setting following Hansen and Richard (1987). We focus on one-period securities as in their original analysis. In the next section, we will provide an extended framework with state variables to accommodate multi-period securities.

We start with a probability space $(\Omega, \mathcal{A}, P)$. We denote the conditioning information as the information available to economic agents at date t by J_t, a subsigma algebra of $\mathcal{A}$. Agents form portfolios of assets based on this information, which includes in particular the prices of these assets. A one-period security purchased at time t has a payoff p at time $(t+1)$. For such securities, an asset pricing model $\pi_t(.)$ defines for the elements p of a set $P_{t+1} \subset J_{t+1}$ of payoffs a price $\pi_t(p) \in J_t$. The payoff space includes the payoffs of primitive assets, but investors can also create new payoffs by forming portfolios.

Assumption 2.1: (Portfolio formation)

$p_1, p_2 \in P_{t+1} \Longrightarrow w_1 p_1 + w_2 p_2 \in P_{t+1}$ for any variables $w_1, w_2 \in J_t$.

Since we always maintain a finite-variance assumption for asset payoffs, P_{t+1} is, by virtue of Assumption 2.1, a pre-Hilbertian vectorial space included in:

$$P_{t+1}^{+} = \{p \in J_{t+1}; E[p^2|J_t] < +\infty\}$$

which is endowed with the conditional scalar product:

$$< p_1, p_2 >_{J_t} = E[p_1 p_2 | J_t] \tag{2.1}$$

The pricing functional $\pi_t(.)$ is assumed to be linear on the vectorial space P_{t+1} of payoffs; this is basically the standard "law of one price" assumption, that is a very weak version of a condition of no-arbitrage.

Assumption 2.2: (Law of one price)

For any p_1 and p_2 in P_{t+1} and any $w_1, w_2 \in J_t$:

$\pi(w_1 p_1 + w_2 p_2) = w_1 \pi(p_1) + w_2 \pi(p_2)$.

The Hilbertian structure (2.1) will be used for orthogonal projections on the set P_{t+1} of admissible payoffs both in the proof of theorem 2.1 below (a conditional version of the Riesz representation theorem) and in section 4. Of course, this implies that we maintain an assumption of closedness for P_{t+1}. Indeed, Assumption 2.2 can be extended to an infinite series of payoffs to ensure not only a property of closedness for P_{t+1} but also a continuity property for $\pi_t(.)$ on P_{t+1} with appropriate notions of convergence for both prices and payoffs. With these assumptions and a technical condition ensuring the existence of a payoff with non-zero price to rule out trivial pricing functions, one can state the fundamental theorem of Hansen and Richard (1987), which is a conditional extension of the Riesz representation theorem.

Theorem 2.1. *:There exists a unique payoff p^* in P_{t+1} that satisfies:*

(i) $\pi_t(p) = E[p^* p | J_t]$ for all p in P_{t+1};
(ii) $P[E[p^{*2}|J_t] > 0] = 1$.

In other words, the particular payoff which is used to characterize any asset price is almost surely non-zero. With an additional non-arbitrage condition, it can be shown to be almost surely positive.

3. Conditioning the discounted payoffs on state variables

We just stated that, given the law of one price, a pricing function $\pi_t(.)$ for a conditional linear space P_{t+1} of payoffs can be represented by a particular payoff p^* such that condition (i) of theorem 2.1 is fulfilled. In this section, we do not focus on the interpretation of the stochastic discount factor as a particular payoff. Instead, we consider a time series $(m_{t+1})_{t\geq 1}$ of *admissible* SDFs or pricing kernels, which means that, at each date t, m_{t+1} belongs to the set $\mathcal{M}_{t+1}$ defined as:

$$\mathcal{M}_{t+1} = \{m_{t+1} \in P^+_{t+1}; \pi_t(p_{t+1}) = E_t[m_{t+1}p_{t+1}|J_t], \forall p_{t+1} \in P_{t+1}\}. \tag{3.1}$$

For a given asset, we will write the asset pricing formula as:

$$\pi_t = E[m_{t+1}p_{t+1}|J_t]. \tag{3.2}$$

For the implementation of such a pricing formula, we need to model the joint probability distribution of (m_{t+1}, p_{t+1}) given J_t. To do this, we will stress the usefulness of factors and state variables. We will suppose without loss of generality[2] that the future payoff is the future price of the asset itself π_{t+1}. The problem is therefore to find the pricing function $\varphi_t(J_t)$ such that:

$$\varphi_t(J_t) = E[m_{t+1}\varphi_t(J_{t+1})|J_t] \tag{3.3}$$

Both factors and state variables are useful to reduce the dimension of the problem to be solved in (3.3). To see this, one can decompose the information J_t into three types of variables. First, one can include asset-specific variables denoted Y_t, which should contain at least the price π_t. Dividends as well as other variables which may help characterize m_{t+1} could be included without really complicating matters. Second, the information will contain a vectorial process F_t of factors. Such factors could be suggested by economic theory or chosen purely on statistical grounds. For example, in equilibrium models, a factor could be the consumption growth process. In factor models, they could be observable macroeconomic indicators or latent factors to be extracted from a universe of asset returns. In both cases these variables are viewed as explanatory factors, possibly latent, of the collection of asset prices at time t. The purpose of these factors is to reduce the cross-sectional dimension of the collection of assets. Third, it is worthwhile to introduce a vectorial process U_t of exogenous state variables in order to achieve a longitudinal reduction of dimension.

[2] As usual, if there are dividends or other cashflows, they may be included in the price by a convenient discounted sum. We will abandon this convenient expositional shortcut when we will refer to more specific assets in subsequent sections.

Two assumptions are made about the conditional probability distribution of $(Y_t, F_t)_{1 \leq t \leq T}$ knowing $U_1^T = (U_t)_{1 \leq t \leq T}$ (for any $T-$ tuplet $t = 1, ..., T$ of dates of interest) to support the claim that the processes making up U_t summarize the dynamics of the processes (Y_t, F_t). First we assume that the state variables subsume all temporal links between the variables of interest.

Assumption 3.1.: The pairs $(Y_t, F_t)_{1 \leq t \leq T}, t = 1, ..., T$ are mutually independent knowing $U_1^T = (U_t)_{1 \leq t \leq T}$.

According to the standard latent factor analysis terminology, Assumption 3.1. means that the TH variables $U_t \in \mathbb{R}^H, t = 1, ..., T$ provide a complete system of factors to account for the relationships between the variables $(Y_t, F_t)_{1 \leq t \leq T}$ (see for example Bartholomew (1987), p. 5). In the original latent variable modeling of Burt (1941) and Spearman (1927) in the early part of the century to study human intelligence, Y_t represented an individual's score to the test number t of mental ability. The basic idea was that individual scores at various tests will become independent (with repeated observations on several human subjects) given a latent factor called general intelligence. In our modeling, t denotes a date. When, with only one observation of the path of $(Y_t, F_t), t = 1, ..., T$, we assume that these variables become independent given some latent state variables, it is clear that we also have in mind a standard temporal structure which provides an empirical content to this assumption. A minimal structure to impose is the natural assumption that only past and present values $U_\tau, \tau = 1, 2, ..., t$ of the state variables matter for characterizing the probability distribution of (Y_t, F_t).

Assumption 3.2.: The conditional probability distribution of (Y_t, F_t) given $U_1^T = (U_t)_{1 \leq t \leq T}$ coincides, for any $t = 1, ..., T$, with the conditional probability distribution given $U_1^t = (U_\tau)_{1 \leq \tau \leq t}$.

Assumption 3.2. is the following conditional independence[3] property assumption:

$$(Y_t, F_t) \amalg (U_{t+1}^T) | (U_1^t) \tag{3.4}$$

for any $t = 1, ..., T$.

Property (3.4) coincides with the definition of non-causality by Sims (1972) insofar as Assumption 3.1. is maintained and means that (Y, F) do not cause U in the sense of Sims[4]. If

[3] See Florens, Mouchart and Rollin (1990) for a systematic study of the concept of conditional independence and Florens and Mouchart (1982) for its relation with non-causality.

[4] This non-causality concept is equivalent to the non-causality notion developed by Granger (1969). Assumption 3.2. can be equivalently replaced by an assumption stating that the state variables U can be optimally forecasted from their own past, with the knowledge of past values of other variables being useless (see Renault (1999)).

we are ready to assume that the joint probability distribution of all the variables of interest is defined by a density function ℓ, assumptions 3.1. and 3.2. are summarized by:

$$\ell[(Y_t, F_t)_{1 \leq t \leq T} | U_1^T] = \prod_{t=1}^{T} \ell[(Y_t, F_t) | U_1^t] \tag{3.5}$$

The framework defined by (3.5) is very general for state-space modeling and extends such standard models as parameter driven models described in Cox (1981), stochastic volatility models as well as the state-space time series models (see Harvey (1989)). Our vector U_t of state variables can also be seen as a hidden Markov chain, a popular tool in nonlinear econometrics to model regime switches introduced by Hamilton (1989).

The merit of assumptions 3.1. and 3.2. for asset pricing is to summarize the relevant conditioning information by the set U_1^t of current and past values of the state variables.

$$\ell[(Y_{t+1}, F_{t+1}, U_{t+1}) | (Y_\tau, F_\tau)_{1 \leq \tau \leq t} U_1^t] = \ell[(Y_{t+1}, F_{t+1}, U_{t+1}) | U_1^t] \tag{3.6}$$

In practice, to make (3.6) useful, one would like to limit the relevant past by a homogeneous Markovianity assumption.

Assumption 3.3.: The conditional probability distribution of $(Y_{t+1}, F_{t+1}, U_{t+1})$ given U_1^t coincides, for any $t = 1, ..., T$, with the conditional probability distribution given U_t. Moreover, this probability distribution does not depend on t.

This assumption implies that the multivariate process U_t is homogeneous Markovian of order one[5].

Given these assumptions, we are allowed to conclude that the pricing function, as characterized by (3.3), will involve the conditioning information only through the current value U_t of the state variables. Indeed, (3.6) can be rewritten:

$$\ell[(Y_{t+1}, F_{t+1}, U_{t+1}) | (Y_\tau, F_\tau)_{1 \leq \tau \leq t} U_1^t] = \ell[(Y_{t+1}, F_{t+1}, U_{t+1}) | U_t] \tag{3.7}$$

We have seen how the dimension reduction is achieved in the longitudinal direction. To arrive at a similar reduction in the cross-sectional direction, one needs to add an assumption about the dimension of the range of m_{t+1}, given the state variables U_t. We assume that this range is spanned by K factors, $F_{kt+1}, k = 1, ..., K$ given as components of the process F_{t+1}.

[5]As usual, since the dimension of the multivariate process U_t is not limited a priori, the assumption of Markovianity of order one is not restrictive with respect to higher order Markov processes. For brevity, we will hereafter term Assumption 3.3 the assumption of Markovianity of the process U_t.

Assumption 3.4.: (SDF spanning)

m_{t+1} is a deterministic function of the variables U_t and F_{t+1}.

This assumption is not as restrictive as it might appear since it can be maintained when there exists an admissible SDF m_{t+1} with an unsystematic part $\varepsilon_{t+1} = m_{t+1} - E[m_{t+1}|F_{t+1}, U_t]$ that is uncorrelated, given U_t, with any feasible payoff $p_{t+1} \in P_{t+1}$. Actually, in this case, $\widehat{m}_{t+1} = E[m_{t+1}|F_{t+1}, U_t]$ is another admissible SDF since $E[m_{t+1}p_{t+1}|U_t] = E[\widehat{m}_{t+1}p_{t+1}|U_t]$ for any $p_{t+1} \in P_{t+1}$ and $\widehat{m}_{t+1}$ is by definition conformable to Assumption 3.4.

In section 4 below, we will consider a linear SDF spanning, even if Assumption 3.4 allows for more general factor structures such as log-linear factor models of interest rates in Duffie and Kan (1996) and Dai and Singleton (1999) or nonlinear APT (see Bansal et al., 1993). The linear benchmark is of interest when, for statistical or economic reasons, it appears useful to characterize the SDF as an element of a particular K-dimensional vector space, possibly time-varying through state variables. This is in contrast with nonlinear factor pricing where structural assumptions make a linear representation irrelevant for structural interpretations, even though it would remain mathematically correct[6]. The linear case is of course relevant when the asset pricing model is based on a linear factor model for asset returns as in Ross (1976) as we will see in the next section.

4. Affine Regression of Payoffs on Factors with Conditioning on State Variables

The longitudinal reduction of dimension through state variables put forward in section 3 will be used jointly with the cross-sectional reduction of dimension through factors in the context of a conditional affine regression of payoffs or returns on factors. More precisely, the factor loadings, which are the regression coefficients on factors and which are often called beta coefficients, will be considered from a conditional viewpoint, where the conditioning information set will be summarized by state variables given (3.7). We will first introduce the conditional beta coefficients and the corresponding conditional beta pricing formulas. We will then revisit the standard asset pricing theory which underpins these conditional beta pricing formulas, namely the arbitrage pricing theory of Ross (1976) stated in a conditional factor analysis setting.

4.1. Conditional Beta Coefficients

We first introduce conditional beta coefficients for payoffs, then for returns.

[6] We will see in particular in Section 5 that a log-linear setting appears justified by a natural log-normal model of returns given state variables.

Definition 4.1. : *The conditional affine regression* $EL_t[P_{t+1}|F_{t+1}]$ *of a payoff* p_{t+1} *on the vector* F_{t+1} *of factors given the information* J_t *is defined by:*

$$EL_t[p_{t+1}|F_{t+1}] = \beta_{0t} + \sum_{k=1}^{K} \beta_{kt} F_{kt+1} \tag{4.1}$$

with: $\varepsilon_{t+1} = p_{t+1} - EL_t[p_{t+1}|F_{t+1}]$ *satisfying:* $E[\varepsilon_{t+1}|J_t] = 0$, $Cov[\varepsilon_{t+1}, F_{t+1}|J_t] = 0$.

Similarly, if we denote by $r_{t+1} = \frac{p_{t+1}}{\pi_t(p_{t+1})}$ the return of an asset with a payoff[7] p_{t+1}, we define the conditional affine regression of the return r_{t+1} on F_{t+1} by:

$$EL_t[r_{t+1}|F_{t+1}] = \beta^r_{0t} + \sum_{k=1}^{K} \beta^r_{kt} F_{kt+1}. \tag{4.2}$$

Of course, the beta coefficients of returns can be related to the beta coefficients of payoffs by:

$$\beta^r_{kt} = \frac{\beta_{kt}}{\pi_t(p_{t+1})} \; for \; k = 0, 1, 2, ..., K. \tag{4.3}$$

Moreover, the characterization of conditional probability distributions in terms of returns instead of payoffs makes more explicit the role of state variables. To see this, let us describe payoffs at time $t+1$ from the price at the same date and a dividend process by[8]:

$$p_{t+1} = \pi_{t+1} + D_{t+1} \tag{4.4}$$

Following Assumption 3.1, we will assume that the rates of growth of dividends[9] are asset-specific variables Y_t and serially uncorrelated given state variables. In other words, $Y_t = \frac{D_t}{D_{t-1}}$,t=1,2,...,T, are mutually independent given U_1^T. Moreover, π_{t+1} in (4.4) has to be interpreted as the price at time (t+1) of the same asset with price π_t at time t defined from the pricing functional (3.3). In other words, the pricing equation (3.3) can be rewritten:

$$\frac{\varphi_t(J_t)}{D_t} = E[m_{t+1} \frac{D_{t+1}}{D_t} (\frac{\varphi_t(J_{t+1})}{D_{t+1}} + 1)|J_t] \tag{4.5}$$

[7]Strictly speaking, the return is not defined for states of nature where $\pi_t(p_{t+1}) = 0$. This may complicate the statement of characterization of the SDF in terms of expected returns as in the main theorem (theorem 4.1) of this section. However, this technical difficulty may be solved by considering portfolios which contain a particular asset with non-zero price in any state of nature. This technical condition ensuring the existence of such a payoff with non-zero price has already been mentioned in Section 2 (see also the sufficient condition 4.11 below when there exists a riskless asset). In what follows, the corresponding technicalities will be neglected.

[8]As announced in section 3, we depart from the expositional shortcut where the price included discounted dividends.

[9]Stationarity (see Assumption 3.3) requires that we include the growth rates of dividends and not their levels in the variables Y_t.

Given assumptions 3.1, 3.2 and 3.3, we are allowed to conclude that, under general regularity conditions[10], equation (4.5) defines a unique time-invariant deterministic function $\varphi(.)$ such that:

$$\varphi(U_t) = E[m_{t+1}\frac{D_{t+1}}{D_t}(\varphi(U_{t+1}) + 1)|U_t] \tag{4.6}$$

In other words, we get the following decomposition formulas for prices and returns:

$$\begin{aligned} \pi_t &= \varphi(U_t)D_t \\ r_{t+1} &= \frac{\pi_{t+1} + D_{t+1}}{\pi_t} = \frac{D_{t+1}}{D_t}\frac{\varphi(U_{t+1}) + 1}{\varphi(U_t)} \end{aligned} \tag{4.7}$$

A by-product of this decomposition is that, by application of (3.7), the joint conditional probability distribution of future factors and returns $(F_\tau, r_\tau)_{\tau>t}$ given J_t depends upon J_t only through U_t in a homogeneous way. In particular, the conditional beta coefficients of returns are fixed deterministic functions of the current value of state variables:

$$\beta^r_{kt} = \beta^r_k(U_t) \ for\ k = 0, 1, 2, ..., K \tag{4.8}$$

4.2. Conditional Beta Pricing

Since the seminal papers of Sharpe (1964) and Lintner(1965) on the unconditional CAPM to the most recent literature on conditional beta pricing (see e.g. Harvey (1991), Ferson and Korajczyk (1995)), beta coefficients with respect to well-chosen factors are put forward as convenient measures of compensated risk which explain the discrepancy between expected returns among a collection of financial assets. In order to document these traditional approaches in the modern setting of SDF, we have to add two fairly innocuous additional assumptions.

Assumption 4.1.: If $p_{F_{t+1}}$ denotes the orthogonal projection (for the conditional scalar product (2.1)) of the constant vector $\boldsymbol{\iota}$ on the space P_{t+1} of feasible payoffs, the set $\mathcal{M}_{t+1}$ of admissible SDF does not contain a variable $\lambda_t p_{F_{t+1}}$ with $\lambda_t \in J_t$.

Assumption 4.2.: Any admissible SDF has a non-zero conditional expectation given J_t.

Without Assumption 4.1, one could write for any $p_{t+1} \in P_{t+1}$:

$$\pi_t(p_{t+1}) = \lambda_t E[p_{Ft+1}p_{t+1}|J_t] = \lambda_t E[p_{t+1}|J_t] \tag{4.9}$$

[10] These regularity conditions amount to the possibility of applying a contraction mapping argument to ensure the existence and unicity of a fixed point $\varphi(.)$ of the functional defining the right hand side of (4.6).

Therefore, all the feasible expected returns would coincide with $1/\lambda_t$. When there is a riskless asset, Assumption 4.1 simply means that an admissible SDF m_{t+1} should be genuinely stochastic at time t, that is not an element of the available information J_t at time t.

Without Assumption 4.2, one could write the price $\pi_t(p_{t+1})$ as:

$$\pi_t(p_{t+1}) = E[m_{t+1}p_{t+1}|J_t] = cov[m_{t+1}p_{t+1}|J_t], \tag{4.10}$$

which would not depend on the expected payoff $E[p_{t+1}|J_t]$. When there is a riskless asset, Assumption 4.2 would be implied by a positivity requirement[11]:

$$P[p > 0] = 1 \Longrightarrow P[\pi_t(p) \leq 0] = 0 \tag{4.11}$$

With these two assumptions, we can state the central theorem of this section, which links linear SDF spanning with linear beta pricing and multibeta models of expected returns.

Theorem 4.2. *:The three following properties are equivalent:*

P1: Linear Beta Pricing: $\exists\ m_{t+1} \in \mathcal{M}_{t+1}$, $\forall p_{t+1} \in P_{t+1}$:

$$\pi_t(p_{t+1}) = \beta_{0t}E[m_{t+1}|U_t] + \sum_{k=1}^{K} \beta_{kt}E[m_{t+1}F_{kt+1}|U_t] \tag{4.12}$$

P2: Linear SDF Spanning:$\exists\ m_{t+1} \in \mathcal{M}_{t+1}, \exists\ \lambda_{kt} \in J_t$, $k = 0, 1, 2, ..., K$

$$\lambda_{kt} = \lambda_k(U_t) \text{ and } m_{t+1} = \lambda_0(U_t) + \sum_{k=1}^{K} \lambda_k(U_t)F_{kt+1} \tag{4.13}$$

P3: Multibeta Model of Expected Returns: $\exists\ \nu_{kt} \in J_t$, $k = 0, 1, 2, ..., K$, for any feasible return r_{t+1}

$$E[r_{t+1}|U_t] = \nu_{0t} + \sum_{k=1}^{K} \nu_{kt}\beta_k^r(U_t). \tag{4.14}$$

Theorem 4.2 can be proved (see Renault, 1999) from three sets of assumptions: assumptions which ensure the existence of admissible SDFs (Section 2), assumptions about the state variables (Section 3), and technical assumptions 4.1 and 4.2.

Three main lessons can be drawn from Theorem 4.2:

(i) It makes explicit what we have called a cross-sectional reduction of dimension through factors, generally conceived to ensure SDF spanning, and more precisely linear SDF spanning,

[11] This positivity requirement implies the continuity of the pricing function $\pi_t(.)$ needed for establishing Theorem 2.1.

which corresponds to the specification (4.13) of the deterministic function referred to in Assumption 3.4. With a linear beta pricing formula, prices $\pi_t(p_{t+1})$ of a large cross-sectional collection of payoffs $p_{t+1} \in P_{t+1}$ can be computed from the prices of $K+1$ particular "assets" :

$$\begin{aligned} \pi_t(\imath) &= E[m_{t+1}|J_t] = E[m_{t+1}|U_t] \\ \pi_t(F_{kt+1}) &= E[m_{t+1}F_{kt+1}|J_t] = E[m_{t+1}F_{kt+1}|U_t], \; k = 1, 2, ..., K \end{aligned} \tag{4.15}$$

If there does not exist a riskless asset or if some factors are not feasible payoffs, one can always interpret suitably normalized factors as returns on particular portfolios called mimicking portfolios. Moreover, since the only property of factors which matters is linear SDF spanning, one may assume without loss of generality that $Var[F_{t+1}|U_t]$ is nonsingular to avoid redundant factors. The beta coefficients are then computed directly by[12]:

$$\begin{aligned} [\beta_{1t}, \beta_{2t}, ..., \beta_{kt}] &= Cov[p_{t+1}, F_{t+1}|J_t] Var[F_{t+1}|U_t]^{-1} \\ \beta_{0t} &= E[p_{t+1}|J_t] - \sum_{k=1}^{K} \beta_{kt} E[F_{t+1}|U_t] \end{aligned} \tag{4.16}$$

to deduce the price:

$$\pi_t(p_{t+1}) = \beta_{0t}\pi_t(\imath) + \sum_{k=1}^{K} \beta_{kt}\pi_t(F_{kt+1}) \tag{4.17}$$

The cross-sectional reduction of dimension consists in computing only $K+1$ factor prices $(\pi_t(\imath), \pi_t(F_{kt+1}))$ to price any payoff. The longitudinal reduction of dimension is also exploited since the pricing formula for these factors (4.15) depends on the conditioning information J_t only through U_t.

(ii) Even though the linear beta pricing formula P1 is mathematically equivalent to the linear SDF spanning property P2, it is interesting to characterize it by a property of the set of feasible returns under the maintained assumption 2.4 of SDF spanning. More precisely, since this assumption allows us to write:

[12]When the payoffs include dividends, the only relevant conditioning information is characterized by state variables:

$$\begin{aligned} Cov[p_{t+1}, F_{t+1}|J_t] &= D_t Cov[\frac{p_{t+1}}{D_t}, F_{t+1}|U_t] \\ E[p_{t+1}|Jt] &= D_t E[\frac{p_{t+1}}{D_t}|U_t]. \end{aligned}$$

$$\pi_t(p_{t+1}) = E[m_{t+1}E[p_{t+1}|F_{t+1}, J_t]|J_t], \tag{4.18}$$

P1 is obtained as soon as a linear factor model of payoffs or returns is assumed (see e.g. Engle, Ng and Rothschild (1990)[13]). It means that the conditional expectation of payoffs given factors and J_t coincide with the conditional affine regression (given J_t) of these payoffs on these factors:

$$E[p_{t+1}|F_{t+1}, J_t] = EL_t[p_{t+1}|F_{t+1}] = \beta_{0t} + \sum_{k=1}^{K} \beta_{kt} F_{kt+1}. \tag{4.19}$$

Such a linear factor model can for instance be deduced from an assumption of joint conditional normality of returns and factors. This is the case when factors are themselves returns on some mimicking portfolios and returns are jointly conditionally gaussian. The standard CAPM illustrates the linear structure that is obtained from such a joint normality assumption for returns.

However, the main implication of linear beta pricing is the zero-price property of idiosyncratic risk (ε_{t+1} in the notation of definition 4.1) since only the systematic part of the payoff p_{t+1} is compensated[14]:

$$\pi_t(p_{t+1}) = \pi_t(EL_t(p_{t+1}|F_{t+1})), \tag{4.20}$$

that is: $\pi_t(\varepsilon_{t+1}) = 0$. As we will see in more details in subsection 4.3 below, this zero-price property for the idiosyncratic risk lays the basis for the APT model developed by Ross (1976). Moreover, if a factor is not compensated because $E[m_{t+1}F_{kt+1}|U_t] = 0$, it can be forgotten in the beta pricing formula. In other words, irrespective of the statistical procedure used to build the factors, only the compensated factors have to be kept:

$$\Delta_{kt} = E[m_{t+1}F_{kt+1}|U_t] \neq 0, \;\; for \;\; k = 1, ..., K. \tag{4.21}$$

(iii) The minimal list of factors that have to be kept may also be characterized by the spanning interpretation P2. In this respect, the number of factors is purely a matter of convention: how many factors do we want to introduce to span the one-dimensional space where evolves the SDF? The existence of the SDF proves that a one-factor model with the SDF itself as the sole factor is always correct. The definition of K factors becomes an issue for reasons such as economic interpretation, statistical procedures or financial strategies. Moreover, this definition can be changed as long as it keeps invariant the corresponding spanned vectorial space. For instance, one may assume that, conditionally to J_t, the factors are mutually uncorrelated, that

[13]However, these authors maintain simultaneously the two assumptions of linear SDF spanning and linear factor model of returns. These two assumptions are clearly redundant as explained above.

[14]The prices of the systematic and idiosyncratic parts are defined, by abuse of notation, by their conditional scalar product with the SDF m_{t+1}.

is $V[F_{t+1}|J_t]$ is a nonsingular diagonal matrix. One may also rescale the factors to obtain unit variance factors (statistical motivation) or unit cost factors (financial motivation). Let us focus on the latter by assuming that:

$$\Delta_{kt} = E[m_{t+1}F_{kt+1}|U_t] = 1,\ for\ k = 1, ..., K. \tag{4.22}$$

By (4.21), the factor F_{kt+1} can be replaced by its scaled value F_{kt+1}/Δ_{kt} to get (4.22) without loss of generality. Each factor can then be interpreted as a return on a portfolio (a payoff of unit price) even though we do not assume that there exists a feasible mimicking portfolio ($F_{kt+1} \in P_{t+1}$). This normalization rule allows us to prove that the coefficients in the multibeta model of expected returns (P3) are given by:

$$\nu_{kt} = E[F_{kt+1}|U_t] - \nu_{0t}\ for\ k = 1, ..., K. \tag{4.23}$$

Since, on the other hand, it is easy to check that:

$$\nu_{0t} = \frac{1}{E[m_{t+1}|U_t]} \tag{4.24}$$

coincides with the risk-free return when there exists a risk-free asset, the multibeta model (P3) of expected returns can be rewritten in the more standard form:

$$E[r_{t+1}|U_t] - \nu_{0t} = \sum_{k=1}^{K} \beta_k^r(U_t)[E[F_{kt+1}|U_t] - \nu_{0t}], \tag{4.25}$$

which gives the risk premium of the asset as a linear combination of the risk premia of the various factors, with weights defined by the beta coefficients viewed as risk quantities. Moreover, (4.25) is very useful for statistical inference in factor models (see in particular subsection 4.3) since it means that the beta pricing formula is characterized by the nullity of the intercept term in the conditional regression of net returns on net factors, given U_t.

4.3. Conditional Factor Analysis

Factor analysis with a cross-sectional point of view has been popularized by Ross (1976) to provide some foundations to multibeta models of expected returns. The basic idea is to start, for a countable sequence of assets $i = 1, 2, ...$ with the decomposition of their payoffs or returns into systematic and idiosyncratic parts with respect to K variables $F_{kt+1}, 1, 2, ..., K$, considered as candidate factors:

$$r_{it+1} = \beta_{i0}^r(U_t) + \sum_{k=1}^{K} \beta_{ik}^r(U_t)F_{kt+1} + \varepsilon_{it+1}$$

$$
\begin{aligned}
E[\varepsilon_{it+1}|U_t] &= 0 \\
Cov[F_{kt+1}, \varepsilon_{it+1}|U_t] &= 0 \ \forall k = 1, 2, ..., K, \ for \ i = 1, 2, ...
\end{aligned} \tag{4.26}
$$

Since, as already explained, the multibeta model (P3) of expected returns amounts to assume that idiosyncratic risks are not compensated, that is:

$$E[m_{t+1}\varepsilon_{it+1}|U_t] = 0 \ for \ i = 1, 2, ..., \tag{4.27}$$

a natural way to look for foundations of this pricing model is to ask why idiosyncratic risk should not be compensated. Ross (1976) provides the following explanation. For a portfolio in the n assets defined by shares θ_{in}, $i = 1, 2, ..., n$ of wealth invested:

$$\sum_{i=1}^{n} \theta_{in=1}, \tag{4.28}$$

the unsystematic risk is measured by:

$$Var[\sum_{i=1}^{n} \theta_{in}\varepsilon_{it+1}|U_t] = \sum_{i=1}^{n} \theta_{in}^2 \sigma_i^2(U_t), \tag{4.29}$$

if we assume that the individual idiosyncratic risks are mutually uncorrelated:

$$Cov[\varepsilon_{it+1}\varepsilon_{jt+1}|U_t] = 0 \ if \ i \neq j, \tag{4.30}$$

and we denote the asset idiosyncratic conditional variances by: $\sigma_i^2(U_t) = Var[\varepsilon_{it+1}|U_t]$.

Therefore, if it is possible to find a sequence $(\theta_{in})_{1 \leq i \leq n}, n = 1, 2, ...$ conformable to (4.28) and (4.31) below:

$$P \lim_{n=\infty} \sum_{i=1}^{n} \theta_{in}^2 \sigma_i^2(U_t) = 0, \tag{4.31}$$

the idiosyncratic risk can be diversified and should not be compensated by a simple no-arbitrage argument. Typically, this result will be valid with bounded conditional variances and equally-weighted portfolios ($\theta_{in} = \frac{1}{n}$ for $i = 1, 2, ...$).

In other words, according to Ross (1976), factors have as a basic property to define idiosyncratic risks which are mutually uncorrelated. This justifies beta pricing with respect to them and provides the following decomposition of the conditional covariance matrix of returns:

$$\Sigma_t = \beta_t \phi_t \beta_t' + D_t \tag{4.32}$$

where Σ_t, β_t, ϕ_t, D_t are matrices of respective sizes n x n, n x k, k x k and n x n defined by:

$$
\begin{aligned}
\Sigma_t &= (Cov(r_{it+1}, r_{jt+1}|U_t))_{1\leq i\leq n, 1\leq j\leq n} \\
\beta_t &= (\beta^r_{ik}(U_t))_{1\leq i\leq n, 1\leq k\leq K} \\
\phi_t &= (Cov(F_{kt+1}, F_{lt+1}|U_t))_{1\leq k\leq K, 1\leq l\leq K} \\
D_t &= (Cov(\varepsilon_{it+1}, \varepsilon_{jt+1}|U_t))_{1\leq i\leq n, 1\leq j\leq n}
\end{aligned} \tag{4.33}
$$

with the maintained assumption that D_t is a diagonal matrix.

In the particular case where returns and factors are jointly conditionally gaussian given U_t, the returns are mutually independent knowing the factors in the conditional probability distribution given U_t. We have therefore specified a Factor Analysis model in a conditional setting. Moreover, if one adopts in such a setting some well-known results in the Factor Analysis methodology, one can claim that the model is fully defined by the decomposition (4.32) of the covariance matrix of returns with the diagonality assumption[15] about the idiosyncratic variance matrix D_t. In particular, this decomposition defines by itself the set of K-dimensional variables F_{t+1} conformable to it with the interpretation (4.33) of the matrices:

$$F_{t+1} = E[F_{t+1}|U_t] + \phi_t\beta_t'\Sigma_t^{-1}(r_{t+1} - E[r_{t+1}|U_t]) + z_{t+1} \tag{4.34}$$

where $r_{t+1} = (r_{it+1})_{1\leq i\leq n}$ and z_{t+1} is a K-dimensional variable assumed to be independent of r_{t+1} given J_t and such that:

$$
\begin{aligned}
E[z_{t+1}|J_t] &= 0 \\
Var[z_{t+1}|J_t] &= \phi_t - \phi_t\beta_t'\Sigma_t^{-1}\beta_t\phi_t
\end{aligned} \tag{4.35}
$$

It means that, up to an independent noise z_t (which represents factor indeterminacy), the factors are rebuilt by the so-called "Thompson Factor scores":

$$\widehat{F}_{t,t+1} = E[F_{t+1}|U_t] + \phi_t\beta_t'\Sigma_t^{-1}(r_{t+1} - E(r_{t+1}|U_t), \tag{4.36}$$

which correspond to the conditional expectation: $\widehat{F}_{t,t+1} = E[F_{t+1}|U_t, r_{t+1}]$ in the particular case where returns and factors are jointly gaussian given U_t.

To summarize, according to Ross (1976) adapted in a conditional setting with latent variables, the question of specifying a multibeta model of expected returns can be addressed in two steps. In a first step, one should identify a factor structure for the family of returns:

[15] Chamberlain and Rothschild (1983) have proposed to take advantage of the sequence model (n→ ∞) to weaken the diagonality assumption on D_t by defining an approximate factor structure. We consider here a factor structure for fixed n.

$$\Sigma_t = \beta_t \phi_t \beta_t' + D_t,$$
$$D_t \text{ diagonal.} \quad (4.37)$$

In a second step, the issue of a multibeta model for expected returns is addressed[16]:

$$E[r_{t+1}|U_t] = \beta_t E[F_{t+1}|U_t]. \quad (4.38)$$

Due to the difficulty of disentangling the dynamics of the beta coefficients in β_t from the one of the factors, both at first order $E[F_{t+1}|U_t]$ in (4.38) and at second order $\phi_t = Var[F_{t+1}|U_t]$ in (4.37), a common solution in the literature is to add the quite restrictive assumption that the matrix β_t of conditional factor loadings is deterministic and time invariant:

$$\beta_t = \beta \text{ for every } t. \quad (4.39)$$

It should be noticed that assumption (4.39) does not imply per se that conditional betas coincide with unconditional ones since unconditional betas are not unconditional expectations of conditional ones. However, since by (4.39):

$$r_{t+1} = E(r_{t+1}|U_t) - \beta E(F_{t+1}|U_t) + \beta F_{t+1} + \varepsilon_{t+1} \quad (4.40)$$

it can be seen that β will coincide with the matrix of unconditional betas if and only if:

$$Cov[E(r_{t+1}|U_t) - \beta E(F_{t+1}|U_t), F_{t+1}|U_t] = 0. \quad (4.41)$$

In particular, if the conditional multibeta model (4.38) of expected returns and the assumption (4.39) of constant conditional betas are maintained simultaneously, the unconditional multibeta model of expected returns can be deduced:

$$Er_{t+1} = \beta E F_{t+1} \quad (4.42)$$

Moreover, this joint assumption guarantees that the conditional factor analytic model (4.40) can be identified by a standard procedure of static factor analysis since:

$$Var(\varepsilon_{t+1}) = E(Var(\varepsilon_{t+1}|U_t)) = E(D_t) \quad (4.43)$$

[16]According to the comments following theorem 4.1, we assume that factors are suitably scaled in order to get the convenient interpretation for the coefficients of the multibeta model of expected returns. Such a scaling can be done without loss of generality since it does not modify the property (4.37). Moreover, in (4.38), returns and factors are implicitly considered in excess of the risk-free rate (net returns and factors).

will be a diagonal matrix as D_t. This remark has been fully exploited by King, Sentana and Wadhwani (1994). However, a general inference methodology for the conditional factor analytic model remains to be stated. First, the restrictive assumption of fixed conditional betas should be relaxed. Second, even with fixed betas, one would like to be able to identify the conditional factor analytic model (4.40) without maintaining the joint hypothesis (4.38) of a multibeta model of expected returns. In this latter case, a factor stochastic volatility approach (see e.g. Meddahi and Renault (1996) and Pitt and Shephard (1999)) should be well-suited. The narrow link between our general state variable setting and the nowadays widespread stochastic volatility model is discussed in the next section.

5. A Dynamic Asset Pricing Model with Latent Variables

In the last section, we analyzed the cross-sectional restrictions imposed by financial asset pricing theories in the context of factor models. While these factor models were conditioned on an information set, the emphasis was not put on the dynamic behavior of asset returns. In this section, we propose an intertemporal asset pricing model based on a conditioning on state variables. Using assumptions spelled out in section 3, we will accommodate a rich intertemporal framework where the stochastic discount factor can represent nonseparable preferences such as recursive utility[17].

5.1. An Equilibrium Asset Pricing Model with Recursive Utility

Many identical infinitely lived agents maximize their lifetime utility and receive each period an endowment of a single nonstorable good. We specify a recursive utility function of the form:

$$V_t = W(C_t, \mu_t), \tag{5.1}$$

where W is an aggregator function that combines current consumption C_t with $\mu_t = \mu(\widetilde{V}_{t+1} \mid J_t)$, a certainty equivalent of random future utility $\widetilde{V}_{t+1}$, given the information available to the agents at time t, to obtain the current-period lifetime utility V_t. Following Kreps and Porteus (1978), Epstein and Zin (1989) propose the CES function as the aggregator function, i.e.

$$V_t = [C_t^\rho + \beta \mu_t^\rho]^{\frac{1}{\rho}}. \tag{5.2}$$

The way the agents form the certainty equivalent of random future utility is based on their risk preferences, which are assumed to be isoelastic, i.e. $\mu_t^\alpha = E[\widetilde{V}_{t+1}^\alpha | I_t]$, where $\alpha \leq 1$ is the

[17]In the proposed intertemporal asset pricing model, we will specify the stochastic discount factor in an equilibrium setting. We will therefore make our stochastic assumptions on economic fundamentals such as consumption and dividend growth rates. In Garcia, Luger and Renault (1999), we make the same types of assumptions directly on the pair SDF-stock returns without reference to an equilibrium model. Similar asset pricing formulas and implications of the presence of leverage effects are obtained in this less specific framework.

risk aversion parameter (1-α is the Arrow-Pratt measure of relative risk aversion). Given these preferences, the following Euler condition must be valid for any asset j if an agent maximizes his lifetime utility (see Epstein and Zin (1989)):

$$E[\beta^{\gamma}(\frac{C_{t+1}}{C_t})^{\gamma(\rho-1)}M_{t+1}^{\gamma-1}R_{j,t+1}|J_t]=1, \tag{5.3}$$

where M_{t+1} represents the return on the market portfolio, $R_{j,t+1}$ the return on any asset j, and $\gamma=\frac{\rho}{\alpha}$. The stochastic discount factor is therefore given by:

$$m_{t+1}=\beta^{\gamma}(\frac{C_{t+1}}{C_t})^{\gamma(\rho-1)}M_{t+1}^{\gamma-1}. \tag{5.4}$$

The parameter ρ is associated with intertemporal substitution, since the elasticity of intertemporal substitution is $1/(1-\rho)$. The position of α with respect to ρ determines whether the agent has a preference towards early resolution of uncertainty ($\alpha<\rho$) or late resolution of uncertainty ($\alpha>\rho$)[18].

Since the market portfolio price, say P_t^M at time t, is determined in equilibrium, it should also verify the first-order condition:

$$E[\beta^{\gamma}(\frac{C_{t+1}}{C_t})^{\gamma(\rho-1)}M_{t+1}^{\gamma}|J_t]=1 \tag{5.5}$$

In this model, the payoff of the market portfolio at time t is the total endowment of the economy C_t. Therefore the return on the market portfolio M_{t+1} can be written as follows:

$$M_{t+1}=\frac{P_{t+1}^M+C_{t+1}}{P_t^M}.$$

Replacing M_{t+1} by this expression, we obtain:

$$\lambda_t^{\gamma}=E\left[\beta^{\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{\gamma\rho}(\lambda_{t+1}+1)^{\gamma}|J_t\right], \tag{5.6}$$

where: $\lambda_t=\frac{P_t^M}{C_t}$. The pricing of assets with price S_t which pay dividends D_t such as stocks will lead us to characterize the joint probability distribution of the stochastic process (X_t,Y_t,J_t) where: $X_t=Log\frac{C_t}{C_{t-1}}$ and $Y_t=Log\frac{D_t}{D_{t-1}}$. As announced in section 3, we define this dynamics through a stationary vector-process of state variables U_t so that:

$$J_t=\vee_{\tau\leq t}[X_\tau,Y_\tau,U_\tau]. \tag{5.7}$$

Given this model structure (with $Log\frac{C_t}{C_{t-1}}$ serving as a factor F_t), we can restate Assumptions 3.1 and 3.2 as:

[18] As mentioned in Epstein and Zin (1991), the association of risk aversion with α and intertemporal sustitution with ρ is not fully clear, since at a given level α of risk aversion, changing ρ affects not only the elasticity of intertemporal sustitution but also determines whether the agent will prefer early or late resolution of uncertainty.

Assumption 5.1.: The pairs $(X_t, Y_t)_{1\leq t\leq T}, t = 1, ..., T$ are mutually independent knowing $U_1^T = (U_t)_{1\leq t\leq T}$.

Assumption 5.2.: The conditional probability distribution of $(X_{t,}Y_t)$ given $U_1^T = (U_t)_{1\leq t\leq T}$ coincides, for any $t = 1, ..., T$, with the conditional probability distribution given $U_1^t = (U_\tau)_{1\leq \tau\leq t}$.

As mentioned in Section 3, Assumptions 5.1 and 5.2 together with Assumption 3.3 and the Markovianity of state variables U_t allow us to characterize the joint probability distribution of the (X_t, Y_t) pairs, t=1,...,T, given U_1^T by:

$$\ell[(X_t, Y_t)_{1\leq t\leq T}|U_1^T] = \prod_{t=1}^{T} \ell[X_t, Y_t|U_t]. \tag{5.8}$$

Proposition 5.1 below provides the exact relationship between the state variables and equilibrium prices.

Proposition 5.1: Under assumptions 5.1 and 5.2 we have:

$$P_t^M = \lambda(U_t)C_{t,} \qquad S_t = \varphi(U_t)D_t,$$

where $\lambda(U_t)$ and $\varphi(U_t)$ are respectively defined by :

$$\lambda(U_t)^\gamma = E\left[\beta^\gamma \left(\frac{C_{t+1}}{C_t}\right)^{\gamma\rho} (\lambda(U_{t+1}) + 1)^\gamma |U_t\right],$$

and

$$\varphi(U_t) = E\left[\beta^\gamma \left(\frac{C_{t+1}}{C_t}\right)^{\gamma\rho-1} \left(\frac{\lambda(U_{t+1}) + 1}{\lambda(U_t)}\right)^{\gamma-1} (\varphi(U_{t+1}) + 1)\frac{D_{t+1}}{D_t} |U_t\right].$$

Therefore, the functions $\lambda(.), \varphi(.)$ are defined on R^P if there are P state variables. Moreover, the stationarity property of the U process together with assumptions 5.1, 5.2 and a suitable specification of the density function (3.6) allow us to make the process (X, Y) stationary by a judicious choice of the initial distribution of (X, Y). In this setting, a contraction mapping argument may be applied as in Lucas (1978) to characterize the functions $\lambda(.)$ and $\varphi(.)$ according to proposition 5.1. It should be stressed that this framework is more general than the Lucas one because the state variables U_t are given by a general multivariate Markovian process (while a Markovian dividend process is the only state variable in Lucas (1978)). Using the return definition for the market portfolio and asset S_t, we can write:

$$LogM_{t+1} = Log\frac{\lambda(U_{t+1})+1}{\lambda(U_t)} + X_{t+1}, \text{and} \tag{5.9}$$

$$LogR_{t+1} = Log\frac{\varphi(U_{t+1})+1}{\varphi(U_t)} + Y_{t+1}.$$

Hence, the return processes (M_{t+1}, R_{t+1}) are stationary as U, X and Y, but, contrary to the stochastic setting in the Lucas (1978) economy, are not Markovian due to the presence of unobservable state variables U.

Given this intertemporal model with latent variables, we will show how standard asset pricing models will appear as particular cases under some specific configurations of the stochastic framework. In particular, we will analyze the pricing of bonds, stocks and options and show under which conditions the usual models such as the CAPM or the Black-Scholes model are obtained.

5.2. Revisiting Asset Pricing Theories for Bonds, Stocks and Options through the Leverage Effect

In this section, we introduce an additional assumption on the probability distribution of the fundamentals X and Y given the state variables U.

Assumption 5.3:

$$\begin{pmatrix} X_{t+1} \\ Y_{t+1} \end{pmatrix} | U_t^{t+1} \sim \aleph \left[\begin{pmatrix} m_{Xt+1} \\ m_{Yt+1} \end{pmatrix}, \begin{bmatrix} \sigma^2_{Xt+1} & \sigma_{XYt+1} \\ \sigma_{XYt+1} & \sigma^2_{Yt+1} \end{bmatrix} \right],$$

where $m_{Xt+1} = m_X(U_1^{t+1}), m_{Yt+1} = m_Y(U_1^{t+1}), \sigma^2_{Xt+1} = \sigma^2_X(U_1^{t+1}), \sigma_{XYt+1} = \sigma_{XY}(U_1^{t+1}), \sigma^2_{Yt+1} = \sigma^2_X(U_1^{t+1})$. In other words, these mean and variance covariance functions are time-invariant and measurable functions with respect to U_t^{t+1}, which includes both U_t and U_{t+1}.

This conditional normality assumption allows for skewness and excess kurtosis in unconditional returns. It is also useful for recovering as a particular case the Black-Scholes formula[19].

5.2.1. The Pricing of Bonds

The price of a bond delivering one unit of the good at time T, $B(t,T)$, is given by the following formula:

$$B(t,T) = E_t[\widetilde{B}(t,T)]. \tag{5.10}$$

[19] It can also be argued that, if one considers that the discrete-time interval is somewhat arbitrary and can be infinitely split, log-normality (conditional on state variables U) is obtained as a consequence of a standard central limit argument given the independence between consecutive (X,Y) given U.

where:

$$\widetilde{B}(t,T) = \beta^{\gamma(T-t)} a_t^T(\gamma) \exp((\alpha-1)\sum_{\tau=t}^{T-1} m_{X\tau+1} + \frac{1}{2}(\alpha-1)^2 \sum_{\tau=t}^{T-1} \sigma^2{}_{X\tau+1}),$$

with: $a_t^T(\gamma) = \prod_{\tau=t}^{T-1} \left[\frac{(1+\lambda(U_{\tau+1})}{\lambda(U_\tau)}\right]^{\gamma-1}$.

This formula shows how the interest rate risk is compensated in equilibrium, and in particular how the term premium is related to preference parameters. To be more explicit about the relationship between the term premium and the preference parameters, let us first notice that we have a natural factorization:

$$\widetilde{B}(t,T) = \prod_{\tau=t}^{T-1} \widetilde{B}(\tau,\tau+1). \tag{5.11}$$

Therefore, while the discount parameter β affect the level of the $\widetilde{B}$, the two other parameters α and γ affect the term premium (with respect to the return-to-maturity expectations hypothesis, Cox, Ingersoll, and Ross (1981)) through the ratio:

$$\frac{B(t,T)}{E_t \prod_{\tau=t}^{T-1} B(\tau,\tau+1)} = \frac{E_t(\prod_{\tau=t}^{T-1} \widetilde{B}(\tau,\tau+1))}{E_t \prod_{\tau=t}^{T-1} E_\tau \widetilde{B}(\tau,\tau+1)}.$$

To better understand this term premium from an economic point of view, let us compare implicit forward rates and expected spot rates at only one intermediary period between t and T:

$$\frac{B(t,T)}{B(t,\tau)} = \frac{E_t \widetilde{B}(t,\tau)\widetilde{B}(\tau,T)}{E_t \widetilde{B}(t,\tau)} = E_t \widetilde{B}(\tau,T) + \frac{Cov_t[\widetilde{B}(t,\tau),\widetilde{B}(\tau,T)]}{E_t \widetilde{B}(t,\tau)}. \tag{5.12}$$

Up to Jensen inequality, equation (5.12) proves that a positive term premium is brought about by a negative covariation between present and future $\widetilde{B}$. Given the expression for $\widetilde{B}(t,T)$ above, it can be seen that for von-Neuman preferences ($\gamma = 1$) the term premium is proportional to the square of the coefficient of relative risk aversion (up to a conditional stochastic volatility effect). Another important observation is that even without any risk aversion ($\alpha = 1$), preferences still affect the term premium through the non-indifference to the timing of uncertainty resolution ($\gamma \neq 1$).

There is however an important sub-case where the term premium will be preference-free because the stochastic discount factor $\widetilde{B}(t,T)$ coincides with the observed rolling-over discount factor (the product of short-term future bond prices, $B(\tau,\tau+1)$, $\tau = t,...,T-1$). Taking equation (5.11) into account, this will occur as soon as $\widetilde{B}(\tau,\tau+1) = B(\tau,\tau+1)$, that is when $\widetilde{B}(\tau,\tau+1)$ is known at time τ. From the expression of $\widetilde{B}(t,T)$ above, it is easy to see that this last property stands if and only if the mean and variance parameters $m_{X\tau+1}$ and $\sigma_{X\tau+1}$ depend on $U_\tau^{\tau+1}$ only through U_τ.

This allows us to highlight the so-called "leverage effect" which appears when the probability distribution of (X_{t+1}) given U_t^{t+1} depends (through the functions m_X, σ_X^2) on the contemporaneous value U_{t+1} of the state process. Otherwise, the non-causality assumption 5.2 can be reinforced by assuming no instantaneous causality from X to U.

In this case, $\ell(X_t|U_1^T) = \ell(X_t|U_1^{t-1})$; it is this property which ensures that short-term stochastic discount factors are predetermined, so the bond pricing formula becomes preference-free:

$$B(t,T) = E_t \prod_{\tau=t}^{T-1} B(\tau, \tau+1).$$

Of course this does not necessarily cancel the term premiums but it makes them preference-free in the sense that the role of preference parameters is fully hidden in short-term bond prices. Moreover, when there is no interest rate risk because the consumption growth rates X_t are iid, it is straightforward to check that constant m_{Xt+1} and σ^2_{Xt+1} imply constant $\lambda(.)$ and in turn $\widetilde{B}(t,T) = B(t,T)$, with zero term premiums.

5.2.2. The Pricing of Stocks

The stock price formula is given by:

$$S_t = E_t \left[\beta^{\gamma(T-t)} \left(\frac{C_T}{C_t} \right)^{\alpha-1} \prod_{\tau=t}^{T-1} \left[\frac{(1+\lambda(U_1^{\tau+1})}{\lambda(U_1^{\tau})} \right]^{\gamma-1} S_T \right].$$

Under conditional log-normality assumption 5.3, we obtain :

$$S_t = E_t\{\beta^{\gamma(T-t)} a_t^T(\gamma) \exp((\alpha-1)\sum_{\tau=t+1}^{T} m_{X\tau} + \frac{1}{2}(\alpha-1)^2 \sum_{\tau=t+1}^{T} \sigma^2{}_{X\tau} + (\alpha-1)\sum_{\tau=t+1}^{T} \sigma_{XY\tau}) S_T\},$$

which can be rewritten as:

$$S_t = E_t \left[\widetilde{B}(t,T) \exp((\alpha-1)\sum_{\tau=t+1}^{T} \sigma_{XY\tau}) S_T \right]. \tag{5.13}$$

As expected, the stock price is expressed as the conditional expectation of its discounted terminal value, where the stochastic discount factor $\widetilde{B}(t,T)$ is risk-adjusted by a CAPM-like term $\exp((\alpha-1)\sum_{\tau=t+1}^{T} \sigma_{XY\tau})$. This term accounts for the covariance risk between the stock and the market portfolio (proportional to the standard CAPM beta risk), weighted by the coefficient of relative risk aversion. In other words, the specific role of time preference parameters β and γ is fully embodied in the stochastic discount factor which characterizes the bond equation. The additional risk premium associated with the stock involves only the risk parameter α.

Another useful way of writing the stock pricing formula is:

$$E_t\left[Q_{XY}(t,T)\right] = 1, \tag{5.14}$$

where:

$$Q_{XY}(t,T) = \widetilde{B}(t,T)\exp((\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau})E[\frac{S_T}{S_t}|U_1^T]. \tag{5.15}$$

To understand the role of the factor $Q_{XY}(t,T)$, it is useful to notice that it can be factorized:

$$Q_{XY}(t,T) = \prod_{\tau=t}^{T-1} Q_{XY}(\tau,\tau+1),$$

and that there is an important particular case where $Q_{XY}(\tau,\tau+1)$ is known at time τ and therefore equal to one by (5.14). This is when there is no leverage effect in the sense that $\ell(X_t, Y_t|U_1^T) = \ell(X_t, Y_t|U_1^{t-1})$. This means that not only there is no leverage effect neither for X nor for Y, but also that the instantaneous covariance σ_{XYt} itself does not depend on U_t. In this case, we have $Q_{XY}(t,T) = 1$. Since we also have $\widetilde{B}(\tau,\tau+1) = B(\tau,\tau+1)$, we can express the conditional expected stock return as:

$$E\left[\frac{S_T}{S_t}|U_1^T\right] = \frac{1}{\prod_{\tau=t}^{T-1} B(\tau,\tau+1)}\exp((1-\alpha)\sum_{\tau=t+1}^{T}\sigma_{XY\tau}).$$

For pricing over one period (t to $t+1$), this formula provides the agent's expectation of the next period return (since in this case the only relevant information is U_1^t):

$$E\left[\frac{S_{t+1}}{S_t}|U_1^t\right] = \frac{1}{B(t,t+1)}\exp[(1-\alpha)\sigma_{XYt+1}].$$

This is a particularly striking result since it is very close to a standard conditional CAPM equation (and unconditional in an iid world), which remains true for any value of the preference parameters α and ρ. While Epstein and Zin (1991) emphasize that the CAPM obtains for $\alpha = 0$ (logarithmic utility) or $\rho = 1$ (infinite elasticity of intertemporal substitution), we stress here that the relation is obtained under a particular stochastic setting for any values of α and ρ. Remarkably, the stochastic setting without leverage effect which produces this CAPM relationship will also produce most standard option pricing models (for example Black and Scholes (1973) and Hull and White (1987)), which are of course preference-free[20].

[20] A similar parallel is drawn in an unconditional two-period framework in Breeden and Litzenberger (1978).

5.2.3. A Generalized Option Pricing Formula

The Euler condition for the price of a European option is given by:

$$\pi_t = E_t\left[\beta^{\gamma(T-t)}\left(\frac{C_T}{C_t}\right)^{\alpha-1}\prod_{\tau=t}^{T-1}\left[\frac{(1+\lambda(U_1^{\tau+1})}{\lambda(U_1^{\tau})}\right]^{\gamma-1} Max[0, S_T - K]\right]. \qquad (5.16)$$

It is worth noting that the option pricing formula (5.16) is path-dependent with respect to the state variables; it depends not only on the initial and terminal values of the process U_t but also on its intermediate values[21]. Indeed, it is not so surprising that when preferences are not time-separable ($\gamma \neq 1$), the option price may depend on the whole past of the state variables.

Using assumptions 5.1 to 5.3, we arrive at an extended Black-Scholes formula:

$$\frac{\pi_t}{S_t} = E_t\left\{Q_{XY}(t,T)\Phi(d_1) - \frac{K\widetilde{B}(t,T)}{S_t}\Phi(d_2)\right\}, \qquad (5.17)$$

where:

$$d_1 = \frac{Log\left[\frac{S_t Q_{XY}(t,T)}{K\widetilde{B}(t,T)}\right]}{(\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2)^{1/2}} + \frac{1}{2}(\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2)^{1/2}, \text{ and}$$

$$d_2 = d_1 - (\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2)^{1/2}.$$

To put this general formula in perspective, we will compare it to the three main approaches that have been used for pricing options: equilibrium option pricing, arbitrage-based option pricing, and GARCH option pricing. The latter pricing model can be set either in an equilibrium framework or in an arbitrage framework. Concerning the equilibrium approach, our setting is more general than the usual expected utility framework since it accommodates non-separable preferences. The stochastic framework with latent variables could also accommodate state-dependent preferences such as habit formation based on state variables.

Of course, the most popular option pricing formulas among practitioners are based on arbitrage rather than on equilibrium in order to avoid in particular the specification of preferences. From the start, it should be stressed that our general formula (5.17) nests a large number of preference-free extensions of the Black-Scholes formula. In particular if $Q_{XY}(t,T) = 1$ and $\widetilde{B}(t,T) = \prod_{\tau=t}^{T-1} B(\tau,\tau+1)$, one can see that the option price (5.17) is nothing but the conditional expectation of the Black-Scholes price, where the expectation is computed with respect to the joint probability distribution of the rolling-over interest rate $\overline{r}_{t,T} = -\sum_{\tau=t}^{T-1}\log B(\tau,\tau+1)$

[21] Since we assume that the state variable process is Markovian, $\lambda(U_1^T)$does not depend on the whole path of state variables but only on the last values U_T.

and the cumulated volatility $\overline{\sigma}_{t,T} = \sqrt{\sum_{\tau=t+1}^{T} \sigma_{Y\tau}^2}$. This framework nests three well-known models. First, the most basic ones, the Black and Scholes (1973) and Merton (1973) formulas, when interest rates and volatility are deterministic. Second, the Hull and White (1987) stochastic volatility extension, since $\overline{\sigma}_{t,T}^2 = Var\left[\log \frac{S_T}{S_t} | U_1^T\right]$ corresponds to the cumulated volatility $\int_t^T \sigma_u^2 du$ in the Hull and White continuous-time setting[22]. Third, the formula allows for stochastic interest rates as in Turnbull and Milne (1991) and Amin and Jarrow (1992). However, the usefulness of our general formula (5.17) comes above all from the fact that it offers an explicit characterization of instances where the preference-free paradigm cannot be maintained. Usually, preference-free option pricing is underpinned by the absence of arbitrage in a complete market setting. However, our equilibrium-based option pricing does not preclude incompleteness and points out in which cases this incompleteness will invalidate the preference-free paradigm. The only cases of incompleteness which matter in this respect occur precisely when at least one of the two following conditions:

$$Q_{XY}(t,T) = 1 \tag{5.18}$$

$$\widetilde{B}(t,T) = \prod_{\tau=t}^{T-1} B(\tau, \tau+1) \tag{5.19}$$

is not fulfilled.

In general, preference parameters appear explicitly in the option pricing formula through $\widetilde{B}(t,T)$ and $Q_{XY}(t,T)$. However, in so-called preference-free formulas, it happens that these parameters are eliminated from the option pricing formula through the observation of the bond price and the stock price. In other words, even in an equilibrium framework with incomplete markets, option pricing is preference-free if and only if there is no leverage effect in the general sense that $Q_{XY}(t,t+1)$ and $\widetilde{B}(t,t+1)$ are predetermined. This result generalizes Amin and Ng (1993a), who called this effect predictability.

It is worth noting that our results of equivalence between preference-free option pricing and no instantaneous causality between state variables and asset returns are consistent with another strand of the option pricing literature, namely GARCH option pricing. Duan (1995) derived it first in an equilibrium framework, but Kallsen and Taqqu (1998) have shown that it could be obtained with an arbitrage argument. Their idea is to complete the markets by inserting the discrete-time model into a continuous time one, where conditional variance is constant between two integer dates. They show that such a continuous-time embedding makes possible arbitrage pricing which is per se preference-free. It is then clear that preference-free

[22] See subsection 5.3 for a detailed comparison between standard stochastic volatility models and our state variable framework.

option pricing is incompatible with the presence of an instantaneous causality effect, since it is such an effect that prevents the embedding used by Kallsen and Taqqu (1994).

5.3. A Comparison with Stochastic Volatility Models

The typical stochastic volatility model (SV model hereafter) introduces a positive stochastic process such that its squared value h_t represents the conditional variance of the value at time $(t+1)$ of a second-order stationary process of interest, given a conditioning information set J_t. In our setting, it is natural to define the conditioning information set J_t by (5.8). It means that the information available at time t is not summarized in general by the observation of past and current values of asset prices, since it also encompasses additional information through state variables U_t. Such a definition is consistent with the modern definition of SV processes (see Ghysels, Harvey and Renault, 1997, for a survey). It incorporates unobserved components that might capture well-documented evidence about conditional leptokurtosis and leverage effects of asset returns (given past and current returns). Moreover, such unobserved components are included in the relevant conditioning information set for option pricing models as in Hull and White (1987). The focus of interest in this subsection are the time series properties of asset returns implied by the dynamic asset pricing model presented in section 5.1. These time series of returns can be seen as stochastic volatility processes by assumption 5.3 on the conditional probability distribution of the fundamentals (X_{t+1}, Y_{t+1}) given J_t. We focus on (X_{t+1}, Y_{t+1}) instead of asset returns since, by (5.9), the joint conditional probability distribution (given U_1^{t+1}) of returns for the two primitive assets is defined by assumption 5.3 up to a shift in the mean.

Let us first consider the univariate dynamics in terms of the innovation process $\eta_{Y_{t+1}}$ of Y_{t+1} with respect to J_t defined as:

$$\eta_{Y_{t+1}} = Y_{t+1} - E[m_Y(U_1^{t+1})|U_1^t]. \tag{5.20}$$

The associated volatility and kurtosis dynamics are then characterized by:

$$\begin{aligned} h_t^Y &= Var[\eta_{Y_{t+1}}|U_1^t] \\ &= Var[m_Y(U_1^{t+1})|U_1^t] + E[\sigma_Y^2(U_1^{t+1})|U_1^t] \end{aligned} \tag{5.21}$$

and

$$\begin{aligned} \mu_{4t}^Y &= E[\eta_{Y_{t+1}}^4|U_1^t] \\ &= 3E[\sigma_Y^4(U_1^{t+1})|U_1^t] \\ &= 3[Var[\sigma_Y^2(U_1^{t+1})|U_1^t] + (E[\sigma_Y^2(U_1^{t+1})|U_1^t])^2] \end{aligned} \tag{5.22}$$

As far as kurtosis is concerned, equations (5.21) and (5.22) provide a representation of the fat-tail effect and its dynamics, sometimes termed heterokurtosis effect. This extends the representation of the standard mixture model, first introduced by Clark (1973) and extended by Gallant, Hsieh and Tauchen (1991). Indeed, in the particular case where:

$$Var[m_Y(U_1^{t+1})|U_1^t] = 0, \tag{5.23}$$

we get the following expression[23] for the conditional kurtosis coefficient:

$$\frac{\mu_{4t}^Y}{(h_t^Y)^2} = 3[1 + (c_t^Y)^2] \tag{5.24}$$

with:

$$c_t^Y = \frac{(Var[\sigma_Y^2(U_1^{t+1})|U_1^t])^{\frac{1}{2}}}{E[\sigma_Y^2(U_1^{t+1})|U_1^t]}. \tag{5.25}$$

This expression emphasizes that the conditional normality assumption does not preclude conditional leptokurtosis with respect to a smaller set of conditioning information. It should be emphasized that formula (5.24) allows for even more leptokurtosis than the standard formula since the probability distributions considered are still conditioned on a large information set, including possibly unobserved components. An additional projection on the reduced information set defined by past and current values of observed asset returns will increase the kurtosis coefficient. In other words, our model allows for innovation terms in asset returns that, even standardized by a genuine stochastic volatility (including a mixture effect), are still leptokurtic. Moreover, condition (5.23) is likely not to hold, providing an additional degree of freedom in our representation of kurtosis dynamics. If we consider the stock return itself instead of the dividend growth, the violation of (5.23) is even more likely since $m_Y(U_1^{t+1})$ is to be replaced by the "expected" return $m_Y(U_1^{t+1}) + \frac{\varphi(U_1^{t+1})}{\varphi(U_1^t)}$. Condition (5.23) will be violated when this expected return will differ from its expected value computed by investors according to our equilibrium asset pricing model, that is $E[m_Y(U_1^{t+1}) + \frac{\varphi(U_1^{t+1})}{\varphi(U_1^t)}|U_1^t]$. We will show now that it is precisely this difference which can produce a genuine leverage effect in stock returns, as defined by Black (1976) and Nelson (1991) for conditionally heteroscedastic returns[24]. This justifies a posteriori the use of the expression leverage effect in Section 5.2 to account for the fact that the probability distribution of (X_{t+1}, Y_{t+1}) given U_1^{t+1} depends (through the functions $m_X, m_Y, \sigma_X, \sigma_Y$ and σ_{XY}) on the contemporaneous value U_{t+1} of the state process[25].

[23] It corresponds to the formula given by Gallant, Hsieh and Tauchen (1991) on page 204.

[24] We will conduct the discussion below in terms of $m_Y(U_1^{t+1})$ but it could be reinterpreted in terms of $m_Y(U_1^{t+1}) + \frac{\varphi(U_1^{t+1})}{\varphi(U_1^t)}$.

[25] The key point is that the mean functions $m_X(U_1^{t+1})$ and $m_Y(U_1^{t+1})$ depend on U_{t+1}. However, if these functions are replaced by the shifted conditional expectations for asset returns according to (5.9), the functions

According to the standard terminology, the stochastic volatility dividend process exhibits a leverage effect if and only if:

$$Cov[\eta_{Y_{t+1}}, h^Y_{t+1}|U^t_1] = Cov[m_Y(U^{t+1}_1), h^Y_{t+1}|U^t_1] < 0 \tag{5.26}$$

Barring the restriction (5.23), if $m_Y(U^{t+1}_1)$ is truly a function of U_{t+1}, the condition in (5.26) amounts to the negativity of the sum of two terms:

$$Cov[m_Y(U^{t+1}_1), Var[m_Y(U^{t+2}_1)|U^{t+1}_1]|U^t_1] \tag{5.27}$$

and:

$$Cov[m_Y(U^{t+1}_1), E[\sigma^2_Y(U^{t+2}_1)|U^{t+1}_1]|U^t_1]. \tag{5.28}$$

In other words, the leverage effect of the stochastic volatility process Y_{t+1} can be produced by any of the two following leverage effects or both[26]. The conditional mean process $m_Y(U^{t+1}_1)$ may be a stochastic volatility process which features a leverage effect defined by the negativity of (5.27). Or the process Y_{t+1} itself may be characterized by a leverage effect and then (5.28) be negative, which means that bad news about expected returns (when $m_Y(U^{t+1}_1)$ is smaller than its unconditional expectations) imply in average a higher expected volatility of Y, that is a value of $E[\sigma^2_Y(U^{t+2}_1)|U^{t+1}_1]$ greater than its unconditional mean. To summarize, Assumption 5.3 not only allows to capture the standard features of a stochastic volatility model (in terms of heavy tails and leverage effects) but also provides for a richer set of possible dynamics. Moreover, we can certainly extend these ideas to multivariate dynamics either for the joint behavior of market and stock returns or for any portfolio consideration. For instance, the dependence of $\sigma_{XY}(U^{t+1}_1)$ on the whole set of state variables offers great flexibility to model the stochastic behavior of correlation coefficients, as recently put forward empirically by Andersen et al. (1999). This last feature is clearly highly relevant for asset allocation or conditional beta pricing models.

6. Conclusion

In this paper, we provided a unifying analysis of latent variable models in finance through the concept of stochastic discount factor (SDF). We extended both the asset pricing factor models and the equilibrium dynamic asset pricing models through a conditioning on state variables. This conditioning enriches the dynamics of asset returns through instantaneous

$\sigma_X(U^{t+1}_1), \sigma_Y(U^{t+1}_1)$ and $\sigma_{XY}(U^{t+1}_1)$ will be reintroduced in these expected returns through the functions $\lambda(U^{t+1}_1)$ and $\varphi(U^{t+1}_1)$ defined by Proposition 5.1.

[26] This decomposition of the leverage effect in two terms is the exact analogue of the decomposition discussed in Fiorentini and Sentana (1998) and Meddahi (1999) for persistence.

causality between the asset returns and the latent variables. Such correlation or leverage effects explain departures from usual CAPM pricing for stocks or Black and Scholes and Hull and White pricing for options. The dependence of conditional covariances on the state variables allows for a rich dynamic stochastic behavior of correlation coefficients which is important for asset allocation or value-at-risk strategies.

The enriched set of empirical implications from such dynamic latent variable models requires to set up a general inference methodology which will account for the inobservability of both cross-sectional factors and longitudinal latent variables. Indirect inference, efficient method of moments or Markov chain Monte Carlo (MCMC) for Bayesian inference are all avenues that can prove useful in this context, since they have been used successfully in stochastic volatility models.

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