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-1 polynômes orthogonaux

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Résumé

Ce mémoire est composé de deux articles qui ont pour but commun de lever le voile et de compléter le schéma d'Askey des q–polynômes orthogonaux dans la limite $q = -1$. L'objectif est donc de trouver toutes les familles de polynômes orthogonaux dans la limite -1, de caractériser ces familles et de les connecter aux autres familles de polynômes orthogonaux -1 déjà introduites. Dans le premier article, une méthode basée sur la prise de limites dans les relations de récurrence est présentée. En utilisant cette méthode, plusieurs nouvelles familles de polynômes orthogonaux sur des intervals continus sont introduites et un schéma est construit reliant toutes ces familles de polynômes -1. Dans le second article, un ensemble de polynômes, orthogonaux sur l'agencement de quatre grilles linéaires, nommé les polynômes de para-Bannai-Ito est introduit. Cette famille de polynômes complète ainsi la liste des para-polynômes.

Mots-clés : polynômes orthogonaux, schéma d'Askey -1, polynômes de Bannai-Ito, polynômes de Bannai-Ito continu, polynômes -1 d'Hahn continu, opérateur de Dunkl, polynôme de para-Bannai-Ito, relation de récurrence, orthogonalité sur quatre grilles linéaires, polynômes orthogonaux bispectraux au-delà des paires tridiagonales.

Abstract

This master thesis contains two articles with the common goal of unveiling and completing the Askey scheme of q–orthogonal polynomials in the $q = -1$ limit. The main objective is to find and characterize new families of -1 orthogonal polynomials and connect them to other already known families. In the first article, a method based on applying limits in recurrence relations is presented. This method is used to find many new families of polynomials orthogonal with respect to continuous measure. A -1 scheme containing them is constructed and a compendium containing the properties of all such families is included. In the second article, a new set of polynomials named the para–Bannai–Ito polynomials is introduced. This new set, orthogonal on a linear quadri–lattice, completes the list of para-polynomials, but it is also a step toward the finalization of the -1 scheme of polynomials orthogonal on finite grids.

Keywords : orthogonal polynomials, -1 Askey scheme, Bannai-Ito polynomials, continuous Bannai-Ito polynomials, Continuous -1 Hahn polynomials, Dunkl operator, para-Bannai-Ito polynomials, recurrence relation, orthogonality on linear quad-lattice, bispectral orthogonal polynomials beyond tridiagonal pair.

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Introduction

La traduction des caractéristiques d'un système physique en équations mathématiques est à la base du travail d'un physicien. C'est cette description mathématique qui lui permet de prévoir l'évolution de ce même système. La précision de cette prévision dépendra bien évidemment de la qualité de la traduction, mais aussi de la possibilité de la résolution de ce modèle. Au prix d'une traduction plus simpliste du système, il est parfois possible de résoudre exactement les équations mathématiques qui le décrivent. On nomme cette catégorie de modèles les systèmes intégrables. On en vient à se questionner sur les caractéristiques qui font qu'un système est intégrable ou non. La réponse repose sur les symétries que possède le système à résoudre. Le théorème de Noether [1] permet de dire que pour chacune de ces symétries, une quantité conservée peut lui être reliée. Si le modèle possède un nombre assez élevé de symétries pour restreindre les degrés de liberté du système, alors on dira que le système est intégrable. La définition de système intégrable peut varier en fonction du contexte, mais l'idée fondamentale reste la même. Cette notion de symétrie est encodée à la base de l'algèbre et de la théorie des groupes, ce qui fait d'eux des outils de choix pour approcher la résolution de ces problèmes. Dans certains de ces cas, il est aussi possible de décrire la dynamique d'un système intégrable explicitement à l'aide de fonctions spéciales qui incorporent ces symétries via la théorie des représentations. Ce cheminement entre les systèmes physiques et les fonctions spéciales est résumé dans le schéma 0.1. L'inversion de ce cheminement peut aussi être utilisée. On commence par la description de nouvelles solutions à partir de celle de solutions connues. Dans le contexte de ce travail, ces solutions sont des familles de polynômes orthogonaux. À partir de ces familles de polynômes orthogonaux, on examine les structures algébriques sous-jacentes, telle que la matrice de Jacobi ou encore l'algèbre de (q-)Zhedanov [2]. Finalement, on tente de lier ces structures à des systèmes physiques.

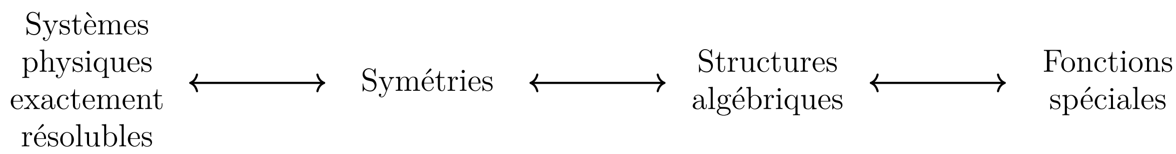

Fig. 0.1. Relation entre les différents domaines qui permettent de passer d'un système physique exactement résoluble aux solutions en fonctions spéciales et vice-versa.

Plus précisément, ce mémoire porte presque uniquement sur la construction et la caractérisation de ces solutions en polynômes orthogonaux. Un avantage de procéder ainsi est qu'il est par la suite possible de rassembler ces solutions dans un recueil. Cela a pour effet de faciliter et d'accélérer la recherche future dans le domaine. C'est donc une étape cruciale dans le cheminement pour obtenir des applications à des systèmes physiques.

La théorie des polynômes orthogonaux est à la base de la résolution de plusieurs systèmes intégrables. Dans les exemples les plus connus, on compte la solution de l'équation d'onde radiale pour l'atome d'hydrogène (polynômes de Laguerre généralisés), la solution de l'équation de Laplace en électrodynamique (polynômes de Legendre) et à la solution de l'oscillateur harmonique quantique en une dimension (polynômes d'Hermite). Les polynômes orthogonaux sont aussi inclus dans la méthode de résolution du réseau de Toda (inverse scattering transformation method) où leur matrice de Jacobi fait partie de la paire de Lax permettant de décrire l'évolution du système [3]. Ils peuvent autrement apparaitre comme élément de matrice de représentation ou encore être impliqués dans l'expression des coefficients de Clebsch–Gordan[4] de certains groupes de symétrie important en physique. Plus récemment, les polynômes orthogonaux se sont retrouvés au centre de l'étude des chaines de spin de type XX qui rendent possible le transfert d'information quantique [5]. Outre les applications aux systèmes intégrables et exactement résolubles, les q-polynômes orthogonaux sont utilisés en théorie des perturbations en théorie de champs quantiques. En physique numérique, les polynômes orthogonaux et la description de leurs zéros permettent de résoudre plus efficacement les problèmes de quadrature et de stabilité des systèmes numériques [6] utile en modélisation.

Pour en arriver à ces applications, une caractérisation et une organisation des résultats dans le domaine des polynômes orthogonaux sont nécessaires. Un exemple est le premier schéma d'organisation des familles de polynômes orthogonaux nommé le schéma d'Askey, présenté suite à la conférence de Bar-le-Duc en 1984 sur les polynômes orthogonaux et leurs applications [7]. Ce tableau lie toutes les familles de polynômes dites classiques. Auparavant, seulement les familles de polynômes de Jacobi, Laguerre et Hermite étaient dites classiques. Toutefois, depuis cette première classification, la définition des familles classiques a changé. Andrews & Askey ont proposé qu'un ensemble de polynômes orthogonaux soit classique s'il correspond à une limite ou une spécialisation des polynômes d'Askey-Wilson ou des polynômes de q-Racah qui trônent au sommet du tableau [8]. Ce tableau contient donc les q–polynômes orthogonaux (analogue q des polynômes orthogonaux), ainsi que les familles correspondant à une limite $q \to 1$ de ces dernières. Toutes les informations sur ces familles ainsi que le tableau d'Askey sont rassemblés dans [9].

Depuis un peu plus d'une dizaine d'années, une nouvelle catégorie de famille de polynômes orthogonaux a été étudié par Vinet & Zhedanov, soit les polynômes -1 [10]. Ils correspondent à une certaine limite $q = -1$ des polynômes d'Askey-Wilson. Cette nouvelle catégorie respecte les critères donnés par Andrews & Askey et elle devrait être considérée classique dans ce sens. Plusieurs travaux ont été effectués dans cette nouvelle branche, autant dans la description des polynômes orthogonaux [11, 12, 13, 14, 15] que dans les algèbres reliées [16]. Ce mémoire continu ce travail et présente une organisation de plusieurs résultats obtenus précédemment.

0.1. Polynômes Orthogonaux

Avant de présenter les objectifs et résultats du mémoire, il est utile de donner un peu plus de détails sur les propriétés des polynômes orthogonaux pour mieux les mettre en contexte. Un ensemble $\{\mathbf{P}_n(x)\}_{n=0}^{\infty}$ de polynômes avec $\mathbf{P}_n(x)$ un polynôme de degré n, $n \in \{0,1,2,...\}$ est dit orthogonal sur E par rapport à la fonction de poids $w(x) \geq 0$ si

$$\int_E \mathbf{P}_n(x)\,\mathbf{P}_m(x)\,w(x)\,dx = h_n\delta_{nm} \qquad n,m \in \{0,1,2,...\}, \tag{0.1.1}$$

Où E est formé d'unions d'intervalles réels sur lesquels $w(x)$ est continu, continu par morceaux ou intégrable. Il est aussi possible de considérer le cas où $\mathrm{supp}(w) = X$ se réduit à un sous-ensemble dénombrable de $\mathbb{R}$, auquel cas l'ensemble $\{\mathbf{P}_n(x)\}_{n=0}^{\infty}$ respecte la relation d'orthogonalité suivante

$$\sum_{x_s \in X} \mathbf{P}_n(x_s)\,\mathbf{P}_m(x_s)\,w(x_s) = h_n\delta_{nm} \qquad n,m \in \{0,1,2,...\}. \tag{0.1.2}$$

Avec cette dernière définition, il est possible de considérer les systèmes de polynômes orthogonaux pour lesquels X est de dimension finie.

$$\sum_{s=0}^{N} \mathbf{P}_n(x_s)\,\mathbf{P}_m(x_s)\,w(x_s) = h_n\delta_{nm} \qquad n,m \in \{0,1,2,...,N\}, \tag{0.1.3}$$

Où les x_s définissent une grille d'orthogonalité. Ceci permet de faire la différence entre deux classes de famille de polynômes, soit les familles à variable continue et les familles à variable discrète. Ce sont les définitions données dans [9] et [17]. En plus de cette relation d'orthogonalité, un ensemble de polynômes orthogonaux respecte automatiquement une relation de récurrence à trois termes. C'est le résultat du théorème de Favard. L'expression de cette dernière relation dépend de la normalisation utilisée. La normalisation utilisée ici est celle dont le coefficient devant x^n de $\mathbf{P}_n(x)$ est 1. Ce faisant, la relation de récurrence se lit:

$$x\mathbf{P}_n(x) = \mathbf{P}_{n+1}(x) + b_n\mathbf{P}_n(x) + u_n\mathbf{P}_{n-1}(x), \tag{0.1.4}$$

où $u_n \geq 0$ et $b_n \in \mathbb{R}$, $n \in \{0,1,2,...\}$ [9]. Si on s'intéresse seulement aux familles classiques, en plus de ces deux relations, les ensembles de polynômes orthogonaux respectent aussi une équation aux valeurs propres de type différentielle (bispectralité). Ils sont aussi exprimés en termes de séries hypergéométriques ou de leur q généralisation, les séries hypergéométriques basiques. Celles-ci sont données respectivement par:

$${}_rF_s\left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_r)_k}{(b_1)_k \ldots (b_s)_k} \frac{z^k}{k!}, \tag{0.1.5}$$

et

$${}_r\phi_s\left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| q; z\right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots (a_r;q)_k}{(b_1;q)_k \ldots (b_s;q)_k} \left[(-1)^k q^{\frac{k(k-1)}{2}}\right]^{1+s-r} \frac{z^k}{(q;q)_k}, \tag{0.1.6}$$

où $(a)_k = a(a+1)\ldots(a+k-1)$ est le symbole de Pochhammer et
$(a;q)_k = (1-aq)(1-aq^2)\ldots\left(1-aq^{k-1}\right)$ est le symbole de q–Pochhammer.

0.2. Chaînes de spin et polynômes orthogonaux de type para

Une chaîne de spin est un modèle composé de N sous-système à niveau qui interagissent seulement avec leurs plus proches voisins. Dans le cas où la chaîne est composée de systèmes à deux niveaux, l'hamiltonien qui représente la chaîne s'écrit:

$$\mathcal{H} = \frac{1}{2}\sum_{i=0}^{N-1} J_{i+1}\left[(1+\lambda)\sigma_x^i\sigma_x^{i+1} + (1-\lambda)\sigma_y^i\sigma_y^{i+1} + \Gamma\sigma_z^i\sigma_z^{i+1}\right] + \frac{1}{2}\sum_{i=0}^{N}\epsilon_i\left(\sigma_z^i + 1\right), \tag{0.2.1}$$

Où les σ_i sont les matrices de Pauli. Il est simple de vérifier que dans le cas d'une chaîne de type XX ($\lambda = 0$ et $\Gamma = 0$), $\left[\mathcal{H}, \sum_{i=1}^{N}\sigma_z^i\right] = 0$ et donc que le nombre de systèmes dans le niveau supérieur d'énergie est conservé lors de l'évolution du système dans le temps. Ce faisant, il est possible d'étudier la chaîne avec un nombre bien défini d'excitations que nous pouvons mettre égal à 1. Une base de ce sous-espace de solutions est composée des états

$$|e_n\rangle = (0,0,...,0,1,0,...0), \quad n \in \{1,2,...,N\}, \tag{0.2.2}$$

Avec l'excitation au $n^{\text{ème}}$ site. On obtient donc que l'application de J, la restriction de $\mathcal{H}$ dans ce sous-espace à une excitation, sur $|e_n\rangle$ est

$$J|e_n\rangle = J_{n+1}|e_{n+1}\rangle + \epsilon_n|e_n\rangle + J_n|e_{n-1}\rangle, \tag{0.2.3}$$

Avec $J_{N+1} = J_0 = 0$. D'un autre côté, on peut s'intéresser à la décomposition des états propres de J ($J\left|s\right\rangle = x_s \left|x_s\right\rangle$) sur la base $\left|e_n\right\rangle$.

$$\left|s\right\rangle = \sum_{n=0}^{N} \sqrt{w_s} P_n\left(x_s\right)\left|e_n\right\rangle . \tag{0.2.4}$$

En combinant 0.2.3 et 0.2.4, on obtient après simplification la relation

$$x_s P_n\left(x_s\right) = J_{n+1} P_{n+1}\left(x_s\right) + \epsilon_n P_n\left(x_s\right) + J_n P_{n-1}\left(x_s\right), \tag{0.2.5}$$

Qui correspond à une relation de récurrence à trois termes pour $P_n\left(x_s\right)$ tel que présenté dans la section précédente. Ceci nous permet de conclure que $P_n\left(x_s\right)$ correspond à une famille de polynômes orthogonaux associée à la matrice de Jacobi J, liant ainsi la physique des chaînes de spin et les polynômes orthogonaux. On obtient alors un modèle analytique pour l'évolution temporelle de l'état de la chaîne. Le lien entre les chaînes de spin et les familles de polynômes orthogonaux de type para devient apparent lorsque l'on désire obtenir un état au dernier site de la chaîne à partir de l'évolution d'un état situé au premier site. Cette transmission de l'état peut donc servir de canal de communication entre les deux bouts. La condition de récupération partielle de l'état initial (FR) à l'autre bout de la chaîne s'écrit

$$U\left(T\right)\left|e_0\right\rangle = e^{-iTJ}\left|e_0\right\rangle = \alpha\left|e_0\right\rangle + \beta\left|e_N\right\rangle, \tag{0.2.6}$$

Avec $|\alpha|^2 + |\beta|^2 = 1$. Si $\alpha = 0$, on a alors une récupération parfaite de l'état initial (PST) à l'autre bout de la chaîne. En utilisant les relations entre la base propre de J et celle d'excitation unique $\left|e_n\right\rangle$, on obtient que la condition de PST est réduite à deux conditions plus simples

$$P_N\left(x_s\right) = (-1)^{N+s} \qquad x_{s+1} - x_s = \frac{\pi}{T} M_s \tag{0.2.7}$$

Où M_s sont des entiers impairs. La première des conditions a été montrée comme étant équivalente à une condition de persymétrie sur la matrice de Jacobi J [18], soit

$$J_{N-n+1} = J_n, \quad \epsilon_{N-n} = \epsilon_n. \tag{0.2.8}$$

L'intérêt des polynômes orthogonaux de type para est que leur matrice de Jacobi est persymétrique ou au minimum une déformation isospectrale d'une matrice persymétrique. Cela fait d'eux des candidats naturels pour la modélisation du transport parfait ou partiel d'information dans les chaînes de spin [19]. De plus, ces familles de polynômes de type para sont orthogonales sur la combinaison de deux grilles, généralisant des résultats plus simples et rajoutant à la richesse des différents cas qui peuvent être traités avec cette modélisation.

L'avantage principal de ce modèle est qu'aucune interaction externe n'est requise pour que l'information contenue dans l'état quantique de départ se retrouve de l'autre côté de la chaîne. Cette transmission est prise en charge par la physique de la chaîne et la façon dont elle a été conçue, ce qui permet de fixer le dispositif lors de la conception. D'autres modèles de transmission d'information quantique existent tels que les protocoles de téléportation quantique et de SWAP. Ces modèles de transfert parfait d'information ont chacun leurs avantages, mais ils requièrent une intervention externe pour fonctionner (application de portes quantique). Un autre avantage du modèle de chaîne de spin est qu'il permet aussi de produire des paires d'états intriqués en utilisant la récupération partielle de l'état avec $|\alpha| = |\beta| = 1/\sqrt{2}$. Ces caractéristiques en font un modèle très intéressant pour la conception de fils quantiques pour la communication entre les différentes parties d'un ordinateur quantique (courte distance), mais aussi pour la conception d'un générateur de paire d'états intriqués à l'intérieur d'un tel ordinateur.

0.3. Objectifs et résultats

Il y a trois objectifs primaires de ce mémoire. Le premier est de compléter la découverte des différentes familles de polynômes -1 orthogonaux par rapport à une fonction de poids continue et de les agencer dans un tableau qui correspond à la limite $q = -1$ du tableau d'Askey. La section du tableau -1 contenant les familles de variable discrète est délaissée pour y revenir dans un travail ultérieur. Le deuxième objectif est de créer un recueil qui rassemble la caractérisation de toutes les familles incluse dans ce nouveau tableau d'Askey -1 continu. Cette caractérisation comprend la relation d'orthogonalité, la relation de récurrence, l'équation différentielle et l'expression explicite en termes de séries hypergéométriques. Le troisième objectif, sensiblement différent des deux premiers, est de caractériser les polynômes de para-Bannai-Ito afin de potentiellement trouver une application dans le transfert d'information quantique.

Ce mémoire est subdivisé en deux chapitres qui contiennent chacun un article. Le premier article couvre les deux premiers objectifs alors que le deuxième couvre le troisième objectif.

Dans le premier article, le tableau d'Askey -1 continu est construit en quatre parties. La partie centrale du tableau, composé des polynômes de Chihara, des polynômes Big -1 Jacobi et toutes leurs spécialisations, est la base du tableau. Une structure en bicolone permet d'observer l'importance des transformations de Christoffel et de Geronimus dans la structure du tableau -1. La deuxième partie du tableau contient les polynômes continus de Bannai-Ito, ainsi que les polynômes -1 de Hahn continu. Les polynômes continus de Bannai-Ito sont reliés aux polynômes big -1 Jacobi, mais aussi aux polynômes -1 de Meixner–Pollaczek qui sont à la tête de la troisième partie du tableau et qui contient les polynômes généralisés d'Hermite et les polynômes d'Hermite. La dernière partie du tableau contient l'équivalent des polynômes continus de Bannai-Ito, mais pour les polynômes de

Bannai-Ito complémentaire. Après cette construction dans le premier article, un recueil est présenté en annexe rassemblant toutes les informations sur chacune des familles comprises dans le tableau -1. Ce recueil sera présenté en annexe de ce mémoire (et non dans les annexes de l'article) pour faciliter la lecture.

Le deuxième papier présente la caractérisation des polynômes de para-Bannai-Ito. Ceux-ci sont présentés comme étant simultanément une troncation non conventionnelle des polynômes généraux de Bannai-Ito (non tronqué) et une limite $q = -1$ des polynômes de q–para–Racah. La relation d'orthogonalité est obtenue, permettant de voir que les polynômes de para-Bannai-Ito sont orthogonaux sur l'agencement de quatre grilles linéaires ou de deux grilles de Bannai-Ito. Il est montré qu'une spécialisation de ces polynômes correspond à une matrice de Jacobi persymétrique. C'est une caractéristique d'importance capitale dans les exemples connus de transfert d'information quantique sur des chaines de spin tel que présenté dans la section précédente. La relation de récurrence, la bispectralité ainsi que l'expression explicite sont aussi obtenues.

0.4. Contributions de l'auteur

Ce projet de recherche a été proposé par le directeur de recherche, Luc Vinet. Initialement, le but du premier article était de découvrir et de caractériser de nouvelles familles correspondant à des limites $q = -1$. Dans cette optique, nous avons décidé de faire un compte rendu de l'avancement de la recherche sur les polynômes -1 continus, qui a mené à plusieurs nouveaux résultats qu'on a pu présenter sous forme de recueil et de tableau. Pour le deuxième article, c'est Luc Vinet qui a énoncé la possibilité d'obtenir un para-polynôme -1 à partir des polynômes de q–para–Racah. Les deux articles inclus dans ce mémoire ont été rédigés majoritairement par l'auteur du mémoire, qui a réalisé tous les calculs et dérivations. Le directeur de recherche ainsi qu'Alexei Zhedanov ont grandement aidé à diriger les efforts. En plus de leurs propositions et leurs conseils pour faire avancer le projet, ils ont aidé à réviser les articles pour en rehausser la qualité.

Chapitre 1

Continuous -1 Hypergeometric Orthogonal Polynomials

Par Jonathan Pelletier, Luc Vinet et Alexei Zhedanov.
Soumis dans *Studies in Applied Mathematics*.

Abstract. The study of -1 orthogonal polynomials viewed as $q \to -1$ limits of the q-orthogonal polynomials is pursued. This paper present the continuous polynomials part of the -1 analog of the q-Askey scheme. A compendium of the properties of all the continuous -1 hypergeometric polynomials and their connections is provided.

1.1. Introduction

In the last decade, attention has been paid to classical orthogonal polynomials that arise as limits when q goes to -1 of polynomials of the q-Askey scheme [**14, 11, 6, 7**]. These are referred to as -1 polynomials. By classical, we mean that in addition to being orthogonal and hence obeying a three-term recurrence relation, they satisfy an eigenvalue equation of the form

$$L\mathbf{P}_n(x) = \lambda_n \mathbf{P}_n(x), \tag{1.1.1}$$

where L is a differential or difference operator. A striking feature of -1 polynomials is that they are eigenfunctions of Dunkl type operators involving reflections [**4**]. A well known example of such polynomials was introduced by Bannai and Ito (Bannai-Ito polynomials) in [**1**], but it was only recently that their eigenvalue equation was identified [**11**] and that their bispectrality was hence established. This eigenvalue equation was found by looking at the most general symmetric first order Dunkl shift operator, that preserves the degree of the polynomials. The Bannai-Ito and the complementary Bannai-Ito polynomials are the most general ones. Being both $q \to -1$ limits of the q–Racah polynomials, they sit at the top of the -1 scheme that is emerging. These families of orthogonal polynomials bring two important concepts in the characterization of the -1 polynomials. The first is the notion of spectral transformations and kernel polynomials. Indeed, these two families are kernel partners, as one can be obtained via a spectral transformation (Christoffel/Geronimus) of the other [**11**]. This type of connection will recur in the description of the continuous part of the -1 scheme. The second concept is the Leonard duality. A monic discrete family with this property is solution to a three-term difference equation in addition to the three-term recurrence relation

$$x\mathbf{P}_n(x) = \mathbf{P}_{n+1}(x) + b_n\mathbf{P}_n(x) + u_n\mathbf{P}_{n-1}(x), \tag{1.1.2}$$

where $b_n,u_n \in \mathbb{R}$ and $u_n > 0$. In the discrete q–Askey scheme, all polynomials enjoy this property. It is not the case for -1 polynomials in general. The complementary Bannai-Ito polynomials and the dual -1 Hahn polynomials [**6, 12**] were found to satisfy a five-term difference equation. This brings new issues in the definition of dual families of orthogonal polynomials under the exchange of the variable and the degree. This dual operation will lead to a five-term recurrence relation, which lies beyond the scope of ordinary orthogonal polynomials (see [**3**]). These two concepts will also appear when considering the continuous part of the -1 scheme.

The main purpose of this paper is to give a classification of the continuous -1 orthogonal polynomials and to organize them in a scheme corresponding to the $q \to -1$ limit of the continuous q-Askey tableau. Four categories of -1 continuous families will be introduced, with many relations connecting them. The characterization of the members of each family, including the explicit expression in terms of hypergeometric series, the three-term recurrence relation, the orthogonality relation, the Dunkl difference or differential equation and the relations to other families of polynomials in the scheme will be provided. The approach is based on the different limits and specializations of the recurrence coefficients of known families of q and -1 orthogonal polynomials. From Favard theorem, it is given that the resulting polynomials will be orthogonal if $b_n,u_n \in \mathbb{R}$ and $u_n > 0$ in (1.1.2). Focusing on continuous polynomials, the corresponding part of the -1 scheme will be constructed. Starting with the most general polynomials with four parameters at the top of this scheme, all other families will be found by cascading down the scheme via specializations or limit processes. Going from a level to the one below, a parameter is lost, until no parameter remains.

The (q-) Askey scheme, with its simplicity, is very useful and practical. Recently, Koornwinder has raised in [**9**] three issues with this scheme that call for possible improvement. These have to do with the completeness of the scheme, the significance (or insignificance) of certain families in the classification and the uniformity of the transformations between families. Following ideas of Verde-Star [**13**], he presented the q–Verde–Star scheme, and more recently, the q–Zhedanov scheme [**10**] focusing on associated algebras. The present work will limit itself to the framework of the q-Askey scheme to determine its -1 analog for continuous polynomials. The methods used to derive the q–Verde–Star and q–Zhedanov schemes are nicely systematic, their application to the $q = -1$ case will however be deferred to future work.

The paper will unfold as follows. In each section, one of the four categories of continuous -1 orthogonal polynomials will be presented. The most general family within this class will be characterized, and its links to the other members of the category will be established through specializations and limit processes. In Section 2, the category of -1 polynomials obtained from direct specialization of the big -1 Jacobi polynomials and the Chihara polynomials is presented. Generalized Gegenbauer and little -1 Jacobi polynomials are also included, as well as the Gegenbauer and special little -1 Jacobi polynomials. The special role played by spectral transformations in structuring the scheme is emphasized in this section. In the next section, we present the -1 polynomials that cascade from the continuous Bannai-Ito polynomials. They include two families: the continuous Bannai-Ito and the continuous -1 Hahn polynomials. The first one is seen to be connected to the big -1 Jacobi and the second, presumably new, is characterized and connected to a $q \to -1$ limit of the continuous q Hahn polynomials. In Section 4, the -1 Meixner-Pollaczek polynomials are introduced as a generalization of the generalized Hermite polynomials via a $q \to -1$ limit of the q-Meixner-Pollaczek polynomials. They are also connected to the continuous -1 Hahn polynomials of Section 3 and to the Chihara polynomials of Section 2. In Section 4, a continuous equivalent of the complementary Bannai-Ito is investigated. These polynomials do not form an orthogonal set, but some specializations of it yield orthogonal families. This leads to the inclusion of the generalized symmetric Bannai-Ito polynomials in the scheme of continuous -1 orthogonal polynomials. Finally, the complete scheme and a compendium of the main properties and connections of the continuous -1 orthogonal polynomials is provided in the appendix.

1.2. Orthogonal Polynomials descending from $q \to -1$ limit of the big q-Jacobi polynomials

In this section, the bulk of the continuous part of the -1 scheme is presented. It is the component that has been the most documented to date (see [**14**], [**15**] and [**7**]). The Chihara polynomials and the big -1 Jacobi polynomials are at the top of this category, with three parameters each. These two sets of polynomials are related by spectral transformations (Christoffel and Geronimus) with spectral parameter equal to 1 [**7**]. They both correspond to $q \to -1$ limit of the same family of polynomials, the big q-Jacobi polynomials. From [**8**], we know that the monic big and little q-Jacobi polynomials are related as follows

$$\mathbf{P}_n^{(\text{big})}(x;a,b,0|q) = (aq)^n \mathbf{P}_n^{(\text{little})}\left(\frac{x}{aq};b,a|q\right) \tag{1.2.1}$$

and we also know that there exists a $q \to -1$ limit of the little q-Jacobi polynomials: the little -1 Jacobi polynomials. This raises a few questions. Does the little q-Jacobi polynomials have a second $q \to -1$ limit and, if so, are those two limits connected via the same spectral transformation? It turns out that it is the case. The recurrence relation of this dilated little q-Jacobi is as follows.

$$x\mathbf{P}_n(x) = \mathbf{P}_{n+1}(x) + (1 - A_n + C_n)\,\mathbf{P}_n(x) + A_{n-1}C_n\mathbf{P}_{n-1}(x), \tag{1.2.2}$$

with $\mathbf{P}_{-1}(x) = 0$ and $\mathbf{P}_0(x) = 1$ and where

$$A_n = \frac{(1-bq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})}, \qquad C_n = \frac{ab^2q^{2n+1}(1-q^n)(1-aq^n)}{(1-abq^{2n})(1-abq^{2n+1})}. \tag{1.2.3}$$

It is straightforward to show that using $q = -e^{\varepsilon}$, $a = -e^{\varepsilon\alpha}$ and $b = -e^{\varepsilon\beta}$ in (1.2.2), one obtains the same definition of the little -1 Jacobi obtained in [**14**] after taking the limit $\varepsilon \to 0$. Moreover, using $q = -e^{\varepsilon}$, $a = -e^{\varepsilon(2\alpha+1)}$ and $b = e^{2\varepsilon\beta}$, one finds the recurrence relation (1.2.4):

$$x\mathbf{G}_n(x) = \mathbf{G}_{n+1}(x) + \sigma_n\mathbf{G}_{n-1}(x), \tag{1.2.4}$$

with $\mathbf{G}_{-1}(x) = 0$ and $\mathbf{G}_0(x) = 1$ and where

$$\sigma_{2n} = \frac{n(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \quad \sigma_{2n+1} = \frac{(n+\alpha+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}. \tag{1.2.5}$$

This set of polynomials is clearly positive definite for all n if $\alpha > -1$ and $\beta > 0$. The recurrence relation is identified as the one for the monic generalized Gegenbauer polynomials [**2**]. When the three-term recurrence relation is presented in the form (1.2.2), it suffices to use the map $A_n \to C_{n+1}$ and $C_n \to A_n$ to take the Christoffel transformation with parameter 1 [**3**]. The generalized Gegenbauer polynomials can also be recovered using this procedure, and so the little -1 Jacobi polynomials are the kernel partner of the generalized Gegenbauer polynomials. The generalized Gegenbauer polynomials are also a specialization of the Chihara polynomials, and thus the structure of the three-parameter row repeats itself at the two-parameter level. This replication of the structure is due to the compatibility of the specializations and of the limits from one row to the lower one. In a diagram, this translates into the commutativity of the sub-diagram containing the two rows. This commutativity puts restrictions on the possible limits that can be used to go down one row in the scheme.

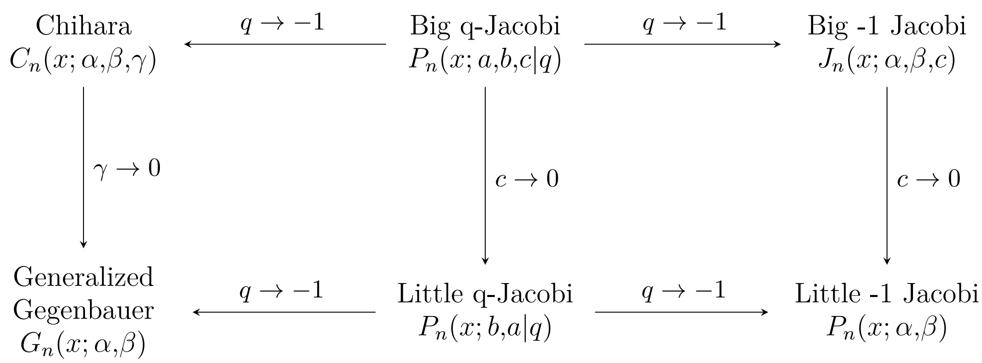

Fig. 1.1. Correspondence between the two different $q \to -1$ limits of the big q-Jacobi and those of the little q-Jacobi

This leads to the inclusion of a third row composed of the Gegenbauer, the special little q-Jacobi and the special little -1 Jacobi polynomials. They correspond to specializations of the two-parameter row using $\alpha = -\frac{1}{2}$, $a = -1$ and $\alpha = 0$ respectively. The special little -1 Jacobi polynomials, even if they were not specifically identified before, were introduced in [**14**] and were seen to be a solution to a Sturm-Liouville equation in addition to the Dunkl-differential equation. This leads to a factorization of the Hamiltonian operator as the square of a first order Dunkl-differential operator. The special little -1 Jacobi and the Gegenbauer polynomials are also kernel partner with parameter 1.

1.3. -1 Orthogonal Polynomials descending from the continuous Bannai-Ito polynomials

In this section, the continuous Bannai-Ito polynomials are shown to be the parent of both the big -1 Jacobi polynomials and the continuous -1 Hahn polynomials. This last family is introduced below, and its main properties are derived through its connection to the continuous Bannai-Ito polynomials.

The continuous Bannai-Ito polynomials were introduced in [**5**] as a continuous version of the Bannai-Ito polynomials. These two families are closely related (using complex parameters and orthogonality on an imaginary axis) and they also correspond to $q \to -1$ limits of the Askey-Wilson polynomials. The explicit expression of the continuous Bannai-Ito $\mathbf{Q}_n(x)$ is:

$$\begin{aligned}\frac{\mathbf{Q}_{2n}\left(x;\alpha,\beta,\gamma,\delta\right)}{\left(-2i\right)^{2n}} = \qquad & (1.3.1)\\ \xi_{2n}\kappa_{n-1}^{(2)}\left(\frac{ix}{2}-\mathfrak{b}-\frac{1}{2}\right){}_4F_3\left(\begin{matrix}-n+1,\ n+g+2,\ \frac{ix}{2}+\mathfrak{b}+1,\ \frac{-ix}{2}+\mathfrak{b}+\frac{3}{2}\\ \frac{5}{2}+\mathfrak{a}+\mathfrak{b},\ 2+\mathfrak{b}+\mathfrak{c},\ 2+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right)+\\ \kappa_n^{(1)}{}_4F_3\left(\begin{matrix}-n,\ n+g+1,\ \frac{ix}{2}+\mathfrak{b},\ \frac{-ix}{2}+\mathfrak{b}+\frac{1}{2}\\ \frac{3}{2}+\mathfrak{a}+\mathfrak{b},\ 1+\mathfrak{b}+\mathfrak{c},\ 1+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right),\end{aligned}$$

$$\frac{\mathbf{Q}_{2n+1}\,(x;\alpha,\beta,\gamma,\delta)}{(-2i)^{2n+1}} = \tag{1.3.2}$$
$$\kappa_n^{(2)}\left(\frac{ix}{2}-\mathfrak{b}-\frac{1}{2}\right){}_4F_3\left(\begin{matrix}-n,\ n+g+2,\ \frac{ix}{2}+\mathfrak{b}+1,\ \frac{-ix}{2}+\mathfrak{b}+\frac{3}{2}\\ \frac{5}{2}+\mathfrak{a}+\mathfrak{b},\ 2+\mathfrak{b}+\mathfrak{c},\ 2+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right)+$$
$$\eta_{2n+1}\kappa_n^{(1)}{}_4F_3\left(\begin{matrix}-n,\ n+g+1,\ \frac{ix}{2}+\mathfrak{b},\ \frac{-ix}{2}+\mathfrak{b}+\frac{1}{2}\\ \frac{3}{2}+\mathfrak{a}+\mathfrak{b},\ 1+\mathfrak{b}+\mathfrak{c},\ 1+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right),$$

where

$$g=\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\mathfrak{d}+1,\qquad \mathfrak{a}=\overline{\mathfrak{d}}=\alpha+i\beta,\qquad \mathfrak{b}=\overline{\mathfrak{c}}=\gamma+i\delta, \tag{1.3.3}$$

$$\begin{aligned}\xi_{2n}&=\frac{n\left(n+\mathfrak{c}+\mathfrak{d}+\frac{1}{2}\right)}{(2n+g)}, & \kappa_n^{(1)}&=\frac{\left(\frac{3}{2}+\mathfrak{a}+\mathfrak{b}\right)_n(1+\mathfrak{b}+\mathfrak{c})_n(1+\mathfrak{b}+\mathfrak{d})_n}{(n+g+1)_n},\\ \eta_{2n+1}&=\frac{(n+\mathfrak{b}+\mathfrak{c}+1)(n+\mathfrak{b}+\mathfrak{d}+1)}{(2n+g+1)}, & \kappa_n^{(2)}&=\frac{\left(\frac{5}{2}+\mathfrak{a}+\mathfrak{b}\right)_n(2+\mathfrak{b}+\mathfrak{c})_n(2+\mathfrak{b}+\mathfrak{d})_n}{(n+g+2)_n}.\end{aligned} \tag{1.3.4}$$

Their recurrence relation reads:

$$x\mathbf{Q}_n(x)=\mathbf{Q}_{n+1}(x)+b_n\mathbf{Q}_n(x)+u_n\mathbf{Q}_{n-1}(x), \tag{1.3.5}$$

where

$$b_n=\begin{cases}2\beta-\frac{(n+4\alpha+2)(\beta-\delta)}{(n+2\alpha+2\gamma+2)}-\frac{n(\beta+\delta)}{(n+2\alpha+2\gamma+1)} & n \text{ even}\\ 2\beta-\frac{(n+4\alpha+4\gamma+3)(\beta+\delta)}{n+2\alpha+2\gamma+2}-\frac{(n+4\gamma+1)(\beta-\delta)}{n+2\alpha+2\gamma+1} & n\text{ odd}\end{cases}, \tag{1.3.6}$$

and

$$u_n=\begin{cases}\frac{n(n+4\alpha+4\gamma+2)\|n+2[\alpha+\gamma+i(\beta+\delta)]+1\|^2}{4(n+2\alpha+2\gamma+1)^2} & n\text{ even}\\ \frac{(n+4\alpha+1)(n+4\gamma+1)\|n+2[\alpha+\gamma+i(\beta-\delta)]+1\|^2}{4(n+2\alpha+2\gamma+1)^2} & n\text{ odd}\end{cases}, \tag{1.3.7}$$

with $\alpha,\beta,\gamma,\delta\in\mathbb{R}^+$.

1.3.1. A Limit to the Big -1 Jacobi Polynomials

As a limit of the Askey-Wilson polynomials with four parameters, the continuous Bannai-Ito polynomials are poised to be at the top of the -1 continuous scheme. We now establish their connection to the big -1 Jacobi polynomials. Let α,β,γ and δ be parametrized as follows

$$\alpha=a_1,\quad \beta=\frac{b_1}{h},\quad \gamma=a_2,\quad \delta=\frac{b_2}{h}, \tag{1.3.8}$$

and denote by

$$\mathbf{Q}_n^{(h)}(x)=\left(\frac{h}{2b_1}\right)^n\mathbf{Q}_n\left(\frac{2b_1x}{h}\right), \tag{1.3.9}$$

the renormalized polynomials resulting from the scaling $x \to \frac{2b_1 x}{h}$. Upon taking the limit $h \to 0$, one obtains the polynomials $\mathbf{Q}_n^{(0)}(x)$ satisfying the recurrence relation

$$x\mathbf{Q}_n^{(0)}(x)) = \mathbf{Q}_{n+1}^{(0)}(x) + (1 - A_n - C_n)\,\mathbf{Q}_n^{(0)}(x) + A_{n-1}C_n\mathbf{Q}_{n-1}^{(0)}(x), \tag{1.3.10}$$

$$A_n = \begin{cases} \frac{\left(1-\frac{b_2}{b_1}\right)(n+4a_1+2)}{2n+4a_1+4a_2+4} & n \text{ even} \\ \frac{\left(1+\frac{b_2}{b_1}\right)(n+4a_1+4a_2+3)}{2n+4a_1+4a_2+4} & n \text{ odd} \end{cases}, \qquad C_n = \begin{cases} \frac{\left(1+\frac{b_2}{b_1}\right)n}{2n+4a_1+4a_2+2} & n \text{ even} \\ \frac{\left(1-\frac{b_2}{b_1}\right)(n+4a_2+1)}{2n+4a_1+4a_2+2} & n \text{ odd} \end{cases}. \tag{1.3.11}$$

Those polynomials $\mathbf{Q}_n^{(0)}(x)$ are identified as the monic big -1 Jacobi polynomials. The continuous Bannai-Ito polynomials are thus the parent of the big -1 Jacobi polynomials, and it is appropriate to have them at the top of the -1 continuous scheme. The exact connection is

$$\lim_{h\to 0}\left(\frac{h}{2b_1}\right)^n \mathbf{Q}_n\left(\frac{2b_1 x}{h}; a_1, \frac{b_1}{h}, a_2, \frac{b_2}{h}\right) = \mathbf{J}_n\left(x; 4a_1+1, 4a_2+1, -\frac{b_2}{b_1}\right). \tag{1.3.12}$$

1.3.2. Continuous -1 Hahn Polynomials

The monic continuous q-Hahn polynomials (with $a = c$, $b = d$ and $x \to (1+q)\,x$ in the definition of [**8**]) are defined through the recurrence relation

$$x\mathbf{P}_n(x) = \mathbf{P}_{n+1}(x) + \frac{1}{2}\left[\frac{a\mathrm{e}^{i\phi} + a^{-1}\mathrm{e}^{-i\phi}}{1+q} - (A_n + C_n)\right]\mathbf{P}_n(x) + \frac{1}{4}A_{n-1}C_n\mathbf{P}_{n-1}(x), \tag{1.3.13}$$

where

$$A_n = \frac{\left(1 - ab\mathrm{e}^{2i\phi}q^n\right)(1-a^2q^n)(1-abq^n)(1-a^2b^2q^{n-1})}{a\mathrm{e}^{i\phi}(1+q)(1-a^2b^2q^{2n-1})(1-a^2b^2q^{2n})}, \tag{1.3.14}$$

and

$$C_n = \frac{a\mathrm{e}^{i\phi}(1-q^n)(1-abq^{n-1})(1-b^2q^{n-1})\left(1-ab\mathrm{e}^{-2i\phi}q^{n-1}\right)}{(1+q)(1-a^2b^2q^{2n-2})(1-a^2b^2q^{2n-1})}, \tag{1.3.15}$$

with $\mathbf{P}_{-1}(x) = 0$ and $\mathbf{P}_0(x) = 1$. Besides the $q \to 1$ limit leading to the continuous Hahn polynomials, a non-trivial $q \to -1$ limit exist and is obtained by using the parametrization

$$q = -e^{\varepsilon}, \quad a = e^{\varepsilon(2\alpha+1)}, \quad b = e^{\varepsilon(2\gamma+1)}, \quad \phi = \frac{\pi}{2} + 2\varepsilon\beta. \tag{1.3.16}$$

Let $\mathbf{K}_n^{(1)}(x)$ be the polynomials obtained when $\varepsilon \to 0$. A direct calculation of the recurrence coefficients gives the relation

$$x\mathbf{K}_n^{(1)}(x) = \mathbf{K}_{n+1}^{(1)}(x) + b_n\mathbf{K}_n^{(1)}(x) + u_n\mathbf{K}_{n-1}^{(1)}(x), \tag{1.3.17}$$

where

$$b_n = \begin{cases} 2\beta - 2\beta\frac{n}{(n+2\alpha+2\gamma+1)} & n \text{ even} \\ 2\beta - 2\beta\frac{(n+4\alpha+4\gamma+3)}{n+2\alpha+2\gamma+2} & n \text{ odd} \end{cases}, \tag{1.3.18}$$

and

$$u_n = \begin{cases} \frac{n(n+4\alpha+4\gamma+2)\|n+2[\alpha+\gamma+2i\beta]+1\|^2}{4(n+2\alpha+2\gamma+1)^2} & n \text{ even} \\ \frac{(n+4\alpha+1)(n+4\gamma+1)\|n+2[\alpha+\gamma]+1\|^2}{4(n+2\alpha+2\gamma+1)^2} & n \text{ odd} \end{cases}, \tag{1.3.19}$$

with $\mathbf{K}_{-1}^{(1)}(x) = 0$ and $\mathbf{K}_1^{(1)}(x) = 1$. It is easy to identify those continuous -1 Hahn polynomials as a direct specialization of the continuous Bannai-Ito presented in (1.3.1)-(1.3.2) with $\delta = \beta$. The complete characterization (expression, difference equation and orthogonality relation) of these continuous -1 Hahn polynomials can be deduced from the properties of the continuous Bannai-Ito ones. In particular, their bispectrality follows from the fact that they satisfy the complex difference eigenvalue equation

$$L\mathbf{K}_n^{(1)}(x) = \lambda_n \mathbf{K}_n^{(1)}(x), \quad \lambda_n = (-1)^n(n + 2\alpha + 2\gamma + 3/2), \tag{1.3.20}$$

$$L = \mathcal{A}\left(S^+R - I\right) + \overline{\mathcal{A}}\left(S^-R - I\right) + (2\alpha + 2\gamma + 3/2)I, \tag{1.3.21}$$

$$\mathcal{A} = \frac{(2\alpha + 1 + i\,[\beta - x])\,(2\gamma + 1 + i\,[\beta - x])}{1 - 2ix}, \tag{1.3.22}$$

where $\overline{\mathcal{A}}$ is the complex conjugate of $\mathcal{A}$, $S^{\pm}f(x) = f(x \pm i)$ and $Rf(x) = f(-x)$. There is in addition another limit resulting from the parametrization

$$q = -e^{\varepsilon}, \quad a = e^{\varepsilon(2\alpha+1)}, \quad b = -e^{\varepsilon(2\gamma+1)}, \quad \phi = \frac{\pi}{2} + 2\varepsilon\beta, \tag{1.3.23}$$

which gives polynomials $\mathbf{K}_n^{(2)}(x)$. These correspond to continuous Bannai-Ito polynomials with $\delta = -\beta$. This second type of continuous -1 Hahn polynomials is very similar to the first type. The differences between the two types do not affect the scheme structure, and both will be represented by one entry in the scheme. More details on this second type are included in the compendium.

1.4. -1 Orthogonal Polynomials descending from the -1 Meixner–Pollaczek polynomials

In this section, a $q \to -1$ limit of the q-Meixner-Pollaczek family is considered to introduce the -1 Meixner-Pollaczek polynomials. These are shown to be a one-parameter extension of the generalized Hermite polynomials.

1.4.1. -1 Meixner-Pollaczek polynomials

The monic q-Meixner-Pollaczek polynomials are defined by the recurrence relation

$$x\mathbf{P}_n(x) = \mathbf{P}_{n+1}(x) + aq^n \cos\phi \mathbf{P}_n(x) + \frac{1}{4}(1 - q^n)\left(1 - a^2q^{n-1}\right)\mathbf{P}_{n-1}(x), \tag{1.4.1}$$

with $\mathbf{P}_{-1}(x) = 0$ and $\mathbf{P}_0(x) = 1$. Multiplying the variable x by $\sqrt{1+q}$, renormalizing the polynomials and using the parametrization

$$q = -e^{-\varepsilon}, \quad a = -e^{-\varepsilon\left(\alpha+\frac{1}{2}\right)}, \quad \phi = \frac{\pi}{2} + \sqrt{\varepsilon}\beta, \tag{1.4.2}$$

we find a set of polynomials that will be denoted $\mathbf{M}_n(x;\alpha,\gamma)$ and that will be called the -1 Meixner-Pollaczek. They verify the recurrence relation

$$x\mathbf{M}_n(x) = \mathbf{M}_{n+1}(x) + (-1)^n\gamma\mathbf{M}_n(x) + u_n\mathbf{M}_{n-1}(x), \tag{1.4.3}$$

where

$$u_{2n} = n, \qquad u_{2n+1} = n + \alpha + \frac{1}{2}, \tag{1.4.4}$$

and $\mathbf{M}_{-1}(x) = 0$ and $\mathbf{M}_0(x) = 1$. Many paths can be used to find the explicit expression of $\mathbf{M}_n(x)$. We will obtain it by first noticing (see below) that the recurrence relation of $\mathbf{M}_n(x)$ is a limit case of the recurrence relation of the continuous -1 Hahn polynomials. Applying the same limit on the explicit expression of these polynomials will yield the expression of $\mathbf{M}_n(x)$. If we define $\hat{\mathbf{K}}_n^{(1)}(x;\alpha,\beta,\gamma) = \frac{1}{(2\alpha)^{n/2}}\mathbf{K}_n^{(1)}(\sqrt{2\alpha}x;\alpha,\beta,\gamma)$ and parametrize as follows

$$\alpha = h, \qquad \beta = \sqrt{\frac{h}{2}}b, \qquad \gamma = \frac{2a-1}{4}, \tag{1.4.5}$$

the recurrence relation (1.3.17) becomes

$$x\hat{\mathbf{K}}_n^{(1)}(x) = \hat{\mathbf{K}}_{n+1}^{(1)}(x) + b_n^{(h)}\hat{\mathbf{K}}_n^{(1)}(x) + u_n^{(h)}\hat{\mathbf{K}}_{n-1}^{(1)}(x), \tag{1.4.6}$$

where

$$b_n^{(h)} = \begin{cases} b\left(1 - \frac{n}{(n+2h+a+1/2)}\right) & n \text{ even} \\ b\left(1 - \frac{n+4h+2a+2}{(n+2h+a+1/2)}\right) & n \text{ odd} \end{cases}, \tag{1.4.7}$$

$$u_n^{(h)} = \begin{cases} \frac{n(n+4h+2a+1)(2h)^2}{4(2h)(n+2h+a+1/2)^2}\|1 + \frac{n+2a+\frac{1}{2}}{2h} + \frac{ib}{\sqrt{2h}}\|^2 & n \text{ even} \\ \frac{(n+4h+1)(n+2a)(2h)^2}{4(2h)(n+2h+a+1/2)^2}\|1 + \frac{n+2a+\frac{1}{2}}{2h}\|^2 & n \text{ odd} \end{cases}. \tag{1.4.8}$$

The $h \to \infty$ limit is then straightforward to compute, and the result is

$$b_n^{(\infty)} = (-1)^n b, \qquad u_n^{(\infty)} = \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{n+2a}{2} & n \text{ odd} \end{cases}, \tag{1.4.9}$$

which coincides with the definition given in (1.4.3). We thus have established that

$$\lim_{h\to\infty} \frac{1}{(2h)^{\frac{n}{2}}}\mathbf{K}_n^{(1)}\left(\sqrt{2h}x, h, \sqrt{\frac{h}{2}}b, \frac{2a-1}{4}\right) = \mathbf{M}_n\left(x;a,b\right). \tag{1.4.10}$$

A very similar relation is also valid for the second type of continuous -1 Hahn polynomials. Using this result with the expressions (1.3.1)-(1.3.2) where δ has been replaced with β, we find the expressions

$$\mathbf{M}_{2n}(x;\alpha,\gamma) = (-1)^n\,(\alpha+1/2)_n\,{}_1F_1\left({-n \atop \alpha+1/2}; x^2-\gamma^2\right),$$
$$\mathbf{M}_{2n+1}(x;\alpha,\gamma) = (-1)^n\,(\alpha+3/2)_n\,(x-\gamma)\,{}_1F_1\left({-n \atop \alpha+3/2}; x^2-\gamma^2\right). \tag{1.4.11}$$

These polynomials were already characterized in [**7**] where their orthogonality relation is derived and the second order Dunkl differential operator that they diagonalize is also given. These are included in the compendium. Even though they were already identified as a limit of the Chihara polynomials, the fact that the -1 Meixner-Pollaczek polynomials are, as expected from the given name, a $q \to -1$ limit of the q-Meixner-Pollaczek polynomials and that they descendant from the continuous -1 Hahn polynomials is new.

The natural way to complete this category of -1 polynomials is by taking $\gamma = 0$ to obtain the generalized Hermite polynomials and then $\alpha = 0$, to obtain the Hermite polynomials. This path of specialization is also coherent with the rest of the scheme, since the limit from the Chihara polynomials to the -1 Meixner-Pollaczek is also valid after the specialization of the Chihara into the generalized Gegenbauer and Gegenbauer polynomials. Then, the generalized Hermite and Hermite polynomials are included as limit cases of the generalized Gegenbauer and Gegenbauer polynomials, respectively.

1.5. Continuous Complementary Bannai-Ito and specialization

In this section, a continuous version of the complementary Bannai-Ito polynomials is investigated with the goal of determining if the Chihara polynomials have a 4-parameters generalization. This family of polynomials is seen to lie beyond the scope of classical orthogonality, since no real three-term recurrence relation can be obtained. Nonetheless, it is shown that some specializations of this family such as the Chihara, the generalized symmetric Bannai-Ito and symmetric Bannai-Ito polynomials possess three-term recurrence relations. The second family is characterized and shown to be related to the Wilson and continuous dual Hahn polynomials.

The continuous Bannai-Ito polynomials were obtained by introducing an imaginary variable and complex parameters in the non-truncated Bannai-Ito polynomials. Using two pairs of complex conjugated parameters, it was shown that the recurrence relation was real and that an infinite family of polynomials orthogonal with respect to a positive continuous measure results from the procedure. The same method will be used here, but with the non-truncated complementary Bannai-Ito polynomials.

Monic complementary Bannai-Ito polynomials with imaginary variable denoted $\tilde{\mathbf{I}}_n(x) = (-i)^n \mathbf{I}_n(ix)$, satisfy the recurrence relation

$$x\tilde{\mathbf{I}}_n(x) = \tilde{\mathbf{I}}_{n+1}(x) - i(-1)^n \rho_2 \tilde{\mathbf{I}}_n(x) + \tau_n \tilde{\mathbf{I}}_{n-1}(x), \tag{1.5.1}$$

$$\begin{aligned} \tau_{2n} &= \tfrac{n(n+\rho_1-r_1+1/2)(n+\rho_1-r_2+1/2)(n-r_1-r_2)}{(2n+g)(2n+g+1)}, \\ \tau_{2n+1} &= \tfrac{(n+g+1)(n+\rho_1+\rho_2+1)(n+\rho_2-r_1+1/2)(n+\rho_2-r_2+1/2)}{(2n+g+1)(2n+g+2)}, \end{aligned} \tag{1.5.2}$$

with $g = \rho_1 + \rho_2 - r_1 - r_2$. It is already clear that ρ_2 needs to be imaginary for the recurrence relation to be real. Consider the following parametrization

$$\rho_1 = a_1 + ib_1 - 1, \qquad \rho_2 = ib_2, \qquad -r_1 = a_3 + ib_3 - \frac{1}{2}, \qquad -r_2 = a_4 + ib_4 - \frac{1}{2}. \tag{1.5.3}$$

Demanding that the family be positive definite leads to three conditions on the coefficients in the recurrence relation (1.5.1):

(1) $b_1 + b_2 + b_3 + b_4 = 0$,

(2) $(n + a_1 + a_3 + i\,[b_1 + b_3])\,(n + a_1 + a_4 + i\,[b_1 + b_4])\,(n + a_3 + a_4 - 1 + i\,[b_3 + b_4]) \in \mathbb{R}$,

(3) $(n + a_3 + i\,[b_2 + b_3])\,(n + a_4 + i\,[b_2 + b_4])\,(n + a_1 + i\,[b_1 + b_2]) \in \mathbb{R}$.

These three conditions are incompatible if $b_2 \neq 0$. Solving those equations with $b_2 = 0$, one finds that $a_3 = a_1$, $b_3 = -b_1$ and $b_4 = -b_2 = 0$. We then define the continuous complementary Bannai-Ito polynomials $\tilde{\mathbf{I}}_n(x; a_1,b_1,a_2,b_2)$ in this framework as the family of polynomials satisfying the recurrence relation

$$x\tilde{\mathbf{I}}_n(x) = \tilde{\mathbf{I}}_{n+1}(x) + (-1)^n b_2 \tilde{\mathbf{I}}_n(x) + \tau_n \tilde{\mathbf{I}}_{n-1}(x), \tag{1.5.4}$$

$$\begin{aligned} \tau_{2n} &= \tfrac{n(n+2a_1-1)(n+a_1+a_2-1+i[b_1-b_2])(n+a_1+a_2-1-i[b_1+b_2])}{(2n+2a_1+a_2-2)(2n+2a_1+a_2-1)}, \\ \tau_{2n+1} &= \tfrac{(n+2a_1+a_2-1)(n+a_2)(n+a_1+i[b_1+b_2])(n+a_1-i[b_1-b_2])}{(2n+2a_1+a_2-1)(2n+2a_1+a_2)}, \end{aligned} \tag{1.5.5}$$

with $\tilde{\mathbf{I}}_{-1}(x) = 0$ and $\tilde{\mathbf{I}}_0(x) = 1$. Although these polynomials are not orthogonal with a positive definite measure on the real line since the recurrence coefficients are not real, many specializations and limits of them are orthogonal. They will be included in the scheme with this caveat as they are part of the overall structure, but they shall of course not be put on equal footing with the other orthogonal families. With l_n normalization factors making them monic, they are expressed as follows in terms of the Wilson polynomials $\mathbf{W}_n(x; a,b,c,d)$.

$$\begin{aligned} \tilde{\mathbf{I}}_{2n}\,(x; a_1,b_1,a_2,b_2) &= l_{2n} \mathbf{W}_n(x^2; ib_2, a_1 + ib_1, a_1 - ib_1, a_2 - ib_2), \\ \tilde{\mathbf{I}}_{2n+1}(x; a_1,b_1,a_2,b_2) &= l_{2n+1}(x - b_2)\mathbf{W}_n(x^2; 1 + ib_2, a_1 + ib_1, a_1 - ib_1, a_2 - ib_2), \end{aligned} \tag{1.5.6}$$

where

$$l_{2n} = \frac{(-1)^n \left(a_1 + i\left[b_2 + b_1\right]\right)_n \left(a_1 + i\left[b_2 - b_1\right]\right)_n (a_2)_n}{(n + 2a_1 + a_2 - 1)_n}, \tag{1.5.7}$$

$$l_{2n+1} = \frac{(-1)^n \left(1 + a_1 + i\left[b_2 + b_1\right]\right)_n \left(1 + a_1 + i\left[b_2 - b_1\right]\right)_n (1 + a_2)_n}{(n + 2a_1 + a_2)_n}. \tag{1.5.8}$$

1.5.1. A Limit to the Chihara Polynomials

The Chihara polynomials can be obtained as a limit of the continuous complementary Bannai-Ito polynomials. Let b_1 and b_2 be parametrized as follows

$$b_1 = hc_1, \qquad b_2 = hc_2, \tag{1.5.9}$$

and denote the renormalized polynomials by

$$\tilde{\mathbf{I}}_n^{(h)}(x) = h^{-n}\left(c_1^2 - c_2^2\right)^{\frac{-n}{2}} \tilde{\mathbf{I}}_n\left(h\sqrt{c_1^2 - c_2^2}x\right). \tag{1.5.10}$$

Upon taking the limit $h \to \infty$, the recurrence relation (1.5.4) goes to

$$x\tilde{\mathbf{I}}_n^{(\infty)}(x) = \tilde{\mathbf{I}}_{n+1}^{(\infty)}(x) + (-1)^n \left(\frac{c_2}{\sqrt{c_1^2 - c_2^2}}\right) \tilde{\mathbf{I}}_n^{(\infty)}(x) + u_n \tilde{\mathbf{I}}_{n-1}^{(\infty)}(x), \tag{1.5.11}$$

$$u_{2n} = \frac{n(n+2a_1-1)}{(2n+2a_1+a_2-2)(2n+2a_1+a_2-1)}, \qquad u_{2n+1} = \frac{(n+2a_1+a_2-1)(n+a_2)}{(2n+2a_1+a_2-1)(2n+2a_1+a_2)}, \tag{1.5.12}$$

which is the recurrence relation of the Chihara polynomials allowing for an affine transformation of the parameters. We conclude that the Chihara polynomials are limits of the continuous complementary Bannai-Ito polynomials. This corresponds to

$$\lim_{h\to\infty} \frac{1}{h^n \left(c_1^2 - c_2^2\right)^{\frac{n}{2}}} \tilde{\mathbf{I}}_n\left(h\sqrt{c_1^2 - c_2^2}x; \frac{\beta+1}{2}, hc_1, \alpha+1, hc_2\right) = \mathbf{C}_n\left(x; \alpha, \beta, \frac{c_2}{\sqrt{c_1^2 - c_2^2}}\right). \tag{1.5.13}$$

1.5.2. Specialization to Generalized Symmetric Bannai-Ito

As mentioned before, taking $b_2 = 0$ in (1.5.4) gives a real recurrence relation for polynomials that are orthogonal with respect to a positive continuous measure when $\tau_n > 0$. These polynomials, that we shall call generalized symmetric Bannai-Ito polynomials and denote $\hat{\mathbf{I}}_n(x)$, are clearly symmetric. They are a made out of Wilson polynomials, and thus the Chihara method for symmetric moment functional [**3**] can be used on the expressions (1.5.6) with $b_2 = 0$. For the remainder of the section, we will use the change of parameter $a = a_1 + ib_1$, $b = a_1 - ib_1$, $c = a_2$ and denote the polynomials by $\hat{\mathbf{I}}_n(x; a, b, c)$. Applying the Chihara procedure, the weight function for the generalized symmetric Bannai-Ito family is found to be

$$\omega(x) = \left|\frac{\Gamma(ix)\,\Gamma(a+ix)\,\Gamma(b+ix)\,\Gamma(c+ix)}{\Gamma(2ix)}\right|^2, \tag{1.5.14}$$

on the interval $[-\infty,\infty]$. When $\mathrm{Re}(a,b,c) > 0$, $a+b+c > 1$ and non-real parameters occur in conjugate pairs, the orthogonality relation reads

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} \omega(x)\hat{\mathbf{I}}_n(x)\hat{\mathbf{I}}_m(x)\mathrm{d}x = \kappa_n\delta_{nm}, \tag{1.5.15}$$

$$\kappa_n = \frac{\Gamma(n+a+b)\,\Gamma(n+a+c)\,\Gamma(n+b+c)\,\Gamma(n+a)\,\Gamma(n+b)\,\Gamma(n+c)\,n!}{\Gamma(2n+a+b+c)\,(n+a+b+c-1)_n}. \tag{1.5.16}$$

The generalized symmetric Bannai-Ito polynomials are eigenfunctions of a modified version of the difference operator of the complementary Bannai-Ito polynomials [**6**] where the shift is on the imaginary axis $S^{\pm}f(x) = f(x \pm i)$ in view of the change of variable introduced in (1.5.1) and where the parameters are taken according to (1.5.3):

$$D_\sigma\hat{\mathbf{I}}_n(x) = \Lambda_n^{(\sigma)}\hat{\mathbf{I}}_n(x), \tag{1.5.17}$$

$$\Lambda_{2n}^{(\sigma)} = n^2 + (a+b+c-1)\,n, \qquad \Lambda_{2n+1}^{(\sigma)} = n^2 + (a+b+c)\,n + \sigma, \tag{1.5.18}$$

$$D_\sigma = D_0 + \frac{\sigma}{2}\,(I - R)\,, \tag{1.5.19}$$

$$D_0 = B(x)S^+ + A(x)S^- + C(x)R - (A(x) + B(x) + C(x))\,I, \tag{1.5.20}$$

$$A(x) = \frac{(ix+a)\,(ix+b)\,(ix+c)}{2\,(2ix+1)}, \tag{1.5.21}$$

$$B(x) = \frac{(ix-a)\,(ix-b)\,(ix-c)}{2\,(2ix-1)}, \tag{1.5.22}$$

$$C(x) = \frac{1}{2}\left(ab + ac + bc - x^2\right) - A(x) - B(x). \tag{1.5.23}$$

This establishes the bispectrality property of the generalized symmetric Bannai-Ito and complete their characterization. It is also interesting to note that the limit (1.5.13) is also valid for $\hat{\mathbf{I}}_n(x)$ with $c_2 = 0$, which relates the generalized symmetric Bannai-Ito family to the generalized Gegenbauer polynomials.

1.5.3. A limit to Symmetric Bannai-Ito polynomials

In view of §1.5.2, the limit from the Wilson to the continuous dual Hahn polynomials can be used similarly for their -1 counterpart. This amount to taking the limit when c goes to ∞. One then obtains a family of orthogonal polynomials called symmetric Bannai-Ito with complex parameters and denoted $\hat{\mathbf{S}}_n(x)$. They were introduced in a different but equivalent way in [**11**]. They are given by

$$\begin{aligned}\hat{\mathbf{S}}_{2n}(x;a,b) &= k_{2n}\mathbf{S}_n(x^2;0,a,b),\\ \hat{\mathbf{S}}_{2n+1}(x;a,b) &= k_{2n+1}x\mathbf{S}_n(x^2;1,a,b),\end{aligned} \tag{1.5.24}$$

with normalization constants k_n ensuring that they are monic and where $\mathbf{S}_n(x)$ are continuous dual Hahn polynomials. These polynomials satisfy the recurrence relation

$$x\hat{\mathbf{S}}_n(x) = \hat{\mathbf{S}}_{n+1}(x) + \tau_n\hat{\mathbf{S}}_{n-1}(x), \tag{1.5.25}$$

$$\tau_{2n} = n\,(n+a+b-1)\,, \qquad \tau_{2n+1} = (n+a)\,(n+b)\,. \tag{1.5.26}$$

When $\mathrm{Re}(a,b) > 0$ and non-real parameters occur in conjugated pairs, the orthogonality relation reads

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\omega(x)\hat{\mathbf{S}}_n(x)\hat{\mathbf{S}}_m(x)\mathrm{d}x = \kappa_n\delta_{nm}, \tag{1.5.27}$$

$$\kappa_n = \Gamma\,(n+a+b)\,\Gamma\,(n+a)\,\Gamma\,(n+b)\,n!, \tag{1.5.28}$$

with

$$\omega(x) = \left|\frac{\Gamma\,(ix)\,\Gamma\,(a+ix)\,\Gamma\,(b+ix)}{\Gamma\,(2ix)}\right|^2. \tag{1.5.29}$$

The difference operator (1.5.17) exists in the limit $c \to \infty$ if we divide on both side by c before taking the limit. The following difference equation is obtained for $\hat{\mathbf{S}}_n(x)$.

$$D_\sigma\hat{\mathbf{S}}_n(x) = \Lambda_n^{(\sigma)}\hat{\mathbf{S}}_n(x), \tag{1.5.30}$$

$$\Lambda_{2n}^{(\sigma)} = n, \qquad \Lambda_{2n+1}^{(\sigma)} = n+\sigma, \tag{1.5.31}$$

$$D_\sigma = D_0 + \frac{\sigma}{2}\,(I-R)\,, \tag{1.5.32}$$

$$D_0 = B(x)S^+ + A(x)S^- + C(x)R - (A(x)+B(x)+C(x))\,I, \tag{1.5.33}$$

$$A(x) = \frac{(ix+a)\,(ix+b)}{2\,(1+2ix)}, \tag{1.5.34}$$

$$B(x) = \frac{(ix-a)\,(ix-b)}{2\,(1-2ix)}, \tag{1.5.35}$$

$$C(x) = \frac{a+b}{2} - A(x) - B(x). \tag{1.5.36}$$

We obtained the symmetric Bannai-Ito by first taking $b_2 = 0$ and then $c \to \infty$. If we reverse the order of these operations on the polynomials, we can observe that the polynomials $\hat{\mathbf{S}}_n$

are some specialization of a $q \to -1$ limit of the continuous dual q-Hahn polynomials. This observation is due to the correspondence between the $c \to \infty$ limit and the limit relation between Askey-Wilson and continuous dual q-Hahn polynomials. It should be noted that the symmetric Bannai-Ito polynomials can also be obtained by setting $\beta = 0$ in the continuous -1 Hahn polynomials of §1.3.2. Finally, the symmetric Bannai-Ito have the generalized Hermite polynomials as a limit. This connection is found in the compendium.

1.6. Conclusion

In this paper, we have constructed the continuous part of the -1 orthogonal polynomials scheme and presented it using four categories. First, we completed the bulk of the scheme, which contains the -1 polynomials obtained from the big q-Jacobi polynomials. The continuous Bannai-Ito polynomials were identified as the top family of the continuous part of the scheme and as parent of the big -1 Jacobi polynomials. Furthermore, they were identified as a generalization of the continuous -1 Hahn polynomials. These polynomials and the continuous Bannai-Ito polynomials formed the second category. In the third part, two new ways of looking at the -1 Meixner-Pollaczek were proposed. They are seen on the one hand as a $q \to -1$ limit of the q-Meixner-Pollaczek polynomials, and on the other as a limit of the novel continuous -1 Hahn polynomials. The -1 Meixner-Pollaczek polynomials, together with the generalized Hermite and Hermite polynomials, made out the third category of -1 continuous polynomials. In the later part, a continuous equivalent to the complementary Bannai-Ito was looked at. Even if the resulting polynomials are not orthogonal, many specializations and limits are. It was found that the Chihara polynomials are descendants of this a family. Another new family was introduced: the generalized symmetric Bannai-Ito polynomials, that are also descendants of the continuous complementary Bannai-Ito polynomials. The explicit expression, orthogonality relation and difference equation were obtained using the properties of the Wilson polynomials. A compendium of the main properties of all the families included in the continuous part of the scheme is presented in the appendix. To arrive at a complete -1 scheme, there remains to determine its discrete part. While many elements of it have been identified, the full picture requires more work. It is our plan to supplement the continuous part of the -1 scheme presented here with its discrete complement in an upcoming report.

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Scheme

of

Continuous -1 Hypergeometric Orthogonal Polynomials

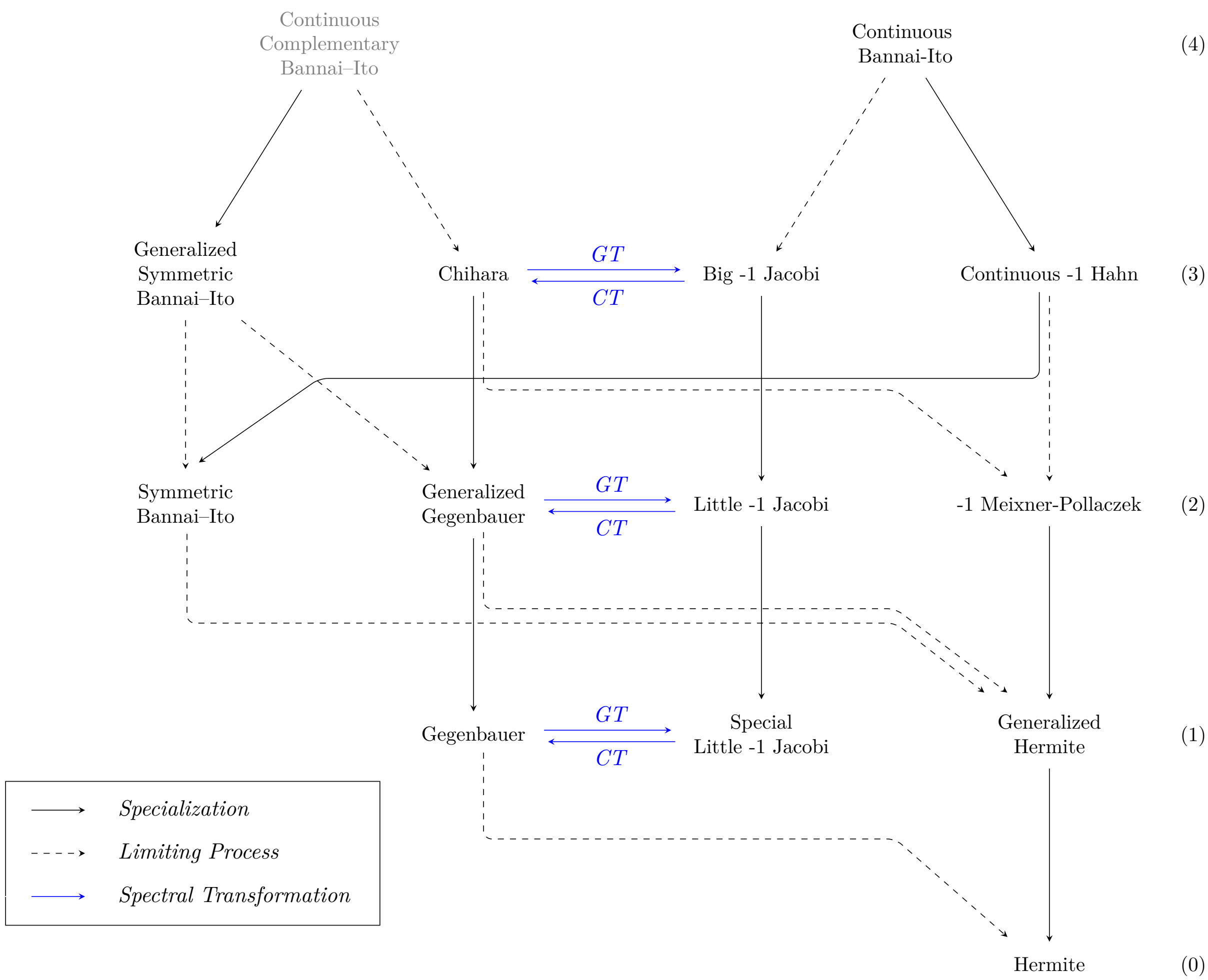

Fig. 1.2. Scheme of continuous -1 hypergeomtric orthogonal polynomials

1.A. A compendium of the properties of the -1 continuous orthogonal polynomials

Cette section a été déplacée en annexe du mémoire pour en faciliter la lecture.

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Chapitre 2

Para-Bannai-Ito Polynomials

Par Jonathan Pelletier, Luc Vinet et Alexei Zhedanov.
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Abstract. New bispectral polynomials orthogonal on a Bannai-Ito bi-lattice (uniform quadri-lattice) are obtained from an unconventional truncation of the untruncated Bannai-Ito and Complementary Bannai-Ito polynomials. A complete characterization of the resulting para-Bannai-Ito polynomials is provided, including a three term recurrence relation, a Dunkl-difference equation, an explicit expression in terms of hypergeometric series and an orthogonality relation. They are also derived as a $q \to -1$ limit of the q-para-Racah polynomials. A connection to the Dual -1 Hahn polynomials is also established.

2.1. Introduction

There are many reasons to be interested in a $q \to -1$ limit of the q-para-Racah polynomials. From a mathematical physics standpoint, since the discovery of the para-Krawtchouk polynomials [**14**], para-polynomials have been linked to perfect state transfer (PST) and fractional revival (FR) in XX-spin chains with nearest neighbour couplings (see also [**9**]). A necessary feature for PST is the persymmetry (i.e. symmetry under anti-diagonal reflections) of the underlying Jacobi matrix and, for FR [**5**], the existence of an isospectral deformation of this persymmetric matrix. The models with these state transport properties that are based on -1 orthogonal polynomials have not been developed. From a mathematical viewpoint, para-polynomials have appeared as basis functions for the finite-dimensional representations of algebras of the Skyanin type [**4, 2, 1**]. The q-para-Racah polynomials arise in particular in connection with the degenerate Sklyanin algebra [**7**]. The $q \to -1$ limits of these q-para-Racah polynomials therefore stand to provide a representation basis for a -1 version of this algebra. The study of the para-Bannai-Ito polynomials is also of interest in the elaboration of the scheme of -1 orthogonal polynomials, but also to obtain the complete set of para-polynomials; Indeed, the bi-Bannai-Ito grid is the only one that has not been studied (appreciating that the quadratic bi-lattice grid leads to the para-Racah polynomials [**10**] and that the q-quadratic bi-lattice grid takes one to the q-para-Racah polynomials [**11**]).

The goal of this paper is to fully characterize the $q \to -1$ limit of the q-para-Racah polynomials. We name them para-Bannai-Ito polynomials, as they are obtained from an unconventional truncation of the general Bannai-Ito [**12**] and general complementary Bannai-Ito polynomials [**6**]. The paper has the following structure. In Section 2, we review some important properties of the general Bannai-Ito and complementary Bannai-Ito polynomials and, their relations via Christoffel/Geronimus transformations. In Section 3, the truncation condition for the general complementary Bannai-Ito with $N = 2j$ (j even) is presented and, are obtained also the three-term recurrence relation, the Dunkl-difference equation, the explicit expression in terms of hypergeometric series and the orthogonality relation. Darboux transformations are used to obtain in Section 4 the recurrence relation of the para-Bannai-Ito polynomials for $N = 2j + 1$ (j even) through the same truncation, albeit in the case of the general Bannai-Ito polynomials. The corresponding cases for j odd are treated in Appendix §2.A and §2.B. In Section 5, the para-Bannai-Ito polynomials are obtained as a $q \to -1$ limit of the q-para-Racah polynomials and a connection to the dual -1 Hahn is presented.

2.2. General Complementary Bannai-Ito and Bannai-Ito

Let us first review some properties of the general untruncated complementary Bannai-Ito. Together with the general Bannai-Ito polynomials, they sit atop the $q \to -1$ limit of the q-Askey scheme and depend on four parameters ρ_1,ρ_2,r_1,r_2. They admit an explicit expression given by

$$\mathbf{I}_{2n}\left(x;\rho_1,\rho_2,r_1,r_2\right) = l_n^{(1)}\left(\rho_1,\rho_2,r_1,r_2\right)\mathbf{W}_n\left(\left(ix\right)^2;\rho_2,\rho_1+1,-r_1+\frac{1}{2},-r_2+\frac{1}{2}\right), \tag{2.2.1}$$

$$\mathbf{I}_{2n+1}\left(x;\rho_1,\rho_2,r_1,r_2\right) = \left(x-\rho_2\right)\frac{l_n^{(2)}}{l_n^{(1)}}\mathbf{I}_{2n}\left(x;\rho_1,\rho_2+1,r_1,r_2\right), \tag{2.2.2}$$

where $l_n^{(i)}\left(\rho_1,\rho_2,r_1,r_2\right)$ are normalizing factor to ensure monicity, $\mathbf{W}_n(x^2)$ are Wilson polynomials given by

$$\mathbf{W}_n\left(x^2;a,b,c,d\right) = {}_4F_3\left(\begin{matrix}-n,\ n+a+b+c+d-1,\ a-ix,\ a+ix\\ a+b,\ a+c,\ a+d\end{matrix};1\right) \tag{2.2.3}$$

$$= \sum_{k=0}^{n} A_{n,k} = \sum_{k=0}^{n}\frac{\left(-n,n+a+b+c+d-1,a-ix,a+ix\right)_k}{\left(1\right)_k\left(a+b,a+c,a+d\right)_k}, \tag{2.2.4}$$

and $\left(a\right)_k = \left(a\right)\left(a+1\right)...\left(a+k-1\right)$ and $\left(a_1,a_2,...,a_n\right)_k = \left(a_1\right)_k\left(a_2\right)_k...\left(a_n\right)_k$ are the standard Pochhammer symbols.

These polynomials obey the three term recurrence relation

$$x\mathbf{I}_n(x) = \mathbf{I}_{n+1}(x) + (\rho_1 - A_n - C_n)\,\mathbf{I}_n(x) + A_{n-1}C_n\mathbf{I}_{n-1}(x), \tag{2.2.5}$$

with recurrence coefficients

$$A_n = \begin{cases} -\frac{(n+2\rho_2-2r_2+1)(n+2\rho_2-2r_1+1)}{4(n+g+1)} & n \text{ even} \\ -\frac{(n+1)(n-2r_1-2r_2+1)}{4(n+g+1)} & n \text{ odd} \end{cases}, C_n = \begin{cases} \frac{(n+2\rho_1-2r_1+1)(n+2\rho_1-2r_2+1)}{4(n+g+1)} & n \text{ even} \\ \frac{(n+2g+1)(n+2\rho_1+2\rho_2+1)}{4(n+g+1)} & n \text{ odd} \end{cases}, \tag{2.2.6}$$

where $g = \rho_1 + \rho_2 - r_1 - r_2$.

The general Bannai-Ito polynomials can be presented as the Geronimus transformation of those $\mathbf{I}_n$ with parameter ρ_1. The general Bannai-Ito are hence given in terms of general complementary Bannai-Ito as follows

$$\mathbf{B}_n(x;\rho_1,\rho_2,r_1,r_2) = \mathbf{I}_n(x;\rho_1,\rho_2,r_1,r_2) - A_{n-1}\mathbf{I}_{n-1}(x;\rho_1,\rho_2,r_1,r_2). \tag{2.2.7}$$

They obey the three term recurrence relation

$$x\mathbf{B}_n(x) = \mathbf{B}_{n+1}(x) + (\rho_1 - A_{n-1} - C_n)\,\mathbf{B}_n(x) + A_{n-1}C_{n-1}\mathbf{B}_{n-1}(x), \tag{2.2.8}$$

where A_n and C_n are as in (2.2.6).

Dunkl-difference equations are presented for the complementary Bannai-Ito and Bannai-Ito polynomials in [**6**] and [**12**] respectfully. It can be seen from formulas (2.2.6) that the positivity condition $A_{n-1}C_n > 0$ cannot be achieved for all $n \in \mathbb{N}$ for the general complementary Bannai-Ito. Nevertheless, it is possible to obtain a finite set of $N+1$ orthogonal polynomials by using as truncation conditions $A_{-1}C_0 = 0$ and $A_N C_{N+1} = 0$. The first condition is always respected, but a parametrization dependent on N and its parity is needed for the second condition, and it can be realized in many ways. If one sets either $(N + 2\rho_2 - 2r_2 + 1) = 0$, $(N + 2\rho_2 - 2r_1 + 1) = 0$ or $(N + 2\rho_1 + 2\rho_2 + 2) = 0$ if N is even or $(N - 2r_1 - 2r_2 + 1) = 0$, $(N + 2\rho_1 - 2r_1 + 2) = 0$ or $(N + 2\rho_1 - 2r_2 + 2) = 0$ if N is odd, then the usual complementary Bannai-Ito polynomials are recovered from the truncation. However, using a parametrization such that

$$(N + 2g + 2) = 0, \tag{2.2.9}$$

when N is even leads to an admissible truncation and to a different result. One should be careful when choosing a parametrization for this last truncation, since a zero is introduced in the denominator of the coefficients A_n and C_n for $n \sim N/2$. The parametrization needs to ensure that a cancellation occurs to have finite expressions in those cases. A very similar truncation arises for the general Bannai-Ito but for N odd, and it reads

$$(N + 2g + 1) = 0. \tag{2.2.10}$$

These truncation conditions lead to a new family of orthogonal polynomials: the para-Bannai-Ito polynomials. Taking $N = 2j$ in (2.2.9) and $N = 2j+1$ in (2.2.10), both truncation conditions are identical and correspond to

$$(j + g + 1) = 0. \tag{2.2.11}$$

. The parametrization then only depends on the parity of j.

2.3. Para-Bannai-Ito polynomials for $N = 2j$, j even

In this section, we obtain the recurrence relation of the para-Bannai-Ito for $N = 2j$, j even, by applying the truncation condition (2.2.11) to the general complementary Bannai-Ito polynomials. An appropriate parametrization is

$$\rho_1 - r_1 = -\frac{j+1}{2} + e_1 t, \qquad \rho_2 - r_2 = -\frac{j+1}{2} + e_2 t, \tag{2.3.1}$$

and (2.2.11) is achieved in the limit $t \to 0$.

2.3.1. Recurrence relation

Inserting (2.3.1) in (2.2.6) and using the change of parameters

$$\rho_1 = \frac{b-j-1+a}{4}, \qquad \rho_2 = \frac{b-j-1-a}{4}, \qquad \frac{e_1}{e_1+e_2} = \alpha, \qquad \frac{e_2}{e_1+e_2} = (1-\alpha), \tag{2.3.2}$$

it is seen that A_j^0 and C_j^0 only depend on e_1 and e_2, through combinations. That can be described in terms of a single parameter α as above. The recurrence relation for the para-Bannai-Ito polynomials $\mathbf{P}_n^{(0)}(x; a,b,\alpha,2j)$ reads

$$x\mathbf{P}_n^{(0)}(x) = \mathbf{P}_{n+1}^{(0)}(x) + \left(\frac{b-j-1+a}{4} - A_n^0 - C_n^0\right)\mathbf{P}_n^{(0)}(x) + A_{n-1}^0 C_n^0 \mathbf{P}_{n-1}^{(0)}(x), \tag{2.3.3}$$

$$A_n^0 = \begin{cases} -\frac{1}{4}(n-j-a) & \begin{matrix} n \text{ even} \\ n \neq j \end{matrix} \\ -\frac{1}{4}\frac{(n+1)(n-j-b)}{(n-j)} & n \text{ odd} \\ \frac{1}{2}(1-\alpha)a & \begin{matrix} n \text{ even} \\ n = j \end{matrix} \end{cases}, \quad C_n^0 = \begin{cases} \frac{1}{4}(n-j+a) & \begin{matrix} n \text{ even} \\ n \neq j \end{matrix} \\ \frac{1}{4}\frac{(n-2j-1)(n-j+b)}{(n-j)} & n \text{ odd} \\ \frac{1}{2}\alpha a & \begin{matrix} n \text{ even} \\ n = j \end{matrix} \end{cases}. \tag{2.3.4}$$

It is now clear that the truncation condition $u_0 = u_{2j+1} = 0$ is achieved. In order to respect the positivity condition $u_n = A_{n-1}^0 C_n^0 > 0$ for $n \in \{1,2,...,2j\}$ and obtain a finite set of orthogonal polynomials, one must choose between the two sets of restrictions on the parameters,

$$\begin{array}{lcl} a \leq -j-1, & & b \geq j, \\ |b| \leq 1, & \text{or} & |a+1| \leq 1, \\ 0 \leq \alpha \leq 1, & & 0 \leq \alpha \leq 1. \end{array} \tag{2.3.5}$$

It is useful to note that the coefficients are mirror symmetric when $\alpha = 1/2$. This mirror symmetry is seen to be tantamount to both conditions $A_n^0 + C_n^0 = A_{2j-n}^0 + C_{2j-n}^0$ and $A_{n-1}^0 C_n^0 = A_{2j-n}^0 C_{2j-n+1}^0$. These conditions are fulfilled when $\alpha = 1/2$ because one can verify that $A_n^0 = C_{2j-n}^0$ and $C_n^0 = A_{2j-n}^0$ which together solve the two conditions. This property will be useful to obtain the orthogonality relation.

2.3.2. Dunkl-Difference equation

It is possible to obtain the Dunkl-difference equation for the para-Bannai-Ito polynomials from the one of the complementary Bannai-Ito polynomials via the truncation (2.3.1). This procedure is straightforward for this equation since no divergence occur. It reads:

$$\mathcal{D}_\beta \mathbf{P}_n^{(0)}(x) = \Lambda_n^{(\beta)} \mathbf{P}_n^{(0)}(x), \tag{2.3.6}$$

with the following eigenvalues

$$\Lambda_{2n}^{(\beta)} = n(n-j), \qquad \Lambda_{2n+1}^{(\beta)} = n(n+1-j) + \beta. \tag{2.3.7}$$

The operator $\mathcal{D}_\beta$ is given by

$$\mathcal{D}_\beta = \mathcal{D}_0 + \beta \frac{(x-\rho_2)}{2x}(I-R), \tag{2.3.8}$$

with

$$\mathcal{D}_0 = A(x)T^+ + B(x)T^- + C(x)R + D(x)T^+R - (A(x)+B(x)+C(x)+D(x))I, \tag{2.3.9}$$

where $T^\pm f(x) = f(x \pm 1)$ and $Rf(x) = f(-x)$ and

$$\begin{aligned} A(x) &= \tfrac{(x+\rho_1+1)(x+\rho_2+1)(2x-2\rho_1-j)(2x-2\rho_2-j)}{8(x+1)(2x+1)}, \\ B(x) &= \tfrac{(x-\rho_2)(x-\rho_1-1)(2x+2\rho_1+j)(2x+2\rho_2+j)}{8x(2x-1)}, \\ C(x) &= \tfrac{(x-\rho_2)\left(4x^2+\omega\right)}{8x} - \tfrac{(x-\rho_2)(x+\rho_1+1)(2x-2\rho_1-j)(2x-2\rho_2-j)}{8x(2x+1)} - B(x), \\ D(x) &= \tfrac{\rho_2(x+\rho_1+1)(2x-2\rho_1-j)(2x-2\rho_2-j)}{8x(x+1)(2x+1)}, \\ \omega &= (2\rho_1+j)(2\rho_2+j) - 4(1+\rho_1)(\rho_1+\rho_2+j). \end{aligned} \tag{2.3.10}$$

One needs to use the change of parameters (2.3.2) in order to obtain the result above. Observe that $\Lambda_n^{(\beta)} = \Lambda_{2j-n}^{(\beta)}$ and that the spectrum is hence degenerate. This means that the para-Bannai-Ito polynomials are bispectral, but not classical.

2.3.3. Explicit expression

The explicit expression for the para-Bannai-Ito is obtained by using the limit described after (2.3.1) in (2.2.1) and (2.2.2). Write the complementary Bannai-Ito polynomials as follows

$$\mathbf{I}_{2n}\left(x\right) = l_n^{(1)} \sum_{k=0}^{n} A_{n,k}^{(I)}, \qquad \mathbf{I}_{2n+1}\left(x\right) = l_n^{(2)}\left(x-\rho_2\right) \sum_{k=0}^{n} B_{n,k}^{(I)}. \tag{2.3.11}$$

We can now define the para-Bannai-Ito as polynomials given by

$$\mathbf{P}_{2n}^{(0)}\left(x\right) = \kappa_n^{(1)} \sum_{k=0}^{n} \lim_{t\to 0} A_{n,k}^{(I)} = \kappa_n^{(1)} \sum_{k=0}^{n} A_{n,k}, \tag{2.3.12}$$

$$\mathbf{P}_{2n+1}^{(0)}\left(x\right) = \kappa_n^{(2)}\left(x-\rho_2\right) \sum_{k=0}^{n} \lim_{t\to 0} B_{n,k}^{(I)} = \kappa_n^{(2)} \left(x - \frac{b-j-1-a}{4}\right) \sum_{k=0}^{n} B_{n,k}, \tag{2.3.13}$$

with the right normalization to make them monic. The summands and the renormalization constants are found to be

$$A_{n,k} = \begin{cases} \frac{(-n)_k (n-j)_k \left(\frac{b-j-1-a}{4}+x\right)_k \left(\frac{b-j-1-a}{4}-x\right)_k}{(1)_k \left(\frac{1+b-j}{2}\right)_k \left(-\frac{j+a}{2}\right)_k \left(-\frac{j}{2}\right)_k} & k \leq \frac{j}{2}, k \leq n \\ \left(\frac{1}{1-\alpha}\right) \frac{(-n)_k \left(\frac{b-j-1-a}{4}+x\right)_k \left(\frac{b-j-1-a}{4}-x\right)_k}{(1)_k \left(\frac{1+b-j}{2}\right)_k \left(-\frac{j+a}{2}\right)_k} \frac{(n-j)_{j-n} (1)_{n+k-j-1}}{\left(-\frac{j}{2}\right)_{\frac{j}{2}} (1)_{k-j/2-1}} & k > \frac{j}{2},, k \leq n \end{cases}, \tag{2.3.14}$$

$$\kappa_n^{(1)} = \begin{cases} \frac{\left(\frac{1+b-j}{2}\right)_n \left(-\frac{j+a}{2}\right)_n \left(-\frac{j}{2}\right)_n}{(n-j)_n} & n \leq \frac{j}{2} \\ (1-\alpha) \frac{\left(\frac{1+b-j}{2}\right)_n \left(-\frac{j+a}{2}\right)_n \left(-\frac{j}{2}\right)_{\frac{j}{2}} (1)_{n-j/2-1}}{(n-j)_{j-n} (1)_{2n-j-1}} & n > \frac{j}{2} \end{cases}, \tag{2.3.15}$$

and

$$B_{n,k} = \begin{cases} \frac{(-n)_k (n+1-j)_k \left(\frac{b-j+3-a}{4}+x\right)_k \left(\frac{b-j+3-a}{4}-x\right)_k}{(1)_k \left(\frac{3+b-j}{2}\right)_k \left(\frac{2-j-a}{2}\right)_k \left(\frac{2-j}{2}\right)_k} & k < \frac{j}{2} \\ \left(\frac{1}{1-\alpha}\right) \frac{(-n)_k \left(\frac{b-j+3-a}{4}+x\right)_k \left(\frac{b-j+3-a}{4}-x\right)_k}{(1)_k \left(\frac{3+b-j}{2}\right)_k \left(\frac{2-j-a}{2}\right)_k} \frac{(n+1-j)_{j-n-1} (1)_{n+k-j}}{\left(\frac{2-j}{2}\right)_{\frac{j-2}{2}} (1)_{k-j/2}} & k \geq \frac{j}{2} \end{cases}, \tag{2.3.16}$$

$$\kappa_n^{(2)} = \begin{cases} \frac{\left(\frac{3+b-j}{2}\right)_n \left(\frac{2-j-a}{2}\right)_n \left(\frac{2-j}{2}\right)_n}{(n+1-j)_n} & n < \frac{j}{2} \\ (1-\alpha) \frac{\left(\frac{3+b-j}{2}\right)_n \left(\frac{2-j-a}{2}\right)_n \left(\frac{2-j}{2}\right)_{\frac{j-2}{2}} (1)_{n-j/2}}{(n+1-j)_{j-n-1} (1)_{2n-j}} & n \geq \frac{j}{2} \end{cases}. \tag{2.3.17}$$

In terms of terminating hypergeometric series, for even degrees we have:

$$\frac{1}{\kappa_n^{(1)}} \mathbf{P}_{2n}^{(0)}\left(x\right) = {}_4F_3\left(\begin{matrix} -n, \ n-j, \ \frac{b-j-1-a}{4}+x, \ \frac{b-j-1-a}{4}-x \\ \frac{1+b-j}{2}, \ -\frac{j+a}{2}, \ -\frac{j}{2} \end{matrix}; 1\right), \tag{2.3.18}$$

if $n \leq \frac{j}{2}$ and,

$$\frac{1}{\kappa_n^{(1)}}\mathbf{P}_{2n}^{(0)}(x) = {}_4F_3\left(\begin{matrix} -n,\ n-j,\ \frac{b-j-1-a}{4}+x,\ \frac{b-j-1-a}{4}-x \\ \frac{1+b-j}{2},\ -\frac{j+a}{2},\ -\frac{j}{2} \end{matrix};1\right) \tag{2.3.19}$$

$$+\frac{(n-j)_{j-n}(-n)_{j/2+1}\left(\frac{b-j-1-a}{4}+x\right)_{j/2+1}\left(\frac{b-j-1-a}{4}-x\right)_{j/2+1}(1)_{n-j/2}}{(1-\alpha)\left(-\frac{j}{2}\right)_{j/2}(1)_{j/2+1}\left(\frac{1+b-j}{2}\right)_{j/2+1}\left(-\frac{j+a}{2}\right)_{j/2+1}} \tag{2.3.20}$$

$$\times\, {}_4F_3\left(\begin{matrix} \frac{j+2}{2}-n,\ n-\frac{j-2}{2},\ \frac{b+j+3-a}{4}+x,\ \frac{b+j+3-a}{4}-x \\ \frac{3+b}{2},\ \frac{2-a}{2},\ \frac{4+j}{2} \end{matrix};1\right) \tag{2.3.21}$$

if $n > \frac{j}{2}$.

For odd degrees, we have

$$\frac{1}{\kappa_n^{(2)}\left(x-\frac{b-j-1-a}{4}\right)}\mathbf{P}_{2n+1}^{(0)}(x) = {}_4F_3\left(\begin{matrix} -n,\ n+1-j,\ \frac{b-j+3-a}{4}+x,\ \frac{b-j+3-a}{4}-x \\ \frac{3+b-j}{2},\ \frac{2-j-a}{2},\ \frac{2-j}{2} \end{matrix};1\right), \tag{2.3.22}$$

if $n < \frac{j}{2}$ and,

$$\frac{1}{\kappa_n^{(2)}\left(x-\frac{b-j-1-a}{4}\right)}\mathbf{P}_{2n+1}^{(0)}(x) = {}_4F_3\left(\begin{matrix} -n,\ n+1-j,\ \frac{b-j+3-a}{4}+x,\ \frac{b-j+3-a}{4}-x \\ \frac{3+b-j}{2},\ \frac{2-j-a}{2},\ \frac{2-j}{2} \end{matrix};1\right) \tag{2.3.23}$$

$$+\frac{(n+1-j)_{j-n-1}(-n)_{j/2}\left(\frac{b-j+3-a}{4}+x\right)_{j/2}\left(\frac{b-j+3-a}{4}-x\right)_{j/2}(1)_{n-j/2}}{(1-\alpha)\left(\frac{2-j}{2}\right)_{j/2-1}(1)_{j/2}\left(\frac{3+b-j}{2}\right)_{j/2}\left(\frac{2-j-a}{2}\right)_{j/2}} \tag{2.3.24}$$

$$\times {}_4F_3\left(\begin{matrix} \frac{j}{2}-n,\ n-\frac{j-2}{2},\ \frac{b+j+3-a}{4}+x,\ \frac{b+j+3-a}{4}-x \\ \frac{3+b}{2},\ \frac{2-a}{2},\ \frac{4+j}{2} \end{matrix};1\right) \tag{2.3.25}$$

if $n \geq \frac{j}{2}$.

2.3.4. Orthogonality relation

To obtain the orthogonality relation, we first need to determine the grid or orthogonality lattice. It is provided by the eigenvalues of the Jacobi matrix or, equivalently, by the zeroes of the characteristic polynomial $\mathbf{P}_{2j+1}^{(0)}(x)$. We can define $\mathbf{P}_{2j+1}^{(0)}(x)$ using the recurrence relation (2.3.3) and inserting the expression of $\mathbf{P}_{2j}^{(0)}(x)$ and $\mathbf{P}_{2j-1}^{(0)}(x)$ given in §2.3.3. An extraction of the zeroes can be done with the help of the Saalschütz summation formula to obtain

$$\mathbf{P}_{2j+1}^{(0)}(x) = \prod_{k=0}^{j}\left(x+(-1)^k\left(\frac{2k-j-b+a}{4}\right)+\frac{1}{4}\right)\prod_{k=0}^{j-1}\left(x+(-1)^k\left(\frac{2k-j-b-a}{4}\right)+\frac{1}{4}\right). \tag{2.3.26}$$

The set of $2j+1$ para-Bannai-Ito polynomials will thus be orthogonal on a Bannai-Ito bi-lattice with $2j+1$ grid points.

$$x_{2s} = -(-1)^s\left(\frac{2s-j-b+a}{4}\right)-\frac{1}{4} \qquad s\in\{0,1,...,j\}, \tag{2.3.27}$$

$$x_{2s+1} = -(-1)^s\left(\frac{2s-j-b-a}{4}\right)-\frac{1}{4} \qquad s\in\{0,1,...,j-1\}. \tag{2.3.28}$$

A depiction of the grid is given in the appendix §2.C. This Bannai-Ito bi-lattice can also be seen as a linear quadri-lattice.

$$x_{4s} = -\left(\frac{4s-j-b+a}{4}\right)-\frac{1}{4} \qquad s\in\{0,1,...,j/2\}, \tag{2.3.29}$$

$$x_{4s+1} = -\left(\frac{4s-j-b-a}{4}\right)-\frac{1}{4} \qquad s\in\{0,1,...,j/2-1\}, \tag{2.3.30}$$

$$x_{4s+2} = \left(\frac{4s+2-j-b+a}{4}\right)-\frac{1}{4} \qquad s\in\{0,1,...,j/2-1\}, \tag{2.3.31}$$

$$x_{4s+3} = \left(\frac{4s+2-j-b-a}{4}\right)-\frac{1}{4} \qquad s\in\{0,1,...,j/2-1\}. \tag{2.3.32}$$

From the theory of orthogonal polynomials [**3**], the weights are given by

$$w_s = \frac{u_1\dots u_{2j}}{\mathbf{P}_{2j}^{(0)}(x_s)\,\mathbf{P}_{2j+1}^{(0)\prime}(x_s)}, \qquad s=0,1,\dots,2j, \tag{2.3.33}$$

in terms of which the orthogonality relation of the para-Bannai-Ito polynomials will read:

$$\sum_{s=0}^{2j} w_s\mathbf{P}_n^{(0)}(x_s)\,\mathbf{P}_m^{(0)}(x_s) = u_1\dots u_n\delta_{nm}. \tag{2.3.34}$$

In [**5**], a method to derive the weight for persymmetric orthogonal polynomials was presented together with a procedure to also achieve that after an isospectral deformation. This is exactly the situation that the para-Bannai-Ito polynomials present. Now, the usual expression

$$\mathbf{P}_N^{(0)}(x_s) = (-1)^{N+s}\sqrt{u_1\dots u_N} \tag{2.3.35}$$

for $\mathbf{P}_N^{(0)}(x_s)$ is predicated on the eigenvalues of the Jacobi matrix ordered in an increasing fashion which is not the case in the presentation of the bi-Bannai-Ito grid given above. Nonetheless, it is possible to alter the increasing order condition and to require instead that between any two zeroes of even index, there must be a zero of odd index and vice versa. Using this modified condition, the interlacing zeroes property and the persymmetry, when $N=2j$, we still find for $\mathbf{P}_N^{(0)}(x_s)$ the expression (2.3.35). In the case of the para-Bannai-Ito polynomials, positivity ensures that the modified condition for the eigenvalues is achieved.

Recalling that the polynomials are persymmetric, we then have that for $\alpha = 1/2$,

$$\tilde{w}_s = \frac{(-1)^s \sqrt{u_1 \dots u_N}}{\mathbf{P}^{(0)\prime}_{2j+1}(x_s)}, \tag{2.3.36}$$

with the u_n also evaluated at $\alpha = 1/2$. A direct computation gives

$$\tilde{w}_{4s} = \frac{h_{2j}}{(-s)_s (1)_{j/2-s} \left(-s - \frac{a}{2}\right)_{j/2} \left(s + \frac{1+a-b-j}{2}\right)_{j/2} \left(s + \frac{1-b-j}{2}\right)_{j/2}}, \tag{2.3.37}$$

$$\tilde{w}_{4s+1} = -\frac{h_{2j}}{(-s)_s (1)_{j/2-s-1} \left(-s + \frac{a}{2}\right)_{j/2+1} \left(s + \frac{1-a-b-j}{2}\right)_{j/2} \left(s + \frac{1-b-j}{2}\right)_{j/2}}, \tag{2.3.38}$$

$$\tilde{w}_{4s+2} = -\frac{h_{2j}}{(-s)_s (1)_{j/2-s-1} \left(-s - \frac{a}{2}\right)_{j/2} \left(s + \frac{1+a-b-j}{2}\right)_{j/2+1} \left(s + \frac{1-b-j}{2}\right)_{j/2}}, \tag{2.3.39}$$

$$\tilde{w}_{4s+3} = \frac{h_{2j}}{(-s)_s (1)_{j/2-s-1} \left(-s + \frac{a}{2}\right)_{j/2} \left(s + \frac{1-a-b-j}{2}\right)_{j/2} \left(s + \frac{1-b-j}{2}\right)_{j/2+1}}, \tag{2.3.40}$$

with

$$h_{2j} = \sqrt{u_1 \dots u_{2j}} = \frac{(1)_j \left(\frac{b+1-j}{2}\right)_j \left(-\frac{j+a}{2}\right)_j}{2^{2j} \left(\frac{1-j}{2}\right)_j}. \tag{2.3.41}$$

It was also shown in [**5**] that after an isospectral deformation, the new weights are related to the persymmetric weights by a multiplicative factor dependent on α in our case,

$$w_s = C\left(1 + \beta(-1)^s\right)\tilde{w}_s. \tag{2.3.42}$$

Using (2.3.33) and (2.3.36) for the first few j, we can solve for C and β. One can verify that $C = 1$ and $\beta = 2\alpha - 1$ which gives

$$w_{2s} = 2\alpha\tilde{w}_{2s} \qquad w_{2s+1} = 2(1-\alpha)\tilde{w}_{2s+1}. \tag{2.3.43}$$

We also have

$$\sum_{s=0}^{j} w_{2s} = \alpha, \quad \sum_{s=0}^{j} w_{2s+1} = 1 - \alpha \tag{2.3.44}$$

which generalize a known result for mirror-symmetric Jacobi matrices. Finally, we give a general expression for $h_n = \sqrt{u_1 \dots u_n}$ which can be squared and inserted in (2.3.34) to complete the characterization of the para-Bannai-Ito for $N = 2j$ with j even.

$$h_{2n} = \frac{\sqrt{(-j)_n (1)_n \left(\frac{1+b-j}{2}\right)_n \left(\frac{1-b-j}{2}\right)_n \left(-\frac{j+a}{2}\right)_n \left(\frac{2+a-j}{2}\right)_n}}{2^{2n}\left(\frac{1-j}{2}\right)_n} \times \begin{cases} 1 & n < \frac{j}{2} \\ \sqrt{2\alpha} & n = \frac{j}{2} \\ 2\sqrt{\alpha(1-\alpha)} & n > \frac{j}{2} \end{cases}, \tag{2.3.45}$$

$$h_{2n+1} = \frac{\sqrt{(-1)(-j)_{n+1} (1)_n \left(\frac{1+b-j}{2}\right)_{n+1} \left(\frac{1-b-j}{2}\right)_n \left(-\frac{j+a}{2}\right)_{n+1} \left(\frac{2+a-j}{2}\right)_n}}{2^{2n+1}\sqrt{\left(\frac{1-j}{2}\right)_n \left(\frac{1-j}{2}\right)_{n+1}}} \times \begin{cases} 1 & n < \frac{j}{2} \\ 2\sqrt{\alpha(1-\alpha)} & n \geq \frac{j}{2} \end{cases}.$$

2.4. Para-Bannai-Ito for $N = 2j+1$, j even

The general complementary Bannai-Ito and the general Bannai-Ito polynomials are related by a Geronimus transformation with parameter ρ_1. The Geronimus transformation and the truncation commute, and therefore, performing the transformations on the para-Bannai-Ito polynomials with $N = 2j+1$, j even, we obtain a set of $N+1 = 2j+2$ polynomials that correspond to the truncation (2.3.1) of the general Bannai-Ito polynomials. The eigenvalue $x_s = \rho_1$ is then added to the spectrum. The treatment of the case $N = 2j+1$, j even, is analogous to the case $N = 2j$, j even. The results are presented below.

2.4.1. Recurrence relation

Using the truncation (2.3.1) in (2.2.8) gives the recurrence relation for $\mathbf{P}_n^{(1)}(x; a,b,\alpha,2j+1)$

$$x\mathbf{P}_n^{(1)}(x) = \mathbf{P}_{n+1}^{(1)}(x) + \left(\frac{b-j-1+a}{4} - A_n^1 - C_n^1\right)\mathbf{P}_n^{(1)}(x) + A_{n-1}^1 C_n^1 \mathbf{P}_{n-1}^{(1)}(x), \tag{2.4.1}$$

$$A_n^1 = \begin{cases} \frac{1}{4}(n-j+a) & n \text{ even}, \ n \neq j \\ \frac{1}{4}\frac{(n-2j-1)(n-j+b)}{(n-j)} & n \text{ odd} \\ \frac{1}{2}\alpha a & n \text{ even}, \ n = j \end{cases}, \quad C_n^1 = \begin{cases} -\frac{1}{4}\frac{n(n-j-1-b)}{(n-j-1)} & n \text{ even} \\ -\frac{1}{4}(n-j-1-a) & n \text{ odd}, \ n \neq j+1 \\ \frac{1}{2}(1-\alpha)a & n \text{ odd}, \ n = j+1 \end{cases}. \tag{2.4.2}$$

The positivity condition is achieved if one chooses the parameters such that

$$|a| \geq j+1, \qquad |b| \leq 1, \qquad 0 \leq \alpha \leq 1. \tag{2.4.3}$$

Again, the polynomials are persymmetric if $\alpha = 1/2$.

2.4.2. Dunkl-Difference equation

Since no divergences appear in the Dunkl-difference equation of the general Bannai-Ito polynomials under the para truncation, it follows that the para-Bannai-Ito polynomials are solutions to the same equation except for a change of variable. It reads

$$L\mathbf{P}_n^{(1)}(x) = \lambda_n \mathbf{P}_n^{(1)}(x), \tag{2.4.4}$$

with the eigenvalues

$$\lambda_{2n} = n, \qquad \lambda_{2n+1} = (j-n)\,. \tag{2.4.5}$$

The Dunkl-Difference operator is

$$L = F(x)(I-R) + G(x)\left(T^{+}R - I\right), \tag{2.4.6}$$

where

$$G(x) = \frac{(4x+1+a-b-j)(4x+1-a-b-j)}{16(2x+1)}, \tag{2.4.7}$$

$$F(x) = \frac{(4x+1+j+a-b)(4x+1+j-a-b)}{32x}. \tag{2.4.8}$$

2.4.3. Explicit expression

Since the para-Bannai-Ito for $N = 2j$ and $N = 2j+1$ are related by a Geronimus transformation, we obtain directly the expression for the case $N = 2j+1$ with j even as a combination of two para-Bannai-Ito polynomials with $N = 2j$. The explicit expression is

$$\mathbf{P}_n^{(1)}(x) = \mathbf{P}_n^{(0)}(x) - C_n^1 \mathbf{P}_{n-1}^{(0)}(x)\,, \tag{2.4.9}$$

where C_n^1 is as in 2.4.1.

2.4.4. Orthogonality relation

We know that the Geronimus transformation will only add $x_s = \rho_1$ to the spectrum provided by the zeros of $\mathbf{P}_{2j+1}^{(0)}$. It turns out that this fits with the grid of the section §2.3.4 and that its points are then given by

$$x_{2s} = -(-1)^s\left(\frac{2s-j-b+a}{4}\right) - \frac{1}{4} \qquad s \in \{0,1,...,j\}, \tag{2.4.10}$$

$$x_{2s+1} = -(-1)^s\left(\frac{2s-j-b-a}{4}\right) - \frac{1}{4} \qquad s \in \{0,1,...,j\}, \tag{2.4.11}$$

or again as a linear quadri-lattice

$$x_{4s} = -\left(\frac{4s-j-b+a}{4}\right) - \frac{1}{4} \qquad s \in \{0,1,...,j/2\}, \tag{2.4.12}$$

$$x_{4s+1} = -\left(\frac{4s-j-b-a}{4}\right) - \frac{1}{4} \qquad s \in \{0,1,...,j/2\}, \tag{2.4.13}$$

$$x_{4s+2} = \left(\frac{4s+2-j-b+a}{4}\right) - \frac{1}{4} \qquad s \in \{0,1,...,j/2-1\}, \tag{2.4.14}$$

$$x_{4s+3} = \left(\frac{4s+2-j-b-a}{4}\right) - \frac{1}{4} \qquad s \in \{0,1,...,j/2-1\}. \tag{2.4.15}$$

The orthogonality relation is again of the form

$$\sum_{s=0}^{2j+1} w_s \mathbf{P}_n^{(1)}(x_s)\,\mathbf{P}_m^{(1)}(x_s) = u_1 \ldots u_n \delta_{nm}, \tag{2.4.16}$$

where

$$w_s = \frac{u_1 \ldots u_{2j+1}}{\mathbf{P}_{2j+1}^{(1)}(x_s)\,\mathbf{P}_{2j+2}^{(1)\prime}(x_s)}, \quad s = 0,1,\ldots,2j+1. \tag{2.4.17}$$

The computation in the persymmetric case followed by the generalization with an isospectral deformation gives the weights

$$w_{4s} = \frac{2\alpha h_{2j+1}}{(-s)_s\,(1)_{j/2-s}\left(-s-\frac{a}{2}\right)_{j/2+1}\left(s+\frac{1+a-b-j}{2}\right)_{j/2}\left(s+\frac{1-b-j}{2}\right)_{j/2}}, \tag{2.4.18}$$

$$w_{4s+1} = -\frac{2(1-\alpha)\,h_{2j+1}}{(-s)_s\,(1)_{j/2-s}\left(-s+\frac{a}{2}\right)_{j/2+1}\left(s+\frac{1-a-b-j}{2}\right)_{j/2}\left(s+\frac{1-b-j}{2}\right)_{j/2}}, \tag{2.4.19}$$

$$w_{4s+2} = -\frac{2\alpha h_{2j+1}}{(-s)_s\,(1)_{j/2-s-1}\left(-s-\frac{a}{2}\right)_{j/2}\left(s+\frac{1+a-b-j}{2}\right)_{j/2+1}\left(s+\frac{1-b-j}{2}\right)_{j/2+1}}, \tag{2.4.20}$$

$$w_{4s+3} = \frac{2(1-\alpha)\,h_{2j+1}}{(-s)_s\,(1)_{j/2-s-1}\left(-s+\frac{a}{2}\right)_{j/2}\left(s+\frac{1-a-b-j}{2}\right)_{j/2+1}\left(s+\frac{1-b-j}{2}\right)_{j/2+1}}, \tag{2.4.21}$$

with

$$h_{2j+1} = \frac{(1)_j\left(\frac{1-b-j}{2}\right)_j\left(-\frac{j+a}{2}\right)_{j+1}}{2^{2j+1}\left(\frac{1-j}{2}\right)_j}. \tag{2.4.22}$$

To complete the characterization for $N = 2j+1$ with j even, we have that the general expressions for the h_n are

$$h_{2n} = \frac{\sqrt{(-j)_n (1)_n \left(\frac{1-b-j}{2}\right)_n \left(\frac{1+b-j}{2}\right)_n \left(-\frac{j+a}{2}\right)_n \left(-\frac{j-a}{2}\right)_n}}{2^{2n}\left(\frac{1-j}{2}\right)_n} \times \begin{cases} 1 & n \leq \frac{j}{2} \\ 2\sqrt{\alpha(1-\alpha)} & n > \frac{j}{2} \end{cases}, \tag{2.4.23}$$

$$h_{2n+1} = \frac{\sqrt{(-1)(-j)_n (1)_n \left(\frac{1-b-j}{2}\right)_n \left(\frac{1+b-j}{2}\right)_n \left(-\frac{j+a}{2}\right)_{n+1} \left(-\frac{j-a}{2}\right)_{n+1}}}{2^{2n+1}\left(\frac{1-j}{2}\right)_n} \times \begin{cases} 1 & n < \frac{j}{2} \\ 2\sqrt{\alpha(1-\alpha)} & n \geq \frac{j}{2} \end{cases}.$$

2.5. Para-Bannai-Ito polynomials as a $q \to -1$ limit of the q-para-Racah polynomials

In [**11**], the q-para-Racah polynomials were introduced as a q generalization of the para-Racah polynomials. In this section, we show that the para-Bannai-Ito polynomials can be obtained as a $q \to -1$ limit of the q-para-Racah polynomials, reinforcing their position in the family of para polynomials. The monic q-para-Racah polynomials for $N = 2j+1$ follow the recurrence relation

$$x\mathbf{R}_n(x) = \mathbf{R}_{n+1}(x) + \left(\frac{c+c^{-1}}{2} - A_n^R - C_n^R\right)\mathbf{R}_n(x) + A_{n-1}^R C_n^R \mathbf{R}_{n-1}(x), \tag{2.5.1}$$

with

$$A_n^R = \begin{cases} \frac{(1-cdq^n)\left(d-cq^{n-j}\right)\left(1-q^{n-2j-1}\right)}{2cd(1-q^{2n-2j-1})(1+q^{n-j})} & n \neq j \\ \frac{\alpha\left(1-cdq^j\right)(d-c)\left(1-q^{-j-1}\right)}{2cd(1-q^{-1})} & n = j \end{cases}, \qquad C_n^R = \begin{cases} \frac{(1-q^n)\left(c-dq^{n-j-1}\right)\left(cd-q^{n-2j-1}\right)}{2cd(1+q^{n-j-1})(1-q^{2n-2j-1})} & n \neq j+1 \\ \frac{(1-\alpha)\left(1-q^{j+1}\right)(c-d)\left(cd-q^{-j}\right)}{2cd(1-q)} & n = j+1 \end{cases}. \tag{2.5.2}$$

In order to recover the para-Bannai-Ito polynomials, one needs to use the parametrization

$$q = -e^{\varepsilon}, \qquad c = i\exp\left(\varepsilon\left[\frac{a+b-j}{2}\right]\right), \qquad d = i\exp\left(\varepsilon\left[\frac{-a+b-j}{2}\right]\right). \tag{2.5.3}$$

After redefining the resulting family of polynomials through the following affine transformation of the variable

$$x \to 2i\varepsilon\left(x + \frac{1}{4}\right), \tag{2.5.4}$$

we have

$$x\tilde{\mathbf{R}}_n(x) = \tilde{\mathbf{R}}_{n+1}(x) + \left(\frac{c+c^{-1}}{4i\varepsilon} - \frac{1}{4} - \frac{A_n^R}{2i\varepsilon} - \frac{C_n^R}{2i\varepsilon}\right)\tilde{\mathbf{R}}_n(x) + \frac{A_{n-1}^R}{2i\varepsilon}\frac{C_n^R}{2i\varepsilon}\tilde{\mathbf{R}}_{n-1}(x). \tag{2.5.5}$$

Using the parametrization (2.5.3) and taking the limit $\varepsilon \to 0$, equivalent to $q \to -1$, one sees that (2.5.5) goes into the recurrence relation of the para-Bannai-Ito polynomials with $N = 2j = 1$ and j even. While this was not discussed in [**11**], the $N = 2j+1$ and $N = 2j$ cases of the q-para-Racah polynomials are related by a Darboux transform, which is the analog of the relation presented before. So the limit process also works for the q-para-Racah with $N = 2j$, and gives the para-Bannai-Ito with $N = 2j$ and j even and the recurrence relation (2.3.4).

2.6. Special cases

2.6.1. Reduction to a simple lattice and connection with the dual -1 Hahn polynomials

Consider the case $\alpha = 1/2$. For $b = 0$, it is observed that the spectrum of the para-Bannai-Ito (i.e. (2.4.10)-(2.4.11)) reduces to a single Bannai-Ito lattice (two linear lattices separated by a gap). This reduction can be observed in the appendix. This restricted grid can be expressed by the sequence

$$2x_k = \frac{(-1)^k}{4}\left(\frac{k}{2} + \frac{1}{4} - \frac{a+j+1}{2}\right) - \frac{1}{4} \qquad k \in \{0,1,...,N\}. \tag{2.6.1}$$

In this setting, we can connect this special case directly to the general complementary Bannai-Ito and to the general Bannai-Ito polynomials.

$$\mathbf{P}_n^{(0)}\left(x; a,0,\frac{1}{2},2j\right) = \mathbf{I}_n\left(x; -\frac{j+1-a}{4}, -\frac{j+1+a}{4}, \frac{j+1-a}{4}, \frac{j+1+a}{4}\right), \tag{2.6.2}$$

$$\mathbf{P}_n^{(1)}\left(x; a,0,\frac{1}{2},2j+1\right) = \mathbf{B}_n\left(x; -\frac{j+1-a}{4}, -\frac{j+1+a}{4}, \frac{j+1-a}{4}, \frac{j+1+a}{4}\right), \tag{2.6.3}$$

where $\mathbf{I}_n$ and $\mathbf{B}_n$ are as in section §2.2

This special case is also connected with the dual -1 Hahn polynomials $\hat{\mathbf{R}}_n\left(x;\alpha,\beta,N\right)$ [**13**] via the relations

$$\mathbf{P}_n^{(0)}\left(x; a,0,\frac{1}{2},2j\right) = \frac{1}{2^{3n}}\hat{\mathbf{R}}_n\left(-\left[2^3x+1\right]; j-a, j-a, 2j\right), \tag{2.6.4}$$

$$\mathbf{P}_n^{(1)}\left(x; a,0,\frac{1}{2},2j+1\right) = \frac{1}{2^{3n}}\hat{\mathbf{R}}_n\left(\left[2^3x+1\right]; a-j-1, a-j-1, 2j+1\right). \tag{2.6.5}$$

2.6.2. Reduction to a simple linear lattice

The simple Bannai-Ito grid can be further specialized by taking $a = -j-1$. This takes it into a simple linear lattice

$$x_k = j - k \qquad k \in \{0,1,...,N\}, \tag{2.6.6}$$

after a relabelling of the grid points. The polynomials are then related to shifted monic Krawtchouk polynomials $\mathbf{K}_n\left(x;p,N\right)$ [**8**].

$$\mathbf{P}_n^{(0)}\left(x;-j-1,0,\frac{1}{2},2j\right)=\frac{1}{2^n}\mathbf{K}_n\left(2x+j;\frac{1}{2},2j\right), \tag{2.6.7}$$

$$\mathbf{P}_n^{(1)}\left(x;-j-1,0,\frac{1}{2},2j+1\right)=\frac{1}{2^n}\mathbf{K}_n\left(2x+j+1;\frac{1}{2},2j+1\right). \tag{2.6.8}$$

2.7. Conclusion

To summarize, the para-Bannai-Ito polynomials have been introduced and characterized. They are obtained by a $q\to-1$ limit of the q-para-Racah polynomials, but also as a special truncation of the general Bannai-Ito and complementary Bannai-Ito polynomials. Their explicit expressions in terms of hypergeometric series was derived as well as their recurrence relation and Dunkl-difference equation. It was shown that the para-Bannai-Ito polynomials are orthogonal on a finite bi-Bannai-Ito grid (linear quadri-lattice). The para-Bannai-Ito have a connection to the dual -1 Hahn polynomials (orthogonal on single Bannai-Ito grid or linear bi-lattice) and to the shifted Krawtchouk polynomials (orthogonal on a linear lattice).

The Jacobi matrix of the para-Bannai-Ito is an isospectral deformation of the persymmetric one when $\alpha=1/2$. Such matrices have been used in the design of spin chains with fractional revival. It would be of interest to examine if models based on the para-Bannai-Ito polynomials exhibit similar properties.

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2.A. Para-Bannai-Ito for $N=2j$, j odd

Truncation from the general complementary Bannai-Ito polynomials

$$\rho_1+\rho_2=-\frac{j+1}{2}+e_1t, \qquad -r_1-r_2=-\frac{j+1}{2}+e_2t. \tag{2.A.1}$$

The para-Bannai-Ito polynomials are obtained through a $t\to0$ limit.

$$\rho_1=\frac{b-j-1+a}{4}, \qquad -r_1=\frac{b-j-1-a}{4}, \qquad \frac{e_1}{e_1+e_2}=\alpha, \qquad \frac{e_2}{e_1+e_2}=(1-\alpha). \tag{2.A.2}$$

Recurrence relation

$$x\mathbf{P}_n^{(2)}(x)=\mathbf{P}_{n+1}^{(2)}(x)+\left(\frac{b-j-1+a}{4}-A_n^2-C_n^2\right)\mathbf{P}_n^{(2)}(x)+A_{n-1}^2C_n^2\mathbf{P}_{n-1}^{(2)}(x)\,, \quad (2.A.3)$$

$$A_n^2=\begin{cases}-\frac{1}{4}\frac{(n-j-a)(n-j-b)}{(n-j)} & n \text{ even}\\ -\frac{1}{4}(n+1) & \begin{matrix}n\text{ odd}\\ n\neq j\end{matrix}\\ -\frac{1}{2}(1-\alpha)(j+1) & \begin{matrix}n\text{ odd}\\ n=j\end{matrix}\end{cases}, \quad C_n^2=\begin{cases}\frac{1}{4}\frac{(n-j+a)(n-j+b)}{(n-j)} & n \text{ even}\\ \frac{1}{4}(n-2j-1) & \begin{matrix}n\text{ odd}\\ n\neq j\end{matrix}\\ -\frac{1}{2}\alpha(j+1) & \begin{matrix}n\text{ odd}\\ n=j\end{matrix}\end{cases}. \quad (2.A.4)$$

Positivity condition

$$\begin{matrix} a\leq -j, & & |a|\leq 1,\\ |b|\leq 1, & \text{or} & b\leq -j,\\ 0\leq\alpha\leq 1, & & 0\leq\alpha\leq 1.\end{matrix} \quad (2.A.5)$$

Dunkl-Difference equation

$$\mathcal{D}_\beta\mathbf{P}_n^{(2)}(x)=\Lambda_n^{(\beta)}\mathbf{P}_n^{(2)}(x)\,, \quad (2.A.6)$$

$$\Lambda_{2n}^{(\beta)}=n(n-j)\,,\qquad \Lambda_{2n+1}^{(\beta)}=n(n+1-j)+\beta, \quad (2.A.7)$$

$$\mathcal{D}_\beta=\mathcal{D}_0+\beta\frac{(4x+b+a+j+1)}{8x}(I-R)\,, \quad (2.A.8)$$

$$\mathcal{D}_0=A(x)T^++B(x)T^-+C(x)R+D(x)T^+R-(A(x)+B(x)+C(x)+D(x))I, \quad (2.A.9)$$

where $T^{\pm}f(x)=f(x\pm1)$ and $Rf(x)=f(-x)$ and where

$$\begin{aligned}
A(x)&=\tfrac{(x+\rho_1+1)\left(x-\rho_1+\frac{1-j}{2}\right)(2x-2r_1+1)(2x+2r_1-j)}{8(x+1)(2x+1)},\\
B(x)&=\tfrac{\left(x+\rho_1+\frac{j+1}{2}\right)(x-\rho_1-1)(2x+2r_1-1)(2x-2r_1+j)}{8x(2x-1)},\\
C(x)&=\tfrac{\left(x+\rho_1+\frac{j+1}{2}\right)\left(4x^2+\omega\right)}{8x}-\tfrac{\left(x+\rho_1+\frac{j+1}{2}\right)(x+\rho_1+1)(2x-2r_1+1)(2x+2r_1-j)}{8x(2x+1)}-B(x),\\
D(x)&=-\tfrac{(2\rho_1+j+1)(x+\rho_1+1)(2x-2r_1+1)(2x+2r_1-j)}{16x(x+1)(2x+1)},\\
\omega&=4(1+\rho_1-r_1)\left(\tfrac{1-j}{2}\right)+4r_1(1-r_1)-j,
\end{aligned} \quad (2.A.10)$$

where the change of parameter (2.A.2) must be used.

Explicit expression

$$\mathbf{P}_{2n}^{(2)}(x) = \kappa_n^{(1)} \sum_{k=0}^{n} A_{n,k}, \tag{2.A.11}$$

$$\mathbf{P}_{2n+1}^{(2)}(x) = \kappa_n^{(2)} \left(x + \frac{b+j+1+a}{4}\right) \sum_{k=0}^{n} B_{n,k}, \tag{2.A.12}$$

where

$$A_{n,k} = \begin{cases} \frac{(-n)_k (n-j)_k \left(-\frac{b+j+1+a}{4}+x\right)_k \left(-\frac{b+j+1+a}{4}-x\right)_k}{(1)_k \left(-\frac{b+j}{2}\right)_k \left(-\frac{j+a}{2}\right)_k \left(\frac{1-j}{2}\right)_k} & k \leq \frac{(j-1)}{2}, k \leq n, \\ \left(\frac{1}{\alpha}\right) \frac{(-n)_k \left(-\frac{b+j+1+a}{4}+x\right)_k \left(-\frac{b+j+1+a}{4}-x\right)_k}{(1)_k \left(-\frac{b+j}{2}\right)_k \left(-\frac{j+a}{2}\right)_k} \frac{(n-j)_{j-n} (1)_{n+k-j-1}}{\left(\frac{1-j}{2}\right)_{\frac{j-1}{2}} (1)_{k-(j+1)/2}} & k > \frac{(j-1)}{2}, k \leq n, \end{cases} \tag{2.A.13}$$

$$\kappa_n^{(1)} = \begin{cases} \frac{\left(-\frac{b+j}{2}\right)_n \left(-\frac{j+a}{2}\right)_n \left(\frac{1-j}{2}\right)_n}{(n-j)_n} & n \leq \frac{(j-1)}{2}, \\ (\alpha) \frac{\left(-\frac{b+j}{2}\right)_n \left(-\frac{j+a}{2}\right)_n \left(\frac{1-j}{2}\right)_{\frac{j-1}{2}} (1)_{n-(j+1)/2}}{(n-j)_{j-n} (1)_{2n-j-1}} & n > \frac{(j-1)}{2}, \end{cases} \tag{2.A.14}$$

And

$$B_{n,k} = \begin{cases} \frac{(-n)_k (n+1-j)_k \left(-\frac{b+j-3+a}{4}+x\right)_k \left(-\frac{b+j-3+a}{4}-x\right)_k}{(1)_k \left(\frac{3-j}{2}\right)_k \left(\frac{2-j-a}{2}\right)_k \left(\frac{2-j-b}{2}\right)_k} & k < \frac{j-1}{2}, \\ \left(\frac{1}{\alpha}\right) \frac{(-n)_k \left(-\frac{b+j-3+a}{4}+x\right)_k \left(-\frac{b+j-3+a}{4}-x\right)_k}{(1)_k \left(\frac{2-j-a}{2}\right)_k \left(\frac{2-j-b}{2}\right)_k} \frac{(n+1-j)_{j-n-1} (1)_{n+k-j}}{\left(\frac{3-j}{2}\right)_{\frac{j-3}{2}} (1)_{k-(j-1)/2}} & k \geq \frac{j-1}{2}, \end{cases} \tag{2.A.15}$$

$$\kappa_n^{(2)} = \begin{cases} \frac{\left(\frac{2-j-a}{2}\right)_n \left(\frac{2-j-b}{2}\right)_n \left(\frac{3-j}{2}\right)_n}{(n+1-j)_n} & n < \frac{j-1}{2}, \\ (\alpha) \frac{\left(\frac{2-j-a}{2}\right)_n \left(\frac{2-j-b}{2}\right)_n \left(\frac{3-j}{2}\right)_{\frac{j-3}{2}} (1)_{n-(j-1)/2}}{(n+1-j)_{j-n-1} (1)_{2n-j}} & n \geq \frac{j-1}{2}. \end{cases} \tag{2.A.16}$$

Orthogonality relation

$$x_{4s} = -\left(\frac{4s-j+a-b}{4}\right) - \frac{1}{4} \qquad s \in \{0,1,...,(j-1)/2\}, \tag{2.A.17}$$

$$x_{4s+1} = \left(\frac{4s-j-a-b}{4}\right) - \frac{1}{4} \qquad s \in \{0,1,...,(j-1)/2\}, \tag{2.A.18}$$

$$x_{4s+2} = \left(\frac{4s+2-j+a-b}{4}\right) - \frac{1}{4} \qquad s \in \{0,1,...,(j-1)/2\}, \tag{2.A.19}$$

$$x_{4s+3} = -\left(\frac{4s+2-j-a-b}{4}\right) - \frac{1}{4} \qquad s \in \{0,1,...,(j-3)/2\}, \tag{2.A.20}$$

$$w_{4s} = \frac{2\alpha h_{2j}}{\left(-s\right)_s \left(1\right)_{(j-1)/2-s} \left(-s + \frac{1-a}{2}\right)_{(j-1)/2} \left(s + \frac{1+a-b-j}{2}\right)_{(j+1)/2} \left(s - \frac{b+j}{2}\right)_{(j+1)/2}}, \quad (2.A.21)$$

$$w_{4s+1} = \frac{2\left(1-\alpha\right) h_{2j}}{\left(-s\right)_s \left(1\right)_{(j-1)/2-s} \left(-s + \frac{1+a}{2}\right)_{(j+1)/2} \left(s + \frac{1-a-b-j}{2}\right)_{(j-1)/2} \left(s - \frac{b+j}{2}\right)_{(j+1)/2}}, \quad (2.A.22)$$

$$w_{4s+2} = -\frac{2\alpha h_{2j}}{\left(-s\right)_s \left(1\right)_{(j-1)/2-s} \left(-s - \frac{1+a}{2}\right)_{(j+1)/2} \left(s + \frac{1+a-b-j}{2}\right)_{(j+1)/2} \left(s + \frac{2-b-j}{2}\right)_{(j-1)/2}}, \quad (2.A.23)$$

$$w_{4s+3} = -\frac{2\left(1-\alpha\right) h_{2j}}{\left(-s\right)_s \left(1\right)_{(j-3)/2-s} \left(-s - \frac{1-a}{2}\right)_{(j+1)/2} \left(s + \frac{1-a-b-j}{2}\right)_{(j+1)/2} \left(s + \frac{2-b-j}{2}\right)_{(j+1)/2}}, \quad (2.A.24)$$

$$h_{2n} = \frac{\sqrt{\left(-j\right)_n \left(1\right)_n \left(\frac{2+a-j}{2}\right)_n \left(-\frac{j+a}{2}\right)_n \left(\frac{2+b-j}{2}\right)_n \left(-\frac{j+b}{2}\right)_n}}{2^{2n}\sqrt{\left(\frac{2-j}{2}\right)_n \left(-\frac{j}{2}\right)_n}} \times \begin{cases} 1 & n \leq \frac{(j-1)}{2} \\ 2\sqrt{\alpha\left(1-\alpha\right)} & n > \frac{(j-1)}{2} \end{cases}, \quad (2.A.25)$$

$$h_{2n+1} = \frac{\sqrt{\left(-1\right)\left(-j\right)_{n+1} \left(1\right)_n \left(\frac{2+a-j}{2}\right)_n \left(-\frac{j+a}{2}\right)_{n+1} \left(\frac{2+b-j}{2}\right)_n \left(-\frac{j+b}{2}\right)_{n+1}}}{2^{2n+1}\sqrt{\left(\frac{2-j}{2}\right)_n \left(-\frac{j}{2}\right)_{n+1}}} \times \begin{cases} 1 & n < \frac{(j-1)}{2} \\ \sqrt{2\alpha} & n = \frac{(j-1)}{2} \\ 2\sqrt{\alpha\left(1-\alpha\right)} & n > \frac{(j-1)}{2} \end{cases}.$$

The orthogonality relation reads

$$\sum_{s=0}^{2j} w_s \mathbf{P}_n^{(2)}\left(x_s\right) \mathbf{P}_m^{(2)}\left(x_s\right) = h_n^2 \delta_{nm}. \quad (2.A.26)$$

2.B. Para-Bannai-Ito for $N = 2j+1$, j odd

Truncation from the general Bannai-Ito polynomials

$$\rho_1 + \rho_2 = -\frac{j+1}{2} + e_1 t, \qquad -r_1 - r_2 = -\frac{j+1}{2} + e_2 t. \quad (2.B.1)$$

The para-Bannai-Ito polynomials are obtained through a $t \to 0$ limit, with

$$\rho_1 = \frac{b-j-1+a}{4}, \qquad -r_1 = \frac{b-j-1-a}{4}, \qquad \frac{e_1}{e_1+e_2} = \alpha, \qquad \frac{e_2}{e_1+e_2} = \left(1-\alpha\right). \quad (2.B.2)$$

Recurrence relation

$$x\mathbf{P}_n^{(3)}(x) = \mathbf{P}_{n+1}^{(3)}(x) + \left(\frac{b-j-1+a}{4} - A_n^3 - C_n^3\right)\mathbf{P}_n^{(3)}(x) + A_{n-1}^3 C_n^3 \mathbf{P}_{n-1}^{(3)}(x), \quad (2.B.3)$$

$$A_n^3 = \begin{cases} \frac{1}{4}\frac{(n-j+a)(n-j+b)}{(n-j)} & n \text{ even} \\ \frac{1}{4}(n-2j-1) & \begin{matrix} n \text{ odd} \\ n \neq j \end{matrix} \\ -\frac{1}{2}\alpha(j+1) & \begin{matrix} n \text{ odd} \\ n = j \end{matrix} \end{cases}, \quad C_n^3 = \begin{cases} -\frac{1}{4}n & \begin{matrix} n \text{ even} \\ n \neq j+1 \end{matrix} \\ -\frac{1}{4}\frac{(n-j-1-a)(n-j-1-b)}{(n-j-1)} & n \text{ odd} \\ -\frac{1}{2}(1-\alpha)(j+1) & \begin{matrix} n \text{ even} \\ n = j+1 \end{matrix} \end{cases}. \quad (2.B.4)$$

Positivity condition

$$\begin{matrix} |a| \geq j, & & |a| \leq 1, \\ |b| \leq 1, & \text{or} & |b| \geq j, \\ 0 \leq \alpha \leq 1, & & 0 \leq \alpha \leq 1. \end{matrix} \quad (2.B.5)$$

Dunkl-Difference equation

$$L\mathbf{P}_n^{(1)}(x) = \lambda_n \mathbf{P}_n^{(1)}(x), \quad (2.B.6)$$

$$\lambda_{2n} = n, \qquad \lambda_{2n+1} = (j-n), \quad (2.B.7)$$

$$L = F(x)(I-R) + G(x)\left(T^+R - I\right), \quad (2.B.8)$$

where

$$G(x) = \frac{(4x+1+b-a-j)(4x+1+a-b-j)}{16(2x+1)}, \quad (2.B.9)$$

$$F(x) = \frac{(4x+j+1-a-b)(4x+j+1+a+b)}{32x}. \quad (2.B.10)$$

Explicit expression

$$\mathbf{P}_n^{(3)}(x) = \mathbf{P}_n^{(2)}(x) - C_n^3 \mathbf{P}_{n-1}^{(2)}(x). \quad (2.B.11)$$

Orthogonality relation

$$x_{4s}=-\left(\frac{4s-j+a-b}{4}\right)-\frac{1}{4}\qquad s\in\{0{,}1{,}...,(j-1)/2\},\tag{2.B.12}$$

$$x_{4s+1}=\left(\frac{4s-j-a-b}{4}\right)-\frac{1}{4}\qquad s\in\{0{,}1{,}...,(j-1)/2\},\tag{2.B.13}$$

$$x_{4s+2}=\left(\frac{4s+2-j+a-b}{4}\right)-\frac{1}{4}\qquad s\in\{0{,}1{,}...,(j-1)/2\},\tag{2.B.14}$$

$$x_{4s+3}=-\left(\frac{4s+2-j-a-b}{4}\right)-\frac{1}{4}\qquad s\in\{0{,}1{,}...,(j-1)/2\},\tag{2.B.15}$$

$$w_{4s}=\frac{2\alpha h_{2j}}{(-s)_s\,(1)_{(j-1)/2-s}\left(-s+\frac{1-a}{2}\right)_{(j+1)/2}\left(s+\frac{1+a-b-j}{2}\right)_{(j+1)/2}\left(s-\frac{b+j}{2}\right)_{(j+1)/2}},\tag{2.B.16}$$

$$w_{4s+1}=\frac{2(1-\alpha)h_{2j}}{(-s)_s\,(1)_{(j-1)/2-s}\left(-s+\frac{1+a}{2}\right)_{(j+1)/2}\left(s+\frac{1-a-b-j}{2}\right)_{(j+1)/2}\left(s-\frac{b+j}{2}\right)_{(j+1)/2}},\tag{2.B.17}$$

$$w_{4s+2}=-\frac{2\alpha h_{2j}}{(-s)_s\,(1)_{(j-1)/2-s}\left(-s-\frac{1+a}{2}\right)_{(j+1)/2}\left(s+\frac{1+a-b-j}{2}\right)_{(j+1)/2}\left(s+\frac{2-b-j}{2}\right)_{(j+1)/2}},\tag{2.B.18}$$

$$w_{4s+3}=-\frac{2(1-\alpha)h_{2j}}{(-s)_s\,(1)_{(j-1)/2-s}\left(-s-\frac{1-a}{2}\right)_{(j+1)/2}\left(s+\frac{1-a-b-j}{2}\right)_{(j+1)/2}\left(s+\frac{2-b-j}{2}\right)_{(j+1)/2}},\tag{2.B.19}$$

$$h_{2n}=\frac{\sqrt{(-j)_n\,(1)_n\left(-\frac{j-a}{2}\right)_n\left(-\frac{j+a}{2}\right)_n\left(-\frac{j-b}{2}\right)_n\left(-\frac{j+b}{2}\right)_n}}{2^{2n}\left(-\frac{j}{2}\right)_n}\times\begin{cases}1 & n<\frac{(j+1)}{2}\\ 2\sqrt{\alpha(1-\alpha)} & n\geq\frac{(j+1)}{2}\end{cases},\tag{2.B.20}$$

$$h_{2n+1}=\frac{\sqrt{(-1)(-j)_n\,(1)_n\left(-\frac{j-a}{2}\right)_{n+1}\left(-\frac{j+a}{2}\right)_{n+1}\left(-\frac{j-b}{2}\right)_{n+1}\left(-\frac{j+b}{2}\right)_{n+1}}}{2^{2n+1}\left(-\frac{j}{2}\right)_{n+1}}\times\begin{cases}1 & n<\frac{(j+1)}{2}\\ 2\sqrt{\alpha(1-\alpha)} & n\geq\frac{(j+1)}{2}\end{cases}.$$

The orthogonality relation reads

$$\sum_{s=0}^{2j+1}w_s\mathbf{P}_n^{(3)}(x_s)\,\mathbf{P}_m^{(3)}(x_s)=h_n^2\delta_{nm}.\tag{2.B.21}$$

2.C. Depiction of the Bannai-Ito bi-lattice

We show here an example of the Bannai-Ito bi-lattice for $a = -(c + j + 1)$ and $c > 0$, which correspond to the first set of restrictions of (2.3.5) with j even. We see that the parameter c, and consequently a, control the gap between the positive and negative part of the grid. As for the parameter b, it controls the gap between consecutive points on each separate side. The first and last points are at the coordinate $x = \pm\frac{2j+b+c}{4}$ and the distances d_3 and d_4 alternate until they reach the end on both side.

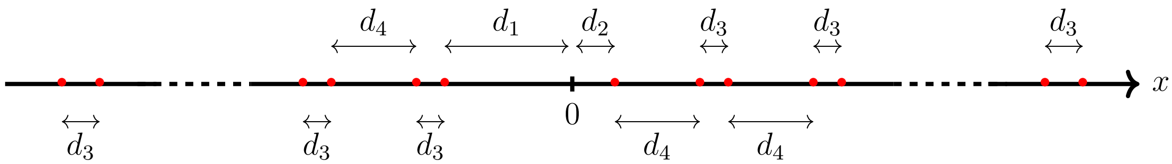

Fig. 2.1. Depiction of a Bannai-Ito bi-lattice.

We have precisely,

$$d_1 = \frac{c + 2 - b}{4}, \qquad d_2 = \frac{b + c}{4}, \qquad d_3 = \frac{1 + b}{2}, \qquad d_4 = \frac{1 - b}{2}. \tag{2.C.1}$$

When we use $b = 0$, we obtain equally distanced points on each side. This is equivalent to a single Bannai-Ito lattice (or two separate linear grid).

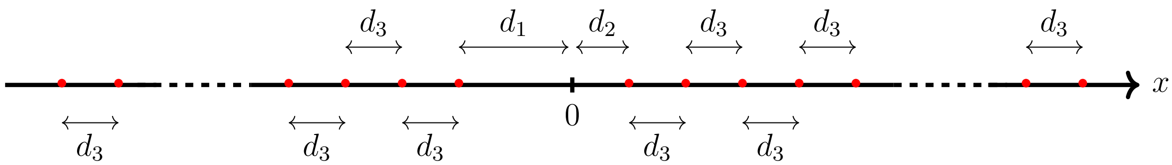

Fig. 2.2. Reduction of the Bannai-Ito bi-lattice to a single Bannai-Ito lattice.

Now, we have

$$d_1 = \frac{c}{4} + \frac{1}{2}, \qquad d_2 = \frac{c}{4}, \qquad d_3 = \frac{1}{2}. \tag{2.C.2}$$

Finally, if we use $c = 0$ (or $a = -j - 1$), we obtain one simple linear grid where the points are all separated by $d = 1/2$.

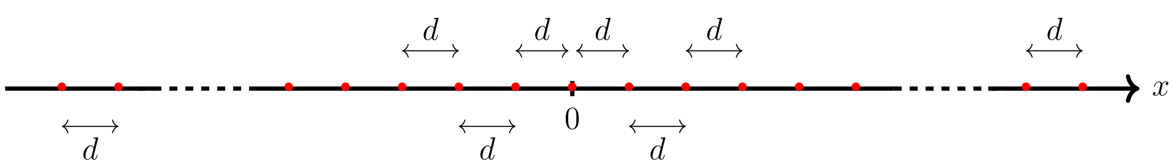

Fig. 2.3. Reduction of the single Bannai-Ito lattice into a single linear lattice.

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Conclusion

Les travaux contenus dans ce mémoire touchent deux sujets reliés: la classification et l'organisation des polynômes orthogonaux -1 d'une variable continue ainsi que la caractérisation d'une nouvelle famille de polynômes orthogonaux nommée polynôme de para-Bannai-Ito.

Dans le premier article, une classification des différentes familles de polynômes correspondant à des limites $q = -1$ de q-polynômes orthogonaux continus a été proposée. Cette classification est constituée de quatre catégories, qui contiennent chacune les spécialisations de la famille en tête. Ensemble, les polynômes de Chihara et les polynômes Big -1 Jacobi sont à la tête de la même catégorie, alors que les polynômes de Bannai-Ito continu, les polynômes -1 de Meixner-Pollaczek, ainsi que les polynômes symétriques généralisés de Bannai-Ito sont chacune à la tête de leur catégorie respective. Cette structure a permis de rassembler toutes les familles de polynômes -1 à variable continue découvertes à ce jour. Un recueil contenant les expressions en série hypergéométrique, les relations de récurrence, les équations aux différences de type Dunkl ainsi que les relations d'orthogonalité pour chacune des familles du tableau -1 a été présenté. Ce tableau ne contient que des familles à variable continue et un travail futur devrait couvrir les familles à variable discrète pour obtenir le schéma -1 au complet. En plus de la partie discrète du schéma -1, il serait intéressant d'étudier les autres racines de l'unité pour q. La recherche sur les autres racines est déjà initiée avec l'idée des polynômes orthogonaux dits tamisés (sieved polynomials) [**34**] et des opérateurs de Dunkl généralisés qui mélange des opérateurs différentiels à des rotations dans le plan complexe [**35**]. Beaucoup de travail reste à faire dans ce domaine. Finalement, il serait utile de développer un équivalent -1 du schéma de q-Verde-Star et du schéma de q-Zhedanov qui propose une méthode de classification plus systématique que celle présentée ici.

Dans le deuxième article, toutes les informations utiles à la caractérisation des polynômes de para–Bannai–Ito sont présentées, et ce, pour tous les différents cas des parités du paramètre j. Cette caractérisation comprend la relation de récurrence, la relation d'orthogonalité, l'expression explicite et une équation aux valeurs propres avec un opérateur différentiel de type Dunkl. Cette famille est observée comme étant orthogonale sur la combinaison de quatre grilles linéaires, mais dont seulement deux s'entrecroisent à la fois. Avec les polynômes de Krawtchouk et de para–Krawtchouk, il a été possible d'observer des relations d'orthogonalité sur une grille linéaire et sur deux grilles linéaires, respectivement. Une question ouverte est de savoir s'il est possible d'obtenir une famille de polynômes qui aurait une relation d'orthogonalité sur l'agencement de huit grilles linéaires. Cette question est hautement liée aux autres cas où q est une racine de l'unité. On peut penser à $q = \pm i$ pour la combinaison de huit grilles. Toutefois, cette grille ne serait pas, à priori, contrainte à l'axe des réels. Dans une optique d'application physique, les polynômes de para-Banna-Ito correspondent à une matrice de Jacobi qui résulte d'une déformation isospectrale d'une matrice persymétrique. Cette caractéristique est très particulière et est nécessaire pour des applications en transfert d'information quantique sur des chaînes de spin. L'idée est d'interpréter la matrice de Jacobi de cet ensemble de polynômes comme étant un Hamiltonien d'une chaîne de spin ayant comme spectre la grille d'orthogonalité. On est alors en mesure d'analyser la possibilité de contrôler le transfert d'information d'un bout à l'autre de la chaîne, ainsi que la création de paires intriquées, comme présenté en introduction. Cette possibilité pour les polynômes de para-Bannai-Ito sera étudiée dans une future publication.

Chapitre A

A compendium of the properties of the -1 continuous orthogonal polynomials

A.1. Continuous Bannai-Ito

Hypergeometric Representation

$$\frac{\mathbf{Q}_{2n}(x;\alpha,\beta,\gamma,\delta)}{(-2i)^{2n}} = \tag{A.1.1}$$
$$\xi_{2n}\kappa_{n-1}^{(2)}\left(\frac{ix}{2}-\mathfrak{b}-\frac{1}{2}\right){}_4F_3\left(\begin{matrix}-n+1,\ n+g+2,\ \frac{ix}{2}+\mathfrak{b}+1,\ \frac{-ix}{2}+\mathfrak{b}+\frac{3}{2}\\ \frac{5}{2}+\mathfrak{a}+\mathfrak{b},\ 2+\mathfrak{b}+\mathfrak{c},\ 2+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right)+$$
$$\kappa_n^{(1)}{}_4F_3\left(\begin{matrix}-n,\ n+g+1,\ \frac{ix}{2}+\mathfrak{b},\ \frac{-ix}{2}+\mathfrak{b}+\frac{1}{2}\\ \frac{3}{2}+\mathfrak{a}+\mathfrak{b},\ 1+\mathfrak{b}+\mathfrak{c},\ 1+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right),$$

$$\frac{\mathbf{Q}_{2n+1}(x;\alpha,\beta,\gamma,\delta)}{(-2i)^{2n+1}} = \tag{A.1.2}$$
$$\kappa_n^{(2)}\left(\frac{ix}{2}-\mathfrak{b}-\frac{1}{2}\right){}_4F_3\left(\begin{matrix}-n,\ n+g+2,\ \frac{ix}{2}+\mathfrak{b}+1,\ \frac{-ix}{2}+\mathfrak{b}+\frac{3}{2}\\ \frac{5}{2}+\mathfrak{a}+\mathfrak{b},\ 2+\mathfrak{b}+\mathfrak{c},\ 2+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right)+$$
$$\eta_{2n+1}\kappa_n^{(1)}{}_4F_3\left(\begin{matrix}-n,\ n+g+1,\ \frac{ix}{2}+\mathfrak{b},\ \frac{-ix}{2}+\mathfrak{b}+\frac{1}{2}\\ \frac{3}{2}+\mathfrak{a}+\mathfrak{b},\ 1+\mathfrak{b}+\mathfrak{c},\ 1+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right),$$

where

$$g=2\alpha+2\gamma+1,\qquad \mathfrak{a}=\overline{\mathfrak{d}}=\alpha+i\beta,\qquad \mathfrak{b}=\overline{\mathfrak{c}}=\gamma+i\delta, \tag{A.1.3}$$

$$\begin{aligned}\xi_{2n}&=\tfrac{n\left(n+\mathfrak{c}+\mathfrak{d}+\frac{1}{2}\right)}{(2n+g)}, & \kappa_n^{(1)}&=\tfrac{\left(\frac{3}{2}+\mathfrak{a}+\mathfrak{b}\right)_n(1+\mathfrak{b}+\mathfrak{c})_n(1+\mathfrak{b}+\mathfrak{d})_n}{(n+g+1)_n},\\ \eta_{2n+1}&=\tfrac{(n+\mathfrak{b}+\mathfrak{c}+1)(n+\mathfrak{b}+\mathfrak{d}+1)}{(2n+g+1)}, & \kappa_n^{(2)}&=\tfrac{\left(\frac{5}{2}+\mathfrak{a}+\mathfrak{b}\right)_n(2+\mathfrak{b}+\mathfrak{c})_n(2+\mathfrak{b}+\mathfrak{d})_n}{(n+g+2)_n}.\end{aligned} \tag{A.1.4}$$

Orthogonality Relation

If α,β,γ,$\delta \in \mathbb{R}^+$

$$\int_{-\infty}^{\infty} W(x)\mathbf{Q}_n(x)\mathbf{Q}_m(x)\mathrm{d}x = 4\pi h_0 \kappa_n \delta_{nm}, \tag{A.1.5}$$

$$W(x) = \left| \frac{\Gamma(\mathfrak{a} + ix/2 + 1)\Gamma(\mathfrak{b} + ix/2 + 1)\Gamma(\mathfrak{c} + ix/2 + 1/2)\Gamma(\mathfrak{d} + ix/2 + 1/2)}{\Gamma(1/2 + ix)} \right|^2, \tag{A.1.6}$$

$$h_0 = \frac{\Gamma(\mathfrak{a} + \mathfrak{b} + 3/2)\Gamma(\mathfrak{a} + \mathfrak{c} + 1)\Gamma(\mathfrak{b} + \mathfrak{c} + 1)\Gamma(\mathfrak{a} + \mathfrak{d} + 1)\Gamma(\mathfrak{b} + \mathfrak{d} + 1)\Gamma(\mathfrak{c} + \mathfrak{d} + 3/2)}{\Gamma(\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} + 2)}, \tag{A.1.7}$$

$$\kappa_{2n} = \frac{4^{2n}\Gamma(n+1)(2\alpha+1)_n(2\gamma+1)_n}{(2\alpha+2\gamma+2)_{2n}(n+2\alpha+2\gamma+2)_n} \times \left[\prod_{k=1}^{n} \|k+\alpha+\gamma+i(\beta-\delta)\|\right]^2 \left[\prod_{k=1}^{n} \|k+\alpha+\gamma+\frac{1}{2}+i(\beta+\delta)\|\right]^2, \tag{A.1.8}$$

$$\kappa_{2n+1} = \frac{4^{2n+1}\Gamma(n+1)(2\alpha+1)_{n+1}(2\gamma+1)_{n+1}}{(2\alpha+2\gamma+2)_{2n+1}(n+2\alpha+2\gamma+2)_{n+1}} \times \left[\prod_{k=1}^{n+1} \|k+\alpha+\gamma+i(\beta-\delta)\|\right]^2 \left[\prod_{k=1}^{n} \|k+\alpha+\gamma+\frac{1}{2}+i(\beta+\delta)\|\right]^2. \tag{A.1.9}$$

Normalized Recurrence Relation

$$x\mathbf{Q}_n(x) = \mathbf{Q}_{n+1}(x) + b_n\mathbf{Q}_n(x) + u_n\mathbf{Q}_{n-1}(x), \tag{A.1.10}$$

$$b_n = \begin{cases} 2\beta - \frac{(n+4\alpha+2)(\beta-\delta)}{(n+2\alpha+2\gamma+2)} - \frac{n(\beta+\delta)}{(n+2\alpha+2\gamma+1)} & n \text{ even} \\ 2\beta - \frac{(n+4\alpha+4\gamma+3)(\beta+\delta)}{n+2\alpha+2\gamma+2} - \frac{(n+4\gamma+1)(\beta-\delta)}{n+2\alpha+2\gamma+1} & n \text{ odd} \end{cases}, \tag{A.1.11}$$

$$u_n = \begin{cases} \frac{n(n+4\alpha+4\gamma+2)\|n+2[\alpha+\gamma+i(\beta+\delta)]+1\|^2}{4(n+2\alpha+2\gamma+1)^2} & n \text{ even} \\ \frac{(n+4\alpha+1)(n+4\gamma+1)\|n+2[\alpha+\gamma+i(\beta-\delta)]+1\|^2}{4(n+2\alpha+2\gamma+1)^2} & n \text{ odd} \end{cases}. \tag{A.1.12}$$

Difference Equation

$$L\mathbf{Q}_n(x) = \lambda_n \mathbf{Q}_n(x), \quad \lambda_n = (-1)^n(n + 2\alpha + 2\gamma + 3/2), \tag{A.1.13}$$

$$L = \mathcal{A}\left(S^{+}R - I\right) + \overline{\mathcal{A}}\left(S^{-}R - I\right) + (2\alpha + 2\gamma + 3/2)I, \tag{A.1.14}$$

$$\mathcal{A} = \frac{(2\alpha + 1 + i\,[\beta - x])\,(2\gamma + 1 + i\,[\delta - x])}{1 - 2ix}, \tag{A.1.15}$$

where $\overline{\mathcal{A}}$ is the complex conjugate of $\mathcal{A}$, $S^{\pm}f(x) = f(x \pm i)$ and $Rf(x) = f(-x)$.

Limit Relations

Continuous Bannai-Ito → Big -1 Jacobi

The Big -1 Jacobi polynomial can be obtained from the Continuous Bannai-Ito polynomials using the parametrization $\beta \to \frac{\beta}{h}$, $\delta \to \frac{\delta}{h}$, the scaling $x \to 2\beta x$ and taking the limit $h \to 0$:

$$\lim_{h\to 0}\left(\frac{h}{2\beta}\right)^n \mathbf{Q}_n\left(\frac{2\beta x}{h};\alpha,\frac{\beta}{h},\gamma,\frac{\delta}{h}\right) = \mathbf{J}_n\left(x; 4\alpha+1, 4\gamma+1, \frac{-\delta}{\beta}\right). \tag{A.1.16}$$

Continuous Bannai-Ito → First type Continuous -1 Hahn

The first type of Continuous -1 Hahn polynomials can be obtained from the Continuous Bannai-Ito polynomials by a specialization:

$$\mathbf{Q}_n\left(x;\alpha,\beta,\gamma,\beta\right) = \mathbf{K}_n^{(1)}\left(x;\alpha,\beta,\gamma\right). \tag{A.1.17}$$

Continuous Bannai-Ito → Second type Continuous -1 Hahn

The second type of Continuous -1 Hahn polynomials can be obtained from the Continuous Bannai-Ito polynomials by a specialization:

$$\mathbf{Q}_n\left(x;\alpha,\beta,\gamma,-\beta\right) = \mathbf{K}_n^{(2)}\left(x;\alpha,\beta,\gamma\right). \tag{A.1.18}$$

A.2. Big -1 Jacobi

Hypergeometric Representation

$$\frac{\mathbf{J}_{2n}\left(x;\alpha,\beta,c\right)}{\eta_{2n}} = {}_2F_1\left(\begin{matrix}-n, & \frac{2n+\alpha+\beta+2}{2} \\ & \frac{\alpha+1}{2}\end{matrix}; \frac{1-x^2}{1-c^2}\right) + \frac{2n(1-x)}{(1+c)(1+\alpha)}\,{}_2F_1\left(\begin{matrix}1-n, & \frac{2n+\alpha+\beta+2}{2} \\ & \frac{\alpha+3}{2}\end{matrix}; \frac{1-x^2}{1-c^2}\right), \tag{A.2.1}$$

$$\frac{\mathbf{J}_{2n+1}(x;\alpha,\beta,c)}{\eta_{2n+1}} = \tag{A.2.2}$$

$${}_2F_1\left(\begin{matrix}-n, & \frac{2n+\alpha+\beta+2}{2} \\ \frac{\alpha+1}{2} & \end{matrix}; \frac{1-x^2}{1-c^2}\right) - \frac{(2n+\alpha+\beta+2)(1-x)}{(1+c)(1+\alpha)} {}_2F_1\left(\begin{matrix}-n, & \frac{2n+\alpha+\beta+4}{2} \\ \frac{\alpha+3}{2} & \end{matrix}; \frac{1-x^2}{1-c^2}\right),$$

where

$$\eta_{2n} = \frac{\left(1-c^2\right)^n \left(\frac{\alpha+1}{2}\right)_n}{\left(\frac{2n+\alpha+\beta+2}{2}\right)_n}, \qquad \eta_{2n+1} = \frac{(1+c)\left(1-c^2\right)^n \left(\frac{\alpha+1}{2}\right)_{n+1}}{\left(\frac{2n+\alpha+\beta+2}{2}\right)_{n+1}}. \tag{A.2.3}$$

Orthogonality Relation

If $\alpha > 0$, $\beta > 0$ and $0 \le c < 1$

$$\int_{\mathcal{C}} \omega(x) \mathbf{J}_n(x) \mathbf{J}_m(x) \mathrm{d}x = \frac{(1-c)\left(1-c^2\right)^{\frac{\alpha+\beta}{2}} \Gamma\left(\frac{\alpha+1}{2}\right) \Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+1\right)} \kappa_n \delta_{nm}, \tag{A.2.4}$$

$$\mathcal{C} = [-1,-c] \bigcup [c,1], \tag{A.2.5}$$

$$\omega(x) = \theta(x) \frac{(1+x)}{(c+x)} \left(1-x^2\right)^{\frac{\alpha-1}{2}} \left(x^2-c^2\right)^{\frac{\beta+1}{2}}, \tag{A.2.6}$$

$$\begin{aligned} \kappa_{2n} &= \frac{\Gamma(n+1)\left(\frac{\alpha+1}{2}\right)_n \left(\frac{\beta+1}{2}\right)_n}{\left(1+\frac{\alpha+\beta}{2}\right)_{2n} \left(n+1+\frac{\alpha+\beta}{2}\right)_n} \left(1-c^2\right)^n, \\ \kappa_{2n+1} &= \frac{\Gamma(n+1)\left(\frac{\alpha+1}{2}\right)_{n+1} \left(\frac{\beta+1}{2}\right)_{n+1}}{\left(1+\frac{\alpha+\beta}{2}\right)_{2n+1} \left(n+1+\frac{\alpha+\beta}{2}\right)_{n+1}} (1+c)^2 \left(1-c^2\right)^n. \end{aligned} \tag{A.2.7}$$

Normalized Recurrence Relation

$$x\mathbf{J}_n(x) = \mathbf{J}_{n+1}(x) + (1 - A_n - C_n)\mathbf{J}_n(x) + A_{n-1}C_n\mathbf{J}_{n-1}(x), \tag{A.2.8}$$

$$A_n = \begin{cases} \frac{(1+c)(n+\alpha+1)}{2n+\alpha+\beta+2} & n \text{ even} \\ \frac{(1-c)(n+\alpha+\beta+1)}{2n+\alpha+\beta+2} & n \text{ odd} \end{cases}, \qquad C_n = \begin{cases} \frac{(1-c)n}{2n+\alpha+\beta} & n \text{ even} \\ \frac{(1+c)(n+\beta)}{2n+\alpha+\beta} & n \text{ odd} \end{cases}. \tag{A.2.9}$$

Differential Equation

$$L\mathbf{J}_n(x) = \lambda_n \mathbf{J}_n(x), \quad \lambda_n = \begin{cases} -2n & n \text{ even} \\ 2(n+\alpha+\beta+1) & n \text{ odd} \end{cases}, \tag{A.2.10}$$

$$\begin{aligned} L = &\left(\frac{(\alpha+\beta+1)x^2 + (c\alpha-\beta)x + c}{x^2}\right)[R-I] \\ &+ \left(\frac{2(1-x)(c+x)}{x}\right)\partial_x R. \end{aligned} \tag{A.2.11}$$

Limit Relations

Big q-Jacobi → Big -1 Jacobi

The big -1 Jacobi polynomials are obtained from the monic big q-Jacobi polynomials by taking the parametrization $q = -e^{\varepsilon}$, $a = -e^{\varepsilon\alpha}$ and $b = -e^{\varepsilon\beta}$:

$$\lim_{\varepsilon\to 0} \mathbf{P}_n(x; a,b,c|q) = \mathbf{J}_n(x; \alpha,\beta,c). \tag{A.2.12}$$

Big -1 Jacobi → Chihara

The Chihara polynomials are obtained by the Christoffel transformation of the big -1 Jacobi polynomials and vice-versa via the Geronimus transformation:

$$(-1)^n \left(1-c^2\right)^{\frac{n}{2}} \mathbf{C}_n\left(\frac{-x}{\sqrt{1-c^2}}\right) = \frac{\mathbf{J}_{n+1}(x) - A_n \mathbf{J}_n(x)}{x-1}, \tag{A.2.13}$$

$$\mathbf{J}_n(x) = (-1)^n \left(1-c^2\right)^{\frac{n}{2}} \left(\mathbf{C}_n\left(\frac{-x}{\sqrt{1-c^2}}\right) + \frac{C_n}{\sqrt{1-c^2}} \mathbf{C}_{n-1}\left(\frac{-x}{\sqrt{1-c^2}}\right)\right), \tag{A.2.14}$$

where A_n and C_n are given by (A.2.9).

$$\frac{1}{(1-c^2)^{\frac{n}{2}}} \mathbf{J}_n(x\sqrt{1-c^2}; \alpha,\beta,c) \underset{\text{CT}}{\overset{\text{GT}}{\rightleftarrows}} \mathbf{C}_n\left(x; \frac{\beta-1}{2}, \frac{\alpha+1}{2}, \frac{-c}{\sqrt{1-c^2}}\right). \tag{A.2.15}$$

Continuous Bannai-Ito → Big -1 Jacobi

The Big -1 Jacobi polynomial can be obtained from the Continuous Bannai-Ito polynomials using the parametrization $\beta \to \frac{\beta}{h}$, $\delta \to \frac{\delta}{h}$, the scaling $x \to 2\beta x$ and taking the limit $h \to 0$:

$$\lim_{h\to 0} \left(\frac{h}{2\beta}\right)^n \mathbf{Q}_n\left(\frac{2\beta x}{h}; \alpha, \frac{\beta}{h}, \gamma, \frac{\delta}{h}\right) = \mathbf{J}_n\left(x; 4\alpha+1, 4\gamma+1, \frac{-\delta}{\beta}\right). \tag{A.2.16}$$

Big -1 Jacobi → Little -1 Jacobi

The little -1 Jacobi polynomial can be obtained from the big -1 Jacobi polynomials by taking c to 0:

$$\mathbf{J}_n\left(x;\alpha,\beta,0\right)=\mathbf{P}_n\left(x;\beta,\alpha\right). \tag{A.2.17}$$

A.3. Chihara

Hypergeometric Representation

$$\mathbf{C}_{2n}\left(x;\alpha,\beta,\gamma\right)=\frac{(-1)^n\left(\alpha+1\right)_n}{\left(n+\alpha+\beta+1\right)_n}\,{}_2F_1\left(\begin{matrix}-n,\ n+\alpha+\beta+1\\ \alpha+1\end{matrix};x^2-\gamma^2\right), \tag{A.3.1}$$

$$\mathbf{C}_{2n+1}\left(x;\alpha,\beta,\gamma\right)=\frac{(-1)^n\left(\alpha+2\right)_n}{\left(n+\alpha+\beta+2\right)_n}\left(x-\gamma\right){}_2F_1\left(\begin{matrix}-n,\ n+\alpha+\beta+2\\ \alpha+2\end{matrix};x^2-\gamma^2\right). \tag{A.3.2}$$

Orthogonality Relation

If $\alpha>-1$ and $\beta>0$

$$\int_{\mathcal{C}}\omega(x)\mathbf{C}_n(x)\mathbf{C}_m(x)\mathrm{d}x=h_n\delta_{nm}, \tag{A.3.3}$$

$$\mathcal{C}=\left[-\sqrt{1+\gamma^2},-|\gamma|\right]\bigcup\left[|\gamma|,\sqrt{1+\gamma^2}\right], \tag{A.3.4}$$

$$\omega(x)=\theta(x)\left(x+\gamma\right)\left(x^2-\gamma^2\right)^{\alpha}\left(1+\gamma^2-x^2\right)^{\beta}, \tag{A.3.5}$$

$$\begin{aligned}h_{2n}&=\tfrac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}\tfrac{n!}{(2n+\alpha+\beta+1)[(n+\alpha+\beta+1)_n]^2},\\ h_{2n+1}&=\tfrac{\Gamma(n+\alpha+2)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)}\tfrac{n!}{(2n+\alpha+\beta+2)[(n+\alpha+\beta+2)_n]^2}.\end{aligned} \tag{A.3.6}$$

Normalized Recurrence Relation

$$x\mathbf{C}_n(x)=\mathbf{C}_{n+1}(x)+(-1)^n\gamma\mathbf{C}_n(x)+\sigma_n\mathbf{C}_{n-1}(x), \tag{A.3.7}$$

$$\sigma_{2n}=\frac{n\left(n+\beta\right)}{\left(2n+\alpha+\beta\right)\left(2n+\alpha+\beta+1\right)},\qquad\sigma_{2n+1}=\frac{\left(n+\alpha+1\right)\left(n+\alpha+\beta+1\right)}{\left(2n+\alpha+\beta+1\right)\left(2n+\alpha+\beta+2\right)}. \tag{A.3.8}$$

Differential Equation

$$L^{(\varepsilon)}\mathbf{C}_n(x) = \lambda_n^{(\varepsilon)}\mathbf{C}_n(x), \quad \begin{cases} \lambda_{2n}^{(\varepsilon)} = n^2 + (\alpha+\beta+1)\,n \\ \lambda_{2n+1}^{(\varepsilon)} = n^2 + (\alpha+\beta+2)\,n + \varepsilon \end{cases}, \tag{A.3.9}$$

$$L^{(\varepsilon)} = S(x)\partial_x^2 + T(x)\partial_x R + U(x)\partial_x + V(x)\left[I - R\right], \tag{A.3.10}$$

$$\begin{aligned} S(x) &= \tfrac{\left(x^2-\gamma^2\right)\left(x^2-\gamma^2-1\right)}{4x^2}, \qquad T(x) = \tfrac{\gamma(x-\gamma)\left(x^2-\gamma^2-1\right)}{4x^3}, \\ U(x) &= \tfrac{\gamma\left(x^2-\gamma^2-1\right)(2\gamma-x)}{4x^3} + \tfrac{\left(x^2-\gamma^2\right)(\alpha+\beta+3/2)}{2x} - \tfrac{\alpha+1/2}{2x}, \\ V(x) &= \tfrac{\gamma\left(x^2-\gamma^2-1\right)(x-3\gamma/2)}{4x^4} - \tfrac{\left(x^2-\gamma^2\right)(\alpha+\beta+3/2)}{4x^2} + \tfrac{\alpha+1/2}{4x^2} + \epsilon\tfrac{x-\gamma}{2x}. \end{aligned} \tag{A.3.11}$$

Limit Relations

Big q-Jacobi → Chihara

The Chihara polynomials are obtained from the monic big q-Jacobi polynomials by taking the parametrization $q = -e^{\varepsilon}$, $a = e^{2\varepsilon\beta}$ and $b = -e^{\varepsilon(2\alpha+1)}$:

$$\lim_{\varepsilon\to 0} \frac{1}{(1-c^2)^{\frac{n}{2}}}\mathbf{P}_n\left(\sqrt{1-c^2}x; a,b,c|q\right) = \mathbf{C}_n\left(x;\alpha,\beta,\frac{-c}{\sqrt{1-c^2}}\right). \tag{A.3.12}$$

Big -1 Jacobi → Chihara

The Chihara polynomials are obtained by the Christoffel transformation of the big -1 Jacobi polynomials and vice-versa via the Geronimus transformation:

$$(-1)^n\left(1-c^2\right)^{\frac{n}{2}}\mathbf{C}_n\left(\frac{-x}{\sqrt{1-c^2}}\right) = \frac{\mathbf{J}_{n+1}(x) - A_n\mathbf{J}_n(x)}{x-1}, \tag{A.3.13}$$

$$\mathbf{J}_n(x) = (-1)^n\left(1-c^2\right)^{\frac{n}{2}}\left(\mathbf{C}_n\left(\frac{-x}{\sqrt{1-c^2}}\right) + \frac{C_n}{\sqrt{1-c^2}}\mathbf{C}_{n-1}\left(\frac{-x}{\sqrt{1-c^2}}\right)\right), \tag{A.3.14}$$

where A_n and C_n are given by (A.2.9).

$$\frac{1}{(1-c^2)^{\frac{n}{2}}}\mathbf{J}_n(x\sqrt{1-c^2};\alpha,\beta,c) \underset{\text{CT}}{\overset{\text{GT}}{\Longleftrightarrow}} \mathbf{C}_n\left(x;\frac{\beta-1}{2},\frac{\alpha+1}{2},\frac{-c}{\sqrt{1-c^2}}\right). \tag{A.3.15}$$

Chihara → -1 Meixner-Pollaczek

The -1 Meixner-Pollaczek polynomials can be obtained from the Chihara polynomials by taking $x \to \beta^{-\frac{1}{2}}x$, $\alpha \to \alpha - \frac{1}{2}$ and $\gamma \to \beta^{-\frac{1}{2}}\gamma$ and letting β go to ∞:

$$\lim_{\beta\to\infty} \mathbf{C}_n\left(\beta^{-\frac{1}{2}}x;\alpha - \frac{1}{2},\beta,\beta^{-\frac{1}{2}}\gamma\right) = \mathbf{M}_n^{(-1)}\left(x;\alpha,\gamma\right). \tag{A.3.16}$$

Chihara → Generalized Gegenbauer

The generalized Gegenbauer polynomials can be obtained from the Chihara polynomials by taking γ to 0:

$$\mathbf{C}_n\left(x;\alpha,\beta,0\right) = \mathbf{G}_n\left(x;\alpha,\beta\right). \tag{A.3.17}$$

A.4. Continuous -1 Hahn type 1

Hypergeometric Representation

$$\frac{\mathbf{K}^{(1)}_{2n}\left(x;\alpha,\beta,\gamma\right)}{(-2i)^{2n}} = \tag{A.4.1}$$
$$\xi_{2n}\kappa^{(2)}_{n-1}\left(\frac{ix}{2}-\mathfrak{b}-\frac{1}{2}\right){}_4F_3\left(\begin{matrix}-n+1,\ n+g+2,\ \frac{ix}{2}+\mathfrak{b}+1,\ \frac{-ix}{2}+\mathfrak{b}+\frac{3}{2}\\ \frac{5}{2}+\mathfrak{a}+\mathfrak{b},\ 2+\mathfrak{b}+\mathfrak{c},\ 2+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right)+$$
$$\kappa^{(1)}_{n}\,{}_4F_3\left(\begin{matrix}-n,\ n+g+1,\ \frac{ix}{2}+\mathfrak{b},\ \frac{-ix}{2}+\mathfrak{b}+\frac{1}{2}\\ \frac{3}{2}+\mathfrak{a}+\mathfrak{b},\ 1+\mathfrak{b}+\mathfrak{c},\ 1+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right),$$

$$\frac{\mathbf{K}^{(1)}_{2n+1}\left(x;\alpha,\beta,\gamma\right)}{(-2i)^{2n+1}} = \tag{A.4.2}$$
$$\kappa^{(2)}_{n}\left(\frac{ix}{2}-\mathfrak{b}-\frac{1}{2}\right){}_4F_3\left(\begin{matrix}-n,\ n+g+2,\ \frac{ix}{2}+\mathfrak{b}+1,\ \frac{-ix}{2}+\mathfrak{b}+\frac{3}{2}\\ \frac{5}{2}+\mathfrak{a}+\mathfrak{b},\ 2+\mathfrak{b}+\mathfrak{c},\ 2+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right)+$$
$$\eta_{2n+1}\kappa^{(1)}_{n}\,{}_4F_3\left(\begin{matrix}-n,\ n+g+1,\ \frac{ix}{2}+\mathfrak{b},\ \frac{-ix}{2}+\mathfrak{b}+\frac{1}{2}\\ \frac{3}{2}+\mathfrak{a}+\mathfrak{b},\ 1+\mathfrak{b}+\mathfrak{c},\ 1+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right),$$

where

$$g=\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\mathfrak{d}+1,\qquad \mathfrak{a}=\overline{\mathfrak{d}}=\alpha+i\beta,\qquad \mathfrak{b}=\overline{\mathfrak{c}}=\gamma+i\beta, \tag{A.4.3}$$

$$\begin{aligned}\xi_{2n} &= \frac{n\left(n+\mathfrak{c}+\mathfrak{d}+\frac{1}{2}\right)}{(2n+g)}, & \kappa^{(1)}_n &= \frac{\left(\frac{3}{2}+\mathfrak{a}+\mathfrak{b}\right)_n(1+\mathfrak{b}+\mathfrak{c})_n(1+\mathfrak{b}+\mathfrak{d})_n}{(n+g+1)_n},\\ \eta_{2n+1} &= \frac{(n+\mathfrak{b}+\mathfrak{c}+1)(n+\mathfrak{b}+\mathfrak{d}+1)}{(2n+g+1)}, & \kappa^{(2)}_n &= \frac{\left(\frac{5}{2}+\mathfrak{a}+\mathfrak{b}\right)_n(2+\mathfrak{b}+\mathfrak{c})_n(2+\mathfrak{b}+\mathfrak{d})_n}{(n+g+2)_n}.\end{aligned} \tag{A.4.4}$$

Orthogonality Relation

If $\alpha,\beta,\gamma\in\mathbb{R}^+$

$$\int_{-\infty}^{\infty} W(x)\mathbf{K}^{(1)}_n(x)\mathbf{K}^{(1)}_m(x)\mathrm{d}x = 4\pi h_0\kappa_n\delta_{nm}, \tag{A.4.5}$$

$$W(x)=\left|\frac{\Gamma(\mathfrak{a}+ix/2+1)\Gamma(\mathfrak{b}+ix/2+1)\Gamma(\mathfrak{c}+ix/2+1/2)\Gamma(\mathfrak{d}+ix/2+1/2)}{\Gamma(1/2+ix)}\right|^2, \quad \text{(A.4.6)}$$

$$h_0=\frac{\Gamma(\mathfrak{a}+\mathfrak{b}+3/2)\Gamma(\mathfrak{a}+\mathfrak{c}+1)\Gamma(\mathfrak{b}+\mathfrak{c}+1)\Gamma(\mathfrak{a}+\mathfrak{d}+1)\Gamma(\mathfrak{b}+\mathfrak{d}+1)\Gamma(\mathfrak{c}+\mathfrak{d}+3/2)}{\Gamma(\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\mathfrak{d}+2)}, \quad \text{(A.4.7)}$$

$$\kappa_{2n}=\frac{4^{2n}\Gamma(n+1)(2\alpha+1)_n(2\gamma+1)_n(1+\alpha+\gamma)_n^2}{(2\alpha+2\gamma+2)_{2n}(n+2\alpha+2\gamma+2)_n}\left[\prod_{k=1}^{n}\|k+\alpha+\gamma+\frac{1}{2}+2i\beta\|\right]^2, \quad \text{(A.4.8)}$$

$$\kappa_{2n+1}=\frac{4^{2n+1}\Gamma(n+1)(2\alpha+1)_{n+1}(2\gamma+1)_{n+1}(1+\alpha+\gamma)_{n+1}^2}{(2\alpha+2\gamma+2)_{2n+1}(n+2\alpha+2\gamma+2)_{n+1}}\left[\prod_{k=1}^{n}\|k+\alpha+\gamma+\frac{1}{2}+2i\beta\|\right]^2. \quad \text{(A.4.9)}$$

Normalized Recurrence Relation

$$x\mathbf{K}_n^{(1)}(x)=\mathbf{K}_{n+1}^{(1)}(x)+b_n\mathbf{K}_n^{(1)}(x)+u_n\mathbf{K}_{n-1}^{(1)}(x), \quad \text{(A.4.10)}$$

$$b_n=\begin{cases}2\beta-2\beta\frac{n}{(n+2\alpha+2\gamma+1)} & n\text{ even}\\ 2\beta-2\beta\frac{(n+4\alpha+4\gamma+3)}{n+2\alpha+2\gamma+2} & n\text{ odd}\end{cases}, \quad \text{(A.4.11)}$$

$$u_n=\begin{cases}\frac{n(n+4\alpha+4\gamma+2)\|n+2\alpha+2\gamma+4i\beta+1\|^2}{4(n+2\alpha+2\gamma+1)^2} & n\text{ even}\\ \frac{(n+4\alpha+1)(n+4\gamma+1)\|n+2\alpha+2\gamma+1\|^2}{4(n+2\alpha+2\gamma+1)^2} & n\text{ odd}\end{cases}. \quad \text{(A.4.12)}$$

Difference Equation

$$L\mathbf{K}_n^{(1)}(x)=\lambda_n\mathbf{K}_n^{(1)}(x), \quad \lambda_n=(-1)^n(n+2\alpha+2\gamma+3/2), \quad \text{(A.4.13)}$$

$$L=\mathcal{A}\left(S^+R-I\right)+\overline{\mathcal{A}}\left(S^-R-I\right)+(2\alpha+2\gamma+3/2)I, \quad \text{(A.4.14)}$$

$$\mathcal{A}=\frac{(2\alpha+1+i[\beta-x])(2\gamma+1+i[\beta-x])}{1-2ix}, \quad \text{(A.4.15)}$$

where $\overline{\mathcal{A}}$ is the complex conjugate of $\mathcal{A}$, $S^{\pm}f(x)=f(x\pm i)$ and $Rf(x)=f(-x)$.

Limit Relations

Continuous Bannai-Ito → First type Continuous -1 Hahn

The first type Continuous -1 Hahn polynomials can be obtained from the Continuous Bannai-Ito polynomials by a specialization:

$$\mathbf{Q}_n\left(x;\alpha,\beta,\gamma,\beta\right)=\mathbf{K}_n^{(1)}\left(x;\alpha,\beta,\gamma\right). \tag{A.4.16}$$

Continuous q-Hahn → Continuous -1 Hahn type 1

The Continuous -1 Hahn type 1 polynomials are obtained from the monic continuous q-Hahn polynomials by taking the parametrization $q=-e^{\varepsilon}$, $a=e^{\varepsilon(2\alpha+1)}$, $b=e^{\varepsilon(2\gamma+1)}$ and $\phi=\frac{\pi}{2}+2\varepsilon\beta$:

$$\lim_{\varepsilon\to 0}\frac{1}{(1+q)^n}\mathbf{P}_n\left((1+q)\,x;a,b,\phi|q\right)=\mathbf{K}_n^{(1)}\left(x;\alpha,\beta,\gamma\right). \tag{A.4.17}$$

Continuous -1 Hahn → -1 Meixner-Pollaczek

The -1 Meixner-Pollaczek polynomials are obtained from any type of continuous -1 Hahn polynomials by the following limit:

$$\lim_{\gamma\to\infty}\frac{1}{(2\gamma)^{\frac{n}{2}}}\mathbf{K}_n^{(i)}\left(\sqrt{2\gamma}x,\frac{2\alpha-1}{4},\sqrt{\frac{\gamma}{2}}\beta,\gamma\right)=\mathbf{M}_n\left(x;\alpha,\beta\right). \tag{A.4.18}$$

Continuous -1 Hahn → Symmetric Bannai-Ito

The symmetric Bannai-Ito polynomials are obtained from any type of continuous -1 Hahn polynomials by the specialization $\beta=0$:

$$\mathbf{K}_n^{(i)}\left(x,\alpha,0,\gamma\right)=\hat{\mathbf{S}}_n\left(x;\alpha,\gamma\right). \tag{A.4.19}$$

A.5. Continuous -1 Hahn type 2

Hypergeometric Representation

$$\frac{\mathbf{K}_{2n}^{(2)}\left(x;\alpha,\beta,\gamma\right)}{(-2i)^{2n}}= \tag{A.5.1}$$

$$\xi_{2n}\kappa_{n-1}^{(2)}\left(\frac{ix}{2}-\mathfrak{b}-\frac{1}{2}\right){}_4F_3\left(\begin{matrix}-n+1,\ n+g+2,\ \frac{ix}{2}+\mathfrak{b}+1,\ \frac{-ix}{2}+\mathfrak{b}+\frac{3}{2}\\ \frac{5}{2}+\mathfrak{a}+\mathfrak{b},\ 2+\mathfrak{b}+\mathfrak{c},\ 2+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right)+$$

$$\kappa_n^{(1)}{}_4F_3\left(\begin{matrix}-n,\ n+g+1,\ \frac{ix}{2}+\mathfrak{b},\ \frac{-ix}{2}+\mathfrak{b}+\frac{1}{2}\\ \frac{3}{2}+\mathfrak{a}+\mathfrak{b},\ 1+\mathfrak{b}+\mathfrak{c},\ 1+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right),$$

$$\frac{\mathbf{K}^{(2)}_{2n+1}\left(x;\alpha,\beta,\gamma\right)}{\left(-2i\right)^{2n+1}} = \tag{A.5.2}$$

$$\kappa^{(2)}_n\left(\frac{ix}{2}-\mathfrak{b}-\frac{1}{2}\right){}_4F_3\left(\begin{matrix}-n,\ n+g+2,\ \frac{ix}{2}+\mathfrak{b}+1,\ \frac{-ix}{2}+\mathfrak{b}+\frac{3}{2}\\ \frac{5}{2}+\mathfrak{a}+\mathfrak{b},\ 2+\mathfrak{b}+\mathfrak{c},\ 2+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right)+$$

$$\eta_{2n+1}\kappa^{(1)}_n{}_4F_3\left(\begin{matrix}-n,\ n+g+1,\ \frac{ix}{2}+\mathfrak{b},\ \frac{-ix}{2}+\mathfrak{b}+\frac{1}{2}\\ \frac{3}{2}+\mathfrak{a}+\mathfrak{b},\ 1+\mathfrak{b}+\mathfrak{c},\ 1+\mathfrak{b}+\mathfrak{d}\end{matrix};1\right),$$

where

$$g=2\alpha+2\gamma+1,\qquad \mathfrak{a}=\overline{\mathfrak{d}}=\alpha+i\beta,\qquad \mathfrak{b}=\overline{\mathfrak{c}}=\gamma-i\beta, \tag{A.5.3}$$

$$\begin{aligned}\xi_{2n} &= \tfrac{n\left(n+\mathfrak{c}+\mathfrak{d}+\frac{1}{2}\right)}{(2n+g)}, & \kappa^{(1)}_n &= \tfrac{\left(\frac{3}{2}+\mathfrak{a}+\mathfrak{b}\right)_n(1+\mathfrak{b}+\mathfrak{c})_n(1+\mathfrak{b}+\mathfrak{d})_n}{(n+g+1)_n},\\ \eta_{2n+1} &= \tfrac{(n+\mathfrak{b}+\mathfrak{c}+1)(n+\mathfrak{b}+\mathfrak{d}+1)}{(2n+g+1)}, & \kappa^{(2)}_n &= \tfrac{\left(\frac{5}{2}+\mathfrak{a}+\mathfrak{b}\right)_n(2+\mathfrak{b}+\mathfrak{c})_n(2+\mathfrak{b}+\mathfrak{d})_n}{(n+g+2)_n}.\end{aligned} \tag{A.5.4}$$

Orthogonality Relation

If $\alpha,\beta,\gamma\in\mathbb{R}^+$

$$\int_{-\infty}^{\infty}W(x)\mathbf{K}^{(2)}_n(x)\mathbf{K}^{(2)}_m(x)\mathrm{d}x=4\pi h_0\kappa_n\delta_{nm}, \tag{A.5.5}$$

$$W(x)=\left|\frac{\Gamma(\mathfrak{a}+ix/2+1)\Gamma(\mathfrak{b}+ix/2+1)\Gamma(\mathfrak{c}+ix/2+1/2)\Gamma(\mathfrak{d}+ix/2+1/2)}{\Gamma(1/2+ix)}\right|^2, \tag{A.5.6}$$

$$h_0=\frac{\Gamma(\mathfrak{a}+\mathfrak{b}+3/2)\Gamma(\mathfrak{a}+\mathfrak{c}+1)\Gamma(\mathfrak{b}+\mathfrak{c}+1)\Gamma(\mathfrak{a}+\mathfrak{d}+1)\Gamma(\mathfrak{b}+\mathfrak{d}+1)\Gamma(\mathfrak{c}+\mathfrak{d}+3/2)}{\Gamma(\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\mathfrak{d}+2)}, \tag{A.5.7}$$

$$\kappa_{2n}=\frac{4^{2n}\Gamma\left(n+1\right)\left(2\alpha+1\right)_n\left(2\gamma+1\right)_n}{\left(2\alpha+2\gamma+2\right)_{2n}\left(n+2\alpha+2\gamma+2\right)_n}\left(\alpha+\gamma+\frac{3}{2}\right)^2_n\left[\prod_{k=1}^{n}\|k+\alpha+\gamma+2i\beta\|\right]^2, \tag{A.5.8}$$

$$\kappa_{2n+1}=\frac{4^{2n+1}\Gamma\left(n+1\right)\left(2\alpha+1\right)_{n+1}\left(2\gamma+1\right)_{n+1}}{\left(2\alpha+2\gamma+2\right)_{2n+1}\left(n+2\alpha+2\gamma+2\right)_{n+1}}\left(\alpha+\gamma+\frac{3}{2}\right)^2_n\left[\prod_{k=1}^{n+1}\|k+\alpha+\gamma+2i\beta\|\right]^2. \tag{A.5.9}$$

Normalized Recurrence Relation

$$x\mathbf{K}^{(2)}_n(x)=\mathbf{K}^{(2)}_{n+1}(x)+b_n\mathbf{K}^{(2)}_n(x)+u_n\mathbf{K}^{(2)}_{n-1}(x), \tag{A.5.10}$$

$$b_n = \begin{cases} 2\beta - 2\beta\frac{(n+4\alpha+2)}{(n+2\alpha+2\gamma+2)} & n \text{ even} \\ 2\beta - 2\beta\frac{(n+4\gamma+1)}{n+2\alpha+2\gamma+1} & n \text{ odd} \end{cases}, \tag{A.5.11}$$

$$u_n = \begin{cases} \frac{n(n+4\alpha+4\gamma+2)\|n+2\alpha+2\gamma+1\|^2}{4(n+2\alpha+2\gamma+1)^2} & n \text{ even} \\ \frac{(n+4\alpha+1)(n+4\gamma+1)\|n+2\alpha+2\gamma+4i\beta+1\|^2}{4(n+2\alpha+2\gamma+1)^2} & n \text{ odd} \end{cases}. \tag{A.5.12}$$

Difference Equation

$$L\mathbf{K}_n^{(2)}(x) = \lambda_n \mathbf{K}_n^{(2)}(x), \quad \lambda_n = (-1)^n(n + 2\alpha + 2\gamma + 3/2), \tag{A.5.13}$$

$$L = \mathcal{A}\left(S^+R - I\right) + \overline{\mathcal{A}}\left(S^-R - I\right) + (2\alpha + 2\gamma + 3/2)I, \tag{A.5.14}$$

$$\mathcal{A} = \frac{(2\alpha + 1 + i\,[\beta - x])\,(2\gamma + 1 - i\,[\beta + x])}{1 - 2ix}, \tag{A.5.15}$$

where $\overline{\mathcal{A}}$ is the complex conjugate of $\mathcal{A}$, $S^{\pm}f(x) = f(x \pm i)$ and $Rf(x) = f(-x)$.

Limit Relations

Continuous Bannai-Ito → Continuous -1 Hahn Type 2

The second type of Continuous -1 Hahn polynomials can be obtained from the Continuous Bannai-Ito polynomials by a specialization:

$$\mathbf{Q}_n\left(x;\alpha,\beta,\gamma,-\beta\right) = \mathbf{K}_n^{(2)}\left(x;\alpha,\beta,\gamma\right). \tag{A.5.16}$$

Continuous q-Hahn → Continuous -1 Hahn type 2

The Continuous -1 Hahn type 2 polynomials are obtained from the monic continuous q-Hahn polynomials by taking the parametrization $q = -e^{\varepsilon}$, $a = e^{\varepsilon(2\alpha+1)}$, $b = -e^{\varepsilon(2\gamma+1)}$ and $\phi = \frac{\pi}{2} + 2\varepsilon\beta$:

$$\lim_{\varepsilon\to 0} \frac{1}{(1+q)^n}\mathbf{P}_n\left((1+q)\,x;a,b,\phi|q\right) = \mathbf{K}_n^{(2)}\left(x;\alpha,\beta,\gamma\right). \tag{A.5.17}$$

Continuous -1 Hahn → -1 Meixner-Pollaczek

The -1 Meixner-Pollaczek polynomials are obtained from any type of continuous -1 Hahn polynomials by the following limit:

$$\lim_{\gamma\to\infty} \frac{1}{(2\gamma)^{\frac{n}{2}}}\mathbf{K}_n^{(i)}\left(\sqrt{2\gamma}x,\frac{2\alpha-1}{4},\sqrt{\frac{\gamma}{2}}\beta,\gamma\right) = \mathbf{M}_n\left(x;\alpha,\beta\right). \tag{A.5.18}$$

Continuous -1 Hahn → Symmetric Bannai-Ito

The symmetric Bannai-Ito polynomials are obtained from any type of continuous -1 Hahn polynomials by the specialization $\beta = 0$:

$$\mathbf{K}_n^{(i)}\left(x,\alpha,0,\gamma\right) = \hat{\mathbf{S}}_n\left(x;\alpha,\gamma\right). \tag{A.5.19}$$

A.6. Generalized Symmetric Bannai-Ito

Hypergeometric Representation

$$\hat{\mathbf{I}}_{2n}\left(x;a,b,c\right) = \eta_{2n}{}_4F_3\left(\begin{matrix}-n,\ n+a+b+c-1,\ ix,\ -ix\\ a,\ b,\ c\end{matrix};1\right) \quad , \tag{A.6.1}$$

$$\hat{\mathbf{I}}_{2n+1}\left(x;a,b,c\right) = \eta_{2n+1}x{}_4F_3\left(\begin{matrix}-n,\ n+a+b+c,\ 1+ix,\ 1-ix\\ 1+a,\ 1+b,\ 1+c\end{matrix};1\right), \tag{A.6.2}$$

where

$$\eta_{2n} = \frac{(-1)^n\left(a\right)_n\left(b\right)_n\left(c\right)_n}{\left(n+a+b+c-1\right)_n}, \tag{A.6.3}$$

$$\eta_{2n+1} = \frac{(-1)^n\left(1+a\right)_n\left(1+b\right)_n\left(1+c\right)_n}{\left(n+a+b+c\right)_n}. \tag{A.6.4}$$

Orthogonality Relation

If $\mathrm{Re}(a,b,c) > 0$, $a+b+c > 1$ and non-real parameters occur in conjugate pairs, then

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\omega(x)\hat{\mathbf{I}}_n(x)\hat{\mathbf{I}}_m(x)\mathrm{d}x = \kappa_n\delta_{nm}, \tag{A.6.5}$$

$$\omega(x) = \left|\frac{\Gamma\left(ix\right)\Gamma\left(a+ix\right)\Gamma\left(b+ix\right)\Gamma\left(c+ix\right)}{\Gamma\left(2ix\right)}\right|^2, \tag{A.6.6}$$

$$\kappa_n = \frac{\Gamma\left(n+a+b\right)\Gamma\left(n+a+c\right)\Gamma\left(n+b+c\right)\Gamma\left(n+a\right)\Gamma\left(n+b\right)\Gamma\left(n+c\right)n!}{\Gamma\left(2n+a+b+c\right)\left(n+a+b+c-1\right)_n}. \tag{A.6.7}$$

Normalized Recurrence Relation

$$x\tilde{\mathbf{I}}_n(x) = \tilde{\mathbf{I}}_{n+1}(x) + \tau_n\tilde{\mathbf{I}}_{n-1}(x), \tag{A.6.8}$$

$$\tau_{2n} = \frac{n(n+a+b-1)(n+a+c-1)(n+b+c-1)}{(2n+a+b+c-2)(2n+a+b+c-1)}, \qquad \tau_{2n+1} = \frac{(n+a+b+c-1)(n+c)(n+a)(n+b)}{(2n+a+b+c-1)(2n+a+b+c)}. \tag{A.6.9}$$

Difference Equation

$$D_\sigma \hat{\mathbf{I}}_n(x) = \Lambda_n^{(\sigma)} \hat{\mathbf{I}}_n(x), \tag{A.6.10}$$

$$\Lambda_{2n}^{(\sigma)} = n^2 + (a+b+c-1)\,n, \qquad \Lambda_{2n+1}^{(\sigma)} = n^2 + (a+b+c)\,n + \sigma, \tag{A.6.11}$$

$$D_\sigma = D_0 + \frac{\sigma}{2}\,(I - R)\,, \tag{A.6.12}$$

$$D_0 = B(x)S^+ + A(x)S^- + C(x)R - (A(x) + B(x) + C(x))\,I, \tag{A.6.13}$$

$$A(x) = \frac{(ix+a)\,(ix+b)\,(ix+c)}{2\,(2ix+1)}, \tag{A.6.14}$$

$$B(x) = \frac{(ix-a)\,(ix-b)\,(ix-c)}{2\,(2ix-1)}, \tag{A.6.15}$$

$$C(x) = \frac{1}{2}\left(ab + ac + bc - x^2\right) - A(x) - B(x). \tag{A.6.16}$$

Limit Relations

Generalized Symmetric Bannai-Ito → Symmetric Bannai-Ito

The symmetric Bannai-Ito polynomials can be obtained from the generalized symmetric Bannai-Ito polynomials by taking the limit $c \to \infty$:

$$\lim_{c\to\infty} \hat{\mathbf{I}}_n\,(x,a,b,c) = \hat{\mathbf{S}}_n\,(x;a,b)\,. \tag{A.6.17}$$

Generalized Symmetric Bannai-Ito → Generalized Gegenbauer

The generalized Gegenbauer polynomials can be obtained from the generalized symmetric Bannai-Ito polynomials by taking the limit:

$$\lim_{h\to\infty} \frac{1}{h^n} \hat{\mathbf{I}}_n \left(hx, \frac{\beta+1}{2} + ih, \frac{\beta+1}{2} - ih, \alpha+1\right) = \mathbf{G}_n\,(x;\alpha,\beta)\,. \tag{A.6.18}$$

A.7. Little -1 Jacobi

Hypergeometric Representation

$$\frac{\mathbf{P}_{2n}\left(x;\alpha,\beta\right)}{\eta_{2n}} = {}_2F_1\left(\begin{matrix}-n, & \frac{2n+\alpha+\beta+2}{2} \\ & \frac{\alpha+1}{2}\end{matrix};x^2\right) + \frac{2nx}{1+\alpha}{}_2F_1\left(\begin{matrix}1-n, & \frac{2n+\alpha+\beta+2}{2} \\ & \frac{\alpha+3}{2}\end{matrix};x^2\right), \tag{A.7.1}$$

$$\frac{\mathbf{P}_{2n+1}\left(x;\alpha,\beta\right)}{\eta_{2n+1}} = {}_2F_1\left(\begin{matrix}-n, & \frac{n+\alpha+\beta+1}{2} \\ & \frac{\alpha+1}{2}\end{matrix};x^2\right) - \frac{(2n+\alpha+\beta+2)x}{1+\alpha}{}_2F_1\left(\begin{matrix}-n, & \frac{2n+\alpha+\beta+4}{2} \\ & \frac{\alpha+3}{2}\end{matrix};x^2\right), \tag{A.7.2}$$

where

$$\eta_{2n} = \frac{\left(\frac{\alpha+1}{2}\right)_n}{\left(\frac{2n+\alpha+\beta+2}{2}\right)_n}, \qquad \eta_{2n+1} = \frac{\left(\frac{\alpha+1}{2}\right)_{n+1}}{\left(\frac{2n+\alpha+\beta+2}{2}\right)_{n+1}}. \tag{A.7.3}$$

Orthogonality Relation

If $\alpha > 0$ and $\beta > 0$

$$\int_{-1}^{1} \omega(x)\mathbf{P}_n(x)\mathbf{P}_m(x)\mathrm{d}x = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+1\right)}\kappa_n\delta_{nm}, \tag{A.7.4}$$

$$\omega(x) = |x|^{\alpha}\left(1-x^2\right)^{\frac{\beta-1}{2}}\left(1+x\right), \tag{A.7.5}$$

$$\kappa_{2n} = \frac{\Gamma(n+1)\left(\frac{\alpha+1}{2}\right)_n\left(\frac{\beta+1}{2}\right)_n}{\left(1+\frac{\alpha+\beta}{2}\right)_{2n}\left(n+1+\frac{\alpha+\beta}{2}\right)_n}, \qquad \kappa_{2n+1} = \frac{\Gamma(n+1)\left(\frac{\alpha+1}{2}\right)_{n+1}\left(\frac{\beta+1}{2}\right)_{n+1}}{\left(1+\frac{\alpha+\beta}{2}\right)_{2n+1}\left(n+1+\frac{\alpha+\beta}{2}\right)_{n+1}}. \tag{A.7.6}$$

Normalized Recurrence Relation

$$x\mathbf{P}_n(x) = \mathbf{P}_{n+1}(x) + \left(1-A_n-C_n\right)\mathbf{P}_n(x) + A_{n-1}C_n\mathbf{P}_{n-1}(x), \tag{A.7.7}$$

$$A_n = \begin{cases} \frac{n+\beta+1}{2n+\alpha+\beta+2} & n \text{ even} \\ \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+2} & n \text{ odd}\end{cases}, \qquad C_n = \begin{cases} \frac{n}{2n+\alpha+\beta} & n \text{ even} \\ \frac{n+\alpha}{2n+\alpha+\beta} & n \text{ odd}\end{cases}. \tag{A.7.8}$$

Differential Equation

$$L\mathbf{P}_n(x) = \lambda_n\mathbf{P}_n(x), \quad \lambda_n = \begin{cases} -2n & n \text{ even} \\ 2\left(n+\alpha+\beta+1\right) & n \text{ odd}\end{cases}, \tag{A.7.9}$$

$$L = \left(\frac{\left(\alpha+\beta+1\right)x^2-\alpha x}{x^2}\right)\left[R-I\right] + 2\left(1-x\right)\partial_x R. \tag{A.7.10}$$

Limit Relations

Little q-Jacobi → Little -1 Jacobi

The little -1 Jacobi polynomials are obtained from the monic little q-Jacobi polynomials by taking the parametrization $q = -e^{\varepsilon}$, $a = -e^{\varepsilon\alpha}$ and $b = -e^{\varepsilon\beta}$:

$$\lim_{\varepsilon\to 0} \mathbf{P}_n\left(x; a, b|q\right) = \mathbf{P}_n\left(x; \alpha, \beta\right). \tag{A.7.11}$$

Big -1 Jacobi → Little -1 Jacobi

The little -1 Jacobi polynomial can be obtained from the big -1 Jacobi polynomials by taking c to 0:

$$\mathbf{J}_n\left(x; \alpha, \beta, 0\right) = \mathbf{P}_n\left(x; \beta, \alpha\right). \tag{A.7.12}$$

Little -1 Jacobi → Generalized Gegenbauer

The generalized Gegenbauer polynomials are obtained by the Christoffel transformation of the little -1 Jacobi polynomials and vice-versa via the Geronimus transformation:

$$\mathbf{G}_n\left(x\right) = \frac{\mathbf{P}_{n+1}(x) - A_n\mathbf{P}_n(x)}{x-1}, \tag{A.7.13}$$

$$\mathbf{P}_n(x) = \mathbf{G}_n\left(x\right) - C_n\mathbf{G}_{n-1}\left(x\right), \tag{A.7.14}$$

where A_n and C_n are given by (A.7.8).

$$\mathbf{P}_n(x; \alpha, \beta) \underset{\text{CT}}{\overset{\text{GT}}{\leftrightarrows}} \mathbf{G}_n\left(x; \frac{\alpha-1}{2}, \frac{\beta+1}{2}\right). \tag{A.7.15}$$

Little -1 Jacobi → Special Little -1 Jacobi

The special little -1 Jacobi polynomial can be obtained from the little -1 Jacobi polynomials by taking α to 0:

$$\mathbf{P}_n\left(x; 0, \beta\right) = \mathbf{P}_n\left(x; \beta\right). \tag{A.7.16}$$

A.8. Generalized Gegenbauer

Hypergeometric Representation

$$\mathbf{G}_{2n}\left(x;\alpha,\beta\right) = \frac{(-1)^n\,(\alpha+1)_n}{(n+\alpha+\beta+1)_n}\,{}_2F_1\left(\begin{matrix}-n,\ n+\alpha+\beta+1\\ \alpha+1\end{matrix};x^2\right), \tag{A.8.1}$$

$$\mathbf{G}_{2n+1}\left(x;\alpha,\beta\right) = \frac{(-1)^n\,(\alpha+2)_n}{(n+\alpha+\beta+2)_n}\,x\,{}_2F_1\left(\begin{matrix}-n,\ n+\alpha+\beta+2\\ \alpha+2\end{matrix};x^2\right). \tag{A.8.2}$$

Orthogonality Relation

If $\alpha > -1$ and $\beta > 0$

$$\int_{-1}^{1}\omega(x)\mathbf{G}_n(x)\mathbf{G}_m(x)\mathrm{d}x = h_n\delta_{nm}, \tag{A.8.3}$$

$$\omega(x) = |x|^{2\alpha+1}\left(1-x^2\right)^{\beta}, \tag{A.8.4}$$

$$\begin{aligned} h_{2n} &= \tfrac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}\tfrac{n!}{(2n+\alpha+\beta+1)[(n+\alpha+\beta+1)_n]^2},\\ h_{2n+1} &= \tfrac{\Gamma(n+\alpha+2)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)}\tfrac{n!}{(2n+\alpha+\beta+2)[(n+\alpha+\beta+2)_n]^2}. \end{aligned} \tag{A.8.5}$$

Normalized Recurrence Relation

$$x\mathbf{G}_n(x) = \mathbf{G}_{n+1}(x) + \sigma_n\mathbf{G}_{n-1}(x), \tag{A.8.6}$$

$$\sigma_{2n} = \frac{n\,(n+\beta)}{(2n+\alpha+\beta)\,(2n+\alpha+\beta+1)}, \qquad \sigma_{2n+1} = \frac{(n+\alpha+1)\,(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)\,(2n+\alpha+\beta+2)}. \tag{A.8.7}$$

Differential Equation

$$L^{(\varepsilon)}\mathbf{G}_n(x) = \lambda_n^{(\varepsilon)}\mathbf{G}_n(x), \qquad \begin{cases}\lambda_{2n}^{(\varepsilon)} = n^2 + (\alpha+\beta+1)\,n\\ \lambda_{2n+1}^{(\varepsilon)} = n^2 + (\alpha+\beta+2)\,n + \varepsilon\end{cases}, \tag{A.8.8}$$

$$L^{(\varepsilon)} = S(x)\partial_x^2 + U(x)\partial_x + V(x)\left[I - R\right], \tag{A.8.9}$$

$$S(x) = \tfrac{x^2-1}{4}, \quad U(x) = \tfrac{x(\alpha+\beta+3/2)}{2} - \tfrac{\alpha+1/2}{2x}, \quad V(x) = -\tfrac{(\alpha+\beta+3/2)}{4} + \tfrac{\alpha+1/2}{4x^2} + \tfrac{\epsilon}{2}. \tag{A.8.10}$$

Limit Relations

Little q-Jacobi → Generalized Gegenbauer

The generalized Gegenbauer polynomials are obtained from the monic little q-Jacobi polynomials by taking the parametrization $q = -e^{\varepsilon}$, $b = e^{2\varepsilon\beta}$ and $a = -e^{\varepsilon(2\alpha+1)}$:

$$\lim_{\varepsilon\to 0} \mathbf{P}_n\left(x; a,b|q\right) = \mathbf{G}_n\left(x; \alpha,\beta\right). \tag{A.8.11}$$

Chihara → Generalized Gegenbauer

The generalized Gegenbauer polynomials can be obtained from the Chihara polynomials by taking γ to 0:

$$\mathbf{C}_n\left(x; \alpha,\beta,0\right) = \mathbf{G}_n\left(x; \alpha,\beta\right). \tag{A.8.12}$$

Little -1 Jacobi → Generalized Gegenbauer

The generalized Gegenbauer polynomials are obtained by the Christoffel transformation of the little -1 Jacobi polynomials and vice-versa via the Geronimus transformation:

$$\mathbf{G}_n\left(x\right) = \frac{\mathbf{P}_{n+1}(x) - A_n\mathbf{P}_n(x)}{x-1}, \tag{A.8.13}$$

$$\mathbf{P}_n(x) = \mathbf{G}_n\left(x\right) - C_n\mathbf{G}_{n-1}\left(x\right), \tag{A.8.14}$$

where A_n and C_n are given by (A.7.8).

$$\mathbf{P}_n(x;\alpha,\beta) \underset{\text{CT}}{\overset{\text{GT}}{\longleftrightarrow}} \mathbf{G}_n\left(x; \frac{\alpha-1}{2}, \frac{\beta+1}{2}\right). \tag{A.8.15}$$

Generalized Symmetric Bannai-Ito → Generalized Gegenbauer

The generalized Gegenbauer polynomials can be obtained from the generalized symmetric Bannai-Ito polynomials by taking the limit:

$$\lim_{h\to\infty} \frac{1}{h^n}\hat{\mathbf{I}}_n\left(hx, \frac{\beta+1}{2} + ih, \frac{\beta+1}{2} - ih, \alpha+1\right) = \mathbf{G}_n\left(x; \alpha,\beta\right). \tag{A.8.16}$$

Generalized Gegenbauer → Generalized Hermite

The Generalized Hermite polynomials can be obtained from the generalized Gegenbauer polynomials by taking $x \to \beta^{-\frac{1}{2}}x$, $\alpha \to \alpha - \frac{1}{2}$ and letting β go to ∞:

$$\lim_{\beta\to\infty} \beta^{\frac{n}{2}}\mathbf{G}_n\left(\beta^{-\frac{1}{2}}x; \alpha - \frac{1}{2}, \beta\right) = \mathbf{H}_n\left(x; \alpha\right). \tag{A.8.17}$$

Generalized Gegenbauer → Gegenbauer

The Gegenbauer polynomials can be obtained from the generalized Gegenbauer polynomials by taking α to $\frac{-1}{2}$:

$$\mathbf{G}_n\left(x; \frac{-1}{2}, \beta - \frac{1}{2}\right) = \mathbf{G}_n\left(x; \beta\right). \tag{A.8.18}$$

A.9. -1 Meixner Pollaczek

Hypergeometric Representation

$$\mathbf{M}_{2n}\left(x; \alpha, \gamma\right) = (-1)^n \left(\alpha + \frac{1}{2}\right)_n {}_1F_1\left(\begin{matrix} -n \\ \alpha + \frac{1}{2} \end{matrix}; x^2 - \gamma^2\right), \tag{A.9.1}$$

$$\mathbf{M}_{2n+1}\left(x; \alpha, \gamma\right) = (-1)^n \left(\alpha + \frac{3}{2}\right)_n (x - \gamma)\, {}_1F_1\left(\begin{matrix} -n \\ \alpha + \frac{3}{2} \end{matrix}; x^2 - \gamma^2\right). \tag{A.9.2}$$

Orthogonality Relation

$$\int_{\mathcal{C}} \omega(x)\mathbf{M}_n(x)\mathbf{M}_m(x)\mathrm{d}x = h_n\delta_{nm}, \qquad \mathcal{C} = (-\infty, -|\gamma|] \bigcup [|\gamma|, \infty), \tag{A.9.3}$$

$$\omega(x) = \theta(x)\left(x + \gamma\right)\left(x^2 - \gamma^2\right)^{\alpha - \frac{1}{2}} e^{-x^2}, \tag{A.9.4}$$

$$h_{2n} = n! e^{-\gamma^2}\Gamma\left(n + \alpha + \tfrac{1}{2}\right), \qquad h_{2n+1} = n! e^{-\gamma^2}\Gamma\left(n + \alpha + \tfrac{3}{2}\right). \tag{A.9.5}$$

Normalized Recurrence Relation

$$x\mathbf{M}_n(x) = \mathbf{M}_{n+1}(x) + (-1)^n\gamma\mathbf{M}_n(x) + u_n\mathbf{M}_{n-1}(x), \tag{A.9.6}$$

$$u_{2n} = n, \qquad u_{2n+1} = n + \alpha + \frac{1}{2}. \tag{A.9.7}$$

Differential Equation

$$L^{(\varepsilon)}\mathbf{M}_n(x) = \lambda_n^{(\varepsilon)}\mathbf{M}_n(x), \quad \begin{cases} \lambda_{2n}^{(\varepsilon)} = n \\ \lambda_{2n+1}^{(\varepsilon)} = n + \varepsilon \end{cases}, \tag{A.9.8}$$

$$L^{(\varepsilon)} = S(x)\partial_x^2 - T(x)\partial_x R + U(x)\partial_x + V(x)\left[I - R\right], \tag{A.9.9}$$

$$\begin{gathered} S\left(x\right) = \tfrac{\gamma^2 - x^2}{4x^2}, \quad T\left(x\right) = \tfrac{\gamma(x-\gamma)}{4x^3}, \\ U\left(x\right) = \tfrac{x}{2} + \tfrac{\gamma}{4x^2} - \tfrac{\gamma^2}{2x^3} - \tfrac{\alpha+\gamma^2}{2x}, \quad V\left(x\right) = \tfrac{3\gamma^2}{8x^4} - \tfrac{\gamma}{4x^3} + \tfrac{\alpha+\gamma^2}{4x^2} + \epsilon\tfrac{x-\gamma}{2x} - \tfrac{1}{4}. \end{gathered} \tag{A.9.10}$$

Limit Relations

q-Meixner-Pollaczek → -1 Meixner-Pollaczek

The -1 Meixner-Pollaczek polynomials are obtained from the monic q-Meixner-Pollaczek polynomials by taking the parametrization $x \to \sqrt{1+q}x$, $q = -e^{\varepsilon}$, $a = -e^{\varepsilon\left(\alpha+\frac{1}{2}\right)}$ and $\phi = \frac{\pi}{2} + \sqrt{\varepsilon}\gamma$:

$$\lim_{\varepsilon\to 0}\mathbf{P}_n\left(x; a,\phi|q\right) = \mathbf{M}_n\left(x;\alpha,\gamma\right). \tag{A.9.11}$$

Chihara → -1 Meixner-Pollaczek

The -1 Meixner-Pollaczek polynomials can be obtained from the Chihara polynomials by taking $x \to \beta^{-\frac{1}{2}}x$, $\alpha \to \alpha - \frac{1}{2}$ and $\gamma \to \beta^{-\frac{1}{2}}\gamma$ and letting β go to ∞:

$$\lim_{\beta\to\infty}\mathbf{C}_n\left(\beta^{-\frac{1}{2}}x; \alpha - \frac{1}{2},\beta,\beta^{-\frac{1}{2}}\gamma\right) = \mathbf{M}_n\left(x;\alpha,\gamma\right). \tag{A.9.12}$$

Continuous -1 Hahn → -1 Meixner-Pollaczek

The -1 Meixner-Pollaczek polynomials are obtained from any type of continuous -1 Hahn polynomials by the following limit:

$$\lim_{\gamma\to\infty}\frac{1}{(2\gamma)^{\frac{n}{2}}}\mathbf{K}_n^{(i)}\left(\sqrt{2\gamma}x, \frac{2\alpha-1}{4},\sqrt{\frac{\gamma}{2}}\beta,\gamma\right) = \mathbf{M}_n\left(x;\alpha,\beta\right) \qquad i = 1,2. \tag{A.9.13}$$

-1 Meixner-Pollaczek → Generalized Hermite

The generalized Hermite polynomials can be obtained from the -1 Meixner-Pollaczek polynomials by taking γ to 0:

$$\mathbf{M}_n\left(x;\alpha,0\right) = \mathbf{H}_n\left(x;\alpha\right). \tag{A.9.14}$$

A.10. Symmetric Bannai-Ito

Hypergeometric Representation

$$\hat{\mathbf{S}}_{2n}\left(x;a,b\right)=\eta_{2n}{}_3F_2\left(\begin{matrix}-n,\ ix,\ -ix\\ a,\ b\end{matrix};1\right), \tag{A.10.1}$$

$$\hat{\mathbf{S}}_{2n+1}\left(x;a,b\right)=\eta_{2n+1}x\,{}_3F_2\left(\begin{matrix}-n,\ 1+ix,\ 1-ix\\ 1+a,\ 1+b\end{matrix};1\right), \tag{A.10.2}$$

where

$$\eta_{2n}=\left(-1\right)^n\left(a\right)_n\left(b\right)_n, \qquad \eta_{2n+1}=\left(-1\right)^n\left(1+a\right)_n\left(1+b\right)_n. \tag{A.10.3}$$

Orthogonality Relation

If Re$(a,b)>0$ and non-real parameters occur in conjugate pairs, then

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\omega(x)\hat{\mathbf{S}}_n(x)\hat{\mathbf{S}}_m(x)\mathrm{d}x=\kappa_n\delta_{nm}, \tag{A.10.4}$$

$$\omega(x)=\left|\frac{\Gamma\left(ix\right)\Gamma\left(a+ix\right)\Gamma\left(b+ix\right)}{\Gamma\left(2ix\right)}\right|^2, \tag{A.10.5}$$

$$\kappa_n=\Gamma\left(n+a+b\right)\Gamma\left(n+a\right)\Gamma\left(n+b\right)n!. \tag{A.10.6}$$

Normalized Recurrence Relation

$$x\tilde{\mathbf{S}}_n(x)=\tilde{\mathbf{S}}_{n+1}(x)+\tau_n\tilde{\mathbf{S}}_{n-1}(x), \tag{A.10.7}$$

$$\tau_{2n}=n\left(n+a+b-1\right), \qquad \tau_{2n+1}=\left(n+a\right)\left(n+b\right). \tag{A.10.8}$$

Difference Equation

$$D_\sigma \hat{\mathbf{S}}_n(x) = \Lambda_n^{(\sigma)} \hat{\mathbf{S}}_n(x), \tag{A.10.9}$$

$$\Lambda_{2n}^{(\sigma)} = n, \qquad \Lambda_{2n+1}^{(\sigma)} = n + \sigma, \tag{A.10.10}$$

$$D_\sigma = D_0 + \frac{\sigma}{2}\left(I - R\right), \tag{A.10.11}$$

$$D_0 = B(x)S^+ + A(x)S^- + C(x)R - \left(A(x) + B(x) + C(x)\right) I, \tag{A.10.12}$$

$$A(x) = \frac{\left(ix + a\right)\left(ix + b\right)}{2\left(1 + 2ix\right)}, \tag{A.10.13}$$

$$B(x) = \frac{\left(ix - a\right)\left(ix - b\right)}{2\left(1 - 2ix\right)}, \tag{A.10.14}$$

$$C(x) = \frac{a + b}{2} - A(x) - B(x). \tag{A.10.15}$$

Limit Relations

Generalized Symmetric Bannai-Ito → Symmetric Bannai-Ito

The symmetric Bannai-Ito polynomials can be obtained from the generalized symmetric Bannai-Ito polynomials by taking the limit:

$$\lim_{c\to\infty} \hat{\mathbf{I}}_n\left(x,a,b,c\right) = \hat{\mathbf{S}}_n\left(x;a,b\right). \tag{A.10.16}$$

Continuous -1 Hahn → Symmetric Bannai-Ito

The symmetric Bannai-Ito polynomials are obtained from any type of continuous -1 Hahn polynomials by the specialization $\beta = 0$:

$$\mathbf{K}_n^{(i)}\left(x,\alpha,0,\gamma\right) = \hat{\mathbf{S}}_n\left(x;\alpha,\gamma\right). \tag{A.10.17}$$

Symmetric Bannai-Ito → Generalized Hermite

The generalized Hermite polynomials can be obtained from the Symmetric Bannai-Ito polynomials by taking $x \to \sqrt{\beta}x$, renormalizing and letting b go to ∞:

$$\lim_{b\to\infty} \frac{1}{b^{\frac{n}{2}}} \hat{\mathbf{S}}_n\left(\sqrt{b}x;\alpha + \frac{1}{2},b\right) = \mathbf{H}_n\left(x;\alpha\right). \tag{A.10.18}$$

A.11. Special Little -1 Jacobi

Hypergeometric Representation

$$\frac{\mathbf{P}_{2n}\left(x;\alpha\right)}{\eta_{2n}} = {}_2F_1\left(\begin{matrix}-n, & \frac{2n+\alpha+2}{2} \\ & \frac{1}{2}\end{matrix}; x^2\right) + 2nx{}_2F_1\left(\begin{matrix}1-n, & \frac{2n+\alpha+2}{2} \\ & \frac{3}{2}\end{matrix}; x^2\right), \tag{A.11.1}$$

$$\frac{\mathbf{P}_{2n+1}\left(x;\alpha\right)}{\eta_{2n+1}} = {}_2F_1\left(\begin{matrix}-n, & \frac{2n+\alpha+2}{2} \\ & \frac{1}{2}\end{matrix}; x^2\right) - (2n+\alpha+2)x{}_2F_1\left(\begin{matrix}-n, & \frac{n+\alpha+3}{2} \\ & \frac{3}{2}\end{matrix}; x^2\right), \tag{A.11.2}$$

where

$$k_{2n} = \frac{\left(\frac{1}{2}\right)_n}{\left(\frac{2n+\alpha+2}{2}\right)_n}, \qquad k_{2n+1} = \frac{\left(\frac{1}{2}\right)_{n+1}}{\left(\frac{2n+\alpha+2}{2}\right)_{n+1}}. \tag{A.11.3}$$

Orthogonality Relation

If $\alpha > 0$

$$\int_{-1}^{1} \omega(x)\mathbf{P}_n(x)\mathbf{P}_m(x)\mathrm{d}x = \left(\frac{\sqrt{\pi}\Gamma\left(\frac{\alpha+1}{2}\right)}{2^{2n}}\right)\left(\frac{\Gamma\left(n+1\right)\Gamma\left(n+1+\alpha\right)}{\Gamma\left(n+1+\frac{\alpha}{2}\right)^2}\right)\frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(1+\alpha\right)}\delta_{nm}, \tag{A.11.4}$$

$$\omega(x) = \left(1-x^2\right)^{\frac{\alpha-1}{2}}\left(1+x\right). \tag{A.11.5}$$

Normalized Recurrence Relation

$$x\mathbf{P}_n(x) = \mathbf{P}_{n+1}(x) + \left(1-A_n-C_n\right)\mathbf{P}_n(x) + A_{n-1}C_n\mathbf{P}_{n-1}(x), \tag{A.11.6}$$

$$A_n = \frac{n+\alpha+1}{2n+\alpha+2}, \qquad C_n = \frac{n}{2n+\alpha}. \tag{A.11.7}$$

Differential Equation

$$L\mathbf{P}_n(x) = \lambda_n\mathbf{P}_n(x), \quad \lambda_n = \begin{cases} -2n & n \text{ even} \\ 2\left(n+\alpha+1\right) & n \text{ odd}\end{cases}, \tag{A.11.8}$$

$$L = (\alpha+1)\left[R-I\right] + 2\left(1-x\right)\partial_x R. \tag{A.11.9}$$

Limit Relations

Little -1 Jacobi → Special Little -1 Jacobi

The special little -1 Jacobi polynomial can be obtained from the little -1 Jacobi polynomials by taking α to 0:

$$\mathbf{P}_n\left(x;0,\beta\right) = \mathbf{P}_n\left(x;\beta\right). \tag{A.11.10}$$

Special Little -1 Jacobi → Gegenbauer

The Gegenbauer polynomials are obtained by the Christoffel transformation of the special little -1 Jacobi polynomials and vice-versa via the Geronimus transformation:

$$\mathbf{G}_n\left(x\right) = \frac{\mathbf{P}_{n+1}(x) - A_n\mathbf{P}_n(x)}{x-1}, \tag{A.11.11}$$

$$\mathbf{P}_n(x) = \mathbf{G}_n\left(x\right) - C_n\mathbf{G}_{n-1}\left(x\right), \tag{A.11.12}$$

where A_n and C_n are given by (A.11.7).

$$\mathbf{P}_n(x;\alpha) \underset{\text{CT}}{\overset{\text{GT}}{\longleftrightarrow}} \mathbf{G}_n\left(x;\frac{\alpha+2}{2}\right). \tag{A.11.13}$$

A.12. Gegenbauer

Hypergeometric Representation

$$\mathbf{G}_{2n}\left(x;\alpha\right) = \frac{(-1)^n\left(\frac{1}{2}\right)_n}{\left(n+\alpha\right)_n}\,{}_2F_1\left(\begin{matrix}-n, & n+\alpha\\ & \frac{1}{2}\end{matrix};x^2\right), \tag{A.12.1}$$

$$\mathbf{G}_{2n+1}\left(x;\alpha\right) = \frac{(-1)^n\left(\frac{3}{2}\right)_n}{\left(n+\alpha+1\right)_n}x\,{}_2F_1\left(\begin{matrix}-n, & n+\alpha+1\\ & \frac{3}{2}\end{matrix};x^2\right). \tag{A.12.2}$$

Orthogonality Relation

$$\int_{\mathcal{C}}\omega(x)\mathbf{G}_n(x)\mathbf{G}_m(x)\mathrm{d}x = h_n\delta_{nm}, \qquad \mathcal{C} = [-1,1], \tag{A.12.3}$$

$$\omega(x) = \left(1-x^2\right)^{\alpha-\frac{1}{2}}, \tag{A.12.4}$$

$$\begin{aligned} h_{2n} &= \frac{\Gamma(n+\frac{1}{2})\Gamma(n+\alpha+\frac{1}{2})}{\Gamma(n+\alpha)}\frac{n!}{(2n+\alpha)[(n+\alpha)_n]^2},\\ h_{2n+1} &= \frac{\Gamma(n+\frac{3}{2})\Gamma(n+\alpha+\frac{1}{2})}{\Gamma(n+\alpha+1)}\frac{n!}{(2n+\alpha+1)[(n+\alpha+1)_n]^2}.\end{aligned} \tag{A.12.5}$$

Normalized Recurrence Relation

$$x\mathbf{G}_n(x) = \mathbf{G}_{n+1}(x) + \sigma_n \mathbf{G}_{n-1}(x), \tag{A.12.6}$$

$$\sigma_n = \frac{n\,(n+2\alpha-1)}{(2n+2\alpha-2)\,(2n+2\alpha)}. \tag{A.12.7}$$

Differential Equation

$$L^{(\varepsilon)}\mathbf{G}_n(x) = \lambda_n^{(\varepsilon)}\mathbf{G}_n(x), \quad \begin{cases} \lambda_{2n}^{(\varepsilon)} = n^2 + \alpha n \\ \lambda_{2n+1}^{(\varepsilon)} = n^2 + (\alpha+1)\,n + \varepsilon \end{cases}, \tag{A.12.8}$$

$$L^{(\varepsilon)} = S(x)\partial_x^2 + U(x)\partial_x + V(x)\,[I-R]\,, \tag{A.12.9}$$

$$S(x) = \tfrac{x^2-1}{4}, \quad U(x) = \tfrac{x(\alpha+1/2)}{2}, \quad V(x) = -\tfrac{(\alpha+1/2)}{4} + \tfrac{\epsilon}{2}. \tag{A.12.10}$$

Limit Relations

Generalized Gegenbauer → Gegenbauer

The Gegenbauer polynomials can be obtained from the generalized Gegenbauer polynomials by taking α to $\frac{-1}{2}$:

$$\mathbf{G}_n\left(x; \frac{-1}{2}, \beta - \frac{1}{2}\right) = \mathbf{G}_n\,(x;\beta)\,. \tag{A.12.11}$$

Special Little -1 Jacobi → Gegenbauer

The Gegenbauer polynomials are obtained by the Christoffel transformation of the special little -1 Jacobi polynomials and vice-versa via the Geronimus transformation:

$$\mathbf{G}_n\,(x) = \frac{\mathbf{P}_{n+1}(x) - A_n\mathbf{P}_n(x)}{x-1}, \tag{A.12.12}$$

$$\mathbf{P}_n(x) = \mathbf{G}_n\,(x) - C_n\mathbf{G}_{n-1}\,(x)\,, \tag{A.12.13}$$

where A_n and C_n are given by (A.11.7).

$$\mathbf{P}_n(x;\alpha) \underset{\text{CT}}{\overset{\text{GT}}{\leftrightarrows}} \mathbf{G}_n\left(x; \frac{\alpha+2}{2}\right). \tag{A.12.14}$$

Gegenbauer → Hermite

The Hermite polynomials can be obtained from the Gegenbauer polynomials by taking $x \to \beta^{-\frac{1}{2}}x$ and letting β go to ∞:

$$\lim_{\alpha\to\infty} \alpha^{\frac{n}{2}} \mathbf{G}_n\left(\alpha^{-\frac{1}{2}}x;\alpha\right) = \mathbf{H}_n\left(x\right). \tag{A.12.15}$$

A.13. Generalized Hermite

Hypergeometric Representation

$$\mathbf{H}_{2n}\left(x;\alpha\right) = (-1)^n \left(\alpha+\frac{1}{2}\right)_n {}_1F_1\left(\begin{matrix}-n\\ \alpha+\frac{1}{2}\end{matrix};x^2\right), \tag{A.13.1}$$

$$\mathbf{H}_{2n+1}\left(x;\alpha\right) = (-1)^n \left(\alpha+\frac{3}{2}\right)_n x {}_1F_1\left(\begin{matrix}-n\\ \alpha+\frac{3}{2}\end{matrix};x^2\right). \tag{A.13.2}$$

Orthogonality Relation

$$\int_{\mathcal{C}} \omega(x)\mathbf{H}_n(x)\mathbf{H}_m(x)\mathrm{d}x = h_n\delta_{nm}, \qquad \mathcal{C} = (-\infty,\infty), \tag{A.13.3}$$

$$\omega(x) = |x|^{2\alpha}e^{-x^2}, \tag{A.13.4}$$

$$h_{2n} = n!\Gamma\left(n+\alpha+\tfrac{1}{2}\right), \qquad h_{2n+1} = n!\Gamma\left(n+\alpha+\tfrac{3}{2}\right). \tag{A.13.5}$$

Normalized Recurrence Relation

$$x\mathbf{H}_n(x) = \mathbf{H}_{n+1}(x) + u_n\mathbf{H}_{n-1}(x), \tag{A.13.6}$$

$$u_{2n} = n, \qquad u_{2n+1} = n+\alpha+\frac{1}{2}. \tag{A.13.7}$$

Differential Equation

$$L^{(\varepsilon)}\mathbf{H}_n(x) = \lambda_n^{(\varepsilon)}\mathbf{H}_n(x), \quad \begin{cases} \lambda_{2n}^{(\varepsilon)} = n \\ \lambda_{2n+1}^{(\varepsilon)} = n + \varepsilon \end{cases}, \tag{A.13.8}$$

$$L^{(\varepsilon)} = S(x)\partial_x^2 + U(x)\partial_x + V(x)\left[I - R\right], \tag{A.13.9}$$

$$S\left(x\right) = \tfrac{-1}{4}, \qquad U\left(x\right) = \tfrac{x}{2} - \tfrac{\alpha}{2x}, \qquad V\left(x\right) = \tfrac{\alpha}{4x^2} + \tfrac{\epsilon}{2} - \tfrac{1}{4}. \tag{A.13.10}$$

Limit Relations

Generalized Gegenbauer → Generalized Hermite

The Generalized Hermite polynomials can be obtained from the generalized Gegenbauer polynomials by taking $x \to \beta^{-\frac{1}{2}}x$, $\alpha \to \alpha - \frac{1}{2}$ and letting β go to ∞:

$$\lim_{\beta\to\infty} \beta^{\frac{n}{2}}\mathbf{G}_n\left(\beta^{-\frac{1}{2}}x;\alpha - \frac{1}{2},\beta\right) = \mathbf{H}_n\left(x;\alpha\right). \tag{A.13.11}$$

-1 Meixner-Pollaczek → Generalized Hermite

The generalized Hermite polynomials can be obtained from the -1 Meixner-Pollaczek polynomials by taking γ to 0:

$$\mathbf{M}_n\left(x;\alpha,0\right) = \mathbf{H}_n\left(x;\alpha\right). \tag{A.13.12}$$

Symmetric Bannai-Ito → Generalized Hermite

The generalized Hermite polynomials can be obtained from the Symmetric Bannai-Ito polynomials by taking $x \to \sqrt{\beta}x$, renormalizing and letting b go to ∞:

$$\lim_{b\to\infty} \frac{1}{b^{\frac{n}{2}}}\hat{\mathbf{S}}_n\left(\sqrt{b}x;\alpha + \frac{1}{2},b\right) = \mathbf{H}_n\left(x;\alpha\right). \tag{A.13.13}$$

Generalized Hermite → Hermite

The Hermite polynomials can be obtained from the generalized Hermite polynomials by taking α to 0:

$$\mathbf{H}_n\left(x;0\right) = \mathbf{H}_n\left(x\right). \tag{A.13.14}$$

A.14. Hermite

Hypergeometric Representation

$$\mathbf{H}_{2n}(x) = (-1)^n \left(\frac{1}{2}\right)_n {}_1F_1\left(\begin{matrix}-n\\ \frac{1}{2}\end{matrix}; x^2\right), \tag{A.14.1}$$

$$\mathbf{H}_{2n+1}(x) = (-1)^n \left(\frac{3}{2}\right)_n x\,{}_1F_1\left(\begin{matrix}-n\\ \frac{3}{2}\end{matrix}; x^2\right). \tag{A.14.2}$$

Orthogonality Relation

$$\int_{\mathcal{C}} \omega(x)\mathbf{H}_n(x)\mathbf{H}_m(x)\mathrm{d}x = h_n\delta_{nm}, \qquad \mathcal{C} = (-\infty,\infty), \tag{A.14.3}$$

$$\omega(x) = e^{-x^2}, \tag{A.14.4}$$

$$h_{2n} = n!\Gamma\left(n+\tfrac{1}{2}\right), \qquad h_{2n+1} = n!\Gamma\left(n+\tfrac{3}{2}\right). \tag{A.14.5}$$

Normalized Recurrence Relation

$$x\mathbf{H}_n(x) = \mathbf{H}_{n+1}(x) + u_n\mathbf{H}_{n-1}(x), \tag{A.14.6}$$

$$u_n = \frac{n}{2}. \tag{A.14.7}$$

Differential Equation

$$L^{(\varepsilon)}\mathbf{H}_n(x) = \lambda_n^{(\varepsilon)}\mathbf{H}_n(x), \quad \begin{cases} \lambda_{2n}^{(\varepsilon)} = n \\ \lambda_{2n+1}^{(\varepsilon)} = n+\varepsilon \end{cases}, \tag{A.14.8}$$

$$L^{(\varepsilon)} = S(x)\partial_x^2 + U(x)\partial_x + V(x)\left[I - R\right], \tag{A.14.9}$$

$$S(x) = \tfrac{-1}{4}, \qquad U(x) = \tfrac{x}{2}, \qquad V(x) = \tfrac{\epsilon}{2} - \tfrac{1}{4}. \tag{A.14.10}$$

Limit Relations

Gegenbauer → Hermite

The Hermite polynomials can be obtained from the Gegenbauer polynomials by taking $x \to \beta^{-\frac{1}{2}}x$ and letting β go to ∞:

$$\lim_{\beta\to\infty} \beta^{\frac{n}{2}} \mathbf{G}_n\left(\beta^{-\frac{1}{2}}x;\beta\right) = \mathbf{H}_n\left(x\right). \tag{A.14.11}$$

Generalized Hermite → Hermite

The Hermite polynomials can be obtained from the generalized Hermite polynomials by taking α to 0:

$$\mathbf{H}_n\left(x;0\right) = \mathbf{H}_n\left(x\right). \tag{A.14.12}$$

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