Université de Montréal

GRAPHES EULÉRIENS ET COMPLÉMENTARITÉ LOCALE

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SOMMAIRE

Nous définissons dans cette thèse la notion de complémentation de graphes bicoloriés et les notions de graphe pur et de graphe inversible. L'étude des graphes purs est motivée par deux conjectures concernant les systèmes de transitions de graphes eulériens et par la conjecture de double recouvrement.

L'utilisation de règles de substitution nous permet de déterminer quand deux suites de complémentation donnent le même graphe. Pour les graphes bicoloriés, ces suites de complémentation font place à des ensembles de complémentation.

Les graphes inversibles (les graphes bicoloriés dont l'ensemble des sommets est un ensemble de complémentation) ont ceci de particulier que leur inverse possède les mêmes automorphismes. L'inversibilité se définit aussi pour les graphes non coloriés en les munissant de leur coloriage naturel.

Il est proposé que la caractérisation des graphes purs permettrait de valider la conjecture de double recouvrement. Nous décrivons comment les graphes purs ont des factorisations essentielles en graphes purs primitifs. Les quatre classes de parité connues de graphes purs primitifs sont présentées. Les listings des programmes ayant permis d'établir la pureté de ces graphes sont inclus.

mots clés : complémentarité locale, systèmes de transitions, graphes purs, graphes inversibles, graphes eulériens, double recouvrement, orthogonalité.

SUMMARY

We define pure graphs, invertible graphs, and the notion of complementation of bicoloured graphs. The study of pure graphs is motivated by two conjectures about the transition systems of eulerian graphs and by the Cycle Double Cover Conjecture.

We show how substitution rules can be used to determine when two complementation words produce the same graph. For bicoloured graphs, complementation words give way to complementation sets.

The invertible graphs (bicoloured graphs whose vertex set is a complementation set) are shown to have the property that their inverse has the same automorphisms. The property of being invertible can also be defined for non-coloured graphs by endowing them with their natural colouring.

It is proposed that a characterization of pure graphs would contribute to establish the truth of the Cycle Double Cover Conjecture. We show how pure graphs have essential factorizations into primitive pure graphs. The four primitive pure parity classes are presented. Included are the listings of the programs used to test graphs for purity.

keywords: local complementation, transition systems, pure graphs, invertible graphs, eulerian graphs, double cycle cover, orthogonality.

DÉDICACE

À Marie, que j'admire.

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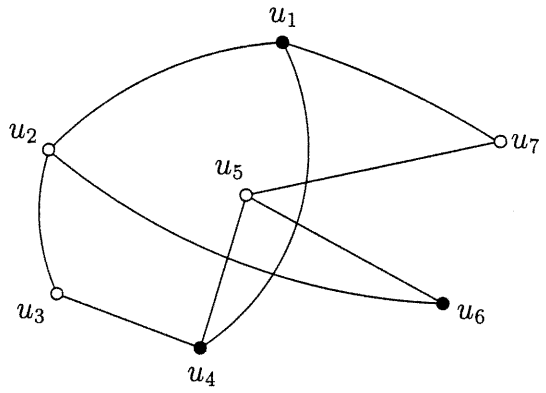
INTRODUCTION

Voici un graphe simple (sans boucles ni arêtes multiples) dont chaque sommet est colorié en blanc ou en noir :

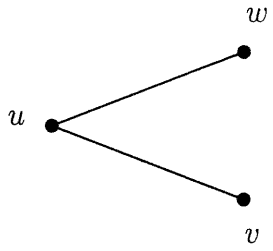


Peut-on trouver un sous-ensemble A des sommets du graphe qui soit un *indépendant* (aucune paire de sommets ne sont adjacents) maximal (tout autre sommet a un voisin dans A) et qui ne contienne que des sommets noirs? Un tel sous-ensemble est appelé une *anticlique noire.*

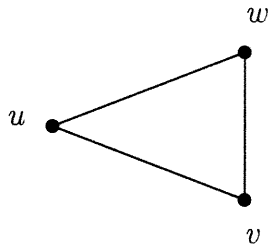
Il est assez facile de se convaincre que, dans l'exemple présenté, le graphe n'admet aucune anticlique noire. En effet, les sommets blancs u_3 et u_7 n'ont chacun qu'un seul sommet noir comme voisin. Toute anticlique noire devrait donc contenir à la fois les sommets u_1 et u_4, mais ceux-ci sont adjacents.

Définissons maintenant un jeu dont le but est encore de trouver une anticlique noire, mais dans lequel il est permis de modifier le graphe de départ en effectuant certaines *complémentations locales.* Complémenter localement un graphe par rapport à un sommet donné, c'est inverser les adjacences entre ses voisins. (c.-à-d. si v et w sont adjacents à u et qu'il y a une arête entre v et w, la complémentation

locale par rapport à u fait disparaître cette arête; s'il n'y en a pas, elle en fait apparaître une). Par exemple, les graphes G et H suivants s'obtiennent l'un de l'autre en complémentant par rapport à u:

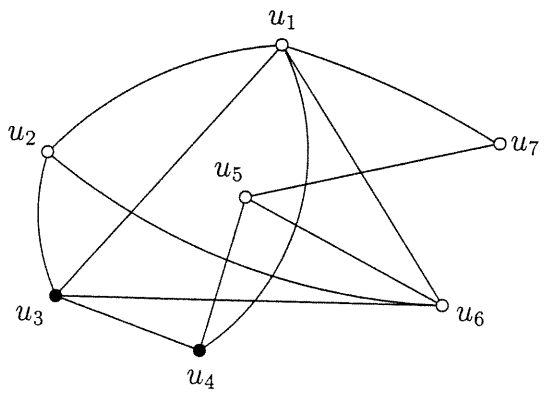


G



H

Dans ce jeu, il est permis de jouer à un sommet blanc, ce qui effectue une complémentation locale par rapport à ce sommet et qui, en plus, inverse la couleur de chacun des voisins. Il est également permis de jouer à une arête incidente avec deux sommets noirs, disons u et v, ce qui produit le même effet que trois complémentations locales successives: d'abord par rapport à u, puis à v et de nouveau à u (ceci est bien défini car inverser les rôles de u et v produit le même résultat). En jouant au sommet u_2 dans l'exemple donné, nous obtenons:



Il ne restera plus qu'à jouer au sommet u_7 pour créer une anticlique noire $A = \{u_3, u_5\}$ dans le graphe résultant.

En permettant ces opérations, l'expérience démontre que le jeu a une solution pour la grande majorité des graphes bicoloriés blanc et noir. C'est-à-dire qu'après

un certain nombre de coups nous avons toutes les chances de faire surgir une anticlique noire. Comme souvent en mathématiques, ce sont les exceptions qui vont nous intéresser : les graphes bicoloriés pour lesquels le jeu ne fait apparaître aucune anticlique noire sont appelés les graphes *purs*.

Il est difficile de croire que ce jeu étrange soit en relation avec les graphes eulériens. C'est pourtant le cas et, lorsqu'on se restreint à un certain type de graphes (les graphes de cordes[1], *circle graphs* en anglais), ce jeu est équivalent à un problème connu. Nous verrons comment l'étude des graphes purs permet de mieux cerner ce problème et jette un éclairage nouveau sur la conjecture de double recouvrement.

Pour mieux manipuler et décrire les graphes qui apparaissent au cours du jeu, il est naturel de se fixer un graphe de départ et d'identifier les autres graphes à l'aide des suites de coups permettant de les obtenir. Dans le premier article, nous chercherons à formaliser ces suites de coups, ce seront les *suites de parité*, et nous verrons comment trouver des suites les plus simples possibles. Ce faisant, nous serons récompensés de notre diligence par un résultat étonnant permettant d'obtenir, à partir de certains graphes que nous appellerons *inversibles*, d'autres graphes ayant les mêmes automorphismes. Dans le second article, nous établirons le lien entre le problème auquel nous faisions allusion plus haut, qui concerne les systèmes de transitions de graphes eulériens, et les graphes purs. Aussi, nous verrons comment les graphes purs ont des *factorisations essentielles* en graphes purs *primitifs*. Enfin, les quatre graphes purs primitifs connus (à transformation par le jeu près) seront présentés.

0.1. DÉFINITIONS

Les définitions les plus importantes sont incluses. Là où elles diffèrent de celles qu'on peut trouver dans la littérature, c'est par souci de concision et de

1. Les graphes de cordes, aussi appelés *graphes d'alternance*, ne doivent pas être confondus aves les graphes *à cordes* (en englais, *chordal graphs*).

clarté dans le cadre du sujet exposé dans cette thèse. Pour les définitions de base qui n'apparaissent pas dans cette section, comme les notions d'isomorphisme de graphes et de connexité, le lecteur peut se référer au livre de Bondy et Murty [**BM**].

Définition 0.1.1. Un *graphe* $G = (V,E)$ est constitué de deux ensembles disjoints V et E dont les éléments sont appelés respectivement les *sommets* et les *arêtes* de G et d'une fonction d'*incidence* qui associe à chaque arête e soit un sommet, auquel cas e est appelée une *boucle*, ou soit une paire non ordonnée de sommets distincts. On dit de ces sommets qu'il sont *incidents* avec l'arête e, ou encore que ce sont les *incidences* de e.

L'*ordre* de G est le nombre de sommets du graphe. Si une arête e a les incidences u et v, on dit que e *relie* ces sommets et que u et v sont *adjacents*. Deux arêtes distinctes ayant une incidence en commun sont également dites *adjacentes*.

Définition 0.1.2. Des arêtes distinctes sont *multiples* si elles ont les mêmes incidences. Un graphe sans boucles ni arêtes multiples est un graphe *simple*.

Dans le cas d'un graphe simple, nous adoptons la convention d'identifier chaque arête avec ses incidences, que nous notons entre crochets (c.-à-d. $[u,v]$ pour l'arête reliant u et v).

Définition 0.1.3. Un *chemin* dans un graphe est une suite $u_0e_1u_1e_2u_2...e_nu_n$ telle que e_i est une arête ayant les incidences u_{i-1} et u_i, $i = 1,...,n$. Le chemin est *fermé* si $u_0 = u_n$.

Définition 0.1.4. Une *chaîne* dans un graphe est une marche dont les sommets sont distincts.

Définition 0.1.5. Un *parcours* dans un graphe est une suite $u_0e_1u_1e_2u_2...e_n$ telle que $u_0e_1u_1e_2u_2...e_nu_0$ est un chemin fermé dont les arêtes sont distinctes.

En général, le début ou la direction du parcours importent peu. Aussi, on dira que e_n et e_1 sont des arêtes successives. Il existe une formalisation des parcours permettant d'éviter de choisir un sommet de départ, ce sont les *permutations*

eulériennes (Sabidussi [**Sa**]). Cependant, dans cette thèse, nous nous limiterons aux parcours tels que définis précédemment.

Définition 0.1.6. Un parcours est *eulérien* s'il contient toutes les arêtes du graphe. Un graphe est *eulérien* s'il admet un parcours eulérien.

Définition 0.1.7. Un graphe $G_1 = (V_1,E_1)$ est un *sous-graphe* du graphe $G_2 = (V_2,E_2)$ si $V_1 \subset V_2$, $E_1 \subset E_2$ et si chaque arête $e \in E_1$ a les mêmes incidences dans G_1 et G_2. C'est un sous-graphe *induit* si toute arête de G_2 ayant ses incidences dans V_1 est dans E_1.

Définition 0.1.8. Le *degré* d'un sommet u est deux fois le nombre de boucles incidentes avec u plus le nombre des autres arêtes incidentes avec u. Un graphe est *d-régulier* (ou simplement *régulier*) si tous ses sommets ont le même degré d. Un sommet de degré nul est dit *isolé*.

Définition 0.1.9. Le sous-graphe de $G = (V,E)$ *induit* par $V' \subset V$ est l'unique sous-graphe induit de G dont l'ensemble de sommets est V'. Le sous-graphe de G *induit* par $E' \subset E$ est le sous-graphe $G' = (V',E')$ où $V' \subset V$ est l'ensemble des incidences des arêtes dans E'.

Définition 0.1.10. Un *cycle* est un graphe non vide, 2-régulier et connexe. Un *cycle* d'un graphe G est un cycle qui est sous-graphe de G. Un *m-cycle* est un cycle d'ordre m (par exemple, un 1-*cycle* est induit par une boucle). Une arête n'appartenant à aucun cycle de G est un *isthme* de G.

Définition 0.1.11. Une *décomposition en cycles (ou en parcours)* d'un graphe G sans sommet isolé est une famille de cycles (parcours) de G telle que chaque arête de G appartient à exactement un de ces cycles (parcours).

0.2. Caractérisation des graphes eulériens

Le théorême élémentaire suivant, dont l'objet est la caractérisation des graphes eulériens, réunit des résultats apparus au cours d'une période de près de deux cents ans, depuis un article d'Euler déposé en 1735 jusqu'à un résultat de Veblen

en 1922. Il est utile de s'y attarder un moment car il restera en filigrane dans le reste de cette thèse.

Théorême 0.2.1. *Soit G un graphe sans sommet isolé et avec un nombre fini de sommets et d'arêtes. Les énoncés suivants sont équivalents:*
(1) G est eulérien;
(2) tous les degrés de G sont pairs et G est connexe;
(3) G a une décomposition en cycles et il est connexe.

DÉMONSTRATION. (1)⇒(2) Soit w un parcours eulérien de G. Comme w relie chaque paire de sommet de G, celui-ci est connexe. Chaque passage du parcours à un sommet u contribue 2 au degré de u (une fois à l'entrée et une fois à la sortie). Donc tous les degrés de G sont pairs.

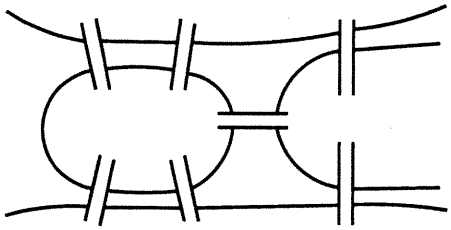
(2)⇒(3) Démontrons l'affirmation plus générale que tout graphe sans sommet isolé dont les degrés sont pairs possède une décomposition en cycles. C'est vrai pour le graphe vide. Soit une chaîne $w = u_0e_1u_1...e_nu_n$ non vide de longueur maximale. Le sommet u_n étant de degré pair, il est incident avec une arête $e_{n+1} \neq e_n$. Si e_{n+1} est une boucle, elle induit un cycle. Sinon, puisque la chaîne est maximale, l'autre incidence de e_{n+1} doit être u_i où $i \in \{0,...,n-1\}$ et dans ce cas, le chemin fermé $u_ie_{i+1}u_{i+1}...e_{n+1}u_i$ coïncide avec un cycle du graphe. En retirant du graphe les arêtes d'un cycle, puis les sommets isolés restants, nous obtenons un graphe dont les degrés sont pairs. Le résultat en découle, par induction sur le nombre d'arêtes.

(3)⇒(1) Si le graphe est vide, le résultat est trivial. Nous allons construire une suite de parcours contenant successivement plus d'arêtes. Soit C_1 un cycle de la décomposition. Choisissons un parcours w de C_1. Si ce parcours ne contient pas toutes les arêtes, puisque G est connexe, il existe un cycle C_2 de la décomposition ayant un sommet en commun avec w. Lors d'un passage du parcours à ce sommet,

on peut interrompre w et parcourir C_2 avant de continuer. De cette façon, on obtient un parcours utilisant les arêtes de C_1 et de C_2. Ce procédé permet de modifier w pour parcourir successivement plus de cycles, jusqu'à ce que tous les cycles aient été parcourus, auquel cas w est eulérien. □

0.3. HISTORIQUE DU PROBLÈME

C'est avec l'article d'Euler sur le problème des ponts de Königsberg que certains associent l'avènement de la théorie des graphes. Il y est question de la cité médiévale aujourd'hui appelée Kaliningrad, située sur la rivière Pregel en Prusse. Euler y demande s'il est possible de trouver un chemin qui traverse chacun des sept ponts une et une seule fois.



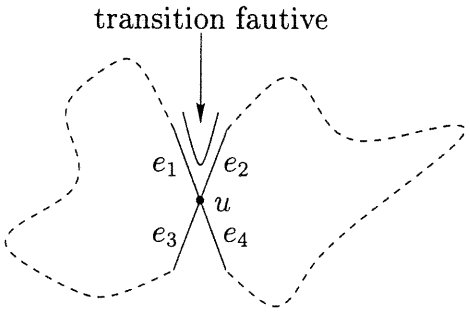
Y a-t-il un chemin traversant les sept ponts de Königsberg?

Pour en donner la réponse, Euler fait une démonstration qui, dans le cas où on demande de revenir au point de départ, se résume à la partie (1)⇒(2) du théorême élémentaire. Pour lui, la réciproque a peu d'intérêt et il faudra attendre en 1871 pour que Hierholzer en fasse la preuve. Une historique intéressante du problème des ponts de Königsberg se retrouve dans Wilson [**W**]. L'équivalence (2)⇔(3) fut d'abord esquissée en 1912 puis établie en 1922 par Veblen [**V1, V2**].

En 1966, Kotzig [**K1**] a introduit la notion d'*orthogonalité* dans les graphes eulériens. Étant donné une décomposition en parcours d'un graphe connexe 4-régulier, il s'est demandé s'il était toujours possible de trouver un parcours eulérien orthogonal à la décomposition, en ce sens qu'aucune paire d'arêtes ne se

succèdent à la fois dans le parcours eulérien et dans un des parcours de la décomposition (nous appelerons des arêtes successives et le sommet entre les deux une *transition* du parcours).

Se limitant aux graphes sans boucles, Kotzig a démontré que oui. Pour le voir, considérons un graphe eulérien dont les degrés sont tous ≥ 4. Choisissons comme point de départ un parcours eulérien w du graphe et modifions ce parcours par étapes, de façon à réduire le nombre de transitions fautives jusqu'à zéro. Soit $\{u,e_1,e_2\}$ une transition fautive de w. Puisque u est de degré ≥ 4, il existe au moins une autre paire d'arêtes incidentes avec u, disons $\{e_3,e_4\}$, qui se succèdent dans le parcours eulérien. À partir de ces deux transitions, on peut modifier le parcours de façon canonique : si possible, on fait se succéder e_1 et e_3 et aussi e_2 et e_4, tout en laissant le reste du parcours inchangé; sinon on fait se succéder e_1 et e_4 et également e_2 et e_3. Il est facile de voir que dans exactement un des deux cas nous obtenons un nouveau parcours eulérien. De plus, le nombre de transitions fautives dans ce parcours aura diminué de 1 ou 2.



Définition 0.3.1. Une *transition* à un sommet u est soit un couple $\{u,e\}$ où e est une boucle incidente avec u ou soit un ensemble $\{u,e_1,e_2\}$ où e_1 et e_2 sont des arêtes distinctes incidentes avec u. Une *transition* d'un parcours w (ou d'un cycle C) à un sommet u est une transition dont les arêtes se succèdent dans w (sont adjacentes dans C). Un *système de transitions* d'un graphe G est l'ensemble des transitions d'une décomposition en parcours ou en cycles de G.

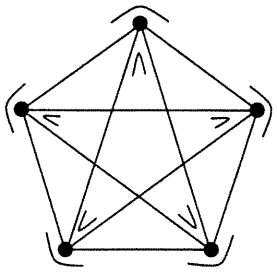
Définition 0.3.2. Deux systèmes de transitions sont *orthogonaux* s'ils n'ont aucune transition en commun. Un parcours eulérien ou une décomposition en cycles sont orthogonaux à un système de transitions donné si les sytèmes de transitions qu'ils induisent le sont.

Le résultat de Kotzig peut donc s'exprimer ainsi : un graphe eulérien G de degré minimal > 2 muni d'un système de transitions S admet un parcours eulérien orthogonal à S si et seulement si pour tout sommet u de degré 4 incident à une boucle e, la transition $\{u,e\}$ est dans S.

Étant donné les différentes caractérisations des graphes eulériens données par le théorême élémentaire, il est naturel de poser la question analogue : quand un graphe eulérien de degré minimal > 2 muni d'un système de transitions S admet-il une décomposition en cycles orthogonale à S?

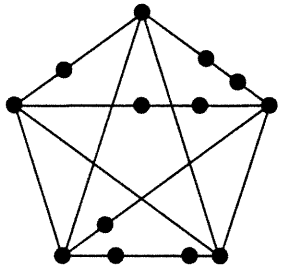
Une condition nécessaire est que S ne contienne aucune transition induite par une boucle (les transitions de type $\{u,e\}$). De plus, tout graphe obtenu en retirant les arêtes e_1 et e_2 d'une transition $\{u,e_1,e_2\} \in S$ doit être connexe. Si ces conditions sont remplies, S est dit *admissible*.

Cependant, il existe des cas où l'on ne peut trouver de décomposition en cycles orthogonale à un système de transitions admissible. Par exemple, le graphe complet sur cinq sommets (K_5) muni d'un système de transitions correspondant à une décomposition en deux 5-cycles :



Un système de transitions sans décomposition en cycles orthogonale.

En 1975, Sabidussi (voir [**Fl1**]) a émis la conjecture qu'une décomposition en cycles orthogonale existe dans le cas où S correspond à un parcours eulérien (conjecture d'orthogonalité). Incidemment, ce sont les travaux qu'il a effectués sur cette conjecture qui ont inspiré les résultats que l'on retrouve dans cette thèse concernant la question plus générale. Une condition suffisante dans le cas général (Fan et Zhang [**FZ**]) est que S soit admissible et que G ne contienne pas de sous-graphe isomorphe à une subdivision de K_5 (une *subdivision* d'un graphe est obtenue en ajoutant des sommets qui «subdivisent» les arêtes du graphe).



Une subdivision de K_5

En particulier, si G est planaire et S est admissible, on peut trouver une décomposition en cycles orthogonale à S.

En ce qui concerne la conjecture d'orthogonalité, Sabidussi [**Sa**] nous dit qu'il suffit de la démontrer pour les graphes bipartis de degrés 4 et 6 (ayant donc un nombre pair de sommets de degré 6). La conjecture est vraie pour les graphes dont les degrés sont divisibles par 4 (Fleischner [**Fl2**], ou voir Jackson [**Jac2**]) et pour les graphes ayant exactement un sommet de degré 6 et les autres de degré 4 (Fleischner [**Fl3**]).

Revenant au théorême de caractérisation des graphes eulériens, on voit qu'il est futile de chercher une décomposition en cycles d'un graphe ayant des sommets de degré impair. Dans ce cas, si on veut recouvrir le graphe par des cycles, il faut

permettre aux arêtes d'apparaître dans plus d'un cycle. C'est peut-être cette réflexion qui est à l'origine de la conjecture de double recouvrement.

Définition 0.3.3. Un *double recouvrement par des cycles* (ou simplement un *double recouvrement*) d'un graphe G est une famille de cycles de G telle que chaque arête de G appartient à exactement deux de ces cycles.

Conjecture 0.3.1 (Conjecture de double recouvrement). *Tout graphe sans isthme possède un double recouvrement.*

La paternité de cette conjecture n'est pas bien établie. Elle est cependant très importante étant donné son lien avec la théorie des flots à valeurs entières et avec les plongements de graphes dans des surfaces (voir Jaeger [**Jae**] et Jackson [**Jac2**]). De plus, cette conjecture est intimement liée au problème d'existence d'une décomposition en cycles orthogonale à un système de transitions. Comme nous le verrons dans le deuxième article, tout contre-exemple à la conjecture de double recouvrement qui serait minimal par rapport au nombre d'arêtes révélerait deux nouveaux graphes purs primitifs (à complémentation près). Fleischner [**Fl3**] a également montré que la conjecture dite «de cycle dominant» et la conjecture d'orthogonalité impliquent, ensemble, la conjecture de double recouvrement.

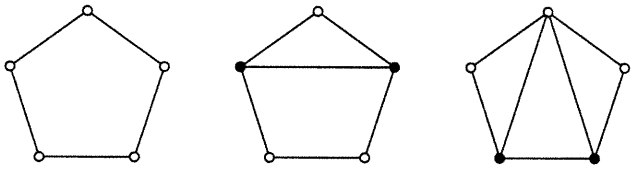
Abordons maintenant le lien entre les systèmes de transitions et le jeu du départ. Étant donné un graphe simple quelconque, colorions en blanc les sommets de degré pair et en noir les sommets impairs. En jouant à un sommet blanc (pair) ou à une arête incidente avec des sommets noirs (impairs), les couleurs du graphe résultant restent en accord avec les parités des sommets, ce qu'on appelle un coloriage *naturel*. À cause de ce rapprochement entre couleur et parité pour une partie des graphes bicoloriés, on appele *classe de parité* de G la famille des graphes bicoloriés pouvant s'obtenir par le jeu à partir d'un graphe bicolorié G.

Sabidussi [**Sa**] a montré que pour chaque graphe eulérien G de degrés 4 et 6 muni d'un parcours eulérien w, il correspond une unique classe de parité de

graphes naturellement coloriés et que G possède une décomposition en cycles orthogonale à w si et seulement si la classe de parité correspondante contient un graphe ayant une anticlique noire. D'où l'intérêt de caractériser les graphes purs.

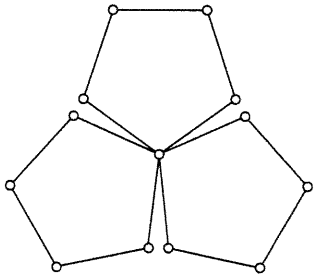
0.4. Les graphes purs

Il est assez facile de vérifier que le pentagone est pur :

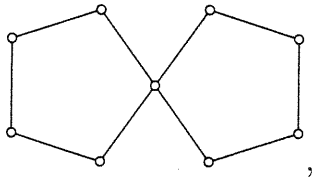


À isomorphisme près, les graphes de la classe de parité du pentagone.

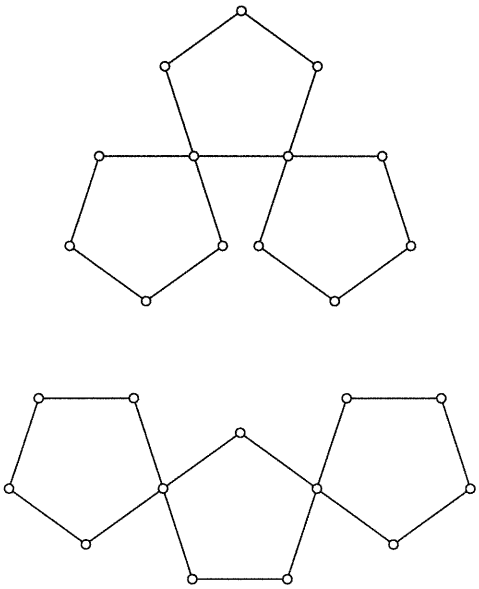
Au moment de commencer mes travaux, les graphes de la classe de parité du pentagone étaient les seuls graphes purs connus non triviaux (un sommet isolé est pur). Sabidussi a alors émis l'hypothèse que, lorsque des pentagones sont identifiés en un sommet, ils forment un graphe pur; comme exemple, voici le trèfle :



Commençant avec le graphe des pentagones siamois :

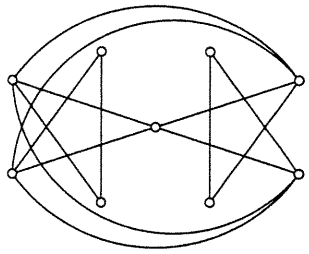


j'ai pu vérifier qu'il était pur en faisant, à la main, la liste exhaustive des graphes non isomorphes de sa classe de parité (cette liste consiste en 60 graphes, voir l'annexe A). La taille de la classe de parité croît apparemment de façon exponentielle au fur et à mesure qu'on augmente le nombre de pentagones : il y a 197 graphes non isomorphes dans la classe du trèfle, ce nombre grimpe à 571 pour quatre pentagones, et pour cinq pentagones, à 1459. De plus, ces classes sont relativement petites : à titre de comparaison, les classes des cycles d'ordre 9, 13, 17 et 21 contiennent respectivement 23, 138, 1034 et 8957 graphes non isomorphes. Déterminer ces classes à la main n'étant pas envisageable, il a fallu mécaniser le procédé et écrire un programme me permettant d'effectuer ce genre de vérifications à l'ordinateur (le listing de ce programme est fourni en annexe C). J'ai ainsi vérifié que le trèfle est pur, tout comme les deux graphes suivants :

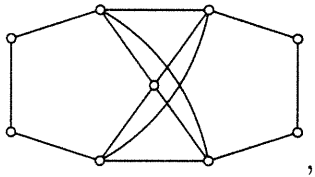


En fait, même en élargissant la classe de parité en permettant les complémentations locales aux sommets noirs (impairs), on ne trouve, pour ces exemples, aucune anticlique noire. Appelant de tels graphes *fortement purs*, on peut montrer que l'identification en un sommet de deux graphes fortement purs donne un graphe fortement pur (toujours en prenant le coloriage naturel).

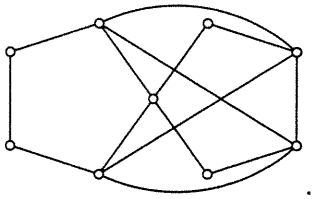
Il n'y a qu'un seul autre graphe eulérien dans la classe de parité des pentagones siamois, c'est



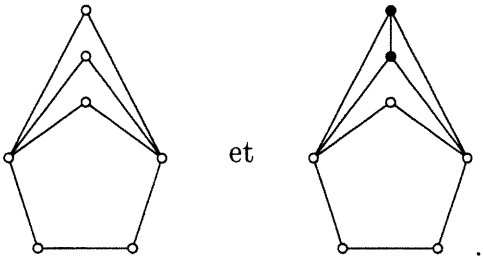
qui est isomorphe à



ce qui suggère de tester le graphe

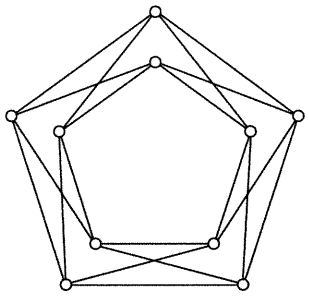


Ce dernier est effectivement pur. Il est intéressant, car c'est le premier exemple que j'ai trouvé d'un graphe pur qui ne soit pas fortement pur. En testant systématiquement tous les graphes connexes (naturellement coloriés) d'ordre ≤ 9, je suis tombé sur d'autres graphes purs non fortement purs, entre autres les graphes :



Ces graphes ont en commun d'avoir une *coupe complète* (une partition $V = V_1 \cup V_2$ telle que $|V_1|,|V_2| \geq 2$ et telle que pour tout $u \in V_1$ adjacent à un sommet

de V_2 et pour tout $v \in V_2$ adjacent à un sommet de V_1, $[u,v] \in E$). C'est ce qui m'a mis sur la piste du théorême de décomposition 3.14 du deuxième article. Dans une première version, ce théorême ne s'appliquait qu'aux graphes naturellement coloriés. Un peu plus tard est apparu le graphe pur



qui, afin d'éviter de le considérer comme un graphe primitif, m'a forcé à élargir mon étude aux bicoloriages non naturels.

Cette généralisation m'a révélé l'existence d'une correspondance biunivoque entre les classes de parité de graphes de cordes arbitrairement bicoloriés et les systèmes de transitions de graphes 4-réguliers connexes. Le deuxième article se conclut sur les implications de ces résultats pour la conjecture de double recouvrement et pose la question : comment généraliser les systèmes de transitions de graphes 4-réguliers connexes afin d'obtenir une correspondance avec les classes de parité (de graphes arbitrairement bicoloriés)?

Chapitre 1

WORD AND SET COMPLEMENTATION OF GRAPHS, INVERTIBLE GRAPHS

François Genest

1.1. ABSTRACT

Local complementation was first introduced as a way to establish relationships between Euler tours of an eulerian graph. It also appears in isotropic systems. We formalize the notion of substitution rules and introduce complementation with respect to sets of vertices in bicolored graphs, a concept intimately related to orthogonality (or compatibility) of transition systems in eulerian graphs and to the Cycle Double Cover Conjecture. Graph inversion is also introduced and we show that the inverse of a graph, when it exists, has the same automorphism group as the initial graph.

1.2. INTRODUCTION

Kotzig [7][8] introduced local complementation when he realized that all Euler tours of a 4-regular graph could be transformed into one another by a sequence of re-routings at vertices. The Euler tours are in correspondence with the members of a *complementation class* where the re-routings become *complementations* at vertices. Sabidussi [10] came upon the notion of a *parity class* while working on his Orthogonality Conjecture (also known as Sabidussi's Compatibility Conjecture). Sabidussi's approach involves looking at the Euler tours of a 4-regular graph that are, in a specified way, *orthogonal* to one particular tour. The resulting subset of

Euler tours is what becomes a parity class when translated into the language of complementation. It is not within the scope of this paper to present the Orthogonality Conjecture or its more famous relative, the Cycle Double Cover Conjecture. The interested reader is referred to the surveys by Jackson [5] and Jaeger [6]; see also the companion paper [4]. Our aim is to describe complementation and parity classes, using words and sets, respectively. A natural question will come up: when are the complementation subsets of a graph in bijection with the graphs in its parity class? First, we need to introduce the concepts of local and global subtitution rules. Fon-der-Flaass [2] gave a tight bound on the diameter of a complementation class. We give an alternative way to obtain this bound, using substitution rules.

In this article all graphs considered will be simple with edges represented by unordered pairs of vertices inside brackets. We will be interested in families of graphs sharing the same vertex set V of some reference graph $G = (V,E)$. Moreover, for $A,B \subseteq V$ with $A \cap B = \emptyset$, K_A will stand for the graph with vertex set V and edge set $\{[u,v] | u,v \in A, u \neq v\}$ and $K_{A,B}$ will have vertex set V and edge set $\{[u,v] | u \in A, v \in B\}$. If $A = \{u\}$ is a singleton, we omit the parentheses and write $K_{u,B}$. The symmetric difference of two graphs with the same vertex set, say $G_1 = (V,E_1)$ and $G_2 = (V,E_2)$, will be $G_1 \triangle G_2 = (V,E_1 \triangle E_2)$.

We will also work with sets of words. Following the notation of semigroup theory, the set of words in an alphabet V is denoted by V^*. The empty word is written ϵ. The letters s,t will be used to represent words while $u,v,...$ will be vertices.

We use the standard notation $N_H(u)$ for the neighbourhood of a vertex u in the graph H, and write $\overline{N_H(u)}$ for the closed neighbourhood of u.

1.3. Local complementation and substitution rules

Definition 1.3.1. The *(local) complement* at a vertex u of a graph $G = (V,E)$ is $Gu = G \triangle K_{N_G(u)}$. In other words, the adjacency relation of Gu coincides with that of G except on $N_G(u)$, where it is replaced by its complement.

Letting $W(G) = V(G)^*$, complementation is extended recursively to words in $W(G)$ by putting $G\epsilon = G$ and $Gsu = (Gs)u$, where $s \in W(G)$ and $u \in V(G)$. The *complementation class* of G is $\mathcal{C}G = \{Gs | s \in W(G)\}$.

Notation 1.3.2.

(i) $V(s)$ is the set of vertices appearing in the word s (the *support* of s);

(ii) $\lambda(s) = |V(s)|$;

(iii) Given a word s beginning with the letter u, $[s]$ is the word su (for example, $[uv] = uvu$).

Different words may complement a graph G in the same way. Accordingly, we define an equivalence relation $\sim_G$ on $W(G)$ by $s \sim_G s'$ if and only if $Gs = Gs'$ so that the resulting quotient set, denoted $\Omega(G)$, is in bijection with $\mathcal{C}G$. Note that $s \sim_G s' \Rightarrow st \sim_G s't$, for any $t \in W(G)$. We are interested in describing $\mathcal{C}G$ using words. When G is finite, so is $\mathcal{C}G$. In that case, an obvious first goal would be to find a finite set of words complementing G to all of $\mathcal{C}G$. To that end, we introduce the notion of a substitution rule.

Substitution rules describe when a subword can be replaced by another, while ensuring that the resulting word is equivalent to the original word. For example, it is clear that complementing with respect to one vertex twice in a row will result in a graph identical to the original graph. Hence, given a graph G and a word $s = s'uus'' \in W(G)$, we know that $s \sim_G s's''$. To put it differently, we can replace the subword uu with the empty word. We want a substitution rule to express this possibility in general terms, without being tied to a particular graph and vertex. To achieve this, the "symbols" u and G appearing in the "rule" $uu \sim_G \epsilon$ are understood to be a vertex variable and a graph variable, respectively. Furthermore, for the rule to make any sense when considering actual values of u and G, we assume that the variables are tied by the relationship $u \in V(G)$. The

following definition attempts to formalize substitution rules just enough for our needs, without resorting to a full description of graphs in terms of logic.

Definition 1.3.3. Let s be a word on a set of vertex variables, let G be a graph variable with $V(s) \subset V(G)$, and let P be a logical formula dependent on G (a property of G). The couple $R : (P,s)$ is a *substitution rule* if

$$P(G) \Rightarrow s \sim_G \epsilon.$$

Such a rule will often be written $R : P \Rightarrow s \sim_G \epsilon$. Unless otherwise specified, we asume rules to be non-trivial, i.e. at least one graph G satisfies P.

Given a substitution rule $R : P \Rightarrow s \sim_G \epsilon$, some fixed graph G and words $s_1 = s'ss''$ and $s_2 = s's''$ such that $P(Gs')$ is true, we deduce that $s_1 \sim_G s_2$. To emphasize that R was used, we sometimes write $s_1 \overset{R}{\sim}_G s_2$. The following are straightforward rules:

$$uu \sim_G \epsilon, \tag{R1}$$

$$u \neq v \text{ and } [u,v] \notin E(G) \Rightarrow uvuv \sim_G \epsilon. \tag{R2}$$

PROOF.

$$(G\triangle K_{N_G(u)})\triangle K_{N_{Gu}(u)} = G\triangle(K_{N_G(u)}\triangle K_{N_G(u)}) = G$$

$$(G\triangle K_{N_G(u)})\triangle K_{N_{Gu}(v)} = G\triangle K_{N_G(u)}\triangle K_{N_G(v)} = G\triangle K_{N_G(v)}\triangle K_{N_G(u)} \tag{1.3.1}$$

$$= (G\triangle K_{N_G(v)})\triangle K_{N_{Gv}(u)} \tag{1.3.2}$$

Thus given $[u,v] \notin E(G)$, we have $uvuv \sim_G vuuv \overset{R1}{\sim}_G vv \overset{R1}{\sim}_G \epsilon$. □

Equations 1.3.1 and 1.3.2 show that rule R2 is really about the commutativity of some local complementations. While we could define substitution rules to allow the form $[u,v] \notin E(G) \Rightarrow uv \sim_G vu$, the definition we chose seems to be more manageable in the handling of proofs.

Defining the inverse of a word s, written s^{-1}, as the word obtained from s by reversing the order of its letters, some direct consequences of R1 are that for

any $s,s',s'',t \in V(G)^*$, we have $tt^{-1} \sim_G \epsilon$ and $s' \sim_{Gs} s'' \iff ss't \sim_G ss''t$. Unfortunately, $s \sim_G s'$ and $t \sim_G t'$ do not guarantee that $st \sim_G s't'$. This means that $\Omega(G)$ cannot be endowed with a group structure using the concatenation operation. It does have an algebraic structure, that of an automaton. However this does not seem to be of any help regarding complementation. The following rule is known [1][10]:

$$[u,v] \in E \Rightarrow [uv][vu] \sim_G \epsilon \tag{R3}$$

This follows from the following lemma, by symmetry between u and v:

Lemma 1.3.4. *If* $[u,v] \in E(G)$, *then*

$$Guvu = G\triangle K_{\{u,v\},V_u\cup V_v}\triangle K_{V_u,V_v}\triangle K_{V_u,V_{uv}}\triangle K_{V_v,V_{uv}},$$

where $V_u = N(u) \setminus \overline{N(v)}$, $V_v = N(v) \setminus \overline{N(u)}$ *and* $V_{uv} = N(u) \cap N(v)$.

PROOF. Partition $E(G)$ into $E(G) = \{[u,v]\} \cup E(K_{u,V_u}\triangle K_{u,V_{uv}}\triangle K_{v,V_v}\triangle K_{v,V_{uv}}) \cup E'$. Since $N(u) = \{v\} \cup V_u \cup V_{uv}$ we have

$$\begin{aligned} E(Gu) &= E(G)\triangle E(K_{v,V_u}\triangle K_{v,V_{uv}}\triangle K_{V_u,V_{uv}}\triangle K_{V_u}\triangle K_{V_{uv}}) \\ &= \{[u,v]\} \cup E(K_{u,V_u}\triangle K_{u,V_{uv}}\triangle K_{v,V_u}\triangle K_{v,V_v}) \cup E' \\ &\quad \triangle E(K_{V_u,V_{uv}}\triangle K_{V_u}\triangle K_{V_{uv}}). \end{aligned}$$

From this we see that $N_{Gu}(v) = \{u\} \cup V_u \cup V_v$ and

$$\begin{aligned} E(Guv) &= E(Gu)\triangle E(K_{u,V_u}\triangle K_{u,V_v}\triangle K_{V_u,V_v}\triangle K_{V_u}\triangle K_{V_v}) \\ &= \{[u,v]\} \cup E(K_{u,V_v}\triangle K_{u,V_{uv}}\triangle K_{v,V_u}\triangle K_{v,V_v}) \cup E' \\ &\quad \triangle E(K_{V_u,V_v}\triangle K_{V_u,V_{uv}}\triangle K_{V_v}\triangle K_{V_{uv}}). \end{aligned}$$

Finally, $N_{Guv}(u) = \{v\} \cup V_v \cup V_{uv}$ and

$$\begin{aligned} E(Guvu) &= E(Guv) \triangle E(K_{v,V_v} \triangle K_{v,V_{uv}} \triangle K_{V_v,V_{uv}}) \\ &= \{[u,v]\} \cup E(K_{u,V_v} \triangle K_{u,V_{uv}} \triangle K_{v,V_u} \triangle K_{v,V_{uv}}) \cup E' \\ &\quad \triangle E(K_{V_u,V_v} \triangle K_{V_u,V_{uv}} \triangle K_{V_v,V_{uv}}) \\ &= E(G) \triangle E(K_{\{u,v\},V_u \cup V_v} \triangle K_{V_u,V_v} \triangle K_{V_u,V_{uv}} \triangle K_{V_v,V_{uv}}). \end{aligned}$$

□

Proposition 1.3.5.

$$[u,v],[v,w],[u,w] \in E \Rightarrow [uv][vw][uw] \sim_G \epsilon. \tag{R4}$$

PROOF. This follows from Figure 1.1 and Lemma 1.3.6. □

Lemma 1.3.6. *Let u,v,w be vertices inducing a triangle in G. Consider the graph $G' = (V',E')$, where $V' = \{u,v,w\} \cup \mathcal{P}(\{u,v,w\})$ and*

$$E' = \{[u,v],[v,w],[u,w]\} \cup \{[x,y] | x \in \{u,v,w\}, y \in \mathcal{P}(\{u,v,w\}), x \in y\}$$

and define

$$\begin{aligned} \Phi \quad : \quad & V \longrightarrow V' \\ & x \longmapsto \begin{cases} x & \text{if } x \in \{u,v,w\} \\ N(x) \cap \{u,v,w\} & \text{if } x \notin \{u,v,w\}. \end{cases} \end{aligned}$$

Then for any word s in the alphabet $\{[uv],[vw],[uw]\}$

$$[x,y] \in E(G \triangle Gs) \iff [\Phi(x),\Phi(y)] \in E(G' \triangle G's)$$

PROOF. An easy induction on the length of a word t in the alphabet $\{u,v,w\}$ shows that

$$N_{Gt}(x) \cap \{u,v,w\} = N_{G't}(\Phi(x)) \cap \{u,v,w\} \quad \text{for all } x \in V. \tag{1.3.3}$$

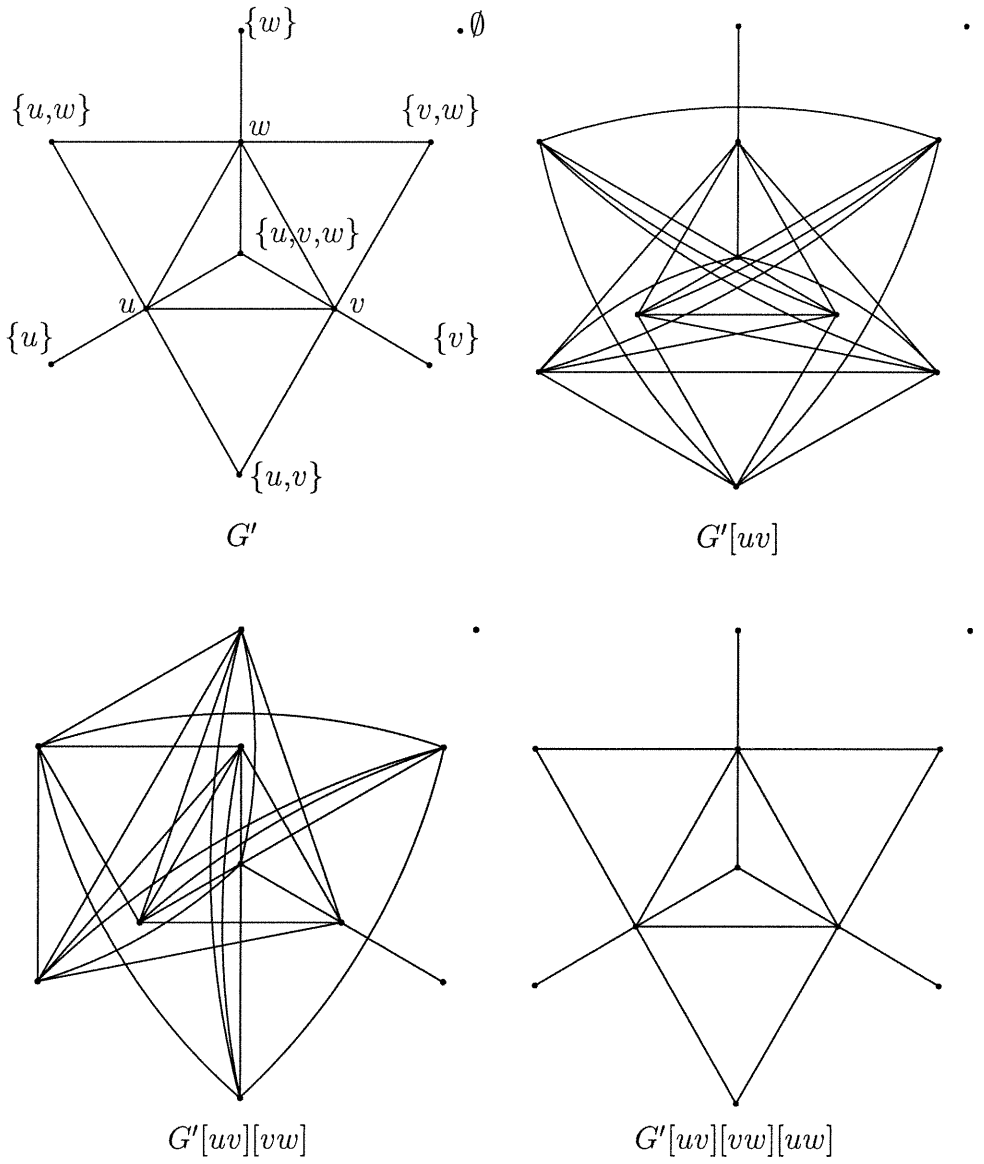


FIG. 1.1 –. *Successive complementations show that* $G' = G'[uv][vw][uw]$.

As a special case, this is also true for t in the alphabet $\{[uv],[vw],[uw]\}$. Now use induction on the length of s. The assertion is true for the empty word. Without loss of generality a non-empty s decomposes into $s'[uv]$ with

$$[x,y] \in E(G\triangle Gs') \iff [\Phi(x),\Phi(y)] \in E(G'\triangle G's'). \qquad (1.3.4)$$

From (1.3.3),

$$N_{Gs'}(x) \cap \{u,v\} = N_{G's'}(\Phi(x)) \cap \{u,v\}$$

and

$$N_{Gs'}(y) \cap \{u,v\} = N_{G's'}(\Phi(y)) \cap \{u,v\}$$

so, using Lemma 1.3.4, we deduce that

$$[x,y] \in E(Gs' \triangle Gs) \iff [\Phi(x),\Phi(y)] \in E(G's' \triangle G's). \tag{1.3.5}$$

The result follows from (1.3.4) and (1.3.5). □

Definition 1.3.7. Given a word $s \in W(G)$ and a set of substitution rules $\mathcal{R}$, $\mathcal{R}_G(s)$ is the set of words which can be deduced to be equivalent to s using the rules in $\mathcal{R}$, i.e., $s' \in \mathcal{R}_G(s) \iff \exists R_1,...,R_n \in \mathcal{R}$ and $s_0,...,s_n \in W(G)$, $n \geq 0$, such that $s_0 = s$, $s_n = s'$, and $s_i \overset{R_{i+1}}{\sim}_G s_{i+1}$, $i = 0,...,n-1$.

Once local substitution rules will have been introduced in Section 1.5, we will see that R1,...,R4 determine all local rules. Accordingly, we write $Loc_G(s)$ instead of $\{R1,R2,R3,R4\}_G(s)$.

Definition 1.3.8. Let $\mathcal{R}$ be a set of substitution rules. The set of substitution rules *generated* by $\mathcal{R}$, written $\langle\mathcal{R}\rangle$, is the set of rules (P,s) such that for every graph G satisfying P, we have $s \in \mathcal{R}_G(\epsilon)$. $\mathcal{R}$ is *independent* if no subset of $\mathcal{R}$ generates $\langle\mathcal{R}\rangle$.

Definition 1.3.9. A word $s \in W(G)$ is *bracket-writable* if $s = s_1 s_2 ... s_n$, where each s_i contains at most one repeated letter, in which case it is of the form $s_i = utu = [ut]$, and no letter appears in different s_i's. Such a word is said to be *reduced* if each s_i either consists of a single vertex or can be expressed as $s_i = [u_{i1}u_{i2}]$, where $[u_{i1},u_{i2}] \in E(Gs_1 s_2 ... s_{i-1})$.

Theorem 1.3.10. *For any $s \in W(G)$ and $u \in V$ there exists a reduced $s' \in Loc_G(s)$ such that $V(s') \subset V(s)$, and u appears in position 1 or 2 of s', if at all.*

PROOF. By way of contradiction, suppose that G and s constitute a counter-example with $\lambda(s)$ minimal. Clearly s is non-empty. Choose $u_0 \in V(s)$, with the

restriction that $u_0 = u$ if $u \in V(s)$. Writing $\rho(v,t)$ for the position of the last occurence of v in t, we can suppose without loss of generality that $\rho(u_0,s) \leq \rho(u_0,s')$ for all $s' \in Loc_G(s)$ such that $V(s') \subset V(s)$. Suppose that $\rho(u_0,s) > 2$. Consider the subwords of s of length 2 and 3 ending with the last occurrence of u_0. It is easy to verify that at least one of the following sequences of substitutions can be performed (in each case, E is meant to be the edge set just prior to complementation with respect to the subword considered):

case 1 $u_0u_0 \sim \epsilon$,

case 2 $uu_0 \overset{R2}{\sim} (u_0uu_0u)uu_0 \overset{R1}{\sim} u_0u$, if $[u,u_0] \notin E$,

case 3 $u_0uu_0 = [u_0u] \overset{R3}{\sim} [uu_0][u_0u][u_0u] \overset{R1}{\sim} [uu_0] = uu_0u$, if $[u,u_0] \in E$,

case 4 $uuu_0 \overset{R1}{\sim} u_0$,

case 5 $vuu_0 \overset{case2}{\sim} uvu_0 \overset{case2}{\sim} uu_0v$, if $[u,v],[v,u_0] \notin E, [u,u_0] \in E$,

case 6 $vuu_0 \overset{R4}{\sim} [uu_0][vu_0][vu]vuu_0 \overset{R1}{\sim} uu_0uv$, if $[u,u_0],[u,v],[v,u_0] \in E$,

case 7 $vuu_0 \overset{R1}{\sim} uuvuu_0 \overset{case6}{\sim} uu_0uv$, if $[u,u_0],[u,v] \in E, [v,u_0] \notin E$,

case 8 $vuu_0 \overset{R1}{\sim} u_0u_0vuu_0 \overset{case6}{\sim} u_0uvu$, if $[u,u_0],[v,u_0] \in E, [u,v] \notin E$.

Since this would contradict the minimality of $\rho(u_0,s)$, we must have $\rho(u_0,s) \leq 2$. By the minimality of $\lambda(s)$, s cannot have the prefix u_0u_0 (if $s = u_0u_0s''$ then $s \sim_G s''$ with $\lambda(s'') < \lambda(s)$). If $s = u_0s''$ with $u_0 \notin V(s'')$ then by the minimality of $\lambda(s)$ we know that s'' can be replaced by a reduced word not containing u_0, resulting in a reduced word equivalent to s, a contradiction. If $s = vu_0s''$ with $[v,u_0] \notin E(G)$ then permuting v and u_0 yields the previous case. Thus s is of the form $s = vu_0s''$ with $[v,u_0] \in E(G)$ and $u_0 \notin V(s'')$. Now consider the graph $G' = Gvu_0$. We can find a reduced word $t \in Loc_{G'}(s'')$ with $V(t) \subset V(s'')$ and v absent from t or in position 1 or 2. However, v cannot be absent from t or else $s \sim_G vu_0t$, a reduced word. If $t = vt'$ then $s \sim_G [vu_0]t'$, a reduced word. The only remaining possibility is $t = wvt'$ with $[v,w] \in E(G')$ (as before, $[v,w] \notin E(G')$ reduces to an earlier case). Knowing that $[v,u_0] \in E(G)$ and $[v,w] \in E(Gvu_0)$,

we must have $[w,u_0] \in E(G)$. Using substitutions as in cases 6 or 7, according to whether $[v,w] \in E(G)$ or not, we have $s \sim_G vu_0wvt' \sim_G wu_0wvvt' \sim_G [wu_0]t'$. In order to avoid $[wu_0]t'$ being reduced, we must have $w \in V(t')$. But t is reduced, so $t = wvwt''$ and thus $s \sim_G [wu_0]wt'' \sim_G wu_0t''$, which is reduced. Therefore, no counter-example exists.

□

Thus to obtain all of $\mathcal{C}G$, we need only look at the reduced words of G, which are finite in number if G is finite. The proof of the following involves a case analysis as in Theorem 1.3.10, and is left to the reader:

Lemma 1.3.11. *Let $s = [uv][wx] \in W(G)$ be reduced (i.e. $[u,v] \in E(G)$ and $[w,x] \in E(G[uv])$), then at least one of $[wx][uv]$,$[wu][vx]$ or $[wv][ux]$ is a reduced word in $Loc_G(s)$ (according to whether $[w,x]$,$[w,u]$ or $[w,v] \in E(G)$, respectively).*

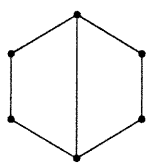
1.4. The diameter of a complementation graph

The structure of $\mathcal{C}G$ can be studied in the *complementation graph* of G. This graph, say H, is defined by

$$V(H) = \mathcal{C}G,$$

$$[G_1,G_2] \in E(H) \iff G_2 = G_1u \text{ for some } u \in V(G)$$

One may ask what is the diameter of H. Fon-der-Flaass [2] found a tight bound of $\max\{|V(G)|+1,10|V(G)|/9\}$ for this diameter when G is connected. This yields an upper bound of $7|V(G)|/6$ in general, which is attained, for example, when all the components of G are of the form:



In this section, we give an alternative proof of Fon-der-Flaass's bound using substitution rules.

Lemma 1.4.1. *Let* $s = [u_{11}u_{12}][u_{21}u_{22}]...[u_{r1}u_{r2}]$ *be a non-empty reduced word in* $W(G)$ *such that* $V(s)$ *induces a connected subgraph of* G. *For any* $u_0 \in V(s)$, *there exists* $s' \in Loc_G(s)$ *such that* $l(s')$ *(= the length of* s'*) is* $\lambda(s)+1$, *and the letter* u_0 *occurs both in the first and last position.*

PROOF. Choose a bracket-writable $s' = u_0u_1u_2...u_ru_0[v_{11}v_{12}]...[v_{m1}v_{m2}] \in Loc_G(s)$ with $V(s') = V(s)$ and r maximal (this exists by Lemma 1.3.11). Again by Lemma 1.3.11 and R3, since $G|_{V(s')}$ is connected, we can suppose that v_{11} is adjacent to one of $u_0,...,u_r$. Substituting repeatedly $u_i[v_{11}v_{12}]$ with $[v_{11}v_{12}]u_i$ (and thus moving v_{11} towards the beginning of the word) as long as $[u_i,v_{11}]$, $[u_i,v_{12}] \notin E(Gu_0u_1...u_{i-1})$, we eventually get to make one of the substitutions

$$u_i[v_{11}v_{12}] \sim v_{11}v_{12}u_i \quad \text{or} \quad u_i[v_{11}v_{12}] \sim v_{12}v_{11}u_i$$

depending on whether or not $[u_i,v_{11}] \in E(Gu_0...u_{i-1})$. This contradicts the maximality of r. Thus s' is of the form $u_0u_1...u_ru_0$ with $u_0,...,u_r$ distinct and $l(s') = \lambda(s)+1$. □

Lemma 1.4.2. *For any* $s \in W(G)$, *there exists* $s' = u_1u_2...u_r[v_{11}v_{12}]...[v_{m1}v_{m2}] \in Loc_G(s)$, *a reduced word such that* $V(s') \subset V(s)$ *and with no edge in* G *between* $\{u_1,...,u_r\}$ *and* $\{v_{11},v_{12},...,v_{m1},v_{m2}\}$.

PROOF. By Theorem 1.3.10, we can choose a reduced $s' \in Loc_G(s)$ such that $V(s') \subset V(s)$. Also, we can impose the condition that the position of the first occurrence of a double occurrence letter is maximal, i.e. s' is bracket-writable as $u_1u_2...u_r[v_{11}v_{12}]s_2...s_m$, where $u_1,...,u_r$ are distinct, each s_i stands for v_i or $[v_{i1}v_{i2}]$, and r is maximal. Suppose $s_{i+1} = v \in V(G)$ for some i. Without loss of generality i is minimal, but then one of the following substitutions can be performed:

$$[v_{i1}v_{i2}]v \sim v[v_{i1}v_{i2}], \quad [v_{i1}v_{i2}]v \sim vv_{i1}v_{i2}, \quad \text{or} \quad [v_{i1}v_{i2}]v \sim vv_{i2}v_{i1}.$$

This contradicts the minimality of i (or the maximality of r, if $i = 1$). Thus $s' = u_1...u_r[v_{11}v_{12}]...[v_{m1}v_{m2}]$. Suppose that s' does not satisfy the desired conditions. Then some u_i and some v_{jk} must be adjacent in $Gu_1...u_r$. By Lemma 1.3.11 and R3, we can suppose that $v_{jk} = v_{11}$. Just as in the proof of Lemma 1.4.1, transform s', in successive steps, by replacing $u_j[v_{11}v_{12}]$ with $[v_{11}v_{12}]u_j$ as long as there is a subword of the form $u_j[v_{11}v_{12}]$, where $[u_j,v_{11}]$,$[u_j,v_{12}] \notin E(Gu_1...u_{l-1})$. Eventually, this produces a subword of the form $u_j[v_{11}v_{12}]$ which can be replaced by $v_{11}v_{12}u_j$ or $v_{12}v_{11}u_j$. The existence of the resulting reduced word contradicts the maximality of r. □

Theorem 1.4.3. *If H is the complementation graph of a connected graph G of finite order $\neq 6$, and d is the diameter of H, then $d \leq 10|V(G)|/9$.*

PROOF. Without loss of generality, we can choose G and $s \in W(G)$ so that Gs is at distance d from G, s is reduced and $\lambda(s)$ is minimal. Because of R2, we can suppose that the components of the subgraph G' of G induced by $V(s)$ have the vertex sets $V(s_1)$,...,$V(s_k)$, respectively, where $s = s_1s_2...s_k$. Note that any permutation of the s_i's would yield an equivalent word. By Lemma 1.4.2, we can also suppose that each s_i either consists of non-repeating letters or satisfies the conditions of Lemma 1.4.1 (in which case $\lambda(s_i)$ is even). In the following, $N(t)$ stands for $N_G(V(t))$. We now describe an algorithm to modify s in steps that preserve equivalence and result in a word s' satisfying $l(s') \leq 10|V(G)|/9$:

Step 1: Set $I = \{1,...,k\}$ and $S = V(G) \setminus V(s)$.

Step 2: For each $i \in I$ such that $l(s_i) = \lambda(s_i)$, modify I by removing i.

Step 3: While there exists an $i \in I$ and $u \in N(s_i) \cap S \setminus \bigcup_{i \neq j \in I} N(s_j)$, replace s_i with an equivalent word given by Lemma 1.4.1 and remove i from I and u from S.

At this point, for $i \in I$ and $u \in N(s_i) \cap S$, there is necessarily a $j \in I \setminus \{i\}$ such that $u \in N(s_j)$. Note that $V(s_i) \cup \{u\}$ induces a connected subgraph of Gu, and by an argument along the lines of the proof of Lemma 1.4.2 there exists a word, which we will denote by s_i', such that $V(s_i') = V(s_i)$, $l(s_i') = \lambda(s_i')$ and $uus_i \sim_G us_i'u$. Given $u \in N(s_i) \cap S$ with $i \in I$, let $I_u = \{j \in I | u \in N(s_j)\}$.

Step 4: If there exists some $u \in N(s_i) \cap S$ with $i \in I$ such that $\sum_{j \in I_u} \lambda(s_j) \geq 8$, bring the corresponding s_i's to the beginning of s using R2, and relabel so that $I_u = \{1,...,l\}$; then make the substitution indicated by

$$s_1 s_2 ... s_l \sim_G us_1'uus_2'u...us_l'u \sim_G us_1's_2'...s_l'u,$$

remove the indices in I_u from I, u from S and go back to step 3.

Given $i \in I$ such that $\lambda(s_i) = 2$ (i.e. $s_i = uvu$ for some $[u,v] \in E(G)$), we must have that $N(s_i) \cap S$ contains at least two vertices. This follows from the minimality of $\lambda(s)$ and by the substitution rules:

$$\text{degree}(u) \leq 1 \Rightarrow u \sim_G \epsilon, \tag{R5}$$

$$N(u) = N(v) \Rightarrow uv \sim_G \epsilon. \tag{R6}$$

Step 5: If there are $i,j \in I$, $i \neq j$, and distinct vertices u and v such that

$$N(s_i) \cap N(s_j) \cap S \setminus \bigcup_{\substack{l \in I \\ i \neq l \neq j}} N(s_l) = \{u,v\},$$

replace s_i and s_j with equivalent words as given by Lemma 1.4.1, remove i and j from I, u and v from S, and go back to step 3.

Step 6: If there exists $i \in I$ with $\lambda(s_i) = 4$, let $u \in N(s_i) \cap S$. Because of step 4, $|I_u| = 2$ and, using R2 and relabeling, we can suppose that $I_u = \{1,2\}$ with $\lambda(s_1) = 4$ and $\lambda(s_2) = 2$. Because of step 5, we can assume that there is

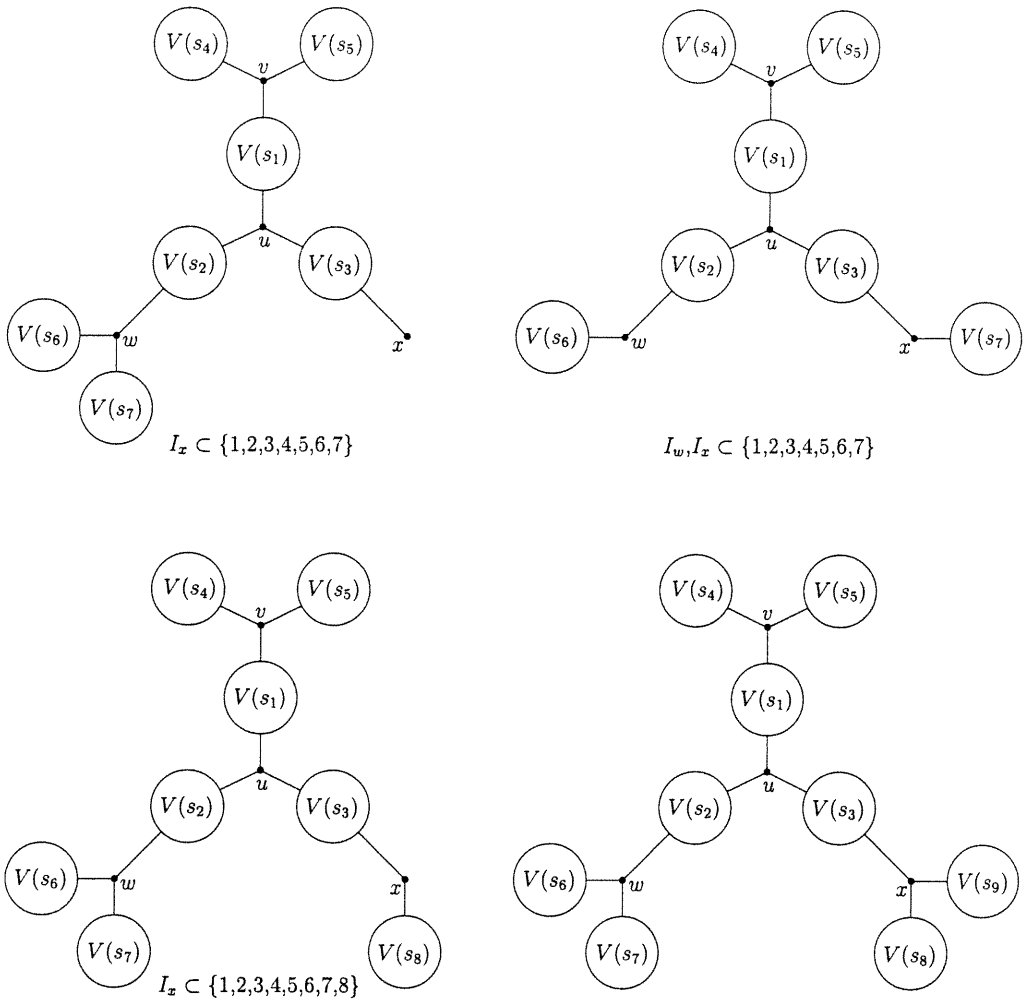


FIG. 1.2 –.

$v \in N(s_2) \cap S$ with $v \neq u$ and $v \notin N(s_1)$. Let $I_v = \{2,3,...,l\}$. We replace $s_1 s_2 ... s_l$ with an equivalent word $s_1 v s_2' ... s_l' v$ in the same manner as noted earlier, remove $1,...,l$ from I, u and v from S, and go back to step 3.

At this point, for each $i \in I$, we have $\lambda(s_i) = 2$ and, for each $u \in N(s_i)$, $|I_u| = 2$ or 3. If for each $i \in I$ and each $u \in N(s_i) \cap S$ we have $|I_u| = 2$, we can stop, since then $|S| \geq |I|$.

Step 7: If for some $i \in I$ and $u \in N(s_i) \cap S$ we have $|I_u| = 2$, we can suppose (with the appropriate use of R2 and relabeling) that $I_u = \{1,2\}$ and $I_v = \{2,3,4\}$, for some v. Now replace $s_1s_2s_3s_4$ with $s_1vs_2's_3's_4'v$, remove 1,2,3,4 from I, u and v from S, and go back to step 3.

Step 8: If there is a vertex $u \in S$ and $i \in I_u$ for which some $v \in N(s_i) \cap S \setminus \{u\}$ has $I_v = I_u$, with say $I_u = \{1,2,3\}$ after an appropriate use of R2 and relabeling, replace $s_1s_2s_3$ with $us_1's_2's_3'u$, remove 1,2,3 from I, u and v from S and go back to step 3.

Step 9: If there is a vertex $u \in S$ and $i \in I_u$ for which some $v \in N(s_i) \cap S \setminus \{u\}$ has $|I_v \cap I_u| = 2$, say, without loss of generality, $I_u = \{1,2,3\}$ and $I_v = \{2,3,4\}$, replace $s_1s_2s_3s_4$ with $us_1's_2's_3'us_4$, remove 1,2,3,4 from I, u and v from S, and go back to step 3.

Now we can suppose that for some $u,v \in S$, we have $I_u = \{1,2,3\}$, $I_v = \{1,4,5\}$ and, because of steps 8 and 9, there are distinct vertices w,x such that $w \in N(s_2) \cap (S \setminus \{u\})$ and $x \in N(s_3) \cap (S \setminus \{u\})$.

Step 10: If $I_w \subset \{1,2,3,4,5\}$ and $I_x \subset \{1,2,3,4,5\}$, replace $s_1s_2s_3s_4s_5$ with $s_2s_3vs_1's_4's_5'v$, remove 1,2,3,4,5 from I, u,v,w and x from S, and go back to step 3.

We can now suppose that $6 \in I_w$.

Step 11: If $I_w,I_x \subset \{1,2,3,4,5,6\}$, replace $s_1s_2s_3s_4s_5s_6$ with $vs_1's_4's_5'vs_2s_3s_6$, remove 1 through 6 from I, u,v,w and x from S, and go back to step 3.

From this point on, we are considering subwords with at least eighteen distinct vertices, so that the length of a replacement word can exceed by two the number of distinct letters and still avoid any possible violation of $\lambda(s') \leq 10|V(G)|/9$.

Step 12: It should be clear by now how to modify our word s in each of the situations depicted in Figure 1.2. In each case, modify I and S appropriately and go back to step 3.

□

Proposition 1.4.4. *Let H be a pair of pentagons sharing a vertex. Let G consist of m copies of H together with a path going through all the cut-vertices, as shown in Figure 1.3. Then the diameter of the complementation graph of G is $10|V(G)|/9$.*

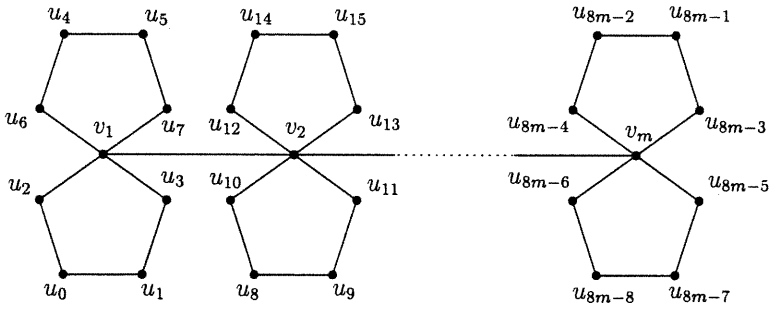
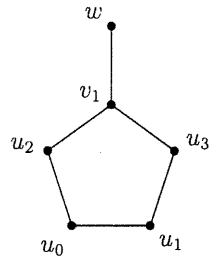


FIG. 1.3 –. *A connected graph G whose complementation graph has diameter $10|V(G)|/9$ (also found in [2]).*

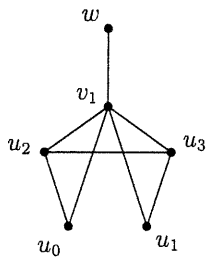
PROOF. Let $s = [u_0u_1][u_2u_3][u_4u_5][u_6u_7]...[u_{8m-4}u_{8m-3}][u_{8m-2}u_{8m-1}]$. Let s' be a shortest word in $W(G)$ such that $s' \sim_G s$. Choose a pentagon of G: for example, the pentagon containing u_0. Identify all the vertices not on the pentagon with a new vertex w. Remove all loops, and identify all multiple edges. We obtain the graph G':



A word in $W(G)$ induces a word in $W(G')$ in a unique way: send ϵ to ϵ, and given $tx \in W(G)$ such that t is sent to t', send tx to

(i) $t'x$ if x is on the pentagon;
(ii) $t'w$ if x is not on the pentagon and $x \in N_{Gt}(\{u_0,u_1,u_2,u_3,v_1\})$;
(iii) t' if x is not on the pentagon and $x \notin N_{Gt}(\{u_0,u_1,u_2,u_3,v_1\})$.

In this way, s' is sent to a word $t \in W(G')$ such that $G't$ is:



It can be verified that for any reduced $t' \in W(G')$ such that $t' \sim_{G'} t$, we have $V(t') = \{u_0,u_1,u_2,u_3\}$ or $V(t') = \{u_0,u_1,u_2,u_3,w\}$. This implies that neither t nor s' contain exactly one occurrence of v_1. If $v_1 \notin V(s')$, then u_0,u_1,u_2,u_3 contribute 5 to the length of s' (i.e., t is one of the words $u_0u_1u_3u_2u_0$, $u_1u_0u_2u_3u_1$, $u_2u_0u_1u_3u_2$, or $u_3u_1u_0u_2u_3$). If v_1 appears at least twice in s', then the vertices of the two pentagons joined at v_1 contribute at least 10 to the length of s'. Therefore, $l(s') \geq 10m$, so the diameter of the complementation graph is precisely $10m = 10|V(G)|/9$. □

1.5. Local and global substitution rules

Going back to the rules R5 and R6 presented in the proof of Theorem 1.4.3, note that each logical formula takes into account adjacencies involving every vertex of the graph, not just the adjacencies within $V(s)$.

Definition 1.5.1. A substitution rule $P \Rightarrow s \sim_G \epsilon$ is *local* if for any two graphs G and H such that G is an induced subgraph of H,

$$P(G) \Rightarrow s \sim_H \epsilon.$$

A non-local rule is *global.*

Thus R1 to R4 are local rules and R5 and R6 are global rules. From the definition follows that:

Proposition 1.5.2. *Local rules generate local rules.*

Proposition 1.5.3. *A substitution rule $P \Rightarrow s \sim_G \epsilon$, where s is a non-empty reduced word, is global.*

Proof. Let $G = (V,E)$ be a graph satisfying P. Let $u,v \notin V$. If s ends with a single occurrence letter w, let $E' = E \cup \{[u,w],[v,w]\}$. If not, we have $s = s'wxw$ with $w,x \notin V(s')$, in which case let $E' = E \cup \{[u,w],[v,x]\}$. By construction, G is an induced subgraph of $H = (V \cup \{u,v\},E')$ but $[u,v] \notin H$ while $[u,v] \in Hs$. Thus $Hs \neq H$. □

Proposition 1.5.4. *The rules*

$$uu \sim_G \epsilon, \tag{R1}$$

$$u \neq v \text{ and } [u,v] \notin E(G) \Rightarrow uvuv \sim_G \epsilon, \tag{R2}$$

$$[u,v] \in E \Rightarrow [uv][vu] \sim_G \epsilon, \tag{R3}$$

$$[u,v],[v,w],[u,w] \in E \Rightarrow [uv][vw][uw] \sim_G \epsilon, \tag{R4}$$

form an independent generating set of the local rules.

PROOF. We first show independence. Consider $G = (\{u,v\},\{[u,v]\})$. Since $\mathcal{C}G = \{G\}$, $\{\text{R1,R2,R4}\}_G(\epsilon) = \{\text{R1}\}_G(\epsilon)$ and any word in $\{\text{R1}\}_G(\epsilon)$ will contain an even number of occurrences of the letter u. Thus any independent generating subset of the four rules must contain R3.

Now let $G = (\{u,v,w\},\{[u,v],[u,w],[v,w]\})$. Any word in $\{\text{R1,R2,R3}\}_G(\epsilon)$ has an even number of letters, so that R4 is also essential. If we let $G = (\{u\},\emptyset)$, then $\{\text{R2,R3,R4}\}_G(\epsilon) = \{\epsilon\}$, thus R1 is essential. Finally, let $G = (\{u,v\},\emptyset)$. Defining the total order $u < v$ on $V(G)$, let the sign of a word $s = u_1u_2...u_n$ in $W(G)$ be $\sigma(s) = (-1)^\alpha$ where $\alpha = \text{card}\{(i,j)|u_i < u_j, 1 \leq i < j \leq n\}$. By induction on the length of words, we can show that for any $s \in \{\text{R1,R3,R4}\}_G(\epsilon) = \{\text{R1}\}_G(\epsilon)$, we have $\sigma(s) = 1$. Since $\sigma(uvuv) = -1$, this completes the proof of independence.

We know from Proposition 1.5.2 that $\langle\{\text{R1,R2,R3,R4}\}\rangle$ is a set of local rules. Let $P \Rightarrow s \sim_G \epsilon$ be a local rule. Consider $V = V(s)$ and $\{G_i\}_{i\in I}$ the family of graphs on the vertex set V satisfying P. By Theorem 1.3.10, for any rule $\mathcal{R}_i : G|_V = G_i \Rightarrow s \sim_G \epsilon$ there exists a rule $\mathcal{R}_i' : G|_V = G_i \Rightarrow s' \sim_G \epsilon$ in $\langle\{\mathcal{R}_i,\text{R1,R2,R3,R4}\}\rangle$ where s' is reduced. Since $\mathcal{R}_i'$ is local, Proposition 1.5.3 forces $s' = \epsilon$. Thus $\mathcal{R}_i' \in \langle\{\text{R1,R2,R3,R4}\}\rangle$, and since every substitution in the proof of Theorem 1.3.10 is reversible, we have $\mathcal{R}_i \in \langle\{\text{R1,R2,R3,R4}\}\rangle$. Since $P \Rightarrow s \sim_G \epsilon$ is generated by the $\mathcal{R}_i$'s, we conclude that it is in $\langle\{\text{R1,R2,R3,R4}\}\rangle$. □

Proposition 1.5.5. *If s,s' are reduced words such that $s' \in Loc_G(s)$, then $V(s) = V(s')$.*

PROOF. Suppose, by way of contradiction, that there is a $u \in V(s) \setminus V(s')$. Then $s's^{-1} \sim_G \epsilon$ and a reduced word $t \in Loc_G(s's^{-1})$ given by Theorem 1.3.10 will contain u. But this would mean that local rules generate a global rule of the form $P \Rightarrow t \sim_G \epsilon$, contradicting Proposition 1.5.2. □

Looking for a new independent (global) rule $P \Rightarrow s \sim_G \epsilon$, we can suppose, by Lemma 1.4.2, that $s = s_1...s_k$ is reduced, where each $V(s_i)$ induces a component of $G|_{V(s)}$, and each s_i is of the form $u_1...u_m$ (non-repeating letters) or

$[u_{11}u_{12}]...[u_{m1}u_{m2}]$ (reduced). Given a component induced by s, say $G' = G|_{V(s_i)}$, we have $s_i \sim_{G'} \epsilon$. Therefore, we ask:

Problem 1.5.1. For which connected graphs $G = (V,E)$ of minimal degree > 1 without twins (vertices u,v such that $N(u) = N(v)$ or $\overline{N(u)} = \overline{N(v)}$), together with a reduced word s such that $V(s) = V$, do we have $s \sim_G \epsilon$?

We will see in section 6 that there is a family of graphs satisfying the conditions of Problem 1.5.1.

1.6. Complementation sets

Notice that if u is of odd degree, then the degree of any vertex of G has the same parity in G and Gu. If u is of even degree, then the parity of the degrees in G and Gu differs precisely over the neighbours of u. Hence, if we colour the vertices of even degree white and the others black, and if we change the colours of the neighbours of u whenever we complement at a white vertex u, then the colours agree with the parity of the degrees for each graph in $\mathcal{C}G$. This will be called the *natural colouring* of G. In the following, a *bicolouring* will always mean a {black, white}-colouring.

Definition 1.6.1. The *(local) complement* of a bicoloured graph G with respect to a vertex u is a bicoloured graph Gu such that

$$V(Gu) = V(G),$$

$$E(Gu) = E(G)\triangle E(K_{N_G(u)}),$$

with its bicolouring defined to be the same as that of G if u is black in G; if u is white in G, then it is obtained from the bicolouring of G by reversing the colours of the vertices in $N_G(u)$.

Complementation with respect to words in the alphabet $V(G)$ is extended in the natural manner.

For the purposes of the next definition, call a set of words $W \subset V(G)^*$ *parity closed* if

(i) W contains the empty word;
(ii) if $s \in W$ and u is a white vertex of Gs, then $su \in W$;
(iii) if $s \in W$ and u,v are adjacent black vertices of Gs, then $s[uv]$,$s[vu] \in W$.

Clearly the intersection of parity closed sets is parity closed, hence there is a smallest parity closed set, denoted by $W^\circ(G)$. The words in $W^\circ(G)$ will be referred to as *parity words.*

Definition 1.6.2. The *parity class* of a bicoloured graph G is

$$[G] = \{Gs | s \in W^\circ(G)\}.$$

Following the idea of section 1.3, two parity words s and s' will be *equivalent* ($s \sim_G s'$) if $Gs = Gs'$. By verifying it for R1 to R4, we can show that local substitution rules are valid for bicoloured graphs (i.e. if G is a bicoloured graph with underlying (uncoloured) graph H and $s' \in Loc_H(s)$, then $Gs = Gs'$).

Theorem 1.6.3. *Two reduced parity words s and t with $V(s) = V(t)$ are equivalent.*

PROOF. Use induction on $\lambda(s)$. If $s = us'$ with $u \notin V(s')$, then u is white in G, and applying Theorem 1.3.10 to t, $t \sim_G ut'$. If s is of the form $[uv]s'$ then u is black in G, and applying Theorem 1.3.10 to t, $t \sim_G [uw]t'$. If $w \neq v$, apply Theorem 1.3.10 again to get $t \sim_G [uw][vx]t''$ and finally, from Lemma 1.3.11, $t \sim_G [uv][wx]t''$. By changing the reference graph to Gu or $G[uv]$ accordingly, the problem reduces to words for which λ is smaller. □

Given a bicoloured graph G, we are now justified to speak about complementation with respect to subsets of V.

Definition 1.6.4. A set $S \subset V(G)$ is a *complementation set* of a bicoloured graph G if there exists a reduced word $s \in W^\circ(G)$ such that $V(s) = S$. In that case, the *complement of G with respect to S* is $GS := Gs$.

1.7. Complementation and symmetry

Definition 1.7.1. A (bicoloured) graph G is *invertible* if $V(G)$ is one of its complementation sets. When no bicolouring is specified, G is assumed to have its natural bicolouring. The *inverse* of an invertible graph G, written G^{-1}, is GS, where $S = V(G)$.

Theorem 1.7.2. *Let $\Phi \in Aut(G)$ (the set of all automorphisms of the bicolored graph G). If Φ stabilizes $S \subset V(G)$ (i.e. $\Phi(S) = S$), then $\Phi \in Aut(GS)$.*

Proof. By definition, Φ preserves colour and $[u,v] \in E(G) \iff [\Phi u,\Phi v] \in E(G)$. By symmetry, u changes colour from G to GS if and only if $\Phi(u)$ changes colour from G to $G\Phi(S) = GS$. Also, we have that $[u,v] \in E(G \triangle GS) \iff [\Phi u,\Phi v] \in E(G \triangle G\Phi(S)) = E(G \triangle GS)$. Thus Φ preserves colour over GS and $[u,v] \in E(GS) \iff [\Phi u,\Phi v] \in E(GS)$. □

Corollary 1.7.3. $Aut(G^{-1}) = Aut(G)$.

Here is a point to watch out for: the two groups $Aut(G)$ and $Aut(G^{-1})$ are in fact identical, not just isomorphic.

Corollary 1.7.4. *The inverse of an invertible vertex-transitive (bicoloured) graph is vertex-transitive.*

We present an explicit formula for the inverse of a cycle:

Proposition 1.7.5. *A cycle* $C_n = \mathrm{Cay}(\mathbb{Z}_n,\{1,-1\})$ *of length* $n \geq 3$ *is invertible if and only if* $n \not\equiv 0 (mod\ 3)$. *Furthermore, for* $m \geq 1$,

$$C_{3m+1}^{-1} = \mathrm{Cay}(\mathbb{Z}_{3m+1},\{1,3,4,6,\ldots,3i-2,3i,\ldots,3m-2,3m\}),$$

$$C_{3m+2}^{-1} = \mathrm{Cay}(\mathbb{Z}_{3m+2},\{2,3,\ldots,3i-1,3i,\ldots,3m-1,3m\}).$$

PROOF. C_3 is not invertible, while $C_4^{-1} = C_4 = \mathrm{Cay}(\mathbb{Z}_4,\{1,3\})$ and $C_5^{-1} = \mathrm{Cay}(\mathbb{Z}_5,\{2,3\})$. By Corollary 1.7.3, the inverse of an invertible circulant (a Cayley graph on $\mathbb{Z}_n$) is also a circulant. For $n \geq 6$, removing vertices $-1,-2$ and -3 from $C_n\{-1,-2,-3\}$ yields C_{n-3} (see Figure 1.4). Thus, C_n is invertible if and only if C_{n-3} is, and given $C_{n-3}^{-1} = \mathrm{Cay}(\mathbb{Z}_{n-3},S)$, we must have $C_n^{-1} = \mathrm{Cay}(\mathbb{Z}_n,S')$ with $S \subset S'$. The result follows by induction. □

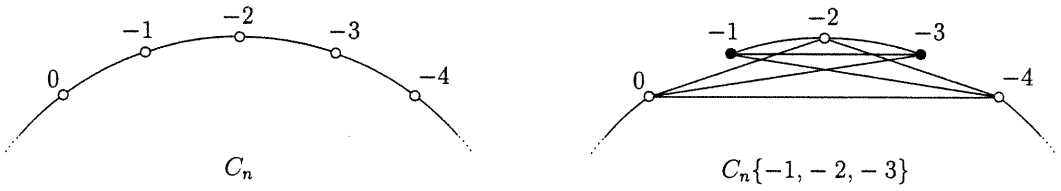


FIG. 1.4 –. *The complementation of* C_n *with respect to* $\{-1,-2,-3\}$.

The remainder of this section is concerned with self-complementary symmetric (i.e. vertex- and edge-transitive) graphs. They were characterized in [11] and classified in [9].

From Theorem 1.7.2, we obtain that:

Corollary 1.7.6. *An invertible self-complementary symmetric graph has either itself or its complement as an inverse.*

Definition 1.7.7. Let $q = p^r$ for some odd prime p, with $q \equiv 1 (\mathrm{mod}\ 4)$. Let Γ be the additive group of the finite field F_q with q elements. Let ω be a primitive

root in F_q and $S = \{\omega^2,\omega^4,...,\omega^{q-1}\}$, the set of non-zero squares. The *Paley graph* of order q is Cay(Γ,S).

Definition 1.7.8. Let $q = p^r$ for some prime $p \equiv 3(\text{mod } 4)$ and some even r. Let Γ and ω be as in the previous definition. Let $S = \{\omega^k | k \equiv 0,1(\text{mod } 4)\}$. The $\mathcal{P}^*$*-graph* of order q is Cay(Γ,S).

Except for one additional special graph $G(23^2)$ on 23^2 vertices, Peisert [9] showed that, up to isomorphism, the self-complementary symmetric graphs are the Paley and $\mathcal{P}^*$-graphs. Just as the other self-complementary symmetric graphs, $G(23^2)$ can be defined as a Cayley graph on the additive group of a field (also in [9]). For our purposes, $G(23^2)$ = Cay(Γ,S), with Γ being the additive group of F_{23^2} and $1, -1 \in S$.

The following definition and theorem are taken from [3].

Definition 1.7.9. A graph is *strongly regular modulo s with parameters* (v,k,λ,μ) if, modulo s, the number of vertices is congruent to v; the degree of each vertex, to k; the number of common neighbours of any two adjacent vertices, to λ; and the number of common neighbours of any two non-adjacent vertices, to μ.

Theorem 1.7.10 (Fon-der-Flaass [3] Theorem 3.6). *$\overline{G} \in \mathcal{C}G$ if and only if G is strongly regular modulo 2 with parameters* (1,0,0,1).

Proposition 1.7.11. *Let G = Cay(Γ,S) of order n be a Paley graph, a $\mathcal{P}^*$-graph or the special graph $G(23^2)$. $\overline{G} \in \mathcal{C}G$ if and only if $n = 1$ or $2 \in S$.*

PROOF. The case $n = 1$ is trivial, so let $n > 1$. Since a self-complementary symmetric graph has order $n \equiv 1(\text{mod } 4)$ and is regular of degree $(n-1)/2$, using a simple counting argument and Theorem 1.7.10, the following are equivalent:

(1) $\overline{G} \in \mathcal{C}G$;

(2) the number of common neighbours of two adjacent vertices is even;

(3) the number of common neighbours of two non-adjacent vertices is odd.

We have $1, -1 \in S$. Therefore G admits the automorphism Φ defined by $\Phi(u) = -u$. The vertices -1 and 1 are adjacent to 0 and, because of Φ, have an odd number of common neighbours. Since 1 and -1 are adjacent if and only if $2 = 1 - (-1) \in S$, the result follows. □

Conjecture 1.7.12. *Given a self-complementary symmetric graph G of order n, $\overline{G} \in \mathcal{C}G$ if and only if $n = 1$ or $n \equiv 5 (mod\ 8)$.*

It is a classic result of number theory that 2 is a quadratic residue modulo a prime p if and only if $p \equiv 1, -1 (\text{mod } 8)$. It can be shown that this remains true if we replace p with a prime power, therefore conjecture 1.7.12 is true for Paley graphs. We omit the proof for $G(23^2)$, which is not trivial but presents no difficulties. For $\mathcal{P}^*$-graphs, there would remain to show that 2 is of the form $2 = \omega^{4k}$, where ω is any primitive root in F_{p^r}.

Given the truth of Conjecture 1.7.12, we would have $G^{-1} = G$ for any invertible self-complementary graph G of order $n \equiv 1 (\text{mod } 8)$.

The following conjectures are suggested by computer testing:

Conjecture 1.7.13. *A self-complementary symmetric graph of order n is invertible if and only if $n = 1$ or $n \equiv 5 (mod\ 8)$, in which case $G^{-1} = \overline{G}$.*

Conjecture 1.7.14. *Let G be the Paley graph of prime order p (i.e., $G = \text{Cay}(\mathbb{Z}_p, S)$, where S is the set of quadratic residues mod p). Let $a,b \in \mathbb{Z}_p^*$ of orders 4 and k, respectively, with $a-1$ being a quadratic residue. Let $S_1 = a\langle b \rangle \cup \langle b \rangle$ and $S_2 = -a\langle b \rangle \cup \langle b \rangle$. Then at most one of S_1 or S_2 is a complementation set of G. Furthermore, if $p \equiv 5 (mod\ 16)$, it cannot be S_1 and if $p \equiv 13 (mod\ 16)$, it cannot be S_2.*

Conjecture 1.7.15. *Let G be the Paley graph on $p = 4q + 1$ vertices with p,q prime. Then $(-2)^q\langle 2^4 \rangle \cup \langle 2^4 \rangle$ is a complementation set of G.*

Conjecture 1.7.14 cannot be strengthened by saying that exactly one of S_1 or S_2 is a complementation set. The first instances of graphs where neither are

complementation sets occur, when $p \equiv 5(\text{mod } 16)$, at $p = 37, 421, 661, 1381, 1621, 2789, 2917, 3061$, and when $p \equiv 13(\text{mod } 16)$, at $p = 2381, 3181, 5437, 5821$.

Proposition 1.7.16. *Let $G = \text{Cay}(\Gamma,S)$ be the Paley graph on $p^r \equiv 5(\text{mod } 8)$ vertices. Let $a \in F_{p^r}^*$ be of order 4. Then $\{0\} \cup \langle a \rangle$ is a complementation set of G.*

PROOF. Since $a \notin S$, we find that $\{0\} \cup \langle a \rangle$ induces a white pentagon in G, which is invertible. □

In Figure 1.5, we can see that $S = \{0,1,-1,8,-8\}$, as well as $3S$ and $-4S$ all induce pentagons. By definition, there exists a word s satisfying $Gs = G^{-1}$. Can we find a method for constructing s that can be generalized to other self-complementary symmetric graphs? This might help with regard to Conjecture 1.7.13, although Proposition 1.7.11 and Conjecture 1.7.12 seem more promising.

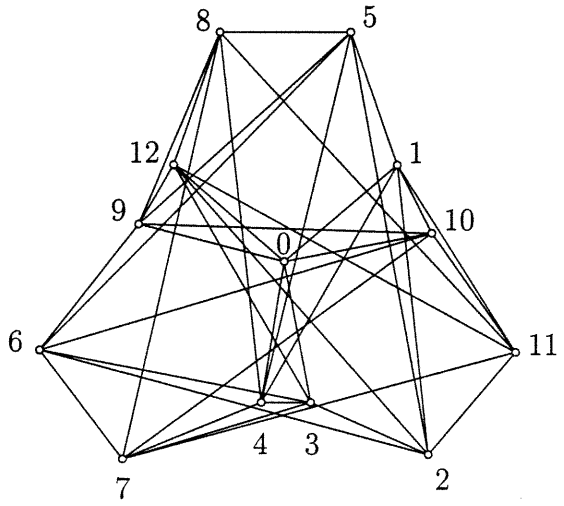


FIG. 1.5 –. *The self-complementary symmetric graph on 13 vertices.*

We can construct many graphs satisfying the conditions of Problem 1.5.1. Given a self-complementary symmetric graph H such that $H^{-1} = \overline{H} = Ht$, with t reduced, form G by adding a new vertex u to H, and joining u to all vertices of H. Choosing $s = tu$ we have $Gs = G$ (by Corollary 1.7.3, $N_{Ht}(u) = N_H(u)$). Since there are an infinite number of primes congruent to 5(mod 8) and assuming

Conjecture 1.7.13 holds, this family of graphs would be infinite. It can be verified that if this same G is provided with its natural bicolouring, then $G^{-1} = Gut^{-1}u \neq G$.

Conjecture 1.7.17. *Given a bicoloured graph G without isolated vertices or twins, $[G]$ is in bijection with the complementation sets of G.*

1.8. References

[1] Bouchet, A., Graphic presentations of isotropic systems, *J. Combin. Theory Ser. B* **45** (1988), 58-76.

[2] Fon-der-Flaass, D. G., Distance between locally equivalent graphs (Russian), *Metody Diskret. Analiz.* **48** (1989), 85-94, 106-107.

[3] Fon-der-Flaass, D. G., Local complementations of simple and oriented graphs (Russian), *Sibirsk. Zh. Issled. Oper.* **1** (1994), 43-62, 87.

[4] Genest, F., Transition systems, orthogonality and local complementation, submitted.

[5] Jackson, B., On circuit covers, circuit decompositions and Euler tours of graphs, *Surveys in combinatorics (Keele, 1993)*, London Math. Soc. Lecture Note Ser. **187**, Cambridge Univ. Press (1993), 191-210.

[6] Jaeger, F., A survey of the cycle double cover conjecture, *Cycles in graphs (Burnaby, 1982)*, North-Holland Math. Stud. **115**, North-Holland, Amsterdam (1985), 1-12.

[7] Kotzig, A., Moves without forbidden transitions in a graph, *Mat. Časopis Sloven. Akad. Vied* **18** (1968), 76-80.

[8] Kotzig, A., *Quelques remarques sur les transformations* κ, séminaire Paris (1977).

[9] Peisert, W., All self-complementary symmetric graphs, *J. Algebra* **240** (2001), 209-229.

[10] Sabidussi, G., *Eulerian walks and local complementation*, D.M.S. 84-21, Dép. de math. et stat., Université de Montréal (1984).

[11] Zhang, H., Self-complementary symmetric graphs, *J. Graph Theory* **16** (1992), 1-5.

Chapitre 2

TRANSITION SYSTEMS, ORTHOGONALITY AND LOCAL COMPLEMENTATION

François Genest

2.1. Abstract

Intimately related to the Cycle Double Cover Conjecture is the problem of finding a cycle decomposition orthogonal to a given transition system in an eulerian graph. One approach consists in finding a black anticlique in the corresponding parity class of bicoloured graphs.

2.2. Introduction

This article discusses problems that seem foreign to each other yet prove to be inextricably linked.

Much effort has been spent on the Cycle Double Cover Conjecture; understandably so, given its many implications (see the surveys by Jackson [10] and Jaeger [11]).

Typical questions about transition systems of eulerian graphs involve finding transition systems that are orthogonal to one another. Fleischner [5][6] showed how the Dominating Circuit Conjecture and Sabidussi's Orthogonality Conjecture imply the Cycle Double Cover Conjecture.

Kotzig [13] demonstrated how κ-transformations of Euler tours in 4-regular connected graphs are essentially the same as local complementations. In a similar way,

Sabidussi [14] made the connection between eulerian graphs with one specified Euler tour and parity classes of simple graphs.

We will go from transition systems to parity classes and back again, showing how the different problems are meshed together. The notion of pure graphs is explored; it is proposed that a characterization of primitive pure graphs would help to settle the Cycle Double Cover Conjecture.

2.3. PRELIMINARIES

The terminology in this article mostly follows Fleischner [6] and Jackson [10]. A *cycle* is a non-empty 2-regular connected graph or subgraph. A *tour* of a graph is a sequence $\alpha = u_0e_1u_1e_2u_2...u_{n-1}e_n$ where e_i is an edge incident with the vertices u_{i-1} and u_i (where subscripts are read modulo n) and where the edges are distinct. Tours are deemed equivalent if one can be transformed into another by successive reversals and cyclic permutations of the sequence. We work on the one hand with eulerian graphs, which may have multiple edges and loops; such a graph will be denoted by Γ. On the other hand, we will carry out local complementations of simple graphs which will be denoted by F,G or H.

A *transition* at a vertex u is either a pair $\{u,e\}$, where e is a loop incident with u, or a set $\{u,e_1,e_2\}$, where e_1 and e_2 are distinct edges incident with u. A *transition system* (TS), intuitively, is a partition of the "half-edges" of the graph into pairs of adjacent ones. To avoid the difficulties of dealing with loops, we define transition systems by way of tour decompositions.

A *tour decomposition* of an eulerian graph Γ is a set of edge-disjoint tours of Γ whose union exhausts all edges. Note that a tour $u_0e_1u_1e_2u_2...e_n$ induces the transitions $\{u_i,e_i,e_{i+1}\}$, $i = 0,...,n-1$. Given a tour decomposition of Γ, the associated *transition system* is the union of the sets of transitions induced by the tours. Since we can reconstruct the tours from the transition system, this is a 1-1 correspondence. Two transition systems (and by extension their corresponding tour decompositions) are *orthogonal* if they are disjoint (we prefer "orthogonal"

to the rather vague "compatible"). Of course, such a comparison makes sense only when there are no vertices of degree two. Since isolated vertices are irrelevant to questions of orthogonality, we will only consider graphs of minimum degree $\delta > 2$. In this setting, a transition system completely determines the graph and we can talk about tours and cycles of transition systems without ambiguity.

Settling a question raised by Nash-Williams, Kotzig [12] showed that for any transition system without loops there exists an orthogonal Euler tour. When loops are present, a necessary and sufficient condition for the existence of an orthogonal Euler tour is for all loop transitions (transitions of the form $\{u,e\}$) incident with vertices of degree 4 to be in the transition system. In [9], Jackson characterizes graphs having three pairwise orthogonal Euler tours.

A natural question arises: when can we find a cycle decomposition orthogonal to a given transition system? A trivial necessary condition is that the transition system must not contain transitions of the form $\{u,e_1,e_2\}$, where u is of degree two within the block (maximal 2-connected subgraph) containing $\{e_1,e_2\}$ (in particular, loop transitions are excluded), because any cycle decomposition of the graph must contain these. A transition system satisfying this condition is said to be *admissible*. Many admissible transition systems admit orthogonal cycle decompositions. A notable exception is the transition system of K_5 decomposed into two 5-cycles. Sabidussi conjectures that if the transition system is induced by an Euler tour (in which case it is necessarily admissible), there always exists an orthogonal cycle decomposition. Much of what follows uses or is inspired by his work on the subject (see [14]). The results on complementation used in this article can be found in [8].

2.4. TRANSITION GRAPHS

Definition 2.4.1. Let Γ be an eulerian graph of minimum degree $\delta > 2$ with transition system $S = \{t_{u,i} | u \in V(\Gamma), i \in \{1,...,\frac{1}{2}d_\Gamma(u)\}$, where each $t_{u,i}$ contains u. A *transition graph* (TG) of S is a graph Γ_S satisfying $V(\Gamma_S) = S$, $E(\Gamma_S) =$

$E(\Gamma) \cup E'$, where $E' = \{e'_{u,i} | u \in V(\Gamma), i = 1,...,\frac{1}{2}d_\Gamma(u)\}$, with incidences defined as follows:

$e \in E(\Gamma)$ is a non-loop edge incident with $t_{u,i}$ and $t_{v,j}$ if $t_{u,i}$ and $t_{v,j}$ are distinct and $e \in t_{u,i} \cap t_{v,j}$; e is a loop incident with $t_{u,i}$ if $t_{u,i} = \{u,e\}$,

$e'_{u,i}$ is incident with $t_{u,i}$ and $t_{u,i+1}$, $i = 1,...,\frac{1}{2}d_\Gamma(u) - 1$; $e'_{u,d_\Gamma(u)/2}$ is incident with $t_{u,1}$ and $t_{u,d_G(u)/2}$.

In Figure 2.1, edges of the original graph are drawn as solid lines while the edges in E' are broken. To identify a transition in the transition graph, simply consider the vertex label together with the incident solid edges.

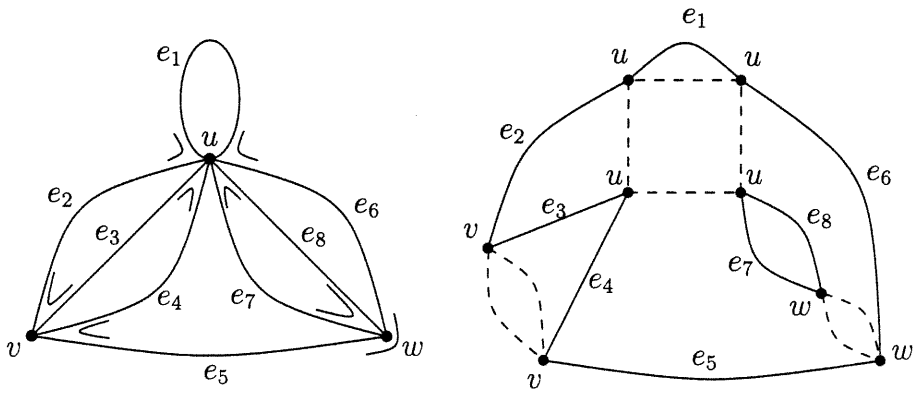


FIG. 2.1 –. *To the left, a graph with TS indicated by arcs. To the right, one of its TG's.*

Note that, in general, different orderings of the transitions of Γ will give rise to different incidences for the edges in E'. In fact, the transition graph will be unique (up to isomorphism) if and only if all vertices of Γ are of degree 4 or 6. Observe also that a transition graph of S has a natural transition system provided by the 2-factors induced by $E(\Gamma)$ and E'. The idea behind constructing a transition graph is that if we can find a cycle decomposition orthogonal to this new transition system, then, contracting the edges in E', we get a cycle decomposition orthogonal

to S. Since all loop transitions of S are in the transition system of Γ_S, Kotzig's result tells us we can find an Euler tour orthogonal to the transition system of Γ_S. Notice that this tour will have successive edges alternating between the two 2-factors (solid and broken edges). Accordingly, it is called an *alternating tour.* When writing it, we omit the edges and, since Γ_S is 4-regular, we are left with a *double occurrence word* (each letter appears exactly twice). At this point, we can construct an alternance (or circle) graph. Alternance graphs have been studied in [3][7][15], for example.

Definition 2.4.2. Given a double occurrence word $w = a_1...a_{2n}$ made up of the letters $u_1,...,u_n$, the *alternance graph* of w is the simple graph with vertex set $\{u_1,...,u_n\}$, and with an edge between u_i and u_j if they alternate in w (i.e. if $a_k = a_l = u_i$ and $a_r = a_s = u_j$ with $k < l, r < s$ then either $k < r < l < s$ or $r < k < s < l$).

Definition 2.4.3. An *anticlique* A of a graph G is a maximal independent set of vertices (i.e. no two vertices of A are adjacent while every vertex in $V(G) \setminus A$ has a neighbour in A).

Theorem 2.4.4 (Sabidussi [14] Theorem 5.1). *Let Γ be a transition graph with $E(\Gamma) = E \cup E'$ being the partition into the two 2-factors. Let w be (the double occurrence word of) an alternating tour of Γ. Every anticlique A of the alternance graph of w determines a cycle decomposition of Γ into $|A| + 1$ cycles. Moreover, if A is odd (i.e. consists of vertices of odd degree), the decomposition is alternating.*

PROOF. Since any cyclic permutation of the letters of w gives rise to the same alternance graph, we can suppose that $w = ua_1a_2...a_ru a_{r+1}...a_{2n-2}$ with $u \in A$. We can further suppose that $a_1,...,a_r \notin A$. Since A is a covering, we must have that $a_1,...,a_r$ are all distinct. Split the tour w into the cycle $ua_1...a_ru$ and the tour $ua_{r+1}...a_{2n-2}$. If r is odd, then both the cycle and the new tour have an even number of edges and remain alternating. While u is not covered by $A' = A \setminus \{u\}$,

it only appears once in the new tour and (replacing A with A') the result follows by induction on $|A|$. □

Given an arbitrary alternating tour, the corresponding alternance graph may or may not admit an odd anticlique. However, Sabidussi [14], extending Kotzig's work on κ-transformations [12], showed that any other alternating Euler tour can be obtained by a sequence of twists and switches.

Definition 2.4.5. The *u-twist* (or *twist* at u) of the Euler tour

$$w = ua_1...a_r ua_{r+1}...a_s$$

is the Euler tour

$$w' = ua_r...a_1 ua_{r+1}...a_s.$$

Note that if w is alternating and r is even (i.e. u is of even degree in the alternance graph), then w' is also alternating.

Definition 2.4.6. Given vertices u and v that alternate in the Euler tour

$$w = ua_1...a_k va_{k+1}...a_l ua_{l+1}...a_r va_{r+1}...a_s,$$

the *uv-switch* of w is the Euler tour

$$w' = ua_1...a_k va_{r+1}...a_s ua_{l+1}...a_r va_{k+1}...a_l$$

If w is alternating and l and $r-k$ are odd (i.e. u,v are of odd degree in the alternance graph) then w' is also alternating. Also, a uv-switch has the same effect as successive twists at u,v and u. Looking at the alternance graph, the result of a u-twist is a local complementation with respect to u.

2.5. LOCAL COMPLEMENTATION AND PURE GRAPHS

In the following, by a *bicoloured* graph is meant a simple graph with a {black, white}-colouring of its vertices. A closer look at local complementation and related results can be found in [8].

Definition 2.5.1. The *(local) complement* of a bicoloured graph G with respect to a vertex u is a bicoloured graph Gu such that

$$V(Gu) = V(G)$$

$$E(Gu) = E(G) \triangle E(K_{N_G(u)}) \text{ (symmetric difference)}$$

where $K_{N_G(u)}$ is the complete graph on the neighbourhood $N_G(u)$. That is to say, the adjacency relation of Gu coincides with that of G except on $N_G(u)$, where it is replaced by its complement. The bicolouring of Gu is defined to be the same as that of G if u is black in G; if u is white in G, then it is obtained from the bicolouring of G by reversing the colours of the vertices in $N_G(u)$.

Complementation is extended recursively to words in the alphabet $V(G)$ by putting $Gsu = (Gs)u$, where $s \in V(G)^*$ (= the set of all words on $V(G)$) and $u \in V(G)$.

The symbol $[uv]$ is short for uvu.

For the purposes of the next definition, call a set a words $W \subset V(G)^*$ *parity closed* if

(i) W contains the empty word;

(ii) if $s \in W$ and u is a white vertex of Gs, then $su \in W$;

(iii) if $s \in W$ and u,v are adjacent black vertices of Gs, then $s[uv]$,$s[vu] \in W$.

Clearly the intersection of parity closed sets is parity closed, hence there is a smallest parity closed set, denoted by $W^\circ(G)$.

Definition 2.5.2. The *parity class* of a bicoloured graph G is

$$[G] = \{Gs | s \in W^\circ(G)\}.$$

An example of a parity class is given in Figure 2.2. There, G is a cycle of length 5 on the vertex set {1,2,3,4,5} with all vertices coloured white. The other graphs in $[G]$ are identified by words in $W^\circ(G)$. These words are not unique: for example, $G13 = G31$.

If a simple graph is given its *natural* colouring, where vertices of even degree are white and vertices of odd degree are black, we can see that the colouring of each graph in its parity class is also natural. We now have that to each transition graph there corresponds a unique parity class, and our search for an alternating cycle decomposition amounts to finding a black anticlique in some graph of the parity class.

Definition 2.5.3. A parity class is *pure* if it contains no graph with an anticlique of black vertices. By extension, its member graphs are also said to be *pure.*

Up to local complementation, the two smallest connected pure graphs are the one-point graph and the pentagon with their natural colourings.

Definition 2.5.4. A *split* in a graph G is a partition of its vertex set into $V = V_1 \cup V_2$, with at least two vertices in each V_i, such that all vertices in $N(V_1) \cap V_2$ are adjacent to all vertices in $N(V_2) \cap V_1$. A splitless graph is said to be *prime.*

It can easily be checked that a split remains a split after local complementations (see [1]), so this is a feature of the whole parity class.

Definition 2.5.5. A *rooted* bicoloured graph with root z is a couple (G,z), where $z \in V(G)$, G being a simple graph with a bicolouring defined on $V(G) \setminus \{z\}$.

Whenever we mention rooted graphs in this paper, it will be understood that we are talking about rooted bicoloured graphs.

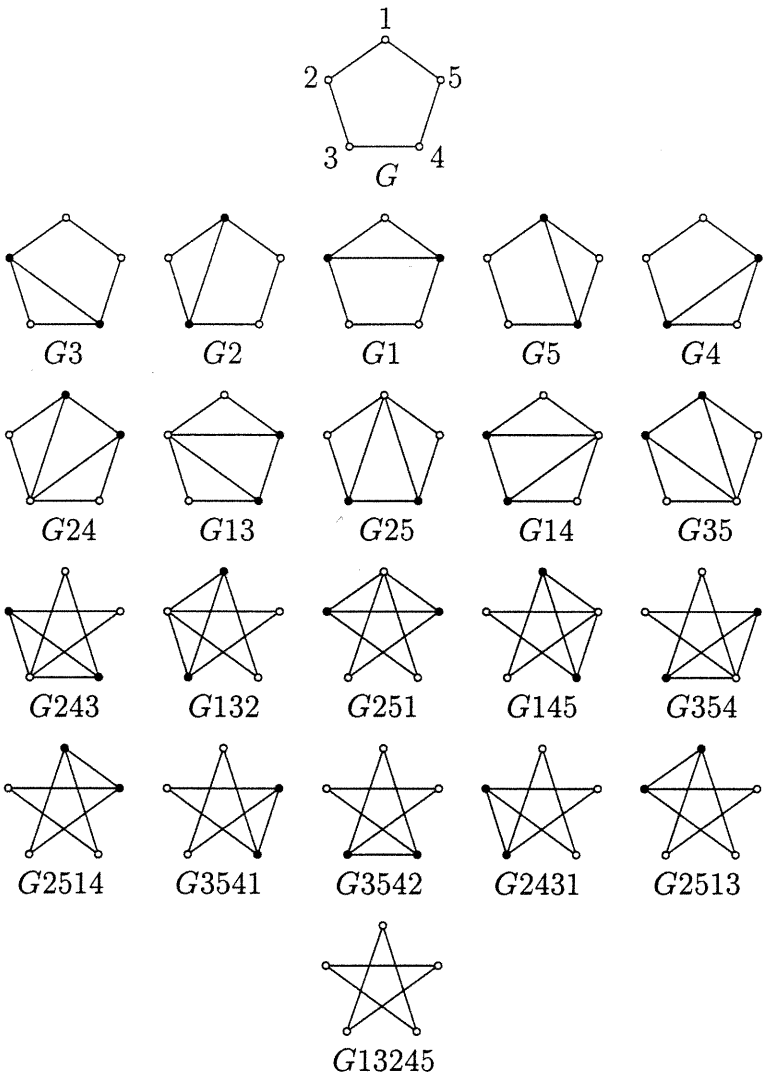


FIG. 2.2 –. *The parity class of the pentagon (with its natural colouring).*

Note that from a split $V = V_1 \cup V_2$ of some bicoloured graph G, we can construct two rooted graphs as follows: let G_1 (respectively G_2) be obtained from G by identifying the vertices of V_2 (respectively V_1) to a single new vertex z_1 (respectively z_2) which will be the root vertex of G_1 (respectively G_2) and removing loops. G_1 and G_2 are said to be *induced* by the split.

Let G be a bicoloured graph with a split $V = V_1 \cup V_2$ inducing the rooted graphs G_1 and G_2. We will construct six pairs of parity classes from G_1 and G_2 and show

that if any one pair consists of two pure parity classes then $[G]$ is also a pure class. Because these constructions and proof are technical and hardly readable by themselves, we will, after giving the necessary definitions, discuss them by means of an example that illustrates all possibilities that may arise.

Definition 2.5.6. Given a vertex v adjacent to a black vertex u of a bicoloured graph G, the *complementation* of G with respect to the set $\{u,v\}$ is

$$G\{u,v\} = \begin{cases} Gvu & \text{if } v \text{ is white,} \\ G[vu] & \text{if } v \text{ is black.} \end{cases}$$

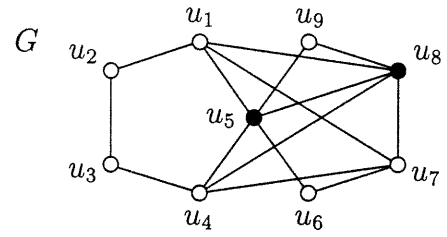
This is well defined because in the case where v is black, $G[vu] = G[uv]$. Complementation with respect to a couple is a special case of set complementation (see [8]).

Definition 2.5.7. Given a rooted graph G with root z, we define the following bicoloured graphs:

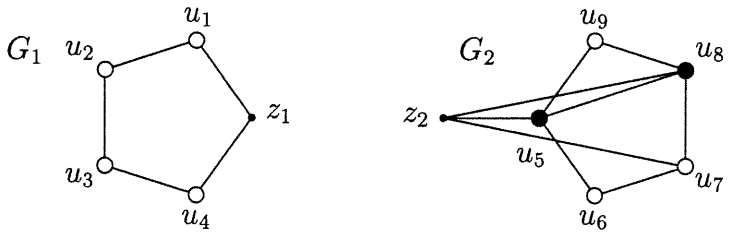
$rw(G)$ is obtained by colouring z white in G,
$rb(G)$ is obtained by colouring z black in G,
$rc(G)$ is obtained by complementing G at z as if z were white and colouring z black,
$lw(G) = G - z$,
$lb(G)$ is obtained by complementing G at z as if z were white and then removing z,
and $lc_v(G)$, given v adjacent to z, is obtained by colouring z black and letting $lc_v(G) = G\{z,v\} - z$.

$rw(G)$,$rb(G)$,$rc(G)$ are called *root* graphs of G and $lw(G)$,$lb(G)$,$lc_v(G)$ are *leaf* graphs of G.

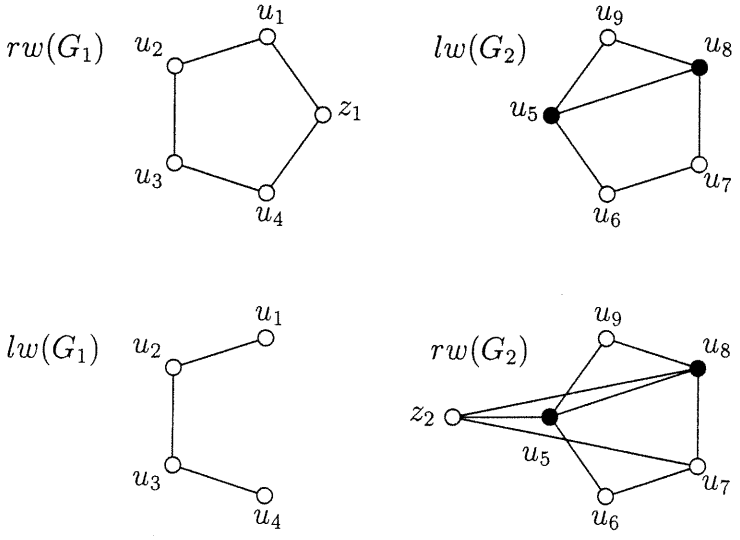
Consider the following bicoloured graph:



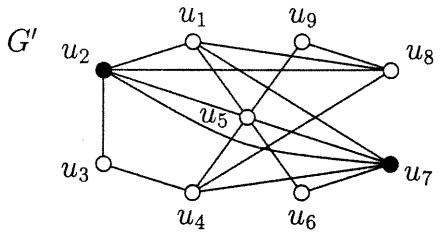
G has only one split (i.e. $V_1 = \{u_1,u_2,u_3,u_4\}$, $V_2 = \{u_5,u_6,u_7,u_8,u_9\}$), which induces:



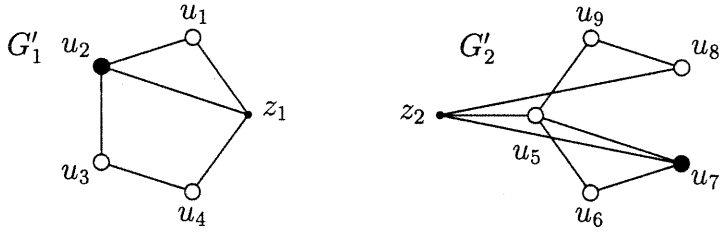
From these rooted graphs, we construct the bicoloured graphs $rw(G_1)$, $lw(G_2)$, $lw(G_1)$ and $rw(G_2)$:



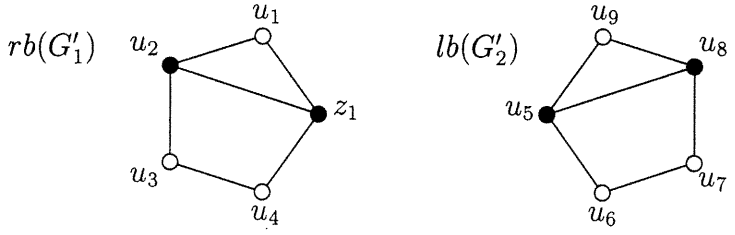
We see that $rw(G_1)$ and $lw(G_2)$ form a pair of pure root and leaf graphs. However, consider now what happens when we start with $G' = Gu_1$:



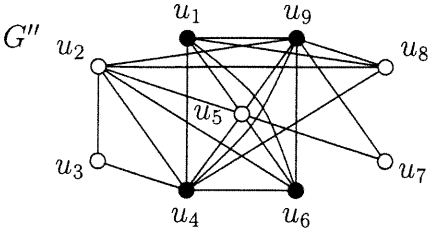
In this case, the split induces:



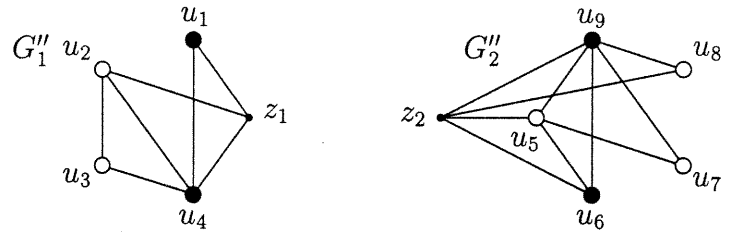
and none of $rw(G'_1)$,$lw(G'_2)$,$rw(G'_2)$,$lw(G'_1)$ are pure. To find pure graphs again, we construct an $\{rb(G'_i),lb(G'_j)\}$ pair, in this case the root graph $rb(G'_1)$ and the leaf graph $lb(G'_2)$:



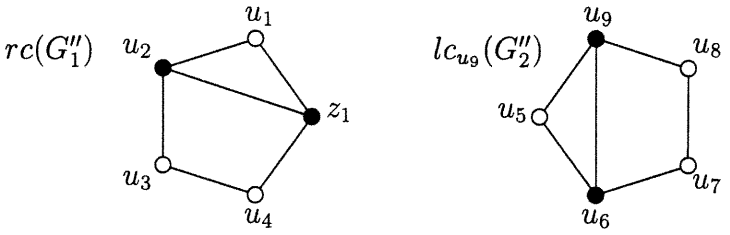
Another situation presents itself when we start with $G'' = Gu_1u_5$:



Now the split induces:



and none of $rw(G_i'')$,$rl(G_j'')$,$rb(G_i'')$,$lb(G_j'')$ are pure, for $\{i,j\} = \{1,2\}$. We have to resort to the last type of construction: for example, the root graph $lc(G_1'')$ and the leaf graph $lc_{u_9}(G_2'')$ form a pure pair:



While these constructions are unappealing, the good news is that we're done: any other graph $H \in [G]$ separated into rooted graphs H_1 and H_2 (corresponding to V_1 and V_2 respectively) by the split will yield a pair of pure graphs of the form $\{rw(H_1),lw(H_2)\}$, $\{rb(H_1),lb(H_2)\}$ or $\{rc(H_1),lc_v(H_2)\}$, as will be seen below.

Definition 2.5.8. Let G be a simple bicoloured graph with a split $V = V_1 \cup V_2$ inducing the rooted graphs (G_1,z_1) and (G_2,z_2). An *essential decomposition* of G along this split is a pair of root and leaf graphs of the form $\{rw(G_i),lw(G_j)\}$, $\{rb(G_i),lb(G_j)\}$ or $\{rc(G_i),lc_v(G_j)\}$, where $\{i,j\} = \{1,2\}$, and v is adjacent to z_j.

Lemma 2.5.9. *Let (G,z) be a rooted graph. If $v,w \in N_G(z)$, then $lc_v(G)$ and $lc_w(G)$ are in the same parity class (i.e. $lc_w(G) \in [lc_v(G)]$).*

PROOF. Let H be the bicoloured graph obtained from G by colouring z black. For $s \in W^\circ(lc_v(G)) \subset W^\circ(H\{v,z\})$, we have $lc_v(G)s = H\{v,z\}s - z$. Using local substitution rules (see [8]), consider the following three cases:

if $v \neq w$ are both white in G, then $[vw] \in W^\circ(lc_v(G))$, and $wz \sim_H vvwz \sim_H vz[vw]$, so that $lc_w(G) = Hwz - z = Hvz[vz] - z = lc_v(G)[vz] \in [lc_v(G)]$;

if v and w are of different colours, say v is black and w is white, then $wv \in W^\circ(lc_v(G))$ and $wz \sim_H wzvv \sim_H [vz]wv$;

if $v \neq w$ are both black, then $[vw] \in W^\circ(lc_v(G))$ and $[wz] \sim_H [vz][vw]$. □

Definition 2.5.10. Given a bicoloured graph $G = (V,E)$ with a split $V = V_1 \cup V_2$ inducing the rooted graphs (G_1,z_1) and (G_2,z_2), the *essential decompositions* of the parity class $[G]$ along this split with respect to G are the pairs $\{[rw(G_i)],[lw(G_j)]\},\{[rb(G_i)],[lb(G_j)]\}$ and, if there is an edge between V_1 and V_2 in G, $\{[rc(G_i)],[lc_v(G_j)]\}$, where $\{i,j\} = \{1,2\}$. For convenience, if there is no edge between V_1 and V_2 in G, the pair $\{[\emptyset],[\emptyset]\}$, where $\emptyset$ stands for the empty graph, is also called an essential decomposition of $[G]$.
Let us write Rw_G,Lw_G,Rb_G,Lb_G for $[rw(G_1)],[lw(G_2)],[rb(G_1)],[lb(G_2)]$ respectively. Let $Rc_G = [rc(G_1)]$ and $Lc_G = [lc_v(G_2)]$ if there is an edge between V_1 and V_2 in G, or else set them equal to the trivial class $[\emptyset]$. To indicate the essential decompositions obtained by interchanging V_1 and V_2, write $Rw'_G,Lw'_G,Rb'_G,Lb'_G,Rc'_G$ and Lc'_G. The proof of the following is left to the reader:

Lemma 2.5.11. *Let G_1,G_2 be identical bicoloured graphs except for a vertex u which is of different colour in G_1 and G_2. Let v be a black vertex adjacent to u. Consider $H_1 = G_1\{u,v\}$ and $H_2 = G_2\{u,v\}$. Then u is black in H_1 and H_2, and each of H_1,H_2 can be obtained from the other by complementing at u and reversing the colouring on $N_{H_1}(u)$ (i.e. by complementing as if u were white). Conversely, if H_1 and H_2 are as described, then for $v \in N_G(u)$, $H_1\{u,v\}$ and $H_2\{u,v\}$ differ only by the colour of u.*

Theorem 2.5.12. *Let G be a bicoloured graph with a split $V = V_1 \cup V_2$. The essential decompositions of $[G]$ along this split with respect to $H \in [G]$ are*

independent of the choice of H. To be more precise, the pairs $\{Rw_H,Lw_H\}$, $\{Rb_H,Lb_H\}$,$\{Rc_H,Lc_H\}$ are identical to $\{Rw_G,Lw_G\}$,$\{Rb_G,Lb_G\}$,$\{Rc_G,Lc_G\}$ up to permutation, and the pairs $\{Rw'_H,Lw'_H\}$,$\{Rb'_H,Lb'_H\}$,$\{Rc'_H,Lc'_H\}$ are identical to $\{Rw'_G,Lw'_G\}$,$\{Rb'_G,Lb'_G\}$,$\{Rc'_G,Lc'_G\}$ up to permutation.

PROOF. When there is no edge between V_1 and V_2 in G, the result is trivial, so suppose V_1 and V_2 are joined by an edge. We have that $H = Gs$ for some $s \in W^\circ(G)$. By induction, it suffices to consider the cases $s = u$, where u is a white vertex of G, and $s = [xy]$, where x,y are adjacent black vertices of G. Let the split induce the rooted graphs (G_1,z_1),(G_2,z_2) from G, and the rooted graphs (H_1,z_1),(H_2,z_2) from H. Let $v \in N_{G_2}(z_2)$ and $w \in N_{H_2}(z_2)$ so that $Lc_G = [lc_v(G_2)]$,$Lc_H = [lc_w(H_2)]$. We consider the following eleven cases.

Case 1) white $u \in V_1 \setminus N_{G_1}(z_1)$.
$rw(H_1) = rw(G_1)u$, $rb(H_1) = rb(G_1)u$, $rc(H_1) = rc(G_1)u$, and since $H_2 = G_2$, $lw(H_2) = lw(G_2)$, $lb(H_2) = lb(G_2)$, and (choosing without loss of generality $w = v$) $lc_w(H_2) = lc_v(G_2)$.

Case 2) white $u \in N_{G_1}(z_1)$.
$rw(H_1) = rb(G_1)u$, $lw(H_2) = lb(G_2)$, $rb(H_1) = rw(G_1)u$, $lb(H_2) = lw(G_2)$. Lemma 2.5.11 gives $rc(H_1) = rc(G_1)[uz_1]$ and, choosing $w = v$, $lc_w(H_2) = lc_v(G_2)$.

Case 3) white $u \in N_{G_2}(z_2)$.
$rw(H_1) = rw(G_1)z_1$, $lw(H_2) = lw(G_2)u$, $rb(H_1) = rc(G_1)$, $rc(H_1) = rb(G_1)$. Choose $w = v = u$, so that $lc_w(H_2) = lb(G_2)$ and $lb(H_2) = lc_v(G_2)$.

Case 4) white $u \in V_2 \setminus N_{G_2}(z_2)$.
$rw(H_1) = rw(G_1)$, $lw(H_2) = lw(G_2)u$, $rb(H_1) = rb(G_1)$, $lb(H_2) = lb(G_2)u$,

$rc(H_1) = rc(G_1)$. We can suppose that $w = v$. To get $lc_w(H_2) = lc_v(G_2)u$, let F be G_2 with z_2 white and verify the following, using local substitution rules: if v is black and $[u,v] \notin E(F)$, $u[vz_2] \sim_F [vz_2]u$; if v is white and $[u,v] \notin E(F)$, $uvz_2 \sim_F vz_2u$; if v is black and $[u,v] \in E(F)$, $uvz_2 \sim_F [vz_2]u$; if v is white and $[u,v] \in E(F)$, $u[vz_2] \sim_F vz_2u$.

Case 5) adjacent black $x,y \in V_1 \setminus N_{G_1}(z_1)$.
$rw(H_1) = rw(G_1)[xy]$, $lw(H_2) = lw(G_2)$, $rb(H_1) = rb(G_1)[xy]$, $lb(H_2) = lb(G_2)$, $rc(H_1) = rc(G_1)[xy]$, and choosing $w = v$, $lc_w(H_2) = lc_v(G_2)$.

Case 6) adjacent black $x,y \in N_{G_1}(z_1)$.
$rw(H_1) = rw(G_1)[xy]$, $lw(H_2) = lw(G_2)$, $rb(H_1) = rb(G_1)[xy]$, $lb(H_2) = lb(G_2)$, $lc_w(H_2) = lc_v(G_2)$, and letting F be G_1 with z_1 white, we get $rc(H_1) = rc(G_1)uw$ by verifying that $[xy]z_1 \sim_F z_1xy$.

Case 7) adjacent black $x,y \in N_{G_2}(z_2)$.
$rw(H_1) = rw(G_1)$, $lw(H_2) = lw(G_2)[xy]$, $rb(H_1) = rb(G_1)$, $rc(H_1) = rc(G_1)$. Let F be G_2 with z_2 white and F' be G_2 with z_2 black. Since $[xy]z_2 \sim_F z_2xy$, $lb(H_2) = lb(G_2)xy$. Choosing $w = x$, since $[xy][xz_2] \sim_{F'} [yz_2]$, $lc_w(H_2) = lc_v(G_2)$.

Case 8) adjacent black $x,y \in V_2 \setminus N_{G_2}(z_2)$.
$rw(H_1) = rw(G_1)$, $lw(H_2) = lw(G_2)[xy]$, $rb(H_1) = rb(G_1)$, $lb(H_2) = lb(G_2)[xy]$, $rc(H_1) = rc(G_1)$, and choosing $w = v$, $lc_w(H_2) = lc_v(G_2)[xy]$.

Case 9) adjacent black $x \in V_1 \setminus N_{G_1}(z_1)$, $y \in N_{G_1}(z_1)$.
$rw(H_1) = rw(G_1)[xy]$, $lw(H_2) = lw(G_2)$, $rb(H_1) = rb(G_1)[xy]$, $lb(H_2) = lb(G_2)$, $rc(H_1) = rc(G_1)yx$, and choosing $w = v$, $lc_w(H_2) = lc_v(G_2)$.

Case 10) adjacent black $x \in V_1$, $y \in V_2$.
By Lemma 2.5.11, $rw(H_1) = rc(G_1)xz_1$, $lb(H_2) = lb(G_2)y$ and $rc(H_1) = rw(G_1)z_1x$. $rb(H_1) = rb(G_1)[xz_1]$. Choosing $w = v = y$, $lc_w(H_2) = lw(G_2)$ and $lw(H_2) = lc_v(G_2)$.

Case 11) adjacent black $x \in N_{G_2}(z_2)$, $y \in V_2 \setminus N_{G_2}(z_2)$.
$rw(H_1) = rw(G_1)$, $lw(H_2) = lw(G_2)[xy]$, $rb(H_1) = rb(G_1)$, $rc(H_1) = rc(G_1)$. Let F be G_2 with z_2 white. Since $[xy]z_2 \sim_F z_2xy$, $lb(H_2) = lb(G_2)xy$. Choose $w = y$ and let F' be G_2 with z_2 black. Since $[xz_2] \sim_{F'} [xy][z_2y]$, $lc_w(H_2) = lc_v(G_2)$.

To sum up, complementing at a white $u \in V_1$ adjacent to V_2 interchanges the sets $\{Rw_G,Lw_G\}$ and $\{Rb_G,Lb_G\}$, complementing at a white $u \in V_2$ adjacent to V_1 interchanges $\{Rb_G,Lb_G\}$ and $\{Rc_G,Lc_G\}$, and complementing with respect to an edge with incident black vertices in V_1 and V_2 interchanges $\{Rw_G,Lw_G\}$ and $\{Rc_G,Lc_G\}$. All other basic (vertex or edge) parity complementations leave the classes unpermuted. □

The proof of the following is left to the reader:

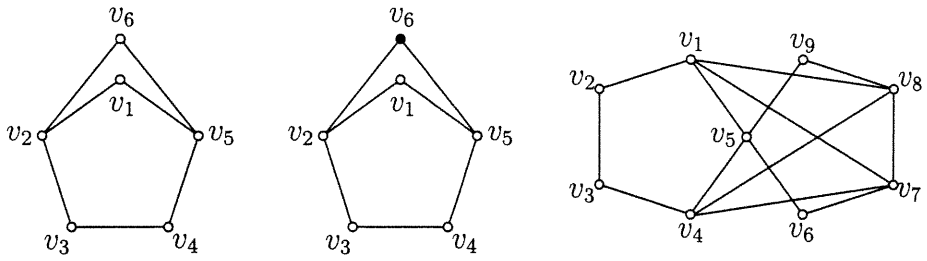
Lemma 2.5.13. *Let A be an independent subset of the black vertices of a bicoloured graph G. Let v be adjacent to $u \in A$. Then $B = A\triangle\{u,v\}$ is an independent subset of the black vertices of $G\{u,v\}$. Furthermore, $N_G(A) \cup A = N_{G\{u,v\}}(B) \cup B$.*

Theorem 2.5.14. *Let G be a simple bicoloured graph with a split $V = V_1 \cup V_2$ inducing the rooted graphs (G_1,z_1) and (G_2,z_2). If any one of the six essential decompositions of $[G]$ along this split is a pair of pure parity classes, then $[G]$ is pure.*

PROOF. If G has no edge between V_1 and V_2, the result is trivial, so let $v \in N_{G_2}(z_2)$. By way of contradiction, suppose that G is not pure. Without loss of generality, G admits a black anticlique A and also one of $\{rw(G_1),lw(G_2)\}$, $\{rb(G_1),lb(G_2)\}$ or $\{rc(G_1),lc_v(G_2)\}$ is a pair of pure graphs. If $A \cap N_{G_1}(z_1) = \emptyset$,

then $A \cap V_2$ is a black anticlique of $lw(G_2)$ and $(A \cap V_1) \cup \{u_1\}$ is a black anticlique of $rb(G_1)$ and $rc(G_1)$. If not, then $A \cap N_{G_2}(z_2) = \emptyset$, $A \cap V_1$ is a black anticlique of $rw(G_1)$ and of $rb(G_1)$, and by Lemma 2.5.13, $(A \cap V_2) \cup \{v\}$ is a black anticlique of $lc_v(G_2)$. □

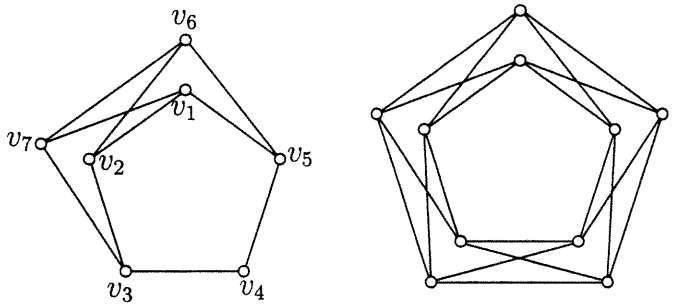
Theorem 2.5.14 allows us to determine numerous pure graphs, for example:



(i) (ii) (iii)

For (i) and (ii), consider the split $V_1 = \{v_2,v_3,v_4,v_5\}$,$V_2 = \{v_1,v_6\}$. For (iii), look at $V_1 = \{v_1,v_2,v_3,v_4\}$,$V_2 = \{v_5,v_6,v_7,v_8,v_9\}$.

From example (i), we find that graph (iv) is pure (take $V_1 = \{v_1,v_3,v_4, v_5,v_6\}$ and $V_2 = \{v_2,v_7\}$) and, by induction, so is graph (v).



(iv) (v) $C_5 \circ \overline{K_2}$

The last of these naturally coloured pure graphs is of special interest, as we will see that it is an intermediate step of a correspondence between C_5 and the transition system of K_5 consisting of two 5-cycles. The following is easily verified:

Proposition 2.5.15. *Given H with split $V = V_1 \cup V_2$ inducing G_1 and G_2, H is an alternance graph if and only if G_1 and G_2 both are.*

Since the pentagon and $\overline{K_2}$ (two isolated vertices) are alternance graphs, so is the lexicographic product $C_5 \circ \overline{K_2}$. Considering the double occurrence word giving rise to $C_5 \circ \overline{K_2}$ as an Euler tour of a transition graph, one of the 2-factors induced by the tour (taking every other edge) is made up of digons (2-cycles), each corresponding to a vertex of C_5. Contracting this 2-factor results in K_5 and determines a transition system made of two 5-cycles. The different steps of the correspondence are depicted in Figure 2.3.

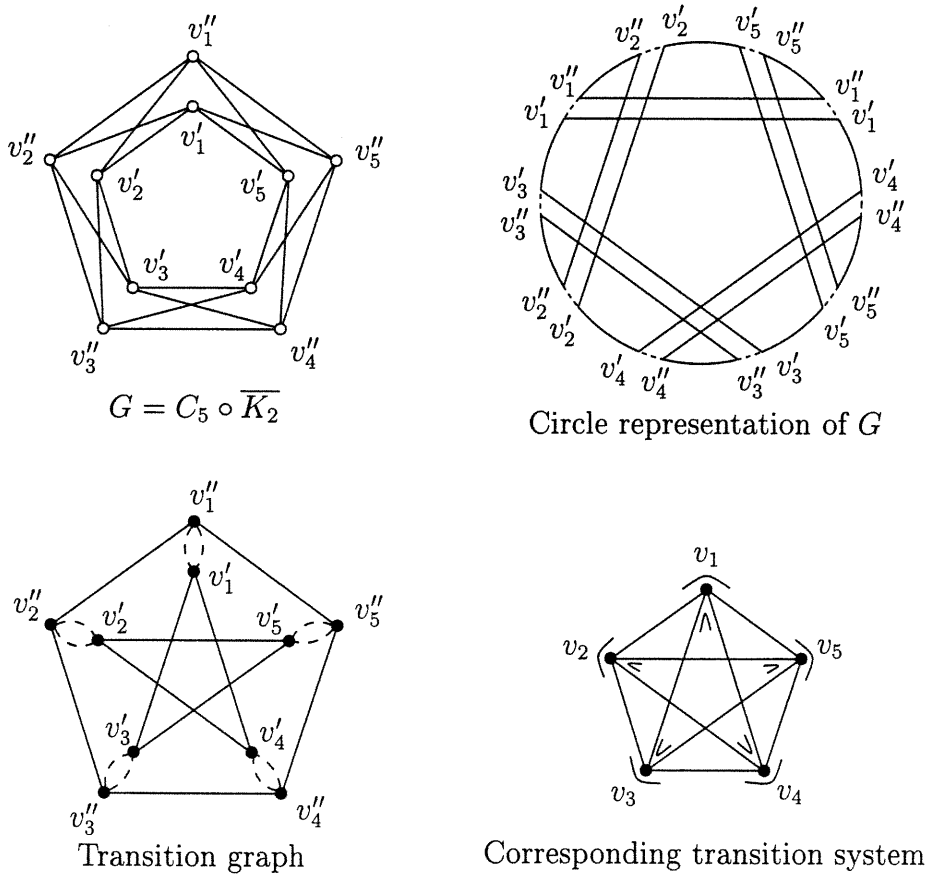


FIG. 2.3 –. *Correspondence between C_5 and K_5 with a transition system made of two 5-cycles.*

In the *circle representation* of $C_5 \circ \overline{K_2}$, the vertices are the chords and the edges are chord intersections. Note that we can read the double occurrence word along the perimeter of the circle. The third graph is obtained by contracting the chords

of the circle representation.

The transformation just described, starting from the pentagon and ending with a transition system of K_5, generalizes naturally to a correspondence between parity classes of alternance bicoloured graphs and transition systems of 4-regular connected graphs.

Definition 2.5.16. The *double* of a bicoloured graph $G = (V,E)$ is the bicoloured graph with vertex set $V \times V(\overline{K_2})$, consisting of $G \circ \overline{K_2}$ and the additional edges $\{[v',v'']|v \text{ black in } G\}$, where v' and v'' stand for (v,u_1) and (v,u_2) and $V(\overline{K_2}) = \{u_1,u_2\}$. The colour of v' and v'' is chosen to be the same as that of v.

Proposition 2.5.17. *A bicoloured graph is pure if and only if its double is pure.*

PROOF. We leave it to the reader to verify that if H is the double of G and u is white in G, or v and w are adjacent black vertices of G, then Hu' is the double of Gu, and $H[v'w']$ is the double of $G[vw]$, respectively. Also, if v is black in G then $H[v'v''] = H$. By the symmetry of $\overline{K_2}$, this shows that $[G]$ is in bijection with $[H]$. Furthermore, a black anticlique of H determines a black anticlique of G using the projection $v',v'' \mapsto v$, and a black anticlique A of G can be used to find a black anticlique of H by taking v' for each $v \in A$. □

Note that a double is necessarily naturally coloured, and that the double of an alternance graph is also an alternance graph. The correspondence we mentioned goes like this:

parity class of a bicoloured alternance graph

↕

parity class of its double

↑

TG with a 2-factor consisting of digons

↕

TS of a 4-regular connected graph

Note that to generalize this to a 1-1 correspondence, we have to associate to each bicolored alternance graph one of its circle representations. Using Proposition 2.5.17, we can now state the following fundamental relationship:

Theorem 2.5.18. *A transition system of a connected 4-regular graph admits no orthogonal cycle decomposition if and only if the corresponding parity class is pure.*

Figure 2.4 shows some examples of pairs (parity class, TS), where the parity class is represented by one of its members. Note that the correspondence induces a bijection between the vertex set of the 4-regular graph and the vertex set of the bicoloured graph. The graphs are drawn to reflect this relationship.

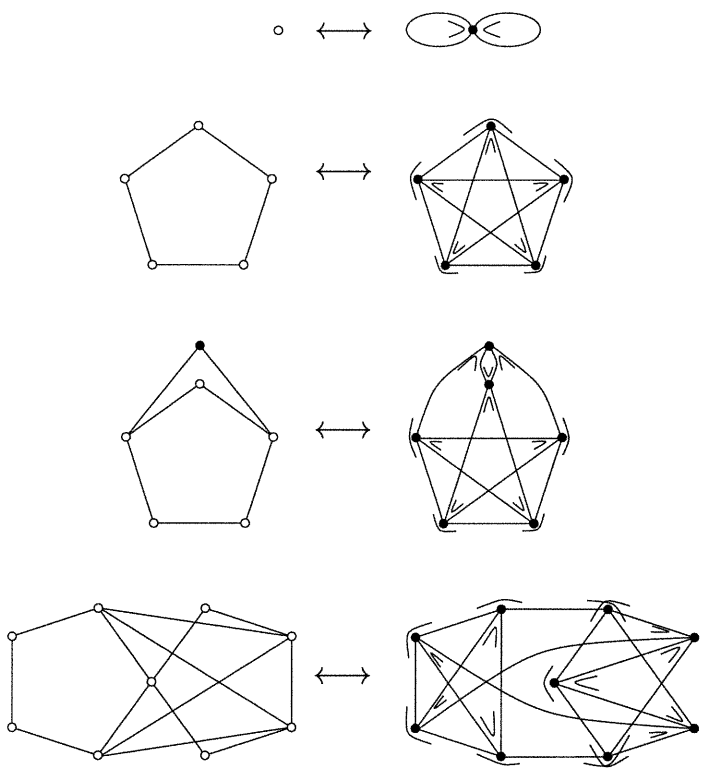


FIG. 2.4 –. *Some pure alternance graphs and corresponding transition systems.*

In the last two pure graphs, hiding on one side of the split, we recognize the pentagon. In the corresponding transition system, this pentagon becomes Fleischner's tetrapus, and to each split, there corresponds a non-trivial edge-cut of size < 6. This is no coincidence.

Proposition 2.5.19. *Let $[G]$ be the parity class of a bicoloured alternance graph, with corresponding transition system defined on Γ. A partition $V(G) = V_1 \cup V_2$ is a split of G if and only if it determines a non-trivial edge-cut $\mathcal{C}$ of Γ of size 2 or 4, and $|V_1|,|V_2| \geq 2$. In this case, $|\mathcal{C}| = 2$ if and only if there is no edge between V_1 and V_2.*

PROOF. Γ being connected and 4-regular, the partition determines an edge cut $\mathcal{C}$ containing an even number of edges. $\mathcal{C}$ is also an edge cut of the transitition graph, partitioning S into $S_1 \cup S_2$, where $t \in S_1 \iff t \cap V \in V_1$. Let w be an alternating tour of the transition graph. Then the following statements are equivalent:

(1) $|\mathcal{C}| = 2$;
(2) the tour is of the form $w = w_1 w_2$ where $w_1 \in S_1^*$,$w_2 \in S_2^*$ (words in the alphabets S_1 and S_2 respectively);
(3) there is no edge between V_1 and V_2 in G.

We can also verify the equivalence of:
(4) $|\mathcal{C}| = 4$;
(5) $w = w_1 w_2 w_3 w_4$ where w_1,$w_3 \in S_1^* \setminus \{\emptyset\}$,$w_2$,$w_4 \in S_2^* \setminus \{\emptyset\}$;
(6) $N_G(V_1) \cap V_2$,$N_G(V_2) \cap V_1$ are non-empty and there is every possible edge between the two sets.

Furthermore, we have the equivalence of:

(7) $\mathcal{C}$ is a trivial edge cut of Γ (say $\mathcal{C} = N_\Gamma(u)$);

(8) without loss of generality $S_1 = \{t_1,t_2\}$ where $u \in t_1 \cap t_2$;

(9) each of w_1,w_3 is either t_1t_2 or t_2t_1;

(10) $V_1 = \{u\}$. □

2.6. An interpretation of Theorem 2.5.14

At this point, let us revisit Definition 2.5.7 and give an interpretation of the root and leaf graph constructions in terms of transition systems when the rooted graph has the structure of an alternance graph:

Definition 2.6.1. Given an eulerian graph Γ of minimum degree $\delta > 2$, a set of transitions S_z of Γ is a *transition system rooted at* $z \in V(\Gamma)$ if no transition of S_z contains z and S_z can be completed to a transition system of Γ by adding transitions at z. Such a set is also called a *z-transition system.*

Note that a z-transition system completely determines Γ so that we can talk about tours and cycles of rooted transition systems. Let (G,z) be a rooted graph such that the underlying simple graph is an alternance graph. Let S be the transition system corresponding to $rw(G)$. First consider the situation where z is not isolated in G. Let $t_1 = \{z,e_1,e_2\}$ and $t_2 = \{z,e_3,e_4\}$ be the transitions of S containing z. We then have that $S_z = S \setminus \{t_1,t_2\}$ is a z-transition system. S_z can be completed to a transition system in three different ways, as shown in Figure 2.5.

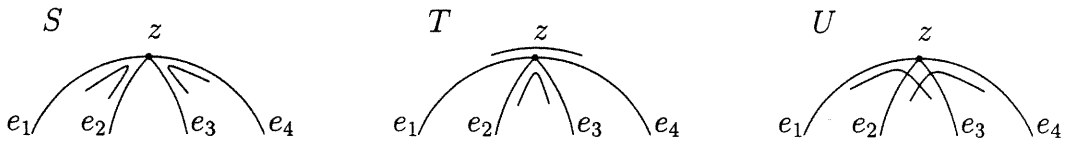


FIG. 2.5 –. *The possible pairs of transitions at z.*

The associated transition graphs differ as follows:

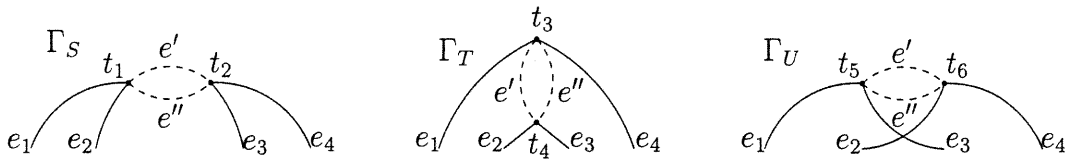


FIG. 2.6 –. *The corresponding transition graphs.*

Let α be an alternating Euler tour of Γ_S corresponding to $rw(G)$. Since z is white in $rw(G)$, α is of the form $t_1e't_2e_3\sigma e_4t_2e''t_1e_2\tau e_1$, where σ and τ are appropriate sequences of vertices and edges. We then have that the alternating Euler tours $\beta = t_3e't_4e_3\sigma e_4t_3e''t_4e_2\tau e_1$ of Γ_T and $\gamma = t_5e't_6e_4\sigma^{-1}e_3t_5e''t_6e_2\tau e_1$ of Γ_U (where σ^{-1} is σ in reverse order) correspond to the root graphs $rb(G)$ and $rc(G)$, respectively. In the case where z is isolated in G, let $t_1 = \{z,e_1,e_3\}$ and $t_2 = \{z,e_2\}$ be the transitions of S at z. The z-transition system $S_z = S \setminus \{t_1,t_2\}$ can be completed to a transition system in only one other way, as seen in Figure 2.7.

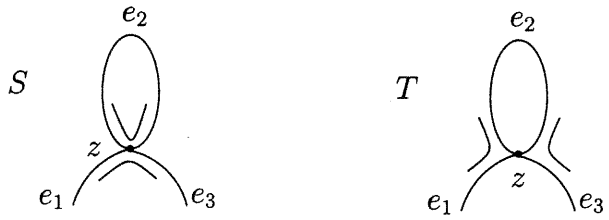


FIG. 2.7 –. *When z is isolated in the bicoloured graph.*

Given an Euler tour $\alpha = t_1e't_2e_2t_2e''t_1e_3\tau e_1$ of Γ_S corresponding to $rw(G)$, $\beta = t_3e't_4e_2t_3e''t_4e_3\tau e_3$ is an Euler tour of Γ_T corresponding to $rb(G)$.

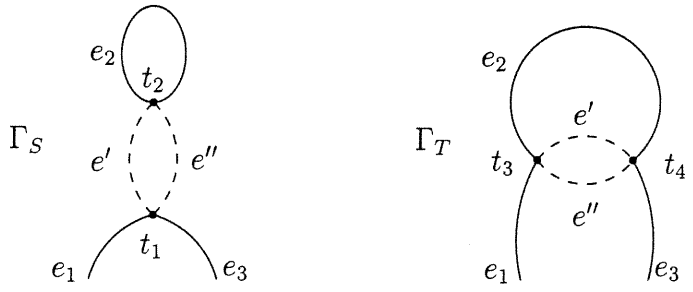


FIG. 2.8 –. *The corresponding transition graphs.*

Thus we see that to each rooted alternance graph (G,z) there corresponds a unique connected 4-regular rooted transition system S_z, and that the root graphs of G are in correspondence with the different ways the rooted transition system can be completed. A similar analysis reveals that the leaf graphs of G are in correspondence with the different transition systems that can be obtained from S_z by identifying the edges incident with z two by two.

We are now in a position to interpret Theorem 2.5.14 in terms of transition systems. Let a bicoloured alternance graph G have an essential decomposition into a pair of pure root and leaf graphs H_1 and H_2, where $V(G) = V_1 \cup V_2$ is a split of G inducing the rooted graphs (G_1,z_1) and (G_2,z_2), such that H_1 is a root graph of G_1, and H_2 is a leaf graph of G_2. Let S_{z_1} and S_{z_2} be the rooted transition systems corresponding to G_1 and G_2. It can be verified that if the transition system corresponding to H_1 is $S_{z_1} \cup \{\{z_1,e_1,e_2\},\{z_1,e_3,e_4\}\}$, then the transition system corresponding to H_2 is obtained from S_{z_2} by identifying e_1 with e_2, and e_3 with e_4.

2.7. PRIMITIVITY OF PURE BICOLOURED GRAPHS

Definition 2.7.1. A family $\mathcal{F}$ of parity classes is *closed*, if for each bicoloured graph G, whenever

i) G admits an essential decomposition $\{H_1,H_2\}$ such that $[H_1],[H_2] \in \mathcal{F}$, or

ii) G has a component G_1 such that $[G_1] \in \mathcal{F}$,

then $[G] \in \mathcal{F}$.

The intersection of closed families of parity classes is closed, hence given any family $\mathcal{F}$ of parity classes, we can define the *closure* $\overline{\mathcal{F}}$ of $\mathcal{F}$ to be the smallest closed family of parity classes containing $\mathcal{F}$ as a subset.

Note that similar definitions can be made when we restrict our attention to parity classes of bicoloured alternance graphs. Accordingly, some of the following results have an equivalent formulation in the context of alternance graphs.

Definition 2.7.2. A pure graph G (or its class $[G]$) is *primitive* if for any family $\mathcal{F}$ of pure classes, $[G] \in \overline{\mathcal{F}}$ implies $[G] \in \mathcal{F}$.

Conjecture 2.7.3. *All primitive pure graphs are prime.*

Recall that from a rooted graph we can construct up to three different root graphs (two if the root vertex is isolated). Of course, any bicoloured graph G can be viewed as a root graph: for a given vertex u of G, let (G,u) be the rooted graph obtained by restricting the colouring of G to $V(G) \setminus \{u\}$, then G is either $rw(G')$ or $rb(G')$. We call the root graphs constructed from (G,u) the root graphs *induced* by u from G.

Definition 2.7.4. Given a pure graph G, a vertex u of G is *critical* if one of the root graphs induced by u is not pure.

Given a bicoloured graph G and $u \in V(G)$ inducing the root graphs G_1,G_2 and G_3, we have, as a corollary of Theorem 2.5.12, that $S = \{[G_1],[G_2],[G_3]\}$ is invariant over $[G]$ (i.e. if u induces the root graphs H_1,H_2 and H_3 from $H \in [G]$, then $\{[H_1],[H_2],[H_3]\} = S$). This gives us:

Proposition 2.7.5. *The set of critical vertices of a pure graph G is invariant over $[G]$.*

Proposition 2.7.6. *Let u be a critical vertex of a pure graph G. Of the root graphs induced by u, only G is pure.*

PROOF. If u is isolated, the result is trivial. Suppose, by way of contradiction, that u is not isolated and that only one root graph induced by u is not pure. Let G' be the rooted graph obtained by restricting the bicolouring of G to $V(G)\setminus\{u\}$. By Proposition 2.7.5, we can suppose that one of $rw(G')$, $rb(G')$ or $rc(G')$ has a black anticlique A. If A is a black anticlique of $rw(G')$, then it is also a black

anticlique of $rb(G')$. If A is a black anticlique of $rb(G')$ but not of $rw(G')$, then it is one of $rc(G')$. If A is a black anticlique of $rc(G')$ but not of $rb(G')$, then it is one of $rw(G')u$. All cases lead to a contradiction. □

Definition 2.7.7. A vertex u of a pure graph G is said to be *tight* if there exists $H \in [G]$ such that $H - u$ has a black anticlique. A pure graph is *tight* if every critical vertex is tight. The parity class is then also said to be *tight.*

Obviously, a tight vertex is also critical. It is easy to verify that the one-point graph and the pentagon with their natural colourings are tight.

Proposition 2.7.8. *A vertex u of a pure graph G is tight if and only if $G - u$ is not pure.*

PROOF. One side is clear. Let S be a complementation set of G (see [8]) such that $GS - u$ has a black anticlique A. If $u \notin S$, then S is a complementation set of $G' = G - u$ and A is a black anticlique of $G'S$ $(= GS - u)$. If $u \in S$, since u is white in GS (by the purity of G), $S' = S \setminus \{u\}$ is a complementation set of G, and then also a complementation set of $G' = G - u$. Since $u \notin N_{GS}(A)$ (again by the purity of G), A is a black anticlique of $G'S'$. □

Consider the closure $\overline{\mathcal{F}}$ of a family $\mathcal{F}$ of parity classes. Given a parity class $[G] \in \overline{\mathcal{F}}$, the following possibilities may arise:

(1) $[G]$ is in $\mathcal{F}$;

(2) G is disconnected, in which case G has a connected component whose parity class is in $\overline{\mathcal{F}}$; or

(3) G has an essential decomposition into two graphs H_1 and H_2 such that $[H_1]$,$[H_2] \in \overline{\mathcal{F}}$.

This analysis can be applied recursively to the resulting smaller graphs ("factors") with parity classes in $\overline{\mathcal{F}}$. The whole process is called an *$\mathcal{F}$-factorization* of G (or more precisely, an *essential $\mathcal{F}$-factorization*). For the purposes of the following proposition, given $[G] \in \overline{\mathcal{F}}$, where $\mathcal{F}$ is a family of pure graphs, call a vertex u of G *factor-essential* if every pure factor H of an $\mathcal{F}$-factorization such that $u \in H$ is connected.

Proposition 2.7.9. *Let $\mathcal{F}$ be a family of parity classes that are primitive, pure and tight. Let $[G] \in \overline{\mathcal{F}}$ be such that all primitive pure graphs of smaller order than G have their parity class in $\mathcal{F}$. If $u \in V(G)$ is factor-essential, then u is a tight vertex of G.*

PROOF. First note that any critical vertex is factor-essential, so that the truth of the statement for a given graph implies that it is tight. We proceed by induction on the order of G. The statement is true if $[G] \in \mathcal{F}$. If G is not connected, then, by the induction hypothesis, u is a tight vertex of the component that contains it. Since the other components are not pure, u is also tight in G. If G is connected but $[G] \notin \mathcal{F}$, then it has an essential decomposition into the pure root graph H_1 and the pure leaf graph H_2 where $[H_1]$,$[H_2] \in \overline{\mathcal{F}}$. Suppose that the corresponding split is $V(G) = V_1 \cup V_2$, inducing the rooted graphs (G_1,z_1) and (G_2,z_2). Without loss of generality, $H_1 = rw(G_1)$ and $H_2 = lw(G_2)$ (because of the presence of white vertices in H_1 and H_2, we can always locally complement G to reach this kind of decomposition).

Consider the case where $u \in V_2$. Since u is factor-essential in G, z_1 is factor-essential in H_1. By the induction hypothesis, z_1 is a tight vertex of H_1 and we can suppose that $H_1 - z_1$ has a black anticlique. Since u has to be factor-essential in H_2, it is a tight vertex of H_2 and we can suppose that $H_2 - u$ has a black anticlique. Taking the union of the anticliques, we see that u is a tight vertex of G.

There remains the case where $u \in V_1$. Since u is factor-essential in G, it is factor-essential in H_1 and thus tight in H_1. Let S be a complementation set of $H_1 - u$ such that $H_1S - u$ has a black anticlique A. We can suppose that $z_1 \notin A$: if it is, take $v \in N_{H_1S}(z_1)$ (which is not empty, since H_1 is connected, and does not contain u, by the purity of H_1) and replace S with $S' = S\triangle\{z_1,v\}$, then $A' = A\triangle\{z_1,v\}$ is a black anticlique of $H_1S' - u$ such that $z_1 \notin A'$. Suppose that either one of $rw(G_2)$ or $rb(G_2)$ is pure. Then the purity of H_2 implies that z_2 is not tight (therefore not critical) and both $rw(G_2)$ and $rb(G_2)$ are pure. However, this would mean that z_2 is not factor-essential in $rw(G_2)$, which would contradict the fact that u is factor-essential in G. Thus neither one of $rw(G_2)$ and $rb(G_2)$ is pure. If $z_1 \in S$, let S' be a complementation set of $F = rw(G_2)$ such that FS' has a black anticlique A'. Without loss of generality (following an argument similar to the z_1 case), $z_2 \notin S'$. By the purity of H_2, $z_2 \in A'$ (i.e. it has changed colour). Then $S'' = S \cup S'$ is a complementation set of G such that $A \cup A' \setminus \{z_2\}$ is a black anticlique of $GS'' - u$, and u is a tight vertex of G. If $z_1 \notin S$, let S' be a complementation set of $F = rb(G_2)$ such that FS' has a black anticlique A'. Without loss of generality, $z_2 \notin S'$. By the purity of H_2, $z_2 \in A'$ (its colour has not changed). Then $S'' = S \cup S'$ is a complementation set of G such that $A \cup A' \setminus \{z_2\}$ is a black anticlique of $GS'' - u$, and u is a tight vertex of G. □

Corollary 2.7.10. *Let $\mathcal{F}$ be a family of parity classes that are primitive, pure and tight. Let G be a smallest primitive pure graph such that $[G] \notin \mathcal{F}$. Then G has no essential decomposition into a root graph H_1 and a leaf graph H_2 such that H_1 is pure.*

PROOF. Suppose, by contradiction, that $V(G) = V_1 \cup V_2$ is a partition inducing the rooted graphs (G_1,z_1) and (G_2,z_2) and such that H_1 is a pure root graph of G_1 and H_2 is the corresponding leaf graph of G_2. The minimality of G forces H_1 to be in $\overline{\mathcal{F}}$. If z_1 is not critical in H_1, then it has an essential factorization such that at least one pure graph contains z_1 but has a pure component not containing

z_1. But then $G \in \overline{\mathcal{F}}$, contradicting the primitivity of G. Therefore z_1 is a tight vertex of H_1, and G is pure if and only if H_2 is pure, contradicting either the purity or the primitivity of G. □

A corresponding statement is obtained by considering only alternance graphs:

Corollary 2.7.11. *Let $\mathcal{F}$ be a family of alternance parity classes (parity classes of bicoloured alternance graphs) that are primitive, pure and tight. Let G be a smallest primitive pure alternance graph such that $[G] \notin \mathcal{F}$. Then G has no essential decomposition into a root graph H_1 and a leaf graph H_2 such that H_1 is pure.*

Theorem 2.7.12. *A smallest primitive pure alternance graph different from the white one-point graph and the graphs in the parity class of the white pentagon must be prime.*

PROOF. Suppose, by way of contradiction, that such a bicoloured graph G has a split $V = V_1 \cup V_2$ inducing the rooted graphs (G_1,z_1) and (G_2,z_2). Let S_{z_1} be the z_1-transition system corresponding to G_1 and S_{z_2} the z_2-transition system corresponding to G_2. We have seen that the transition system S corresponding to G has an edge-cut of size 2 or 4 separating V_1 from V_2. In fact, since G is primitive (and thus connected), this cut contains four edges. Clearly, these are the edges, say e_1,e_2,e_3 and e_4, that are incident with z_1 in S_{z_1} and incident with z_2 in S_{z_2}. Consider the number $n(S_z)$ of partitions into pairs of the edges incident with z that can be extended to an orthogonal cycle decomposition of the rooted transition system S_z. By Corollary 2.7.11, $n(S_{z_1}),n(S_{z_2}) > 1$. By the pigeonhole principle, at least one partition of the set $\{e_1,e_2,e_3,e_4\}$ into pairs of edges can be extended to an orthogonal cycle decomposition of S. □

The following is essentially Conjecture 12 in [6], due to Fleischner and Jackson:

Conjecture 2.7.13. *If $\Gamma \neq K_5$ (with $\delta > 2$) has no non-trivial edge cut of size < 6, then any transition system of Γ admits an orthogonal cycle decomposition.*

Conjecture 2.7.14. *The only connected prime pure alternance graphs are the white one-point graph and those in the parity class of the white pentagon.*

The two conjectures are equivalent: a connected prime pure alternance graph gives rise to a 4-regular transition system without non-trivial edge cuts of size < 6, while a counter-example to Conjecture 2.7.13 would imply the existence of at least one new primitive pure alternance graph, the smallest of which must be prime. In addition to providing a characterization of transition systems with orthogonal cycle decompositions, the truth of Conjectures 2.7.13 and 2.7.14 would imply the following:

Conjecture 2.7.15 (Cycle Double Cover Conjecture). *For every bridgeless graph (i.e. without edge-cuts of size one) there is a family of cycles of this graph such that every edge belongs to exactly two cycles of the family.*

The idea is that a counter-example G, minimal with respect to the number of edges, would be 3-regular and cyclically 6-edge-connected. Contracting a 1-factor (which exists by Petersen's Theorem) and choosing the transition system induced by the remaining 2-factor of G, we get a 4-regular graph without non-trivial edge cuts of size < 6. By Conjecture 2.7.13, we obtain a cycle decomposition orthogonal to the transition system. Completing it with edges of the 1-factor, this decomposition extends to a family of cycles of G covering the 1-factor twice and the 2-factor once. Throwing in the cycles of the 2-factor, we get a cycle double cover of G. Alternatively, as pointed out by Jaeger (see [11]), we can take the line graph of G with transition system inducing the triangles corresponding to the vertices of G. This is another transition system of a 4-regular graph without non-trivial edge cuts of size < 6. A cycle decomposition orthogonal to this transition system extends naturally to a cycle double cover of G, and vice versa.

To illustrate the preceding argument, here is how we can rule out the Petersen graph as a counter-example to the Cycle Double Cover Conjecture: contracting a 1-factor yields our familiar K_5 with a 5-cycle decomposition, corresponding to the

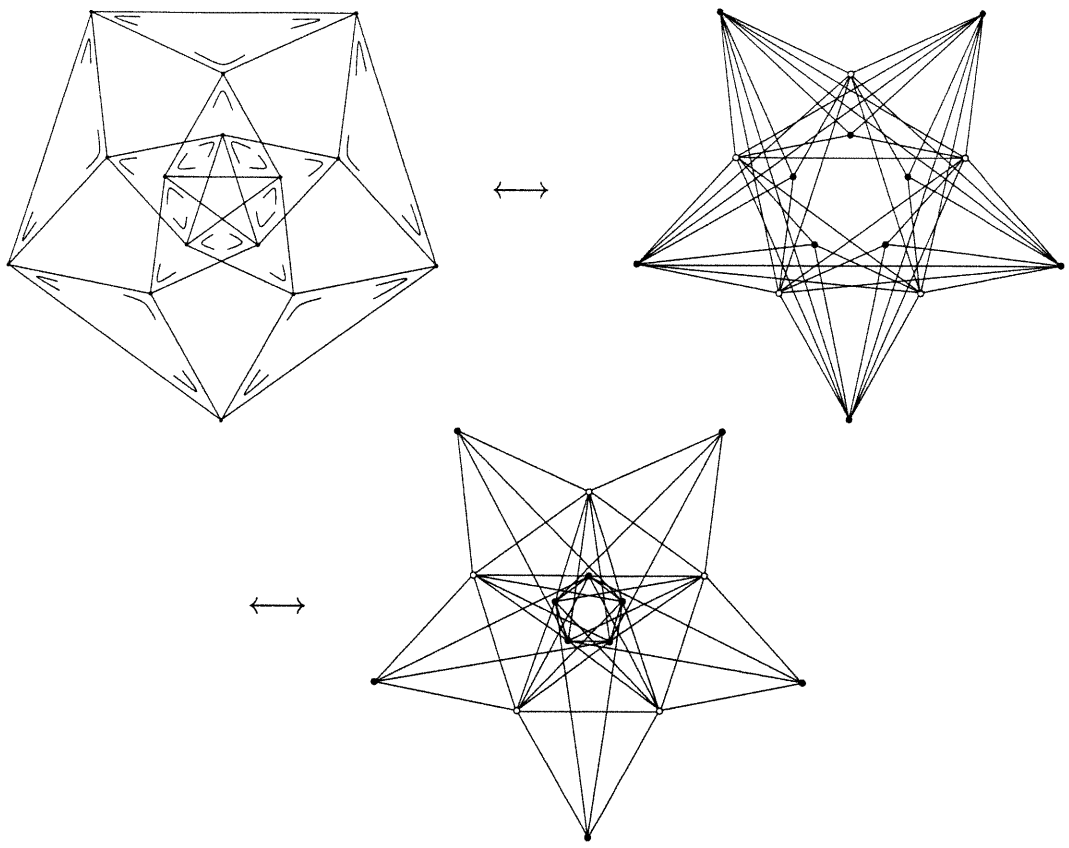


FIG. 2.9 –. *A transition system constructed from the Petersen graph and some associated bicoloured graphs.*

pure class of the pentagon. However, taking the line graph approach, we obtain a parity class which admits black anticliques.

In Figure 2.9, we give two highly symmetric representatives of this class. Black anticliques can be found by taking the inner five vertices of the first graph or the outer five vertices of the second one.

It is easy to see from the correspondence that it suffices to prove Conjecture 2.7.13 for 4-regular graphs. Furthermore, any counter-example must contain a K_5-minor (Fan and Zhang [4]).

Two more primitive pure classes are known. We have that $[Z_{13}]$ and $[Z_{17}]$ are pure classes, where Z_{13} and Z_{17} are the naturally coloured Cayley graphs $\mathrm{Cay}(\mathbb{Z}_{13}, \pm$

$\{1,3,4\}$) and Cay($\mathbb{Z}_{17}, \pm \{1,5\}$). However, while prime, the members of $[Z_{13}]$ and $[Z_{17}]$ are not alternance graphs.

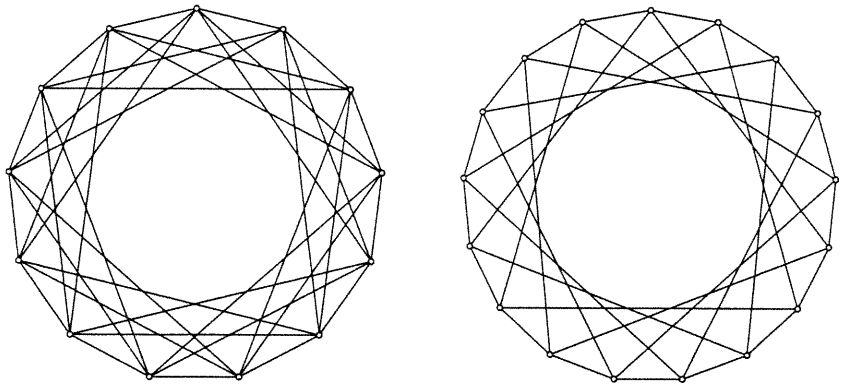


FIG. 2.10 –. *The primitive pure graphs Z_{13} and Z_{17}.*

$[Z_{13}]$ and $[Z_{17}]$ contain 39 and 1069 graphs respectively, up to isomorphism. Many more graphs were tested by computer, with the focus on vertex-transitive graphs. I tend to believe these four primitive pure classes are exhaustive but this is definitely an open question. Parity classes are subsets of complementation classes, in which complementation is allowed at black vertices as well as at white vertices. Isotropic systems are a generalization of 4-regular graphs and dual binary matroids. Bouchet [2] showed how the set of isotropic systems can be put in bijection with the set of complementation classes of naturally coloured graphs. Perhaps a new algebraic object akin to isotropic systems can be put in bijection with parity classes. This might help to characterize primitive pure graphs and settle the Cycle Double Cover Conjecture.

2.8. REFERENCES

[1] Bouchet, A., Reducing prime graphs and recognizing circle graphs, *Combinatorica* **7** (1987), 243-254.

[2] Bouchet, A., Graphic presentations of isotropic systems, *J. Combin. Theory Ser. B* **45** (1988), 58-76.

[3] Bouchet, A., Circle graph obstructions, *J. Comb. Theory Series B* **60** (1994), 107-144.

[4] Fan, G.; Zhang, C.-Q., Circuit decompositions of eulerian graphs, *J. Combin. Theory Ser. B.* **78** (2000), 1-23.

[5] Fleischner, H., Cycle decompositions, 2-coverings, removable cycles, and the four-color-disease, *Progress in graph theory (Waterloo, 1982)*, Academic Press, Toronto (1984), 233-246.

[6] Fleischner, H., Some blood, sweat, but no tears in eulerian graph Theory, *Congr. Numer.* **63** (1988), 8-48.

[7] de Fraysseix, H., A characterization of circle graphs, *European J. Combin.* **5** (1984), 223-238.

[8] Genest, F., Word and set complementation of graphs, invertible graphs, submitted.

[9] Jackson, B., A characterization of graphs having three pairwise compatible Euler tours, *J. Combin. Theory Ser. B* **53** (1991), 80-92.

[10] Jackson, B., On circuit covers, circuit decompositions and Euler tours of graphs, *Surveys in Combinatorics (Keele, 1993)*, London Math. Soc. Lecture Note Ser. **187**, Cambridge Univ. Press, Cambridge (1993), 191-210.

[11] Jaeger, F., A survey of the cycle double cover conjecture, *Cycles in graphs (Burnaby, 1982)*, North-Holland Math. Stud. **115**, North-Holland, Amsterdam (1985), 1-12.

[12] Kotzig, A., Eulerian lines in finite 4-valent graphs and their transformations, *Theory of Graphs (Tihany, 1966)*, Proc. Colloqium on Graph Theory Tihany 1966, Academic Press (1968), 219-230.

[13] Kotzig, A., *Quelques remarques sur les transformations* κ, séminaire Paris (1977).

[14] Sabidussi, G., *Eulerian walks and local complementation*, D.M.S. 84-21, Dép. de math. et stat., Université de Montréal (1984).

[15] Spinrad, J., Recognition of circle graphs, *J. Algorithms* **16** (1994), 264-282.

CONCLUSION

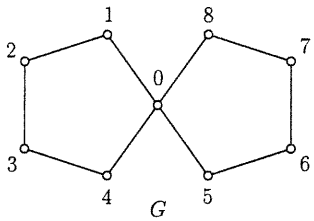
Le théorême de décomposition essentielle (théorême 2.5.14 du deuxième article) est le théorême fondamental de la théorie des graphes purs. Grâce à lui, l'étude des graphes purs se résume à l'étude des graphes purs primitifs. Nous proposons que ces derniers sont premiers (conjecture 2.7.3 du même article).

Étant donné la correspondance canonique entre les systèmes de transitions 4-réguliers connexes et les classes de parité de graphes de cordes bicoloriés, l'étude des systèmes de transitions sans décompositions en cycles est équivalente à l'étude des graphes de cordes purs primitifs.

Une meilleure compréhension de la pureté passe probablement par une généralisation des systèmes de transitions 4-réguliers connexes. Le principal intérêt d'une caractérisation des graphes purs (ne serait-ce que celle des graphes de cordes purs) réside dans son éventuelle contribution à la résolution de la conjecture de double recouvrement.

Annexe A

CLASSE DE PARITÉ DES PENTAGONES SIAMOIS



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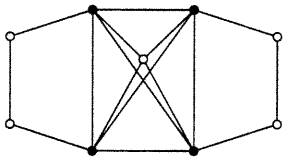

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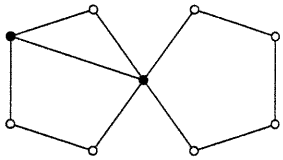

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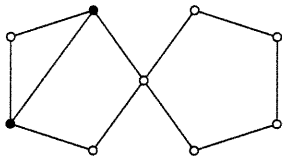

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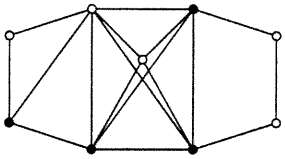

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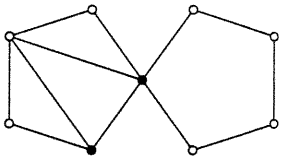

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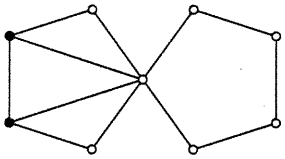

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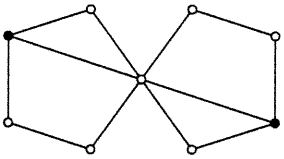

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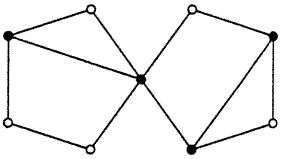

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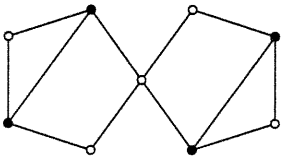

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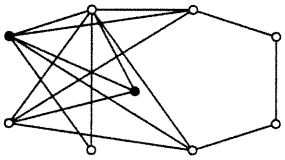

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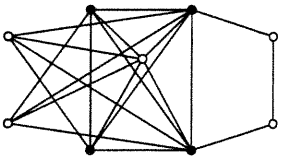

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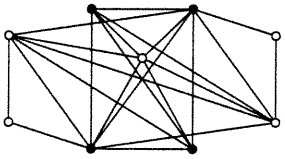

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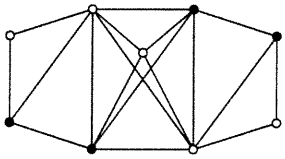

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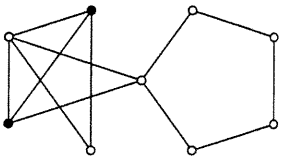

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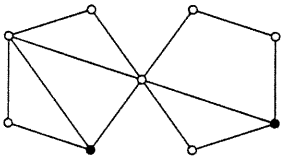

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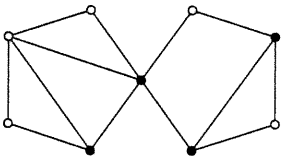

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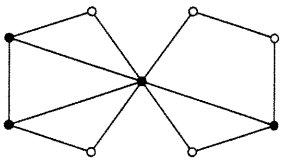

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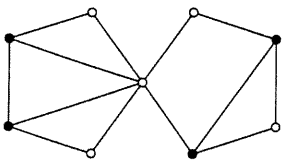

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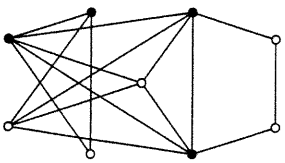

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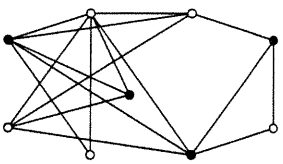

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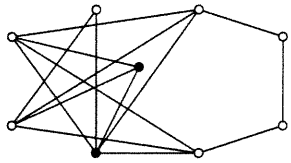

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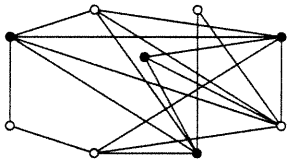

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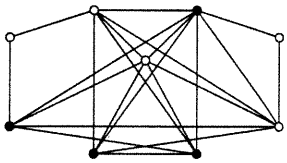

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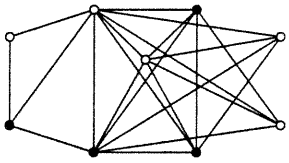

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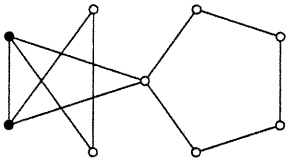

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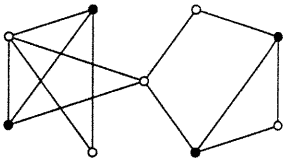

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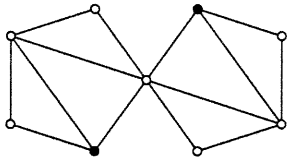

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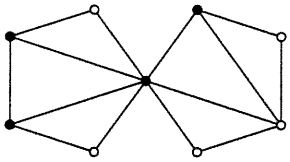

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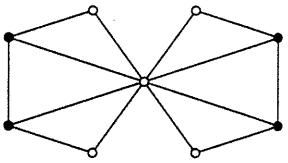

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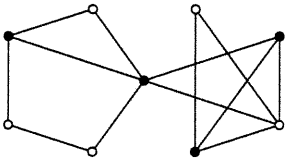

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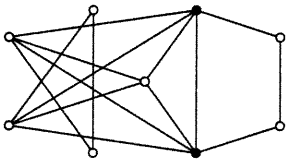

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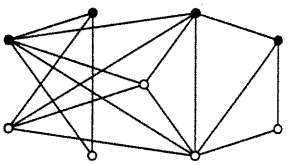

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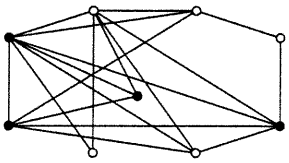

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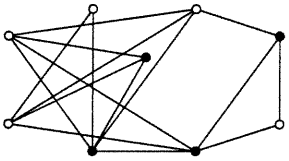

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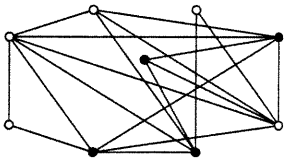

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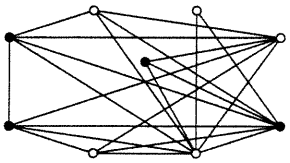

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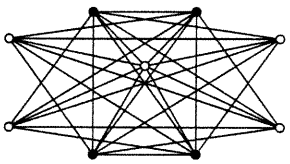

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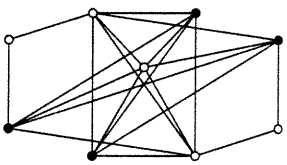

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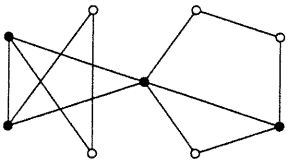

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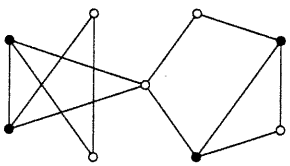

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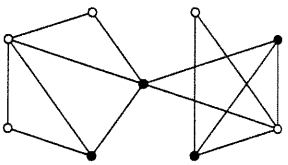

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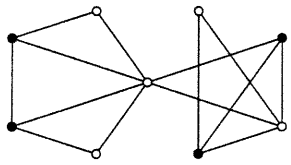

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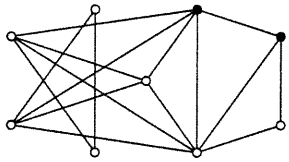

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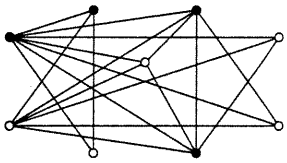

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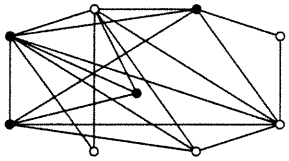

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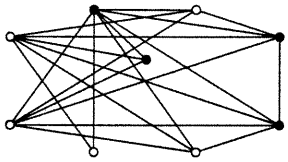

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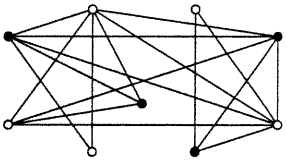

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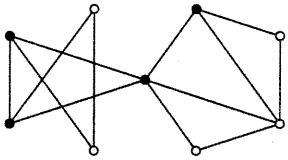

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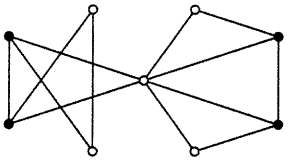

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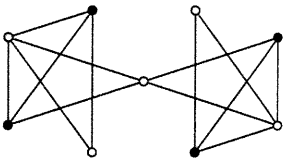

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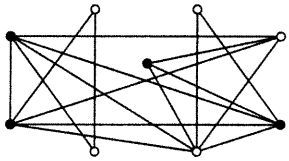

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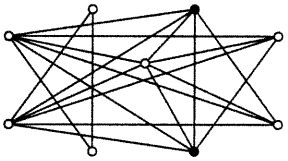

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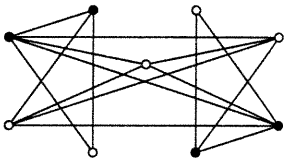

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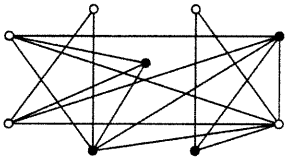

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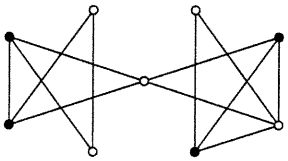

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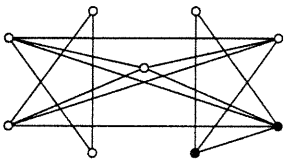

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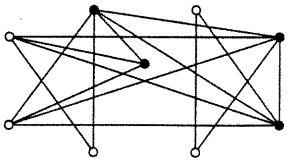

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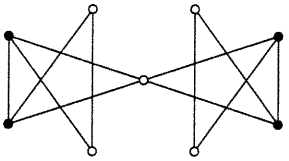

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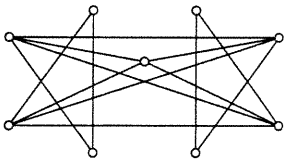
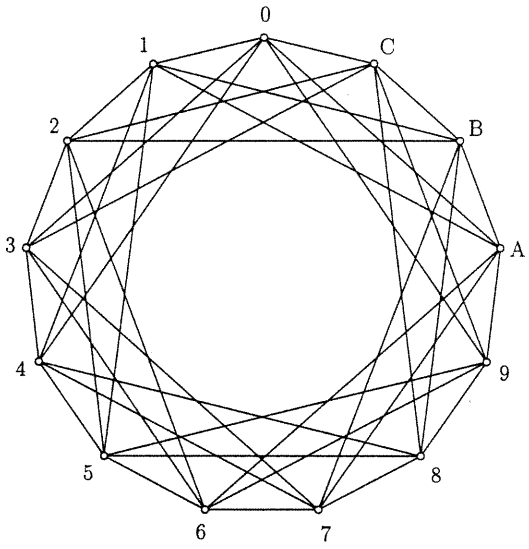


$G132457680$

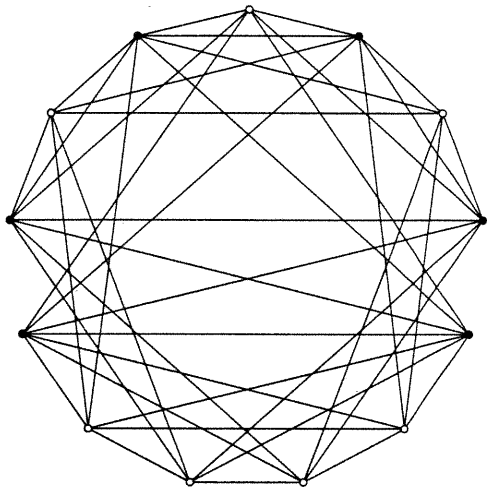
Tout autre graphe de la classe de parité de G est isomorphe à un de ces 60 représentants non isomorphes. On peut vérifier que la classe est pure en s'assurant qu'aucun de ces graphes n'a d'anticlique noire.

Annexe B

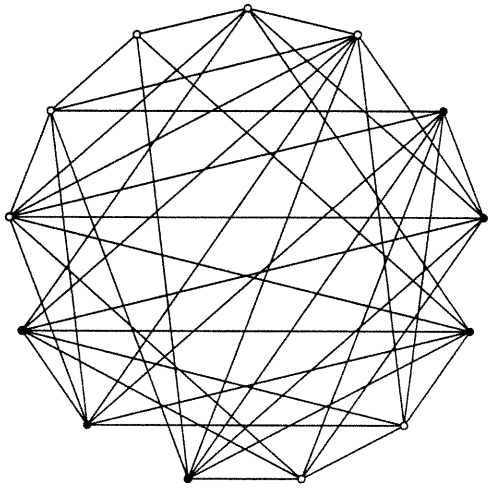
CLASSE DE PARITÉ DE $\mathrm{CAY}(\mathbb{Z}_{13}, \pm\{1,3,4\})$



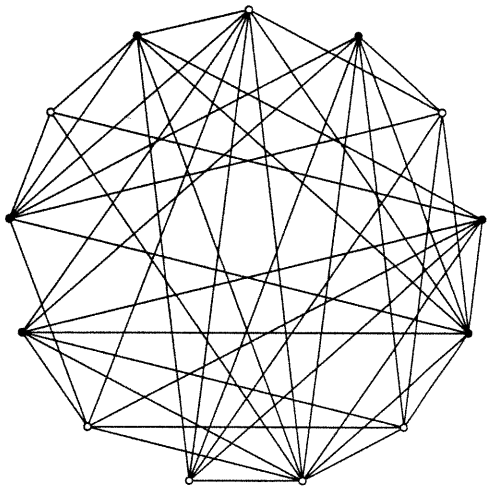
G



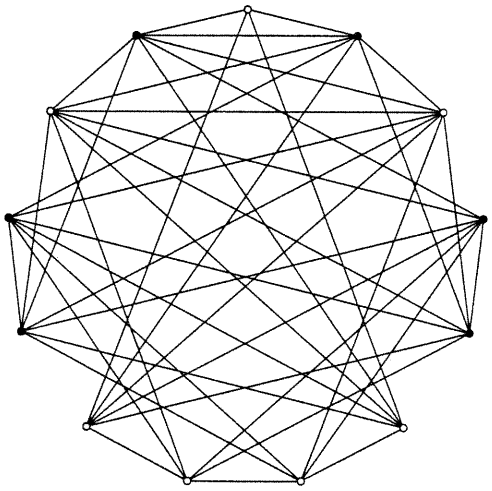
$G0$



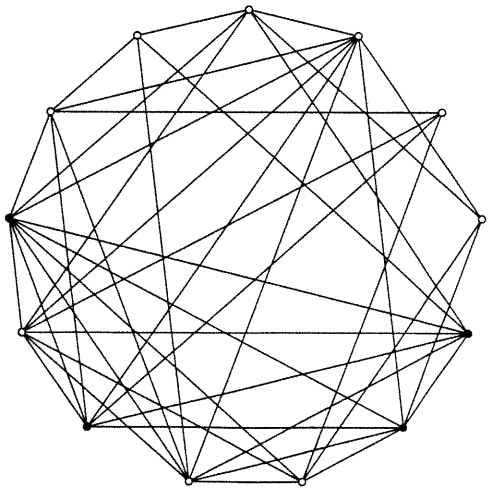
$G02$



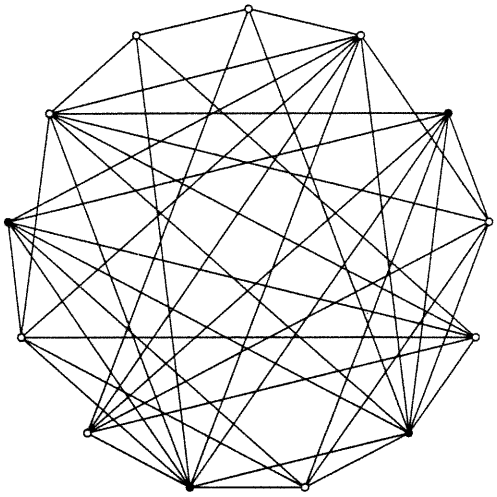
$G130$



$G1C0$



$G027$



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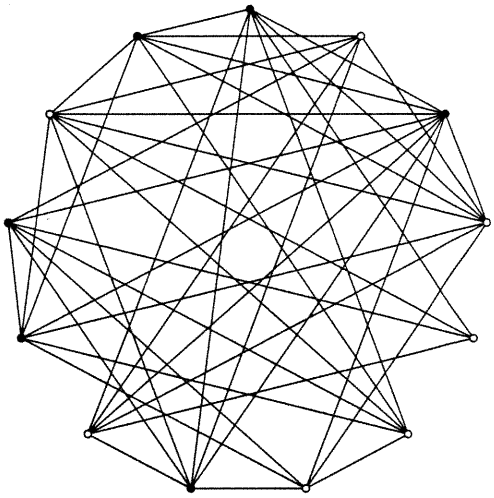
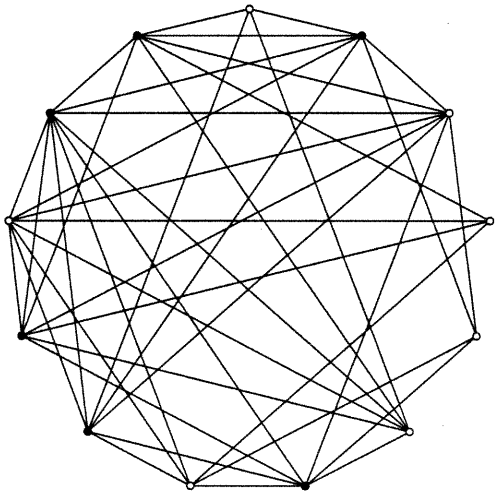


$G051C$



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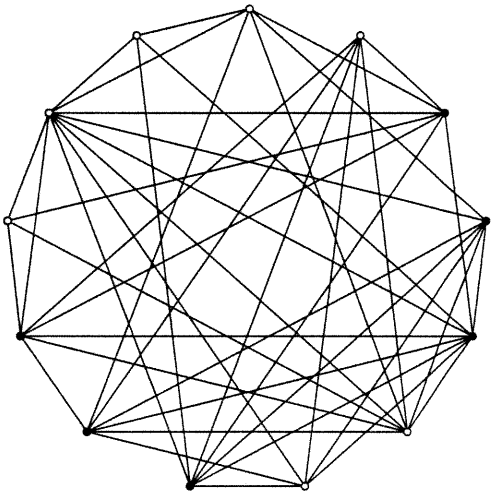

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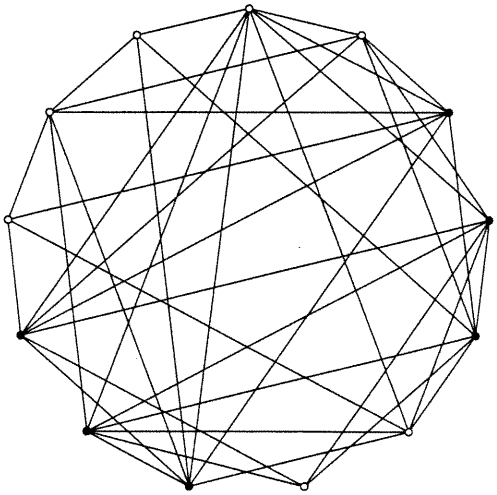

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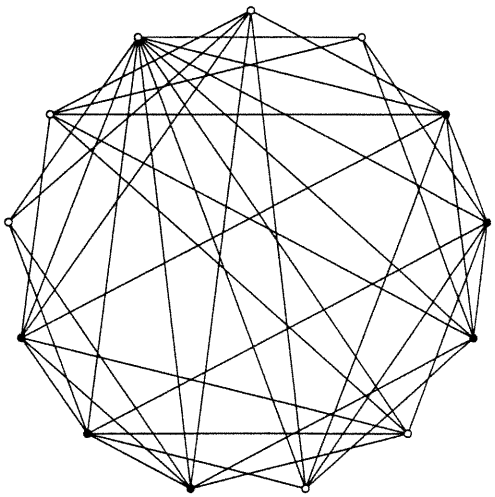

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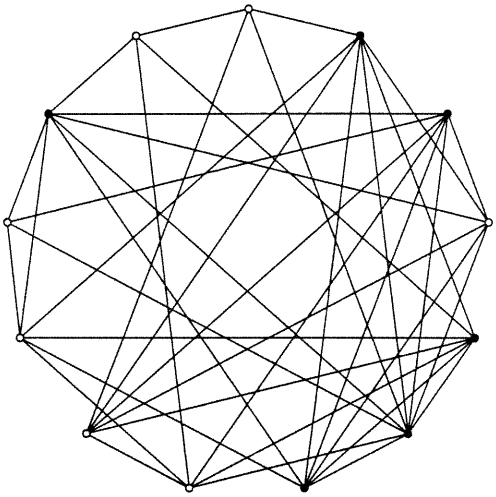
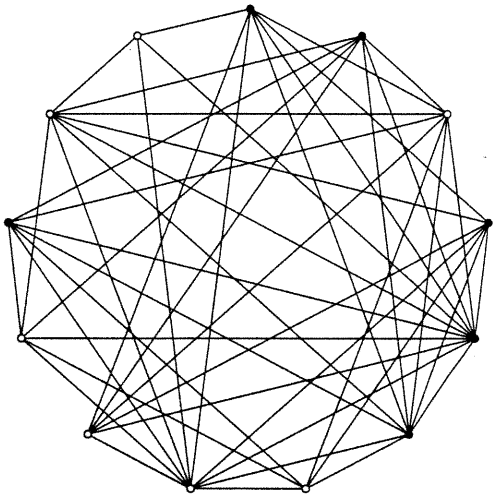
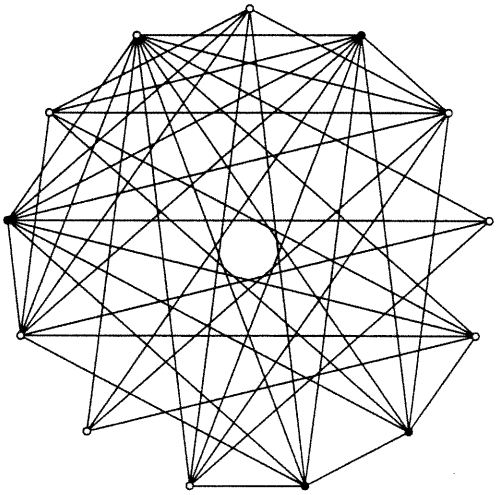


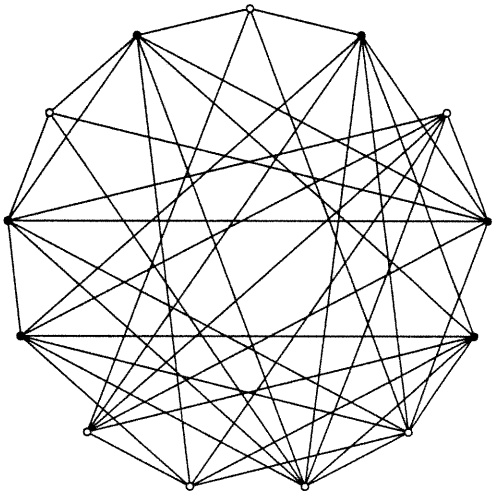
$G1C024$



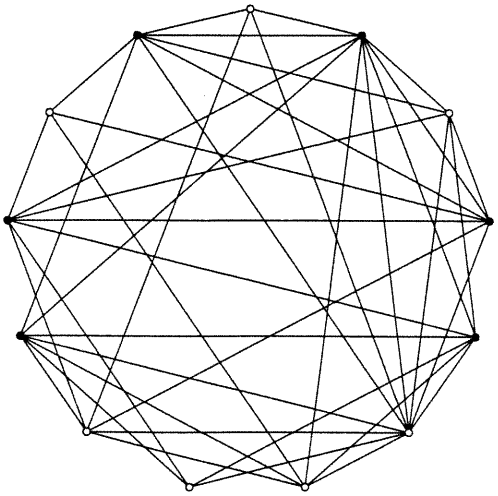
$G051C2$



$G02169$



$G134C0$



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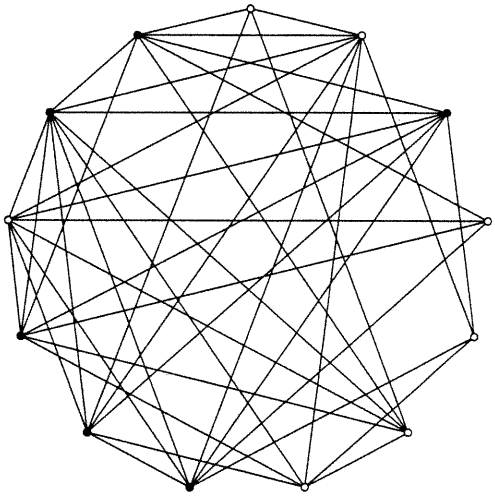

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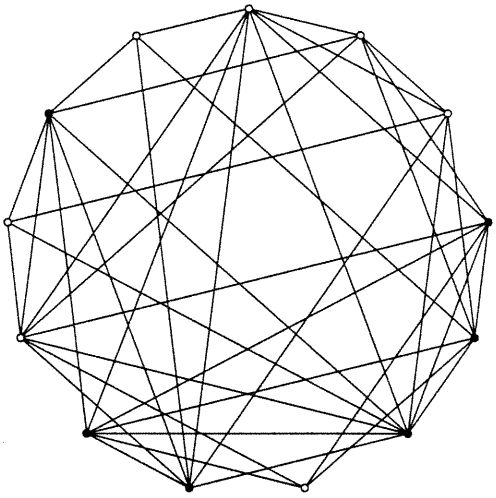

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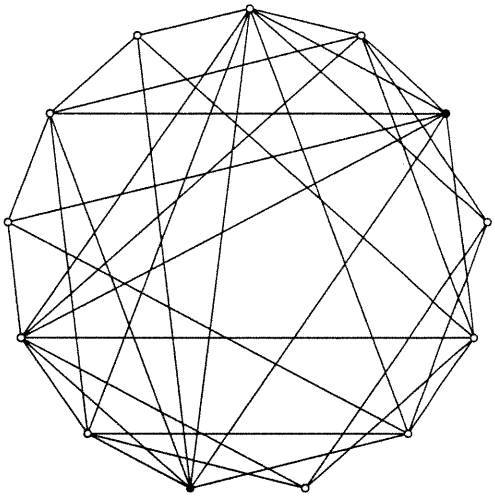

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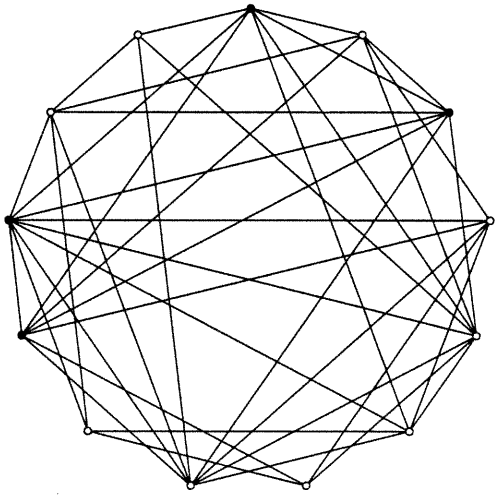

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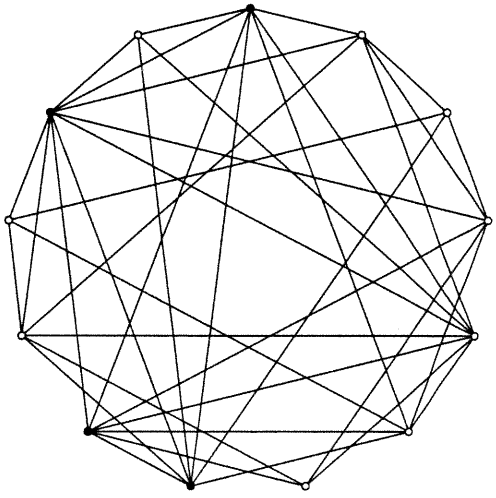

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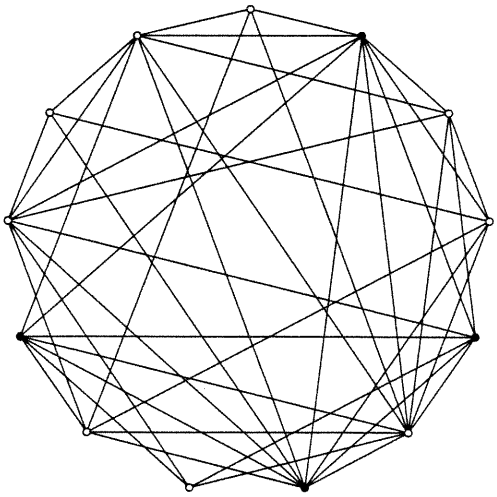

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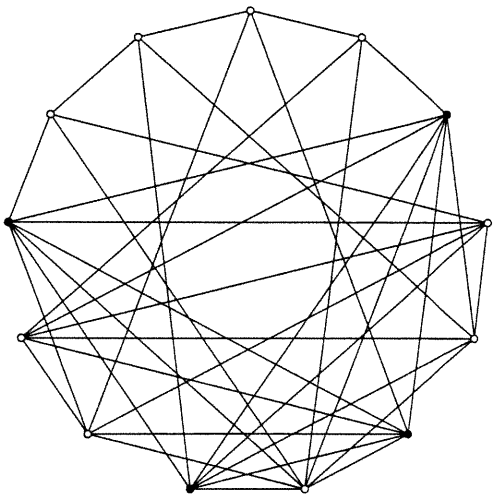

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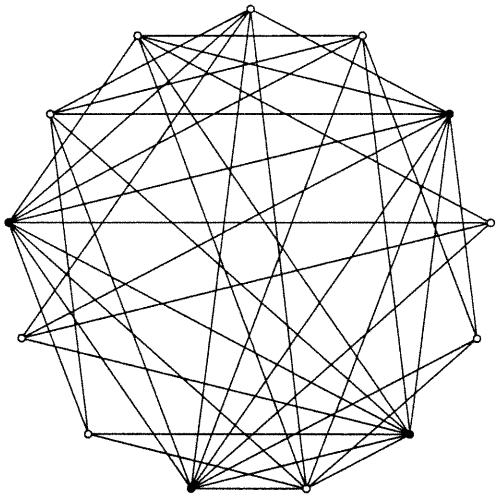

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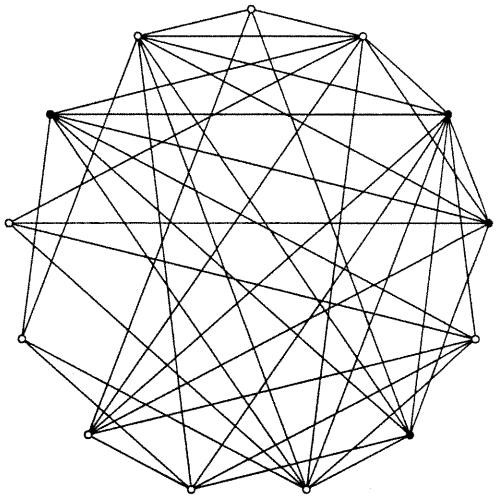

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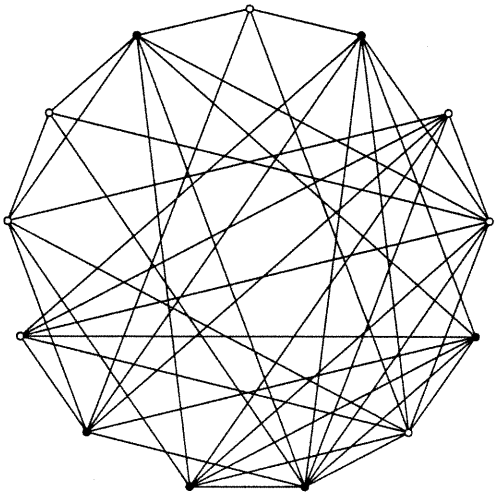

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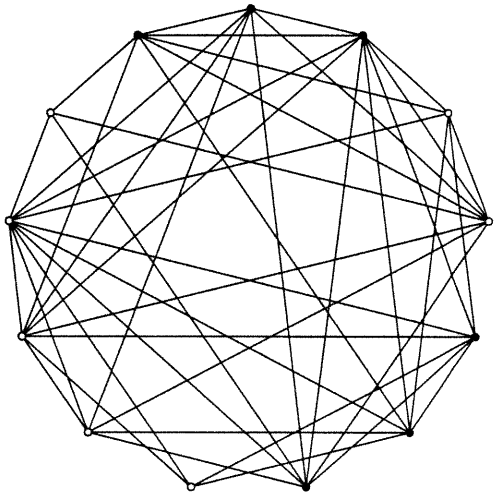

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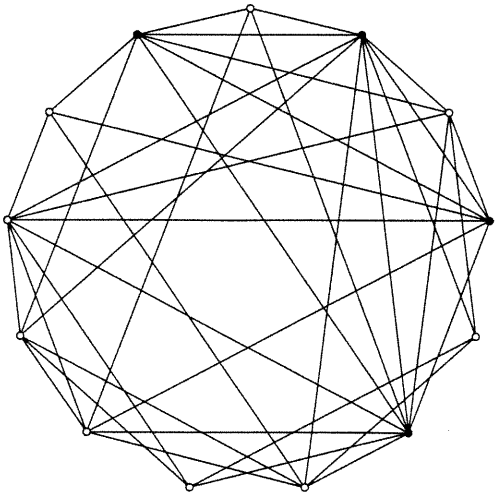

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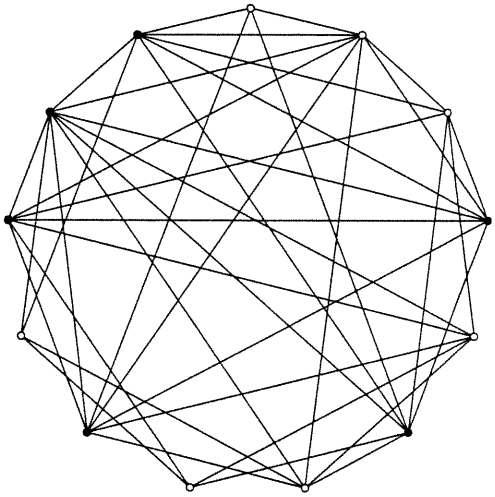

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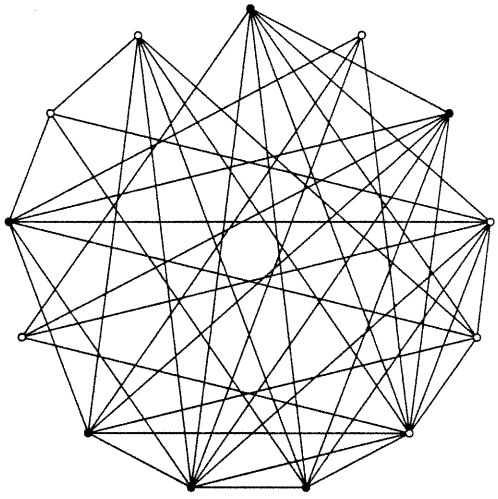

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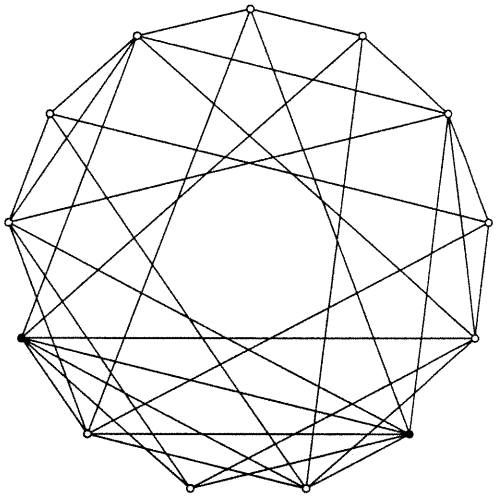

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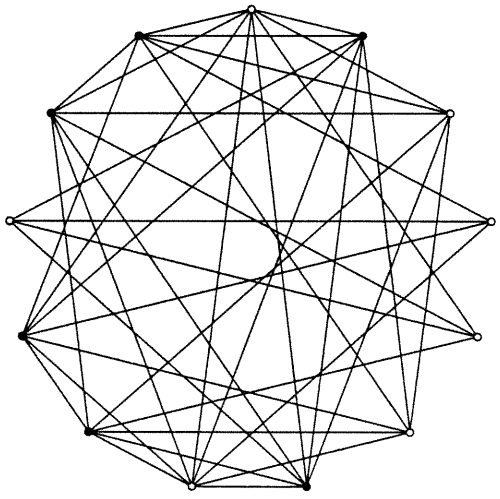

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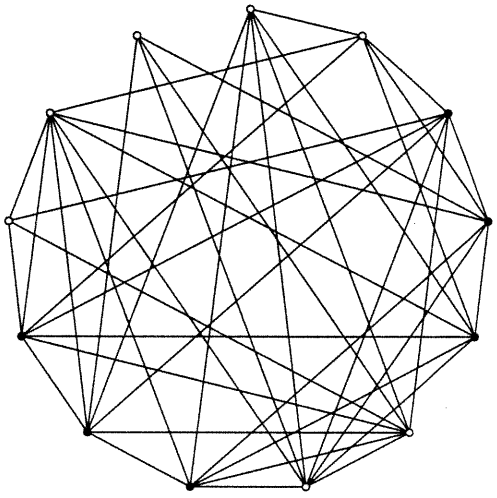

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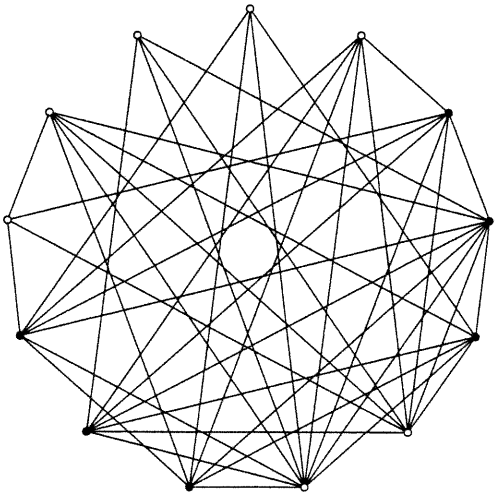

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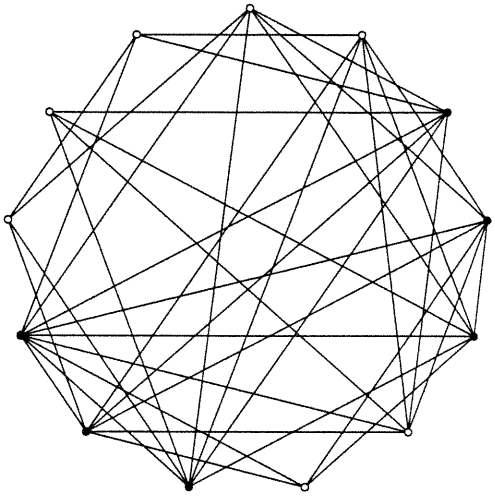

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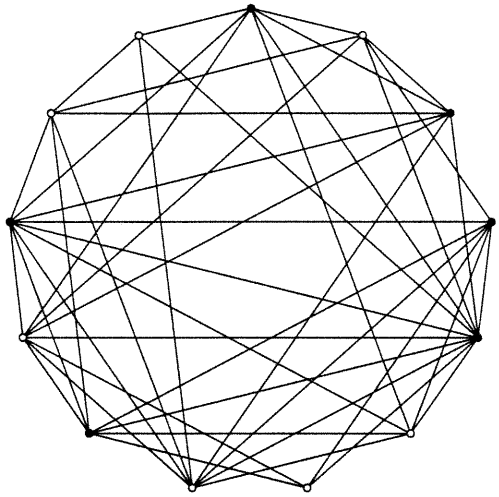

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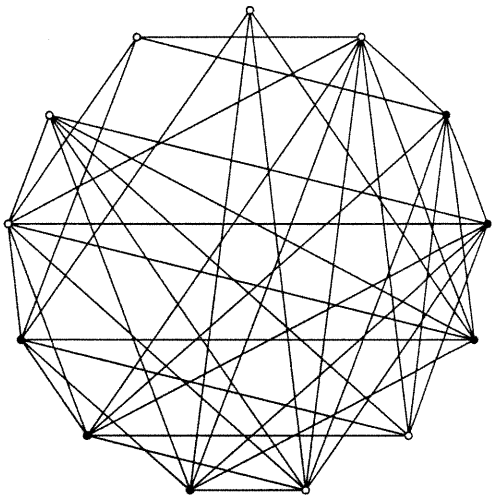

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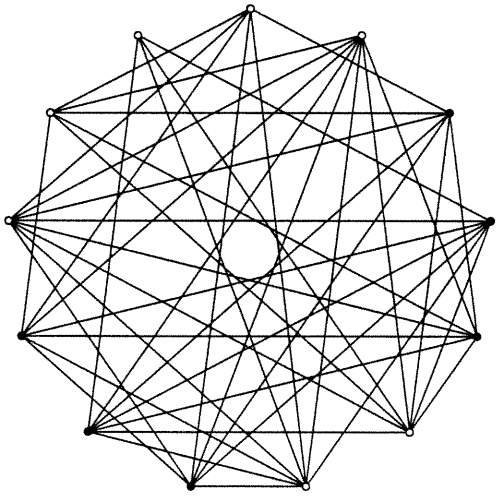

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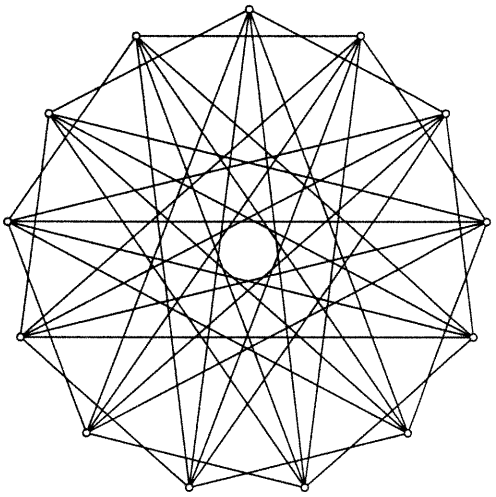

*G*021354697A8BC

Les trente-neuf premiers graphes sont des représentants non isomorphes de la classe de parité de G, le graphe pur sommet transitif naturellement colorié d'ordre treize. Le dernier graphe est l'inverse de G.

Annexe C

PROGRAMMES

Cette annexe comprend les listings des différents logiciels qui ont été écrits dans le cadre de cette thèse. Ces programmes ont été conçus afin de générer des graphes et d'en tester la pureté.
Les structures de données standard utilisées sont tirées de [**AHU**]. La structure «coloriage» et ses méthodes sont basés sur l'algorithme de test d'isomorphisme que l'on retrouve dans [**Ba**].

makefile

```
# Pour créer les fichiers exécutables pur, isomorphes, inverse et circulante,
    faites la commande make all dans le répertoire contenant tous les fichiers
    source. Pour utiliser un de ces programme, disons circulante, entrez la
    commande ./circulante.

CC=g++
CFLAGS = -O3

.SUFFIXES : .cpp .o

.cpp.o :
  $(CC) $(CFLAGS) -c $<

OBJS = graphe.o vertex.o
SOURCES = graphe.hpp

all :
  make pur isomorphes inverse circulante "CFLAGS=$(CFLAGS)"

graphe.o : $(SOURCES)
vertex.o : $(SOURCES)
```

```
bg_liste.o : $(SOURCES) bg_liste.hpp keytype.hpp
keytype.o : $(SOURCES) keytype.hpp
bigraphe.o : $(SOURCES)
coloriage.o : $(SOURCES)
vset.o : $(SOURCES)
pur.o : $(SOURCES) classe_de_parite.hpp
isomorphes.o : $(SOURCES)
inverse.o : $(SOURCES)
circulante.o : $(SOURCES)

pur : $(OBJS) bigraphe.o coloriage.o keytype.o bg_liste.o classe_de_parite.o vset.o pur.o
  $(CC) -o $(@) $(OBJS) bigraphe.o coloriage.o keytype.o bg_liste.o classe_de_parite.o
      vset.o pur.o

isomorphes : $(OBJS) bigraphe.o coloriage.o vset.o isomorphes.o
  $(CC) -o $(@) $(OBJS) bigraphe.o coloriage.o vset.o isomorphes.o

inverse : $(OBJS) bigraphe.o coloriage.o vset.o inverse.o
  $(CC) -o $(@) $(OBJS) bigraphe.o coloriage.o vset.o inverse.o

circulante : $(OBJS) bigraphe.o coloriage.o vset.o circulante.o
  $(CC) -o $(@) $(OBJS) bigraphe.o coloriage.o vset.o circulante.o

clean :
  - rm -f pur isomorphes inverse circulante *.o *~
```

circulante.cpp

```
/*****************************************************************************
```

Programme circulante

Description: L'utilisateur fournit l'ordre du graphe (valeur n) et un ensemble de générateurs S (des éléments non nuls de $\mathbb{Z}_n$). Le programme construit la circulante Cay($\mathbb{Z}_n$,S) *et la place dans le fichier de sortie spécifié par l'utilisateur sous forme de matrice d'adjacence.*

Note: Lors de la saisie des générateurs, un nombre entier est remplacé par l'élement de $\mathbb{Z}_n$ déterminé par l'homomorphisme naturel

$$\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}_n.$$

Les inverses sont ajoutés automatiquement (pour que $S = -S$).

```
*****************************************************************************/

#include <stdlib.h>
#include <iostream.h>
```

```
#include <fstream.h>
#include "graphe.hpp"

graphe G;
unsigned int n;
int m;

main()
{ char nom [40];
  cout « "\nordre de la circulante : ";
  cin » n;
  vset S(n);
  cout « "\nentrez chaque générateur suivi de <ENTER>(0 pour terminer) :\n";
  m=0;
  do
  { cout « "générateur : ";
    cin » m;
    while (m<0) m+=n;
    while (m>=n) m-=n;
    if (m!=0) S.ajoute(m);
  } while (m!=0);
  cout «"\nnom du fichier de sortie : ";
  cin » nom;
  ofstream sortie (nom, ios::out);
  if (! sortie) { cout « "\nL'ouverture du fichier de sortie a échoué.\n"; exit (1) ; }
  G = cayley (n, S);
  sortie « G;
  sortie . close () ;
}
```

inverse.cpp

```
/*****************************************************************************

Programme inverse

Ce programme nécessite un fichier contenant le graphe (naturellement colorié) à
    inverser, sous forme de matrice d'adjacence.

Description : Étant donné le graphe à inverser G, ce programme vérifie si G est
    inversible. Si c'est le cas, son inverse H est placé dans un fichier de sortie
    spécifié par l'utilisateur, sous forme de matrice d'adjacence.

*****************************************************************************/

#include <stdlib.h>
#include <fstream.h>
#include "graphe.hpp"
```

```
graphe G,H;

main()
{ char nom [40];
  cout « "\nfichier contenant le graphe à inverser: ";
  cin » nom;
  ifstream entree (nom, ios::in);
  if (!entree) { cout « "\nL'ouverture du fichier a échoué.\n"; exit (1); }
  entree » G;
  entree.close();
  if (G.nul())
  { cout « "\nC'est le graphe nul (sans sommets): son inverse est nul.\n";
    exit(1);
  }
  vset S(G);
  S.remplir();
  H = G.complementaire(S);
  if (H.nul())
  { cout « "Ce graphe n'est pas inversible.\n";
    return non;
  }
  cout « "Ce graphe est inversible.\n";
  cout « "Enregistrer ce graphe dans: ";
  cin » nom;
  ofstream sortie (nom, ios::out);
  if (!sortie) { cout « "\nL'ouverture du fichier a échoué.\n"; exit (1); }
  sortie « H;
  sortie.close();
  return oui;
}
```

isomorphes.cpp

```
/*****************************************************************************

Programme isomorphes

Ce programme nécessite deux fichiers contenant les graphes (naturellement
    coloriés) à comparer sous forme de matrices d'adjacence.

Description: Ce programme détermine si les graphes fournis G et H sont
    isomorphes.

*****************************************************************************/

#include <stdlib.h>
#include <iostream.h>
```

```
#include <fstream.h>
#include "graphe.hpp"

graphe G,H;
ifstream entree;

main()
{ char nom [40];
  cout « "\nfichier contenant le premier graphe : ";
  cin » nom;
  entree.open (nom, ios::in);
  if (!entree) { cout « "\nL'ouverture du fichier a échoué.\n"; exit (1); }
  entree » G;
  entree.close();
  if (G.nul())
  { cout « "\nC'est le graphe nul (sans sommets): il n'y a rien à comparer.\n";
    exit(1);
  }
  cout « "\nfichier contenant le second graphe : ";
  cin » nom;
  entree.open (nom, ios::in);
  if (!entree) { cout « "\nL'ouverture du fichier a échoué.\n"; exit (1); }
  entree » H;
  entree.close();
  if (H.nul())
  { cout « "\nC'est le graphe nul (sans sommets): il n'y a rien à comparer.\n";
    exit(1);
  }
  if (G==H)
    cout « "\nLes graphes sont isomorphes.\n";
  else
    cout « "\nLes graphes ne sont pas isomorphes.\n";
}
```

pur.cpp

```
/*****************************************************************************
```

Programme pur

Ce programme nécessite un fichier contenant le graphe (naturellement colorié) à tester, sous forme de matrice d'adjacence.

Description : Étant donné le graphe G, ce programme construit le graphe bicolorié H égal à G muni de son coloriage naturel. Au choix de l'utilisateur, le programme peut vérifier ou non si le graphe est pur et peut afficher ou non les graphes trouvés en cours de calcul. Par exemple, si on veut la liste complète

des graphes d'une classe de parité non pure à isomorphisme près, il faut demander l'affichage et refuser le test de pureté.

**********/

```
#include <stdlib.h>
#include <iostream.h>
#include <fstream.h>

#include "graphe.hpp"
#include "classe_de_parite.hpp"

#define    oui    1
#define    non    0

bigraphe  G,F;
graphe    H;
int       pur, chercher_anticlique_impaire, afficher ; // variables booléenes (vrai/faux)
unsigned int nombre_de_graphes;
vertex    u,v;
vset      S;

int main ()
{ char nom [40];
  cout « "\nfichier contenant le graphe à tester : ";
  cin » nom;
  ifstream entree (nom, ios::in);
  if (!entree) {  cout « "\nL'ouverture du fichier a échoué.\n"; exit (1) ; }
  entree » H;
  entree.close();
  if (H.nul())
  { cout « "\nC'est le graphe nul (sans sommets): il n'est pas pur.\n";
    exit(1);
  }
  cout « "Vérifier si le graphe est pur? (oui/non) ";
  cin » nom;
  if ((nom[0] == 'n')||(nom[0]=='N'))
    chercher_anticlique_impaire = non;
  else
    chercher_anticlique_impaire = oui;
  cout « "Afficher les graphes trouvés? (oui/non) ";
  cin » nom;
  if ((nom[0] == 'n')||(nom[0]=='N'))
    afficher = non;
  else
    afficher = oui;

  G = H;
  classe_de_parite C(G);
  pur = oui;
```

```
  nombre_de_graphes = 1;
  S = vset(G);
  while (C.prochain(F,S))
  { nombre_de_graphes++;
    if ( afficher )
    { cout « "\n// ensemble de complémentation : " « S « "\n" « F;
    }
    if (chercher_anticlique_impaire && AnticliqueImpaire(F))
    { cout « "\n//Ce graphe a une anticlique noire: la classe de parité n'est pas
        pure.\n";
      return 0;
    }
  }
  cout « "\n//Classe de : " « H;
  cout « "//Nombre de graphes : " « nombre_de_graphes « "\n";
  if (chercher_anticlique_impaire && pur)
    cout « "Cette classe de parité est pure.\n";
  return 0;
}
```

bg_liste.hpp

```
/*****************************************************************************
```

Objet : **bg_liste**

Méthodes : **insert, membre, deletemin, vide, echangelistes**

Description : Cette liste est tenue sous forme d'un «arbre deux-trois» (deux-trois car chaque sommet intermédiaire a deux ou trois enfants) de listes chaînées de graphes bicoloriés. Chacune des listes chaînées contient des graphes ayant le même type **keytype**. *La recherche d'un graphe à l'intérieur d'une de ces listes chaînée est de complexité linéaire (*$\mathbf{O}(n)$*). La structure d'arbre deux-trois assure que le temps d'ajout, de retrait ou de recherche d'une liste chaînée ne dépasse pas la complexité logarithmique (*$\mathbf{O}(\log n)$*). La structure globale est donc efficace en autant que les listes chaînées contiennent chacune une petite fraction des graphes. Ceci peu être contrôlé en raffinant le type* **keytype** *sans avoir à réécrire les méthodes de* **bg_liste**.

Voilà pour la structure de liste. En plus de contenir des graphes bicoloriés, **bg_liste** *conserve pour chaque graphe un ensemble de sommets. Cette information est utilisée par l'objet* **classe_de_parite** *pour tenir compte des graphes déjà trouvés au cours de son développement : si pour le graphe en cours de traitement une complémentation par rapport à un sommet blanc ou par rapport a une arête incidente à deux sommets noirs donne un graphe déjà trouvé (mais pas encore traité), on ajoute le ou les sommets à l'ensemble associé à ce graphe non traité. Ainsi, rendu à ce dernier graphe, nous n'avons*

pas besoin de considérer les complémentations par rapport à des sommets dans cet ensemble, ni par rapport à des arêtes dont au moins un sommet est dans cet ensemble (de telles complémentations donneraient nécessairement des graphes déjà trouvés).

**********/

```
#ifndef _BG_LISTE_HPP
#define _BG_LISTE_HPP

#include <stdlib.h>
#include <iostream.h>
#include <fstream.h>
#include "graphe.hpp"
#include "keytype.hpp"

#define oui 1
#define non 0

class bg_liste
{
public:
  struct link
  { bigraphe g;
    vset S;
    link * next;
  };

  struct listType
  { keytype key;
    link *   list ;
  };

private:
  struct node                          // type d'un sommet de l'arbre
  { listType Elem0, Elem1;
    node *left , *mid, *right;
  };

// données :

  node * racine;                       // racine de l'arbre deux-trois

// fonctions d'insertion et de lecture dans une liste chaînée de graphes :

  int drop (bigraphe &, keytype &, vset &, listType &);
  int drop (bigraphe &, keytype &, listType &);

// fonctions privées de l'arbre deux-trois :
```

```
  int insert_sub (bigraphe & G, keytype & key, vset & S, node * & returnpos, listType &
      returnlist, node * & insertpos);

  int insert_sub (bigraphe & G, keytype & key, node * & returnpos, listType & returnlist,
      node * & insertpos);

  void  scrapliste (listType &);
  void scrapchildren (node *);
  friend void swaplists (listType &, listType &);
  int deletemin_sub (node * insertpos, listType & Elem);
  int membre_sub (bigraphe & G, listType &);
  int membre_sub (bigraphe & G, listType &, vset & S);

public:
  bg_liste () { racine = NULL; }
  bg_liste (bigraphe & G);
  ~bg_liste ();
  int vide () {return racine == NULL;}    // retourne oui si l'arbre est vide
  int insert (bigraphe & G, keytype & K, vset & S);
  int insert (bigraphe & G, keytype & K);
  int insert (bigraphe & G);
  bigraphe deletemin (keytype & clef, vset & S);
  bigraphe deletemin (keytype & clef);
  bigraphe deletemin ();
  int membre (bigraphe & G, keytype & K);
  int membre (bigraphe & G, vset & S);
  friend void echangelistes (bg_liste &, bg_liste &);
};

#endif // _BG_LISTE_HPP
```

classe_de_parite.hpp

```
/*****************************************************************************
```

Objet : **classe_de_parite**

Méthodes : **prochain, developper**

Description: L'initialisation demande un graphe bicolorié de la classe de parité. Chaque appel de la méthode **prochain** *retourne* 1 *si un nouveau graphe bicolorié de la classe est trouvé (un graphe non isomorphe aux graphes déjà trouvés), auquel cas ce graphe et un ensemble de complémentation donnant ce graphe sont également retournés.*

Tant que la classe n'est pas complètement développée, on conserve quatre listes de graphes : **a_voir** *et* **vus** *contiennent les graphes non isomorphes*

à la même «distance» du graphe de départ, où la distance est le nombre de complémentations effectuées (une complémentation par rapport à une arête incidente à deux sommets noirs compte comme une seule complémentation dans ce cas-ci). Les graphes plus éloignés d'un coup (du graphe de départ) que ceux dans **a_voir** *et* **vus** *sont placés dans* **suivante**; *les graphes plus proches d'un coup se retrouvent dans* **precedente**.

La méthode **developper** *développe complètement la classe.*

```
**********************************************************************************/

#ifndef _CLASSE_DE_PARITE_HPP
#define _CLASSE_DE_PARITE_HPP

#include <stdlib.h>
#include <iostream.h>
#include "graphe.hpp"
#include "keytype.hpp"
#include "bg_liste.hpp"

#define oui 1
#define non 0

class classe_de_parite
{ bg_liste precedente, a_voir, vus, suivante;
  vertex u, v;
  vertex n;
  bigraphe G1, G2;
  keytype clef_de_G1;
  vset ensemble_comp_G1;
  vset_liste L;

public:
  classe_de_parite (bigraphe &);
  int prochain (bigraphe &, vset & ensemble_de_complementation);
  void developper (); // les graphes trouvés sont envoyés à la sortie standard
};
#endif // _CLASSE_DE_PARITE_HPP
```

graphe.hpp

```
/**********************************************************************************
```

Objets : **sommet, vertexlist, vset, vset_liste, vertexqueue, graphe, bigraphe, coloriage**

Ce fichier contient les définitions et méthodes nécessaires à la manipulation de graphes. Pour plus de précisions, se référer aux sections spécifiques.

```
*****************************************************************************/

#ifndef _GRAPHE_HPP
#define _GRAPHE_HPP

#include <stdlib.h>
#include <iostream.h>
#include <fstream.h>

#define oui                1
#define non                0
#define NombreMaximalDeSommets    255    //
```

Cette limite, qui peut être augmentée au besoin, ne s'applique que dans les situations où on se sert des méthodes d'isomorphisme (== et **iso***) ou bien si on cherche une anticlique noire (*__AnticliqueImpaire__*).*

```
void error () ; // message d'erreur en cas de manque de mémoire

class graphe;
class bigraphe;
class vset;
class vertexlabel;

/*****************************************************************************
```

Objet : **sommet**

Méthodes : Les sommets sont représentés par des entiers non négatifs et toutes les opérations sur les entiers sont utilisables (par exemple, les sommets peuvent être comparés à l'aide de l'opérateur <).

```
*****************************************************************************/

typedef unsigned int vertex;

/*****************************************************************************
```

Objet : **vertexlist**

*Méthodes : Les méthodes habituelles d'une liste (*__premier, suivant, ajoute, enleve__*,...) et certaines qui sont plus spécifiques aux graphes.*

Description : Une liste chaînée comme celle-ci est idéale si on n'a pas besoin de tester souvent l'appartenance de sommets à la liste (méthode **element***) et si on ne veut pas fixer d'avance le nombre maximal de sommets qui pouront entrer dans la liste. Chaque* **vertexlist** *possède un signet (le pointeur* **current***) qui est placé au debut la liste à chaque utilisation des méthodes* **premier**, **enleve**, **enleve_voisins_et**. *Ce signet est déplacé pour indiquer l'élément suivant à chaque utilisation de* **suivant**.

```
**********************************************************************************/

class vertexlist
{
protected:
  struct link
  { vertex sommet;
    link * next;
  };
  link * first , * current;

public:
   vertexlist () { first = current = NULL; }
  ~vertexlist ();
  int vide () { return (first == NULL); }
  int premier (vertex &);
  int suivant (vertex &);
  int ajoute (vertex);
  int element (vertex);
  int enleve (vertex);
  void enleve_voisins_et (vertex, graphe &); // enlève le sommet et ses voisins de la
       liste.
  int adjacent (vertex, graphe &);      // vérifie si le sommet donné est adjacent à un
       sommet de la liste
  vertex  degre (vertex u, graphe &); // retourne le nombre de voisins de u contenus
       dans la liste
   vertexlist (const vertexlist &);
   vertexlist & operator = (const vertexlist & L);
   vertexlist (vset &);
};
ostream & operator « (ostream & sortie, vertexlist & L);

/*********************************************************************************
```

Objet : **vertexqueue**

Méthodes : Les méthodes habituelles d'une structure **queue**.

Description : Les sommets sont extraits par **dequeue** *dans l'ordre selon lequel ils ont été entrés par* **ajoute**. *Cette classe est construite à partir de l'objet* **vertexlist**.

```
**********************************************************************************/

class vertexqueue : private vertexlist
{
public:
  vertexqueue () : vertexlist () {}
  int vide () { return (first == NULL); }
```

```
  int dequeue (vertex &);
  void ajoute (vertex);
};

/*****************************************************************************
```

Objet : **vset**

Méthodes : les méthodes habituelles d'une structure **set** *plus celle d'une liste (grâce à l'ordre total des sommets).*

Description : On a choisi ici la structure compacte d'un champ de bits contigus. Il faut connaître l'ordre n du graphe à l'initialisation. On utilise un bit par sommet pour représenter cet ensemble. Le bit 0 est egal à 1 si l'ensemble contient le sommet 0, ... , le bit $n-1$ est égal a 1 si l'ensemble contient le sommet $n-1$.

```
*****************************************************************************/

class vset
{
unsigned int  o;          // ordre du graphe
unsigned int n;           // taille de set

unsigned char * set;

public:
  vset () { o = n = 0; set = NULL; }
  vset (vertex nombre_de_sommets_maximal);
  vset (graphe &);
  ~vset ();
  void  initialiser (vertex nombre_de_sommets_maximal);
  void remplir () ;     // place tous les sommets dans l'ensemble
  int plein () ;        // retourne oui si tous les sommets sont dans l'ensemble
  vset (const vset &);
  void vider () ;
  int vide () ;
  friend vset V (vertex u, graphe & G);       // retourne l'ensemble des voisins de u
      dans G
  vset & operator = (const vset &);
  friend int operator == (vset &, vset &);
  friend int operator < (vset &, vset &);      // ordre total d'ensemble induit par
      l'ordre des sommets
  friend ostream & operator « (ostream & sortie, vset &);
  void ajoute (vertex);
  void enleve (vertex);
  void inverse (vertex);          // inverse l'appartenance (ajoute le sommet s'il
      n'est pas déjà dans l'ensembe et l'enlève sinon)
  int element (vertex);
```

```
  vset & operator -= (vset & S);  // retire de l'ensemble les sommets contenus
      dans S
  int premier (vertex &);         // selon l'ordre des sommets
  int suivant (vertex & u);       // remplace u par le prochain élément dans
      l'ensemble, retourne oui s'il y en a bien un
  int adjacent (vertex u, graphe &);   // vérifie si u a un voisin dans l'ensemble
  vertex degre (vertex u, graphe &);  // retourne le nombre de voisins de u dans
      l'ensemble
  vertex ordre() {return o;}
  vset & operator += (const vset &); // différence symétrique
  friend vset relabel (vset & S, vertexlabel & L);
};

/*****************************************************************************
```

Objet : **vset_liste**

Méthodes : **ajouter, vider**

Description : Cet objet est utilisé par **classe_de_parite** *pour tenir compte des ensembles de complémentation déjà considérés. Si l'ensemble que* **ajouter** *tente d'insérer dans la liste s'y trouve deja, cette méthode retourne* **non** *(zéro). Cette liste ordonnée est sous forme d'arbre binaire, ce qui donne une complexité d'accès ou d'insertion d'ordre logarithmique en moyenne, ce qui est bon en autant que les ajouts ne soient pas trop ordonnés à l'insertion.*

```
*****************************************************************************/

class vset_liste
{
public:
  struct node
  { vset S;
    node * left;
    node * right;
  };

  node * racine;

private:
  void deleterecurs (node *);
  int ajouterrecurs (vset &, node * &);

public:
  vset_liste () { racine = NULL; }
  ~vset_liste ();
  void vider();
  int ajouter (vset &); // retourne oui si l'ajout a été fait, non si l'ensemble est
      déjà présent
};
```

```
/*****************************************************************************
```

Objet : vertexlabel

Description : Cet objet établit un lien entre les sommets d'un graphe et les membre d'une liste de sommets. Il est utilisé par des structures de données en cours de développement.

```
*****************************************************************************/

class vertexlabel
{ vertex n, sub_n; // ordres du graphe, de la liste
public:
  vertex * sub, * sur; // sub[i]:= n si i n'est pas dans L

  vertexlabel () { n=0; sub_n = 0; sub = NULL; sur = NULL; }
  vertexlabel ( vertexlist & L, vertex ordre, int supprimer = non);
  ~vertexlabel () ;
  vertexlabel (const vertexlabel &);
  vertexlabel & operator = (const vertexlabel &);
  friend ostream & operator « (ostream & sortie, vertexlabel & L);
  vertexlist list_labels () ;
  vertexlist list_labels ( vertexlist & L);
  vertexlist sublist ( vertexlist & L);
  void relabel (vertex u, vertex label);
  friend vset relabel (vset & S, vertexlabel & L);
};

/*****************************************************************************
```

Objet : coloriage

Description : À l'usage exclusif de l'opérateur == de relation d'isomorphisme entre graphes ordinaires ou bicoloriés, cet objet est initialisé à l'aide d'un graphe auquel on associe un coloriage de ses sommets (qui peut faire intervenir plus de deux couleurs). Des raffinements successifs de ce coloriage aboutissent à un coloriage canonique, qui induit un ordre partiel sur les graphes bicoloriés. Lors de la comparaison de deux graphes, si on obtient le même coloriage canonique et qu'une classe de couleur possède au moins deux sommets, on peut recolorier un des sommets de chaque graphe avec une nouvelle couleur et faire de nouveaux raffinements de facon récursive. Tous les choix possibles doivent être explorés, mais cet algorithme minimise le nombre de choix et donne une complexité d'ordre faiblement exponentielle.

```
*****************************************************************************/

class   coloriage
{
public:
```

```
  struct link
  { vertex sommet;
    link * next;
  };

private:
  struct couleur
  { vertex  clef;
    char drapeau; // drapeau = oui ⟺ cette couleur est à traiter
    vertex n;     // cardinalité de l'orbite
    link * orbite;
    couleur * next;
  };

  couleur * racine;
  graphe * g;

  int insert (couleur *, vertex sommet, vertex degre);
  void scrap ();
  int meilleursommet (couleur * &); // Cette méthode trouve le «meilleur» choix de
      sommet à distinguer par une nouvelle couleur. couleur et link serviront de
      paramètres pour nouvellecouleur. La valeur de retour contient la «color-
      valence» de ce sommet.
  coloriage nouvellecouleur (couleur *, link *); // Cette méthode donne une nouvelle
      couleur au sommet indiqué. Si link == NULL c'est le premier sommet de
      la couleur fournie, sinon c'est le sommet en link->next
  friend int costabilize (coloriage &, coloriage &);
  friend int operator *= (coloriage &, coloriage &); // retourne oui si les coloriages
      raffinés sont identiques

  coloriage (graphe & G);
  coloriage (bigraphe & G);
  ~coloriage();
  friend ostream & operator « (ostream &, coloriage &);
  int raffiner ();
  int ordonne ();    //vérifie si les sommets ont tous des couleurs différentes
  coloriage & operator = (const coloriage &);
  coloriage (const coloriage &);
  friend int operator == (coloriage, coloriage); //isomorphes

public:
  friend int operator == (graphe &, graphe &);
  friend int operator == (bigraphe &, bigraphe &);
};

/*****************************************************************************
```

Objet : graphe

Méthodes : beaucoup de méthodes applicables aux graphes

Description : En entrée et en sortie, les graphes sont représentés par leurs matrices d'adjacence. En mémoire, on se limite à un champ de $n(n-1)/2$ bits, chacun de ces bits correspondant à l'existence d'une arête entre deux sommets (s'il vaut 1) ou une non (s'il vaut 0).

*Note : Pour nos besoins, une limite d'une centaine de sommets s'est avérée suffisante lorsqu'on cherche à comparer des graphes (c.-à-d. l'existence ou non d'un isomorphisme) ou à tester l'existence d'une anticlique noire (***AnticliqueImpaire***). Pour pouvoir tester l'isomorphisme ou la pureté de graphes d'ordre plus élevé, il suffit d'augmenter la constante* **NombreMaximalDeSommets** *au début du fichier et de recompiler.*

```
**********************************************************************************/

class  graphe
{ static vertex isomorphisme [NombreMaximalDeSommets];

protected:
  vertex n; // ordre du graphe
  char * adj;

  friend int  AntiRaffine (graphe &G, vertexlist *blancs,  vertexlist  *noirs,
            vertexlist  *independant);
  friend void AntiSub (graphe &G, vertexlist *anticlique, unsigned int & compte, int max
      , vertexlist *blancs, vertexlist *noirs,  vertexlist  *independant, vertex * sommet, int
       * flag, int compter);
  friend int AnticliqueImpaire (graphe & G, vertexlist *  anticlique , int max = 1, int
      compter = oui);

public:
  void enleve (vertex i, vertex j);
  void ajoute (vertex i, vertex j);
  void complemente (vertex i, vertex j);
  graphe (vertex = 0, char * = NULL);
  ~graphe () { if (adj != NULL) delete adj; }
  graphe (vertex n, unsigned char *);
  graphe (const graphe &);
  graphe & operator = (const graphe &);
  graphe & operator ++ ();     // ajout d'un sommet isolé
  graphe CompletionLocale (vertex);
  friend int operator == (graphe &, graphe &); //retourne oui si un isomorphisme
      est trouvé
  static int iso (vertex);  // Si G==H, iso est un isomorphisme qui envoie V(H)
      sur V(G).
  friend int operator *= (graphe &, graphe &); //retourne oui s'il y a égalité stricte
      entre les graphes
  vertex degre (vertex);
  friend int Eulerien (graphe &);   // retourne oui si tous les degres sont pairs (peu
      importe la connexité)
```

```
  friend ostream & operator « (ostream &, graphe &);
  friend istream & operator » (istream &, graphe &);
  vertex ordre () ;   // retourne l'ordre du graphe
  int arete (vertex, vertex); // retourne oui s'il y a une arête entre les sommets
  int nul () {return (n == 0);} // retourne oui si le graphe n'a pas de sommets
  friend int operator == (coloriage, coloriage);
  friend int operator == (bigraphe &, bigraphe &);
  void reunion (graphe & G); // ajoute les arêtes de G au graphe
  graphe complementaire (); // retourne le complementaire du graphe
  graphe operator * (graphe & G); // retourne le produit lexicographique du graphe
      avec le graphe G
  friend graphe cayley (vertex ordre, vset & L); //retourne le graphe de Cayley
      correspondant
  graphe complementaire (vset & L); // retourne le complementaire par rapport à
      l'ensemble de complementation L
  graphe supprimer (vertex); // retourne le graphe obtenu en enlevant ce sommet
  graphe supprimer (vset &);
  graphe supprimer (vertexlist &);
  graphe induire (vset &);   // retourne le sous-graphe induit par l'ensemble
  graphe induire ( vertexlist &);
  friend int AnticliqueImpaire (graphe & G); //existence d'une anticlique de sommets
      de degré impair
};

vset composante (vertex u, bigraphe & G); // retourne l'ensemble des sommets d'une
    composante connexe de G


/*****************************************************************************
```

Objet : bigraphe

*Méthodes : En plus des méthodes de graphe, bigraphe permet d'assigner un nouveau bicoloriage (***colorier***), de tester l'inversibilite (***inversible***) et de tester si un ensemble de sommets est un ensemble de complémentation du graphe (***comp_ensemble***).*

Description : Le bicoloriage est donné par l'ensemble **noirs** *qui contient les sommets noirs du graphe. Un graphe bicolorié initialisé à partir d'un graphe sans bicoloriage reçoit le coloriage naturel de ce graphe.*

```
*****************************************************************************/

class bigraphe : public graphe
{
public:
  vset  noirs;      // contient les sommets noirs du graphe

  bigraphe (vertex = 0);
  bigraphe (vertex, char *, vset & ensemble);
```

```
  bigraphe (graphe &);
  bigraphe (const bigraphe &);
  bigraphe CompletionLocale (vertex);
  bigraphe CompletionLocale (vertex, vset &);
  void  colorier  (vset & S);
  friend ostream & operator « (ostream &, bigraphe &);
  friend istream & operator » (istream &, bigraphe &);
  friend int AnticliqueImpaire (bigraphe & G); //existence d'une anticlique noire
  bigraphe supprimer (vertex);
  bigraphe supprimer (vertexlist &);
  bigraphe induire ( vertexlist &);
  bigraphe induire (vset & L);
  bigraphe parite ( vertexlist & L);
  bigraphe complementaire (vset & L);
  int comp_ensemble (vset & S);
  int  inversible  () ;
};

#endif // _GRAPHE_HPP
```

keytype.hpp

```
/*****************************************************************************
```

Objet : keytype

Méthodes : Initialisés à partir de graphes, les objets de type **keytype** *sont totalement ordonnés (ils se comparent avec les opérateurs <et ==).*

Description : Cet objet sert à établir un ordre partiel sur les graphes. L'ordre est total pour les objets de type **keytype**, *mais deux graphes non isomorphes peuvent déterminer des objets* **keytype** *identiques. Les* **keytype** *sont des suites finies d'entiers en ordre croissant, déterminées par les suites de degrés des graphes correspondants. À l'initialisation, les degrés sont ordonnés à l'aide de l'algorithme standard* **quicksort**.

Remarque : Le code source de **keytype** *est séparé de celui de l'objet graphe afin de faciliter l'amélioration ou le remplacement de* **keytype** *sans avoir a réécrire les méthodes de l'objet* **graphe**.

```
*****************************************************************************/

#ifndef _KEYTYPE_HPP
#define _KEYTYPE_HPP

#include <stdlib.h>
#include <iostream.h>
```

```
#include <new.h>
#include "graphe.hpp"

class keytype
{ vertex n;
  vertex * degres;

public:
  keytype () {n = 0; degres = NULL;}
  keytype (graphe &);
  void nullify () ;
  ~keytype ();
  keytype (const keytype &);
  keytype & operator *= (keytype &);  // attention! simple copie de pointeur
  keytype & operator = (const keytype &);
  friend int operator == (keytype &, keytype &);
  friend int operator < (keytype &, keytype &);
};

#endif // _KEYTYPE_HPP
```

bg_liste.cpp

```
#include "keytype.hpp"
#include "bg_liste.hpp"

void swaplists (bg_liste::listType & E, bg_liste::listType & F)
{ keytype K; bg_liste::link * L;
  K *= E.key; E.key *= F.key; F.key *= K; K.nullify();
  L = E.list ; E. list = F. list ; F. list = L;
}

bg_liste::bg_liste (bigraphe & G)
{ racine = new (node);
  if (racine == NULL) {error(); exit(1);}
  keytype K;
  K = keytype(G);
  racine->Elem0.key = keytype (G);
  racine->Elem0.list = new (link);
  if (racine->Elem0.list == NULL) {error(); exit(1);}
  racine->Elem0.list->g = G;
  racine->Elem0.list->S = vset(G);
  racine->Elem0.list->next = NULL;
  racine->Elem1.list = NULL;
  racine->left = racine->mid = racine->right = NULL;
}

void bg_liste:: scrapliste (listType & Elem)
```

```
{ link * templink;
  while (Elem.list != NULL)
  { templink = Elem.list;
    Elem.list = Elem.list->next;
    delete templink;
  }
}

void bg_liste::scrapchildren (node * N)
{ if (N->left != NULL)
  { scrapchildren (N->left); delete N->left;
    scrapchildren (N->mid); delete N->mid;
    if (N->Elem1.list != NULL) { scrapchildren (N->right); delete N->right; }
  }
  scrapliste (N->Elem0); scrapliste (N->Elem1);
}

bg_liste::~bg_liste ()
{ if (racine == NULL) return;
  node * tempnode = racine->mid;
  scrapliste (racine->Elem0);
  delete racine;
  if (tempnode != NULL)
  { scrapchildren (tempnode);
    delete tempnode;
  }
}

int bg_liste::drop (bigraphe & G, keytype & key, vset & S, listType & Elem)
// retourne oui si l'insertion a eu lieu
{ link *  templink;

  if (Elem.list == NULL)
  { Elem.key = key;
    Elem.list = new (link);
    if (Elem.list == NULL) {error(); exit(1);}
    Elem.list->g = G;
    Elem.list->S = S;
    Elem.list->next = NULL;
    return oui;
  }
  templink = Elem.list;
  while (templink != NULL)
  { if (templink->S == S)
    return non;
    templink = templink->next;
  }
  templink = Elem.list;
  if (templink->g == G)
    return non;
```

```
  while (templink->next != NULL)
  { templink = templink->next;
    if (templink->g == G)
      return non;
  }
  templink->next = new (link);
  if (templink->next == NULL) {error(); exit(1);}
  templink->next->g = G;
  templink->next->S = S;
  templink->next->next = NULL;
  return oui;
}

int bg_liste::drop (bigraphe & G, keytype & key, listType & Elem)
// retourne oui si l'insertion a eu lieu
{ link * templink;

  if (Elem.list == NULL)
  { Elem.key = key;
    Elem.list = new (link);
    if (Elem.list == NULL) {error(); exit(1);}
    Elem.list->g = G;
    Elem.list->next = NULL;
    return oui;
  }
  templink = Elem.list;
  if (templink->g == G)
    return non;
  while (templink->next != NULL)
  { templink = templink->next;
    if (templink->g == G)
      return non;
  }
  templink->next = new (link);
  if (templink->next == NULL) {error(); exit(1);}
  templink->next->g = G;
  templink->next->next = NULL;
  return oui;
}

int bg_liste::insert_sub (bigraphe & G, keytype & key, vset & S, node * & returnpos,
    listType & returnlist, node * & insertpos)

{ node * finger;
  node * newnode;
  listType templist;
  char ordre;
  int insertion;

// lecture du sommet à partir duquel on veut insérer G
```

```
  if (key == racine->Elem0.key)
  { insertion = drop (G, key, S, racine->Elem0);
    returnpos = NULL;
  }

// cas où le sommet est une feuille
  else if (racine->mid == NULL)
  { if (racine->Elem1.list == NULL)
    { insertion = drop (G, key, S, racine->Elem1);
      if (key < racine->Elem0.key)
      { swaplists (racine->Elem0, racine->Elem1); //permuter les deux listes
        insertpos = racine;  // au cas où la nouvelle liste aurait
      }                      // la plus petite clé
      returnpos = NULL;
    }
    else
    { returnlist .key. nullify () ; returnlist . list = NULL;
      if (key < racine->Elem0.key)
      { returnpos = new (node);
        if (returnpos == NULL) {error(); exit(1);}
  returnpos->Elem0.list = NULL; returnpos->Elem1.list = NULL;
  returnpos->left = returnpos->right = returnpos->mid = NULL;
  insertion = drop (G, key, S, returnpos->Elem0);
  swaplists ( returnlist , racine->Elem0);
  swaplists (racine->Elem0, returnpos->Elem0);
  swaplists (returnpos->Elem0, racine->Elem1);
      }
      else
      { if (key == racine->Elem1.key)
  { insertion = drop (G, key, S, racine->Elem1);
    returnpos = NULL;
    return insertion;
  }
        returnpos = new (node);
  if (returnpos == NULL) {error(); exit(1);}
  returnpos->Elem0.list = NULL; returnpos->Elem1.list = NULL;
  returnpos->left = returnpos->right = returnpos->mid = NULL;
  insertion = drop (G, key, S, returnpos->Elem0); // insertion
  if (key < racine->Elem1.key)
  { swaplists ( returnlist , returnpos->Elem0);
    swaplists (returnpos->Elem0, racine->Elem1);
  }
  else swaplists ( returnlist , racine->Elem1);
      }
      insertpos = racine;
    }
  }

// cas général
  else
```

```
{ finger = racine;        //sauvegarder le sommet actuel
  if (key < racine->Elem0.key)
  { ordre = 1; racine = racine->left;  // choisissons le sous-arbre de gauche
  }
  else
  { if ((racine->Elem1.list == NULL) || (key < racine->Elem1.key))
    { ordre = 2; racine = racine->mid; // ... du milieu
    }
    else
    { if (key == racine->Elem1.key)
{ insertion = drop (G, key, S, racine->Elem1);
  returnpos = NULL;
  return insertion;
}
      ordre = 3; racine = racine->right; // ... de droite
    }
  }
  insertion = insert_sub (G,key,S,returnpos,returnlist,insertpos);
  racine = finger;
  if (returnpos != NULL)
  { if (racine->Elem1.list == NULL)
    { if (ordre == 2)
{ swaplists (racine->Elem1, returnlist);
  racine->right = returnpos;
}
      else if (ordre == 1)
{ swaplists (racine->Elem1, racine->Elem0);
  swaplists (racine->Elem0, returnlist);
  racine->right = racine->mid; racine->mid = returnpos;
}
      returnpos = NULL; return insertion;
    }
    newnode = new (node);
    if (newnode == NULL) {error(); exit(1);}
    newnode->Elem0.list = NULL; newnode->Elem1.list = NULL;
    newnode->right = NULL;
    if (ordre == 2)
    { swaplists (newnode->Elem0, racine->Elem1);
      newnode->mid = racine->right;
newnode->left = returnpos;
    }
    else if (ordre == 1)
    { swaplists ( returnlist , racine->Elem0);
      swaplists (newnode->Elem0, racine->Elem1);
newnode->mid = racine->right; newnode->left = racine->mid;
racine->mid = returnpos;
    }
    else
    { swaplists (newnode->Elem0, returnlist);
      swaplists ( returnlist , racine->Elem1);
```

```
  newnode->mid = returnpos; newnode->left = racine->right;
      }
      racine->right = NULL;
      returnpos = newnode;
    }
  }
  return insertion;
}

int bg_liste::insert_sub (bigraphe & G, keytype & key, node * & returnpos, listType &
    returnlist, node * & insertpos)
{ node *     finger;
  node *     newnode;
  listType     templist;
  char       ordre;
  int        insertion;

// lecture du sommet à partir duquel on veut insérer G
  if (key == racine->Elem0.key)
  { insertion = drop (G, key, racine->Elem0);
    returnpos = NULL;
  }

// cas où le sommet est une feuille
  else if (racine->mid == NULL)
  { if (racine->Elem1.list == NULL)
    { insertion = drop (G, key, racine->Elem1);
      if (key < racine->Elem0.key)
      { swaplists (racine->Elem0, racine->Elem1); //permuter les deux listes
        insertpos = racine;  // au cas où la nouvelle liste aurait
      }                      // la plus petite clé
      returnpos = NULL;
    }
    else
    { returnlist.key.nullify(); returnlist.list = NULL;
      if (key < racine->Elem0.key)
      { returnpos = new (node);
        if (returnpos == NULL) {error(); exit(1);}
  returnpos->Elem0.list = NULL; returnpos->Elem1.list = NULL;
  returnpos->left = returnpos->right = returnpos->mid = NULL;
  insertion = drop (G, key, returnpos->Elem0);
  swaplists ( returnlist , racine->Elem0);
  swaplists (racine->Elem0, returnpos->Elem0);
  swaplists (returnpos->Elem0, racine->Elem1);
      }
      else
      { if (key == racine->Elem1.key)
        { insertion = drop (G, key, racine->Elem1);
          returnpos = NULL;
    return insertion;
```

```
  }
        returnpos = new (node);
  if (returnpos == NULL) {error(); exit(1);}
  returnpos->Elem0.list = NULL; returnpos->Elem1.list = NULL;
  returnpos->left = returnpos->right = returnpos->mid = NULL;
  insertion = drop (G, key, returnpos->Elem0); // insertion
  if (key < racine->Elem1.key)
  { swaplists ( returnlist , returnpos->Elem0);
          swaplists (returnpos->Elem0, racine->Elem1);
  }
  else swaplists ( returnlist , racine->Elem1);
      }
      insertpos = racine;
    }
  }

// cas général
  else
  { finger = racine;        //sauvegarder le sommet actuel
    if (key < racine->Elem0.key)
    { ordre = 1;
      racine = racine->left; // choisissons le sous-arbre de gauche
    }
    else
    { if ((racine->Elem1.list == NULL) || (key < racine->Elem1.key))
      { ordre = 2; racine = racine->mid; // ... du milieu
      }
      else
      { if (key == racine->Elem1.key)
  { insertion = drop (G, key, racine->Elem1);
          returnpos = NULL;
    return insertion;
  }
        ordre = 3; racine = racine->right; // ... de droite
      }
    }
    insertion = insert_sub (G,key,returnpos,returnlist,insertpos);
    racine = finger;
    if (returnpos != NULL)
    { if (racine->Elem1.list == NULL)
      { if (ordre == 2)
  { swaplists (racine->Elem1, returnlist);
          racine->right = returnpos;
  }
        else if (ordre == 1)
  { swaplists (racine->Elem1, racine->Elem0);
          swaplists (racine->Elem0, returnlist);
    racine->right = racine->mid; racine->mid = returnpos;
        }
        returnpos = NULL; return insertion;
```

```
        }
        newnode = new (node);
        if (newnode == NULL) {error(); exit(1);}
        newnode->Elem0.list = NULL; newnode->Elem1.list = NULL;
        newnode->right = NULL;
        if (ordre == 2)
        { swaplists (newnode->Elem0, racine->Elem1);
          newnode->mid = racine->right;
    newnode->left = returnpos;
        }
        else if (ordre == 1)
        { swaplists ( returnlist , racine->Elem0);
          swaplists (newnode->Elem0, racine->Elem1);
    newnode->mid = racine->right; newnode->left = racine->mid;
    racine->mid = returnpos;
        }
        else
        { swaplists (newnode->Elem0, returnlist);
          swaplists ( returnlist , racine->Elem1);
    newnode->mid = returnpos; newnode->left = racine->right;
        }
        racine->right = NULL;
        returnpos = newnode;
      }
    }
    return insertion;
}

int bg_liste:: insert (bigraphe & G, keytype & key, vset &S)
// retourne oui si l'insertion a eu lieu
{ node      *returnpos, *insertpos , * finger , *newnode;
  listType     returnlist ;
  int      insertion ;

// cas où l'arbre est nul
  if (racine == NULL)
  { racine = new (node);
    if (racine == NULL) {error(); exit(1);}
    racine->Elem0.key = keytype(G);
    racine->Elem0.list = new (link);
    if (racine->Elem0.list == NULL) {error(); exit(1);}
    racine->Elem0.list->g = G;
    racine->Elem0.list->S = S;
    racine->Elem0.list->next = NULL;
    racine->Elem1.list = NULL;
    racine->left = racine->right = racine->mid = NULL;
    return oui;
  }

// si G a la meme clef que la racine inserer G dans la liste de la racine
```

```
  if (key == racine->Elem0.key)
    return drop (G, key, S, racine->Elem0);

// si l'arbre ne contient qu'un seul sommet ajouter une node
  if (racine->mid == NULL)
  { racine->mid = new (node);
    if (racine->mid == NULL) {error(); exit(1);}
    newnode = racine->mid;
    newnode->Elem1.list = NULL;
    newnode->left = newnode->right = newnode->mid = NULL;
    newnode->Elem0.key = key;
    newnode->Elem0.list = new (link);
    if (newnode->Elem0.list == NULL) {error(); exit(1);}
    newnode->Elem0.list->g = G;
    newnode->Elem0.list->S = S;
    newnode->Elem0.list->next = NULL;
    if (key < racine->Elem0.key)
    { racine->mid = NULL; // permuter les deux nodes
      newnode->mid = racine;
      racine = newnode;
    }
    return oui;
  }

// cas général
  finger = racine;
  racine = racine->mid;
  returnlist . list = NULL;
  insertion = insert_sub (G, key, S, returnpos, returnlist , insertpos);
  racine = finger;
  if (returnpos != NULL)
  { newnode = new (node);
    if (newnode == NULL) {error(); exit(1);}
    newnode->Elem0.list = NULL; newnode->Elem1.list = NULL;
    newnode->left = racine->mid;
    newnode->mid = returnpos;
    newnode->right = NULL;
    swaplists (newnode->Elem0, returnlist);
    racine->mid = newnode;
  }

// si la nouvelle liste a la plus petite clé il faut l'échanger avec
// la liste du premier sommet:
  if (key < racine->Elem0.key)
    swaplists (insertpos->Elem0, racine->Elem0);

  return insertion;
}

int bg_liste:: insert (bigraphe & G, keytype & key)
```

```
// retourne oui si l'insertion a eu lieu
{ node      *returnpos, *insertpos, * finger , *newnode;
  listType     returnlist ;
  int      insertion ;

// cas où l'arbre est nul
  if (racine == NULL)
  { racine = new (node);
    if (racine == NULL) {error(); exit(1);}
    racine->Elem0.key = keytype(G);
    racine->Elem0.list = new (link);
    if (racine->Elem0.list == NULL) {error(); exit(1);}
    racine->Elem0.list->g = G;
    racine->Elem0.list->next = NULL;
    racine->Elem1.list = NULL;
    racine->left = racine->right = racine->mid = NULL;
    return oui;
  }

// si G a la meme clef que la racine inserer G dans la liste de la racine
  if (key == racine->Elem0.key)
    return drop (G, key, racine->Elem0);

// si l'arbre ne contient qu'un seul sommet ajouter une node
  if (racine->mid == NULL)
  { racine->mid = new (node);
    if (racine->mid == NULL) {error(); exit(1);}
    newnode = racine->mid;
    newnode->Elem1.list = NULL;
    newnode->left = newnode->right = newnode->mid = NULL;
    newnode->Elem0.key = key;
    newnode->Elem0.list = new (link);
    if (newnode->Elem0.list == NULL) {error(); exit(1);}
    newnode->Elem0.list->g = G;
    newnode->Elem0.list->next = NULL;
    if (key < racine->Elem0.key)
    { racine->mid = NULL; // permuter les deux nodes
      newnode->mid = racine;
      racine = newnode;
    }
    return oui;
  }

// cas général
  finger = racine;
  racine = racine->mid;
  returnlist . list = NULL;
  insertion = insert_sub (G, key, returnpos, returnlist , insertpos);
  racine = finger;
  if (returnpos != NULL)
```

```
  { newnode = new (node);
    if (newnode == NULL) {error(); exit(1);}
    newnode->Elem0.list = NULL; newnode->Elem1.list = NULL;
    newnode->left = racine->mid;
    newnode->mid = returnpos;
    newnode->right = NULL;
    swaplists (newnode->Elem0, returnlist);
    racine->mid = newnode;
  }

// si la nouvelle liste a la plus petite clé il faut l'échanger avec
// la liste du premier sommet:
  if (key < racine->Elem0.key)
    swaplists (insertpos->Elem0, racine->Elem0);

  return insertion;
}

int bg_liste:: insert (bigraphe & G)
{ keytype K = keytype(G);
  return insert(G, K);
}

int   bg_liste:: deletemin_sub (node * currentnode, listType & Elem)
{ if (currentnode->mid == NULL) // si la node est une feuille
  { swaplists (Elem, currentnode->Elem0);
    if (currentnode->Elem1.list == NULL) return oui;
    swaplists (currentnode->Elem0, currentnode->Elem1);
    return non;
  }
  if (deletemin_sub (currentnode->left, Elem))
  { if (currentnode->mid->Elem1.list != NULL)
    { swaplists (currentnode->left->Elem0, currentnode->Elem0);
      swaplists (currentnode->Elem0, currentnode->mid->Elem0);
      swaplists (currentnode->mid->Elem0, currentnode->mid->Elem1);
      currentnode->left->left = currentnode->left->mid;
      currentnode->left->mid = currentnode->mid->left;
      currentnode->mid->left = currentnode->mid->mid;
      currentnode->mid->mid = currentnode->mid->right;
      return non;
    }
    swaplists (currentnode->mid->Elem1, currentnode->mid->Elem0);
    swaplists (currentnode->mid->Elem0, currentnode->Elem0);
    currentnode->mid->right = currentnode->mid->mid;
    currentnode->mid->mid = currentnode->mid->left;
    currentnode->mid->left = currentnode->left->mid;
    delete currentnode->left;
    if (currentnode->Elem1.list == NULL) return oui;
    swaplists (currentnode->Elem0, currentnode->Elem1);
    currentnode->left = currentnode->mid;
```

```
    currentnode->mid = currentnode->right;
    currentnode->right = NULL;
  }
  return non;
}

bigraphe bg_liste::deletemin (keytype & K, vset & S)
{ if (racine == NULL) return bigraphe (); //graphe nul
  bigraphe G = racine->Elem0.list->g;
  S = racine->Elem0.list->S;
  K = racine->Elem0.key;
  link * templink = racine->Elem0.list;
  racine->Elem0.list = templink->next;
  delete templink;
  if (racine->Elem0.list == NULL)
  { if (racine->mid == NULL)
    { delete racine;
      racine = NULL;
    }
    else
    { listType Elem; Elem.list = NULL;
      if (deletemin_sub (racine->mid, Elem))
      { node * tempnode = racine;
        racine = racine->mid;
  delete tempnode;
  racine->left = NULL;
      }
      swaplists (racine->Elem0, Elem);
    }
  }
  return G;
}

bigraphe bg_liste::deletemin (keytype & K)
{ if (racine == NULL) return bigraphe (); //graphe nul
  bigraphe G = racine->Elem0.list->g;
  K = racine->Elem0.key;
  link * templink = racine->Elem0.list;
  racine->Elem0.list = templink->next;
  delete templink;
  if (racine->Elem0.list == NULL)
  { if (racine->mid == NULL)
    { delete racine;
      racine = NULL;
    }
    else
    { listType Elem; Elem.list = NULL;
      if (deletemin_sub (racine->mid, Elem))
      { node * tempnode = racine;
        racine = racine->mid;
```

```
    delete tempnode;
    racine->left = NULL;
        }
        swaplists (racine->Elem0, Elem);
      }
    }
    return G;
}

bigraphe bg_liste::deletemin ()
{ keytype K;
  return deletemin(K);
}

int bg_liste::membre_sub (bigraphe & G, listType & Elem)
{ if (Elem.list == NULL)
  { cout « "\nerreur dans membre_sub : liste vide"; exit(1);}
  link * currentlink = Elem.list;

  while (currentlink != NULL)
  { if (currentlink->g == G)
    { return oui;
    }
    currentlink = currentlink->next;
  }
  return non;
}

int bg_liste::membre_sub (bigraphe & G, listType & Elem, vset & S)
{  if (Elem.list == NULL)
  { cout « "\nerreur dans membre_sub : liste vide"; exit(1);}
    link * currentlink = Elem.list;
    while (currentlink != NULL)
    { if (currentlink->g == G)
      { S = currentlink->S;
        return oui;
      }
      currentlink = currentlink->next;
    }
    return non;
  }

int bg_liste::membre (bigraphe & G, keytype & K)
{ node * currentnode = racine;
  while (currentnode != NULL)
  { if (currentnode->Elem0.key == K) return membre_sub(G,currentnode->Elem0);
    if (K < currentnode->Elem0.key) currentnode = currentnode->left;
    else if (currentnode->Elem1.list == NULL) currentnode = currentnode->mid;
    else if (K < currentnode->Elem1.key) currentnode = currentnode->mid;
```

```
    else if (K == currentnode->Elem1.key) return membre_sub (G,currentnode->
        Elem1);
    else currentnode = currentnode->right;
  }
  return non;
}

int bg_liste::membre (bigraphe & G, vset & S)
{ node * currentnode = racine;
  keytype K(G);
  while (currentnode != NULL)
  { if (currentnode->Elem0.key == K) return membre_sub(G,currentnode->Elem0, S);
    if (K < currentnode->Elem0.key) currentnode = currentnode->left;
    else if (currentnode->Elem1.list == NULL) currentnode = currentnode->mid;
    else if (K < currentnode->Elem1.key) currentnode = currentnode->mid;
    else if (K == currentnode->Elem1.key) return membre_sub (G,currentnode->
        Elem1, S);
    else currentnode = currentnode->right;
  }
  return non;
}

void echangelistes (bg_liste & L, bg_liste & M)
{ bg_liste::node * temp = L.racine;
  L.racine = M.racine; M.racine = temp;
}
```

bigraphe.cpp

```
#include "graphe.hpp"

bigraphe::bigraphe (vertex ordre) : graphe (ordre)
{ noirs = vset(ordre);
}

bigraphe::bigraphe (vertex ordre, char * matrice, vset & S) : graphe (ordre, matrice)
  // matrice = "" indique un graphe sans arêtes
{ vertex u;
  if (S.ordre() == 0)
  { noirs = vset(ordre);
    for (u=0; u<n; u++)
      if ((degre(u)&1)==1)
        noirs.ajoute(u);
  }
  else if (S.ordre() != ordre)
  { cout « "erreur dans constructeur de bigraphe: coloriage d'ordre différent de
      celui du graphe\n";
    exit(1);
```

```
  }
  else
    noirs = S;
}

bigraphe::bigraphe (const bigraphe & G)
{
  n = G.n;
  unsigned long size = n;
  size = (((size*(size-1))/2)+7)/8;
  if (G.adj == NULL)
    adj = NULL;
  else
  { adj = new char [size];
    if (adj == NULL) {error(); exit(1);}
    for (unsigned long i=0; i<size; i++)
      adj [i] = G.adj [i];
  }
  noirs = G.noirs;
}

bigraphe::bigraphe (graphe & G) : graphe (G)
{ vertex u;
  noirs = vset(n);
  for (u=0; u<n; u++)
    if (G.degre(u)&1)
      noirs.ajoute(u);
}

int AnticliqueImpaire (bigraphe & G, vertexlist * anticlique , int max, int compter)
// anticlique est un tableau pouvant contenir un nombre max d'objets de type
    vset.
{ vertex u;
  unsigned int compte = G.ordre();
  static vertexlist blancs[NombreMaximalDeSommets];
  static vertexlist noirs[NombreMaximalDeSommets];
  static vertexlist independant[NombreMaximalDeSommets];
  static vertex sommet[NombreMaximalDeSommets];
  static int flag[NombreMaximalDeSommets];

  vertexlist vide;
  blancs[0] = vide;
  noirs[0] = vide;
  independant[0] = vide;

  for (u=0; u<compte; u++)
  { if (G.noirs.element(u))
      noirs [0]. ajoute (u);
    else
      blancs [0]. ajoute (u);
```

```
  }
  if (AntiRaffine (G, blancs, noirs, independant))
    return 0;
  compte = 0;
  AntiSub (G, anticlique, compte, max, blancs, noirs, independant, sommet, flag, compter);
  return compte;
}

int AnticliqueImpaire (bigraphe & G)
{ unsigned int compte = 0;
  vertexlist L;
  compte = AnticliqueImpaire (G, &L, 1, non);
  return (compte != 0);
}

bigraphe bigraphe::CompletionLocale (vertex s)
{
  vertex * voisins;
  voisins = new vertex [n-1];
  if (voisins == NULL) {error(); exit(1);}
  vertex i,j,k;
  bigraphe H = * this;
  int s_est_noir = noirs.element(s);

  for (i = 0, k = 0; i < n; i++)
    if (H.arete (s, i))
    { voisins [k++] = i; // voisins[0..k-1] := voisinage de s
      if (!s_est_noir) // s est blanc
        H.noirs.inverse (i);
    }
  for (i = 0; i < k; i++)
    for (j = i+1; j < k; j++)
      H.complemente (voisins[i], voisins [j]);
            // inverser la relation d'adjacence entre les voisins
  delete voisins;
  return H;
}

bigraphe bigraphe::CompletionLocale (vertex s, vset & S)
{
  vertex * voisins;
  voisins = new vertex [n-1];
  if (voisins == NULL) {error(); exit(1);}
  vertex i,j,k;
  bigraphe H = * this;
  int s_est_noir = noirs.element(s);

  k=0;
  for (int succes = S.premier(i); succes; succes = S.suivant(i))
    if (H.arete (s, i))
```

```
    { voisins [k++] = i; // voisins[0..k-1] := voisinage de s
      if (!s_est_noir) // s est blanc
        H.noirs.inverse (i);
    }
  for (i = 0; i < k; i++)
    for (j = i+1; j < k; j++)
      H.complemente (voisins[i], voisins [j]);
            // inverser la relation d'adjacence entre les voisins
  delete voisins;
  return H;
}

int operator == (bigraphe & G, bigraphe & H)
{ coloriage :: couleur * pointeurG, * pointeurH;
  if (G.n != H.n) return non;
  coloriage ColG(G);
  coloriage ColH(H);
  pointeurG = ColG.racine;
  pointeurH = ColH.racine;
  while (pointeurG != NULL)
  { if (pointeurH == NULL) return non;
    if (G.noirs.element(pointeurG->orbite->sommet)^H.noirs.element(pointeurH->orbite
        ->sommet))
      return non;
    pointeurG = pointeurG->next;
    pointeurH = pointeurH->next;
  }
  return (ColG == ColH);
}

void bigraphe:: colorier (vset & S)
{ if (S.ordre() != n)
  { cout « "erreur dans bigraphe::colorier : coloriage d'ordre différent de celui du
      graphe\n";
    exit(1);
  }
  noirs = S;
}

ostream & operator « (ostream & sortie, bigraphe & G)
{ vertex u;
  graphe H(G);
  sortie « H;
  sortie « "\n//sommets noirs:\n";
  for (u=0; u<G.n; u++)
    if (G.noirs.element(u))
      sortie « u « " ";
  cout « '\n';
  return sortie;
}
```

```
istream & operator » (istream & entree, bigraphe & G)
{ graphe H;
  entree » H;
  G = bigraphe(H);
  return entree;
}

bigraphe bigraphe::parite ( vertexlist & L)
// retourne le graphe vide si la suite S n'est pas valide
{ bigraphe G = *this;
  int succes, trouve;
  vertex u,v;
  if (L.vide())
    return G;
  while (!L.vide())
  { succes = L.premier(u);
    trouve = non;
    while (succes && !trouve)
    { if (! G.noirs.element(u))
      { G = G.CompletionLocale(u);
        L.enleve(u);
        succes = non;
        trouve = oui;
      }
      else
        succes = L.suivant(u);
    }
    if (! trouve)
    { L.premier(u);
      succes = L.suivant(v);
      while (succes && (!G.arete(u,v)))
        succes = L.suivant(v);
      if (succes)
      { G = G.CompletionLocale(u);
        G = G.CompletionLocale(v);
        G = G.CompletionLocale(u);
        L.enleve(u);
        L.enleve(v);
        trouve = oui;
      }
      else
        return bigraphe();
    }
  }
  return G;
}

bigraphe bigraphe::induire (vset & L)
{ vertex v,w;
```

```
  if (n==0)
    return bigraphe();
  vertex * u = new vertex[n];
  if (u == NULL) {error(); exit(1);}
  vertex indice = 1;
  if (!L.premier(v)) { delete u; return bigraphe(); }
  if (v >= n) { cout « "erreur dans bigraphe::induire : sommet non valide\n"; exit
      (1);}
  u[0] = v;
  while (L.suivant(v))
  { if (v >= n) { cout « "erreur dans bigraphe::induire : sommet non valide\n"; exit
      (1);}
    u[indice++] = v;
  }
  bigraphe G(indice);
  for (v=0; v<indice; v++)
    for (w=v+1; w<indice; w++)
      { if (arete(u[v],u[w]))
          G.ajoute (v,w);
      }
  for (v=0; v<indice; v++)
    if (noirs.element(u[v]))
      G.noirs.ajoute (v);
  delete u;
  return G;
}

bigraphe bigraphe::complementaire (vset & S)
{ int total = 0;
  bigraphe G = * this;
  if (S.vide()) return G;
  vertexlist L,M;
  vertex u,v;
  S.premier(u);
  L.ajoute(u);
  while (S.suivant(u))
    L.ajoute(u);
  int succes, trouve;

  while (!L.vide())
  { succes = L.premier(u);
    trouve = non;
    while (succes && !trouve)
    { if (!G.noirs.element(u))
      { G = G.CompletionLocale(u);
        L.enleve(u);
        succes = non;
        trouve = oui;
        total++;
      }
```

```
        else
        { M = L;
          succes = M.premier(v);
          while (succes)
          { if ( (! G.arete(u,v))  || (! G.noirs.element(v)) )
              succes = M.suivant(v);
            else
            { G = G.CompletionLocale(u);
              G = G.CompletionLocale(v);
              G = G.CompletionLocale(u);
              L.enleve(u);
              L.enleve(v);
              succes = non;
              trouve = oui;
              total+=2;
            }
          }
        }
        if (!trouve)
          succes = L.suivant(u);
      }
      if ((!trouve)&&(!L.vide()))
      { G = bigraphe();
        return G;
      }
    }
    return G;
}

int  bigraphe::comp_ensemble (vset & S)
{ if (S.vide()) return oui;
  vset L,M;
  vertex u,v;
  int succes, trouve;
  unsigned int compte = 0;
  bigraphe G = (* this).induire(S); // il suffit de vérifier si S est un ensemble de
      complémentation du bigraphe induit par S
  L = vset(G);
  L.remplir();
  unsigned int total = L.ordre();
  while (!L.vide())
    { if (( total > 500) && (compte > (total+1)/2) )
          // optimisation (supposée) obtenue en réduisant périodiquement le sous-
              graphe induit considéré; le minimum de 500 sommets pour faire cette
              opération est arbitraire; la complexité de cet algorithme n'a pas été
              étudiée
      { G = G.induire(L);
        total -= compte;
        compte = 0;
        L = vset(G);
```

```
      L.remplir();
    }
    succes = L.premier(u);
    trouve = non;
    while (succes && !trouve)
    { if (!G.noirs.element(u))
      { G = G.CompletionLocale(u,L);
        L.enleve(u);
        succes = non;
        trouve = oui;
        compte++;
      }
      else
      { M = L;
        succes = M.premier(v);
        while (succes)
        { if ( (!G.arete(u,v)) || (!G.noirs.element(v)) )
            succes = M.suivant(v);
          else
          { G = G.CompletionLocale(u,L);
            G = G.CompletionLocale(v,L);
            G = G.CompletionLocale(u,L);
            L.enleve(u);
            L.enleve(v);
            succes = non;
            trouve = oui;
            compte+=2;
          }
        }
      }
      if (!trouve)
        succes = L.suivant(u);
    }
    if ((!trouve)&&(!L.vide()))
      return non;
  }
  return oui;
}

int bigraphe::inversible () // moins efficace que comp_ensemble
{ bigraphe G = * this;
  vertex u,v;
  int trouve;

  trouve = oui;
  while (trouve)
  { trouve = non;
    for (u=0; (!trouve) && (u<G.n); u++)
    { if (!G.noirs.element(u))
      { G = G.CompletionLocale(u);
```

```
        G = G.supprimer(u);
        trouve = oui;
      }
      else
      { for (v=u+1; (!trouve) && (v<G.n); v++)
        { if (!G.noirs.element(v))
          { G = G.CompletionLocale(v);
            G = G.supprimer(v);
            trouve = oui;
          }
          else if (G.arete(u,v))
          { G = G.CompletionLocale(u);
            G = G.CompletionLocale(v);
            G = G.CompletionLocale(u);
             vertexlist  L;
            L.ajoute (u);
            L.ajoute (v);
            G = G.supprimer(L);
            trouve = oui;
          }
        }
      }
    }
  }
  return (G.nul());
}
```

classe_de_parite.cpp

```
#include "classe_de_parite.hpp"

classe_de_parite::classe_de_parite (bigraphe & G)
{ n = G.ordre();
  G1 = G; ensemble_comp_G1 = vset(G1); clef_de_G1 = keytype(G1);
  u = 0; v = 1;
}

int classe_de_parite::prochain (bigraphe & H, vset & ensemble_de_complementation)
{ keytype clef_de_G2;
  vset ensemble_comp_G2;
  int nouveau, admissible;
  do
  {  while (!G1.nul())
     { while (u < n)
       { if (ensemble_comp_G1.element(u))
   { u++; v = u+1;
   }
         else
```

```
{ ensemble_comp_G2 = ensemble_comp_G1;
  ensemble_comp_G2.ajoute(u);
  admissible = non;
  if (!G1.noirs.element(u) && L.ajouter(ensemble_comp_G2)) // si le sommet est
      blanc et crée une nouvelle suite
  { G2 = G1.CompletionLocale(u);
    admissible = oui;  u++; v = u+1;
  }
  else if (G1.noirs.element(u))
  { while (v < n && (!G1.noirs.element(v) || ensemble_comp_G1.element(v) || !G1.arete
      (u,v)))
    v++;
    if (v<n)
    { ensemble_comp_G2.ajoute(v);
      if (L.ajouter(ensemble_comp_G2)) // si [u,v] est une arête noire qui crée une
          nouvelle suite
      { G2 = G1.CompletionLocale(u).CompletionLocale(v).CompletionLocale(u);
        admissible = oui;  v++;
      }
    }
  }
  if (admissible)
  { clef_de_G2 = keytype(G2);
    if (G1 == G2)
      nouveau = non;
    else if (precedente.membre(G2, clef_de_G2) || a_voir.membre(G2, clef_de_G2) ||
        vus.membre(G2, clef_de_G2))
      nouveau = non;
    else
      nouveau = suivante.insert (G2, clef_de_G2, ensemble_comp_G2);

    if (nouveau)
    { ensemble_de_complementation = ensemble_comp_G2;
      H = G2;
      return oui;
    }
  }
  else
  { u++; v = u+1;
  }
}
    }
    u = 0; v = 1;
    vus.insert (G1, clef_de_G1, ensemble_comp_G1);
    G1 = a_voir.deletemin (clef_de_G1, ensemble_comp_G1);
  }
  if (suivante.vide())
  { H = bigraphe();
    return non;
  }
```

```
      bg_liste VIDE; echangelistes (precedente, VIDE);
      echangelistes (precedente, vus); echangelistes (a_voir, suivante);
      G1 = a_voir.deletemin (clef_de_G1, ensemble_comp_G1);
      L.vider();
   } while (oui);
}

void classe_de_parite::developper ()
{ bigraphe  G;
  vset ens_comp_G;
  while (prochain (G, ens_comp_G))
    cout « "\n//ensemble de complémentation : " « ens_comp_G « "\n" « G;

}
```

coloriage.cpp

```
#include "graphe.hpp"

coloriage :: coloriage (bigraphe & G)
{ vertex  n = G.ordre();
  vertex  u, degre;
  couleur * pointeur, * suivant;
  link  * templink;
  int insertion;

  g = & G;
  if (n==0) {cout « "\nerreur dans coloriage: graphe nul"; exit(1);}
  racine = new couleur;
  if (racine == NULL) {error(); exit(1);}
  racine->clef = G.degre(0);
  racine->drapeau = oui;
  racine->n = 1;
  racine->orbite = new link;
  if (racine->orbite == NULL) {error(); exit(1);}
  racine->orbite->sommet = 0;
  racine->orbite->next = NULL;
  racine->next = NULL;
  for (u=1; u<n; u++)
  { degre = G.degre(u);
    if (racine->clef > degre)
    { pointeur = racine;
      racine = new couleur;
      if (racine == NULL) {error(); exit(1);}
      racine->clef = degre;
      racine->drapeau = oui;
      racine->n = 1;
      racine->orbite = new link;
```

```
  if (racine->orbite == NULL) {error(); exit(1);}
  racine->orbite->sommet = u;
  racine->orbite->next = NULL;
  racine->next = pointeur;
}
else
{ pointeur = racine;
  while (pointeur->clef < degre && pointeur->next != NULL
    && pointeur->next->clef <= degre)
    pointeur = pointeur->next;
  suivant = pointeur->next;
  if (pointeur->clef < degre)// alors suivant == NULL ou suivant- >clef
      >degre
  { pointeur->next = new couleur;
    if (pointeur->next == NULL) {error(); exit(1);}
    pointeur = pointeur->next;
    pointeur->clef = degre;
    pointeur->drapeau = oui;
    pointeur->n = 1;
    pointeur->orbite = new link;
    if (pointeur->orbite == NULL) {error(); exit(1);}
    pointeur->orbite->sommet = u;
    pointeur->orbite->next = NULL;
    pointeur->next = suivant;
  }
  else // (alors pointeur->clef == degre)
  { insertion = !(G.noirs.element(u) ^ G.noirs.element(pointeur->orbite->sommet));
    // c.-à-d. vrai si u est de la meme couleur que les sommet de
       pointeur- >orbite
    if (! insertion && suivant != NULL && suivant->clef == degre)
    { insertion = oui;
      pointeur = suivant;
    }
    if ( insertion )
    { (pointeur->n)++;
      templink = pointeur->orbite;
      pointeur->orbite = new link;
      if (pointeur->orbite == NULL) {error(); exit(1);}
      pointeur->orbite->sommet = u;
      pointeur->orbite->next = templink;
    }
    else
    { pointeur->next = new couleur;
      if (pointeur->next == NULL) {error(); exit(1);}
      pointeur->next->next = suivant;
      suivant = pointeur->next;
      suivant->clef = degre;
      suivant->drapeau = oui;
      templink = new link;
      if (templink == NULL) {error(); exit(1);}
```

```
            templink->sommet = u;
            templink->next = NULL;
            if (G.noirs.element(u))
            { suivant->n = 1;
              suivant->orbite = templink;
            }
            else
            { suivant->n = pointeur->n;
              suivant->orbite = pointeur->orbite;
              pointeur->n = 1;
              pointeur->orbite = templink;
            }
          }
        }
      }
    }
}

int coloriage :: insert (couleur * pointeur, vertex v, vertex degre)
{ link * templink;
  couleur * tempcouleur;
  if (pointeur->clef == degre)
  { (pointeur->n)++;
    templink = pointeur->orbite;
    pointeur->orbite = new link;
    if (pointeur->orbite == NULL) {error(); exit(1);}
    pointeur->orbite->sommet = v;
    pointeur->orbite->next = templink;
    return non; //pas de nouvelle couleur
  }
  tempcouleur = pointeur->next;
  if ((tempcouleur != NULL)&& !(tempcouleur->clef > degre))
    return (insert (tempcouleur, v, degre));
  pointeur->next = new couleur;
  if (pointeur->next == NULL) {error(); exit(1);}
  pointeur->next->clef = degre;
  pointeur->next->drapeau = oui;
  pointeur->next->n = 1;
  pointeur->next->orbite = new link;
  if (pointeur->next->orbite == NULL) {error(); exit(1);}
  pointeur->next->orbite->sommet = v;
  pointeur->next->orbite->next = NULL;
  pointeur->next->next = tempcouleur;
  return oui;//ajout d'une nouvelle couleur
}

coloriage :: coloriage (graphe & G)
{ vertex n = G.ordre();
  g = & G;
  if (n==0) {cout « "\nerreur dans coloriage: graphe nul"; exit(1);}
```

```
    racine = new couleur;
    if (racine == NULL) {error(); exit(1);}
    racine->clef = G.degre(0);
    racine->drapeau = oui;
    racine->n = 1;
    racine->orbite = new link;
    if (racine->orbite == NULL) {error(); exit(1);}
    racine->orbite->sommet = 0;
    racine->orbite->next = NULL;
    racine->next = NULL;
    vertex i, degre;
    couleur * tempcouleur;
    for (i = 1; i<n; i++)
    { degre = G.degre(i);
      if (racine->clef > degre)
      { tempcouleur = racine;
        racine = new couleur;
        if (racine == NULL) {error(); exit(1);}
        racine->clef = degre;
        racine->drapeau = oui;
        racine->n = 1;
        racine->orbite = new link;
        if (racine->orbite == NULL) {error(); exit(1);}
        racine->orbite->sommet = i;
        racine->orbite->next = NULL;
        racine->next = tempcouleur;
      }
      else
        insert (racine, i, degre);
    }
}

void coloriage::scrap() // désallouer toutes les listes du coloriage en préparation de
    sa destruction ou d'une recopie.
{ couleur * tempcouleur;
  link * templink;
  while (racine != NULL)
  { tempcouleur = racine;
    while (racine->orbite != NULL)
    { templink = racine->orbite;
      racine->orbite = templink->next;
      delete templink;
    }
    racine = tempcouleur->next;
    delete tempcouleur;
  }
}

coloriage::~coloriage()
{ scrap(); }
```

```
ostream & operator « (ostream & sortie, coloriage & C)
{ coloriage :: couleur * tempcouleur;
  coloriage :: link * templink;
  tempcouleur = C.racine;
  while (tempcouleur != NULL)
  { sortie « "\ncouleur " « tempcouleur->clef « " contenant " « tempcouleur->n « "
      sommets : ";
    templink = tempcouleur->orbite;
    if (templink != NULL)
    { sortie « templink->sommet;
      templink = templink->next;
    }
    while (templink != NULL)
    { sortie « "," « templink->sommet;
      templink = templink->next;
    }
    tempcouleur = tempcouleur->next;
  }
  sortie « "\n";
  return sortie;
}

int coloriage :: raffiner ()
{ couleur * CouleurATraiter, * boucle, * tempcouleur, * suite;
  couleur * loop;
  link * ListeDeSommets, * templink;
  vertex sommet, DegreRelatif; int notdone;
  CouleurATraiter = racine;
  while ((CouleurATraiter != NULL) && (CouleurATraiter->drapeau == non))
    CouleurATraiter = CouleurATraiter->next;
  if (CouleurATraiter == NULL) return (non); // c'est un coloriage stable, aucun
      raffinement effectué
  boucle = racine;
  CouleurATraiter->drapeau = non;
  while (boucle != NULL)
  { if (boucle == CouleurATraiter) boucle = boucle->next;
    else
    { boucle->clef = 0;
      sommet = boucle->orbite->sommet;
      templink = CouleurATraiter->orbite;
      while (templink != NULL)
      { boucle->clef += (*g).arete (sommet, templink->sommet);
        templink = templink->next;
      }
      ListeDeSommets = boucle->orbite->next;
      boucle->orbite->next = NULL;
      suite = boucle->next;
      boucle->next = NULL;
      boucle->n = 1;
```

```
while (ListeDeSommets != NULL)
{ sommet = ListeDeSommets->sommet;
  DegreRelatif = 0;
  templink = CouleurATraiter->orbite;
  while (templink != NULL)
  { DegreRelatif += (*g).arete (sommet, templink->sommet);
    templink = templink->next;
  }
  templink = ListeDeSommets;
  ListeDeSommets = ListeDeSommets->next;
  if (DegreRelatif == boucle->clef)
  { templink->next = boucle->orbite;
    boucle->orbite = templink;
    (boucle->n)++;
  }
  else
  { boucle->drapeau = oui;
    if (DegreRelatif < boucle->clef)
    { tempcouleur = boucle->next;
      boucle->next = new couleur;
      if (boucle->next == NULL) {error(); exit(1);}
      boucle->next->clef = boucle->clef;
      boucle->next->drapeau = oui;
      boucle->next->n = boucle->n;
      boucle->next->orbite = boucle->orbite;
      boucle->next->next = tempcouleur;
      boucle->clef = DegreRelatif;
      boucle->drapeau = oui;
      boucle->n = 1;
      templink->next = NULL;
      boucle->orbite = templink;
    }
    else // DegreRelatif >boucle->clef
    { notdone = oui;
      loop = boucle;
      while (notdone && (loop->next != NULL))
      { if (DegreRelatif == loop->next->clef)
        { templink->next = loop->next->orbite;
          loop->next->orbite = templink;
          (loop->next->n)++;
          notdone = non;
        }
        else if (DegreRelatif < loop->next->clef)
        { tempcouleur = loop->next;
          loop->next = new couleur;
          if (loop->next == NULL) {error(); exit(1);}
          loop->next->next = tempcouleur;
          loop->next->clef = DegreRelatif;
          loop->next->drapeau = oui;
          loop->next->n = 1;
```

```
                templink->next = NULL;
                loop->next->orbite = templink;
                notdone = non;
              }
              else loop = loop->next;
            }
            if (notdone)
            { loop->next = new couleur;
              if (loop->next == NULL) {error(); exit(1);}
              loop->next->next = NULL;
              loop->next->clef = DegreRelatif;
              loop->next->drapeau = oui;
              loop->next->n = 1;
              templink->next = NULL;
              loop->next->orbite = templink;
            }
          }
        }
      } // fin de la partition de l'ensemble pointé par boucle
      while (boucle->next != NULL) boucle = boucle->next;
      boucle->next = suite;
      boucle = suite;
    }
  } // fin de la partition de tous les ensembles
  return oui;
}

coloriage & coloriage::operator = (const coloriage & C)
{ couleur * tempcouleur, * boucle;
  link * templink, * loop;
  if (this != &C)
  { scrap(); // désallouer anciennes listes
    g = C.g;
    if (C.racine == NULL)
    { racine = NULL;
      return *this;
    }
    racine = new couleur;
    if (racine == NULL) {error(); exit(1);}
    boucle = racine;
    for (tempcouleur = C.racine; tempcouleur != NULL;
                        tempcouleur = tempcouleur->next)
    { boucle->clef = tempcouleur->clef;
      boucle->n = tempcouleur->n;
      boucle->drapeau = tempcouleur->drapeau;
      templink = tempcouleur->orbite;
      boucle->orbite = new link;
      if (boucle->orbite == NULL) {error(); exit(1);}
      loop = boucle->orbite;
      loop->sommet = templink->sommet;
```

```
        while (templink->next != NULL)
        { templink = templink->next;
          loop->next = new link;
          if (loop->next == NULL) {error(); exit(1);}
          loop = loop->next;
          loop->sommet = templink->sommet;
        }
        loop->next = NULL;
        if (tempcouleur->next != NULL)
        { boucle->next = new couleur;
          if (boucle->next == NULL) {error(); exit(1);}
          boucle = boucle->next;
        }
      }
      boucle->next = NULL;
    }
    return * this;
}

coloriage :: coloriage (const coloriage & C)
{ racine = NULL;
  *this = C;
}

int operator *= (coloriage & C, coloriage & D) //possiblement isomorphes
{ coloriage :: couleur * tempcouleur = C.racine;
  coloriage :: couleur * boucle = D.racine;
  while (tempcouleur != NULL)
  { if ((boucle == NULL) || (boucle->clef != tempcouleur->clef) ||
                            (boucle->n != tempcouleur->n))
      return non;
    tempcouleur = tempcouleur->next;
    boucle = boucle->next;
  }
  return (boucle == NULL);
}

int costabilize (coloriage & C, coloriage &D)
{ int iso = oui;
  if (!( C*=D)) iso = non;
  while (iso && (D.raffiner()))
  { if (!( C. raffiner ())) iso = non;
    else if (!( C*=D)) iso = non;
  }
  if (! iso) return non;
  return oui;
}

int coloriage :: ordonne () //vérifie si les sommets ont tous des couleurs différentes
{ for (couleur * curseurcouleur = racine; curseurcouleur != NULL;
```

```
    curseurcouleur = curseurcouleur->next)
  if (curseurcouleur->n > 1) return non;
  return oui;
}

int operator == (coloriage C, coloriage D)
{ coloriage :: couleur * curseurcouleurC = C.racine;
  coloriage :: couleur * curseurcouleurD = D.racine;
  if (!( costabilize (C,D))) return non;//stabilise les graphes coloriés
  if (C.ordonne()) //si les sommets ont tous des couleurs différentes
  { while (curseurcouleurC != NULL)
    { graphe::isomorphisme [curseurcouleurD->orbite->sommet] =
      curseurcouleurC->orbite->sommet;
      curseurcouleurC = curseurcouleurC->next;
      curseurcouleurD = curseurcouleurD->next;
    }
    vertex i,j; vertex n = C.g->ordre();
    for (i=0; i<n; i++)
      for (j=i+1; j<n; j++)
        if ((*(D.g)).arete(i,j) &&
                        !((*(C.g)).arete(graphe::iso(i),graphe::iso(j))))
    return non;
    return oui;
  }
  coloriage :: couleur * bestcouleurC, *bestcouleurD;
  C.meilleursommet (bestcouleurC);
  D.meilleursommet (bestcouleurD);
  coloriage nouveauC = C.nouvellecouleur(bestcouleurC, NULL);
  if (nouveauC == D.nouvellecouleur(bestcouleurD, NULL))
    return oui;
  coloriage :: link * curseursommetD = bestcouleurD->orbite;
  while (curseursommetD->next != NULL)
  { if (nouveauC == D.nouvellecouleur(bestcouleurD, curseursommetD))
    return oui;
    curseursommetD = curseursommetD->next;
  }
  return non;
}

int coloriage :: meilleursommet (couleur * & bestcouleur)
// trouve le meilleur choix de sommet à distinguer par une nouvelle couleur.
     couleur servira de paramètre pour nouvellecouleur la valeur de retour
     contient la «color-valence» des sommets.
{ int colorvalence, valence, covalence;
  vertex sommet;
  int maxcolorvalence = -1;
  bestcouleur = racine;
  //bestcouleur = NULL; cette ligne est inutile si le graphe est connexe
  couleur * curseurcouleur, * autrecouleur;
  link * curseursommet;
```

```
  for (curseurcouleur = racine; curseurcouleur != NULL; curseurcouleur = curseurcouleur
      ->next)
  { if (curseurcouleur->n > 1)
    { sommet = curseurcouleur->orbite->sommet;
      for (autrecouleur = racine; autrecouleur != NULL; autrecouleur = autrecouleur->
          next)
      { valence = 0;
        for (curseursommet = autrecouleur->orbite; curseursommet != NULL;
            curseursommet = curseursommet->next)
          valence += (*g).arete (sommet, curseursommet->sommet);
        covalence = (autrecouleur->n) - valence;
        if (valence < covalence) colorvalence = valence;
        else colorvalence = covalence;
        if (( colorvalence > maxcolorvalence) ||
        ((maxcolorvalence == colorvalence) && ((autrecouleur->n) < (bestcouleur->n))))
        { maxcolorvalence = colorvalence;
          bestcouleur = curseurcouleur;
        }
      }
    }
  }
  return maxcolorvalence;
}

coloriage coloriage::nouvellecouleur (couleur * bestcouleur, link * precedent)
// donne une nouvelle couleur au sommet indiqué et retourne ce nouveau
    coloriage.
{ coloriage C = * this;
  couleur * curseurcouleur, * tempcouleur;
  couleur * autrecouleur = C.racine;
  link * curseursommet;
  for (curseurcouleur = racine; curseurcouleur != bestcouleur;
                                curseurcouleur = curseurcouleur->next)
    autrecouleur = autrecouleur->next;
  tempcouleur = autrecouleur->next;
  autrecouleur->next = new couleur;
  if (autrecouleur->next == NULL) {error(); exit(1);}
  autrecouleur->next->clef = autrecouleur->clef;
  autrecouleur->next->drapeau = oui;
  autrecouleur->next->n = (autrecouleur->n)-1;
  autrecouleur->n = 1;
  autrecouleur->drapeau = oui;
  autrecouleur->next->next = tempcouleur;
  if (precedent == NULL)
  { autrecouleur->next->orbite = autrecouleur->orbite->next;
    autrecouleur->orbite->next = NULL;
    return C;
  }
  link * autresommet = autrecouleur->orbite;
```

```
  for (curseursommet = bestcouleur->orbite; curseursommet != precedent; curseursommet
       = curseursommet->next)
    autresommet = autresommet->next;
  autrecouleur->next->orbite = autrecouleur->orbite;
  autrecouleur->orbite = autresommet->next;
  autresommet->next = autresommet->next->next;
  autrecouleur->orbite->next = NULL;
  return C;
}
```

graphe.cpp

```
#include "graphe.hpp"

void error ()
{ cout « "\nerreur: mémoire insuffisante";}

vertex graphe::isomorphisme [NombreMaximalDeSommets];

graphe::graphe (vertex ordre, char * matrice) // matrice = "" indique un graphe sans
    arêtes
{ unsigned long i,j,k;
  unsigned long size = ordre;
  size = (((size*(size-1))»1)+7)»3;
  int vide = oui;
  unsigned char bit = 1;
  n = ordre;
  if (matrice == NULL) adj = NULL;
  else
  { adj = new char [size];
    if (adj == NULL) {error(); exit(1);}
    adj [0] = 0; k = 0;
    for (i=0; i<ordre; i++)
      for (j=i+1; j<ordre; j++)
      { if (bit == 0)
        { k++; adj[k]=0; bit = 1;
        }
        if (matrice [i*ordre+j] == '1')
        {adj [k] |= bit ; vide = non;
        }
        bit «= 1;
      }
    if (vide) {delete adj; adj = NULL;}
  }
}

graphe::graphe (vertex ordre, unsigned char * info)
{ n = ordre;
```

```
  unsigned long size = ordre;
  size = (((size*(size-1))/2)+7)/8;
  adj = new char [size];
  if (adj == NULL) {error(); exit(1);}
  for (unsigned long i = 0; i < size; i++)
    adj [i] = info [i];
}

graphe::graphe (const graphe & g)
{ n = g.n;
  unsigned long size = n;
  size = (((size*(size-1))/2)+7)/8;
  if (g.adj == NULL) adj = NULL;
  else
  { adj = new char [size];
    if (adj == NULL) {error(); exit(1);}
    for (unsigned long i=0; i<size; i++)
      adj [i] = g.adj [i];
  }
}

graphe & graphe::operator = (const graphe & G)
{ unsigned long size, byte;
  if (this != &G)
  { n = G.n;
    if (adj != NULL)
    { delete adj;
      adj = NULL;
    }
    if (G.adj != NULL)
    { size = n;
      size = (((size*(size-1))/2)+7)/8;
      adj = new char [size];
      if (adj == NULL) {error(); exit(1);}
      for (byte = 0; byte < size; byte++)
        adj [byte] = G.adj [byte];
    }
  }
  return *this;
}

void graphe::enleve (vertex i, vertex j)
{ unsigned long byte; int vide = oui; vertex temp;
  if (i==j) return;
  if (j<i) { temp = i; i = j; j = temp; } // échanger i et j
  unsigned long indice = ((((long(n)«1)-i-1)*i)»1)+j-i-1;
  byte = indice»3;
  unsigned char mask = 1«char(indice&7);
  adj [byte] |= mask; adj [byte] ^= mask;
  unsigned long size = n;
```

```
  size = (((size*(size-1))/2)+7)/8;
  for (byte = 0; byte < size; byte++)
    if (adj[byte] != 0)
      vide = non;
  if (vide) {delete adj; adj = NULL;}
}

void graphe::ajoute (vertex i, vertex j)
{ unsigned long byte, b; vertex temp;
  if (i==j) return;
  if (j<i) { temp = i; i = j; j = temp; } // échanger i et j
  unsigned long indice = ((((long(n)«1)-i-1)*i)»1)+j-i-1;
  byte = indice»3;
  unsigned char mask = 1«char(indice&7);
  if (adj == NULL)
  { unsigned long size = n;
    size = (((size*(size-1))/2)+7)/8;
    adj = new char [size];
    if (adj == NULL) {error(); exit(1);}
    for (b = 0; b < size; b++)
      adj [b] = 0;
  }
  adj [byte] |= mask;
}

void graphe::complemente (vertex i, vertex j)
{ unsigned long byte; vertex temp;
  if (i==j) return;
  if (j<i) { temp = i; i = j; j = temp; } // échanger i et j
  unsigned long indice = ((((long(n)«1)-i-1)*i)»1)+j-i-1;
  byte = indice»3;
  unsigned char mask = 1«char(indice&7);
  adj [byte] ^= mask;
}

graphe & graphe::operator ++ ()
{ int fini = non;
  vertex i = 0;
  while ((!fini) && (i<n-1))
  { if (arete (i, n-1))
    { enleve (i, n-1); i++;
    }
    else
    { ajoute (i, n-1); fini = oui;
    }
  }
  return * this;
}

graphe graphe::CompletionLocale (vertex s)
```

```
{ vertex * voisins;
  voisins = new vertex [n-1];
  if ( voisins == NULL) {error(); exit(1);}
  vertex i,j,k;
  graphe H = * this;

  for (i = 0, k = 0; i < n; i++)
    if (H.arete (s, i))
      voisins [k++] = i; // voisins[0..k - 1] := voisinage de s

// inverser la relation d'adjacence entre les voisins
  for (i = 0; i < k; i++)
    for (j = i+1; j < k; j++)
      H.complemente (voisins[i], voisins [j]);
  delete voisins;
  return H;
}

int operator == (graphe & G, graphe & H)
{ if (G.n != H.n) return non;
  coloriage ColG(G);
  coloriage ColH(H);
  return (ColG == ColH);
}

int graphe::iso (vertex i)
{ return (isomorphisme [i]);
}

int operator *= (graphe & G, graphe & H)
{ if (G.n != H.n) return non;
  unsigned int size = G.n;
  size = (((size*(size-1))/2)+7)/8;
  for (int i = 0; i < size; i++)
    if (G.adj [i] != H.adj [i]) return non;
  return oui;
}

int AntiRaffine (graphe &G, vertexlist *blancs, vertexlist *noirs, vertexlist *independant
    )
// retourne oui s'il y a un sommet blanc non couvert par un sommet noir
{ vertex u, v;
  int succes;
  vertex degre;
  succes = (*blancs).premier (u);
  while (succes)
  { degre = (*noirs).degre(u, G);
    if (degre == 0)
      return oui;
    if (degre == 1)
```

```
    { succes = (*noirs).premier (v);
      while (succes)
      { if (G.arete(u,v))
          succes = non;
        else
          succes = (*noirs).suivant (v);
      }
      (*noirs).enleve_voisins_et (v,G);
      (*blancs).enleve_voisins_et (v,G);
      (*independant).ajoute(v);
      succes = (*blancs).premier (u);
    }
    else
      succes = (*blancs).suivant (u);
  }
  // à présent tout sommet blanc est adjacent à au moins 2 sommets noirs
  succes = (*noirs).premier (u);
  while (succes)
  { degre = (*noirs).degre(u, G);
    if (degre == 0)
    { (*noirs).enleve_voisins_et (u,G);
      (*blancs).enleve_voisins_et (u,G);
      (*independant).ajoute(u);
      succes = (*noirs).premier (u);
    }
    else
      succes = (*noirs).suivant (u);
  }
  // à present tout sommet noir est adjacent à au moins un autre sommet noir
  return non;
}

void AntiSub (graphe &G, vertexlist *anticlique, unsigned int & compte, int max,
    vertexlist *blancs, vertexlist *noirs, vertexlist *independant, vertex * sommet, int *
    flag, int compter)
{ vertex u, v;
  unsigned int degre, maxdegre;
  unsigned int niveau = 0;
  int drapeau = 0;
  int succes;
  do
  { if (drapeau == 0)
    { if (noirs[niveau].vide())
      { if (compte < max)
          anticlique[compte] = independant[niveau];
        compte++;
        if ((compter == non)||(niveau == 0)) return;
        niveau--;
        drapeau = flag[niveau];
        v = sommet[niveau];
```

```
    }
    else
    { maxdegre = 0;
      if (blancs[niveau].vide())
        for (succes = noirs[niveau].premier (u); succes;
                                  succes = noirs[niveau].suivant(u))
        { degre = noirs[niveau].degre(u,G);
          if (degre > maxdegre)
          { v = u;
            maxdegre = degre;
          }
        }
      else
        for (succes = noirs[niveau].premier (u); succes;
                                  succes = noirs[niveau].suivant(u))
        { degre = blancs[niveau].degre(u,G);
          if (degre > maxdegre)
          { v = u;
            maxdegre = degre;
          }
        }
      // à ce point du programme v a un nombre de voisins maximal dans blancs
      flag[niveau] = 1;
      sommet[niveau] = v;
      niveau++;
      if (niveau == NombreMaximalDeSommets)
      {error(); cout « "tableaux pleins dans AntiSub\n Essayez d'augmenter la
          constante NombreMaximalDeSommets\n"; exit(1);}
      blancs[niveau] = blancs[niveau-1];
      noirs[niveau] = noirs[niveau-1];
      independant[niveau] = independant[niveau-1];
      noirs[niveau].enleve_voisins_et (v,G);
      blancs[niveau].enleve_voisins_et (v,G);
      independant[niveau].ajoute (v);
      if (AntiRaffine(G, &(blancs[niveau]), &(noirs[niveau]), &(independant[niveau])))
      { if (niveau == 0)
          return;
        niveau--;
        drapeau = flag[niveau];
        v = sommet[niveau];
      }
    }
  }
  else if (drapeau == 1)
  { flag[niveau] = 2;
    niveau++;
    drapeau = 0;
    blancs[niveau] = blancs[niveau-1];
    noirs[niveau] = noirs[niveau-1];
    independant[niveau] = independant[niveau-1];
```

```
        noirs[niveau].enleve(v);
        blancs[niveau].ajoute(v);
        if (AntiRaffine(G, &(blancs[niveau]), &(noirs[niveau]), &(independant[niveau])))
        { if (niveau == 0)
            return;
          niveau--;
          drapeau = flag[niveau];
          v = sommet[niveau];
        }
      }
      else
      { if (niveau == 0)
          return;
        niveau--;
        drapeau = flag[niveau];
        v = sommet[niveau];
      }
    } while (1);
}

int AnticliqueImpaire (graphe & G, vertexlist * anticlique , int max, int compter)
// anticlique est un tableau pouvant contenir un nombre max d'objets de type
     vset
{ vertex u;
  unsigned int compte = G.ordre();
  static vertexlist blancs       [NombreMaximalDeSommets];
  static vertexlist noirs        [NombreMaximalDeSommets];
  static vertexlist independant [NombreMaximalDeSommets];
  static vertex   sommet     [NombreMaximalDeSommets];
  static int      flag       [NombreMaximalDeSommets];

  vertexlist vide;
  blancs[0] = vide;
  noirs[0] = vide;
  independant[0] = vide;

  for (u=0; u<compte; u++)
  { if ((G.degre(u)&1) == 0)
      blancs[0].ajoute (u);
    else
      noirs[0].ajoute (u);
  }
  if (AntiRaffine (G, blancs, noirs, independant))
    return 0;
  compte = 0;
  AntiSub (G, anticlique, compte, max, blancs, noirs, independant,
           sommet, flag, compter);
  return compte;
}
```

```
int AnticliqueImpaire (graphe & G)
{ unsigned int compte = 0;
   vertexlist  L;
  compte = AnticliqueImpaire (G, &L, 1, non);
  return (compte != 0);
}

vertex graphe::degre (vertex sommet)
{ vertex degre = 0;
  for (int i=0; i < n; i++)
    if (arete (sommet, i)) degre++;
  return degre;
}

int Eulerien (graphe & G)
{ for (vertex i=0; i< G.n; i++)
  if (G.degre(i) & 1)
    return non;
  return oui;
}

ostream & operator « (ostream & sortie, graphe & G)
{ vertex i,j;
  sortie « "\n";
  for (i=0; i<G.n; i++)
  { for (j=0; j<G.n; j++)
    if (G.arete (i,j))
      sortie « '1';
    else sortie « '0';
    sortie « "\n";
  }
  return sortie;
}

istream & operator » (istream & entree, graphe & G)
{ vertex i, j, n;
  int k;
  int okay = oui;
  char c; char buffer [NombreMaximalDeSommets];
  unsigned char bit = 1;
  if (G.adj != NULL)
  { delete G.adj;
    G.adj = NULL;
  }
  entree » ws;
  do
  { entree.getline (buffer, NombreMaximalDeSommets-1);
  }
  while (entree && ((buffer [0] == '/') ||( buffer [0] == '\0')));
  // le caractère '/' permet l'affichage de commentaires
```

```
  for (n=0; buffer [n] != '\0'; n++); //compter l'ordre du graphe
  G.n = n;
  unsigned int size = n;
  size = (((size*(size-1))/2)+7)/8;
  if (n==0)
    okay = non;
  else
  { G.adj = new char [size];
    if (G.adj == NULL) {error(); exit(1);}
    k = 0; G.adj[0] = 0;
    for (i=1; (i<n) && (buffer [i] == '0' || buffer [i] == '1'); i++)
    { if (bit == 0) { k++; G.adj[k]=0; bit = 1;}
      if (buffer [i] == '1') G.adj[k] |= bit;
      bit «= 1;
    }
    if (i<n)
      okay = non;
    else
      for (i=1; (i<n) && okay; i++)
      { for (j=0; (j<n) && okay; j++)
  { entree.get(c);
    if (c != '0' && c != '1')
      okay = non;
    else if (j>i)
    { if (bit == 0) {k++; G.adj[k]=0; bit=1;}
      if (c == '1') G.adj[k] |= bit;
      bit «= 1;
    }
  }
        entree.get(c);
  if (c != '\n') okay = non;
      }
  }
  if (!okay)
  { delete G.adj; G.adj = NULL;
    entree.clear (ios::badbit | entree.rdstate());
  }
  return entree;
}

vertex  graphe::ordre ()
{ return n;
}

int graphe::arete (vertex i, vertex j)
{ if ((adj == NULL) || (i==j)) return non;
  vertex temp; unsigned long byte;
  if (j<i) { temp = i; i = j; j = temp; } // échanger i et j
  unsigned long indice = ((((long(n)«1)-i-1)*i)»1)+j-i-1;
  byte = indice»3;
```

```
  unsigned char mask = 1«char(indice&7);
  return ((adj[byte] & mask) != 0);
}

void graphe::reunion (graphe & G)
{ if (this == &G) return;
  if (n<G.n)
  { graphe H(G.n);
    vertex i,j;
    for (i=0; i<n; i++)
      for (j=i+1; j<n; j++)
        if (arete(i,j))
          H.ajoute(i,j);
    *this = H;
  }
  if (G.adj == NULL) return;
  if (n==G.n)
  { unsigned int size = n;
    size = (((size*(size-1))/2)+7)/8;
    int i;
    if (adj == NULL)
    { adj = new char [size];
      if (adj == NULL) {error(); exit (1);}
      for (i = 0; i<size; i++)
        adj[i] = 0;
    }
    for (i=0; i<size; i++)
      adj[i] |= G.adj[i];
    return;
  }
  vertex i,j;
  for (i=0; i<G.n; i++)
    for (j=i+1; j<G.n; j++)
      if (G.arete(i,j))
        ajoute(i,j);
}

graphe graphe::complementaire ()
{ vertex  u,v;
  graphe  H(n);

  for (u = 0; u<n; u++)
    for (v = u+1; v<n; v++)
      if (!arete(u,v))
        H.ajoute(u,v);
  return H;
}

graphe graphe::complementaire (vset & S)
{ int total =0;
```

```
graphe G = * this;
if (S.vide()) return G;
vertexlist L,M;
vertex u,v;
S.premier(u);
L.ajoute(u);
while (S.suivant(u))
  L.ajoute(u);
int succes, trouve;

while (!L.vide())
{ succes = L.premier(u);
  trouve = non;
  while (succes && !trouve)
  { if ((G.degre(u)&1)==0)
    { G = G.CompletionLocale(u);
      L.enleve(u);
      succes = non;
      trouve = oui;
      total++;
    }
    else
    { M = L;
      succes = M.premier(v);
      while (succes)
      { if ( (! G.arete(u,v)) || (( G.degre(v)&1)==0) )
          succes = M.suivant(v);
        else
        { G = G.CompletionLocale(u);
          G = G.CompletionLocale(v);
          G = G.CompletionLocale(u);
          L.enleve(u);
          L.enleve(v);
          succes = non;
          trouve = oui;
          total+=2;
        }
      }
    }
    if (! trouve)
      succes = L.suivant(u);
  }
  if ((! trouve)&&(!L.vide()))
  { G = graphe();
    return G;
  }
}
return G;
}
```

```
graphe graphe::operator * (graphe & G)
{ vertex ordre = n*G.n;
  vertex u,v;
  graphe H(ordre);
  for (u=0; u<ordre; u++)
    for (v=u+1; v<ordre; v++)
      if ((arete(u/G.n,v/G.n))||((u/G.n==v/G.n)&&(G.arete(u%G.n,v%G.n))))
        H.ajoute(u,v);
  return H;
}

graphe cayley (vertex ordre, vset & L)
{ vertex u,v;
  if (L.vide())
    return graphe();
  graphe G(ordre);
  L.premier(v);
  for (u=0; u<ordre; u++)
    G.ajoute (u,(u+v)%ordre);
  while (L.suivant(v))
  { for (u=0; u<ordre; u++)
      G.ajoute (u, (u+v)%ordre);
  }
  return G;
}

graphe graphe::supprimer (vertex u)
{ vertex v,w,a,b;

  if (!(u<n))
  { cout « "erreur dans supprimer: sommet non valide\n";
    return * this;
  }
  graphe G(n-1);
  for (v=0, a=0; v<n; v++)
  { if (v==u)
      a = 1;
    else
    { for (w=v, b=a; w<n; w++)
      { if (w==u)
          b = 1;
        else
        { if (arete(v,w))
            G.ajoute(v-a,w-b);
        }
      }
    }
  }
  return G;
}
```

```
bigraphe bigraphe::supprimer (vertex u)
{ vertex v,w,a,b;

  if (!( u<n))
  { cout « "erreur dans supprimer: sommet non valide\n";
    return * this;
  }
  bigraphe G(n-1);
  for (v=0, a=0; v<n; v++)
  { if (v==u)
      a = 1;
    else
    { for (w=v, b=a; w<n; w++)
      { if (w==u)
          b = 1;
        else
        { if (arete(v,w))
            G.ajoute(v-a,w-b);
        }
      }
      if (noirs.element(v))
          G.noirs.ajoute (v-a);
    }
  }
  return G;
}

graphe graphe::supprimer (vset & L)
{ vertex v;
  if (L.vide())
    return *this;
  char * u = new char[n];
  if (u == NULL)
    cout « "erreur dans supprimer: manque de mémoire\n";
  for (v=0; v<n; v++)
    u[v] = non;
  L.premier(v);
  u[v] = oui;
  while (L.suivant(v))
    u[v] = oui;
  graphe G = * this;
  for (v = n; v!=0; v--)
    if (u[v-1])
      G = G.supprimer(v-1);
  delete u;
  return G;
}

graphe graphe::supprimer (vertexlist & L)
```

```
{ vertex v;
  if (L.vide())
    return *this;
  char * u = new char[n];
  if (u == NULL)
    cout « "erreur dans supprimer: manque de mémoire\n";
  for (v=0; v<n; v++)
    u[v] = non;
  L.premier(v);
  u[v] = oui;
  while (L.suivant(v))
    u[v] = oui;
  graphe G = * this;
  for (v = n; v!=0; v--)
    if (u[v-1])
      G = G.supprimer(v-1);
  delete u;
  return G;
}

bigraphe bigraphe::supprimer (vertexlist & L)
{ vertex v;
  if (L.vide())
    return *this;
  char * u = new char[n];
  if (u == NULL)
    cout « "erreur dans supprimer: manque de mémoire\n";
  for (v=0; v<n; v++)
    u[v] = non;
  L.premier(v);
  u[v] = oui;
  while (L.suivant(v))
    u[v] = oui;
  bigraphe G = * this;
  for (v = n; v!=0; v--)
    if (u[v-1])
      G = G.supprimer(v-1);
  delete u;
  return G;
}

graphe graphe::induire (vset & L)
{ vertex v;
  if (L.vide())
    return graphe();
  char * u = new char[n];
  if (u == NULL)
    cout « "erreur dans induire (vset): manque de mémoire\n";
  for (v=0; v<n; v++)
    u[v] = oui;
```

```
  L.premier(v);
  u[v] = non;
  while (L.suivant(v))
    u[v] = non;
  graphe G = * this;
  for (v = n; v!=0; v--)
    if (u[v-1])
      G = G.supprimer(v-1);
  delete u;
  return G;
}

graphe graphe::induire ( vertexlist & L)
{ vertex v;
  if (L.vide())
    return graphe();
  char * u = new char[n];
  if (u == NULL)
    cout « "erreur dans induire (vertexlist) : manque de mémoire\n";
  for (v=0; v<n; v++)
    u[v] = oui;
  L.premier(v);
  u[v] = non;
  while (L.suivant(v))
    u[v] = non;
  graphe G = * this;
  for (v = n; v!=0; v--)
    if (u[v-1])
      G = G.supprimer(v-1);
  delete u;
  return G;
}

bigraphe bigraphe::induire ( vertexlist & L)
{ vertex v;
  if (n == 0)
    return bigraphe();
  char * u = new char[n];
  if (u == NULL)
    cout « "erreur dans induire (vertexlist) : manque de mémoire\n";
  for (v=0; v<n; v++)
    u[v] = oui;
  L.premier(v);
  if (v >= n) { cout « "erreur dans bigraphe::induire(vertexlist) : sommet non
      valide\n"; exit (1);}
  u[v] = non;
  while (L.suivant(v))
  { if (v >= n) { cout « "erreur dans bigraphe::induire : sommet non valide\n"; exit
      (1);}
    u[v] = non;
```

```
  }
  bigraphe G = * this;
  for (v = n; v!=0; v--)
    if (u[v-1])
      G = G.supprimer(v-1);
  delete u;
  return G;
}

vset composante (vertex u, bigraphe & G)
{ vset S(G);
  vertexqueue Q;
  vertex n = G.ordre();
  char * visite = new char [n];
  vertex v,w;
  for (v=0; v<n; v++)
    visite [v] = non;
  visite [u] = oui;
  Q.ajoute (u);
  S.ajoute(u);
  while (!Q.vide())
  { Q.dequeue(v);
    for (w=0; w<n; w++)
      if (! visite [w] && G.arete(v,w))
      { Q.ajoute(w);
        S.ajoute(w);
        visite [w] = oui;
      }
  }
  return S;
}
```

keytype.cpp

```
#include "keytype.hpp"

void quicksort (vertex n, vertex * degres)
{ vertex  temp;
  if (n == 2)
  { if (degres [1] < degres [0])
    { temp = degres [0]; degres [0] = degres [1]; degres [1] = temp;
    }
    return;
  }
  else if (n == 3)
  { if (degres [1] < degres [0])
    { temp = degres [0]; degres [0] = degres [1]; degres [1] = temp;
    }
```

```
      if (degres [2] < degres [1])
      { temp = degres [2]; degres [2] = degres [1];
        if (temp < degres [0])
        { degres [1] = degres [0]; degres [0] = temp;
        }
        else
          degres [1] = temp;
      }
    }
    else
    { vertex  i , j , k, mid;
      vertex * mergelist = new vertex[n];
      if (mergelist == NULL) {error(); exit(1);}
      mid = n/2;
      quicksort (mid, degres);
      quicksort (n-mid, &(degres[mid]));
      i=0; j=mid; k = 0;
      while ((i<mid) && (j<n))
      { if (degres[j] < degres[i])
          mergelist [k] = degres [j++];
        else
          mergelist [k] = degres [i++];
        k++;
      }
      if (i == mid)
        while (j<n)
          mergelist [k++] = degres [j++];
      else
        while (i<mid)
          mergelist [k++] = degres [i++];
      for (i=0; i<n; i++)
        degres [i] = mergelist[i];
      delete mergelist;
    }
}

keytype::keytype (graphe & G)
{ n = G.ordre();
  vertex i;
  degres = new vertex [n];
  if (degres == NULL) {error(); exit(1);}
  for (i=0; i<n; i++) degres [i] = G.degre(i);
  if (n>1)
    quicksort (n, degres);
}

void keytype::nullify ()
{ degres = NULL;
}
```

```
keytype::~keytype ()
{ if (degres != NULL)
    delete degres;
}

keytype & keytype::operator *= (keytype & K)
{ n = K.n;
  degres = K.degres;
  return * this;
}

keytype & keytype::operator = (const keytype & K)
{ if (this != &K)
  { vertex i;
    n = K.n;
    if (degres != NULL)
      delete degres;
    if (K.degres == NULL)
      degres = NULL;
    else
    { degres = new vertex [n];
      if (degres == NULL) {error(); exit(1);}
      for (i=0; i<n; i++)
        degres [i] = K.degres [i];
    }
  }
  return * this;
}

keytype::keytype (const keytype & K)
{ degres = NULL;
  * this = K;
}

int operator == (keytype & K, keytype & L)
{ vertex i; vertex n = K.n;
  if (L.n != n) return non;
  for (i=0; i<n; i++)
    if (K.degres[i] != L.degres[i]) return non;
  return oui;
}

int operator < (keytype & K, keytype & L)
{ vertex i; vertex n = K.n;
  if (n<L.n)
    return oui;
  if (L.n<n)
    return non;
  for (i=0; i<n; i++)
  { if (K.degres[i] < L.degres[i])
```

```
      return oui;
    if (K.degres[i] > L.degres[i])
      return non;
  }
  return non;
}
```

vertex.cpp

```
#include "graphe.hpp"

vertexlist ::~ vertexlist ()
{ link * temp1, * temp2;
  temp1 = first;
  while (temp1 != NULL)
  { temp2 = temp1;
    temp1 = temp1->next;
    delete temp2;
  }
}

int vertexlist :: premier (vertex & u)
{ if ( first == NULL) return non;
  current = NULL;
  u = first->sommet;
  return oui;
}

int vertexlist :: suivant (vertex & u)
{ if ( first == NULL) return non;
  if (current == NULL)
    current = first ;
  else
    current = current->next;
  if (current->next == NULL) return non;
  u = current->next->sommet;
  return oui;
}

int vertexlist :: ajoute (vertex u)
{ link * temp;
  if (current == NULL)
  { temp = first;
    first = new link;
    if ( first == NULL) {error(); exit (1);}
    first ->sommet = u;
    first ->next = temp;
  }
```

```
  else
  { temp = current->next;
    current->next = new link;
    if (current->next == NULL) {error(); exit (1);}
    current->next->sommet = u;
    current->next->next = temp;
  }
  return oui;
}

int vertexlist ::element (vertex u)
{ link * temp;
  temp = first;
  while (temp != NULL)
  { if (temp->sommet == u) return oui;
    temp = temp->next;
  }
  return non;
}

int vertexlist ::enleve (vertex u)
{ link * temp1, * temp2;
  current = NULL;
  temp1 = first;
  if (temp1 == NULL) return non;
  if (temp1->sommet == u)
  { first = temp1->next;
    delete temp1;
    return oui;
  }
  while ((temp1->next != NULL) && (temp1->next->sommet != u))
    temp1 = temp1->next;
  if (temp1->next == NULL) return non;
  temp2 = temp1->next;
  temp1->next = temp2->next;
  delete temp2;
  return oui;
}

void vertexlist ::enleve_voisins_et (vertex u, graphe & G)
{ vertex v;
  if (vide()) return;
  premier (v);
  do
  { if ((v == u) || G.arete(u,v))
    { enleve(v);
      if (vide()) return;
      premier(v);
    }
    else
```

```
      if (!suivant(v)) return;
  } while (1);
}

int vertexlist ::adjacent (vertex u, graphe & G)
{ link * temp;
  temp = first;
  while (temp != NULL)
  { if (G.arete(temp->sommet, u)) return oui;
    temp = temp->next;
  }
  return non;
}

vertex  vertexlist ::degre (vertex u, graphe & G)
{  vertex   d;
  link    * temp;
  d = 0;
  temp = first;
  while (temp != NULL)
  { if (G.arete(temp->sommet, u)) d++;
    temp = temp->next;
  }
  return d;
}

vertexlist :: vertexlist (const vertexlist & L)
{ link * temp1, * temp2;
  temp1 = L.first;
  if (temp1 == NULL)
  { first = NULL;
    current = NULL;
    return;
  }
  current = NULL;
  first = new (link); if ( first == NULL) {error(); exit(1);}
  temp2 = first;
  temp2->sommet = temp1->sommet;
  if (L.current == temp1)
    current = temp2;
  while (temp1->next != NULL)
  {  temp2->next = new (link); if (temp2->next == NULL) {error(); exit(1);}
    temp2 = temp2->next;
    temp1 = temp1->next;
    temp2->sommet = temp1->sommet;
    if (L.current == temp1)
      current = temp2;
  }
  temp2->next = NULL;
}
```

```
vertexlist & vertexlist ::operator = (const vertexlist & L)
{ link * temp1, * temp2;
  if (this != &L)
  { temp1 = first;
    while (temp1 != NULL)
    { temp2 = temp1->next;
      delete temp1;
      temp1 = temp2;
    }
    temp1 = L.first;
    if (temp1 == NULL)
    { first = NULL;
      current = NULL;
      return *this;
    }
    current = NULL;
    first = new (link); if ( first == NULL) {error(); exit(1);}
    temp2 = first;
    temp2->sommet = temp1->sommet;
    if (L.current == temp1)
      current = temp2;
    while (temp1->next != NULL)
    { temp2->next = new (link); if (temp2->next == NULL) {error(); exit(1);}
      temp2 = temp2->next;
      temp1 = temp1->next;
      temp2->sommet = temp1->sommet;
      if (L.current == temp1)
        current = temp2;
    }
    temp2->next = NULL;
  }
  return *this;
}

vertexlist :: vertexlist (vset & S)
{ vertex u;
  link * ptr;
  current = NULL;
  if (!S.premier(u))
  { first = NULL;
    return;
  }
  first = new (link); if ( first == NULL) {error(); exit(1);}
  ptr = first ;
  ptr->sommet = u;
  for (int succes = S.suivant(u); succes; succes = S.suivant(u))
  { ptr->next = new(link); if (ptr == NULL) {error(); exit(1);}
    ptr = ptr->next;
    ptr->sommet = u;
```

```
  }
  ptr->next = NULL;
}

ostream & operator « (ostream & sortie, vertexlist & L)
{ vertex u;
  int succes;
  for (succes = L.premier(u); succes; succes = L.suivant(u))
    sortie « int(u) « " ";
  sortie « '\n';
  return sortie;
}

int vertexqueue::dequeue (vertex & u)
{ if (vide())
    return non;
  u = first->sommet;
  link * temp = first;
  first = first->next;
  delete temp;
  return oui;
}

void vertexqueue::ajoute (vertex u)
{ if ( first == NULL)
  { first = new link; if ( first == NULL) {error(); exit(1);}
    first->sommet = u;
    first->next = NULL;
    current = first ;
  }
  else
  { current->next = new link; if (current->next == NULL) {error(); exit(1);}
    current = current->next;
    current->sommet = u;
    current->next = NULL;
  }
}

vertexlabel :: vertexlabel ( vertexlist & L, vertex ordre, int supprimer)
{ vertex u,v,compte;
  n = ordre;
  sub = new vertex [n]; if (sub == NULL) {error(); exit(1);}
  if (supprimer)
  { compte = n;
    for (u=0; u<ordre; u++)
      sub[u] = 0;
    for (int succes=L.premier(u); succes; succes=L.suivant(u))
    { if (u >= ordre) {cout « "erreur dans vertexlabel: sommet >= ordre\n"; exit(1)
        ;}
      sub[u] = ordre;
```

```
      compte--;
    }
  }
  else
  { compte=0;
    for (u=0; u<ordre; u++)
      sub[u] = ordre;
    for (int succes=L.premier(u); succes; succes=L.suivant(u))
    { if (u >= ordre) {cout « "erreur dans vertexlabel: sommet >= ordre\n"; exit(1)
        ;}
      sub[u]=0;
      compte++;
    }
  }
  sub_n = compte;
  sur = new vertex [compte]; if (sub == NULL) {error(); exit(1);}
  for (u=0,v=0; u<ordre; u++)
  { if (sub[u]==0)
    { sub[u]=v;
      sur[v++]=u;
    }
  }
}

vertexlabel::~vertexlabel ()
{ delete sub;
  delete sur;
}

vertexlabel & vertexlabel::operator = (const vertexlabel & L)
{ vertex u;
  if (&L != this)
  { n = L.n;
    sub_n = L.sub_n;
    delete sur;
    delete sub;
    sub = new vertex [n]; if (sub == NULL) {error(); exit(1);}
    sur = new vertex [sub_n]; if (sur == NULL) {error(); exit(1);}
    for (u = 0; u < sub_n; u++)
    { sub[u] = L.sub[u];
      sur[u] = L.sur[u];
    }
    while (u<n)
    { sub[u] = L.sub[u];
      u++;
    }
  }
  return * this;
}
```

```
ostream & operator « (ostream & sortie, vertexlabel & L)
{ for (vertex u=0; u<L.sub_n; u++)
     sortie « L.sur[u ] « " ";
  sortie « '\n';
  return sortie;
}

vertexlist vertexlabel :: list_labels ()
{ vertexlist L;
  for (vertex u = 0; u<sub_n; u++)
    L.ajoute (sur[u]);
  return L;
}

vertexlist vertexlabel :: list_labels ( vertexlist & L)
{ vertexlist M;
  vertex u;
  for (int succes = L.premier(u); succes; succes = L.suivant(u))
  { if (u >= sub_n)
    { cout « "erreur dans vertexlabel::list_labels(vertexlist) : sommet fautif\n"; exit
        (1);}
    M.ajoute (sur[u]);
  }
  return M;
}

vertexlist vertexlabel :: sublist ( vertexlist & L)
{ vertexlist M;
  vertex u;
  for (int succes = L.premier(u); succes; succes = L.suivant(u))
  { if (u >= n || sub[u] == n)
    { cout « "erreur dans vertexlabel::sublist(vertexlist) : sommet fautif\n"; exit(1)
        ; }
    M.ajoute (sub[u]);
  }
  return M;
}

void vertexlabel:: relabel (vertex u, vertex label)
{ if (u >= sub_n || label >= n || sub[label] != n)
  { cout « "erreur dans vertexlabel::relabel: (u,label) non valide\n" ; exit (1) ; }
  sub[label] = u;
  sub[sur[u]] = n;
  sur[u] = label;
}
```

vset.cpp

```
#include "graphe.hpp"

vset :: vset (vertex nombre)
{ o = nombre;
  n = (nombre+7)»3;   // nombre d'octets pour contenir tous les sommets
  set = NULL;
}

vset :: vset (graphe & G)
{ o = G.ordre();
  n = (o+7)»3;
  set = NULL;
}

vset::~vset ()
{ delete (set);
}

void vset:: initialiser (vertex nombre)
{ delete (set);
  set = NULL;
  o = nombre;
  n = (nombre+7)»3;
}

void vset::remplir ()
{ unsigned int byte;
  unsigned char mask;
  if (n == 0) return;
  if (set == NULL)
  { set = new unsigned char [n];
    if (set == NULL)
      { error(); exit(1); }
  }
  for (byte=0; byte<n-1; byte++)
    set[byte] = 255;
  mask = o&7;
  if (mask == 0)
    set[byte] = 255;
  else
    set[byte] = (1u «mask)-1;
}

int vset :: plein ()
{ unsigned int byte;
  unsigned char mask;
  if (set == NULL)
    return non;
  for (byte=0; byte<n-1; byte++)
    if (set[byte]!=255)
```

```
      return non;
  mask = o&7;
  if (mask == 0)
    return (set[byte]==255);
  else
    return (set[byte]==(1u«mask)-1);
}

vset :: vset (const vset & K)
{ set = NULL;
  * this = K;
}

void vset::vider()
{ delete (set);
  set = NULL;
}

int vset::vide()
{ unsigned int i;
  if (set == NULL)
    return oui;
  for (i=0; i<n; i++)
    if (set[i]!=0)
      return non;
  delete (set);
  set = NULL;
  return oui;
}

vset  V (vertex u, graphe & G)
{ vset S(G);
  vertex v;
  unsigned char mask = 1;
  unsigned int byte;
  S.set = new unsigned char [S.n];
  if (S.set == NULL) {error(); exit(1);}
  for (byte=0; byte < S.n; byte++)
    S.set[byte] = 0;
  byte = 0;
  for (v=0; v<S.o; v++)
  { if (G.arete(u,v))
      S.set[byte] |= mask;
    mask «= 1;
    if (mask == 0)
    { mask = 1;
      byte++;
    }
  }
  return S;
```

```
}

vset & vset::operator = (const vset & K)
{ if (this != &K)
  { vertex i;
    o = K.o;
    n = K.n;
    if (set != NULL) delete set;
    if (K.set == NULL) set = NULL;
    else
    { set = new unsigned char [n];
      if (set == NULL) {error(); exit(1);}
      for (i=0; i<n; i++)
        set [i] = K.set [i];
    }
  }
  return * this;
}

int operator == (vset & K, vset & L)
// on compare des suites du même graphe
{  vertex i; vertex n = K.n;
  if (L.o != K.o)
  { cout « "erreur dans vset::operator ==: ordres différents\n";
  cout « "L.o = " « L.o « " et K.o = " « K.o « "\n";
  exit(1);}
  for (i=0; i<n; i++)
    if (K.set[i] != L.set[i]) return non;
  return oui;
}

int operator < (vset & K, vset & L) // ordre naturel
{  unsigned int byte; vertex n = K.n;
  if (L.o != K.o)
  { cout « "erreur dans vset::operator <: ordres différents\n"; exit(1);}
  for (byte=n; byte>0; byte--)
  { if (K.set[byte-1] < L.set[byte-1]) return oui;
    if (K.set[byte-1] > L.set[byte-1]) return non;
  }
  return non;
}

void vset::ajoute (vertex u)
{ unsigned int i;
  if (u>=o)
  { cout « "erreur dans vset::ajoute: sommet non valide\n"; exit (1); }
  if (set == NULL)
  { set = new unsigned char [n];
    if (set == NULL) {error(); exit(1);}
    for (i=0; i<n; i++) set [i] = 0;
```

```
  }
  set[u»3] |= char(1u«(u&7));
}

void vset::enleve (vertex u)
{ unsigned int byte;
  unsigned char mask;
  if (u>=o)
  { cout « "erreur dans enleve (vset) : sommet non valide\n"; exit (1) ; }
  if (set == NULL)
    return;
  byte = u»3;
  mask = char (1u«(u&7));
  set[byte] |= mask;
  set[byte] ^= mask;
}

void vset :: inverse (vertex u)
{ unsigned int byte;
  if (u>=o)
  { cout « "erreur dans vset::inverse : sommet non valide\n"; exit(1) ; }
  if (set == NULL)
  { set = new unsigned char [n];
    if (set == NULL) { error(); exit(1); }
    for (byte=0; byte<n; byte++)
      set[byte] = 0;
  }
  set[u»3]^=char(1u«(u&7));
}

int vset :: element (vertex u)
{ if ((set == NULL)||(u>=o))
    return non;
  unsigned int byte = u»3;
  return ((set[byte] & char(1«(u&7)))!=0);
}

ostream & operator « (ostream & sortie, vset & K)
{ vertex u;
  for (u=0; u<K.o; u++)
    if (K.element(u))
      cout « u « " ";
  return sortie;
}

vset & vset::operator -= (vset & T)
{ unsigned int max;
  if ((set != NULL)&&(T.set != NULL))
  { max = T.n;
    if (n < max) max = n;
```

```
    for (unsigned int i = 0; i<max; i++)
    { set[i] |= T.set[i];
      set[i] ^= T.set[i];
    }
  }
  return * this;
}

int vset::premier (vertex & u)
{ unsigned int byte;
  unsigned char mask = 1;
  if (set == NULL) return non;
  for (byte = 0; byte < n; byte++)
    if (set[byte]!=0)
    { u = byte « 3;
      while ((set[byte]&mask)==0)
      { u++;
        mask «=1;
      }
      if (u < o)
        return oui;
    }
  return non;
}

int vset::suivant (vertex & u)
{ unsigned int byte;
  unsigned char mask;
  if (set == NULL) return non;
  u++;
  byte = u » 3;
  mask = 1 « (u & 7);
  while (u < o)
  { if ((set[byte]&mask)!=0)
      return oui;
    u++;
    mask «= 1;
    if (mask == 0)
    { mask = 1;
      byte++;
      while ((byte < n)&&(set[byte]==0))
      { u+=8;
        byte++;
      }
    }
  }
  return non;
}

int vset::adjacent (vertex u, graphe & G)
```

```
{ vertex v;
  if (!premier(v)) return non;
  do
  { if (G.arete(u,v))
      return oui;
  } while (suivant(v));
   return non;
}

vertex  vset::degre (vertex u, graphe & G)
{ vertex v,d;
  if (!premier(v)) return 0;
  d = 0;
  do
  { if (G.arete(u,v))
      d++;
  } while (suivant(v));
  return d;
}

vset & vset::operator += (const vset & S)
{ if (S.o != o) {cout « "erreur dans vset::+=\n"; exit(1);}
  for (vertex u=0; u<n; u++)
    set[u] ^= S.set[u];
  return * this;
}


vset relabel (vset & S, vertexlabel & L)
{ vertex u;
  vset T(L.n);
  if (S.o != L.sub_n)
    { cout « "erreur dans relabel(vset, vertexlabel) : "
           « " vset et vertexlabel incompatibles\n"; exit(1);}
  for (u=0; u < S.o; u++)
    if (S.element(u))
      T.ajoute (L.sur[u]);
  return T;
}

void vset_liste:: deleterecurs (node * L)
{ if (L == NULL) return;
  deleterecurs (L->left);
  deleterecurs (L->right);
  delete L;
}

void vset_liste:: vider ()
{ deleterecurs (racine);
  racine = NULL;
```

```
}

vset_liste::~vset_liste ()
{ if (racine == NULL) return;
   deleterecurs (racine->left);
  deleterecurs (racine->right);
  delete racine;
}

int vset_liste :: ajouterrecurs (vset & S, node * & L)
{ if (L == NULL)
  { L = new node;
    if (L == NULL) {error(); exit (1);}
    L->S = S;
    L->left = NULL;
    L->right = NULL;
    return oui;
  }
  if (S < L->S) return ajouterrecurs (S, L->left);
  if (S == L->S) return non;
  return ajouterrecurs(S, L->right);
}

int vset_liste :: ajouter (vset & S)
{ if (racine == NULL)
  { racine = new node;
    if (racine == NULL) {error(); exit (1);}
    racine->S = S;
    racine->left = NULL;
    racine->right = NULL;
    return oui;
  }
  if (S < racine->S) return ajouterrecurs(S, racine->left);
  if (S == racine->S) return non;
  return ajouterrecurs(S, racine->right);
}
```

BIBLIOGRAPHIE

[AHU] AHO, A. V.; HOPCROFT, J. E.; ULLMAN, J. D., *Data structures and algorithms*, Addison-Wesley Series in Computer Science and Information Processing, Addison-Wesley Publishing Co., Reading (1983).

[Ba] BABAI, L., Moderately exponential bound for graph isomorphism, *Fundamentals of computation theory (Szeged, 1981)*, Lecture Notes in Comput. Sci. **117**, Springer, Berlin-New York (1981), 34-50.

[BM] BONDY, J. A.; MURTY, U. S. R., *Graph theory with applications*, American Elsevier Publishing Co., New York (1976).

[Bo1] BOUCHET, A., Reducing prime graphs and recognizing circle graphs, *Combinatorica* **7** (1987), 243-254.

[Bo2] BOUCHET, A., Graphic presentations of isotropic systems, *J. Combin. Theory Ser. B* **45** (1988), 58-76.

[Bo3] BOUCHET, A., κ-transformations, local complementations and switching, *Cycles and rays (Montreal, 1987)*, NATO Adv. Sci. Inst. Ser. C **301**, Kluwer Acad. Publ., Dordrecht (1990), 41-50.

[Bo4] BOUCHET, A., Recognizing locally equivalent graphs, Combinatorics and algorithms (Jerusalem, 1988), *Discrete Math.* **114** (1993), 75-86.

[Bo5] BOUCHET, A., Circle graph obstructions, *J. Combin. Theory Ser. B* **60** (1994), 107-144.

[FZ] FAN, G.; ZHANG, C.-Q., Circuit decompositions of eulerian graphs, *J. Combin. Theory Ser. B.* **78** (2000), 1-23.

[Fl1] FLEISCHNER, H., Eulersche Linien und Kreisüberdeckungen, die vorgegebene Durchgänge in den Kanten vermeiden (allemand), *J. Combin. Theory Ser. B* **29** (1980), 145-167.

[Fl2] FLEISCHNER, H., Cycle decompositions, 2-coverings, removable cycles, and the four-color-disease, *Progress in graph theory (Waterloo, 1982)*, Academic Press, Toronto (1984), 233-246.

[Fl3] FLEISCHNER, H., Some blood, sweat, but no tears in eulerian graph theory, 250th Anniversary Conference on Graph Theory (Fort Wayne, 1986), *Congr. Numer.* **63** (1988), 8-48.

[Fo1] FON-DER-FLAASS, D. G., On local complementations of graphs, Combinatorics (Eger, 1987), *Colloq. Math. Soc. János Bolyai* **52**, North-Holland, Amsterdam (1988), 257-266.

[Fo2] FON-DER-FLAASS, D. G., Distance between locally equivalent graphs (russe), *Metody Diskret. Analiz.* **48** (1989), 85-94, 106-107.

[Fo3] FON-DER-FLAASS, D. G., Local complementations of simple and oriented graphs (russe), *Sibirsk. Zh. Issled. Oper.* **1** (1994), 43-62, 87.

FON-DER-FLAASS, D. G., Local complementations of simple and oriented graphs (traduction anglaise), *Discrete Analysis and Operations Research*, Kluwer Acad. Publ., Dordrecht (1996), 15-34.

[Fr] DE FRAYSSEIX, H., A characterization of circle graphs, *European J. Combin.* **5** (1984), 223-238.

[Jac1] JACKSON, B., A characterisation of graphs having three pairwise compatible Euler tours, *J. Combin. Theory Ser. B* **53** (1991), 80-92.

[Jac2] JACKSON, B., On circuit covers, circuit decompositions and Euler tours of graphs, *Surveys in combinatorics (Keele, 1993)*, London Math. Soc. Lecture Note Ser. **187**, Cambridge Univ. Press, Cambridge (1993), 191-210.

[Jae] JAEGER, F., A survey of the cycle double cover conjecture, *Cycles in graphs (Burnaby, 1982)*, North-Holland Math. Stud. **115**, North-Holland, Amsterdam (1985), 1-12.

[K1] KOTZIG, A., Moves without forbidden transitions in a graph, *Mat. Časopis Sloven. Akad. Vied* **18** (1968), 76-80.

[K2] KOTZIG, A., Eulerian lines in finite 4-valent graphs and their transformations, *Theory of Graphs (Tihany, 1966)*, Proc. Colloq. on Graph Theory Tihany 1966, Academic Press, New York (1968), 219-230.

[K3] KOTZIG, A., *Quelques remarques sur les transformations* κ, séminaire Paris (1977).

[P] PEISERT, W., All self-complementary symmetric graphs, *J. Algebra* **240** (2001), 209-229.

[Sa] SABIDUSSI, G., *Eulerian walks and local complementation*, D.M.S. 84-21, Dép. de math. et stat., Université de Montréal (1984).

[Sp] SPINRAD, J., Recognition of circle graphs, *J. Algorithms* **16** (1994), 264-282.

[V1] VEBLEN, O., An application of modular equations in analysis situs, *Ann. Math.* **14** (1912), 86-94.

[V2] VEBLEN, O., *Analysis situs*, Amer. Math. Soc. Colloq. Publ., Vol. V, Part II (1922, 1931).

[W] WILSON, R. J., An Eulerian trail through Königsberg, *J. Graph Theory* **10** (1986), 265-275.

[Z] ZHANG, H., Self-complementary symmetric graphs, *J. Graph Theory* **16** (1992), 1-5.