Université de Montréal

SYSTÈMES DE HITCHIN GÉNÉRALISÉS ET VARIÉTÉS SYMPLECTIQUES

par

Mounia Kjiri

Département de mathématiques et statistique
Faculté des arts et des sciences

Thèse présentée à la Faculté des études supérieures
en vue de l'obtention du grade de
Philosophiæ doctor (Ph.D.)
en mathématiques
option mathématiques fondamentales

Janvier 2000

Université de Montréal
Faculté des études supérieures

Cette thèse intitulée

SYSTÈMES DE HITCHIN GÉNÉRALISÉS ET VARIÉTÉS SYMPLECTIQUES

présentée par

Mounia Kjiri

a été évalué par un jury composé des personnes suivantes:

Yvan Saint-Aubin

(président)

Jacques C. Hurtubise

(directeur de recherche)

Eyal Markman

(examinateur externe)

Christiane Rousseau

(membre du jury)

David London

(représentant du doyen)

Thèse acceptée le:

À mes parents,
en témoignage de toute mon affection.

SOMMAIRE

Cette thèse porte sur l'étude de certains aspects des systèmes de Hitchin généralisés, définis par Bottacin et Markman.

Ces systèmes sont définis sur l'espace des paires stables qui consistent en un fibré vectoriel holomorphe et une 1-forme méromorphe à valeurs dans les endomorphismes du fibré, les pôles étant les points correspondant à un diviseur D fixé.

Nous obtenons dans un premier temps une description algébro-géométrique de ces systèmes en terme de courbes spectrales et de fibrés en droite, puis nous construisons des coordonnées de Darboux naturelles pour ces systèmes. Ces coordonnées permettent de déterminer une correspondance "birationnelle" entre ces systèmes et le produit symétrique d'une surface symplectique qui leur est naturellement associée.

Dans des cas particuliers (en genre zéro et en genre un), les fibrés sont rigides, et ces systèmes peuvent être exprimés en utilisant des matrices R classiques qui sont rationnelles, elliptiques ou trigonométriques. Les systèmes obtenus sont alors les modèles de Gaudin respectivement rationnels, elliptiques ou trigonométriques. Les formules sont explicitées pour ces trois cas.

Par la suite, nous considérons cette fois les systèmes de Hitchin généralisés pour un groupe réductif complexe G arbitraire. Nous démontrons alors que la structure locale de ce système intégrable est celle d'une fibration $Pr \to U$ par des variétés de Prym généralisées au-dessus d'un ouvert U, qui paramétrise une famille de courbes W-invariantes, W étant le groupe de Weyl de G.

Les systèmes de Hitchin généralisés satisfont à une condition de rang 2 généralisée, ce qui permet d'en déduire une variété X qui va les caractériser.

Soit $\mathfrak{h}$ la sous-algèbre de Cartan de $\mathfrak{g}$ et K le fibré canonique au-dessus de la surface de Riemann Σ. On démontre que X est l'éclatement $\widetilde{K_\Sigma[D]} \otimes \mathfrak{h}$ de l'espace total du fibré vectoriel $K_\Sigma[D] \otimes \mathfrak{h}$ sur Σ. Les points éclatés sont les points d'intersection de la courbe spectrale S et l'image inverse dans $K_\Sigma[D] \otimes \mathfrak{h}$ du diviseur D dans Σ.

Il y a une correspondance locale bijective entre ces systèmes intégrables et les variétés éclatées $\widetilde{K_\Sigma[D]} \otimes \mathfrak{h}$ munies de 2-formes appropriées à valeurs dans $\mathfrak{h}$.

TABLE DES MATIERES

REMERCIEMENTS

J'aimerais avant tout exprimer ma reconnaissance et ma gratitude à mon directeur de thèse, le Professeur Jacques Hurtubise, pour son soutien à la fois scientifique et financier, ainsi que pour sa disponibilité illimitée durant la préparation de ce travail.

Je souhaiterais également remercier les organismes qui m'ont accordés des bourses au cours de mon doctorat: l'Agence Canadienne de Développement International, l'Institut des Sciences Mathématiques du Québec, la Faculté des Études Supérieures, le Syndicat Général des Professeurs de l'Université de Montréal ainsi que le Ministère de l'Éducation Nationale du Royaume du Maroc.

Enfin, j'aimerais surtout remercier mon mari, Hassan Aurag, qui m'a encouragé durant toute ma thèse, m'offrant même à mi-chemin le plus beau des cadeaux, notre fils Adam.

INTRODUCTION

Les systèmes de Hitchin ont fait leur apparition en 1987 à travers les articles [Hi1] et [Hi2]. Les espaces de modules de fibrés vectoriels stables au-dessus d'une surface de Riemann avaient été étudiés jusque là sous différents aspects, mais Nigel Hitchin a apporté une toute nouvelle vision de ces espaces en faisant appel à la géométrie symplectique de leurs fibrés cotangents. Il démontra que l'on obtenait alors un système intégrable d'une façon très naturelle. Dès lors, ces systèmes sont devenus le théâtre d'activités scientifiques très intenses. De nombreux mathématiciens et physiciens se sont effectivement penchés sur ces systèmes, chacun y apportant ainsi son grain de sel, et permettant d'obtenir de plus en plus de propriétés.

Mais les choses ne pouvant s'arrêter là, cinq années plus tard, une généralisation de ces systèmes naquit. On la doit à E. Markman et à F. Bottacin.

Indépendamment ([M], [Bo]), et sous des aspects quelque peu différents, ils ont tous deux démontré que là encore, on obtenait un système intégrable. Il fallait donc à présent vérifier si les différents résultats obtenus pour les systèmes de Hitchin restaient valides pour les systèmes de Hitchin généralisés. A travers les deux articles qui constituent le corps de cette thèse, je tente d'éclaircir le point sur certaines de ces questions.

Avant de donner davantage de précisions sur le travail qui a été réalisé, je vais faire un bref rappel des notions de base qui seront utilisées au cours de ce travail.

Tout d'abord de quel espace de modules s'agit-il ? [Hi2]

Soit Σ une surface de Riemann, K_Σ son fibré cotangent holomorphe, G un groupe réductif complexe connexe et E un G-fibré au-dessus de Σ.

Soit $\mathcal{M}$ l'espace de modules des G-fibrés holomorphes stables au-dessus de Σ. (La stabilité est une condition technique nécessaire à l'existence d'une "bonne" structure d'espace.)

L'espace cotangent $T^*\mathcal{M}$ sera donc l'ensemble des paires (E, ϕ), où E est un G-fibré holomorphe stable et ϕ une section holomorphe de $ad(E) \otimes K_\Sigma$, c'est-à-dire une 1-forme à valeurs dans le fibré adjoint associé à E.

Hitchin a démontré que pour les groupes classiques (Gl(r), Sp(r), SO$(2r)$ et SO$(2r+1)$), l'espace cotangent $T^*\mathcal{M}$ de l'espace de modules des fibrés stables au-dessus d'une surface de Riemann possède un système complètement intégrable naturel. Faltings a étendu ceci aux groupes algébriques linéaires connexes sur un corps de caractéristique zéro.

Généralisant ce système, on peut considérer l'espace $\mathcal{N}_D$ des paires (E, ϕ), où E est un G-fibré holomorphe et ϕ une section holomorphe de $ad(E) \otimes K_\Sigma[D]$, c'est-à-dire une section *méromorphe* de $ad(E) \otimes K_\Sigma$ admettant des pôles en un diviseur D fixé. $\mathcal{N}_D$ est l'espace de phase des systèmes de Hitchin généralisés. Dans les deux cas, les hamiltoniens peuvent se décrire comme coefficients des équations définissant une courbe spectrale. Pour chaque paire (E, ϕ) de $\mathcal{N}_D$ (le cas $T^*\mathcal{M}$ correspondant au cas où $D = 0$), on considère la courbe spectrale S de ϕ qui est dans l'espace total $\mathcal{K}_D \otimes \mathfrak{h}$ du fibré vectoriel $K(D) \otimes \mathfrak{h}$ au-dessus de Σ, $\mathfrak{h}$ étant la sous-algèbre de Cartan de $\mathfrak{g}$. Elle est découpée par les équations $p_i(h) = a_i(z)$, où les p_i forment une base homogène de polynômes W-invariants sur $\mathfrak{h}$ et les a_i sont donc des sections de $(K_\Sigma(D))^{\otimes deg(p_i)}$ au-dessus de Σ. En exprimant les a_i dans une base de $H^0(\Sigma, (K(D) \otimes \mathfrak{h})^{\otimes deg p_i})$, l'espace des sections holomorphes globales de $(K(D) \otimes \mathfrak{h})^{\otimes deg p_i}$, on obtient des fonctions qui commutent sous le crochet de Poisson et qui définissent un système complètement intégrable sur $\mathcal{N}_D$.

Dans le cas de $\mathrm{Gl}(r)$, un point de $\mathcal{N}_D$ peut être représenté par un fibré vectoriel stable E de rang r et une section $\phi \in H^0(\Sigma, End(E) \otimes K(D))$ admettant des pôles aux points du diviseur D. Pour chaque paire (E, ϕ), on considère la courbe spectrale S définie par $det(\phi - \zeta I) = 0$ dans l'espace total $\mathcal{K}_D$ du fibré vectoriel $K(D)$, ainsi que le fibré en droite déterminé par la suite exacte:

$$0 \to \pi^* E \otimes K^*_\Sigma(-D) \xrightarrow{\phi - \zeta I} \pi^* E \to L \to 0.$$

où $\pi : \mathcal{K}_D \to \Sigma$ est la projection.

A partir de la paire (S, L), on peut reconstruire la paire (E, ϕ), et ainsi considérer $\mathcal{N}_D$ comme l'ensemble des paires (S, L). Les hamiltoniens seront alors les coefficients de l'équation de la courbe spectrale. S est une constante du mouvement et le flot de L est linéaire sur la jacobienne.

Considérons le cas où $\Sigma = \mathbb{P}^1$ et $G = Gl(r)$. Le fibré E, s'il est semi-stable et de degré 0, est alors isomorphe à $\mathcal{O}^{\oplus r}$.

Dans le cas des systèmes de Hitchin, $T^*\mathcal{M}$ est un point. Par contre, pour les systèmes de Hitchin généralisés ($D \neq 0$), le ϕ, sur E trivial, devient une fonction rationnelle à valeurs dans $gl(r)$.

En identifiant les 1-formes holomorphes sur $\mathbb{P}^1$ avec les fonctions ayant un zéro double à l'infini, on a:

$$\mathcal{N}_{D+2\infty} = \{f : \mathbb{P}^1 \to gl(n) \text{admettant des poles aux points du diviseur } D\}.$$

Si $D = \Sigma_i \alpha_i$ avec $\alpha_i \neq \alpha_j$ si $i \neq j$ alors:

(*) $$\mathcal{N}_{D+2\infty} = \{N(\lambda) = \Sigma_i \frac{\mu_i}{\lambda - \alpha_i}\}.$$

La variété $\mathcal{N}_{D+2\infty}$ est une variété de Poisson, dont les feuilles symplectiques sont obtenues en fixant l'orbite coadjointe $\mathcal{O}_{\mu_i}$ des μ_i. La formule (*) met en évidence un symplectomorphisme entre les feuilles symplectiques de $\mathcal{N}_{D+2\infty}$ et les produits des orbites coadjointes $\mathcal{O}_{\mu_1} \times \ldots \times \mathcal{O}_{\mu_n}$. Pour obtenir les hamiltoniens, il suffit alors de considérer le polynôme

$$p(\lambda, \zeta) = det(N(\lambda) - \zeta I) = 0$$

et de déterminer les coefficients $p_{ij}(\lambda) = \mathrm{res}(\mathrm{tr}\lambda^i N^j(\lambda))$ de cette équation.

Les équations du flot sont alors données par:

$$\dot{N} = [(\mathrm{res}(\mathrm{tr}\lambda^i N^{j-1}))_+, N]$$

où $+$ dénote la partie polaire du développement de l'expression.

Considérons comme exemple, l'oscillateur de Neumann: on a $\Sigma = \mathbb{P}^1$, $G = Gl(2)$ et

$$N(\lambda) = \lambda^{-1} \begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix}$$

$$+ \frac{\lambda^{-1}}{2} \begin{pmatrix} -\Sigma_{i=1}^n x_i y_i/(\lambda - \alpha_i) & -\Sigma_{i=1}^n y_i^2/(\lambda - \alpha_i) \\ \Sigma_{i=1}^n x_i^2/(\lambda - \alpha_i) & \Sigma_{i=1}^n x_i y_i/(\lambda - \alpha_i) \end{pmatrix}.$$

Le flot correspondant à l'hamiltonien de N est

$$\dot{N} = [B, N] \text{ avec } B = \begin{pmatrix} x^T y & \lambda + y^T y \\ -x^T x & -x^T y \end{pmatrix}.$$

On peut linéariser le flot en passant à des coordonnées elliptiques λ_μ avec des impulsions correspondantes ζ_μ. Elles sont données par les équations suivantes:

$$\Sigma_{i=1}^n \frac{x_i^2}{\lambda - \alpha_i} = \frac{\Pi_{\mu=1}^n(\lambda - \lambda_\mu)}{a(\lambda)}, \zeta_\mu = \frac{1}{2}\Sigma_{i=1}^n \frac{x_i y_i}{\lambda_\mu - \alpha_i},$$

et on démontre que (λ_μ, ζ_μ) définissent un système de coordonnées de Darboux. Ces deux équations sont équivalentes à

$$[(\zeta I - N(\lambda))]_{adj}(1, 0)^T = 0,$$

où adj représente la matrice des cofacteurs.

Dans le cas plus général de $G = Gl(r)$ sur $\mathbb{P}^1(\mathbb{C})$, on a la suite exacte:

$$0 \to \mathcal{O}(-n)^{\oplus r} \xrightarrow{\zeta I - N(\lambda)} \mathcal{O}^{\oplus r} \xrightarrow{\pi} L \to 0$$

et on démontre encore [AHH] que les coordonnées de Darboux (λ_μ, ζ_μ), sont données explicitement par $[(\zeta I - N(\lambda))]_{adj}(1, 0, ..., 0)^T = 0$.

Par ailleurs, d'un point de vue algébro-géométrique, le fait que (λ_μ, ζ_μ) soient des coordonnées de Darboux se traduit de la façon suivante: les variations de la courbe S dans l'espace total $\mathcal{T}$ correspondent à des sections du fibré normal N à la courbe S dans $\mathcal{T}$, c'est-à-dire des éléments de $H^0(S, N)$. L'espace des variations permises dans une feuille symplectique peut ensuite être identifié à $H^0(S, K_\Sigma)$ où K_Σ est le fibré canonique. Par ailleurs, vu que les variations du fibré en droite L correspondent au groupe de cohomologie $H^1(S, \mathcal{O})$, on peut décomposer l'espace tangent de l'orbite comme la somme:

$$H^0(S, K) \oplus H^1(S, \mathcal{O}).$$

Par une relation de Serre, les deux espaces sont duaux l'un de l'autre. On obtient une forme symplectique naturelle ω_S sur l'espace tangent. On démontre qu'il s'agit en fait de la forme de Kostant-Kirillov. Ceci peut être vu comme une "abélianisation". On réduit la forme de Kostant-Kirillov à une expression concernant des fibrés de rang 1, en passant à un revêtement $S \to \Sigma$. On peut ensuite calculer la forme ω_S, et on trouve que (λ_μ, ζ_μ) sont des coordonnées de Darboux

Dans le premier article "Separating coordinates for the generalized Hitchin systems and the classical R-matrices", qui constitue le Chapitre 1 de cette thèse, ces résultats pour $\mathbb{P}_1$ sont généralisés aux cas d'une surface de Riemann quelconque. Dans cet article, une attention particulière est apportée aux deux cas suivants (le cas où la surface de Riemann est la sphère de Riemann ayant été traité par M.R. Adams, J. Harnad et J.C. Hurtubise):

- la surface de Riemann est une courbe elliptique,
- la surface de Riemann est une courbe nodale rationnelle.

Ce sont des cas où le fibré E est rigide. On obtient alors respectivement les systèmes de la matrice R elliptique et trigonométrique. Les coordonnées explicites sont obtenues à partir du cas général des systèmes de Hitchin généralisés.

Ce travail a été réalisé pour le cas où le groupe considéré initialement est $\mathrm{Gl}(r)$. La question qui vient inévitablement à l'esprit est: Que se passe-t-il lorsque l'on considère un groupe réductif complexe quelconque? L'objectif de l'article "The generalized Hitchin systems and Prym varieties", qui constitue le Chapitre 2 de cette thèse, est de répondre à cette question. À travers cet article, il est démontré, dans un premier temps, que ce système peut être perçu comme l'ensemble des paires constituées d'une courbe spectrale S qui vit dans l'espace total de $K[D] \otimes \mathfrak{h}$ et d'un H-fibré au-dessus de S (H étant le sous-groupe de Cartan de G et $\mathfrak{h}$ l'algèbre de Lie qui lui est associée).

Nous démontrons alors que la structure locale de ce système intégrable est celle d'une fibration $Pr \to U$ par des variétés de Prym généralisées au-dessus d'un ouvert U, qui paramétrise une famille de courbes W-invariantes, W étant le groupe de Weyl de G. Cette structure symplectique permet d'obtenir une nouvelle fois une abélianisation (réduction de G à H).

Ces systèmes satisfont à une condition de rang 2 généralisée qui permet de mettre en évidence une variété X qui va les caractériser. Cette variété X est l'éclatement $\widetilde{K_\Sigma[D]} \otimes \mathfrak{h}$ de l'espace total du fibré vectoriel $K_\Sigma[D] \otimes \mathfrak{h}$ sur Σ, les points éclatés étant les points d'intersection de la courbe spectrale S et l'image inverse dans $K_\Sigma[D] \otimes \mathfrak{h}$ du diviseur D dans Σ.

Il y a une correspondance locale bijective entre ces systèmes intégrables et les variétés éclatées $\widetilde{K_\Sigma[D]} \otimes \mathfrak{h}$ munies de 2-formes appropriées à valeurs dans $\mathfrak{h}$.

CHAPITRE I

Article 1: "Separating coordinates for the generalized Hitchin systems and the classical R-matrices"

Cet article, écrit en collaboration avec le professeur J.C. Hurtubise, sera publié dans la revue "Communications in Mathematical Physics" (2000).

Separating coordinates for the generalized Hitchin systems and the classical r-matrices

J.C. HURTUBISE AND M. KJIRI

ABSTRACT: We exhibit natural Darboux coordinates for the generalized Hitchin systems studied by Bottacin and Markman. These systems are defined on spaces of stable pairs consisting of a vector bundle and a form-valued meromorphic endomorphism of the bundle. In special cases (genus zero, genus one), the bundles are rigid and one has the rational, trigonometric and elliptic Gaudin systems. Explicit formulae are given in these cases.

1. Introduction

Integrable Hamiltonian systems occur in a wide variety of contexts in mathematical physics, ranging from the very classical problems of 19th century mechanics to the systems occuring in Seiberg-Witten theory. One general class of system which appears in all these guises is the system, due to Markman [Ma] and Bottacin [Bo], which is also known as the generalized Hitchin system. It is defined on a moduli space of pairs (holomorphic vector bundles over a Riemann surface, meromorphic section of the adjoint bundle). Specializing to various cases, mostly over the Riemann sphere, gives the classical examples (tops, geodesics on the ellipsoid, etc.) as well as many interesting and important integrable systems of current interest (Gaudin model, Landau-Lifschitz, and others). More precisely (see the book [FT], the survey [RS2], and the references therein, or the articles [M, AvM, RS1, AHP, HH]):

- Over rational curves, and in some cases, over elliptic curves, and their degenerations into nodal curves, one has that the bundle is rigid, and one is dealing with endomorphisms of a fixed bundle. The systems can then be expressed in terms of classical r-matrices, either rational, elliptic or trigonometric, and the systems one obtains are often referred to as the rational, elliptic or trigonometric Gaudin model.
- Specializing further, one can fix the curve to be rational, fix the rank, and

The authors of this article would like to thank NSERC and FCAR for their support

of the classical systems: the Neumann oscillator, the various tops, as well as finite gap solutions to the KdV, the NLS, the CNLS and the Boussinesq equations.

- In the elliptic case, one can also further specialize, for example, to the Landau-Lifschitz equation, or the Steklov top.

One natural question in integrable systems is of course solving the equations and finding the flows, and this usually involves some form of separation of variables. This note is devoted to the question of separation of variables for the generalized Hitchin systems, and we will find that there are separating Darboux coordinates which are very natural from a geometric viewpoint, corresponding to the standard algebro-geometric description of these systems in terms of curves and line bundles. This can then of course be specialized to all the cases alluded to above, and in this specialization, one obtains quite detailed formulae.

The coordinates also define a "birational" map between the systems and a symmetric product of a symplectic surface naturally associated to each system. (More properly, rather than a symmetric product, one should be saying a Hilbert scheme of 0-cycles). Other systems with such coordinates ("rank two systems") were studied in [Hu1].

In the special cases of interest to mathematical physics corresponding to when the bundle over the Riemann surface is rigid under deformations, there are, as we mentioned above, three cases. When the Riemann surface is the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, one has the rational r-matrix systems, and the separation of variables was given in [AHH1], as a consequence of a direct calculation. Here we finish the problem and treat the case when the curve is elliptic (elliptic r-matrix) or a nodal rational curve (trigonometric r-matrix), and the explicit coordinates will follow from the general considerations on the Bottacin-Markman systems; such a procedure can also be used to give another derivation of the results of [AHH1]. Similar coordinates were produced in the rank two elliptic case by

Sklyanin [S].

Section two of the paper will begin by recalling some facts about the generalized systems, following [Ma]. In section three, we will show how the coordinates arise, show that they are Darboux coordinates, and explain how they lead to an easy integration of the equations of motion. Section four specializes to the special case of an elliptic curve, and section five to the nodal curve. Finally, there is another context in which the same r-matrices are used, and that is in defining integrable systems over Poisson-Lie groups. In section 6 we will explain how the results of the paper should extend to cover this case.

2. The Bottacin-Markman or generalized Hitchin systems.

Let Σ be a closed Riemann surface of genus γ, D a positive divisor of degree n on Σ. We consider over Σ the moduli spaces $\mathcal{M}(r, D, d)$ of *Higgs pairs* (E, ϕ), where

- E is a degree d rank r holomorphic vector bundle over Σ.
- ϕ, the *Higgs field*, is a holomorphic section of the associated adjoint bundle $\mathrm{End}(E)$, twisted by $K_\Sigma(D)$, where K_Σ is the canonical bundle of Σ: $\phi \in H^0(\Sigma, \mathrm{End}(E) \otimes K_\Sigma(D))$. Alternately, ϕ is a meromorphic $\mathrm{End}(E)$-valued 1-form, with poles at the divisor D. The pairs must satisfy an appropriate stability condition; see [Bo],[Ma]. The case considered by Hitchin in [Hi1],[Hi2], is that of $D = 0$.

The first result is that the $\mathcal{M}(r, D, d)$ are Poisson. The Poisson structure can be defined directly ([Bo] or [Ma], section 7), but it is easiest to obtain it by Poisson reduction of a larger space, the cotangent bundle of the moduli space of bundles with level structure at D.

Following [Ma], we consider the moduli space $\mathcal{U}(r, D, d)$ of vector bundles with level structure at D, that is the moduli space of pairs (E, tr) where E is rank r vector bundle over Σ, and tr is a trivialization of E over the divisor D, that is an isomorphism between $E|_D$ and $\mathcal{O}_D^{\oplus r}$. Again, there is an appropriate stability condition one must impose to get a good moduli space. The tangent space to $\mathcal{U}(r, D, d)$ at (E, tr) is canonically isomorphic to $H^1(\Sigma, \mathrm{End}(E)(-D))$; dually the cotangent space is $H^0(\Sigma, \mathrm{End}(E) \otimes K_\Sigma(D))$. The cotangent bundle $T^*\mathcal{U}(r, D, d)$ is then identified with a space of triples (E, tr, ϕ), with E, tr as above and $\phi \in H^0(\Sigma, \mathrm{End}(E) \otimes K_\Sigma(D))$.

There is a natural action of $Gl(r, D)$, the invertible $\mathcal{O}_D$-valued $r \times r$ matrices, on $\mathcal{U}(r, D, d)$, simply by modification of the trivialization tr. The action lifts to a symplectic action on the cotangent bundle, and one has:

PROPOSITION (2.1) [Ma] *1) The action of $Gl(r, D)$ has as moment map*

$$\begin{aligned} \mu : T^*\mathcal{U}(r, D, d) &\to gl(r, D)^* \\ (E, tr, \phi) &\mapsto \hat{\phi} \end{aligned} \tag{2.2}$$

where $\hat{\phi}$ is the expression of the polar part of ϕ over D in the tr-trivialization, and $gl(r, D)^*$ is identified with $gl(r, D) \otimes_{\mathcal{O}_D} (K_\Sigma)|_D$ by a trace-residue pairing.

2) The quotient $T^*\mathcal{U}(r, D, d)/Gl(r, D)$ is then Poisson, and is naturally identified over an open dense set with $\mathcal{M}(r, D, d)$. Its symplectic leaves are obtained as inverse images under μ of coadjoint orbits.

The next step is to define the integrable system on $\mathcal{M}(r, D, d)$, that is to specify the ring of Hamiltonians. This is given by considering for each pair (E, ϕ), the *spectral curve* S of ϕ. This curve lies in the total space $\mathcal{K}_D$ of the line bundle $K(D)$ over Σ. It is cut out by the equation

$$\det(\phi - \zeta \mathbb{I}) = 0. \tag{2.3}$$

Here ζ represents the tautological section of $\pi^* K(D)$ over $\mathcal{K}_D$, where $\pi : \mathcal{K}_D \to \Sigma$ is the projection. The adjunction formula tells us that the genus of S is

$$g = r^2(\gamma - 1) + \frac{(r-1)rn}{2} + 1. \tag{2.4}$$

We expand (2.3) in powers of ζ:

$$\zeta^r + a_1\zeta^{r-1} + a_2\zeta^{r-2} + \ldots + a_r = 0. \tag{2.5}$$

We have that the $a_i = a_i(E,\phi)$ lie in $H^0(\Sigma, (K(D))^{\otimes i})$. These spaces have dimension $d_i = (2i+1)(\gamma-1)+in$. Let $v_{1,i}, \ldots v_{d_i,i}$ be a basis for $H^0(\Sigma, (K(D))^{\otimes i})$. Expanding $a_i(E,\phi)$ as

$$a_i(E,\phi) = \sum_{j=1}^{d_i} f_{j,i}(E,\phi)v_{j,i} \tag{2.6}$$

gives one functions $f_{j,i}$ on $\mathcal{M}(r,D,d)$.

PROPOSITION (2.7) [Bo, Ma]*1) The functions $f_{j,i}$ Poisson commute, and define a completely integrable system on $\mathcal{M}(r,D,d)$. Joint level sets of the $f_{j,i}$ are given by fixing the spectral curve S, so that the spectral curve map $\mathcal{M}(r,D,d) \to$ (family of spectral curves) defines a Lagrangian foliation.*

2) The symplectic leaves of the Poisson structure on $\mathcal{M}(r,D,d)$ correspond to fixing the intersection of the spectral curve with the divisor $\pi^{-1}(D)$.

3) On the generic symplectic leaf, the leaf of the Lagrangian foliation at a smooth spectral curve S is a Zariski open set of the Jacobian of S.

The leaf of the Lagrangian foliation at S is thus a family of line bundles on S. The line bundle L corresponding to (E,ϕ) is defined via the exact sequence of sheaves over the surface $\mathcal{K}_D$:

$$0 \to \pi^*E \otimes K^*_\Sigma(-D) \xrightarrow{\phi-\zeta\mathbb{I}} \pi^*E \to L \to 0. \tag{2.8}$$

When the spectral curve is smooth, L is a line bundle supported on the spectral curve.

PROPOSITION (2.9) [Hu1] *One can then reconstruct (E,ϕ) from (S,L):*

- $E = \pi_*(L)$,

- *ϕ is the map induced on E by multiplication by the tautological section ζ on L.*

Reduction to $SL(r, \mathbb{C})$

In more generality, one can consider similar structures for arbitrary reductive groups G. The bundle E then gets replaced by a principal G-bundle P, and the bundle $\mathrm{End}(E)$ gets replaced by $ad(P)$. We will not consider these structures in such generality; see however [Hu2], and the references therein. We consider the case $G = SL(r, \mathbb{C})$. One then has a moduli space $\mathcal{M}(SL(r, \mathbb{C}), D, d)$ of pairs (E, ϕ), with E a rank r vector bundle with $\Lambda^r(E)$ holomorphically trivial, and ϕ a meromorphic $sl(E)$-valued 1-form, with poles at the divisor D. We now exhibit how these spaces can be obtained from a symplectic reduction, at least up to an r-fold covering.

The group $Pic_0(\Sigma)$ of degree zero line bundles on the base curve Σ acts on $\mathcal{M}(r, D, d)$ by

$$\begin{aligned} Pic_0(\Sigma) \times \mathcal{M}(r, D, d) &\to \mathcal{M}(r, D, d) \\ (V, (E, \phi)) &\mapsto (E \otimes V, \phi) \end{aligned} . \tag{2.10}$$

Alternately,

$$(V, (S, L)) \mapsto (S, L \otimes \pi^* V). \tag{2.11}$$

This action is symplectic, and is indeed Hamiltonian, being the flow of the Hamiltonians

$$\mathrm{tr}(\phi) \in H^0(\Sigma, K_\Sigma(D)). \tag{2.12}$$

If we take the reduction at $0 \in H^0(\Sigma, K_\Sigma(D))$ with respect to the action of this group, one fixes the trace of ϕ to be zero, then quotients out the action on E of tensoring with a line bundle. Up to an r-th root of the trivial bundle, one can achieve this by fixing the maximal exterior power of E to be a fixed line bundle V, giving:

Proposition (2.13) *The space of pairs*

$$\mathcal{M}_V(r, D, d) = \{(E, \phi) \in \mathcal{M}(r, D, d) | \Lambda^r(E) = V, \mathrm{tr}(\phi) = 0\} \tag{2.14}$$

embeds in $\mathcal{M}(r,D,d)$, symplectically over its smooth locus. It is a covering of the quotient $\mathrm{tr}^{-1}(0)/Pic_0(\Sigma)$.

If we consider the case $d=0$, V trivial, then E is an $SL(r,\mathbb{C})$-bundle, and then $\mathcal{M}_{\mathcal{O}}(r,D,0)=\mathcal{M}(SL(r,\mathbb{C}),D,0)$.

3. Symplectic geometry of $\mathcal{M}(r,D,d)$

We are thus in a situation in which we have two Lagrangian fibrations: the first, on the cotangent bundle $T^*\mathcal{U}(r,D,d)$, is given by projection to $\mathcal{U}(r,D,d)$, and the second, on the reduced space $\mathcal{M}(r,D,d)$, by the integrable system, that is, a map to the space of spectral curves.

Corresponding to the first fibration, we have that the tangents to the fibers are given by elements of $H^0(\Sigma,\mathrm{End}(E)\otimes K(D))$; on the base, deformations of the bundles, along with the level structure, are given to first order by elements of $H^1(\Sigma,\mathrm{End}(E)(-D))$. One then has an exact sequence:

$$0\to H^0(\Sigma,\mathrm{End}(E)\otimes K(D))\to T(T^*\mathcal{U}(r,D,d))\to H^1(\Sigma,\mathrm{End}(E)(-D))\to 0. \tag{3.1}$$

We would like to split this sequence at (E,tr,ϕ), allowing us to write:

$$T(T^*\mathcal{U}(r,D,d))\simeq H^1(\Sigma,\mathrm{End}(E)(-D))\oplus H^0(\Sigma,\mathrm{End}(E)\otimes K(D)). \tag{3.2}$$

Cover Σ by $n+1$ open sets, $U_0=\Sigma-\mathrm{support}(D)$ and $U_i, i=1,..,n$ disjoint discs centered at the points p_i of D. Choose trivializations of E on U_0, and also trivializations on the U_i compatible with tr at p_i, and let $F_{0,i}$ be the transition functions of E from U_i to U_0 for these trivializations. Now let V be a subspace of the space of cocycles for $\mathrm{End}(E)(-D)$, mapping isomorphically to $H^1(\Sigma,\mathrm{End}(E)(-D))$. The (E',t') near (E,tr) can be obtained from transition functions $F_{0,i}\cdot exp(v_{0,i})$, with $(v_{0,i})=v\in V$. This defines a parametriszation

$$V\to\mathcal{U}(r,D,d),$$

and so a symplectic map

$$V\times V^*\to T^*\mathcal{U}(r,D,d). \tag{3.3}$$

This gives a splitting, but it is not the one that we want. We represent a one parameter family of elements $(E(t), tr(t), \phi(t))$ of $T^*\mathcal{U}(r, D, d)$ by $(F_{0,i}(t), \phi_0(t), \phi_i(t))$, with $\phi_0(t)) = F_{0,i}(t)\phi_i(t)F_{0,i}(t)^{-1}$ on the overlaps $U_0 \cap U_i$, and choose $F_{0,i}(0) = F_{0,i}$. At $t = 0$, the corresponding tangent vectors are given by $v_{0,i} = F_{0,i}^{-1}\dot{F}_{0,i}, \dot{\phi}_0, \dot{\phi}_i$, with

$$\dot{\phi}_0 = F_{0,i} \cdot ([v_{0,i}, \phi_i] + \dot{\phi}_i) \cdot F_{0,i}^{-1}.$$

One can split (3.1) as follows: one can write the Serre duality pairing explicitly as

$$< v, \phi >= \sum_i tr(res_i(v_{0,i} \cdot \phi_i)).$$

For any sections ψ_i over U_i^* define $\psi^{\&} \in H^0(\Sigma, End(E) \otimes K(D))$ by asking that for all $w \in V$

$$< w, \psi^{\&} >= \sum_i tr(res_i(w_{0,i} \cdot \dot{\psi}_i). \qquad (3.4)$$

Applying this to our vectors $\dot{\phi}_i$ defines the splitting of (3.1). Composing with the differential at the origin of the map (3.3), we have a map from $V \times V^*$ to $H^1(\Sigma, \mathrm{End}(E)(-D)) \oplus H^0(\Sigma, \mathrm{End}(E) \otimes K(D))$ which is given by

$$(v, \dot{\phi}) \to (v, \dot{\phi} + \frac{1}{2}[v, \phi]^{\&}).$$

Using this, we find that the symplectic form with respect to our splitting of (3.1) is given by:

$$\Omega((v, \dot{\phi}), (v^*, \dot{\phi}^*)) =< v, \dot{\phi}^* > - < v^*, \dot{\phi} > + < [v, v^*], \phi > \qquad (3.5)$$

From the point of view of the second Lagrangian fibration, the first order deformations of the spectral curve at a fixed spectral curve S are given by sections of the normal bundle N_S, that is, via the adjunction formula, the bundle $K_S \otimes K^*_{\mathcal{K}_D}$. We note that the canonical bundle of $\mathcal{K}_D$ is $\pi^*\mathcal{O}(-D)$, so that $N_S = K_S(D)$. If one is interested in the deformations of the spectral curve which have fixed intersection with $\pi^*(D)$ (so that in $\mathcal{M}(r, D, d)$ one is moving along a symplectic leaf $\mathcal{L}$), we then have that our infinitesimal deformation space for the curves is given by sections of K_S.

In turn, noting that deformations of a line bundle on S are given by the cohomology group $H^1(S, \mathcal{O})$, we have that the tangent spaces at (S, L) to $\mathcal{M}(r, D, d)$ and to the leaf $\mathcal{L}$ in $\mathcal{M}(r, D, d)$ fit into exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & H^1(S,\mathcal{O}) & \to & T(\mathcal{M}(r,D,d)) & \to & H^0(S,K_S(D)) & \to & 0 \\ 0 & \to & H^1(S,\mathcal{O}) & \to & T(\mathcal{L}) & \to & H^0(S,K_S) & \to & 0. \end{array} \tag{3.6}$$

Again, we want to split this last sequence at (S, L): the geometric way of doing this is to extend the line bundle to a neighbourhood of S in $\mathcal{K}_D$, giving us a way of moving the curve while keeping the line bundle fixed. One then has

$$T(\mathcal{L}) \simeq H^1(S, \mathcal{O}) \oplus H^0(S, K_S). \tag{3.7}$$

On this sum there is again a natural skew form Ω_S, as the summands are again Serre duals. The extension of the line bundle and the splitting (3.7) it produces are not unique, but the splittings all define the same symplectic form, as a consequence of (3.8) below.

b) Abelianization: $\Omega_S = \Omega_{\Sigma,red}$

Our first result "abelianises" the symplectic form Ω_Σ by lifting to the curve S:

PROPOSITION (3.8) *On the leaves $\mathcal{L}$ in $\mathcal{M}(r, D, d)$, over the locus of smooth curves, $\Omega_S = \Omega_{\Sigma,red}$, the reduction of the form on $T^*\mathcal{U}(r, D, d)$.*

PROOF: It suffices to prove the identity on a dense set, and so we will make the assumption that the spectral curve over the divisor D is unramified. The symplectic reduction by $Gl(r, D)$ from $T^*\mathcal{U}(r, D, d)$ to $\mathcal{L}$ can then be thought of as a two step process: one first restricts to the subset $\mathcal{T}$ in $T^*\mathcal{U}(r, D, d)$ of elements (E, tr, ϕ) such that ϕ is diagonal over D in the tr-trivialization, then takes the symplectic quotient under the residual action of the torus $T(r, D)$. Let us then take a two parameter family $A(x,y) = (E, tr, \phi)(x,y)$ of elements of $\mathcal{T}$ lying in the inverse image of $\mathcal{L}$, and compute the form $\Omega_\Sigma(A_x, A_y)$ on this family at $(x,y) = (0,0)$. Corresponding to $A(x,y)$, there is a family of curves $\pi : S(x,y) \to \Sigma$, and line bundles $L(x,y)$ over $S(x,y)$; the trivialization of E at D in an eigenbasis of ϕ gives a trivialization of L at $\pi^{-1}(D)$.

We again cover the base curve Σ by open sets $U_0 = \Sigma - \text{support}(D)$, and $U_i, i = 1, ..., n$ non-intersecting discs around the points p_i in D, so that the curves $S(x,y)$ are unramified over U_i. Let λ_i be coordinates on the U_i centred at p_i, so that $d\lambda_i$ is a trivialization of K; let ρ be a trivialization of K over U_0. Over U_i, we let the r branches of the curves $S(x,y)$ have coordinates in $\mathcal{K}$ given by forms $\zeta_{i,j}(x,y,\lambda_i)d\lambda_i, j = 1, ..., r$; these have poles at $\lambda_i = 0$. Similarliy, in the ρ-trivializations, let the r branches of the curve be given by $\hat{\zeta}_{i,j}(x,y,\lambda_i)\rho$ Choose trivializations of the $L(x,y)$ over the open sets $\pi^{-1}(U_i), i = 0, .., n$, in such a way that they are compatible with the trivializations over $\pi^{-1}(D)$, and let the transition functions for $L(x,y)$ from $\pi^{-1}(U_0)$ to $\pi^{-1}(U_i)$ be given by an r-tuple of functions $f_j(\lambda_i) = f_j(\lambda_i, x, y)$, one for each branch of the curve.

Over U_i, given the trivializations of L, we have a natural basis for $E = \pi_*(L)$, whose j-th element is given by a section which is only non-zero on the j-th branch of S over Σ, and coincides with the trivialization on that branch. In this basis,

$$\phi(x,y,\lambda_i) = \text{diag}(\zeta_{i,j}(x,y,\lambda_i))d\lambda_i. \tag{3.9}$$

On the open set U_0, using the section ρ of $K \simeq K_\Sigma$, we have an identification $L(x,y) \simeq \pi^*(K_\Sigma(D))$. The tautological section ζ of $K_\Sigma(D)$ over $\mathcal{K}_D$ then gets identified with a global section of L over the spectral curve, which identifies sections of $E = \pi_*(L)$ as polynomials in ζ of degree $r-1$ with coefficients in $\mathcal{O}_\Sigma$, essentially by Lagrange interpolation. In the basis $1, \zeta, \zeta^2, ...$, the matrix of ϕ is in rational canonical form. The transition function for E from this rational canonical basis to the diagonal basis over U_i is then given in terms of the Vandermonde matrix $\widehat{VD}_{j,k} = (\hat{\zeta}_{i,j}(x,y,\lambda_i))^{k-1}, j,k = 1,..,r$ by

$$F_{0,i} = \widehat{VD}^{-1} \cdot \mathrm{diag}_j(f_{i,j}) \frac{\rho}{d\lambda_i}, \tag{3.10}$$

where $f_{i,j} = f_j(\lambda_i)$.

Now let us take derivatives along our family parametrised by tr. The cocycle representing the variation in the bundle E in the x direction at $(x,y) = (0,0)$ is given in the U_i trivialization (setting $F = F_{0,i}$) by

$$F^{-1}F_x = \mathrm{diag}_j((\ln(f_{i,j}))_x) + \mathrm{diag}_j(f_{i,j}^{-1}) \cdot \mathrm{diag}_j((\hat{\zeta}_{i,j}(0,0,\lambda_i))_x) \cdot \widehat{VD}' \cdot \widehat{VD}^{-1} \cdot \mathrm{diag}_j(f_{i,j}), \tag{3.11}$$

where $\widehat{VD}'_{j,k} = (j-1)(\zeta_{i,j}(x,y,\lambda_i))^{k-2}$. There is a similar expression for $F^{-1}F_y$. The derivatives of ϕ in the U_i trivializations are given by

$$\phi_x = \mathrm{diag}_j((\zeta_{i,j}(0,0,\lambda_i))_x) d\lambda_i. \tag{3.12}$$

With this in place, the evaluation of $\Omega_\Sigma(A_x, A_y)$ is given by

$$\sum_i \mathrm{res}_{p_i}(\mathrm{tr}((F^{-1}F_x) \cdot (\phi_y) - (F^{-1}F_y) \cdot (\phi_x)) + ([F^{-1}F_x, F^{-1}F_y] \cdot \phi))). \tag{3.13}$$

Now we can substitute the values of (3.11,3.12), and get

$$\begin{array}{l} \sum_{i,j} \mathrm{res}[(\ln(f_{i,j}))_x (\zeta_{i,j}(0,0,\lambda_i))_y - (\ln(f_{i,j}))_y (\zeta_{i,j}(0,0,\lambda_i))_x] + \\ \sum_i \mathrm{restr}([\mathrm{diag}_j((\hat{\zeta}_{i,j}(0,0,\lambda_i))_x) \cdot \widehat{VD}' \cdot \widehat{VD}^{-1}, \mathrm{diag}_j((\hat{\zeta}_{i,j}(0,0,\lambda_i))_y) \cdot \widehat{VD}' \cdot \widehat{VD}^{-1}], \phi) \end{array} . \tag{3.14}$$

The second term, however, vanishes, as one can replace $\hat{\zeta}_{i,j}$ by $\lambda_i^{\mathrm{ord}_{p_i}(D)} \zeta_{i,j}$ and replace $\widehat{VD}, \widehat{VD}'$ by the corresponding VD, VD' defined using $\lambda_i^{\mathrm{ord}_{p_i}(D)} \zeta_{i,j}$ instead of $\hat{\zeta}_{i,j}$. The derivatives $(\lambda_i^{\mathrm{ord}_{p_i}(D)}(D)\zeta_{i,j})_x, (\lambda_i^{\mathrm{ord}_{p_i}(D)} \zeta_{i,j})_y$ vanish at the origin to order $\mathrm{ord}_{p_i}(D)$, since we are taking the symplectic reduction. This gives a trivial residue. The evaluation of $\Omega_\Sigma(A_x, A_y)$ then reduces to

$$\sum_{i,j} \mathrm{res}[(\ln(f_{i,j}))_x (\zeta_{i,j}(0,0,\lambda_i))_y - (\ln(f_{i,j}))_y (\zeta_{i,j}(0,0,\lambda_i))_x]. \tag{3.15}$$

From the explicit version of the Serre duality pairing

$$H^1(S,\mathcal{O})\otimes H^0(S,K_S)\to H^1(S,K_S)\to\mathbb{C},$$

this is, however, exactly $\Omega_S(A_x,A_y)$.

REMARK (3.16): The proof given above shows us that a trivialization of L over $\pi^{-1}(U)$ gives us a "Lagrange interpolation basis" $1,\zeta,\zeta^2$... for $E=\pi_*(L)$, and so a basis for the line bundle $det(E)$. If, in particular, L is the line bundle associated to some divisor C supported away from the branch locus of $\pi:\mathbb{S}\to\Sigma$, this gives us in a straightforward fashion:

$$det(E)=[\pi(C)]\otimes(K_\Sigma^*(-D))^{\otimes\frac{r(r-1)}{2}}. \tag{3.17}$$

c) Divisor coordinates for Ω_S.

The pairs (curve S of fixed genus g, line bundle L on the curve of fixed degree d) parametrize the symplectic leaves of the moduli space. Let us fix a spectral curve S_0 and a line bundle L_0, and let (S,L) denote a nearby point. Choose a line bundle K_0 of degree $g-d$ on a neighbourhood of S_0 such that the line bundles $\hat{L}=K_0\otimes L$ (which are then of degree g) on the nearby curves have a one-dimensional space of sections. Corresponding to such generic $\hat{L}$, there is then a well defined divisor $\sum_\mu q_\mu$. These points lie in the curve S, and so in the surface $\mathcal{K}_D$. The point of this section is that when these points are distinct, they can be thought of as providing Darboux coordinates for the varieties $\mathcal{M}$.

Indeed, the surface $\mathcal{K}_D$ comes equipped with a standard meromorphic two-form ω, with poles at the inverse image in $\mathcal{K}_D$ of the divisor D. Choosing again a two parameter family $A(x,y)=(S(x,y),L(x,y))$, with $(S(0,0),L(0,0))=(S_0,L_0)$, we can take the derivatives $(q_\mu)_x,(q_\mu)_y$ of the corresponding curves $q_\mu(x,y)$ in $\mathcal{K}_D$. We have:

PROPOSITION (3.18)

$$\sum_\mu\omega((q_\mu)_x,(q_\mu)_y)=\Omega_S(A_x,A_y). \tag{3.19}$$

PROOF: Let us write a local equation for the curves $S(x,y)$ as $g(x,y,\lambda,\zeta)=0$. With respect to some suitable covering of the curves by open sets, we can suppose that the zeroes of the sections of $\hat{L}(x,y)$ are cut out by $s(x,y,\lambda,\zeta)=0$, so that the q_μ are given by the simultaneous vanishing of g and s. We note that transition functions for L over S are given by the function s on punctured disks surrounding the zeroes of s. We have:

$$\omega((q_\mu)_x,(q_\mu)_y) = \mathrm{Res}\left((s_y g_x - s_x g_y)\frac{\omega}{g\cdot s}\right), \tag{3.20}$$

where Res denotes the two-dimensional residue; if ω is $f(\lambda,\zeta)d\lambda\wedge d\zeta$, then this expression is simply:

$$f(\lambda,\zeta)\cdot(s_y g_x - s_x g_y)\cdot(s_\zeta g_\lambda - s_\lambda g_\zeta)^{-1}. \tag{3.21}$$

Over the curve S, the "Poincaré residue" reduces this to a residue on the curve:

$$\mathrm{res}_{q_\mu}\left(\frac{s_y}{s}\ P.R(\frac{g_x\omega}{g}) - \frac{s_x}{s}\ P.R(\frac{g_y\omega}{g})\right). \tag{3.22}$$

The terms $\rho_x = P.R(\frac{g_x\omega}{g}), \rho_y = P.R(\frac{g_y\omega}{g})$ are simply the expressions of the sections of the normal bundle giving the deformations as a 1-form, under the various identifications which come into play, giving us

$$\sum_\mu(\omega((q_\mu)_x,(q_\mu)_y)) = \sum_\mu \mathrm{res}_{q_\mu}\left(\frac{s_y}{s}\ \rho_x - \frac{s_x}{s}\ \rho_y\right), \tag{3.23}$$

which is the Serre duality form Ω_S on (3.7), applied to A_x, A_y.

From this, if one chooses Darboux coordinates (z,ζ) for the form ω on $\mathcal{K}_D$, then expressing q_μ in these coordinates as pairs (z_μ,ζ_μ), one has Darboux coordinates on $\mathcal{M}$.

One can then linearize the flows by a standard Liouville generating function technique. Let $C_1,...C_s$ denote a basis for the Casimir functions amongst the Hamiltonians, and choose a complementary basis $H_1,...H_g$ for the rest of the

Hamiltonians. Fixing C_i determines a symplectic leaf $\mathcal{L}$, and fixing the H_i as well determines a spectral curve S, and so defines ζ implicitly in terms of z and the H_i, C_j: $\zeta = \zeta(z, H_i, C_j)$. We set

$$F(z_\mu, H_i, C_j) = \sum_\mu \int^{z_\mu} \zeta(z, H_i, C_j) dz. \tag{3.24}$$

Since $\partial F/\partial z_\mu = \zeta_\mu$, the Liouville generating technique tells us that the derivatives $Q_i = \partial F/\partial H_i$ provide linearizing coordinates for the H_i flows. Setting $P(z,\zeta) = \det(\phi(z) - \zeta\mathbb{I})$, we have

$$Q_i = \sum_{\mu=1}^{g} \int^{z_\mu} \frac{(\partial P/\partial H_i)}{(\partial P/\partial z)} dz. \tag{3.25}$$

The integrands, as H_i varies, give a basis of the Abelian differentials over the spectral curve. This is a consequence of the Poincaré residue formula and the corresponding exact sequence for differentials over the surface $\tilde{\mathcal{K}}_D$ (see, e.g.[GH]). This gives the linearization one expects from the algebro-geometric picture.

d) The systems as symmetric products of surfaces

These Darboux coordinates are a particular manifestation of a more general phenomenon. Indeed, suppose that we have a local integrable system of Jacobians, that is a Lagrangian fibration

$$\mathcal{H} : \mathbb{J} \to U. \tag{3.26}$$

where U is a ball in $\mathbb{C}^g$, and $\mathbb{J}$ is $2g$-dimensional, symplectic (with form Ω). The fibers are Jacobians of smooth genus g curves, and so, corresponding to $\mathbb{J}$ there is a family of curves $\mathbb{S}$, with

$$\mathcal{H}' : \mathbb{S} \to U. \tag{3.27}$$

The Abel map gives us an embedding

$$A : \mathbb{S} \hookrightarrow \mathbb{J}\ . \tag{3.28}$$

This map is not unique, but depends on the choice of a base-point in the fibre J_h for each h in U. One has

THEOREM (3.29) [Hul]

(i) Let $A^*\Omega \wedge A^*\Omega = 0$. Under the embedding A, the variety $\mathbb{S}$ is coisotropic. Quotienting by the null foliation, one obtains, restricting U if necessary, a surface Q to which the form $A^*\Omega$ projects, defining a symplectic form ω on Q. The curves S_h all embed in Q.

(ii) If A, $\tilde{A}$ are two Abel maps with $A^*\Omega \wedge A^*\Omega = 0$, $\tilde{A}^*\Omega \wedge \tilde{A}^*\Omega = 0$, then $A^*\Omega = \tilde{A}^*\Omega$, when $g \geq 3$, and so Q depends only on $\mathbb{S}$ and not on the particular Abel map chosen. For $g = 2$, $A^*\Omega \wedge A^*\Omega$ is always zero.

(iii) There is a symplectic isomorphism

$$\Phi : \widetilde{SP}^g(Q, \omega) \to \mathbb{J},$$

defined over a Zariski open set, between a desingularisation $\widetilde{SP}^g(Q, \omega) = \widetilde{SP}^g(Q)$ of the g-fold symmetric product $SP^g(Q)$ of Q and $\mathbb{J}$. The symmetric product $SP^g(S_h)$ of the curves is Lagrangian in $\widetilde{SP}^g(Q)$, and the restriction of Φ to $SP^g(S_h)$ is the Abel map

$$SP^g(S_h) \to J_h \ .$$

$\widetilde{SP}^g(Q)$ is the Hilbert scheme of length g 0-dimensional subschemes of Q.

The case studied here is an example of this phenomenon. Indeed, in our case, the spectral curves are all embedded in the surface $\mathcal{K}_D$, which has a canonical meromorphic two-form ω, with poles along D. On the other hand, the spectral curves on the symplectic leaves also have fixed intersection with D. Blowing up the surface at these intersection points gives a surface $\tilde{\mathcal{K}}_D$ in which the curves move freely, and in which the lift of the ω is holomorphic.

Proposition (3.18) is in effect expressing the moduli space as the symmetric product of $\tilde{\mathcal{K}}_D$. Similar theorems can be proven for integrable systems of Prym varieties; see [HM].

4) The elliptic Gaudin system.

a) Elliptic Lie-Poisson structures.

We first recall the elliptic Lie-Poisson structures and the integrable elliptic Gaudin systems, following [RS2]. Let $q = \exp(2\pi i/r)$, and set

$$I_1 = \mathrm{diag}(1, q, q^2, ..., q^{r-1}), \qquad I_2 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{4.1}$$

Then

$$I_1 I_2 I_1^{-1} = q^{-1} I_2. \tag{4.2}$$

We consider the algebra $\mathcal{L}_\nu$ of semi-infinite Laurent series in the variable $z - \nu$ with values in $sl(r, \mathbb{C})$:

$$\mathcal{L}_\nu = \{ \sum_{i=-k}^{\infty} \phi_i (z-\nu)^i, k \in \mathbb{Z},\ \phi_i \in sl(r, \mathbb{C})\}. \tag{4.3}$$

Let $\mathcal{L}_\nu^+$ be the subalgebra of series with $\phi_i = 0$ for $i < 0$. Now let D_{red} represent a sum $\nu_1 + \nu_2 + ... + \nu_n$ of distinct points ν_i in the fundamental domain of an elliptic curve

$$\Sigma = \mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}). \tag{4.4}$$

Set

$$\mathcal{L} = \oplus_i \mathcal{L}_{\nu_i}, \quad \mathcal{L}^+ = \oplus_i \mathcal{L}_{\nu_i}^+,$$

and define the subalgebra $\mathcal{T} \in \mathcal{L}$ of meromorphic functions with values in $sl(r, \mathbb{C})$ and with poles only at the translates of the ν_i, satisfying the quasiperiodicity relations:

$$\phi(z + \omega_i) = I_i \phi(z) I_i^{-1}, i = 1, 2. \tag{4.5}$$

One can split $\mathcal{L}$ as a sum $\mathcal{L} = \mathcal{L}^+ \oplus \mathcal{T}$. Furthermore, we have on $\mathcal{L}$ a bilinear form given by $(a,b) \mapsto \mathrm{tr}(\sum_i \mathrm{res}_{\nu_i}(ab))$. This identifies $\mathcal{T}$ with the dual of $\mathcal{L}^+$; we equip $\mathcal{T}$ with the canonical Lie Poisson bracket; one has that along symplectic leaves, the order of the poles at D_{red} and the conjugacy class at D_{red} are both fixed.

Following either the theorem of Adler, Kostant and Symes [AKS] , or by using the r-matrix formalism as in [RS2], one has that the functions on $\mathcal{T}$ defined as the coefficients of the equation of the spectral curve:

$$\det(\phi(z) - \zeta \mathbb{I}) = 0$$

Poisson commute on $\mathcal{T}$, and define an integrable system. The flows are given by Lax equations:

$$\dot{\phi}(z) = [P(f(\phi(z), z)), \phi(z)], \tag{4.6}$$

where P is the projection from $\mathcal{L}$ to $\mathcal{T}$, and f is a function depending on the choice of Hamiltonian.

b) Bundles on an elliptic curve

Vector bundles on an elliptic curve were classified by Atiyah [A]. As a consequence of his results, one has:

PROPOSITION (4.7) *Let $(r,d) = 1$.*

(a) The stable bundles E of rank r, degree d on Σ are classified by their top exterior power $\Lambda^r(E)$. One has that $E \otimes L$, $L \in Pic^0(\Sigma)$ is isomorphic to E if and only if $L^r \simeq \mathcal{O}$.

b) For $0 < d < r$, $h^0(\Sigma, E) = d$.

By the theorem of Narasimhan and Seshadri[NS] , stable bundles correspond to irreducible representations of a $\mathbb{Z}$-central extension of the fundamental group; the center, for bundles of degree d, rank r , is mapped to q^d. In our case, that of bundles over an elliptic curve, we are looking at a central extension of $\mathbb{Z} \times \mathbb{Z}$.

We fix the degree to be one. The generators T_1, T_2 of $\mathbb{Z} \times \mathbb{Z}$ satisfy the relation

$$T_1 T_2 T_1^{-1} = q^{-1} T_2. \tag{4.8}$$

From this, one sees that T_2^r commutes with T_1, T_2 and so for an irreducible representation, must be central. Similarily, T_1^r is also central.

Multiplication of T_1, T_2 by scalars corresponds to tensoring the vector bundle by a line bundle, and so one might as well begin by classifying irreducible representations satisfying $T_1^r = T_2^r = 1$. One can begin by conjugating T_1 to the diagonals:

$$T_1 = \mathrm{diag}(q^{m_1}, q^{m_2}, ..., q^{m_r}), \tag{4.9}$$

with $0 \leq m_1 \leq m_2 \leq m_3 ... \leq r-1$. The relation (4.8) tells us then that $(T_2)_{ij} = 0$ unless $m_i - m_j = -1$, modulo r. From this, one sees that the only way to avoid having an invariant non-trivial proper subspace is to have $m_i = i - 1$ (we had arranged the m_i in increasing order) . One can then choose the basis so that

$$T_2 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & & . \\ . & . & . & & . \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{4.10}$$

In short, we can set $T_i = I_i$.The stable bundle E determined by the *monodromy matrices* I_i over an elliptic curve is then unique, up to tensoring by a line bundle. It has a non-zero section, which we will compute. In terms of the matrices I_i, sections will be given by functions on $\mathbb{C}$ with suitable automorphy properties when one translates by a lattice point. Also, as the degree is one, the functions must get multiplied by q as one winds around a fixed puncture in the curve [AB]. We normalize the periods ω_i, so that the elliptic curve Σ be given as $\mathbb{C}/(\frac{1}{r}\mathbb{Z} \oplus \frac{\tau}{r}\mathbb{Z})$, with a projection $\Pi : \mathbb{C} \to \Sigma$. Let p be $\Pi((1+\tau)/2r))$. We have:

PROPOSITION (4.11)*(a) A section of the bundle E is given by an r-tuple $\Psi = (\psi_1, ..., \psi_r)$ of functions which are r-valued over the inverse image in $\mathbb{C}$ of the punctured curve $\Sigma - \{p\}$; These functions must satisfy*

$$\Psi(z + 1/r) = I_1 \cdot \Psi(z), \ \ \Psi(z + \tau/r) = I_2 \cdot \Psi(z), \tag{4.12}$$

and be of the form

$$(z - ((1 + \tau)/2r))^{\frac{-1}{r}}(\text{holomorphic}) \tag{4.13}$$

near the inverse images in $\mathbb{C}$ of the puncture.

(b) Let L_c be the line bundle determined by the monodromy factors 1, $exp(2\pi ic)$; then $E_c = E \otimes L_c$ has monodromy matrices $I_1, exp(2\pi ic)I_2$; $E_c \simeq E_{c'}$ iff $c - c' = (n + m\tau)/r$ for some integers n, m.

(c) A section of $\mathrm{End}(E_c)$ *is given by a matrix M of functions on $\mathbb{C}$, satisfying*

$$M(z + 1/r) = I_1 \cdot M(z) \cdot I_1^{-1}, \ \ M(z + \tau/r) = I_2 \cdot M(z) \cdot I_2^{-1}. \tag{4.14}$$

The sections of $sl(E)$ with poles at D_{red} are then precisely the elements of the subalgebra $\mathcal{T}$ defined above. Now recall that our bundle E is rigid, up to tensoring by a line bundle. If one reduces by the action of $Pic_0(\Sigma) \simeq \Sigma$, the class of a pair (E, ϕ) is determined by ϕ, which is of trace zero, and corresponds by (4.5) to an element of $\mathcal{T}$. We have the following result of Markman, referring to (2.13):

PROPOSITION (4.15) [Ma] *Let $\mathcal{T}_D$ be the Poisson subspace of $\mathcal{T}$ of functions whose polar divisor is bounded by D. (a) The open set U of $\mathcal{M}(r, D, 1)$ for which the bundle is a stable bundle of the form E_c (and so for which the sections ϕ are given by the automorphy relations (4.14)) is Poisson isomorphic to the space $\Sigma \times \mathbb{C} \times \mathcal{T}_D$ by the map*

$$(E_c, \phi) \mapsto (c, tr(\phi)/r, \phi - tr(\phi)/r)$$

(b) Reducing U by the action of $Pic_0(\Sigma)$, referring to (2.13), gives a subspace U_V of $\mathcal{M}_V(r, D, 1)$ isomorphic to $\mathcal{T}_D$

The isomorphism intertwines the Hamiltonians of the integrable systems defined on the two subspaces.

The proof proceeds by remarking that for both spaces, there is a well defined Poisson embedding into a product of the duals of finite dimensional Lie algebras, given in both cases by taking polar parts at the divisor D.

c) Sections of the bundle E.

Let us take the r-th powers $f_i = \Psi_{i+1}^r$ of the components of Ψ, so that:

$$\Psi_i = f_{i-1}^{\frac{1}{r}}. \tag{4.16}$$

We would then like to find an r-tuple F of functions $(f_0, ..., f_{r-1})$, which are of the form $z^{-1}(\text{holomorphic})^r$ near the punctures, satisfying

$$F(z + 1/r) = F(z),\ F(z + \tau/r) = I_2 \cdot F(z), \tag{4.17}$$

and are such that the r-th roots along the real and imaginary axes satisfy

$$\begin{aligned} (f_i)^{\frac{1}{r}}(z + 1/r) &= q^i (f_i)^{\frac{1}{r}}(z) \\ (f_i)^{\frac{1}{r}}(z + \tau) &= (f_i)^{\frac{1}{r}}(z). \end{aligned} \tag{4.18}$$

Since $I_1^r = I_2^r = 1$, one is dealing with functions over the elliptic curve $\Sigma' = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$; let θ be the standard theta function for this curve; recall that it has a simple zero at the points $((1+\tau)/2) + \mathbb{Z} + \tau\mathbb{Z}$, and is otherwise non-zero and holomorphic. We distinguish two cases:

Case 1: r is odd.

Let

$$\theta_{k,j}(z) = \theta(z + \frac{(k + j\tau)}{r}), 0 \leq k, j \leq (r-1). \tag{4.19}$$

We have the relations:

$$
\begin{aligned}
\theta_{k,j}(z+m) =& \theta_{k,j}(z),\\
\theta_{k,j}(z+m\tau) =& exp(-\pi i m^2\tau - 2\pi i m(z+\frac{(k+j\tau)}{r}))\theta_{k,j}(z),\\
\theta_{k,j}(z+\frac{1}{r}) =& \theta_{k+1,j}(z),\\
\theta_{k,j}(z+\frac{\tau}{r}) =& \theta_{k,j+1}(z), 0 \leq j < (r-1),\\
\theta_{k,r-1}(z+\frac{\tau}{r}) =& \theta_{k,0}(z) exp(-\pi i\tau - 2\pi i(z+\frac{k}{r})).
\end{aligned}
\tag{4.20}
$$

where m is an integer. Now if

$$
\rho_j = (\frac{r-1}{2} - j),
\tag{4.21}
$$

we set

$$
f_j(z) = exp\left(2\pi i\tau\left(\tfrac{-jr(r-1)}{2} + \tfrac{(r-1)j(j+1)}{2}\right)\right)\prod_{k=0}^{r-1}\left(\frac{\theta_{k,j}^{r-2}(z)\theta_{k,j}(z+\rho_j\tau)}{\prod_{\ell=0,\ell\neq j}^{r-1}\theta_{k,\ell}(z)}\right)
\tag{4.22}
$$

Using the relations given for the $\theta_{k,j}$, one checks that it has the correct form near the punctures, and that (4.17) holds. Now let τ be imaginary. Let us consider the involutions $f(z) \mapsto f(-z), f(z) \mapsto f(\bar{z})$. Both these involutions preserve the poles and zeros of f_0. From this, one has that f_0 must be even, as $f_0(0) \neq 0$. Using the second involution, one can then multiply f_0 by a constant c so that $cf_0(0)$ is real. The function is then real on both imaginary and real axes, and has no zeros. From this, one has that (4.18) holds for f_0. From the relations (4.20), (4.18) follows for the other f_i. Deforming, the same then must hold for arbitrary τ.

Case 2: r is even

We then set

$$
\xi_{k,j}(z) = \theta(z + \frac{(k+j\tau)}{r} - \frac{(1+\tau)}{2r}), 0 \leq k,j \leq (r-1).
\tag{4.23}
$$

We have the relations:

$$\begin{aligned}
\xi_{k,j}(z+m) &= \xi_{k,j}(z), \\
\xi_{k,j}(z+m\tau) &= exp(-\pi i m^2\tau - 2\pi i m(z + \frac{(k+j\tau)}{r} - \frac{(1+\tau)}{2r}))\xi_{k,j}(z), \\
\xi_{k,j}(z+\frac{1}{r}) &= \xi_{k+1,j}(z), \\
\xi_{k,j}(z+\frac{\tau}{r}) &= \xi_{k,j+1}(z), 0 \leq j < (r-1), \\
\xi_{k,r-1}(z+\frac{\tau}{r}) &= \xi_{k,0}(z) exp(-\pi i \tau - 2\pi i(z + \frac{k}{r} - \frac{(1+\tau)}{2r})).
\end{aligned} \tag{4.24}$$

where m is an integer. We then define

$$\rho_j = \frac{\tau}{2} - j \tag{4.25}$$

and set

$$f_j(z) = (-1)^j exp\left(2\pi i\tau\left(\frac{-j(r-1)}{2}\right)\right)\prod_{k=0}^{r-1}\left(\frac{\theta_{k,j}^{r-1}(z)\theta_{k,j}(z+\rho_j\tau)}{\prod_{\ell=0}^{r-1}\xi_{k,\ell}(z)}\right). \tag{4.26}$$

Again, the r-th roots of the f_j define our section.

In a similar way we can compute a section Ψ_c of E_c; this complicates the formulae somewhat..

d) Darboux coordinates and integration of the system

Given Ψ_c, we can then simply apply our theorems of section 3, and obtain separating coordinates for our integrable systems (for the $Gl(n,\mathbb{C})$ case) on $\mathcal{M}(r, D, 1)$; we then reduce to the $Sl(n,\mathbb{C})$-case; for this case, one only needs Ψ.

The canonical line bundle of the elliptic curve is trivial, and we can write the total space $\mathcal{K}$ as $\mathbb{C}/(\frac{1}{r}\mathbb{Z}\oplus\frac{\tau}{r}\mathbb{Z})\ \times\mathbb{C}$, with corresponding coordinates z,ζ. The symplectic form on $\mathcal{K}$ can then be written as $dz\wedge d\zeta$. We can then use the map $\mathcal{K}\to\mathcal{K}_D$ to transport these coordinates over to $\mathcal{K}_D$, at the same time trivializing the bundle $K(D)$ with a singularity over D. The coordinates (z,ζ) are Darboux coordinates on the blow-up $\tilde{\mathcal{K}}_D$.

Our coordinates were defined as the zeroes (z_μ, ζ_μ) of a suitably normalized section of the line bundle L of (2.8). Recall that L is the cokernel of $(\phi - \zeta\mathbb{I})$; it is then a quotient of E, and so there is a natural map ρ of E to L. Both E and L here have a one-dimensional space of sections, and projecting the section Ψ of E gives that of L. The projection $\rho(\Psi)$ to E vanishes iff Ψ lies in the image of $(\phi - \zeta\mathbb{I})$. Let $\hat{\Phi}(z,\zeta)$ be the matrix of cofactors of $(\phi - \zeta\mathbb{I})$, so that

$$\hat{\Phi}(z,\zeta) \cdot (\phi - \zeta\mathbb{I}) = \det(\phi - \zeta\mathbb{I}) \cdot \mathbb{I}.$$

If the spectral curve is smooth, then all of the eigenspaces are of dimension one [AHH2], and one has that $\rho(\Psi) = 0$ iff

$$\hat{\Phi}(z,\zeta) \cdot \Psi_c(z) = 0. \tag{4.27}$$

To summarise: starting from the matrix ϕ, one computes the matrix of cofactors $\hat{\Phi}$, then solves (4.27), using the fixed section computed in section c). There are generically g distinct solutions (z_μ, ζ_μ) to this equation ([AHH1],[AHH2]) and these are our Darboux coordinates on $\mathcal{M}(r, D, 1)$.

For the coordinates on $\mathcal{T} = \mathcal{M}_V(r, D, 1)$, we must reduce. One restricts first to $c = 0$, so fixing the bundle to $E = E_0$, and in particular fixing its determinant. Recall from (3.16) that the line bundle $det(E)$ corresponds to the divisor on Σ given by $D' + \sum_\mu (z_\mu)$, for some fixed divisor D'. It follows that the centre of mass in Σ of the coordinates z_μ must be a constant. The reduction to $\mathcal{T}$ can then be effected explicitly by normalising the ζ_μ also to $\sum_\mu \zeta_\mu = 0$, as one does for centre of mass coordinates.

e) A rank two example

Formulae (4.16), (4.22), (4.26) and (4.27) give an explicit way of determining the Darboux coordinates from ϕ. We briefly exhibit the formulae for the case $r = 2$. This case was treated in [S]. It is difficult to see whether the coordinates obtained are the same, though they seem to have common features.

For $r = 2$, we write the matrix ϕ as

$$\phi = \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix}.$$

The periodicity relations work out to:

$$\begin{aligned} a(z+\frac{1}{2}) &= a(z), \quad b(z+\frac{1}{2}) = -b(z), \quad c(z+\frac{1}{2}) = -c(z), \\ a(z+\frac{\tau}{2}) &= -a(z), \quad b(z+\frac{\tau}{2}) = c(z). \end{aligned}$$

Let $D = \sum \nu_i$. The coefficients a, b, c are then linear combinations of, respectively, elliptic functions a_i, b_i, c_i with poles only at $\nu_i, \nu_i + 1/2, \nu_i + \tau/2, \nu_i + (1+\tau)/2$ and their translates. These are fairly straightforward to write out in terms of theta-functions. For example, setting $\rho_i = \nu_i + \frac{1}{4}$

$$a_i = \prod_{k,j=0,1} \frac{\theta_{k,j}(z-\rho_i)}{\theta_{k,j}(z-\nu_i)}.$$

The matrix of cofactors of $\phi - \zeta \mathbb{I}$ is given by:

$$\hat{\Phi} = \begin{pmatrix} -a(z) - \zeta & -b(z) \\ -c(z) & a(z) - \zeta \end{pmatrix}.$$

The section Ψ is given by

$$\begin{aligned} \Psi_1 &= \left[\prod_{k=0}^{1} \left(\frac{\theta_{k,0}(z)\theta_{k,0}(z+\tau)}{\xi_{k,0}(z)\xi_{k,1}(z)} \right) \right]^{1/2} \\ \Psi_2 &= -e^{-\pi i \tau} \left[\prod_{k=0}^{1} \left(\frac{\theta_{k,1}(z)\theta_{k,1}(z)}{\xi_{k,0}(z)\xi_{k,1}(z)} \right) \right]^{1/2} \end{aligned}.$$

The coordinates (z_μ, ζ_μ) are then the solutions to the equations:

$$\begin{aligned} (-a(z) - \zeta)\Psi_1(z) - b(z)\Psi_2(z) &= 0 \\ -c(z)\Psi_1(z) + (a(z) - \zeta)\Psi_2(z) &= 0 \end{aligned}.$$

5) The trigonometric case

We now exhibit another set of classical systems associated to rigid vector bundles over a curve: those associated to "trigonometric" r- matrices. Our curve here will be a singular nodal rational curve, that is a Riemann sphere $\mathbb{P}^1$ with two points identified. Such curves arise as degeneracies of elliptic curves, and we shall see many common features with the elliptic case.

The rigid bundle E we consider is obtained from the sum of line bundles $E_0 = \mathcal{O}(1) \oplus \mathcal{O} \oplus \mathcal{O} ... \oplus \mathcal{O}$ on $\mathbb{P}^1$ with transition matrix

$$\mathrm{diag}(\mathrm{z}^{-1}, 1, 1, ..., 1) \tag{5.1}$$

from $U_0 = \{z \neq \infty\}$ to $U_1 = \{z \neq 0\}$. E is built by identifying the fiber of E_0 over 0 with that over ∞ via a non-singular matrix M. The bundle E_0 is rigid, and we shall see that the glueing matrix M is essentially unique for a stable bundle, once one adjusts by a suitable automorphism of E_0, and fixes the top exterior power of E.

Indeed, consider $E_0 = \mathcal{O}(1) \oplus \mathcal{O}^{r-1}$ on $\mathbb{P}_1$. We would like to find conditions on M to get stability of the bundle. A destabilising bundle is obtained by glueing from $F_0 = \mathcal{O}(1) \oplus \mathcal{O}^k$, and one finds that one has a subbundle for the glueing iff the vector $e_1 = (1, 0, 0, .., 0)$ belongs to an M-invariant subspace of dimension less than r. Thus, the bundle is stable iff e_1 is cyclic for M. So if the bundle is stable one can write the matrix M as

$$M = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & 0 & \dots & 0 & a_1 \\ \cdot & \cdot & \cdot & & \cdot & \\ \cdot & \cdot & \cdot & & \cdot & \\ 0 & 0 & 0 & \dots & 1 & a_{r-1} \end{pmatrix} \tag{5.2}.$$

Now, one can modify M by automorphisms of $\mathcal{O}(1) \oplus \mathcal{O}^{r-1}$, and in particular, by the following automorphism:

$$\begin{aligned} e_1 &\longrightarrow e_1 \\ e_j &\longrightarrow e_j + b_j e_1 \end{aligned}, \quad b_j \in \mathbb{C}. \tag{5.3}.$$

When we compute $\det(M - z\mathbb{I})$, we can see that with a suitable choice of b_i, we can get $\det(M - z\mathbb{I}) = (-1)^{r+1}a_0$. In other words, by an automorphism, we can set $a_1 = a_2 = ... = a_{r-1} = 0$, and the glueing only depends on a_0. The determinant $a_0 \in \mathbb{C}^*$ represents the highest power $\wedge^r E$ of the bundle E in Pic. Thus, when the determinant is fixed, the bundle is rigid since the bundle $\mathcal{O}(1) \oplus \mathcal{O}^{r-1}$ on $\mathbb{P}_1$ is infinitesimally rigid and the glueing is rigid.

Let us set $a_0 = 1$; we will take the $Sl(r, \mathbb{C})$ moduli space and so consider ϕ's which are traceless.

We can change trivializations so that the transition matrix becomes:

$$T(z) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ z^{-1} & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{5.4}$$

instead of the diagonal matrix (5.1). The glueing matrix M is then the identity. The unique (up to scale) section of E is then represented over U_0 by the vector of functions

$$(1 + z, 1, 1, ..., 1)^T.$$

Conjugating T by

$$S = \mathrm{diag}(1, z^{\frac{1}{r}}, z^{\frac{2}{r}}, ..., z^{\frac{r-1}{r}})$$

transforms it to

$$STS^{-1} = z^{\frac{-1}{r}} I_2,$$

with I_2 the matrix of (4.1). This can be thought of as a multi-valued change of trivialization. The section is then represented by

$$(1 + z, z^{\frac{1}{r}}, z^{\frac{2}{r}}, ..., z^{\frac{r-1}{r}}). \tag{5.5}$$

More generally, meromorphic sections of E are then represented by vectors of functions $F = (f_0, ... f_{r-1})$ on $\mathbb{C}^*$ such that

- $f_i = z^{\frac{i}{r}} \cdot$(meromorphic),

- $\lim_{z\to 0} F(z)$ exists,

- $\lim_{z\to\infty} z^{\frac{-1}{r}} I_2 F(z)$ exists,

- and the two limits coincide.

Changing variables by $z = \exp(2\pi i\ rx)$, one is still dealing with functions $f_i(x)$ satisfying $f_i(x+\frac{1}{r}) = q^i f_i(x)$, that is,

$$F(x+\frac{1}{r}) = I_1 F(x), \tag{5.6}$$

the other matrix I_2 now being used for the boundary conditions at $ix \to \pm\infty$. Similarily, sections of the End(E) get represented by matrices $M(z)$ with

$$M(x+\frac{1}{r}) = I_1 M(x) I_1^{-1}.$$

As in section 4 , there is a splitting of a loop algebra $\mathcal{L}$ of the periodic functions satisfying the appropriate boundary conditions into a $\mathcal{L}^+$ of positive series and a $\mathcal{T}$ of meromorphic functions satisfying only one periodicity condition. Corresponding to this, there is an integrable system, as for the elliptic case. Our construction above shows us that the elements of $\mathcal{T}$ can be interpreted as sections of a rigid bundle E, and so Markman's result (extended to handle singular base curves) gives us a Poisson isomorphism between the coadjoint orbits in $\mathcal{T}$ and the symplectic leaves in the corresponding moduli of Higgs pairs. One then obtains, as in the elliptic case, suitable separating coordinates for these systems.

6) Poisson-Lie groups

There are three main cases of a curve with a rigid bundle, yielding a corresponding splitting of the loop algebra of matrices into a sum of two subalgebras which are dual to each other and so allowing us to define an integrable system, using either the Adler Kostant Symes theorem, or more generally the r-matrix formalism. The curves are either rational, elliptic or nodal rational, and correspond to the rational, elliptic and trigonometric r-matrices respectively. These r-matrices can also be used to define quite different, if related, Poisson structures, the quadratic or Sklyanin bracket. Again these come in three types, rational, elliptic and trigonometric. They are obtained, very roughly, by thinking of (generically invertible) matrix valued functions as elements of a group, rather than an algebra, and applying the formalism for constructing Poisson Lie groups, as in [RS2]. Again one has integrable systems, defined again in terms of spectral curves: the Lagrangian foliations for both the Lie Poisson and Sklyanin structures share the same leaves. The symplectic leaves of the two types of structure are however quite different. In any case, one can ask if there is any analogue of the separating coordinates in the quadratic case.

The answer for the rational quadratic bracket is yes, and can be found in [Sc]. One has, as for the rational Lie Poisson case, a spectral curve, line bundle, and section giving a divisor Σp_ν on the spectral curve. This curve lies in the same surface as in the Lie Poisson case, that is the total space of the line bundle $\mathcal{K}(D)$ over the curve $\mathbb{P}^1$. The divisor then gives an isomorphism I of the symplectic leaves with a symmetric product of this surface. Let z be a coordinate on $\mathbb{P}_1$, and ζ the corresponding cotangent coordinate, and let $a(z) = 0$ cut out the divisor D. In the Lie Poisson setting, the form $a(z)^{-1}dz\wedge d\zeta$ on $\mathcal{K}(D)$ induce a symplectic form on the symmetric product and turn the isomorphism I into a symplectic one. In the rational quadratic case, one has the same result, but with the form $\zeta^{-1}dz \wedge d\zeta$.

We conjecture that a similar result holds for the quadratic or Sklyanin bracket in both the elliptic and the trigonometric cases. One again has divisor coordinates on a surface $\mathcal{K}(D)$ defined over an elliptic or a nodal curve; instead of taking the symplectic form on $\mathcal{K}(D)$ with poles along $\pi^{-1}(D)$, one chooses the form with a pole along the zero-section. Choosing Darboux coordinates (z, ζ) for this form, and expressing the divisor corresponding to the pair (line bundle, curve) in these coordinates as a sum $\Sigma(z_\mu, \zeta_\mu)$ should give us separating Darboux coordinates. The validity of this result could be checked with a direct but probably rather difficult calculation, as in [Sc], but does not seem to be feasible with the methods of this paper.

References

[A] M. Atiyah, *Vector bundles over an elliptic curve*, Proc. Lond. Math. Soc **7** (1957), 414–452.

[AB] M. Atiyah, R. Bott, *The Yang-Mills equations over Riemann surfaces,* Phil. Trans. R. Soc. Lond. A, **308**(1982), 523-615.

[AHH1] M.R. Adams,J. Harnad, and J. Hurtubise, *Darboux coordinates and Liouville -Arnold integration in loop algebras*, Comm. Math. Phys. **155** (1993), no. 2, 385–413.

[AHH2] M.R. Adams,J. Harnad, and J. Hurtubise, *Isospectral Hamiltonian flows in finite and infinite dimensions* II. *Integration of flows*, Comm. Math. Phys. **134** (1990), 555–585.

[AHP] M.R. Adams, J. Harnad, and E. Previato, *Isospectral Hamiltonian flows in finite and infinite dimensions* I. *Generalised Moser systems and moment maps into loop algebras*, Comm. Math. Phys. **117** (1988), no. 3, 451–500.

[AKS] M. Adler, *On a trace functional for formal pseudo-differential operators and the symplectic structure of the Korteweg-de Vries equation*, Invent. Math. **50** (1978/79), no. 3, 219–248; B. Kostant, *The solution to a generalized Toda lattice and representation theory*, Adv. Math. **34** (1979), no. 3, 195–338; W. Symes, *Systems of Toda type, inverse spectral problems and representation theory*, Invent. Math. **59** (1980), 13–51.

[AvM] M. Adler and P. van Moerbeke, *Completely integrable systems, Euclidean Lie algebras, and curves*, Adv. in Math. **38** (1980), no. 3, 267-317; *Linearization of Hamiltonian systems, Jacobi varieties and representation the-*

ory, ibid. **38** (1980), no. 3, 318–379.

[Bo] F. Bottacin, *Symplectic geometry on moduli spaces of stable pairs,* Ann. Sci. Ecole Norm. Sup. (4) **28** (1995), no. 4, 391-433.

[FT] L.D. Faddeev and L.A. Takhtajan, *Hamiltonian methods in the theory of solitons*, eds., Springer-Verlag, Berlin, 1987.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, New-York, (1978).

[HH] J. Harnad and J. Hurtubise, *Generalised tops and moment maps into loop algebras*, J. Math. Phys. **37** (1991), no. 7, 1780–1787.

[Hi1] N.J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 59–126.

[Hi2] N.J. Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987), no. 1, 91–114.

[HM] J. Hurtubise and E. Markman, *Rank* 2 *integrable systems of Prym varieties*, Adv. Theo. Math. Phys.,**2** (1998), 633-695.

[Hu1] J. Hurtubise, *Integrable systems and algebraic surfaces*, Duke Math. J. **83** (1996), no. 1, 19–50.

[Hu2] J. Hurtubise, *The algebraic geometry of the Kostant-Kirillov form*, J. Lond. Math.Soc. (2) **56** (1997), 504-518.

[M] Moser, J., "Geometry of Quadrics and Spectral Theory", *The Chern Symposium, Berkeley, June 1979*, 147-188, Springer, New York, (1980).

[Ma] E. Markman, *Spectral curves and integrable systems*, Compositio Math. **93**, 255-290, (1994).

[NS] M. Narasimhan and C.S. Seshadri, *Stable and Unitary vector bundles on a compact Riemann surface*, Annals of Math **82**, 540-567.

[RS1] A.G. Reiman and M.A. Semenov-Tian-Shansky, *Reduction of Hamiltonian systems, affine Lie algebras and lax equations* I, II, Invent. Math. **54** (1979), no. 1, 81–100; ibid. **63** (1981), no. 3, 423–432.

[RS2] A.G. Reiman and M.A. Semenov-Tian-Shansky, *Integrable Systems II*, chap.2, in "Dynamical Systems VII", Encyclopaedia of Mathematical Sciences, vol 16., V.I. Arnold and S.P.Novikov, eds., Springer-Verlag, Berlin, 1994.

[S] E.K. Sklyanin, *Poisson structure of a periodic classical XYZ-chain*, J. Sov. Math. , 46 (1989) 1664-1683.

[Sc] D.R.D. Scott, *Classical functional Bethe ansatz for $SL(N)$:separation of variables for the magnetic chain*, J. Math. Phys. bf 35, 5831-5843 (1994)

J. C. Hurtubise
Centre de Recherches Mathématiques
Université de Montréal
and Department of Mathematics
McGill University
email: hurtubis@crm.umontreal.ca

M.Kjiri
Centre de Recherches Mathématiques
Université de Montréal
email: kjirim@dms.umontreal.ca

CHAPITRE II

Article 2: "The G-generalized Hitchin systems and Prym varieties"

Cet article a été soumis à la revue "Journal of Mathematical Physics".

The G-generalized Hitchin systems and Prym varieties

M. Kjiri

ABSTRACT: In this article, we consider the generalized Hitchin systems, introduced by Bottacin and Markman, for an arbitrary reductive complex group G; they have the structure of Lagrangian fibrations $\mathbb{P}r \to U$ by generalized Prym varieties over sets U parameterizing families of Weyl-invariant curves. The G-generalized Hitchin systems satisfy a rank two condition and one can find invariant varieties X which distinguish these systems. It is then shown that there is a correspondence between these integrable systems and the varieties $X = \widetilde{K_\Sigma[D] \otimes \mathfrak{h}}$, which are equipped with an appropriate two form with values in the Cartan subalgebra $\mathfrak{h}$.

1. Introduction.

The moduli spaces of stable vector bundles over a Riemann surface have been studied for many years from different aspects. In 1987, Hitchin introduced a new way of seeing them, that is through the symplectic geometry of their cotangent bundle. Through the article "Stable bundles and integrable systems" [Hi1], he proved that there were natural algebraically integrable Hamiltonian systems on the cotangent bundle, that is complex integrable systems, such that the joint level sets of the Hamiltonians when compactified and desingularized are Abelian varieties, in such a way that the linear structure given by the Hamiltonian flow is that of the Abelian variety.

Since then, a lot of work has been done on these systems -called the Hitchin systems- and many different properties of them have been proven. In 1994 a generalization of these systems was introduced independently by E. Markman [Ma] and F. Bottacin [Bo], which has been a source of considerable interest [F, Sc1, Sc2, HK]. The question was essentually to try to see if the results obtained for the Hitchin sytems could be generalized.

There are versions of these systems for any reductive complex group G. One considers the moduli space of stable pairs (P, ϕ), where P is a G-bundle over

Σ, and ϕ a meromorphic $ad(P)$-valued 1-form with poles at a divisor D. One would like to know if there is some invariant that distinguishes these integrable systems.

When $G = Gl(r, \mathbb{C})$, these systems are integrable sytems of Jacobians and the question was studied in [Hu1] for the Hitchin systems and in [HK] for the $Gl(r, \mathbb{C})$-generalized systems. One has a Lagrangian fibration $\mathbb{J} \to U$ of Jacobians, where U is an open set in $\mathbb{C}^g$, a corresponding family $\mathbb{S} \to U$ of Riemann surfaces and an Abel map $\mathcal{A} : \mathbb{S} \to \mathbb{J}$. A first invariant is the rank of the pull-back $\mathcal{A}^*(\Omega)$ of the symplectic form on $\mathbb{J}$. When the rank is two, that is $\mathcal{A}^*(\Omega) \wedge \mathcal{A}^*(\Omega) = 0$, there is a null foliation of dimension $g-1$ on $\mathbb{S}$ which one can quotient out to obtain a symplectic surface $\mathcal{Q}$. The system, at least in a neighborhood of any Lagrangian leaf, is birational to the g-th symmetric product of $\mathcal{Q}$ so that the invariant is first the rank, and secondly the surface $\mathcal{Q}$; at least locally, these invariants are complete.

Now, what happens if G is any reductive complex group?

The question was studied in [HM] for the Hitchin systems. The authors have shown that the systems can then be seen as rank-2 integrable systems of Prym varieties for a suitably generalized notion of rank, and found an appropriate variety X equipped with a two form with values in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

The aim of this article is to show that those results extend to the G-generalized Hitchin systems.

In section 2, we will summarize the most relevant results for what follows from the article by J.C. Hurtubise and E. Markman, "Rank two integrable systems of Prym varieties".

In section 3, we will consider the generalized Hitchin systems, and we will see how to the Higgs pair (P,ϕ), where P is a G-principal bundle over a Riemann surface Σ and ϕ a meromorphic $ad(P)$-valued 1-form with poles at a divisor D, one can associate a spectral curve S lying in the total space X of the vector bundle $K_\Sigma(D)\otimes\mathfrak{h}$ and an H-bundle over S, $\mathfrak{h}$ being the Cartan subalgebra of $\mathfrak{g}$ and H the corresponding Lie group; we will see that both the curve and the bundle are invariant under the action of the Weyl group W.

In section 4, we will see that the local structure of the integrable system is that of a fibration $\mathbb{P}r \to U$ by generalized Prym varieties over a set U parameterizing a family of W-invariant curves. These Prym varieties parameterize W-invariant H-bundles over the spectral curve S. We will then prove the following theorem:

Theorem 1.1. *The G-generalized Hitchin system for an arbitrary reductive group satisfies a rank two condition. The variety associated to it is a blow up $\widetilde{K_\Sigma[D]}\otimes\mathfrak{h}$ of $K_\Sigma[D]\otimes\mathfrak{h}$ and it comes equipped with an appropriate $\mathfrak{h}$-valued two form.*

2. Rank two integrable system of Prym varieties.

In this section, we will define a rank two condition for integrable systems of Prym varieties and review some results obtained by J.C. Hurtubise and E. Markman in [HM] that will allow us to see later that the generalized Hitchin systems can be viewed as rank two integrable systems of Prym varieties.

One considers for a fixed finite group W, a family U of W-invariant curves, and a representation of the group on a finite dimensional lattice $\chi=\mathbb{Z}^v$, inducing a representation on the complex vector space $V=\chi\otimes_{\mathbb{Z}}\mathbb{C}$. One has an action on the Jacobians J_u of the curves and so a diagonal action on $J_u\otimes_{\mathbb{Z}}\chi$. The generalized Prym varieties will be the connected component of the origin of the fixed point set $(J_u \otimes_{\mathbb{Z}} \chi)^W$.

Let $\mathbb{P}r \to U$ be the associated fibration of such Pryms : we will assume that $\mathbb{P}r$ is symplectic and this fibration is Lagrangian. We will define what it means for this system to have rank 2 and see that under some genericity conditions, there will be a $(v+1)$-dimensional complex manifold X, into which all the curves S_u embed. This variety comes equipped with a generically non degenerate V^*-valued two form Ω_V.

We will see that under some genericity hypothesis, this induces a 1-1 correspondence between rank 2-integrable systems of Prym varieties and appropriate $(v+1)$-folds X with a $(\chi^* \otimes \mathbb{C})$-valued two form.

This section will be then divided in three parts: a first part where we will fix the notation that will be used and recall some definitions, a second part where we will consider the different types of closed two-forms that we have and define the rank two condition, and a third part in which we will review the most relevant theorems for the purpose of this paper.

a) Definitions and properties.

Let:

- W be a finite group which will here be a Weyl group,
- χ, a free $\mathbb{Z}$-module of rank v on which W acts linearly,
- $V = \chi \otimes_{\mathbb{Z}} \mathbb{C}$, the corresponding complex representation of W,
- $U \subset \mathbb{C}^d$, an open ball
- $\sigma : \mathbb{S} \to U$, a holomorphic submersion whose fiber at $u \in U$ is a compact Riemann surface S_u of genus g. We suppose that W acts on $\mathbb{S} \to U$,

inducing a trivial action on U, so that W acts on each S_u. We suppose that W embeds into $Aut(S_u)$.

- $\rho : \mathbb{J} \to U$, the corresponding fibration of Jacobians on which W also acts.

For each Jacobian J_u, we consider $J_u \otimes_{\mathbb{Z}} \chi$ which is isomorphic to the cartesian product $(J_u)^v$. Let $\rho_\chi : \mathbb{J} \otimes_{\mathbb{Z}} \chi \to U$ be the associated fibrewise tensor product.

Definition 2.1. *We can associate to the group action on* $(J_u \otimes_{\mathbb{Z}} \chi)^W$ *a generalized Prym variety*

$$Pr_u = (J_u \otimes_{\mathbb{Z}} \chi)^W_0 ,$$

the connected component of the identity of the fixed point set of the diagonal W*-action on* $J_u \otimes_{\mathbb{Z}} \chi$*. Let* $\mathbb{P}r \to U$ *be the corresponding fibration: since* U *is contractible, it is a component of the fixed point set of the* W*-action on* $\mathbb{J} \otimes_{\mathbb{Z}} \chi$*.*

b) Symplectic structures and rank two systems.

We consider the following closed two-forms:

1) a V^*-valued two form $\Omega_V \in H^0(\mathbb{J}, (\Lambda^2 T^* \mathbb{J}) \otimes_{\mathbb{C}} V^*)$ on the fibration $\mathbb{J} \to U$,

2) an ordinary two form Ω on the fibration $J \otimes_{\mathbb{Z}} \chi \to U$,

3) the restriction ω of Ω to the fibration $\mathbb{P}r \to U$.

We assume that Ω_V, Ω and ω are isotropic on their respective fibrations, as well as on their zero-sections.

Let $e_1, ..., e_v$ and $e^1, ..., e^v$ be arbitrary dual basis of χ and χ^* respectively.

We have the contractions:

$$\begin{tikzcd} \mathbb{J} \otimes_{\mathbb{Z}} \chi \arrow[r, "\pi_{e^i}"] \arrow[dr] & \mathbb{J} \arrow[d] \\ & U \end{tikzcd}$$

and the tensoring maps:

$$\begin{tikzcd} \mathbb{J} \arrow[r, "\eta_{e_i}"] \arrow[dr] & \mathbb{J} \otimes_{\mathbb{Z}} \chi \arrow[d] \\ & U \end{tikzcd}$$

Proposition 2.2. *[HM]*

a) *There is a canonical one to one correspondence between the forms* Ω_V *and* Ω *which is given by:*

$$\Omega = \sum_{i=1}^{v} (\pi_{e^i})^*(A(e_i)\Omega_V), \qquad \Omega_V = \sum_{i=1}^{v} [\eta^*_{e_i}(\Omega)] \otimes e^i,$$

where A *is the contraction.*

b) *The form* Ω_V *is invariant with respect to the joint* W*-action on* $\mathbb{J}$ *and* V^* *if and only if the form* Ω *is invariant under the diagonal action on* $J \otimes_{\mathbb{Z}} \chi$.

c) *There is a canonical one to one correspondance given by restriction between the* W*-invariant forms* Ω *and the forms* ω.

Definition 2.3. *Let* $\mathcal{A}$ *be the Abel map, we say that the system has rank two, if*

$$\mathcal{A}^*(\Omega_V) \wedge \mathcal{A}^*(\Omega_V) = 0,$$

as a section of $\Lambda^4(T^*\mathbb{S}) \otimes V^* \otimes V^*$.

For a basis e_i of V, let Ω_i be the contraction of Ω_V with e_i. The system has rank two iff

$$\mathcal{A}^*(\Omega_i) \wedge \mathcal{A}^*(\Omega_j) = 0,$$

for all i, j.

If the $\mathcal{A}^*(\Omega_i)$ are non-vanishing, using a theorem of Darboux, we can say that there exist functions x_i, y_i with non-vanishing differentials such that $\mathcal{A}^*(\Omega_i) = dx_i \wedge dy_i$. Note that then, $dx_i \wedge dy_i \wedge dx_j \wedge dy_j = 0$

c) Rank two integrable system of Prym varieties and associated $(v+1)$*-varieties.*

Genericity Assumption A. *The pull-back* $\mathcal{A}^*(\Omega_V)$ *is nowhere vanishing on* $\mathbb{S}$ *and its null-space is everywhere transverse to the curves* S_u.

Note that the rank two condition tells us that the span of $dx_1, dx_2, \ldots$ is at most $(v+1)$-dimensional.

Genericity Assumption B. *The span of* $dx_1, \ldots, dx_v, dy_1, \ldots, dy_v$ *is everywhere* $(v+1)$*-dimensional. More invariantly, the null-space of* Ω_V *has codimension* $(v+1)$ *everywhere.*

Proposition 2.4. *[HM] Locally, under the genericity assumptions A and B, there exists forms* $\phi_0, \phi_1, \ldots, \phi_v$ *on* X *such that* $\mathcal{A}^*(\Omega_i) = \phi_0 \wedge \phi_i$.

Genericity Assumption B'. *The null-space of* Ω_V *is of codimension* $(v+1)$ *over a dense open set* O. *Over the set* O, *this null space defines a vector bundle which extends to a globally defined* W*-invariant subbundle of the tangent* *bundle. Similarly, the subbundle of the tangent bundle defined over O as the kernel of ϕ_0 extends to all of $\mathbb{S}$ as a W-invariant subbundle of the tangent bundle.*

Theorem 2.5. *[HM] Let the system $\mathbb{P}r \to U$ have rank two, and assume that it satisfies genericity conditions A and B'. Restricting U if necessary and quotienting by the null foliation of V,*

(i) There is a $v+1$ dimensional complex manifold X into which the curves S_u all embed. It is equipped with a generically non-degenerate V^*-valued two forms Ω_V. The group W acts on X, preserving Ω_V.

(ii) Let $v > 2$. The manifold X comes equipped with a codimension 1 W-invariant foliation (cut out by $\phi_0 = 0$). The form defines a bundle map between the tangent spaces to the leaves, and the tensor product of the conormal bundle to the leaves with V^*.

(iii) X admits a W-invariant fibration to a closed curve Σ. The quotient curves S_u/W are sections of $X/W \to \Sigma$. All the quotient curves S_u/W are isomorphic to Σ.

The following theorem gives us a converse to Theorem 2.5 (the appropriate definitions are in [HM]):

Theorem 2.6. *[HM] Let X be a $v+1$ dimensional complex manifold, with a submersion onto a closed curve Σ. Let X be equipped with a minimally non-degenerate V^*-valued two form Ω_V, such that the group W acts on X, preserving Ω_V and the fibers of the map to Σ. Assume that there is a smooth W-invariant curve S_0 in X, on which W acts generically freely with quotient Σ. Then, deforming S_0 in X, the family of smooth W-invariant curves S_u defines a rank-2 integrable system of Prym varieties.*

3. The generalized Hitchin systems.

The motivating example of this work was introduced by Hitchin [Hi1,Hi2], and generalized for $Gl(n)$ by Bottacin [Bo] and Markman [Ma]. Many authors have been working on those systems, among them we have Faltings [F] and Scognamillo [Sc1,Sc2]. In [HM], the authors showed how the results described in section 2 could be applied for the case of the Hitchin systems. In section 4, we will see how this work extands to the more general case.

But first, let us recall through this section some facts about those systems. This section will be divided in two parts. In section 3.a we will recall the $Gl(n)$-case and in section 3.b we will see how they generalize to an arbitrary reductive group G.

a) The $Gl(n)$-case.

We first summarize the notation that will be used. Let:

- Σ be a closed Riemann surface of genus g,
- D a positive divisor of degree n on Σ,
- E a vector bundle over Σ of fixed degree d and rank r,
- ϕ a meromorphic $End(E)$-valued 1-form, with poles at the divisor D: $\phi \in H^0(\Sigma, End(E) \otimes K_\Sigma(D))$ where K_Σ is the canonical bundle of Σ.

Definition 3.1. *A $K_\Sigma(D)$-twisted Higgs pair (E, ϕ) is a pair consisting of a vector bundle E and a section $\phi \in H^0(\Sigma, End(E) \otimes K(D))$*

We consider over Σ the moduli spaces $\mathcal{M}(r, D, d)$ of stable $K_\Sigma(D)$-twisted *Higgs pairs* (E, ϕ). See [Bo],[Ma]. If $D = 0$, we get the case considered by

Hitchin in [Hi1],[Hi2]. There is a Poisson structure on the $\mathcal{M}(r, D, d)$ which can be defined directly or by Poisson reduction of a larger space, the cotangent bundle of the moduli space of bundles with level structure at D. See [Bo],[Ma],[HK].

Definition 3.2. *Let E a vector bundle over Σ, and D a divisor on E. A D-level structure on E is an isomorphism $\eta \in \mathrm{Isom}_{\mathcal{O}_D}(E|_D, \oplus_{i=1}^r \mathcal{O}_D)$, i.e. a trivialization of E over D.*

Let $\mathcal{U}(r, D, d)$ be the moduli space parameterizing rank r, degree d, δ-stable vector bundles with D-level structure and let $Gl(r, D)$ be the invertible $\mathcal{O}_D$-valued $r \times r$ matrices. There is a natural action of $Gl(r, D)$ on $\mathcal{U}(r, D, d)$, simply by modification of the trivialization t, which lifts to a symplectic action on the cotangent bundle $T^*\mathcal{U}(r, D, d)$.

This action has as moment map

$$\begin{aligned} \mu : T^*\mathcal{U}(r, D, d) &\to gl(r, D)^* \\ (E, t, \phi) &\mapsto \hat{\phi} \end{aligned} \tag{3.3}$$

where $\hat{\phi}$ is the expression of the polar part of ϕ over D in the t-trivialization, and $gl(r, D)^*$ is identified with $gl(r, D) \otimes_{\mathcal{O}_D} (K_\Sigma)|_D$ by a trace-residue pairing.

The quotient $T^*\mathcal{U}(r, D, d)/Gl(r, D)$ is Poisson, and is naturally identified over a open dense set with $\mathcal{M}(r, D, d)$, [Ma]. The symplectic leaves are obtained as inverse images under μ of coadjoint orbits.

For each pair (E, ϕ), we can consider the *spectral curve* S of ϕ which lies in the total space $\mathcal{K}_D$ of the line bundle $K(D)$ over Σ. It is cut out by the equation

$$\det(\phi - \zeta \mathbb{I}) = 0. \tag{3.4}$$

Here ζ represents the tautological section of $\pi^*K(D)$ over $\mathcal{K}_D$, where $\pi : \mathcal{K}_D \to \Sigma$ is the projection. If we expand (3.4) in powers of ζ, we get

$$\zeta^r + a_1\zeta^{r-1} + a_2\zeta^{r-2} + \ldots + a_r = 0, \tag{3.5}$$

where the $a_i = a_i(E,\phi)$ lie in $H^0(\Sigma, (K(D))^{\otimes i})$.

Let d_i be the dimension of $H^0(\Sigma, (K(D))^{\otimes i})$, and $u_{1,i}, \ldots u_{d_i,i}$ be a basis for $H^0(\Sigma, (K(D))^{\otimes i})$. We can then write $a_i(E,\phi)$ as

$$a_i(E,\phi) = \sum_{j=1}^{d_i} f_{j,i}(E,\phi)u_{j,i}.$$

This gives functions $f_{j,i}$ on $\mathcal{M}(r,D,d)$ which Poisson commute and define a completely integrable system on $\mathcal{M}(r,D,d)$. Joint level sets of the $f_{j,i}$ are given by fixing the spectral curve S, so that the spectral curve map $\mathcal{M}(r,D,d) \to$ (family of spectral curves) defines a Lagrangian foliation.

The leaf of the Lagrangian foliation at S is a family of line bundles on S. The sheaf L corresponding to (E,ϕ) is defined via the exact sequence of sheaves over the surface $\mathcal{K}_D$:

$$0 \to E \otimes K^*_\Sigma(-D) \xrightarrow{\phi - \zeta\mathbb{I}} E \to L \to 0. \tag{3.6}$$

When the spectral curve is smooth, L is a line bundle supported on the spectral curve. For more details about all this, one can refer to [HK].

Proposition 3.7. *[Hu1] One can then reconstruct (E,ϕ) from (S,L):*

- $E = \pi_*(L)$,

- ϕ is the map induced on E by multiplication by the tautological section ζ on L.

One can then think of the moduli space $\mathcal{M}(r, D, d)$ as the space of pairs (S, L).

In the following part, we will see that similar results can be obtained if we consider the more general case for arbitrary reductive groups G. The bundles E then gets replaced by a principal G-bundle P, and the bundle $\mathrm{End}(E)$ gets replaced by $ad(P)$. We will see that to the pair (P, ϕ), one can associate a pair consisting of a spectral curve and an H-bundle.

b) The generalized Hitchin systems for general reductive groups.

We will keep the same notations as in Part 3.a, adding to them and modifying some of them as follows:

- G a reductive connected complex group of rank r with Lie algebra $\mathfrak{g}$ and Weyl group W,
- P a G-principal bundle over Σ,
- ϕ a meromorphic $ad(P)$-valued 1-form, with poles at the divisor D: $\phi \in H^0(\Sigma, ad(P) \otimes K_\Sigma(D))$ where K_Σ is the canonical bundle of Σ.

We now consider the moduli space $\mathcal{M}(G, D, d)$ of stable Higgs pairs (P, ϕ) of fixed degree d (in fact d corresponds to an element of $\pi_1(G)$).

Let $\mathfrak{h}$ be the r-dimensional Cartan subalgebra of $\mathfrak{g}$ with corresponding group H. To any element f of $\mathfrak{g}$, one can associate the Weyl group orbit of elements which lie in the intersection of the closure of the G-orbit of f and $\mathfrak{h}$. After choosing a trivialization of P over $V \subset \Sigma$, one can do this for $f = \phi(p)$, over every point p of Σ: one then obtains a Weyl invariant curve over V. This curve does not depend on the trivialization of P. Invariantly, over Σ, one has a Weyl invariant curve S lying in the total space of the rank r bundle $K_\Sigma[D] \otimes \mathfrak{h}$

over Σ.

Let B be a Borel subgroup of G which contains H. We consider the lift q^*P of P to S. It has a reduction P_B to B, such that $q^*\phi$ lies in $ad(P_B) \otimes K[D]$, and the image of $q^*\phi$ via the map from $ad(P_B) \otimes K[D]$ to $K[D] \otimes \mathfrak{h}$ gives S. [Sc2]

Now since we have a projection from B to H, we can associate to P_B a bundle P_H. This bundle is not Weyl-invariant; it can however be twisted in a standard fashion to give rise to a bundle $\tilde{P_H}$ which is invariant under W [Sc2]. So to the pair (P, ϕ) we have associate a pair $(S, \tilde{P_H})$.[F, Sc2]

We are now going to define an integrable system on $\mathcal{M}(G, D, d)$. To do so, we need to specify the ring of Hamiltonians. As in a), for each pair (P, ϕ), we can consider the spectral curve S of ϕ which lies in the total space $\mathcal{K}_D \otimes \mathfrak{h}$ of the vector bundle $K(D) \otimes \mathfrak{h}$ over Σ. It is cut out by equations

$$p_i(h) = a_i(z),$$

where p_i forms a homogeneous basis of the W-invariant polynomials on $\mathfrak{h}$; the a_i are then sections of $(K_\Sigma(D))^{\otimes \deg(p_i)}$ over Σ. Expressing the a_i in a basis of $H^0(\Sigma, (K(D) \otimes \mathfrak{h})^{\otimes \deg p_i})$ gives rise to functions which Poisson commute and define a completely integrable system on $\mathcal{M}(G, D, d)$.

One notes that if $Pr(S)$ is the connected component containing the trivial bundle of the set of the W-invariant bundles, $\tilde{P_H}$ does not necessarily lie in $Pr(S)$ but rather in a translate of it.

Let $\widetilde{K_\Sigma[D]} \otimes \mathfrak{h}$ be the blow up of total space of the vector bundle $K(D) \otimes \mathfrak{h}$ over Σ. In fact, we blow up at the points of intersection of the spectral curve S and the inverse image in $K(D) \otimes \mathfrak{h}$ of the divisor D on Σ.

Let $\mathcal{N}$ be the moduli space consisting of pairs (P, tr), where:

- P is a holomorphic principal G-bundle of degree d

- tr is a trivialization of P over D.

The cotangent bundle $T^*\mathcal{N}$ is then the space of triples (P, tr, ϕ), where P and tr are as defined above and ϕ is a section in $H^0(\Sigma, ad(P) \otimes K_\Sigma(D))$.

Proposition 3.8. *[Ma] The moment map for the action of the level group* $G_D := [Aut(\oplus_1^n \mathcal{O}_D)]/\mathbb{C}^*$ *on* $T^*\mathcal{N}$ *is given by the polar part of* ϕ *in the trivialization* tr. $T^*\mathcal{N}/\mathcal{G}$ *is identified with* $\mathcal{M}(G, D, d)$. *The symplectic leaves are then given by fixing a coadjoint orbit* $\mathcal{O}$ *in the Lie algebra of* G_D *and asking that the polar parts of* ϕ *lie in this orbit.*

Let $\mathcal{M} = \mathcal{M}_\mathcal{O}$ be a symplectic leaf in $\mathcal{M}(G, D, d)$. Theorem 3.9 below shows that under some conditions, one can reconstruct the Higgs pair (P, ϕ) from the pair $(S, \tilde{P_H})$.

Theorem 3.9. *[F,Hi1,Hi2,Sc2] Let* S' *be an* W*-invariant deformation of* S *in* $K[D] \otimes \mathfrak{h}$ *fixed at* D*, and* $\tilde{P_H}$ *an* W*-invariant* H*-bundle over* S' *lying in* $Pr(S')$*. The variety of such pairs* $(S', \tilde{P_H})$ *is locally isomorphic to* $\mathcal{M}$ *and the projection* $(S', \tilde{P_H}) \to S$ *defines a Lagrangian foliation of an open subset of* $\mathcal{M}$*.*

One must note that curves which are deformations of S in the blow up $\widetilde{K_\Sigma[D]} \otimes \mathfrak{h}$ of $K_\Sigma[D] \otimes \mathfrak{h}$ are deformations of S in $K_\Sigma[D] \otimes \mathfrak{h}$ fixed at D.

4. The G-Hitchin systems as a rank 2 integrable system of Prym varieties .

For $Gl(n)$, given the moduli spaces $\mathcal{M}(r, D, d)$ of stable *Higgs pairs* (E, ϕ), where E is a vector bundle over Σ of fixed degree d and rank r, and ϕ is a meromorphic $End(E)$-valued 1-form, with poles at the divisor D, we can define a completely integrable system on it. In [HK], it is proved that this system (which can be seen as a rank 2 integrable system of Jacobians) corresponds to a symmetric product of a symplectic surface.

In this section, we will see following [Hi] that in the more general case, once we consider the moduli spaces $\mathcal{M}(r, D, d)$ of stable pairs (P, ϕ), where P is a holomorphic G-bundle over Σ and ϕ is a holomorphic section of $ad(P) \otimes K_\Sigma(D)$, then one have a correspondence between this system (which can be seen as a rank 2 integrable system of Pryms) and the particular variety $X = \widetilde{K_\Sigma[D]} \otimes \mathfrak{h}$.

This section will then be divided in two parts. In the first part, we will describe the Lagrangian foliation coming from the integrable system at a generic point of the moduli space. Then we will see that we also have a Lagrangian foliation on the cotangent bundle of the moduli space parameterizing rank r, degree d, δ-stable principal bundles with D-level structure. We will relate the two, then using them, we will compute the symplectic form. In the second part, we will see that the G-Hitchin system is a rank two integrable system of Prym varieties, and we will show that the variety X corresponding to it is $\widetilde{K_\Sigma[D]} \otimes \mathfrak{h}$ and compute the form Ω_V that should be associated to it.

a) Symplectic structure.

In this section, we will keep the notation introduced in section 3b). We are now going to compute the form.

Let N be the normal bundle to the curves in the space $\widetilde{K_\Sigma[D]} \otimes \mathfrak{h}$. Corresponding to the Lagrangian foliation of the Poisson manifold $\mathcal{M}(r, D, d)$, we have:

$$0 \to H^1(S, \mathcal{O} \otimes \mathfrak{h})^W \to T\mathcal{M} \to H^0(S, N)^W \to 0, \tag{4.1}$$

where the W-superscript denotes invariance.

The deformations of the spectral curve correspond to sections of the normal bundle. Since we have a natural $\mathfrak{h}$-valued two-form on the total space of $\widetilde{K_\Sigma[D]} \otimes \mathfrak{h}$ with poles at D, obtained from the cotangent structure on K, we get a map:

$$N \to K[D] \otimes \mathfrak{h},$$

$$n \to \omega(n, .)$$

which induced the following isomorphism (see [H]):

$$H^0(S, N)^W \simeq H^0(S, K_S[D] \otimes \mathfrak{h})^W$$

which allow us to write (4.1) as:

$$0 \to H^1(S, \mathcal{O} \otimes \mathfrak{h})^W \to T\mathcal{M} \to H^0(S, K_S[D] \otimes \mathfrak{h})^W \to 0. \tag{4.2}$$

We have that the tangent spaces at (S, L) to the leaf $\mathcal{L}$ in $\mathcal{M}(r, D, d)$ fit into exact sequence:

$$0 \to H^1(S, \mathcal{O} \otimes \mathfrak{h})^W \to T(\mathcal{L}) \to H^0(S, K_S \otimes \mathfrak{h})^W \to 0, \tag{4.3}$$

since we are interested in deformations of the spectral curve which have fixed intersection with $\pi^*(D)$, so that in $\mathcal{M}(r, D, d)$ one is moving along a symplectic leaf $\mathcal{L}$.

To split the sequence (4.3), one can extend the line bundle to a neighborhood of S, getting then a way of fixing the bundle while the curve varies. We then have

$$T(\mathcal{L}) = H^1(S, \mathcal{O} \otimes \mathfrak{h})^W \oplus H^0(S, K_S \otimes \mathfrak{h})^W. \tag{4.4}$$

We then define a form Ω_S by using the Serre duality pairing between $H^0(S, K_S \otimes \mathfrak{h})$ and $H^1(S, \mathcal{O} \otimes \mathfrak{h})$.

We also have a Lagrangian foliation on the cotangent bundle $T^*\mathcal{U}(r, D, d)$ given by the projection to $\mathcal{U}(r, D, d)$. The tangents to the fibers are given by elements of $H^0(\Sigma, ad(P) \otimes K_\Sigma(D))$ and on the base, deformation of the bundle along with the level structure are given to first order by elements of $H^1(\Sigma, ad(P)(-D))$. We then have the following exact sequence:

$$0 \to H^0(\Sigma, ad(P) \otimes K_\Sigma(D)) \to T(T^*\mathcal{U}(r, D, d)) \to H^1(\Sigma, ad(P)(-D)) \to 0. \tag{4.5}$$

We would like to get the following splitting:

$$T(T^*\mathcal{U}(r, D, d)) \simeq H^0(\Sigma, ad(P) \otimes K(D)) \oplus H^1(\Sigma, ad(P)(-D)). \tag{4.6}$$

To do so, we cover Σ by $n+1$ open sets, $U_0 = \Sigma - \text{support}(D)$ and $U_i, i = 1, .., n$ disjoint discs centered at the points p_i of D. Choose trivializations of P on U_0, and also trivializations on the U_i compatible with t at p_i, and let $F_{0,i}$ be the transition functions of P from U_i to U_0 for these trivializations. Now let V be a subspace of the space of cocycles for $\mathrm{ad}(P)(-D)$, mapping isomorphically to $H^1(\Sigma, \mathrm{ad}(P)(-D))$. The (P', t') near (P, t) can be obtained from transition functions $F_{0,i} \cdot exp(v_{0,i})$, with $(v_{0,i}) = v \in V$. This defines a parametrization

$$V \to \mathcal{U}(r, D, d),$$

and so a symplectic map

$$V \times V^* \to T^*\mathcal{U}(r, D, d). \tag{4.7}$$

We represent a one parameter family of elements $(E(s), t(s), \phi(s))$ of the cotangent space $T^*\mathcal{U}(r, D, d)$ by $(F_{0,i}(s), \phi_0(s), \phi_i(s))$, with

$$\phi_0(s) = \mathrm{Ad}_{F_{0,i}(s)} \phi_i(s)$$

on the overlaps $U_0 \cap U_i$, and choose $F_{0,i}(0) = F_{0,i}$. At $t = 0$, the corresponding tangent vectors are given by $v_{0,i} = F_{0,i}^{-1} \dot{F}_{0,i}, \dot{\phi}_0, \dot{\phi}_i$, with

$$\dot{\phi}_0 = F_{0,i} \cdot ([v_{0,i}, \phi_i] + \dot{\phi}_i) \cdot F_{0,i}^{-1}.$$

One can write the Serre duality pairing as

$$< v, \phi >= \sum_i tr(res_i(v_{0,i} \cdot \phi_i)),$$

where tr denotes the Killing form.

For any sections ψ_i over U_i^* define $\psi^{\&} \in H^0(\Sigma, End(E) \otimes K(D))$ by asking that for all $w \in V$

$$< w, \psi^{\&} >= \sum_i tr(res_i(w_{0,i} \cdot \psi_i). \tag{4.8}$$

Applying this to our vectors $\dot{\phi}_i \to \phi_i^{\&}$ defines the splitting (4.6).

Let $(v', \phi'), (v'', \phi'')$ represent two elements of (4.6). Using the same computations as those made in [HK], we find that the symplectic form with respect to our splitting of (4.4) is given by:

$$\Omega_\Sigma((v', \phi'), (v'', \phi'')) =< v', \phi'' > - < v", \phi' > + < [v', v''], \phi > \tag{4.9}$$

where $<, >$ denotes Serre duality on Σ.

Theorem 4.10. *Assuming S is smooth, we have $\Omega_S = \Omega_{\Sigma,\text{reduced}}$.*

PROOF: Let us then take a two parameter family $A(x, y) = (P, t, \phi)(x, y)$ of elements of $\mathcal{T}$ lying in the inverse image of $\mathcal{L}$ under the quotient $\mathcal{U} \to \mathcal{M}$, and compute the form $\Omega_\Sigma(A_x, A_y)$ on this family at $(x, y) = (0, 0)$. The fact that S is smooth implies that $\phi(\lambda)$ is regular. We can assume by genericity that the polar parts of ϕ are semi-simple.

We again cover the base curve Σ by $n+1$ open sets $U_0 = \Sigma - \text{support}(D)$, and $U_i, i = 1, ..., n$ non-intersecting discs around the points p_i in D. We assume that the curves $S(x, y)$ are unramified over U_i.

Choose trivializations of P on the U_i compatible with t, and let $F_{0,i}$ be the transition functions of P for these trivializations.

The Higgs fields are then represented by the Lie algebra valued forms $\phi_0 = \mathrm{Ad}_{F_{0,i}}(\phi_i)$ on the overlaps. The symplectic form is given by

$$\Omega_\Sigma(A_x, A_y) = \sum_i \mathrm{res}_{p_i}(\mathrm{tr}((F_i^{-1}F_{ix})\cdot(\phi_{iy}) - (F_i^{-1}F_{iy})\cdot(\phi_{ix})) + ([F_i^{-1}F_{ix}, F_i^{-1}F_{iy}]\cdot\phi))) \tag{4.11}$$

where tr denotes the Killing form and $F_i = F_{0,i}$.

Instead of computing Ω on Σ, we can lift it to the spectral curve ($\pi : S \to \Sigma$) and compute there.

Now, on the spectral curve, we have a reduction to the Borel subgroup B in such a way that ϕ lies in $\mathfrak{b}$. We have the following sequence of groups

$$0 \to N \to B \to H \to 0$$

where N is the unipotent subgroup. We then get the corresponding sequence of Lie algebras:

$$0 \to \mathfrak{n} \to \mathfrak{b} \to \mathfrak{h} \to 0$$

We fix a principal nilpotent element e in $\mathfrak{n}$, then any element $h \in \mathfrak{h}$ has a unique representative in $\mathfrak{b}$ of the form $e+h$, up to the action of the Weyl group. We then choose a trivialization of P over U_0 such that $\phi(z) = e + h_0(z)$, $z \in S$; this requires a genericity assumption on P which is implied by smoothness.

On the disc U_i, restricting if necessary, we can conjugate to $\mathfrak{h}$, and then write ϕ as $h_i(z) \in \mathfrak{h}$. On the overlap of U_0 and U_i, we have

$$h_i(z) = F_{0,i}(z)(e + h_0(z)).$$

$F_{0,i} \in B$ can be split as $F_{0,i}(z) = F_{0,i}^N(z)F_{0,i}^H(z), F_{0,i}^H \in H, F_{0,i}^N \in N$, and we then get:

$$F_i^{-1}F_{ix} = (F_i^H)^{-1}(F_{ix}^H) + (F_i^H)^{-1}(F_i^N)^{-1}(F_{ix}^N)^{-1}F_i^H. \tag{4.12}$$

The second term in (4.12) lies in $\mathfrak{n}$, and so gives zero when paired with elements of $\mathfrak{h}$. In (4.11),$F_i^{-1}F_{ix}$ and $F_i^{-1}F_{iy}$ both lies in $\mathfrak{b}$, so the commutator lies in $\mathfrak{n}$ and will then give zero when paired with elements of $\mathfrak{h}$.

Using this, we can evaluate the symplectic form. We have

$$< a_1, b_2 >_S - < a_2, b_1 >_S = res_p[tr[(h_1)_x((F_i^H)^{-1}F_{iy}^H) - (h_1)_y((F_i^H)^{-1})F_{ix}^H]] \tag{4.13}$$

The formula (4.13) gives the explicit expression of Ω_S on $H^0(S, K_S \otimes \mathfrak{h})^W \oplus H^1(S, \mathcal{O} \otimes \mathfrak{h})^W$, and so proves the theorem.

We can then write the moduli space as an integrable system of Prym varieties with the symplectic form Ω_S defined above.

b) The G-Hitchin systems as a rank 2 integrable system of Prym variety.

The local structure of the integrable systems is that of the fibration $\mathbb{P}r \to U$ by generalized Prym varieties over a set U parameterizing a family of W-invariant curves.

In this part, we will prove the following theorem:

Theorem 4.14. *The generalized Hitchin system for an arbitrary reductive group is of rank two at every smooth spectral curve, and the variety associated to it is $K_\Sigma\widetilde{[D]} \otimes \mathfrak{h}$: it comes equipped with a $\mathfrak{h}$-valued two form $\Omega_\mathfrak{h}$.*

Now, fix a spectral curve S_0 and a bundle P_H^0. Let (S, P_H) be a nearby point. Choose an extension of P_H to a neighborhood of S_0 in $K_\Sigma[D] \otimes \mathfrak{h}$. We can then write all H-bundles as $P_H^0 \otimes P'_H$, were P'_H has degree zero and is W-invariant. The curves lie in the total space of $K_\Sigma[D] \otimes \mathfrak{h}$ which comes equipped with a meromorphic 2-form ω with poles at D.

We have previously seen that our symplectic forms Ω_Σ and Ω_S, where Ω_S is defined using the Serre duality between $H^0(S, K \otimes \mathfrak{h})^W$ and $H^1(S, \mathcal{O} \otimes \mathfrak{h})^W$ are equal on $\mathbb{P}r$. We then get equality of the corresponding $\mathfrak{h}$-valued two form $\Omega_{V,\Sigma}$ and $\Omega_{V,S}$ on the corresponding associated family of Jacobians. Set $\Omega_V = \Omega_{V,S}$.

We then take the pull back $\mathcal{A}(\Omega_V)$ of this form to the space $\mathbb{S} \to U$ under the Abel map. Let p_0 be the intersection of S_u with a fixed fiber $\pi^{-1}(\lambda)$, where $\pi : K_\Sigma[D] \otimes \mathfrak{h} \to \Sigma$, and a fixed Weyl chamber. We choose p_0 to be our base point for S_u. Through the Abel map, we associate to a point p away from the branch points in a curve S_u, the line bundle corresponding to the divisor $p - p_0$ and averaging under the Weyl group. The projection of S to Σ gives uniform coordinates on all curves S_u and allows us to split $T\mathbb{S}$ as $TU \oplus T\Sigma$ away from the branch points of $S_u \to \Sigma$. We identify $T\Sigma$ with $T\mathbb{S}_u$. On the other hand, TU is in fact $H^0(S, N)^W$. Let $X = \widetilde{K_\Sigma[D]} \otimes \mathfrak{h}$ be the blow up of the vector bundle $K(D) \otimes \mathfrak{h}$ over Σ at the points of intersection of the spectral curve S and the inverse image in $K(D) \otimes \mathfrak{h}$ of the divisor D on Σ. The space $\mathbb{S}$ maps to X, and corresponding by, there is a map of normal bundles of the curves S_u, which is simply the evaluation map:

$$H^0(S, N)^W \to N_{S_u}, \tag{4.15}$$

where N_{S_u} is the normal bundle of S_u in X.

We thus map TU to the normal bundle N_{S_u}. The symplectic form is obtained by first mapping vectors in TU to N_{S_u}, then using the $\mathfrak{h}$-valued symplectic form on $X = \widetilde{K_\Sigma[D]} \otimes \mathfrak{h}$ to map to $K_{S_u}[D] \otimes \mathfrak{h}$, and finally pairing with TS_u. In particular, the vectors in the kernel of (4.15) are isotropic, showing that the $\mathfrak{h}$-valued symplectic form on $\mathbb{S}$ is lifted from X.

Acknowledgments: I would like to express my gratitude to my PhD advisor, Professeur J. Hurtubise, for suggesting this problem and for many helpful discussions that we had during the preparation of this article. I would like to thank Professor E. Markman for valuable suggestions.

References

[AHH] M.R. Adams,J. Harnad, and J. Hurtubise, *Isospectral Hamiltonian flows in finite and infinite dimensions* II. *Integration of flows*, Comm. Math. Phys. **134** (1990), 555–585.

[Bo] F. Bottacin, *Symplectic geometry on moduli spaces of stable pairs,* Ann. Sci. Ecole Norm. Sup. (4) **28** (1995), no. 4, 391-433.

[F] G. Faltings, Jour. Alg. Geom. **2** (1993), 507.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry,* Wiley, New-York, (1978).

[Hi1] N.J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 59–126.

[Hi2] N.J. Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987), no. 1, 91–114.

[HK] J. Hurtubise and M. Kjiri, *Separating coordinates for the generalized Hitchin systems and the classical r-matrices*, To appear in Comm. Math. Phys. (2000).

[HM] J. Hurtubise and E. Markman, *Rank* 2 *integrable systems of Prym varieties*, Adv. Theor. Math. Phys. **2** 633–695 (1998).

[Hu1] J. Hurtubise, *Integrable systems and algebraic surfaces*, Duke Math. J. **83** (1996), no. 1, 19–50.

[Hu2] J. Hurtubise, *The algebraic geometry of the Kostant-Kirillov form*, Jour.

London. Math. Soc. (2)**56** (1997), 504–518.

[M] Moser, J., "Geometry of Quadrics and Spectral Theory", *The Chern Symposium, Berkeley, June 1979*, 147–188, Springer, New York, (1980).

[Ma] E. Markman, *Spectral curves and integrable systems*, Compositio Math. **93**, 255-290, (1994).

[NS] M. Narasimhan and C.S. Seshadri, *Stable and Unitary vector bundles on a compact Riemann surface*, Annals of Math **82**, 540-567.

[RS1] A.G. Reiman and M.A. Semenov-Tian-Shansky, *Reduction of Hamiltonian systems, affine Lie algebras and lax equations* I, II, Invent. Math. **54** (1979), no. 1, 81–100; ibid. **63** (1981), no. 3, 423–432.

[RS2] A.G. Reiman and M.A. Semenov-Tian-Shansky, *Integrable Systems II*, chap.2, in "Dynamical Systems VII", Encyclopaedia of Mathematical Sciences, vol 16., V.I. Arnold and S.P.Novikov, eds., Springer-Verlag, Berlin, 1994.

[Sc1] R. Scognamillo, *Prym-Tjurin varieties and the Hitchin map*, Math. Ann. **303**, 47-62, (1998).

[Sc2] R. Scognamillo, *An Elementary Approach to the Abelianization of the Hitchin System for Arbitrary reductive Groups*, Compositio Math. **110**, 17-37, (1998).

M.Kjiri

Centre de Recherches Mathématiques

Université de Montréal

email: kjiri@crm.umontreal.ca

CONCLUSION

Une remarque tout d'abord se rapportant à un théorème démontré par Ron Donagi et Eyal Markman dans "Spectral covers, algebraically completely integrable Hamiltonian systems, and moduli of bundles", dont l'énoncé est le suivant:

Soit X une variété munie d'une 2-forme ω qui admet des pôles en un diviseur D. Soit B un sous-espace qui paramétrise les courbes irréductibles sur X qui ne sont pas dans D. Les auteurs ont démontré que le fibré Picard relatif $h : \mathcal{M} \to B$ était muni d'une structure de Poisson et que l'application $h : \mathcal{M} \to B$ était une fibration Lagrangienne. Les feuilles symplectiques sont alors obtenues en fixant l'intersection d'une courbe S avec D.

Les résultats du Chapitre II nous donnent précisément cette situation: dans notre cas, $X = \mathcal{K}(D)$ et les variétés Lagrangiennes sont les courbes spectrales.

En conclusion, nous pourrions revenir sur certaines questions soulevées à travers cette thèse et auxquelles nous pourrions nous attaquer.

À travers l'article "Separating coordinates for the generalized Hitchin systems and the classical R-matrices", nous avons porté une attention particulière aux cas où la surface de Riemann est une courbe elliptique ou une courbe nodale rationnelle, le cas où la surface de Riemann est la sphère de Riemann ayant déjà été traité [AHH]. Ces trois cas mènent à une décomposition de l'algèbre de lacets des matrices en une somme de deux sous-algèbres qui sont duales l'une de l'autre, permettant ainsi de définir un système intégrable en utilisant soit le théorème de Adler-Kostant-Symes, soit le formalisme de la matrice R. Selon que la courbe soit rationnelle, elliptique ou nodale rationelle, on obtient respectivement les matrices R rationnelle, elliptique ou trigonométrique. Ces matrices R peuvent être utilisées pour définir de nouvelles structures de Poisson, le crochet quadratique ou de Sklyanin. Nous en obtenons encore trois sortes: rationnel, elliptique ou trigonométrique.

On définit une nouvelle fois un système intégrable en utilisant les courbes spectrales: les feuilles des fibrations Lagrangiennes sont les même que ce soit pour la structure de Poisson ou la structure de Sklyanin. Les feuilles symplectiques sont cependant différentes.

La question que l'on se pose alors, est de savoir s'il existe un analogue de la séparation des variables pour le cas quadratique.

Dans le cas où le crochet est quadratique rationnel, le problème a été résolu par D.R.D. Scott [Sc]. On considère une courbe spectrale, un fibré en droite et une section qui pemet d'obtenir un diviseur Σp_μ sur la courbe spectrale. Cette courbe vit dans le même espace total que dans le cas du crochet de Poisson, soit l'espace total du fibré en droite $\mathcal{K}(D)$ au-dessus de la courbe $\mathbb{P}^1$. Le diviseur permet alors d'obtenir un isomorphisme I entre les feuilles symplectiques et un produit symétrique de cette surface. Soit z une coordonnée sur $\mathbb{P}^1$, ζ la coordonnée cotangente correspondante, et soit $a(z)$ telle que $a(z) = 0$ rencontre le diviseur D. Dans le cas du crochet de Lie-Poisson, la forme $a(z)^{-1}dz \wedge d\zeta$ sur $\mathcal{K}(D)$ induit une forme symplectique sur le produit symétrique et l'isomorphisme I est alors symplectique. Dans le cas du crochet rationnel quadratique, on obtient des résultat similaires mais avec la forme $\zeta^{-1}dz \wedge d\zeta$.

Nous devrions obtenir des résultats analogues pour le crochet de Sklyanin dans les cas elliptiques et trigonométriques. On devrait encore avoir des coordonnées de diviseurs sur une surface $\mathcal{K}(D)$ definie au dessus d'une courbe elliptique ou nodale. Au lieu de considérer la forme symplectique sur $\mathcal{K}(D)$ avec des pôles le long de $\pi^{-1}(D)$, on choisit la forme avec un pôle le long de la zéro-section. En prenant des coordonnées de Darboux (z, ζ) pour cette forme, puis en écrivant le diviseur qui correspond à la paire (courbe, fibré en droite) en fonction de ces coordonnées comme une somme $\Sigma(z_\mu, \zeta_\mu)$, nous devrions obtenir les coordonnées de Darboux séparées.

BLIBLIOGRAPHIE

[A] M. Atiyah, *Vector bundles over an elliptic curve*, Proc. Lond. Math. Soc **7** (1957), 414–452.

[AB] M. Atiyah, R. Bott, *The Yang-Mills equations over Riemann surfaces,* Phil. Trans. R. Soc. Lond. A, **308**(1982), 523-615.

[AHH1] M.R. Adams, J. Harnad and J. Hurtubise, *Darboux coordinates and Liouville-Arnold integration in loop algebras*, Comm. Math. Phys. **155** (1993), no. 2, 385–413.

[AHH2] M.R. Adams, J. Harnad and J. Hurtubise, *Isospectral Hamiltonian flows in finite and infinite dimensions* II. *Integration of flows*, Comm. Math. Phys. **134** (1990), 555–585.

[AHP] M.R. Adams, J. Harnad and E. Previato, *Isospectral Hamiltonian flows in finite and infinite dimensions* I. *Generalised Moser systems and moment maps into loop algebras*, Comm. Math. Phys. **117** (1988), no. 3, 451–500.

[AKS] M. Adler, *On a trace functional for formal pseudo-differential operators and the symplectic structure of the Korteweg-de Vries equation*, Invent. Math. **50** (1978/79), no. 3, 219–248; B. Kostant, *The solution to a generalized Toda lattice and representation theory*, Adv. Math. **34** (1979), no. 3, 195–338;

W. Symes, *Systems of Toda type, inverse spectral problems and representation theory*, Invent. Math. **59** (1980), 13–51.

[AvM] M. Adler and P. van Moerbeke, *Completely integrable systems, Euclidean Lie algebras, and curves*, Adv. in Math. **38** (1980), no. 3, 267-317; *Linearization of Hamiltonian systems, Jacobi varieties and representation theory*, ibid. **38** (1980), no. 3, 318–379.

[Bo] F. Bottacin, *Symplectic geometry on moduli spaces of stable pairs,* Ann. Sci. Ecole Norm. Sup. (4) **28** (1995), no. 4, 391-433.

[DM] R. Donagi and E. Markman, *Spectral Covers, Algebraically Completely Integrable Hamiltonian Systems, and Moduli of Bundles,* Lect. Notes in Math. **1620** (1996), 1, Springer 1996.

[F] G. Faltings, *Stable G-bundles and projective connections,* Jour. Alg. Geom. **2** (1993), 507.

[FT] L.D. Faddeev and L.A. Takhtajan, *Hamiltonian methods in the theory of solitons*, eds., Springer-Verlag, Berlin, 1987.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, New-York, (1978).

[HH] J. Harnad and J. Hurtubise, *Generalised tops and moment maps into loop algebras*, J. Math. Phys. **37** (1991), no. 7, 1780–1787.

[Hi1] N.J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 59–126.

[Hi2] N.J. Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987), no. 1, 91–114.

[HK] J. Hurtubise and M. Kjiri, *Separating coordinates for the generalized Hitchin systems and the classical r-matrices*, To appear in Comm. Math. Phys. (2000).

[HM] J. Hurtubise and E. Markman, *Rank* 2 *integrable systems of Prym varieties*, Adv. Theo. Math. Phys., **2** (1998), 633–695.

[Hu1] J. Hurtubise, *Integrable systems and algebraic surfaces*, Duke Math. J. **83** (1996), no. 1, 19–50.

[Hu2] J. Hurtubise, *The algebraic geometry of the Kostant-Kirillov form*, J. Lond. Math. Soc. (2), **56** (1997), 504–518.

[M] Moser, J., "Geometry of Quadrics and Spectral Theory", *The Chern Symposium, Berkeley, June 1979*, 147-188, Springer, New York, (1980).

[Ma] E. Markman, *Spectral curves and integrable systems*, Compositio Math. **93**, 255-290, (1994).

[NS] M. Narasimhan and C.S. Seshadri, *Stable and Unitary vector bundles on a compact Riemann surface*, Annals of Math **82**, 540-567.

[RS1] A.G. Reiman and M.A. Semenov-Tian-Shansky, *Reduction of Hamiltonian systems, affine Lie algebras and lax equations* I, II, Invent. Math. **54** (1979), no. 1, 81–100; ibid. **63** (1981), no. 3, 423–432.

[RS2] A.G. Reiman and M.A. Semenov-Tian-Shansky, *Integrable Systems II*, chap.2, in "Dynamical Systems VII", Encyclopaedia of Mathematical Sciences, vol 16., V.I. Arnold and S.P.Novikov, eds., Springer-Verlag, Berlin, 1994.

[S] E.K. Sklyanin, *Poisson structure of a periodic classical XYZ-chain*, J. Sov. Math. , 46 (1989) 1664-1683.

[Sc] D.R.D. Scott, *Classical functional Bethe ansatz for $SL(N)$:separation of variables for the magnetic chain*, J. Math. Phys. bf 35, 5831-5843 (1994)

[Sc1] R. Scognamillo, *Prym-Tjurin varieties and the Hitchin map*, Math. Ann. **303**, 47-62, (1998).

[Sc2] R. Scognamillo, *An Elementary Approach to the Abelianization of the Hitchin System for Arbitrary reductive Groups*, Compositio Math. **110**, 17-37, (1998).