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Holomorphic Mappings: Expansion, Distortion and Approximation

par

Mohamad Reza Pouryayevali

Départment de mathématiques et de statistique
Faculté des arts et sciences

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présentée par

Mohamad Reza Pouryayevali

a été évaluée par un jury composé des personnes suivantes:

(Jean Marc Terrier)
(président-rapporteur)

(Paul M. Gauthier)
(directeur de recherche)

(Qazi Ibadur Rahman)
(membre du jury)

(Manfred Stoll)
(examinateur externe)

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Sommaire

Les applications holomorphes sont étudiées entre des variétés surtout equidimensionelles en considérant le comportement des vecteurs tangents sous l'action des différentielles de ces applications. Les métriques différentielles définies sur ces variétés pour mesurer les longueurs des vecteurs tangents représentent le language le plus efficace dans l'étude des expansions et des distortions des applications holomorphes. En particulier, dans l'étude des applications holomorphes avec expansions bornés, quelques théorèmes classiques de la théorie des fonctions complexe à une variable sont généralisés. Une sous-classe de fonctions à expansion bornée, appelée fonctions semi-Bloch, est caractérisée dans des domaines de $\mathbb{C}^n$. Dans l'étude des applications à distortion bornée, en exhibant un contre-exemple, nous apportons une réponse négative à la question de l'existence d'un théorème de Bloch pour cette classe. Dans le cas d'une variable, les applications (fonctions) locales sont approchées par des applications globales avec zéros prescrits et une propriété d'injectivité.

Abstract

Holomorphic mappings are studied between mostly equidimensional manifolds considering the behavior of the tangent vectors under the differential of these mappings. The differential metrics defined on these manifolds to measure the length of the tangent vectors is the most efficient tool in the study of expansion and distortion of holomorphic mappings. In particular in the study of holomorphic mappings with bounded expansion, certain classical theorems in complex function theory of one variable are generalized. A subclass of holomorphic functions with bounded expansion, called semi-Bloch functions is characterized on domains of $\mathbb{C}^n$. In the study of mappings with bounded distortion, by giving a counterexample, the question of the existence of Bloch's theorem for this class, will be answered negatively. In the one variable case, local mappings (functions) are approximated by global ones with prescribed zeros and a certain injectivity property.

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Contents

Introduction

The Riemman mapping theorem is one of the most important theorems in complex analysis. This theorem asserts that except for $\mathbb{C}$, every simply connected domain in $\mathbb{C}$ is biholomorphic to the unit disk. Poincaré proved that for $n > 1$, the n-polydisk $\mathbb{D}\times\mathbb{D}\times\cdots\times\mathbb{D}$ and the unit ball in $\mathbb{C}^n$ are not biholomorphic. This shows that for $n > 1$, the topological property simple connectedness is not sufficient to guarantee that, for example, bounded domains in $\mathbb{C}^n$ are biholomorphic.

In 1926 C. Carathéodory introduced a pseudo-distance C on domains of $\mathbb{C}^n$, which is "invariant" under biholomorphic mappings. Applying this object one can decide whether two given domains are biholomorphic. By a distance we mean a metric as we have on metric spaces. The words "metric" and "differential metric" will be used for other objects such as Hermitian metrics.

Carathéodory proved that if $\Omega_1 \subset \mathbb{C}^n$, $\Omega_2 \subset \mathbb{C}^m$ are domains and $f : \Omega_1 \longrightarrow \Omega_2$ is holomorphic, then f is distance decreasing, i. e. for each x, $y \in \Omega_1$

$$C(f(x), f(y)) \leq C(x, y).$$

In particular, if f is biholomorphic on $\Omega_1 \subset \mathbb{C}^n$, then $C(f(x), f(y)) = C(x, y)$. This means Carathéodory pseudo-distance C is invariant under biholomorphic mappings.

In 1967 S. Kobayashi, by dualizing Carathéodory's construction, introduced a new pseudo-distance K. On the unit disk $\mathbb{D}$ in the complex plane, these two pseudo-distances coincide with the Poincaré -Bergman distance ρ which is invariant under automorphisms of the disk. Kobayashi pseudo-distance K is the largest pseudo-distance on $\Omega \subset \mathbb{C}^n$, such that all holomorphic mappings $f : (\mathbb{D}, \rho) \longrightarrow (\Omega, K)$ are distance decreasing.

The pseudo-distance K has applications in a number of areas, including diophantine geometry [40] and value distribution theory of holomorphic functions [47]. For example in generalizing Picard's theorems to higher dimensions, this invariant distance has been applied as a principal tool.

A Riemann surface is called hyperbolic if its universal covering space is the unit disk. Most Riemann surfaces are hyperbolic. In particular every compact Riemann surface of genus ≥ 2 is hyperbolic. On the other hand the Kobayashi pseudo-distance K can be defined on every manifold. For a compact Riemann surface X of genus ≥ 2, K_X coincides with the distance on X obtained from a Hermitian metric of negative curvature. This was the motivation for Kobayashi to call a complex manifold M "hyperbolic".

A complex manifold M is said to be hyperbolic if the pseudo-distance K_M is a distance on M. The infinitesimal form k_M of K_M which is called the Kobayashi-Royden differential metric, was introduced by Kobayashi [36] and Royden [52]. Invariant pseudo-distances, their infinitesimal versions and their applications have been investigated for three decades [37], [38].

The Poincaré-Bergman metric is a powerful tool in function theory on the unit disk in the complex plane. Applying the natural generalization of this metric, we wish to extend various concepts in the classical case on the unit disk to bounded domains in $\mathbb{C}^n$ and hyperbolic manifolds.

Bloch and normal functions are holomorphic functions defined on the unit disk having bounded expansion with respect to the ordinary metric on $\mathbb{C}$ and spherical metric on the Riemann sphere $\mathbb{C}_\infty$, respectively. The literature on Bloch and normal functions in one complex variable is extensive. But in several complex variables it is less developped.

K. T. Hahn applied Hermitian metrics on the unit ball in $\mathbb{C}^n$ [28], [29] and on Hermitian manifolds [30] to investigate the extension of certain problems of classical function theory. R. M. Timony generalized the notion of Bloch functions in several complex variables to functions on bounded homogeneous domains with Bergman metric [55], [56]. Hahn [31] studied the asymptotic behavior of normal mappings of several complex variables. He applied the Kobayashi-Royden metric on hyperbolic manifolds to define normal mappings. S. Krantz [39] considered Bloch functions on strongly pseudoconvex domains with Kobayashi-Royden metric.

We define Bloch and normal mappings on hyperbolic manifolds by considering mappings of bounded expansion on hyperbolic manifolds. We distinguish between Bloch and normal mappings according to non-compactness or compactness of the target manifold.

In 1925 Bloch proved that there exists a positive number β such that every holomorphic function on the unit disk $\mathbb{D}$ with the normalization $f'(0) = 1$ maps some subdomain of $\mathbb{D}$ biholomorphically onto a disk of radius β. The "largest" such β is called the Bloch constant. Many investigations have been done to find the precise value of the Bloch constant [2], [4], [11], [13].

We study the qualitative aspect of Bloch's theorem in several complex variables. Bochner initiated the study of Bloch's theorem in higher dimensions. He showed that one has a Bloch constant if one imposes additional restrictions on the class of functions considered. In fact the Bloch constant in higher dimensions for general normalized holomorphic mappings does not exist. H. Wu in [60] among other things reproved that, for a subclass of holomorphic mappings on $\mathbb{B}^n$, called quasiregular holomorphic mappings, a Bloch constant exists. S. S. Chern [14], applying differential geometric techniques, considered this problem for certain Hermitian manifolds. R. Greene and H. Wu [27] proved the existence of a Bloch constant for meromorphic functions. K. T. Hahn [28], [29], [30] considered different classes of holomorphic mappings such as the family of bounded holomorphic mappings and he estimated the Bloch constant for these families. D. Minda [43] studied holomorphic and meromorphic functions on hyperbolic Riemann surfaces using conformal metrics and gave upper and lower estimates for Bloch constants in this context.

X. Liu [41] considered Bloch mappings on the unit ball in $\mathbb{C}^n$ and applying Bonk's method [11], obtained lower and upper bounds for Bloch constants for various subfamilies of Bloch mappings defined on $\mathbb{C}^n$. C. FitzGerald and S. Gong [19] studied such holomorphic mappings on classical domains.

In 1928 Grotzsch introduced quasiconformal mappings in the complex plane, which were a generalization of the notion of conformal mappings. Quasiregular mappings in $\mathbb{R}^n$ are a generalization of analytic functions of one complex variable. A more general situation is the case that a mapping $f : \Omega \longrightarrow \mathbb{C}^n$ is simultaneously holomorphic and quasiregular, where $\Omega \subset \mathbb{C}^n$. Bloch's theorem fails for the class $\mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$ of general holomorphic mappings, for $n > 1$. Bochner proved that Bloch's theorem does hold for the subclass of quasiregular holomorphic mappings.

On the space of holomorphic mappings on the unit ball in $\mathbb{C}^n$, we define a function β and we prove that this function is lower semi-continuous on the space of quasiregular holomorphic mappings. As an application of this theorem, we will prove the existence of a Bloch constant for quasiregular holomorphic mappings. By a counterexample we will prove directly that Bloch's theorem does not hold for the class of quasiregular smooth mappings defined on the unit disk $\mathbb{D}$ in the complex plane.

Bloch's theorem leads to strong geometric information on how entire functions must behave. For example, it implies Picard's theorem. In order to prove the sharpness of many theorems, one needs to know how an entire function may behave. That is, one needs to show the existence of entire functions having prescribed behaviour. One of the most powerful tools for this purpose is the theory of complex approximation. Using approximation, we shall show the existence of an entire holomorphic function having an arbitrarily prescribed sequence of "biholomorphic disks" in its image.

In chapter four we will consider the problem of approximation of functions defined on a closed set $F \subset \mathbb{C}$ by meromorphic functions with prescribed poles and zeros outside the set of approximation.

The spectacular theorem of Picard may be the prime mover, historically underlying the theorems of this thesis: hyperbolicity, expansion, distortion, and approximation.

Indeed, the study of Bloch functions (functions of bounded expansion) leads to Bloch's theorem which implies Picard's theorem. The introduction of hyperbolic manifolds gave rise to a beautiful generalization of Picard's theorem to higher dimensions. Picard's theorem was highly refined by value distribution theory (Nevanlinna theory). The problem of showing that this refinement was "sharp" (in some sense, ultimate), was known as the inverse problem of Nevanlinna theory. To attack the inverse problem Arakelian developed the theory of approximation on unbounded sets. Finally, the inverse problem was completely settled by Drasin [17] using quasiregular mappings (mappings of bounded distortion).

Chapter 1

Preliminaries

1.1 Basic properties of hyperbolic manifolds

The manifolds considered throughout this dissertation are connected and second countable. Our main references for this preliminary chapter are [40], [47] and [60].

Let M be a complex manifold and TM be its complex tangent bundle. A differential metric is a mapping $F : TM \longrightarrow \mathbb{R}_{\geq 0}$ satisfying the following conditions:

$$F(aX_x) = |a|F(X_x), \text{ for any } X_x \in T_xM \text{ and } a \in \mathbb{C}.$$

Moreover, if F is continuous and for each tangent vector $X_x \in T_x - \{0_x\}$, $F(X_x) > 0$, then we call F a Finsler metric.

On complex connected manifolds of dimension one, that is, on Riemann surfaces a differential metric is a conformal metric.

Indeed, let $\{(U_i, z_i) : i \in I\}$ be a covering of a Riemann surface M by coordinate neighbourhoods. For each $j \in I$ consider the holomorphic local frame

$$s_j(x) := \frac{d}{dz_j}|_x \qquad \text{on } U_j.$$

If $U_i \cap U_j \neq \phi$, then $T_{ij} := dz_i/dz_j$ is holomorphic non-vanishing and

$$s_j(x) = T_{ij}(x)s_i(x).$$

Let $F : TM \longrightarrow \mathbb{R}_{\geq 0}$ be a differential metric on the Riemann surface M. Then define $\rho_i : U_i \longrightarrow \mathbb{R}_{\geq 0}$ by $\rho_i(x) := F(s_i(x))$, for each $i \in I$. It follows that, for each $x \in U_i \cap U_j \neq \phi$,

$$\begin{aligned} \rho_j(x) = F(s_j(x)) &= F(T_{ij}(x)s_i(x)) \\ &= |T_{ij}(x)|F(s_i(x)) \\ &= |T_{ij}(x)|\rho_i(x). \end{aligned} \tag{1.1}$$

Sometimes the last equality in (1.1) is expressed by $\rho_j(z_j) = |dz_i/dz_j|\rho_i(z_i)$. In the literature the conformal metric ρ is often denoted by $\rho(z)|dz|$.

The following example is one of the most important differential metrics on a complex manifold.

Let M be a complex manifold and consider $k_M : TM \longrightarrow \mathbb{R}_{\geq 0}$ as follows. For each $X_x \in T_xM$, define

$$k_M(X_x) := \inf\{a > 0 : \exists \varphi : \mathbb{D} \longrightarrow M, \varphi(0) = x \text{ and } \varphi'(a(\frac{\partial}{\partial z})_0) = X_x\},$$

where $\mathbb{D}$ is the unit disk in $\mathbb{C}$.

The proof of the following lemma can be found in [47] or in [52].

Lemma 1.1.1 *The mapping $k_M : TM \longrightarrow \mathbb{R}_{\geq 0}$ is a differential metric.*

k_M is called the Kobayashi-Royden differential metric. On the unit disk $\mathbb{D}$ in the complex plane, $k_{\mathbb{D}}$ coincides with the Poincaré metric

$$\rho(z, \xi) = \frac{|\xi|}{1 - |z|^2}.$$

For general complex manifolds the continuity of k_M is not known, but we have the following theorem proved by Royden [52].

Theorem 1.1.2 *Let M be a complex manifold. Then the Kobayashi-Royden differential metric k_M is upper semicontinuous on TM, i. e. for each $\epsilon > 0$ and $X \in TM$, there is a neighbourhood U of X in TM such that, for each $Y \in U$, $k_M(Y) < k_M(X) + \epsilon$.*

Note that a Hermitian metric on the tangent bundle of a complex manifold gives a differential metric, in fact, if $h : TM \times TM \longrightarrow \mathbb{C}$ is a Hermitian metric, then h is positive definite and smooth, so

$$\tilde{h}(X_p) := h(X_p, X_p)^{1/2},$$

is a Finsler metric. We will consider this Finsler metic on TM, when we refer to a Hermitian manifold M.

On every bounded domain $\Omega \subset \mathbb{C}^n$, there exists a canonical Hermitian metric called the Bergman metric, namely, we have $h : T\Omega \times T\Omega \longrightarrow \mathbb{C}$, defined by

$$h_z(\xi, \eta) := \sum_{i,j=1}^{n} h_{i,j}(z)\xi_i\bar{\eta}_j, \qquad z \in \Omega,\ \xi, \eta \in \mathbb{C}^n,$$

where the entries of the Hermitian matrix $[h_{i,j}(z)]$ with respect to the usual basis of $\mathbb{C}^n$ are

$$h_{i,j} := \frac{1}{2}\frac{\partial^2}{\partial z_j \partial \bar{z}_j} \log B(z, \bar{z}),$$

for $1 \leq i, j \leq n$, where $B(z, \bar{z})$ is the Bergman kernel function for Ω. The kernel function is defined as

$$B(z, \bar{z}) = \sum_{n=1}^{\infty} \phi_n(z)\overline{\phi_n(z)},$$

where $\{\phi_n\}_{n=1}^{\infty}$ is an arbitrary orthonormal basis for the Hilbert space

$$\{f : \Omega \longrightarrow \mathbb{C} :\ f \text{ holomorphic },\ \int_{\Omega} |f(z)|^2 d\mu < \infty\}.$$

μ is the Lebesgue measure on $\mathbb{C}^n$. $B(z, \bar{z})$ is continuous and positive on Ω.

The following theorem asserts that the Kobayashi-Royden differential metric is "contracted" by holomorphic mappings and hence invariant under biholomorphic mappings.

Theorem 1.1.3 *Let M and N be complex manifolds and $f : M \longrightarrow N$ a holomorphic mapping. Then $k_N(f'(X_x)) \leq k_M(X_x)$. In particular, if f is biholomorphic, then $k_N(f'(X_x)) = k_M(X_x)$*

Let $\gamma : [0,1] \longrightarrow M$ be a piecewise smooth curve. Define

$$L_M(\gamma) := \int_0^1 k_M(\gamma'(t))dt,$$

where $\gamma'(t) := \gamma_*((\frac{d}{dt})_t)$ is the differential of γ at the point t. Since $k_M : TM \longrightarrow \mathbb{R}_{\geq 0}$ is upper semicontinuous $k_M(\gamma'(t))$ is Lebesgue integrable and $L_M(\gamma)$ is finite.

For $x, y \in M$, we define

$$K_M(x,y) := \inf\{L_M(\gamma) : \gamma \text{ is a piecewise smooth curve joining } x \text{ and } y\}.$$

We have:

- $K_M(x,x) = 0,$
- $K_M(x,y) = K_M(y,x),$
- $K_M(x,z) \leq K_M(x,y) + K_M(y,z).$

Therefore K_M is a pseudo-distance.

This pseudo-distance is called the Kobayashi pseudo-distance. From Theorem 1.1.3 we have the following distance decreasing principle which is one of its most important properties.

Theorem 1.1.4 *Let M and N be complex manifolds and $f : M \longrightarrow N$ a holomorphic mapping. Then for each $x, y \in M$ we have*

$$K_N(f(x), f(y)) \leq K_M(x,y).$$

In particular, if f is biholomorphic, then $K_N(f(x), f(y)) = K_M(x,y)$.

A consequence of this theorem is that the Kobayashi pseudo-distance is continuous.

Definition 1.1.5 *A complex manifold M is called a hyperbolic manifold if the Kobayashi pseudo-distance K_M is a distance. Moreover, if K_M is a complete distance, then M is called a complete hyperbolic manifold.*

Let M be a Riemann surface. The universal covering space $\tilde{M}$ of M is a simply connected Riemann surface, so it is $\mathbb{C}_\infty$, $\mathbb{C}$ or the unit disk $\mathbb{D}$. Consider a Riemann surface M with the universal covering $\mathbb{D}$ and the covering projection $\pi : \mathbb{D} \longrightarrow M$. The Kobayashi-Royden metric k_M is obtained by projecting the Poincaré metric ρ to M [47, p. 7]. Let $x, y \in M$ be arbitrary points and take $\tilde{x} \in \mathbb{D}$ with $\pi(\tilde{x}) = x$. Then we have

$$K_M(x,y) = \inf\{K_{\mathbb{D}}(\tilde{x},\tilde{y}) : \ \tilde{y} \in \mathbb{D}, \ \pi(\tilde{y}) = y\},$$

and K_M is a distance and M is complete [47]. Hence such a Riemann surface is hyperbolic. Compact Riemann surfaces of genus ≥ 2 are hyperbolic and if there exists a Green function on a Riemann surface then it is hyperbolic and non-compact.

The next theorem proved by Royden, states criteria for hyperbolicity [52].

Theorem 1.1.6 *For a complex manifold M the following conditions are equivalent.*

(i) M is hyperbolic.

(ii) The topology induced by K_M is equivalent to the original topology on M.

(iii) Let $F : TM \longrightarrow \mathbb{R}_{\geq 0}$ be a Finsler metric. Then for any $x \in M$, there is a neighbourhood U of x and a constant $c > 0$ such that $k_M(X_y) \geq cF(X_y)$ for all $X_y \in T_yM$ with $y \in U$.

(iv) Let $F : TM \longrightarrow \mathbb{R}_{\geq 0}$ be a Finsler metric. Then, for any $X \in TM$, there is a neighbourhood W of X and a constant $c > 0$ such that $k_M \geq cF$ on W.

Let M, N be complex manifolds. The set of all continuous and holomorphic mappings from M to N will be denoted by $\mathcal{C}(M, N)$ and $\mathcal{H}(M, N)$, respectively. We equipe $\mathcal{C}(M, N)$ with the compact open topology. A sequence $\{f_n\} \subset \mathcal{C}(M, N)$ is called compactly divergent if given any compact set K in M, and compact K' in N, there exists $n_0 \in \mathbb{N}$ such that $f_n(K) \cap K' = \phi$, for all $n \geq n_0$.

Definition 1.1.7 *$\mathcal{F} \subset \mathcal{C}(M, N)$ is called normal if every sequence in $\mathcal{F}$ contains a subsequence which is either convergent in $\mathcal{C}(M, N)$ or compactly divergent.*

The following theorem [38] will clarify the relationship among different concepts such as normality and equicontinuty.

Theorem 1.1.8 (Arzela-Ascoli) *Let X be a locally compact, separable space and Y a locally compact metric space. Then, a family $\mathcal{F} \subset \mathcal{C}(X, Y)$ is relatively compact in $\mathcal{C}(X, Y)$ (i.e., every sequence of mappings $f_n \in \mathcal{F}$ contains a subsequence which converges in $\mathcal{C}(X, Y)$) if and only if*

(a) $\mathcal{F}$ is equicontinuous at every point $x \in X$;

(b) for every $x \in X$, the set $\{f(x) : f \in \mathcal{F}\}$ is relatively compact in Y.

Let us state the classical theorem of Montel, which will be applied frequently in this dissertation.

Theorem 1.1.9 (Montel) *Let M be a complex manifold. Then a locally uniformly bounded family of holomorphic mappings from M into $\mathbb{C}^n$ is equicontinuous and hence normal in $\mathcal{H}(M, \mathbb{C}^n)$.*

A complex manifold M is called taut if the family $\mathcal{H}(\mathbb{D}, M)$ is a normal family.

Theorem 1.1.10 *[47] Let M be a complete hyperbolic manifold then the family $\mathcal{H}(\mathbb{D}, M)$ is a normal family.*

Therefore, every complete hyperbolic manifold is taut. On the other hand tautness provides us with valuable information about hyperbolicity of a manifold and continuity of the Kobayashi-Royden metric.

Theorem 1.1.11 *[52] If M is taut, then it is hyperbolic and k_M is continuous on TM.*

Chapter 2

Bloch and normal mappings on manifolds

2.1 Definitions and basic properties

In this chapter, we assume that M and N are connected complex manifolds of dimensions m and n with differential metrics F_M and F_N, respectively.

Definition 2.1.1 *We say that a mapping $f \in \mathcal{H}(M, N)$ is of bounded expansion if it satisfies*

$$\|f'\| := \sup\{\|f'(p)\| : p \in M\} < \infty, \tag{2.1}$$

where

$$\|f'(p)\| := \sup\{F_N(f'(p)X_p) : X_p \in T_pM, \; F_M(X_p) = 1\}. \tag{2.2}$$

Usually, M will be hyperbolic and F_M will be the Kobayashi-Royden metric k_M while N will be Hermitian and F_N will be the Hermitian metric h_N. To avoid confusion, when there are different differential metrics on M and N, we will sometimes use the notations $\|f'(p)\|_{F_M}^{F_N}$ and $\|f'\|_{F_M}^{F_N}$, to denote (2.1) and (2.2) respectively. The class of mappings of bounded expansion will be denoted by $\mathcal{E}(M, N)$.

Let F_1 and F_2 be two differential metric such that for each $0_p \neq X_p \in T_pM$

$$F_i(X_p) \neq 0 \qquad i = 1, 2.$$

Then

$$\begin{aligned} \sup_{X_p \neq 0_p} \frac{h_N(f'(p)X_p)}{F_1(X_p)} &= \sup_{F_2(X_p)=1} \frac{h_N(f'(p)X_p)}{F_1(X_p)} \\ &= \sup_{F_1(X_p)=1} h_N(f'(p)X_p). \end{aligned} \tag{2.3}$$

If M is hyperbolic and (p, ξ) is the representation of a tangent vector $X_p \in T_pM$ in a coordinate neighbourhood of $p \in M$, then, by Theorem 1.1.6 (iii), $k_M(p, \xi) \neq 0$ for $\xi \neq 0$, and by (2.3) we have

$$\|f'(p)\| = \sup_{k_M(p,\xi)\neq 0} \frac{h_N(f(p), f'(p)\xi)}{k_M(p, \xi)},$$

$$= \sup_{|\xi|=1} \frac{h_N(f(p), f'(p)\xi)}{k_M(p, \xi)}, \tag{2.4}$$

where $| \quad |$ is the Euclidean norm in $\mathbb{C}^m$.

Note that (2.4) does not depend on the coordinate neighbourhood around $p \in M$ and the representation (p, ξ) of the tangent vector $X_p \in T_pM$.

If N is noncompact, we refer to mappings of bounded expansion as Bloch mappings and if N is compact we refer to them as normal mappings.

The sets of all Bloch and normal mappings from M to N will be denoted by $\mathcal{B}(M, N)$ and $\mathcal{N}(M, N)$, respectively.

For the case of normal mappings our definition is consistant with Hahn's definition of normal mappings in [30], [31].

Example 2.1.2 Consider $\mathbb{D}$ with Poincaré metric

$$\rho(z, \xi) = \frac{|\xi|}{1 - |z|^2},$$

and $\mathbb{C}$ with Euclidean metric $e(z,\xi) = |\xi|$. Also on the Riemann sphere $\mathbb{C}_\infty$, considering two charts (φ_0, U_0) and $(\varphi_\infty, U_\infty)$ where $U_0 := \mathbb{C}_\infty \backslash \{\infty\}$ and $U_\infty := \mathbb{C}_\infty \backslash \{0\}$ with $\varphi_0(z) = z$ and $\varphi_\infty(z) = 1/z$, the chordal metric σ is defined by

$$\sigma(w,\xi) = \frac{|\xi|}{1 + |\varphi_i(w)|^2}, \qquad \forall w \in U_i, \qquad i = 0, \infty.$$

Then a function $f : \mathbb{D} \longrightarrow \mathbb{C}$ is Bloch if and only if

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)|f'(z)| < \infty,$$

and a function $f : \mathbb{D} \longrightarrow \mathbb{C}_\infty$ is normal if and only if

$$\sup_{z\in\mathbb{D}} (1 - |z|^2) f^{\#}(z) < \infty.$$

where, $f^{\#}(z)$ is $|f'(z)|/[1 + |f(z)|^2]$ or $|(1/f)'(z)|/[1 + |(1/f)z|^2]$ if $f(z) \neq \infty$ or $f(z) = \infty$, respectively.

This shows that our definitions coincide with the classical definitions of Bloch and normal functions, respectively.

Definition 2.1.3 *Let N and $\tilde{N}$ be connected complex manifolds. If $\tilde{N}$ is compact and $N \subset \tilde{N}$ is open, then we call $\tilde{N}$ a compact extension of N.*

Examples 2.1.4 .

(a) Consider the n-dimensional complex projective space $\mathbb{P}^n$ with the canonical projection $\pi : \mathbb{C}^{n+1} \backslash \{0\} \longrightarrow \mathbb{P}^n$. Let $W_i := \{z = (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : z_i = 1\}$ and $U_i := \pi(W_i)$. Then $\mathbb{P}^n$ is a compact extension of $\mathbb{C}^n$, because the neighbourhood $U_1 \subset \mathbb{P}^n$ is isomorphic to $\mathbb{C}^n$.

(b) Set

$$\overline{\mathbb{C}^n} := \underbrace{\mathbb{C}_\infty \times \mathbb{C}_\infty \cdots \times \mathbb{C}_\infty}_{n-times},$$

where $\mathbb{C}_\infty$ is the Riemann sphere. In some references $\overline{\mathbb{C}^n}$ is called the Osgood closure of $\mathbb{C}^n$. Consider the two canonical charts (φ_0, U_0) and $(\varphi_\infty, U_\infty)$ of each factor

$\mathbb{C}_\infty$, which have been introduced in Example 2.1.2. Then we obtain coordinates on $\overline{\mathbb{C}^n}$ by considering

$$U_{i_1,i_2,..,i_n} := U_{i_1} \times U_{i_2} \times \dots U_{i_n}$$

and

$$\varphi_{i_1 i_2..,i_n} := (\varphi_{i_1}, \varphi_{i_2}, ..., \varphi_{i_n}) : U_{i_1,i_2..,i_n} \longrightarrow \mathbb{C}^n,$$

where $i_\mu \in \{0, \infty\}$. Obviously $\overline{\mathbb{C}^n}$ is compact and $\mathbb{C}^n \simeq U_{0,0,..,0} \subset \overline{\mathbb{C}^n}$ is an open subset. Hence $\overline{\mathbb{C}^n}$ is a compact extension of $\mathbb{C}^n$.

We are going to verify whether various properties of Bloch and normal functions in the classical case are still satisfied in this more general setting.

It is well known that, in the classical case, every Bloch function is normal. If N and $\tilde{N}$ are Hermitian, with N open in $\tilde{N}$ and $h_{\tilde{N}} \leq h_N$ on TN and M is hyperbolic, then clearly any mapping $f \in \mathcal{H}(M, N)$ of bounded expansion is also of bounded expansion, considered as a mapping in $\mathcal{H}(M, \tilde{N})$. If, moreover $\tilde{N}$ is a compact extension of N, we may then write $\mathcal{B}(M, N) \subset \mathcal{N}(M, \tilde{N})$. Thus, every Bloch mapping to N is a normal mapping to $\tilde{N}$.

Moreover, if $f \in \mathcal{B}(M, N)$ then the family $\{f \circ \varphi | \ \varphi \in \text{Aut}M\}$ as a family of holomorphic mappings to $\tilde{N}$ is a normal family, where Aut$M$ denotes the group of holomorphic automorphisms of M. In fact, since the Kobayashi-Royden differential metric is invariant under $\varphi \in \text{Aut}M$, i.e., for each $p \in M$ and $\xi \in \mathbb{C}^n$ we have $k_M(p, \xi) = k_M(\varphi(p), \varphi'(p)\xi)$, so

$$\frac{h_N((f \circ \varphi)(p), (f \circ \varphi)'(p)\xi)}{k_M(p, \xi)} = \frac{h_N(f(\varphi(p)), f'(\varphi(p))\varphi'(p)\xi)}{k_M(\varphi(p), \varphi'(p)\xi)}.$$

Therefore

$$\sup_{p \in M} \|(f \circ \varphi)'(p)\| = \sup_{p \in M} \|f'(\varphi(p))\| = \sup_{q \in M} \|f'(q)\|. \tag{2.5}$$

Since f is Bloch, there exists α such that for all $\varphi \in \text{Aut}M$ and $p \in M$ we have $\|(f \circ \varphi)'(p)\| < \alpha$. Hence for each $p \in M$ and $\xi \in \mathbb{C}^m$

$$h_N((f \circ \varphi)(p), (f \circ \varphi)'(p)\xi) \leq \alpha k_M(p, \xi).$$

Integrating along any C^1 curve connecting any two points p_1 and p_2 in M implies

$$d_{\tilde{N}}((f \circ \varphi)(p_1), (f \circ \varphi)(p_2)) \leq \alpha K_M(p_1, p_2), \qquad \forall \varphi \in \text{Aut}M.$$

Thus the family $\{f \circ \varphi | \ \varphi \in \text{Aut}M\}$ is equicontinuous. Since $\tilde{N}$ is compact, by the Arzela-Ascoli Theorem the family $\{f \circ \varphi | \ \varphi \in \text{Aut}M\}$ is a normal family.

Note that the property $h_{\tilde{N}} \leq h_N$ in the classical case is fulfilled for $\mathbb{C} \subset \mathbb{C}_\infty \simeq \mathbb{P}^1$. This is also the case for $\mathbb{C}^n$ and projective space $\mathbb{P}^n$ with the Fubini-Study Hermitian metric:

$$h_{\mathbb{P}^n}(z, \xi) = \frac{[(1 + |z|^2)|\xi|^2 - |(z, \xi)|^2]^{1/2}}{1 + |z|^2},$$

where z represents inhomogeneous coordinates of a point in $\mathbb{P}^n$, $\xi \in \mathbb{C}^n$ and $(\ , \ )$ is the usual inner product in $\mathbb{C}^n$.

The next theorem is a characterisation of normal mappings on complex manifolds due to K.T. Hahn [31].

Theorem 2.1.5 *Assume that N is compact and M is hyperbolic and homogeneous, i.e., the group* Aut$M$ *of holomorphic automorphisms of M is transitive. Then, for $f \in \mathcal{H}(M, N)$ the following statements are equivalent:*

(a) $f \in \mathcal{N}(M, N)$;

(b) the family $\{f \circ \varphi \mid \varphi \in \text{Aut}M\}$ forms a normal family;

(c) $f \in \mathcal{H}(M, N)$ is uniformly continuous, i.e., for each $\epsilon > 0$ there exists a $\delta > 0$ such that for all $p, q \in M$ with $K_M(p, q) < \delta$ we have

$$d_N(f(p), f(q)) < \epsilon,$$

where $K_M(p, q)$ is the Kobayashi distance on M.

To prove our next theorem we need two lemmas, which indicates the relation between a distance and the " derivative" of that distance.

Lemma 2.1.6 *Let Ω be a domain in $\mathbb{C}^n$, $\delta : \Omega \times \mathbb{C}^n \longrightarrow \mathbb{R}_{\geq 0}$ a Hermitian metric and let d be its corresponding distance. Then, for each $a \in \Omega$ and $\xi \in \mathbb{C}^n$,*

$$\lim_{t \to 0+} \frac{1}{t} d(a, a + t\xi) = \delta(a, \xi). \tag{2.6}$$

Proof: By definition $d(a, b) := \inf\{\int_0^1 \delta(\alpha(\tau), \alpha'(\tau))d\tau\}$ where the infimum is taken over all piecewise smooth curves $\alpha : [0, 1] \longrightarrow \Omega$ joining a and b. Fix $a \in \Omega$ and $\xi \in \mathbb{C}^n$, $\xi \neq 0$. Taking $\alpha(\tau) := a + t\tau\xi$, we see that

$$\begin{aligned}
\limsup_{t \to 0+} \frac{1}{t} d(a, a + t\xi) &\leq \limsup_{t \to 0+} \frac{1}{t} \int_0^1 \delta(a + t\tau\xi, t\xi)d\tau \\
&= \limsup_{t \to 0+} \frac{1}{t} \int_0^t \delta(a + s\xi, \xi)ds \\
&\leq \delta(a, \xi).
\end{aligned}$$

So far, we have used the fact that δ is a differential metric and upper semicontinuous.

Conversely, fix $0 < \theta < 1$. Since δ is continuous, for $\eta_0 \in \mathbb{C}^n$ with $|\eta_0| = 1$, there exist neighbourhoods $U_{\eta_0} \subset \Omega$ of a and $V_{\eta_0} \subset \mathbb{C}^n$ of η_0 such that $\delta(z, \eta) \geq \theta\delta(a, \eta)$ for each $z \in U_{\eta_0}$ and $\eta \in V_{\eta_0}$. Since

$$\mathbb{S}^{n-1} := \{\eta \in \mathbb{C}^n : \ |\eta| = 1\},$$

is compact there exist $\eta_1, \ldots, \eta_N \in \mathbb{S}^{n-1}$ such that $\mathbb{S}^{n-1} \subset \cup_{i=1}^N V_{\eta_i}$. Take $U := \cap_{i=1}^N U_{\eta_i}$. Then U is a neighbourhood of a and $\delta(z, \eta) \geq \theta\delta(a, \eta)$ for each $z \in U$ and $\eta \in \mathbb{S}^{n-1}$. Since $\delta(z, .)$ is a $\mathbb{C}$-norm, it follows that for each $z \in U$ and $\eta \in \mathbb{C}^n$, $\delta(z, \eta) \geq \theta\delta(a, \eta)$.

For small $t > 0$, there exists a geodesic $\gamma_t : [0,1] \longrightarrow \tilde{U} \subset U$, with $\gamma_t(0) = a$, $\gamma_t(1) = a + t\xi$. Therefore

$$\begin{aligned} d(a, a+t\xi) &= \int_0^1 \delta(\gamma_t(\tau), \gamma_t'(\tau))d\tau, \\ &\geq \theta \int_0^1 \delta(a, \gamma_t'(\tau))d\tau, \\ &\geq \theta\delta(a, \int_0^1 \gamma_t'(\tau)d\tau), \\ &= \theta\delta(a, \gamma_t(1) - \gamma_t(0)), \\ &= \theta\delta(a, t\xi). \end{aligned}$$

The second inequality follows from the fact that $\delta(a, .)$ is a complex norm. It follows that

$$\liminf_{t\to 0^+} \frac{1}{t} d(a, a+t\xi) \geq \theta\delta(a, \xi).$$

Hence, the proof is complete.

□

Lemma 2.1.7 *Let Ω and δ be as in the previous lemma, $\Omega_1 \subset \mathbb{C}^m$ and $f : \Omega_1 \longrightarrow \Omega$ a holomorphic mapping. Then, for $a \in \Omega_1$ and $\xi \in \mathbb{C}^m$,*

$$\lim_{t\to 0^+} \frac{1}{t} d(f(a), f(a+t\xi)) = \delta(f(a), f'(a)\xi).$$

Proof: We shall prove that for each $b \in \Omega$, there exist $M, r > 0$, such that $d(z', z'') \leq M\|z' - z''\|$, for all $z', z'' \in \mathbb{B}_r(b) \subset G$. To see this, fix $b \in \Omega$ and choose $r > 0$ such that $\bar{\mathbb{B}}_r(b) \subset \Omega$. Since δ is upper semicontinuous, there exists $M > 0$ such that $\delta(z, \xi) \leq M\|\xi\|$, for each $z \in \bar{\mathbb{B}}_r(b)$ and each $\xi \in \mathbb{C}^n$. It follows that for $z', z'' \in \mathbb{B}_r(b)$ we have

$$d(z', z'') \leq \int_0^1 \delta(z' + t(z'' - z'), z'' - z')dt \leq M\|z' - z''\|. \tag{2.7}$$

We claim that

$$\delta(b, \xi_0) = \lim_{\substack{t\to 0+ \\ \xi\to\xi_0}} \frac{1}{t} d(b, b+t\xi). \tag{2.8}$$

Choose M and r as above. Then for $0 < t < (r/2)/(\|\xi_0\| + r/2)$ and $\xi \in \mathbb{B}_{r/2}(\xi_0)$ we have $\|b + t\xi - b\| < r/2$. Hence, by (2.7),

$$d(b + t\xi_0, b + t\xi) \le Mt\|\xi - \xi_0\|.$$

Therefore,

$$\begin{aligned} d(b, b + t\xi) &\le d(b, b + t\xi_0) + d(b + t\xi_0, b + t\xi) \\ &\le d(b, b + t\xi_0) + Mt\|\xi - \xi_0\|. \end{aligned}$$

Similarly,

$$d(b, b + t\xi_0) \le d(b, b + t\xi_0) + Mt\|\xi - \xi_0\|. \tag{2.9}$$

Applying the previous lemma we have,

$$\lim_{\substack{t \to 0+ \\ \xi \to \xi_0}} \frac{1}{t} d(b, b + t\xi) = \lim_{t \to 0^+} \frac{1}{t} d(b, b + t\xi_0) = \delta(b, \xi_0). \tag{2.10}$$

Hence our claim in (2.8) is proved. With the help of this fact, for $a \in \Omega_1$ and $\xi \in \mathbb{C}^m$ we have,

$$\begin{aligned} \lim_{t \to 0^+} \frac{1}{t} d(f(a), f(a + t\xi)) &= \lim_{t \to 0^+} \frac{1}{t} d(f(a), f(a) + t(f'(a)\xi + o(1))), \\ &= \delta(f(a), f'(a)\xi). \end{aligned}$$

□

For the next theorem we will give two proofs. The first proof is based on the application of Lemmas 2.1.6 and 2.1.7, while the second proof is a direct proof.

Theorem 2.1.8 *Let M and N be hyperbolic and Hermitian manifolds, respectively. Then the function $f \longmapsto \|f'\|$ from the class of holomorphic bounded expansion mappings $\mathcal{E}(M, N)$ equipped with the compact open topology to $\mathbb{R}$ is lower semicontinuous.*

Proof (1) : Let $\{f_n\}$ be a sequence in $\mathcal{E}(M,N)$ which converges to a holomorphic mapping f uniformly on compact subsets of M. Let $\{\|f'_{n_k}\|\}$ be a subsequence of $\{\|f'_n\|\}$ converging to α. For a given $\varepsilon > 0$ and $p,\ q \in M,\ p \neq q$, there exists $n_\varepsilon \in \mathbb{N}$ such that for $n_k > n_\varepsilon$ we have

$$| \ \|f'_{n_k}\| - \alpha| < \frac{\varepsilon}{2K_M(p,q)}, \tag{2.11}$$

and

$$d_N(f_{n_k}(p), f(p)) \le \varepsilon/4, \qquad d_N(f_{n_k}(q), f(q)) \le \varepsilon/4. \tag{2.12}$$

On the other hand, since for each $n \in \mathbb{N},\ f_n \in \mathcal{N}(M,N)$, according to the definitions of $\|f'_n(x)\|$ and $\|f'_n\|$, for each $x \in M$ and $\xi \in \mathbb{C}^m$

$$h_N(f_n(x), f'_n(x)\xi) \le \|f'_n\| k_M(x,\xi).$$

By integrating along any C^1 curve connecting p to q, we have

$$d_N(f_n(p), f_n(q)) \le \|f'_n\| K_M(p,q).$$

Hence employing (2.11) and (2.12), for $n_k > n_\varepsilon$ we obtain

$$\begin{aligned} d_N(f(p), f(q)) &\le d_N(f(p), f_{n_k}(p)) + d_N(f_{n_k}(p), f_{n_k}(q)) \\ &\quad + d_N(f_{n_k}(q), f(q)) \\ &\le \varepsilon/2 + \|f'_{n_k}\| K_M(p,q) < \varepsilon + \alpha K_M(p,q). \end{aligned}$$

Therefore

$$d_N(f(p), f(q)) \le \alpha K_M(p,q). \tag{2.13}$$

Now applying Lemma 2.1.7 and also the first part of Lemma 2.1.6 locally, for each $p \in M$ and each $\xi \in \mathbb{C}^m$, we have

$$\begin{aligned} h_N(f(p), f'(p)\xi) &= \lim_{t\to 0^+} \frac{1}{t} d_N(f(p), f(p+t\xi)) \\ &\le \alpha \limsup_{t\to 0^+} \frac{1}{t} K_M(p, p+t\xi) \\ &\le \alpha k_M(p,\xi). \end{aligned}$$

It follows that $f \in \mathcal{E}(M,N)$ and $\|f'\| \leq \alpha$. The assertion is thus proved.

Proof (2) : Let $\{f_n\}$ be a sequence in $\mathcal{E}(M,N)$ which converges to a holomorphic mapping f uniformly on compact subsets of M. We wish to show that $\liminf_{n\to\infty} \|f_n'\| \geq \|f'\|$.

Since $\{f_n\}$ is a sequence of holomorphic mappings and $\lim_{n\to\infty} f_n = f$, it follows that $\lim_{n\to\infty} f_n' = f'$. So for each $p \in M$, we have $f_n'(p) \longrightarrow f'(p)$. Fix $X_p \in T_pM$. Then $f_n'(p)X_p \longrightarrow f'(p)X_p$. Since h_N is continuous,

$$\lim_{n\to\infty} h_N(f_n'(p)X_p) = h_N(f_n'(p)X_p).$$

Thus, if $K_M(X_p) = 1$, $\|f_n'(p)\| \geq h_N(f_n'(p)X_p)$ and so

$$\liminf_{n\to\infty} \|f_n'(p)\| \geq h_N(f'(p)X_p).$$

Taking the supremum over $X_p \in T_pM$ with $k_M(X_p) = 1$, we have

$$\liminf_{n\to\infty} \|f_n'(p)\| \geq \|f'(p)\|, \qquad \forall p \in M.$$

Hence, for each $p \in M$, $\liminf_{n\to\infty} \|f_n'\| \geq \|f'(p)\|$. Thus

$$\liminf_{n\to\infty} \|f_n'\| \geq \|f'\|.$$

□

2.2 Bloch mappings to $\mathbb{C}^n$

Bloch functions on the unit disk have been studied by many authors, see for example [6]. In several complex variables they have been studied by R. Timony [55] on homogeneous bounded domains, and on strongly pseudoconvex domains by S. Krantz and D. Ma. Applying the Kobayashi-Royden metric, we will study Bloch mappings from hyperbolic manifolds to Euclidian space $\mathbb{C}^m$.

Applying a lemma due to Hahn [31], we shall give a Marty type criterion on hyperbolic manifolds.

Lemma 2.2.1 *[31] Let M be hyperbolic and N compact. A family $\mathcal{F} \subset \mathcal{H}(M, N)$ is normal if and only if for each compact $K \subset M$ there exists a constant $C(K)$ such that, for each $p \in K$,*

$$\sup\{\|f'(p)\| : \ f \in \mathcal{F}\} \leq C(K) \tag{2.14}$$

Theorem 2.2.2 *Let M be a hyperbolic manifold. A family $\mathcal{F} \subset \mathcal{H}(M, \mathbb{C})$ is normal as a family of holomorphic functions to $\mathbb{C}_\infty$ if and only if for each compact $K \subset M$ there exists $C(K)$ such that, for each $p \in K$,*

$$\sup\{\|f'(p)\|_k^e/(1 + |f(p)|^2) : \ f \in \mathcal{F}\} \leq C(K), \tag{2.15}$$

where $\|f'(p)\|_k^e := \sup\{|f'(p)X_p| : k_M(X_p) = 1\}$.

Proof: Suppose $\mathcal{F} \subset \mathcal{H}(M, \mathbb{C})$ is normal, so that by the previous lemma, for $K \subset M$ compact, there exists a constant $C(K) > 0$ such that for each $p \in K$ and each $f \in \mathcal{F}$,

$$\sup\{\|f'(p)\| : \ f \in \mathcal{F}\} \leq C(K).$$

Hence, for each $p \in K$ and each $X_p \in T_pM$ with $k_M(X_p) = 1$ we have,

$$\frac{|f'(p)X_p|}{1 + |f(p)|^2} \leq C(K).$$

Therefore $|f'(p)X_p| \leq C(K)(1 + |f(p)|^2)$. It follows that for each $p \in K$ and each $f \in \mathcal{F}$,

$$\frac{\|f'(p)\|_k^e}{1 + |f(p)|^2} \leq C(K).$$

Conversely, let

$$\frac{\|f'(p)\|_k^e}{1 + |f(p)|^2} \leq C(K), \qquad \forall p \in K \quad \forall f \in \mathcal{F}.$$

Thus for each $p \in K$ and each $X_p \in T_pM$ with $k_M(X_p) = 1$, we obtain $|f'(p)X_p| \leq C(K)(1 + |f(p)|^2)$. Therefore

$$\sup_{k_M(X_p)=1} \sigma_{\mathbb{C}_\infty}(f'(p)X_p) \leq C(K),$$

where σ is the chordal metric on $\mathbb{C}_\infty$. Now Lemma 2.2.1 will complete the proof.

□

The following theorem will confirm that several equivalent statements concerning Bloch functions [6] in the classical case, are valid in a more general setting.

We will consider Bloch mappings from hyperbolic manifolds to $\mathbb{C}^n$, however it would also be interesting to consider the case where the target space N is a Hermitian manifold.

Timony in his paper [55] considered Bloch functions on bounded homogenous domains with Bergman metric. This metric is a Hermitian metric. We will consider hyperbolic manifolds for which the Kobayashi-Royden metric is continuous. From this point of view our construction is more general than Timony's work. However as Krantz pointed out in [39], for the case of strongly pseudoconvex domains, the Bergman, Kobayashi, and Carathéodory metrics are "equivalent" in the sense that the Bloch spaces defined in terms of the three metrics are the same.

For the next theorem we need to define the notion of sequence of P-points. This version of this notion was introduced by P. M. Gauthier [22] and was applied by Hahn [31] in studying normal mappings on hyperbolic spaces.

Definition 2.2.3 *Let d_M and d_N be distances on complex manifolds M and N, respectively. A sequence $\{p_n\}$ of points in M is called a sequence of P-points of $f \in \mathcal{H}(M,N)$ if there exists a sequence $\{q_n\}$ in M such that*

$$\lim_{n\to\infty} d_M(p_n, q_n) = 0, \qquad \textit{and} \qquad \lim_{n\to\infty} d_N(f(p_n), f(q_n) > 0.$$

Theorem 2.2.4 *Let M be a hyperbolic homogenous manifold, f be a holomorphic function on M and $p_0 \in M$. Then the following statements are equivalent:*

(a) $f \in \mathcal{B}(M, \mathbb{C})$.

(b) $f : (M, K_M) \longrightarrow (\mathbb{C}, |\ .\ |)$ is uniformly continuous.

(c) $f \in \mathcal{H}(M, \mathbb{C})$ has no sequence of P-points in M.

(d) The family

$$\{(f \circ \varphi)(p) - (f \circ \varphi)(p_0) : \varphi \in \mathrm{Aut}M\}$$

is normal as a family of holomorphic functions to $\mathbb{C}_\infty$.

(e)

$$\sup\{\|(f \circ \varphi)'(p_0)\|_k^e : \varphi \in \mathrm{Aut}M\} < \infty.$$

Proof: $(a) \Rightarrow (b)$. If (a) holds then there exists a constant α such that for each $p \in M$ and $\xi \in \mathbb{C}^m$,

$$|f'(p)\xi| \leq \alpha k_M(p, \xi).$$

Integrating along any C^1 curve γ connecting any two points p_1 and p_2 in M we deduce

$$|f(p_1) - f(p_2)| \leq \alpha K_M(p_1, p_2).$$

Hence f is uniformly continuous in the designated metrics.

$(b) \Leftrightarrow (c)$. As in Hahn [31], this follows directly from the definition of uniform continuity and the definition of a sequence of P-points.

$(b) \Rightarrow (d)$. Since by Theorem 1.1.4 holomorphic mappings are distance decreasing in the Kobayashi pseudo-distance, for each $p, q \in M$ and $\varphi \in \mathrm{Aut}M$, we have, $K_M(p, q) = K_M(\varphi(p), \varphi(q))$. Hence from the uniform continuity of f it follows that the family $\{(f(\varphi(p)) - f(\varphi(p_0)) : \varphi \in \mathrm{Aut}M\}$ is equicontinuous. Therefore, by the Arzela-Ascoli Theorem this family is normal.

$(d) \Rightarrow (e)$. This follows from Theorem 2.2.2 with $K = \{p_0\}$.

$(e) \Rightarrow (a)$. We have seen in (2.5) that $\|(f \circ \varphi)'(p_0)\|_k^e = \|f'(\varphi(p_0))\|_k^e$ and since $\text{Aut}M$ is transitive we have $\sup\{\|f'(p)\|_k^e : p \in M\} < \infty$, as required.

□

Note that, by Proposition 2.3.19 in ([1, p. 204]), since M is hyperbolic and homogenous, k_M is continuous. However, we did not use this fact in our proof.

We can apply the previous theorem to Bloch mappings from a hyperbolic manifold M to $\mathbb{C}^n$. Indeed a mapping $f \in \mathcal{H}(M, \mathbb{C}^n)$ is a Bloch mapping if and only if each component $f_i \in \mathcal{H}(M, \mathbb{C})$ is Bloch. This obviously can be deduced from the fact that

$$|f_i'(p)X_p| \leq |f'(p)X_p| \leq \sqrt{n} \max_j |f_j'(p)X_p|,$$

for each $p \in M$, and $X_p \in T_pM$.

We shall give an example to show that the homogeneity condition on M cannot be waived, for the equivalence $(a) \iff (d)$ in Theorem 2.2.4. For this example we consider a hyperbolic manifold M of dimension one. That is, M is a Riemann surface for which the unit disk $\mathbb{D}$, is its universal covering space. To state the example we need to consider another criterion for Bloch functions on hyperbolic Riemann surfaces.

Definition 2.2.5 *Let M be a Riemann surface, $f \in \mathcal{H}(M, \mathbb{C})$ and $p \in M$. By a schlicht disk at $f(p)$ (or unramified disk), we mean an open disk $\mathbb{D}_r(f(p))$ in $f(M)$ with the property that there exists a neighbourhood V of p in M such that $f|_V$ is injective onto $\mathbb{D}_r(f(p))$.*

It is well known [6] that $f \in \mathcal{H}(\mathbb{D}, \mathbb{C})$ is Bloch if and only if the radii of schlicht disks in the range of f is bounded above. we will prove the analogous theorem on hyperbolic Riemann surfaces.

Theorem 2.2.6 *Let M be a hyperbolic Riemann surface. Then $f \in \mathcal{H}(M, \mathbb{C})$ is Bloch if and only if the radii of the schlicht disks in the range of f are bounded above.*

Proof: Let $\pi : \mathbb{D} \longrightarrow M$ be the universal covering map. Then, for $p \in M$ there exists V open in $\mathbb{D}$ and $z_0 \in V$ such that, $\pi(z_0) = p$ and $\pi|_V : V \longrightarrow U$ is biholomorphic. With the help of the remark after Definition 1.1.5 and considering the chart $(\varphi := (\pi|_V)^{-1}, U)$ around p, we have

$$\begin{aligned} \|f'(q)\| &= \sup_{|\xi|=1} \frac{|f'(q)\xi|}{k_M(q,\xi)}, \\ &= \frac{|(f \circ \varphi^{-1})'(z)|}{\frac{1}{1-|z|^2}}, \\ &= (1 - |z|^2)|(f \circ \pi)'(z)|, \qquad q \in U,\ \varphi(q) = z. \end{aligned} \tag{2.16}$$

Therefore $f \in \mathcal{H}(M, \mathbb{C})$ is Bloch if and only if $f \circ \pi \in \mathcal{H}(\mathbb{D}, \mathbb{C})$ is Bloch.

For $f \in \mathcal{H}(M, \mathbb{C})$ and $x \in M$, set

$$\beta(p, f) := \sup\{r > 0 : \ \mathbb{D}_r(f(p)) \text{ is a schlicht disk at } f(p) \text{ contained in} f(M)\}.$$

We take $\beta(p, f) = 0$ if there is no schlicht disk at $f(p)$.

Hence by the above remark $f \circ \pi \in \mathcal{H}(\mathbb{D}, \mathbb{C})$ is Bloch if and only if

$$\beta(f \circ \pi) := \sup\{\beta(z, (f \circ \pi)) \ : \ z \in \mathbb{D} \},$$

is bounded above. By the monodromy theorem $\beta(z, f \circ \pi) = \beta(\pi(z), f)$. Therefore $f \in \mathcal{H}(M, \mathbb{C})$ is Bloch if and only if

$$\beta(f) := \sup\{\beta(p, f) \ : \ p \in M\},$$

is bounded above. The assertion is thus proved.

□

As we mentioned, the following example shows that the homogeneity condition on M cannot be waived, for the equivalence $(a) \Longleftrightarrow (d)$ in Theorem 2.2.4.

Example 2.2.7 Consider the annulus

$$A_{r,R} = \{z \in \mathbb{C} : \ r < |z| < R\},$$

where $0 < r < R < \infty$.

One can easily prove that the group $\mathrm{Aut}(A_{r,R})$ consists only of the rotations $\mu(z) = uz$, for u a unimodular constant and reflection $\sigma(z) = Rr/z$, and compositions of these two types of functions. Therefore $A_{r,R}$ is not homogenous. In fact if $z_1, z_2 \in A_{r,R}$ with $|z_1| \neq |z_2|$ and if $|z_1| \neq Rr/|z_2|$, then there is no automorphism of $A_{r,R}$ which sends z_1 to z_2.

Since $\mathbb{D}$ is hyperbolic, $A_{r,R}$ is hyperbolic. Indeed we can find the universal covering map $\pi : \mathbb{D} \longrightarrow A_{r,R}$, explicitly. It is defined by

$$\pi(z) := (Rr)^{1/2} \exp -i\frac{\ln R/r}{\pi} \ln(\frac{1-z}{1+z}).$$

Now consider $f : A_{r,R} \longrightarrow \mathbb{C}$ defined by

$$f(z) := \frac{R-z}{R+z},$$

f is not a Bloch function since the radii of schlicht disks in the range of $f \circ \pi$ are not bounded above.

On the other hand for a fixed holomorphic function $f : A_{r,R} \longrightarrow \mathbb{C}$, consider the family

$$\{(f \circ \varphi)(z) - (f \circ \varphi)(z_0); \ \varphi \in \mathrm{Aut}(A_{r,R})\},$$

for some $z_0 \in A_{r,R}$. Let K be a compact subset in $A_{r,R}$ which contains z_0 and Q_1 a closed concentric annulus in $A_{r,R}$ containing K. Then $Q_2 =: \sigma(Q_1)$ is compact and by definition of σ, $\sigma(Q_2) = Q_1$. Consider the compact set $Q := Q_1 \cup Q_2$, we have $\sigma(Q) = Q$ and hence for every $\varphi \in \mathrm{Aut}(A_{r,R})$, $\varphi(Q) \subset Q$. Therefore

$$\cup\{\varphi(K) : \ \varphi \in \mathrm{Aut}(A_{r,R})\} \subset Q.$$

It follows that, for each $\varphi \in \mathrm{Aut}(A_{r,R})$,

$$\sup_{z\in K} |(f \circ \varphi)(z)| \leq \sup_{z\in Q} |f(z)| \equiv M_K.$$

Hence, for each $\varphi \in \mathrm{Aut}(A_{r,R})$ and $z \in K$,

$$|(f \circ \varphi)(z) - (f \circ \varphi)(z_0)| \leq 2M_K,$$

and by Montel's theorem

$$\{(f \circ \varphi)(z) - (f \circ \varphi)(z_0) : \; \varphi \in \mathrm{Aut}(A_{r,R})\},$$

is normal.

2.3 Semi-Bloch functions

A function $g \in \mathcal{H}(\mathbb{D}, \mathbb{C})$ is called a semi-Bloch function if for each complex number λ, the function $f_\lambda := \exp(\lambda g(z))$ is a normal function, as a function to C_∞. This function space on the unit disk $\mathbb{D}$, was introduced by F. Colonna [15]. She proved that a semi-Bloch function is a normal function and showed that there exist semi-Bloch functions which are not Bloch functions. She also asked whether the sum of two semi-Bloch functions is a semi-Bloch function. This question was answered negatively by R. Aulaskari and P. Lappan [8]. They gave analytic and geometric characterizations for semi-Bloch functions on the unit disk in the complex plane. With the help of the Kobayashi-Royden metric, we will define semi-Bloch functions on bounded domains in $\mathbb{C}^n$ and give a characterization of these functions.

Definition 2.3.1 *Let Ω be a bounded domain in $\mathbb{C}^n$. Then $g \in \mathcal{H}(\Omega, \mathbb{C})$ is called a semi-Bloch function, if for each $\lambda \in \mathbb{C}$, $f_\lambda(z) = \exp(\lambda g(z))$, as a function from Ω with Kobayashi-Royden metric to $\mathbb{C}_\infty$, is a normal function.*

Let Ω be a bounded domain in $\mathbb{C}^n$ and $g \in \mathcal{B}(\Omega, \mathbb{C})$. Set $f(z) := \exp(g(z))$. Then considering $f \in \mathcal{H}(\Omega, \mathbb{C}_\infty)$, we have

$$\|f'(z)\| := \sup_{|\xi|=1} \frac{\sigma_{\mathbb{C}_\infty}(f(z), f'(z)\xi)}{k_\Omega(z, \xi)}, \tag{2.17}$$

where $\sigma_{\mathbb{C}_\infty}(w,\eta)$ is the chordal metric on $\mathbb{C}_\infty$. Thus,

$$\|f'(z)\| = \sup_{|\xi|=1} \frac{|\exp(g(z))||g'(z)\xi|}{1+|\exp(g(z))|^2}\frac{1}{k_\Omega(z,\xi)},$$

$$= \frac{|\exp(g(z))|}{1+|\exp(g(z))|^2}\|g'(z)\|. \tag{2.18}$$

Therefore f is a normal function. This shows that every Bloch function is a semi-Bloch function.

Before stating the main result of this section, we remark that if Ω is a bounded domain in $\mathbb{C}^n$, then one can easily see that for each constant $c \in \mathbb{C}$, $f \in \mathcal{H}(\Omega, \mathbb{C})$ is a semi-Bloch function if and only if $f + c$ is a semi-Bloch function. We will follow Aulaskari and Lappan [8] to prove the next theorem.

Theorem 2.3.2 *Let Ω be a bounded domain in $\mathbb{C}^n$ and $g \in \mathcal{H}(\Omega, \mathbb{C})$. Then g is a semi-Bloch function if and only if, for each line L in the complex plane,*

$$\sup\{\|g'(z)\| : \ g(z) \in L\} = C_L < \infty.$$

Proof: Suppose $g \in \mathcal{H}(\Omega, \mathbb{C})$ is a semi-Bloch function and L is the imaginary axis. Set $f(z) := \exp(g(z))$.

For $z \in g^{-1}(L)$, we have $|f(z)| = 1$ and, considering $f \in \mathcal{H}(\Omega, \mathbb{C}_\infty)$,

$$\|f'(z)\| = \frac{|\exp(g(z))|}{1+|\exp(g(z))|^2}\|g'(z)\| = \frac{1}{2}\|g'(z)\|.$$

Since f is normal, it follows that

$$\sup\{\|g'(z)\| \ z \in g^{-1}(L)\} = C_L < \infty.$$

Now let L be any line in the complex plane. There exist complex numbers β and γ such that $\tilde{L} := \{\beta z + \gamma : \ z \in L\}$ is the imaginary axis. Moreover, by the above remark $\tilde{g} = \beta g + \gamma$ is a semi-Bloch function. Therefore, considering $\tilde{g}$, we can complete this part of the proof.

Conversely, suppose $\lambda \neq 0$ is given. Let L be be a line in the complex plane such that $\{\lambda z : z \in L\}$ is the imaginary axis. Consider $f_\lambda(z) := \exp(\lambda g(z))$. Then, for $z \in g^{-1}(L)$, we have, considering $f_\lambda \in \mathcal{H}(\Omega, \mathbb{C}_\infty)$,

$$\|f'_\lambda(z)\| = \frac{|\lambda|}{1 + |\exp(\lambda g(z))|^2}\|g'(z)\| = \frac{|\lambda|}{2}\|g'(z)\|.$$

Thus,

$$\sup\{\|f'_\lambda(z)\| : \ z \in g^{-1}(L)\} = \frac{|\lambda|}{2}C_L < \infty.$$

The line L has (many more than) 5 distinct elements. Therefore, by Corollary 3.12 in ([34, p.360]) f_λ is normal and the proof is complete. Note that in Corollary 3.12 in [34] the authors used the notion of uniformly normal, which is based on Proposition 1.6 (3) and Example 2.22 of this paper; this notion is equivalent to the normal property of f.

□

2.4 A characterization of Bloch mappings

Lohwater and Pommerenke [42] studied necessary and sufficient conditions for a function to be non-normal. Their idea was extended in [31] to normal mappings $f \in \mathcal{H}(\mathbb{D}, N)$, where N is a compact Hermitian manifold. Minda [44] by adapting their idea, established a necessary and sufficient condition for a function $f \in \mathcal{H}(\Omega, \mathbb{C})$ to be non-Bloch, where Ω is a hyperbolic region. In this section following [44] we will study this problem for $f \in \mathcal{H}(\Omega, \mathbb{C}^n)$, where Ω is a domain in $\mathbb{C}^m$. In the next theorem $\delta_\Omega(p) := d(p, \partial\Omega)$, denotes the Euclidean distance from p to $\partial\Omega$.

Theorem 2.4.1 *Let Ω be a bounded domain in $\mathbb{C}^m$. If $f \in \mathcal{H}(\Omega, \mathbb{C}^n)$ satisfies*

$$\sup_{p\in\Omega} \|f'(p)\|_e^e \delta_\Omega(p) = \infty, \tag{2.19}$$

then there are sequences $\{p_i\}$ in Ω and $\{r_i\}$, $r_i > 0$ with

$$\lim_{i\to\infty} \frac{r_i}{\delta_\Omega(p_i)} = 0, \tag{2.20}$$

such that $\{f(p_i + r_i\omega) - f(p_i)\}$ converges locally uniformly in $\mathbb{C}^m$ to a nonconstant holomorphic mapping $g \in \mathcal{H}(\mathbb{C}^m, \mathbb{C}^n)$.

Proof: Let $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, where for each i, Ω_i is a domain in $\mathbb{C}^m$, $\bar{\Omega}_i$ is compact and $\bar{\Omega}_i \subset \Omega_{i+1}$. Consider

$$M_i = \max\{\|f'(p)\|_e^e \delta_{\Omega_i}(p) : \; p \in \bar{\Omega}_i\}.$$

We claim that there exists $p_i \in \Omega_i$ such that $M_i = \|f'(p_i)\|_e^e \delta_{\Omega_i}(p_i)$, for each i. Otherwise, since $\delta_{\Omega_i}(p) = 0$ for $p \in \partial\Omega_i$, we have $\|f'(p)\|_e^e \delta_{\Omega_i}(p) = 0$ for $p \in \Omega_i$. But $\delta_{\Omega_i}(p) \neq 0$ on Ω_i, hence $\|f'(p)\|_e^e = 0$ for $p \in \Omega_i$. Since f is holomorphic, $f' \equiv 0$ on Ω which contradicts (2.19).

We also claim that $\lim_{i\to\infty} M_i = \infty$. If not, since $M_i \subset M_{i+1}$, there exists $M > 0$ such that for each i, $M_i \leq M$. Thus

$$\|f'(p)\|_e^e \delta_{\Omega_i}(p) \leq M, \tag{2.21}$$

for all $p \in \Omega_i$ and all i. Now take $q_0 \in \Omega$. Then $q_0 \in \Omega_i$, for all sufficiently large i and $\lim_{i\to\infty} \delta_{\Omega_i}(q_0) = \delta_\Omega(q_0)$. Applying (2.21) we deduce that $\|f'(q_0)\|_e^e \delta_\Omega(q_0) \leq M$ which is a contradiction with (2.19).

Define $r_i := \delta_{\Omega_i}(p_i)/M_i$. Since $r_i/\delta_\Omega(p_i) \leq r_i/\delta_{\Omega_i}(p_i) = 1/M_i$, it follows that $\lim_{i\to\infty} r_i/\delta_\Omega(p_i) = 0$. If $|\omega| < M_i = \delta_{\Omega_i}(p_i)/r_i$, then $|r_i\omega| \leq \delta_{\Omega_i}(p_i)$. Therefore

$$p_i + r_i\omega \in \mathbb{B}_{\delta_{\Omega_i}(p_i)}(p_i) := \{z \in \mathbb{C}^m : \; |z - p_i| < \delta_{\Omega_i}(p_i)\}.$$

Since $\mathbb{B}_{\delta_{\Omega_i}(p_i)}(p_i) \subset \bar{\Omega}_i$, we have $p_i + r_i\omega \in \bar{\Omega}_i$ and the holomorphic mapping $g_i(\omega) := f(p_i + r_i\omega) - f(p_i)$ is well defined on $\mathbb{B}_{M_i}(0) := \{\omega \in \mathbb{C}^m : |\omega| < M_i\}$. Moreover $g_i(0) = 0$ and $\|g_i'(0)\|_e^e = r_i\|f'(p_i)\|_e^e = 1$, for each i.

Let K be a compact subset in $\mathbb{C}^m$. Since $M_i \longrightarrow \infty$, there exists $N \in \mathbb{N}$ such that $K \subset \mathbb{B}_{M_i}(0)$, for all $i > N$, and for $\omega \in K$,

$$\|g_i'(\omega)\|_e^e = r_i\|f_i'(p_i + \omega r_i)\|_e^e \leq r_i \frac{M_i}{\delta_{\Omega_i}(p_i + \omega r_i)} = \left(\frac{\delta_{\Omega_i}(p_i + \omega r_i)}{\delta_{\Omega_i}(p_i)}\right)^{-1}$$
$$\leq \left(1 - |1 - \frac{\delta_{\Omega_i}(p_i + \omega r_i)}{\delta_{\Omega_i}(p_i)}|\right)^{-1}.$$

Note that δ_{Ω_i} is a Lipschitz function of order 1. So for each i,

$$|\delta_{\Omega_i}(p_i) - \delta_{\Omega_i}(p_i + \omega r_i)| \leq |\omega| r_i.$$

Hence $\|g_i'(\omega)\|_e^e \leq (1 - |\omega|/M_i)^{-1}$.

Since $|\omega|$ is uniformly bounded on K and $\lim_{i\to\infty} M_i = \infty$ so $\{\|g_i'\|_e^e : i \in \mathbb{N}\}$ is uniformly bounded on K. We also have $g_i(0) = 0$, for each i. Thus $\{g_i : i \in \mathbb{N}\}$ is locally uniformly bounded on $\mathbb{C}^m$. Therefore by Montel's Theorem 1.1.9 the family $\{g_i : i \in \mathbb{N}\}$ is a normal family. Hence by passing to a subsequence, which will be denoted again by $\{g_i : i \in \mathbb{N}\}$, we can suppose that it converges uniformly on compact sets of $\mathbb{C}^m$ to a holomorphic mapping $g \in \mathcal{H}(\mathbb{C}^m, \mathbb{C}^n)$. Since $\|g'(0)\|_e^e = 1$, g is not constant and the proof is complete.

□

The following theorem is a "weak" converse of the previous theorem. The proof is almost the same as that of Theorem 6.1 in [31] with a modification by applying the implication $(a) \Rightarrow (c)$ in Theorem 2.2.4 in the second statement.

Theorem 2.4.2 *Let Ω be any bounded domain in $\mathbb{C}^m$ and $f \in \mathcal{H}(\Omega, \mathbb{C}^n)$. Suppose that there exist sequences $\{p_n\}$ in Ω and $\{r_n\}$, $r_n > 0$ with the property (2.20), such that $\{f(p_n + r_n\omega) - f(p_n)\}$, converges locally uniformly to a nonconstant holomorphic mapping $g \in \mathcal{H}(\mathbb{C}^m, \mathbb{C}^n)$. Then there exists a sequence of P-points for f. Moreover, if Ω is a homogeneous domain in $\mathbb{C}^m$, then* $\sup\{\|f'(p)\|_k^e : p \in \Omega\} = \infty$ *and f is non-Bloch.*

Now we consider the case where $m = 1$ and Ω is the open unit disk $\mathbb{D}$ in $\mathbb{C}$. Then

$$\|f'(z)\|_k^e = \sup_{|\xi|=1} |f'(z)\xi|(1 - |z^2|).$$

Since

$$\delta_{\mathbb{D}}(z) \leq 1 - |z^2| \leq 2\delta_{\mathbb{D}}(z),$$

(2.19) and (2.20) will become $\sup_{z\in\mathbb{D}} \|f'(z)\|_k^e = \infty$ and $\lim_{i\to\infty} r_i/(1 - |z_i|) = 0$, respectivly. Therefore we can deduce the following theorem:

Theorem 2.4.3 *The map $f \in \mathcal{H}(\mathbb{D}, \mathbb{C}^n)$ is not Bloch if and only if there exist sequences $\{z_i\}$ in $\mathbb{D}$, $\{r_i\}$, $r_i > 0$ such that*

$$\lim_{i\to\infty} \frac{r_i}{1 - |z_i|} = 0,$$

and $\{f(z_i + r_i\omega) - f(z_i)\}$ converges locally uniformly in $\mathbb{C}$ to a nonconstant holomorphic mapping $g \in \mathcal{H}(\mathbb{C}, \mathbb{C}^n)$.

Chapter 3

Quasiregular mappings

3.1 Quasiregular holomorphic mappings

In this section the quasiregular holomorphic mappings and their relation to Bloch constants will be studied. Our basic refrence for quasireqular mappings is [48], however our approach in this section is holomorphic rather than the smooth approach in [48]. The applications of this class of holomorphic mappings in value distribution theory has been studied in [61].

Let V and W be finite dimenional inner product spaces (Euclidean spaces). The inner products on both V and W will be denoted by $< \ , \ >$ and $|x|^2 :=< x, x >$, will denote the square of the length of a vector x. We also denote by $\mathcal{L}(V, W)$ the set of all linear transformation of V to W. For a linear mapping $L \in \mathcal{L}(V, W)$, let

$$\|L\| = \sup_{|x|=1} |L(x)|.$$

The following lemma can be found in [48].

Lemma 3.1.1 *Let V, W be finite-dimensional Euclidean spaces and $T \in \mathcal{L}(V, W)$, $T \neq 0$. Then there exist orthonormal systems of vectors $u_1, \ldots, u_k$ in V and $v_1, \ldots, v_k$ in W and numbers $\lambda_i > 0$ such that $\{u_1, \ldots, u_k\}$ is a basis of* $\mathrm{Im}T^*$ *and*

$\{v_1, \dots, v_k\}$ is basis of $\mathrm{Im}T$ *and for* $1 \le i \le k$,

$$Tu_i = \lambda_i v_i, \;\; T^* v_i = \lambda_i u_i,$$

where T^* is the adjoint of T.

The quantities $\lambda_1, \lambda_2, \dots, \lambda_k$ are called the principal dilatation coefficients or singular numbers of T.

Now let $\dim V = \dim W = n$. As the previous lemma shows, for every $T \in \mathcal{L}(V, W)$ we can find the square root of eigenvalues of TT^*. Let us arrange these quantities in an increasing order:

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n.$$

If there exists $K > 0$ such that $\lambda_n \le K\lambda_1$, then we say that T is K-quasiregular.

Let $\dim V = \dim W = n$ and the mapping $T \in \mathcal{L}(V, W)$ be nonsingular. Then $\dim \mathrm{Im}T = \dim V = n$ and all principal dilatation coefficients of T are positive. We want to determine the image of the sphere

$$\mathbb{S}_1(0) := \{x = x_1u_1 + x_2u_2 + \cdots + x_nu_n \in V \;:\; x_1^2 + x_2^2 + \cdots + x_n^2 = 1\},$$

in V, under the nonsingular linear mapping $T \in \mathcal{L}(V, W)$. A point $x = x_1v_1 + \cdots + x_nv_n$ is in $T(\mathbb{S}_1(0))$ if and only if,

$$T^{-1}(x) = \frac{x_1}{\lambda_1}u_1 + \frac{x_2}{\lambda_2}u_2 + \cdots + \frac{x_n}{\lambda_n}u_n \in \mathbb{S}_1(0).$$

Hence $x \in T(\mathbb{S}_1(0))$ if and only if,

$$\frac{x_1^2}{\lambda_1^2} + \frac{x_2^2}{\lambda_2^2} + \cdots + \frac{x_n^2}{\lambda_n^2} = 1.$$

This means that $T(\mathbb{S}_1(0))$ is an ellipsoid with axes of lengths λ_i, $1 \le i \le n$.

Consider $x = x_1u_1 + \cdots + x_nu_n \in V$ with $|x| = 1$. Then $T(x) = \lambda_1x_1v_1 + \cdots + \lambda_nx_nv_n$, so we have

$$\begin{aligned}|T(x)|^2 =< T(x), T(x) > &= \lambda_1^2x_1^2 + \cdots + \lambda_n^2x_n^2 \\ &\le \lambda_n^2(x_1^2 + \cdots + x_n^2) = \lambda_n|x|^2 = \lambda_n^2.\end{aligned}$$

Thus if $|x| = 1$, then $|T(x)| \leq \lambda_n$ and if $x = u_n$ then $|T(x)| = \lambda_n$, hence

$$||T|| := \sup_{|x|=1} |T(x)| = \lambda_n. \tag{3.1}$$

Similarly, we can prove

$$l(T) := \inf_{|x|=1} |T(x)| = \lambda_1. \tag{3.2}$$

Let $V = W$ and $T \in \mathcal{L}(V, V)$ be nonsingular. Then by Lemma 3.1.1 there exist $\{u_1, \ldots, u_n\}$, $\{v_1, \ldots, v_n\}$ bases of $\mathrm{Im}T^*$ and $\mathrm{Im}T$, respectively and principal dilatation coefficients λ_i of T with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ such that, for $1 \leq i \leq n$,

$$Tu_i = \lambda_i v_i, \;\; T^* v_i = \lambda_i u_i.$$

Suppose φ is an orthogonal transformation on V with $\varphi(v_i) = u_i$. Then for $L := T\varphi$ we have $Lv_i = \lambda_i v_i$. Since $\det L = \lambda_1 \lambda_2 \ldots \lambda_n$ and $\det L = \det T \det \varphi$, we have

$$\lambda_1 \lambda_2 \ldots \lambda_n = |\det T|. \tag{3.3}$$

If $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$, then for all $x \in V$, $L(x) = \lambda x$. So L is a homothety and $T = L\varphi^{-1}$.

Conversely if $T = L\varphi$ where L is a homothety and φ is an orthogonal, transformation, then one can deduce that all principal dilatation coefficients of T are equal.

For every nonsingular operator $T \in \mathcal{L}(V, V)$, define

$$K(T) := \lambda_n^n / \lambda_1 \lambda_2 \ldots \lambda_n = ||T||^n / |\det T|. \tag{3.4}$$

Then, $K(T) \geq 1$, and $K(T) = 1$ if and only if $T = L\varphi$, where φ is orthogonal and L is a homothety. Thus an orthonormal transformation $T \in \mathcal{L}(V, V)$ carries every sphere in V into another sphere and the quantity $K(T)$ for an arbitrary nonsingular mapping $T \in \mathcal{L}(V, V)$, characterizes the degree of nonorthogonality of the mapping T.

The following definition is the general definition of a quasiregular holomorphic mapping on a Hermitian manifold, however we will consider the flat Hermitian metric on $\mathbb{C}^n$.

Let M be a complex Hermitian manifold of dimension n, so there exists a Hermitian bilinear form $h : TM \times TM \longrightarrow \mathbb{C}$, which is positive definite and smooth, i.e., in a coordinate neighbourhood $U \subset M$, we have

$$h_p(\xi, \eta) = \sum_{i,j=1}^{n} h_{i,j}(p) dz_i(\xi) \overline{dz_j(\eta)},$$

where $h_{i,j} = h(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})$ is Hermitian and C^∞ on U.
$g := \Re h$ is symmetric and defines a Riemannian metric on M, as a $2n$-dimensional real manifold. Let us denote by RTM the real tangent bundle to M.

Definition 3.1.2 *Let M and N be Hermitian manifolds of the same dimension n, then $f \in \mathcal{H}(M, N)$ is called holomorphic K-quasiregular if for each $p \in M$, $f'(p) : RT_pM \longrightarrow RT_{f(p)}N$ is K-quasiregular, that is for each $p \in M$, $\lambda_{2n}(p) \leq K\lambda_1(p)$.*

As we have shown in (3.1) and (3.2), when X_p ranges over the unit vectors in RT_pM,

$$\begin{aligned} \|f'(p)\| &:= \sup_{X_p} |f'(p)X_p| = \lambda_{2n}(p) \\ l(f'(p)) &:= \inf_{X_p} |f'(p)X_p| = \lambda_1(p). \end{aligned} \tag{3.5}$$

Note that :

- Quasiregular mappings are defined more naturally between Riemannian manifolds.
- For $n = 1$, all holomorphic functions $f \in \mathcal{H}(M, N)$ are 1-quasiregular.
- For $n > 1$, holomorphic K-quasiregular mappings are locally biholomorphic (see [49]).

We can consider the finite dimensinal complex inner product spaces (unitary spaces) V, W, to define the principal dilatation coefficients for a mapping $T \in \mathcal{L}(V, W)$.

Now let M and N be complex Hermitian manifolds of the same dimension n and $f \in \mathcal{H}(M, N)$. For each $p \in M$, consider the holomorphic differential $f'(p) : T_pM \longrightarrow T_{f(p)}N$, with principal dilatation $0 \leq \tilde{\lambda}_1(p) \leq \tilde{\lambda}_2(p) \leq \cdots \leq \tilde{\lambda}_n(p)$, where T_pM and $T_{f(p)}N$ are n-dimensional unitary spaces. Applying (3.1) and (3.2), we have $\|f'(p)\| = \tilde{\lambda}_n(p)$ and $l(f'(p)) = \tilde{\lambda}_1(p)$. This means $f \in \mathcal{H}(M, N)$ is K-quasiregular if and only if $\tilde{\lambda}_n(p) \leq K\tilde{\lambda}_1(p)$. Greene and Wu [27] and Hahn [30] used this fact to define the K-quasiregular holomorphic mappings on Hermitian manifolds.

3.2 Bloch constants

Consider a nonconstant map $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$, where $\mathbb{B}^n$ is the unit ball in $\mathbb{C}^n$ centered at 0. Recall that a ball with center $f(a)$ is called a schlicht ball of f at $f(a)$, if f maps an open subset of $\mathbb{B}^n$ containing a biholomorphically onto this ball. For $z \in \mathbb{B}^n$ let $\beta(z, f)$ be the supremum of radii of schlicht balls centered at $f(z)$, and

$$\beta_f = \sup\{\beta(z, f) : z \in \mathbb{B}^n\}. \tag{3.6}$$

The Bloch constant relative to a family $\mathcal{F} \subset \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$, is defined by

$$\beta(\mathcal{F}) := \inf\{\beta_f : f \in \mathcal{F}\}.$$

We will prove that the function $f \mapsto \beta_f$ from $\mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$ equipped with the compact open topology to $[0, \infty]$ is lower semicontinuous.

To prove our next lemma we need to state Rouché's theorem in several complex variables.

Let Ω be a domain in $\mathbb{C}^n$, $f \in \mathcal{H}(\Omega, \mathbb{C}^n)$ and E_f denote the set of zeros of f. If $f'(a) \neq 0$ at an isolated point a of E_f, then a is called a simple zero of f. By Proposition 2.1 in [5, p. 19], if the closure of a neighbourhood U_a of a zero a of the mapping $f \in \mathcal{H}(\Omega, \mathbb{C}^n)$ does not contain other zeros, then there exists an $\epsilon > 0$ such that for almost all $\zeta \in \mathbb{B}_\epsilon(0)$ the mapping

$$\omega = f(z) - \zeta \tag{3.7}$$

has only simple zeros in U_a, and their number depends neither on ζ nor on the choice of the neighbourhood U_a. The number of zeros of (3.7) in U_a is called the multiplicity of the zero a of the mapping f.

Note that, since E_f is an analytic set, if f is holomorphic on the closure of the bounded domain Ω and $\partial\Omega$ does not contain zeros of f, then f has only isolated zeros in Ω.

Theorem 3.2.1 *(**Rouché principle**) Let Ω be a bounded domain in $\mathbb{C}^n$, $f, g \in \mathcal{H}(\bar{\Omega}, \mathbb{C}^n)$ and suppose the boundary $\partial\Omega$ does not contain zeros of f. If for each point $z \in \partial\Omega$ there is at least one index j, $(j = 1, 2 \ldots n)$ such that*

$$|g_j(z)| < |f_j(z)|,$$

then the mapping $f + g$ has the same number of zeros in Ω (counting multiplicity) as f.

The proof of this theorem can be found in [5, p. 20].

Lemma 3.2.2 *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $\{f_j\}$ be a sequence in $\mathcal{H}(\Omega, \mathbb{C}^n)$ which converges to an injective function f uniformly on compact subsets of Ω, then for every compact $K \subset \Omega$, there exists $J_K \in \mathbb{N}$, such that for every $j > J_K$, f_j is injective on K.*

Proof: Taking G such that $K \subset G \subset \Omega$, if necessary, we can suppose that $\partial\Omega$ is smooth, $\bar{\Omega}$ is compact and f, f_j are holomorphic on a neigbourhood of $\bar{\Omega}$. Since

$\partial\Omega$ and K are compact and f is injective, there exists μ such that,

$$0 < \mu \leq |f(a) - f(z)| \qquad \forall a \in K \qquad \forall z \in \partial\Omega.$$

$\{f_j\}$ converges uniformly to f on K and $\partial\Omega$, so there exists $J_K \in \mathbb{N}$ such that, for every $a \in K$, $z \in \partial\Omega$ and $j > J_K$ we have,

$$|f_j(z) - f(z)| < \mu/2 \quad \text{and} \quad |f_j(a) - f(a)| < \mu/2.$$

Hence, for $a \in K$ and $z \in \partial\Omega$,

$$|f_j(z) - f(z) - (f_j(a) - f(a))| < \mu, \quad \forall j \in J_K. \tag{3.8}$$

Fix $a \in K$ and define $F_j(z) := f_j(z) - f_j(a)$ and $F(z) := f(z) - f(a)$ on $\bar{\Omega}$. By (3.8), for $z \in \partial\Omega$, we have

$$|F_j(z) - F(z)| < \mu \leq |f(z) - f(a)| = |F(z)|, \quad \forall j \in J_K.$$

Therefore by Theorem 3.2.1 $F(z)$ and $F_j(z)$ have the same number of zeros on Ω. Since $F(z) = f(z) - f(a)$ has only one zero on Ω, thus for $j > J_K$, $F_j(z) = f_j(z) - f(a)$ has only one zero $z = a$. Therefore, f_j assumes the value $f_j(a)$ once on Ω and hence once on K.

□

Theorem 3.2.3 *Let Ω be a domain in $\mathbb{C}^n$. Then the function $b : \mathcal{H}(\Omega, \mathbb{C}^n) \longrightarrow [0, \infty]$ defined by $b(f) := \beta_f$ is lower semicontinuous.*

Proof: Without loss of generality we suppose that $\beta_f > 0$. Let $f \in \mathcal{H}(\Omega, \mathbb{C}^n)$ and $0 < r < \beta_f$. Choose r_1, r_2 such that $r < r_2 < r_1 < \beta_f$.
According to the definition of β_f there is a domain $G_1 \subset \Omega$ such that $f|G_1 : G_1 \longrightarrow \mathbb{B}_{r_1}$ is biholomorphic, where $\mathbb{B}_{r_1}$ is an open ball in $\mathbb{C}^n$ with radius r_1.

Consider $\mathbb{B}_{r_2}$ (with the same center as $\mathbb{B}_{r_1}$) and $G_2 := (f|G_1)^{-1}(\mathbb{B}_{r_2})$ so $f|G_2 : G_2 \longrightarrow \mathbb{B}_{r_2}$ is biholomorphic and $\partial f(G_2) = f(\partial G_2)$, with the same argument for $\mathbb{B}_r$, we can define $G_3 := (f|G_1)^{-1}(\mathbb{B}_r)$.

Now let $f_j \in \mathcal{H}(\Omega, \mathbb{C}^n)$ where $\{f_j\}$ converges to f uniformly on compact subsets of Ω. For $\epsilon < r_2 - r$ there exists $J \in \mathbb{N}$ such that for every $j > J$ and for every $z \in \partial G_2$

$$|f_j(z) - f(z)| < \epsilon,$$

By Lemma 3.2.2 there exists $J_2 \in \mathbb{N}$ such that for every $j > J_2$, f_j is injective on $\bar{G}_2$, so f_j is open. Therefore $f_j(\partial G_2) \supseteq \partial f_j(G_2)$.

Since $\{f_j\}$ converges to f uniformly on ∂G_2, for $\epsilon < r_2 - r$ there exists $J_3 > J_1$ such that for every $n > J_3$ and $z \in \partial G_2$,

$$|f_j(z) - f(z)| < \epsilon.$$

For $z \in \partial G_2$ one has

$$\begin{aligned} d(f_j(z), \partial \mathbb{B}_{r_1}) &\leq d(f_j(z), f(z)) + d(f(z), \partial \mathbb{B}_{r_1}) < \epsilon + r_1 - r_2 \\ &< r_1 - r = d(\partial \mathbb{B}_r, \partial \mathbb{B}_{r_1}). \end{aligned}$$

So $f_j(\partial G_2) \cap \bar{\mathbb{B}}_r = \phi$, but $\partial f_j(G_2) = f_j(\partial G_2)$, hence $\partial f_j(G_2) \cap \bar{\mathbb{B}}_r = \phi$, and $f_j(G_2) \supset \mathbb{B}_r$. Therefore $\beta_j := \beta_{f_j} \geq r$ and $\liminf_{j \to \infty} \beta_j \geq r$, for every $r < \beta_f$. So $\liminf_{j \to \infty} \beta_j \geq \beta_f$.

□

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ and consider the family of normalized holomorphic functions

$$\mathcal{H} := \{f \in \mathcal{H}(\mathbb{D}, \mathbb{C}) : f'(0) = 1\}.$$

In 1925 Bloch proved that $\beta := \beta(\mathcal{H}) > 0$. Since then, many efforts have been made to find the precise value of this constant. The following upper and lower estimates for β were shown by Ahlfors and Grunsky [4] and [2].

$$.43 \cdots = \sqrt{3}/4 \leq \beta \leq \frac{\Gamma(1/3)\Gamma(11/12)}{\Gamma(1/4)(1 + \sqrt{3}^2)} = .47 \ldots$$

There was no improvement in this estimate for over half a century. In 1990 Bonk [11] improved the lower estimate to $\sqrt{3}/4+10^{-14} < \beta$. Recently Chen and Gauthier [13] improved this lower bound to $\sqrt{3}/4 + 2 \cdot 10^{-4} < \beta$.

Applying Theorem 3.2.3 for $n = 1$ and a lemma of Brody [12], we will prove Bloch's theorem: the existence of a Bloch constant for normalized holomorphic functions on the disk.

For $n > 1$, Bloch's theorem for the family of holomorphic mappings from $\mathbb{B}^n$ to $\mathbb{C}^n$, with normalization $|\det f'(0)| = 1$, does not hold [60]. One may argue that the correct generalization of the normalization $f'(0) = 1$ to several variables is $f'(0) = I$, where I is the identity mapping. However, for this case there also exist counterexamples [16]. Therefore, special classes of holomorphic mappings have to be considered. Bloch's theorem for the holomorphic classes of quasiregular, bounded and Bloch mappings has been proved by Wu [60], and Liu [41], respectively.

Applying Theorem 3.2.3 we will modify Wu's proof [60] for the existence of a Bloch constant for quasiregular holomorphic mappings in case $n > 1$.

The following lemma of Brody is an important lemma not only in complex hyperbolic space and value distribution theory [40] but also in complex dynamics [20]. This lemma reparametrizes holomorphic mappings from disks in $\mathbb{C}$ to Hermitian manifolds in such a way that the reparametrized mappings enjoy certain properties. We state Brody's lemma only for holomorphic mappings from the unit disk to $\mathbb{C}^n$ and then extend it to holomorphic mappings from the unit ball in $\mathbb{C}^n$ to $\mathbb{C}^n$.

Lemma 3.2.4 *(***Brody***)[12] Given $f \in \mathcal{H}(\mathbb{D}, \mathbb{C}^n)$ with $\|f'(0)\| \geq c \geq 0$, then there exists $\tilde{f} \in \mathcal{H}(\mathbb{D}, \mathbb{C}^n)$ with*

$$\sup_{z\in\mathbb{D}} \|\tilde{f}'(z)\|(1 - |z^2|) = \|\tilde{f}'(0)\| = c. \tag{3.9}$$

Proposition 3.2.5 *Let* $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$ *with* $|\det f'(0)| = 1$. *Then, there exists* $\varphi \in \mathrm{Aut}\mathbb{B}^n$ *and* $0 < t_0 < 1$ *such that* $\tilde{f} := f_{t_0} \circ \varphi$ *satisfies*

$$\sup_{z \in \mathbb{B}^n} |\det \tilde{f}'(z)|(1-|z|^2)^{\frac{n+1}{2}} = |\det \tilde{f}'(0)| = 1,$$

where $f_{t_0} : \bar{\mathbb{B}}^n \longrightarrow \mathbb{C}^n$ *is defined by* $f_{t_0}(z) := f(t_0 z)$.

Proof: For $t \in [0,1)$, consider $f_t : \bar{\mathbb{B}}^n \longrightarrow \mathbb{C}^n$, $f_t(z) := f(tz)$. Define

$$s(t) := \sup_{z \in \bar{\mathbb{B}}^n} |\det f_t'(z)|(1-|z|^2)^{\frac{n+1}{2}},$$

and

$$\kappa(z) := \frac{1}{(1-|z|^2)^{\frac{n+1}{2}}}.$$

Hence

$$\frac{|\det f_t'(z)|}{\kappa(z)} = t^n \frac{(1-|z|^2)^{\frac{n+1}{2}}}{(1-|tz|^2)^{\frac{n+1}{2}}} \frac{|\det f'(tz)|}{\kappa(tz)}. \tag{3.10}$$

Therefore we have:

- For fixed t, $0 \le t < 1$, $|\det f_t'(z)|/\kappa(z)$ is continuous on the closure of $\mathbb{B}^n$, so it is bounded on this compact set and hence $0 \le s(t) < \infty$. Note that by (3.10) $|\det f_t'(z)|/\kappa(z) = 0$ for $|z| = 1$. Therefore the supremum of this function is at a point $z_t \in \mathbb{B}^n$. Hence

$$s(t) = \sup_{z \in \mathbb{B}^n} |\det f_t'(z)|(1-|z|^2)^{\frac{n+1}{2}}.$$

- Since the function $|\det f_t'(z)|/\kappa(t)$ is continuous on $[0,1) \times \bar{\mathbb{B}}^n$, $s(t)$ is continuous on $[0,1)$.

- Let $0 \le t_1 < t_2 < 1$. Then we have $z_1 \in \mathbb{B}^n$ such that

$$s(t_1) = \sup_{z \in \mathbb{B}^n} \frac{|\det f_{t_1}'(z)|}{\kappa(z)} = {t_1}^n \frac{(1-|z_1|^2)^{\frac{n+1}{2}}}{(1-|t_1 z_1|^2)^{\frac{n+1}{2}}} \frac{|\det f'(t_1 z_1)|}{\kappa(t_1 z_1)}.$$

  Set $z_2 = t_1 z_1 / t_2$. By a straightforward calculation, we obtain

$$s(t_1) = \sup_{z \in \mathbb{B}^m} \frac{|\det f_{t_1}'(z)|}{\kappa(z)} \le \frac{|\det f_{t_2}'(z_2)|}{\kappa(z_2)} \le s(t_2).$$

- Since $f_0 = f(0)$, s(0)=0. On the other hand

$$s(t) \geq t^n \frac{|\det f'(0)|}{\kappa(0)} = t^n.$$

Therefore $\liminf_{t\to 1} s(t) \geq 1$. Hence there exists t_0, $0 < t_0 < 1$, such that $s(t_0) = 1$, or $s(t) < 1$, for all t, $0 < t < 1$. In the latter case, $\limsup_{t\to 1} s(t) \leq 1$, and hence $\lim_{t\to 1} s(t) = 1$. Thus, since

$$|\det f'(z)|(1-|z|^2)^{\frac{n+1}{2}} = \lim_{t\to 1} \frac{|\det f'(tz)|}{\kappa(tz)},$$

it follows from (3.10) that we may take $\tilde{f} = f$. If $s(t_0) = 1$ for $0 < t_0 < 1$, then as we explained above, there exists a $z_0 \in \mathbb{B}^n$ with

$$s(t_0) = \frac{|\det f'_{t_0}(z_0)|}{\kappa(z_0)} = 1.$$

Let $\varphi \in \mathrm{Aut}\mathbb{B}^n$ with $\varphi(0) = z_0$. Set $\tilde{f} = f_{t_0} \circ \varphi$. Since

$$|\det \varphi'(0)| = (1-|z_0|^2)^{\frac{n+1}{2}},$$

$|\det f'_{t_0}(\varphi(0))||\det \varphi'(0)| = |\det(f_{t_0} \circ \varphi)'(0)| = 1$. On the other hand,

$$\begin{aligned}\sup_{z\in\mathbb{B}^n} |\det f'_{t_0}(z)|(1-|z|^2)^{\frac{n+1}{2}} &= \sup\{|\det(f_{t_0} \circ \psi)'(0)| : \psi \in \mathrm{Aut}\mathbb{B}^n\} \\ &= \sup\{|\det(f_{t_0} \circ \varphi \circ \psi)'(0)| : \psi \in \mathrm{Aut}(\mathbb{B}^n)\} \\ &= \sup_{z\in\mathbb{B}^n} |\det(f_{t_0} \circ \varphi)'(z)|(1-|z|^2)^{\frac{n+1}{2}}.\end{aligned}$$

Therefore

$$\sup_{z\in\mathbb{B}^n} |\det \tilde{f}'(z)|(1-|z|^2)^{\frac{n+1}{2}} = |\det \tilde{f}'(0)| = 1.$$

□

Lemma 3.2.6 *Let Ω and D be domains in $\mathbb{C}^n$. Let $\varphi \in \mathcal{H}(\Omega, D)$ and $f \in \mathcal{H}(D, \mathbb{C}^n)$. Then*

$$\beta_{f\circ\varphi} \leq \beta_f.$$

Proof: Suppose $f \circ \varphi$ maps a domain $G \subset \Omega$ biholomorphically onto a domain $B \subset \mathbb{C}^n$. Then φ is injective on G and hence φ maps G biholomorphically onto $\varphi(G)$. Thus, $f = f \circ \varphi|_G \circ \varphi|_G{}^{-1}$ maps $\varphi(G)$ biholomorphically onto B.

□

Theorem 3.2.7 *There exists a normalized f on $\mathbb{D}$ such that $\beta_f = \beta$.*

Proof: Since $\beta = \inf\{\beta_f : |f'(0)| = 1\}$, for every $n \in \mathbb{N}$ there exists $f_n \in \mathcal{H}(\mathbb{D})$ such that $|f_n'(0)| = 1$ and $\beta_{f_n} = \beta_n < \beta + \frac{1}{n}$.

By Proposition 3.2.5 there exist $0 < t_n < 1$ and $\varphi_n \in \text{Aut}\mathbb{D}$, such that $\tilde{f}_n := f_{t_n} \circ \varphi_n \in \mathcal{H}(\mathbb{D})$ satisfies

$$\sup_{z\in\mathbb{D}} |\tilde{f}_n{}'(z)|(1 - |z|^2) = |\tilde{f}_n{}'(0)| = 1,$$

where $f_{t_n}(z) = f_n(t_n z)$. So for every $z \in \mathbb{D}$, $|\tilde{f}_n{}'(z)| \leq 1/(1 - |z|^2)$. Therefore the sequence $\{\tilde{f}_n{}'\}$ is uniformly bounded on compact subsets of $\mathbb{D}$.

By defining $\tilde{g}_n(z) := \tilde{f}_n(z) - \tilde{f}_n(0)$ if necessary, we can suppose $\tilde{f}_n(0) = 0$. Hence applying the Cauchy theorem, we have

$$|\tilde{f}_n(z)| \leq |\int_0^z \tilde{f}_n{}'(z)\, dz|,$$

so $\tilde{f}_n$ is uniformly bounded on compact subsets of $\mathbb{D}$. Therefore we can suppose $\tilde{f}_n$ converges to $\tilde{f} \in \mathcal{H}(\mathbb{D})$ uniformly on compact subsets of $\mathbb{D}$ and $|\tilde{f}'(0)| = 1$.

We shall prove that $\beta_{\tilde{f}_n} \leq \beta_{f_n}$. In fact if $\mathbb{D}_\beta(\tilde{f}_n(z_0))$ is a schlicht disk for $\tilde{f}_n$, then for constant $0 < t_n < 1$, $\mathbb{D}_\beta(f_n(t_n\varphi_n(z_0)))$ is a schlicht disk for f_n. Indeed, if $\tilde{S} \subset \mathbb{D}$ is a domain and $\tilde{f}_n|_{\tilde{S}} : \tilde{S} \longrightarrow \mathbb{D}_\beta(\tilde{f}_n(z_0))$ is biholomorphic, then considering $S := \{\xi \in \mathbb{D} : \frac{1}{t_n}\varphi_n^{-1}(\xi) \in \tilde{S}\}$, and $f_n|_S : S \longrightarrow \mathbb{D}_\beta(f_n(t_n\varphi_n(z_0)))$, one can prove that $f_n|_S$ is biholomorphic.

Since b is lower semicontinuous, we have $\beta_{\tilde{f}} \leq \liminf_{n\to\infty} \tilde{\beta}_n \leq \liminf_{n\to\infty} \beta_n = \lim_{n\to\infty} \beta_n = \beta$. On the other hand $\beta \leq \beta_{\tilde{f}}$ so $\beta = \beta_{\tilde{f}}$.

□

Corollary 3.2.8 *The Bloch constant for normalized holomorphic functions on $\mathbb{D}$ is positive.*

This is Bloch's theorem.

Lemma 3.2.9 *Let $\varphi_j : \mathbb{B}^n \longrightarrow \mathbb{B}^n$ be a sequence of holomorphic mappings such that φ_j converges to I in the compact open topology on $\mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$. Then $\beta_{f\circ\varphi_j}$ converges to β_f, for every $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$.*

Proof: Since $f_j := f \circ \varphi_j$ converges to f uniformly on compact subsets of $\mathbb{B}^n$, and by Theorem 3.2.3, b is lower semicontinuous, so $\beta_f \leq \liminf_{j\to\infty} \beta_{f_j}$. On the other hand by Lemma 3.2.6 $\limsup_{j\to\infty} \beta_{f_j} \leq \beta_f$, so $\lim_{j\to\infty} \beta_{f_j} = \beta_f$.

□

As a consequence, we have that the Bloch constant for many families of normalized holomorphic mappings on the open ball (if it exists) is the same as for the closed ball. Indeed, if f is a normalized holomorphic mapping in $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$ and $\varphi_j(z) = (1 - 1/j)z$, then $g_j(z) = f \circ \varphi_j(z)/(1 - 1/j)$ is a normalized mapping in $\mathcal{H}(\bar{\mathbb{B}}^n, \mathbb{C}^n)$ and $\{g_j\}$ converges to f. This explains why, in the literature, the classical Bloch theorem for holomorphic functions in the unit disk is stated sometimes for the open disk and sometimes for the closed disk. The statements are equivalent.

Theorem 3.2.10 *[60] For every sequence of K-quasiregular mappings $\{f_j\}$ in $\mathcal{H}(\bar{\mathbb{B}}^n, \mathbb{C}^n)$ with $|\det f'(0)| = 1$ there exist sequences $\{b_j\}$ in $\mathbb{B}^n$ and $\{r_j\}$, $0 < r_j \leq 1$ such that the sequence $\{\tilde{f}_{j_k}\} \subset \mathcal{H}(\bar{\mathbb{B}}^n, \mathbb{C}^n)$, defined by $\tilde{f}_j(z) := f_j(b_j + (r_j/2)z) - f_j(b_j)$, is a sequence of K-quasiregular mappings which converges uniformly on $\bar{\mathbb{B}}^n$ to a holomorphic map $\tilde{f}$ with $\beta_{\tilde{f}} \neq 0$.*

Recall that the normalization which we will consider on quasireqular mappings f is $|\det f'(0)| = 1$.

Theorem 3.2.11 *[60] The Bloch constant for the family of holomorphic normalized K-quasiregular mappings on $\bar{\mathbb{B}}^n$ is positive.*

Proof: Applying the definition of β, for each $j \in \mathbb{N}$ there exists a normalized K-quasiregular holomorphic mapping f_j, such that $\beta_j := \beta_{f_j} < \beta + 1/j$.

By Theorem 3.2.10 there exists $b_j \in \mathbb{B}^n$ and $0 < r_j \leq 1$ such that the sequence $\{\tilde{f}_j\}$ converges uniformly on $\bar{\mathbb{B}}^n$, to a holomorphic map $\tilde{f}$, where

$$\tilde{f}_j(z) = f_j(b_j + \frac{r_j}{2}z) - f_j(b_j).$$

Let $\tilde{f}_j|_{\tilde{S}} : \tilde{S} \longrightarrow \mathbb{B}_\alpha(\tilde{f}_j(z_0))$ be a biholomorphism. Then consider

$$S = \{\xi \in \mathbb{B}^n : \frac{2}{r_j}\xi - b_j \in \tilde{S}\}.$$

We can prove that $f_j|_S : S \longrightarrow \mathbb{B}_\alpha(f_j(r_j/2)z_0 + b_j)$ is a biholomorphism, so $\tilde{\beta}_j := \beta_{\tilde{f}_j} \leq \beta_j$, for each $j \in \mathbb{N}$. Therefore $\tilde{\beta}_j \leq \beta_j < \beta + 1/j$. Since by Theorem 3.2.3 the function b is lower semicontinuous, $\beta_{\tilde{f}} \leq \liminf_{j\to\infty} \tilde{\beta}_j$. Hence

$$\beta_{\tilde{f}} \leq \liminf_{j\to\infty} \tilde{\beta}_j \leq \liminf_{j\to\infty} \beta_j = \lim_{j\to\infty} \beta_j \leq \beta.$$

On the other hand $\beta \leq \beta_{\tilde{f}}$. Therefore $\beta = \beta_{\tilde{f}} > 0$.

□

3.3 Bloch constants and smooth quasiregular mappings

Quasiregular mappings can be defined, not only on Hermitian manifolds, but also on Riemannian manifolds. In contrast to quasiregular holomorphic mappings, this class has been studied for a long time. (See for example [3], [49], [57], [58], [59]).

As we mentioned in the previous section, for $\mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$, Bloch's theorem fails when $n > 1$. Despite this fact one might ask whether Bloch's theorem holds for

$\mathcal{H}(\mathbb{C}^n, \mathbb{C}^n)$. Rosay and Rudin [50] proved that, for this case also, Bloch's theorem does not hold. Similarly one might ask the same kind of questions for quasiregular mappings. For example one may ask whether Bloch's theorem holds for quasiregular mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$. This question was answered by Eremenko [18] positively. We ask whether Bloch's theorem holds for K-quasiregular mappings from the unit ball in $\mathbb{R}^n$ to $\mathbb{R}^n, n \geq 1$. In the next example we will answer this question negatively.

In the following example for every $K > 1$, we will construct a family of K-quasiregular mappings on the unit disk in $\mathbb{R}^2$, (which we identify with $\mathbb{C}$) for which Bloch's theorem does not hold.

Example 3.3.1 Fix $t_0 \in (0,1)$ we define

$$h(x) = \begin{cases} x, & \text{if } x \leq \ln t_0; \\ (1 - \frac{1}{K}) \ln t_0 + \frac{1}{K} x, & \text{if } \ln t_0 < x \leq 0. \end{cases}$$

We also define

$$f_{t_0}(z) := \exp(h(\ln |z|)) \exp(i \arg z),$$

so if $|z| \leq t_0$, then $|z| = \exp(\ln |z|) \leq t_0$. Hence $h(\ln |z|) = \ln |z|$ and

$$f_{t_0}(z) = \exp(\ln |z|) \exp(i \arg z) = z.$$

Also for $t_0 < |z| \leq 1$, we have $h(\ln |z|) = (1 - 1/K) \ln t_0 + (1/K) \ln |z|$. Thus

$$\begin{aligned} f_{t_0}(z) &= \exp[(1 - 1/K) \ln t_0 + (1/K) \ln |z|] \exp(i \arg z) \\ &= {t_0}^{(1-1/K)} |z|^{(1/K)-1} z. \end{aligned}$$

The function h is continuous in $(-\infty, 0]$, since

$$\lim_{x \to \ln t_0{}^+} h(x) = \lim_{x \to \ln t_0{}^+} (1 - 1/K) \ln t_0 + (1/K) x = \ln t_0.$$

Of course h is not smooth and so f_{t_0} is not smooth.

Let $\alpha = \ln t_0$. For $\epsilon > 0$ sufficiently small, define $g(x)$, a linear function, such that $g(\alpha-\epsilon) = 0$ and $g(\alpha+\epsilon) = (1/K) - 1$. Let $\hat{h}(x) = x + \int_{\alpha-\epsilon}^{x} g(t)dt$. Therefore $\hat{h} : [\alpha-\epsilon, \alpha+\epsilon] \longrightarrow \mathbb{R}$ has the following properties:

(a) $\hat{h}(\alpha-\epsilon) = \alpha - \epsilon$.

(b) $\hat{h}(\alpha+\epsilon) = \alpha + (1/K)\epsilon$.

(c) $\hat{h}'_{+}(\alpha-\epsilon) = 1$ and $\hat{h}'_{-}(\alpha+\epsilon) = 1/K$.

(d) $\hat{h}'(x)$ is decreasing and by (c), $1/K \leq \hat{h}'(x) \leq 1 < K$.

Now if we redefine h as,

$$h(x) = \begin{cases} x, & \text{if } \ x \leq \alpha - \epsilon; \\ \hat{h}(x) & \text{if } \alpha - \epsilon \leq x \leq \alpha + \epsilon; \\ (1-\frac{1}{K})\alpha + \frac{1}{K}x, & \text{if } \ \alpha + \epsilon \leq x \leq 0. \end{cases}$$

Then h is smooth and therefore f_{t_0} is smooth.

We shall prove that f_{t_0}, as a function from $\bar{\mathbb{D}} \subset \mathbb{R}^2$ to $\mathbb{R}^2$ is quasiregular. For $z = r\exp(i\theta)$, $f_{t_0} = (\exp(h(\ln r))\cos\theta, \exp(h(\ln r))\sin\theta)$. Let us define

$$\begin{aligned} u(r,\theta) &:= \exp(h(\ln r))\cos\theta, \\ v(r,\theta) &:= \exp(h(\ln r))\sin\theta. \end{aligned} \tag{3.11}$$

Thus writing $x = r\cos\theta$, $y = r\sin\theta$ we have

$$\det f'_{t_0} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x},$$

which is equal to $(1/r^2)h'(\ln r)\exp(2h(\ln r))$. Since $0 \neq 1/K < h'$, it follows that f'_{t_0} is a bijection.

Let λ_1 and λ_2 be the principal dilatation coefficients of f'_t, $0 < \lambda_1 \leq \lambda_2$. In order to prove f is K-quasiregular we must show that for each $z \in \mathbb{D}$, $\lambda_2(z) \leq K\lambda_1(z)$. Recall that by (3.2) and (3.1) for each z, $l(f'_t(z)) = \lambda_1(z)$ and $\|f'_t(z)\| = \lambda_2(z)$.

Let $X = \exp(i\beta)$ be a unit vector. Then by Euler's formulas $2f'(z)X =$

$p(z)\exp(i\beta) + q(z)\exp(-i\beta)$, where

$$p = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}), \tag{3.12}$$

and

$$q = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}). \tag{3.13}$$

Consequently $2|f'(z)X| = |p(z) + q(z)\exp(-2i\beta)|$. From this, we have $2\|f'(z)\| = |p(z)| + |q(z)|$ and $2l(f'(z)) = ||p(z)| - |q(z)||$. So we have to prove that, for each $z \in \mathbb{D}$,

$$H(z) := \frac{|p(z)| + |q(z)|}{||p(z)| - |q(z)||} \leq K. \tag{3.14}$$

Since $0 < 1/K \leq h' \leq 1 < K$, by applying (3.11) in (3.12) and (3.13), we obtain

$$|p(z)| = \frac{1}{r}(h'(\ln r) + 1)\exp h(\ln r),$$

and

$$|q(z)| = \frac{1}{r}(1 - h'(\ln r))\exp h(\ln r).$$

Therefore $H(z) = 1/h'(\ln r) \leq K$, as required.

The image of the unit disk under f is the disk of radius $a = t^{1-1/K}$. For any fixed $K > 1$, if we take t small we may make a as small as we like. This shows that for the family $\{f_t : 0 < t < 1\}$ the Bloch theorem does not hold.

Chapter 4

Holomorphic functions with Mittag-Leffler constraints

In this chapter we will consider the problem of approximation of functions defined on a closed set F by meromorphic functions with prescribed poles and zeros outside the set of approximation and then we will consider simultaneous approximation and interpolation by entire functions.

A. Sauer [53] considered the problem of approximation by entire functions which possess certain asymptotic expansions. We will generalize his main result applying a different approach.

4.1 Background

For $F \subset \mathbb{C}$, we denote by $\mathcal{H}(F)$ and $\mathcal{M}(F)$, the set of all holomorphic functions and meromorphic functions on F, respectively. We also denote the set of all functions continuous on F and holomorphic on F° by $\mathcal{A}(F)$, where F° is the interior of F.

Definition 4.1.1 *Let F be a closed subset of $\mathbb{C}$. A speed on F is a positive, continuous function on F. If ε is a speed on F, then F is called a set of ε-approximation, provided that for each $f \in \mathcal{A}(F)$ and each constant $\lambda > 0$, there is* *a function* $g \in \mathcal{H}(\mathbb{C})$ *such that for* $z \in F$,

$$|f(z) - g(z)| < \lambda\varepsilon(z).$$

Definition 4.1.2 *A closed subset* F *of* $\mathbb{C}$ *is called a set of uniform approximation if* F *is a set of* ε*-approximation for some (hence for each) positive constant* ε.

Note that, if F is a set of ε-approximation and the speed ε is bounded then F is, *a fortiori*, a set of uniform approximation.
A characterization of sets of uniform approximation is due to Arakelyan [7].

Theorem 4.1.3 *A set* F *is a set of uniform approximation if and only if* $\mathbb{C}_\infty \setminus F$ *is connected and locally connected at* ∞.

A strong form of approximation where F is a closed subset in $\mathbb{C}$ such that for every speed on F and for every $f \in \mathcal{A}(F)$ there exists $g \in \mathcal{H}(\mathbb{C})$ with $|f(z) - g(z)| < \varepsilon(z)$ on F, was characterized by Gauthier [23] and Nersesyan [45]. On open Riemann surfaces this characterization is due to Boivin [10].

Let us state another Theorem of Arakelyan which will be useful in this paper (see [21, p. 39]).

Theorem 4.1.4 *Let* $\varepsilon : [0, \infty) \longrightarrow (0, \infty)$ *be continuous and decreasing such that*

$$\int_1^\infty t^{-3/2} \log \varepsilon(t) dt > -\infty. \tag{4.1}$$

Then for every set of uniform approximation F *and for every function* $f \in \mathcal{A}(F)$ *there exists an entire function* g *such that*

$$|f(z) - g(z)\ | < \varepsilon(|z|),$$

for all $z \in F$.

We may extend any continuous function $\varepsilon : [0, \infty) \longrightarrow (0, \infty)$ to a function continuous on all of $\mathbb{C}$ by setting $\varepsilon(z) = \varepsilon(|z|)$. Let us call such a function ε satisfying the conditions in Theorem 4.1.4 a canonical speed. As an example, we can consider $\varepsilon : [0, \infty) \longrightarrow (0, \infty)$ defined by

$$\varepsilon(t) := \exp(-t^{1/3}).$$

Then ε is a canonical speed satisfying $\varepsilon < 1$.

4.2 Basic results

We will show in the next example that the decreasing condition in Theorem 4.1.4 cannot be waived. First we will state the Two Constants Theorem which will be used in this example (see [32]).

Theorem 4.2.1 *Let D be a Jordan domain such that the boundary of D consists of two arcs α and β disjoint except for their end points. Then, for each compact set $K \subset D$ there exists a constant $0 < \omega < 1$, such that if $f \in \mathcal{H}(D)$ is bounded by M and $\limsup_{z \to \alpha} |f(z)| \leq m$, then*

$$\max_{z \in K} |f(z)| \leq M^{1-\omega} m^{\omega}.$$

Example 4.2.2 Consider the set of uniform approximation

$$F = \{z \in \mathbb{C} : \Re z \geq 0\}.$$

We will construct $\varepsilon : [0, \infty) \longrightarrow (0, 1]$ continuous satisfying (4.1), with $\lim_{t \to \infty} \varepsilon(t) = 0$, in such a way that F is not a set of ε-approximation.

For each $n \in \mathbb{N}_1 := \mathbb{N} \backslash \{1\}$, set $F_n = \{z \in \mathbb{C} : |z| \leq n, \Re z \geq 0\}$ and $\alpha_n = \{z \in \mathbb{C} : |z| = n, \Re z > 0\}$. By the previous theorem there exists a decreasing sequence $\{\varepsilon_n : 0 < \varepsilon_n < 1, n \in \mathbb{N}\}$, such that if $f \in \mathcal{A}(F_n)$, $|f| \leq 1$ and $|f| \leq \varepsilon_n$ on α_n. Then

$$\max_{z \in K} |f(z)| < \frac{1}{n},$$

where $K = \{z \in \mathbb{C} : |z-1| \leq \frac{1}{3}\}$. Therefore if $f \in \mathcal{A}(F)$, $|f| \leq \varepsilon_n$ on α_n and $|f| \leq 1$, then $f(z) = 0$ for every $z \in K$, so $f \equiv 0$. Hence if $\varepsilon : [0,\infty) \longrightarrow (0,1]$ is any continuous function such that for $n \in \mathbb{N}_1$, $\varepsilon(n) \leq \varepsilon_n$, then $f \in \mathcal{A}(F)$ and $|f| \leq \varepsilon$ implies $f \equiv 0$. This shows that F is not a set of ε-approximation. Indeed, consider $f(z) := 1/(z+1), f \in \mathcal{A}(F)\backslash\mathcal{H}(\mathbb{C})$ and suppose there exists $g \in \mathcal{H}(\mathbb{C})$ such that on F

$$|f-g| < \varepsilon.$$

Thus $f = g$ on F, so $f = g$ on $\mathbb{C}\backslash\{-1\}$ which is a contradiction.

Among all continuous functions $\varepsilon : [0,\infty) \longrightarrow (0,1]$ with $\varepsilon(n) \leq \varepsilon_n, n \in \mathbb{N}_1$, we construct one which satisfies (4.1) and $\lim_{t\to\infty} \varepsilon(t) = 0$.

Let $\tilde{\varepsilon} : [0,\infty) \longrightarrow (0,1]$ be a continuous decreasing function satisfying (4.1). For $n > 1$, choose ε_n as above, and decreasing so rapidly that $\varepsilon_n < \tilde{\varepsilon}(n+1)$, and choose $0 < \eta_n < 1/2$ such that

$$\int_{n-\eta_n}^{n+\eta_n} t^{-3/2} \log \varepsilon_n dt > -\frac{1}{2^n}.$$

Now we define a continuous function ε as follows: on $[n, n+\eta_n]$, it is the segment from point (n, ε_n) to the point $(n+\eta_n, \tilde{\varepsilon}(n+\eta_n))$, on $[n+\eta_n, n+1-\eta_{n+1}]$, it is equal to $\tilde{\varepsilon}(t)$ and on $[n+1-\eta_{n+1}, n+1]$, it is the segment from point $(n+1-\eta_{n+1}, \tilde{\varepsilon}(n+1-\eta_{n+1}))$ to the point $(n+1, \varepsilon_{n+1})$, for each $n \geq 1$. We may define ε on $[0,1]$ by $\varepsilon(t) = \varepsilon_1$. Thus, considering $I_n := [n-\eta_n, n+\eta_n]$, we deduce

$$\begin{aligned}\int_1^{\infty} t^{-3/2} \log \varepsilon(t) dt &= \int_{[0,\infty)\backslash\bigcup_{n=1}^{\infty} I_n} t^{-3/2} \log \varepsilon(t) dt + \int_{\bigcup_{n=1}^{\infty} I_n} t^{-3/2} \log \varepsilon(t) dt \\ &\geq \int_1^{\infty} t^{-3/2} \log \tilde{\varepsilon}(t) dt + \sum_{n=1}^{\infty} \int_{I_n} t^{-3/2} \log \varepsilon_n dt \\ &> -\infty,\end{aligned}$$

as required.

To prove the next theorem we need two lemmas.

Lemma 4.2.3 *Let F be a set of uniform approximation and U an open neighbourhood of F. Then, there exists a simply connected open neighbourhood U_s of F such that $F \subset U_s \subset U$.*

Proof: Let $\mathcal{W} := \{W_j : j \in J\}$ be the class of all bounded components of $\mathbb{C} \backslash U$. Using triangulation we may assume that ∂U is locally polygonal, so $\mathcal{W}$ is locally finite.

For each $j \in J$, let $\widetilde{W_j}$ be the component of $\mathbb{C} \backslash F$ containing W_j. $\widetilde{W_j}$ is unbounded because $\mathbb{C}_\infty \backslash F$ is connected.

By Theorem 4.1.3, $\mathbb{C}_\infty \backslash F$ is locally connected at ∞, so by a characterization of the local connnectedness of $\mathbb{C}_\infty \backslash F$ at ∞, for every neighbourhood G_1 of ∞ there exists a neighbourhood $G_2 \subset G_1$ of ∞ with the property that each point $z \in G_2 \backslash F$, $z \neq \infty$ can be connected to ∞ in $\mathbb{C}$ by a continuous curve $\gamma \subset G_1 \backslash F$. This means that the continuous function $\gamma : [0,1] \longrightarrow G_1 \backslash F$ with $\gamma(0) = z$ has the property that for any given compact set $K \subset \mathbb{C}$ there is a t_K such that, for each $t > t_K$, $\gamma(t) \notin K$. Therefore there is a basis $\{V_j : j \in J\}$ of open neighbourhoods of ∞ such that, for each j, $V_{j+1} \subset V_j$ and each $w \in V_{j+1}$ can be connected to ∞ by a curve in $V_j \backslash F$.

Hence for each $j \in J$, there exists a curve σ_j in $\mathbb{C} \backslash F$ from a point $w_j \in W_j$ to ∞ and we may assume that the family $\{\sigma_j : j \in J\}$ is locally finite. Let B_j be a connected polygonal neighbourhood of σ_j which does not intersect F. We may also assume that $\{\bar{B}_j : j \in J\}$ is a locally finite family of closed sets. Hence $\cup_{j \in J} \bar{B}_j$ is closed. Set $U_s = U \backslash \cup_{j \in J} \bar{B}_j$, thus $F \subset U_s$. Then, $\mathbb{C}_\infty \backslash U = \cup_{j \in J} W_j \cup W_\infty$, where W_∞ is the component of $\mathbb{C}_\infty \backslash U$ which contains ∞, so

$$\mathbb{C}_\infty \backslash U_s = (\cup_{j \in J} W_j \cup B_j) \cup \{\infty\} \cup W_\infty,$$

which is connected, therefore U_s is simply connected.

$\square$

Lemma 4.2.4 *Let F be a set of uniform approximation and $f \in \mathcal{A}(F)$ without zeros on F. Then there exists a branch of* $\ln f$ *in $\mathcal{A}(F)$.*

Proof: Considering a continuous extension of f on $\mathbb{C}$ and applying the previous lemma, we can suppose the existence of a continuous nonvanishing extension of f on a simply connected neighbourhood U_s of F. Thus there exists $f_s : U_s \longrightarrow \mathbb{C}_0$ continuous such that $f_s \circ i = f$ where $\mathbb{C}_0 := \mathbb{C}\backslash\{0\}$ and $i : F \longrightarrow U_s$ is the identity map. Let $\tilde{\mathbb{C}}_0 = \mathbb{C}$ be the universal covering of $\mathbb{C}_0$ and $\tilde{f} : U_s \longrightarrow \tilde{\mathbb{C}}_0$ be the lift of f_s, so $\exp \circ \tilde{f} = f_s$ and $\exp \circ \tilde{f} \circ i = f$.

We will prove that $\tilde{f}$ is holomorphic on F°. Let $z_\circ \in F^\circ$, $p_\circ = \tilde{f}(z_\circ)$ and $\tilde{U}_0$ a neighbourhood of $p_\circ$ such that $\exp(\tilde{U}_0)$ is biholomorphic to $U_0 \subset \mathbb{C}_0$. Suppose V_0 is a neighbourhood of $z_\circ$, $V_0 \subset F^\circ$ small enough such that $\tilde{f}(V_0) \subset \tilde{U}_0$ and $f(V_0) \subset U_0$. Hence $\exp \circ \tilde{f}|_{V_0} = f|_{V_0}$ and $\tilde{f}|_{V_0} = (\exp|_{U_0})^{-1} \circ f|_{V_0}$. Therefore $\tilde{f}$ is holomorphic on F°.

□

A divisor on $\mathbb{C}$ is a function $D : \mathbb{C} \longrightarrow \mathbb{Z}$, such that the set of points $z \in \mathbb{C}$ where $D(z) \neq 0$ is a discrete set. we denote a divisor D by a formal sum

$$D := \sum_{\zeta \in \mathbb{C}} D(\zeta)\zeta.$$

Suppose $\Omega \subseteq \mathbb{C}$, $f \in \mathcal{M}(\Omega)$ and $\zeta \in \Omega$ and let the Laurent series expansion of f in a neighbourhood of ζ be

$$f(z) = \sum_{j=n}^{\infty} a_j (z - \zeta)^j$$

and $a_n \neq 0$. Then the order of f at ζ is n and will be denoted by $\mathrm{ord}_\zeta(f)$. Clearly f has a zero of order n at ζ if $\mathrm{ord}_\zeta(f) = n \geq 1$, f has a pole of order n at ζ if $\mathrm{ord}_\zeta(f) = n < 0$, and f has neither zero nor pole at ζ if $\mathrm{ord}_\zeta(f) = 0$. By the divisor of $f \in M(\mathbb{C})$, $f \neq 0$, we mean the divisor

$$D := \sum_{\zeta \in \mathbb{C}} \mathrm{ord}_\zeta(f)\zeta.$$

We will call a sequence $\{z_n\}$ (possibly finite or empty) of distinct points in $\mathbb{C}$ admissible (with respect to a set F) if $\{z_n\}$ has no finite accumulation point and all z_n are contained in $\mathbb{C}\backslash F$.

Theorem 4.2.5 *Let F be a set of ε-approximation, $\varepsilon \leq 1$, $\{z_n\}$ an admissible sequence and $\{o_n\}$ a sequence in $\mathbb{Z}_0 := \mathbb{Z}\backslash\{0\}$. Further, let $\varphi \in \mathcal{A}(F)$ without zeros and bounded on F. Then there exists $f \in \mathcal{M}(\mathbb{C})$ such that $D = \sum_n o_n z_n$ is the divisor of f and $|\varphi - f| < \varepsilon$ on F.*

Proof: We remark that

$$|1 - e^w| \leq e\,|w|\,, \qquad\qquad if \quad |w| \leq 1.$$

Consider $h \in \mathcal{M}(\mathbb{C})$ such that the divisor of h is D (see [33]). Since F is a set of uniform approximation, by Lemma 4.2.4 there exists a simply connected neighbourhood of F containing no z_n and branches H and Φ of $\ln h$ and $\ln\varphi$ respectively, in $\mathcal{A}(F)$. By hypothesis there exists $G \in \mathcal{H}(\mathbb{C})$ such that on F,

$$|H - (G + \Phi)| < \frac{\varepsilon}{e\tilde{M}} < 1,$$

where $M = \sup_{z\in F} |\varphi(z)|$ and $\tilde{M} = \max\{1, M\}$.

Set $g = e^{-G}$ and $f = gh$, so $f \in \mathcal{M}(\mathbb{C})$, and the divisor of f is D. On F we have

$$\begin{aligned}
|\varphi - f| &= |1 - \frac{f}{\varphi}||\varphi| \\
&= |1 - \frac{gh}{\varphi}||\varphi| \\
&\leq e\,|H - G - \Phi|\,|\varphi| \\
&< e\frac{\varepsilon}{e\tilde{M}}\tilde{M} = \varepsilon.
\end{aligned}$$

□

Corollary 4.2.6 *Let F, ε, $\{z_n\}$ be as in the previous theorem and $\{o_n\}$ a sequence in $\mathbb{N}$. Then there exists $f \in \mathcal{H}(\mathbb{C})$ with exactly the zeros z_n of order o_n and $|1 - f| < \varepsilon$.*

We wish to apply the results of approximation theory to a study of asymptotic expansions.

Definition 4.2.7 *Let F be an unbounded set in $\mathbb{C}$. A function $f : F \longrightarrow \mathbb{C}$ has an asymptotic expansion in F if there exists a complex sequence $\{a_n\}$ such that for all $n \in \mathbb{N}$*

$$z^n \left(f(z) - \sum_{i=0}^{n-1} a_i z^{-i} \right) \to a_n$$

as $z \to \infty$ in F . We denote $f(z) \sim \sum_{i=0}^{\infty} a_i z^{-i}$.

For $n = 1, 2, ...$, we set

$$R_n(f, z) := f(z) - \sum_{i=0}^{n-1} a_i z^{-i}.$$

Then $f(z) \sim \sum_{i=0}^{\infty} a_i z^{-i}$ is equivalent to $R_n(f, z) = O(|z|^{-n})$ for all $n \in \mathbb{N}$. Note that the asymptotic expansion of f need not converge and is therefore a formal power series in $1/z$.

As a particular case if a is a constant, we have that $f \sim a$ if and only if $f(z) - a = O(|z|^{-n})$, for all $n \in \mathbb{N}$.

The next corollary will show that Corollary 4.2.6 is a generalization of the main result of [53].

Corollary 4.2.8 *If F is a set of uniform approximation, $\{z_n\}$ an admissible sequence, and $\{o_n\}$ a sequence in $\mathbb{N}$, then there exists $f \in \mathcal{H}(\mathbb{C})$ with exactly the zeros z_n of order o_n and $f \sim 1$ on F.*

Proof: Taking the canonical speed $\varepsilon(z) := e^{-|z|^{\frac{1}{3}}}$ on F and applying Corollary 4.2.6 implies $f \sim 1$.

□

Theorem 4.2.9 *Let F be a set of uniform approximation, $\{z_n\}$ an admissible sequence, $\{o_n\}$ a sequence in $\mathbb{Z}_0$, $\varepsilon \leq 1$ a canonical speed. Then, there exists $f \in \mathcal{M}(\mathbb{C})$ such that $D := \sum_n o_n z_n$ is the divisor of f and $|f| < \varepsilon$ on F.*

Proof: Since F is a set of uniform approximation and ε is a canonical speed, there exists a nonvanishing function $\varphi \in \mathcal{H}(\mathbb{C})$ such that

$$|\varphi(z)| < \frac{1}{2}\varepsilon(z),$$

for all $z \in F$, (see [21], p. 40).

By Theorem 4.2.5, there exists $f \in \mathcal{M}(\mathbb{C})$ with divisor D such that for $z \in F$,

$$|\varphi(z) - f(z)| < \frac{1}{2}\varepsilon(z),$$

so

$$|f(z)| < \varepsilon(z),$$

for all $z \in F$.

□

Corollary 4.2.10 *Let F be an unbounded set of uniform approximation, $\{z_n\}$ an admissible sequence and $\{o_n\}$ a sequence in $\mathbb{Z}_0$. Then there exists $f \in \mathcal{M}(\mathbb{C})$ such that the divisor of f is $\sum_n o_n z_n$ and $f \sim 0$ on F.*

By a left tail at $\zeta \in \mathbb{C}$ (see [24]), we mean a series of the form

$$\sum_{j=-\infty}^{J} a_j (z - \zeta)^j,$$

for any integer J. If a left tail is convergent in some deleted neighbourhood of ζ then we say that it is an admissible left tail. If the coefficients of a left tail at ζ coincide with the corresponding Laurent coefficients of a function f holomorphic in a deleted neighbourhood of ζ, then we say that the left tail is a left tail of f at the point ζ. For the special case where $J = -1$, we call it a p-tail.

Lemma 4.2.11 *Let F be a set of ε-approximation and $Z:= \{z_n\}$ an admissible sequence. Moreover, for each n let t_n be an admissible left p-tail at z_n. Then for $f \in \mathcal{A}(F)$, there exists a function g holomorphic in $\mathbb{C}\backslash Z$ such that t_n is a left tail of g at z_n and $|f - g| < \varepsilon$ on F.*

Proof: By Theorem 4 in [24], there exists a function f_∞ holomorphic on $\mathbb{C}$ except for isolated (possible artificial) singularities at the points of Z such that for each n, t_n is a left tail of f_∞ at z_n. Since Z is an admissible sequence, $f - f_\infty \in \mathcal{A}(F)$. On the other hand F is a set of ε-approximation so there exists $g_0 \in \mathcal{H}(\mathbb{C})$ such that on F,

$$|f - f_\infty - g_0| < \varepsilon.$$

Set $g := f_\infty + g_0$. Then g is holomorphic on $\mathbb{C}$ except for isolated singularities at the points of Z, such that for each n, t_n is a left tail of g at z_n.

□

Theorem 4.2.12 *Let F be a set of ε-approximation, $\varepsilon \leq 1$, $Z := \{z_n\}$ an admissible sequence, and*

$$t_n(z) := \sum_{j=-\infty}^{j_n} w_n^j (z - z_n)^j,$$

an admissible left tail at z_n. Then for $f \in \mathcal{A}(F)$ there exists a function g holomorphic in $\mathbb{C}\backslash Z$ such that, t_n is a left tail of g at z_n and for $z \in F$,

$$|f(z) - g(z)| < \varepsilon(z). \tag{4.2}$$

Proof: Corollary 4.2.6 of Theorem 4.2.5 implies that there exists an entire function $\tilde{f}$ with zeros exactly at z_n of order $o_n > j_n$ and on F,

$$|1 - \tilde{f}(z)| < \frac{\varepsilon(z)}{4}.$$

For each n let g_n be the p-tail of the function $t_n/\tilde{f}$ at z_n so $t_n/\tilde{f} = g_n + \varphi_n$ locally at z_n with φ_n holomorphic at z_n.

By Lemma 4.2.11, there exists a function γ holomorphic on $\mathbb{C}\backslash Z$ such that g_n is a left tail of γ at z_n and for $z \in F$,

$$|\gamma(z)| < \frac{\varepsilon(z)}{4},$$

for all z in F.

Define $h := \gamma\tilde{f}$. Since $\tilde{f}$ is an entire function so locally $h = (g_n + q_n)\tilde{f}$ with q_n holomorphic at z_n. In a neighbourhood of z_n,

$$\begin{aligned} h = (g_n + q_n)\tilde{f} &= (\frac{t_n}{\tilde{f}} + q_n - \varphi_n)\tilde{f} \\ &= t_n + (q_n - \varphi_n)\tilde{f}. \end{aligned}$$

Since $(q_n - \varphi_n)\tilde{f}$ is holomorphic at z_n with zero of order at least o_n, so t_n is a left tail of h at z_n.

On F we have

$$\begin{aligned} |h(z)| &= |\tilde{f}(z)\gamma(z)| \\ &\leq (|\tilde{f}(z) - 1| + 1)|\gamma(z)| \\ &< \frac{\varepsilon(z)}{4} + \frac{\varepsilon(z)}{4} \\ &< \frac{\varepsilon(z)}{2}. \end{aligned}$$

By Corollary 4.2.6, there exists an entire function ω having zeros of order o_n at z_n and near 1 on F. Multiplying by a constant, we may assume that on F, ω is bounded and $|\omega| > 1$. Since F is a set of ε-approximation, there is an entire function ψ such that for $z \in F$,

$$|\psi(z) - \frac{f}{\omega}(z)| < \frac{\varepsilon}{2|\omega(z)|}.$$

Set $\tilde{g} := \omega\psi$. Then $|\tilde{g} - f| < \varepsilon/2$ on F and $\tilde{g}$ has zeros of order at least o_n at z_n.

Set $g := h + \tilde{g}$. Then g is a holomorphic function on $\mathbb{C}\backslash Z$ such that for each n, t_n is a left tail of g at z_n and for $z \in F$ it satisfies (4.2).

□

Of course what we have done in Lemma 4.2.11 can be done for principal parts

$$\sum_{j=N}^{-1} a_j(z-\zeta)^j,$$

and also Theorem 4.2.12 can be applied to

$$p_n(z) := \sum_{j=i_n}^{j_n} w_{n,j}(z-z_n)^j,$$

where, for each n, i_n is an integer, $i_n < j_n$.

Corollary 4.2.13 *Let F be a set of ε-approximation, $\varepsilon \leq 1$ and $\{z_n\}$ an admissible sequence. Further, let $w_{n,j}$, $n \in \mathbb{N}$, $j = 0, 1, 2, ..., j_n$, be complex numbers. Then for $f \in \mathcal{A}(F)$ there exists $g \in \mathcal{H}(\mathbb{C})$ such that for each $z \in F$,*

$$|f(z) - g(z)| < \varepsilon(z),$$

and for $j = 0, 1, 2, ...j_n$, $n \in \mathbb{N}$,

$$g^{(j)}(z_n) = w_{n,j}.$$

Proof: By the previous theorem for

$$t_n = w_{n,0} + w_{n,1}(z-z_n) + \frac{1}{2!}w_{n,2}(z-z_n)^2 + ... + \frac{1}{j_n!}w_{n,j_n}(z-z_n)^{j_n},$$

we can find $g \in \mathcal{H}(\mathbb{C})$ which satisfies the desired properties.

□

Applying Theorem 4.1.4 in Lemma 4.2.11 and Theorem 4.2.12, analogous results can be deduced for sets of uniform approximation F and canonical speeds ε.

Theorem 4.2.14 *Let F be a set of ϵ-approximation, $\epsilon \leq 1$, $\{z_n\}$ an admissible sequence and $\{w_n\}$, $\{r_n\}$ arbitrary sequences of complex and positive numbers, respectively. Then for $f \in \mathcal{A}(F)$, there exists $g \in \mathcal{H}(\mathbb{C})$ such that, for $z \in F$, $|f(z) - g(z)| < \epsilon$ and for each n, $g(z_n) = w_n$ and $g(\mathbb{C})$ contains a schlicht disk of radius r_n.*

Proof: By Corollary 4.2.13, there exists $g \in \mathcal{H}(\mathbb{C})$ such that for each n, $g(z_n) = w_n$, $g'(z_n) = 1$ and on F, $|f(z) - g(z)| < \epsilon(z)$.

For each n, consider the holomorphic function $h_n(z) := g(\tilde{r}_n z + z_n)$ on the unit disk $\mathbb{D}$ with $\tilde{r}_n := r_n/\beta$, where β is the Bloch constant of normalized holomorphic functions on $\mathbb{D}$. By Bloch's Theorem, $h_n(\mathbb{D})$ contains a schlicht disk of radius $\beta\tilde{r}_n$. It follows that for each n, $g(\mathbb{C})$ contains a schlicht disk of radius $\beta\tilde{r}_n = r_n$.

$\square$

Chapter 5

Conclusions

In this dissertation, we considered the class of bounded expansion mappings from hyperbolic manifolds to Hermitian manifolds. According to compactness and non-compactness of the target manifold we distinguished two cases and called them normal and Bloch mappings, respectively. These are natural generalizations of the classical normal and Bloch functions on the unit disk $\mathbb{D}$ in the complex plane to the Riemann sphere $\mathbb{C}_\infty$ and $\mathbb{C}$, respectively.

We proved that the function $f \mapsto \|f'\|$ from the class $\mathcal{E}(M, N)$ of mappings of bounded expansion equipped with the compact open topology to $\mathbb{R}$ is lower semicontinuous. Some equivalent statements concerning Bloch mappings to $\mathbb{C}^n$ had been proved. The natural question is whether these statements are equivalent for the general case where N is any Hermitian manifold.

Semi-Bloch functions on bounded domains of $\mathbb{C}^n$ were defined and an analytic characterization for this class was obtained. In [8] for $n = 1$, a geometric characterization for semi-Bloch functions has been obtained. We don't know if such a geometric characterization exists for $n > 1$.

In chapter three we proved that the function b from $\mathcal{H}(\Omega, \mathbb{C}^n)$ to $[0, \infty]$ defined by $b(f) := \beta_f$ is lower semicontinuous, where β_f is the supremum of radii of schlicht balls in the range of f. We also considered the Brody lemma on $\mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$, where $\mathbb{B}^n$ is the unit ball in $\mathbb{C}^n$. It would be interesting to prove the existence of a Bloch constant for quasiregular holomorphic mappings, with the help of this modified Brody reparametrization lemma. At the end of this chapter, for $K > 1$, we constructed a family of K-quasiregular mappings on the unit disk in $\mathbb{R}^2$, for which Bloch's theorem does not hold.

In chapter four we considered the problem of approximation of functions defined on a closed set F by meromorphic functions with prescribed poles and zeros outside F and then by considering interpolation we proved that a holomorphic function on a closed set can be approximated by entire holomorphic functions in such a way that in the range of that function there exist schlicht disks of arbitrary radii. We ask whether the higher dimensional generalization of such a problem is still valid. A preprint of this chapter as a paper with a slight modification is being prepared for publication [26].

Bibliography

[1] Abate, M.; *Iteration Theory of Holomorphic Maps on Taut Manifolds,* Mediterranean Press, Italy, (1989).

[2] Ahlfors, L. V.; *An extension of Schwarz's lemma,* Trans. Amer. Math. Soc. **43** (1938), 359-364.

[3] Ahlfors, L. V.; *On quasiconformal mappings,* J. Analyse Math. **3** (1954), 1-58.

[4] Ahlfors, L. V., Grunsky, H.; *Über die Blochsche Konstante,* Math. Z. **42** (1937), 671-673.

[5] Aĭzenberg, L. A., Yuzhakov, A. P.; *Integral Representation and Residues in Multidimensional Complex Analysis,* Transl. of Math. Monogr. **58**, Amer. Math. Soc., Providence, RI, (1983).

[6] Anderson, J. M., Clunie, J. G. Pommerenke, Ch.; *On Bloch functions and normal functions,* J. Reine Angew. Math. **270** (1974), 12-37.

[7] Arakelyan, N. U.; *Approximation complexe et propriétés des fonctions analytiques,* Actes Congrès Inter. Math. Tome **2**, (1970), 595-600.

[8] Aulaskari, R., Lappan, P.; *Some characterizations of semi-Bloch functions,* Proc. Amer. Math. Soc. **119** (1993), 1233-1238.

[9] Barth, T.; *Taut and tight complex manifolds,* Proc. Amer. Math. Soc. **24** (1970), 429-431.

[10] Boivin, A.; *Carleman approximation on Riemann surfaces,* Math. Ann. **276**, (1986), 57-70.

[11] Bonk, B.; *On Bloch's constant,* Proc. Amer. Math. Soc. **110** (1990), 889-894.

[12] Brody, R.; *Compact manifolds and hyperbolicity,* Trans. Amer. Math. Soc. **235** (1978), 213-219.

[13] Chen, H., Gauthier, P. M.; *On Bloch's constant,* J. Analyse Math. **69** (1996), 275-291.

[14] Chern S. S.; *On holomorphic mappings of Hermitian manifolds of the same dimension,* Proc. Symp. Pure Math. **11**, 157-170, *Entire Functions and Related parts of Analysis,* Amer. Math. Soc., (1968).

[15] Colonna, F.; *Bloch and normal functions and their relations,* Rend. Circ. Mat. Palermo **38** (1989), 161-180.

[16] Duren, P., Rudin, W.; *Distortion in several variables,* Complex Variables **5** (1986), 323-326.

[17] Drasin, D.; *The inverse problem of the Nevanlinna theory,* Acta Math. **138** (1977), 83-151.

[18] Eremenko, A.; *Bloch radius, normal famillies and quasiregular mappings,* Proc. Amer. Math. Soc. to appear.

[19] FitzGerald, C. H.; Gong, S. *The Bloch theorem in several complex variables,* J. Geom. Analysis **4**, (1994), 35-58.

[20] Fornæss, J. E.; *Dynamics in several complex variables,* CBMS Regional Conference Series in Math. **87**, Amer. Math. Soc., Providence, RI, (1994).

[21] Fuchs, W. H. J.; *Théorie de l'Approximation des Fonctions d'une Variable Complexe,* Séminaire de Mathématiques Supérieures (**26**), Les Presses de l'Université de Montréal, (1967).

[22] Gauthier, P. M.; *A criterion for normalcy,* Nagoya Math. J. **32**, (1968), 277-282.

[23] Gauthier, P. M.; *Tangential approximation by entire functions and functions holomorphic in a disc,* Izv. Akad. Nauk Armyan. SSR Ser. Mat. **4** (1969), 319-326.

[24] Gauthier, P. M.; *Mittag-Leffler theorems on Riemann surfaces and Riemannian manifolds,* Canad. J. Math. **50**, (1998), 1-16.

[25] Gauthier, P. M.; Hengartner, W.; *Complex approximation and simultaneous interpolation on closed sets,* Canad. J. Math. **29**, (1977), 701-706.

[26] Gauthier, P. M.; Pouryayevali, M. R.; *Approximation by meromorphic functions with Mittag-Leffler type constraints,* preprint.

[27] Greene, R. E., Wu, H. H.; *Bloch's theorem for meromorphic functions,* Math. Z. **116**, (1970), 247-257.

[28] Hahn, K. T.; *Higher dimensional generalizations of the Bloch constant and their lower bounds,* Trans. Amer. Math. Soc. **179**, (1973), 263-274.

[29] Hahn, K. T.; *Holomorphic mappings of the hyperbolic space into complex Euclidean space and the Bloch theorem,* Canad. J. Math. **27**, (1975), 446-458.

[30] Hahn, K. T.; *Quantitative Bloch's theorem for certain classes of holomorphic mappings of the ball into $P_n(\mathbb{C})$,* J. Reine Angew. Math. **283**, (1976), 99-109.

[31] Hahn, K. T.; *Asymptotic behavior of normal mappings of several complex variables,* Canad. J. Math. **36**, (1984), 718-746.

[32] Hille, E.; *Analytic Function Theory,* Vol. II, Ginn and Company, (1962).

[33] Hörmander, L.; *An Introduction to Complex Analysis in Several Variables,* North Holand, (1990).

[34] Joseph, J.; Kwack, M. H.; *Some classical theorems and families of normal maps in several complex variables,* Complex Variables **29**, (1996), 343-362.

[35] Kerzman, N.; Rosay, J. P.; *Fonctions plurisousharmoniques d'exhaustion bornées et domaines taut,* Math. Ann. **257**, (1982), 171-184.

[36] Kobayashi, S.; *Invariant distances on complex manifolds and holomorphic mappings,* J. Math. Soc. Japan **19**, (1967), 460-480.

[37] Kobayashi, S.; *Hyperbolic Manifolds and Holomorphic Mappings,* Marcel Dekker, New York, (1970).

[38] Kobayashi, S.; *Hyperbolic Complex Spaces,* Springer, Berlin, (1998).

[39] Krantz, S. G., Ma D.; *Bloch functions on strongly pseudoconvex domains,* Indiana Univ. Math. J. **37**, (1988), 145-163.

[40] Lang, S.; *Introduction to Complex Hyperbolic Spaces,* Springer-Verlag, New York, (1987).

[41] Liu, X.; *Bloch functions of several complex variables,* Pacific J. Math. **152**, (1992), 347-363.

[42] Lohwater, A. J., Pommerenke, Ch.; *On normal meromorphic functions,* Ann. Acad. Sci. Fenn. Ser. A.I., No. 550 (1973), 12pp.

[43] Minda, D.; *Bloch constants,* J. Analyse Math. **41**, (1982), 54-84.

[44] Minda, D.; *Bloch and normal functions on general planar regions,* Vol. I, Math. Sci. Res. Inst. Publ. **10**, (1986), *Holomorphic Functions and Moduli,* 101-110.

[45] Nersesyan, A. A.; *On Carleman sets,* Izv. Akad. Nauk Armyan. SSR Ser. Mat. **6**, (1971), 465-471; English translation: Transl. of Math. Monographs **44**, Amer. Math. Soc., Providence, RI, (1974).

[46] Nersesyan, A. A.; *On uniform and tangential approximation by meromorphic functions,* Izv. Akad. Nauk Armyan. SSR Ser. Mat. **7**, (1972), 405-412.

[47] Noguchi, J., Ochiai, T.; *Geometric Function Theory in Several Complex Variables,* Transl. of Math. Monogr. **80**, Amer. Math. Soc., Providence, RI, (1990).

[48] Reshetnyak, Yu. G.; *Space Mappings with Bounded Distortion,* Transl. of Math. Monographs, **73**, Amer. Math. Soc., Providence, RI, (1989).

[49] Rickman, S.; *Quasiregular Mappings,* Springer-Verlag, Berlin, (1993).

[50] Rosay, J. P., Rudin, W.; *Holomorphic maps from* $\mathbb{C}^n$ *to* $\mathbb{C}^n$*,* Trans. Amer. Math. Soc. **310**, (1985), 47-86.

[51] Roth, A.; *Uniform and tangential approximations by meromorphic functions on closed sets,* Canad. J. Math. **28**, (1976), 104-111.

[52] Royden, H.; *Remarks on the Kobayashi metric,* Lecture Notes in Math., **185**, 369-383, *Proc. Maryland Conference on Several Complex Variables,* Springer-Verlag, Berlin, (1971).

[53] Sauer, A.; *Meromorphic functions with prescribed asymptotic behaviour, zeros and poles and applications in complex approximation,* Canad. J. Math. **51**, (1999), 117–129.

[54] Shabat, B. V.; *Introduction to complex analysis, part II, Functions of Several Variables,* Transl. of Math. Monographs, **110**, Amer. Math. Soc., Providence, RI, (1992).

[55] Timony, R. M.; *Bloch functions in several complex variables, I,* Bull. London Math. Soc. **12**, (1980), 241-267.

[56] Timony, R. M.; *Bloch functions in several complex variables, II,* J. Reine Angew Math. **319**, (1980), 1-22.

[57] Väisälä, J.; *Lectures on n-Dimensional Quasiconformal Mappings,* Lecture Notes in Math., **229**, Springer-Verlag, Berlin, (1971).

[58] Vuorinen, M.; *Conformal Geometry and Quasiregular mapping,* Lecture Notes in Math., **1319**, Springer-Verlag, Berlin, (1988).

[59] Vuorinen, M.; *Quasiconformal Space Mapping,* Lecture Notes in Math., **1508**, *A Collection of Surveys 1960-1990,* Springer-Verlag, Berlin, (1992).

[60] Wu, H.; *Normal families of holomorphic mappings,* Acta Math. **119**, (1967), 193-233.

[61] Wu, H.; *Remarks on the first main theorem in equidistribution theory. IV,* J. Differential Geom. **3** (1969), 433-446.