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Maximal Abelian Subalgebras of Pseudoeuclidean Real Lie Algebras and Their Application in Physics

par

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Maximal Abelian Subalgebras of Pseudoeuclidean Real Lie Algebras and Their Application in Physics

présentée par

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To my mom Zorka
and my grandparents
Oľga and Matej

Sommaire

Nous construisons les classes de conjugaison de sous-algèbres maximales abéliennes (SAMAs) des algèbres de Lie pseudo-euclidiennes réelles $e(p,q)$ sous l'action des groupes de Lie pseudo-euclidiens correspondants. L'algèbre $e(p,q)$ est la somme semi-directe de l'algèbre pseudo-orthogonale $o(p,q)$ et de l'ideal abélien de translations $T(p+q)$. Nous utilisons tout d'abord cette structure particulière pour écrire les SAMAs "splitting" comme sommes directes de sous-algèbres de $o(p,q)$ et $T(p+q)$. Les SAMAs "splitting" permettent alors de construire les SAMAs "nonsplitting" d'algèbres $e(p,q)$. Les résultats pour $q = 0, 1$ et 2 sont explicites. Les SAMAs d'algèbres $e(p,0)$ et $e(p,1)$ sont utilisées pour construire les systèmes de coordonnées qui sont non-équivalents sous la transformation conforme et pour lesquels l'équation d'ondes et l'équation d'Hamilton-Jacobi permettent la séparation des variables.

La réduction par symétrie de deux équations aux dérivées partielles est donnée comme illustration de la classification des sous-algèbres. Les solutions analytiques de l'équation de Landau-Lifshitz et de l'équation de diffusion nonlinéaire sont obtenues par la méthode de la reduction par symétrie. Les groupes de symétrie des deux équations sont obtenus et tous les sous-groupes de dimension deux sont classifiés. Les sous-groupes sont alors utilisés pour réduire ces deux équations en équations différentielles ordinaires, qui sont résolues en termes de fonctions elliptiques.

Abstract

We construct the conjugacy classes of maximal abelian subalgebras (MASAs) of the real pseudoeuclidean Lie algebras $e(p,q)$ under the conjugation by the corresponding pseudoeuclidean Lie groups $E(p,q)$. The algebra $e(p,q)$ is a semi-direct sum of the pseudoorthogonal algebra $o(p,q)$ and the abelian ideal of translations $T(p+q)$. We use this particular structure to construct first the splitting MASAs, which are themselves direct sums of subalgebras of $o(p,q)$ and $T(p+q)$. Splitting MASAs give rise to the nonsplitting MASAs of $e(p,q)$. The results for $q=0,1$ and 2 are entirely explicit. MASAs of $e(p,0)$ and $e(p,1)$ are used to construct conformally nonequivalent coordinate systems in which the wave equation and Hamilton-Jacobi equations allow the separation of variables.

As an application of subgroup classification we perform symmetry reduction for two nonlinear partial differential equations. The method of symmetry reduction is used to obtain analytical solutions of the Landau-Lifshitz and a nonlinear diffusion equations. The symmetry group is found for both equations and all two-dimensional subgroups are classified. These are used to reduce both equations to ordinary differential equations, which are solved in terms of elliptic functions.

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Introduction

This thesis deals with maximal abelian subalgebras (MASAs) of pseudoeuclidean real Lie algebras and their application in physics. It consists of three articles [1, 2, 3]. The first one is published as a preprint, CRM-2615, and is submitted to *Linear Algebra and Its Applications*, the second one will appear in the July 1998 issue of *Journal of Mathematical Physics* and the third one is already published in *Journal of Physics A - Mathematical and General.* Each chapter of the thesis contains one article. The material in the articles is selfcontained and therefore we do not repeat here the general notions and facts concerning MASAs. All necessary definitions and basic theorems are given in the introductory and general comments sections in Chapters 1 and 2, corresponding to articles [1] and [2], respectively.

There is an extensive literature devoted to the classification of MASAs of semi-simple Lie algebras. Cartan subalgebras are a special type of MASAs. They are self-normalizing and consist of nonnilpotent elements. Cartan subalgebras have been completely classified and constructed by several authors [4, 5, 6]. Over the field of complex numbers only one class of Cartan subalgebra of given semi-simple Lie algebra exists. However, for the real semi-simple Lie algebras the number of conjugacy classes is finite, not necessarily equal to one.

Another important type of MASAs are maximal abelian nilpotent subalgebras (MANSs) - consisting entirely of nilpotent elements. A MANS can be represented by nilpotent matrices in any finite-dimensional representation. Basic results on MANSs of $sl(n,\mathbb{C})$ and $sl(n,\mathbb{R})$ were obtained by Kravchuk [7] and further developed in book on commutative matrices by Suprunenko and Tyshkevich [8]. MASAs of maximal dimension for all complex simple finite-dimensional Lie algebras were studied by Maltsev [9]. Those of minimal dimension were studied by Gerstenhaber [10] and Laffey [11].

More recently a series of articles by P. Winternitz and collaborators was devoted to study of MASAs of classical simple Lie algebras namely, symplectic $sp(n, C)$ and $sp(n, \mathbb{R})$ [12], pseudounitary $su(p, q)$ [13], orthogonal $o(n, \mathbb{C})$ [14] and pseudoorthogonal ones $o(p, q)$ [15].

Recently, the study of MASAs was extended to the finite dimensional affine Lie algebras. The first study was done for the complex Euclidean Lie algebras $e(n, \mathbb{C})$ [16]. The next step was to consider real pseudoeuclidean Lie algebras $e(p, q)$, first only for the small values of q ($q = 0, 1$) [2], and then to study MASAs of $e(p, q)$ for any values of p, q [1]. The results of these studies are the content of the presented thesis.

There are several reasons for studying MASAs of a given Lie algebra. Of mathematical interest is the classification of all subalgebras of a given Lie algebra. The classification of MASAs is part of such a program. It is an extension of a different problem: finding the conjugacy classes of elements of a Lie algebra under the action of the corresponding Lie group.

Apart from purely mathematical interest in classification of subalgebras there is also a motivation coming from mathematical and especially physical applications. These applications are for example a systematic study of symmetry breaking (spontaneous or explicit) [17, 18, 19, 20], the construction of complete sets of commuting operators and maximal sets of additive quantum numbers in quantum-mechanical problems and the construction of integrals of motion in involution in classical systems.

Other applications are related to differential equations. Let G be the symmetry group of a differential equation (or of a system of differential equations). Then subgroups of G can be used to construct group invariant solutions [21, 22]. In particular, abelian subgroups for linear partial differential equations (PDEs) are related to the separation of variables in coordinate systems with a maximal number of ignorable variables (variables which do not figure in the metric tensor) [23, 24, 25, 26]. For nonlinear PDEs abelian subgroups provide the simplest way of symmetry reduction (reducing the number of independent variables).

In the present thesis we contribute to the classification problem by classifying the MASAs of the real pseudoeuclidean Lie algebra [1]. Also, we give the examples of application by constructing conformally nonequivalent coordinate systems in the Minkowski space-time [2]. Another example of application is symmetry reduction. We use the method of symmetry reduction to obtain analytical solution of (2+1) dimensional spin systems [3].

The first chapter deals with MASAs of $e(p,q)$. The MASAs of $e(p,q)$ are classified under the action of the group $E(p,q)$. The general procedure uses the fact that $e(p,q)$ is a semi-direct sum of a pseudoorthogonal Lie algebra $o(p,q)$ and translation algebra $T(p+q)$. First we construct "splitting" subalgebras which are direct sums of subalgebras of $o(p,q)$ and subalgebras of $T(p+q)$. Section 4 of this chapter contains complete and explicit results for splitting MASAs of $e(p,q)$. The complementary "nonsplitting" MASAs are constructed in Section 5. They are constructed explicitly for free-rowed MANSs. The problem of classifying non-splitting MASAs containing non-free-rowed MANSs is more complicated. There exist several series of non-free-rowed MANSs of $o(p,q)$. For two of these series we construct all MASAs of the corresponding $e(p,q)$. Section 7 of this chapter contains a detailed study of MASAs of $e(p,2)$. The algebra $e(p,2)$ is already rich enough to contain all possible types of orthogonally indecomposable subalgebras of $o(p,2)$ and still simple enough to provide completely explicit results.

The second chapter is devoted to the detailed study of MASAs of $e(p,0)$ and $e(p,1)$. They are classified into the conjugacy classes under the action of the corresponding Lie groups $E(p,0)$ and $E(p,1)$, respectively. Also, we classify MASAs under the action of larger group, namely the conformal groups $O(p+1,1)$ and $O(p+1,2)$ of Euclidean and Minkowski spaces, respectively. The results are used to show (for $q=0$ or 1) which MASAs of $e(p,q)$ are also MASAs of $o(p+1,q+1)$, and which MASAs that are inequivalent under $E(p,q)$ become equivalent (conjugate) under the conformal group $O(p+1,q+1)$. These MASAs provide us with conformaly nonequivalent separable coordinate systems in Euclidean and Minkowski space, respectively. These coordinate systems allow the separation of variables in the Laplace and wave equation with a maximal number of ignorable variables [23, 24, 25, 26].

In the third chapter we use group theoretical methods of symmetry reduction [21, 22, 27, 28] to find the solutions of the Landau-Lifshitz [29] and nonlinear diffusion equation [30]. Although both equations are physically important [31, 32, 33, 34], there are only few analytical results for them [35, 36]. We used a MACSYMA package [37] to find the symmetry groups of both equations; in each case we obtained three different symmetry groups (depending on the values of parameters in the equations). In general, to perform a symmetry reduction systematicaly one needs to classify all subgroups of a given symmetry group. In this case we classified all two-dimensional subgroups of the symmetry group for each equation. In each case four of these subgroups were abelian and they were present for all three symmetry groups. We used all two-dimensional subgroups to reduce the equations to ordinary differential equations. These were solved in terms of elliptic functions when possible. In other cases we obtained the equation for the Painlevé transcendent P_V.

Chapter 1

Maximal Abelian Subalgebras of e(p,q) algebras

Maximal Abelian Subalgebras of e(p,q) algebras

Z. Thomova and P. Winternitz

CRM - 2615

Abstract

Maximal abelian subalgebras of one of the classical real inhomogeneous Lie algebras are constructed, namely those of the pseudoeuclidean Lie algebra $e(p,q)$. Use is made of the semidirect sum structure of $e(p,q)$ with the translations $T(p+q)$ as an abelian ideal. We first construct splitting MASAs that are themselves direct sums of abelian subalgebras of $o(p,q)$ and of subalgebras of $T(p+q)$. The splitting subalgebras are used to construct the complementary nonsplitting ones. We present general decomposition theorems and construct indecomposable MASAs for all algebras $e(p,q)$, $p \geq q \geq 0$. The case of $q=0$ and 1 were treated earlier in a physical context. The case $q=2$ is analyzed here in detail as an illustration of the general results.

Les sous-algèbres maximales abéliennes (SAMAs) d'une algèbre réelle classique non-homogène sont construites, en particulier, celles d'algèbre de Lie pseudo-euclidienne $e(p,q)$. On utilise la structure de la somme semi-directe de $e(p,q)$ avec les translations $T(p+q)$ qui représente un idéal abélien. Nous avons construit, en premier, les SAMAs "splitting", qui sont des sommes directes des sous-algèbres abéliennes de $o(p,q)$ et de sous-algèbres de $T(p+q)$. Les sous-algèbres "splitting" sont utilisées pour construire les sous-algèbres complementaire -"nonsplitting". Nous présentons les théorèmes généraux de décomposition et nous construisons les SAMAs indécomposables pour toutes les algèbres $e(p,q)$, $p \geq q \geq 0$. Les cas de $q=0$ et 1 sont déjà traités dans un context physique. Le cas $q=2$ est analysé ici en détail comme une illustration des résultats généraux.

1 Introduction

The purpose of this article is to present a classification of the maximal abelian subalgebras (MASAs) of the pseudoeuclidean Lie algebra $e(p,q)$. Since this Lie algebra can be represented by a specific type of real matrices of dimension $(p+q+1)\times(p+q+1)$, the subject of this article is placed squarely within a classical problem of linear algebra, the construction of sets of commuting matrices.

Most of the early papers in this direction [1–3] as well as more recent ones [4–8], were devoted to commuting matrices within the set of all matrices of a given dimension. In other words, they studied abelian subalgebras of the Lie algebras $gl(n,\mathbb{C})$ and $gl(n,\mathbb{R})$. For a historical review with many references see the book by Suprunenko and Tyshkevich [9].

Maltsev constructed all maximal abelian subalgebras of maximal dimension for all complex finite-dimensional simple Lie algebras [10]. An important subclass of MASAs are Cartan subalgebras, *i.e.* self-normalizing MASAs [11]. The simple complex Lie algebras, as well as the compact ones, have just one conjugacy class of Cartan subalgebras. The real noncompact forms of the simple Lie algebras can have several conjugacy classes of them. They have been classified by Kostant [12] and Sugiura [13].

This article is part of a series, the aim of which is to construct all MASAs of the classical Lie algebras. Earlier articles were devoted to the classical simple Lie algebras, such as $sp(2n,\mathbb{R})$ and $sp(2n,\mathbb{C})$ [14], $su(p,q)$ [15], $o(n,\mathbb{C})$ [16] and $o(p,q)$ [17]. General results for MASAs of classical simple Lie algebras are presented in [18]. More recently MASAs of some inhomogeneous classical Lie algebras were studied, namely those of $e(n,\mathbb{C})$ [19], $e(p,0)$ and $e(p,1)$ [20]. Here we consider $e(p,q)$ for all $p\geq q\geq 0$. The two special cases, $q=0$ and $q=1$, treated earlier, are of particular importance in physics and are also much simpler than the general case.

The motivation for a study of MASAs was discussed in previous articles [14–20]. As a mathematical problem the classification of MASAs is an extension of the classification of individual elements of Lie algebras into conjugacy classes [21–23]. A classification of MASAs of classical Lie algebras is an important ingredient in the classification of all subalgebras of these algebras.

In applications in the theory of partial differential equations, MASAs provide coordinate systems in which invariant equations allow the separation of variables. More specifically, they provide "ignorable variables" not figuring in the corresponding metric tensors, when considering Laplace-Beltrami or Hamilton-Jacobi equations. In quantum physics they provide complete sets of commuting operators. In classical physics they provide integrals of motion in involution.

The classification problem is formulated in Section 2, where we also present some necessary definitions and explain the classification strategy. Section 3 contains a brief summary of the known results on MASAs of $o(p,q)$ [17]. They are needed in the rest of this article and we reproduce them in a condensed form to make the article self-contained. Section 4 is devoted to splitting subalgebras of $e(p,q)$, *i.e.* subalgebras that are direct sums of subalgebras of the algebra $o(p,q)$ and those of the translation algebra $T(p+q)$. The complementary case of non-splitting MASAs of $e(p,q)$ is the subject of Section 5. The results on MASAs of $e(p,q)$ obtained in Sections 4 and 5 are reformulated in terms of a decomposition of the underlying linear space $S(p,q)$ in Section 6. Indecomposable MASAs of $e(p,q)$ are described in the same section. Section 7 is devoted to a special case in which all results are entirely explicit, namely MASAs of $e(p,2)$.

2 General formulation

2.1 Some definitions

The pseudoeuclidean Lie algebra $e(p,q)$ is the semidirect sum of the pseudoorthogonal Lie algebra $o(p,q)$ and an abelian algebra $T(n)$ of translations

$$e(p,q) = o(p,q) \niplus T(n), \qquad n = p+q. \tag{2.1}$$

We will make use of the following matrix representation of the Lie algebra $e(p,q)$ and the corresponding Lie group $E(p,q)$. We introduce an "extended metric"

$$K_e = \begin{pmatrix} K & 0 \\ 0 & 0_1 \end{pmatrix}, \tag{2.2}$$

where K satisfies

$$K = K^T \in \mathbb{R}^{n\times n}, \quad n = p+q, \quad detK \neq 0, \tag{2.3}$$

$$sgnK = (p,q), \qquad p \geq q \geq 0. \tag{2.4}$$

Here $sgnK$ denotes the signature of K, where p and q are the numbers of positive and negative eigenvalues, respectively. Then $X_e \in e(p,q)$ and $H \in E(p,q)$ are represented as

$$X_e(X,\alpha) \equiv X_e = \begin{pmatrix} X & \alpha^T \\ 0 & 0 \end{pmatrix}, \qquad X \in \mathbb{R}^{n\times n}, \qquad \alpha \in \mathbb{R}^{1\times n}, \tag{2.5}$$

$$H = \begin{pmatrix} G & a^T \\ 0 & 1 \end{pmatrix}, \qquad G \in \mathbb{R}^{n\times n}, \qquad a \in \mathbb{R}^{1\times n}, \tag{2.6}$$

$$XK + KX^T = 0, \quad GKG^T = K, \qquad X_eK_e + K_eX_e^T = 0. \tag{2.7}$$

The vector $\alpha \in \mathbb{R}^{1\times n}$ represents the translations. We say that the translations are positive, negative or zero (isotropic) length if

$$\alpha K \alpha^T > 0, \quad \alpha K \alpha^T < 0, \quad \alpha K \alpha^T = 0, \tag{2.8}$$

respectively.

We will be classifying maximal abelian subalgebras of the pseudoeuclidean Lie algebra $e(p,q)$ into conjugacy classes under the action of the pseudoeuclidean Lie group $E(p,q)$. Let us define some basic concepts.

Definition 2.1 *The centralizer $cent(L_0, L)$ of a Lie algebra $L_0 \subset L$ is a subalgebra of L consisting of all elements in L, commuting elementwise with L_0*

$$cent(L_0, L) = \{e \in L | [e, L_0] = 0\}. \tag{2.9}$$

Definition 2.2 *A maximal abelian subalgebra L_0 (MASA) of L is an abelian subalgebra, equal to its centralizer*

$$[L_0, L_0] = 0, \quad cent(L_0, L) = L_0. \tag{2.10}$$

Definition 2.3 *A normalizer group $Nor(L_0, G)$ in the group G of the subalgebra $L_0 \subseteq L$ is*

$$Nor(L_0, G) = \{g \in G | gL_0g^{-1} \subseteq L_0\}. \tag{2.11}$$

Definition 2.4 *A splitting subalgebra L_0 of the semidirect sum*

$$L = F \ni N, \; [F,F] \subseteq F, \; [F,N] \subseteq N, \; [N,N] \subseteq N \tag{2.12}$$

is itself a semidirect sum of a subalgebra of F and a subalgebra of N

$$L_0 = F_0 \ni N_0, \; F_0 \subseteq F, \; N_0 \subseteq N. \tag{2.13}$$

All other subalgebras of $L = F \ni N$ are called *nonsplitting subalgebras*.

An *abelian splitting subalgebra* of $L = F \ni N$ is a direct sum

$$L_0 = F_0 \oplus N_0, \quad F_0 \subseteq F, \quad N_0 \subseteq N. \tag{2.14}$$

Definition 2.5 *A maximal abelian nilpotent subalgebra (MANS) M of a Lie algebra L is a MASA, consisting entirely of nilpotent elements, i.e. it satisfies*

$$[M, M] = 0, \quad [[[L, M]M]\ldots]_m = 0 \tag{2.15}$$

for some finite number m (we commute M with L m-times). A MANS is represented by nilpotent matrices in any finite dimensional representation.

2.2 Classification strategy

The classification of MASAs of $e(p,q)$ is based on the fact that $e(p,q)$ is the semidirect sum of the Lie algebra $o(p,q)$ and an abelian ideal $T(n)$ (the translations). We use here a procedure related to one used earlier [19] for $e(n, C)$ and [20] for $e(p, 1)$. It proceeds in five steps.

1. Classify subalgebras $T(k_+, k_-, k_0)$ of $T(n)$. They are characterized by a triplet (k_+, k_-, k_0), where k_+, k_- and k_0 are the number of positive length, negative length and isotropic vectors, respectively.

2. Find the centralizer $C(k_+, k_-, k_0)$ of $T(k_+, k_-, k_0)$ in $o(p,q)$

$$C(k_+, k_-, k_0) = \{X \in o(p,q) | [X, T(k_+, k_-, k_0)] = 0\}. \tag{2.16}$$

3. Construct all MASAs $M(k_+, k_-, k_0)$ of $C(k_+, k_-, k_0)$ and classify them under the action of normalizer $Nor[T(k_+, k_-, k_0), G]$ of $T(k_+, k_-, k_0)$ in the group $G \sim E(p,q)$.

4. Obtain a representative list of all splitting MASAs of $e(p,q)$ as direct sums

$$M(k_+, k_-, k_0) \oplus T(k_+, k_-, k_0) \tag{2.17}$$

   and keep only those amongst them that are indeed maximal (and mutually inequivalent).

5. Construct all nonsplitting MASAs from splitting ones as described below in Section 5.1.

3 Results on MASAs of $o(p, q)$

3.1 General results

Let us briefly sum up some known [17] results on MASAs of $o(p, q)$ that we shall need below. We shall represent these MASAs by matrix sets $\{X, K\}$ with notations as in (2.3) ... (2.7).

Definition 3.1 *A MASA of $o(p, q)$ is called orthogonally decomposable (OD) if all matrices in the set $\{X, K\}$ can be simultaneously represented by block diagonal matrices with the same decomposition pattern. It is called orthogonally indecomposable (OID) otherwise.*

Proposition 3.1 *Every OD MASA of $o(p, q)$ can be represented by a matrix set*

$$\begin{gathered}
X = diag(X_1, X_2, \dots, X_k), \qquad K = diag(K_{p_1,q_1}, K_{p_2,q_2}, \dots, K_{p_k,q_k}), \\
X_j K_{p_j,q_j} + K_{p_j,q_j} X_j^T = 0, \qquad X_j, K_{p_j,q_j} \in \mathbb{R}^{(p_j+q_j)\times(p_j+q_j)}, \\
K_{p_j,q_j} = K_{p_j,q_j}^T, \qquad sgn K_{p_j,q_j} = (p_j, q_j), \\
det K_{p_j,q_j} \neq 0, \qquad 1 \leq j \leq k, \qquad 2 \leq k \leq \left[\tfrac{p+q+1}{2}\right], \\
\textstyle\sum_{j=1}^k p_j = p, \quad \sum_{j=1}^k q_j = q, \quad p_1 + q_1 \geq p_2 + q_2 \geq \dots \geq p_k + q_k \geq 1,
\end{gathered} \tag{3.1}$$

where:

i) For each j, the matrix set $\{X_j, K_{p_j,q_j}\}$ represents an OID MASA of $o(p_j, q_j)$; let us call it M_{p_j,q_j}.

ii) At most one of the MASAs M_{p_j,q_j} is a maximal abelian nilpotent subalgebra (MANS) of $o(p_j, q_j)$. In particular only one pair (p_j, q_j) can satisfy $p_j + q_j = 1$. The corresponding pair $\{X, K\}$ is $(0, 1)$ and represents a MANS of $o(1, 0)$ or $o(0, 1)$.

To obtain representatives of all $O(p, q)$ classes of OD MASAs of $o(p, q)$ we let M_{p_j,q_j}, for all j, run independently through all representatives of $O(p_j, q_j)$ conjugacy classes of OID MASAs of $o(p_j, q_j)$, subject to the restriction (ii). Conversely, each such matrix set represents a conjugacy class of OD MASAs of $o(p, q)$.

The problem of classifying MASAs of $o(p, q)$ is thus reduced to the classification of OID MASAs. Under the field extension from $\mathbb{R}$ to $\mathbb{C}$ an OID MASA can remain OID, or become orthogonally decomposable. In the first case we call it *absolutely orthogonally decomposable* (AOID) in the second *nonabsolutely orthogonally indecomposable* (NAOID). The following types of orthogonally indecomposable MASAs of $o(p,q)$ exist:

1. Maximal abelian nilpotent subalgebras (MANSs). They exist for all values of (p,q), $min(p,q) \geq 1$. They are discussed below in Section 3.2. They are AOID MASAs.

2. MASAs that are decomposable but not orthogonally decomposable (AOID but D). They stay OID when considered over $\mathbb{C}$. They exist for all values of $p = q \geq 1$. Their canonical form is

$$M = \left\{ X_{p,p} = \begin{pmatrix} A & \\ & -A^T \end{pmatrix}, \quad K = \begin{pmatrix} & I_p \\ I_p & \end{pmatrix} \right\}, \tag{3.2}$$

where $A = \mathbb{R}I_p \oplus$ MANS of $sl(p,\mathbb{R})$.

3. MASAs that are indecomposable over $\mathbb{R}$ but become orthogonally decomposable after field extension to $\mathbb{C}$ (NAOID, ID but NAID). They exist for $p = 2k$, $q = 2l$, $min(k,l) \geq 1$. Their canonical form is

$$M = \mathbb{R}Q \oplus \text{MANSs of } su(k,l), \qquad K = \begin{pmatrix} I_{2k} & \\ & -I_{2l} \end{pmatrix},$$

$$Q = diag(F_2, \ldots, F_2) \in \mathbb{R}^{2(k+l)\times 2(k+l)}, \qquad F_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.3}$$

4. MASAs that are indecomposable over $\mathbb{R}$ and decomposable over $\mathbb{C}$ (but not orthogonally decomposable even over $\mathbb{C}$) (OID, AOID but NAID). They exist for $p = q = 2k$, $k \geq 1$. Their canonical form is

$$M = \mathbb{R}Q \oplus \text{OID but D MASAs of } su(k,k)$$

with Q as in eq.(3.3).

An exception is the case of $o(2)$, itself abelian. Thus, for $p = 2$, $q = 0$ or $p = 0$, $q = 2$, $o(2)$ is AOID but NAID.

5. Decomposable MASAs that become orthogonally decomposable over $\mathbb{C}$ (NAOID and D). They occur only for $p = q = 2k$, $k \geq 1$. Their canonical form is

$$M = \left\{ X = \begin{pmatrix} A & \\ & -A^T \end{pmatrix}, \quad K = \begin{pmatrix} & I_{2k} \\ I_{2k} & \end{pmatrix} \right\}, \tag{3.4}$$

where

$$A = \mathbb{R}Q_{2k} \oplus \text{MANSs of } sl(2k, \mathbb{C}).$$

3.2 MANSs of o(p, q)

A MANS M of a classical Lie algebra is characterized by its Kravchuk signature, which we will denote KS [3, 9, 17, 18]. It is a triplet of integers

$$(\lambda\ \mu\ \lambda), \qquad 2\lambda + \mu = n, \qquad \mu \geq 0, \qquad 1 \leq \lambda \leq q \leq p, \tag{3.5}$$

where λ is the dimension of the kernel of M, equal to the codimension of the image of M. A MANS can be transformed into the Kravchuk normal form

$$N = \begin{pmatrix} 0 & A & Y \\ 0 & S & -\tilde{K}A^T \\ 0 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} & & I_\lambda \\ & \tilde{K} & \\ I_\lambda & & \end{pmatrix},$$
$$A \in \mathbb{R}^{\lambda\times\mu}, \qquad Y = -Y^T \in \mathbb{R}^{\lambda\times\lambda}, \qquad S\tilde{K} + \tilde{K}S^T = 0, \tag{3.6}$$
$$S \in \mathbb{R}^{\mu\times\mu}, \quad \tilde{K} = \tilde{K}^T \in \mathbb{R}^{\mu\times\mu}, \quad sgn\tilde{K} = (p - \lambda, q - \lambda)$$

and S nilpotent.

There are two types of MANS of $o(p, q)$:

i) Free-rowed MANS. The first row of A has μ free real entries. All other entries in A and S depend linearly on those μ free entries.

ii) Non-free-rowed MANS. Any combination of rows of A contains less than μ free real entries.

The results on free-rowed MANS of $o(p, q)$ [17] are stated in the following proposition.

Proposition 3.2 *A representative list of $O(p, q)$ conjugacy classes of free-rowed MANSs of $o(p, q)$ with Kravchuk signature $(\lambda\ \mu\ \lambda)$ is given by the matrix sets*

$$N = \begin{pmatrix} 0 & A & Y \\ 0 & 0 & -\tilde{K}A^T \\ 0 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} & & I_\lambda \\ & \tilde{K} & \\ I_\lambda & & \end{pmatrix}, \tag{3.7}$$

$$A = \begin{pmatrix} \alpha Q_1 \\ \alpha Q_2 \\ \vdots \\ \alpha Q_\lambda \end{pmatrix}, \qquad \begin{array}{c} \alpha \in \mathbb{R}^{1\times\mu} \\ Y = -Y^T \in \mathbb{R}^{\lambda\times\lambda}, \end{array} \tag{3.8}$$

$$Q_i \in \mathbb{R}^{\mu\times\mu}, \quad Q_i\tilde{K} = \tilde{K}Q_i^T, \quad [Q_i, Q_j] = 0, \tag{3.9}$$
$$Q_1 = I, \quad TrQ_i = 0, \qquad 2 \leq i \leq \lambda.$$

The entries in α and Y are free. The matrices Q_i are fixed and form an abelian subalgebra of the Jordan algebra $jo(p-\lambda, q-\lambda)$. In the case $\lambda = 2$ we must have $Q_2 \neq 0$. There exists a $\lambda_1 \in \mathbb{Z}, 1 \leq \lambda_1 \leq \lambda$ such that $Q_1, \ldots, Q_{\lambda_1}$ are linearly independent and $Q_\nu = 0$, $\lambda_1 + 1 \leq \nu \leq \lambda$.

Proofs of the Propositions 3.1 and 3.2 and details about MASAs of $o(p,q)$ are given in Ref. [17]. The results on non-free-rowed MANS of $o(p,q)$ are less complete and we shall not reproduce them here [17].

4 Splitting MASAs of e(p,q)

4.1 General comments on MASAs of e(p,q)

A MASA of $e(p,q)$ will be represented by a matrix set $\{X_e, K_e\}$

$$X_e = \begin{pmatrix} N & & & & & & \xi^T \\ & X_{p_1,q_1} & & & & & \delta_1^T \\ & & \ddots & & & & \vdots \\ & & & X_{p_j,q_j} & & & \delta_j^T \\ & & & & 0_{k_+} & & x^T \\ & & & & & 0_{k_-} & y^T \\ & & & & & & 0_1 \end{pmatrix}, \tag{4.1}$$

$$K_e = \begin{pmatrix} K_0 & & & & & & \\ & K_{p_1,q_1} & & & & & \\ & & \ddots & & & & \\ & & & K_{p_j,q_j} & & & \\ & & & & I_{k_+} & & \\ & & & & & -I_{k_-} & \\ & & & & & & 0_1 \end{pmatrix}, \tag{4.2}$$

$$p = p_0 + k_0 + \sum_{i=1}^{j} p_i + k_+, \qquad q = q_0 + k_0 + \sum_{i=1}^{j} q_i + k_-, \tag{4.3}$$

where $M_{p_i,q_i} = \{X_{p_i,q_i}, K_{p_i,q_i}\}$, $i = 1, \ldots j$ is an OID MASA of $o(p_i, q_i)$, that is not a MANS. The vector ξ has the following form

$$\xi = \begin{pmatrix} z^T \\ \beta^T \\ \gamma^T \end{pmatrix}, \qquad \begin{array}{l} z, \gamma \in \mathbb{R}^{1\times k_0} \\ \beta \in \mathbb{R}^{1\times(p_0+q_0)} \end{array} \tag{4.4}$$

and N is a MANS of $o(p_0 + k_0, q_0 + k_0)$ with Kravchuk signature $(k_0\ p_0{+}q_0\ k_0)$ and is given by

$$N = \begin{pmatrix} 0_{k_0} & A & Y \\ 0 & S & -K_{p_0,q_0}A^T \\ 0 & 0 & 0_{k_0} \end{pmatrix}, K_0 = \begin{pmatrix} 0 & 0 & I_{k_0} \\ 0 & K_{p_0,q_0} & 0 \\ I_{k_0} & 0 & 0 \end{pmatrix} \tag{4.5}$$

$$\begin{gathered} Y = -Y^T, \qquad SK_{p_0,q_0} + K_{p_0,q_0}S^T = 0 \\ A \in \mathbb{R}^{k_0\times(p_0+q_0)}, \qquad S \in \mathbb{R}^{(p_0+q_0)\times(p_0+q_0)}, \qquad Y \in \mathbb{R}^{k_0\times k_0}, \\ K_{p_0,q_0} = K^T_{p_0,q_0}, \qquad sgnK_{p_0,q_0} = (p_0, q_0) \end{gathered} \tag{4.6}$$

The entries in z, x and y are free and represent the positive, negative and zero length translations contained in $T(k_+, k_-, k_0)$. The entries in β, γ and δ_i are linearly dependent on the free entries in A, Y and X_{p_i,q_i}. If they are nonzero (and cannot be annulled by an $E(p,q)$ transformation), we have a nonsplitting MASA. This case will be discussed in Section 5.

4.2 Basic results on splitting MASAs

In this section we shall construct all splitting MASAs of $e(p,q)$.

Theorem 4.1 *Every splitting MASA of $e(p,q)$ is characterized by a partition*

$$\begin{array}{ll} p = p_0 + k_+ + k_0 + \sum_{i=1}^{j} p_i, & q = q_0 + k_- + k_0 + \sum_{i=1}^{j} q_i \\ k_0 + k_+ + k_- \neq p + q - 1, & 0 \leq k_0 \leq q. \end{array} \tag{4.7}$$

A representative list of $E(p,q)$ conjugacy classes of MASAs of $e(p,q)$ is given by the matrix sets $\{X_e, K_e\}$ of eq.(4.1) and (4.2) with

$$\delta_i = 0, \quad i = 1, \ldots j, \qquad \xi = \begin{pmatrix} z^T \\ 0 \\ 0 \end{pmatrix}. \tag{4.8}$$

If $k_0 = 0$ then the MANS N is absent. M_{p_i,q_i} is an orthogonally indecomposable MASA of $o(p_i, q_i)$ which is not a MANS. Running through all possible partitions, all MANSs $\{N, K_0\}$ and all MASAs M_{p_i,q_i} we obtain a representative list of all splitting MASAs of $e(p,q)$.

Proof. We start by choosing a subalgebra $T(k_+, k_-, k_0)$. Calculating the centralizer of $T(k_+, k_-, k_0)$ in $o(p,q)$ gives us

$$C(k_+, k_-, k_0) = \begin{pmatrix} \tilde{M} & & \\ & 0_{k_+} & \\ & & 0_{k_-} \end{pmatrix}, \qquad K = \begin{pmatrix} \tilde{K} & & \\ & I_{k_+} & \\ & & -I_{k_-} \end{pmatrix}, \tag{4.9}$$
$$sgn\tilde{K} = (p - k_+, q - k_-).$$

$\tilde{M}$ is a subalgebra of $o(p - k_+, q - k_-)$ which commutes with the translations corresponding to $\xi = (z, 0)$, $\xi \in \mathbb{R}^{1\times(p+q-k_+-k_-)}$, $z \in \mathbb{R}^{1\times k_0}$, and with no other translations. To obtain a MASA of $e(p,q)$ we must complement $T(k_+, k_-, k_0)$ by a MASA $F(k_+, k_-, k_0)$ of the centralizer $C(k_+, k_-, k_0)$. $F(k_+, k_-, k_0)$ must not commute with any further translations, hence $F(k_+, k_-, k_0)$ is either a MANS of $o(p - k_+, q - k_-)$ with KS $(k_0,\ \ p-k_+-k_0+q-k_--k_0,\ \ k_0)$ or an orthogonally decomposable MASA containing a MANS N with KS $(k_0\ \mu\ k_0)$. For $k_0 = 0$ the MANS N is absent. This leads to eq. (4.8) and each $M_{p_i,q_i} = \{X_{p_i,q_i}, K_{p_i,q_i}\}$ is an OID MASA of $o(p_i, q_i)$ of the type 2,3,4, or 5, listed in Section 3.1. □

5 Nonsplitting MASAs of e(p,q)

5.1 General comments

First we describe the general procedure for finding nonsplitting MASAs of $e(p,q)$.

Every nonsplitting MASA $M(k_+, k_-, k_0)$ of $e(p, q)$ is obtained from a splitting one by the following procedure:

1. Choose a basis for $F(k_+, k_-, k_0)$ and $T(k_+, k_-, k_0)$ e.g. $F(k_+, k_-, k_0) \sim \{B_1, \ldots, B_J\}$, $T(k_+, k_-, k_0) \sim \{X_1, \ldots X_L\}$.

2. Complement the basis of $T(k_+, k_-, k_0)$ to a basis of $T(n)$.

$$T(n)/T(k_+, k_-, k_0) = \{Y_1, \ldots, Y_N\}, \qquad L + N = n.$$

3. Form the elements

$$\tilde{B}_a = B_a + \sum_{j=1}^{N} \tilde{\alpha}_{aj} Y_j, \qquad a = 1, \ldots, J, \tag{5.1}$$

where the constants $\tilde{\alpha}_{aj}$ are such that $\tilde{B}_a$ form an abelian Lie algebra $[\tilde{B}_a, \tilde{B}_b] = 0$. This provides a set of linear equations for the coefficients $\tilde{\alpha}_{aj}$. Solutions $\tilde{\alpha}_{aj}$ are called 1-cocycles and they provide abelian subalgebras $\tilde{M}(k_+, k_-, k_0) \sim \{\tilde{B}_a, X_b\} \subset e(p, q)$.

4. Classify the subalgebras $\tilde{M}(k_+, k_-, k_0)$ into conjugacy classes under the action of the group $E(p, q)$. This can be done in two steps.

   i) Generate trivial cocycles t_{aj}, called coboundaries, using the translation group $T(n)$

   $$e^{\theta_j P_j} \tilde{B}_a e^{-\theta_j P_j} = \tilde{B}_a + \theta_j [P_j, \tilde{B}_a] = \tilde{B}_a + \sum_j t_{aj} P_j. \tag{5.2}$$

   The coboundaries should be removed from the set of the cocycles. If we have $\tilde{\alpha}_{aj} = t_{aj}$ for all (a, j) the algebra is splitting (*i.e.* equivalent to a splitting one).

   ii) Use the normalizer of the original splitting subalgebra in the group $O(p, q)$ to further simplify and classify the nontrivial cocycles.

The general form of a nonsplitting MASA of $e(p, q)$ is $M_e = \{X_e, K_e\}$ given by eq. (4.1) and (4.2). Requiring commutativity $[X_e, X_e'] = 0$ leads to

$$\begin{aligned} X_{p_i,q_i} \delta_i'^T &= X'_{p_i,q_i} \delta_i^T \\ N \xi'^T &= N' \xi^T. \end{aligned} \tag{5.3}$$

From the eq.(5.3) we see that the entries in δ_i depend linearly only on X_{p_i,q_i}, *i.e.* only on the MASA M_{p_i,q_i} of $o(p_i,q_i)$.

Each M_{p_i,q_i} belongs to one of the four types of OID MASAs of $o(p_i,q_i)$ which were listed in the Section 3.1 - AOID but D MASAs, AOID but NAID MASAs, NAOID ID but NAID MASAs or NAOID but D MASAs.

We will make use of the following result:

Lemma 5.1 *If M is a MASA of $o(p,q)$ when considered over $\mathbb{R}$, then it will also be a MASA of $o(n,\mathbb{C})$, $n=p+q$, when considered over $\mathbb{C}$.*

If any of the vectors δ_i were non zero then after field extension we would obtain a nonsplitting MASA of $e(n,\mathbb{C})$ of a type that does not exist [19]. This implies that all of the $\delta_i's$ are zero.

Any further study of nonsplitting MASAs of $e(p,q)$ is reduced to studying the matrices

$$X_e = \begin{pmatrix} N & & & & & & \xi^T \\ & M_{p_1,q_1} & & & & & 0 \\ & & \ddots & & & & \vdots \\ & & & M_{p_j,q_j} & & & 0 \\ & & & & 0_{k_+} & & 0 \\ & & & & & 0_{k_-} & 0 \\ & & & & & & 0_1 \end{pmatrix} \tag{5.4}$$

with ξ and N as in eq.(4.4) and (4.5), respectively. Further, we can see from eq.(5.3) and (5.4) that the study of nonsplitting MASAs is in fact reduced to the study of nonsplitting MASAs of $e(p_0+k_0, q_0+k_0)$ for which the projection onto the subalgebra $o(p_0+k_0, q_0+k_0)$ is a MANS with Kravchuk signature $(k_0\ \mu\ k_0)$, $\mu = p_0+q_0$. Further classification is performed under the group $E(p_0+k_0, q_0+k_0)$.

The MASAs of $e(p_0+k_0, q_0+k_0)$ to be considered will thus be represented by the matrix sets $\{X_e, K_e\}$

$$X_e = \begin{pmatrix} 0_{k_0} & A & Y & z^T \\ 0 & S & -K_{p_0,q_0}A^T & \beta^T \\ 0 & 0 & 0_{k_0} & \gamma^T \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_e = \begin{pmatrix} & & I_{k_0} & \\ & K_{p_0,q_0} & & \\ I_{k_0} & & & \\ & & & 0_1 \end{pmatrix}, \tag{5.5}$$

where $Y = -Y^T$, and $\beta \in \mathbb{R}^{1\times\mu}$, $\gamma \in \mathbb{R}^{1\times k_0}$ depend linearly on the free entries in A and Y. Using the commutativity $[X_e, X_e'] = 0$ we obtain

$$\begin{aligned} A\beta'^T + Y\gamma'^T &= A'\beta^T + Y'\gamma^T \\ S\beta'^T - K_{p_0,q_0}A^T\gamma'^T &= S'\beta^T - K_{p_0,q_0}A'^T\gamma^T \end{aligned} \tag{5.6}$$

The translations

$$\Pi = \begin{pmatrix} 0_{k_0} & 0 & 0 & 0 \\ 0 & 0_{p_0,q_0} & 0 & \tau^T \\ 0 & 0 & 0_{k_0} & \zeta^T \\ 0 & 0 & 0 & 0_1 \end{pmatrix}, \qquad \tau \in \mathbb{R}^{1\times\mu}, \zeta \in \mathbb{R}^{1\times k_0}. \tag{5.7}$$

will be used to remove coboundaries from β and γ and the remaining cocycles will be classified under the action of the normalizer of the MANS N in the group $O(p_0 + k_0, q_0 + k_0)$.

The situation will be very different for free-rowed and non-free-rowed MANS of $o(p_0 + k_0, q_0 + k_0)$. The two cases will be treated separately.

5.2 Nonsplitting MASAs of $\mathbf{e}(p_0 + k_0, q_0 + k_0)$ related to free-rowed MANSs

Let N be a free-rowed MANS of $o(p_0+k_0, q_0+k_0)$. The corresponding nonsplitting MASAs of $e(p_0 + k_0, q_0 + k_0)$ can be represented as follows.

Theorem 5.1 *A nonsplitting MASA of $e(p, q)$ must contain a MANS of $o(p_0 + k_0, q_0 + k_0)$ with $1 \le k_0 \le q$, $min(p_0 + k_0, q_0 + k_0) \ge 1$. All nonsplitting MASAs of $e(p_0 + k_0, q_0 + k_0)$ for which the projection onto $o(p_0 + k_0, q_0 + k_0)$ is a free rowed MANS N with Kravchuk signature $(k_0\ \mu\ k_0)$, $\mu = p_0 + q_0$ can be represented by the matrix sets $\{X_e, K_e\}$ of eq.(5.5) with $S = 0$ and A and Y as in eq.(3.8).*
1. For $k_0 \ge 3$ we have

$$\beta = a\Lambda, \qquad \gamma = 0 \tag{5.8}$$

$\Lambda \in \mathbb{R}^{\mu\times\mu}$ satisfies the following conditions:

$$\Lambda = \Lambda^T, \qquad Q_j\Lambda K_{p_0,q_0}^{-1} = \Lambda K_{p_0,q_0}^{-1} Q_j. \tag{5.9}$$

2. $k_0 = 2,\ \mu \geq 2$. Λ *satisfies eq.(5.9) for* $j = 2$ *and*

$$\beta = \alpha\Lambda + y\rho, \qquad \gamma = \begin{pmatrix} 0 \\ \alpha\rho^T \end{pmatrix}, \qquad \rho = (1, 0, \ldots, 0) \tag{5.10}$$

for Q *following*

$$Q = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \qquad K_{p_0,q_0} = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & K_{p_0-1,q_0-1} \end{pmatrix}. \tag{5.11}$$

For all the other Q

$$\beta = \alpha\Lambda, \qquad \gamma = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.12}$$

3. $k_0 = 2,\ \mu = 1$

$$\beta = y\rho, \qquad \gamma^T = \begin{pmatrix} 0 \\ a\rho + p_2 y \end{pmatrix}, \tag{5.13}$$

where (ρ, p_2) *is* $(1,0), (0,1)$, *or* $(1,1)$.

4. $k_0 = 2,\ \mu = 0$, *there is no* β *and we have*

$$\gamma^T = \begin{pmatrix} y \\ 0 \end{pmatrix}. \tag{5.14}$$

5. $k_0 = 1,\ \mu \geq 2$

$$\beta = \alpha\Lambda, \qquad \Lambda = \Lambda^T, \qquad \gamma = 0. \tag{5.15}$$

6. $k_0 = 1,\ \mu = 1$

$$\beta = 0, \qquad \gamma = a. \tag{5.16}$$

The case $k_0 = 1,\ \mu = 0$ *is not allowed.*

Two free-rowed non-splitting MASAs of $e(p_0 + k_0, q_0 + k_0)$, $M(p_0, q_0, k_0, \Lambda)$ *and* $M'(p_0, q_0, k_0, \Lambda')$, *are* $E(p_0 + k_0, q_0 + k_0)$ *conjugated (for cases 1 and 5) if the matrices* Λ, Λ' *characterizing them satisfy:*

$$\Lambda' = \frac{1}{g_1} G_2 (\Lambda - \sum_{k=1}^{k_0} \theta_k Q_k K_{p_0,q_0}) G_2^T \tag{5.17}$$

for some $g_1, g_j \in \mathbb{R}$, $\theta_k \in \mathbb{R}$, $G_2 \in o(p_0, q_0)$ *such that*

$$Q_j = \frac{1}{g_1} g_j G_2 Q_j G_2^{-1}. \tag{5.18}$$

Proof: 1. $k_0 \geq 3$ We start from a free-rowed MANS in eq.(5.5). Requiring commutativity $[X_e, X_e'] = 0$ leads to the following equations

$$\begin{aligned} (\alpha Q_j)\beta'^T + y_{ja}\gamma_a' &= (\alpha' Q_j)\beta^T + y_{ja}'\gamma_a \\ (Q_j\alpha^T)\gamma_j' &= (Q_j\alpha'^T)\gamma_j. \end{aligned} \tag{5.19}$$

The entries in β, γ are linearly dependent on those in Y and α, *i.e.*

$$\begin{aligned} \beta &= \alpha\Lambda + \textstyle\sum_{1\leq i<k\leq k_0} y_{ik}\rho_{ik}, & \Lambda \in \mathbb{R}^{\mu\times\mu}, \rho_{ik} \in \mathbb{R}^{1\times\mu} \\ \gamma &= \alpha W + \textstyle\sum_{1\leq i<k\leq k_0} y_{ik}P_{ik}, & W \in \mathbb{R}^{\mu\times k_0}, P_{ik} \in \mathbb{R}^{1\times k_0} \end{aligned} \tag{5.20}$$

We substitute β and γ into eq. (5.19) and compare coefficients of $\alpha_i\alpha_j'$, for i and j fixed. First consider the case $j = 1$. We obtain that

$$\begin{gathered} \Lambda = \Lambda^T; \qquad P_{ik,a} = 0, \quad 2 \leq i < k, \quad 1 < a; \qquad P_{1k,a} = P_{1a,k}, \\ \rho_{ik} = 0, \quad 2 \leq i < k; \qquad W_a = \rho_{1a}, \quad a \geq 2, \\ Q_j\Lambda K_{p_0,q_0}^{-1} = \Lambda K_{p_0,q_0}^{-1} Q_j. \end{gathered} \tag{5.21}$$

For $j = 2$ we obtain

$$\begin{aligned} P_{ik,1} &= 0 \quad 3 \leq i < k, & P_{12,a} &= -P_{2a,1} \\ \rho_{1k} &= 0 \quad k \geq 3, & W_1 &= -Q_2\rho_{12}^T. \end{aligned} \tag{5.22}$$

And for $j = 3$ we get

$$W = 0, \qquad \rho_{ik} = 0, \qquad P_{ik} = 0 \qquad \text{for} \quad k_0 \geq 3, \tag{5.23}$$

Using the translations we obtain the coboundaries θ_i

$$e^{\theta_i P_i} Z e^{-\theta_i P_i} = Z - \theta_i[Z, P_i]. \tag{5.24}$$

This leads to replacing Λ by

$$\Lambda' = \Lambda - \sum_{k=1}^{k_0} \theta_k Q_k K_{p_0,q_0}. \tag{5.25}$$

All θ_i are free and can be used to remove all coboundaries. In particular if K_{p_0,q_0} is chosen to satisfy $TrK_{p_0,q_0} \neq 0$ we can use θ_1 to make Λ traceless. Equation (5.17) corresponds to transformations of Λ using the normalizer of N in $E(p,q)$.

2. $k_0 = 2$, $\mu \geq 2$

Here there is only one matrix $Q = Q_2$, the vector γ is $\gamma = (\gamma_1, \gamma_2)$ and $Y = \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}$. We have

$$\beta = \alpha\Lambda + y\rho, \qquad \rho \in \mathbb{R}^{1\times\mu} \tag{5.26}$$

$$\gamma_1 = \alpha w_1^T + p_1 y, \quad \gamma_2 = \alpha w_2^T + p_2 y, \quad w_1, w_2 \in \mathbb{R}^{1\times\mu}, \quad p_1, p_2 \in \mathbb{R} \tag{5.27}$$

From the $[X_e, X_e'] = 0$ we obtain that

$$\Lambda = \Lambda^T, \qquad Q\Lambda K^{-1}_{p_0,q_0} = \Lambda K^{-1}_{p_0,q_0} Q \tag{5.28}$$

$$\beta = \alpha\Lambda + y\rho, \qquad \gamma = \begin{pmatrix} -\alpha Q\rho^T \\ \alpha\rho^T \end{pmatrix}. \tag{5.29}$$

Equation (5.19) for $j = 2$ leads to

$$[Q^T(\alpha^T\alpha' - \alpha'^T\alpha) + (\alpha'^T\alpha - \alpha^T\alpha')Q]\rho^T = 0. \tag{5.30}$$

Writing eq.(5.30) in components and choosing α and α' such that $\alpha_a = 1, \alpha'_b = 1$ and all other components vanish, we obtain

$$(Q^T)_{ia}\rho_b - (Q^T)_{ib}\rho_a + \sum_{k=1}^{\mu}(\delta_{ib}Q_{ak} - \delta_{ia}Q_{bk})\rho_k = 0, \qquad \forall i, a, b. \tag{5.31}$$

This provides us with two types of relations

$$Q_{ai}\rho_b - Q_{bi}\rho_a = 0, \qquad a \neq i, \quad b \neq i \tag{5.32}$$

$$-Q_{ii}\rho_a + Q_{ai}\rho_i + \sum_{k=1}^{\mu} Q_{ak}\rho_k = 0, \qquad a \neq i. \tag{5.33}$$

The matrix Q is block diagonal,

$$Q = diag(J_1, J_2, \ldots, J_r), \qquad \textstyle\sum_{i=1}^{r} dim J_i = \mu$$
$$dim J_1 \geq dim J_2 \geq \ldots \geq dim J_r \geq 1, \tag{5.34}$$

where each J_i is an indecomposable element of a Jordan algebra $jo(p_i, q_i)$, $p_i + q_i = dim J_i$ (see e.g Ref. [23]). The matrix K_{p_0,q_0} has the same block structure. Possible forms of elementary blocks in Q are

$$J_i(q_i) = \begin{pmatrix} q_i & 1 & & \\ & q_i & 1 & \\ & & \ddots & 1 \\ & & & q_i \end{pmatrix},$$

$$J_i(r_i + s_i) = \begin{pmatrix} r_i & s_i & 1 & 0 & & & \\ -s_i & r_i & 0 & 1 & & & \\ & & \ddots & & \ddots & & \\ & & & & & 1 & 0 \\ & & & & & 0 & 1 \\ & & & & & r_i & s_i \\ & & & & & -s_i & r_i \end{pmatrix}. \tag{5.35}$$

After complexification the second type of block reduces to the first one, so it actually suffices to consider the first type of block only (see Lemma 5.1).

Let us first assume $dim J_1 \geq 3$. Writing relation (5.33) for $i = 1$ and $2 \leq a \leq r$ we obtain $\rho_3 = \rho_4 = \ldots = \rho_\mu = 0$. Taking $a = 1, i = 2$ in (5.32) we then obtain $\rho_2 = 0$. Taking $a = 1, b = 2, i = 3$ in (5.33) we obtain $\rho_1 = 0$. Thus, if the largest block $J_1(q)$ satisfies $dim J_1(q) \geq 3$, we have $\rho = 0$.

Now let us assume $dim J_1(q) = 2$ so that all other blocks have dimension 2 or 1. By the same argument we have $\rho_3 = \rho_4 = \ldots = \rho_\mu = 0$ and also $\rho_2 = 0$. If Q has the form (5.11), then all relations (5.32) and (5.33) are satisfied and ρ_1 remains free. If any of the other diagonal elements, say Q_{33} is not zero, then relation (5.33) for $i = 3, a = 1$ implies $\rho_1 = 0$. If we have $q \neq 0$ in $J_1(q)$, then at least one other diagonal element of Q must satisfy $Q_{aa} \neq 0, a \geq 3$, since we have $TrQ = 0$.

Finally, let Q be diagonal. We have $Q \neq 0, TrQ = 0$, hence at least two diagonal elements are nonzero. Relations (5.32) and (5.33) then imply $\rho_i = 0, i = 1, \ldots \mu$.

Using the normalizer $G = diag(g_1, g_2, G_2, g_1^{-1}, g_2^{-1})$ we normalize ρ_1 to $\rho_1 = 1$ for $\rho_1 \neq 0$.

3. $k_0 = 2$, $\mu = 1$

There is no matrix Q and we have

$$\beta = \lambda a + \rho y, \qquad \lambda \in \mathbb{R} \tag{5.36}$$

$$\gamma_1 = aw_1 + p_1 y, \qquad \gamma_2 = aw_2 + p_2 y \qquad w_1, w_2, p_1, p_2 \in \mathbb{R}.$$

Condition $[X_e, X_e'] = 0$ implies $w_1 = 0, p_1 = 0$ and after removing the coboundaries we obtain

$$\beta = \rho y, \qquad \gamma_1 = 0, \qquad \gamma_2 = a\rho + p_2 y. \tag{5.37}$$

Using the normalizer $G = diag(g_1, g_2, g_3, g_4, g_5, 1)$, satisfying $G\tilde{K}_0 G^T = \tilde{K}_0$, we can normalize (ρ, p_2) to one of the following: $(1, 0), (1, 1), (0, 1)$.

4. $k_0 = 2$, $\mu = 0$

Using the normalizer $G = diag(g_1, G_2, \frac{1}{g_1}, 1)$ we obtain eq. (5.14).

5. $k_0 = 1$, $\mu \geq 2$

In this case $Y = 0$ and $A = \alpha \in \mathbb{R}^{1\times\mu}$ in eq. (5.5). Then we have

$$\beta = \alpha\Lambda, \qquad \gamma = \alpha w^T, \quad \beta \in \mathbb{R}^{1\times\mu}, \quad \gamma \in \mathbb{R}. \tag{5.38}$$

From the $[X_0, X_0'] = 0$ we obtain that

$$\Lambda = \Lambda^T, \qquad w = 0. \tag{5.39}$$

Removing the coboundaries leads to replacing Λ by

$$\Lambda' = \Lambda - \theta K_{p_0, q_0}, \tag{5.40}$$

where θ can be chosen to annul trace of Λ (if $TrK_{p_0,q_0} \neq 0$).

6. $k_0 = 1$, $\mu = 1$

The proof is trivial and can be found in Ref.[20].

Using the normalizer of the splitting MASA (4.8)in the group $E(p_0 + k_0, q_0 + k_0)$ we can simplify Λ further. The normalizer is represented by block diagonal matrices

$$G = diag(G_1, G_2, G_1^{-1}, 1). \tag{5.41}$$

Choosing $G_1 = diag(g_1, \ldots, g_{k_0})$, G_2 satisfying $G_2 K_{p_0,q_0} G_2^T = K_{p_0,q_0}$ leads to equations (5.17) and (5.18).

This completes the proof of the Theorem 5.1. □

5.3 Nonsplitting MASAs of $\mathbf{e}(p_0 + k_0, q_0 + k_0)$ related to non-free-rowed MANSs

The general study of non-free rowed MASAs of $o(p,q)$ is less well developed. Many different series of MASAs of $o(p,q)$ exist. We will consider only two of them, which we denote $A(2k+1,0)$ and $A(2k+1,1)$, by analogy with series of non-free-rowed MANSs of $o(n,\mathbb{C})$ [16].

1. The series $A(2k+1,0)$ of $o(p,q)$ is represented by the matrix set

$$X = \begin{pmatrix} 0 & a_1 & 0 & a_2 & \ldots & a_k & 0 \\ & \ddots & \ddots & \ddots & \ddots & & a_k \\ & & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & \ddots & a_2 \\ & & & & \ddots & \ddots & 0 \\ & & & & & \ddots & a_1 \\ & & & & & & 0 \end{pmatrix}, \tag{5.42}$$

$$K = F_{2k+1} = \begin{pmatrix} & & & & \epsilon \\ & & & -\epsilon & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & -\epsilon & & & \\ \epsilon & & & & \end{pmatrix}, \tag{5.43}$$

where all $a_i's$ are free.

Thus for $\epsilon = 1$ we have $M \subset \begin{cases} o(k+1,k) & \text{for } k \text{ even} \\ o(k,k+1) & \text{for } k \text{ odd} \end{cases}$

and for $\epsilon = -1$ we have $M \subset \begin{cases} o(k+1,k) & \text{for } k \text{ odd} \\ o(k,k+1) & \text{for } k \text{ even.} \end{cases}$

The splitting MASA of $e(p,q)$ for this series (in accordance with Theorem 4.1) is written as follows:

$$X_e = \begin{pmatrix} 0 & a_1 & 0 & a_2 & \dots & a_k & 0 & \alpha \\ & \ddots & \ddots & \ddots & \ddots & & a_k & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & \ddots & a_2 & 0 \\ & & & & \ddots & \ddots & 0 & 0 \\ & & & & & \ddots & a_1 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{pmatrix} . \tag{5.44}$$

Theorem 5.2 *Every nonsplitting MASA of $e(p,q)$ corresponding to the splitting MASA (5.44) is $E(p,q)$ conjugate to the following one*

$$X_e = \begin{pmatrix} 0 & a_1 & 0 & a_2 & \dots & \dots & a_k & 0 & \alpha \\ & \ddots & \ddots & \ddots & \ddots & & & a_k & 0 \\ & & \ddots & \ddots & \ddots & \ddots & & 0 & a_k \\ & & & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & \ddots & \ddots & a_2 & 0 \\ & & & & & \ddots & \ddots & 0 & a_2 \\ & & & & & & \ddots & a_1 & 0 \\ & & & & & & & 0 & a_1 \\ & & & & & & & & 0 \end{pmatrix}, \quad K_e = \begin{pmatrix} F_{2k+1} & \\ & 0 \end{pmatrix} \tag{5.45}$$

where all entries in X_e are free.

Proof: We will construct a nonsplitting MASA from the splitting one (5.44)

$$X'_e = \begin{pmatrix} 0 & a_1 & 0 & a_2 & \dots & & a_k & 0 & \alpha \\ & 0 & a_1 & 0 & a_2 & \dots & & a_k & \beta_2 \\ & & \ddots & \ddots & \ddots & \ddots & & 0 & \beta_3 \\ & & & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & \ddots & \ddots & a_2 & \beta_{2k-2} \\ & & & & & \ddots & \ddots & 0 & \beta_{2k-1} \\ & & & & & & \ddots & a_1 & \beta_{2k} \\ & & & & & & & 0 & \beta_{2k+1} \\ & & & & & & & & 0 \end{pmatrix}, \tag{5.46}$$

where $\beta' s$ are linearly dependent on $a_i' s$. Before imposing commutation relations we will remove the coboundaries.

Consider one element of the algebra (5.46)

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & & \dots & 0 & 0 \\ & 0 & 1 & 0 & \dots & & \dots & 0 & \alpha_{1,2} \\ & & \ddots & \ddots & \ddots & & & 0 & \alpha_{1,3} \\ & & & \ddots & \ddots & \ddots & & \vdots & \vdots \\ & & & & \ddots & \ddots & \ddots & 0 & \alpha_{1,2k-2} \\ & & & & & \ddots & \ddots & 0 & \alpha_{1,2k-1} \\ & & & & & & \ddots & 1 & \alpha_{1,2k} \\ & & & & & & & 0 & \alpha_{1,2k+1} \\ & & & & & & & & 0 \end{pmatrix}, \tag{5.47}$$

where $\alpha_{1,l}, l = 2, \dots, 2k+1$ represent the translations. We note that $\alpha_{1,l} \dots \alpha_{1,2k}$ correspond to coboundaries and can be eliminated by conjugation by the translation group. Thus only $\alpha_{1,2k+1}$ is left in A_1.

Now consider an element A_i of algebra (5.46), obtained by setting $a_i = \delta_{ij}$,

$j \geq 2$

$$A_i = \begin{pmatrix} 0 & 0 & 0 & 1 & \dots & & \dots & 0 & 0 \\ & 0 & 0 & 0 & 1 & & \dots & 0 & \alpha_{i,2} \\ & & \ddots & \ddots & \ddots & \ddots & & 0 & \alpha_{i,3} \\ & & & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & \ddots & \ddots & 1 & \alpha_{i,2k-2} \\ & & & & & \ddots & \ddots & 0 & \alpha_{i,2k-1} \\ & & & & & & \ddots & 0 & \alpha_{i,2k} \\ & & & & & & & 0 & \alpha_{i,2k+1} \\ & & & & & & & & 0 \end{pmatrix}. \tag{5.48}$$

Commuting A_1 with all $A_i, i = 2, \dots k$ we obtain that $\alpha_{j,2k-2j+3} = \alpha_{1,2k+1}$, $j = 2, \dots, k$ and all other $\alpha_{i,j}$ have to be zero.

Using the normalizer G of the form

$$G = \left(g_k^k, \dots, g_k^2, g_k, 1, g_k^{-1}, \dots, g_k^{-k}\right) \tag{5.49}$$

we can normalize $\alpha_{1,2k+1}$ to $\alpha_{1,2k+1} = 1$. This leads to the MASA (5.45) and completes the proof of Theorem 5.2. □

2. The series $A(2k+1,1)$ of $o(p,q)$ is represented by the following matrix set:

$$X = \begin{pmatrix} 0 & a_1 & 0 & a_2 & \dots & a_k & 0 & b \\ & \ddots & \ddots & \ddots & \ddots & & a_k & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & \ddots & a_2 & 0 \\ & & & & \ddots & \ddots & 0 & 0 \\ & & & & & \ddots & a_1 & 0 \\ & & & & & & 0 & 0 \\ & & & & & 0 & -\epsilon b & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} F_{2k+1} & \\ & 1 \end{pmatrix}, \tag{5.50}$$

where all $a_i's$ and b are free. The corresponding metric is

$$K = \begin{pmatrix} F_{2k+1} & \\ & 1 \end{pmatrix} = \begin{pmatrix} & & & & \epsilon & 0 \\ & & & -\epsilon & & 0 \\ & & \cdot^{\cdot^{\cdot}} & & & \vdots \\ & -\epsilon & & & & 0 \\ \epsilon & & & & & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix} \tag{5.51}$$

Thus for $\epsilon = 1$ we have $M \subset \begin{cases} o(k+1, k+1) & \text{for } k \text{ odd} \\ o(k+2, k) & \text{for } k \text{ even} \end{cases}$

and for $\epsilon = -1$ we have $M \subset \begin{cases} o(k+1, k+1) & \text{for } k \text{ even} \\ o(k+2, k) & \text{for } k \text{ odd.} \end{cases}$

Theorem 5.3 *Every nonsplitting MASA corresponding to the splitting MASA (5.50) is $E(p,q)$ conjugated to the MASA of the form*

$$X_e = \begin{pmatrix} 0 & a_1 & 0 & a_2 & \dots & & a_k & 0 & b & \alpha \\ & \ddots & \ddots & \ddots & \ddots & & & a_k & 0 & \lambda b \\ & & \ddots & \ddots & \ddots & \ddots & & 0 & 0 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & & & \ddots & \ddots & \ddots & a_2 & 0 & 0 \\ & & & & & \ddots & \ddots & 0 & 0 & 0 \\ & & & & & & \ddots & a_1 & 0 & 0 \\ & & & & & & & 0 & 0 & 0 \\ & & & & & & & -\epsilon b & 0 & \lambda a_1 + \mu b \\ & & & & & & & 0 & 0 & 0_1 \end{pmatrix} \tag{5.52}$$

with the metric as in (5.51). The entries a_i, b and α are free. Parameters λ and μ are one of the following sets:

$$(\lambda, \mu) = \begin{cases} (0,1) \\ (0,-1) \\ (1,\mu), \quad \mu \in \mathbb{R}. \end{cases} \tag{5.53}$$

Proof: The proof is similar to that of Theorem 5.2 and we omit it here. □

6 Decomposition properties of MASAs of $\mathbf{e}(p,q)$

The results of Sections 4 and 5 can be formulated in terms of a decomposition of the underlying pseudoeuclidean space $S(p,q)$. Both splitting and nonsplitting MASAs have been represented by matrix sets $\{X_e, K_e\}$ as in eq.(5.4), (4.2). We shall call a MASA of $e(p,q)$ *decomposable* if the metric K_e in (4.2) consists of 2 or more blocks. The projection of such a MASA onto the $o(p,q)$ subalgebra is then an orthogonally decomposable MASA of $o(p,q)$. Let $M_e(p,q)$ be a decomposable MASA of $e(p,q)$. The space $S(p,q)$ then splits into a direct sum of subspaces

$$S(p,q) = \bigoplus_{i=1}^{l} S(p_i,q_i), \qquad \sum_{i=1}^{l} p_i = p, \qquad \sum_{i=1}^{l} q_i = q \tag{6.1}$$

and each indecomposable component of the decomposable MASA of $e(p,q)$ acts independently in one of the spaces $S(p_i,q_i)$. We shall write

$$M_e(p,q) = \bigoplus_{i=1}^{l} M_e(p_i,q_i). \tag{6.2}$$

Each individual indecomposable MASA $M_e(p_i,q_i) \subset e(p_i,q_i)$ can then be considered separately.

Consider the matrix set $\{X_e, K_e\}$, X_e given by eq.(5.4), K_e as in eq.(4.2), where each block is indecomposable. The blocks to be considered consist of a block on the diagonal in X_e, plus an entry from the right hand column in X_e.

The following types of indecomposable MASAs $M_e(p_i,q_i) \subset e(p_i,q_i)$ exist.

- $dimS = 1$. The MASAs are pure positive or negative length translations.

$$M_e(1,0) = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \quad x \in \mathbb{R}, \quad K_e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \tag{6.3}$$

$$M_e(0,1) = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}, \quad y \in \mathbb{R}, \quad K_e = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \right\}. \tag{6.4}$$

  A MASA $M_e(p,q)$ of $e(p,q)$ contains k_+ of the first ones and k_- of the second.

- $dimS = 2$. The MASAs are $o(2)$ rotations in a $(++)$, or $(--)$ type subspace, or $o(1,1)$ pseudorotations in a $(+-)$ space:

$$M_e(2,0) = \left\{ \begin{pmatrix} 0 & x & 0 \\ -x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_e = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \right\} \tag{6.5}$$

$$M_e(0,2) = \left\{ \begin{pmatrix} 0 & x & 0 \\ -x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_e = \begin{pmatrix} -I_2 & 0 \\ 0 & 0 \end{pmatrix} \right\} \tag{6.6}$$

$$M_e(1,1) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_e = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}. \tag{6.7}$$

- $dimS = k \geq 3$. There are two possible types of indecomposable MASAs of $e(p,q)$ for $p+q \geq 3$. Both of them have $k_+ = k_- = 0$ (no nonisotropic translations).

  i) $M_e(p,q)$ contains k_0 isotropic translations with $k_0 \geq 1$. The projection of $M_e(p,q)$ onto $o(p,q)$ is then a MANS of $o(p,q)$ with Kravchuk signature $(k_0, p{+}q{-}2k_0, k_0)$. The MANS can be free-rowed or non-free-rowed. The MASA of $e(p,q)$ can be splitting , or nonsplitting. Such MASAs exist for any $p+q \geq 3$, $min(p,q) \geq 1$. They were treated in Sections 4 and 5.

  ii) $M(p,q)$ is an orthogonally indecomposable MASA of $o(p,q)$ that is not a MANS. It gives rise to a splitting MASA of $e(p,q)$ which contains no translations $(k_0 = 0)$. As reviewed in Section 3 such MASAs of $o(p,q)$ exist only for $p+q$ even.

7 A special case: MASAs of e(p,2)

The case $q = 2$, like $q = 1$ and $q = 0$, presented earlier [20] is simpler than that of $q \geq 3$. All MASAs can be presented explicitly, in particular those involving non-free-rowed MANS of $o(p,2)$.

The possible decomposition patterns (6.2) for MASAs of $e(p,2)$ are

$$M_e(p,2) \;=\; M_e(p_1,2) \oplus l_+ M_e(2,0) + k_+ M_e(1,0) \tag{7.8}$$

$$
\begin{aligned}
& p_1 = 1, \quad or \quad p_1 \geq 2, \qquad p_1 + 2l_+ + k_+ = p \\
M_e(p,2) &= M_e(p_1,1) \oplus M_e(p_2,1) \oplus l_+ M_e(2,0) + k_+ M_e(1,0) \qquad (7.9) \\
& p_1 + p_2 + 2l_+ + k_+ = p \\
M_e(p,2) &= M_e(0,2) \oplus l_+ M_e(2,0) + k_+ M_e(1,0) \qquad (7.10) \\
& 2l_+ + k_+ = p.
\end{aligned}
$$

The algebras $M_e(2,0)$, $M_e(0,2)$ and $M_e(1,0)$ are already abelian (and one dimensional) as are $M_e(0,1)$ and $M_e(1,1)$. The MASAs $M_e(p,1)$ of $e(p,1)$, $p \geq 2$ were studied in our earlier article [20].

Thus, we need to treat only indecomposable MASAs of $e(p,2)$. As was stated in Section 6 for general $e(p,q)$, two cases arise, namely $k_0 = 0$ and $1 \leq k_0 \leq min(p,q)$, where k_0 is the number of linearly independent translation generators present.

1. $k_0 = 0$

Then $M(p,2)$ is an orthogonally indecomposable MASA of $o(p,2)$ that is not a MANS. These exist only when p is even ($p \geq 2$).

For $p = 2$ three inequivalent OID MASAs that are not MANS exist and the corresponding splitting MASAs of $e(p,2)$ are given by the following matrix sets:

i) $M(2,2)$ is AOID but D

$$
X_e = \begin{pmatrix} a & b & & & 0 \\ 0 & a & & & 0 \\ & & -a & 0 & 0 \\ & & -b & -a & 0 \\ & & & & 0_1 \end{pmatrix}, \quad K_e = \begin{pmatrix} & I_2 & \\ I_2 & & \\ & & 0_1 \end{pmatrix} \qquad (7.11)
$$

ii) $M(2,2)$ is AOID, ID but NAID

$$
X_e = \begin{pmatrix} 0 & a & 0 & b & 0 \\ -a & 0 & -b & 0 & 0 \\ & & 0 & a & 0 \\ & & -a & 0 & 0 \\ & & & & 0_1 \end{pmatrix} \qquad (7.12)
$$

with K_e same as in *i)*.

iii) $M(2,2)$ is NAOID but D

$$X_e = \begin{pmatrix} a & b & & & 0 \\ -b & a & & & 0 \\ & & -a & b & 0 \\ & & -b & -a & 0 \\ & & & & 0_1 \end{pmatrix} \tag{7.13}$$

with K_e same as in *i)*.

For $p = 2l$, $l \geq 2$ we have just one OID MASA of $o(p,2)$ (NAOID, ID but NAID), namely $M = RQ \oplus$MANS of $su(l,1)$. The corresponding splitting MASA of $e(p,2)$ is represented as following matrix set

$$X_e = \begin{pmatrix} 0 & b & a_1 & 0 & \dots & a_{l-1} & 0 & 0 & c & 0 \\ -b & 0 & 0 & a_1 & \dots & 0 & a_{l-1} & -c & 0 & 0 \\ & & 0 & b & & & & -a_1 & 0 & 0 \\ & & -b & 0 & & & & 0 & -a_1 & 0 \\ & & & & \ddots & & & \vdots & \vdots & \vdots \\ & & & & & 0 & b & -a_{l-1} & 0 & \vdots \\ & & & & & -b & 0 & 0 & -a_{l-1} & 0 \\ & & & & & & & 0 & b & 0 \\ & & & & & & & -b & 0 & 0 \\ & & & & & & & & & 0_1 \end{pmatrix},$$

$$K_e = \begin{pmatrix} & & I_2 & \\ & I_{2l-2} & & \\ I_2 & & & \\ & & & 0_1 \end{pmatrix}. \tag{7.14}$$

2. $k_0 = 1$

The projection of $M_e(p,2)$ onto $o(p,2)$ will be a MANS of $o(p,2)$ with Kravchuk signature $(1\ p\ 1)$. This MANS can be free-rowed, or non-free-rowed, so we obtain two splitting MASAs of $e(p,2)$ represented, respectively, by

i) free-rowed

$$X_e = \begin{pmatrix} 0 & \alpha & 0 & z \\ 0 & 0 & -K_0\alpha^T & 0 \\ 0 & 0 & 0 & 0 \\ & & & 0_1 \end{pmatrix}, \quad K_e = \begin{pmatrix} 0 & 0 & 1 \\ 0 & K_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{7.15}$$

where K_0 has signature $(p-1,1)$, $\alpha \in \mathbb{R}^{1\times p}$, $1 \leq p$

ii) non-free rowed

$$\begin{aligned} X_e &= \begin{pmatrix} 0 & a & \alpha & 0 & b & 0 & z \\ & 0 & 0 & a & 0 & -b & 0 \\ & & 0 & 0 & 0 & -\alpha^T & \vdots \\ & & & 0 & -a & 0 & 0 \\ & & & & 0 & -a & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0_1 \end{pmatrix}, \\ K_e &= \begin{pmatrix} & & & & 1 & 0 \\ & & & 1 & 0 & 0 \\ & & I_{\nu+1} & 0 & 0 & \vdots \\ & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0_1 \end{pmatrix} \end{aligned} \tag{7.16}$$

$\alpha \in \mathbb{R}^{1\times\nu}$, $1 \leq \nu$ and $\nu = p - 3$.

The MASA (7.15) gives rise to three different nonsplitting MASAs for $p \geq 2$ which can be expressed as

$$X_e = \begin{pmatrix} 0 & \alpha & 0 & z \\ 0 & 0 & -K_0\alpha^T & BK_0\alpha^T \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_e = \begin{pmatrix} & & 1 & \\ & K_0 & & \\ 1 & & & \\ & & & 0_1 \end{pmatrix}. \tag{7.17}$$

K_0 is the same as in (7.15) and B satisfies the condition $BK_0 = K_0B^T$, *i.e.* B is an element of the Jordan algebra $jo(p-1,1)$. A classification of the elements Jordan algebras was performed in the paper by Djokovic et al [23] and the couple

$\{B, K_0\}$ can have one of the three different following forms (keeping in mind the signature of K_0):

i)

$$B = \begin{pmatrix} a & \\ & B_0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} -1 & \\ & I \end{pmatrix} \tag{7.18}$$

ii)

$$B = \begin{pmatrix} a & 0 & \\ 1 & a & \\ & & B_0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & I \end{pmatrix} \tag{7.19}$$

iii)

$$B = \begin{pmatrix} a & 0 & 0 & \\ 1 & a & 0 & \\ 0 & 1 & a & \\ & & & B_0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0 & 0 & 1 & \\ 0 & 1 & 0 & \\ 1 & 0 & 0 & \\ & & & I \end{pmatrix}, \tag{7.20}$$

where B_0 is a diagonal matrix.

For $p = 1$ the nonsplitting MASA corresponding to eq.(7.15) is

$$X_e = \begin{pmatrix} 0 & a & 0 & z \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_e = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_1 \end{pmatrix} \tag{7.21}$$

The MASA (7.16) for $\nu \geq 2$ gives rise to one type of nonsplitting MASA that can be represented as

$$X_e = \begin{pmatrix} 0 & a & \alpha & 0 & b & 0 & z \\ & 0 & 0 & a & 0 & -b & \alpha\rho^T \\ & & 0 & 0 & 0 & -\alpha^T & a\rho^T + \Lambda\alpha^T \\ & & & 0 & -a & 0 & 0 \\ & & & & 0 & -a & 0 \\ 0 & & & & & 0 & 0 \\ & & & & & & 0_1 \end{pmatrix} \tag{7.22}$$

with $\Lambda = \Lambda^T$. Using the normalizer $G = diag(g, g_1, G_2, g_3, 1/g_1, g, 1)$, $G_2 \in \mathbb{R}^{\nu\times\nu}$, $g, g_1, g_3 \in \mathbb{R}$, satisfying $G_2 G_2^T = I_\nu$, $g^2 = g_3^2 = 1$ we can transform $\Lambda, \rho)$ into

$$\Lambda' = \frac{1}{g} G_2 \Lambda G_2^T, \qquad \rho' = \frac{1}{g_1 g_3} G_2 \rho. \tag{7.23}$$

We can use G_2 either to diagonalize Λ, or to rotate ρ into *e.g.* $\rho = (\rho_1, 0, \ldots, 0)$.
3. $k_0 = 2$
The projection of $M_e(p, 2)$ onto $o(p, 2)$ is a free-rowed MANS with Kravchuk signature $(2\ p-2\ 2)$. The corresponding splitting MASA of $e(p, 2)$ is given in Theorem 5.1 with $q = k_0 = 2$ and $K_{p_0, q_0} = I_{p-2}$. In this case Q_2 can be chosen as $Q_2 = diag(1, q_2, \ldots, q_\mu)$, $q_1 = 1 \geq |q_2| \geq \ldots \geq |q_\mu|$. This MASA in turn gives rise to the following non-splitting MASAs.

$$X_e = \begin{pmatrix} 0 & 0 & \alpha & 0 & y & z_1 \\ 0 & 0 & \alpha Q & -y & 0 & z_2 \\ 0 & 0 & 0 & -\alpha^T & -Q\alpha^T & \Lambda^T\alpha^T \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_1 \end{pmatrix}. \tag{7.24}$$

Here Λ is a diagonal matrix, $Tr\Lambda = 0$ and K_e is same as in eq.(5.5).

8 Conclusions

The main conclusion is that we have presented guidelines for constructing all MASAs of $e(p, q)$ for any fixed values of p and q. Some of the results are entirely explicit, such as Theorem 4.1 describing all splitting MASAs of $e(p, q)$, and

Theorem 5.1 presenting nonsplitting MASAs containing a free-rowed MANS of $o(p_0 + k_0, q_0 + k_0) \subset o(p, q)$. The results on MASAs of $e(p, q)$ involving non-free-rowed MANS of $o(p_0 + k_0, q_0 + k_0)$ are less complete and amount to specific examples (see Theorems 5.2 and 5.3). The decomposition results of Section 6 allow us to restrict all considerations to indecomposable MASAs of $e(p, q)$, both splitting and non-splitting ones. The results for $e(p, 2)$ presented in Section 7 are complete and explicit, like those given earlier for $e(p, 0)$ and $e(p, 1)$ [20]. In particular we have constructed all MASAs related to non-free-rowed MANSs.

Work concerning the application of MASAs of $e(p, q)$ is in progress. In particular, we use MASAs of $e(p, q)$ to construct the coordinate systems in which certain partial differential equations (Laplace-Beltrami, Hamilton-Jacobi) allow the separation of variables.

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Chapter 2

Maximal Abelian Subgroups of the Isometry and Conformal Groups of Euclidean and Minkowski Spaces

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Maximal Abelian subgroups of the isometry and conformal groups of Euclidean and Minkowski spaces

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Abstract. The maximal Abelian subalgebras (MASAs) of the Euclidean $e(p,0)$ and pseudo-euclidean $e(p,1)$ Lie algebras are classified into conjugacy classes under the action of the corresponding Lie groups $E(p,0)$ and $E(p,1)$, and also under the conformal groups $O(p+1,1)$ and $O(p+1,2)$, respectively. The results are presented in terms of decomposition theorems. For $e(p,0)$ orthogonally indecomposable MASAs exist only for $p=1$ and $p=2$. For $e(p,1)$, on the other hand, orthogonally indecomposable MASAs exist for all values of p. The results are used to construct new coordinate systems in which wave equations and Hamilton–Jacobi equations allow the separation of variables.

Résumé. Les sous-algèbres maximales abéliennes (SAMAs) d'algèbres Euclidiennes $e(p,0)$ et pseudo-euclidiennes $e(p,1)$ sont classifiées en classes de conjugasion sous l'action des groupes de Lie correspondants $E(p,0)$ et $E(p,1)$. Elles sont aussi classifiées sous l'action des groupes conformes $O(p+1,1)$ et $O(p+1,2)$. Les résultats sont presentés dans des théoremes de decompositions. Pour $e(p,0)$, les SAMAs orthogonallement indecomposables existent seulement pour $p=1$ et $p=2$. Pour $e(p,1)$, les SAMAs orthogonalement indecomposables existent pour toutes les valeurs de p. Les résultats sont utilisés pour construire des nouveau systèmes de coordonnées, dans lesquelles les équations d'onde et les équations de Hamilton–Jacobi admettent la separation de variables.

1. Introduction

The stage for much of mathematical physics is the real flat space $\mathbb{R}^n$ with a non-degenerate indefinite metric of signature (p,q). We shall denote this space $M(p,q)$ with $p+q=n$. The isometry group of this space is the pseudo-euclidean group $E(p,q)$ and the conformal group is $C(p,q)\sim O(p+1,q+1)$ (the pseudo-orthogonal group in $p+q+2$ dimensions, acting locally and nonlinearly on $M(p,q)$).

The purpose of this article is to present a classification of the maximal Abelian subalgebras (MASAs) of the real Euclidean and pseudo-euclidean Lie algebras $e(p,0)\equiv e(p)$ and $e(p,1)$. The classification is first performed with respect to conjugation under the corresponding Lie groups $E(p,0)\equiv E(p)$ and $E(p,1)$, respectively, and it also provides a classification of the connected maximal Abelian subgroups of the corresponding groups $E(p)$ and $E(p,1)$. We also present a classification of MASAs of the corresponding conformal algebras $c(p,0)\sim o(p+1,1)$ and $c(p,1)\sim o(p+1,2)$ under the corresponding groups $O(p+1,1)$ and $O(p+1,2)$. This classification is used to show (for $q=0$ or 1) which MASAs of $e(p,q)$ are also MASAs of $o(p+1,q+1)$ and which MASAs that are inequivalent under $E(p,q)$ are nevertheless mutually conjugated under the larger conformal group $O(p+1,q+1)$.

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The classification of the MASAs of $e(p, q)$ $(q = 0, 1)$ will be used to address a physical problem: the separation of variables in Laplace–Beltrami and Hamilton–Jacobi equations in the corresponding spaces $M(p, q)$.

The motivation for our study of subgroups of Lie groups and subalgebras of Lie algebras is multifold. For instance, consider any physical problem leading to a system of differential, difference, algebraic, integral or other equations. Let the set of all solutions of the system be invariant under some Lie group G, the 'symmetry group'. Special solutions, corresponding to special boundary, or initial conditions, can be constructed as 'invariant solutions', invariant under some subgroup of the group G [1, 2]. For linear equations, or for Hamilton–Jacobi type equations, solutions obtained by separation of variables are examples of invariant solutions. While all types of subgroups $G_0 \subset G$ are relevant to this problem, Abelian subgroups provide particularly simple reductions and particularly simple coordinate systems. Indeed, each one-dimensional subalgebra of an Abelian symmetry algebra will provide an 'ignorable' variable [3–8], i.e. a variable that does not figure in the metric tensor (a 'cyclic' variable in classical mechanics).

Another example of the application of maximal Abelian subgroups of an invariance group is in any quantum theory, where Abelian subalgebras provide sets of commuting operators that characterize states of a physical system. The system itself is characterized by the Casimir operators of the group G. Complete information about possible quantum numbers would be provided by constructing MASAs of the enveloping algebra of the Lie algebra L of G. MASAs of the Lie algebra itself provide additive quantum numbers.

A third application is in the theory of integrable systems, both finite and infinite dimensional, where MASAs of any underlying Lie algebra provide integrals of motion in involution, commuting flows, and other basic information about the systems.

A series of earlier papers was devoted to MASAs of the classical Lie algebras, such as $sp(2n, R)$ and $sp(2n, C)$ [9], $su(p, q)$ [10], $so(n, C)$ [11] and $so(p, q)$ [12]. In all MASAs of simple and semisimple Lie algebras Cartan subalgebras on the one hand, and maximal Abelian nilpotent algebras (MANSs) on the other, play a special role. The Cartan subalgebras are their own normalizers [13] and consist entirely of non-nilpotent elements. For a complex semisimple Lie algebra there is, up to conjugacy, only one Cartan subalgebra. For real semisimple Lie algebras they were classified by Kostant [14] and Sugiura [15]. Maximal Abelian nilpotent subalgebras consist entirely of nilpotent elements (represented by nilpotent matrices in any finite dimensional representation). They were studied by Kravchuk for $sl(n, C)$ and his results are summed up in book form [16]. Maltsev obtained all MANSs of maximal dimension for the simple Lie algebras [17]. Those of minimal dimension have also been studied [18].

More recently, the study of MASAs was extended to inhomogeneous classical Lie algebras, or finite dimensional affine Lie algebras, starting from the complex Euclidean Lie algebras $e(n, C)$ [19].

The next natural step is to consider the real Euclidean and pseudo-euclidean algebras $e(p, q)$ for $p \geqslant q \geqslant 0$. This study is initiated in the present paper, where we concentrate on the values $q = 0$ and 1. On the one hand, these are the most important in physical applications, since they include the Lie algebras of the groups of motions $E(p)$ of Euclidean spaces and $E(p, 1)$ of Minkowski spaces. On the other, they are the simplest ones to treat, so all results are entirely explicit. The general case of $q \geqslant 2$ will be treated separately and is more complicated from a mathematical point of view.

The classification strategy and some general results on the MASAs of $e(p, q)$ are presented in section 2. The real Euclidean algebra $e(p)$ is treated in section 3, where we also list the MASAs of $o(p, 1)$ and the classification of MASAs of $e(p)$ under the action of the group $O(p+1,1)$. Section 4 then treats MASAs of $e(p,1)$. Section 5 lists results on MASAs of $o(p,2)$ and the classification of MASAs of $e(p,1)$ under the action of the conformal group $O(p+1,2)$ of the compactified Minkowski space $M(p,1)$. In other words, certain MASAs not conjugated under $E(p,1)$ are conjugated under the larger group $O(p+1,2)$. MASAs of $e(p,1)$ are used in section 6 to obtain the maximal Abelian subgroups of $E(p,1)$. These in turn provide us with all separable coordinate systems in the Minkowski space $M(p,1)$ with a maximal number of ignorable variables. Some conclusions are drawn in section 7.

2. General formulation

2.1. Some definitions

We will be classifying maximal Abelian subalgebras of the pseudo-euclidean Lie algebra $e(p,q)$ into conjugacy classes under the action of the pseudo-euclidean Lie group $E(p,q)$. A convenient realization of this algebra and this group is by real matrices Y and H, satisfying

$$Y(X,\alpha) \equiv Y = \begin{pmatrix} X & \alpha \\ 0 & 0 \end{pmatrix} \qquad X \in \mathbb{R}^{n\times n} \quad \alpha \in \mathbb{R}^{n\times 1} \tag{2.1}$$

$$H = \begin{pmatrix} G & a \\ 0 & 1 \end{pmatrix} \qquad G \in \mathbb{R}^{n\times n} \quad a \in \mathbb{R}^{n\times 1} \tag{2.2}$$

respectively, where X and G satisfy

$$\begin{aligned} &XK + KX^{\mathrm{T}} = 0 && GKG^{\mathrm{T}} = K \\ &K = K^{\mathrm{T}} \in \mathbb{R}^{n\times n} && n = p+q \quad det K \neq 0 \\ &\operatorname{sgn} K = (p,q) && p \geqslant q \geqslant 0 \end{aligned} \tag{2.3}$$

respectively. Here $\operatorname{sgn} K$ denotes the signature of K, with p the number of positive eigenvalues of K and q the number of negative ones. We shall also make use of an 'extended' matrix $K_{\mathrm{e}} \in \mathbb{R}^{(n+1)\times(n+1)}$ satisfying

$$K_{\mathrm{e}} = \begin{pmatrix} K & 0 \\ 0 & 0_1 \end{pmatrix} \qquad YK_{\mathrm{e}} + K_{\mathrm{e}}Y^{\mathrm{T}} = 0. \tag{2.4}$$

A convenient basis for the algebra $e(p,q)$ is provided by n translations P_μ and $n(n-1)/2$ rotations and pseudorotations $L_{\mu\nu}$. The commutation relations for this basis are

$$\begin{aligned} &[L_{ik}, L_{ab}] = \delta_{ka}L_{ib} - \delta_{kb}L_{ia} - \delta_{ia}L_{kb} + \delta_{ib}L_{ka} \\ &[L_{\alpha\beta}, L_{\gamma\delta}] = \delta_{\beta\gamma}L_{\alpha\delta} - \delta_{\beta\delta}L_{\alpha\gamma} - \delta_{\alpha\gamma}L_{\beta\delta} + \delta_{\alpha\delta}L_{\beta\gamma} \\ &[L_{ik}, L_{a\beta}] = \delta_{ka}L_{i\beta} - \delta_{ia}L_{k\beta} \\ &[L_{i\alpha}, L_{\beta\gamma}] = \delta_{\alpha\beta}L_{i\gamma} - \delta_{\alpha\gamma}L_{i\beta} \\ &[L_{a\beta}, L_{i\mu}] = \delta_{\beta\mu}L_{ai} + \delta_{ai}L_{\beta\mu} \end{aligned} \tag{2.5}$$

where $i,k,a,b \leqslant p$ and $p < \alpha,\beta,\gamma,\delta,\mu \leqslant q$

$$\begin{aligned} &[P_\alpha, L_{\mu\nu}] = g_{\alpha\mu}P_\nu - g_{\alpha\nu}P_\mu \\ &[P_\mu, P_\nu] = 0 \end{aligned} \tag{2.6}$$

for $0 < \alpha, \mu, \nu \leqslant p+q$,

$$g_{11} = g_{22} = \cdots = g_{pp} = -g_{p+1,p+1} = \cdots = -g_{p+q,p+q} = 1$$
$$g_{\mu\nu} = 0 \qquad \text{for } \mu \neq \nu.$$

A standard realization of this basis in terms of differential operators is given by

$$P_\mu = \frac{\partial}{\partial x_\mu} \qquad L_{ik} = x_i \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_i} \tag{2.7}$$

for $1 \leqslant i < k \leqslant p$ or $p+1 \leqslant i < k \leqslant p+q$ and

$$L_{ik} = -\left(x_k \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_k} \right) \qquad 1 \leqslant i \leqslant p \quad p+1 \leqslant k \leqslant p+q.$$

From the above discussion we see that the pseudo-euclidean Lie algebra is the semidirect sum of the pseudo-orthogonal Lie algebra $o(p,q)$ and an Abelian algebra $T(n)$ of translations.

Since $T(n)$ is an ideal in $e(p,q)$, we can consider the adjoint representation of $o(p,q)$ on $T(n)$. Abusing notation, we use the same letters $P_1, \ldots, P_p, P_{p+1}, \ldots, P_{p+q}$ for basis vectors in this representation. The metric tensor $g_{\mu\nu}$ defined above provides an invariant scalar product on the representation space

$$(P, Q) = g_{\mu\nu} P_\mu Q_\nu. \tag{2.8}$$

We shall call vectors satisfying $P^2 > 0$, $P^2 < 0$ and $P^2 = 0$ $(P \neq 0)$ positive length, negative length and isotropic, respectively.

We also need to define some basic algebraic concepts.

Definition 2.1. The centralizer cent(L_0, L) of a Lie algebra $L_0 \in L$ is a subalgebra of L consisting of all elements in L, commuting elementwise with L_0:

$$\text{cent}(L_0, L) = \{e \in L | [e, L_0] = 0\}. \tag{2.9}$$

Definition 2.2. A maximal Abelian subalgebra L_0 (MASA) of L is an Abelian subalgebra, equal to its centralizer

$$[L_0, L_0] = 0 \qquad \text{cent}(L_0, L) = L_0. \tag{2.10}$$

Definition 2.3. A splitting subalgebra L_0 of the semidirect sum

$$L = F \rhd N \qquad [F, F] \subseteq F \quad [F, N] \subseteq N \quad [N, N] \subseteq N \tag{2.11}$$

is itself a semidirect sum of a subalgebra of F and a subalgebra of N:

$$L_0 = F_0 \rhd N_0 \qquad F_0 \subseteq F \quad N_0 \subseteq N \tag{2.12}$$

(or conjugate to such a semidirect sum).

All other subalgebras of $L = F \rhd N$ are called *non-splitting subalgebras.*

An *Abelian splitting subalgebra* of $L = F \rhd N$ is a direct sum

$$L_0 = F_0 \oplus N_0 \qquad F_0 \subseteq F \quad N_0 \subseteq N. \tag{2.13}$$

Definition 2.4. A maximal Abelian nilpotent subalgebra (MANS) M of a Lie algebra L is a MASA, consisting entirely of nilpotent elements, i.e. it satisfies

$$[M, M] = 0 \qquad [[[L, M]M] \cdots]_m = 0 \tag{2.14}$$

for some finite number m (we commute M with L m times).

Let us now consider the pseudo-euclidean space $M(p, q)$, i.e. $\mathbb{R}^n, n = p + q$ with an invariant quadratic form given by the matrix K of equation (2.3):

$$ds^2 = dx^T K dx. \tag{2.15}$$

The group and Lie algebra actions are given by

$$x' = Gx + a \qquad x' = Xx + \alpha \tag{2.16}$$

respectively, with (X, α) and (G, a) as in equations (2.1) and (2.2).

Definition 2.5. A subalgebra $L_0 \subset e(p, q)$ is orthogonally decomposable if it preserves an orthogonal decomposition of $M(p, q)$

$$M(p, q) = M(p_1, q_1) \oplus M(p_2, q_2) \qquad p_1 + p_2 = p \quad q_1 + q_2 = q \tag{2.17}$$

into two (or more) non-empty subspaces. It is called orthogonally indecomposable otherwise.

2.2. *Classification strategy*

The classification of MASAs of $e(p, q)$ is based on the fact that $e(p, q)$ is the semidirect sum of the Lie algebra $o(p, q)$ and an Abelian ideal $T(n)$ (the translations). We use here a modification of a procedure described earlier [19] for $e(n, C)$. We proceed in five steps.

1. Classify subalgebras $T(k_+, k_-, k_0)$ of $T(n)$. They are characterized by a triplet of non-negative integers (k_+, k_-, k_0) where k_+, k_- and k_0 are the numbers of positive, negative and isotropic vectors in an orthogonal basis, respectively.
2. Find the centralizer $C(k_+, k_-, k_0)$ of $T(k_+, k_-, k_0)$ in $o(p, q)$:

$$C(k_+, k_-, k_0) = \{X \in o(p, q) | [X, T(k_+, k_-, k_0)] = 0\}. \tag{2.18}$$

3. Construct all MASAs of $C(k_+, k_-, k_0)$ and classify them under the action of normalizer $Nor[T(k_+, k_-, k_0), G]$ of $T(k_+, k_-, k_0)$ in the group $G \sim E(p, q)$.
4. Obtain a list of splitting MASAs of $e(p, q)$ by forming the direct sums

$$C(k_+, k_-, k_0) \oplus T(k_+, k_-, k_0) \tag{2.19}$$

and dropping all such algebras that are not maximal from the list.
5. Complement the basis of $T(k_+, k_-, k_0)$ to a basis of $T(n)$ in each case and construct all non-splitting MASAs. The procedure is described below in subsection 4.2.

This general strategy can also be expressed in terms of sets of matrices of the form (2.1)–(2.4).

The subalgebra $T(k_+, k_-, k_0)$ can be represented by the matrices

$$\Pi = \begin{pmatrix} 0_{k_0} & & & & & \xi \\ & 0_{p+q-2k_0-k_+-k_-} & & & & 0 \\ & & 0_{k_0} & & & 0 \\ & & & 0_{k_+} & & x \\ & & & & 0_{k_-} & y \\ & & & & & 0_1 \end{pmatrix} \tag{2.20}$$

$$K_e = \begin{pmatrix} & & I_{k_0} & & & 0 \\ & K_0 & & & & \vdots \\ I_{k_0} & & & & & \vdots \\ & & & I_{k_+} & & 0 \\ & & & & -I_{k_-} & 0 \\ & & & & & 0_1 \end{pmatrix} \tag{2.21}$$

where K_0 has the signature $(p-k_+-k_0, q-k_--k_0)$.

The centralizer $C(k_+, k_-, k_0)$ of $T(k_+, k_-, k_0)$ will then be represented by the block diagonal matrices

$$C = \begin{pmatrix} \tilde{M} & & & \\ & 0_{k+} & & \\ & & 0_{k-} & \\ & & & 0_1 \end{pmatrix} \qquad \tilde{M} = \begin{pmatrix} 0_{k_0} & \tilde{A} & \tilde{Y} \\ 0 & \tilde{S} & -\tilde{K}\tilde{A}^{\mathrm{T}} \\ 0 & 0 & 0_{k_0} \end{pmatrix} \tag{2.22}$$

$$\tilde{Y} = -\tilde{Y}^{\mathrm{T}} \qquad \tilde{S}\tilde{K} + \tilde{K}\tilde{S}^{\mathrm{T}} = 0.$$

The Lie algebra of matrices $\{\tilde{M}\}$ represents a subalgebra of $o(p-k_+, q-k_-)$ and we need to classify the MASAs of $o(p-k_+, q-k_-)$ contained in $\{\tilde{M}\}$. Such MASAs were studied elsewhere [12] and we shall recall some basic facts here.

A MASA of $o(p, q)$ is characterized by a set of matrices X and a 'metric' matrix K, satisfying equation (2.3). A MASA can be orthogonally indecomposable (OID), or orthogonally decomposable (OD). If it is OD, we decompose it, i.e. transform it, together with K, into block diagonal form. Each block is an OID MASA of some $o(p_i, q_i)$, $\sum p_i = p, \sum q_i = q$. At most one of the blocks is a MANS.

From the above we can see that the MASA of $e(p, q)$ will have the following general form:

$$M = \begin{pmatrix} 0_{k_0} & A & Y & & & & \xi \\ & S & -K_{p_1q_1}A^{\mathrm{T}} & & & & \\ & & 0_{k_0} & & & & \\ & & & M_1 & & & \\ & & & & 0_{k_+} & & x \\ & & & & & 0_{k_-} & y \\ & & & & & & 0_1 \end{pmatrix} \tag{2.23}$$

$$K_e = \begin{pmatrix} & & I_{k_0} & & & & \\ & K_{p_1q_1} & & & & & \\ I_{k_0} & & & & & & \\ & & & K_{p_2q_2} & & & \\ & & & & I_{k_+} & & \\ & & & & & -I_{k_-} & \\ & & & & & & 0_1 \end{pmatrix} \tag{2.24}$$

where M_1 is a MASA of $o(p_2, q_2)$ not containing a MANS, $p = p_1 + p_2 + k_+ + k_0$ and $q = q_1 + q_2 + k_- + k_0$. The MASA M_1 can be absent (when $p_2 = q_2 = 0$). It may be orthogonally decomposable.

The block

$$M_0 = \begin{pmatrix} 0_{k_0} & A & Y \\ 0 & S & -K_{p_1q_1}A^{\mathrm{T}} \\ 0 & 0 & 0_{k_0} \end{pmatrix} \qquad (2.25)$$

$$Y + Y^{\mathrm{T}} = 0 \qquad SK_{p_1q_1} + K_{p_1q_1}S^{\mathrm{T}} = 0$$

represents a MANS of $o(p_1 + k_0, q_1 + k_0)$, so $S \in \mathbb{R}^{(p_1+q_1)\times(p_1+q_1)}$ is a nilpotent matrix. For $k_0 = 0$ the MANS M_0 is absent.

2.3. Embedding into the conformal Lie algebra

The algebra $o(p + 1, q + 1)$ contains the rotations and pseudorotations $L_{\alpha\beta}$, translations P_μ, the dilation D and the proper conformal transformations C_μ. The realization of the additional basis elements in terms of differential operators is given by

$$D = x_\alpha \frac{\partial}{\partial x_\alpha} \qquad C_a = g_{aa}x_a x_\alpha \frac{\partial}{\partial x_\alpha} - \frac{1}{2}(x_\alpha g_{\alpha\beta} x_\beta)\frac{\partial}{\partial x_0}. \qquad (2.26)$$

They satisfy the following commutation relations:

$$\begin{aligned} &[P_\mu, C_\alpha] = 2g_{\mu\alpha}D - 2g_{\alpha\alpha}L_{\mu\alpha} \\ &[C_\alpha, L_{\mu\nu}] = g_{\alpha\mu}C_\mu - g_{\alpha\nu}C_\mu \\ &[D, L_{\mu\nu}] = 0 \\ &[P_\mu, D] = P_\mu \\ &[C_\mu, D] = -C_\mu. \end{aligned} \qquad (2.27)$$

A matrix representation of $o(p + 1, q + 1)$ is

$$M_C = \begin{pmatrix} d & \alpha & 0 \\ \beta^{\mathrm{T}} & X_0 & -K_0\alpha^{\mathrm{T}} \\ 0 & -\beta K_0 & -d \end{pmatrix} \qquad K_C = \begin{pmatrix} & & 1 \\ & K_0 & \\ 1 & & \end{pmatrix} \qquad (2.28)$$

$$X_0K_0 + K_0X_0^{\mathrm{T}} = 0$$

where α, β, d, X_0 represent translations, conformal transformations, the dilation, rotations and pseudorotations, respectively. K_0 has the signature (p, q). We have

$$M_CK_C + K_CM_C^{\mathrm{T}} = 0. \qquad (2.29)$$

We see that in equation (2.28) the algebra $e(p, q)$ is embedded as a subalgebra of one of the maximal subalgebras of $o(p+1, q+1)$, namely the similitude algebra $\mathrm{sim}(p, q)$ obtained by setting $\beta = 0$ in (2.28). The MASAs of $e(p, q)$ are thus embedded into $o(p+1, q+1)$. In each case we shall determine whether a MASA of $e(p, q)$ is also maximal in $o(p+1, q+1)$. Conversely this representation can be used to determine whether a MASA of $o(p+1, q+1)$ is contained in $e(p, q)$. Finally, we shall use it to establish possible conformal equivalences between MASAs of $e(p, q)$ that are inequivalent under $E(p, q)$.

3. MASAs of $e(p, 0)$ and $o(p, 1)$

3.1. Classification of all MASAs of $e(p, 0) \equiv e(p)$

The metric is positive definite and, hence, a subspace of the translations is completely characterized by its dimension.

A basis for $e(p)$ is given by L_{ik}, $1 \leqslant i < k \leqslant p$, and $P_1, \ldots, P_p$.

Theorem 3.1. Every MASA of $e(p,0)$ splits into the direct sum $M(k) = F(k) \oplus T(k)$ and is $E(p,0)$ conjugate to precisely one subalgebra with

$$F(k) = \{L_{12}, L_{34}, \ldots, L_{2l-1,2l}\} \qquad T(k) = \{P_{2l+1}, \ldots, P_p\}$$

where k is such that $p-k$ is even $(p-k=2l)$.

Proof. We take $T(k) = \{P_{p-k+1}, \ldots, P_p\}$. Its centralizer in $o(p,0)$ is $o(p-k,0)$. This algebra has just one class of MASAs, namely the Cartan subalgebra:

1. $\tilde{F}_k = \{L_{12}, L_{34}, \ldots, L_{p-k-1,p-k}\}$ if $p-k$ is even;

2. $\tilde{F}_k = \{L_{12}, L_{34}, \ldots, L_{p-k-2,p-k-1}\}$ if $p-k$ is odd.

The splitting MASAs would then be $T(k) \oplus \tilde{F}_k$, but for $p-k$ odd, the subalgebra is not maximal. The elements of a non-splitting MASA would have the form $X = L_{a,a+1} + \sum_{j=1}^{p-k} \alpha_{a,j} P_j$ where $a = 1, 3, \ldots, p-k-1$. After imposing the commutation relations $[X,Y]=0$ we obtain that all $\alpha_{a,j}=0$. There are no non-splitting MASAs. □

3.2. MASAs of o(p, 1)

We present here some results from [12] on MASAs of $o(p,1)$. A MASA of $o(p,1)$ can be

1. Orthogonally decomposable. Two decomposition patterns are possible, namely:
   (a) $l(2,0) \oplus (k,1)$ for $k = 0, 1, \ldots, p-2$ $\quad (l \geqslant 1)$ where $(k,1)$ is a MANS;
   (b) $(1,1) \oplus (1,0) \oplus l(2,0)$.
2. Orthogonally indecomposable. Then the MASA is a MANS of $o(p,1)$.

A representative list of $O(p,1)$ conjugacy classes of MANSs of $o(p,1)$ is given by the matrix sets

$$X = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & -\alpha^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} \qquad K = \begin{pmatrix} & & 1 \\ & I_\mu & \\ 1 & & \end{pmatrix} \qquad \alpha = (a_1, \ldots, a_\mu) \quad a_j \in \mathbb{R}. \tag{3.1}$$

The entries in α are free, and the dimension of M is hence

$$\dim M = p - 1 = \mu. \tag{3.2}$$

The algebra $o(2l+1,1)$ has a single (non-compact) Cartan subalgebra, corresponding to the orthogonal decomposition $l(2,0) \oplus (1,1)$. The algebra $o(2l,1)$ has two inequivalent Cartan subalgebras, corresponding to the decompositions $l(2,0) \oplus (0,1)$ (compact) and $(1,0) \oplus (1,1) \oplus l(2,0)$ (non-compact).

The situation is illustrated in figure 1.

3.3. Behaviour of MASAs of e(p, 0) under the action of the group O(p + 1, 1)

Theorem 3.2. All MASAs of $e(p,0)$ inequivalent under $E(p,0)$ are also inequivalent under the action of the group $O(p+1,1)$ and are also MASAs of $o(p+1,1)$.

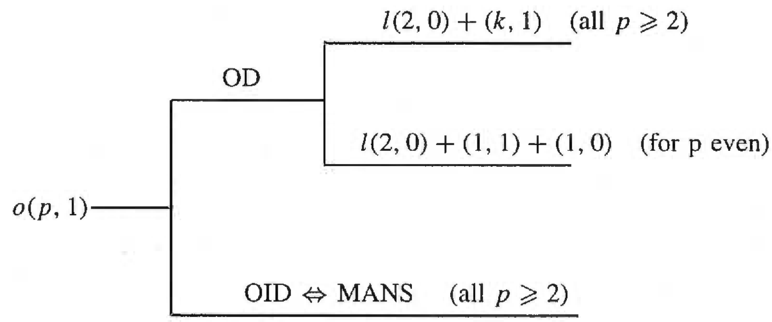

Figure 1. MASAs of $o(p, 1)$.

Proof. A MASA of $e(p, 0)$ can be represented in matrix form as follows:

$$M_e = \begin{pmatrix} M_1 & & & & 0 \\ & \ddots & & & \vdots \\ & & M_l & & 0 \\ & & & 0_{k_+} & x^T \\ & & & & 0_1 \end{pmatrix} \qquad M_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix} \qquad i = 1, \dots, l \quad a_i \in \mathbb{R}$$

$$K_e = \begin{pmatrix} I_{2l} & & \\ & I_{k_+} & \\ & & 0_1 \end{pmatrix} \tag{3.3}$$

which corresponds in $o(p+1, 1)$ to the following matrix realization:

$$M_e = \begin{pmatrix} M_1 & & & & & 0 \\ & \ddots & & & & \vdots \\ & & M_l & & & 0 \\ & & & 0 & x & 0 \\ & & & & 0_{k_+} & -x^T \\ & & & & & 0 \end{pmatrix}$$

$$K_e = \begin{pmatrix} I_{2l} & & & \\ & & & 1 \\ & & I_{k_+} & \\ & 1 & & \end{pmatrix} \tag{3.4}$$

which is an orthogonally decomposable MASA of $o(p+1, 1)$ with decomposition $l(2, 0) \oplus$ MANS of $o(p - 2l + 1, 1)$ (realized as in equation (3.1)). □

3.4. Summary of MASAs of e(p, 0)

The classification of MASAs of $e(p, 0)$ can be summed up in terms of orthogonal decompositions of the Euclidean space $M(p, 0) \equiv M(p)$.

Theorem 3.3. 1. Orthogonally indecomposable MASAs exist only for $p = 1$ and $p = 2$. Namely

$$p = 1 \qquad \{P_1\} \tag{3.5}$$

$$p = 2 \qquad \{M_{12}\}. \tag{3.6}$$

2. All MASAs of $e(p, 0)$ are obtained by orthogonally decomposing the space $M(p)$ according to a pattern

$$M(p) = lM(2) \oplus kM(1) \qquad p = 2l + k \tag{3.7}$$

and taking a MASA of type (3.6) in each $M(2)$ space and type (3.5) in each $M(1)$ space.
3. For each partition $p = 2l + k, 0 \leqslant l \leqslant [p/2]$ we have precisely one conjugacy class of MASAs, both under the isometry group $E(p, 0)$ and the conformal group $O(p + 1, 1)$.

4. MASAs of $e(p, 1)$

4.1. Splitting MASAs of e(p, 1)

For $e(p, 1)$ only the values $k_- = 0, 1$ and $k_0 = 0, 1$ are allowed, while $0 \leqslant k_+ \leqslant p$. We can write a MASA in the following form:

$$M(k_+, k_-, k_0) \equiv M = \begin{pmatrix} M_0 & & & & & \gamma^{\mathrm{T}} \\ & M_1 & & & & 0 \\ & & \ddots & & & \vdots \\ & & & M_l & & 0 \\ & & & & 0_{k_+} & x^{\mathrm{T}} \\ & & & & & 0_1 \end{pmatrix} \tag{4.1}$$

$$K_e = \begin{pmatrix} K_0 & & & \\ & I_{2l} & & \\ & & I_{k_+} & \\ & & & 0_1 \end{pmatrix} \qquad \operatorname{sgn} K_0 = (p - k_+ - 2l, 1)$$

where

$$M_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix} \qquad x \in \mathbb{R}^{1\times k_+}.$$

From now on we will only write the form of M_0, γ and K_0 together with conditions on the values l and k_+. The complete MASA can be obtained by substituing the appropriate M_0, γ and K_0 in equation (4.1). We denote the dimensions of these MASAs as $\dim M(k_+, k_-, k_0) \equiv d$.

Theorem 4.1. Three different kinds of splitting MASAs exist. They are characterized by the triplet (k_+, k_-, k_0):

(A) $M(k_+, 1, 0)$, $0 \leqslant k_+ \leqslant p$:

$$M_0 = 0 \in \mathbb{R} \qquad \gamma^{\mathrm{T}} = z \in \mathbb{R} \qquad \text{and} \qquad K_0 = -1 \tag{4.2}$$

$p-k_+$ is even, $0 \leqslant l \leqslant \frac{1}{2}(p-k_+)$, $d = \dim M(k_+, 1, 0) = 1+l+k_+$, $\left[\frac{1}{2}(p+3)\right] \leqslant d \leqslant p+1$;

(B) $M(k_+, 0, 0), 0 \leqslant k_+ \leqslant p-1$:

$$M_0 = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \qquad \gamma^{\mathrm{T}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad K_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.3}$$

where $p-k_+$ is odd, $0 \leqslant l \leqslant \frac{1}{2}(p-k_+-1)$, $d = \dim M(k_+, 0, 0) = 1+l+k_+$, $\left[\frac{1}{2}(p+2)\right] \leqslant d \leqslant p$;

(C) $M(k_+, 0, 1), 0 \leqslant k_+ \leqslant p-2$:

$$M_0 = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & -\alpha^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} \qquad \gamma^{\mathrm{T}} = \begin{pmatrix} z \\ 0_\mu \\ 0 \end{pmatrix} \qquad K_0 = \begin{pmatrix} & & 1 \\ & I_\mu & \\ 1 & & \end{pmatrix} \tag{4.4}$$

where $1 \leqslant \mu \leqslant p-1$ and $0 \leqslant l \leqslant \frac{1}{2}(p-k_+-2)$, $z \in \mathbb{R}$, $\alpha \in \mathbb{R}^{1\times\mu}$, $d = \dim M(k_+, 0, 1) = \mu + l + k_+ + 1$, $\left[\frac{1}{2}(p+3)\right] \leqslant d \leqslant p$.

All entries a_i, x, z, α and c are free.

Proof. Let us use the representation (2.1) of $e(p, 1)$. The translations are represented by the matrix Y with $X = 0$. We run through the three translation subalgebras T fixed in theorem 4.1 and for each of them find their centralizer $C(T)$ in $o(p, 1)$, i.e. the set of matrices X and Y, such that we have

$$[Y(X, 0), Y(0, \alpha)] = 0 \tag{4.5}$$

for the chosen set of the translations α. We must then determine all MASAs of $C(T)$ such that they commute only with T and with no other translations.

(A) For $T = T(k_+, 1, 0)$ we have $C(T) \sim o(p-k_+, 0)$ which has only one MASA: the Cartan subalgebra. The condition $p - k_+$ being even is needed, otherwise the MASA will commute with $k_+ + 1$ positive length vectors. We thus arrive at eq.(4.2).

(B) For $T = T(k_+, 0, 0)$ we obtain $C(T) \sim o(p-k_+, 1)$. The MASAs of $o(p-k_+, 1)$ are known (see section 3.2 above and also [12]). Any MASA of $o(p-k_+, 1)$ containing a nilpotent element will also commute with an isotropic vector in T, not contained in $T(k_+, 0, 0)$. Hence we need only to consider a Cartan subalgebra of $o(p-k_+, 1)$. Moreover, it must be non-compact, or it will commute with a negative length vector in T. Finally, if $p - k_+$ is even, the MASA will commute with $k_+ + 1$ positive length vectors in T. We arrive at the result in (4.3).

(C) Take $T = T(k_+, 0, 1)$. We obtain $C(T) \sim e(p-k_+-1, 0)$, an Euclidean Lie algebra realized as a subalgebra of $o(p-k_+, 1)$, e.g. by the matrices

$$Z = \begin{pmatrix} 0 & \nu & 0 \\ 0 & R & -\nu^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} \tag{4.6}$$

where $R + R^{\mathrm{T}} = 0$, $R \in \mathbb{R}^{(p-k_+-1)\times(p-k_+-1)}$, $\nu \in \mathbb{R}^{1\times(p-k_+-1)}$.

Applying theorem 3.1 we obtain the result given in (4.4). The results concerning the dimensions of the MASAs are obvious; they amount to counting the number of free parameters in M_0, M_i, γ and x in the matrix (4.1). □

4.2. Non-splitting MASAs of e(p, 1)

First we describe the general procedure for finding non-splitting MASAs of $e(p, q)$.

Every non-splitting MASA $M(k_+, k_-, k_0)$ of $e(p, q)$ is obtained from a splitting MASA by the following procedure.

1. Choose a basis for $C(k_+, k_-, k_0)$ and $T(k_+, k_-, k_0)$ e.g. $C(k_+, k_-, k_0) \sim \{B_1, \dots, B_J\}$, $T(k_+, k_-, k_0) \sim \{X_1, \dots, X_L\}$.
2. Complement the basis of $T(k_+, k_-, k_0)$ to a basis of $T(n)$.

$$T(n)/T(k_+, k_-, k_0) = \{Y_1, \dots, Y_N\} \qquad L + N = n.$$

3. Form the elements

$$\tilde{B}_a = B_a + \sum_{j=1}^{N} \tilde{\alpha}_{aj} Y_j \qquad a = 1, \dots, J \tag{4.7}$$

where the constants $\tilde{\alpha}_{aj}$ are such that $\tilde{B}_a$ form an Abelian Lie algebra $[\tilde{B}_a, \tilde{B}_b] = 0$. This provides a set of linear equations for the coefficients $\tilde{\alpha}_{aj}$. The solutions $\tilde{\alpha}_{aj}$ are called 1-cocycles and they provide the Abelian subalgebras $\tilde{M}(k_+, k_-, k_0) \sim \{\tilde{B}_a, X_b\} \subset e(p, q)$.
4. Classify the subalgebras $\tilde{M}(k_+, k_-, k_0)$ into conjugacy classes under the action of the group $E(p, q)$. This can be done in two steps.

(i) Generate trivial cocycles t_{aj}, called coboundaries, using the translation group $T(n)$

$$e^{p_j P_j} \tilde{B}_a e^{-p_j P_j} = \tilde{B}_a + p_j [P_j, \tilde{B}_a] = \tilde{B}_a + \sum_j t_{aj} P_j. \tag{4.8}$$

The coboundaries should be removed from the set of cocycles. If we have $\tilde{\alpha}_{aj} = t_{aj}$ for all (a, j) the algebra is splitting (i.e. equivalent to a splitting algebra).

(ii) Use the normalizer of the splitting subalgebra in the group $O(p, q)$ to further simplify and classify the non-trivial cocycles.

Theorem 4.2. Non-splitting MASAs of $e(p, 1)$ are obtained from splitting ones of type C in theorem 4.1 and are conjugate to precisely one MASA of the form

(i) for $\mu \geqslant 2$:

$$M_0 = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & -\alpha^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} \qquad \gamma^{\mathrm{T}} = \begin{pmatrix} z \\ A\alpha^{\mathrm{T}} \\ 0 \end{pmatrix} \tag{4.9}$$

where A is a diagonal matrix with $a_1 = 1 \geqslant |a_2| \geqslant \cdots \geqslant |a_\mu| \geqslant 0$ and $\mathrm{Tr}\, A = 0$, K_0 is as in (4.4)

(ii) for $\mu = 1$ we have a special case for which the non-splitting MASA has the form

$$M_0 = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix} \qquad \gamma^{\mathrm{T}} = \begin{pmatrix} z \\ 0 \\ a \end{pmatrix} \qquad K_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.10}$$

No other non-splitting MASAs of $e(p, 1)$ exist.

Proof. The non-splitting MASA is represented in general as follows:

$$Z_e = \begin{pmatrix} M_0 & & & & & \beta_0^{\mathrm{T}} \\ & M_1 & & & & \beta_1^{\mathrm{T}} \\ & & \ddots & & & \vdots \\ & & & M_l & & \beta_l^{\mathrm{T}} \\ & & & & 0_{k_+} & x^{\mathrm{T}} \\ & & & & & 0_1 \end{pmatrix} \tag{4.11}$$

where $\beta_0 \in R^{1\times(p-k_+-2l)}$ and $\beta_i \in \mathbb{R}^{1\times 2}, i = 1, \ldots, l$, depend linearly on the free entries in the MASA of $o(p, 1)$, i.e. the matrices $M_i, 0 \leqslant i \leqslant l$. We impose the commutativity $[Z_e, Z_e'] = 0$ and obtain

$$M_i \beta_i'^{T} = M_i' \beta_i^{\mathrm{T}} \qquad i = 0, \ldots, l. \tag{4.12}$$

From equation (4.12) we see that vectors β_i depends linearly on the matrices M_i only. The block (M_i, β_i), $\beta_i = (a_i, a_{i+1})$ for $i = 1, \ldots, l$ represents elements of the type

$$L_{i,i+1} + a_i P_i + a_{i+1} P_{i+1} \qquad 1 \leqslant i \leqslant p.$$

In all cases the coefficients a_i are coboundaries, since we have

$$\exp(\alpha_i P_i + \alpha_{i+1} P_{i+1}) L_{i,i+1} \exp(-\alpha_i P_i - \alpha_{i+1} P_{i+1}) = L_{i,i+1} + \alpha_i P_{i+1} - \alpha_{i+1} P_i. \tag{4.13}$$

The coefficients α_i can be chosen so as to annul a_i and a_{i+1}. Thus we have

$$\beta_j = 0 \qquad 1 \leqslant j \leqslant l \tag{4.14}$$

for all non-splitting MASAs of $e(p, 1)$. Hence for case (A) in theorem 4.1 there are no non-splitting MASAs. In case (B) the block (M_0, β_0) represents the element of the type $L_{p,p+1} + a_p P_p + a_{p+1} P_{p+1}$. Here again the coefficients a_i are coboundaries, since we have

$$\exp(\alpha_p P_p + \alpha_{p+1} P_{p+1}) L_{p,p+1} \exp(-\alpha_p P_p - \alpha_{p+1} P_{p+1}) = L_{p,p+1} + \alpha_p P_{p+1} + \alpha_{p+1} P_p \tag{4.15}$$

and the coefficients α_i can be chosen so as to annul a_p and a_{p+1}. We have that $\beta_0 = 0$, and there are no non-splitting MASAs. In case (C) the non-splitting part of M_0 is as follows:

$$Z_0 = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & -\alpha^{\mathrm{T}} & \beta_0^{\mathrm{T}} \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0_1 \end{pmatrix}. \tag{4.16}$$

Commutativity $[Z_e, Z_e'] = 0$ gives us the following conditions:

$$\alpha \beta_0'^{T} = \alpha' \beta_0^{\mathrm{T}} \tag{4.17}$$

$$\alpha^{\mathrm{T}} y' = \alpha'^{T} y \qquad y \in \mathbb{R} \tag{4.18}$$

which gives

$$\beta_0^{\mathrm{T}} = A \alpha^{\mathrm{T}} \tag{4.19}$$

$$y = \mu \alpha^{\mathrm{T}} \tag{4.20}$$

where A is a matrix and μ is a row vector.

Looking again at the commutativity condition with equation (4.20) satisfied, we find that

$$A = A^{\mathrm{T}} \qquad \text{and} \qquad \mu = 0. \tag{4.21}$$

The symmetric matrix A represents the 1-cocycles. The coboundaries are represented by the matrix δI and we use them to set $\mathrm{Tr}\, A = 0$. For further simplification and classificaation we use the normalizer of the splitting MASA in the group $o(p, 1)$. The normalizer is represented by block diagonal matrices of the same block structure as in (4.1). The part acting on M_0 is represented by

$$G = \mathrm{diag}(g, G_0, g^{-1}, 1) \qquad \text{satisfying } G_0 G_0^{\mathrm{T}} = I. \tag{4.22}$$

Computing

$$G M_0 G^{-1} = M_0' \tag{4.23}$$

gives the following transformation of A:

$$A' = \frac{1}{g}(G_0 A G_0^{\mathrm{T}}). \tag{4.24}$$

We use the matrix G_0 to diagonalize A and to order the eigenvalues. The normalization $a_1 = 1$ is due to a choice of g. The proof of case (ii) is almost identical to the previous one and we omit it here. The dimension of the non-splitting subalgebra is the same as the dimension of the corresponding splitting subalgebra. □

4.3. A decomposition theorem for MASAs of e(p, 1)

Again, all the results of this section can be summed up in a decomposition theorem.

Theorem 4.3. 1. Indecomposable MASAs of $e(p, 1)$ exist for all values of p, namely

$$p = 0: \qquad \{P_0\} \tag{4.25}$$

$$p = 1: \qquad \{L_{01}\} \tag{4.26}$$

$$p = 2: \qquad \{P_0 - P_1, L_{02} - L_{12} + \kappa(P_0 + P_1)\} \qquad \kappa = 0, \pm 1 \tag{4.27}$$

$$p \geqslant 3: \qquad \{P_0 - P_1, L_{0j} - L_{1j} + a_j P_j)\} \qquad j = 2, \ldots, p$$

$$a_2 = 1 \geqslant |a_3| \geqslant \cdots \geqslant |a_p| \geqslant 0 \quad \sum a_i = 0 \tag{4.28}$$

$$\text{or } a_2 = a_3 = \cdots = a_p = 0.$$

MASAs corresponding to different values of κ, or different sets $(a_2, \ldots, a_p)$ are mutually inequivalent under the connected component of $E(p, 1)$. If the entire group $E(p, 1)$ is allowed (containing $O(p, 1)$, rather than only $SO(p, 1)$), then $\kappa = -1$ is equivalent to $\kappa = 1$ and can be omitted.

2. All MASAs of $e(p, 1)$ are obtained by orthogonally decomposing the Minkowski space $M(p, 1)$ according to the pattern

$$M(p, 1) = M(k, 1) \oplus l M(2, 0) \oplus m M(1, 0)$$

$$p = k + 2l + m \quad 0 \leqslant k \leqslant p \quad 0 \leqslant l \leqslant \left[\frac{p}{2}\right] \tag{4.29}$$

and taking a MASA of the type (3.5) for each $M(1)$, of the type (3.6) for each $M(2)$ and of the type (4.25), (4.26), (4.27) or (4.28) for $M(k, 1)$.

3. Each decomposition (4.29) and each choice of constants κ and $\{a_j\}$, respectively, provides a different MASA (mutually inequivalent under the group $E(p, 1)$).

5. Embedding of MASAs of $e(p, 1)$ into the conformal algebra $o(p+1, 2)$

5.1. Introductory comments

Let us realize the algebra $o(r, 2)$ by matrices X satisfying

$$XK + KX^{\mathrm{T}} = 0 \quad K, X \in \mathbb{R} \qquad K = K^{\mathrm{T}} \qquad \operatorname{sgn} K = (r, 2). \tag{5.1}$$

A MASA of $o(r, 2)$ will be called *orthogonally decomposable* (OD) if all matrices representing the MASA can be simultaneously transformed by some matrix G, together with the matrix K, into block diagonal sets of the form

$$\tilde{X} = \begin{pmatrix} X_1 & & & \\ & X_2 & & \\ & & \ddots & \\ & & & X_j \end{pmatrix} \qquad \tilde{K} = \begin{pmatrix} K_1 & & & \\ & K_2 & & \\ & & \ddots & \\ & & & K_j \end{pmatrix} \tag{5.2}$$

$$\tilde{X} = GXG^{-1} \qquad \tilde{K} = GKG^{\mathrm{T}} \qquad G \in GL(r+2, \mathbb{R}).$$

If no such matrix G exists, the MASA is *orthogonally indecomposable* (OID).

A MASA can be orthogonally indecomposable, but *not absolutely indecomposable* (OID, but NAOID). This means it is orthogonally decomposable after complexification of the ground field.

Let us now present some results on MASAs of $o(r, 2)$ which can be extracted from [12].

5.2. MASAs of o(r, 2)

We shall first consider $r \geqslant 3$, then treat the case $r = 2$ separately.

Proposition 5.1. Precisely three types of MASAs exist for $r = 2k \geqslant 4$, 2 for $r = 2k+1 \geqslant 3$:

1. Orthogonally decomposable MASAs (any r).
2. Absolutely orthogonally indecomposable MASAs (any r).
3. Orthogonaly indecomposable, but not absolutely orthogonally indecomposable MASAs ($r = 2k$).

Proposition 5.2. Every orthogonally decomposable MASA of $o(r, 2)$ can be represented in the form (5.2) where each $\{X_i, K_i\}$ represents an orthogonally indecomposable MASA of lower dimension. The allowed decomposition patterns are

1. $$(r, 2) = (s, 2) + l(2, 0) \qquad r = s + 2l \quad l \geqslant 1$$
2. $$(r, 2) = (s, 2) + (1, 1) + l(2, 0) \qquad r = s + 2l + 1.$$

A *maximal Abelian nilpotent subalgebra* (MANS) of $o(p, q)$ is characterized by its Kravchuk signature $(\lambda\ \mu\ \lambda)$, a triplet of non-negative integers satisfying

$$2\lambda + \mu = p + q \qquad \mu \geqslant 0 \quad 1 \leqslant \lambda \leqslant q \leqslant p. \tag{5.3}$$

For a given MANS M the positive integer λ is the dimension of the kernel of M and also the codimension of the image space of M. For a given signature $(\lambda\ \mu\ \lambda)$ the MANS M can be transformed into Kravchuk normal form, namely

$$X = \begin{pmatrix} 0 & A & Y \\ 0 & S & -K_0A^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} \qquad K = \begin{pmatrix} & & I_\lambda \\ & K_0 & \\ I_\lambda & & \end{pmatrix}$$

$$A \in \mathbb{R}^{\lambda\times\mu} \quad Y = -Y^{\mathrm{T}} \in \mathbb{R}^{\lambda\times\lambda} \quad SK_0 + K_0S^{\mathrm{T}} = 0$$
$$S \in \mathbb{R}^{\mu\times\mu} \quad K_0 = K_0^{\mathrm{T}} \in \mathbb{R}^{\mu\times\mu} \quad \operatorname{sgn} K_0 = (p-\lambda, q-\lambda). \tag{5.4}$$

The matrix S is nilpotent, the matrix K_0 fixed. The classification of the MANSs of $o(p,q)$ reduces to a classification of matrices A, S and Y satisfying the commutativity relation $[X, X'] = 0$:

$$AK_0A'^{T} = A'K_0A^{\mathrm{T}} \qquad AS' = A'S \qquad [S, S'] = 0. \tag{5.5}$$

Two types of MANSs of $o(p,q)$ exist:

1. *Free-rowed MANS.* There exists a linear combination of the λ rows of the matrix A in (5.4) that contains μ free real entires.
2. *Non-free-rowed MANS.* No linear combination of the λ rows of A contains more than $\mu - 1$ real free entries.

Proposition 5.3. An absolutely orthogonally indecomposable MASA of $o(r, 2)$ is a MANS. Three types of MANSs of $o(r, 2)$ exists. Using the metric

$$K = \begin{pmatrix} & & 1 \\ & K_0 & \\ 1 & & \end{pmatrix} \qquad K_0 = \begin{pmatrix} & & 1 \\ & I_{r-2} & \\ 1 & & \end{pmatrix} \tag{5.6}$$

they can be written as follows.

1. Kravchuk signature $(1\ r\ 1)$, free rowed

$$X = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & -K_0\alpha^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} \qquad \alpha \in \mathbb{R}^{1\times r}. \tag{5.7}$$

2. Kravchuk signature $(1\ r\ 1)$, non-free rowed

$$X = \begin{pmatrix} 0 & a & \alpha & 0 & b & 0 \\ & 0 & 0 & a & 0 & -b \\ & & 0 & 0 & 0 & -\alpha^{\mathrm{T}} \\ & & & 0 & -a & 0 \\ & & & & 0 & -a \\ & & & & & 0 \end{pmatrix} \qquad a, b \in \mathbb{R} \quad \alpha \in \mathbb{R}^{1\times(r-3)}. \tag{5.8}$$

3. Kravchuk signature $(2\ r-2\ 2)$, free rowed

$$X = \begin{pmatrix} 0 & 0 & \alpha & x & 0 \\ 0 & 0 & \alpha Q & 0 & -x \\ & & & -Q\alpha^{\mathrm{T}} & -\alpha^{\mathrm{T}} \\ & & & 0 & 0 \\ & & & 0 & 0 \end{pmatrix} \qquad \alpha \in \mathbb{R}^{1\times(r-2)}$$

$$Q = \operatorname{diag}(q_1, \ldots, q_{r-2}) \neq 0 \qquad \sum_{j=1}^{r-2} q_j = 0$$
$$1 = q_1 \geqslant |q_2| \geqslant \cdots \geqslant |q_{r-2}| \geqslant 0. \tag{5.9}$$

Proposition 5.4. The algebra $o(2k,2), k \geqslant 2$ has precisely one class of orthogonally indecomposable, but not absolutely indecomposable MASAs. It can be represented by the set of matrices $\{X, K\}$

$$X = \begin{pmatrix} 0 & a & b_1 & b_1 & & b_{k-1} & b_{k-1} & 0 & c \\ -a & 0 & b_1 & -b_1 & & b_{k-1} & -b_{k-1} & -c & 0 \\ & & 0 & a & & & & -b_1 & -b_1 \\ & & -a & 0 & & & & -b_1 & b_1 \\ & & & & & & & & \\ & & & & & 0 & a & -b_{k-1} & -b_{k-1} \\ & & & & & -a & 0 & -b_{k-1} & b_{k-1} \\ & & & & & & & 0 & a \\ & & & & & & & -a & 0 \end{pmatrix} \qquad (5.10)$$

$$K = \begin{pmatrix} & & & 1 & \\ & & & & 1 \\ & & I_{2k-2} & & \\ 1 & & & & \\ & 1 & & & \end{pmatrix}. \qquad (5.11)$$

The algebra $o(2,2)$ is exceptional for two reasons, namely we have $p = q =$ even and moreover, it is semisimple rather than simple. Two orthogonal decompositions exist, namely those of proposition 5.2 with $s = 0, l = 1$ in the first case, and $s = 1, l = 0$ in the second. The MANS of equation (5.7) also exists in this case, as does the MASA (5.10); however, those of (5.8) and (5.9) do not. On the other hand, two further MASAs exist, both decomposable, but not orthogonally decomposable. In terms of matrices, they are represented by

$$X = \begin{pmatrix} a & b & & \\ & a & & \\ & & -a & -b \\ & & & -a \end{pmatrix} \qquad K = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (5.12)$$

and

$$X = \begin{pmatrix} a & b & & \\ -b & a & & \\ & & -a & -b \\ & & b & -a \end{pmatrix} \qquad K = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \qquad (5.13)$$

respectively. Thus $o(2,2)$ has six classes of MASAs. Propositions 5.1–5.4, as well as the results for $o(2,2)$, are proved in [12].

Let us now sum up the results on MASAs of $o(p,2)$ in terms of the 'physical' basis (2.7), (2.26), starting from the orthogonally indecomposable ones.

1. The MANS, equation (5.7), of $o(r,2)$ corresponds to the translations

$$\{P_0, P_1, \ldots, P_{r-1}\} \qquad (5.14)$$

and is contained in $e(r-1,1)$.

2. The MANS, equation (5.8), of $o(r,2)$ corresponds to

$$\{P_0 - P_1, L_{02} - L_{12} + P_0 + P_1, P_3, \ldots, P_{r-1}\} \tag{5.15}$$

and is contained in $e(r-1,1)$.
3. The MANS, equation (5.9), of $o(r,2)$ corresponds to

$$\{P_0 - P_1, P_k + q_k(L_{0k} - L_{1k}), k = 2, \ldots, r-1\} \tag{5.16}$$

and is contained in $e(r-1,1)$.
4. The MANS, equation (5.10), of $o(2k,2)$ corresponds to

$$\{2(L_{23} + L_{45} + \cdots + L_{2k-2,2k-1}) + (P_0 - P_1) - (C_0 + C_1),$$
$$P_j + P_{j+1} + L_{0j} + L_{1j} - L_{0,j+1} - L_{1,j+1}, j = 2, \ldots, 2k-2, P_0 + P_1\} \tag{5.17}$$

and is not contained in $e(r-1,1)$.
5. For the $o(2,2)$ case, equations (5.12) correspond to

$$\{P_0 - P_1, D - L_{01}\} \tag{5.18}$$

and equations (5.13) correspond to

$$\{D - L_{01}, P_0 - P_1 + (C_0 + C_1)\}. \tag{5.19}$$

They are not contained in $e(1,1)$.

In the orthogonally decomposable MASAs each component is an orthogonally indecomposable MASA of one of the types listed above.

5.3. MASAs of e(p, 1) classified under the group O(p + 1, 2)

Let us make use of the realization (2.28) of the algebra $o(p+1,2)$ and choose K_0 as in (4.4). The algebra $e(p,1) \subset o(p+1,2)$ is represented as follows:

$$X = \begin{pmatrix} 0 & p_+ & \alpha & p_- & 0 \\ 0 & k & \beta & 0 & -p_- \\ 0 & -\gamma^{\mathrm{T}} & R & -\beta^{\mathrm{T}} & -\alpha^{\mathrm{T}} \\ 0 & 0 & \gamma & -k & 0 \\ 0 & 0 & 0 & -p_+ & 0 \end{pmatrix} \qquad p_-, p_+, k \in \mathbb{R}$$
$$\alpha, \beta, \gamma \in \mathbb{R}^{1\times(p-1)} \qquad R = -R^{\mathrm{T}} \in \mathbb{R}^{(p-1)\times(p-1)}. \tag{5.20}$$

In equation (5.20) R represents rotations in the subspace $\mathbb{R}^{p-1}$, and furthermore, we have

$$p_- \sim P_0 - P_1 \qquad p_+ \sim P_0 + P_1 \qquad \alpha \sim (P_2, \ldots, P_k)$$
$$k \sim L_{01} \qquad \beta \sim (L_{02} - L_{12}, \ldots, L_{0p} - L_{1p}) \tag{5.21}$$
$$\gamma \sim (L_{02} + L_{12}, \ldots, L_{0p} + L_{1p}).$$

We shall use a transformation represented by a matrix $G \in O(p,2)$, $G \in E(p,1)$, namely

$$G = \begin{pmatrix} G_0 & & \\ & I_{p-1} & \\ & & G_0 \end{pmatrix} \qquad GXG^{-1} = X' \qquad GKG^{\mathrm{T}} = K. \tag{5.22}$$

The transformation (5.22) with $G_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ leaves R and $P_0 - P_1$ invariant, interchanges α and β, i.e. P_j and $L_{0j} - L_{1j}$ $(j = 2, \ldots, p)$ and takes L_{01}, $P_0 + P_1$ and $L_{0j} + L_{1j}$ out of the $o(p, 1)$ subalgebra that we will use to conjugate different MASAs of $e(p, 1)$ that are inequivalent under $E(p, 1)$.

Let us now consider the individual decompositions of the space $M(p, 1)$ listed in equation (4.29) of theorem 4.3.

First of all we note that the presence of $o(2)$ subalgebras acting in the $M(2, 0)$ subspaces (for $l \geqslant 1$) implies an orthogonal decomposition of the corresponding MASA of $o(p+1, 2)$. We are then dealing with Abelian subalgebras (ASA) of the form

$$\mathrm{ASA}[o(p+1, 2)] = l[o(2)] \oplus \mathrm{ASA}[o(j+1, 2)] \qquad j + 2l = p. \tag{5.23}$$

From now on we only need to consider subalgebras of $e(j, 1) \subset o(j+1, 2)$ and their possible conjugacy under $O(j+1, 2)$. These MASAs of $o(j+1, 2)$ contain no rotations L_{ik}. The following situations arise.

1. $k = 0, m = p - 2l$ in (4.29) and $j = m$. The MASA of $e(j, 1)$ consists of translations only:$\{P_0, P_1, \ldots, P_j\}$. This is the free-rowed MANS of $o(j+1, 2)$ with Kravchuk signature $(1\ \ j{+}1\ \ 1)$ as in (5.7) and (5.14).
2. $k = 1, m = p - 2l - 1$ in (4.29) and $j = m + 1$. The MASA of $e(j, 1)$ is an orthogonally decomposable MASA of $o(j+1, 2)$ of the form

$$\mathrm{MASA}[o(j+1, 2)] = o(1, 1) \oplus \mathrm{MANS}[o(j, 1)]$$

where the MANS of $o(j, 1)$ has the Kravchuk signature $(1\ \ j-1\ \ 1)$ as in (3.1). In the physical basis it is $\{L_{01}, P_2, \ldots, P_j\}$.
3. $k = 2, m = p - 2l - 2$ in (4.29) and $j = m + 2$, $\kappa \neq 0$ in (4.27). We have the MASA $\{L_{02} - L_{12} \pm (P_0 + P_1), P_0 - P_1, P_2, \ldots, P_j\}$. This is a non-free-rowed MANS of $o(j+1, 2)$ with Kravchuk signature $(1\ \ j{+}1\ \ 1)$ as in (5.8) and (5.15).
4. $k = 2, m = p - 2l - 2$ in (4.29) and $j = m + 2$, $\kappa = 0$ in (4.27). We have the MASA $\{L_{02} - L_{12}, P_0 - P_1, P_3, \ldots, P_j\}$. The transformation (5.22) takes this algebra into $\{P_0 - P_1, P_2, L_{03} - L_{13}, \ldots, L_{0j} - L_{1j}\}$. Thus, if we are interested in conformally inequivalent MASAs, we must impose, for $\kappa \neq 0$, $j \geqslant 3$, i.e. $m \geqslant 1$ in (4.29). This MASA is a free-rowed MANS of $o(j+1, 2)$ with Kravchuk signature $(2\ \ j{-}2\ \ 2)$ as in (5.9) and (5.16).
5. $k \geqslant 3, m = p - 2l - k$ in (4.29) and $j = m + k$, $a_2 = a_3 = \cdots = a_j = 0$ in (4.28). The MASA is $\{P_0 - P_1, L_{02} - L_{12}, \ldots, L_{0k} - L_{1k}, P_{k+1}, \ldots, P_j\}$ and is conformally equivalent to $\{P_0 - P_1, P_2, \ldots, P_k, L_{0,k+1} - L_{1,k+1}, \ldots, L_{0j} - L_{1j}\}$. It is a free-rowed MANS of $o(j+1, 2)$ with Kravchuk signature $(2\ \ j{-}1\ \ 2)$ as in (5.9) and (5.16).
6. $k \geqslant 3, m = p - 2l - k$ in (4.29) so $j = m + k$, $|a_2| = 1 \geqslant |a_3| \geqslant \cdots \geqslant |a_j|$in ((4.28). The MASA is $\{P_0 - P_1, L_{02} - L_{12} + a_2 P_2, \ldots, L_{0k} - L_{1k} + a_k P_k, P_{k+1}, \ldots, P_j\}$. Again we have a free-rowed MANS of $o(j+1, 2)$ with Kravchuk signature $(2\ \ j{-}1\ \ 2)$ as in (5.9) and (5.16).

We see that the MASAs listed above in cases 4, 5 and 6 are all related. Indeed, let us fix some value of j and consider the MANS, equation (5.9), of $o(j+1, 2)$. Cases 4 and 5 correspond to the first two rows in (5.9) being

$$\begin{pmatrix} 0 & 0 & \alpha & x & 0 \\ 0 & 0 & \beta & 0 & -x \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha_2 & \cdots & \alpha_k & 0 & \cdots & 0 & x & 0 \\ 0 & 0 & 0 & \cdots & 0 & \beta_{k+1} & \cdots & \beta_j & 0 & -x \end{pmatrix}. \tag{5.24}$$

The transformation (5.22) with

$$G_0 = \begin{pmatrix} 1 & 1 \\ a & b \end{pmatrix} \tag{5.25}$$

puts (5.24) in the standard form with

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}\alpha_2 & \cdots & \alpha_k & \beta_{k+1} & \cdots & \beta_j\\ a\alpha_2 & \cdots & a\alpha_k & b\beta_{k+1} & \cdots & b\beta_j\end{pmatrix} \tag{5.26}$$

with $j-1$ free entries in row 1 and $Q=\mathrm{diag}(aI_{k-1}, bI_{j-k})$, with

$$(k-1)a+(j-k)b=0 \qquad b\neq a. \tag{5.27}$$

An exception occurs when $m=0$. The algebra then is $\{P_0-P_1, L_{02}-L_{12}, \ldots, L_{0j}-L_{1j}\}$. This is equivalent to $\{P_0+P_1, P_2, \ldots, P_j\}$ and is hence not maximal in $o(j+1)$ (it would correspond to $Q=0$ in ((5.9), which is not allowed).

Case 6 can also be transformed into the MASA of equation (5.9), i.e. equation (5.16) by a transformation of the form (5.22) with G_0 satisfying

$$G_0=\begin{pmatrix}b & 1\\ c & d\end{pmatrix} \qquad b+a_1\neq 0 \qquad (k-1)c+d(a_2+\cdots+a_k)+md=0. \tag{5.28}$$

Thus, all MASAs of $e(k,1)$ discussed above in cases 4, 5 and 6 are special cases of the free-rowed MASA (5.9) of $o(j+1,2)$ with Kravchuk signature (2 j–1 2). To determine the decomposition of the space $M(j,1)$, consider a general transformation of the type (5.22). The entries depending on α in the first two rows of X transform as

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\alpha\\ \alpha Q\end{pmatrix}=\begin{pmatrix}\alpha(a+bQ)\\ \alpha(c+dQ)\end{pmatrix} \qquad ad-bc\neq 0. \tag{5.29}$$

We have

$$a+bQ=\mathrm{diag}(a+bq_1, a+bq_2, \ldots, a+bq_{j-2}) \tag{5.30}$$

To obtain a decomposition we must annul as many as possible of the elements in the diagonal matrix (5.30) by an appropriate choice of a and b. This number is equal to the highest multiplicity of an eigenvalue of the matrix Q. Since we have $\mathrm{Tr}\, Q=0$, the multiplicity is at most $j-3$. Let us order the eigenvalues in such a manner that the last entry in Q has the highest multiplicity equal to r. We then choose a and b in ((5.30) so that the matrix in (5.29) has the form

$$\begin{pmatrix}\alpha'\\ \beta'\end{pmatrix}=\begin{pmatrix}\alpha_2 & \cdots & \alpha_s & 0 & \cdots & 0\\ r_2\alpha_2 & \cdots & r_s\alpha_s & \beta_1 & \cdots & \beta_r\end{pmatrix} \qquad r+s=j \tag{5.31}$$

i.e. the MASAs

$$\{P_0-P_1, P_2+r_2(L_{02}-L_{12}), \ldots, P_s+r_s(L_{0s}-L_{1s}), P_{s+1}, \ldots, P_{s+r}\}$$

$$r_j\neq 0 \quad 2\leqslant j\leqslant s \quad \sum_{i=2}^{s} r_i=0$$

$$r_2=1\geqslant |r_3|\geqslant\cdots\geqslant |r_s|>0. \tag{5.32}$$

Each integer s and set of numbers $(r_2,\ldots,r_s)$ corresponds to an $O(p+1,2)$ conjugacy class of MASAs of $e(p,1)$.

Finally, let us sum up the above results as a theorem.

Theorem 5.1. A representative list of maximal Abelian subalgebras of the pseudo-euclidean Lie algebra $e(p,1)$ that are mutually inequivalent under the action of the conformal group $O(p+1,2)$ coincides with a list of the MASAs of $o(p+1,2)$ of the form

$$\mathrm{MASA}[e(p,1)]\sim l[o(2)]\oplus M_j \qquad j=p-2l \tag{5.33}$$

where M_j is a MASA of $o(j+1,2)$ contained in the subalgebra $e(j,1)$. Specifically we have the following.

1. $M_j \sim o(1,1) \oplus M_0$ where M_0 is a free-rowed MANS of $o(j,1)$ with Kravchuk signature $(1\ j-1\ 1)$ as in (3.1). The MASA of $e(p,1)$ is

$$\{L_{12}, L_{34}, \ldots, L_{2l-1,2l}\} \oplus \{P_{2l+1}, \ldots, P_{p-1}\} \oplus \{L_{0p}\}. \tag{5.34}$$

2. M_j is a free-rowed MANS of $o(j+1,2)$ with Kravchuk signature $(1\ j{+}1\ 1)$ as in (5.7). The MASA of $e(p,1)$ is

$$\{L_{12}, L_{34}, \ldots, L_{2l-1,2l}\} \oplus \{P_0, P_{2l+1}, \ldots P_p\}. \tag{5.35}$$

3. M_j is a non-free-rowed MANS of $o(j+1,2)$ with Kravchuk signature $(1\ j{+}1\ 1)$ as in (5.8). The MASA of $e(p,1)$ is

$$\begin{aligned}&\{L_{12}, \ldots, L_{2l-1,2l}\} \oplus \{L_{0,2l+1} - L_{p,2l+1} + \epsilon(P_0 + P_p), P_0 - P_p,\\ &\qquad P_{2l+2}, \ldots, P_{p-1}\} \qquad \epsilon = \pm 1.\end{aligned} \tag{5.36}$$

4. M_j is a free-rowed MANS of $o(j+1,2))$ with Kravchuk signature $(2\ j-1\ 2)$ as in (5.9). The MASA of $e(p,1)$ is

$$\begin{aligned}&\{L_{12}, \ldots, L_{2l-1,2l}\} \oplus \{P_{2l+1} + q_{2l+1}(L_{0,2l+1} - L_{p,2l+1}),\\ &\qquad \ldots, P_{p-1} + q_{p-1}(L_{0,p-1} - L_{p,p-1}), P_0 - P_p\}.\end{aligned} \tag{5.37}$$

The algebra (5.34) is conformally equivalent to

$$\begin{aligned}&\{L_{12}, \ldots, L_{2l-1,2l}\} \oplus \{P_0 - P_p, (L_{0,2l+1} - L_{p,2l+1}) + a_{2l+1}P_{2l+1},\\ &\qquad \ldots, (L_{0s} - L_{ps}) + a_s P_s, P_{s+1}, \ldots, P_{p-1}\}\end{aligned} \tag{5.38}$$

$$r + s = j \qquad \sum_{k=2l+1}^{s} a_k = 0 \qquad a_{2l+1} = 1 \geqslant |a_{2l+2}| \geqslant \cdots \geqslant |a_s| > 0 \tag{5.39}$$

where $p - s - 1$ is the highest multiplicity of any of the numbers $q_{2l+1}, \ldots, q_p$.

Let us give some examples of the last case in theorem 5.1 for $e(5,1)$.

(i) $\{P_0 - P_1, L_{02} - L_{12}, L_{03} - L_{13}\} \oplus L_{45}$ $(j = 3)$. It can be represented as follows:

$$M = \begin{pmatrix} 0 & a & & & & & & \\ -a & 0 & & & & & & \\ & & 0 & 0 & 0 & 0 & d & 0 \\ & & 0 & 0 & b & c & 0 & -d \\ & & & & 0 & 0 & -b & \\ & & & & 0 & 0 & -c & \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \end{pmatrix} \tag{5.40}$$

$$K = \begin{pmatrix} I_2 & & & \\ & & & J_2 \\ & & I_2 & \\ & J_2 & & \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which is equivalent under $O(6,2)$ to

$$M' = \begin{pmatrix} 0 & a & & & & & & \\ -a & 0 & & & & & & \\ & & 0 & 0 & b & c & -d & 0 \\ & & 0 & 0 & 0 & 0 & 0 & d \\ & & & & 0 & 0 & 0 & -b \\ & & & & 0 & 0 & 0 & -c \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \end{pmatrix}. \qquad (5.41)$$

Here K is as in (5.40). This algebra is $\{L_{45}, P_0 - P_1, P_2, P_3\}$ and is not maximal in $e(5,1)$ since we can add $\{P_0 + P_1\}$.

(ii) $\{P_0 - P_1, L_{02} - L_{12}, L_{03} - L_{13}\} \oplus \{P_4, P_5\}$ $(j = 5)$. It can be represented as

$$M = \begin{pmatrix} 0 & 0 & a & b & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & c & d & 0 & -e \\ & & 0 & 0 & 0 & 0 & 0 & -a \\ & & 0 & 0 & 0 & 0 & 0 & -b \\ & & & & 0 & 0 & -c & \\ & & & & 0 & 0 & -d & \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \end{pmatrix} \qquad K = \begin{pmatrix} & & J_2 \\ & I_4 & \\ J_2 & & \end{pmatrix}. \qquad (5.42)$$

This is equivalent under $O(6,2)$ to

$$M' = \begin{pmatrix} 0 & 0 & a & b & c & d & e & 0 \\ 0 & 0 & -a & -b & c & d & 0 & -e \\ & & 0 & 0 & 0 & 0 & a & -a \\ & & 0 & 0 & 0 & 0 & b & -b \\ & & & & 0 & 0 & -c & -c \\ & & & & 0 & 0 & -d & -d \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \end{pmatrix} \qquad (5.43)$$

and $M' \sim \{P_0 - P_1, L_{02} - L_{12} - P_2, L_{03} - L_{13} - P_3, L_{04} - L_{14} + P_4, L_{05} - L_{15} + P_5\}$. We see that here we have a free-rowed MANS of $o(6,2)$ with Kravchuk signature (2 4 2).

(iii) $\{P_0 - P_1, L_{02} - L_{12} + P_2, L_{03} - L_{13} + aP_3, L_{04} - L_{14} - (1+a)P_4, L_{05} - L_{15}\} \sim M$. This algebra is conformally equivalent to $M' \sim \{P_0 - P_1, P_2 + L_{02} - L_{12}, P_3 + a(L_{03} - L_{13}), P_4 - (1+a)(L_{04} - L_{14})\}$ and hence not figure in the list given in theorem 5.1 (i.e. M' will figure, but M will not).

6. Separation of variables in Laplace and wave operators

6.1. MASAs and ignorable variables

Let us consider an n-dimensional Riemannian, or pseudo-Riemannian space with metric

$$ds^2 = g_{ik}(x)dx^i\,dx^k \qquad (6.1)$$

and isometry group G. The Laplace–Beltrami equation on this space is

$$\Delta_{\mathrm{LB}}\Psi = E\Psi$$
$$\Delta_{\mathrm{LB}} = g^{-1/2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x^j}g^{1/2}g^{ij}\frac{\partial}{\partial x^i} \qquad g = \det(g_{ij}) \tag{6.2}$$

and the Hamilton–Jacobi equation is

$$g^{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j} = E. \tag{6.3}$$

We shall be interested in multiplicative separation of variables for equation (6.2) and additive separation for equation (6.3), i.e. in solutions of the form

$$\Psi(x) = \prod_{i=1}^{n}\psi_i(x_i, c_1, \ldots, c_n) \tag{6.4}$$

$$S(x) = \sum_{i=1}^{n} S_i(x_i, c_1, \ldots, c_n) \tag{6.5}$$

respectively. Here the c_j are parameters, the separation constants and ψ_i and S_i obey ordinary differential equations.

A variable x_j is *ignorable* [8] if it does not figure in the metric tensor g_{ik}. Ignorable variables are directly related to elements of the Lie algebra L of the isometry group G [7]. Indeed, let $X_1, \ldots, X_l \in L$ be a basis for an Abelian subalgebra of L. We can represent these elements by vector fields on M expressed in terms of the coordinates x. Let us further assume that these vector fields are linearly independent at a generic point $x \in M$. We can then introduce coordinates (locally) on M

$$(x_1, \ldots, x_n) \longrightarrow (\alpha_1, \ldots, \alpha_l, s_1, \ldots, s_k) \qquad l + k = n \tag{6.6}$$

which 'straighten out' this algebra

$$X_i = \frac{\partial}{\partial \alpha_i} \qquad i = 1, \ldots, l. \tag{6.7}$$

The variables α_i are the ignorable separable variables [7, 8]. Each MASA of the isometry algebra L will provide a maximal set of ignorable variables, both for the Laplace–Beltrami and Hamilton–Jacobi equations.

Specifically, for the spaces $M(p, q)$ of this paper, we generate the coordinates as follows. We use the realization (2.2) of the group $E(p, q)$ but restrict H to be a maximal Abelian subgroup of $E(p, q)$. We have $G = \langle \exp X \rangle$, where X is one of the MASAs we have constructed. We then write

$$\begin{pmatrix} x \\ 1 \end{pmatrix} = \mathrm{e}^X \begin{pmatrix} s \\ 1 \end{pmatrix} \qquad s \in \mathbb{R}^{p+q} \tag{6.8}$$

where s represents a vector in a subspace of $M(p, q)$ parametrized by non-ignorable variables $(s_1, \ldots, s_k)$, and X is a MASA of $e(p, q)$, parametrized by a set of ignorable variables.

6.2. Ignorable variables in Euclidean space M(p)

For Euclidean space the above considerations are entirely trivial. In Cartesian coordinates we have

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \tag{6.9}$$

In view of theorem 3.3 we split the space $M(p)$ into a direct sum of one and two-dimensional spaces. In each $M(1)$ we have a Cartesian coordinate x_i, corresponding to the translation P_i. In each subspace $M(2)$ we have polar coordinates, e.g. $M_{12} = \partial/\partial\alpha_1$ corresponds to

$$\begin{aligned} x_1 &= s_1 \cos\alpha_1 \\ x_2 &= s_1 \sin\alpha_1 \end{aligned} \tag{6.10}$$

with α_1 ignorable.

6.3. Ignorable variables in Minkowski space M(p, 1).

Here the situation is much more interesting. In Cartesian coordinates we have

$$\begin{aligned} &\Box_{p,1}\Psi = E\Psi \\ &\Delta_{\mathrm{LB}} \equiv \Box_{p,1} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_0^2}. \end{aligned} \tag{6.11}$$

Consider the decomposition (4.29) in theorem 4.3. In each indecomposable subspace we introduce a separable system of coordinates with a maximal number of ignorable variables. Each space $M(1,0)$ corresponds to a Cartesian coordinate, $M(2,0)$ to a polar coordinate as in equation (6.9). Now let us consider the coordinates corresponding to $M(k,1)$.

$M(0,1)$: x_0

$M(1,1)$: $x_0 = \rho\cosh\alpha \qquad x_1 = s\sinh\alpha$

$x_0 = \rho\sinh\alpha \qquad x_2 = s\cosh\alpha$

(for $x_0^2 - x_1^2 = \pm s^2$, respectively).

$M(2,1)$: the algebra (4.27) with $\kappa = 1$ provides two ignorable variables, z and a and we have

$$\begin{aligned} x_0 + x_1 &= r\sqrt{2} + 2a \\ x_0 - x_1 &= ra^2\sqrt{2} + \tfrac{2}{3}a^3 - z\sqrt{2} \\ x_2 &= -a^2 - ar\sqrt{2}. \end{aligned} \tag{6.12}$$

The coordinates (6.12) were obtained using equation (6.8) with

$$G = \mathrm{e}^X \qquad X = \begin{pmatrix} 0 & a\sqrt{2} & 0 & z\sqrt{2} \\ 0 & 0 & -a\sqrt{2} & 0 \\ 0 & 0 & 0 & a\sqrt{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad s = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}. \tag{6.13}$$

We then have

$$P_0 - P_1 = -\frac{\partial}{\partial z} \qquad L_{02} - L_{12} + P_0 + P_1 = \frac{\partial}{\partial a} \tag{6.14}$$

and the operator in this $M(2, 1)$ subspace of $M(p, 1)$ is

$$\Box_{2,1} = \sqrt{2}\frac{\partial^2}{\partial r\partial z} + \frac{1}{2}\frac{1}{r^2}\frac{\partial^2}{\partial a^2} + \frac{1}{r^2}\frac{\partial}{\partial r^2} - \frac{\sqrt{2}}{r^2}\frac{\partial^2}{\partial r\partial a} + \frac{1}{\sqrt{2}}\frac{1}{r}\frac{\partial}{\partial z} - \frac{1}{r^3}\frac{\partial}{\partial r} + \frac{1}{\sqrt{2}}\frac{1}{r^3}\frac{\partial}{\partial a}. \tag{6.15}$$

The separated solutions of the wave equation (6.11) have the form

$$\Psi = R_{Eml}(r)\mathrm{e}^{mz}\mathrm{e}^{la}. \tag{6.16}$$

The equation for $R_{Eml}(r) \equiv R$ has the form

$$R'' + \tilde{p}(r)R' + \tilde{q}(r)R = 0. \tag{6.17}$$

Using the transformation

$$R(r) = f(r)W(\rho)$$
$$f(r) = r^{\frac{1}{2}(2-\lambda-\lambda')}\exp\left(-\frac{mr^3}{3} + \frac{lr}{\sqrt{2}}\right) \qquad \rho = r^{-2} \tag{6.18}$$

we obtain the equation

$$W'' + p(\rho)W' + q(\rho)W = 0 \tag{6.19}$$

where $p(\rho)$ and $q(\rho)$ are

$$p(\rho) = \frac{1-\lambda-\lambda'}{r^{-2}} \qquad q(\rho) = -k^2 + 2\alpha r^2 + \lambda\lambda' r^4 \tag{6.20}$$

$$\lambda' = \frac{(A-1) \pm \sqrt{(a-1)^2 + 4m^2}}{2} \qquad 1-\lambda-\lambda' = A \quad A = 3 \text{ or } \tfrac{1}{2} \quad 2\alpha = lm\sqrt{2} - E. \tag{6.21}$$

The solution of (6.19) is a confluent hypergeometric series [20].

Let us consider the space $M(k, 1)$ with $k \geqslant 2$ and the splitting MASA (4.28) with $a_2 = a_3 = \cdots = a_k = 0$. The corresponding matrix realization is given by equation (4.1) with M_0 and γ as in equation (4.4) and all the M_i and x absent. Applying equation (6.8) with

$$X = \begin{pmatrix} 0 & \alpha & 0 & z \\ 0 & 0 & -\alpha^{\mathrm{T}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad s = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ r \end{pmatrix} \qquad r \in \mathbb{R} \tag{6.22}$$

we obtain the coordinates

$$\begin{aligned} x_k + x_0 &= r\sqrt{2} \\ x_k - x_0 &= -r\alpha\alpha^{\mathrm{T}}\frac{1}{\sqrt{2}} + z\sqrt{2} \\ x_1 &= -r\alpha_1 \\ &\vdots \\ x_{k-1} &= -r\alpha_{k-1}. \end{aligned} \tag{6.23}$$

The wave operator in these coordinates is

$$\Box_{k,1} = 2\frac{\partial^2}{\partial z\partial r} + \frac{k-1}{r}\frac{\partial}{\partial z} + \frac{1}{r^2}\sum_{i=1}^{k-1}\frac{\partial^2}{\partial\alpha_i^2}. \tag{6.24}$$

The variables z and α_i are ignorable (only r figures in equation (6.24)) and indeed we have

$$P_0 - P_k = -\sqrt{2}\frac{\partial}{\partial z} \qquad L_{0i} - L_{ki} = \sqrt{2}\frac{\partial}{\partial \alpha_i}. \tag{6.25}$$

The solution of the wave equation then separates

$$\psi = R(r)e^{mz}\prod_{i=1}^{k-1} e^{b_i\alpha_i} \tag{6.26}$$

with $R(r)$ as follows:

$$R(r) = r^{-k/2}\exp\left(\frac{1}{r}\frac{\sum_{i=1}^{k-1} b_i^2}{2m}\right)\exp\left(\frac{Er}{2m}\right). \tag{6.27}$$

We have shown in subsection 5.3 that this MASA is conformaly equivalent to a subalgebra of the algebra of translations, namely to $(P_0 - P_k, P_1, \ldots, P_{k-1})$. A consequence of this is that we can relate these coordinates to a set of Cartesian ones. Indeed, we can rewrite equation (6.24) as

$$\Box_{k,1} = (y_0 + y_k)^{\frac{1}{2}(k-1)}(y_0 + y_k)^2\left[\frac{\partial^2}{\partial y_0^2} - \frac{\partial^2}{\partial y_1^2} - \cdots - \frac{\partial^2}{\partial y_k^2}\right](y_0 + y_k)^{-\frac{1}{2}(k-1)} \tag{6.28}$$

with

$$\begin{aligned} x_1 + x_0 &= -\frac{1}{y_0 + y_k}\sqrt{2} \\ x_1 - x_0 &= -\frac{1}{\sqrt{2}}\frac{1}{y_0 + y_k}(y_0^2 - y_1^2 - \cdots - y_k^2) \\ x_j &= \frac{y_j}{y_0 + y_k} \qquad j = 1, \ldots, k-1. \end{aligned} \tag{6.29}$$

We note, however, that the wave equation separates in coordinates (r, z, α_i) but not in $(y_0, y_1, \ldots, y_k)$.

Now consider the space $M(k, 1)$ for $k \geqslant 3$ and the non-splitting MASA (4.28) with $a_i \neq 0$. The coordinates we obtain are

$$\begin{aligned} x_k + x_0 &= r\sqrt{2} \\ x_k - x_0 &= \frac{1}{\sqrt{2}}(2z - r\alpha\alpha^{\mathrm{T}} + \alpha A\alpha^{\mathrm{T}}) \\ x_1 &= (q_1 - r)\alpha_1 \\ &\vdots \\ x_{k-1} &= (q_{k-1} - r)\alpha_{k-1}. \end{aligned} \tag{6.30}$$

The wave operator is

$$\Box_{k,1} = 2\frac{\partial^2}{\partial z\partial r} - \left(\sum_{i=1}^{k-1}\frac{1}{(q_i - r)}\right)\frac{\partial}{\partial z} + \sum_{i=1}^{k-1}\frac{1}{(q_i - r)^2}\left(\frac{\partial^2}{\partial \alpha_i^2}\right). \tag{6.31}$$

We see that α_k, z are ignorable variables. The solution of the wave equation then separates and we have

$$\Psi = R(r)e^{mz}\prod_{i=1}^{k-1} e^{a_i\alpha_i} \tag{6.32}$$

with $R(r)$ equal to

$$R(r) = \prod_{i=2}^{k} (q_i - r)^{-\frac{1}{2}} \exp\left(-\frac{1}{2m}\sum_{i=2}^{k}\frac{b_i^2}{q_i - r}\right) \exp\left(\frac{Er}{2m}\right). \tag{6.33}$$

We mention that the three new coordinates systems, equations (6.12), (6.23) and (6.30) are all non-orthogonal, hence the cross terms (mixed derivatives) in the corresponding forms of the wave operator.

7. Conclusions

The classification of MASAs of $e(p, 0)$ and $e(p, 1)$ performed in this paper is complete, entirely explicit and the results are reasonably simple. Indeed, they are summed up in theorems 3.1, 3.2 and 3.3 for $e(p, 0)$ and theorems 4.1, 4.2, 4.3 and 5.1 for $e(p, 1)$.

In section 6 we have presented a first application of this classification. Namely, we have constructed the coordinate systems (6.12), (6.23) and (6.30) which allow the separation of variables in the wave equation and have the maximal number of ignorable variables. In turn, these coordinate systems have further applications.

Thus, instead of the wave equation itself, let us consider a more general equation, namely

$$[\Box + V(x)]\Psi = E\Psi. \tag{7.1}$$

First of all, it is possible to choose the potential $V(x)$ to be such that equation (7.1) allows the separation of variables in one of the above coordinate systems. The obtained equation will be integrable in that there will exist a complete set of p second-order operators commuting with $H = \Box + V$ and with each other. They will be of the form $X_i^2 + f_i(x_i)$ where $\{X_i\}$ is the corresponding MASA and $f_i(x_i)$ is a function of the corresponding ignorable variable. The actual form of f depends on the separable potential $V(x)$ [21, 22].

The coordinates (6.30) have been used to construct equations of the type (7.1) that obey the Huygens principle [23]. The Crum–Darboux transformation [24–26] can be used to generate specific potentials $V(x)$ (depending on one ignorable variable in a given separable coordinate system) that have specific integrability properties. In particular this provides a method for constructing overcomplete commutative rings of partial differential operators and 'algebraically integrable' dynamical systems [27–29].

The reason we bring this up here is that Crum–Darboux transformations have traditionally been performed in Cartesian or polar coordinates. The fact that they can be applied to other types of coordinates, associated with other types of MASAs, opens new possibilities.

Work is in progress on the classification of MASAs of $e(p, q)$ for $p \geqslant q \geqslant 2$ [30].

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Chapter 3

Solutions of (2+1)-dimensional spin systems

Solutions of (2+1)-dimensional spin systems

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ABSTRACT

We use the methods of group theory to reduce the equations of motion of two spin systems in (2+1) dimensions to sets of coupled ordinary differential equations. We present solutions of some classes of these sets and discuss their physical significance.

Les méthodes de la théorie des groupes sont utilisées pour réduire les équations du mouvement de deux systèmes de spins de dimensions (2+1) à des systèmes d'équations différentielles ordinaires. Les solutions de certaines classes de ces systèmes sont presentées et les aspects physiques sont discutés.

1. Introduction

In this paper we look for solutions of the equations of the Landau-Lifshitz model (with, perhaps, nonvanishing anisotropy) and of a nonlinear vector diffusion equation. The equations are given, respectively, by

$$\frac{\partial \vec{\phi}}{\partial t} = \vec{\phi} \times \vec{F} \tag{1.1}$$

and

$$\frac{\partial \vec{\phi}}{\partial t} = \vec{F} - \vec{\phi}(\vec{\phi}\,.\,\vec{F}), \tag{1.2}$$

where $\vec{F}$ is given by

$$\vec{F} = \Delta\vec{\phi} + (A\phi_3 + B)\vec{e}_3, \tag{1.3}$$

where $\vec{e}_3$ is a unit vector in the 3^{rd} direction in the $\vec{\phi}$ space and $\vec{\phi}$ satisfies $\vec{\phi}\,.\,\vec{\phi} = 1$. A and B are possible anisotropy coefficients.

The motivation for this work comes from the original observation made by Landau and Lifshitz[1] in their study of the ferromagnetic continuum. They pointed out that for phenomena for which substantial spatial variations occur only over a large number of lattice spacings, we can use the continuum approximation. They showed that a ferromagnetic medium is characterized by the magnetization vector $\vec{M}$ (like the vector $\vec{\phi}$ above) which precesses around the effective magnetic field and so obeys, what is now called the Landau-Lifshitz equation, namely (1.1). Since the original work of Landau and Lifshitz many papers have been written on the subject[2] and the equation has been modified by the inclusion of various additional terms to $\vec{F}$. It has been used to describe the dynamics of magnetic bubbles in a ferromagnetic continuum and also of vortices in HeII or in a superconductor[2]. Various studies of the dynamics of such topological soliton-like structures have been performed both theoretically and experimentally[3][4] and they have exhibited many interesting, and perhaps unexpected, phenomena - like the skew deflection of these structures under the influence of a magnetic field gradient which resembles the more familiar Hall motion of electrons in external magnetic and electric fields[5].

A recent work of Papanicolaou and Tomaras[2], as well as some earlier work of other people[6] has shown that many experimentally observed facts can indeed be explained using the Landau-Lifshitz equation. Much of the work involved deriving various conserved quantities describing these structures and then using them to restrict the description of the dynamics. All this work has provided further evidence as to the relevance of the Landau-Lifshitz equation to the description of physical phenomena. However, as the Landau-Lifshitz equation is quite complicated, only some results were obtained in an analytical form. Most more recent studies[7] involved numerical simulations.

The vector nonlinear diffusion equation (1.2) has less obvious physical applications but it has been used[8] in the study of phase ordering kinetics where one investigates the time evolution of a system quenched from the disordered into an ordered phase. This topic has attracted considerable attention in recent years[9]. In fact, it has been shown that many features of phase ordering in systems supporting topologically stable defects (for example, in systems described by the O(N) vector model in d dimensions with $d \leq N$[10][11]), or in two and three-dimensional nematic liquid crystals[12] can be understood theoretically by investigating the dynamics of the numerous topological defects generated during the quench. A special and interesting case is that of the O(3) model system in 2 spatial dimensions. It supports topologically stable, but non-singular objects which, in the condensed matter community language, are called topological textures. Such systems were studied numerically in ref. [8].

Given the paucity of analytical results for both equations (1.1) and (1.2) (especially involving the dynamics) one of the aims of this paper is to see what time dependent solutions can be found using the group theoretical method of symmetry reduction[13][14][15]. This method exploits the symmetry of the original equations to find solutions invariant under some subgroup (the classic example one can give here involves seeking solutions in three dimensions which are rotationally invariant). The method puts all such attempts on a unified footing and it has been applied with success to many equations[16] . The method gives equations whose solutions represent specific solutions of the full equations; the solutions are determined locally and the method does not tell us whether these solutions are stable or not with respect to any perturbations.

In a recent paper[17], two of us (PW and WJZ) together with M. Grundland, have applied this technique to looking for solutions of the relativistic CP^1 model.

In this paper we investigate solutions of (1.1) and (1.2). We are particularly interested in time dependent solutions; all time independent solutions of (1.1) and (1.2) (when there is no anisotropy) are also the time independent solutions of the relativistic model and so can be found in ref [17].

Like in the relativistic CP^1 model studied before, in order to perform the symmetry reductions, we have to decide what variables to use. To avoid having to use the constrained variables ($\vec{\phi}$) it is convenient to use the W formulation of the model which involves the stereographic projection of the sphere $\vec{\phi}\cdot\vec{\phi}=1$ onto the complex plane. In this formulation instead of using the $\vec{\phi}$ fields, we express all the dependence on $\vec{\phi}$ in terms of their stereographic projection onto the complex plane W. The $\vec{\phi}$ fields are then related to W by

$$\phi_1 = \frac{W+W^*}{1+|W|^2}, \quad \phi_2 = i\frac{W-W^*}{1+|W|^2}, \quad \phi_3 = \frac{1-|W|^2}{1+|W|^2}. \tag{1.4}$$

To perform our analysis it is convenient to use the polar version of the W variables; *i.e.* to put $W = R\exp iQ$ and then study the equations for R and Q. The advantage of this approach is that the equations become simple; the disadvantage comes from having to pay attention that R is real and Q should be periodic with a period of 2π. (If the period is not 2π then the solution may become multi-valued) Thus if we find solutions that do not obey these restrictions, then these solutions, however interesting they may be, cannot be treated as solutions of the original model.

In the case of the Landau-Lifshitz equation the equations for R and Q take the form

$$\partial_t R - 2\frac{(1-R^2)}{(1+R^2)}\Big(\partial_x Q \partial_x R + \partial_y Q \partial_y R\Big) - R(\partial_{xx} Q + \partial_{yy} Q) = 0 \tag{1.5}$$

and

$$\begin{aligned} \partial_t Q &= B + A\frac{1-R^2}{1+R^2} - \frac{\partial_{xx} R + \partial_{yy} R}{R} \\ +\frac{(1-R^2)}{(1+R^2)}&\Big((\partial_x Q)^2 + (\partial_y Q)^2\Big) + \frac{2}{(1+R^2)}\Big((\partial_x R)^2 + (\partial_y R)^2\Big), \end{aligned} \tag{1.6}$$

while for the diffusion case they are respectively

$$\partial_t Q - 2\frac{(1-R^2)}{R(1+R^2)}\Big(\partial_x Q \partial_x R + \partial_y Q \partial_y R\Big) - (\partial_{xx} Q + \partial_{yy} Q) = 0 \tag{1.7}$$

and

$$\begin{aligned} \partial_t R &+ BR + AR\frac{1-R^2}{1+R^2} - \partial_{xx} R - \partial_{yy} R \\ +\frac{(1-R^2)R}{(1+R^2)}&\Big((\partial_x Q)^2 + (\partial_y Q)^2\Big) + \frac{2R}{(1+R^2)}\Big((\partial_x R)^2 + (\partial_y R)^2\Big) = 0. \end{aligned} \tag{1.8}$$

Note, that, in the Landau-Lifshitz case, if we put $R = 1$ the equations become $\Delta Q = 0$ and $\partial_t Q = B$ which have a very simple solution, and in the diffusion case, we have to set $B = 0$ and then we end up with $\partial_t Q - \Delta Q = 0$ as the equation for Q. The latter case is the nonrelativistic analogue of what was found in the relativistic case where $R = 1$ reduced the equation for Q to the linear wave equation for the phase Q.

In the next section we determine the symmetry group of our equations (1.6) , (1.5) and of (1.7) and (1.8) . In the following sections we solve the derived equations and discuss their solutions.

2. The Symmetry Group and its Two Dimensional Subgroups

The symmetry group of our systems of equations, respectively (1.6) and (1.5) and (1.7) and (1.8) , can be calculated using the standard methods[13] [14] [15] [16] . We actually made use of a MACSYMA package[18] that provides a simplified and partially solved set of determining equations.

Solving the determining equations we find that three different cases must be distinguished:

1. $A = B = 0$, *i.e.* the anisotropy is absent. The Landau-Lifshitz equation and the diffusion equation have isomorphic symmetry groups, consisting of translations in space and time directions, rotations in the x, y plane, dilations and a group of $O(3)$ rotations between the components of the field $\vec{\phi}$. The corresponding Lie algebra L_1 has the structure of a direct sum

$$L_1 = s(2,1) \quad \oplus \quad o(3). \tag{2.1}$$

Bases for these two algebras are given by the following vector fields, acting on space-time and on the fields in the $\{R, Q\}$ realization of eq. (1.5)-(1.8):

$$\begin{aligned} s(2,1): \qquad & P_0 = \partial_t, \quad P_1 = \partial_x, \quad P_2 = \partial_y, \\ & L = -x\partial_y + y\partial_x, \quad D = 2t\partial_t + x\partial_x + y\partial_y. \end{aligned} \tag{2.2}$$

$$\begin{aligned} o(3): \qquad & X = \frac{1}{2}\Big(sinQ(R - \frac{1}{R})\partial_Q + cosQ(R^2+1)\partial_R\Big), \\ & Y = \frac{1}{2}\Big(cosQ(R - \frac{1}{R})\partial_Q - sinQ(R^2+1)\partial_R\Big), \\ & Z = \partial_Q. \end{aligned} \tag{2.3}$$

2. $A \neq 0$.

The symmetry algebra for both equations is reduced to

$$L_2 = \{P_0, P_1, P_2, L\} \quad \oplus \quad \{Z\}, \tag{2.4}$$

i.e. the dilations are absent and the only $\vec{\phi}$ rotations left are those around the third axis (*i.e.* around ϕ_3).

3. $A = 0, \quad B \neq 0.$

The symmetry algebra for the dissipative equations (1.8) and (1.7) is still L_2, as in (2.4). That of the Landau-Lifshitz equation is

$$L_3 = \{P_0, P_1, P_2, L, \tilde{D}\} \quad \oplus \quad \{X, Y, Z\}, \tag{2.5}$$

with

$$\tilde{D} = 2t\partial_t + x\partial_x + y\partial_y + 2Bt\partial_Q \tag{2.6}$$

and Q is replaced by $Q - Bt$ in X and Y, eq. (2.3) .

In order to perform symmetry reduction we need to classify the subalgebras of the symmetry algebras L_1, L_2 and L_3. We wish to reduce equations (1.5)-(1.8) to ordinary differential equations. To do this, we will require that the solutions are invariant under a two-dimensional subgroup of the symmetry group. In order to do this systematically we need to derive a classification of the two dimensional subalgebras of the symmetry algebra. Moreover, we can restrict ourselves to subalgebras, all elements of which act nontrivially on space-time, *i.e.* which do not contain any rotations in $\vec{\phi}$ space.

The subalgebra classification can be done in an algorithmic way[14]; the results are quite simple and we present them without a proof.

1. $A = B = 0$. Every two-dimensional subalgebra of L_1, each element of which acts nontrivially on space-time, is conjugate under the action of the group of

inner automorphisms to one of the following ones

$$\begin{aligned}
A_{2,1} =&\{P_1 + aZ, P_2 + bZ\}, \\
A_{2,2} =&\{L + aZ, P_0 + bZ\}, \\
A_{2,3} =&\{P_0 + aZ, P_2 + bZ\}, \\
A_{2,4} =&\{P_0 - vP_1 + aZ, P_2 + bZ\}, \quad v \neq 0, \\
A_{2,5} =&\{D + bL + aZ, P_0\}, \quad b \neq 0, \\
A_{2,6} =&\{D + aZ, P_0\}, \\
A_{2,7} =&\{D + aZ, L + bZ\}, \\
A_{2,8} =&\{D + aZ, P_2\}.
\end{aligned} \tag{2.7}$$

The parameters a, b and v are arbitrary real numbers. In some cases their ranges can be further constrained but that is not important for our purposes.

2. $A \neq 0$.

Every two-dimensional subalgebra of the considered type is conjugate to one listed above as $A_{2,1}, ...A_{2,4}$.

3. $A = 0, \quad B \neq 0$.

For the dissipative equations (1.8) and (1.7) the subalgebra classes are represented by $A_{2,1}, ...A_{2,4}$. For the Landau-Lifshitz equations (1.5) and (1.6) they are represented by $A_{2,1}, ...A_{2,8}$ with D replaced by $\tilde{D}$ and P_0 replaced by $P_0 = P_0 + bZ$.

We can now proceed to perform various reductions. We are particularly interested in reductions that do not result in time independence as these were already studied in ref [17].

3. Solutions of the Landau-Lifshitz Equation

3.1. General Procedure

Our aim is to solve the Landau-Lifshitz equations (1.5) and (1.6) , using the method of symmetry reduction. This involves assuming that a solution is invariant under a subgroup G_0 of the symmetry group G, namely one of the two dimensional groups corresponding to the algebras $A_{2,1}, \ldots, A_{2,8}$ of (2.7). The assumption makes it possible to reduce the partial differential equations (1.5) and (1.6) to a pair of coupled ordinary differential equations. Whenever possible, we decouple them and find explicit solutions for the functions R and Q, hence for W, and finally for the vector $\vec{\phi}$ figuring in (1.1) .

For all 8 algebras in (2.7) the invariant solution will have the form

$$R(x,y,t) = R(\xi), \qquad Q(x,y,t) = \alpha(\xi) + \beta(x,y,t) \tag{3.1}$$

where ξ and β are explicitly given and $R(\xi)$ and $\alpha(\xi)$ satisfied coupled ordinary differential equations obtained by substituting (3.1) into (1.5) and (1.6) .

The reduced equation (1.5) is

$$(\nabla\xi)^2\alpha_{\xi\xi} + [2\frac{(1-R^2)}{R(1+R^2)}(\nabla\xi)^2 R_\xi + \Delta\xi]\alpha_\xi = \frac{R_\xi}{R}\xi_t - 2\frac{(1-R^2)}{R(1+R^2)}R_\xi(\nabla\xi, \nabla\beta) - \Delta\beta. \tag{3.2}$$

For algebra $A_{2,1}$ we have

$$\nabla\xi^2 = \Delta\xi = \Delta\beta = 0, \qquad \xi_t = 1 \tag{3.3}$$

and so (3.2) reduces to $R_\xi = 0$.

In all other cases we have $(\nabla\xi)^2 \neq 0$. Eq. (3.2) is a first order linear inhomogeneous equation for α_ξ. We can integrate it explicitly and obtain α_ξ in terms of

R, whenever the functions ξ and β satisfy

$$\frac{d}{d\xi}\left(\frac{h}{(\nabla\xi)^2}\right) = 0, \qquad \frac{d}{d\xi}\left[h\frac{(\nabla\xi, \nabla\beta)}{(\nabla\xi)^2}\right] - \frac{h\Delta\beta}{(\nabla\xi)^2} = 0 \tag{3.4}$$

where

$$\begin{aligned} h(\xi) = 1 \quad &\text{for} \quad \Delta\xi = 0, \\ \frac{h'(\xi)}{h(\xi)} = \frac{\Delta\xi}{(\nabla\xi)^2} \quad &\text{for} \quad \Delta\xi \neq 0. \end{aligned} \tag{3.5}$$

Conditions (3.4) are always satisfied for the algebras $A_{2,2} \ldots A_{2,6}$, not however for $A_{2,7}$ and $A_{2,8}$. When conditions (3.4) are satisfied, we can integrate eq. (3.2) once to obtain

$$\alpha_\xi = \frac{S}{h}\frac{(1+R^2)^2}{R^2} + \mu\frac{1+R^2}{R^2} + \nu \tag{3.6}$$

where S is an arbitrary real integration constant and where we have

$$\begin{array}{llll} \mu = -\frac{v}{2}, & \nu = 0, & \text{for} & A_{2,4} \\ \mu = 0, & \nu = -\frac{a}{b^2+1}, & \text{for} & A_{2,5} \\ \mu = 0, & \nu = \frac{a\xi}{1+\xi^2}, & \text{for} & A_{2,6} \\ \mu = 0, & \nu = 0, & \text{for} & A_{2,2}, A_{2,3}. \end{array} \tag{3.7}$$

Equation (1.6) for algebras $A_{2,2}, \ldots, A_{2,8}$ is reduced to a second order differential equation for $R(\xi)$, that also involves $\alpha_\xi(\xi)$. For reductions corresponding to Lie algebras $A_{2,2}, \ldots, A_{2,6}$ we can substitute α_ξ from (3.6) , to obtain an ordinary differential equation for $R(\xi)$ alone. To transform this equation to a standard form we put

$$R(\xi) = \sqrt{-U(\eta)}, \quad \eta = \int h^{-1}(\xi)d\xi. \tag{3.8}$$

The equation for $U(\eta)$ is then written as

$$U_{\eta\eta} = (\frac{1}{2U} + \frac{1}{U-1})U_\eta^2 - \frac{2S^2}{U}(U+1)(U-1)^3 + p\frac{U(U+1)}{U-1} + qU + m(U-1)^2. \tag{3.9}$$

Equation (3.9) can be integrated in terms of elliptic functions if p, q and m are constants. This is always the case for algebras $A_{2,3}, \ldots, A_{2,6}$. In the case of algebra $A_{2,2}$ this is true if we set $A = 0, B = b$.

Eq. (3.9) has a first integral that we can write as

$$U_\eta^2 = -4S^2U^4 + K_1U^3 + KU^2 + K_2U + K_3 \tag{3.10}$$

where K is an integration constant, and the constants K_1, K_2 and K_3 are related to the coefficients S, p, q and m in (3.9) .

In this article we restrict ourselves to solutions of the Landau-Lifshitz equation that are obtained by solving (3.10) .

We shall first discuss solutions of (3.10) in general, then run through algebras $A_{2,2}, \dots, A_{2,6}$ and specify the values of the coefficients in (3.10) in each case, as well as the independent variable η.

Algebra $A_{2,1}$ leading to a first order equation, will be treated separately.

3.2. Solutions of the elliptic function equation

We shall call (3.10) the "elliptic function equation". Its solutions are of course well known[19]. We shall however list those that are relevant in the context of solving (3.9) , and more importantly, the Landau-Lifshitz equation.

Several comments are in order here:

1. The functions $R(\eta)$ must be real (and nonnegative), hence $U(\eta)$ must be real and nonpositive.
2. For $S \neq 0$ the coefficient of the highest power of U in (3.10) is nonnegative. this means that all real solutions of (3.10) are nonsingular.
3. For $S = 0$, $K_1 \neq 0$ in (3.10) the real solutions of (3.10) can be singular. Since we are really interested in the fields ϕ_i we note that singular solutions of U will give regular functions ϕ_i.
4. In general, equation (3.10) is solved in terms of Jacobi elliptic functions. However, these reduce to elementary functions whenever the polynomial on the right hand side has multiple roots, or when $S = K_1 = 0$.

Let us run through individual cases.

I. $S \neq 0$

We rewrite (3.10) as

$$U_\eta^2 = -4S^2(U-U_1)(U-U_2)(U-U_3)(U-U_4) \tag{3.11}$$

1. $U_1 \leq U \leq U_2 = U_3 = U_4 < 0$

$$U(\eta) = U_2 - \frac{U_2 - U_1}{1 + S^2(U_2-U_1)^2(\eta-\eta_0)^2} \tag{3.12}$$

this is an algebraic solitary wave, equal to U_2 for $\eta \to \pm\infty$, and dipping down to U_1 for $\eta = \eta_0$.

2. $U_1 = U_2 = U_3 < U \leq U_4 \leq 0$

$$U(\eta) = U_1 + \frac{U_4 - U_1}{1 + S^2(U_4-U_1)^2(\eta-\eta_0)^2} \tag{3.13}$$

Also an algebraic solitary wave, rising to $U = U_4$ for $\eta = \eta_0$, equal to U_1 for $\eta \to \pm\infty$.

3. $U_1 \leq U < U_2 = U_3 < U_4, \;\; U_2 \leq 0$

$$\begin{aligned} U(\eta) =& U_2 - \frac{(U_4-U_2)(U_2-U_1)}{(U_4-U_1)\cosh^2\mu(\eta-\eta_0) - (U_2-U_1)} \\ \mu =& S\sqrt{(U_4-U_2)(U_2-U_1)} \end{aligned} \tag{3.14}$$

4. $U_1 < U_2 = U_3 < U \leq U_4 \leq 0$

$$U(\eta) = U_3 + \frac{(U_3-U_1)(U_4-U_3)}{(U_4-U_1)\cosh^2\mu(\eta-\eta_0) - (U_4-U_3)} \tag{3.15}$$

with μ as in (3.14) .

The last two solutions are solitons, the first one a well, the second a bump.

5. $U_1 \leq U \leq U_2 < U_3 = U_4,\ U_2 \leq 0$

$$\begin{aligned} U(\eta) =& U_4 - \frac{(U_4 - U_2)(U_4 - U_1)}{(U_2 - U_1)\sin^2 \mu(\eta - \eta_0) + U_4 - U_2} \\ \mu =& S\sqrt{(U_4 - U_2)(U_4 - U_1)} \end{aligned} \tag{3.16}$$

6. $U_1 = U_2 < U_3 \leq U \leq U_4 \leq 0$

$$\begin{aligned} U(\eta) =& U_1 + \frac{(U_4 - U_1)(U_3 - U_1)}{(U_4 - U_3)\sin^2 \mu(\eta - \eta_0) + U_3 - U_1} \\ \mu =& S\sqrt{(U_4 - U_1)(U_3 - U_1)} \end{aligned} \tag{3.17}$$

7. $U_1 \leq U \leq U_2 < U_3 < U_4,\ U_2 \leq 0$

$$\begin{aligned} U(\eta) =& U_4 - \frac{(U_4 - U_2)(U_4 - U_1)}{(U_2 - U_1)sn^2(\mu(\eta - \eta_0), k) + U_4 - U_2} \\ \mu =& S\sqrt{(U_4 - U_2)(U_3 - U_1)}, \quad k^2 = \frac{(U_4 - U_3)(U_2 - U_1)}{(U_4 - U_2)(U_3 - U_1)} \end{aligned} \tag{3.18}$$

8. $U_1 < U_2 < U_3 \leq U \leq U_4 < 0$

$$U(\eta) = U_1 + \frac{(U_4 - U_1)(U_3 - U_1)}{(U_4 - U_3)sn^2\mu(\eta - \eta_0) + U_3 - U_1} \tag{3.19}$$

with k^2 and μ as in (3.18) .

9. $U_1 \leq U \leq U_2 \leq 0,\ U_{3,4} = p \pm iq,\ q > 0$

$$\begin{aligned} U(\eta) =& \frac{(MU_1 - NU_2)cn(\mu(\eta - \eta_0), k) + MU_1 + NU_2}{(M - N)cn(\mu(\eta - \eta_0), k) + M + N} \\ M^2 =& (U_2 - p)^2 + q^2, \quad N^2 = (U_1 - p)^2 + q^2 \\ k^2 =& \frac{(U_2 - U_1)^2 - (M - N)^2}{4MN}, \quad \mu = 2S\sqrt{MN} \end{aligned} \tag{3.20}$$

Solutions (3.16) , . . . , (3.20) are periodic. All the elementary solutions can be viewed as limits of solutions (3.18) , (3.19) and (3.20) .

II. $S = 0, \;\; K_1 \neq 0$

Set

$$\mu = \frac{1}{2}\sqrt{|K_1|(U_3 - U_1} \tag{3.21}$$

1. $K_1 < 0$, $U_1 = U_2 < U \leq U_3 \leq 0$

$$U = U_3 - (U_3 - U_2)\tanh^2 \mu(\eta - \eta_0) \tag{3.22}$$

2. $K_1 < 0, U < U_1 = U_2 < U_3, U_1 \leq 0$

$$U = U_3 - \frac{(U_3 - U_1)}{\tanh^2 \mu(\eta - \eta_0)} \tag{3.23}$$

3. $K_1 < 0, U < U_1 = U_2 = U_3 \leq 0$

$$U = U_1 - \sqrt{\frac{2}{-K_1}}\frac{1}{(\eta - \eta_0)^2} \tag{3.24}$$

4. $K_1 < 0, U \leq U_1 < U_2 = U_3, U_1 \leq 0$

$$U = U_3 - \frac{U_3 - U_1}{\sin^2 \mu(\eta - \eta_0)} \tag{3.25}$$

5. $K_1 < 0, U_1 < U_2 < U \leq U_3 \leq 0$

$$U = U_3 - (U_3 - U_2)sn^2(\mu(\eta - \eta_0), k), \qquad k^2 = \frac{U_3 - U_2}{U_3 - U_1} \tag{3.26}$$

6. $K_1 < 0, U \leq U_1 < U_2 < U_3, U_1 \leq 0$

$$U = U_3 - \frac{U_3 - U_1}{sn^2(\mu(\eta - \eta_0), k)} \tag{3.27}$$

k as in (3.26)

7. $K_1 > 0, U_1 < U < U_2 = U_3 = 0$

$$U = U_1 \frac{1}{\cosh^2 \mu(\eta - \eta_0)} \tag{3.28}$$

8. $K_1 > 0, U_1 < U < U_2 < 0 < U_3$

$$U = (U_2 - U_1) sn^2(\mu(\eta - \eta_0), k), \qquad k^2 = \frac{U_2 - U_1}{U_3 - U_1} \tag{3.29}$$

9. $K_1 < 0, U_1 \leq 0, U_{2,3} = p \pm iq, q > 0$

$$\begin{aligned} U =& U_1 + A - \frac{2A}{1 - cn(\mu(\eta - \eta_0), k)} \\ A^2 =& (p - U_1)^2 + q^2, \quad k^2 = \frac{A - p + U_1}{2A}, \quad \mu = \sqrt{|K_1| A} \end{aligned} \tag{3.30}$$

$III.$ $S = 0, K_1 = 0, K \neq 0$

1. $K > 0, U \leq U_1 < 0 < U_2$

$$U = U_1 - (U_2 - U_1) \sinh^2 \frac{\sqrt{K}}{2}(\eta - \eta_0) \tag{3.31}$$

2. $K > 0, U < U_1 = U_2 = 0$

$$U = - \exp(-\sqrt{K}(\eta - \eta_0)) \tag{3.32}$$

3. $K < 0, U_1 < U < U_2 \leq 0$

$$U = U_1 + (U_2 - U_1) \cos^2 \frac{\sqrt{-K}}{2}(\eta - \eta_0) \tag{3.33}$$

$IV.$ $S = K_1 = K = 0, K_2 \neq 0$

$$U = -\frac{K_3}{K_2} + \frac{K_2}{4}(\eta - \eta_0)^2, \quad K_2 < 0, \quad K_3 < 0 \tag{3.34}$$

$V.$ $S = K_1 = K = K_2 = 0$

$$U = \sqrt{K_3}(\eta - \eta_0), \quad K_3 > 0 \tag{3.35}$$

3.3. Individual reductions

1. Algebra $A_{2,1}$.

This is an exceptional case when (3.2) implies $R_\xi = 0$. We find that the only solution for W of (1.4) is

$$W = R_0 e^{iQ}, \quad Q = ax + by + \big(B + \frac{1 - R_0^2}{1 + R_0^2}(a^2 + b^2 + A)\big)t + \alpha_0 \tag{3.36}$$

where R_0 and α_0 are integration constants.

2. Algebra $A_{2,2}$.

We find

$$W = R(\rho) \exp i[\alpha(\rho) + a\phi + bt], \qquad \xi = \rho \tag{3.37}$$

where ρ and ϕ are polar coordinates. The singlevaluedness of W requires a to be an integer. The phase $\alpha(\rho)$ and variable η satisfy

$$\alpha_\rho(\rho) = S\frac{(1 + R^2)^2}{\rho R^2}, \quad \eta = \ln \rho \tag{3.38}$$

(see (3.6)). For the function $U(\eta)$ of (3.8) we obtain the elliptic function equation if and only if we set

$$A = 0, \qquad b = B \tag{3.39}$$

(A and B are defined in (1.6)).

We have

$$K_1 = K_2 = 2a^2 + 4S^2 - \frac{K}{2}, \quad K_3 = -4S^2 \tag{3.40}$$

in (3.10) .

For $S \neq 0$ eq. (3.40) implies that we can have two negative and two positive roots in eq. (3.11) or two negative roots and two complex conjugate ones. These cases lead to real solutions, namely (3.16) , (3.18) and (3.20) . Note that all of them are periodic. In particular, for $a = 0$ eq. (3.11) always has a double root $U_3 = U_4 = 1$ and reduces to

$$U_\eta^2 = -4S^2[U^2 + (1 + \frac{K}{8S^2})U + 1] \tag{3.41}$$

For $S = 0$, $K \neq 4a^2$ we obtain the equation

$$\begin{aligned} U_\eta^2 &= 2(a^2 - \frac{K}{4})U(U - U_1)(U - U_2), \\ U_1U_2 = 1, \quad U_1 + U_2 &= \frac{2K}{K - 4a^2}. \end{aligned} \tag{3.42}$$

The relevant solutions of (3.42) in this case are:

1. $2a^2 \leq K < 4a^2, U_1 < U_2 < 0$
   solution (3.29) (with $U_3 = 0$). For $K = 2a^2$ we have $U = -1$.
2. $K > 4a^2, U_1 \leq U_2 < 0$
   Solutions (3.26) , (3.27) , (3.22) and (3.23) (all with $U_3 = 0$).
3. $K > 4a^2, 0 < U_1 < U_2$
   Solutions (3.26) (with $U_3 \to U_2$, $U_2 \to U_1, U_1 = 0$).
4. $K > 4a^2, U_{1,2} = p \pm iq, q > 0$ solution (3.30) .

For $S = 0, K = 4a^2$ (3.10) reduces to an elementary one and its solution is

$$R(\rho) = -R_0^2\rho^{\pm 2a} \tag{3.43}$$

where R_0 is an integration constant.

3. Algebra $A_{2,3}$ and $A_{2,4}$

The reduction formulas in both of these cases are

$$W = R(\xi)\exp i[\alpha(\xi) - at - by], \quad \xi = x + vt, \quad \eta = \xi \tag{3.44}$$

with $v = 0$ and $v \neq 0$ for the algebras $A_{2,3}$ and $A_{2,4}$ respectively. Since the Landau-Lifshitz equation is not Galilei invariant, we cannot change the value of v by a group transformation. The transformation (3.8) leads to (3.10) with

$$\begin{aligned} K_1 &= -\frac{K}{2} + 4S^2 + 2(A + b^2) - \frac{v^2}{2} + 2vS + 2(-B + a) \\ K_2 &= -\frac{K}{2} + 4S^2 + 2(A + b^2) + \frac{3v^2}{2} - 6vS + 2(B - a) \\ K_3 &= -(2S - v)^2. \end{aligned} \tag{3.45}$$

Eq. (3.6) in this case gives

$$\alpha_\xi = S\frac{(1 + R^2)^2}{R^2} - \frac{v}{2}\frac{1 + R^2}{R^2} \tag{3.46}$$

For $S \neq 0$ we obtain (3.11) with the constraint

$$U_1U_2U_3U_4 = \left(1 - \frac{v}{2S}\right)^2 \tag{3.47}$$

imposed on these roots. Hence, only even number of roots can be negative (0, 2 or 4). This however means that all solutions (3.12) ... (3.20) can occur, though in some cases we must impose $U_4 \leq 0$ ($U_4 = 0$ is allowed for $v = 2S$).

For $S = 0$, all solutions (3.22) , ..., (3.30) can occur.

4. Algebra $A_{2,5}$

The reduction formula is

$$W = R(\xi)\exp i[\alpha(\xi) + \frac{a}{b}\phi + Bt], \quad \xi = \ln\rho + \frac{1}{b}\phi \tag{3.48}$$

and we must set $A = 0$ in the Landau-Lifshitz equation. The phase $\alpha(\xi)$ satisfies

$$\alpha_\xi = S\frac{(1 + R^2)^2}{R^2} - \frac{a}{b^2 + 1} \tag{3.49}$$

(see (3.6)).

The function $U = -R^2(\xi)$ satisfies (3.10) and for $S \neq 0$ we have

$$K_1 = K_2 = -\frac{K}{2} + 4S^2 - \frac{2a^2}{(b^2+1)^2}, \quad K_3 = -4S^2. \tag{3.50}$$

The solutions that can occur in this case are (3.14) , . . ., (3.20). However, there are constraints between various parameters of the solution which follow from the requirement of singlevaluedness of W.

For $a = 0$ eq. (3.11) again reduces to (3.41) and we only obtain the elementary periodic solutions (3.16) .

For $S = 0$, $K_1 \neq 0$ we obtain the equation

$$\begin{aligned} U_\xi^2 =& K_1 U(U - U_1)(U - U_2), \\ U_1 U_2 = 1, \qquad & U_1 + U_2 = 2\big[1 + \frac{2a^2}{K_1(b^2+1)}\big]. \end{aligned} \tag{3.51}$$

For $-\frac{a^2}{b^2+1} < K_1 < 0$ we have $U_1 \leq U_2 < 0$ and solutions (3.22) , (3.23) , (3.26) and (3.27) are obtained.

For $K_1 > 0$ we have $0 < U_1 \leq U_2$ but we obtain no real solutions.

For $K_1 < -a^2/(b^2+1)$ we obtain solutions (3.30) .

Finally, for $S = 0$, $K_1 = 0$ the solution is $U = -R_0^2 \exp(\pm\sqrt{K}\xi)$ and hence

$$R = R_0 \exp\big[\pm\frac{1}{2}\sqrt{K}(\ln\rho + \frac{1}{b}\phi)\big] \tag{3.52}$$

For $S = 0$, $K_1 \neq 0$ equation (3.10) reduces to

$$U_\xi^2 = K_1 U(U^2 + \frac{K}{K_1}U + 1), \quad K_1 = -\frac{K}{2} - \frac{2a^2}{(b^2+1)^2} \tag{3.53}$$

Real solutions are obtained only for $K_1 < 0$. More specifically, solutions (3.27) and (3.30) can occur for any $K_1 < 0$. Solution (3.26) for K_1 in the range $-\frac{2a^2}{(b^2+1)^2} \leq K_1 < 0$, (3.22) for $K_1 = -\frac{a^2}{(b^2+1)^2}$ and (3.23) either for $a = 0$, or $K_1 = -\frac{a^2}{(b^2+1)^2}$.

5. Algebra $A_{2,6}$

The reduction formula is

$$W = R(\xi)\exp i[\alpha(\xi) + Bt + a\ln x],\ \xi = \frac{y}{x},\ \eta = \arctan\frac{y}{x} = \phi \tag{3.54}$$

and α satisfies

$$\alpha_\xi = \frac{S}{1+\xi^2}\frac{(1+R^2)^2}{R^2} + \frac{a\xi}{1+\xi^2} \tag{3.55}$$

and $U(\phi) = -R^2(\xi)$ satisfies (3.11) with $K_1 = K_2 = -\frac{K}{2} + 4S^2 + 2a^2, K_3 = -4S^2$. For $a = 0$ the equation again reduces to (3.41) .

A real solution is obtained only for $K > 8S^2$, namely (3.16) . It is periodic in ϕ and hence singlevalued when μ is an integer.

For $S = 0$ we get a real solution for $K_1 < 0$, namely solution (3.25) which in this case reduces to

$$U(\phi) = -\tan^2\frac{1}{2}\sqrt{|K_1|}(\phi - \phi_0). \tag{3.56}$$

This is a singlevalued function whenever $\sqrt{|K_1|}$ is an integer.

6. Algebras $A_{2,7}$ and $A_{2,8}$

The corresponding reductions lead to equations that we cannot decouple without introducing higher derivatives, so we will not treat them here.

4. Solutions of the Nonlinear Diffusion Equation

4.1. General Procedure

Let us consider the system (1.7) and (1.8) , the NDLE for short. We impose that the solution be invariant under one of the Lie groups generated by the algebra in eq. (2.7) . The functions R and Q will then have the form (3.1) with ξ and β as in Section 3 (different for each subalgebra $A_{2,1},\ldots,A_{2,8}$).

As for the LL equation, algebra $A_{2,1}$ must be treated separately.

For $A_{2,2},\ldots,A_{2,8}$ we always have $(\nabla\xi)^2 \neq 0$. Eq. (1.7) and (1.8) reduce to

$$\alpha_{\xi\xi} = -2\frac{(1-R^2)}{1+R^2}\frac{R_\xi}{R}\alpha_\xi + \frac{f_\xi}{f}\alpha_\xi - 2fm\frac{(1-R^2)}{1+R^2}\frac{R_\xi}{R} + hf \tag{4.1}$$

$$R_{\xi\xi} = \frac{f_\xi}{f}R_\xi + \frac{2R}{1+R^2}R_\xi^2 + \frac{1-R^2}{1+R^2}R\left[\alpha_\xi^2 + 2mf\alpha_\xi + g^2\right] + \frac{1}{(\nabla\xi)^2}\left[BR + AR\frac{1-R^2}{1+R^2}\right] \tag{4.2}$$

The functions $f(\xi), m(\xi), h(\xi)$ and $g(\xi)$ are defined by the relations

$$\begin{aligned} \frac{f_\xi}{f} &= \frac{\xi_t - \Delta\xi}{(\nabla\xi)^2}, \quad & m &= \frac{(\nabla\xi,\nabla\beta)}{(\nabla\xi)^2 f} \\ h &= \frac{\beta_t - \Delta\beta}{(\nabla\xi)^2 f}, \quad & g^2 &= \frac{(\nabla\beta)^2}{(\nabla\xi)^2} \end{aligned} \tag{4.3}$$

In order to decouple equations (4.1) and (4.2) , we impose a restriction on the functions defined in (4.3) namely

$$m_\xi + h = 0 \tag{4.4}$$

Eq. (4.1) can then be integrated once to give

$$\alpha_\xi = \left[S\frac{(1+R^2)^2}{R^2} - m\right]f(\xi) \tag{4.5}$$

where S is an integration constant. We substitute (4.5) into (4.2) and put

$$R(\xi) = \sqrt{-U(\eta)}, \qquad \eta = \int f(\xi) d\xi. \tag{4.6}$$

The equations are decoupled and the one for $U(\eta)$ is already in a standard form[20] namely

$$U_{\eta\eta} = (\frac{1}{2U} + \frac{1}{U-1})U_\eta^2 + 2S^2\frac{(1+U)(1-U)^3}{U} + M\frac{1+U}{1-U}U + NU \tag{4.7}$$

with

$$\begin{aligned} M =& \frac{2}{f^2}(g^2 - m^2 f^2 + \frac{A}{(\nabla\xi)^2}) \\ N =& \frac{2}{f^2}\frac{B}{(\nabla\xi)^2}. \end{aligned} \tag{4.8}$$

We now make a further restriction, namely, that M and N, defined in (4.8) are constants. Eq. (4.7) then has a first integral K and we obtain the elliptic equation (3.10) with

$$\begin{aligned} K_1 =& -\frac{K}{2} + 4S^2 + M + N \\ K_2 =& -\frac{K}{2} + 4S^2 + M - N \\ K_3 =& -4S^2. \end{aligned} \tag{4.9}$$

In many cases we have $M = N = 0$ and the polynomial on the right hand side of (3.10) has a double root at $U_3 = U_4 = 1$. The solution we obtain for $S \neq 0$ is (3.16) with $U_4 = 1$ ie:

$$\begin{aligned} U(\eta) =& 1 - \frac{(1 - U_2)(1 - U_1)}{(U_2 - U_1)\sin^2 \mu(\eta - \eta_0) + 1 - U_2} \\ \mu =& S\sqrt{(1 - U_2)(1 - U_1)}, \quad U_1 \leq U \leq U_2 < 0 \end{aligned} \tag{4.10}$$

For $S = 0$, $K > 0$ we obtain solution (3.25) i.e

$$U = -\tan^2 \sqrt{\frac{K}{8}}(\eta - \eta_0) \tag{4.11}$$

4.2. INDIVIDUAL REDUCTIONS

1. Algebra $A_{2,1}$

We have

$$R = R(t), \qquad Q = \alpha_0 + ax + by \tag{4.12}$$

where α_0 is a constant and R_t satisfies:

$$R_t = -\frac{R(1-R^2)}{1+R^2}(A + a^2 + b^2) - BR \tag{4.13}$$

Equation (4.13) can easily be integrated (differently depending on whether $(A + B + a^2 + b^2)(A - B + a^2 + b^2)$ vanishes, or not) and we obtain a transcendental equation for $R(t)$.

2. Algebra $A_{2,2}$

The reduction formula is (3.37) . We have $m = 0$ and (4.4) requires $b = 0$ so the solutions are static ones. The variable η and constants involved satisfy

$$\begin{aligned} \eta = \ln \rho, \quad M = 2a^2 \\ A = B = N = b = 0 \end{aligned} \tag{4.14}$$

Since we have $b = 0$, the solutions are static ones. All solutions (3.12) , . . . , (3.30) can occur.

3. Algebra $A_{2,3}$

We have

$$\eta = x, \quad M = 2(b^2 + A), \quad N = 2B, \quad a = 0 \tag{4.15}$$

and again, the solutions are static ones, since the reduction formula is (3.44). All solutions of Section 3.2 can occur.

4. Algebra $A_{2,4}$

   The reduction formula is (3.44) and we have

$$\begin{aligned} \eta =& \frac{1}{v}(e^{v(x+vt)} - 1), \quad a = 0, \quad A = -b^2, \quad B = 0 \\ M =& N = 0, \quad v \neq 0 \end{aligned} \tag{4.16}$$

   The obtained solutions are (4.10) and (4.11) and they are t-dependent.

5. Algebra $A_{2,5}$

   We have eq. (3.48) with

$$\begin{aligned} \eta = \xi = & \ln \rho + \frac{1}{b}\phi, \quad A = B = N = 0, \\ M = & \frac{2a^2b^2}{(b^2+1)^2}, \quad b \neq 0 \end{aligned} \tag{4.17}$$

   Since we have $M \geq 0$ we obtain the solutions (3.16), (3.18), (3.20), (3.22), (3.23), (3.26), (3.27) and (3.30). The parameters of these solutions must satisfy, however, certain constraints in order for the solutions to be single-valued. For $K = 2M$ we obtain

$$U = -R_0^2 \rho^{\frac{2|ab|}{b^2+1}} e^{\frac{2|a|}{b^2+1}\phi} \tag{4.18}$$

   a solution that is not singlevalued.

6. Algebra $A_{2,6}$

   The reduction formula is (3.54) and we have

$$\eta = \int \frac{d\xi}{1+\xi^2} = \phi, \quad A = B = a = M = N = 0 \tag{4.19}$$

   so the relevant solutions are (4.10) and (4.11) The solutions are static and they are singlevalued for μ or $\sqrt{K/8}$ being integers.

7. Algebra $A_{2,7}$

   We put

$$
\begin{aligned}
W =& R(\xi) \exp i[\alpha(\xi) + \frac{a}{2} \ln t + b\phi] \\
\xi =& \frac{x^2 + y^2}{t}, \quad \eta = \int \frac{1}{\xi} e^{-\frac{\xi}{4}} d\xi = Ei(-\frac{1}{4}\xi)
\end{aligned}
\tag{4.20}
$$

   where $Ei(x)$ is the exponential integral function. Moreover, we have $b = a = A = B = M = N = 0$ and the relevant solutions are (4.10) and (4.11)

8. Algebra $A_{2,8}$

   We have

$$
\begin{aligned}
W =& R(\xi) \exp i[\alpha(\xi) + \frac{a}{2} \ln t], \quad \xi = \frac{x}{\sqrt{t}} \\
\eta =& \int e^{-\xi^2/4} d\xi = \sqrt{\pi}\Phi(\xi)
\end{aligned}
\tag{4.21}
$$

   where $\Phi(x)$ is the probability integral. We have $M = N = A = B = a = 0$ and so we obtain solutions (4.10) and (4.11)

We see that time-dependent solutions are obtained for the algebras $A_{2,1}$, $A_{2,4}$, $A_{2,7}$ and $A_{2,8}$. For $A_{2,5}, A_{2,7}$ and $A_{2,8}$ the solutions are trigonometric ones.

The phases $\alpha(\xi)$ can be calculated by direct integration, since we have

$$
\frac{d\alpha}{d\eta} = -S\frac{(1-U)^2}{U} \tag{4.22}
$$

and U is already known.

Thus for U given by (4.10) we get

$$
\begin{aligned}
\alpha(\eta) = & \sqrt{\frac{(U_1 - 1)(U_2 - 1)}{U_1 U_2}} \arctan\left\{\sqrt{\frac{U_2(U_1 - 1)}{U_1(U_2 - 1)}} \cot \mu(\eta - \eta_0)\right\} \\
& - \sqrt{(U_1 - 1)(U_2 - 1)} \arctan\left\{\sqrt{\frac{(U_1 - 1)}{(U_2 - 1)}} \cot \mu(\eta - \eta_0)\right\} + \alpha_0,
\end{aligned}
\tag{4.23}
$$

while for U given by (4.11) we have

$$\alpha = 4\sqrt{\frac{2}{K}}S\,\cot\{\sqrt{2K}(\eta - \eta_0)\} + \alpha_0. \tag{4.24}$$

5. Conclusions

The Landau-Lifshitz equation (1.1)has received quite a bit of previous attention, mainly in the context of continuum Heisenberg ferromagnetic spin systems[21] [22] [23] [24] [25] [26] [27]. The anisotropy coefficients A and B of eq. (1.3)were usually set equal to zero. Use was made of the fact that eq (1.1)is integrable, at least in the one-dimensional case, or in the two-dimensional, spherically symmetric one.

A one-soliton solution has been obtained[23] [25]a radially symmetric one. In our variables R, Q of eq. (1.5), (1.6)this solution corresponds to

$$R = \frac{4t + \alpha_1}{4t + \alpha_1 + \cosh^2\left[\frac{4t+\alpha_1}{(4t+\alpha_1)^2+\alpha_2^2}(x^2 + y^2)\right]} \tag{5.1}$$

where α_1 and α_2 are arbitrary real constants. As noted by Lakshamanan and Porsezian[25]the soliton spreads in time.

The solution (5.1) is not among the invariant solutions obtained in this article, nor can it be obtained from such a solution by applying transformations from the symmetry group. As often happens[28], the method of symmetry reduction that does not rely on integrability, provides different solutions for integrable equations, than the use of Lax pairs, or Backlund transformations.

We note that eq. (1.1) with $A \neq 0$ is not integrable.

In general, we have reduced the LL equation to the ordinary differential equation (3.9). We have integrated eq. (3.9) in terms of elliptic functions whenever p, q and m are constants. For the algebras $A_{2,3}, \ldots, A_{2,6}$ this was always the case.

For algebra $A_{2,2}$ we obtained eq. (3.10) only for $A = 0, B = b$. Let us briefly consider the case when the anisotropy coefficient A does not vanish. We then return to the original variable $\xi = \rho = \sqrt{x^2 + y^2} = exp\eta$ and transform eq. (3.9) into

$$U_{\rho\rho} = (\frac{1}{2U} + \frac{1}{U-1})U_\rho^2 - \frac{1}{\rho}U_\rho + \frac{2S^2}{\rho^2}(U-1)^2(-U + \frac{1}{U}) - \frac{2(a^2 + A^2\rho^2)}{\rho^2} + 2(B-b) \tag{5.2}$$

For $a = 0, b = B$ this is the equation for the fifth Painlevé transcendent P_V [20]. However, for $B \neq b$ eq (5.2) does not have the Painlevé property. According to the Painlevé conjecture[29] [30], eq. (1.1) is hence, in general, not integrable.

This has not stopped us from obtaining numerous solutions, both in integrable and nonintegrable cases. The algebra $A_{2,2}$ (cylindrical symmetry) for $A = 0, B = b$ leads to periodic solutions, as discussed in Section 3.3. The periodicity is in the radial variable ρ. The time dependence is restricted to the phase Q, as is seen in eq. (3.37). Moreover the time-dependence is entirely due to the presence of the external field B (we have $b = B$) that generates a rotation between the components ϕ_1 and ϕ_2 of the original vector $\vec{\phi}$.

Some elementary nonperiodic solutions that we can extract from Section 3 are

$$R^2 = \frac{U_1 - U_2 S^2(U_2 - U_1(ln\rho/\rho_0)^2}{1 + S^2(U_2 - U_1)^2(ln\rho/\rho_0)^2} \tag{5.3}$$

$$R^2 = \frac{4R_0^2 U_1(U_2 - U_1) - U_2(U_4 - U_1)[2R_0^2 + \rho^{2\mu} + R_0^4\rho^{-2\mu}]}{(U_4 - U_1)[2R_0^2 + \rho^{2\mu} + R_0^4\rho^{-2\mu}] - 4(U_2 - U_1)R_0^2} \tag{5.4}$$

with S, R_0, ρ_0, U_i constants and

$$Q = S\int \frac{(1+R^2)^2}{\rho R^2} d\rho + a\phi + Bt \tag{5.5}$$

in both cases.

For $S = 0$ we have for instance

$$R = [-U_1 + \sqrt{\frac{2}{-K_1}}(ln\frac{\rho}{\rho_0})^{-1}]^{1/2} \tag{5.6}$$

$$R = \frac{2\sqrt{-U_1}R_0}{\rho^{\mu} + R_0^2\rho^{-\mu}} \tag{5.7}$$

with

$$Q = a\phi + Bt + Q_0 \tag{5.8}$$

For $A \neq 0$, as mentioned above , solutions are obtained in terms of $P_V(\rho)$. Their time dependence is again given by the term Bt in the phase Q.

For algebras $A_{2,3}$ and $A_{2,4}$ we obtain eq. (3.10)and a multitude of explicit solutions for all values of a, b, A and B . Note that for

$$S \neq 0, \quad B = a, \quad A = -b^2 \tag{5.9}$$

in particular for the one dimensional ($b = 0$), static ($a = 0$) with no external fields ($A = B = 0$), two of the roots in eq. (3.11)coincide and the equation reduces to

$$U_\eta^2 = -4S^2(U-1)^2[U^2 + (-\frac{K_1}{4S^2} + 2)U - \frac{K_3}{4S^2}] \tag{5.10}$$

Eq. (5.10) only allows elementary solutions like (3.12), ...(3.17), not however the elliptic function ones. These occur when the fields A and B are such that (5.9) is not satisfied.

To our knowledge, the NLDE (1.2) has not been investigated from the point of view of its integrability and we have no solutions to compare ours to.

We have derived many explicit exact solutions of both equations. Looking at them we note that most of them have infinite energy. They can describe coherent phenomena in various solid state and condensed matter applications.

Looking first at the solutions of the LL equation we note that some of our solutions have finite energy. In particular, this is the case for (3.43). This solution is obtained from the familiar static solution describing n solitons "on top of each other" [31]. Its time dependence is given by the factor e^{iBt} which thus describes a rotation of this static solution in the ϕ_1, ϕ_2 plane with the angular frequency given by the anisotropy B. The other solutions of this class correspond to the static elliptic solutions discussed in ref [17] again rotated by e^{iBt}.

The solutions corresponding to algebras $A_{2,1}$, $A_{2,3}$ and $A_{2,4}$ have infinite energies. As such, they describe various waves in the medium (generalizations of plane waves). These can for instance be spin waves; the energy per period is finite.

An interesting class of solutions are those corresponding to algebras $A_{2,5}$ and $A_{2,6}$. Given the choice of parameters, these solutions can be of finite energy; however, due to their dependence on the variable $\phi = \arctan \frac{y}{x}$, they may become singular when x and y vanish. They can be used to describe media with defects.

Most of the comments made above apply also to the solutions of the NLDE. The static solutions in both cases are of course the same. When we consider non-static solutions, the most interesting from the physical point of view, are perhaps solutions corresponding to algebras $A_{2,1}$, $A_{2,7}$ and $A_{2,8}$. All of them have infinite energies. The solution corresponding to $A_{2,1}$ describes a structure shrinking towards the origin (or expanding to infinity - depending on the values of the parameters). The other solutions describe field configurations evolving in time. They can be used in the description of some physical phenomena in condensed matter or solid state physics.

Among the questions that we plan to return to, we mention the study of "partially invariant" solutions[32] [33] [34] of eq. (1.1)and (1.2), and also "conditionally invariant" ones[35]. A study of solutions involving Painlevé transcendents is also warranted.

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