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Bose-Einstein Condensation and Superfluidity of Excitons in Semiconductors

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Abstract

In this thesis, we study the possibility for excitons to form a highly correlated state which can be associated with Bose-Einstein Condensation and has finite extent in space. In particular, we study solitonic mechanisms of exciton superfluidity. We provide a theoretical explanation of recent experiments on the propagation of exciton packets in semiconductors. In these experiments, the excitonic transport under the action of a laser pulse has been studied. It turned out that under certain conditions this transport becomes anomalous and the excitons propagate through the crystal in a wave packet without diffusion. We propose a model for this phenomenon which relies on the presence of an exciton-phonon interaction and the formation of exciton-phonon condensate. In this model, the subsonic exciton propagation is described by soliton solutions of the nonlinear Schrödinger equation. The theory predicts two critical velocities for propagation of the packet, and this is in a good qualitative agreement with experimental data. In addition, we explain the results of experiments on strong nonlinear interactions between moving excitonic packets by introducing the exciton-phonon droplets with Bose-cores inside them. Such cores are characterized by a finite correlation length and can be considered as a kind of Bose-Einstein condensate.

We also study a model of a nonideal Bose-gas moving in a channel. It is known that the vortex model of superfluid dissipation cannot predict correctly the value of the critical velocity of superfluidity in planar geometry. We show that the existence of superfluidity in the Bose-gas can depend on the strength of the boundary interactions with channel walls. Indeed, if the dilute moving Bose-gas interacts with the walls via hard-core repulsion, boundary (boson-phonon) excitations can be introduced. They can reduce the value of the critical velocity of the superfluid. Such surface modes seem to exist in "soft matter" containers with flexible walls; they can be one of the sources of friction in anomalous excitonic transport in semiconductor heterostructures as well.

Keywords:

Bose-Einstein condensation, superfluidity, condensate, elementary excitations, mean field approximation, Bogoliubov transform, critical velocity, coherence, exciton, cuprous oxide, paraexcitons, excitonic transport, Davydov soliton, exciton-exciton interaction, exciton-phonon interaction, nonlinear Schrödinger equation, nonlinear wave equation, bright soliton, stability.

Résumé

Dans cette thèse, nous étudions la condensation de Bose-Einstein (CBE) des excitons dans les semi-conducteurs. Contrairement aux gaz atomiques dilués dans un piège, la recherche concernant la CBE des excitons et des excitons-polaritons est encore en cours. Il est généralement admis que l'état extrêmement corrélé qui correspond à la condensation de Bose-Einstein peut être obtenu. Toutefois, contrairement à la condensation de Bose-Einstein conventionnelle dans l'espace des moments, un condensat pourrait avoir une étendue finie dans l'espace réel et, de plus, se déplacer dans un cristal. Les paquets en mouvement ont un avantage évident: l'exciton ne peut pas être converti en un photon directement, et donc il pourrait y avoir un temps suffisant pour développer la cohérence de Bose-Einstein dans le gaz excitonique. De plus, les techniques expérimentales modernes permettent de prolonger la durée de vie des excitons à une échelle telle qu'ils peuvent parcourir des distances macroscopiques à travers un cristal.

En particulier, nous avons étudié les mécanismes solitoniques de suprafluidité des excitons. Nous présentons une explication théorique des expériences récentes sur la propagation des paquets d'excitons dans les semi-conducteurs. Dans ces expériences, le transport excitonique provoqué par des impulsions laser a été étudié dans des monocristaux de cuprite (Cu_2O). Il se trouve que dans certains cas, ce transport devient anormal et les excitons se propagent à travers le cristal sous forme de paquet d'onde sans diffusion. Les vitesses balistiques enregistrées pour tels paquets excitoniques s'avèrent être toujours moindres, mais relativement proches de la vitesse de son longitudinal dans le cristal, $v < c_s$. De plus, l'interaction entre les paquets indique une sorte d'interférence constructive lorsque les paquets se chevauchent. Il faut mentioner que la forme des tels paquets correspond bien avec la fonction solitonique $\cosh^{-2}(x)$, avec une longue trainée en arrière du soliton.

Nous proposons un modèle de ce phénomène qui est basé sur la présence d'une

interaction exciton-phonon et la formation du condensat correspondant. Afin de comprendre la physique du transport excitonique anormal, nous posons comme hypothèse que la fonction d'onde macroscopique $\Psi_0 \sim \phi_o\, e^{i\varphi_c}$ peut être associée à la partie cohérente du paquet excitonique.

Dans ce modèle, la propagation subsonique des excitons est décrite par les solutions solitoniques de l'équation non-linéaire de Schrödinger.

À la température limite inférieure ($T \to 0$), l'interaction entre le cœur Bosien du paquet et le nuage non-condensé peut être ignorée. Les équations suivantes concernant la fonction d'enveloppe, $\phi_o(x)$, peuvent alors être écrites:

$$-|\mu|\, \phi_o(x) = -(\hbar^2/2m_x)\, \partial_x^2 \phi_o(x) - |\tilde{\nu}_0|\, \phi_o^3(x) + \tilde{\nu}_1\, \phi_o^5(x). \tag{1}$$

Ici, μ est le potential chimique efficace du gaz d'exciton, m_x est la masse d'un exciton, et $\tilde{\nu}_j$, $j = 0, 1$ sont des intensités d'interaction rénormalisés qui proviennent des interactions exciton-exciton et exciton-phonon. La partie cohérente du champs de déplacement, $\mathbf{u}_0 = (u_o,\, 0,\, 0)$, s'écrit alors,

$$\partial_x u_o(x) \approx -\mathrm{const}_0\, \phi_o^2(x) + \mathrm{const}_1\, \phi_o^4(x). \tag{2}$$

Dans l'étape suivante, nous prenons en compte les fluctuations du condensat et introduisons ses excitations élementaires. En restant dans l'approximation semi-classique, nous estimons le spectre d'énergie de ces excitations. En gros, celles-ci peuvent être divisées en des excitations à l'intérieur et à l'extérieur du condensat (trainée).

La théorie prédit deux vitesses critiques, v_o et v_{cr}, qui sont importantes pour comprendre la propagation des paquets, ce qui est en accord qualitatif avec les données expérimentales. La première vitesse critique, v_o, résulte de la renormalisation de l'interaction de deux particules exciton-exciton due aux phonons. Puis, l'état de soliton "clair" de l'équation (1) peut être formé à $v > v_o$. Le plus important paramètre qui contrôle la forme et la largeur caractéristique de la fonction d'onde du condensat est le potentiel chimique adimensionnel, $|\mu|/\mu^* < 1$, avec

$\mu^* = (3/16)\,|\tilde{\nu}_0|^2/\tilde{\nu}_1$. La deuxième vitesse, $v_{\rm cr}$, dérive de l'utilisation des arguments de Landau pour évaluer l'instabilité / stabilité dynamique du condensat en mouvement. Il est possible d'introduire l'échelle d'énergie correspondante, $\mu_{\rm cr}$, et l'émission des excitons est alors controlée par le valeur de $|\mu|/\mu_{\rm cr}$.

En ce qui concerne l'approximation semi-classique pour les excitations du condensat, nous avons trouvé que plus v est proche de $c_{\rm s}$ ($\mu_{\rm cr} < |\mu| < \mu^*$), plus le paquet cohérent est stable. De plus, nous discutons de la possibilité d'observation d'un régime instable dans lequel le condensat peut être formé sous une forme inhomogène avec $v \neq 0$, mais avec $v_{\rm o} < v < v_{\rm cr}$ (ce qui correspond à $|\mu| < \mu_{\rm cr}$). Un tel condensat doit disparaitre pendant son passage à travers un monocristal pur utilisé lors des expériences. Du fait que la forme du paquet en mouvement dépend du temps, la forme du signal enregistré peut dépendre de la longueur du cristal et changer du profil solitonique au profil de densité de diffusion standard. Ainsi, la théorie fournit une description qualitative des observations ainsi que des valeurs raisonnables pour les vitesses critiques, par exemple, $v_{\rm o} \approx (0.5 - 0.6)\,c_{\rm s}$.

En fait, le condensat d'exciton-phonon auto-cohérent représente seulement une partie du paquet réel en mouvement. La partie non-cohérente correspondante, à savoir les excitons non-condensés $\Delta n(x,t)$ et le vent de phonons uni-directionnel $\Delta u(x,t)$, affecte la propagation du condensat. Nous traitons la question de la possibilité de la diminution (et idéalement l'extinction) du vent de phonons après la formation du condensat d'exciton-phonon en mouvement. Une géométrie spécifique du cristal prenant en compte le paquet d'exciton-phonon en mouvement est proposée. En conséquence, nous nous attendons à ce que les excitons non-condensés diffusibles puissent être retardés, et que le signal cohérent et le signal non-cohérent soient séparés dans le temps. De plus, nous expliquons les résultats des expériences sur les interactions fortes non-linéaires entre les paquets excitoniques en mouvement par l'introduction des gouttes d'excitons-phonons comportant des cœurs Bosiens. De tels cœurs sont caractérisés par une longueur de corrélation finie et peuvent être considérés comme une sorte de condensat de

Bose-Einstein.

Pour récapituler, nous mentionnons que les phonons jouent un rôle crucial dans presque tous les modèles actuels qui visent à expliquer ou prédire le comportement cohérent des excitons dans des semi-conducteurs. A l'aide du modèle solitonique du condensat d'excitons-phonons, la largeur des paquets d'excitons-phonons quand $T \to 0$ peut être prédite, mais il faut que le modèle soit généralisé dans le cas où $T \neq 0$. Par exemple, les excitons thermiques (e.g., la trainée faible qui est observée derrière le soliton) et les phonons thermiques du cristal doivent être pris en compte.

Nous avons aussi étudié un modèle de gaz Bosien non-idéal se déplaçant dans un canal. Ne perdons pas de vue que plusieurs "vieux" problèmes dans la Physique de la Matière Condensée, tels que la cinétique de la condensation de Bose-Einstein, la nature de suprafluidité, le problème de la vitesse critique, restent encore des sujets très étudiés. Par exemple, il est bien connu que le modèle de tourbillons de la dissipation suprafluide ne peut pas prédire correctement la valeur de la vitesse critique de suprafluidité en géométrie plane. Ainsi, l'étude des mécanismes de dissipation de la suprafluidité dans des canaux planaires est important.

Nous avons montré que l'existence de suprafluidité du gaz Bosien peut dépendre de l'importance des interactions aux limites avec les parois du canal. Mathématiquement, l'observation-clé est qu'une des catégories de solutions périodiques (à savoir les solutions elliptiques) de l'équation non-linéaire de Schrödinger,

$$\mu\phi_{\mathrm{o}}(x) = -(\hbar^2/2m_{\mathrm{x}})\partial_x^2\phi_{\mathrm{o}}(x) + \nu_0\,\phi_{\mathrm{o}}^3(x), \quad \phi_{\mathrm{o}}(x=0) = \phi_{\mathrm{o}}(x=L) = 0, \tag{3}$$

peut être utilisée pour décrire le comportement des excitations non-homogènes dans le système "condensat suprafluide + parois du canal". Ici, L est la largeur de canal. Une étude détaillée des propriétés analytiques de ces solutions aboutit à une conclusion théorique intéressante: une valeur non-nulle de la vitesse critique dans les systèmes étudiés peut être obtenue seulement au-delà de l'approximation semi-classique. En fait, si le gaz Bosien dilué en mouvement interagit avec les parois via la répulsion de cœur dur, des excitations (bosons-phonons) peuvent être introduites aux limites. Elles peuvent réduire la valeur de la vitesse critique d'un suprafluide. De tels modes de surface semblent exister dans des matériaux en "matière molle" avec des parois flexibles. Ils peuvent aussi être une des sources de friction lors du transport excitonique anormal dans les structures hétérogènes semi-conductrices.

Mots clés:
condensation de Bose-Einstein, superfluidité, vitesse critique, excitations élémentaires, transformation de Bogoliubov, spectre d'énergie, stabilité, dissipation, semi-conducteurs, excitons, cuprite, paraexcitons, transport anormal, condensat stationnaire, longueur de corrélation, interaction exciton-phonon, interaction exciton-exciton, équation non-linéaire de Schrödinger, soliton de Davydov, soliton clair.

Summary

In this thesis, we study the Bose-Einstein condensation (BEC) of excitons in semiconductors. Unlike the dilute atomic gases in a trap, the search for the BEC of excitons and exciton-polaritons is still in progress. It is believed that the highly correlated state which can be associated with Bose-Einstein Condensation can be achieved. However, unlike the conventional Bose-Einstein condensation in momentum space, a condensate may have a finite extent in the real space and, moreover, move in a crystal. The moving packets have an obvious advantage: the exciton cannot be converted into a photon directly, so it could be enough time to develop the Bose-Einstein coherence in the excitonic gas. Moreover, modern experimental technique allow to extend the lifetime of excitons to such a scale that they can pass macroscopic distances through a crystal.

In particular, we study solitonic mechanisms of exciton superfluidity. We provide a theoretical explanation of recent experiments on the propagation of exciton packets in semiconductors. In these experiments, the excitonic transport under the action of a laser pulse has been studied in the cuprous oxide, Cu_2O. It turned out that under certain conditions this transport becomes anomalous, and the excitons propagate through the crystal in a wave packet without diffusion. The registered ballistic velocities of such excitonic packets turn out to be always less, but relatively close to the longitudinal sound speed of the crystal, $v < c_s$. In addition, the interaction between the packets hints at a kind of constructive interference when the packets overlap. Note that the shape of such packets fits well into the solitonic $\cosh^{-2}(x)$ function with the long-lasting tale behind the soliton. We propose a model for this phenomenon which relies on the presence of exciton-phonon interaction and formation of exciton-phonon condensate. To understand the physics of anomalous excitonic transport, we assume that the macroscopic wave function $\Psi_0 \sim \phi_0\, e^{i\varphi_c}$ can be associated with the coherent part of the excitonic packet.

In this model, the subsonic exciton propagation is described by soliton solutions of the nonlinear Schrödinger equation. In the low temperature limit, $T \to 0$, one can disregard the interaction between the Bose-core of the packet and the non-condensate cloud and write down the following equation on the envelope function $\phi_{\mathrm{o}}(x)$:

$$-|\mu|\,\phi_{\mathrm{o}}(x) = -(\hbar^2/2m_{\mathrm{x}})\,\partial_x^2\phi_{\mathrm{o}}(x) - |\tilde{\nu}_0|\,\phi_{\mathrm{o}}^3(x) + \tilde{\nu}_1\,\phi_{\mathrm{o}}^5(x). \tag{1}$$

Here, μ is the effective chemical potential of the exciton gas, m_{x} is the exciton mass, and $\tilde{\nu}_j$, $j = 0, 1$ are the renormalized interaction vertices originated from exciton-exciton and exciton-phonon interactions. The coherent part of the displacement field $\mathbf{u}_0 = (u_{\mathrm{o}},\, 0,\, 0)$ can be written as follows:

$$\partial_x u_{\mathrm{o}}(x) \approx -\mathrm{const}_0\,\phi_{\mathrm{o}}^2(x) + \mathrm{const}_1\,\phi_{\mathrm{o}}^4(x). \tag{2}$$

As a next step, we take into account the fluctuations of the condensate and introduce the elementary excitations of it. Within the semiclassical approximation, we estimate the energy spectrum of these excitations. Roughly, they can be divided into inside- and outside-condensate excitations.

The theory predicts two critical velocities, v_{o} and v_{cr}, which are important to understand the propagation of the packet, and this is in a good qualitative agreement with experimental data. The first critical velocity, v_{o}, comes from the renormalization of two particle exciton-exciton interaction due to phonons. Then, the "bright" soliton state of Eq. (1) can be formed at $v > v_{\mathrm{o}}$. The important parameter that controls the shape and the characteristic width of the condensate wave function is the dimensionless chemical potential, $|\mu|/\mu^* < 1$. Here, $\mu^* = (3/16)\,|\tilde{\nu}_0|^2/\tilde{\nu}_1$. The second velocity, v_{cr}, comes from use of Landau arguments for investigation of the dynamic stability / instability of the moving condensate. It is possible to introduce the corresponding energy scale, μ_{cr}, and the emission of excitations is controlled by the value of $|\mu|/\mu_{\mathrm{cr}}$.

Within the semiclassical approximation for the condensate excitations, we found that more close v is to c_{s} ($\mu_{\mathrm{cr}} < |\mu| < \mu^*$) more stable the coherent packet is. We also discuss the possibility of observation of the instability regime in which the condensate can be formed in the inhomogeneous state with $v \neq 0$, but with $v_o < v < v_{cr}$. (In other words, we have $|\mu| < \mu_{cr}$.) Such a condensate has to disappear during its move through a single pure crystal used for experiments. As the shape of the moving packet depends on time, the form of the registered signal may depend on the crystal length. Then, with the same initial conditions, one can obtain the solitonic profile for a short crystal and the standard diffusion density profile for a relatively long crystal. Thus, the theory yields a qualitative description of the experiments and reasonable values for the critical velocities, e.g., $v_o \approx (0.5 - 0.6)\, c_s$.

In fact, the self-consistent exciton-phonon condensate is only a part of the real moving packet. The noncoherent part of it, namely, the noncondensed excitons $\Delta n(x,t)$ and the unidirectional phonon wind $\Delta u(x,t)$, effects the propagation of the condensate. We address the question on whether it is possible to diminish (ideally, to turn off) the phonon wind after the moving exciton-phonon condensate has been formed. A special geometry of the crystal with the moving exciton-phonon packet is proposed. As a result, we expect that the diffusive noncondensed excitons can be delayed, and the coherent signal and the noncoherent one are separated in time. In addition, we explain the results of experiments on strong nonlinear interaction between moving excitonic packets by introducing the exciton-phonon droplets with Bose-cores inside them. Such cores are characterized by a finite correlation length and can be considered as a kind of the Bose-Einstein condensate.

To sum up, we note that the phonons play a crucial role in almost all the current models aimed to explain or predict coherent behavior of excitons in semiconductors. Within the solitonic model of the exciton-phonon condensate, one can predict the width of the exciton-phonon packet at $T \to 0$, but the model has to be generalized to the case of $T \neq 0$. For example, one has to take into account the thermal excitons (e.g., the weak tail that is always observed behind the soliton) and the thermal phonons of the crystal.

We also study a model of a nonideal Bose-gas moving in a channel. Recall that some "old" problems in Condensed Matter Physics – such as, the kinetics of Bose-Einstein condensation, the nature of superfluidity, the critical velocity problem, etc. – still remain the subject under consideration. For example, it is known that the vortex model of superfluid dissipation cannot predict correctly the value of the critical velocity of superfluidity in planar geometry. Therefore, the study of dissipation mechanisms of superfluidity in planar channels is important.

We show that the existence of superfluidity of the Bose-gas can depend on the strength of boundary interactions with channel walls. Mathematically, the key observation is that a certain class of periodic (so-called *elliptic*) solutions of the nonlinear Schrödinger equation,

$$\mu\phi_{\rm o}(x) = -(\hbar^2/2m_{\rm x})\partial_x^2\phi_{\rm o}(x) + \nu_0\,\phi_{\rm o}^3(x), \quad \phi_{\rm o}(x=0) = \phi_{\rm o}(x=L) = 0, \qquad (3)$$

can be employed to describe the behavior of inhomogeneous excitations in the system *Superfluid Condensate* + *Channel Walls*. Here, L is the width of the channel. A detailed study of analytic properties of these solutions then leads to an interesting theoretical conclusion: a nonzero value of the critical velocity in systems under consideration can be obtained only beyond the semiclassical approximation. Indeed, if the dilute moving Bose-gas interacts with the walls via hard-core repulsion, boundary (boson-phonon) excitations can be introduced. They can reduce the value of the critical velocity of a superfluid. Such surface modes seem to exist in "soft matter" containers with flexible walls; they can be one of the sources of friction in anomalous excitonic transport in semiconductor heterostructures as well.

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Introduction

Six articles published during the years 1997-2000 are included into this thesis. All of them are devoted to the problem of Bose-Einstein condensation (BEC). Recently, this correlated state of matter was achieved in dilute Bose-gases (e.g., Li, Rb) contained within artificially prepared traps [1]. In the context of Solid State Physics, the search for the BEC of excitons and exciton-polaritons in semiconductors is still in progress.

At near band-gap excitations of a semiconductor and small electric fields, excitons can be introduced as the quanta of the induced polarization field [2], and, roughly, they are boson-like excitations of a crystal. Nowadays, there is a lot of experimental evidence that excitons in semiconducting crystals and heterostructures can form strongly correlated states. In some cases, they can be assigned to the excitonic Bose-Einstein condensate (BEC) [3]. As a rule, the excitons are created by the electromagnetic radiation (e.g., laser light), and we expect that the cloud of excitons will reach the ground state with the zero average quasi-momentum, $\langle \hbar\, \mathbf{k} \rangle \simeq 0$. (In many cases, they are in a quasi-equilibrium state with some effective temperature $T^*(t)$ cooling down toward the crystal temperature T.) In the low density limit, it is a good approximation to consider the excitons as a dilute Bose-gas [4]. In the high density limit, the fermionic nature of the exciton can be important but this case is not discussed in this thesis.

The optically active excitons are unstable against conversion into photons, i.e., such excitons have a finite life time. Usually, the state of Bose-Einstein condensation is looked among these excitons. However, the time scale necessary for the well-defined coherent phase to be developed in the stationary cloud with the average concentration larger than the critical one (i.e., $n_{\mathrm{x}} > n_c$ and the temperature $T^* = T < T_c$, and $n_{\mathrm{x}}(t) = \mathrm{const}$) seems to be strongly renormalized in the case of the non-stationary cloud [5],[6]. In the later case, we have $n_{\mathrm{x}}(t = 0) = n_0 > n_c$

and the number of photons $n_{\rm ph}(t=0) \approx 0$, but $\partial_t n_{\rm x}(t) < 0$ and $\partial_t n_{\rm ph}(t) > 0$. If we consider the excitons as an ideal three dimensional Bose-gas, the critical concentration n_c and temperature T_c of the Bose-Einstein condensation are related to each other as follows [7]:

$$(\hbar^2/2\,m_{\rm x})\,n_c^{2/3} \simeq T_c.$$

However, the Bose-Einstein condensation of interacting bosons immersed into a crystal lattice can be different from the conventional BEC of a dilute Bose-gas.

Recall that one can think of the Bose-Einstein condensation as the existence of a macroscopically occupied state in the following decomposition of the density matrix of the system [8]:

$$\rho(\mathbf{x},\,\mathbf{x}',\,t) = \langle \hat{\psi}^\dagger(\mathbf{x},t)\,\hat{\psi}(\mathbf{x}',t)\rangle = \sum_j n_j(t)\,\xi_j^*(\mathbf{x},t)\,\xi_j(\mathbf{x}',t). \qquad (1)$$

Here, $\hat{\psi}^\dagger(\mathbf{x})$ is the creation operator of a boson of the Bose-gas of $N_{\rm x}$ particles,

$$\Big\langle \int d\mathbf{x}\,\hat{\psi}^\dagger(\mathbf{x},t)\,\hat{\psi}(\mathbf{x},t)\Big\rangle = N_{\rm x},$$

and $\xi_j(\mathbf{x},t)$ is the wave function of the one-particle basis set $\{\,\xi_j\,\}$, and $n_j(t)$ is the corresponding occupation number. The standard commutation relations are

$$[\,\hat{\psi}(\mathbf{x},t),\,\hat{\psi}^\dagger(\mathbf{x}',t)\,] = \delta(\mathbf{x}-\mathbf{x}') \quad \text{and} \quad [\,\hat{\psi}(\mathbf{x},t),\,\hat{\psi}(\mathbf{x}',t)\,] = 0.$$

In the case of $T=0$, the brackets $\langle\cdots\rangle$ mean the ground state average, and, in the case of $T \neq 0$, they imply the standard temperature average,

$$\mathrm{Tr}\,(\cdots\exp(-\hat{H}/T))\,/\,\mathrm{Tr}\,\exp(-\hat{H}/T).$$

In the non-equilibrium case, the brackets mean averaging the Heisenberg operators inside them with the initial density matrix of the Bose-gas, $\mathrm{Tr}(\cdots\hat{\rho})/\mathrm{Tr}\,\hat{\rho}$.

If $n_{j=0}(t) \simeq N_{\rm x} \gg 1$ and all the other $n_j(t) \simeq 1$, see Eq. (1), one can call the corresponding $\xi_{j=0}(\mathbf{x},t)$ the macroscopic wave function of the system. It is better, however, to renormalize $\xi_0(\mathbf{x},t)$ $\;(\int d\mathbf{x}\,|\xi_0(\mathbf{x},t)|^2 = 1)\;$ by introducing another wave function, namely, $\Psi_0(\mathbf{x}, t) = \sqrt{N_0(t)}\, \xi_0(\mathbf{x}, t)$. Here, $\int d\mathbf{x}\, |\Psi_0(\mathbf{x}, t)|^2 = N_0(t)$ and $N_0(t)$ is the condensate occupation number. Then, we can rewrite the density matrix as follows

$$\rho(\mathbf{x},\, \mathbf{x}',\, t) = \Psi_0^*(\mathbf{x}, t)\, \Psi_0(\mathbf{x}', t) + \sum_{j\neq 0} n_j(t)\, \xi_j^*(\mathbf{x}, t)\, \xi_j(\mathbf{x}', t). \tag{2}$$

Usually, the conclusion about the presence of Bose-Einstein correlations among the excitons (actually, a low-energy problem within the scale of the exciton Rydberg energy) is based on unusual properties of the direct photoluminescence signal (high-energy photons) from the excitonic cloud [3]. However, such an interpretation is not always convincing, and the non-equilibrium Bose-Einstein condensation is far from being completely understood [8].

Almost all the articles included into this thesis are motivated by experimental data on the transport properties of excitons that *move* in 3D crystals, such as the cuprous oxide Cu_2O [9], or 2D sheets in BiI_3 [10], or exciton-polaritons in a semiconductor microcavity [11]. The moving packets have an obvious advantage: the exciton cannot be converted into a photon directly, so it could be enough time to develop the Bose-Einstein coherence in the excitonic gas. Moreover, modern experimental technique allow to extend the lifetime of excitons to such a scale that they can pass macroscopic distances through the crystal. Therefore, instead of unusual photoluminescence properties, one can expect unusual transport properties of the correlated excitons at $T < T_c$. Then, *spatial* characteristics of a packet and its stability, and details of the correlation properties of the excitons are among the subjects to investigate.

It is important to prepare the following initial conditions: a relatively dense cloud of excitons with the density of $n_{\mathrm{x}} > n_c(T)$ to be in a moving state with $\langle \hbar\, \mathbf{k}_{\mathrm{x}} \rangle \neq 0$. (For cuprous oxide, $n_c(T = 2\,\mathrm{K}) \approx 8.7 \times 10^{16}\,\mathrm{cm}^{-3}$ and the gas parameter $n_c\, a_{\mathrm{x}}^3 \ll 1, \quad a_{\mathrm{x}} \simeq 6 - 7\,\text{Å}$.) For example, one can expect non-trivial physical effects if

$$\langle \hbar\, \mathbf{k}_{\mathrm{x}} \rangle \simeq m_{\mathrm{x}}\, c_s,$$

where c_s is the sound speed. In the case of Cu_2O crystals, these conditions can be achieved because there is approximately the same number of long wavelength acoustic phonons being formed in the same place as the excitons, $N_{ph} \simeq N_x$ and $\varepsilon_{ph} \simeq 1 - 5\,\mathrm{meV}$. Due to the exciton-phonon interaction and (important!) quasi-one dimensional geometry of the initial conditions, the acoustic phonons and excitons can form a packet with the nonzero average quasi-momenta, $\langle \hbar\, \mathbf{k}_{ph} \rangle \neq 0$ and $\langle \hbar\, \mathbf{k}_x \rangle \neq 0$, so that

$$\langle \hbar\, \mathbf{k}_x \rangle \parallel \langle \hbar\, \mathbf{k}_{ph} \rangle \parallel Ox \tag{3}$$

and $\varepsilon_x = m_x v^2/2 < m_x c_s^2/2 \sim 10^{-3}\, E_x$. Here, $c_s = 4.5 \times 10^5\,\mathrm{cm/s}$ is the (longitudinal) speed of sound, and $E_x \simeq 0.15\,\mathrm{eV}$ is the exciton Rydberg, and $m_x \simeq 1.5\, m_e$ is the exciton mass. Note that if the excitation area is smaller (usually, much smaller) than the surface area of the crystal, Eq. (3) is not valid. A classical analog of the quasi-one dimensional initial conditions could be the excitation of solitons on the shallow water surface in pipes and channels used in laboratory experiments on the surface waves. The simplest theoretical model to describe such a case is the (1D) KdV equation [12].

In recent experiments on excitons an interesting phenomenon has been observed: the excitons created under the action of a laser pulse form a quasi-one dimensional soliton-like packet and propagate without diffusion through a crystal when the intensity of pulse is high enough or the temperature of the crystal is low enough, see Figs. 1, 2, and 3. It seems that the exciton-phonon interaction is crucial to understand this phenomenon.

We assume that this is a critical phenomenon, namely, a Bose-correlated exciton-phonon core can be formed inside the exciton-phonon packet under these conditions. Then, such a packet can move ballistically through the whole crystal at $T < T_c$, and the coherence of the Bose-core can be revealed by the packet-packet interaction or by stimulated scattering into the packet with the condensate. (And such experiments were done with the paraexcitons in cuprous oxide, see [13] for a detailed account.) If $T > T_c$ or $n_x < n_c(T)$, the exciton-phonon packet exhibits the standard diffusive behavior [9]. Thus, the 3D droplets that could contain the excitonic Bose-Einstein condensate are found in a spatially inhomogeneous state with the well-defined characteristic width $L_{\rm ch}$ in the direction of motion. Note that this width $L_{\rm ch}$ can be different from the Bose-Einstein correlation length of the Bose-core L_0. (For example, $\langle \psi^{\dagger}(x,t)\, \psi(x',t)\rangle \approx 0$ at $|x-x'| > L_0$ and $L_0 < L_{\rm ch}$). The estimates $L_0 \sim 10^{-1}\, L_{\rm ch}$ and, for the corresponding duration in time,

$$\tau_0 \simeq 50 - 60\,\mathrm{ns} \sim 10^{-1}\,\tau_{\rm ch}, \quad \tau_{\rm ch} \simeq 0.2 - 0.4\,\mu\mathrm{s}, \tag{4}$$

can be extracted from the experiments on two packet interaction at $T < T_c$ [9]. Here, we used the simple formulas, $L_{\rm ch} = c_{\rm s}\,\tau_{\rm ch}$ and $L_0 = c_{\rm s}\,\tau_0$, to introduce the characteristic length and time scales. Indeed, one can expect strong nonlinear interaction between the two co-moving packets if the distance between them is $x_{0,1} - x_{0,2} \leq L_0$. Meanwhile, the interaction is week if $x_{0,1} - x_{0,2} \simeq L_{\rm ch}$ although the packets are well overlapped.

The registered ballistic velocities of such excitonic packets turn out to be always less, but relatively close to the longitudinal sound speed of the crystal, $v < c_{\rm s}$. Note that the paraexcitons in pure Cu_2O crystals have an extremely large lifetime, $\tau \gg 13\,\mu$s, and a moving exciton with $\hbar k_x \sim m_{\rm x}\, c_{\rm s}$ cannot be converted into a photon directly. Then, one can exclude the photons from simple models describing the transport of a *single* packet of excitons in a periodic medium and the interaction between two packets in the comoving regime. Moreover, we neglect the ortho-para exciton conversion inside the *formed* packet of the moving paraexcitons in Cu_2O crystals. We assume that all the orthoexcitons $(1S)$ are down-converted into the paraexcitons $(J = 0)$ during the initial stage of the packet formation. This means $N_{\rm x,\,para} \simeq \mathrm{const} > N_c$, $N_{\rm x,\,ortho} \simeq 0$, for discussion see [14]. Thus, details of the inner structure of an excitons are not important in almost all the problems discussed in this thesis. That is why we employ the creation / destruction operators $\hat{\psi}^{\dagger}(\mathbf{x})$, $\hat{\psi}(\mathbf{x})$ in the real space without any inner degrees of freedom. (Roughly, $\mathbf{x}$ is the coordinate of the center-of-mass of an exciton taken as a structureless boson.)

Note that it is common and certainly convenient to treat the gas of excitons and lattice phonons by use of the language of Bose creation and annihilation operators in the $\mathbf{k}$-space from the beginning, i.e. from writing out the corresponding Hamiltonian(s), see [15] and [5] for a review. Then, the macroscopically occupied mode is chosen to be (roughly) the $\mathbf{k} = 0$ state [16],[17], and the exciton and phonon operators are normalized to the same volume V equal to the volume of the crystal. Here, we mention some works which do not follow this standard way. For example, V. Gergel' and the co-workers [18] described the Bose-condensed *repulsive* excitons by the nonlinear Schrödinger equation (NLS) with an effective friction term. (As the excitons have a finite life time, the condensate decays.) They discuss whether the superfluidity of such excitons is possible. However, E. Hanamura [19] proposed that the careful investigation of the exciton-exciton interaction can lead to an effective attraction, and the sign of an effective interaction term in the NLS can be negative. Later, A. R. Vasconcellos et al. [20] generalized the above-mentioned works to investigate the dynamics of coherent exciton packets.

To understand the physics of anomalous excitonic transport, we also assume that the macroscopic wave function $\Psi_0(\mathbf{x},t) \simeq \phi_{\rm o}\, e^{i\varphi_{\rm c}}$ can be associated with the coherent part of the excitonic packet at $T < T_c$. Here, $\varphi_{\rm c}$ is the coherent phase of the condensate. Indeed, the experimental results [9],[10] suggest the following decomposition of the density of excitons in the packet:

$$n(\mathbf{x},t) = n_{\rm coh}(\mathbf{x},t) + \Delta n(\mathbf{x},t), \tag{5}$$

where $n_{\rm coh}(\mathbf{x},t) \approx n_{\rm core}(x - vt)$ is the ballistic (superfluid) part of the packet,

$$n_{\rm core}(x - vt) \approx |\Psi_0|^2(x - vt) = \phi_{\rm o}^2(x - vt), \tag{6}$$

and $\Delta n(\mathbf{x},t)$ is the non-condensed part of it.

The following decomposition can be written for the out-of-condensate part:

$$\Delta n(\mathbf{x},t) = \langle \delta\hat{\psi}^\dagger \delta\hat{\psi}(\mathbf{x},t)\rangle \approx \delta n_{\rm cloud}(\mathbf{x},t) + \delta n_{\rm tail}(\mathbf{x},t). \tag{7}$$

The challenging problem is how to describe the spatially inhomogeneous state of the moving droplet with the excitonic BEC inside in terms of $\Psi_0(\mathbf{x}, t)$ and $\delta\hat{\psi}(\mathbf{x}, t)$, where $\delta\hat{\psi}$ is the "fluctuating" part of the exciton Bose-field. For example, within the quasistationary approximation and in the comoving frame, one has to calculate

$$n_{\rm core}(x) = \phi_{\rm o}^2(x/L_0) \quad \text{and} \quad \Delta n(x) \simeq \delta n_{\rm o,\,cloud}(x/L_{\rm ch}) + \delta n_{\rm tail}(x), \tag{8}$$

and understand how the different characteristic lengths and coherence properties could appear in the theory with the Bose-condensate at $T \neq 0$. Note that if the excitonic packet moves in a crystal (or another semiconductor structure), it interacts with thermal phonons and noncondensed excitons ($T \neq 0$), impurities, and other imperfections of the lattice. Then, there is always a noise factor acting on the system. Although the coherent core of the packet is assumed to be in a quasistable state during the observation time, the fluctuations of $\phi_{\rm o}(x - vt)$ and, especially, $\varphi_{\rm c}(x, t)$ can be of a great importance to interpret the experimental data.

Exciton-Phonon Condensate

To obtain the necessary density of excitons $n_{\rm x}$ in the excitonic cloud and, thus, meet the BEC conditions, the crystals are irradiated by laser pulses with $\hbar\omega_L \gg E_{\rm gap}$, and the temperature of the crystal is $T \simeq 1 \sim 5\,\mathrm{K}$. If the cross-section area S of an excitation spot on the surface of the crystal can be made large enough, such as $S \approx S_{\rm surf}$, the hot droplet of paraexcitons can acquire an average momentum during its thermalization process ($T^*(t) \to T$). Indeed, the phonon wind, or the flow of nonequilibrium phonons, blows unidirectionally from the surface into the bulk [34] and transfers the nonzero momentum to the excitonic cloud,

$$\mathbf{P}_{\rm x} \simeq N_{\rm x}\,\langle \hbar\,\mathbf{k}_0 \rangle \neq 0 \;\text{ and }\; \mathbf{P}_{\rm x} \perp S_{\rm surf}, \tag{9}$$

see Figs. 1, 2, and 3. As a result, the packet of moving excitons and nonequilibrium phonons of the phonon wind ($N_{\rm ph} \simeq N_{\rm x}$) is actually the system that undergoes the transition toward developing the Bose-Einstein correlations at $T^* < T_c$.

Let us assume that the condensate has been already formed inside the moving excitonic droplet, and the following representation of the exciton Bose-field holds:

$$\hat{\psi} = \Psi_0 + \delta\hat{\psi}, \quad \langle\delta\hat{\psi}\rangle = 0.$$

Note that the kinetics of condensate formation will not be explored, so we assume $T_{\rm cloud} \simeq T$, and the simplest case to begin the discussion is $T \to 0$. For the displacement field of the crystal $\hat{\mathbf{u}}$, we introduce a nontrivial coherent part too, i.e.,

$$\hat{\mathbf{u}} = \mathbf{u}_0 + \delta\hat{\mathbf{u}}, \quad \langle\delta\hat{\mathbf{u}}\rangle = 0,$$

and $\mathbf{u}_0 \neq 0$. Recall that introducing anomalous averages (in our case, Ψ_0 and $\mathbf{u}_0$, and also $\langle\delta\hat{\psi}\delta\hat{\psi}\rangle$, $\langle\delta\hat{\psi}^\dagger\delta\hat{\psi}^\dagger\rangle$, etc.) is the easiest way to describe a system with the Bose-Einstein condensate [21]. Then, the question of the presence of condensate is the question of the long-range structure of the corresponding classical fields, $\Psi_0(\mathbf{x}, t)$ and $\mathbf{u}_0(\mathbf{x}, t)$ [22].

We can write the exciton density matrix in the following form (compare with Eq. (2)):

$$\rho_{\rm x}(\mathbf{x},\, \mathbf{x}',\, t) = \Psi_0^*(\mathbf{x}, t)\, \Psi_0(\mathbf{x}', t) + \langle\delta\hat{\psi}^\dagger(\mathbf{x}, t)\, \delta\hat{\psi}(\mathbf{x}', t)\rangle, \tag{10}$$

and the similar formula can be written for the phonon density matrix. For example, $\rho_{{\rm ph},\,xx}(\mathbf{x},\, \mathbf{x}',\, t) = \langle\hat{u}_x(\mathbf{x}, t)\, \hat{u}_x(\mathbf{x}', t)\rangle$ has the following classical part,

$$\rho_{{\rm ph},\,xx}(\mathbf{x},\, \mathbf{x}',\, t) = u_0(\mathbf{x}, t)\, u_0(\mathbf{x}', t) + \langle\delta\hat{u}_x(\mathbf{x}, t)\, \delta\hat{u}_x(\mathbf{x}', t)\rangle.$$

This means that while the macroscopic number of excitons occupies the state Ψ_0 the phonons form a coherent state which is analogous to the phonon part of the Davydov soliton [23]. Recall that the Davydov soliton describes an one dimensional system of (Frenkel) excitons in a chain of atoms, and, within the long

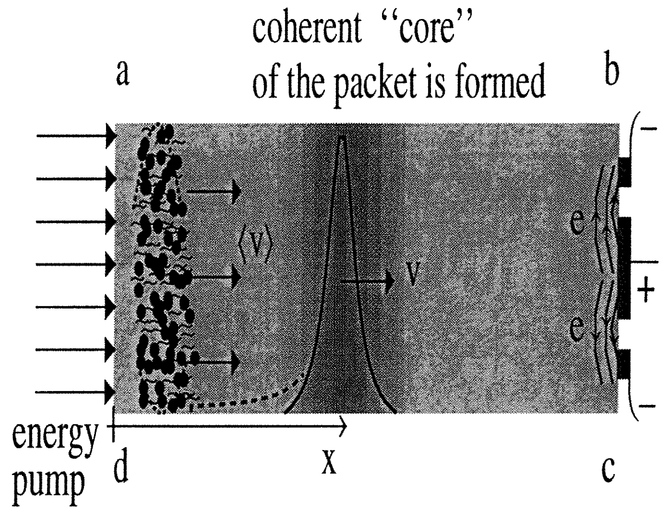


Figure 1: Excitonic packet moving in a crystal.

A medium, in which the exciton-phonon droplet can propagate, is presented in the form of the channel 'abcd' on this Figure. After some amount of energy has been pumped into the medium during a short time interval and absorbed near a boundary, a localized excited state is formed near the face 'ad'. It is schematically shown here as a mixture of excitons and phonons. If there is a mechanism of the momentum transfer to the excited state, the droplet begins to move toward the opposite face 'bc' with the velocity $\langle v \rangle$. Then, such conditions can favor the appearance of an inhomogeneous coherent state inside the droplet if the average density of the excitons $n_{\mathrm{x}} > n_c(T)$. Moreover, a sort of Bose-condensate can appear because of the effective attraction among the bosons (excitons) at $T < T_c$. The profile of the excitonic part of it, $n_{\mathrm{core}}(x,t) \simeq |\Psi_{\mathrm{o}}(x,t)|^2$, is shown by the bold line and the intensity of the elastic (phonon) part, $\partial_x u_{\mathrm{o},x}(x,t)$, is represented by changements of the intensity of the background color. When the packet reaches the face 'bc', the total density of excitons $n(x)$ $(N_{\mathrm{x}} > N_{\mathrm{core}})$ is converted into a measurable electric current, $i(t)$. It is possible to create an electric field near the surface 'bc', (**e** is depicted on the Figure). It can be strong enough to break the coming exciton into holes and electrons.

wavelength approximation, it can be written as the following trial wave function:

$$|\Theta\rangle \sim \int dx\, \xi_{\rm o}(x,t)\, \hat{\Psi}^{\dagger}(x) \times$$

$$\exp\Big(-i/\hbar \int dx\, {\rm const}\, \tilde{u}_{\rm o}(x,t)\, \hat{\pi}_x - {\rm const}'\, \partial_t \tilde{u}_{\rm o}(x,t)\, \hat{u}_x\Big)\, |0:\ {\rm ex}, {\rm phon}\rangle. \qquad (11)$$

Here, $\xi_{\rm o}(x,t)$ is the exciton wave function and the Davydov soliton is the one-exciton state, $\int dx\, |\xi_{\rm o}(x,t)|^2 = 1$. Meanwhile, $\tilde{u}_{\rm o}(x,t) = \langle \hat{u}_x \rangle$ and $\rho\, \partial_t \tilde{u}_{\rm o}(x,t) = \langle \hat{\pi}_x \rangle$ are the amplitudes of the displacement field of the medium and its conjugated momentum, and ρ is the mass density. In fact, Eq. (11) is the many-phonon state of the displacement field of the crystal, a kind of the Glauber coherent state [24]. The dynamic equations on $\xi_{\rm o}(x,t)$ and $\tilde{u}_{\rm o}(x,t)$ turn out to be a system of coupled Schrödinger and wave equations. For example, the simplest model has the following form:

$$i\hbar\, \partial_t \xi_{\rm o}(x,t) = -\frac{\hbar^2}{2m}\, \partial_x^2 \xi_{\rm o}(x,t) + \sigma_0\, \partial_x \tilde{u}_{\rm o}(x,t)\, \xi_{\rm o}(x,t),$$

$$(\partial_t^2 - c_l^2 \partial_x^2)\, \tilde{u}_{\rm o}(x,t) = \rho^{-1} \sigma_0\, \partial_x \{|\xi_{\rm o}|^2(x,t)\}. \qquad (12)$$

In the case of many excitons and many phonons, we can write the following ansatz for the many-particle wave function of the coherent state:

$$|\Theta\rangle \sim \exp\Big(\int dx\, \Psi_{\rm o}(x,t)\, \hat{\psi}^{\dagger}(x)\Big) \times$$

$$\exp\Big(-i/\hbar \int dx u_{\rm o}(x,t)\, \hat{\pi}_x - {\rm const}'\, \partial_t u_{\rm o}(x,t)\, \hat{u}_x\Big)\, |0:\ {\rm ex}, {\rm phon}\rangle. \qquad (13)$$

Then, the dynamic equations on $\Psi_{\rm o}(x,t)$ and $u_{\rm o}(x,t)$ can be written out as a system of coupled nonlinear Schrödinger and nonlinear wave equations. The nonlinearity has its origin in the many-particle nature of the considered coherent phenomenon, namely, in the exciton-exciton interaction (scattering processes), e.g., $\hat{H}_{\rm x-x} \sim \int d\mathbf{x}\, (\nu_0/2)\, \psi^{\dagger}\psi^{\dagger}\, \psi\psi$, and the phonon-phonon interaction (cubic anharmonicity of the crystal), e.g., $\hat{H}_{\rm ph-ph} \sim \int d\mathbf{x}\, (\kappa_3/3)\, (\nabla \mathbf{u})^3$. Note that the two-particle exciton-exciton interaction is described by the (oversimplified) contact interaction, $U(\mathbf{x},\mathbf{x}') = \nu_0\, \delta(\mathbf{x}-\mathbf{x}')$. Here, $\nu_0 = {\rm const} > 0$, $[\nu_0] = {\rm energy}\, L^3$,

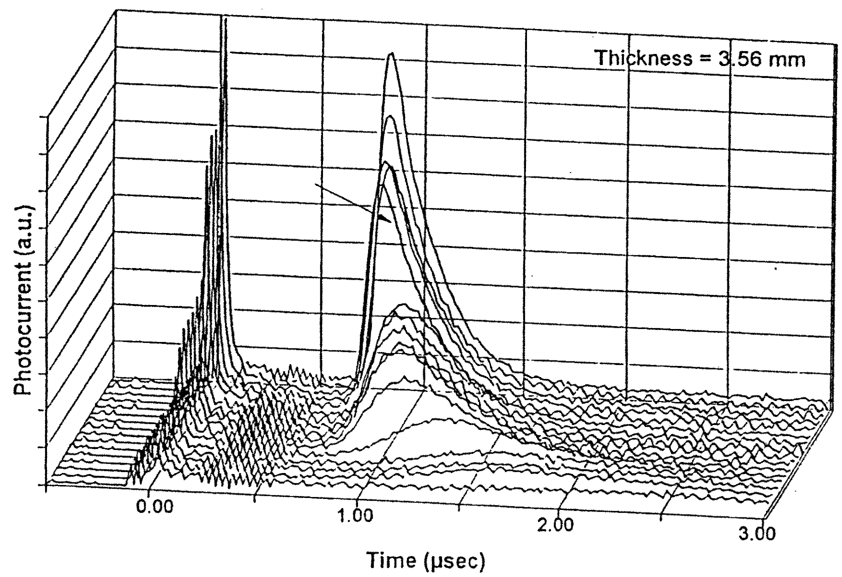


Figure 2: Time-resolved photocurrent at a constant temperature.

Time-resolved photocurrent obtained in a Cu_2O sample following intense laser excitation of the opposite surface for different excitation intensities and a constant temperature of 1.85 K. The intensity increases from the bottom to the top curve from 3.75×10^4 to 1.5×10^6 W/cm^2. The curve marked by the arrow, which shows a dramatic difference from the curve with slightly less intensity, is for an intensity of 6×10^5 W/cm^2 (from Ref. 9).

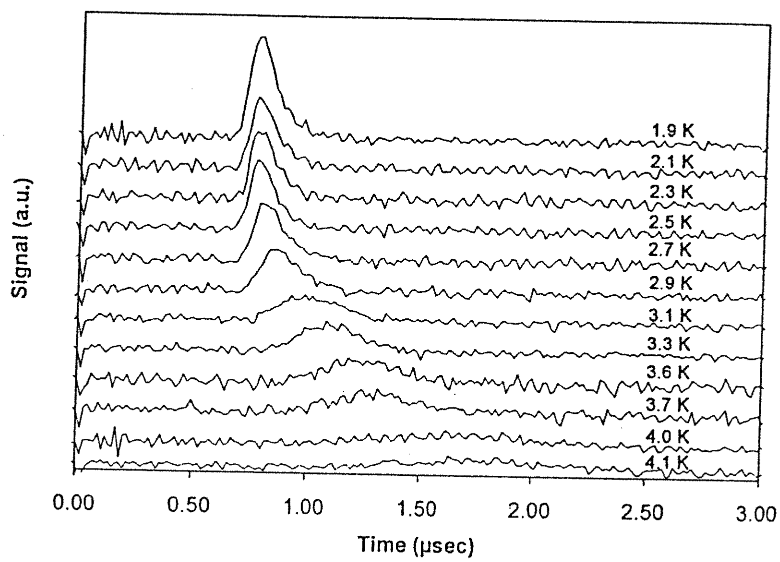


Figure 3: Time-resolved photocurrent at a constant power.

Time-resolved photocurrent from the same sample as in Figure 1A, at different temperatures and a constant illumination intensity of 1.2×10^{6} W/cm^{2} (from Ref. 9).

is the corresponding interaction vertex proportional to the exciton-exciton scattering length. The coupling between the exciton and phonon fields comes from the exciton-(LA) phonon interaction in the form of Deformation Potential, for example, $\hat{H}_{\mathrm{x-ph}} \sim \int d\mathbf{x}\, \sigma_0\, \psi^\dagger \psi\, (\nabla \mathbf{u})$ with the positive constant σ_0. Note that the exciton-phonon interaction is also described by the (oversimplified) contact interaction, $\mathcal{U}(\mathbf{x}, \mathbf{x}') = \sigma_0\, \delta(\mathbf{x} - \mathbf{x}')$, $[\sigma_0] = \mathrm{energy}$.

As the two particle interaction ($\sim \nu_0\, (\psi^\dagger)^2 \psi^2$) can be strongly renormalized because of interaction with other fields, we include an additional hard core interaction term modeled by repulsion $\sim \nu_1\, (\psi^\dagger)^3 \psi^3$ in $\hat{H}_{\mathrm{x-x}}$. We assume that the "bare" characteristic energies of the particle-particle interactions satisfy the following inequality:

$$0 < \nu_1 / a_{\mathrm{x}}^6 < (\ll)\, \nu_0 / a_{\mathrm{x}}^3 \simeq \mathrm{const}\, \mathrm{E_x}, \quad \mathrm{const} \sim 10,$$

see [25] for discussion. Here, a_{x} and E_{x} are the exciton Bohr radius and characteristic Rydberg energy, respectively. However, it is the coupling σ_0 that is responsible for formation of the moving exciton-phonon coherent state in the form of the "bright" soliton. Note that it is crucial to write out everything in the coordinate space. We could write the moving condensate in terms of the standard exciton and phonon operators defined in the momentum space, e.g.,

$$|\Theta\rangle \sim \exp\Big(\Psi_0\, \hat{a}^\dagger_{k_0}\Big) \times \exp\Big(\sum_q u_q(t)\, \hat{b}_q - u_q^*(t)\, \hat{b}_q^\dagger\Big)\, |0 : \mathrm{ex, phon}\rangle$$

with $\Psi_0 \sim \sqrt{N_{\mathrm{o}}}$, but such a representation does not give any insight. Note that there is a better approximation of the ground state of the moving *interacting* bosons occupying the volume V [26]:

$$|\Theta\rangle \sim \exp\Big(\Psi_0\, \hat{a}^\dagger_{k_0} + (1/2) \sum_{q \neq 0} \phi_q\, \hat{a}^\dagger_{k_0+q} \hat{a}^\dagger_{k_0-q}\Big)\, |0\rangle, \quad |\Psi_0| \gg 1, \quad |\phi_q| < 1.$$

To discuss the inhomogeneous state of bosons, we can rewrite this wave function as follows:

$$|\Theta_{\mathrm{x}}\rangle \sim \exp\Big(\int dx\, \Psi_{\mathrm{o}}(x,t)\, \hat{\psi}^\dagger(x) + (1/2) \int dx\, dx'\, \Psi_1(x, x', t)\, \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(x')\Big) |0 : \mathrm{ex}\rangle.$$

It includes the quantum depletion of the condensate $\propto |\Psi_1(x,x',t)|^2$. In the same manner, one can include fluctuations of the displacement field into the coherent phonon part of the exciton-phonon condensate (13), for example, [27],

$$|\Theta_{\rm ph}\rangle \sim \exp\Big((1/2)\int dx\, dx'\, \mathcal{D}_1(x,x',t)\, \hat{u}(x)\, \hat{u}(x')\Big)|0:\,{\rm ph}\rangle.$$

If the depletion of the condensate is small, one can start from the coherent state approximation alone, see Eq. (13), and calculate the functions $\Psi_1(x,x',t)$ and $\mathcal{D}_1(x,x',t)$ as small corrections to the known $\Psi_{\rm o}(x,t)$ and $u_{\rm o}(x,t)$. However, in this thesis, we use the mean field theory to describe the out-of-condensate part.

Thus, in the limit of $T \to 0$, we obtain the following system:

$$i\hbar\, \partial_t \Psi_{\rm o}(x,t) = \tilde{E}_g\, \Psi_{\rm o}(x,t) - \frac{\hbar^2}{2m}\, \partial_x^2 \Psi_{\rm o}(x,t) +$$

$$+ \Big(\nu_0\, |\Psi_{\rm o}|^2\, \Psi_{\rm o}(x,t) + \nu_1\, |\Psi_{\rm o}|^4\, \Psi_{\rm o}(x,t)\Big) + \sigma_0\, \partial_x u_{\rm o}(x,t)\, \Psi_{\rm o}(x,t),$$

$$(\partial_t^2 - c_l^2 \partial_x^2)\, u_{\rm o}(x,t) - c_l^2 \kappa_3\, \partial_x\{\partial_x u_{\rm o}(x,t)\}^2 = \rho^{-1}\sigma_0\, \partial_x\{|\Psi_{\rm o}|^2(x,t)\}. \tag{14}$$

This system can be considered as a generalization of the Zakharov system which appeared in Plasma Physics first, [28]. The main difference is that in the case of collective phenomena in Condensed Matter Physics Eqs. (14) define the main part of the moving packet. There is always a depletion of the condensate. In fact, in the case of $T \neq 0$ and, generally, in all the nonstationary cases, these two equations have to be coupled with another system of equations on the so-called out-of-condensate excitations [21]. The dynamics of these excitation states can strongly influence the dynamics of the "parent" condensate: for example, one can expect effective dissipation terms to appear in Eqs. (14).

The important property of the exciton-phonon condensate is a kind of self-consistency condition. Roughly, we have

$$\nabla \mathbf{u}_0(x - vt) \approx \partial_x u_{\rm o}(x - vt) \propto |\Psi_0(x - vt)|^2, \tag{15}$$

as it is for the Davydov soliton within Eq. (12). (In fact, this is only the first term in an expansion of a complicated formula, $\partial_x u_{\rm o} = F(|\Psi_0|)$.) Thus, the moving packet contains both the macroscopically occupied exciton-phonon condensate, or the Bose-core $\Psi_0(\mathbf{x},t)\cdot\mathbf{u}_0(\mathbf{x},t)$, and out-of-condensate excitons and phonons. The macroscopic wave function of excitons $\Psi_0(x,t)$ is normalized as follows:

$$\int |\Psi_0|^2(x,t)\,d\mathbf{x} = S_\perp \int \phi_o^2(x/L_0)\,dx = N_o \gg 1, \tag{16}$$

where N_o is the macroscopic number of condensed excitons, and, generally, $N_o(T) < N_x$. Here, $S_\perp$ is the cross section area of the crystal. Usually, within the quasistationary approximation at $T \neq 0$, the following assumption simplifies the theory:

$$N_o(T) = \mathrm{const} \gg \delta N(T) = N_x - N_o = \mathrm{const}', \quad T \ll T_c.$$

Obviously, this implies stability of the moving packet with the Bose-core during the finite observation time and, in fact, this is questionable. As we showed in [30], it is more realistic to obtain (see Fig. 4 (a) and (b))

$$\partial_t N_o < 0 \quad \text{and} \quad \partial_t n_{\mathrm{tail}} > 0,$$

during the packet moves through the crystal, although this process is relatively slow. This means that the energy of the Bose-core of the packet is not conserved.

Recall that solitons are known to move ballistically through a medium without changement of their shape [29]. In practice, the shape of moving inhomogeneous states changes, and a long lasting tail appears behind the soliton. Such a behavior can be described within a model of exciton-phonon condensate as a dynamic effect in soliton transport. For example, the coupling between bosonic excitations of the medium, such as excitons, and elastic modes of it, such as phonons, can be responsible for these effects. Then the following coupling terms can appear in the Heisenberg equations describing the exciton field:

$$\tilde{E}_g\hat{\psi} \rightarrow \tilde{E}_g\hat{\psi} + \sigma_0\,\partial_j\hat{u}_j\,\hat{\psi}, \tag{17}$$

$$-(\hbar^2/2m_x)\,\Delta\hat{\psi} \rightarrow -(\hbar^2/2m_x)\,\Delta\hat{\psi} - \vartheta_0\,\partial_j\hat{u}_j\,\Delta\hat{\psi}. \tag{18}$$

Here, $\tilde{E}_g = E_{\mathrm{gap}} - E_x$, and σ_0 and ϑ_0 are the coupling constants of the exciton-phonon interaction written in the form of Deformation Potential. Developing a theory, we have a freedom with the sign of ϑ_0; its value, however, can be roughly estimated as $|\vartheta_0| \simeq \hbar^2/2m_{\rm x}$ whereas $\sigma_0 \simeq E_g$.

In terms of the lattice analog of the medium Hamiltonian, we made the hopping matrix element t, t_{ij} in $\hat{H}_{\rm gas} \sim \sum t_{ij}\,\hat{\psi}_i^\dagger \hat{\psi}_j$, of the lattice boson $\hat{\psi}_j$ to be dependent on the lattice deformation field $\hat{u}_j$,

$$t_{ij} \to t_{ij}(u_i,\, u_j) \approx t_{ij} + \tilde{\vartheta}_0\,(u_i - u_j).$$

Meanwhile, the energy on a site, $\varepsilon_{0,\,i} = \varepsilon_0$ in $\hat{H}_{\rm gas} \sim \sum \varepsilon_{0,\,i}\,\hat{\psi}_i^\dagger \hat{\psi}_i$, depends on the lattice displacements too, for example,

$$\varepsilon_{0,\,i} \to \varepsilon_0 + \tilde{\sigma}_0\,(u_{i+1} - u_{i-1}),$$

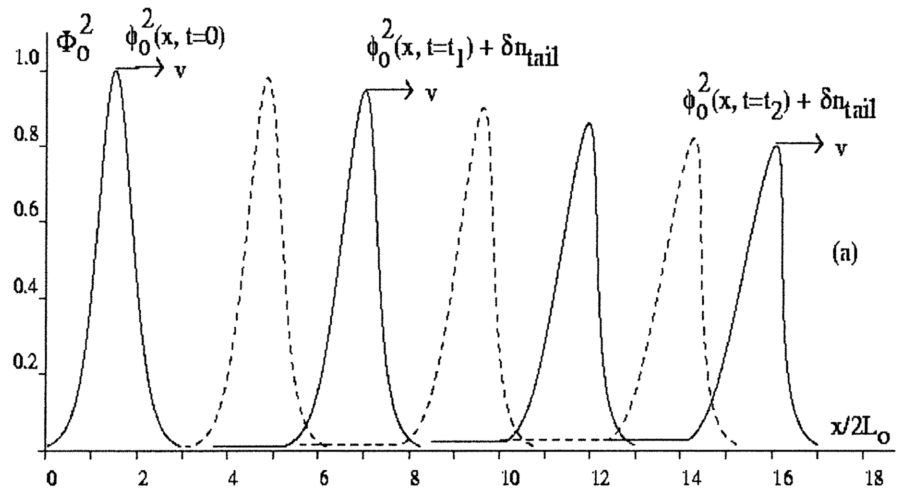
see [31], [32] for discussion. In the model with $\vartheta_0 \neq 0$, the simple ballistic ansatz, e.g.,

$$\psi(x,t) \sim \exp(ik_0 x - i\omega_0 t + \varphi_c)\,\phi(x - vt)$$

with $\hbar k_0 = m_{\rm x}\,v$ and $\omega_0(k_0) \propto k_0^2$, and the steady envelope function $\phi(x/L_0)$, is not a good one anymore.

Unlike the "true" solitons and kinks, it is possible to introduce inhomogeneous corrections to the ballistic velocity $v \to v + \delta v(x,t)$ and the coherent phase of the condensate $\varphi_c \to \varphi_c + \delta\varphi(x,t)$ that control the changement of the packet shape. Then, the amplitudes of the soliton and kink, $\Phi_{\rm o}$ and $2Q_{\rm o}$, respectively, change in time as well, see Fig. 4, pp. 17, 18. And the tail starts growing behind the localized packet. The total packet can be associated with an exciton-phonon "comet" with the quasistable coherent Bose-core and incoherent tail moving in the medium. Note that such an image, i.e. the "comet" with the non-trivial nucleus, coma, and tail, can be also used in attempts to interpret the latest experiments on selective amplification of the moving exciton-phonon packets in Cu_2O [33].

However, the quasistationary approximation is the simplest to deal with, and we explore a model based on the coupled Nonlinear Schrödinger equation (NLS) and Nonlinear Wave equation (NLW) to describe the exciton-phonon condensate.



These two components of the model have different covariances: the first one is Galilean invariant while the second one is Lorentz invariant ($c \to c_s$). That is why the effective exciton-exciton interaction depends on the velocity of the packet: it becomes attractive in the propagation direction if the velocity exceeds a critical one. Then, the packet propagation can be described by the soliton solution of the effective NLS. The critical velocity known from the theory of superfluidity appears from the investigation of stability of the moving soliton.

To model the ballistic motion of a single packet, we use the following ansatz to describe the Bose-core of the packet:

$$\Psi_0(\mathbf{x}, t) = \mathrm{e}^{-i(\tilde{E}_g + m_x v^2/2 - |\mu|)\, t/\hbar}\, \mathrm{e}^{i(\varphi_c + k_0 x)}\, \phi_o(x - vt), \tag{19}$$

$$u_{0\,j}(\mathbf{x}, t) = u_o(x - vt)\, \delta_{1j}, \quad (x, y, z = 1, 2, 3). \tag{20}$$

where $\tilde{E}_g = E_{\text{gap}} - E_x$, $\varphi_c = \text{const}$ is the macroscopic phase of the condensate, $\hbar k_0 = m_x v$, and $\mu = \mu(N_o) < 0$ is the effective chemical potential of the condensate. Note that by use of this simple ansatz we *prescribe* the "rigid" phase φ_c to the condensate and, for the superfluid velocity, we have $v_s(x, t) \propto \partial_x \varphi_c(x, t) = v$. At $T \ll T_c$, one can disregard the interaction between the Bose-core and the out-of-condensate cloud and write down the following one-dimensional equations on

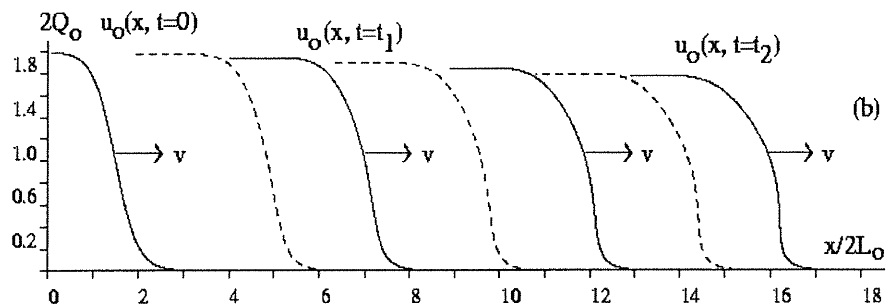


Figure 4: Soliton and kink with a leakage into their tail.

To model transport properties of the boson-phonon soliton in the case of effective dissipation (the "leakage"), we started from the symmetric soliton without a tail as an initial condition at $t = 0$. Dynamics of the boson (exciton) part of the packet is presented on Fig. 4 (a), page 17, in the form of moving $|\Psi_{\rm o}(x - v(x,t)t)|^2 + \delta n(x,t)$, where $\delta n(x,t) \simeq \delta n_{\rm tail}$ at $\bar{x} < -(2-3)\,L_0$. The coherent phonon part (a moving kink of the displacement field) is depicted on Fig. 2 (b); the phonon part of the tail $\langle(\partial_x \delta\hat{u}_x)^2\rangle(t) \neq 0$ is not presented on this figure. The initial value of the interaction parameter $\zeta(v)\,\Phi_{\rm o}^2 \simeq \delta v_{\rm top}/v$ is taken to be $+0.05$. Here, ζ is proportional to the exciton-phonon interaction strength $(\zeta \propto \vartheta_0\,\sigma_0 > 0)$ and $\delta v_{\rm top} = v_{\rm top} - v$. Then, the visible changements occur after the packet has traveled the distance of $(20-30)\,L_0$, which corresponds to the effective value of $\delta v_{\rm top}(t)\,\Delta t/L_0(t) \sim 1$. Note that the energy of the total moving packet (coherent part + tail) is conserved at $T \to 0$.

the envelope function $\phi_o(x/L_0)$ and the coherent part of the displacement field $u_o(x/L_0)$ (see Eqs. (14)):

$$-|\mu|\,\phi_o(x) = -(\hbar^2/2m_x)\,\partial_x^2\phi_o(x) - |\tilde{\nu}_0|\,\phi_o^3(x) + \tilde{\nu}_1\,\phi_o^5(x),$$

$$\partial_x u_o(x) \approx -\mathrm{const}_0\,\phi_o^2(x) + \mathrm{const}_1\,\phi_o^4(x). \tag{21}$$

Here, $\tilde{\nu}_j = \tilde{\nu}_j(v)$ and $\mathrm{const}_j = \mathrm{const}_j(v)$, $j = 0, 1$, are the renormalized interaction vertices originated from exciton-exciton, exciton-phonon, and phonon-phonon interactions, see Eqs. (14), so that tildes are used to emphasize this fact. They can be calculated *exactly* at $T = 0$.

At $T \neq 0$, $T < T_c$, we choose the quasistationary approximation to write out the decomposition of the exciton and phonon fields of the moving droplet,

$$\hat{\psi}_0(\mathbf{x}, t) = \exp(i\varphi_c(x, t)\,)\{\phi_o(x - vt) + \delta\hat{\psi}_o(x - vt,\, \mathbf{x}_\perp,\, t)\},$$

$$\hat{u}_{0,j}(\mathbf{x}, t) = u_o(x - vt)\,\delta_{1,j} + \delta\hat{u}_{o,j}(x - vt,\, \mathbf{x}_\perp,\, t). \tag{22}$$

Then, the following correlation functions have to be included into an analog of Eq. (21): the "anomalous" ones, such as $\tilde{\mathrm{m}}(x) = \langle\delta\hat{\psi}_o\,\delta\hat{\psi}_o\rangle$, the exciton-phonon correlators, such as $\tilde{q}_j = \langle\partial_j\delta\hat{u}_{o,j}\,\delta\hat{\psi}_o(x,\, \mathbf{x}_\perp,\, t)\rangle$, and the out-of-condensate density of the excitons and phonons,

$$\delta n_o(x) = \langle\delta\hat{\psi}_o^\dagger\delta\hat{\psi}_o(x,\, \mathbf{x}_\perp,\, t)\rangle \;\text{ and }\; Q_{xx}(x) = \langle(\partial_x\delta\hat{u}_{o,x})^2\rangle.$$

As a result, we have the following system (compare with Eqs. (14)):

$$i\hbar\,\partial_t\Psi_o(x, t) = \tilde{E}_g\,\Psi_o(x, t) - \frac{\hbar^2}{2m}\,\partial_x^2\Psi_o(x, t) +$$

$$+\Big(\nu_0\,|\Psi_o|^2\,\Psi_o(x, t) + \nu_1\,|\Psi_o|^4\,\Psi_o(x, t) +$$

$$+\,6\nu_1\,\delta n_o(\mathbf{x}, t)\,|\Psi_o|^2\,\Psi_o(x, t) + \nu_1\,\mathrm{m}^*(\mathbf{x}, t)\,\Psi_o^2\,\Psi_o(x, t) + 3\nu_1\,\mathrm{m}(\mathbf{x}, t)\,|\Psi_o|^2\,\Psi_o^*(x, t)\Big) +$$

$$+\,2\nu_0\,\delta n_o(\mathbf{x}, t)\,\Psi_o(x, t) + \nu_0\,\mathrm{m}(\mathbf{x}, t)\,\Psi_o^*(x, t) + \sigma_0\,\partial_x u_o(x, t)\,\Psi_o(x, t) + \sigma_0\,(q_x + q_y + q_z),$$

$$(\partial_t^2 - c_l^2\partial_x^2)\,u_o(x, t) - c_l^2\kappa_3\,\partial_x\{\partial_x u_o(x, t)\}^2 =$$

$$= \rho^{-1}\sigma_0\, \partial_x\{|\Psi_{\rm o}|^2(x,t) + \delta n_{\rm o}(\mathbf{x},t)\} + \kappa_3\, c_l^2\, \partial_x Q_{xx}. \tag{23}$$

At this stage one can predict that the bare vertex $\nu_0 > 0$ in the exciton-exciton interaction term $\nu_0\, |\Psi_{\rm o}|^2\, \Psi_{\rm o}(x,t)$ will be renormalized not only because of the coherent part of the displacement field $\partial_x u_{\rm o}(x,t)$ but also because of $\mathrm{m}(\mathbf{x},t) \neq 0$ and $\delta n_{\rm o}(\mathbf{x},t) \neq 0$ being taken into account at $T < T_c$. (And the same fact seems to be true for the vertex $\nu_1 > 0$.)

We substitute the ballistic ansatz (22) into Eqs. (23) to rewrite them in the comoving frame of reference. Thus, it is possible to generalize Eqs. (21) to the case of $T \neq 0$. Here, we present the simplest version of such a generalization:

$$-(\,|\mu| + \delta\mu(x)\,)\,\phi_{\rm o}(x) = -(\hbar^2/2m^*)\,\partial_x^2\phi_{\rm o}(x) + (\tilde{\nu}_0 + \delta\nu_0(x))\,\phi_{\rm o}^3(x) +$$

$$+\,(\tilde{\nu}_1 + \delta\nu_1(x))\,\phi_{\rm o}^5(x), \tag{24}$$

$$\partial_x u_{\rm o}(x) \approx -\mathrm{const}_0'\,\phi_{\rm o}^2(x) + \mathrm{const}_1'\,\phi_{\rm o}^4(x) - |\mathrm{const}_{\rm tail}|, \;\; (x<0). \tag{25}$$

To a first approximation, we disregard all the dissipation terms (and the noise factors), and assume $T^* \simeq T$, the crystal temperature. Then, $\Psi_{\rm o} = \phi_{\rm o}\, \mathrm{e}^{i\varphi_c}$ can be interpreted again as a macroscopically occupied state of the excitons with the defined ("rigid") envelope function and phase. To simplify the nonlinear equation on the condensate wave function coupled with the inside- and outside-excitations by $\delta\mu(x)$, $\delta\nu_0(x)$, etc., we have to make some assumptions on the asymptotic behavior of the so-called u- and v-wave functions of the inside- and outside-excitations. In the homogeneous case, they are $\sim \mathrm{const}_k\, \exp(i\mathbf{k}\,\mathbf{x})/\sqrt{V}$ (recall the Bogoliubov transform). In fact, the $\mathrm{u}(x)$ and $\mathrm{v}(x)$ functions define the spatial behavior of the unknown correlation functions in Eqs. (23) in the inhomogeneous case. For example, these assumptions can lead to the following formulas:

$$\delta n_{\rm o,\,in}(x) \to \cdots + \mathrm{const}_n^{(1/2,1/2)}\,\phi_{\rm o}(x) + \mathrm{const}_n^{(1,1)}\,\phi_{\rm o}^2(x) + \cdots,$$

$$\tilde{\mathrm{m}}(x) \to \cdots + \mathrm{const}_{\rm m}^{(1/2,1/2)}\,\phi_{\rm o}(x) + \mathrm{const}_{\rm m}^{(1,1)}\,\phi_{\rm o}^2(x) + \cdots,$$

$$\delta n_{\text{o, out}}(x) \to \text{const} \ \text{ and } \ Q_{xx,\text{out}}(x) \to \text{const}' \ \ x < 0,$$

where $\text{const}_n^{(1,1)}$ and $\text{const}_{\text{m}}^{(1,1)}$ are dimensionless and real. Then, the corrections to the effective chemical potential $|\mu|$ and the exciton-exciton and exciton-phonon interaction vertices, $\tilde{\nu}_j$ and const_j in Eqs. (24) and (25), can be reduced to the constants which depend on the temperature T through the above mentioned correlation functions.

This means Eqs. (24) and (25) with the *unknown* constant interaction vertices can be solved formally: again, they can be reduced to the well-known NLS. In fact, they have to be solved together with the equations on the out-of-condensate excitons and phonons, $\delta\hat{\psi}_{\text{o}}$ and $\delta\hat{\psi}_{\text{o}}^{\dagger}$, and $\delta\hat{u}_{\text{o},j}$ and $\delta\hat{\pi}_{\text{o},j}$. Usually, these equations are simplified by use of linearization so that they can be diagonalized by the generalized *u*-*v* Bogoliubov transform. For example, the following formula can be used to calculate the depletion of the condensate

$$\delta n_{\text{o}}(x) \approx \sum_{s,j} (|\text{u}_s(x)|^2 + |\text{v}_s(x)|^2)\, n_{\text{BE}}(\hbar\tilde{\omega}_s/T) + |\text{v}_s(x)|^2, \tag{26}$$

where $n_{\text{BE}}(x) = (\exp(x) - 1)^{-1}$ and $\text{u}_s(x)$ and $\text{v}_s(x)$ are the Bogoliubov – de Gennes amplitudes; the energies of the elementary excitations $\hbar\tilde{\omega}_s$ have to be taken in the laboratory frame. A similar formula can be written for $\tilde{\text{m}}(x)$. Thus, the assumption on the asymptotic behavior of $\delta n_{\text{o, in}}(x/L_{\text{ch}})$ and $\tilde{\text{m}}_{\text{in}}(x/L_{\text{ch}})$ can be justified. To obtain some insight on how to deal with the inhomogeneous part of the elementary excitations at $T < T_c$, one has to study the case of $T = 0$ in detail.

At $T \to 0$, we can write out the linear equations on the out-of-condensate part, and all the coefficients and vertices are known. (Eqs. (21) were solved before.) We have the following system in the comoving frame of reference:

$$\begin{aligned} i\hbar\,\partial_t\,\delta\hat{\psi}_{\text{o}}(x,\mathbf{x}_\perp,t) &= -(\hbar^2/2m)\,\Delta\delta\hat{\psi}_{\text{o}}(x,\mathbf{x}_\perp,t) + |\mu|\,\delta\hat{\psi}_{\text{o}} + \\ &+ \Big\{ (\nu_0 + \tilde{\nu}_0)\,\phi_{\text{o}}^2(x) + (2\nu_1 + \tilde{\nu}_1)\,\phi_{\text{o}}^4(x) \Big\}\,\delta\hat{\psi}_{\text{o}} + \Big\{ \nu_0\,\phi_{\text{o}}^2(x) + 2\nu_1\,\phi_{\text{o}}^4(x) \Big\}\,\delta\hat{\psi}_{\text{o}}^{\dagger}(x,\mathbf{x}_\perp,t) + \\ &+ \sigma_0\,\phi_{\text{o}}(x)\,\partial_j\delta\hat{u}_{\text{o},j}(x,\mathbf{x}_\perp,t), \end{aligned} \tag{27}$$

$$-i\hbar\,\partial_t\,\delta\hat{\psi}^\dagger_{\rm o}(x,\mathbf{x}_\perp,t) = -(\hbar^2/2m)\,\Delta\delta\hat{\psi}^\dagger_{\rm o}(x,\mathbf{x}_\perp,t) + |\mu|\,\delta\hat{\psi}^\dagger_{\rm o} +$$

$$+\Big\{\,(\nu_0+\tilde{\nu}_0)\,\phi^2_{\rm o}(x)+(2\nu_1+\tilde{\nu}_1)\,\phi^4_{\rm o}(x)\,\Big\}\,\delta\hat{\psi}^\dagger_{\rm o}+\Big\{\nu_0\,\phi^2_{\rm o}(x) + 2\nu_1\,\phi^4_{\rm o}(x)\,\Big\}\,\delta\hat{\psi}_{\rm o}(x,\mathbf{x}_\perp,t) +$$

$$+\,\sigma_0\,\phi_{\rm o}(x)\,\partial_j\delta\hat{u}_{{\rm o},\,j}(x,\mathbf{x}_\perp,t). \tag{28}$$

$$\{\,(\partial_t - v\partial_x)^2 - c_{\rm s}^2\partial_x^2 - c_{\rm s}^2\partial_\perp^2\}\,\delta\hat{u}_{{\rm o},\,j}(x,\mathbf{x}_\perp,t) \;=$$

$$=\;\rho^{-1}\sigma_0\,\partial_j\Big\{\phi_{\rm o}(x)\,(\delta\hat{\psi}^\dagger_{\rm o}(x,\mathbf{x}_\perp,t) + \delta\hat{\psi}_{\rm o}(x,\mathbf{x}_\perp,t)\,)\Big\},\quad j = 2,3\,(\equiv\perp), \tag{29}$$

$$\{\,(\partial_t - v\partial_x)^2 - c_{\rm s}^2\,\partial_x^2 - c_{\rm s}^2\partial_\perp^2\}\,\delta\hat{u}_{{\rm o},\,x}(x,\mathbf{x}_\perp,t) -$$

$$-\,\delta c^2_{\rm in}(x)\,\partial_x^2\delta\hat{u}_{{\rm o},\,x}(x,\mathbf{x}_\perp,t) - \{\,\partial_x\delta c^2_{\rm in}(x)\,\}\partial_x\delta\hat{u}_{{\rm o},\,x}(x,\mathbf{x}_\perp,t) \;=$$

$$=\;\rho^{-1}\sigma_0\,\partial_x\Big\{\phi_{\rm o}(x)\,(\delta\hat{\psi}^\dagger_{\rm o}(x,\mathbf{x}_\perp,t) + \delta\hat{\psi}_{\rm o}(x,\mathbf{x}_\perp,t)\,)\Big\}. \tag{30}$$

Here, $\delta c^2_{\rm in}(x) \approx c_{\rm s}^2\,2\gamma(v)\,|\kappa_3|\,(\sigma_0/Mc_{\rm s}^2)\,\{a_l^3\,\phi^2_{\rm o}(x)\} \propto \phi^2_{\rm o}(x)$, and $\gamma(v) = v^2/(c_{\rm s}^2 - v^2)$, and $M = \rho\,a_l^3$ is the mass of an elementary cell of the crystal. This approach is no more than the standard mean-field approximation [35], and only the condensate is included into the mean field at $T = 0$. We can introduce elementary excitations of the exciton-phonon condensate. For example, the two branches of them can be described by the Bose-operators $\hat{\alpha}_{j,s}$ and $\hat{\alpha}^\dagger_{j,s}$, $j = 1, 2$, and

$$[\hat{\alpha}_{j,s},\,\hat{\alpha}^\dagger_{q,p}] = \delta_{jp}\,\delta_{sp}\quad\text{and}\quad[\hat{\alpha}_{j,s},\,\hat{\alpha}_{q,p}] = 0. \tag{31}$$

One can write the generalized Bogoliubov transform for $\alpha_{j,s}$ [1],

$$\hat{\alpha}_{j,s} = \int d\mathbf{x}\,\Big(\,U_{j,s}(\mathbf{x})\,\delta\hat{\psi}_{\rm o}(\mathbf{x}) + V_{j,s}(\mathbf{x})\,\delta\hat{\psi}^\dagger_{\rm o}(\mathbf{x}) + Y^i_{j,s}(\mathbf{x})\,\delta\hat{u}_{{\rm o},\,i}(\mathbf{x}) + Z^i_{j,s}(\mathbf{x})\,\delta\hat{\pi}_{{\rm o},\,i}(\mathbf{x})\,\Big)\,,$$

and, roughly, $s \equiv \mathbf{k} = (k_x,\,\mathbf{k}_\perp)$. On the other hand, we can write out the back transform, for example

$$\delta\hat{\psi}_{\rm o}(\mathbf{x}) = \sum_{1,s}\mathrm{u}_{1,s}(\mathbf{x})\,\hat{\alpha}_{1,s} \,+\, \mathrm{v}^*_{1,s}(\mathbf{x})\,\hat{\alpha}^\dagger_{1,s} \,+\, \sum_{2,s}\mathrm{u}_{2,s}(\mathbf{x})\,\hat{\alpha}_{2,s} \,+\, \mathrm{v}^*_{2,s}(\mathbf{x})\,\hat{\alpha}^\dagger_{2,s}, \tag{32}$$

$$\delta\hat{u}_{{\rm o},\,r}(\mathbf{x}) = \sum_{1,s}\mathrm{C}^r_{1,s}(\mathbf{x})\,\hat{\alpha}_{1,s} + \mathrm{C}^{r^*}_{1,s}(\mathbf{x})\,\hat{\alpha}^\dagger_{1,s} \,+\, \sum_{2,s}\mathrm{C}^r_{2,s}(\mathbf{x})\,\hat{\alpha}_{2,s} + \mathrm{C}^{r^*}_{2,s}(\mathbf{x})\,\hat{\alpha}^\dagger_{2,s}. \tag{33}$$

In the comoving frame of reference, one has to rewrite the phonon part of the exciton-phonon Lagrangian as follows:

$$\delta\mathcal{L}_{2,\,\mathrm{phon}} = \int d\mathbf{x}\, \rho/2\, \{\, (\partial_t - v\partial_x)\delta\hat{u}_{\mathrm{o},j}\,\}^2 - \rho\, c_l^2/2\, (\partial_s \delta\hat{u}_{\mathrm{o},j})^2 + \cdots .$$

Then, we have $\delta\hat{\pi}_{\mathrm{o},j}(\mathbf{x},t) = \rho\,(\partial_t - v\partial_x)\delta\hat{u}_{\mathrm{o},j}(\mathbf{x},t)$, and it is easy to represent $\delta\hat{\pi}_{\mathrm{o},j}$ as a sum of $\hat{\alpha}_{j,s}$, $\hat{\alpha}^\dagger_{j,s}$, see Eq. (33). Indeed, the effective Hamiltonian that leads to Eqs. (28) – (30) is supposed to be diagonal in terms of $\hat{\alpha}$, $\hat{\alpha}^\dagger$,

$$\delta\hat{H}_2[\,\delta\hat{\psi}_{\mathrm{o}},\, \delta\hat{\psi}^\dagger_{\mathrm{o}},\, \delta\hat{u}_{\mathrm{o}},\, \delta\hat{\pi}_{\mathrm{o}}\,] \to \sum_{j,s} \hbar\,\omega_{j,s}\, \hat{\alpha}^\dagger_{j,s}\hat{\alpha}_{j,s}, \tag{34}$$

and we have

$$\hat{\alpha}_{j,s}(t) = \hat{\alpha}_{j,s}\, \exp(-i\,\omega_{j,s}t) \;\; \text{and} \;\; \hat{\alpha}^\dagger_{j,s}(t) = \hat{\alpha}^\dagger_{j,s}\, \exp(i\,\omega_{j,s}t). \tag{35}$$

Obviously, one has to divide the excitations of the condensate into the inside-condensate excitation (i.e., they are localized and move with the condensate) and the outside-condensate ones (roughly, the tail without any characteristic length) to proceed. For example, the tail consists of the *uncoupled* excitons ($j = 1$) and phonons ($j = 2$) moving in the laboratory frame of reference behind the condensate. (Indeed, $\phi_{\mathrm{o}}(x) = \Phi_{\mathrm{o}}\, f(x/L_0) \to 0$ at $x < -(3 \sim 4)\, L_0$ and $\varepsilon_{\mathrm{x}} \approx \tilde{E}_g + \hbar^2\, \mathbf{k}^2/2m_{\mathrm{x}} \gg \hbar\omega_{\mathrm{ph}}(\mathbf{k})$ in the laboratory frame.) To describe the inside-condensate excitations, we can introduce smooth envelope functions as follows:

$$\mathrm{u}_{j,s}(\mathbf{x}) \sim u_{j,s}\, f(x/L_0)\, \exp(i\varphi_{j,s}(\mathbf{x})),\;\; \mathrm{v}_{j,s}(\mathbf{x}) \sim v_{j,s}\, f(x/L_0)\, \exp(i\varphi_{j,s}(\mathbf{x})),$$

$$\mathrm{C}_{j,s}(\mathbf{x}) \sim C_{j,s}\, f^2(x/L_0)\, \exp(i\varphi_{j,s}(\mathbf{x})), \tag{36}$$

Here, $\varphi_{j,s}(\mathbf{x}) \to \varphi_{j,\mathbf{k}}(\mathbf{x}) = \varphi_{\mathbf{k}} + \mathbf{k}\,\mathbf{x}$ is used to describe the quickly oscillating parts of the inside-excitations and $u_{j,s}$, $v_{j,s}$, and $C_{j,s}$ are constants. It is worthy of note that the Bose commutation relations (31) lead to the important orthogonality conditions for the functions $\mathrm{u}_{j,s}$, $\mathrm{v}_{j,s}$, and $\mathrm{C}_{j,s}$, for example,

$$\int d\mathbf{x}\, (\mathrm{u}^*_{1,s}\mathrm{u}_{1,s'}(\mathbf{x}) - \mathrm{v}^*_{1,s}\mathrm{v}_{1,s'}(\mathbf{x})\,) + (i/\hbar) \sum_{r=1,2,3} \int d\mathbf{x}\, \left(\mathrm{C}^{r*}_{1,s}\, \rho\, (-i\omega_{1,s'} - v\partial_x)\mathrm{C}^r_{1,s'}(\mathbf{x}) - \right.$$

$$- \{\rho\,(i\omega_{1,s} - v\partial_x)\mathrm{C}^{\tau *}_{1,s}\}\,\mathrm{C}^{\tau}_{1,s'}(\mathbf{x})\,\big) = \delta_{ss'}.$$

At $T \neq 0$, it is possible to write out almost the same system (see Eqs. (28) – (30)) although after a lot of assumptions. The most important assumptions concern the asymptotic behavior of inside- and outside-excitations mentioned before and the possibility to neglect the interaction $\delta\hat{H}_{\text{x-ph}} = \int d\mathbf{x}\,\sigma_0\,\delta\hat{\psi}^{\dagger}_{\text{o}}\delta\hat{\psi}_{\text{o}}\,\partial_j\delta\hat{u}_{\text{o},j}$. As a result, one can "hide" the unknown correlation functions appearing in the (formally) linear equations on the out-of-condensate part. Again, this will be no more than the simplified mean-field approximation at $T \neq 0$ so that one can use Eqs.(32) and (33) to introduce the elementary excitations and diagonalize the corresponding $\delta\hat{H}_2$, see Eq. (34). Note that the interaction vertices staying at the inhomogeneous coefficients of Eqs. (27)–(30) become now the unknown parameters which depend on temperature like the vertices in the simplified Eqs. (24) and (25). The calculation of the out-of-condensate correlation functions will make the presented approach self-concistent.

Alternatively, it is possible to develop a kind of the Landau-Ginzburg Ψ-theory with the phenomenological constants $\mu(T)$ and $\nu_j(T)$, e.g., $\nu_0(T) \propto -(1-T/T_c)^{\beta}$. In this case, $\Psi(x,t)$ is the order parameter [36] to describe the solitonic part of the packet.

It is important that at $T \neq 0$, $T < T_c$ the effective interaction vertices in Eq. (24) are strongly renormalized in comparison with the "bare" ones staying in the Hamiltonian of the exciton-phonon system. As a result, the localized solution for $\phi_{\text{o}}(x/L_0)$ (a kind of the "bright" soliton of the cubic-quintic NLS equation) exists due to the effect of such a strong renormalization, $\tilde{\nu}_0 + \delta\nu_0 < 0$ and $\tilde{\nu}_1 + \delta\nu_1 > 0$. However, unlike the cubic NLS equation, the value of $|\tilde{\nu}_0 + \delta\nu_0|$ has to be larger than some critical one for the "bright" soliton to exist. (This is a way how a correction to the critical temperature of the non-ideal Bose-gas could appear in our theory of the excitonic soliton formed by the interacting bosons.) For comparison, the BEC can be achieved in an atomic gas system with the attractive two particle interaction (the scattering length $a_{\text{sc}} < 0$ and $\nu_0 \propto a_{\text{sc}}$) if the dilute Bose-gas was put into a trap. However, the number of particles N has to be $N < N_c$ for the condensate to be (quasi)stable [8]. Moreover, one can change the sign of the two particle scattering length from, e.g., $a_{\rm sc} > 0$ to $a_{\rm sc} < 0$ and back by changing the strength of an external magnetic field. (Experimentally, a system of ^{85}Rb atoms was investigated near the so-called Feshbach resonance conditions [37].)

Note that the dependence of $n_{\rm o}(x/L_{\rm o})$ on $N_{\rm o}$ is a highly nonlinear one. As a rough estimate, we can write $n_{\rm o} \propto N_{\rm o}^2$ $(T \to 0)$. This suggests that the estimates presented in [38] without any assumption on Bose-Einstein correlations among the excitons have to be revisited. In fact, if $T_{\rm cloud} < T_c$, a delicate balance between the two energies $E_{\rm cloud} = E(N_{\rm x},\, N_{\rm ph},\, v)$ and $E_{\rm coh.\,cloud} = E(N_{\rm coh} = N_{\rm o} + \delta N_{\rm o},\, v,\, |\mu|)$ (it has to be a free energy of the coherent packet at $T \neq 0$) defines the coherent and noncoherent components of the moving packet. This means that after the ordering process we can write $E_{\rm cloud} \to E_{\rm coh.\,cloud} + \Delta E$. For example, for the moving non-correlated exciton-phonon packet, we can estimate

$$E_{\rm cloud}(T=0) \simeq N_{\rm x}\{\tilde{E}_g + (\hbar^2/2m_{\rm x})\langle \mathbf{k}_0^2 \rangle + 2\nu_0\, n_{\rm x}\,\} + N_{\rm ph}\,\langle \hbar\omega'_{\rm ac} \rangle,$$

where $\hbar\langle k_{0\,x}\rangle = m_{\rm x}\langle v\rangle < m_{\rm x}\, c_{\rm s}$, $\langle k_{0\,y}\rangle = \langle k_{0\,y}\rangle \approx 0$, $\nu_0 > 0$ is the "bare" exciton-exciton interaction strength, and $n_{\rm x}$ is the average 3D density of excitons in the packet, and, for the phonon energies of the acoustic and optic branches, $\hbar\omega'_{\rm ac} < \hbar\omega_{\rm opt,\,min}$. Recall that the average $\langle k_{0\,x}\rangle \neq 0$ is the result of interaction between the paraexcitons and the phonon wind, and $N_{\rm ph} \simeq N_{\rm x}$.

At $T \ll T_c$, we can use the following estimate for the energy of the moving condensate ($N_{\rm coh} = N_{\rm o}$):

$$\begin{aligned} E_{\rm coh.\,cloud} &\to E_{\rm o}(N_{\rm o},\, \mu,\, v) = E_{\rm x,o} + E_{\rm int,o} + E_{\rm ph,o} \approx \\ &\approx N_{\rm o}\left(\tilde{E}_g + \frac{m_{\rm x}\, v^2}{2}\right) - N_{\rm o}\left(|\tilde{\mu}| + \nu_0\,\Phi_{\rm o}^2/3\right) + \\ &+ N_{\rm o}\left\{\frac{M\,(c_{\rm s}^2 + v^2)}{2}\,\gamma^2(v)\,\left(\frac{\sigma_0}{Mc_{\rm s}^2}\right)^2\,\frac{2}{3}\,(a_l^3\,\Phi_{\rm o}^2)\right\}. \end{aligned} \tag{37}$$

Here, Φ_o is the dimensional amplitude of $\phi_o(x) = \Phi_o\, f(x)$. As a rough estimate, we have (assuming the interaction vertices $\nu_1 \to 0$ and $\kappa_3 \to 0$)

$$|E_{\text{int, o}}|/N_o \approx |\widetilde{\nu_0}\,(v)|\, \Phi_o^2/2 + \nu_0\, \Phi_o^2/3 \simeq (\nu_0/a_x^3)\left(a_x^3\Phi_o^2\right) \ll E_x, \tag{38}$$

$$E_{\text{ph, o}}/N_o \approx \frac{M\,(c_s^2+v^2)}{2}\, \vartheta(N_o,\, v) \simeq M c_s^2\, \vartheta(N_o,\, v), \tag{39}$$

where

$$\vartheta(N_o,\, v) = \left(\gamma(v)\, \frac{\sigma_0}{M c_s^2}\right)^2 \frac{2}{3}\, (a_l^3\, \Phi_o^2) \ll 1, \tag{40}$$

σ_0 is the strength of exciton-phonon interaction. Within the harmonic approximation for lattice displacement field and the s-wave scattering approximation for the exciton-exciton interaction, we can write $a_x^3\Phi_o^2 \propto \left(N_o/N_o^*\right)^2$, where N_o^* is a large number, $N_o^* \sim S_\perp/a_x^2$. Thus, if the condensate is formed, the scale of the energy changement per exciton is very small, so the most important thing is phasing among the particles. Some qualitative results obtained for the coherent exciton-phonon packets are presented on Fig. 5.

In this thesis, we discuss another transport problem in the theory of superfluidity: the problem of critical velocity. It is known that the vortex model of superfluid dissipation cannot predict correctly the value of the critical velocity of superfluidity in planar geometry. To address this problem, we discuss a model based on the observation that the existence of superfluidity of the Bose-gas can depend on the strength of boundary interactions with channel walls. Obviously, the moving superfluid liquid interacts with the channel which contains it. In the simplest case, for example, the dilute moving Bose-gas interacts with the walls via hard-core repulsion and boundary excitations can be introduced. In particular, the coherent boson-phonon excitations can exist in the system "gas + walls". Mathematically, the key observation is that a certain class of periodic (so-called *elliptic*) solutions of the Nonlinear Schrödinger equation,

$$\mu\, \phi_o(x) = -(\hbar^2/2m_x)\, \partial_x^2\phi_o(x) + \nu_0\, \phi_o^3(x), \quad \phi_o(x=0) = \phi_o(x=L) = 0, \tag{41}$$

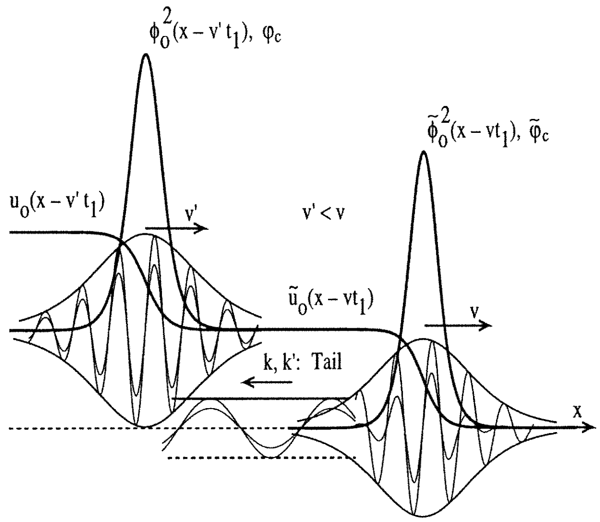


Figure 5: Two comoving ballistic packets.

Two ballistic packets with two exciton-phonon condensates ($e^{i\varphi_c(x,t)}\phi_o(x-x_0-vt)\cdot u_o(x-x_0-vt)\,\delta_{1j}$) inside were created with the same concentration of excitons and the same ballistic velocity, v, in a crystal. (The face 'ad' on Fig. 1 was irradiated by the same laser pulse two times.) The time delay between the pulses, $\delta\tau$, is a free parameter. Roughly, $\delta\tau \simeq (x_{0,1} - x_{0,2})\,c_s$, see Eq. (4). Then, two different interaction regimes are possible. The first one corresponds to the case in which the Bose-cores of the packets overlap. This is a strong interaction case, and the packets can merge into one droplet. The second regime, in which the Bose-cores do not overlap, is the case of weak interaction between the packets. It is depicted on this Figure. However, the second moving packet (the left one) can "feel" the first packet (the right one) through the interaction with the exciton-phonon tail of the first one. As a result, the second packet slows down, $v' < v$, and becomes more broad.

can be employed to describe the behavior of inhomogeneous excitations in the system "Superfluid Condensate + Channel Walls". Here, L is the width of the channel, and the condensate wave function is

$$\Psi_{\mathrm{o}}(\mathbf{x}, t) \sim \exp(-i\omega_0 t)\, \exp(ik_0 y)\, \phi_{\mathrm{o}}(x).$$

The walls occupy the area outside the band (x, y): $0 < x < L$. The boson-phonon excitations can reduce the value of the critical velocity of a superfluid. Such surface modes seem to exist in "soft matter" containers with flexible walls; they can be one of the sources of friction in anomalous excitonic transport in semiconductor heterostructures as well.

List of articles included in the present thesis

The first two articles (Ch. 1) were written by Dr. I. Loutsenko and me (D.R.), and our contributions can be considered as being equal. Meanwhile, I am the principal author of the other articles (Chs. 2 –4) in the list.

Chapter 1

1. Critical Velocities in Exciton Superfluidity, **I. Loutsenko, D. Roubtsov**, Physical Review Letters 78, (1997) pp. 3011-3014

2. Loutsenko and Roubtsov Reply on Tikhodeev's Comment, **I. Loutsenko, D. Roubtsov**, Physical Review Letters 84, (2000) p. 3503

Chapter 2

1. Moving stationary state of Exciton-Phonon Condensate in Cu_2O, **D. Roubtsov, Y. Lépine**, Physica Status Solidi B 210, (1998) pp. 127-143

2. Bosons in a Lattice: Exciton-Phonon Condensate in Cu_20, **D. Roubtsov, Y. Lépine**, Physical Review B 61, (2000) pp. 5237-5253

Chapter 3

1. On Bose-Einstein condensate inside moving exciton-phonon droplets, **D. Roubtsov, Y. Lépine, I. Loutsenko**, Physics Letters A 265, (2000) pp. 145-152

Chapter 4

1. On reduction of critical velocity in a model of superfluid Bose-gas with boundary interactions **D. Roubtsov, Y. Lépine**, Physics Letters A 246, (1998) pp. 139-147

Chapter 1

Solitons and exciton superfluidity

The first article, in which a many-particle generalization of the Davydov soliton was introduced, is included into this chapter. The quasi-one dimensional model for the excitonic soliton is derived and solved at $T \to 0$, but only the outside-condensate excitations were taken into account to investigate the stability of the soliton. It turns out that to describe the moving condensate of excitons one can use the nonlinear Schrödinger equation known in the theory of BEC as Gross-Pitaevskii equation. Note that the repulsive excitons can form the bright soliton state ($\propto \exp(ik_0x)/\cosh(x-vt)$, [29]) due to the coherent phonons of the exciton-phonon packet.

This article raised discussion whether a moving exciton-phonon packet could contain a condensate so that a macroscopic wave function of the correlated excitons could be introduced. For example, S. Tikchodeev and the co-authors proposed to describe the moving packet of excitons by a kind of diffusion equation [34],[5]. To obtain the soliton-like shape of the concentration of excitons, $n(x,t) \simeq n(x-vt,\, t) \propto 1/\cosh^2(x-vt)$ instead of $n(x,t) \simeq \exp(-x^2/Dt)$ typical for the diffusion process, they introduce a special drift term in their dynamic model. This term is responsible for the interaction between the two packets, the excitons and phonon wind.

The arguments supporting the interpretation in terms of the Bose-Einstein condensation are presented in the end of this chapter in the form of a short reply to Tikchodeev's criticism [38]. For example, the physical phenomena sensitive to the phasing of the excitons in the condensate could be observed; they are absent within any model based on the classical diffusion or KdV equations, see also [39].

Critical Velocities in Exciton Superfluidity

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Abstract

The presence of exciton phonon interactions is shown to play a key role in the exciton superfluidity. It turns out that there are essentially two critical velocities in the theory. Within the range of these velocities the condensate can exist only as a bright soliton. The excitation spectrum and differential equations for the wave function of this condensate are derived.

PACS numbers: 05.30.Jp, 63.20.Ls, 64.60.Ht, 67.40.Db

The problem of critical velocities in the theory of superfluidity arose a long time ago when the experiments with liquid He showed a substantial discrepancy with quantum-mechanical predictions. Later, the effect was analyzed and its phenomenological description was given (e.g. see [1]). The fact that liquid He could not be treated as a weakly non-ideal Bose gas was believed to be the main reason for the inconsistency between the microscopic theory and experimental data.

For a long time He has been the only substance where superfluidity can be observed. The recent experiments with a dilute gas of excitons [2], [3] provide new possibilities for studying different types of superfluidity.

In this series of experiments a Cu_2O crystal was irradiated with laser light pulses of several ns duration. At low intensities of the laser beam (low concentration of excitons) the system revealed a typical diffusive behavior of exciton gas. Once the intensity of the beam exceeds some value, the majority of particles move together in the packet. Their common propagation velocity is close to the longitudinal sound velocity, and the packet evolves as a bright soliton.

Some alternative explanations of the phenomena are known. One of them [2] implies that the bright soliton is a one-dimensional traveling wave which satisfies the Gross-Pitaevskii (nonlinear Schrödinger) equation [4] for the Bose-condensate wave function $\Psi(\mathbf{x}, t)$

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + \nu\Psi^*\Psi^2 \tag{1}$$

with attractive potential of exciton-exciton interaction $\nu < 0$.

A quantitative treatment given in [5] provides an iterative solution for the Heisenberg equation with the use of perturbational methods. In this picture the second order interactions, neglected in the Bogoliubov approximation, contribute to the negative value of ν. However, the influence of exciton-phonon interactions on the dynamics of the condensed excitons is not treated [5].

Another interpretation is based on a classical model [6] where the normal exciton gas is pushed towards the interior of a sample by the phonon wind emanating from the surface. Such an explanation seems to be in discrepancy with the experiment, because the signal observed is one order of magnitude longer than the excitation pulse duration [3].

In this study we give an alternative and, in our opinion, more intrinsic interpretation of these phenomena. We argue that it is a propagation of a superfluid *exciton-phonon condensate* which is observed experimentally. The presence of exciton-phonon interactions is crucial for a "soliton-like superfluidity". This interaction plays a key role when the propagation velocity approaches the longitudinal sound velocity.

We start with the Hamiltonian of the exciton-phonon system

$$H = H_{\mathrm{ex}} + H_{\mathrm{ph}} + H_{\mathrm{int}},$$

$$H_{\mathrm{ex}} = -\frac{\hbar^2}{2m}\int \hat{\Psi}^*(\mathbf{x})\Delta\hat{\Psi}(\mathbf{x})\mathrm{d}\mathbf{x} + \frac{1}{2}\int \hat{\Psi}^*(\mathbf{x})\hat{\Psi}^*(\mathbf{y})\,\nu(\mathbf{x}-\mathbf{y})\,\hat{\Psi}(\mathbf{x})\hat{\Psi}(\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y},$$

$$H_{\mathrm{ph}} = \int \left\{\frac{1}{2\rho}\hat{\pi}(\mathbf{x})^2 + \frac{c^2\rho}{2}\left(\nabla\,\hat{\mathbf{u}}(\mathbf{x})\right)^2\right\}\mathrm{d}\mathbf{x},$$

$$H_{\mathrm{int}} = \int \sigma(x-y)\hat{\Psi}^*(\mathbf{x})\hat{\Psi}(\mathbf{x})\left(\nabla\,\hat{\mathbf{u}}(\mathbf{y})\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}, \qquad (2)$$

where $\hat{\Psi}$ and $\hat{\mathbf{u}}$ are the operators of the exciton and phonon fields correspondingly, c is the longitudinal sound velocity and ρ denotes the mass density of the crystal. The field variables obey the following commutation relations

$$\left[\hat{\Psi}(\mathbf{x}), \hat{\Psi}^*(\mathbf{y})\right] = \delta(\mathbf{x}-\mathbf{y}), \quad [\hat{\pi}_i(\mathbf{x}), \hat{u}_j(\mathbf{y})] = -i\hbar\,\delta_{ij}\,\delta(\mathbf{x}-\mathbf{y}),\ i,j = 1,2,3.$$

In (2) we omit the terms with the transverse sound velocity, since the interaction of excitons with transverse sound waves is much weaker than with the longitudinal ones.

It is convenient to change the reference system when we consider a uniform motion of the Bose gas. The transition to the reference system moving uniformly with the velocity $\mathbf{v} = (v, 0, 0)$ is immediate. In new coordinates the classical field equations become:

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\Delta + \frac{mv^2}{2} - \int \nu(\mathbf{x}-\mathbf{y})|\psi(\mathbf{y},t)|^2\mathrm{d}y^3\right)\psi(\mathbf{x},t)$$

$$= \psi(\mathbf{x}, t) \int \sigma(\mathbf{x} - \mathbf{y}) \, (\nabla \mathbf{u}(\mathbf{y}, t)) \, d\mathbf{y} \tag{3}$$

$$\left(\frac{\partial^2}{\partial t^2} - 2v \frac{\partial^2}{\partial t \partial x_1} + v^2 \frac{\partial^2}{\partial x_1^2} - c^2 \Delta \right) \mathbf{u}(\mathbf{x}, t) = \frac{1}{\rho} \nabla \int \sigma(\mathbf{x} - \mathbf{y}) |\psi(\mathbf{y}, t)|^2 d\mathbf{y}, \tag{4}$$

where $\psi(\mathbf{x}, t) = \Psi(x_1 + vt, x_2, x_3, t) \exp(-imvx_1/\hbar)$.

The l.h.s. of Equation (3) is Galileian invariant, while the l.h.s. of (4) is Lorentz invariant. As a result, the system (3), (4) is neither Galileian nor Lorentz invariant. As we will see later, it is due to this noninvariance that the effective potential of exciton-exciton interactions depends on velocity.

Let us consider slowly varying solutions of the system (3), (4). In this (long wavelength) limit one can replace $\nu(\mathbf{x})$ and $\sigma(\mathbf{x})$ by $\nu_0 \delta(\mathbf{x})$ and $\sigma_0 \delta(\mathbf{x})$, where $\nu_0 \, (> 0)$ and σ_0 denote the zero-mode Fourier components of the corresponding potentials.

Solving (4), one can express the bounded at infinity time-independent solution $\mathbf{u}(x)$ in terms of $\psi(\mathbf{x})$. The effective potential of the exciton-exciton interaction is obtained after substituting this expression into (3). The phonon field makes this potential long-range, anisotropic and v-dependent. The potential becomes asymptotically attractive along the $\mathbf{v}$-direction and asymptotically repulsive in directions perpendicular to $\mathbf{v}$. It follows that stability of the corresponding solutions $\psi = \phi(x_1) \exp(-i\omega_0 t)$, $u_i = \delta_{i1} q(x_1)$ is preserved under the one-dimensional reduction of the system (3), (4). The functions $\phi(x_1)$, $q(x_1)$ obey the following equations

$$\left(\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_1^2} - \lambda \right) \phi(x_1) = \left(\nu_0 - \frac{\sigma_0^2}{(c^2 - v^2)\rho} \right) \phi(x_1)^3, \quad \lambda = -\hbar\omega_0 - \frac{mv^2}{2} + C\sigma_0, \tag{5}$$

$$\frac{\partial q(x_1)}{\partial x_1} = C - \frac{\sigma_0 \phi(x_1)^2}{(c^2 - v^2)\rho}, \tag{6}$$

where the integration constant C is fixed by the condition $q \to \text{const}$ as $|x_1| \to \infty$. In the last equations ϕ assumed to be real. This choice does not change the result but simplifies our calculations.

It follows from (5) that the effective potential becomes attractive when v exceeds the critical velocity

$$v_0 = \sqrt{c^2 - (\sigma_0^2/\nu_0\rho)}. \tag{7}$$

But if v exceeds the sound velocity c, the potential becomes repulsive again. As for the solution varying in the direction $\mathbf{n} = (n_1, n_2, n_3)$, $\psi = f(\mathbf{n}\,\mathbf{x})\exp(-i\omega_0 t)$, $\mathbf{u} = \mathbf{n}q(\mathbf{n}\,\mathbf{x})$, the critical velocity is $v_0(\mathbf{n}) = v_0/\cos(\theta)$, $|\cos(\theta)| > v_0/c$, where θ is the angle between $\mathbf{n}$ and $\mathbf{v}$.

When v is less than the critical velocity (7), equations (5),(6) have the following stable stationary solutions

$$\text{(i)} \quad \phi = \phi_0 = \sqrt{N/V} = \text{const}, \quad \mathbf{u} = \text{const}, \quad \text{and}$$

$$\text{(ii)} \quad \phi = \phi_0 \tanh\left(\beta\phi_0(x_1 - a)\right), \frac{\partial q(x_1)}{\partial x_1} = -\frac{\sigma_0\phi_0^2}{(c^2 - v^2)\rho}\frac{1}{\cosh^2\left(\beta\phi_0(x_1 - a)\right)}, \tag{8}$$

$$\beta = \sqrt{\frac{m\nu_0}{\hbar^2}\frac{|v_0^2 - v^2|}{|c^2 - v^2|}}, \lambda = \left(\frac{\sigma_0^2}{(c^2 - v^2)\rho} - \nu_0\right)\phi_0^2 = \nu_0\phi_0^2\frac{v^2 - v_0^2}{c^2 - v^2}, \; C = \frac{\sigma_0\phi_0^2}{(c^2 - v^2)\rho}.$$

In (8,i) N and V stand for the number of particles in the condensate and the volume of the system.

When v exceeds v_0, we have only one stable stationary solution

$$\phi = \frac{\phi_0}{\cosh\left(\beta\phi_0(x_1 - a)\right)}, \quad \frac{\partial q(x_1)}{\partial x_1} = -\frac{\sigma_0\phi_0^2}{(c^2 - v^2)\rho}\frac{1}{\cosh^2\left(\beta\phi_0(x_1 - a)\right)},$$

$$\lambda = \frac{\phi_0^2}{2}\left(\frac{\sigma_0^2}{(c^2 - v^2)\rho} - \nu_0\right) = \frac{\nu_0\phi_0^2}{2}\frac{v^2 - v_0^2}{c^2 - v^2}, \quad C = 0. \tag{9}$$

To find the excitation spectrum of the system we expand the field operators near the proper classical solutions:

$$\hat{\psi}(\mathbf{x}, t) = \left(\phi(x_1) + \hat{\chi}(\mathbf{x}, t)\right)e^{-i\omega_0 t}, \quad \hat{u}_i(\mathbf{x}, t) = \delta_{i1}q(x_1) + \hat{\eta}_i(\mathbf{x}, t).$$

The Hamiltonian of the system can be written as follows

$$H = H_0 + \hbar H_2 + \ldots, \tag{10}$$

where $H_0 = H(\phi e^{-i\omega_0 t}, q)$ stands for the classical part of H. It is important that H_2 is bilinear in $\hat{\chi}(\mathbf{x},t)$, $\hat{\eta}(\mathbf{x},t)$, whereas the linear terms are absent in (10) (since the classical fields satisfy the stationary equations (5),(6)). From now on we are working in quasiclassical approximation and neglecting the terms of power greater than one (in $\hbar$).

The quasiclassical Hamiltonian (10) is reduced to the normal form

$$H_2 = \sum_i \omega_i \hat{b}_i^* \hat{b}_i + \text{const}, \quad \left[\hat{b}_i, \hat{b}_j^*\right] = \delta_{ij}, \left[\hat{b}_i, \hat{b}_j\right] = 0. \tag{11}$$

Indeed, since H_2 is a bilinear function of $\hat{\chi}$, $\hat{\eta}$, the equations of motion are linear in field operators. They coincide with the corresponding classical equations (i.e. equations (3),(4) linearized around $\psi(\mathbf{x},t) = \phi(x_1)\exp(-i\omega_0 t)$, $u_i(\mathbf{x},t) = \delta_{i1}q(x_1)$):

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\Delta - \lambda + C\sigma_0 + \left\{\frac{\sigma_0^2}{(c^2-v^2)\rho} - 2\nu_0\right\}\phi(x)^2\right)\chi$$

$$-\nu_0\phi(x)^2\chi^* - \sigma_0\phi(x)(\nabla\,\boldsymbol{\eta}) = 0, \tag{12}$$

$$\left(c^2\Delta - v^2\frac{\partial^2}{\partial x_1^2} + 2v\frac{\partial^2}{\partial t\partial x_1} - \frac{\partial^2}{\partial t^2}\right)\boldsymbol{\eta} + \frac{\sigma_0}{\rho}\nabla\left(\phi(x)(\chi+\chi^*)\right) = 0. \tag{13}$$

The quantities ω_i in (11) are characteristic frequencies of the system (12), (13).

Let us consider the homogeneous Bose gas moving uniformly with velocity $v < v_0$. The condensate wave function is given by (8,i). The differential equations (12),(13) have constant coefficients so that the characteristic frequencies $\omega(\mathbf{k})$ are determined as roots of the following characteristic polynomial

$$(\Omega^2 - c^2k^2)\left[\hbar^2(\Omega + vk_1)^2 - \frac{\hbar^2k^2}{2m}\left(\frac{\hbar^2k^2}{2m} + \left\{2\nu_0 - \frac{\sigma_0^2}{(c^2-v^2)\rho}\right\}\phi_0^2\right)\right] -$$

$$\frac{\hbar^2k^2}{2m}\frac{\sigma_0^2\phi_0^2k^2}{\rho} = 0, \tag{14}$$

where $\Omega = \omega(\mathbf{k}) - vk_1$ are the excitation frequencies in the crystal reference frame. In the limit $\sigma_0 \to 0$ one gets the Bogoliubov [7] spectrum $\hbar\omega(k) = \sqrt{\frac{\hbar^2k^2}{2m}\left(\frac{\hbar^2k^2}{2m} + 2\nu_0\phi_0^2\right)}$ for the exciton gas as well as the free phonon spectrum

$\Omega = ck$. When we switch on an exciton-phonon interaction, the spectrum $\omega(\mathbf{k})$ becomes $\mathbf{v}$-dependent.

The quantization near the translationally noninvariant classical solution (8,ii) in the region $v < v_0$ yields the same continuous spectrum $\omega(\mathbf{k})$. The only new feature is that there appears a bounded state at $\omega = 0$ in the $\mathbf{v}$-direction. This fact has a simple explanation: the family of the solutions (8,ii) contains an arbitrary translation parameter a, which, in fact, is a collective coordinate. Differentiation of (8,ii) with respect to a gives then necessary time independent solution of (12),(13). This bounded state does not affect the quasiclassical excitation spectrum and contributes only to highest approximations (e.g. see [8]).

If the velocity v exceeds (7), the characteristic polynomial (14) has complex roots and there is no stable constant solutions. The condensate (i.e. classical) wave function turns into the (bright) soliton (9) of the one-dimensional nonlinear Schrödinger equation (5). This solution decreases exponentially. This allows us to obtain the continuous spectrum from asymptotics of (12), (13). We have

$$\hbar\omega(\mathbf{k}) = \lambda + \frac{\hbar^2 k^2}{2m}$$

for the exciton branch of the model, and

$$\omega(\mathbf{k}) = ck + vk_1$$

for the phonon branch. As in the previous case we get a bounded state at zero energy. We skip the question of existence of other bound states, since it is not essential for our purposes.

The spectrum now has a *gap* in the exciton branch which is equal to λ. In a sense, the situation is similar to the BCS theory: the exciton-phonon interaction makes the effective exciton-exciton potential attractive, and the excitation spectrum acquires a gap.

The transition to the ballistic regime is accompanied by the symmetry breakdown: a new condensate wave function (9) is no more translationally invariant.

However, it contains a free translation parameter. We can interpret this as a phase transition of the second order.

The value ϕ_0 is readily computed from the normalization condition $\int \phi(\mathbf{x})^2 \mathrm{d}\mathbf{x} = N$, and λ is then obtained from (9)

$$\lambda = \frac{m\nu_0^2 N^2}{8\hbar^2 S^2} \left(\frac{v^2 - v_0^2}{c^2 - v^2} \right)^2 . \tag{15}$$

In (15) S denotes the packet cross-section in $x_2 x_3$-plane. When v approaches the longitudinal sound velocity c, the gap magnitude increases and the soliton becomes more stable. The soliton energy can be estimated from (2)

$$E = N \left\{ \frac{m\nu_0^2 N^2}{24\hbar^2 S^2} \frac{(v^2 - v_0^2)}{(c^2 - v^2)^3} \left(v^4 + 3v^2 c^2 + v_0^2 c^2 - 5 v_0^2 v^2 \right) + \frac{mv^2}{2} \right\} + \ldots .$$

It follows from the last formula that $E \to \infty$ as $v \to c$. Roughly speaking, the soliton effective mass tends to infinity when its speed approaches the longitudinal sound velocity. Then its motion is less subjected to the external forces.

The onset of ballistical regime is determined by the condition $v > v_0$. It is easy to see that the solution (9) is the most stable in the class of one-dimensional traveling waves moving uniformly with given $v (> v_0)$ and N. We argue that (9) is also the most stable solution in the class of all solutions with given $v (> v_0)$ and N, because the effective exciton-exciton potential is attractive in the $\mathbf{v}$-direction and repulsive in the perpendicular directions. We would like to stress that effective one-dimensional solutions of three-dimensional nonlinear Schrödinger equations (1) with attractive potentials do not have the similar properties. In particular, the stability of such solutions is doubtful [9].

In the present work we have discussed the properties of the system at zero temperature. The extension of our results to finite temperatures seems to be a more difficult problem.

We hope that the similar approach (involving solitonic mechanisms) can be applied to the solution of the general problem of critical velocities in the superfluidity of liquid helium.

References

[1] Zygmunt M.Galasiewicz, Helium 4, Pergamon Press, 1971

[2] E.Fortin, S.Fafard, A.Mysyrowicz, Phys.Rev.Lett. 70, 3951 (1993)

[3] E.Benson, E.Fortin, A.Mysyrowicz, Phys.Stat.Sol.B 191, 345 (1995)

[4] E.P.Gross, Nuovo Cimento 20, 454 (1961)

[5] E.Hanamura, Sol.Stat.Com. 91 889 (1994)

[6] A.E.Bulatov, S.G.Tichodeev, Phys.Rev.B 46, 15058 (1992)

[7] N.N.Bogoliubov, J.Phys.USSR 11, 23 (1947)

[8] Christ N.H., Lee T.D., Phys.Rev.D 12, 1606 (1975)

[9] Kuznetsov E.A., Rubenchik A.M., and Zakharov V.E., Phys.Lett.C 142, 103 (1986)

Loutsenko and Roubtsov Reply: The model proposed in [1] can be considered as a model describing the anomalous transport of nonrelativistic bosons in a medium with the linear dispersion law at zero temperature.

The main idea of Letter [1] is that the effective two-particle interaction between bosons switches from being repulsive to the attractive one if the velocity of collective propagation of the bosons in the medium exceeds some critical speed. This effect occurs due to different covariance of the dynamic equations that describe the nonrelativistic subsystem of bosons and the linearly dispersive medium. As a result, the boson subsystem "collapses" into a soliton wave packet above this threshold.

Note that the effect of such an anomalous propagation is the quantum one, since Bose-Einstein condensation (more exactly, occurrence of the exciton-phonon Bose condensate) is essential in our consideration.

The theory developed in [1] was applied to interpret the experiments on excitonic propagation in semiconducting crystals (Cu_20) at nonzero temperature [2]. In this series of experiments, a crystal was irradiated by laser pulses. At low intensities of the laser beam (i.e. at low concentration of the excitons), the system revealed a typical *diffusive* behavior. Once the intensity of the beam exceeded some value, the majority of particles moved together in a sharp *solitonic* packet with the ballistic velocity exceeding some critical speed. This coincides with the main result of our theory.

Although this theory yields a qualitative description of the experiments and a reasonable value for the critical velocity, the estimate of the width of the condensate at zero temperature is in a strong disagreement with experimental data [2] for the total exciton-phonon packet (at finite temperature). Indeed, we obtain [1]

$$L_{\mathrm{ch}}(N_{\mathrm{o}},\, v) \simeq 4\sqrt{\frac{\hbar^2}{m_{\mathrm{x}}\,|\tilde{\nu}_0(v)|}\,\Phi_{\mathrm{o}}^{-2}(N_{\mathrm{o}},\, v)}\,, \tag{1}$$

i.e., $L_{\mathrm{ch}} = \mathcal{F}\, a_{\mathrm{x}}$, where the large factor $\mathcal{F}$ can vary as $10^2 \sim 10^4$, and a_{x} is the exciton Bohr radius. Thus, the duration of the condensate can be estimated as

$t_{\rm ch} \simeq 2 \times (10^{-11} - 10^{-9})\,$s (compare with the corresponding estimate in [3]).

On the other hand, $\Delta t \simeq 5 \times 10^{-7}\,$s was obtained experimentally [2].

This fact was pointed out by S. G. Tikhodeev in Comment [3]. In our opinion, the crucial question is whether the Bose-Einstein condensate, or, better, any macroscopically occupied coherent mode, exists inside the exciton-phonon packet at $T < T_c$. If yes, one can ask, for example, how many excitons form the coherent core of the packet, and the value of $N_{\rm o}(T)/N_{\rm tot}$ has to be estimated at $T < T_c$.

Indeed, localized moving solutions for the exciton concentration $n(x,t)$ can be obtained within classical models (see, e.g., [4]), in which the Bose-Einstein condensate of excitons is absent and the excitonic cloud is dragged by the sound wave of a large amplitude created under the action of the strong laser pulse.

Here, we list several facts that support the idea that the localized excitonic condensate has to be taken into account to interpret experimental data [2].

- Experiments are conducted at nonzero temperature, and the excitonic condensate (if it appears) can constitute of a relatively small fraction of the total number of particles. We considered the system bosons + medium at zero temperature. Experimental data, however, show strong dependence of the packet length on the temperature (at least, an order of magnitude in the range of $2 - 5\,$K). Thus, no conclusion can be made until our theory is extended to nonzero temperatures or experiments are conducted at much more lower temperatures.

- All the results on nonlinear interaction between two packets [2] point out to a kind of coherent interaction. This is expected within our quantum model of the moving exciton-phonon droplet with the "Bose-nucleus". It is an open question whether such a behavior can be the case within any classical model.

- If, according to experiments, the phonon source is generated at a surface by a strongly absorbed radiation, the phonon wind [4] does not effect strongly the excitons with a density just below the threshold one, and no appreciable effect is detected. On the contrary, the injection of cold excitons distributed throughout the volume leads to the appearance of a localized packet in the above conditions.

We believe that the future theory needs both the exciton-phonon condensate and the proper incorporation of non-condensed excitons and phonons. It can be tested by further experiments as well. For example, one can set the crystal geometry in such a way that the sound wave of the phonon wind theory can be dumped out, and only the coherent part of the packet (if it exists) will continue to propagate.

In conclusion, we note that the phonons play a crucial role in almost all current models aimed to explain or predict coherent behavior of excitons in semiconductors, see, f.ex., [5],[6],[7]. The fact that our simple model fails to predict the width of the packet correctly does not mean our approach is wrong. Indeed, one has to take into account the thermal excitons (e.g., the weak tail that is always observed behind the soliton [2]) and the thermal phonons of the crystal to make the model with *condensate* more realistic.

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References

[1] I. Loutsenko, D. Roubtsov, Phys. Rev. Lett. **78**, 3011 (1997).

[2] A. Mysyrowicz, E. Benson, and E. Fortin, Phys. Rev. Lett. **77**, 896 (1996); E. Benson, E. Fortin, A. Mysyrowicz, Sol. Stat. Comm. **101**, 313, (1997); E. Benson, E. Fortin, B. Prade, and A. Mysyrowicz, Europhys. Lett. **40**, 311 (1997).

[3] S. G. Tikhodeev, preceding Comment, Phys. Rev. Lett. **84**, 3502 (2000).

[4] A. E. Bulatov, S. G. Tichodeev, Phys. Rev. B **46**, 15058 (1992); G. A. Kopelevich, S. G. Tikhodeev, and N. A. Gippius, JETP **82**, 1180 (1996).

[5] Yu. E. Lozovik, A. V. Poushnov, JETP **88**, 747 (1999).

[6] W. Zhao, P. Stenius, A. Imamoğlu, Phys. Rev. B **56**, 5306 (1997).

[7] H. Haug, A. L. Ivanov, and L. V. Keldysh, *Nonlinear Optical Phenomena in Semiconductors and Semiconductor Microstructures*, (World Scientific, 1999).

Chapter 2

Exciton-Phonon packets with condensate: detailed description

In this chapter, we included two articles which generalize the approach presented in Chapter 1. Almost all the important formulas are derived and discussed. In the first article, we show how the quasi-one dimensional (toy) model follows from a difficult three-dimensional one. In addition, inside-condensate excitations are introduced. In the limit of $T \to 0$, the inside-excitations have the same characteristic length as the moving condensate. Indeed, the equations to solve are a kind of the Bogoliubov-de Gennes equations which are linear and the coefficients depend on the condensate envelope function, $\propto \text{const}\, \phi_{\text{o}}^{n}(x/L_{\text{o}})$, $n = 2, 4$.

It turns out that within the model with contact interactions (both exciton-exciton and exciton-phonon ones) we need the higher interaction terms, $\hat{H}_{\text{x-x}} \simeq \int d\mathbf{x}\, (\nu_1/3)\, (\psi^{\dagger})^3\, (\psi)^3$ and $\hat{H}_{\text{ph-ph}} \simeq \int d\mathbf{x}\, (\kappa_3/3)\, (\nabla \mathbf{u})^3$ to avoid the problems with the low-$\mathbf{k}$ part of the inside-excitation spectrum. Then, instead of the standard NLS equation, we have the so-called "subcritical" NLS equation to describe the coherent state of excitons and phonons with macroscopic occupancy. Note that in Condensed Matter Physics it is used in the theory of superfluidity and Bose-Einstein condensation [40]. It is also applied in nonlinear optics for light pulses in the medium with a cubic-quintic nonlinearity [41], and it is known as the Lienard equation in the theory of exactly solvable nonlinear equations [42]. Similar to the model considered in Chapter 1, the repulsive excitons can form the bright soliton state ($\propto \exp(ik_0 x)\, (a + b\, \cosh^2(x - vt))^{-1/2}$) due to the coherent phonons of the exciton-phonon packet.

The stability of the ballistic transport against the emission of excitations is studied in the second article in detail. The Landau arguments are based on calculation of the energy and momentum of the moving condensate with an excitation (or several excitations) so that $\delta E \simeq \hbar\, (\omega_{j,\mathbf{k}} + k_x v) > 0$. Here, they are employed to estimate the value of the critical velocity. As for the interference of two condensates, (recall that we prescribe them the coherent phases, $\varphi_{c,1}$ and $\varphi_{c,2}$), it is the nonlinear terms of the NLS equation that makes such an interference more like the soliton-soliton interaction than the constructive / deconstructive interference of the linear quantum mechanics. However, we are mostly interested in the stage of the strong nonlinear interaction itself, i.e., what happens during the time when the condensate wave functions overlap.

Moving Stationary State of Exciton - Phonon Condensate in Cu_2O

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Abstract

We explore a simple theoretical model to describe the properties of Bose condensed para-excitons in Cu_2O. Taking into account the exciton - phonon interaction and introducing a coherent phonon part of the moving condensate, we derive the dynamic equations for the exciton - phonon condensate. Within the Bose approximation for excitons, we discuss the conditions for the moving inhomogeneous condensate to appear in the crystal. We calculate the condensate wave function and energy and a collective excitation spectrum in the semiclassical approximation. The stability conditions of the moving condensate are analyzed by use of Landau arguments, and two critical velocities appear in the theory. Finally, we apply our model to describe the recently observed interference between two coherent exciton - phonon packets in Cu_2O.

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1 Introduction

Excitons in semiconductor crystals [1] and nanostructures [2] are a very interesting and challenging system to search for the process of Bose Einstein condensation (BEC). Nowadays there is a lot of experimental evidence that the optically inactive para-excitons in Cu_2O can form a highly correlated state, or the excitonic Bose Einstein condensate [1],[3],[4]. A moving condensate of para-excitons in a 3D Cu_2O crystal turns out to be spatially inhomogeneous in the direction of motion, and the registered velocities of coherent exciton packets turn out to be always less, but approximately equal to the longitudinal sound speed of the crystal [5].

Analyzing recent experimental [3],[5],[6] and theoretical [7]-[10] studies of BEC of excitons in Cu_2O, we can conclude that there are essentially two different stages of this process. The first stage is the kinetic one, with the characteristic time scale of $10 \sim 20\,\mathrm{ns}$. At this stage, a condensate of long-living para-excitons begins to be formed from a quasi-equilibrium degenerate state of the gas of excitons ($\mu \neq 0$, $T_{\mathrm{eff}} > T_{\mathrm{latt}}$) when the concentration and the effective temperature of excitons in a cloud, T_{eff}, meet the conditions of Bose-Einstein Condensation [1]. Note that we do not discuss here the behavior of ortho-excitons (with the lifetime $\tau_{\mathrm{ortho}} \simeq 30\,\mathrm{ns}$) and their influence on the para-exciton condensation process. For more details about the ortho-excitons in Cu_2O, ortho-to-para-exciton conversion, etc. see [3],[4],[11].

The most intriguing feature of the kinetic stage is that formation of the para-exciton condensate and the process of momentum transfer to the para-exciton cloud are happening simultaneously. It seems that nonequilibrium acoustic phonons (appearing at the final stage of exciton cloud cooling) play the key role in the process of momentum transfer. Indeed, the theoretical results obtained in the framework of the "phonon wind" model [8],[12] and the experimental observations [3],[4],[5] are strong arguments in favor of this idea. To the authors' knowledge, there are no realistic theoretical models of the kinetic stage of para-exciton condensate formation where quantum degeneracy of the initial exciton state and possible coherence of nonequilibrium phonons pushing the excitons would be taken into account. Indeed, the condensate formation and many other processes involving it are essentially nonlinear ones. Therefore, the condensate, or, better, the macroscopically occupied mode, can be different from $n(\mathbf{k} = 0) \gg 1$, and the language of the states in $\mathbf{k}$-space and their occupation numbers $n(\mathbf{k})$ may be not relevant to the problem, see [13].

In this study, we will not explore the stage of condensate formation. Instead, we investigate the second, quasi-equilibrium stage, in which the condensate has already been formed and it moves through a crystal with some constant velocity and characteristic shape of the density profile. In theory, the time scale of this "transport" stage, Δt_{tr}, could be determined by the para-exciton lifetime ($\tau_{\text{para}} \simeq$ 13 μs [1]). In practice, it is determined by the characteristic size ℓ of a high-quality single crystal available for experiments:

$$\Delta t_{\text{tr}} \simeq \ell / c_l \simeq 0.5 - 1.5\,\mu\text{s} \ll \tau_{\text{para}},$$

where c_l is the longitudinal sound velocity.

We assume that at the "transport" stage, the temperature of the moving packet (condensed + noncondensed particles) is equal to the lattice temperature of the crystal,

$$T_{\text{eff}} = T_{\text{latt}} < T_c.$$

Then we can consider first the simplest case of $T = 0$ and disregard the influence of all sorts of *nonequilibrium* phonons (which appear at the stages of exciton formation, thermalization [8]) on the formed moving condensate.

Any theory of the exciton BEC in Cu_2O has to point out some physical mechanism(s) by means of which the key experimental facts can be explained. (For example, the condensate moves without friction within a narrow interval of velocities localized near c_l, and the shape of the stable macroscopic wave function of excitons resembles soliton profiles [6].) Here we explore a simple model of exciton-phonon condensate. In this case, the general structure of the Hamiltonian of the moving exciton packet and the lattice phonons is the following:

$$\hat{H} = H_{\rm ex}(\hat{\psi}^\dagger, \hat{\psi}) - \mathbf{v}\mathbf{P}_{\rm ex}(\hat{\psi}^\dagger, \hat{\psi}) + H_{\rm ph}(\hat{\mathbf{u}}, \hat{\pi}) - \mathbf{v}\mathbf{P}_{\rm ph}(\hat{\mathbf{u}}, \hat{\pi}) + H_{\rm int}(\hat{\psi}^\dagger\hat{\psi},\, \partial_j \hat{u}_k). \quad (1)$$

Here $\hat{\psi}$ is the Bose-field operator describing the excitons, $\hat{\mathbf{u}}$ is the field operator of lattice displacements, $\hat{\pi}$ is the momentum density operator canonically conjugate to $\hat{\mathbf{u}}$, $\mathbf{v}$ is the exciton packet velocity and, finally, $\mathbf{P}$ is the momentum operator. Note that the Hamiltonian (1) is written in the reference frame moving with the exciton packet, i.e. $\mathbf{x} \to \mathbf{x} - \mathbf{v}t$ and $\mathbf{v} = \text{const}$ is the packet velocity.

2 3D Model of Moving Exciton-Phonon Condensate

To derive the equations of motion of the field operators (and generalize these equations to the case of $T \neq 0$), it is more convenient, however, to start from the Lagrangian. In the proposed model, the Lagrangian density has the form

$$\begin{aligned}
\mathcal{L} = \frac{i\hbar}{2}(\hat{\psi}^\dagger\, \partial_t \hat{\psi} - \partial_t \hat{\psi}^\dagger\, \hat{\psi}) + v\frac{i\hbar}{2}(\partial_x \hat{\psi}^\dagger\, \hat{\psi} - \hat{\psi}^\dagger \partial_x \hat{\psi}) - \\
-E_g \hat{\psi}^\dagger \hat{\psi} - \frac{\hbar^2}{2m}\nabla\hat{\psi}^\dagger \nabla\hat{\psi} - \frac{\nu_0}{2}(\psi^\dagger(\mathbf{x},t))^2(\psi(\mathbf{x},t))^2 - \frac{\nu_1}{3}(\psi^\dagger(\mathbf{x},t))^3(\psi(\mathbf{x},t))^3 + \\
+\frac{\rho}{2}(\partial_t \hat{\mathbf{u}})^2 - \frac{\rho c_l^2}{2}\partial_j \hat{u}_s\, \partial_j \hat{u}_s - \rho v \partial_t \hat{\mathbf{u}}\, \partial_x \hat{\mathbf{u}} + \frac{\rho v^2}{2}(\partial_x \hat{\mathbf{u}})^2 - \\
-\sigma_0 \hat{\psi}^\dagger(\mathbf{x},t)\hat{\psi}(\mathbf{x},t)\nabla\hat{\mathbf{u}}(\mathbf{x},t), \qquad (2)
\end{aligned}$$

where m is the exciton "bare" mass ($m = m_e + m_h \simeq 3m_e$ for 1s excitons in Cu_2O), ν_0 is the exciton-exciton interaction constant ($\nu_0 > 0$ that corresponds to the repulsive interaction between para-excitons in Cu_2O [14]), ρ is the crystal density, σ_0 is the exciton-longitudinal phonon coupling constant, and $\mathbf{v} = (v, 0, 0)$. The energy of a free exciton is $E_g + \hbar^2\mathbf{k}^2/2m$. Although the validity of the condition [15] $n\tilde{a}_{\rm B}^3 \ll 1$ ($\tilde{a}_{\rm B}$ is the exciton Bohr radius) makes it possible to disregard all the multiple-particle interactions with more than two participating particles in $\hat{H}_{ex}$, we leave the hard-core repulsion term originated from the three-particle interaction in $\mathcal{L}$, i.e. $\nu_1 \neq 0$ and $0 < \nu_1 \ll \nu_0$. For simplicity's sake, we take all the interaction terms in $\mathcal{L}$ in the *local* form and disregard the interaction between the excitons and transverse phonons. The packet velocity $v \equiv v_s$ is one of the parameters of the theory, and we will not take into account the excitonic normal component and velocity, $(T = 0)$.

The equations of motion can be easily derived by the standard variational method from the following condition:

$$\delta S = \delta \int dt\, d\mathbf{x}\, \mathcal{L}(\hat{\psi}^\dagger(\mathbf{x},t),\ \hat{\psi}(\mathbf{x},t),\ \hat{\mathbf{u}}(\mathbf{x},t)) = 0.$$

Indeed, after transforming the Bose-fields $\hat{\psi}^\dagger$ and $\hat{\psi}$ by

$$\hat{\psi}(\mathbf{x},t) \to \exp(-iE_g t/\hbar)\exp(imvx/\hbar)\hat{\psi}(\mathbf{x},t),$$

we can write these equations as follows:

$$(i\hbar\partial_t + mv^2/2)\hat{\psi}(\mathbf{x},t) =$$

$$= \Big(-\frac{\hbar^2}{2m}\Delta + \nu_0\hat{\psi}^\dagger\hat{\psi}(\mathbf{x},t) + \nu_1\hat{\psi}^{\dagger 2}\hat{\psi}^2(\mathbf{x},t)\Big)\hat{\psi}(\mathbf{x},t) + \sigma_0\nabla\hat{\mathbf{u}}(\mathbf{x},t)\,\hat{\psi}(\mathbf{x},t), \qquad (3)$$

$$(\partial_t^2 - c_l^2\Delta - 2v\partial_t\partial_x + v^2\partial_x^2)\hat{\mathbf{u}}(\mathbf{x},t) = \rho^{-1}\sigma_0\nabla(\hat{\psi}^\dagger\hat{\psi}(\mathbf{x},t)). \qquad (4)$$

We assume that the condensate of excitons exists. This means that the following representation of the exciton Bose-field holds: $\hat{\psi} = \psi_0 + \delta\hat{\psi}$. Here $\psi_0 \neq 0$ is the classical part of the field operator $\hat{\psi}$ or, in other words, the condensate wave function, and $\delta\hat{\psi}$ is the fluctuational part of $\hat{\psi}$, which describes noncondensed particles.

One of the important objects in the theory of BEC is the correlation functions of Bose-fields. The standard way to calculate them in this model (the excitonic function $<\psi(\mathbf{x},t)\psi^\dagger(\mathbf{x}',t')>$, for example,) can be based on the effective action or the effective Hamiltonian approaches [16]. Indeed, one can, first, integrate over the phonon variables $\mathbf{u}$, get the expression for $S_{\text{eff}}(\psi,\psi^\dagger)$ and, second, use S_{eff} (or $\hat{H}_{\text{eff}}$) to derive the equations of motion for ψ_0, $\delta\hat{\psi}$, correlation functions, etc..

In this work we do not follow that way; instead, we treat excitons and phonons equally [9],[17]. This means that the displacement field $\hat{\mathbf{u}}$ can have a nontrivial classical part too, i.e. $\hat{\mathbf{u}} = \mathbf{u}_0 + \delta\hat{\mathbf{u}}$ and $\mathbf{u}_0 \neq 0$, and the actual moving condensate can be an exciton-phonon one, i.e. $\psi_0(\mathbf{x},t)\cdot\mathbf{u}_0(\mathbf{x},t)$. Then the equation of motion for the classical parts of the fields $\hat{\psi}$ and $\hat{\mathbf{u}}$ can be derived by use of the variational method from $\mathcal{L} = \mathcal{L}(\psi,\psi^*,\mathbf{u})$, where all the fields can be considered as the classical ones. Eventually we have

$$(i\hbar\partial_t + mv^2/2\,)\psi_0(\mathbf{x},t) =$$

$$=\Big(-\frac{\hbar^2}{2m}\Delta + \nu_0|\psi_0|^2(\mathbf{x},t) + \nu_1|\psi_0|^4(\mathbf{x},t)\,\Big)\psi_0(\mathbf{x},t) + \sigma_0\nabla\mathbf{u}_0(\mathbf{x},t)\,\psi_0(\mathbf{x},t)\,, \quad (5)$$

$$(\partial_t^2 - c_l^2\Delta - 2v\partial_t\partial_x + v^2\partial_x^2)\mathbf{u}_0(\mathbf{x},t) = \rho^{-1}\sigma_0\nabla(|\psi_0|^2(\mathbf{x},t)). \quad (6)$$

Notice that deriving these equations we disregarded the interaction between the classical (condensate) and the fluctuational (noncondensate) parts of the fields. That is certainly a good approximation for $T=0$ and $T \ll T_c$ cases [18].

In this article a steady-state of the condensate is the object of the main interest. In the co-moving frame of reference, the condensate steady-state is just the stationary solution of Eqs. (5), (6) and it can be taken in the form

$$\psi_0(\mathbf{x},t) = \exp(-i\mu t/\hbar)\exp(i\varphi)\phi_{\rm o}(\mathbf{x}), \quad u_0(\mathbf{x},t) = \mathbf{q}_{\rm o}(\mathbf{x}),$$

where $\phi_{\rm o}$ and $\mathbf{q}_{\rm o}$ are the real number functions, and $\varphi = \text{const}$ is the (macroscopic) phase of the condensate wave function. (This phase can be taken zeroth if only a single condensate is the subject of interest.) Then, the following equations have to be solved ($\tilde{\mu} = \mu + mv^2/2$):

$$\tilde{\mu}\,\phi_{\rm o}(\mathbf{x}) =\Big(-\frac{\hbar^2}{2m}\Delta + \nu_0\phi_{\rm o}^2(\mathbf{x}) + \nu_1\phi_{\rm o}^4(\mathbf{x})\,\Big)\phi_{\rm o}(\mathbf{x}) + \sigma_0\nabla\mathbf{q}_{\rm o}(\mathbf{x})\,\phi_{\rm o}(\mathbf{x})\,, \quad (7)$$

$$-(\,(c_l^2 - v^2)\partial_x^2 + c_l^2\partial_y^2 + c_l^2\partial_z^2\,)\mathbf{q}_{\rm o}(\mathbf{x}) = \rho^{-1}\sigma_0\nabla\phi_{\rm o}^2(\mathbf{x}). \quad (8)$$

Indeed, the last equation can be solved relative to $\nabla\mathbf{q}_{\rm o}$. If $v < c_l$, the corresponding solution can be represented as follows:

$$\nabla\mathbf{q}_{\rm o}(\mathbf{x}) = (-\frac{\lambda^2}{3} - \frac{2}{3})\frac{\sigma_0}{\rho c_l^2}\,\phi_{\rm o}^2(\mathbf{x}) \;+\; \frac{\lambda^3}{4\pi}(1-\lambda^{-2})\int\mathcal{F}(\mathbf{x}-\mathbf{x}')\frac{\sigma_0}{\rho c_l^2}\,\phi_{\rm o}^2(\mathbf{x}')\,d\mathbf{x}', \quad (9)$$

where $\lambda^2 = c_l^2/(c_l^2 - v^2)$, and $\mathcal{F}$ can be expressed in terms of the Green function of Eq. (8). Substituting $\nabla \mathbf{q}_o$ in Eq. (7), we rewrite the latter in the following form:

$$\tilde{\mu}\phi_o(\mathbf{x}) = \Big(-\frac{\hbar^2}{2m}\Delta + \nu_0 \phi_o^2(\mathbf{x}) + \int U_{\text{eff}}(\mathbf{x}-\mathbf{x}')\phi_o^2(\mathbf{x}')\, d\mathbf{x}' + \nu_1 \phi_o^4(\mathbf{x}) \Big) \phi_o(\mathbf{x}), \quad (10)$$

where the effective exciton-exciton interaction U_{eff} is induced by the lattice ($q_o \neq 0$). It can be represented as follows:

$$U_{\text{eff}}(\mathbf{x}) = \Big(-\frac{\lambda^2}{3} - \frac{2}{3} \Big) \frac{\sigma_0^2}{\rho c_l^2} \delta(\mathbf{x}) +$$

$$+ \frac{\sigma_0^2}{4\pi\rho(c_l^2 - v^2)} \frac{v^2}{c_l(c_l^2 - v^2)^{1/2}} \Big(\frac{3\lambda^2 x^2}{(\lambda^2 x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(\lambda^2 x^2 + y^2 + z^2)^{3/2}} \Big). \quad (11)$$

The first (isotropic) term in (11) causes the renormalization of the exciton-exciton interaction constant $\nu_0 > 0$:

$$\nu_0 \to \nu_{\text{eff}} = \nu(v;\, c_l, \sigma_0). \quad (12)$$

Note that ν_{eff} can be positive or negative depending on the value of v. The second term in the effective potential (11) is strongly anisotropic. Moreover, on the cylinder $y^2 + z^2 = \varepsilon^2$, its value is negative in the vicinity of $x = 0$ and positive at the large scales of x $(Ox \| \mathbf{v})$.

The possibility of the existence of specific and to some extent unexpected solutions of Eq. (10) follows from the fact that the effective two particle interaction between excitons can be attractive at small distances between the particles and repulsive at large distances. For example, the wave function $\phi_o(\mathbf{x})$ may become strongly localized in 3D space because of this attraction.

If $v > c_l$, the formulas for $\nabla \mathbf{q}_o(\mathbf{x})$ and $U_{\text{eff}}(\mathbf{x})$ are very similar to those obtained in the case of $v < c_l$. For instance, the solution of Eq. (8) can be written as follows:

$$\nabla \mathbf{q}_o(\mathbf{x}) = (\frac{\tilde{\lambda}^2}{3} - \frac{2}{3}) \frac{\sigma_0}{\rho c_l^2} \phi_o^2(\mathbf{x}) + \frac{\tilde{\lambda}}{2\pi}(1 + \tilde{\lambda}^2) \int \mathcal{F}'(\mathbf{x} - \mathbf{x}') \frac{\sigma_0}{\rho c_l^2} \phi_o^2(\mathbf{x}')\, d\mathbf{x}'. \quad (13)$$

(Here $\tilde{\lambda}^2 = c_l^2/(v^2 - c_l^2)$). Again, the renormalization (12) takes place and a nonisotropic part of $U'_{\text{eff}}(\mathbf{x})$ appears in Eq. (10). However, the effective two-particle interaction between excitons remains repulsive, i.e. $\nu_0 \delta(\mathbf{x}) + U'_{\text{eff}}(\mathbf{x}) > 0$.

3 Effective 1D Model for the Condensate Wave Function

Solving Eqs. (7),(8) in 3D space seems to be a difficult problem (see Eqs. (10),(11)). However, these equations can be essentially simplified if we assume that the condensate is inhomogeneous along the x-axis only, that is $\phi_o(\mathbf{x}) = \phi_o(x)$ and $\mathbf{q}_o(\mathbf{x}) = (q_o(x), 0, 0)$. Such an effective reduction of dimensionality transforms the difficult integral-differential equation (10) into a rather simple differential one, and obtained in this way the effective 1D model for the condensate wave function $\phi_o \circ q_o$ conserves all the important properties of the "parent" 3D model. Indeed, if $v < c_l$, the following equations stand for the condensate:

$$\tilde{\mu}\phi_o(x) = (-(\hbar^2/2m)\partial_x^2 + \nu_{\text{eff}}\phi_o^2(x) + \nu_1\phi_o^4(x))\phi_o(x), \tag{14}$$

$$\partial_x q_o(x) = \text{const} - (\sigma_0/\rho(c_l^2 - v^2))\,\phi_o^2(x), \tag{15}$$

where $\nu_{\text{eff}} = \nu_0 - \sigma_0^2/\rho(c_l^2 - v^2)$. If $v > c_l$, Eq. (14) describes the excitonic part of the condensate, but with the enhanced effective repulsion $\nu'_{\text{eff}} = \nu_0 + \sigma_0^2/\rho(v^2 - c_l^2)$.

The effective two-particle interaction constant ν_{eff} is negative if the velocity of the condensate lies inside the interval $v_o < v < c_l$, where

$$v_o = \sqrt{c_l^2 - (\sigma_0^2/\rho\nu_0)} \tag{16}$$

can be called the first 'critical' velocity in the model. (Note that outside this interval $\nu_{\text{eff}} > 0$ [9].)

The estimate value of the threshold velocity v_o can be obtained from the following formula:

$$v_o \simeq c_l\sqrt{1 - (C_c - C_v)^2/(8\pi\,\text{Ry}^*\,\text{Ry}\gamma^3)},$$

where $C_c - C_v$ is the relative volume deformation potential of a semiconductor, ($\sigma_0 \simeq C_c - C_v$), Ry* and Ry are the exciton and atom Rydberg energies, $\gamma = \tilde{a}_{\text{B}}/a_l$ and a_l is the lattice constant. (The repulsive exciton-exciton interaction is taken in the form $\nu_0 \simeq 4\pi\hbar^2 a_{\text{ex}}/m$). In the case of Cu_2O oxide, we have $v_o \simeq (0.5 \sim 0.7)c_l$.

In this study we will consider the case of $v_o < v < c_l$ in detail. If the sign of the effective two-particle interaction can vary, an extra (repulsive) term should be included into the Hamiltonian of a many-particle system to insure the finiteness of a steady-state wave function or the absence of wave function collapse in dynamic processes. In our case, it is the term $\nu_1(\Psi^\dagger)^3\Psi^3$, and $\nu_1 > 0$ is supposed to be the smallest energy parameter in the theory. In the framework of the considered 1D model, however, a finite steady-state solution of Eqs. (14),(15) can be obtained without accounting for the "hard core" repulsion term $\nu_1\phi_o^4$. Indeed, we can write out the corresponding solution as follows:

$$\phi_o(x) = \Phi_o/\cosh(\beta\Phi_o x), \quad \partial_x q_o(x) = -(\sigma_0/\rho(c_l^2 - v^2))\Phi_o^2/\cosh^2(\beta\Phi_o x), \tag{17}$$

$$\tilde{\mu} = \nu_0\Phi_o^2/2 - (\sigma_0^2/\rho(c_l^2 - v^2))\Phi_o^2/2 < 0, \tag{18}$$

$$\beta = \beta(v) = \sqrt{\frac{m\nu_0}{\hbar^2}\frac{v^2 - v_o^2}{c_l^2 - v^2}}. \tag{19}$$

The amplitudes of the exciton and phonon parts of the condensate, the characteristic width of the condensate, $L_0 = (\beta(v)\Phi_o)^{-1}$, and the value of the effective chemical potential $\tilde{\mu}$ depend on the normalization of the exciton wave function $\phi_o(x)$. We normalize it in 3D space assuming that the characteristic width of the packet in the (y, z) plane is finite and the cross-section area of the packet can be made equal to the cross-section area S of a laser beam. Then we can write this condition as follows:

$$\int |\psi_0|^2(x,t)\, d\mathbf{x} = S\int \phi_o^2(x)\, dx = N_o, \tag{20}$$

where N_o is the number of condensed excitons. Immediately, we get the following results:

$$\Phi_o = \frac{N_o}{2S}\beta(v), \quad L_0 = \left(\frac{N_o}{2S}\beta^2(v)\right)^{-1}, \tag{21}$$

$$\tilde{\mu} = -\frac{\nu_0}{2}\frac{v^2 - v_o^2}{c_l^2 - v^2}\left(\frac{N_o}{2S}\beta(v)\right)^2 = -\frac{\hbar^2}{2m}L_0^{-2}. \tag{22}$$

The important "nonlinear" property of the obtained solution (17) is the dependence of the amplitudes of exciton and phonon parts of the condensate on the velocity v and the number of excitons in the condensate, N_o . For example,

$$\Phi_o^2 = \Phi_o^2(N_o, v) \sim N_o^2\,(v^2 - v_o^2)/(c_l^2 - v^2)$$

stands for the exciton amplitude. Notice that the characteristic width of the condensate and its velocity are not independent parameters, see (21). For estimates, the formula for L_0 can be rewritten as follows:

$$L_0^{-1} \simeq 2\tilde{a}_{\mathrm{B}}^{-1}(n\tilde{a}_{\mathrm{B}}^3)^{1/2}\left(\frac{v^2 - v_o^2}{c_l^2 - v^2}\right)^{1/2},$$

where n is the average density of excitons in the soliton state. Although $\tilde{a}_{\mathrm{B}}^{-1}$ in this formula is multiplied by a small factor, this factor is not small enough to show quantitative agreement with the experimentally observed value of L_0, $2L_{\mathrm{exp}} \simeq 10^{-1} \sim 10^{-2}$ cm. It seems to be reasonable that a theory with nonzero temperature ($T > |\tilde{\mu}|$) will provide a more realistic value of the effective size of the packet.

Returning to the laboratory reference frame, we can write the condensate wave function in the form:

$$\psi_0(x,t)\cdot u_0(x,t)\delta_{1j} = \exp\left(-i\left(E_g + \frac{mv^2}{2} - |\tilde{\mu}|\right)t/\hbar\right)\exp(imvx/\hbar)\times$$

$$\times\Phi_o/\cosh(L_0^{-1}(x-vt))\cdot\Big(Q_o\ -\ Q_o\tanh(L_0^{-1}(x-vt))\Big), \tag{23}$$

where we count the exciton energy from the bottom of the crystal valence band; $2Q_o(N_o, v)$ is the amplitude of the phonon part of condensate and $Q_o \propto \Phi_o$. To calculate the energy of the moving condensate within the Lagrangian approach, (see Eq. (2)), we have to integrate the zeroth component of the energy-momentum tensor $\mathcal{T}_0^0$ over the spatial coordinates,

$$\mathcal{T}_0^0(\mathbf{x},t) = E_g\psi^\dagger\psi + \frac{\hbar^2}{2m}\nabla\psi^\dagger\,\nabla\psi + (\nu_0/2)(\psi^\dagger)^2\psi^2 + \frac{\rho}{2}(\partial_t\mathbf{u})^2 + \frac{\rho c_l^2}{2}\,\partial_j u_k\partial_j u_k + \sigma_0\psi^\dagger\psi\,\nabla\mathbf{u}.$$

We do not take into account the small correction to this energy due to the quantum depletion of the condensate as well as the term $(\nu_1/3)\,\phi_o^6$ in $\mathcal{T}_0^0$. Then the result reads:

$$E_o = \int d\mathbf{r}\, \mathcal{T}_0^0 = E_{\rm ex} + E_{\rm int} + E_{\rm ph} =$$

$$= N_o\left(E_g + \frac{mv^2}{2}\right) - N_o\left(|\tilde{\mu}| + (\nu_0/3)\Phi_o^2\right) + \frac{c_l^2+v^2}{3(c_l^2-v^2)}\,N_o\frac{\sigma_0^2}{\rho(c_l^2-v^2)}\Phi_o^2.$$

The value $|\tilde{\mu}|$ is a rather small parameter,

$$|\tilde{\mu}| \simeq 6{\rm Ry}^*(n\,a_{\rm ex}^3)\,(v^2-v_0^2)/(c_l^2-v^2),$$

and the energy of the phonon part of the condensate is estimated as $E_{\rm ph} \lesssim (5 \sim 6)N_o|\tilde{\mu}|$. However, the back surface of the crystal will experience some pressure when the condensate reaches this surface and the excitons are destroyed near it. The estimation of the maximum value of the pressure in a pulse is as follows:

$$p_m \simeq 10\sigma_0\Phi_o^2 \simeq 10^{-2} \sim 10^{-3}\,{\rm J/cm}^{-3}$$

and the main contribution to this value comes from the phonon part. Indeed, one can see that the exciton-phonon condensate carries a nonzeroth momentum $P_{o\,x} = P_{{\rm ex},\,x} + P_{{\rm ph},\,x}$:

$$P_{o\,x} = \int d\mathbf{x}(\hbar/2i)(\phi_0^*(x,t)\,\partial_x\phi_0(x,t) - \partial_x\phi_0^*(x,t)\,\phi_0(x,t)\,) - \rho\partial_t u_0(x,t)\,\partial_x u_0(x,t) =$$

$$= \int d\mathbf{x}\, mv\,\Phi_o^2(x) + \rho v\left(\frac{\sigma_0}{(c_l^2-v^2)\rho}\Phi_o^2(x)\right)^2.$$

4 Low-Lying Excitations of Exciton-Phonon Condensate

Although the condensate wave function $\phi_o(x)\cdot q_o(x)$ was obtained in the framework of the effective 1D model, (but normalized in 3D space), we will use this solution as a classical part in the 3D field operator decomposition:

$$\hat{\psi}(\mathbf{x},t) = \exp(-i\mu t)(\phi_o(x) + \delta\hat{\psi}(\mathbf{x},t)\,), \tag{24}$$

$$\hat{u}_j(\mathbf{x},t) = q_o(x)\delta_{1j} + \delta\hat{u}_j(\mathbf{x},t), \tag{25}$$

where $\mu = \tilde{\mu} - mv^2/2$. Substituting the field operators of the form (24),(25) into the Lagrangian density (2), we can write the later in the following form:

$$\mathcal{L} = \mathcal{L}_o(\mathrm{e}^{-i\mu t}\phi_o(x),\, q_o(x)\delta_{1j}\,) + \mathcal{L}_2(\delta\hat{\psi}^\dagger(\mathbf{x},t), \delta\hat{\psi}(\mathbf{x},t),\, \delta\hat{\mathbf{u}}(\mathbf{x},t)\,) + ..., \tag{26}$$

where $\mathcal{L}_o$ stands for the classical part of $\mathcal{L}$, and $\mathcal{L}_2$ is the bilinear form in the δ-operators. As the classical parts of the field operators satisfy the equality $\delta S_o[\psi_0^*, \psi_0,\, u_0] = 0$, the linear form in the "δ-operators" vanishes in (26).

In the simplest (Bogoliubov) approximation [19],[20], $\mathcal{L} \approx \mathcal{L}_o + \mathcal{L}_2$ and, hence, the bilinear form $\mathcal{L}_2$ defines the equations of motion for the fluctuating parts of the field operators. (To derive them we use the variational method: $\delta S_2 = \delta \int \mathcal{L}_2(\delta\psi^\dagger, \delta\psi, \delta\mathbf{u})\, d\mathbf{x}dt = 0$.) As a result, these equations are linear and can be written as follows:

$$i\hbar\partial_t\, \delta\hat{\psi}(\mathbf{x},t) = \left(-\frac{\hbar^2}{2m}\Delta + |\tilde{\mu}| + \left\{ 2\nu_0 - \frac{\sigma_0^2}{\rho(c_l^2 - v^2)} \right\} \phi_o^2(x) + 3\nu_1\phi_o^4(x) \right) \delta\hat{\psi}(\mathbf{x},t) +$$

$$+ (\nu_0\phi_o^2(x) + 2\nu_1\phi_o^4(x))\delta\hat{\psi}^\dagger(\mathbf{x},t) + \sigma_0\phi_o(x)\nabla\delta\hat{\mathbf{u}}(\mathbf{x},t), \tag{27}$$

$$(\partial_t^2 - c_l^2\Delta - 2v\partial_t\partial_x + v^2\partial_x^2)\delta\hat{\mathbf{u}}(\mathbf{x},t) = \rho^{-1}\sigma_0\nabla\left(\phi_o(x)\left(\delta\hat{\psi}(\mathbf{x},t) + \delta\hat{\psi}^\dagger(\mathbf{x},t)\right)\right) \tag{28}$$

The same approximation can be performed within the Hamiltonian approach. Indeed, decomposition of the field operators near their nontrivial classical parts leads to the decomposition of the Hamiltonian (1) itself, and – as it was done with the Lagrangian – only the classical part of $\hat{H}$, H_o, and the bilinear form in the fluctuating fields, $\hat{H}_2$, are left for examination:

$$\hat{H} \approx H_o(\psi_0^*, \psi_0,\, \pi_0, u_0) + H_2(\delta\hat{\psi}^\dagger,\, \delta\hat{\psi},\, \delta\hat{\pi}, \delta\hat{u}). \tag{29}$$

In this approximation, the Hamiltonian (29) can be diagonalized and rewritten in the form:

$$\hat{H} = H_o(\mathrm{e}^{-i\mu t}\phi_o(x),\, q_o(x)\,) + \delta E_o + \sum_s \hbar\omega_s\, \hat{\alpha}_s^\dagger\hat{\alpha}_s. \tag{30}$$

Here, δE_o is the quantum correction to the energy of the condensate and the index s labels the elementary excitations of the system. The operators $\hat{\alpha}_s^\dagger$, $\hat{\alpha}_s$ are the Bose ones, and they can be represented by linear combinations of the exciton and displacement field operators:

$$\hat{\alpha}_s = \int d\mathbf{x} \left(U_s(\mathbf{x})\, \delta\hat{\psi}(\mathbf{x}) + V_s^*(\mathbf{x})\, \delta\hat{\psi}^\dagger(\mathbf{x}) + X_{s,j}(\mathbf{x})\, \delta\hat{u}_j(\mathbf{x}) + Y_{s,j}(\mathbf{x})\, \delta\hat{\pi}_j(\mathbf{x}) \right) \tag{31}$$

$$\hat{\alpha}_s^\dagger = \int d\mathbf{x} \left(U_s^*(\mathbf{x})\, \delta\hat{\psi}^\dagger(\mathbf{x}) + V_s(\mathbf{x})\, \delta\hat{\psi}(\mathbf{x}) + X_{s,j}^*(\mathbf{x})\, \delta\hat{u}_j(\mathbf{x}) + Y_{s,j}^*(\mathbf{x})\, \delta\hat{\pi}_j(\mathbf{x}) \right) \tag{32}$$

Note that by analogy with the exciton-polariton modes in semiconductors [21] the excitations of the condensate (23) can be considered as a mixture of the exciton- and phonon-type modes, but in this model the phonons come from fluctuations of the $u_0(x,t)$-part of the condensate.

Since the α-operators (see (30)) evolve in time as simply as

$$\hat{\alpha}_s(t) = \mathrm{e}^{-i\omega_s t}\hat{\alpha}_s \quad \hat{\alpha}_s^\dagger(t) = \mathrm{e}^{i\omega_s t}\hat{\alpha}_s^\dagger,$$

these operators (and the frequencies $\{\omega_s\}$) are the eigenvectors (and, correspondingly, the eigenvalues) of the equations of motion (27),(28) obtained within the Lagrangian method. Then, the time dependent "δ-operators" in (27),(28) can be represented by the following linear combinations of the α-operators:

$$\delta\hat{\psi}(\mathbf{x},t) = \sum_s \mathrm{u}_s(\mathbf{x})\, \hat{\alpha}_s \mathrm{e}^{-i\omega_s t} + \mathrm{v}_s^*(\mathbf{x})\, \hat{\alpha}_s^\dagger \mathrm{e}^{i\omega_s t}, \tag{33}$$

$$\delta\hat{\psi}^\dagger(\mathbf{x},t) = \sum_s \mathrm{u}_s^*(\mathbf{x})\, \hat{\alpha}_s^\dagger \mathrm{e}^{i\omega_s t} + \mathrm{v}_s(\mathbf{x})\, \hat{\alpha}_s \mathrm{e}^{-i\omega_s t}, \tag{34}$$

$$\delta\hat{u}_j(\mathbf{x},t) = \sum_s C_{s,j}(\mathbf{x})\, \hat{\alpha}_s \mathrm{e}^{-i\omega_s t} + C_{s,j}^*(\mathbf{x})\, \hat{\alpha}_s^\dagger \mathrm{e}^{i\omega_s t}. \tag{35}$$

Substituting this ansatz (which is a generalization of the u-v Bogoliubov transformation) into Eqs. (27),(28), we obtain the following coupled eigenvalue equations [9]:

$$\left(\hat{L}(\Delta) - \hbar\omega_s\right) \mathrm{u}_s(\mathbf{x}) + \left(\nu_0 \phi_o^2(x) + 2\nu_1 \phi_o^4(x)\right) \mathrm{v}_s(\mathbf{x}) + \sigma_0 \phi_o(x) \nabla \mathbf{C}_s(x) = 0 \tag{36}$$

$$\left(\nu_0\phi_{\rm o}^2(x) + 2\nu_1\phi_{\rm o}^4(x)\right)\mathrm{u}_s(\mathbf{x}) + \left(\hat{L}(\Delta) + \hbar\omega_s\right)\mathrm{v}_s(\mathbf{x}) + \sigma_0\phi_{\rm o}(x)\nabla\mathbf{C}_s(\mathbf{x}) = 0 \qquad (37)$$

$$-\rho^{-1}\sigma_0\nabla\Big(\phi_{\rm o}(x)\,\mathrm{u}_s(\mathbf{x})\Big) - \rho^{-1}\sigma_0\nabla\Big(\phi_{\rm o}(x)\,\mathrm{v}_s(\mathbf{x})\Big) + \left[(-i\omega_s - v\partial_x)^2 - c_l^2\Delta\right]\mathbf{C}_s(\mathbf{x}) = 0, \qquad (38)$$

where $\hat{L}(\Delta) = (-\hbar^2/2m)\Delta + |\tilde{\mu}| + \left[\,2\nu_0 - \sigma_0^2/\rho(c_l^2 - v^2)\,\right]\phi_{\rm o}^2(x) + 3\nu_1\phi_{\rm o}^4(x)$.

To simplify investigation of the characteristic properties of the different possible solutions of Eqs. (36)-(38), we subdivide the excitations (33)-(35) into two major parts, the *inside*-excitations and the *outside*-ones. The *inside*-excitations are localized merely inside the packet area, i.e. $|\mathbf{x}| < L_0$ and $\phi_{\rm o}^2(x) \approx$ const, whereas the *outside*-excitations propagate merely in the outside area, i.e. $|\mathbf{x}| > (1 \sim 2)L_0$ and $\phi_{\rm o}^2(x) \simeq 4\,\Phi_{\rm o}^2\exp(-2|x|/L_0) \to 0$.

4.1 Outside-Excitations

For the outside collective excitations, the asymptotics of the low-lying energy spectrum can be found easily. Indeed, if we assume that $\phi_{\rm o}^2(x) \approx 0$ in the outside packet area, the equations (27) and (28) begin to be uncoupled. Then, Eq. (27) describes the excitonic branch of the outside-excitations with the following dispersion law in the co-moving frame:

$$\hbar\omega_{\rm ex}(\mathbf{k}) \approx |\tilde{\mu}| + (\hbar^2/2m)k^2, \quad (\mathrm{u}_{\mathbf{k}}(\mathbf{x}) \sim \mathrm{e}^{i\mathbf{kx}},\;\; \mathrm{v}_{\mathbf{k}}(\mathbf{x}) \approx 0), \qquad (39)$$

and Eq. (28) describes the phonon branch and yields the spectrum $\omega_{ph}(\mathbf{k}) = c_l|\mathbf{k}|$ in the laboratory frame of reference. Then, the exciton field operator, which describes the exciton condensate with the one outside excitation, has the following form:

$$\psi(\mathbf{x},t) \simeq \exp(-i(E_g + mv^2/2 - |\tilde{\mu}|)t/\hbar\,)\exp(imvx/\hbar)\,\phi_{\rm o}(x - vt) +$$

$$+\exp(-i(E_g + mv^2/2 - |\tilde{\mu}|)t/\hbar\,)\exp(imvx/\hbar)\times$$

$$\times\,\left\{\exp(-i(|\tilde{\mu}| + \hbar\mathbf{k}^2/2m + k_x v)t/\hbar\,)\exp(i\mathbf{kx})\,u_{\mathbf{k}}\,\right\}$$

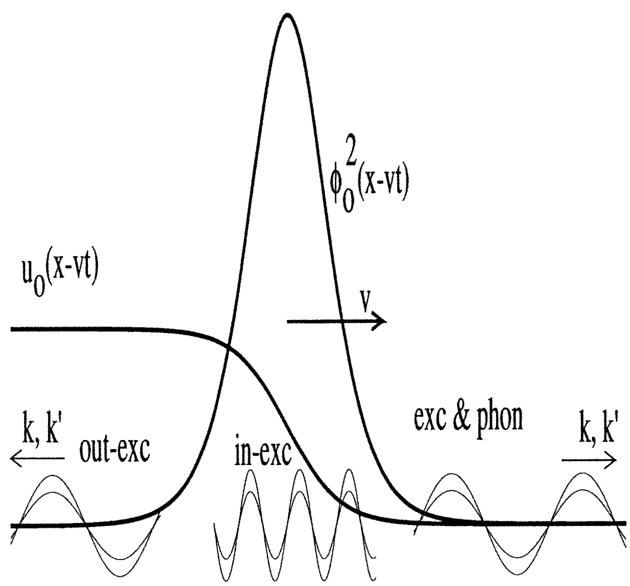

Figure 1: Moving exciton-phonon condensate.
Moving exciton-phonon condensate, $\phi_{\rm o}(x - vt) \cdot u_{\rm o}(x - vt)\delta_{1j}$, and inside- and outside-excitations of the condensate. (Longitudinal exciton-phonon excitations, $\mathbf{k} \,||\, Ox$, are schematically depicted.)

It is easy to see that such a collective excitation, $\omega_{\rm ex} = |\tilde{\mu}|/\hbar + \hbar\mathbf{k}^2/2m + k_x v$, can be interpreted as an exciton with the energy $E_g + \hbar^2\tilde{\mathbf{k}}^2/2m$, where $\hbar\tilde{k}_j = \hbar k_j + mv\delta_{1j}$. Then we can compare the condensate energy $E_{\rm o}(N_{\rm o})$ and the energy of the condensate with one outside excitation,

$$E_{\rm o}(N_{\rm o}-1)+E_{\rm exc}(\tilde{k})+E_{\rm ph}(k') \approx E_{\rm o}(N_{\rm o})-\partial_N E_{\rm o}(N_{\rm o})+(E_g+\hbar^2\tilde{k}^2/2m)+\hbar c_l|k'| \approx$$

$$\approx E_{\rm o}(N_{\rm o})+(\hbar^2\tilde{k}^2/2m - mv^2/2)+3(|\tilde{\mu}|+(\nu_0/3)\Phi_{\rm o}^2)-\frac{c_l^2+v^2}{c_l^2-v^2}\frac{\sigma_0^2}{\rho(c_l^2-v^2)}\Phi_{\rm o}^2+\hbar c_l|k'|$$

$$> E_{\rm o}(N_{\rm o}) \text{ in the } k\to 0 \text{ limit } (k'\neq 0 \text{ for the phonon part}).$$

Note that the energy (and the momentum) of the phonon part of the condensate changes after exciton emission. We assume that the transformation $N_{\rm o} \to N_{\rm o}-1$ (or emission of an outside exciton) corresponds to the situation when the outside exciton and the outside phonon(s) appear together, and the phonon is emitted with the energy compensating the changement of $-\delta E_{\rm ph}$ in the phonon part of the condensate.

However, the condensate collective excitations are uncoupled only in the $k \to 0$ limit, i.e. $\lambda = 2\pi/k \gg L_0$. It follows from the structure of Eqs. (36)-(38) that the coupling between the *outside* phonon and exciton branches is originated from the condensate "surface" area, i.e. from the scale $L_o < |x| < 3L_0$. Indeed, $\phi_{\rm o}(x)$ and $\phi_{\rm o}'(x) \simeq \pm\phi_{\rm o}(x)L_0^{-1}$ cannot be put equal zero in this area, and the "particle"- and the "hole"-type components of the exciton operators, namely $\mathrm{u}_s \sim \mathrm{e}^{ikx}$ and $\mathrm{v}_s^* \sim \mathrm{e}^{-ikx}$, should be both different from zero and spatially modulated in the surface area, at least for the excitations with $\lambda < 2L_0$. We left this question for future investigations.

4.2 Inside-Excitations

To simplify the calculation of *inside*-excitation spectrum (see Eqs. (36)-(38)) we will use the semiclassical approximation [20]. In this approximation, the excitations can be labeled by the wave vector $\mathbf{k}$ in the co-moving frame, and the following representation holds:

$$\mathrm{u}_s(\mathbf{x}) = \mathrm{u}_{\mathbf{k}}(\mathbf{x})\mathrm{e}^{i\varphi_{\mathbf{k}}(\mathbf{x})}, \;\; \mathrm{v}_s(\mathbf{x}) = \mathrm{v}_{\mathbf{k}}(\mathbf{x})\mathrm{e}^{i\varphi_{\mathbf{k}}(\mathbf{x})}, \;\; C_{s,j}(\mathbf{x}) = C_{\mathbf{k},j}(\mathbf{x})\mathrm{e}^{i\varphi_{\mathbf{k}}(\mathbf{x})}, \tag{40}$$

where the phase $\varphi_{\mathbf{k}}(\mathbf{x}) \approx \varphi_o + \mathbf{kx}$, and $\mathrm{u}_{\mathbf{k}}(\mathbf{x})$, $\mathrm{v}_{\mathbf{k}}(\mathbf{x})$, and $C_{j,\mathbf{k}}(\mathbf{x})$ are assumed to be smooth functions of $\mathbf{x}$ in the inside condensate area. Notice that the $\mathbf{k}$- and $\mathbf{x}$-representations are mixed here. This means that the operator nature of the fluctuating fields is factually dismissed within the semiclassical approximation, However, the orthogonality relations between u_s and $\mathrm{v}_{s'}$, and, hence, between $\mathrm{u}_{\mathbf{k}}$ and $\mathrm{v}_{\mathbf{k}'}$ come from the Bose commutation relations between the operators α_s and $\alpha^\dagger_{s'}$ [19],[20]. For example, Eq. (33) is modified as follows:

$$\delta\psi(\mathbf{x},t) \simeq \int \frac{d\mathbf{k}}{(2\pi)^3} \left(\mathrm{u}_{\mathbf{k}}(\mathbf{x})\mathrm{e}^{i\varphi_{\mathbf{k}}(\mathbf{x})}\mathrm{e}^{-i\omega_{\mathbf{k}}(\mathbf{x})t} + \mathrm{v}^*_{\mathbf{k}}(\mathbf{x})\mathrm{e}^{-i\varphi_{\mathbf{k}}(\mathbf{x})}\mathrm{e}^{i\omega_{\mathbf{k}}(\mathbf{x})t}\right), \tag{41}$$

and the *inside*-excitation part of the elementary excitation term in (30), $\sum_s \ldots \approx \sum_{s,\,\mathrm{out}} + \sum_{s,\,\mathrm{surf}} + \sum_{s,\,\mathrm{in}} \ldots$, can be written in the form

$$\sum_{s,\,\mathrm{in}} \hbar\omega_s \hat{\alpha}^\dagger_s \hat{\alpha}_s \simeq \int \frac{d\mathbf{k}\, d\mathbf{x}}{(2\pi)^3}\, \hbar\omega_{\mathbf{k}}(\mathbf{x})\, n_{\mathbf{k}}(\mathbf{x}). \tag{42}$$

Note that the semiclassical energy $\hbar\omega_{\mathbf{k}}(x)$ of the inside-excitation mode is supposed to be a smooth function of x, (i.e. at least as smooth as $\phi_o^2(x) \approx$ const in the "inside" approximation).

Although the low-lying excitations cannot be properly described within the semiclassical approximation, we apply it here to calculate the low energy asymptotics of the spectrum. In fact, all the important properties of these excitations can be understood within this approach.

Substituting (40) into Eqs. (36)-(38), we transform these differential equations into the algebraic ones ($L(-\mathbf{k}^2) = \hat{L}(\Delta \to -\mathbf{k}^2)$):

$$\left(L(-\mathbf{k}^2) - \hbar\omega_{\mathbf{k}}\right)\mathrm{u}_{\mathbf{k}}(\mathbf{x}) + \left(\nu_0\phi_o^2(\mathbf{x}) + 2\nu_1\phi_o^4(\mathbf{x})\right)\mathrm{v}_{\mathbf{k}}(\mathbf{x}) + \sigma_0\phi_o(\mathbf{x})i\mathbf{k}\mathbf{C}_{\mathbf{k}}(\mathbf{x}) = 0, \tag{43}$$

$$\left(\nu_0\phi_o^2(\mathbf{x}) + 2\nu_1\phi_o^4(\mathbf{x})\right)\mathrm{u}_{\mathbf{k}}(\mathbf{x}) + \left(L(-\mathbf{k}^2) + \hbar\omega_{\mathbf{k}}\right)\mathrm{v}_{\mathbf{k}}(\mathbf{x}) + \sigma_0\phi_o(\mathbf{x})i\mathbf{k}\mathbf{C}_{\mathbf{k}}(\mathbf{x}) = 0 \tag{44}$$

$$\rho^{-1}\sigma_0\phi_o(x)ik_j\,\mathrm{u}_\mathbf{k}(\mathbf{x}) + \rho^{-1}\sigma_0\phi_o(x)ik_j\,\mathrm{v}_\mathbf{k}(\mathbf{x}) + \left[(\omega_\mathbf{k} + vk_x)^2 - c_l^2\mathbf{k}^2\right]\,C_{\mathbf{k},j}(\mathbf{x}) = 0. \tag{45}$$

After some straightforward algebra, we can write out the equation for the exciton-phonon excitation spectrum:

$$\begin{gathered}\left((\omega_\mathbf{k} + vk_x)^2 - c_l^2\mathbf{k}^2\right) \times \\ \times\left[(\hbar\omega_k)^2 - \Big(L(-\mathbf{k}) - \big(\nu_0\phi_o^2(\mathbf{x}) + 2\nu_1\phi_o^4(\mathbf{x})\big)\Big)\Big(L(-\mathbf{k}) + \big(\nu_0\phi_o^2(\mathbf{x}) + 2\nu_1\phi_o^4(\mathbf{x})\big)\Big)\right] \\ = \Big(L(-\mathbf{k}) - \big(\nu_0\phi_o^2(\mathbf{x}) + 2\nu_1\phi_o^4(\mathbf{x})\big)\Big)\,\frac{2\sigma_0^2}{\rho c_l^2}\phi_o^2(x)\,(c_l^2\mathbf{k}^2).\end{gathered} \tag{46}$$

Note that within the semiclassical description of the *inside*-excitations, the low energy limit means $k \to k_0$ where k_0 is the momentum cut-off,

$$(\hbar^2/2m)\mathbf{k}_0^2 \simeq |\tilde{\mu}| = (\hbar^2/2m)L_0^{-2}.$$

The inequality $k > k_0 \simeq L_0^{-1}$ ensures the function $\hbar\omega_\mathbf{k}(\mathbf{x})$ in (46) to be real and positive. Indeed, only the excitations with the wave lengths $\lambda < (2 \sim 3)L_0$ can be considered as the *inside* ones. The presence of the "hard core" terms, const $\nu_1\phi_o^4(x)$, leads to a slight renormalization of the value of the momentum cut-off. However, this renormalization does not factually change the characteristic properties of the possible solutions of Eq. (46). We will mark the "hard core" terms by $\epsilon_+ > 0$. For example, in the low energy limit,

$$k \approx k_0 + \delta k, \quad \delta k \to 0,$$

the exciton part of the l.h.s. of Eq. (46) – i.e. the formula inside the square brackets – can be reduced to the form:

$$(\hbar\omega_\mathbf{k})^2 - \left(\frac{\hbar^2(\mathbf{k}^2 - \mathbf{k}_0^2)}{2m} + F(x) + \epsilon_+\right)\left(\frac{\hbar^2(\mathbf{k}^2 - \mathbf{k}_0^2)}{2m} + F(x) + 2\nu_0\phi_o^2(x) + \epsilon_+\right), \tag{47}$$

where $F(x) = (\sigma_0^2/\rho(c^2 - v^2) - \nu_0)\,(\Phi_o^2 - \phi_o^2(x))$, and the following two estimates hold: $F(x) \simeq 2|\tilde{\mu}|(2x/L_0)^2$ at $x \sim 0$ and $F(x) \simeq 2|\tilde{\mu}|$ at $x > \pm 2L_0$.

There are two different types of the inside-excitations, the longitudinal excitations and the transverse ones. The later have the wave vectors $\mathbf{k}$ perpendicular to the $x\,(v)$-direction. Although Eq. (46) can be solved exactly for the transverse excitation spectrum [22], taking into account the coupling term changes the values of excitation energies slightly, and the excitations can be approximately considered as of the pure exciton or phonon types. Then we have the acoustic phonon dispersion law for the phonon branch and the following spectrum for the exciton branch:

$$(\hbar\omega_{\mathrm{ex},\,k_\perp})^2 \simeq \left(\frac{\hbar^2}{2m}(k^2-k_0^2)+\epsilon_+\right)\left(\frac{\hbar^2}{2m}(k^2-k_0^2)+2\nu\phi_o^2(x)+\epsilon_+\right) \approx$$

$$\approx \frac{\hbar^2}{2m}(k^2-k_0^2)\,2\nu\phi_o^2(x)+2\nu\phi_o^2(x)\epsilon_+ < \left(\,|\tilde{\mu}|+(\hbar^2/2m)k_\perp^2\,\right)^2. \qquad (48)$$

The smooth function $\omega_{k_\perp}(x)>0$ has a gap when $k\to k_0$, but unlike the case of the outside excitations, the gap value is determined by the "hard core" repulsion term and is much less than $|\mu|$. Furthermore, if we let (formally) the x coordinate in (48) change in the area of $|x|>L_0$, the dispersion law $\hbar\omega_{\mathrm{ex},\,k_\perp}(x)$ reproduces the outside excitation asymptotics, $|\tilde{\mu}|+(\hbar^2/2m)k_\perp^2$. However, inside the condensate, we obtain a strong deviation of the collective excitation spectrum from the simple excitonic one.

In the case of the longitudinal excitations, the mode interaction is non-negligible in the low energy limit. For instance, the "exciton" root of Eq. (46) with the nonzeroth r.h.s. exists if $k_x>2.5\,L_0^{-1}$. Therefore there are no distinct exciton and phonon modes, and the cases $k_x>0$ and $k_x<0$ are different because of different position of the "bare" phonon root on the energy axis. Then, on the energy axis $\hbar\omega$, the modified phonon spectrum is located higher than the phonon frequencies and the modified exciton spectrum is located lower then $\hbar\omega^{(o)}_{\mathrm{ex},\,k_x}$. Yet, like the case of transverse excitations, the same inequality and the same (formal) asymptotics are valid for the lower branch of the spectrum:

$$0<\omega_{k_x}<|\tilde{\mu}|+(\hbar^2/2m)k_x^2.$$

The approximate formulas for the longitudinal spectrum are too cumbersome to be presented here. However, the phonon component of the excitonic longitudinal excitation, $C_{k,1}(x) \neq 0$, can be found approximately from Eq. (45) by use of the Bogoliubov form for the wave functions $\mathrm{u}_k^2(x)$ and $\mathrm{v}_k^2(x)$ [20]:

$$\mathrm{u}_k^2(x) \approx \left(\frac{1}{V_{\text{eff}}}\right) \frac{L(-k^2) + \hbar\omega_k}{2\hbar\omega_k}, \quad \mathrm{v}_k^2(x) \approx \left(\frac{1}{V_{\text{eff}}}\right) \frac{L(-k^2) - \hbar\omega_k}{2\hbar\omega_k}, \tag{49}$$

where $L(-k^2) \approx \hbar^2(k^2 - k_0^2)/2m + \nu\phi_{\text{o}}^2(x) + \epsilon_+$. The effective condensate volume $V_{\text{eff}} \simeq 4SL_0$ is used to normalize the u- and v-wave functions of the inside excitations, $\int d\mathbf{r}(|\mathrm{u}_k|^2 - |\mathrm{v}_k|^2) = 1$. Subsequently, we get

$$C_{k,1}(x) \approx \frac{\rho^{-1}\sigma_0\phi_{\text{o}}(x) ik_x \left(\mathrm{u}_k(x) + \mathrm{v}_k(x)\right)}{c_l^2 k^2 - (\omega_{k_x} + vk_x)^2}. \tag{50}$$

One can use the zeroth approximation, Eq. (48), for ω_{k_x} in (49),(50). Then we can roughly estimate the maximum value of $|C_k(x)|^2$ in the low energy limit ($k \to 2.5L_0^{-1}$, $\omega \ll c|k_x|$):

$$|C_k|^2 \simeq \left(|\tilde{\mu}|/\rho(c_l^2 - v^2)V_{\text{eff}}\right) L_0^2 \lll L_0^2. \tag{51}$$

To investigate the stability of the moving condensate in relation to the creation of inside excitations, we can calculate the energy of the condensate with the *one* inside excitation described by the following set: k, ω_k, u_k and v_k, and C_k. Although such an excitation was defined in the co-moving frame, calculations should be done in the laboratory frame. Returning to the lab frame, we represent the exciton and phonon field functions as follows:

$$\phi_{\text{o}}(x-vt,\, t) \to \phi_{\text{o}}(x-vt,\, t) + \exp(-i(E_g + mv^2/2 - |\tilde{\mu}|)t)\exp(imvx)\,\delta\tilde{\Psi}(\mathbf{x}, t), \tag{52}$$

$$\delta\tilde{\Psi}(\mathbf{x}, t) = \mathrm{u}_k(x - vt)e^{i\mathbf{kx}}e^{-i(\omega_k + k_x v)t} + \mathrm{v}_k(x - vt)e^{-i\mathbf{kx}}e^{+i(\omega_k + k_x v)t},$$

$$u_{\text{o}}(x - vt) \to u_{\text{o}}(x - vt) + C_k(x - vt)\exp(i\mathbf{kx})\exp(-i(\omega_k + k_x v)t) + \text{c.c.}. \tag{53}$$

As the inside excitation is considered as an fluctuation, the number of particles in the condensate and its energy can be estimated as $N_{\text{o}} - \int d\mathbf{x}\,\delta\psi^\dagger\delta\psi$ and $E_{\text{o}}(N_{\text{o}}) - \partial_N E_{\text{o}} \int d\mathbf{x}\,\delta\psi^\dagger\delta\psi$, respectively.

The zeroth component of the energy-momentum tensor can be represented in the form

$$\mathcal{T}_0^0 = \mathcal{T}_0^0(\phi_o,\, u_o) + \mathcal{T}_0^{0\,(2)}(\delta\Psi^\dagger, \delta\Psi, \delta u \,|\, \phi_o,\, u_o),$$

where the first part gives the condensate energy E_o and the second part will give the energy of the inside excitation, E_{in}. After substitution of (52),(53) into $E_{\text{in}} = \int d\mathbf{x}\, \mathcal{T}_0^{0\,(2)}$, it can be rewritten as follows:

$$E_{\text{in}} = \int d\mathbf{x}(E_g + mv^2/2 - |\tilde{\mu}|)\delta\tilde{\psi}^\dagger\delta\tilde{\psi} +$$

$$+ \int d\mathbf{x}\, \hbar(\omega_k + k_x v)(|\mathrm{u}_k|^2 - |\mathrm{v}_k|^2) + 2\rho|C_k|^2(\omega_k + vk_x)^2. \tag{54}$$

The first term appearing in (54), $\delta\tilde{\psi}^\dagger\delta\tilde{\psi} \to |\mathrm{u}_k|^2 + |\mathrm{v}_k|^2$, vanishes in the following formula for the total energy:

$$E_o + E_{\text{in}} \approx E_o(N_o) + (2|\tilde{\mu}| + \nu_0\Phi_o^2) \int d\mathbf{x}\, \delta\tilde{\psi}^\dagger\delta\tilde{\psi} + \int d\mathbf{x}\, \hbar(\omega_k + k_x v)(|\mathrm{u}_k|^2 - |\mathrm{v}_k|^2) -$$

$$- \frac{c_l^2 + v^2}{c_l^2 + v^2}\frac{\sigma_0^2}{\rho(c_l^2 - v^2)}\Phi_o^2 \int d\mathbf{x}\, \delta\tilde{\psi}^\dagger\delta\tilde{\psi} + \int d\mathbf{x}\, 2\rho|C_k|^2(\omega_k + vk_x)^2. \tag{55}$$

Here the last two terms compensate each other approximately, and all the interesting effects come form the exciton part of (55). Although $\omega_k + k_x v$ can be negative, its negative value can be compensated by the term $2|\tilde{\mu}| + \nu_0\Phi_o^2$ if the velocity of the condensate is close to c_l or the exciton concentration is high enough. Therefore, there is a second critical velocity v_c in the theory. If the velocity of the condensate is less then the velocity v_c and $v_o < v_c < c_l$, the condensate is unstable in relation to creation of the inside excitations, i.e. $E_o + E_{\text{in}}(k_x, v) < E_o$ in the lab frame. To estimate the value of v_c, we solve the following equation:

$$(3 \sim 5)\hbar v L_0^{-1}(v) \simeq \left(\sigma_0^2/(c_l^2 - v^2)\rho\right) \Phi_o^2(v),$$

which can be reduced to $(3 \sim 5)\hbar v \simeq \left(\sigma_0^2/(c_l^2 - v^2)\rho\right)(N_o/2S)$. For example, for the packets with the exciton concentration of $n \simeq 10^{14} \sim 10^{15}$ cm^{-3}, the estimate is as follows:

$$\frac{c_l - v_c}{c_l} \simeq \frac{\sigma_0^2}{\rho c_l^2}\frac{N_o}{2S}\frac{0.1}{\hbar c_l} \simeq 0.1 \sim 0.3. \tag{56}$$

5 Interference between Two Moving Packets

There are at least two interesting problems that can be examined in the framework of the proposed model (Eqs. (5,6)). The first problem is the investigation of the condensate steady-state and its stability, calculation of the low-lying excitation spectrum, etc.. (Some part of this program was presented in the previous sections.) The second one is the investigation of interference between two moving packets [23]. In this case the problem is essentially nonstationary. Indeed, the amplitude and the shape of the resultant moving packet are expected to change in time [24]. These effects, however, can be considered theoretically by a numerical solving of Eqs. (5),(6) with the proper initial conditions. (We will disregard the influence of noncondensed particles, the condensate depletion and nonequilibrium phonons on the dynamic processes being considered.) For example, if two "input" packets have the same velocity ($\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}$) and shape, we can write the initial conditions in the form

$$\psi_0(\mathbf{x}, t=0) \cdot \mathbf{u}_0(\mathbf{x}, t=0) = \phi_o(\mathbf{x}) \cdot \mathbf{q}_o(\mathbf{x}) + e^{i\delta\varphi} \phi_o(\mathbf{x} + \mathbf{v}\tau) \cdot \mathbf{q}_o(\mathbf{x} + \mathbf{v}\tau), \quad (57)$$

where $\delta\varphi = \text{const}$ is the macroscopical phase difference between the two "input" condensates, and $\tau = \text{const}$ is the time delay between them.

In the simplest (quasi-1D) model, the following equations govern the dynamics of the two "input" packets:

$$(\, i\hbar\partial_t + mv^2/2\,)\psi_0(x,t) = (-\frac{\hbar^2}{2m}\partial_x^2 + \nu_0|\psi_0|^2 + \nu_1|\psi_0|^4)\,\psi_0(x,t) + \sigma_0\,\partial_x u_0(x,t)\,\psi_0(x,t) \quad (58)$$

$$(\partial_t^2 - (c_l^2 - v^2)\partial_x^2 - 2v\partial_t\partial_x\,)u_0(x,t) = \rho^{-1}\sigma_0\partial_x|\psi_0|^2(x,t). \quad (59)$$

Then the initial conditions (57) can be written in the explicit 1D form by using the exact solution (17) of the model (14,15). Note that the amplitudes of $\phi_o(x)$ and $\partial u_o(x)$ are defined from the normalization condition and depend on the values of v and N_o, and, hence, the amplitudes of the "input" condensates in (57) have the same values.

Some predictions of the form of the expected solution can be made easily. Indeed, the shape and other characteristics of the steady-state solution of (58), (57) will depend mainly on the value of the parameter x_0/L_0 where $x_0 = v\tau$ and L_0 is the characteristic condensate width. If $x_0/L_0 < 1 \sim 2$, the nonlinear interaction between the packets plays an important role in the process of total wave function formation. (Notice that the lower limit of x_0, $x_0^* = \tau^* v$, is defined by the time scale of formation of a condensate wave function.) It is reasonable to assume that in the limit of strong interaction between condensates, the $N_{\text{o}} + N_{\text{o}}$ excitons can form the *single* condensate wave function (17) in the steady-state regime. Then this wave function can be written (in the laboratory frame) as follows:

$$\psi_0(x,t) \cdot u_0(x,t) \simeq \exp(-i(\tilde{\mu}(2N_{\text{o}}, \tilde{v}) + m\tilde{v}^2/2)t) \exp(im\tilde{v}x) \times$$

$$\times \exp(i\tilde{\varphi})\phi_{\text{o}}(x - \tilde{v}t;\, 2N_{\text{o}}) \cdot q_{\text{o}}(x - \tilde{v}t;\, 2N_{\text{o}}).$$

As the dynamic equations (58),(59) conserve the energy,

$$E_{\text{in}}(N, v;\, x_0/L_0) = E_{\text{out}}(2N, \tilde{v}),$$

$\tilde{v}$ cannot be equal to v in theory. Moreover, if the parameter x_0/L_0 is small enough, it can be only the approximate equality, $\tilde{v} \approx v$.

In the case of $x_0/L_0 \gg 1$, one can disregard the influence of the mutual nonlinear interaction on the dynamics of the packets. In this approximation, the packet moving in the crystal can be modeled by the following formula:

$$\psi_0(x,t) \cdot u_0(x,t) \simeq \exp(-i(\tilde{\mu}(N_{\text{o}}) - mv^2/2)t)\phi_{\text{o}}(x;\, N_{\text{o}}) \cdot q_{\text{o}}(x;\, N_{\text{o}}) +$$

$$+ \exp(i\delta\varphi) \exp(-i(\tilde{\mu}(N_{\text{o}}) - mv^2/2)t)\phi_{\text{o}}(x + x_0;\, N_{\text{o}}) \cdot q_{\text{o}}(x + x_0;\, N_{\text{o}}).$$

An interesting and noninvestigated case in the interference problem is the condensate dynamics after posing nonsymmetric initial conditions. In fact, the amplitude and the velocity of the "input" packets can be different, for example, $N_2 > N_1$ and $v_2 > v_1$, $\mathbf{v}_2 \| \mathbf{v}_1$. We use here the experimental result [5] that at $v > v_0$, the velocity of the condensate depends on the laser power or, equivalently, on the initial concentration of excitons, i.e. $v = v(N)$. (Note that in theory the exciton amplitude Φ_o is the function of N_o and v.)

If the exciton concentration in the first packet, n_1, is close to the Bose condensation threshold value and the exciton concentration in the second packet, n_2, can be made $\gg n_1$, the velocity difference between condensates can reach $(0.2 \sim 0.3)c_l$. Then, in the reference frame moving with the first packet, the initial conditions can be taken as the following:

$$\psi_0(\mathbf{x}, t=0) \cdot \mathbf{u}_0(\mathbf{x}, t=0) = \phi_o(\mathbf{x}; N_1) \cdot \mathbf{q}_o(\mathbf{x}; N_1) +$$

$$+ \exp(i\delta\varphi) \exp(im\, \delta\mathbf{v}\, \mathbf{x})\, \phi_o(\mathbf{x} + \mathbf{x}_0; N_2) \cdot \mathbf{q}_o(\mathbf{x} + \mathbf{x}_0; N_2), \qquad (60)$$

where $\delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$, $x_o = v_1\tau$, and the second packet moves in this frame of reference. In the case of such the initial conditions, the regime of strong nonlinear interaction between the condensates is unavoidable. Following the logic of the soliton theory, we speculate that the steady-state solution may consist of two packets moving with different velocities and with different exciton concentrations. Roughly speaking, the "input" packets could exchange their places, i.e. the *both* packets survive after collision, and the first packet arrives at the "detecting" boundary of a crystal after the second one:

$$\psi_0(x,t) \cdot u_0(x,t) \simeq \exp(-i(\tilde{\mu}(N_1) - mv_1^2/2)t\,)\, \phi_o(x; N_1) \cdot q_o(x; N_1) +$$

$$+ \exp(-i(\tilde{\mu}(N_2) - mv_1^2/2 + m\delta v^2/2)t\,) \exp(i\delta\tilde{\varphi}) \exp(im\, \delta v\, x) \times$$

$$\times \phi_o(x + x_0 - \delta v\, t; N_2) \cdot q_o(x + x_0 - \delta v\, t; N_2).$$

However, the hypothesis about the solitonic character of packet collisions in 3D needs both numerical and experimental evidence.

6 Conclusion

In this study, we considered a model within which the inhomogeneous excitonic condensate with a nonzero momentum can be investigated. The important physics we include in our model is the exciton-phonon interaction and the appearance of a coherent part of the crystal displacement field that makes the moving condensate of the exciton-phonon one. Then, the condensate wave function and its energy can be calculated exactly in the simplest quasi-1D model. We believe that the transport and other unusual properties of the coherent para-exciton packets in Cu_2O can be described in the framework of the proposed model properly generalized to meet more realistic conditions.

As the exciton-phonon interaction is very important in any processes involving excitons in lattices [26], we can speculate about the possibility of use of piezoelectrical transducers to pump acoustic waves into the system condensate + lattice. Moreover, the transducers could be used to register the phonon part of the coherent packet in experiments in which the condensate is formed by optically inactive excitons and phonons.

We showed that there are two critical velocities in the theory, v_0 and v_c. The first one, v_0, comes from the renormalization of two particle exciton-exciton interaction due to phonons, and the inhomogeneous soliton state can be formed if $v > v_0$. The second one, v_c, comes from use of Landau arguments [19] for investigation of the dynamical stability / instability of the moving condensate. Within the semiclassical approximation for the condensate excitations, we found the condensate is unstable if $v < v_c$. It is interesting to discuss the possibility of observation of such an instability when the condensate can be formed in the inhomogeneous state with $v \neq 0$, but with $v_0 < v < v_c(n, v)$. Such a coherent packet has to disappear during its move through a single pure crystal used for experiments. As the shape of the moving packet depends on time, the form of the registered signal may depend on the crystal length changing from the solitonic to the standard diffusion density profile.

We did not concentrate on detailed investigation of excited states of the moving exciton-phonon condensate in this study. First, the possibility of their observation is an unclear question itself. Second, the stability conditions of the moving condensate – in relation to the creation of condensate excitations – can be derived from the low energy asymptotics of the excitation spectra at $T \ll T_c$. However, the stability problem is not without difficulties [22],[25]. One can easily imagine the situation when the condensate moves in a very high quality crystal, but with some impurity region carefully prepared in the middle of the sample. Then the impurities could bound the noncondensed excitons, which always accompany the condensate, and could mediate, for instance, the emission of the outside excitations. The last process may lead to depletion of the condensate and, perhaps, some other observable effects, such as damping, bound exciton PL, etc.. On the other hand, the inside excitations could manifest themselves at $T \neq 0$ by the effective enlargement of the packet length, $L_0 \to L_{\text{eff}}$, $T \neq 0$, or by interaction with external acoustic waves.

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References

[1] J. P. Wolfe, J. L. Lin, D. W. Snoke and by A. Mysyrowicz in *Bose-Einstein Condensation*, edited by A. Griffin, D. W. Snoke and S. Stringari (Cambridge University Press, Cambridge, 1995).

[2] L. V. Butov, A. Zrenner, G. Abstreiter, G. Bohm, and G. Weimann, Phys. Rev. Lett **73**, 304, (1994); Physics–Uspekhi **39**, 751 (1996).

[3] J. L. Lin, J. P. Wolfe, Phys. Rev. Lett. **71**, 122 (1993).

[4] T. Goto, M. Y. Shen, S. Koyama, T. Yokouchi, Phys. Rev. B **55**, 7609 (1997).

[5] E. Fortin, S. Fafard, A. Mysyrowicz, Phys. Rev. Lett. **70**, 3951 (1993).

[6] E. Benson, E. Fortin, A. Mysyrowicz, Phys. Stat. Sol. B **191**, 345 (1995); Sol. Stat. Comm. **101**, 313, (1997).

[7] E. Hanamura, Sol. Stat. Comm. **91**, 889 (1994);
J. Inoue, E. Hanamura, Sol. Stat. Comm. **99**, 547 (1996);

[8] A. E. Bulatov, S. G. Tichodeev, Phys. Rev. B **46**, 15058 (1992);
G. A. Kopelevich, S. G. Tikhodeev, and N. A. Gippius, JETP **82**, 1180 (1996).

[9] I. Loutsenko, D. Roubtsov, Phys. Rev. Lett. **78**, 3011 (1997).

[10] A. Imamoḡlu and R. J. Ram, Phys. Lett. A **214**, 193 (1996);
W. Zhao, P. Stenius, A. Imamoḡlu, Phys. Rev. B **56**, 5306 (1997).

[11] G. M. Kavoulakis, G. Baym, and J. P. Wolfe, Phys. Rev. B **53**, 7227 (1996);
G. M. Kavoulakis, Y.-C. Chang, and G. Baym, Phys. Rev. B **55**, 7593 (1997);

[12] S. G. Tichodeev, Phys. Rev. Lett. **78**, 3225 (1997);
A. Mysyrowicz, *ibidem*, 3226 (1997).

[13] A. L. Ivanov, C. Ell, and H. Haug, Phys. Rev. B **57**, 9663 (1998).

[14] S. Schmitt-Rink, D. S. Chemla, and D. A. B. Miller, Adv. Phys. **38**, 89 (1989).

[15] L. V. Keldysh and A. N. Kozlov, Sov. Phys. JETP **27**, 521 (1968);
E. Hanamura, H. Haug, Phys. Rep. C **33**, 209 (1977).

[16] V. N. Popov, *Functional Integrals and Collective Modes*, (Cambridge University Press, N.Y., 1987).

[17] M. F. Miglei, S. A. Moskalenko, and A. V. Lelyakov, Phys. Stat. Sol. **35**, 389 (1969);
V. M. Nandkumaran and K. P. Sinha, Zeit. für Phys. B **22**, 173 (1975).

[18] A. Griffin, Phys. Rev. B **53**, 9341 (1996).

[19] Al. L. Fetter and J. D. Waleska, *Quantum Theory of Many-Particle System*, (McGrav-Hill, New York, 1971).

[20] S. Giorgini, L. P. Pitaevskii, S. Stringari, Phys. Rev. A **54**, 4633 (1996);
F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, submitted to Rev. Mod. Phys.; cond-mat/9806038.

[21] J.J. Hopfield, Phys. Rev. **112**, 1555 (1958).

[22] S. A. Moskalenko, D. W. Snoke, *Bose Condensation of Excitons and Coherent Nonlinear Optics*, (Cambridge University Press, in press).

[23] E. Benson, E. Fortin, B. Prade, and A. Mysyrowicz, to appear in Europhys. Lett..

[24] A. Mysyrowicz, E. Benson, and E. Fortin, Phys. Rev. Lett. **77**, 896 (1996).

[25] D. Roubtsov, Y. Lépine, to appear in Phys. Lett. A.; cond-mat/9807023.

[26] H. Stolz, *Time-Resolved Light Scattering from Excitons*, (Springer-Verlag, 1994).

Bosons in a Lattice: Exciton - Phonon Condensate in Cu_2O

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Abstract

We explore a nonlinear field model to describe the interplay between the ability of excitons to be Bose-condensed and their interaction with other modes of a crystal. We apply our consideration to the long-living para-excitons in Cu_2O. Taking into account the exciton-phonon interaction and introducing a coherent phonon part of the moving condensate, we derive the dynamic equations for the exciton-phonon condensate. These equations can support localized solutions, and we discuss the conditions for the moving inhomogeneous condensate to appear in the crystal. We calculate the condensate wave function and energy, and a collective excitation spectrum in the semiclassical approximation; the inside-excitations were found to follow the asymptotic behavior of the macroscopic wave function exactly. The stability conditions of the moving condensate are analyzed by use of Landau arguments, and Landau critical parameters appear in the theory. Finally, we apply our model to describe the recently observed interference and strong nonlinear interaction between two coherent exciton - phonon packets in Cu_2O.

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1 Introduction

Excitons in semiconductor crystals [1] and nanostructures [2],[3] are a very interesting and challenging system to search for the process of Bose-Einstein condensation (BEC). Nowadays there is a lot of experimental evidence that the optically inactive para-excitons in Cu_2O can form a highly correlated state, or the excitonic Bose Einstein condensate [1],[4],[5]. A moving condensate of para-excitons in a 3D Cu_2O crystal turns out to be spatially inhomogeneous in the direction of motion, and the registered velocities of coherent exciton packets turn out to be always less, but approximately equal to the longitudinal sound speed of the crystal [6].

Analyzing recent experimental [4],[6],[7] and theoretical [8]-[13] studies of BEC of excitons in Cu_2O, we can conclude that there are essentially two different stages of this process. The first stage is the kinetic one, with the characteristic time scale of $10 \sim 20\,\mathrm{ns}$. At this stage, a condensate of long-living para-excitons begins to be formed from a quasi-equilibrium degenerate state of excitons ($\mu \neq 0$, $T_{\mathrm{eff}} > T_{\mathrm{latt}}$) when the concentration and the effective temperature of excitons in a cloud, T_{eff}, meet the conditions of Bose-Einstein Condensation [1]. Note that we do not discuss here the behavior of ortho-excitons (with the lifetime $\tau_{\mathrm{ortho}} \simeq 30\,\mathrm{ns}$) and their influence on the para-exciton condensation process. For more details about the ortho-excitons in Cu_2O, ortho-para-exciton conversion, etc. see [4],[5], [14], [15].

The most intriguing feature of the kinetic stage is that formation of the para-exciton condensate and the process of momentum transfer to the para-exciton cloud are happening simultaneously. If the diameter of an excitation spot on the crystal surface is large enough, $S_{\mathrm{spot}} \simeq S_{\mathrm{surf}}$, and the energy of a laser beam satisfies $\epsilon_{\mathrm{phot}} \gg E_{\mathrm{gap}}$, nonequilibrium acoustic phonons may play the key role in the process of momentum transfer. As a result, the mode with macroscopical occupancy of the excitons appears to be with $\langle \mathbf{k} \rangle \neq 0$, where $\hbar \langle k_x \rangle = m_{\mathrm{x}}\, v$ and v is the packet velocity.

Indeed, the theoretical results obtained in the framework of the "phonon wind" model [10],[16] and the experimental observations [4],[5],[6] are strong arguments in favor of this idea. To the authors' knowledge, there are no realistic theoretical models of the kinetic stage of para-exciton condensate formation where quantum degeneracy of the appearing exciton state and possible coherence of nonequilibrium phonons pushing the excitons would be taken into account. Indeed, the condensate formation and many other processes involving it are essentially nonliner ones. Therefore, the condensate, or, better, the *macroscopically occupied mode*, can be different from $n(\mathbf{k} = 0) \gg 1$, and the language of the states in $\mathbf{k}$-space and their occupation numbers $n(\mathbf{k})$ may be not relevant to the problem, see [17].

In this study, we will not explore the stage of condensate formation. Instead, we investigate the second, quasi-equilibrium stage, in which the condensate has already been formed and it moves through a crystal with some constant velocity and characteristic shape of the density profile. In theory, the time scale of this "transport" stage, Δt_{tr}, could be determined by the para-exciton lifetime ($\tau_{\mathrm{para}} \simeq$ 13 μs [1]). In practice, it is determined by the characteristic size ℓ of a high-quality single crystal available for experiments:

$$\Delta t_{\mathrm{tr}} \simeq \ell / c_l \simeq 0.5 - 2\,\mu\mathrm{s} \ll \tau_{\mathrm{para}},$$

where c_l is the longitudinal sound velocity.

We assume that at the "transport" stage, the temperature of the moving packet (condensed + noncondensed particles) is approximately equal to the lattice temperature,

$$T_{\mathrm{eff}} = T_{\mathrm{latt}} < T_c.$$

Then we can consider the simplest case of $T = 0$ and disregard the influence of all sorts of *nonequilibrium* phonons (which appear at the stages of exciton formation, thermalization [10]) on the formed moving condensate.

Any theory of the exciton BEC in Cu_2O has to point out some physical mechanism(s) by means of which the key experimental facts can be explained. (For example, the condensate moves without friction within a narrow interval of velocities localized near c_l, and the shape of the stable macroscopic wave function of excitons resembles soliton profiles [7].) Here we explore a simple model of the ballistic exciton-phonon condensate. In this case, the general structure of the Hamiltonian of the moving exciton packet and the lattice phonons is the following:

$$\hat{H} = H_{\text{ex}}(\hat{\psi}^\dagger, \hat{\psi}) - \mathbf{v}\mathbf{P}_{\text{ex}}(\hat{\psi}^\dagger, \hat{\psi}) + H_{\text{ph}}(\hat{\mathbf{u}}, \hat{\pi}) - \mathbf{v}\mathbf{P}_{\text{ph}}(\hat{\mathbf{u}}, \hat{\pi}) + H_{\text{int}}(\hat{\psi}^\dagger\hat{\psi},\, \partial_j \hat{u}_k). \quad (1)$$

Here $\hat{\psi}$ is the Bose-field operator describing the excitons, $\hat{\mathbf{u}}$ is the field operator of lattice displacements, $\hat{\pi}$ is the momentum density operator canonically conjugate to $\hat{\mathbf{u}}$, and $\mathbf{P}$ is the momentum operator. Note that the Hamiltonian (1) is written in the reference frame moving with the exciton packet, i.e. $\mathbf{x} \to \mathbf{x} - \mathbf{v}t$ and $\mathbf{v} = \text{const}$ is the ballistic velocity of the packet.

2 3D Model of Moving Exciton-Phonon Condensate

To derive the equations of motion of the field operators (and generalize these equations to the case of $T \neq 0$), it is more convenient to start from the Lagrangian. In the proposed model, the Lagrangian density has the following form in the co-moving frame

$$\mathcal{L} = \frac{i\hbar}{2}(\hat{\psi}^\dagger\, \partial_t\hat{\psi} - \partial_t\hat{\psi}^\dagger\, \hat{\psi}) + v\frac{i\hbar}{2}(\partial_x\hat{\psi}^\dagger\, \hat{\psi} - \hat{\psi}^\dagger\partial_x\hat{\psi}) -$$

$$-\tilde{E}_g\hat{\psi}^\dagger\hat{\psi} - \frac{\hbar^2}{2m}\nabla\hat{\psi}^\dagger\nabla\hat{\psi} - \frac{\nu_0}{2}(\psi^\dagger(\mathbf{x},t))^2(\psi(\mathbf{x},t))^2 - \frac{\nu_1}{3}(\psi^\dagger(\mathbf{x},t))^3(\psi(\mathbf{x},t))^3 +$$

$$+\frac{\rho}{2}(\partial_t\hat{\mathbf{u}})^2 - \frac{\rho c_l^2}{2}(\,\partial_j\hat{u}_s\,)^2 - \frac{\rho c_l^2}{3}\,\kappa_3\,(\partial_j\hat{u}_s\,)^3 - \frac{\rho v}{2}\,(\partial_t\hat{\mathbf{u}}\,\partial_x\hat{\mathbf{u}} + \partial_x\hat{\mathbf{u}}\,\partial_t\hat{\mathbf{u}}\,) + \frac{\rho v^2}{2}(\partial_x\hat{\mathbf{u}})^2 -$$

$$-\sigma_0\hat{\psi}^\dagger(\mathbf{x},t)\hat{\psi}(\mathbf{x},t)\nabla\hat{\mathbf{u}}(\mathbf{x},t), \quad (2)$$

where m is the exciton "bare"mass ($m = m_e + m_h \simeq 3m_e$ for 1s excitons in Cu_2O), ν_0 is the exciton-exciton interaction constant ($\nu_0 > 0$ corresponds to the repulsive interaction between para-excitons in Cu_2O [18]), ρ is the crystal density, σ_0 is the exciton-longitudinal phonon coupling constant, and $\mathbf{v} = (v, 0, 0)$. The energy of a free exciton is $\tilde{E}_g + \hbar^2\mathbf{k}^2/2m$. Although the validity of the condition $n\,\tilde{a}_{\mathrm{B}}^3 \ll 1$ ($\tilde{a}_{\mathrm{B}}$ is the exciton Bohr radius) makes it possible to disregard all the multiple-particle interactions with more than two participating particles in $\hat{H}_{\mathrm{ex}}$ [19], we include the hard-core repulsion term originated from the three-particle interaction in $\mathcal{L}$, i.e. $\nu_1 \neq 0$ and

$$0 < \nu_1/\tilde{a}_{\mathrm{B}}^6 \ll \nu_0/\tilde{a}_{\mathrm{B}}^3 \simeq \mathrm{const}\,\mathrm{Ry}^*.$$

(For 3D case, one has to take const $\simeq 10$ because $\nu_0 = 4\pi\,(\hbar^2/m)\,a_{\mathrm{sc}}$ and $a_{\mathrm{sc}} \simeq (1 \sim 3)\,\tilde{a}_{\mathrm{B}}$; see the discussion in [20].)

Moreover, in the Lagrangian of the displacement field, we include the first nonlinear term $\propto \kappa_3\,(\partial u)^3$. (The dimensionless parameter κ_3 originates from Taylor's expansion of an interparticle potential $U(|r_i - r_j|)$ of the medium atoms.) Assuming that a dilute excitonic packet moves in a weakly nonlinear medium, we will not take into account more higher nonlinear terms in (2).

For simplicity's sake, we take all the interaction terms in $\mathcal{L}$ in the *local* form and disregard the interaction between the excitons and transverse phonons of the crystal. Note that the ballistic velocity v is one of the parameters of the theory, and we will not take into account the excitonic normal component and velocity, i.e. $v = v_s \sim \nabla\varphi_{\mathrm{c}}$, $(T = 0)$. This means that we choose the spatial part of the coherent phase of the packet, $\varphi_{\mathrm{c}}(\mathrm{x})$, to be in the simplest form,

$$\exp(i\varphi_{\mathrm{c}}(\mathrm{x})) = \exp(i(\varphi + k_0 x)), \quad \varphi = \mathrm{const}, \quad \hbar k_0 = mv. \tag{3}$$

The equations of motion can be easily derived by the standard variational method from the following condition:

$$\delta S = \delta \int dt\, d\mathbf{x}\, \mathcal{L}(\hat{\psi}^\dagger(\mathbf{x}, t),\, \hat{\psi}(\mathbf{x}, t),\, \hat{\mathbf{u}}(\mathbf{x}, t)) = 0.$$

Indeed, after transforming the Bose-fields $\hat{\psi}^\dagger$ and $\hat{\psi}$ by

$$\hat{\psi}(\mathbf{x}, t) \to \exp(-i\tilde{E}_g t/\hbar)\exp(imvx/\hbar)\hat{\psi}(\mathbf{x}, t),$$

we can write these equations as follows:

$$(i\hbar\partial_t + mv^2/2)\hat{\psi}(\mathbf{x},t) =$$

$$=\Big(-\frac{\hbar^2}{2m}\Delta + \nu_0\hat{\psi}^\dagger\hat{\psi}(\mathbf{x},t) + \nu_1\hat{\psi}^{\dagger\,2}\hat{\psi}^2(\mathbf{x},t)\Big)\hat{\psi}(\mathbf{x},t) + \sigma_0\nabla\hat{\mathbf{u}}(\mathbf{x},t)\,\hat{\psi}(\mathbf{x},t), \tag{4}$$

$$(\partial_t^2 - c_l^2\Delta - v(\partial_t\partial_x + \partial_x\partial_t) + v^2\partial_x^2)\hat{u}_s(\mathbf{x},t) - c_l^2\sum_j 2\kappa_3\,\partial_j^2\hat{u}_s\,\partial_j\hat{u}_s(\mathbf{x},t) =$$

$$= \rho^{-1}\sigma_0\partial_s(\hat{\psi}^\dagger\hat{\psi}(\mathbf{x},t)). \tag{5}$$

We assume that the condensate of excitons *exists*. This means that the following representation of the exciton Bose-field holds: $\hat{\psi} = \psi_0 + \delta\hat{\psi}$. Here $\psi_0 \neq 0$ is the classical part of the field operator $\hat{\psi}$ or, in other words, the condensate wave function, and $\delta\hat{\psi}$ is the fluctuational part of $\hat{\psi}$, which describes out-of-condensate particles.

One of the important objects in the theory of BEC is the correlation functions of Bose-fields. The standard way to calculate them in this model (the excitonic function $\langle\psi(\mathbf{x},t)\psi^\dagger(\mathbf{x}',t')\rangle$, for example,) can be based on the effective action or the effective Hamiltonian approaches [21]. Indeed, one can, first, integrate over the phonon variables $\mathbf{u}$, get the expression for $S_{\text{eff}}(\psi,\psi^\dagger)$ and, second, use S_{eff} (or $\hat{H}_{\text{eff}}$) to derive the equations of motion for ψ_0, $\delta\hat{\psi}$, correlation functions, etc..

In this work we do not follow that way; instead, we treat excitons and phonons equally [11],[22],[23]. This means that the displacement field $\hat{\mathbf{u}}$ can have a non-trivial coherent part too, i.e. $\hat{\mathbf{u}} = \mathbf{u}_0 + \delta\hat{\mathbf{u}}$ and $\mathbf{u}_0 \neq 0$, and the actual moving condensate can be an exciton-phonon one, i.e. $\psi_0(\mathbf{x},t)\cdot\mathbf{u}_0(\mathbf{x},t)$. Then the equation of motion for the classical parts of the fields $\hat{\psi}$ and $\hat{\mathbf{u}}$ can be derived by use of the variational method from $\mathcal{L} = \mathcal{L}(\psi,\psi^*,\mathbf{u})$, in which all the fields can be considered as the classical ones. Eventually we have

$$(\,i\hbar\partial_t + mv^2/2\,)\psi_0(\mathbf{x},t) =$$

$$=\Big(-\frac{\hbar^2}{2m}\Delta + \nu_0|\psi_0|^2(\mathbf{x},t) + \nu_1|\psi_0|^4(\mathbf{x},t)\Big)\psi_0(\mathbf{x},t) + \sigma_0\nabla\mathbf{u}_0(\mathbf{x},t)\,\psi_0(\mathbf{x},t)\,, \tag{6}$$

$$(\partial_t^2 - c_l^2\Delta - 2v\partial_t\partial_x + v^2\partial_x^2)u_{0\,s}(\mathbf{x},t) - c_l^2\sum_j 2\kappa_3\,\partial_j^2 u_{0\,s}\,\partial_j u_{0\,s}(\mathbf{x},t) =$$
$$= \rho^{-1}\sigma_0\partial_s(|\psi_0|^2(\mathbf{x},t)). \qquad (7)$$

Notice that deriving these equations we disregarded the interaction between the classical (condensate) and the fluctuational (noncondensate) parts of the fields. That is certainly a good approximation for $T = 0$ and $T \ll T_c$ cases [24].

In this article, a steady-state of the condensate is the object of the main interest. In the co-moving frame of reference, the condensate steady-state is just the stationary solution of Eqs. (6), (7) and it can be taken in the form

$$\psi_0(\mathbf{x},t) = \exp(-i\mu t)\exp(i\varphi)\phi_o(\mathbf{x}), \quad u_0(\mathbf{x},t) = \mathbf{q}_o(\mathbf{x}),$$

where ϕ_o and $\mathbf{q}_o$ are the real number functions, and $\varphi =$ const is the coherent phase of the condensate wave function in the co-moving frame, see Eq. (3). (This phase can be taken equal to zero if only a single condensate is the subject of interest.)

Then, the following equations have to be solved, ($\mu = \tilde{\mu} - mv^2/2$, $s = 1,2,3$):

$$\tilde{\mu}\phi_o(\mathbf{x}) = \Big(-\frac{\hbar^2}{2m}\Delta + \nu_0\phi_o^2(\mathbf{x}) + \nu_1\phi_o^4(\mathbf{x}) \Big)\phi_o(\mathbf{x}) + \sigma_0\nabla\mathbf{q}_o(\mathbf{x})\,\phi_o(\mathbf{x})\,, \qquad (8)$$

$$-\{\,(c_l^2 - v^2)\partial_x^2 + c_l^2\partial_y^2 + c_l^2\partial_z^2\,\}q_{o\,s}(\mathbf{x}) - c_l^2\sum_{j=x,y,z} 2\kappa_3\,\partial_j^2 q_{o\,s}\,\partial_j q_{o\,s}(\mathbf{x}) = \rho^{-1}\sigma_0\partial_s\phi_o^2(\mathbf{x}). \qquad (9)$$

Note that in order to simplify Eq. (7) to Eq. (9), we assumed only $\mathbf{u}_0(\mathbf{x},t) = \mathbf{q}_o(\mathbf{x})$. In this model, it is enough to obtain localized solutions for the displacement field.

3 Effective 1D Model for the Condensate Wave Function

Solving Eqs. (8),(9) in 3D space seems to be a difficult problem. However, these equations can be simplified if we assume that the condensate is inhomogeneous along the x-axis only, that is

$$\phi_{\rm o}(\mathbf{x}) = \phi_{\rm o}(x) \quad \text{and} \quad \mathbf{q}_{\rm o}(\mathbf{x}) = (q_{\rm o}(x),\, 0, 0).$$

Note that the cross-section area of an excitation spot, S, has to be basically constant across the sample cross-section. In this case, the problem can be considered as an effectively one-dimensional one.

Such an effective reduction of dimensionality transforms a difficult (nonlocal differential) equation for the condensate wave function into a rather simple differential one, and obtained in this way the effective 1D model for the condensate wave function $\phi_{\rm o} \cdot q_{\rm o}$ conserves all the important properties of the "parent" 3D model.

Indeed, if $v < c_l$, the following equations stand for the condensate ($y(x) = \partial_x q_{\rm o}(x)$):

$$\tilde{\mu}\phi_{\rm o}(x) = \big(-(\hbar^2/2m)\partial_x^2 + \nu_0\phi_{\rm o}^2(x) + \nu_1\phi_{\rm o}^4(x)\big)\phi_{\rm o}(x) + \sigma_0\, y(x)\, \phi_{\rm o}(x), \tag{10}$$

$$-(c_l^2 - v^2)\partial_x y(x) - 2c_l^2\kappa_3\, \partial_x y\, y(x) = \rho^{-1}\sigma_0\, \partial_x\phi_{\rm o}^2(x). \tag{11}$$

The last equation can be easily integrated,

$$y(x) + \tilde{\kappa}_3\, y^2(x) = \Phi(x) + \text{const}, \tag{12}$$

and solved relative to $y(x)$. Here

$$\tilde{\kappa}_3 = \frac{c_l^2}{c_l^2 - v^2}\,\kappa_3 \equiv \gamma(v)\,\kappa_3, \qquad \Phi(x) = -\frac{\sigma_0}{\rho(c_l^2 - v^2)}\,\phi_{\rm o}^2(x) \equiv -\gamma(v)\,\frac{\sigma_0}{\rho\, c_l^2}\,\phi_{\rm o}^2(x).$$

Note that the medium nonlinearity parameter κ_3 can be enhanced by the factor of the order of $4 \sim 10$ if the value of v is less, but close to c_l. (For spatially localized solutions, $\partial_x q_{\rm o}(x) \simeq 0$ and $\phi_{\rm o}^2(x) \simeq 0$ at $|x| \gg L_{\rm ch}$, so that $\text{const} = 0$.)

If $\kappa_3 < 0$, we can always represent the solution of (11),(12) in the following form

$$y(x) = \Phi(x) + |\tilde{\kappa}_3|\, \Phi^2(x) + 2\,\tilde{\kappa}_3^2\, \Phi^3(x) + \cdots. \tag{13}$$

(Indeed, the parameters of medium nonlinearity can be chosen as $\kappa_3 < 0$ and $\kappa_4 > 0$ [25].) After substitution of (13) into Eq. (10), Eqs. (10), (11) can be rewritten as follows:

$$\tilde{\mu}\,\phi_o(x) = (\,-(\hbar^2/2m)\,\partial_x^2 + \tilde{\nu_0}\,\phi_o^2(x) + \tilde{\nu_1}\,\phi_o^4(x) + \epsilon_2\,)\phi_o(x), \tag{14}$$

$$\partial_x\, q_o(x) = \Phi(x) + |\tilde{\kappa}_3|\,\Phi^2(x) + \epsilon_2', \tag{15}$$

where the interparticle interaction constants are renormalized as follows

$$\tilde{\nu_0} = \nu_0 \;-\; \sigma_0\,\frac{\sigma_0}{\rho(c_l^2 - v^2)}, \qquad \tilde{\nu_1} = \nu_1 \;+\; \sigma_0\,\frac{\sigma_0^2}{(\,\rho(c_l^2 - v^2)\,)^2}\,|\tilde{\kappa}_3|, \tag{16}$$

and more higher nonlinear terms are designated by ϵ_2. A small parameter in Eq. (16) comes from the term

$$\sigma_0/\rho(c_l^2 - v^2) = \gamma(v)\,(\,\sigma_0/Mc_l^2\,)\,a_l^3,$$

where $\gamma(v) = c_l^2/(c_l^2 - v^2)$ and M is the mass of the crystal elementary cell.

The effective two-particle interaction constant $\tilde{\nu_0}\,(v)$ can be **negative** if the velocity of the condensate lies inside the interval $v_o < v < c_l$, where

$$v_o = \sqrt{c_l^2 - (\sigma_0^2/\rho\nu_0)}. \tag{17}$$

Outside this interval, $\tilde{\nu_0}\,(v) > 0$ [11] and the velocity v_o can be called the first 'critical' velocity in the model. The meaning of this velocity can be clarified by rewriting (16) in the dimensionless form,

$$\frac{\tilde{\nu_0}}{\sigma_0\,a_l^3} = \frac{\nu_0}{\sigma_0\,a_l^3} - \gamma(v)\left(\frac{\sigma_0}{Mc_l^2}\right), \tag{18}$$

If $v > v_o$,

$$\gamma(v)\left(\frac{\sigma_0}{Mc_l^2}\right) > \frac{\nu_0}{\sigma_0\,a_l^3} \simeq \frac{\mathrm{const\,Ry}^*}{\sigma_0}\,\frac{\tilde{a}_B^3}{a_l^3}, \tag{19}$$

where Ry^* and $\tilde{a}_B^2$ are the characteristic energy and the cross-section of two-particle collisions in the exciton subsystem. The following inequalities are true for excitons in a crystal

$$\tilde{a}_B^3 > (\gg)\,a_l^3 \quad \text{and} \quad \mathrm{const\,Ry}^* < (\ll)\,\sigma_0,$$

and, usually, the value of the parameter $\nu_0/\sigma_0\, a_l^3$ is > 1.

For para-excitons in Cu_20, however, we assume the (effective) value of $\nu_0/\sigma_0\, a_l^3$ can be estimated as $0.3 \sim 0.6 < 1$, whereas the value of $\sigma_0/Mc_l^2 \simeq 0.1 \sim 0.3$. This makes the inequality (19) valid at, say, $v \approx (0.8 \sim 0.9)\, c_l$, or $\gamma(v) \simeq 5$. Thus, within the effective 1D model, the critical factor $\gamma_o = \gamma(v_o)$ is the following ratio

$$\gamma_o = \left(\frac{\nu_0}{\sigma_0\, a_l^3}\right) / \left(\frac{\sigma_0}{Mc_l^2}\right),$$

and, for the substances with $\nu_0/\sigma_0\, a_l^3 < 1$, the regime with $\widetilde{\nu_0} < 0$ can be obtained at velocities *reasonably close* but not equal to c_l, for example, beginning from some $\gamma_o < 10$, $(\gamma(0.95\, c_l) \approx 10)$.

On the other hand, the effective three-particle interaction constant $\widetilde{\nu_1}\,(v)$ is always positive for crystals with $\kappa_3 < 0$. It can be represented in the dimensionless form as follows

$$\frac{\widetilde{\nu_1}}{\sigma_0 (a_l^3)^2} \simeq \text{const}' \left(\frac{\nu_0}{\sigma_0\, a_l^3}\right) \frac{\tilde{a}_B^3}{a_l^3} + \gamma(v)\, |\kappa_3| \left(\gamma(v)\, \frac{\sigma_0}{Mc_l^2}\right)^2 . \tag{20}$$

Here we estimated the "bare" vertex of the three-particle collisions as

$$\nu_1 \simeq \text{const}\, \text{Ry}^* \tilde{a}_B^6 \simeq \text{const}'\, \nu_0\, \tilde{a}_B^3, \quad \text{const}' \leq 1,$$

and the same Ry^* can be taken as a characteristic energy of collisions. The effective vertex $\widetilde{\nu_1} > 0$ is enhanced by the medium nonlinearity, and both terms in the r.h.s. of (20) can be equally important at $\gamma(v) > \gamma_o$.

Note that in the case of *strongly nonlinear lattices* with excitons, the effective interaction vertices in (14) ($\widetilde{\nu_1}$, $\widetilde{\nu_2}$, etc.) depend on the velocity v and the parameters of medium nonlinearity (κ_3, κ_4, etc.). Then the effective exciton-exciton interaction can be strongly renormalized at sufficiently large gamma-factors $\gamma(v)$ and the vertices may change their signs as it can happen with $\widetilde{\nu_0}\,(v)$. In this article, however, we consider the case of *weakly nonlinear medium* with excitons (e.g., a crystal with long living excitons). More accurately, this means that at velocities $v \to c_l$ the effective vertex $\widetilde{\nu_0}\,(v)$ becomes < 0, while the more higher vertices, such as $\widetilde{\nu_1}\,(v)$ and $\widetilde{\nu_2}\,(v)$, do not change their sign; they remain > 0 at $\gamma(v) > \gamma_0$. Finally, to describe the weakly nonlinear case, it is enough to take into account the parameters $\nu_1 > 0$ and $\kappa_3 < 0$ and neglect more higher nonlinearities (ϵ_2 and ϵ_2' in (14), (15)).

In this study, we will consider the case of $v_o < v < c_l$ in detail. Indeed, in the case of $\widetilde{\nu_0}\,(v) < 0$ and $\widetilde{\nu_1}\,(v) > 0$, some localized solutions of Eqs. (14),(15) do exist. For example, the so-called 'bright soliton' solution of (14) exists if the generalized chemical potential is negative, $\tilde{\mu} < 0$, and $|\tilde{\mu}| < \mu^*$. Here

$$\mu^* = \frac{|\widetilde{\nu_0}|^2}{(16/3)\,\widetilde{\nu_1}} \approx 0.2\,\sigma_0\,\frac{(\,|\widetilde{\nu_0}|/\sigma_0\,a_l^3\,)^2}{(\,\widetilde{\nu_1}/\sigma_0\,a_l^6\,)}. \tag{21}$$

For $|\kappa_3| \sim 1$ and $\gamma(v) > \gamma_o \simeq 3 \sim 5$, we can roughly estimate the effective vertex $\widetilde{\nu_1}\,(v)$ as

$$\widetilde{\nu_1}\,(v)/\sigma_0\,a_l^6 \simeq (1 \sim 10)\,(\nu_0/\sigma_0\,a_l^3).$$

Then $\mu^*(v) \simeq (10^{-1} \sim 10^{-2})\,\mathrm{Ry}^*$, and the larger is the value of $|\kappa_3|$ the smaller is the value of $\mu^*(v)$.

The 'bright soliton' solution of Eq. (14) can be represented in the following form

$$\phi_o(x) = \Phi_o\, f(\,\beta(\Phi_o)\,x,\,\eta_1(\Phi_o)\,), \quad \beta(\Phi_o) = \sqrt{\frac{2m}{\hbar^2}\,|\tilde{\mu}|(\Phi_o)}. \tag{22}$$

Here $\eta_1(\Phi_o)$ is some dimensionless parameter, and the generalized chemical potential $\tilde{\mu} < 0$ is given by the formula

$$|\tilde{\mu}| = |\tilde{\mu}|(\Phi_o) = |\widetilde{\nu_0}|\,\Phi_o^2/2 - \widetilde{\nu_1}\,\Phi_o^4/3. \tag{23}$$

Like the chemical potential $|\tilde{\mu}|$, the amplitude of the bright soliton, Φ_o, satisfies

$$\Phi_o^2 < (\Phi_o^*)^2 = |\widetilde{\nu_0}|/(\,4/3\,\widetilde{\nu_1}),$$

and $\mu^* = |\widetilde{\nu_0}|\,\Phi_o^{*\,2}/4$.

For $|\tilde{\mu}|/\mu^* \ll 1$, the following approximation is valid

$$\eta_1 \approx \frac{1}{4}\,(|\tilde{\mu}|/\mu^*) + \frac{1}{8}\,(|\tilde{\mu}|/\mu^*)^2 \ll 1, \tag{24}$$

and this formula can be used up to $|\tilde{\mu}|/\mu^* \simeq 0.5$. Then we can *approximate* the solution of (14) by the following formulas

$$\phi_o(x) \approx \Phi_o \left(\sqrt{1-\eta_1(\Phi_o)}\, \cosh(\beta(\Phi_o)\, x) + (1 - \sqrt{1-\eta_1(\Phi_o)}) \right)^{-1}, \tag{25}$$

$$\phi_o(x) \simeq 2\Phi_o \exp(-\beta(\Phi_o)\, |x|\,)/\sqrt{1-\eta_1} \quad \text{for} \quad |x| > 2\beta(\Phi_o)^{-1}, \quad |\mu| \ll \mu^*. \tag{26}$$

The amplitudes of the exciton and phonon parts of the condensate, the characteristic width of the condensate, and the value of the effective chemical potential $\tilde{\mu}$ depend on the normalization of the exciton wave function $\phi_o(x)$. We normalize it in 3D space assuming that the characteristic width of the packet in the (y,z)-plane is sufficiently large, i.e. the cross-section area of the packet $S_\perp$ can be made equal to the cross-section area S of a laser beam and

$$S_\perp \simeq S \simeq S_{\text{surf}}.$$

Then we can write this condition as follows:

$$\int |\psi_0|^2(x,t)\, d\mathbf{x} = S \int \phi_o^2(x)\, dx = N_o, \tag{27}$$

where N_o is the number of condensed excitons, and, generally, $N_o \neq N_{\text{tot}}$.

Applying this normalization condition, we get the following results

$$\Phi_o^2 \approx \frac{|\widetilde{\nu_0}\,(v)|}{2\,(N_o^*/N_o)^2\, x\, \text{Ry}^*\, \tilde{a}_B^6 \,+\, 2\,\widetilde{\nu_1}\,(v)}. \tag{28}$$

Here we used the following notations, $N_o^* = 2S/\tilde{a}_B^2$, $\hbar^2/m = 2x\, \text{Ry}^*\, \tilde{a}_B^2$, where $\text{Ry}^* = \hbar^2/2\mu_{\text{exc}}\tilde{a}_B^2$ and $x = \mu_{\text{exc}}/m$. The formula (28) is valid for $|\tilde{\mu}|/\mu^* < 0.3 \sim 0.4$. We assume that, at $N_o^*/N_o = \bar{n}_o > 10$ (this is the important parameter!), we always have

$$2\,\bar{n}_o^2\, x\, \text{Ry}^*\, \tilde{a}_B^6 \gg \widetilde{\nu_1}\,(v) = \tilde{\varepsilon}_1(|\kappa_3|, v)\, \tilde{a}_B^6 \simeq (1 \sim 10)\, \text{Ry}^*\, \tilde{a}_B^6.$$

Then, the following inequalities are valid: $\Phi_o^2(N_o,\, v) \ll \Phi_o^{*\,2}$ and

$$|\tilde{\mu}|(N_o,\, v) \approx \frac{|\widetilde{\nu_0}\,(v)|^2}{2\,\{2\,\bar{n}_o^2\, x\, \text{Ry}^*\, \tilde{a}_B^6 \,+\, 4\,\widetilde{\nu_1}\,(v)\,\}} \ll \mu^* = \frac{|\widetilde{\nu_0}\,(v)|^2}{5.3\,\widetilde{\nu_1}\,(v)}. \tag{29}$$

The characteristic length of the packet can be estimated from Eq. (22) as follows ($\widetilde{\nu_0}\,(v) = \tilde{\varepsilon}_0(v)\,\tilde{a}_B^3$):

$$L_{\rm ch}^{-1}(N_{\rm o},\,v) \simeq \frac{1}{4}\,\beta(\Phi_{\rm o}) \approx \frac{1}{4}\frac{|\tilde{\varepsilon}_0(v)|}{(2\,x\,{\rm Ry}^*\,\tilde{a}_B)}\,\frac{1}{\sqrt{\bar{n}_{\rm o}^2\,+\,(\tilde{\varepsilon}_1(v)/x\,{\rm Ry}^*)}} \simeq$$

$$\simeq \frac{1}{8}\,\frac{|\tilde{\varepsilon}_0(v)|}{x\,{\rm Ry}^*}\,\frac{1}{\tilde{a}_B\,\bar{n}_{\rm o}} \quad \text{at} \quad \bar{n}_{\rm o} > 10. \tag{30}$$

Therefore, at $\gamma(v) \simeq 2\gamma_{\rm o}$, we can roughly estimate

$$L_{\rm ch}(N_{\rm o},\,v) \simeq 4\,\frac{{\rm Ry}^*}{|\tilde{\varepsilon}_0(v)|}\,\bar{n}_{\rm o}\,\tilde{a}_B \sim 4\,(10^1\,\bar{n}_{\rm o})\,\tilde{a}_B, \tag{31}$$

and, for the average concentration of condensed excitons in the packet, $n_{\rm o}$, we have

$$n_{\rm o}\,\tilde{a}_B^3 \approx (N_{\rm o}/SL_{\rm ch})\,\tilde{a}_B^3 \simeq 1/\bar{n}_{\rm o}^2 \ll 1.$$

Recall that the second part of the condensate, the displacement field $q_{\rm o}(x)$, is of the same importance as the first part, the exciton wave function $\phi_{\rm o}(x)$. The displacement field $\partial_x q_{\rm o}(x)$ can be represented as follows

$$\partial_x q_{\rm o}(x) = -\gamma(v)\left(\frac{\sigma_0}{Mc_l^2}\right)\left(a_l^3\,\phi_{\rm o}^2(x)\right) + \gamma(v)\,|\kappa_3|\left(\gamma(v)\,\frac{\sigma_0}{Mc_l^2}\right)^2\left(a_l^3\,\phi_{\rm o}^2(x)\right)^2. \tag{32}$$

To estimate its amplitude, $\partial_x q_{\rm o}$, we have to estimate the parameter $a_l^3\,\Phi_{\rm o}^2$ first.

For $\bar{n}_{\rm o} > 10$ and $|\tilde{\varepsilon_0}\,(v)| \simeq (10^{-1} \sim 1)\,{\rm Ry}^*$ (i.e. $\gamma(v) \geq 2\gamma_{\rm o}$), we obtain

$$a_l^3\,\Phi_{\rm o}^2 \simeq \frac{a_l^3}{\tilde{a}_B^3}\,\frac{|\tilde{\varepsilon}_0(v)|}{x\,{\rm Ry}^*}\,\frac{1}{2\,\bar{n}_{\rm o}^2} \sim \frac{a_l^3}{\tilde{a}_B^3}\,\frac{1}{2\,\bar{n}_{\rm o}^2} \propto N_{\rm o}^2.$$

If this parameter is small enough, such as $a_l^3\,\Phi_{\rm o}^2(N_{\rm o},\,v) \simeq 10^{-3} \sim 10^{-5}$, we can neglect the nonlinear corrections to the amplitude $\partial_x q_{\rm o} < 0$ and to the shape of $\partial_x q_{\rm o}(x)$ as well,

$$\partial_x q_{\rm o} = -\gamma(v)\,\frac{\sigma_0}{Mc_l^2}\left(a_l^3\,\Phi_{\rm o}^2\right)\left\{1\,-\,\gamma(v)\,|\kappa_3|\left(\gamma(v)\,\frac{\sigma_0}{Mc_l^2}\right)\left(a_l^3\,\Phi_{\rm o}^2\right)\right\} \approx$$

$$\approx -\gamma(v)\Big(\frac{\sigma_0}{Mc_l^2}\Big)(a_l^3\,\Phi_{\rm o}^2). \tag{33}$$

Thus, due to the validity of $n_o\,\tilde{a}_B^3 \ll 1$, there is almost no difference between the approximation

$$\phi_o(x)\cdot\partial_x q_o(x) \approx \Phi_o\,\cosh^{-1}(\beta(\Phi_o)\,x)\cdot(-|\partial_x q_o|\,)\,\cosh^{-2}(\beta(\Phi_o)\,x), \tag{34}$$

where

$$\beta(\Phi_o) \equiv L_0^{-1} \approx \frac{|\tilde{\varepsilon}_0(v)|}{x\,\mathrm{Ry}^*}\,\frac{1}{2\,\bar{n}_o\,\tilde{a}_B}, \qquad N_o = N_o^*/\bar{n}_o, \quad \bar{n}_o \gg 1, \tag{35}$$

and the exact solution of the weakly nonlinear case with $\nu_1 > 0$ and $\kappa_3 < 0$. For $S \simeq (10^{-2} \sim 10^{-3})\,\mathrm{cm}^2$, $\tilde{a}_B^2 \simeq (25 \sim 50)\,10^{-16}\,\mathrm{cm}^2$, we estimate $N_o^* \simeq 10^{13} \sim 10^{14}$. Although the approximate solutions we used in this study are valid for $N_o \ll N_o^*$, they can be used at $N_o < N_o^*$ for estimates.

Note that the effective chemical potential is a rather small parameter in this model,

$$|\tilde{\mu}|(N_o,\,v) \approx \frac{|\widetilde{\nu_0}\,(v)|^2}{4\,\bar{n}_o^2\,x\,\mathrm{Ry}^*\,\tilde{a}_B^6} \approx \frac{|\tilde{\varepsilon}_0(v)|}{4\,\bar{n}_o^2}\left(\frac{|\tilde{\varepsilon}_0(v)|}{x\,\mathrm{Ry}^*}\right). \tag{36}$$

That is why the characteristic length, $L_{\mathrm{ch}} \propto |\tilde{\mu}|^{-1/2}$, see (30), can be estimated as $(10^2 \sim 10^4)\,\tilde{a}_B$ within the validity of approximations (28), (29). Moreover, $|\tilde{\mu}|/\mu^* \leq 10^{-2}$ and the parameter $\eta_1(\Phi_o)$ in (22) can be estimated as $\sim 10^{-2}$. In this case, one can neglect it in Eq. (25).

Returning to the laboratory reference frame, we can write the condensate wave function in the form (see Fig. 1):

$$\psi_0(x,t)\cdot u_0(x,t)\delta_{1j} \approx \exp\left(-i\left(\tilde{E}_g + \frac{mv^2}{2} - |\tilde{\mu}|\right)t\right)\exp(i(\varphi + mvx)\,)\times$$
$$\times\Phi_o\cosh^{-1}(L_0^{-1}(x-vt)\,)\cdot\Big(Q_o\;-\;Q_o\tanh(L_0^{-1}(x-vt))\Big), \tag{37}$$

where we count the exciton energy from the bottom of the crystal valence band, $(\tilde{E}_g < E_{\mathrm{gap}})$, and $2Q_o(N_o,v)$ is the amplitude of the phonon part of condensate,

$$Q_o \approx \gamma(v)\left(\frac{\sigma_0}{Mc_l^2}\right)\left(\frac{a_l^2}{\tilde{a}_B^2}\,\frac{1}{\bar{n}_o}\right)\,a_l \ll a_l.$$

To calculate the energy of the moving condensate within the Lagrangian approach, (see Eq. (2)), we have to integrate the zeroth component of the energy-momentum tensor $\mathcal{T}_0^0$ over the spatial coordinates. Consequently, we have the following formula

$$\mathcal{T}_0^0(\mathbf{x},t) = \tilde{E}_g \phi_o^* \phi_o + \frac{\hbar^2}{2m} \nabla\phi_o^* \nabla\phi_o + \frac{\nu_0}{2}(\phi_o^*)^2 \phi_o^2 + \frac{\nu_1}{3}(\phi_o^*)^3 \phi_o^3 +$$

$$+ \frac{\rho}{2}(\partial_t q_o)^2 + \frac{\rho c_l^2}{2}(\partial_x q_o)^2 + \frac{\rho c_l^2}{2}\kappa_3 (\partial_x q_o)^3 + \sigma_0\, \phi_o^* \phi_o\, \partial_x q_o.$$

Here we do not take into account a small correction to this energy due to the quantum depletion of the condensate ($\langle\langle\delta\Psi^\dagger\delta\Psi(\mathbf{x})\rangle_{T=0} \neq 0$ and $\langle(\partial_x \delta u_j)^2\rangle_{T=0} \neq 0$). Then the result reads

$$E_o(N_o,\, v) = \int d\mathbf{x}\, \mathcal{T}_0^0 = E_{\rm ex} + E_{\rm int} + E_{\rm ph} \approx$$

$$\approx N_o \left(\tilde{E}_g + \frac{m\, v^2}{2}\right) - N_o \left(|\tilde{\mu}| + \nu_0\, \Phi_o^2/3\right)$$

$$+ N_o \left\{ \frac{M\,(c_l^2 + v^2)}{2}\, \gamma^2(v) \left(\frac{\sigma_0}{M c_l^2}\right)^2 \frac{2}{3}\,(a_l^3\, \Phi_o^2) \right\}. \tag{38}$$

We will disregard the terms $\sim N_o\, \nu_1 \Phi_o^4$ in $E_{\rm int} < 0$, and the corrections $\propto |\kappa_3|$ in $E_{\rm ph}$. Then we can write

$$|E_{\rm int}|/N_o \approx |\widetilde{\nu_0}\,(v)|\, \Phi_o^2/2 \,+\, \nu_0\, \Phi_o^2/3 \,\simeq\, (\nu_0/\tilde{a}_B^3)\left(\tilde{a}_B^3 \Phi_o^2\right) < {\rm Ry}^* \tag{39}$$

and

$$E_{\rm ph}/N_o \approx \frac{M\,(c_l^2 + v^2)}{2}\, \vartheta(N_o,\, v) \,\simeq\, M c_l^2\, \vartheta(N_o,\, v), \tag{40}$$

where

$$\vartheta(N_o,\, v) = \left(\gamma(v)\, \frac{\sigma_0}{M c_l^2}\right)^2 \frac{2}{3}\,(a_l^3\, \Phi_o^2) \ll 1. \tag{41}$$

Note that the parameter $\vartheta(N_o,\, v)$ is a rather small one, $\vartheta \sim a_l^3\, \Phi_o^2 \simeq 10^{-3} \sim 10^{-5}$, so that the value of $E_{\rm ph}/N_o$ can be $< {\rm Ry}^*$, and, roughly, $\Phi_o^2 \propto N_o^2$.

One can see that the exciton-phonon condensate carries a non-zeroth momentum,

$P_{o\,x} = P_{{\rm ex},\,x} + P_{{\rm ph},\,x}$:

$$P_{o\,x} = \int d\mathbf{x}\, (\hbar/2i)(\phi_0^*(x,t)\, \partial_x \phi_0(x,t) - \partial_x \phi_0^*(x,t)\, \phi_0(x,t)\,) - \rho\, \partial_t u_0(x,t)\, \partial_x u_0(x,t) =$$

$$= \int d\mathbf{x}\, mv\, \phi_o^2(x) \,+\, \rho v \left(\gamma(v)\, \frac{\sigma_0}{Mc_l^2}\, a_l^3\, \phi_o^2(x) \right)^2 \approx N_o\, mv \,+\, N_o\, Mv\, \vartheta(N_o,\, v) \equiv$$

$$\equiv N_o\, m \left\{ 1 + (M/m)\vartheta(N_o,\, v) \right\} v. \qquad (42)$$

Thus, we obtain $m_{\text{eff}} = m\,\{1 + (M/m)\vartheta(N_o,\, v)\}$ and estimate the parameter $(M/m)\,\vartheta(N_o,\, v) \simeq 1 \sim 5$ at $\gamma(v) \geq 2\gamma_o,\ \bar{n}_o \geq 10$.

4 Low-Lying Excitations of Exciton-Phonon Condensate

To consider the stability of the exciton-phonon condensate moving in a lattice, one has to couple the excitons with different sources of perturbation, such as impurities, thermal lattice phonons, surfaces, etc.. In this work, however, we will not specify any source. Instead, we consider the stability conditions in relation to creation (emission) of the condensate excitations that can be found in the framework of investigation of the low-energy excitations of the condensate itself.

Although the condensate wave function $\phi_o(x)\cdot q_o(x)$ was obtained in the framework of the effective 1D model, we normalized it in 3D space. Therefore, we can use this solution as a classical part in the following decomposition of 3D field operators in the co-moving frame:

$$\hat{\psi}(\mathbf{x}, t) = \exp(-i\mu t)\, (\phi_o(x) + \delta\hat{\psi}(\mathbf{x}, t)\,), \qquad (43)$$

$$\hat{u}_j(\mathbf{x}, t) = q_o(x)\, \delta_{1j} + \delta\hat{u}_j(\mathbf{x}, t), \qquad (44)$$

where $\mu = \tilde{\mu} - mv^2/2$. Substituting the field operators of the form (43),(44) into the Lagrangian density (2), we can write the later in the following form:

$$\mathcal{L} = \mathcal{L}_o(\mathrm{e}^{-i\mu t}\phi_o(x),\, q_o(x)\delta_{1j}\,) \,+\, \mathcal{L}_2(\delta\hat{\psi}^\dagger(\mathbf{x}, t),\, \delta\hat{\psi}(\mathbf{x}, t),\, \delta\hat{\mathbf{u}}(\mathbf{x}, t)\,) \,+ \ldots, \qquad (45)$$

where $\mathcal{L}_o$ stands for the classical part of $\mathcal{L}$, and $\mathcal{L}_2$ is the bilinear form in the δ-operators.

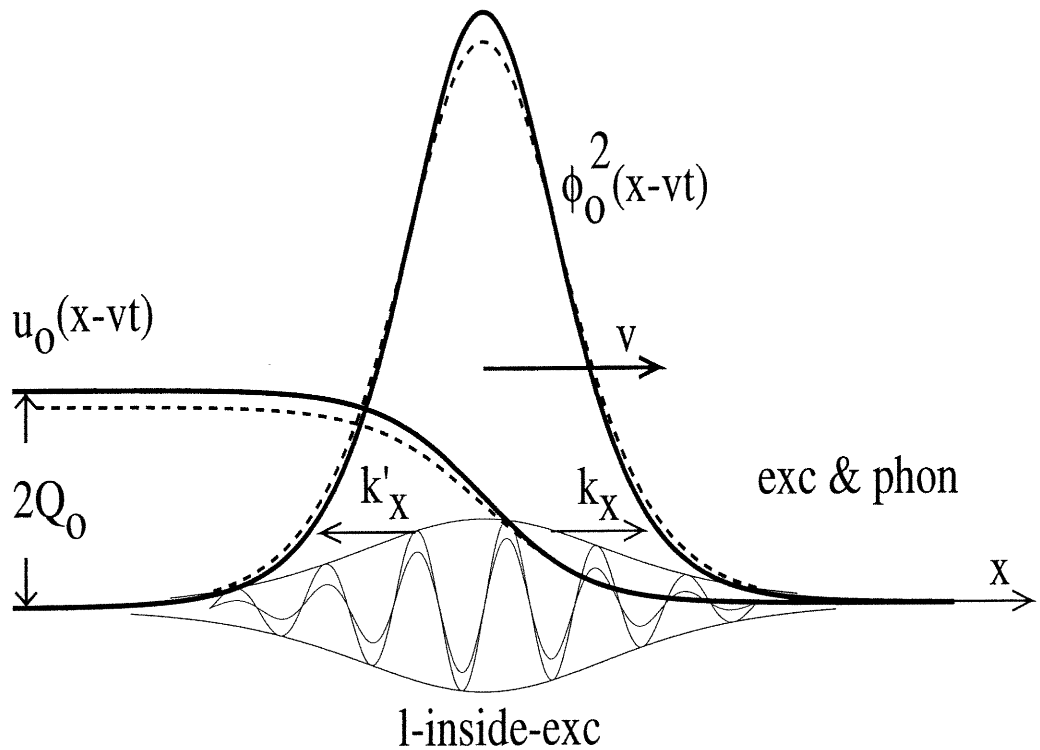


Figure 1: Moving exciton-phonon condensate with excitations.
Moving exciton-phonon condensate, as it appears in the quasi-stationary model, $\phi_o(x-vt) \cdot u_o(x-vt)\delta_{1j}$, is presented by bold lines on this Figure. Here, $2Q_o$ is the amplitude of the coherent phonon state $u_o(x-vt)$, and Φ_o is the amplitude of the macroscopic wave function of excitons. Longitudinal exciton-phonon excitations $(\mathbf{k}\,||\,Ox)$ of the condensate are schematically depicted. Under transformation $N_o \to N_o - \delta N$, the condensate wave function is changed as it is presented by dashed lines.

In the simplest (Bogoliubov) approximation [26],[27], $\mathcal{L} \approx \mathcal{L}_{\rm o} + \mathcal{L}_2$ and, hence, the bilinear form $\mathcal{L}_2$ defines the equations of motion for the fluctuating parts of the field operators. As a result, these equations are *linear* and can be written as follows:

$$i\hbar\partial_t\, \delta\hat{\psi}(\mathbf{x},t) =$$
$$= \left(-\frac{\hbar^2}{2m}\Delta + |\tilde{\mu}| + \{\nu_0 + \widetilde{\nu_0}(v)\}\, \phi_{\rm o}^2(x) + \{2\nu_1 + \widetilde{\nu_1}(v)\}\, \phi_{\rm o}^4(x) \right) \delta\hat{\psi}(\mathbf{x},t) +$$
$$+ \left(\nu_0\phi_{\rm o}^2(x) + 2\nu_1\phi_{\rm o}^4(x)\right) \delta\hat{\psi}^\dagger(\mathbf{x},t) + \sigma_0\phi_{\rm o}(x)\nabla\delta\hat{\mathbf{u}}(\mathbf{x},t), \tag{46}$$

and

$$\left(\partial_t^2 - c_l^2\Delta - v(\partial_t\partial_x + \partial_x\partial_t) + v^2\partial_x^2\right) \delta\hat{u}_j(\mathbf{x},t) =$$
$$= \rho^{-1}\sigma_0\partial_j\Big(\phi_{\rm o}(x)(\delta\hat{\psi}(\mathbf{x},t) + \delta\hat{\psi}^\dagger(\mathbf{x},t))\Big), \quad j = 2,3(\equiv\perp), \tag{47}$$
$$\left(\partial_t^2 - c_l^2\Delta - v(\partial_t\partial_x + \partial_x\partial_t) + v^2\partial_x^2\right) \delta\hat{u}_x(\mathbf{x},t) -$$
$$-c_l^2 2\kappa_3\left(\partial_x q_{\rm o}(x)\right)\partial_x^2\delta\hat{u}_x(\mathbf{x},t) - c_l^2 2\kappa_3\left(\partial_x^2 q_{\rm o}(x)\right)\partial_x\delta\hat{u}_x(\mathbf{x},t) =$$
$$= \rho^{-1}\sigma_0\, \partial_x\Big(\phi_{\rm o}(x)\left(\delta\hat{\psi}(\mathbf{x},t) + \delta\hat{\psi}^\dagger(\mathbf{x},t)\right)\Big), \quad j = 1(\equiv x). \tag{48}$$

The same approximation can be performed within the Hamiltonian approach. Indeed, decomposition of the field operators near their nontrivial classical parts leads to the decomposition of the Hamiltonian (1) itself, and – as it was done with the Lagrangian – only the classical part of $\hat{H}$, $H_{\rm o}$, and the bilinear form in the fluctuating fields, $\hat{H}_2$, are left for examination:

$$\hat{H} \approx H_{\rm o}(\psi_0^*, \psi_0, \pi_0, u_0) + H_2(\delta\hat{\psi}^\dagger,\, \delta\hat{\psi},\, \delta\hat{\pi}, \delta\hat{u}). \tag{49}$$

In the co-moving frame, $\hat{\pi}_j = \rho\,\partial_t\hat{u}_j - \rho v\,\partial_x\hat{u}_j$, i.e.

$$\pi_{\rm oj}(x) = -\rho v\,\partial_x q_{\rm o}(x)\,\delta_{1j} \neq 0 \quad \text{and} \quad \delta\hat{\pi}_j = \rho\,\partial_t\delta\hat{u}_j - \rho v\,\partial_x\delta\hat{u}_j,$$

and the standard commutation relation, $[\,\delta\hat{u}_j(\mathbf{x},t),\ \delta\hat{\pi}_s(\mathbf{x}',t)\,]$, has the form

$$[\,\delta\hat{u}_j(\mathbf{x},t),\ \rho\,\partial_t\delta\hat{u}_s(\mathbf{x}',t) - \rho v\,\partial_x\delta\hat{u}_s(\mathbf{x}',t)\,] = i\hbar\,\delta(\mathbf{x}-\mathbf{x}')\,\delta_{js}. \tag{50}$$

However, the Hamiltonian (49) can be diagonalized and rewritten in the form:

$$\hat{H} = H_o(e^{-i\mu t}\phi_o(x),\, q_o(x)\,) + \delta E_o + \sum_{1,s} \hbar\omega_{1,s}\, \hat{\alpha}^{\dagger}_{1,s}\hat{\alpha}_{1,s} + \sum_{2,s} \hbar\omega_{2,s}\, \hat{\alpha}^{\dagger}_{2,s}\hat{\alpha}_{2,s}. \quad (51)$$

Here, δE_o is the quantum correction to the energy of the condensate and the indexes $1, s$ and $2, s$ label the elementary excitations of the system. We assume the operators $\hat{\alpha}^{\dagger}_{j,s}$, $\hat{\alpha}_{j,s}$ are the Bose ones. These operators describe two different branches of the excitations, $j = 1, 2$, and they can be represented by the following linear combinations of the "delta- operators":

$$\hat{\alpha}_{j,s} = \int d\mathbf{x} \left(U_{j,s}(\mathbf{x})\, \delta\hat{\psi}(\mathbf{x}) + V_{j,s}(\mathbf{x})\, \delta\hat{\psi}^{\dagger}(\mathbf{x}) + Y^{i}_{j,s}(\mathbf{x})\, \delta\hat{u}_i(\mathbf{x}) + Z^{i}_{j,s}(\mathbf{x})\, \delta\hat{\pi}_i(\mathbf{x}) \right), \quad (52)$$

$$\hat{\alpha}^{\dagger}_{j,s} = \int d\mathbf{x} \left(U^{*}_{j,s}(\mathbf{x})\, \delta\hat{\psi}^{\dagger}(\mathbf{x}) + V^{*}_{j,s}(\mathbf{x})\, \delta\hat{\psi}(\mathbf{x}) + Y^{i*}_{j,s}(\mathbf{x})\, \delta\hat{u}_i(\mathbf{x}) + Z^{i*}_{j,s}(\mathbf{x})\, \delta\hat{\pi}_i(\mathbf{x}) \right). \quad (53)$$

Note that by analogy with the exciton-polariton modes in semiconductors [28],[29] the excitations of the condensate (37) can be considered as a mixture of exciton- and phonon-type modes. However, in this model, the phonons are fluctuations of the $(\pi_0(x,t),\, u_0(x,t)\,)$-part of the condensate. The commutation relations between α-operators are Bose ones, so that

$$[\,\hat{\alpha}_{1,s},\ \hat{\alpha}^{\dagger}_{1,s'}\,] = \delta_{ss'}$$

lead to the following orthogonality condition

$$\int d\mathbf{x} \left(U_{1,s}U^{*}_{1,s'}(\mathbf{x}) - V_{1,s}V^{*}_{1,s'}(\mathbf{x}) \right) + (i\hbar) \sum_{r=1,2,3} \int d\mathbf{x} \left(Y^{r}_{1,s}Z^{r*}_{1,s'}(\mathbf{x}) - Z^{r}_{1,s}Y^{r*}_{1,s'}(\mathbf{x}) \right) = \delta_{ss'}.$$

Since the α-operators (see Eq. (51)) evolve in time as simply as

$$\hat{\alpha}_{j,s}(t) = e^{-i\omega_{j,s}t}\, \hat{\alpha}_{j,s}, \quad \hat{\alpha}^{\dagger}_{j,s}(t) = e^{i\omega_{j,s}t}\, \hat{\alpha}^{\dagger}_{j,s},$$

these operators (and the frequencies $\{\omega_{j,s}\}$) are the eigenvectors (and, correspondingly, the eigenvalues) of the equations of motion (46),(47) obtained within the

Lagrangian method. Then, the time dependent "δ-operators" in (46),(47) can be represented by the following linear combinations of the α-operators:

$$\delta\hat{\psi}(\mathbf{x},t) = \sum_{1,s} \mathrm{u}_{1,s}(\mathbf{x})\,\hat{\alpha}_{1,s}\mathrm{e}^{-i\omega_{1,s}t} + \mathrm{v}^{*}_{1,s}(\mathbf{x})\,\hat{\alpha}^{\dagger}_{1,s}\mathrm{e}^{i\omega_{1,s}t} +$$

$$+ \sum_{2,s} \mathrm{u}_{2,s}(\mathbf{x})\,\hat{\alpha}_{2,s}\mathrm{e}^{-i\omega_{2,s}t} + \mathrm{v}^{*}_{2,s}(\mathbf{x})\,\hat{\alpha}^{\dagger}_{2,s}\mathrm{e}^{i\omega_{2,s}t}, \tag{54}$$

$$\delta\hat{u}_r(\mathbf{x},t) = \sum_{1,s} C^r_{1,s}(\mathbf{x})\,\hat{\alpha}_{1,s}\mathrm{e}^{-i\omega_{1,s}t} + C^{r^*}_{1,s}(\mathbf{x})\,\hat{\alpha}^{\dagger}_{1,s}\mathrm{e}^{i\omega_{1,s}t} +$$

$$+ \sum_{2,s} C^r_{2,s}(\mathbf{x})\,\hat{\alpha}_{2,s}\mathrm{e}^{-i\omega_{2,s}t} + C^{r^*}_{2,s}(\mathbf{x})\,\hat{\alpha}^{\dagger}_{2,s}\mathrm{e}^{i\omega_{2,s}t}, \tag{55}$$

For $\delta\hat{\pi}_r(\mathbf{x},t)$, one has to change $C^r_{j,s}(\mathrm{x})$ to $D^r_{j,s} = \rho(-i\omega_{j,s} - v\partial_x)C^r_{j,s}(\mathrm{x})$ in (35). Note that this *ansatz* is, in fact, a generalization of the u-v Bogoliubov transformation.

Then we can rewrite Eqs. (52),(53) as follows ($j = 1,2$)

$$\hat{\alpha}_{j,s} =$$

$$\int d\mathbf{x}\left(\mathrm{u}^{*}_{j,s}(\mathbf{x})\,\delta\hat{\psi}(\mathbf{x}) - \mathrm{v}^{*}_{j,s}(\mathbf{x})\,\delta\hat{\psi}^{\dagger}(\mathbf{x}) - (i/\hbar)D^{r^*}_{j,s}(\mathbf{x})\,\delta\hat{u}_r(\mathbf{x}) + (i/\hbar)C^{r^*}_{j,s}(\mathbf{x})\,\delta\hat{\pi}_r(\mathbf{x}) \right), \tag{56}$$

$$\hat{\alpha}^{\dagger}_{j,s} =$$

$$\int d\mathbf{x}\left(\mathrm{u}_{j,s}(\mathbf{x})\,\delta\hat{\psi}^{\dagger}(\mathbf{x}) - \mathrm{v}_{j,s}(\mathbf{x})\,\delta\hat{\psi}(\mathbf{x}) + (i/\hbar)D^{r}_{j,s}(\mathbf{x})\,\delta\hat{u}_r(\mathbf{x}) - (i/\hbar)C^{r}_{j,s}(\mathbf{x})\,\delta\hat{\pi}_r(\mathbf{x}) \right), \tag{57}$$

and one of the orthogonality relations has the form ($s = s'$)

$$\int d\mathbf{x}\,(|\mathrm{u}_{1,s}(\mathbf{x})|^2 - |\mathrm{v}_{1,s}(\mathbf{x})|^2\,) +$$

$$+ (i/\hbar)\sum_{r=1,2,3}\int d\mathbf{x}\,\left(C^{r^*}_{1,s}\,\rho(-i\omega_{1,s} - v\partial_x)C^r_{1,s}(\mathbf{x}) + \rho(-i\omega_{1,s} + v\partial_x)C^{r^*}_{1,s}\,C^r_{1,s}(\mathbf{x})\,\right) = 1. \tag{58}$$

The question we want to clarify is whether coupling between excitonic excitations and phonon excitations is important for understanding the condensate excitations. Substituting *ansatz* (54)-(55) into Eqs. (46),(47), we obtain the following coupled eigenvalue equations [11]:

$$\left(\hat{L}(\Delta) - \hbar\omega_{j,s}\right) \mathrm{u}_{j,s}(\mathbf{x}) + \left(\nu_0\phi_{\mathrm{o}}^2(x) + 2\nu_1\phi_{\mathrm{o}}^4(x)\right) \mathrm{v}_{j,s}(\mathbf{x}) + \sigma_0\phi_{\mathrm{o}}(x)\partial_r C_{j,s}^r(\mathbf{x}) = 0, \quad (59)$$

$$\left(\nu_0\phi_{\mathrm{o}}^2(x) + 2\nu_1\phi_{\mathrm{o}}^4(x)\right) \mathrm{u}_{j,s}(\mathbf{x}) + \left(\hat{L}(\Delta) + \hbar\omega_{j,s}\right) \mathrm{v}_{j,s}(\mathbf{x}) + \sigma_0\phi_{\mathrm{o}}(x)\partial_r C_{j,s}^r(\mathbf{x}) = 0, \quad (60)$$

$$\begin{aligned} -\rho^{-1}\sigma_0\, \partial_r \Big(\phi_{\mathrm{o}}(x)\, \mathrm{u}_{j,s}(\mathbf{x})\Big) - \rho^{-1}\sigma_0\, \partial_r \Big(\phi_{\mathrm{o}}(x)\, \mathrm{v}_{j,s}(\mathbf{x})\Big) + \\ + \left[(-i\omega_{j,s} - v\partial_x)^2 - c_l^2\Delta\right] C_{j,s}^r(\mathbf{x}) = 0, \quad r = 2,3, \end{aligned} \quad (61)$$

$$\begin{aligned} -\rho^{-1}\sigma_0\, \partial_x \Big(\phi_{\mathrm{o}}(x)\, \mathrm{u}_{j,s}(\mathbf{x})\Big) - \rho^{-1}\sigma_0\, \partial_x \Big(\phi_{\mathrm{o}}(x)\, \mathrm{v}_{j,s}(\mathbf{x})\Big) + \\ \left[(-i\omega_{j,s} - v\partial_x)^2 - c_l^2(1 + |\kappa_3| F_3(x))\, \partial_x^2 - c_l^2\partial_\perp^2 - c_l^2|\kappa_3|(\partial_x F_3(x))\, \partial_x\right] C_{j,s}^x(\mathbf{x}) = 0. \end{aligned} \quad (62)$$

Here we used the following notations

$$\hat{L}(\Delta) = (-\hbar^2/2m)\Delta + |\tilde{\mu}| + \{\nu_0 + \widetilde{\nu_0}\,(v)\,\}\phi_{\mathrm{o}}^2(x) + \{2\nu_1 + \widetilde{\nu_1}\,(v)\}\phi_{\mathrm{o}}^4(x),$$

$$F_3(x) = 2\,\gamma(v)\left(\sigma_0/Mc_l^2\right) a_l^3\phi_{\mathrm{o}}^2(x).$$

To simplify investigation of the characteristic properties of the different solutions of Eqs. (59)-(61), we subdivide the excitations (54)-(55) into two major parts, the *inside*-excitations and the *outside*-ones. The *inside*-excitations are localized merely inside the packet area, i.e. $|\mathbf{x}| < 2L_0$ and $\phi_{\mathrm{o}}^2(x) \approx \Phi_{\mathrm{o}}^2$, whereas the *outside*-excitations propagate merely in the outside area, i.e. $|\mathbf{x}| > 2L_0$ and $\phi_{\mathrm{o}}^2(x) \simeq 4\Phi_{\mathrm{o}}^2 \exp(-2|x|/L_0) \to 0$.

4.1 Outside-Excitations

For the outside collective excitations, the asymptotics of the low-lying energy spectrum can be found easily. Indeed, if we assume that $\phi_{\mathrm{o}}^2(x) \approx 0$ and $\partial_x q_{\mathrm{o}}(x) \approx 0$ in the outside packet area, the equations (46) and (47) are (formally) uncoupled.

Then, Eq. (46) describes the excitonic branch of the outside-excitations with the following dispertion low in the co-moving frame:

$$\hbar\omega_{\rm ex}(\mathbf{k}) \approx |\tilde{\mu}| + (\hbar^2/2m)k^2, \quad (\mathrm{u}_{\mathbf{k}}(\mathbf{x}) \approx u_{\mathbf{k}}\, \mathrm{e}^{i\mathbf{k}\mathbf{x}}, \;\; \mathrm{v}_{\mathbf{k}}(\mathbf{x}) \approx 0), \tag{63}$$

and $\omega_{\rm ph}(\mathbf{k}') = c_l|\mathbf{k}'|$ in the laboratory frame of reference.

Then the exciton field operator, which describes the exciton condensate with *one* long-wavelength outside-excitation, has the following form:

$$\begin{aligned} \psi(\mathbf{x},t) \simeq \exp(-i(\tilde{E}_g + mv^2/2 - |\tilde{\mu}|)t\,) \exp(i(\varphi + mvx)\,)\, \phi_{\rm o}(x - vt) + \\ + \exp(-i(\tilde{E}_g + mv^2/2 - |\tilde{\mu}|)t\,) \exp(i(\varphi + mvx)\,)\times \\ \times \left\{\exp(-i(|\tilde{\mu}| + \hbar\mathbf{k}^2/2m + k_x v)t\,)\, u_{\mathbf{k}} \exp(i\mathbf{k}\mathbf{x})\,\right\}. \end{aligned} \tag{64}$$

It is easy to see that such a collective excitation,

$$\hbar\omega_{\rm ex}(\mathbf{k}) = |\tilde{\mu}| + \hbar^2\mathbf{k}^2/2m + \hbar k_x v,$$

can be interpreted as a free exciton with the energy and the (quasi)momentum

$$\varepsilon_{\rm x}(\tilde{k}) = \tilde{E}_g + \hbar^2\tilde{\mathbf{k}}^2/2m \quad \text{and} \quad \hbar\tilde{k}_j = \hbar k_j + mv\,\delta_{1j}.$$

Note that the condition $\hbar\omega_{\rm ex} > 0$ can be violated at the velocities close to $v_{\rm o}$, whereas $\varepsilon_{\rm x}$ is always positive. Then the question is whether $\hbar\omega_{\rm ex} < 0$ really means the condensate instability in relation to the creation of outside excitations. For example, being unstable, the condensate could continuously emit outside-excitations, which form a sort of 'tail' behind the localized packet.

Recall that the particle number $\hat{N}$ is not conserved in quantum states with a condensate, and $\langle\delta N^2\rangle \simeq N_{\rm o}$. However, for $N_{\rm o} \geq 10^{10}$ and $T \ll T_c$, the following estimate is valid

$$\sqrt{\langle\delta N^2\rangle}/N \simeq 1/\sqrt{N_{\rm o}} \leq 10^{-5}.$$

Therefore, we can compare the condensate energy $E_{\rm o}(N_{\rm o},\, v)$ and the energy of the condensate that emits excitons, or, equivalently, the condensate with outside excitations, $\langle u_{\mathbf{k}} \rangle \sim \sqrt{\delta N}$. For simplicity's sake, we consider δN different wave vectors, $\{\tilde{\mathbf{k}}_j\}$, to be close to each other, so that the values of $\langle \tilde{k} \rangle^2$ and $\langle \tilde{k}_x \rangle$ are well-defined. (This is a model of how the instability tail could be formed.) We obtain (see Eqs. (38)-(41))

$$\begin{aligned} &E_{\rm o}(N_{\rm o} - \delta N,\, v) + E_{\rm x}(\langle \tilde{k} \rangle, \delta N) + E_{\rm ph}(\langle k' \rangle,\, \delta N) \approx \\ &\approx E_{\rm o}(N_{\rm o},\, v) + 3 \left(|\tilde{\mu}| + \frac{\nu_0\, \Phi_{\rm o}^2}{3} \right) \delta N - 3\, \frac{M\,(c_l^2 + v^2)}{2}\, \vartheta(N_{\rm o},\, v)\, \delta N + \\ &+ \left(\frac{\hbar^2 \langle \tilde{k} \rangle^2}{2m} - \frac{mv^2}{2} \right) \delta N + \hbar c_l |\langle k' \rangle|\, \delta N. \end{aligned} \tag{65}$$

For the momentum of the moving condensate with the outside excitations, we have

$$\begin{aligned} &P_{{\rm o}\,x}(N_{\rm o} - \delta N,\, v) + \hbar \langle \tilde{k}_x \rangle \delta N + \hbar \langle k'_x \rangle \delta N \approx \\ &\approx P_{{\rm o}\,x}(N_{\rm o},\, v) + (\hbar \langle \tilde{k}_x \rangle - mv)\, \delta N + (\hbar \langle k'_x \rangle - 3\, Mv\, \vartheta(N_{\rm o},\, v))\, \delta N. \end{aligned} \tag{66}$$

Note that the energy and the momentum of the phonon part of the condensate change after exciton emission. We hypothesize that the transformation $N_{\rm o} \to N_{\rm o} - \delta N$ (with the emission of outside excitons, see Figs. 1,2) corresponds to the case in which the outside exciton and the outside acoustic phonon appear *together*. Indeed, in the $k \to 0$ limit (i.e. $\lambda = 2\pi/k \gg L_0$), we approximately considered the condensate collective excitations as being uncoupled. However, the phonon $\hbar\mathbf{k}'$ can be emitted with the energy compensating the changement of $\delta E_{\rm ph} = -(3/2)\, M(c_l^2 + v^2)\, \vartheta(N_{\rm o},\, v)$ in the phonon part of the condensate energy. Moreover, the order of value of $|\delta E_{\rm ph}|$ is typical for the low-energy acoustic phonons, $\sim 1\,{\rm meV}$. If $\hbar k'_x > 0$, the emitted phonon can compensate the changement of $\delta P_{{\rm ph}\,x} = -3\, Mv\, \vartheta(v)$ as well.

The most interesting case is the backward emission of excitons, i.e. $\hbar k_j = \hbar k\, \delta_{1j} < 0$ in the co-moving frame. Then we can rewrite (65) as follows

$$E_{\rm o}(N_{\rm o} - \delta N,\, v) + \langle \hbar\omega_{\rm ex}(\tilde{k}) \rangle\, \delta N + \langle \hbar\omega_{\rm ph}(k') \rangle\, \delta N \approx$$

$$\approx E_{\mathrm{o}}(N_{\mathrm{o}},\, v) + \big(2|\tilde{\mu}| + \nu_0\, \Phi_{\mathrm{o}}^2\big)\delta N + \Big\{|\tilde{\mu}| + \frac{\hbar^2 \langle k\rangle^2}{2m} - \hbar|\langle k\rangle|\, v\Big\}\delta N. \tag{67}$$

The moving condensate can be considered as a stable one in relation to emission of the outside excitations ($\delta N > \sqrt{\langle \delta N^2\rangle}$) if such an emission gains energy,

$$E_{\mathrm{o}}(N_{\mathrm{o}} - \delta N,\, v) + E_{\mathrm{exc}}(\langle \tilde{k}\rangle,\, \delta N) + E_{\mathrm{ph}}(\langle k'\rangle,\, \delta N) > E_{\mathrm{o}}(N_{\mathrm{o}},\, v).$$

This means that the following inequality has to be valid

$$\Big\{|\tilde{\mu}| + \frac{\hbar^2 \langle k\rangle^2}{2m} - \hbar|\langle k\rangle|\, v\Big\} + (2|\tilde{\mu}| + \nu_0\, \Phi_{\mathrm{o}}^2\,) > 0. \tag{68}$$

This condition can be rewritten in the dimensionless form as follows:

$$\Big(3 + 2\,\frac{\nu_0}{|\widetilde{\nu_0}\,(v)|} + \Big(\frac{2\pi}{\langle z\rangle}\Big)^2\Big) - \frac{2\pi}{\langle z\rangle}\,\frac{v}{c_l}\,\sqrt{\frac{(4m/M)\, Mc_l^2/2}{|\tilde{\mu}|(N_{\mathrm{o}},\, v)}} > 0. \tag{69}$$

We argue that, even for velocities close to v_{o} (where $|\widetilde{\nu_0}\,(v)|$ can be $\sim 0.1\,\nu_0$, and the instability could appear as $|\tilde{\mu}|(N_{\mathrm{o}},\, v) + \hbar^2 k_x^2/2m - \hbar|k_x|v < 0$), inequality (68) seems to be always true in the long-wavelength approximation, $k = (2\pi/z)\, L_0^{-1} \le 10^{-1}\, L_0^{-1}$. On Fig. 2, the stable ballistic condensate is shown with its long-wavelength outside-excitations.

Note that the stability against large-k modes cannot be properly described within approximation (63),(64). However, we can discuss this case within the inside-approximation.

4.2 Inside-Excitations

To simplify the calculation of *inside*-excitation spectrum (see Eqs. (59)-(61)) we will use the semiclassical approximation [27]. In this approximation, the excitations can be labeled by the wave vector $\mathbf{k}$ in the co-moving frame, and the following representation holds:

$$\mathrm{u}_{j,s}(\mathbf{x}) = \mathrm{u}_{j,\mathbf{k}}(\mathbf{x})\, \mathrm{e}^{i\varphi_{\mathbf{k}}(\mathbf{x})},\;\; \mathrm{v}_{j,s}(\mathbf{x}) = \mathrm{v}_{j,\mathbf{k}}(\mathbf{x})\, \mathrm{e}^{i\varphi_{\mathbf{k}}(\mathbf{x})},\;\; C^{r}_{j,s}(\mathbf{x}) = C^{r}_{j,\mathbf{k}}(\mathbf{x})\, \mathrm{e}^{i\varphi_{\mathbf{k}}(\mathbf{x})}, \tag{70}$$

where the phase $\varphi_{\mathbf{k}}(\mathbf{x}) \approx \varphi_{\mathrm{o}} + \mathbf{k}\mathbf{x}$, and $\mathrm{u}_{\mathbf{k}}(\mathbf{x})$, $\mathrm{v}_{\mathbf{k}}(\mathbf{x})$, and $C_{\mathbf{k}}(\mathbf{x})$ are assumed to be smooth functions of $\mathbf{x}$ in the inside condensate area. Notice that the $\mathbf{k}$-

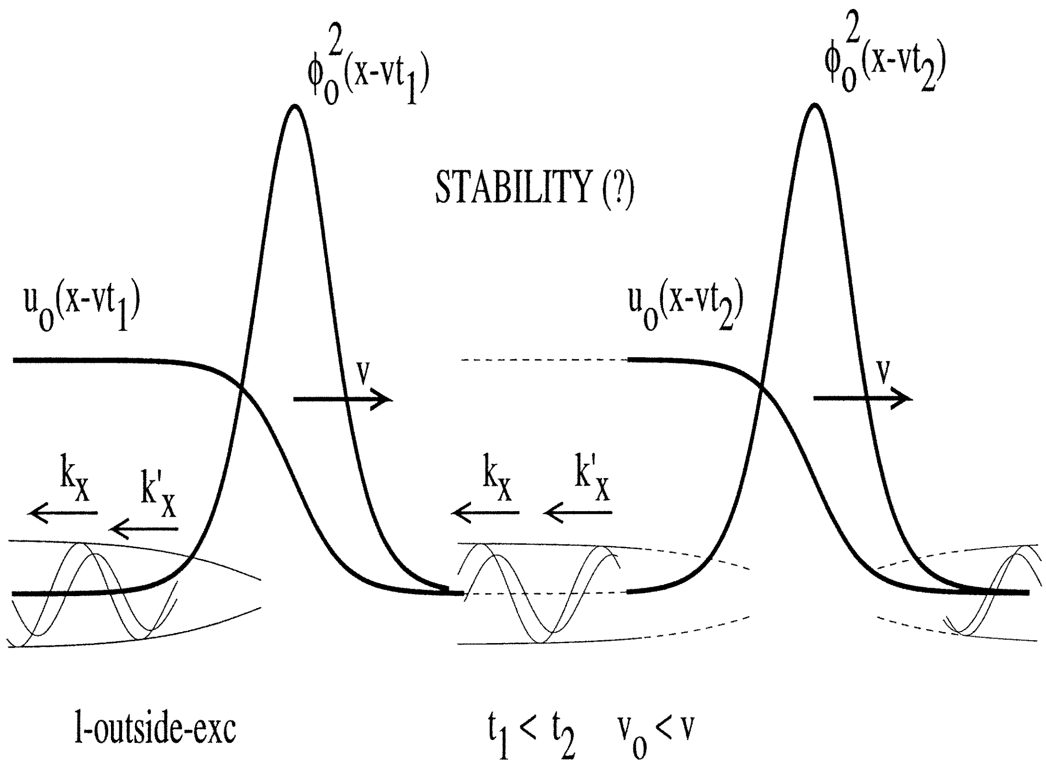


Figure 2: Stability regime.
The ballistic condensate, $\phi_{\text{o}}(x - vt) \cdot u_{\text{o}}(x - vt)\delta_{1j}$, is stable in relation to emission of the outside exciton-phonon excitations. (We consider the backward emission in the long-wavelength limit.) The outside-excitations presented on this figure are labeled by the wave vectors, k_x, $k'_x < 0$ in the co-moving frame. In the first approximation, the outside-excitations can be described in terms of free excitons and free (acoustic) phonons emitted from the condensate coherently.

and $\mathbf{x}$-representations are mixed here. This means that the operator nature of the fluctuating fields is actually dismissed within the semiclassical approximation. Howevever, the orthogonality relations among $\mathrm{u}_{j,s}$, $\mathrm{v}_{j,s'}$, and $C_{j,s}$, $C^*_{j,s'}$, and, hence, among $\mathrm{u}_{j,\mathbf{k}}$, $\mathrm{v}_{j,\mathbf{k}'}$, and $C_{j,\mathbf{k}}$, $C^*_{j,\mathbf{k}'}$ come from the Bose commutation relations between the operators $\alpha_{j,s}$ and $\alpha^\dagger_{j,s'}$ [26],[27]. For example, Eq. (54) is modified as follows:

$$\delta\psi(\mathbf{x},t) \simeq \int \frac{d\mathbf{k}}{(2\pi)^3}\, \mathrm{u}_\mathbf{k}(\mathbf{x}) \mathrm{e}^{i\varphi_\mathbf{k}(\mathbf{x})} \mathrm{e}^{-i\omega_\mathbf{k}(\mathbf{x})t} + \mathrm{v}^*_\mathbf{k}(\mathbf{x}) \mathrm{e}^{-i\varphi_\mathbf{k}(\mathbf{x})} \mathrm{e}^{i\omega_\mathbf{k}(\mathbf{x})t}, \tag{71}$$

and the *inside*-excitation part of the elementary excitation term in Eq. (51), $\sum_{j,s} \cdots \approx \sum_{j,s,\mathrm{out}} + \sum_{j,s,\mathrm{surf}} + \sum_{j,s,\mathrm{in}} \cdots$, can be written as

$$\sum_{1,s,\mathrm{in}} \hbar\omega_{1,s} \hat{\alpha}^\dagger_{1,s} \hat{\alpha}_{1,s} \simeq \int \frac{d\mathbf{k}\, d\mathbf{x}}{(2\pi)^3}\, \hbar\omega_{1,\mathbf{k}}(\mathbf{x})\, n_{1,\mathbf{k}}(\mathbf{x}). \tag{72}$$

Note that the semiclassical energy $\hbar\omega_{j,\mathbf{k}}(x)$ of the inside-excitation mode $j,\mathbf{k}$ is supposed to be a smooth function of x as well, (at least, as smooth as $\phi^2_\mathrm{o}(x)$, which is taken constant in the inside-approximation).

Although the low-lying excitations cannot be properly described within the semiclassical approximation, we apply it here to calculate the low-energy asymptotics of the spectrum. In fact, within the approximation $\bar{H} \approx H_\mathrm{o} + \bar{H}_2$, all the important properties of such excitations can be understood by use of the semiclassical approach.

There are two different types of the inside-excitations, the longitudinal excitations and the transverse ones. The later have the wave vectors $\mathbf{k}$ perpendicular to the x (v)-direction. For the sake of simplicity, we choose $\mathbf{k} \parallel Oy$. Then the vector $C^r_{j,\mathbf{k}}$ has one nonzeroth component for such transverse excitations, $C^y_{j,k} \neq 0$.

Substituting *ansatz* (70) with $k_r = k_\perp \delta_{2,r}$ and $C^r_{j,\mathbf{k}} = C^y_{j,k}\, \delta_{2,r}$ into Eqs. (59)-(61), we transform these differential equations into the algebraic ones (within the inside-approximation $\hat{L}(\Delta) \to L(-\mathbf{k}^2)$):

$$(L(-k^2_\perp) - \hbar\omega_{j,k})\, \mathrm{u}_k(\mathbf{x}) + \left(\nu_0\phi^2_\mathrm{o}(x) + 2\nu_1\phi^4_\mathrm{o}(x)\right) \mathrm{v}_{j,k}(\mathbf{x}) + \sigma_0\phi_\mathrm{o}(x)\, ik_\perp C^y_{j,k}(\mathbf{x}) = 0, \tag{73}$$

$$\left(\nu_0\phi_o^2(x) + 2\nu_1\phi_o^4(x)\right) \mathrm{u}_{j,k}(\mathbf{x}) + \left(L(-k_\perp^2) + \hbar\omega_{j,k}\right) \mathrm{v}_{j,k}(\mathbf{x}) + \sigma_0\phi_o(x)\, ik_\perp C^y_{j,k}(\mathbf{x}) = 0 \tag{74}$$

$$\rho^{-1}\sigma_0\phi_o(x)\, ik_\perp\, \mathrm{u}_{j,k}(\mathbf{x}) + \rho^{-1}\sigma_0\phi_o(x)\, ik_\perp\, \mathrm{v}_{j,k}(\mathbf{x}) + \left[\omega_{j,k}^2 - c_l^2 k_\perp^2\right] C^y_{j,k}(\mathbf{x}) = 0. \tag{75}$$

After some straightforward algebra, we can write out the equation that defines the spectrum of transverse exciton-phonon excitations in the inside-approximation:

$$\begin{aligned}&\left(\omega_{j,k}^2 - c_l^2 {k_\perp}^2\right) \times \\ &\left[(\hbar\omega_{j,k})^2 - \left(L(-k_\perp^2) - \nu_0\phi_o^2(x) - 2\nu_1\phi_o^4(x)\right)\left(L(-k_\perp^2) + \nu_0\phi_o^2(x) + 2\nu_1\phi_o^4(x)\right)\right] \\ &\quad = \left(L(-k_\perp^2) - \nu_0\phi_o^2(x) - 2\nu_1\phi_o^4(x)\right) \frac{2\,\sigma_0^2}{\rho\, c_l^2}\, \phi_o^2(x)\, (c_l^2 k_\perp^2).\end{aligned} \tag{76}$$

Taking into account the momentum cut-off k_0, which is defined as

$$(\hbar^2/2m)\, k_0^2 \approx |\tilde{\mu}| = (\hbar^2/2m)\, L_0^{-2}, \quad k > k_0,$$

we can rewrite Eq. (76) as follows

$$\begin{aligned}&\left(\omega_{j,k}^2 \;-\; c_l^2 k_\perp^2\right) \times \\ &\left[(\hbar\omega_{j,k})^2 - \left(\frac{\hbar^2}{2m}[k_\perp^2 - k_0^2] + F(x) + \epsilon_+\right)\left(\frac{\hbar^2}{2m}[k_\perp^2 - k_0^2] + F(x) + 2\nu_0\phi_o^2(x) + \epsilon'_+\right)\right] \\ &= \left(\frac{\hbar^2}{2m}[k_\perp^2 - k_0^2] + F(x) + \epsilon_+\right) 2\frac{\sigma_0}{Mc_l^2}\, (\sigma_0 a_l^3)\, \Phi_o^2\, (c_l^2 k_\perp^2), \quad |x| < L_0.\end{aligned} \tag{77}$$

Here $F(x) = |\widetilde{\nu_0}\,(v)|(\Phi_o^2 - \phi_o^2(x)\,) > 0$, i.e. $F(x) \simeq 0$ inside the condensate, and

$$\epsilon_+ \approx \widetilde{\nu_1}\, \phi_o^4(x), \quad \epsilon'_+ \approx 5\nu_1\, \phi_o^4(x), \quad k_\perp > k_0.$$

Although Eq. (77) can be solved exactly for the transverse excitation spectrum [30], taking into account the coupling term in the r.h.s. of (77) changes the values of excitation energies slightly, and the excitations can be approximately considered as of the pure excitonic ($\hbar\omega_{1,k} = \hbar\omega_{\mathrm{ex},\,k_\perp}$) or the pure phonon ($\hbar\omega_{2,k} = \hbar\omega_{\mathrm{ph},\,k_\perp}$) types.

It is also useful to investigate *asymptotics* of the transverse inside excitations. For definiteness sake, we investigate the left side asymptotics of these excitations here,

$$\mathrm{u}_{j,s}(\mathbf{x}) = \exp(\ell_u x/L_0)\, \mathrm{e}^{ik_\perp y}\, u_{j,k}, \qquad \mathrm{v}_{j,s}(\mathbf{x}) = \exp(\ell_v x/L_0)\, \mathrm{e}^{ik_\perp y}\, v_{j,k}, \quad x<0, \quad (78)$$

$$C^y_{j,s}(\mathbf{x}) = \exp(\ell_c x/L_0)\, \mathrm{e}^{ik_\perp y}\, C^y_{j,k}, \quad C^x_{j,s}(\mathbf{x}) = \exp(\ell_c x/L_0)\, \mathrm{e}^{ik_\perp y}\, C^x_{j,k}, \quad x<0. \quad (79)$$

Here $u_{j,k}$, $v_{j,k}$, $C^y_{j,k}$, and $C^x_{j,k}$ are smooth functions of $\mathbf{x}$ at $|x|>L_0$. Note that we introduced two components of $C^r_{j,s} \sim \mathrm{e}^{ik_\perp y}$ to make Eqs. (59)-(62) self-consistent. Let the equalities

$$\ell_u = \ell_v = 1 \quad \text{and} \quad 1+\ell_u = \ell_c = 2 \qquad (80)$$

be valid. Then the system of differential equations (59)-(62) can be reduced to a system of algebraic ones, which are analogous to Eqs. (73)-(75). Consequently, we can write out the equation for $\omega_{j,k}(x)$ valid at $|x|>L_0$,

$$\begin{gathered}\left(\left[\,\omega_{j,k} - iv\,(2/L_0)\,\right]^2 \,-\, c_l^2[k_\perp^2 - (2/L_0)^2]\right) \times \\ \left[(\hbar\omega_{j,k})^2 - \left(\frac{\hbar^2}{2m}[k_\perp^2 - \tilde{k}_0^2] + \tilde{F}(x)\right)\left(\frac{\hbar^2}{2m}[k_\perp^2 - \tilde{k}_0^2] + \tilde{F}(x) + 2\nu_0(2\Phi_o\exp(x/L_0))^2\right)\right] \\ = \left(\frac{\hbar^2}{2m}[k_\perp^2 - \tilde{k}_0^2] + \tilde{F}(x)\right)\, 2\frac{\sigma_0}{Mc_l^2}\,(\sigma_0 a_l^3)\,(2\Phi_o\exp(x/L_0)\,)^2\, c_l^2[\,k_\perp^2 - (2/L_0)^2], \quad (81)\end{gathered}$$

where $\tilde{k}_0 = \sqrt{2}/L_0 \simeq k_0$, $k_\perp > \tilde{k}_0$, and

$$\tilde{F}(x) = |\widetilde{\nu_0}\,(v)|\,\left(\Phi_o^2 - (2\Phi_o\exp(x/L_0)\,)^2\right) \to 2|\tilde{\mu}| - \epsilon \quad \text{at} \quad |x| \gg L_0.$$

(We neglected the terms, such as $\widetilde{\nu_1}\,\phi_o^4(x)\,\mathrm{u}_{j,s}(\mathbf{x})$ and $\nu_1\phi_o^4(x)\,\mathrm{v}_{j,s}(\mathbf{x}) \sim \exp(\,(4+\ell_u)x/L_0\,)$ in Eqs. (59)-(60), and the terms $\propto \kappa_3$ in Eq. (62) as well.)

Obviously, the structure of Eqs. (77) and (81) is the same. As the coupling between exciton and phonon branches is weak for the transverse inside-excitations (see the r.h. sides of Eqs. (77) and (81)) and the effect of the finite width L_0 can be taken into account as the spatial dependence of the important parameters in $\hbar\omega_{\rm ex}$, we use the following formula to estimate the low-energy excitation spectrum:

$$(\hbar\omega_{\mathrm{ex},\,k_\perp})^2 \simeq \left(\frac{\hbar^2}{2m}(k_\perp^2 - k_0^2) + F(x) + \epsilon_+\right)\left(\frac{\hbar^2}{2m}(k_\perp^2 - k_0^2) + F(x) + 2\nu_0\phi_o^2(x) + \epsilon'_+\right)$$

$$\sim \frac{\hbar^2}{2m}(k_\perp^2 - k_0^2)\, 2\nu_0 \Phi_\mathrm{o}^2 + 2\nu_0 \Phi_\mathrm{o}^2\, \epsilon_+ \quad \text{at} \quad k_\perp \to k_0. \tag{82}$$

Here we take the inside-condensate-asymptotics of $F(x)$, $\phi_\mathrm{o}^2(x)$, and $\epsilon_+ \approx \widetilde{\nu_1}(v)\,\phi_\mathrm{o}^4(x)$ to estimate $\hbar\omega_\mathrm{ex}$. Note that, for the inside-condensate excitations, the the low-energy limit means

$$(\hbar^2/2m)(k_\perp^2 - k_0^2) \simeq (1 \sim 10)\,|\tilde{\mu}|.$$

Then, in the co-moving frame, the low-energy excitation spectrum $\hbar\omega_{\mathrm{ex}\,k_\perp}$ may develop a gap of the order of $|\tilde{\mu}|$, (see Eq. (64) for comparison). Thus, inside the condensate, we obtain a strong deviation of the collective excitation spectrum from both the simple excitonic one, $|\tilde{\mu}| + (\hbar^2/2m)k_\perp^2$, and the Bogoliubov-Landau spectrum $\propto |k_\perp|$.

4.3 Longitudinal inside-excitations

The case of the longitudinal excitations, $k_r = k_x\,\delta_{1,r}$, $C^r_{j,\mathbf{k}} = C^x_{j,k}\,\delta_{1,r}$, is more difficult to analyze because the mode interaction is non-negligible in the low-energy limit. (On Fig. 1., a longitudinal inside-excitation is shown with the two possible directions of the wave vector $\mathbf{k} \parallel Ox$.) Recall that the "bare" phonon modes, which can be written in the laboratory frame as

$$u_x(x,t) \simeq q_\mathrm{o}(x - vt) + C^x_k(x - vt)\exp(ik_x x - i\omega_\mathrm{ph} t) + c.c.$$

with $\omega_\mathrm{ph} = c_l|k_x|$ and $C^x_k(x) \sim \phi_\mathrm{o}^2(x)$, will be considered in the co-moving frame, $x - vt \to x$. Then, within the inside-condensate approximation, the following equation stands for the excitation spectrum:

$$\big((\omega_{j,k} + vk_x)^2 \;-\; c_l^2 k_x^2\big) \times$$

$$\left[(\hbar\omega_{j,k})^2 - \left(\frac{\hbar^2}{2m}[k_x^2 - k_0^2] + F(x) + \epsilon_+\right)\left(\frac{\hbar^2}{2m}[k_x^2 - k_0^2] + F(x) + 2\nu_0\phi_\mathrm{o}^2(x) + \epsilon'_+\right)\right]$$

$$= \left(\frac{\hbar^2}{2m}[k_x^2 - k_0^2] + F(x) + \epsilon_+\right) 2\frac{\sigma_0}{Mc_l^2}\,(\sigma_0 a_l^3)\,\Phi_\mathrm{o}^2\,(c_l^2 k_x^2), \quad |x| < L_0. \tag{83}$$

It is important to note that, unlike the case of transverse excitations, the values of

$$\left(\hbar\omega^{(0)}_{\mathrm{ph},\,k_x}\right)^2 \simeq \hbar^2\,(c_l - v)^2((3\sim 7)\,k_0)^2$$

and

$$\left(\hbar\omega^{(0)}_{\mathrm{ex},\,k_x}\right)^2 \simeq \left(\frac{\hbar^2}{2m}(10\sim 40)k_0^2 + \epsilon_+\right)\left(\frac{\hbar^2}{2m}(10\sim 40)k_0^2 + 2\,\nu_0\,\phi_{\mathrm{o}}^2(x) + \epsilon'_+\right)$$

are of the same order of value at $k_x \simeq (3\sim 8)\,k_0$, and the inequality $(\hbar\omega^{(0)}_{\mathrm{ex},\,k_x})^2 > (\hbar\omega^{(0)}_{\mathrm{ph},k_x})^2$ is valid in the low-energy limit. Moreover, the two cases, $k_x > 0$ (+-case) and $k_x < 0$ (−-case), are different as it can be seen from the l.h.s. of Eq. (83). In the low-energy limit, we can write the approximate solution of (83) as follows

$$\left(\hbar\omega^{(\pm)}_{\mathrm{ex},\,k_x}\right)^2 \approx \left(\frac{\hbar^2}{2m}[k_x^2 - k_0^2] + F(x) + \epsilon_+\right) \times$$
$$\times\left(\frac{\hbar^2}{2m}[k_x^2 - k_0^2] + F(x) + 2\nu_0\,\phi_{\mathrm{o}}^2(x) \pm 2\,q_\pm\gamma(v)\,\frac{\sigma_0}{Mc_l^2}\,(\sigma_0 a_l^3)\,\phi_{\mathrm{o}}^2(x) + \epsilon'_+\right),$$

where $q_+ \sim 1$ and $0 < q_- < 1$. Note that $\hbar\omega^{(+)}_{\mathrm{ex},\,k_x} > \hbar\omega^{(0)}_{\mathrm{ex},\,k_x}$, whereas, for the phonon-type branch, $\omega^{(+)}_{\mathrm{ph},\,k_x} < (c_l - v)\,k_x$. For $k_x < 0$, we have the following inequality for the excitonic branch,

$$\left(\frac{\hbar^2}{2m}[k_x^2 - k_0^2] + F(x) + \epsilon_+\right) \times$$
$$\times\left(\frac{\hbar^2}{2m}[k_x^2 - k_0^2] + F(x) + 2\,\widetilde{\nu}_0(v)\,\phi_{\mathrm{o}}^2(x) + \epsilon'_+\right) < \left(\hbar\omega^{(-)}_{\mathrm{ex},\,k_x}\right)^2 < \left(\hbar\omega^{(0)}_{\mathrm{ex},\,k_x}\right)^2, \quad (84)$$

where $2\,\widetilde{\nu}_0(v)\,\phi_{\mathrm{o}}^2(x) \simeq -4\,|\tilde{\mu}|$ within the inside-approximation, and, for the phonon-type branch, we obtain $\omega^{(-)}_{\mathrm{ph},\,k_x} > (c_l + v)\,|k_x|$.

To derive the formulas for the amplitudes $\mathrm{u}_k(x)$, $\mathrm{v}_k(x)$, and $C_k^{\mp}(x)$ of the excitonic branch, we use the following approximations

$$L_\pm(-k_x^2) = L(-k_x^2) + \frac{\rho^{-1}\sigma_0^2\phi_{\mathrm{o}}^2(x)\,k_x^2}{(\omega_{\mathrm{ex},\,k_x} \pm v|k_x|)^2 - c_l^2k_x^2} \approx L(-k_x^2) \pm q_\pm\gamma(v)\,\frac{\sigma_0}{Mc_l^2}\,(\sigma_0 a_l^3)\,\phi_{\mathrm{o}}^2(x),$$

and $B = \nu_0\phi_{\mathrm{o}}^2(x) + 2\nu_1\phi_{\mathrm{o}}^4(x)$ is modified as

$$B_\pm \approx \nu_0\phi_{\mathrm{o}}^2(x) + 2\nu_1\phi_{\mathrm{o}}^4(x) \pm q_\pm\gamma(v)\,\frac{\sigma_0}{Mc_l^2}\,(\sigma_0 a_l^3)\,\phi_{\mathrm{o}}^2(x).$$

Then we can rewrite the formulas for the excitonic excitation spectrum as follows

$$(\hbar\omega^{(\pm)}_{\mathrm{ex},\,k_x})^2 \approx L^2_\pm(-k^2_x) - B^2_\pm = \Big(L(-k^2_x) - B\Big)\Big(L_\pm(-k^2_x) + B_\pm\Big).$$

Recall that the orthogonality relation (58) can be used to normalize the amplitudes. Within the inside-approximation, Eq. (58) can be rewritten as follows ($\delta_{ss} = 1 \to \delta_{kk} = 1$)

$$\int d\mathbf{x}\,(|\mathrm{u}_k(x)|^2 - |\mathrm{v}_k(x)|^2) \,+\, (1/\hbar)\int d\mathbf{x}\, 2\rho\,(\omega_{\mathrm{ex},\,k_x} + vk_x)|C^x_k(x)|^2 = 1, \qquad (85)$$

and we have for the excitonic amplitudes

$$\left|\mathrm{u}^{(\pm)}_k(x)\right|^2 \approx \left(\frac{\Upsilon_\pm}{V_{\mathrm{eff}}}\right)\frac{L_\pm(-k^2_x) + \hbar\omega^{(\pm)}_{\mathrm{ex},\,k_x}}{2\,\hbar\omega^{(\pm)}_{\mathrm{ex},\,k_x}}, \quad \left|\mathrm{v}^{(\pm)}_k(x)\right|^2 \approx \left(\frac{\Upsilon_\pm}{V_{\mathrm{eff}}}\right)\frac{L_\pm(-k^2_x) - \hbar\omega^{(\pm)}_{\mathrm{ex},\,k_x}}{2\,\hbar\omega^{(\pm)}_{\mathrm{ex},\,k_x}},$$

$$\mathrm{u}^{(\pm)\,*}_k\mathrm{v}^{(\pm)}_k(x) \approx -\left(\frac{\Upsilon_\pm}{V_{\mathrm{eff}}}\right)\frac{B_\pm}{2\,\hbar\omega^{(\pm)}_{\mathrm{ex},\,k_x}}. \qquad (86)$$

Here the effective condensate volume $V_{\mathrm{eff}} \simeq 2SL_0$ is used to normalize the u- and v-wave functions of the inside excitations, and $\int d\mathbf{r}\,(|\mathrm{u}_k|^2 - |\mathrm{v}_k|^2) = \Upsilon_\pm < 1$.

Subsiquently, we get for

$$C^x_k(x) = -\frac{\rho^{-1}\sigma_0\,\phi_o(x)\,ik_x\,(\mathrm{u}_k(x) + \mathrm{v}_k(x)\,)}{(\omega_{\mathrm{ex},\,k_x} + vk_x)^2 - c^2_l k^2} \qquad (87)$$

the following approximate formulas

$$C^{x\,(\pm)}_k(x) \approx \mp\, q_\pm\,\gamma(v)\,\frac{\sigma_0}{Mc^2_l}\sqrt{a^3_l\,\phi^2_o(x)}\,\frac{i}{k_x}\,\sqrt{a^3_l}\,(\mathrm{u}^{(\pm)}_k(x) + \mathrm{v}^{(\pm)}_k(x)\,).$$

To estimate the characteristic value of $C^{x\,(\pm)}_k(x)$, we use $\mathrm{u}_k(x) \simeq \mathrm{v}_k(x) \sim \sqrt{\Upsilon_\pm/V_{\mathrm{eff}}}$ and obtain

$$|C^{x\,(\pm)}_k| \simeq q_\pm\,\gamma(v)\,\frac{\sigma_0}{Mc^2_l}\sqrt{a^3_l\,\Phi^2_o}\,\sqrt{\frac{\Upsilon_\pm\,a_l\,L_0}{2S}}\,\frac{a_l}{(3\sim 7)} \lll a_l.$$

The parameters $\Upsilon_\pm$ characterize the relative weight of excitonic degrees of freedom in the considered branch of excitations. As the parameter $\hbar vk_x/\hbar\omega^{(+)}_{\mathrm{ex},\,k_x} <$

1 at $k_x \simeq (4 \sim 8)\, k_0$, the parameter $\Upsilon_+(k_x)$ can be estimated as $0.5 \sim 0.7$. For $k_x < 0$, we obtain the following equation from (85),

$$\Upsilon_- \left(1 + \frac{(a_l^3\, \Phi_o^2)}{m/M}\, q_-^2\, \gamma^2(v) \left(\frac{\sigma_0}{M c_l^2}\right)^2 \left(1 - \frac{\hbar v |k_x|}{\hbar \omega^{(-)}_{\mathrm{ex},\, k_x}}\right) \frac{L(-k_x^2) - B}{\hbar^2 k_x^2/2m}\right) \approx 1.$$

Within the stability area (see the next subsection for an extended discussion), we estimate the ratio $\hbar v|k_x|/\hbar\omega^{(-)}_{\mathrm{ex},\, k_x} \simeq 1/2 \sim 1/3$ at $|k_x| \simeq (4 \sim 8)\, k_0$. Then, $\Upsilon_-(k_x) > 0$ and $\Upsilon_- \simeq 0.6 \sim 0.8$.

To go beyond the inside-approximation, the effect of inhomogeneous behavior of the longitudinal excitations can be considered. We use the following *ansatz* for the left side asymptotics (see Fig. 2)

$$\mathrm{u}_{j,s}(\mathbf{x}) = \exp(\ell_u x/L_0)\, \mathrm{e}^{ik_x x}\, u_k, \qquad \mathrm{v}_{j,s}(\mathbf{x}) = \exp(\ell_v x/L_0)\, \mathrm{e}^{ik_x x}\, v_k, \tag{88}$$

$$C^x_{j,s}(\mathbf{x}) = \exp(\ell_c x/L_0)\, \mathrm{e}^{ik_x x}\, C^x_k, \quad x < 0. \tag{89}$$

where $\ell_u = \ell_v = 1$ and $\ell_c = 2$. Then, like the case of transversal excitations (see Eq. (81)), we can write the equation for $\omega_{j,k_x}(x)$ valid at $|x| > L_0$,

$$\begin{aligned}
&\Big(\, (\omega_{j,k_x} + v\, \tilde{k}_x)^2 \;-\; c_l^2 \tilde{k}_x^2 \Big) \times \\
&\left[(\hbar\omega_{j,k_x})^2 \;-\; \left(\frac{\hbar^2}{2m}[\bar{k}_x^2 - k_0^2] + \tilde{F}(x)\right) \times \right.\\
&\left. \times \left(\frac{\hbar^2}{2m}[\bar{k}_x^2 - k_0^2] + \tilde{F}(x) + 2\nu_0 (2\Phi_o \exp(x/L_0)\,)^2\right)\right] \;= \\
&= \left(\frac{\hbar^2}{2m}[\bar{k}_x^2 - k_0^2] + \tilde{F}(x)\right)\, 2\frac{\sigma_0}{M c_l^2}\, (\sigma_0 a_l^3)\, (2\Phi_o \exp(x/L_0)\,)^2\, c_l^2 \tilde{k}_x^2,
\end{aligned} \tag{90}$$

where $k_x \to \tilde{k}_x = k_x - i(2/L_0)$ in the phonon parts of this equation and $k_x \to \bar{k}_x = k_x - i(1/L_0)$ in the exciton parts of it ($x < 0$). It is easy to see that Eqs. (83) and (90) are in the continuity correspondence, i.e. they describe the same object. For example, the (left side) asymptotic behavior of $\hbar\omega^{(\pm)}_{\mathrm{ex},\, k_x}(x)$ can be obtained from the inside-condensate formulas by the substitute

$$k_x \to \bar{k}_x, \quad F(x) \to \tilde{F}(x) \to 2|\tilde{\mu}|,$$

and

$$\phi_o^2(x) \to (2\Phi_o \exp(x/L_0)\,)^2 \to 0.$$

As a result, we obtain $\hbar\omega_{\mathrm{ex},\,k_x} \simeq |\tilde{\mu}| + \hbar^2 \bar{k}_x^2/2m$ that corresponds to the outside-excitation spectrum, Eq. (63).

The excitonic input into $\langle \delta\hat{\Psi}^\dagger \delta\hat{\Psi}(\mathbf{x})\rangle_{T=0}$, the quantum depletion of the moving condensate, can be calculated by $\sum_{1,s} |\mathrm{v}_{1,s}(x)|^2$ [20]. To estimate this value, one can approximate $|\mathrm{v}_k(x)|^2$ as follows

$$\left|\mathrm{v}_{k_x}^{(\pm)}(x)\right|^2 \simeq \left(\frac{\Upsilon_\pm}{V_{\mathrm{eff}}}\right) \frac{B_\pm^2}{4L_\pm^2(-k_x^2)}.$$

However, the summation $\sum_{1,s}$ implies $\int dk_x\, d^2k_\perp/(2\pi)^3$ within the semiclassical approximation. Assuming that such an integration makes the difference among $\mathrm{v}_{k_x}^{(-)}$, $\mathrm{v}_{k_x}^{(+)}$ and $\mathrm{v}_{k_\perp}$ not essentially important, we use the following estimate for $|\mathrm{v}_{\mathbf{k}}(x)|^2$,

$$\left|\mathrm{v}_{\mathbf{k}}(x)\right|^2 \simeq \left(\frac{1}{V_{\mathrm{eff}}}\right) \frac{B^2}{4L^2(-\mathbf{k}^2)}.$$

Then, the integration $\int dk_x\, d^2k_\perp/(2\pi)^3$ can be reduced to $\int_{k_0} k^2 dk/2\pi^2$, and the main input ($\sim \phi_o^2(x)$) can be estimated from the following formula

$$\langle \delta\hat{\Psi}^\dagger \delta\hat{\Psi}(\mathbf{x})\rangle_{T=0} \simeq \frac{1}{8\pi^2\, L_0^3}\,\frac{\nu_0\phi_o^2(x)}{2|\tilde{\mu}|} + \frac{\epsilon}{8\pi^2\, L_0^3}\,\frac{(\nu_0\phi_o^2(x)\,)^2}{2|\tilde{\mu}|^2} \sim \frac{1}{8\pi^2\, L_0^3}\,\frac{\nu_0\Phi_o^2\cosh^{-2}(x/L_0)}{|\widetilde{\nu_0}|\;\Phi_o^2}.$$

Using this density, we speculate that the *localized* depletion of condensate – i.e. the number of particles that are out of the condensate but move with it coherently – seems to be a small value. We obtain the following estimate ($\nu_0 = \varepsilon_0\, \tilde{a}_B^3$ and $|\widetilde{\nu_0}(v)| = |\tilde{\varepsilon}_0(v)|\, \tilde{a}_B^3$):

$$\delta N_0 = \int d\mathbf{x}\, \langle \delta\hat{\Psi}^\dagger \delta\hat{\Psi}(\mathbf{x})\rangle_{T=0} \simeq \frac{1}{16\,\pi^2}\,\frac{\varepsilon_0}{|\tilde{\mu}|}\,\frac{\tilde{a}_B^3}{L_0^3}\, N_o,$$

where the factor before N_o can be estimated as

$$\frac{\varepsilon_0}{|\tilde{\mu}|(N_o,\,v)}\,\frac{\tilde{a}_B^3}{L_0^3(N_o,\,v)} \simeq \frac{\varepsilon_0\,|\tilde{\varepsilon}_0(v)|}{2\,x\,(\mathrm{Ry}^*)^2}\,\frac{1}{\bar{n}_o} \sim (10^{-1} \sim 10^{-2})\,\bar{n}_o^{-1}.$$

(Here we used Eqs. (35),(36) within the approximation $\bar{n}_o \gg 10$.) Note that there is no small $\mathbf{k}$ input to the estimate of $\sum_{1,s}$ because, first, such excitations belong to the outside-excitation branch in our model, and, second, we use the approximation $\mathrm{v}_{\mathbf{k}}(x) \approx 0$ for them.

4.4 Stability of the Moving Condensate

To investigate the stability of the moving condensate in relation to the creation of inside-excitations, we calculate the energy of the condensate with the *one* inside excitation, $\langle\alpha^{\dagger}_{1,s}\alpha_{1,s}\rangle = 1$, described by the following set: k, ω_k, and u_k, v_k, C_k. In this study, we analyze the stability inside the excitonic sector of our model.

Although the excitations were defined in the co-moving frame, calculations should be done in the laboratory frame. Returning to the lab frame, we represent the exciton and phonon field functions as follows:

$$\phi_{\rm o}(x-vt,\, t) \to \phi_{\rm o}(x-vt,\, t)+\exp(-i(\tilde{E}_g+mv^2/2-|\tilde{\mu}|)t\,)\exp(i(\varphi+mvx)\,)\,\delta\tilde{\Psi}(\mathbf{x},t), \tag{91}$$

where

$$\delta\tilde{\Psi}(\mathbf{x},t) = \mathrm{u}_k(x-vt)\,\mathrm{e}^{i(\varphi_0+\mathbf{kx})}\,\mathrm{e}^{-i(\omega_k+k_xv)t} + \mathrm{v}_k(x-vt)\,\mathrm{e}^{-i(\varphi_0+\mathbf{kx})}\,\mathrm{e}^{i(\omega_k+k_xv)t},$$

and

$$u_{\rm o}(x-vt) \to u_{\rm o}(x-vt)+C_k(x-vt)\,\exp(i(\varphi_0+\mathbf{kx}))\,\exp(-i(\omega_k+k_xv)\,t)+\text{c.c.}, \tag{92}$$

see Eq. (64) for comparison. In this analysis, the inside excitations are not considered as fluctuations, and the (average) number of particles in the condensate and its energy are changed as $N_{\rm o} - \int d\mathbf{x}\,\delta\psi^{\dagger}\delta\psi$ and $E_{\rm o}(N_{\rm o}) - (\partial_N E_{\rm o})\int d\mathbf{x}\,\delta\psi^{\dagger}\delta\psi$, respectively. However, these changes are not important if the number of excitation in a system is less than $\sqrt{\delta N^2} \simeq \sqrt{N_{\rm o}}$. They could be important in the case of instability of the moving condensate.

The zeroth component of the energy-momentum tensor can be represented in the form

$$\mathcal{T}^0_0 = \mathcal{T}^0_0(\phi_{\rm o},\, u_{\rm o}) + \mathcal{T}^{0\,(2)}_0(\delta\Psi^{\dagger}, \delta\Psi, \delta u,\, \partial_t\delta u\,|\,\phi_{\rm o},\, u_{\rm o}),$$

where the first part corresponds to the condensate energy $E_{\rm o}$ and the second part gives the energy of inside-excitations, $E_{\rm in}$. After substitution of (91),(92) into $E_{\rm in} = \int d\mathbf{x}\, \mathcal{T}_0^{0\,(2)}$, we have for the total energy

$$E_{\rm o} + E_{\rm in-ex} \approx E_{\rm o}(N_{\rm o}) + \delta E_{\rm o}(N_{\rm o}) +$$

$$+ \int d\mathbf{x}\, \hbar(\omega_k(x) + k_x v)\, \left\{|\mathrm{u}_k|^2 - |\mathrm{v}_k|^2 + (2/\hbar)\rho(\omega_k(x) + v k_x)|C_k|^2\right\} =$$

$$= E_{\rm o}(N_{\rm o}) + \delta E_{\rm o}(N_{\rm o}) + \hbar\,\langle\omega_k + k_x v\rangle, \tag{93}$$

where

$$\delta E_{\rm o}(N_{\rm o}) = (2|\tilde{\mu}| + \nu_0 \Phi_{\rm o}^2) \int d\mathbf{x}\, \delta\tilde{\psi}^\dagger \delta\tilde{\psi} - \frac{M(c_l^2 + v^2)}{2}\, 3\vartheta(N_{\rm o},\, v) \int d\mathbf{x}\, \delta\tilde{\psi}^\dagger \delta\tilde{\psi}, \tag{94}$$

$$\delta\tilde{\psi}^\dagger \delta\tilde{\psi}(x) \to |\mathrm{u}_k|^2 + |\mathrm{v}_k|^2,$$

see Eqs. (65),(67) for comparison.

In this study, we discuss qualitatively the stability of the condensate in relation to the backward emission of inside-excitations, (i.e. $k_x < 0$ in the co-moving frame). To begin with, we consider the standard criterion,

$$\hbar(\omega^{(-)}_{\text{ex},\,k} - |k_x| v) > 0 \quad \text{at} \quad |k_x| \simeq z\, L_0^{-1}, \tag{95}$$

where $z \simeq 3 \sim 10$ corresponds to the low-lying excitations. The value of $\omega^{(-)}_{\text{ex},\,k}(x)$ is taken within the inside-approximation, see Eq. (84), so that $\hbar\omega^{(-)}_{\text{ex},\,k} \simeq f(z)\,|\tilde{\mu}|$ and $z/f(z) \simeq 0.1 \sim 0.3$. Then, it is easy to conclude that the following inequality

$$\frac{\hbar|k_x|v}{\hbar\omega^{(-)}_{\text{ex},\,k}} \simeq \frac{z}{f(z)}\,\frac{v}{c_l}\,\sqrt{\frac{(4m/M)\,Mc_l^2/2}{|\tilde{\mu}|(N_{\rm o},\,v)}} < 1 \tag{96}$$

is valid in the low-energy limit if the effective chemical potential (36) is large enough, see Eq. (69) for comparison.

More precisely, the ballistic velocity v and the number of particles in the condensate, $N_{\rm o}$, have to be large enough, f.ex., $|\tilde{\varepsilon}_{\rm o}(v)| \simeq (10^{-1} \sim 1)\,\mathrm{Ry}^*$ and $\bar{n}_{\rm o} \simeq 10$, in order to the inequality

$$\frac{(4m/M)\,Mc_l^2/2}{|\tilde{\mu}|(N_{\rm o},\,v)} \simeq \frac{(4m/M)\,Mc_l^2/2}{|\tilde{\varepsilon}_{\rm o}(v)|^2\,/\,4\bar{n}_{\rm o}^2\,x\,\mathrm{Ry}^*} < 10 \sim 20 \tag{97}$$

can be satisfied. Thus, for

$$|\tilde{\mu}|(N_o,\, v) > \mu_{\rm cr} \simeq 10^{-1}\,(4m/M)\,Mc_l^2/2, \tag{98}$$

where $\mu_{\rm cr} \sim 10^{-4}\,$eV, one can expect conditions (96),(97) to be valid.

Despite the condensate can be formed near $\gamma(v) \approx \gamma_o$ in theory, f.ex., with $|\widetilde{\nu_0}(v)| \simeq 0.1\,\nu_0$ and $\bar{n}_o \gg 10$, such a ballistic state seems to be unstable against the creation of inside-excitations. Note that the critical (Landau) velocity, $v_{\rm cr}$, can be found as a solution of Eq. (98) and $v_o < v_{\rm cr}(N_o) < c_l$. In fact, the parameter $|\tilde{\mu}|/\mu_{\rm cr}$ controls the stability/instability of the condensate, see Eqs. (69),(96).

Analyzing (95), we did not take into account Eq. (94). However, if the instability regime takes place, more than $\sqrt{N_o}$ inside-excitations can appear. As the changes in $u_o(x - vt)$ because of $N_o \to N_o - \delta N$ are nonlocal (in spite of creation of the localized excitations, see Figs. 1 and 3), a free acoustic phonon can appear in the system lattice + excitons *together* with a appearance of the localized excitation $\hbar\omega^{(-)}_{{\rm ex},\,k}(x)$. Like the case of outside-excitations, we assume that $(3/2)\,M(c_l^2 + v^2)\,\vartheta(N_o,\, v) \;\sim\; \hbar c_l k_{\rm ph}$, see Eq. (94). Then, only the term $\propto 2|\tilde{\mu}| + \nu_0\Phi_o^2$ is important. In fact, this term leads to some renormalization of the values of the critical parameters, $\mu_{\rm cr}$ and $v_{\rm cr}$.

5 Interference Between Two Moving Packets

In this section, we address the problem of interaction between two moving condensates. This problem is essentially nonstationary, especially if the initial ballistic velocities of packets are different. Within the quasi-1D conserving model, the following equations govern the dynamics of the two input packets (we choose the reference frame moving with the slow packet):

$$\Big(i\hbar\partial_t + \frac{m(v')^2}{2}\Big)\psi_0(x,t) = \Big(-\frac{\hbar^2}{2m}\partial_x^2 + \nu_0|\psi_0|^2 + \nu_1|\psi_0|^4\Big)\,\psi_0(x,t) + \\ + \sigma_0\,\partial_x u_0(x,t)\,\psi_0(x,t), \tag{99}$$

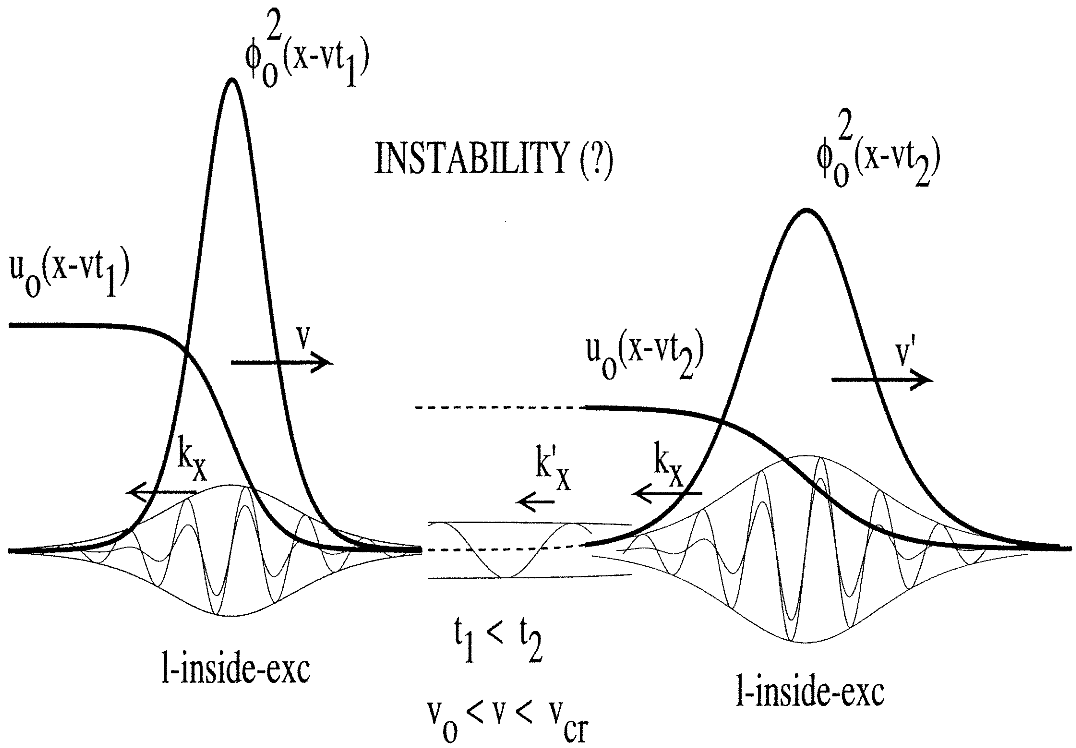


Figure 3: Instability regime.

The ballistic condensate, $\phi_o(x - vt) \cdot u_o(x - vt)\delta_{1j}$, can be unstable in relation to emission of the inside-excitations if the effective chemical potential $|\tilde{\mu}|(N_o, v) < \mu_{cr}$. In terms of Landau critical velocity, this means $v_o < v < v_{cr}(N_o)$. If such an instability takes place, the emission of inside-excitations can be accompanied by the emission of outside-excitations of the condensate. The longitudinal inside-excitations are labeled by the wave vector $k_x < 0$ on this figure, whereas the outside-excitation is labeled by the wave vector $k'_x < 0$.

$$((\partial_t - v'\,\partial_x)^2 - c_l^2\partial_x^2)\,u_0(x,t) - c_l^2\,2\kappa_3\,\partial_x^2 u_0(x,t)\,\partial_x u_0(x,t) = \rho^{-1}\sigma_0\,\partial_x|\psi_0|^2(x,t). \tag{100}$$

Then, the initial conditions can be written in the explicit 1D form by using the exact solution of the model (10),(11). Note that the amplitudes of the stationary ballistic state, $\phi_o(x - vt) \cdot \partial_x u_o(x - vt)$, were defined from the normalization condition and depend on the values of v and N_o. Hence, the amplitudes of the "input" condensates for Eqs. (99),(100) may not have the same values.

In this study, we approach the problem of strong interaction between the condensates. Therefore, we choose the nonsymmetric initial conditions, i.e. the amplitude and the velocity of the "input" packets are different, for example, $v > v'$ and $\mathbf{v}||\mathbf{v}'$, see Fig. 4. Here, we reply on the experimental observation [5] that, at $v > v_o$, the ballistic velocity of the condensate depends on the power of a laser beam irradiating the crystal. If the exciton concentration in the first packet, $n_{o\,1}$, is close to the value of the Bose condensation threshold, the exciton concentration in the second packet, $n_{o\,2} > n_{o\,1}$, the velocity difference between condensates can reach $(0.1 \sim 0.3)\,c_l$. Then, in the reference frame moving with the first (slow) packet, the initial conditions can be taken as the following:

$$\psi_0(x, t=0) \cdot u_0(x, t=0) = \phi_o(x; N_{o\,1}) \cdot q_o(x; N_{o\,1}) +$$

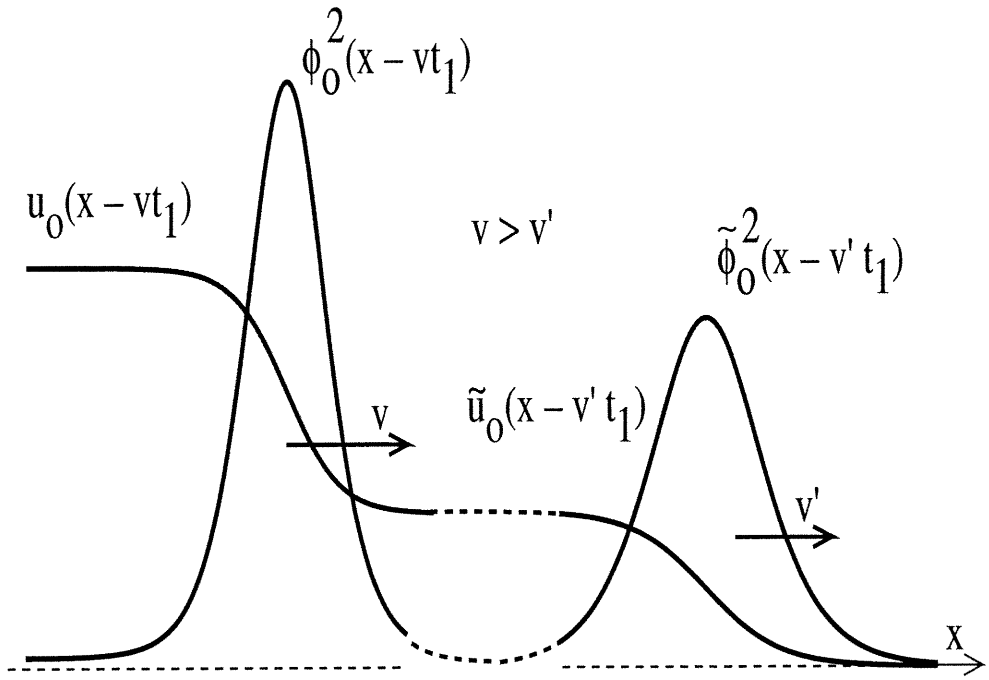
$$+ \exp(i(\delta\varphi + m\,\delta v\,x))\,\phi_o(x + x_0; N_{o\,2}) \cdot q_o(x + x_0; N_{o\,2}), \tag{101}$$

where $\delta\varphi = \varphi - \varphi'$, $\delta v = v - v'$, $x_0 = v'\tau$, and τ is the (initial) time delay. As the second packet moves in this frame of reference, the regime of strong nonlinear interaction between the condensates is (theoretically) unavoidable. Note that, even before collision, a time-dependent interference term in $|\psi_o(x,t)|^2$ begins to influence the packet dynamics, see Fig. 4. For example, the r.h.s. of (100) contains

$$\sim \partial_x \left\{ 2\cos(m\,\delta v x - \delta\omega\,t + \delta\varphi)\,\phi_o(x; N_{o\,1})\,\phi_o(x - \delta v t + x_0; N_{o\,2}) \right\}, \tag{102}$$

where

$$\hbar\delta\omega = m\,\delta v(v + v')/2 - (\,|\tilde{\mu}|(N_{o\,2}, v) - |\tilde{\mu}|(N_{o\,1}, v')\,)$$



interference area

Figure 4: Interaction of packets as an overtaking collision.
Two ballistic condensates move with different velocities, $v - v' \simeq (0.1 \sim 0.3)\, c_l$, and $t = t_1$ before the "collision", or, the strong interaction process. If one can prescribe the coherent phase to each of the participating condensates, e.g., $\varphi_c(\mathbf{x}) \approx \varphi + (mv/\hbar)\, x$, the interference area appears between them. (The interference area is marked by bold dashed lines on this figure.) As $v \neq v'$, the fringes are non-stationary, and the outside-excitations can actually be excited in this area.

and $|\tilde{\mu}| \propto (N_{\rm o}/N_{\rm o}^*)^2\,|\tilde{\varepsilon}_0(v)|^2$. The ratio $|\tilde{\mu}_2|/|\tilde{\mu}_1|$ can be of the order of 10^1, and the characteristic scale of fringes (102) is

$$\frac{\pi(\hbar/m\,\delta v)}{L_0} \simeq (10 \sim 30)\sqrt{\frac{|\tilde{\mu}|(N_{\rm o\,1}, v)}{(m/M)\,Mc_l^2/2}} \simeq 5 \sim 10,$$

that is they are of the long-wavelength nature.

To answer the question which model (conserving (99),(100) or kinetic [19],[32] one) is more adequate to describe the packet collision, we have to compare the estimate of interaction time,

$$\tau^* \simeq L_{\rm ch\,2}/\delta v \sim 10^3\,\tilde{a}_B/0.2\,c_l \simeq 10^{-9} \sim 10^{-10}\,{\rm s},$$

and characteristic time scales of the processes

$$|N_{\rm o\,2}(t), v\rangle \cdot |N_{\rm o\,1}(t), v'\rangle \;\rightarrow\; |N_{\rm o\,2}(t) \pm \delta N, v\rangle \cdot |N_{\rm o\,1}(t) \mp \delta N, v'\rangle \qquad (103)$$

driven by phonons or by $x-x$ interaction. Note that some thermal phonons have to be excited in the system to assist such transitions, and the value of τ^* is of the order of scattering time of the exciton-LA-phonon interaction (although without any macroscopical occupancy) [17].

If processes (103) are driven by the lattice phonons, two phonons are necessary to satisfy the laws of conservation. For instance, we choose $\delta N = +1$ in (103) and obtain (see Eqs. (65),(66))

$$\hbar k_{1,x} = m\delta v + 3Mv\,\vartheta(N_{\rm o\,2},\, v) - 3Mv'\,\vartheta(N_{\rm o\,1},\, v') + \hbar k_{2,x},$$

$$\hbar c_l|k_{1,x}| = m\delta v\,(v{+}v')/2 - 3\left(|\tilde{\mu}|(N_{\rm o\,2},\, v) - |\tilde{\mu}|(N_{\rm o\,1},\, v') + \nu_0\,\Phi_{\rm o\,2}^2/3 - \nu_0\,\Phi_{\rm o\,1}^2/3\right) +$$
$$+\,3/2\,M(c_l^2 + v^2)\,\vartheta(N_{\rm o\,2},\, v) - 3/2\,M(c_l^2 + v'^2)\,\vartheta(N_{\rm o\,1},\, v') + \hbar c_l|k_{2,x}|.$$

Although the second packet moves faster, $m\delta v > 0$, this state can be considered as a more stable (and, thus, more preferable) one for the excitons of the slow packet. Indeed, the following inequality for the effective difference between the generalized chemical potentials seems to be valid

$$m\delta v\,(v + v')/2 - 3\left(|\tilde{\mu}|(N_{\rm o\,2},\, v) - |\tilde{\mu}|(N_{\rm o\,1},\, v') + \nu_0\,\Phi_{\rm o\,2}^2/3 - \nu_0\,\Phi_{\rm o\,1}^2/3\right) < 0, \quad (104)$$

and the absolute value of the l.h.s. of (104) is $\sim |\tilde{\mu}|(N_{\text{o}\,2}\, v)$. Thus, within the quantum kinetic model, the relevant transition probabilities have to be calculated at least in the second order of perturbation theory, f.ex.,

$$|N_{\text{o}\,2}, v\rangle \cdot |N_{\text{o}\,1}, v'\rangle \xrightarrow{\text{ph}_1} |N_{\text{o}\,2}, v\rangle \cdot |N_{\text{o}\,1},\, \hbar\omega_{\text{ex},\,k_j},\, v'\rangle \xrightarrow{\text{ph}_2} |N_{\text{o}\,2} + 2, v\rangle \cdot |N_{\text{o}\,1} - 2, v'\rangle.$$

As a result, the system of two quantum Boltzmann equations will describe the interaction process.

Unlike the Boltzmann equations, Eqs. (99, 100) contain information about the quantum coherence between two condensates explicitly and, moreover, can describe the case of strong interaction. Unlike the 1D NLS equation that supports many-soliton solutions, Eqs. (99),(100) are, in fact, quasi-1D ones, and $\nu_0 > 0$ in (99). Therefore, it is an open question what happens with two (excitonic) solitons after they collide in the crystal.

In this work, we assume that the dominant process(es) of condensate interactions is that one(s) leading to $\partial_t N_{\text{o}\,2} > 0$. Then, at the time scales $\gg \tau^*$, one solitonic packet can appear as a result of these processes. Such a resultant ballistic packet can be approximately described by the steady-state one-soliton solution of Eqs. (8),(9) with $\tilde{N}_{\text{o}} \approx N_{\text{o}\,2} + N_{\text{o}\,1}$ and the low of energy conservation,

$$E(N_{\text{o}\,1}, v') + E(N_{\text{o}\,2}, v) \approx E(\tilde{N}_{\text{o}} = N_{\text{o}\,2} + N_{\text{o}\,1}, \tilde{v}). \tag{105}$$

If $\tilde{N}_{\text{o}} < N_{\text{o}}^*$, all the approximate solutions having been found in this study are valid to describe the resultant packet.

As we prescribed the value of $\tilde{N}_{\text{o}}$, we have to estimate the value of $\tilde{v}$ from Eq. (105), (generally, $\tilde{v} \neq v$). Moreover, we have to assume that the total momentum of the condensates, $P_x(N_{\text{o}\,1}, v') + P_x(N_{\text{o}\,2}, v)$, may not be conserved because of lattice participation in such a condensate "merger". However, the challenging question of the results of coherent packet collision needs further theoretical and experimental efforts.

6 Conclusion

In this study, we considered a model within which the inhomogeneous excitonic *condensate* with a nonzero momentum can be investigated. The important physics we include in our model is the exciton-phonon interaction and the appearance of a coherent part of the crystal displacement field, which renormalizes the x-x interaction vertices. Then, the condensate wave function and its energy can be calculated exactly in the simplest quasi-1D model, and the solution is a sort of Davydov's soliton [23]. We believe that the transport and other unusual properties of the coherent para-exciton packets in Cu_2O can be described in the framework of the proposed model properly generalized to meet more realistic conditions.

We showed that there are two critical velocities in the theory, namely, v_o and v_{cr}. The first one, v_o, comes from the renormalization of two particle exciton-exciton interaction due to phonons, and the bright soliton state can be formed if $v > v_o$. Then, the important parameter, which controls the shape and the characteristic width of the condensate wave function, is $|\tilde{\mu}|/\mu^*$, see Eqs. (29),(36). The second velocity, v_{cr}, comes from use of Landau arguments [26] for investigation of the dynamic stability / instability of the moving condensate. In fact, the important parameter, which controls the emission of excitations, is $|\tilde{\mu}|/\mu_{cr}$, see Eq. (98). Then, within the semiclassical approximation for the condensate excitations, we found more close v is to c_l ($\mu_{cr} < |\tilde{\mu}|(N_o, v) < \mu^*$) more stable the coherent packet is. It is interesting to discuss the possibility of observation of an instability when the condensate can be formed in the inhomogeneous state with $v \neq 0$, but with $v_o < v < v_{cr}(N_o)$, or, better, $|\tilde{\mu}|(N_o, v) < \mu_{cr}$. Such a coherent packet has to disappear during its move through a single pure crystal used for experiments. As the shape of the moving packet depends on time, the form of the registered signal may depend on the crystal length changing from the solitonic to the standard diffusion density profile.

We found that the excited states of the moving exciton-phonon condensate can be described by use of the language of elementary excitations. Although the possibility of their direct observation is an unclear question itself (and, thus, the question about the gap in the excitation spectrum is still an open one), the stability conditions of the moving condensate can be derived from the low-energy asymptotics of the excitation spectra at $T \ll T_c$. However, the stability problem is not without difficulties [30],[33]. One can easily imagine the situation when the condensate moves in a very high quality crystal, but with some impurity region prepared, f.ex., in the middle of the sample. In this case, the excitonic superfluidity can be examined by impurity scattering of the ballistic condensate. Indeed, such impurities could bound the noncondensed excitons, which always accompany the condensate, and could mediate, for instance, the emission of the outside excitations. The last process may lead to depletion of the condensate and, perhaps, some other observable effects, such as damping, bound exciton PL, etc.. On the other hand, the inside excitations could manifest themselves at $T \neq 0$ by the effective enlargement of the packet length, $L_0 \to L_{\text{eff}}$, $T \neq 0$, or by interaction with external acoustic waves.

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References

[1] J. P. Wolfe, J. L. Lin, D. W. Snoke and by A. Mysyrowicz in *Bose-Einstein Condensation*, edited by A. Griffin, D. W. Snoke and S. Stringari (Cambridge University Press, Cambridge, 1995).

[2] L. V. Butov, A. Zrenner, G. Abstreiter, G. Bohm, and G. Weimann, Phys. Rev. Lett **73**, 304, (1994); Physics–Uspekhi **39**, 751 (1996);
L. V. Butov, A. I. Filin, Phys. Rev. B **58**, 1980 (1998).

[3] V. Negoita, D. W. Snoke, and K. Eberl, cond-mat/9901088

[4] J. L. Lin, J. P. Wolfe, Phys. Rev. Lett. **71**, 122 (1993).

[5] T. Goto, M. Y. Shen, S. Koyama, T. Yokouchi, Phys. Rev. B **55**, 7609 (1997).

[6] E. Fortin, S. Fafard, A. Mysyrowicz, Phys. Rev. Lett. **70**, 3951 (1993).

[7] E. Benson, E. Fortin, A. Mysyrowicz, Phys. Stat. Sol. B **191**, 345 (1995); Sol. Stat. Comm. **101**, 313, (1997).

[8] E. Hanamura, Sol. Stat. Comm. **91**, 889 (1994);
J. Inoue, E. Hanamura, Sol. Stat. Comm. **99**, 547 (1996);

[9] J. Fernández-Rossier and C. Tejedor, Phys. Rev. Lett. **78**, 4809 (1997);
J. Fernández-Rossier, C. Tejedor, R. Merlin, cond-mat/9909232.

[10] A. E. Bulatov, S. G. Tichodeev, Phys. Rev. B **46**, 15058 (1992);
G. A. Kopelevich, S. G. Tikhodeev, and N. A. Gippius, JETP **82**, 1180 (1996).

[11] I. Loutsenko, D. Roubtsov, Phys. Rev. Lett. **78**, 3011 (1997);
D. Roubtsov, Y. Lépine, Phys. Stat. Sol. (B) **210**, 127 (1998); cond-mat/9807140.

[12] Th. Östreich, K. Schönhammer, L. J. Sham, Phys. Rev. Lett. **74**, 4698 (1995); cond-mat/9807135.

[13] A. Imamoḡlu and R. J. Ram, Phys. Lett. A **214**, 193 (1996); W. Zhao, P. Stenius, A. Imamoḡlu, Phys. Rev. B **56**, 5306 (1997).

[14] G. M. Kavoulakis, G. Baym, and J. P. Wolfe, Phys. Rev. B **53**, 7227 (1996); G. M. Kavoulakis, Y.-C. Chang, and G. Baym, Phys. Rev. B **55**, 7593 (1997);

[15] Yu. E. Lozovik, A. V. Poushnov, JETP **88**, 747 (1999); cond-mat/9803318.

[16] S. G. Tichodeev, Phys. Rev. Lett. **78**, 3225 (1997); A. Mysyrowicz, *ibidem*, 3226 (1997).

[17] A. L. Ivanov, C. Ell, and H. Haug, Phys. Rev. E **55**, 6363 (1997); Phys. Rev. B **57**, 9663 (1998).

[18] S. Schmitt-Rink, D. S. Chemla, and D. A. B. Miller, Adv. Phys. **38**, 89 (1989).

[19] L. V. Keldysh and A. N. Kozlov, Sov. Phys. JETP **27**, 521 (1968); E. Hanamura, H. Haug, Phys. Rep. C **33**, 209 (1977).

[20] J. Schumway and D. M. Ceperley, cond-mat/9907309, submitted to Phys. Rev. B.

[21] V. N. Popov, *Functional Integrals and Collective Modes*, (Cambridge University Press, N.Y., 1987).

[22] M. F. Miglei, S. A. Moskalenko, and A. V. Lelyakov, Phys. Stat. Sol. **35**, 389 (1969); V. M. Nandkumaran and K. P. Sinha, Zeit. für Phys. B **22**, 173 (1975).

[23] A. S. Davydov, *Solitons in Molecular Systems*, (Reidel, 1984).

[24] A. Griffin, Phys. Rev. B **53**, 9341 (1996); cond-mat/9901172.

[25] G. Huang and B. Hu, Phys. Rev. B **58**, 9194 (1998).

[26] Al. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle System*, (McGrav-Hill, New York, 1971).

[27] S. Giorgini, L. P. Pitaevskii, S. Stringari, Phys. Rev. A **54**, 4633 (1996); F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999); cond-mat/9806038.

[28] J.J. Hopfield, Phys. Rev. **112**, 1555 (1958).

[29] A. L. Ivanov, H. Haug, and L. V. Keldysh, Phys. Reports **296**, 237 (1998); H. Haug, A. L. Ivanov, and L. V. Keldysh, *Nonlinear Optical Phenomena in Semiconductors and Semiconductor Microstructures*, (World Scientific, 1999).

[30] S. A. Moskalenko, D. W. Snoke, *Bose Condensation of Excitons and Coherent Nonlinear Optics*, (Cambridge University Press, in press).

[31] E. Benson, E. Fortin, B. Prade, and A. Mysyrowicz, Europhys. Lett. **40**, 311 (1997).

[32] A. Mysyrowicz, E. Benson, and E. Fortin, Phys. Rev. Lett. **77**, 896 (1996).

[33] D. Roubtsov, Y. Lépine, Phys. Lett. A **246**, 139 (1998); cond-mat/9807023.

[34] H. Stolz, *Time-Resolved Light Scattering from Excitons*, (Springer-Verlag, 1994).

Chapter 3

How to switch off the phonon wind

In this chapter, we included a paper in which we discuss the nature of the exciton-phonon "comet" moving in a monocrystal of Cu_2O. We raise a question whether it is possible to separate the coherent part of the comet and the non-coherent part of it. As a first step, we propose to switch off a phonon wind, i.e., the remaining non-coherent phonons of the packet.

Actually, it is an open question to which extent the coma and the tail of the exciton-phonon comet are non-coherent. Within the standard approach of interacting bosons at $T < T_c$, we have the Bose-condensate in equilibrium and its depletion. The depletion is considered as a collection of the elementary excitations of the condensate ($T \neq 0$), and these excitations do not have the same phase in contrast with the Bose-condensate. If we apply this approach to the moving exciton-phonon condensate, the tail is a collection of the (non-coherent) outside-excitations of the condensate. However, those excitons and phonons that did not take part in the formation of the coherent core from the beginning can be also found in the tail. It would be an interesting task to think how to remove this part from the moving packet.

On Bose-Einstein condensate inside moving exciton-phonon droplets

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Abstract

We explore a nonlinear field model to describe the interplay between the ability of excitons to be Bose condensed and their interaction with other modes of a crystal. We apply our consideration to the long-living paraexcitons in Cu_2O. Taking into account the exciton-phonon interaction and introducing a coherent phonon part of the moving condensate, we solve the quasi-stationary equations for the exciton-phonon condensate. These equations support localized solutions, and we discuss the conditions for the inhomogeneous condensate to appear in the crystal. Allowable values of the characteristic width of ballistic condensates are estimated in the limit $T \to 0$. The stability conditions of the moving condensate are analyzed by use of Landau arguments, and Landau critical parameters appear in the theory. It follows that, under certain conditions, the condensate can move through the crystal as a stable droplet. To separate the coherent and non-coherent parts of the exciton-phonon packet, we suggest to turn off the phonon wind by the changes in design of the 3D crystal and boundary conditions for the moving droplet.

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Keywords: Bose Einstein condensation; Excitons; Cuprous oxide; Exciton-Phonon condensate; Phonon Wind; Stability

1 Introduction

Nowadays, there is a lot of experimental evidence that paraexcitons in Cu_2O crystals can form a strongly correlated state, which can be assigned to the excitonic Bose Einstein condensate (BEC) [1],[2],[3].

Surprisingly enough, a cloud of excitons that seems to contain the condensate can be prepared in a moving state, and such a packet moves ballistically through the crystal at $T < T_c$. At $T > T_c$, however, the excitonic packet exhibits the standard diffusive behavior. Thus, in the three dimensional Cu_2O crystals, the excitonic Bose Einstein condensate is found to be in a spatially inhomogeneous state with the well defined characteristic width $L_{\rm ch}$ in the direction of motion [1]. The registered ballistic velocities of the coherent exciton packets turn out to be always less, but approximately equal to the longitudinal sound speed of the crystal, $v < c_{\rm s}$. Note that paraexcitons in the pure $Cu_2 0$ crystals have the extremely large lifetime, $\tau \simeq 13\,\mu$s, and, at the conditions we discuss in this work, they are optically inactive [4]. Although transferring of the moving coherent excitonic field into a coherent photon field is not a hopeless task [5],[6], no convincing experimental results are obtained yet, [2],[7].

To understand the physics of anomalous excitonic transport, we accept [4],[8],[9] that the macroscopic wave function $\Psi_0 \sim \phi_{\rm o}\, e^{i\varphi_{\rm c}}$ can be associated with the coherent part of the excitonic packet. (Here $\varphi_{\rm c}$ is the coherent phase of the condensate.) In other words, the experimental results [1] suggest the following decomposition of the density of excitons in the packet,

$$n(\mathbf{x},t) = n_{\rm coh}(\mathbf{x},t) + \Delta n(\mathbf{x},t), \tag{1}$$

where $n_{\rm coh}(\mathbf{x},t) \approx n_{\rm coh}(x - vt)$ is the ballistic (*superfluid*) part of the packet,

$$n_{\rm coh}(x - vt) \simeq |\Psi_0|^2(x - vt), \tag{2}$$

and $\Delta n(\mathbf{x},t)$ is the diffusive (non-condensed) part of it,

$$\Delta n(\mathbf{x},t) \simeq \langle \delta\hat{\psi}^\dagger \delta\hat{\psi}(\mathbf{x},t)\rangle. \tag{3}$$

Therefore, the problem is how to describe a spatially inhomogeneous state of the excitonic BEC in terms of $\Psi_0(\mathbf{x},t)$ and $\delta\hat{\psi}(\mathbf{x},t)$, where $\delta\hat{\psi}$ is the "fluctuating" part of the exciton Bose field. Indeed, if a coherent excitonic packet moves in a crystal (or another semiconductor structure), it interacts with phonons, non-condensed excitons, impurities and other imperfections of the lattice, etc., (see Ref. [10]).

All this makes the problem of superfluidity of the Bose-condensed excitons a rather complicated and challenging one.

In this Letter, we start from the simplest possible approximation: we describe the moving condensate only and show that a sort of Gross Pitaevskii equation [11],[12] does make sense at $T \ll T_c$. In this case, the finite characteristic length of the condensate (but not of the total exciton-phonon packet) appears naturally in the framework of the effective 1D nonlinear Schrödinger equation, by which we model the real 3D conditions.

2 Exciton-Phonon Condensate

To obtain the necessary density of excitons in the excitonic cloud and, thus, meet the BEC conditions, the Cu_20 crystals were irradiated by laser pulses with $\hbar\omega_L \gg E_{\rm gap}$ at $T \simeq 2 \sim 5\,\mathrm{K}$. Note that the cross-section area of an excitation spot on a surface of the crystal, S, can be made large enough, so that $S \simeq S_{\rm surf}$. Then, the classical "phonon wind", or the flow of nonequilibrium phonons from the surface into the bulk [4], can transfer the nonzero momentum to the excitonic cloud, $\mathbf{P}_{\rm exc} \simeq N_{\rm x}\,\langle \hbar\,\mathbf{k}_0\rangle \neq 0$ and $\mathbf{P}_{\rm exc} \perp S_{\rm surf}$.

As a result, the packet of moving excitons and nonequilibrium phonons of the phonon wind ($N_{\rm ph} \simeq N_{\rm x}$) is actually the system that undergoes the transition toward the Bose Einstein condensation. Then, one can estimate the energy of the packet without any condensate as follows

$$E \simeq N_{\rm x}\{(\hbar^2/2m_{\rm x})\langle \mathbf{k}_0^2\rangle + 2\nu_0 n_{\rm x}\} + N_{\rm ph}\,\langle \hbar\omega'_{\rm ac}\rangle,$$

where $\hbar\langle k_{0\,x}\rangle = m_{\rm x}\langle v\rangle \simeq m_{\rm x}\,c_{\rm s}$ and $\nu_0 > 0$ is the exciton-exciton interaction strength, and n_x is the average density of excitons in the packet.

At $T < T_c(n_x)$, or, equivalently, $n_x > n_c(T)$ [22], the condensate can be formed inside the excitonic droplet and the following representation of the exciton Bose-field holds, $\hat{\psi} = \Psi_0 + \delta\hat{\psi}$. (As the kinetics of condensate formation is not a subject of this paper, we assume $T_{\rm cloud} \simeq T$ and $T \to 0$.) Moreover, for the displacement field of the crystal, $\hat{\mathbf{u}}$, we can introduce a nontrivial coherent part too, i.e. $\hat{\mathbf{u}} = \mathbf{u}_0 + \delta\hat{\mathbf{u}}$ and $\mathbf{u}_0 \neq 0$. Then, a moving coherent packet can contain the exciton-phonon condensate, which is the self-consistent exciton-phonon field, $\Psi_0(\mathbf{x}, t) \cdot \mathbf{u}_0(\mathbf{x}, t)$.

The macroscopic wave function of excitons, $\Psi_0(\mathbf{x}, t) \approx \Psi_0(x, t)$, is normalized as follows

$$\int |\Psi_0|^2(x,t)\, d\mathbf{x} = S \int \phi_o^2(x)\, dx = N_o, \tag{4}$$

where N_o is the (macroscopic) number of condensed excitons, and, generally, $N_o \neq N_x$. To model the ballistic motion of $n_{\rm coh}(x, t)$, we use the following *ansatz*,

$$\Psi_0(x,t) = \mathrm{e}^{-i(\tilde{E}_g + m_x v^2/2 - |\mu|)\, t}\, \mathrm{e}^{i(\varphi + k_0 x)}\, \phi_o(x - vt), \tag{5}$$

$$u_{0j}(x,t) = u_o(x - vt)\, \delta_{1j}, \tag{6}$$

where $\tilde{E}_g = E_{\rm gap} - E_x$, E_x is the exciton Rydberg, $\varphi = $ const, $\hbar k_0 = m_x v$, and μ is the effective chemical potential of the condensate. Note that there is no difference between the ballistic velocity of the exciton-phonon packet and the superfluid velocity of the condensate if we start from ansatz (5).

For the envelope function $\phi_o(x)$ and the phonon direct current $u_o(x)$, we obtain the following stationary equations [8],[9]

$$-|\mu|\, \phi_o(x) = (-(\hbar^2/2m_x)\partial_x^2 + \tilde{\nu}_0\, \phi_o^2(x) + \tilde{\nu}_1\, \phi_o^4(x))\, \phi_o(x), \tag{7}$$

$$\partial_x u_o(x) \approx \mathrm{const}_1\, \phi_o^2(x) + \mathrm{const}_2\, \phi_o^4(x). \tag{8}$$

Here, the "bare" exciton vertices, both the two-particle $\nu_0 > 0$ and the three-particle $\nu_1 > 0$, can be strongly renormalized because of exciton-phonon interaction. We choose it in the simplest form of Deformation Potential,

$$\tilde{E}_g\, \hat{\psi}^\dagger \hat{\psi} \to (\tilde{E}_g + \sigma_0\, \partial_j \hat{u}_j)\, \hat{\psi}^\dagger \hat{\psi}, \quad \sigma_0 > 0. \tag{9}$$

Yet the first (cubic) anharmonicity $\kappa_3 \neq 0$ is taken into account to model the lattice. Note that we consider the case $v \simeq c_{\rm s}$, and, at first glance, the (adiabatic) assumption of coherent propagation of the crystal deformation field and the excitonic condensate might not be valid. However, the nonlinear lattices support some localized excitations that can consist of two parts, namely, the direct current and alternating one,

$$\hat{u} \simeq u_{\rm o}(x - \tilde{v}t) + \delta\hat{u}(x - \tilde{v}t,\, t).$$

Such excitations can move with the (group) velocity $\tilde{v} \simeq c_{\rm s}$ [14]. Therefore, Eq. (8) can be used to describe the exciton-phonon condensate with $v \simeq c_{\rm s}$.

If the following conditions

$$\tilde{\nu}_0 = \tilde{\nu}_0(\,\nu_0,\, \sigma_0,\, v/c_{\rm s}\,) < 0 \quad \text{and} \quad \tilde{\nu}_1 = \tilde{\nu}_1(\,\nu_1,\, \sigma_0,\, v/c_{\rm s},\, \kappa_3\,) > 0 \tag{10}$$

can be valid, the localized (solitonic) solution of Eq. (7) exists. It can be written in the following form:

$$\phi_{\rm o}(x) = \Phi_{\rm o}\, f(\beta(\Phi_{\rm o})x,\, \text{const}(\Phi_{\rm o})\,), \qquad \beta(\Phi_{\rm o}) = \sqrt{(2m_{\rm x}/\hbar^2)\,|\mu|(\Phi_{\rm o})}\,, \tag{11}$$

where $\Phi_{\rm o}$ is the amplitude of the soliton and $1/\beta(\Phi_{\rm o}) \equiv L_0$ is its width. Then, the characteristic length of the condensate can be estimated as

$$L_{\rm ch} \simeq (2 \sim 4)\, L_0 \propto |\mu|^{-1/2},$$

and the effective chemical potential of the condensate, $|\mu|(N_{\rm o},\, v)$, defines the value of $L_{\rm ch}$ in this model.

We can use the simplest approximation,

$$|\mu|(\Phi_{\rm o}) \approx |\tilde{\nu}_0|\, \Phi_{\rm o}^2/2 \quad \text{and} \quad |\mu|(N_{\rm o},\, v) \propto N_{\rm o}^2, \tag{12}$$

which turns out to be valid at $\bar{n}_{\rm o} > 5 - 10$. Here $\bar{n}_{\rm o} = N_{\rm o}^*/N_{\rm o}$, and $N_{\rm o}^* = 2\,S/a_{\rm x}^2$ is a macroscopically large parameter ($a_{\rm x}$ is the exciton Bohr radius),

$$N_{\rm o}^* \simeq 10^{13} \sim 10^{14}.$$

Indeed, one can estimate $|\mu|$ more accurately,

$$|\mu|(\Phi_o) = |\tilde{\nu}_0|\,\Phi_o^2/2 \,-\, \tilde{\nu}_1\,\Phi_o^4/3 \quad \text{and} \quad |\mu|(N_o,\,v) \propto \Big(2\,\bar{n}_o^2\,\mathrm{x}\,E_\mathrm{x}\,a_\mathrm{x}^6 \,+\, 4\,\tilde{\nu}_1(v/c_\mathrm{s})\Big)^{-1}. \tag{13}$$

Here, $\mathrm{x} = \mu_\mathrm{x}/m_\mathrm{x}$ and the last estimate is valid for $N_o < N_o^*$.

We choose approximation (12), and the characteristic length of the condensate can be estimated as follows ($\tilde{\nu}_0 = \tilde{\varepsilon}_0(v/c_\mathrm{s})\,a_\mathrm{x}^3$ and $\nu_0 = \varepsilon_0\,a_\mathrm{x}^3$):

$$L_\mathrm{ch}(N_o,\,v) \simeq 4\sqrt{\frac{\hbar^2}{m_\mathrm{x}\,|\tilde{\nu}_0|}\,\Phi_o^{-2}(N_o,\,v)} \;\simeq\; 4\,\frac{E_\mathrm{x}}{|\tilde{\varepsilon}_0(v/c_\mathrm{s})|}\,\bar{n}_o\,a_\mathrm{x} \,\propto\, N_o^{-1}. \tag{14}$$

At $v = v_o$, or, equivalently, at $\gamma(v) = \gamma_o$, where

$$\gamma(v) = c_\mathrm{s}^2/(c_\mathrm{s}^2 - v^2) \quad \text{and} \quad v < c_\mathrm{s},$$

we have $\tilde{\varepsilon}_0(v_o/c_\mathrm{s}) = 0$ and $\gamma_o \simeq 3 \sim 5$ [1]. Therefore, solitonic solution (11) and estimate (14) are valid at $v_o < v < c_\mathrm{s}$, or $\gamma(v) > \gamma_o$. For example, at $\gamma(v) \approx 2\,\gamma_o \simeq 6 \sim 10$, $(\gamma(v = 0.95\,c_\mathrm{s}) \approx 10)$, we obtain $\tilde{\nu}_0 < 0$ and estimate $\tilde{\varepsilon}_0(v/c_\mathrm{s}) \approx -\varepsilon_0$.

Thus, at $\gamma(v) > \gamma_o$ and $\bar{n}_o > 10$ (for estimates we take $|\tilde{\varepsilon}_0(v/c_\mathrm{s})| \simeq (10^{-2} \sim 10^{-1})\,E_\mathrm{x}$ and $\bar{n}_o \simeq 10^1 \sim 10^2$, i.e. $N_o \simeq 10^{11} \sim 10^{12}$), we obtain a large factor multiplied by a_x as an estimate of L_ch [9], e.g.,

$$L_\mathrm{ch} = \mathcal{F}\,a_\mathrm{x} \gg a_\mathrm{cr}.$$

Here $a_\mathrm{cr} \simeq 4\,\text{Å}$ is the lattice constant and $a_\mathrm{x}^3 \simeq (3 \sim 4)\,a_\mathrm{cr}^3$ for the paraexcitons in Cu_2O.

Within the quasi-stationary approximation (5)-(7) at $T = 0$, this is a reasonable result, $\mathcal{F} \simeq 10^2 \sim 10^4$, and the duration of the condensate can be estimated as $t_\mathrm{ch} \simeq 2\cdot(10^{-11} - 10^{-9})\,\mathrm{s}$. Although these results are in a qualitative agreement with the average 3D densities of the Bose-condensed excitons in moving packets at $T \neq 0$, $n_o \simeq 10^{17} \sim 10^{18}\,\mathrm{cm}^{-3}$ [1], the characteristic duration of such localized packets is $\Delta t \simeq (4 \sim 8)\cdot 10^{-7}\,\mathrm{s}$ experimentally.

3 Stability against Outside Excitations

Analyzing experimental results [1], one can notice that a long-lasting "tail" of excitons is followed by the coherent (localized) excitonic packet. This tail might be explained by instability of the exciton-phonon condensate moving in the lattice. Indeed, such an instability can lead to continuous emission of the excitons out from the condensate. As $\partial_t N_{\mathrm{o}} < 0$, we have $\partial_t L_{\mathrm{ch}}(t) > 0$ and, as a result, nonstationary transport of the ballistic condensate. Alternatively, the condensate is stable, and diffusive propagation of the non-condensed excitons, $\Delta N \simeq N_{\mathrm{x}} - N_{\mathrm{o}}$, is responsible for the tail in $n(x,t)$. We argue that the second scenario seems to be true.

For outside collective excitations (see Fig. 1), the asymptotics of the low-lying energy spectrum can be found easily. Indeed, if we assume that $\phi_{\mathrm{o}}^2(x) \approx 0$ and $\partial_x u_{\mathrm{o}}(x) \approx 0$ in the outside packet area, the excitonic and phonon branches are (formally) uncoupled. Then, for the excitonic branch, we obtain in the co-moving frame

$$\hbar\omega_{\mathrm{ex}}(\mathbf{k}) \approx |\mu| + (\hbar^2/2m)k^2, \quad \mathrm{u}_{\mathbf{k}}(\mathbf{x}) \approx u_{\mathbf{k}}\, \mathrm{e}^{i\mathbf{kx}}, \quad \mathrm{v}_{\mathbf{k}}(\mathbf{x}) \approx 0, \quad |x| \gg L_0, \tag{15}$$

where $\mathrm{u}_{\mathbf{k}}$ and $\mathrm{v}_{\mathbf{k}}$ are Bogoliubov-deGennes amplitudes, and $\omega_{\mathrm{ph}}(\tilde{\mathbf{k}}) = c_{\mathrm{s}}|\tilde{\mathbf{k}}|$ in the laboratory frame of reference. Note that the condition $\hbar\omega_{\mathrm{ex}}(\mathbf{k}) + \hbar\, k_x\, v > 0$ can be violated if the velocities get close to v_{o}. This is the hint that the instability regime can occur [8].

We assume that the exciton and phonon can be emitted from the condensate coherently. Then the emission of δN excitons out of the coherent packet can be described as an appearance of δN outside collective excitations in the system condensate plus medium, see Fig. 1. Therefore, one can use Landau arguments to analyze the stability of the condensate. It turns out that the moving condensate is stable against the direct emission of outside excitations. However, this is not the case for the inside excitations. In the laboratory frame, these excitations can be represented in the form

$$\delta\psi(\mathbf{x},t) \sim \mathrm{u}_k(x-vt)\, \mathrm{e}^{i(\varphi_0+\mathbf{kx})}\, \mathrm{e}^{-i(\omega_k+k_x v)t} + \mathrm{v}_k(x-vt)\, \mathrm{e}^{-i(\varphi_0+\mathbf{kx})}\, \mathrm{e}^{i(\omega_k+k_x v)t},$$

and

$$\delta u(\mathbf{x},t) \sim C_k(x-vt)\,\exp(i(\varphi_0+\mathbf{kx}))\,\exp(-i(\omega_k+k_x v)\,t)+\text{c.c.}, \tag{16}$$

and the following asymptotic behavior is valid for the Bogoliubov-deGennes amplitudes ($\phi_{\rm o}(x/L_0) \sim \exp(-|x|/L_0)$):

$$\mathrm{u}_k(x) \sim \mathrm{v}_k(x) \sim \phi_{\rm o}(x/L_0), \quad C_k(x) \sim \phi_{\rm o}^2(x/L_0) \;\; \text{at} \;\; |x| > L_0.$$

We can start from the standard criterion,

$$\hbar(\omega^{(-)}_{\mathrm{ex},\,k} - |k_x|v) > 0 \quad \text{at} \quad |k_x| \simeq z\,L_0^{-1}, \tag{17}$$

where the superscript $(-)$ means $k_x < 0$ and $z \simeq 3 \sim 10$ corresponds to the low-lying inside excitations, and find under which conditions inequality (17) is valid.

It turns out the analog of Landau criterion for Eq. (17) is rather simple [9]

$$|\tilde{\mu}|(N_{\rm o},\,v) > \mu_{\rm cr} \simeq \text{const}\,(2\,m_{\rm x}c_{\rm s}^2), \tag{18}$$

where $m_{\rm x} \simeq 2.7\,m_{\rm e}$, $c_{\rm s} \simeq 4.5\cdot 10^5$ cm/s. For $z \simeq 3 \sim 10$, we estimate const(z) in Eq. (18) as of the order of 10^{-1}, and we obtain the following result

$$\mu_{\rm cr} \simeq 10^{-5} - 10^{-4}\,\text{eV} \simeq (10^{-4} - 10^{-3})\,E_{\rm x}. \tag{19}$$

Qualitatively, inequality (18) is valid for the relatively high velocities and numbers of the condensed particles. In theory, one can fix the number $\bar{n}_{\rm o} = N_{\rm o}^*/N_{\rm o}$ and obtain an analog of Landau critical velocity, but $v_{\rm cr} < v < c_{\rm s}$ is the stability criterion in this case.

However, within approximation (12),(14) and with $|\tilde{\varepsilon}_0(v/c_{\rm s})| \simeq 10^{-1}\,E_{\rm x}$ and $\bar{n}_{\rm o} \simeq 10^1$, we have $|\mu| \simeq \mu_{\rm cr}$ and $L_{\rm ch} \simeq 4\cdot 10^2\,a_{\rm x} \simeq (2-3)\cdot 10^3$ Å. We speculate that such a ballistic condensate can be considered as a stable one in the limit of $T \to 0$, see Fig. 1. For comparison, the condensates with $|\tilde{\varepsilon}_0| \simeq 10^{-2}\,E_{\rm x}$ and

$\bar{n}_o \geq 50 - 100$ seems to be unstable against the inside excitations. If this is the case, the continuous generation of the low-lying inside excitation takes place,

$$N_o \to N_o - \delta N(t), \quad \partial_t\, \delta N_{\text{in-ex}}(t) > 0, \quad \delta N_{\text{in-ex}}(t) > \sqrt{N_o}.$$

This process can be accompanied by the emission of outside excitations as well.

To conclude, more accurate investigation of the stability problem is necessary. Indeed, the characteristic width of the quasi-stationary solution near the threshold of stability is of $(1-3)\cdot 10^3$ Å, whereas the typical length of the crystal used for experiments is of $(2-4)\cdot 10^{-1}$ cm. We speculate that the ballistic (superfluid) propagation of the condensate is more than changing $x \to x - vt$ and adding $\exp(ik_0 x)$ to the solution of Eqs. (7) and (8). For example, the localized solitonic solution (5),(6) can be used as a reasonable initial condition for modeling of how the coherent part of exciton-phonon packet actually moves in the presence of thermal phonons and/or point scattering centers, etc..

4 Discussion

Recall that the self-consistent exciton-phonon condensate seems to be only a part of the real moving packet. The noncoherent part of it, the noncondensed excitons $\Delta n(x,t)$ and the unidirectional phonon wind $\Delta u(x,t)$, effects the propagation of the condensate. Here we address the question on whether it is possible to diminish (ideally, to turn off) the phonon wind after the moving exciton-phonon condensate has been formed. As a result, the diffusive noncondensed excitons can be delayed, and the coherent signal and the noncoherent one are separated in time.

We consider a system consisting of a crystal (semiconductor) and two ideal conductors (metals). The geometry of such a system is shown on Fig. 2. Let $l = |\text{db}| = |\text{ac}|$ be the distance between two conducting planes surrounding part A′B of the crystal. Let λ be a wavelength of the light emitted during the exciton recombination. (Usually, λ lies in the visible wavelength area, $\simeq 500\,\text{nm}$.)

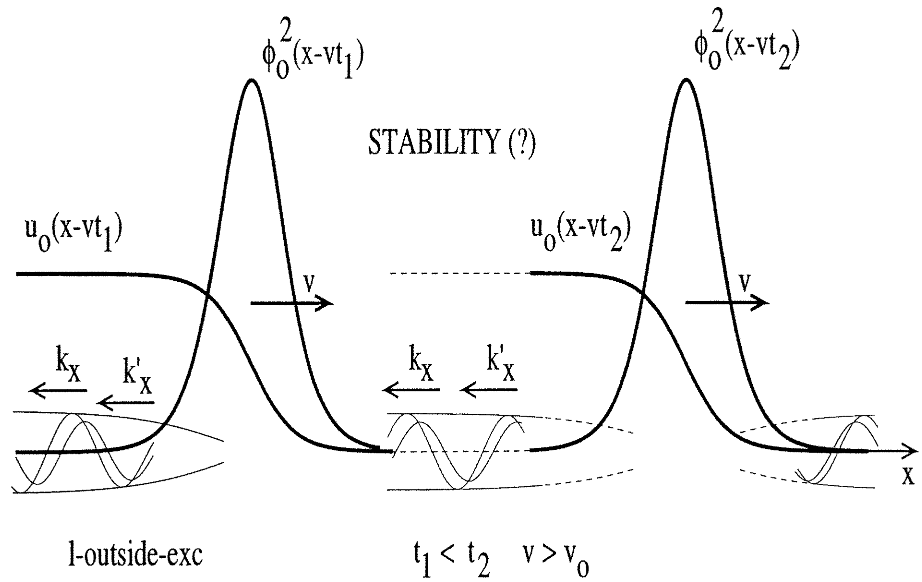

Figure 1: Stability against creation of excitations

The ballistic condensate, $\phi_o(x - vt) \cdot u_o(x - vt)\delta_{1j}$, is stable in relation to the direct emission of outside exciton-phonon excitations. (We consider the backward emission in the long-wavelength limit, $|k_x| \simeq z\, L_0^{-1}$ and $z \ll 1$.) The outside excitations presented on this figure are labeled by the wave vectors, k_x, $k'_x < 0$ in the co-moving frame. To a first approximation, the outside excitations can be described in terms of free excitons and free (acoustic) phonons emitted from the condensate coherently.

If the relation $l \leq \lambda \ll D$ can be realized by appropriate changes in the growing technique, an interesting physics of penetration into a channel appears. Here $D \simeq 1 - 2\,\mathrm{mm}$ is the diameter of an excitation spot on the surface A of the crystal, $D \simeq \sqrt{S}$.

Note that due to boundary conditions on the conducting walls, it is impossible to emit light with the wave vector less than π/l and the quantum energy less than $\hbar c\,\pi/l$. If the last value is larger than the energy of the electron-hole recombination, the latter is suppressed in the region abcd. Then the lifetime of excitons (both free and bound ones) can be increased by several orders of magnitude.

We are interested in evolution of the exciton-phonon system when the cloud reaches $x =$ B (see Fig. 2), and the excitons begin to move in channel abcd. Roughly speaking, we have such a connection problem, in which the boundary conditions and the characteristic length scales (L_y in our case) can be quite different for the 3D crystal and the channel, respectively. As boundaries ab and cd of the channel are metallized, the boundary conditions of the displacement field $\mathbf{u}(x, y, t) \approx (u_x, 0, 0)$ can be taken zero along them.

We consider transition of the phonon wind from part AB of the crystal to the channel by posing the time-dependent boundary condition at the connection B | bd (see Fig. 2):

$$\partial_x u(x = \mathrm{B}, y, t) = f(y, t) \sim \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-i\omega t}\, f(\omega), \tag{20}$$

where $f(\omega)$ is localized inside $|\omega| < \omega^*$. For the displacement field inside the channel, we can write the following representation,

$$u(x, y, t) = \sum_{n=\mathrm{odd}}^{\infty} u_n(x, t) \cos\frac{\pi n y}{l}, \quad |y| < l/2, \quad u_n(x, t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \tilde{A}_n(\omega)\, \mathrm{e}^{ik(\omega)x - i\omega t}. \tag{21}$$

Obviously, all the modes in Eq. (21) with frequencies less than $\omega_0 \simeq c_{\mathrm{s}}\pi/l$ are dumped inside the channel. Let the following inequality be valid

$$\omega^* \simeq \frac{2\pi}{\tau} \simeq \frac{2\pi c_{\mathrm{s}}}{\Lambda} < (\ll)\, \omega_0 \simeq \frac{\pi c_{\mathrm{s}}}{l}.$$

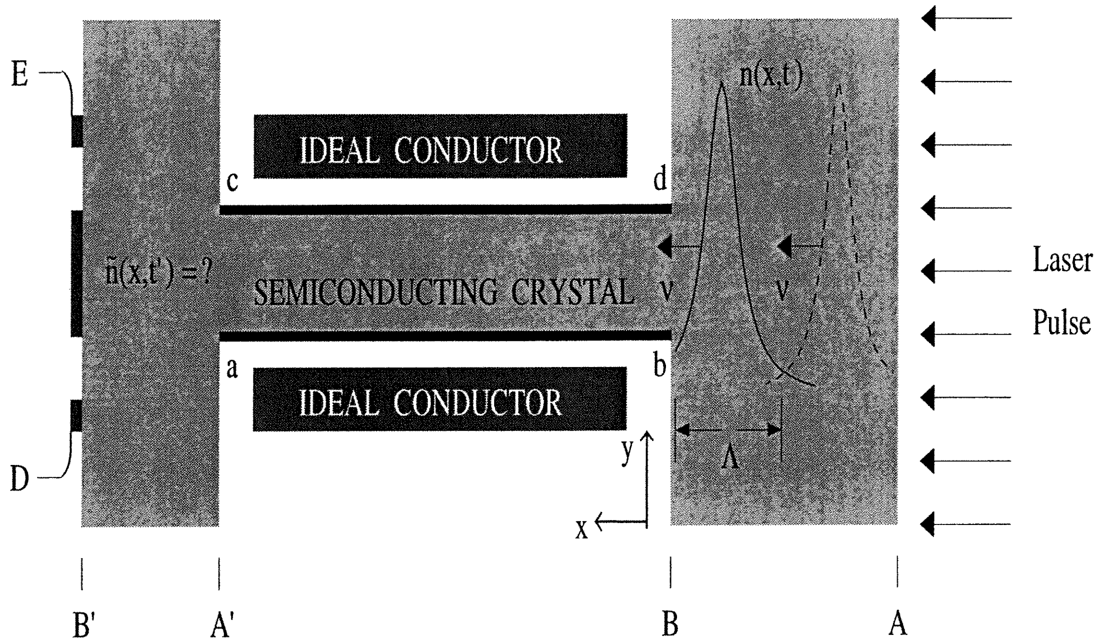

Figure 2: Packet enters the channel of a special geometry.

In order to separate the coherent and noncoherent components of the total exciton-phonon packet at $T \ll T_c$, it is crucial to turn off a phonon wind. (The phonon wind, or a sequence of nonequilibrium acoustic phonons, "blows" unidirectionally from surface A of the crystal.) This can happen when the packet $n(x,t)$ enters the channel abcd of the width l in the y direction. The characteristic lengths of the total packet and the condensate, Λ and L_0, respectively, satisfy the inequality $L_0 < l < \Lambda$. Then, the condensate could move ballistically through the channel with the velocity v, whereas the non-condensed excitons will diffuse (almost) freely inside the channel because the phonon wind cannot penetrate into it. As a result, the (total) excitonic current $\tilde{n}(x,t') \approx \tilde{n}_o(x - vt') + \Delta\tilde{n}(x,t')$ from the output of the channel ($x = \text{ac}$) can be converted into the electric current near the surface B'; $|\text{A}'\text{B}'| \ll |\text{AB}|$.

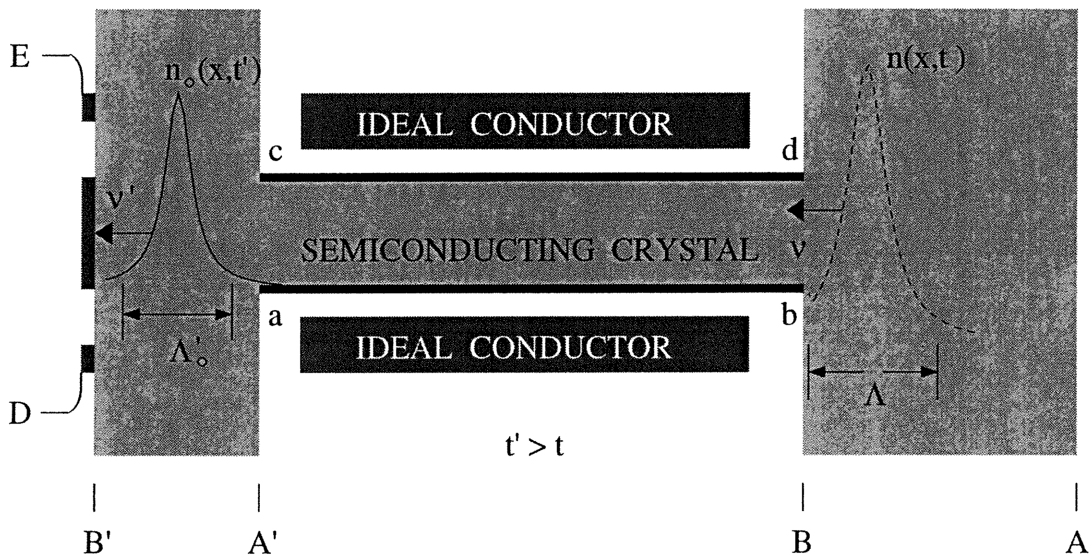

Figure 3: Survival of the coherent part of the packet.
We assume that the transition from the 3D crystal AB to the channel abcd can be done smoothly enough in order not to create turbulence in the condensate penetrating the channel. Then, one can expect the solitonic shape of the coherent part of the droplet at the end of the channel. (The noncoherent part at $t = t'$ is not shown on this figure.)

Here, τ and Λ can be estimated from the duration and the characteristic width of the (total) exciton-phonon packet driven by the phonon wind, $\tau \simeq (1 \sim 5) \cdot 10^{-7}\,\mathrm{s}$ and $\Lambda \simeq 10^{-2}\,\mathrm{cm}$. Then, only the "tails" of $f(\omega)$ are actually transfered into the amplitudes $\tilde{A}_n(\omega)$ of the moving packet $u_n(x,t)$. Hence, the phonon drag force ($\propto \partial_x u(x,y,t)$) can be essentially suppressed inside the channel, and the non-condensed excitons can diffuse almost freely there.

We speculate that the exciton-phonon condensate being formed in AB part of the 3D crystal can, first, penetrate into and, second, pass through the channel (see Fig. 3). This problem seems to be not a quasi-stationary one. Indeed, within approximation $\Psi_0 = \psi_{\mathrm{o}}(x,y,t)$ and $\mathbf{u}_0 = (u_{\mathrm{o}\,x}(x,y,t),\, 0,\, 0)$, we have the following system to describe the dynamics at $x > \mathrm{B}$:

$$i\hbar\partial_t\psi_{\mathrm{o}}(x,y,t) = \Big(-\frac{\hbar^2}{2m_{\mathrm{x}}}\Delta_{x,y} + \nu_0|\psi_{\mathrm{o}}|^2(x,y,t) + \nu_1|\psi_{\mathrm{o}}|^4(x,y,t) \Big)\psi_{\mathrm{o}}(x,y,t) + \tag{22}$$
$$+\sigma_0\,\partial_x u_{\mathrm{o}\,x}(x,y,t)\,\psi_{\mathrm{o}}(x,y,t),$$

$$(\partial_t^2 - c_{\mathrm{s}}^2\Delta_{x,y})u_{\mathrm{o}\,x}(x,y,t) - c_{\mathrm{s}}^2\sum_{j=x,y} 2\kappa_3\,\partial_j^2 u_{\mathrm{o}\,x}\,\partial_j u_{\mathrm{o}\,x}(x,y,t) = \rho^{-1}\sigma_0\,\partial_x(|\psi_{\mathrm{o}}|^2(x,y,t)), \tag{23}$$

$$\psi_{\mathrm{o}}(x,-l/2,t) = \psi_{\mathrm{o}}(x,l/2,t) = 0, \quad u_{\mathrm{o}\,x}(x,-l/2,t) = u_{\mathrm{o}\,x}(x,l/2,t) = 0. \tag{24}$$

Boundary conditions (24) model the metallized surfaces of the channel.

Here, we model a kind of smooth penetration, for example, no vortexes are generated near the 3D-channel intersection. Recall that the width of the condensate in our quasi-1D model is $L_0 \simeq (10^2 - 10^3)\,a_{\mathrm{x}} < \Lambda$, and $\tau_0 \simeq L_0/c_s \simeq 10^{-10} - 10^{-9}\,\mathrm{s} < \tau$. As $L_0 < l < \Lambda$ and $\omega^* < \omega_0 < \tau_0^{-1}$, we assume that the localized nonlinear condensate can penetrate into the beginning of the channel without the loss of its coherence. Then, instead of solving Eq. (22)-(24) with "pumping", (i.e., time dependent) boundary conditions at $x = \mathrm{B}$, one can transform the quasi-stationary solution (5) and (6) into the initial condition inside the channel and choose $t = 0$. For example, one can use the following trial functions:

$$\psi_o(x,y,0)\cdot u_o(x,y,0)\,\delta_{1j} =$$

$$= \exp(im_x vx)\,\Phi_o\cosh^{-1}(L_0^{-1}x)\,\bar{\phi}(y)\cdot\Big(Q_o\,-\,Q_o\tanh(L_0^{-1}x)\Big)\,\bar{Q}(y), \qquad (25)$$

where $\bar{\phi}(y)$ and $\bar{Q}(y)$ have to satisfy the boundary conditions (24) and be self-consistent as well. We assume that near the boundaries they have the same coherence length $L_{0\,y}$. It can be estimated as [15]

$$L_{0\,y}^{-1} \simeq \sqrt{\nu_0\,|\phi_o(x)|^2\,m_x}/\hbar\;\to\;\sqrt{(2m_x/\hbar^2)\,\nu_0\,\Phi_o^2/2}\,\cosh^{-1}(L_0^{-1}x\,). \qquad (26)$$

Near the soliton maximum ($x\,=\,0$), we obtain $L_{0\,y}\,\simeq\,L_0$ (see Eq. (14) and $\nu_0 \simeq |\tilde{\nu}_0|$ is valid in the stability area).

We estimate that as many as $\simeq N_o\,(l/D)$ of the N_o original Bose condensed excitons can penetrate from the crystal into the channel if only $l/2 > L_{0\,y}(x)$ for $|x| < (2 \sim 4)L_0$. Then, the BEC conditions can be saved for the droplet inside the channel at $T \neq 0$, and the exciton-phonon condensate can move through the channel *ballistically* (Fig. 3). However, the question of stability of such a motion remains open [16]. Experimentally, the task is to register the signal from the output of the channel, $x =$ ac. For example, a small 3D part could be attached to the channel in order to convert the excitonic mass current into a measurable electric signal, see Figs. 2 and 3.

5 Conclusion

In conclusion, we note that the phonons play a crucial role in almost all the current models aimed to explain or predict coherent behavior of excitons in semiconductors, see, f.ex., [10],[17],[18]. Although our model of the exciton-phonon condensate fails to predict the width of the exciton-phonon packet correctly, the theory yields a qualitative description of the experiments and a reasonable value for the critical velocity. However, one has to take into account the thermal excitons (e.g., the weak tail that is always observed behind the soliton [1]) and the thermal phonons of the crystal to make the model *with condensate* more realistic.

References

[1] A. Mysyrowicz, E. Benson, and E. Fortin, Phys. Rev. Lett. 77, 896 (1996); E. Benson, E. Fortin, and A. Mysyrowicz, Sol. Stat. Comm. 101, 313 (1997); E. Benson, E. Fortin, B. Prade, and A. Mysyrowicz, Europhys. Lett. 40, 311 (1997).

[2] J. L. Lin, J. P. Wolfe, Phys. Rev. Lett. 71, 122 (1993).

[3] T. Goto, M. Y. Shen, S. Koyama, T. Yokouchi, Phys. Rev. B 55, 7609 (1997).

[4] J. P. Wolfe, J. L. Lin, D. W. Snoke and by A. Mysyrowicz in: Bose Einstein condensation, A. Griffin, D. W. Snoke and S. Stringari, eds. (Cambridge, New York, 1995).

[5] J. Fernández-Rossier and C. Tejedor, Phys. Rev. Lett. 78, 4809 (1997); J. Fernández-Rossier, C. Tejedor, R. Merlin, cond-mat/9909232.

[6] T. Portengen, Th. Östreich, and L.J. Sham, Solid State Commun. 100, 325 (1996).

[7] L. V. Butov, A. I. Filin, Phys. Rev. B 58, 1980 (1998).

[8] I. Loutsenko, D. Roubtsov, Phys. Rev. Lett. 78, 3011 (1997).

[9] D. Roubtsov, Y. Lépine, Phys. Stat. Sol. B 210, 127 (1998); D. Roubtsov, Y. Lépine, Phys. Rev. B, (2000); cond-mat/9910079.

[10] Yu. E. Lozovik, A. V. Poushnov, JETP 88, 747 (1999).

[11] A. Griffin, Phys. Rev. B 53, 9341 (1996); cond-mat/9901172.

[12] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).

[13] G. A. Kopelevich, S. G. Tikhodeev, and N. A. Gippius, JETP 82, 1180 (1996); S. G. Tichodeev, Phys. Rev. Lett. 78, 3225 (1997).

[14] G. Huang and B. Hu, Phys. Rev. B 58, 9194 (1998).

[15] Al. L. Fetter and J. D. Walecka, Quantum Theory of Many-Particle System, (McGrav-Hill, New York, 1971).

[16] D. Roubtsov, Y. Lépine, Phys. Lett. A 246, 139 (1998); cond-mat/9807023.

[17] W. Zhao, P. Stenius, A. Imamoḡlu, Phys. Rev. B **56**, 5306 (1997).

[18] H. Haug, A. L. Ivanov, and L. V. Keldysh, *Nonlinear Optical Phenomena in Semiconductors and Semiconductor Microstructures*, (World Scientific, 1999).

Chapter 4

Critical velocity in the theory of superfluidity

In this chapter, we discuss a new model of the dissipation of a superflow. We were motivated by the ballistic movement of the exciton-phonon condensate discussed in the previous chapters. In general, it is hard to believe that vortexes could be responsible for dissipation in the anomalous transport in crystal structures interpreted in terms of the superfluidity of excitons [43].

Recall that the superfluidity as a frictionless flow ceases to exist if the velocity of the moving liquid exceeds some critical value. The creation of vortexes is believed to be the main mechanism of dissipation in the case if the superfluid liquid moves relatively fast [44].

Instead of the Bose-liquid, we consider a Bose-gas with the two-particle interaction $U(\mathbf{x}, \mathbf{x}') = \nu_0\, \delta(\mathbf{x} - \mathbf{x}')$ in this chapter. It can also move without friction in a channel as it was shown by N. N. Bogoliubov long time ago [21]. However, the coupling between long-wavelength elementary excitations of the coherent bosons of the Bose-gas (restricted by the channel walls within which it moves) and the surface phonons of these walls, e.g., $\hat{H}_{\mathrm{x-ph}} \sim \int d\mathbf{x}\, d\mathbf{x}'\, \mathcal{U}(\mathbf{x},\, \mathbf{x}')\, \psi^{\dagger}\psi(\mathbf{x})\, \nabla\mathbf{u}(\mathbf{x}')$, can lead to the existence of a new branch of excitations. This branch is explored in the article included into this chapter.

On Reduction of Critical Velocity in a Model of Superfluid Bose-gas with Boundary Interactions

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Abstract

The existence of superfluidity in a 3D Bose-gas can depend on boundary interactions with channel walls. We study a simple model where the dilute moving Bose-gas interacts with the walls via hard-core repulsion. Special boundary excitations are introduced, and their excitation spectrum is calculated within a semiclassical approximation. It turns out that the state of the moving Bose-gas is unstable with respect to the creation of these boundary excitations in the system gas + walls, i.e. the critical velocity vanishes in the semiclassical (Bogoliubov) approximation. We discuss how a condensate wave function, the boundary excitation spectrum and, hence, the value of the critical velocity can change in more realistic models, in which "smooth" attractive interaction between the gas and walls is taken into account. Such a surface mode could exist in "soft matter" containers with flexible walls.

Keywords: Nonideal Bose-gas, Boundary Interactions, Surface Excitations, Critical Velocity, Superfluidity.

PACS numbers: 05.30.Jp, 64.60.Ht, 67.40.Db

1 Introduction

Recent experiments on Bose-Einstein Condensation (BEC) in magnetically trapped gases [1],[2], and excitons in semiconductor crystals and nanostructures [3],[4],[5] have made the subject of BEC more vital, more interdisciplinary. Many new questions appeared naturally as the understanding of the process of BEC progressed [6]. However, some "old" problems – such as the kinetics of BEC, the nature of superfluidity, the critical velocity problem, etc. – still remain the subject under consideration [7], (especially for new physical objects where the state of BEC was recently demonstrated [6]). Although the critical velocities are one of the first difficulties to be encountered in the study of superfluidity, they are still the least understood aspect in the theory [8],[9],[10].

This article is motivated mainly by the problem of critical velocity(-ies) in exciton superfluidity. It has been found experimentally [4],[11],[12] that a cloud of condensed excitons moves through a crystal with some constant velocity and some characteristic shape of the density profile. Several theoretical explanations of this anomalous transport have been put forward [13],[14], [15]. In spite of the fact that these explanations are based on different assumptions, there are several common ideas in the background of all these theories. For instance, it is the notion that interaction with a lattice is very important (if not to say crucial) in the BEC of excitons [15],[16],[17].

However, there are many outstanding questions that remain the subject of discussion [18]. One of such questions, for example, is the superfluid nature of exciton anomalous transport. In fact, it is not clear how the exciton condensate "feels" the boundary of the crystal (via interaction with surface phonons, e.g.,) or the impurities and other lattice imperfections that can bound an exciton. Generally speaking, clarification of the role of these friction sources may be essential for the understanding of the exciton superfluidity and the nature of critical velocities.

We approach the critical velocity problem by working out a simple model in which dilute 3D Bose-gas moves in a channel and interacts with the walls of this channel. The walls are modeled as two 3D solid bodies with well-defined boundaries. Although we take into account repulsive interaction between the particles of the gas, the proposed model cannot describe, for example, the superfluid He, which is a Bose liquid with strong interparticle interaction. Yet this is not the aim of this article. The main goal of this study is to explore the space of manæuvre appearing in the framework of the well known simple models, such as the weakly nonideal Bose-gas, if we switch on the gas-wall boundary interaction.

We show that the existence of the repulsive interactions between the Bose-gas and the channel walls leads to the essential reduction of the critical velocity of the superflow. The finiteness of the Landau critical velocity for the bulk (i.e. Bose-gas) excitations turns out not to be a sufficient condition of superfluidity. Note that in the present work we investigate the boundary excitations. Although the breakdown of the superfluidity is assumed to be accompanied by vortex emission, we leave for future studies the questions of the vortex formation and their dynamics in the case when interaction with walls is taken into account.

2 Critical Velocity Problem

A closed system cannot undergo an inner macroscopic motion in thermodynamic equilibrium. Once such a motion is present, the system must evolve toward an equilibrium state. However, unless this transition is kinematically prohibited, (i.e. incompatible with conservation laws), the macroscopic motion is sustained. Such is the case with a small object moving without any viscous drag in stationary superfluid [19],[20]. This object is assumed to have no inner degrees of freedom, so that its momentum and energy depend only on the velocity $\mathbf{v}$ of the center of mass.

If the conservation laws for the creation of an excitation with the energy $\epsilon_{\text{gas}}(k)$ and the momentum $\mathbf{k}$ in a fluid (gas) lead to [19]-[21]

$$\epsilon_{\text{gas}}(k) > 0, \text{ in the object reference frame,} \tag{1}$$

the particle will continue to move without any experience of drag forces. Condition (1) is known as the Landau criterion of superfluidity. In fact, Eq. (1) holds that [19]-[21]

$$v < v_{\mathbf{L}} = (\epsilon_{\text{gas}}(k)/k)_{\text{min}} \,. \tag{2}$$

Here $v_{\mathbf{L}}$ is the Landau critical velocity.

Formula (2) is in agreement with experiments performed with the semimicroscopic objects moving in the liquid helium [21].

Formula (1), (taken in the channel reference frame), is employed as a criterion of Bose-gas superfluid flow in channels. In that case, the channel walls are regarded as a massive macroscopic body in the above consideration (i.e. the walls act as some source of perturbations on the gas flow), and the final result is formulated in the form (2).

On experiments with liquid helium flow, however, the registered values of the critical velocities turn out to be much smaller than $v_{\mathbf{L}}$. Moreover, the critical velocity depends on the channel dimensions [22]. The fact that the liquid superfluid helium could not be treated as a dilute Bose-gas is believed to be the main reason for this discrepancy. It is generally assumed that the critical velocities are related to the appearance of quantized vortex lines in the superfluid. The Landau criterion (1) applied to the vortex excitations [23] can explain the critical effects in circular geometries. However, it cannot account for the drastically different critical velocities for rotation and linear flows [24].

A superflow of a dilute Bose gas, described by nonlinear Schrödinger equation [25], has been studied recently in different geometries with the use of direct numerical methods [26]-[27]. It has been observed that the distinct critical velocity is linked to the emission of vortices. This velocity turns out to be equal to the Bogoliubov [28] velocity of sound propagation in the gas. It is in good agreement with the criterion (1) since the Dirichlet boundary conditions (for Bose-gas wave function) are imposed on the channel walls [27]. This means that the walls, being considered as rigid immovable bodies, have no degrees of freedom. As a consequence, the arguments leading to formulas (1),(2) can be used.

In reality, the channel walls have a large number of degrees of freedom. Indeed, short- and long-range forces between a particle and a surface, boundary and interface phonons are well known subjects in the surface physics [29],[30]. This means that the (superfluid) Bose-gas is coupled with the channel, in which the gas moves. Then special boundary excitations can exist in the system of Bose-gas + channel walls because of the coupling between, say, the surface phonons of the walls and the Bogoliubov phonons of the Bose-gas. Therefore, the Landau criterion in the form (1) cannot be applied; it has to be modified. To get an analog of it we use the laws of conservation, taking the (inner) walls' degrees of freedom into consideration. The superflow can exist, provided the following condition holds:

$$\epsilon(k) > 0, \text{ in the channel (laboratory) reference frame,} \tag{3}$$

where $\epsilon(k)$ is the energy of *any* elementary excitation of the *whole* (gas + walls) system.

Condition (3) means that the state of the system can not be changed, since the occurrence of any number of elementary excitations leads to the increase of a total energy, but the latter is prohibited by the low of conservation. The excess of the momentum is "absorbed" by the motion of the center of mass of the walls. The energy is not actually changed by this motion (in the channel reference frame) because of a large mass of the walls and their zero initial velocity.

3 Bose-Gas with Boundary Interactions

We study the model of the dilute Bose-gas in the channel of the width $2l$ (in y-direction) and of infinite length (in x- and z-directions). The channel walls occupy the $|y| > l$ part of space (see Fig. 1). The general structure of the Hamiltonian is the following:

$$\hat{H} = H_{\text{gas}}(\hat{\psi},\, \hat{\psi}^{\dagger}) + H_{\text{ph}_1}(\hat{q}_1,\, \hat{\pi}_1) + H_{\text{int}1}(\hat{q}_1,\, \hat{\psi}^{\dagger}\hat{\psi})+$$

$$+H_{\text{ph}_2}(\hat{q}_2,\, \hat{\pi}_2) + H_{\text{int}2}(\hat{q}_2,\, \hat{\psi}^\dagger\hat{\psi}), \tag{4}$$

where $\hat{\psi}$ is the Bose-gas field operator, $\hat{q}$ is the displacement field operator of a wall, $\hat{\pi}$ is the momentum density operator conjugate to $\hat{q}$, and the indexes 1,2 correspond to the upper and lower part of the channel respectively. In the model being considered, the Bose-gas Hamiltonian has the following form:

$$H_{\text{gas}} = \int \hat{\psi}^\dagger(\mathbf{r}) \left(-\frac{\hbar^2}{2m}\Delta\right) \hat{\psi}(\mathbf{r})\, d\mathbf{r} + \int \frac{\nu}{2}\, \delta(\mathbf{r}-\mathbf{r}')\, \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r}')\hat{\psi}(\mathbf{r})\, d\mathbf{r}d\mathbf{r}',$$

where $\nu > 0$ is the interparticle interaction constant [20], and the wall Hamiltonian can be written as follows

$$H_{\text{ph}} = \int \frac{\hat{\boldsymbol{\pi}}^2(\mathbf{r})}{2\rho} + \partial_j\hat{q}_k(\mathbf{r})\lambda_{jkln}\partial_l\hat{q}_n(\mathbf{r})\, d\mathbf{r},$$

where the tensor λ_{jkln} describes the elastic properties of the channel walls.

We derive the excitation spectrum using the techniques of semiclassical approximation (cf. [28],[31],[32],[33]). Expanding the field operators near certain classical solutions, i.e. $\hat{\psi} = \psi_0 + \delta\hat{\psi}$ and $\hat{q} = q_0 + \delta\hat{q}$, we represent the Hamiltonian in the form

$$\hat{H} = H_0 + \hbar\hat{H}_2 + \dots, \tag{5}$$

where H_0 stands for the classical part of $\hat{H}$. Note that $\psi_0 \neq 0$ indicates the existence of a condensate in the moving Bose-gas, whereas $q_0 \neq 0$ appears in this model mainly to satisfy the boundary conditions (see below). The Hamiltonian $\hat{H}_2$ in (5) is bilinear with respect to the field operators. As a consequence this (semiclassical) Hamiltonian can be reduced to the normal form

$$\hat{H}_2 = \sum_i \omega_i\hat{b}_i^\dagger\hat{b}_i + \text{const}, \; \left[\hat{b}_i, \hat{b}_j^\dagger\right] = \delta_{ij}, \; \left[\hat{b}_i, \hat{b}_j\right] = 0. \tag{6}$$

The quantum Heisenberg and the classical Poisson-Hamilton equations of motion for the field operators (functions),

$$i\hbar\,(d\hat{b}_i/dt) = \left[\hat{b}_i, \hat{H}\right], \quad (db_i/dt) = \{b_i, H\}, \tag{7}$$

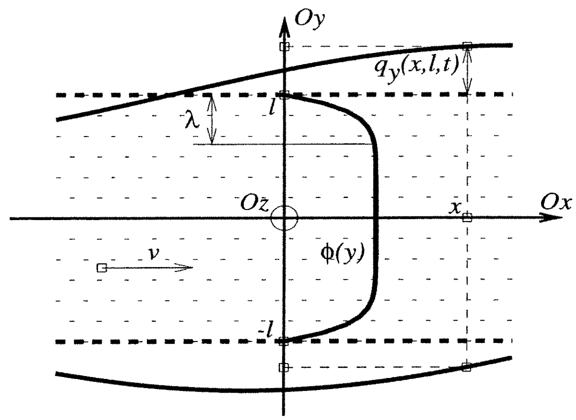

Figure 1: Bose-gas inside a channel.
The Bose gas moves with the velocity $\mathbf{v}$ in a channel of the width $2l$. The profile of the stationary wave function $\phi(y) = |\psi_0(x, y, t)|$ is shown in the center, where λ is the coherence length. The top and bottom curves depict the boundaries of the walls interacting with the gas, while the horizontal bold dashed lines correspond to the unperturbed boundaries, and $q_y(x, \pm l, t)$ are deviations from this equilibrium.

have the same form, if we neglect the terms of power in $\hbar$ greater than one in Eq. (5). These equations are linear with respect to the field variables.

It follows from (6), (7) that the excitation energies ω_i are equal to the characteristic frequencies of these equations. Thus, to determine the semiclassical energy spectrum of the system we need to find the characteristic frequencies of the classical field equations for $\delta\psi$ and δq linearized around a proper stationary solution ψ_0, q_0. (This means that $\delta\psi$ and δq, originally of the operator nature, can be treated as a c-number and a real number function respectively). Then the gas state can be characterized by the Bose wave function $\psi(\mathbf{r}, t) = \psi(x, y, t)$, while the state of the walls is determined by the displacement field $\mathbf{q}(\mathbf{r}, t) = (q_x(x, y, t),\ q_y(x, y, t),\ 0)$. For simplicity, the system is asumed to be homogeneous in the z-direction.

The interaction between the wall and gas atoms is given in our model by the sum of hard-core repulsion and some "smooth" potential $\mathcal{U} = \mathcal{U}(\mathbf{r},\, \psi^*\psi,\, \mathbf{q})$. The hard-core repulsion makes the walls impenetrable for the gas. It follows that the wave function of the Bose-gas vanishes along the actual boundaries $y_\pm = \pm l + q_y(x, \pm l, t)$ (see Fig. 1).

$$\psi\left(x, \pm l + q_y(x, \pm l, t), t\right) = 0. \tag{8}$$

The other boundary conditions, namely

$$\sigma_{yy}(x, \pm l, t) = \frac{\hbar^2}{m} \partial_y \psi^*\, \partial_y \psi(x, y = \pm l, t), \tag{9}$$

$$\sigma_{xy}(x, \pm l, t) = 0, \tag{10}$$

correspond to the equality of the forces between the solid and gas on the boundary. In (9), (10) σ_{ij} denotes the wall stress tensor and m is a mass of the gas atom. The pressure of the gas (r.h.s. of (9)) is equal to the normal component of the stress tensor (l.h.s. of (9)). Eq. (10) follows from the fact that the tangent stress vanishes on the boundary.

The wave function ψ of the repulsive Bose-gas satisfies the nonlinear Schrödinger equation [25]

$$\left\{ i\hbar\partial_t + \frac{\hbar^2}{2m}\Delta - \nu\psi^*\psi \right\} \psi = V(\mathbf{r},\, \partial_j q_k)\, \psi, \quad |y| < l, \tag{11}$$

while the wall dynamics obeys the hyperbolic equation for the displacement field $\mathbf{q}$ [34],

$$\rho\partial_t^2 q_i = \partial_j \sigma_{ij} - \partial_i W(\mathbf{r},\, \psi^*\psi), \quad |y| > l, \tag{12}$$

(ρ denotes the wall mass density). The potential V and the potential density W depend on the "smooth" part of the gas-wall interaction $\mathcal{U}$; they must vanish if $\mathcal{U} = 0$. To make our model as simple as possible, we set $\mathcal{U} = 0$ *a priori*. Then, the dynamics is described by equations (11), (12) with constant coefficients, whereas the hard-core interactions fix the boundary conditions (8)-(10). Note that these conditions imply that the walls have a finite compressibility, $K = -V(\partial_V p)_S < \infty$. Therefore, even though $\mathcal{U} = 0$, the wall and Bose-gas excitations can be coupled because $(\partial_y \psi)\, q_y(x, \pm l)$ in (8) and $\sigma_{yy} \sim K \partial_y q_y(x, \pm l)$ in (9) are finite and time dependent.

We suppose that the walls are isotropic (this assumption does not affect the results qualitatively, simplifying our calculations), so that [34]

$$\sigma_{ij} = \rho c_t^2 \left(\partial_j q_i + \partial_i q_j + (\beta - 1)\delta_{ij}(\nabla \mathbf{q})\right), \ \beta = (c_l^2/c_t^2) - 1, \tag{13}$$

where c_l, c_t are longitudinal and transversal sound velocities respectively. It is convenient to rescale variables in such a way that the spatial coordinates are expressed via the coherence length λ units while the flow velocity is measured in terms of the Bogoliubov sound velocity c_B [28]:

$$\lambda = \hbar/\sqrt{\nu \rho_\mathrm{gas}}, \quad c_\mathrm{B} = \sqrt{\nu \rho_\mathrm{gas}}/m.$$

Here, ρ_gas stands for the bulk gas density. Then Eqs. (11),(12) become

$$(i\partial_t + (1/2)\Delta - \psi^* \psi)\, \psi(x, y, t) = 0, \tag{14}$$

$$\left\{c_t^{-2}\partial_t^2 - \Delta - \beta \nabla(\nabla \cdot)\right\} \mathbf{q}(x, y, t) = 0, \tag{15}$$

where the sound velocity, time and $\mathbf{q}$ are measured in the units of c_B, λ/c_B and λ respectively.

In the stationary regime, the Bose gas moves uniformly in the x-direction with the velocity v. As the system is homogeneous in the x and z directions, $|\psi_0(\mathbf{r}, t)| = \phi(y)$ and $\mathbf{q}_0(\mathbf{r}, t) = \mathbf{q}_0(y)$. The walls are deformed only in the y-direction, since the tangent stress vanishes in the stationary regime $\mathbf{q}_0(y) = \big(0,\, q_{0_y}(y),\, 0\big)$. The corresponding solution of (14), (15) is given by

$$\psi_0(x, y, t) = \phi(y) \mathrm{e}^{ivx}\, \mathrm{e}^{-i\Omega t}, \ \ \phi(\pm l) = 0, \ \ \phi'(\pm l) \neq 0, \tag{16}$$

$$q_{0_x} = 0, \ \ q_{0_y} \equiv Q(y) = \pm \mathrm{const}(y \pm l), \ (y \to \pm l), \ \ Q(\pm l) = 0, \ \ Q'(\pm l) \neq 0 \tag{17}$$

It follows from (16), (14) that $\Omega = \tilde{\mu} + \frac{v^2}{2}$, $\tilde{\mu} = 1$ (this corresponds to the value of a chemical potential $\mu = \nu \rho_\mathrm{gas}/m$ at $T = 0$), and $\phi(y)$ satisfies the following equation [35]:

$$-\frac{1}{2}\phi'' + \phi^3 = \phi, \quad \phi(y) = \phi(-y). \tag{18}$$

The parity of ϕ in (18) and the boundary conditions in (16) are obtained from (8)-(10) (see Fig. 1).

We follow the procedure (5)-(7) expanding the field variables around the stationary solution (16), (17)

$$\psi = \Big(\phi(y) + \xi(x - vt, y, t)\Big)\mathrm{e}^{ivx}\,\mathrm{e}^{-i(\tilde{\mu}+v^2/2)t},$$
$$q_x = \zeta_x(x - vt, y, t),\; q_y = Q(y) + \zeta_y(x - vt, y, t).$$

Substituting these expansions into (14),(15) we get the following linear differential equations for the fluctuations $\xi, \boldsymbol{\zeta}$,

$$\left\{i\partial_t + \frac{1}{2}\Delta + 1 - 2\phi(y)^2\right\}\xi - \phi(y)^2\xi^* = 0,\ |y| < l, \tag{19}$$
$$\left\{c_t^{-2}(\partial_t - v\partial_x)^2 - \Delta - \beta\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot)\right\}\boldsymbol{\zeta} = 0, \quad |y| > l, \tag{20}$$
$$\xi = \xi(x, y, t), \quad \boldsymbol{\zeta} = \boldsymbol{\zeta}(x, y, t),$$

written in the reference frame moving with the Bose-gas, $x \to x' = x - vt$.

One of the advantages of setting $\mathcal{U} = 0$ is the possibility of using the exact solution of Eq. (18) with the boundary conditions (16),

$$\phi(y) = \mathrm{sn}(y + l,\ \varrho),$$

where $\mathrm{sn}(y, \varrho)$ is the elliptic sine [36], the parameter ϱ is chosen to fit the boundary conditions and the following condition holds $\mathrm{sn}(y, \varrho) \to \tanh(y)$ if $l \to \infty$. (Notice that the dimensional condensate wave function can be written in the form $\phi_d(y) = \mathrm{const}\sqrt{\rho_{\mathrm{gas}}/m}\,\mathrm{sn}(\,(y + l)/\lambda,\ \varrho\,)$.

Nontrivial excitations can not propagate over the wall region far from the boundary. Indeed, the equation (20) has constant coefficients and, hence, the dispersion law for such excitations coincides with the phonon one (i.e., corresponding asymptotical solutions describe propagation of the ordinary sound waves far from the boundaries). Therefore, we have to look for the solution of (20) decreasing in $y \to \pm\infty$ directions. We assume that the experimentally discovered dependence of the critical velocity on the canal width [22], $v_c \sim l^{-n}$, $n \simeq 2$, is a hint to search for the special type of excitations, in which two boundaries of the canal can contribute *coherently.*

Such a solution of (19), (20) can be written in the form

$$\xi = \chi_1(y)\sin(kx-\omega t) + i\chi_2(y)\cos(kx-\omega t),\ \ |y|<l, \tag{21}$$

$$\zeta_x = r_1(y)\cos(kx-\omega t),\ \ \zeta_y = r_2(y)\sin(kx-\omega t),\ \ |y|>l \tag{22}$$

with

$$r_i(y) = A_i\exp(-\kappa|y|) + B_i\exp(-\eta|y|). \tag{23}$$

Two exponential terms in (23) correspond to the different polarizations of the boundary excitations. The characteristic values

$$\kappa = \sqrt{k^2-\omega^2/c_t^2} \ \text{ and } \ \eta = \sqrt{k^2-\omega^2/c_l^2}$$

are eigenvalues of the ordinary linear equations obtained by substitution of (22) into (20). Note that the ansatz (21), (22) is equivalent to the Bogoliubov u-v transformation generalized to a nonuniform case [28],[32],[33] and coupling with the surface phonons:

$$\begin{aligned} e^{i\mu t}\delta\psi(x,y,t) &= u_k(y)\,\mathrm{e}^{i(kx-\omega(k)t)} + v_k^*(y)\,\mathrm{e}^{i(-kx+\omega(k)t)} \\ \delta\mathbf{q}(x,y,t) &= \mathbf{C}_k(y)\,\mathrm{e}^{i(kx-\omega(k)t)} + \text{c.c..} \end{aligned}$$

Then the operators $b_k^\dagger$, (b_k) that create (annihilate) the boundary excitations (21-23) in the diagonalized Hamiltonian (6) can be represented by the linear combinations of the Bose-gas field operators, $\delta\hat\psi$ and $\delta\hat\psi^\dagger$, and the displacement field operators, $\delta\hat q$ and $\delta\hat\pi$.

We linearize the boundary conditions (8)-(10) according to the method of the semiclassical approximation (5), neglecting the terms of power greater than one in $\xi, \boldsymbol{\zeta}$. Together with the proper solution (21),(22) of Eqs. (19), (20), conditions (8)-(10) determine values $A = A(k,\omega)$, $B = B(k,\omega)$ in (23) and the boundary conditions for $\chi_{1,2}$:

$$\left(\frac{\chi_1'}{\chi_1}\right)_{y=\pm l} = \mp k\,\gamma(z),\ \ \chi_2(\pm l) = 0, \quad z = \frac{\epsilon(k)^2}{(c_t k)^2}, \tag{24}$$

$$\gamma(z) = \frac{\rho c_t^2/\rho_{\rm gas} c_{\rm B}^2}{(\phi'(l))^2} \frac{1}{2z} \left(4\sqrt{1-z} - \frac{(2-z)^2}{\sqrt{1 - c_t^2 z/c_l^2}} \right) \gg 1.$$

The value $\epsilon(k)$ in (24)

$$\epsilon(k) = \omega(k) - kv$$

equals the boundary excitation energy in the channel reference system (3).

It follows from (21) and (19) that the variables $\chi_{1,2}(y)$ and ω satisfy the following system:

$$\begin{aligned} &L_1 \chi_1(y) + 2\omega \chi_2(y) = 0, \quad L_1 = \partial_y^2 - k^2 + 2 - 6\phi(y)^2, \\ &L_2 \chi_2(y) + 2\omega \chi_1(y) = 0, \quad L_2 = \partial_y^2 - k^2 + 2 - 2\phi(y)^2. \end{aligned} \tag{25}$$

Note that in view of the parity of $\phi(y)$ (see (18)) and the boundary conditions (24), solutions of (25) can be either symmetric or antisymmetric with respect to y.

According to the criterion (3), a breakdown of superfluidity occurs at such a value v if there exists such a $\tilde{k} \neq 0$ that $\epsilon(\tilde{k}) = 0$. Then the argument z of γ in (24) vanishes and the boundary conditions do not depend explicitly on ω, i.e. $(\chi_1'/\chi_1)|_{y=\pm l} = \mp \tilde{k}\gamma(0) = \mp$const. In principle, this fact makes it possible to calculate v_c without finding any final expression of $\omega(k)$. Indeed, one has to solve Eqs. (25) with $\omega = \tilde{k} v_c$ and fixed boundary conditions.

In this simple model, however, it is possible to calculate the dispersion relation $\omega(k)$, at least in the $k \to 0$ limit [37]. We look for the symmetric solution of (25), expanding $\chi_{1,2}(y)$ in powers of ω^2:

$$\chi_1 = \chi_1^{(0)} + \omega^2 \chi_1^{(1)} + \ldots, \quad \chi_2 = \omega(\chi_2^{(0)} + \omega^2 \chi_2^{(1)} + \ldots).$$

The functions $\chi_{1,2}^{(i)}$ symmetric in y satisfy the following recurrence relations

$$\begin{aligned} &L_1 \chi_1^{(0)} = 0, \ \chi_1^{(0)}(\pm l) = 1, \\ &L_1 \chi_1^{(i)} + \chi_2^{(i-1)} = 0, \ \chi_1^{(i)}(\pm l) = 0, \ i = 1, 2, \ldots \\ &L_2 \chi_2^{(i)} + \chi_1^{(i)} = 0, \ \chi_2^{(i)}(\pm l) = 0, \ i = 0, 1, 2, \ldots. \end{aligned} \tag{26}$$

The analytic study of (26) seems to be difficult. Instead we proceed numerically [38]; we obtain the eigenvalues $\omega = \omega(k)$ by solving (26) recursively and imposing the boundary conditions (24) on χ.

The result reads

$$\omega(k) = \alpha\sqrt{(\gamma+\delta)k^3}, \quad \text{as} \quad k \to 0, \tag{27}$$

where $\alpha > 0$ and $\delta > 0$ are some bounded functions of l, $\gamma = \gamma(0) \simeq \rho c_t^2/\rho_{\text{gas}} c_{\text{B}}^2$. The dependence of $\log(\omega(k))$ on $\log(k)$ is shown on Fig. 2. Note that only the solutions with finite $\gamma > 1$ have the physical meaning. For "conventional" systems, such as a dilute Bose gas inside a solid container, the value of $\gamma(0)$ is very large, $\gamma(0) \simeq 10^{7\sim 8}$ or even more. Moreover, the validity of the long-wavelength / low-energy approximation implies $k\gamma < 1$ and $\omega \ll 1$, and the relevant wavelengths are unphysically huge. However, for the "soft matter" substances with flexible walls, γ can be of the order of $10^{2\sim 3}$ and the distance between the walls can be $2l > \lambda$. Then, beginning from the wavelengths of the order of $(10^{2.5\sim 3})\lambda$, we are within the '$k \to 0$' limit and $\omega(k) < 10^{-3}\tilde{\mu}$. (On Fig. 2, we present also the curves with $\gamma < 1$ because of similarity between our result and the dispersion relation of capillary waves on the interface between liquid and gaseous He, the so-called "ripplons" [39].) If the channel walls were rigid and incompressible, that corresponds to $\gamma = \infty$ in (27), the inhomogeneous surface excitations introduced in this study just do not exist as a well-defined object.

It is easy to conclude that the semiclassical critical velocity is *zero* in this *model*, since for any v there exist $k \neq 0$ such that $\epsilon(k) = \omega(k) - kv < 0$. This means that the model, in which all the gas–boundary interactions are reduced to the hard-core repulsion, predicts (in the semiclassical (Bogoliubov) approximation) an instability of the Bose-gas current state in relation to occurrence of boundary excitations. However, whether the damping of superflow can happen via the energy transfer from the 3D condensate to the boundary localized modes is an open question, which cannot be answered in the framework of models with the superfluid density $\rho_s = \rho_{\text{gas}}$, $T = 0$. More sophisticated models of superfluidity,

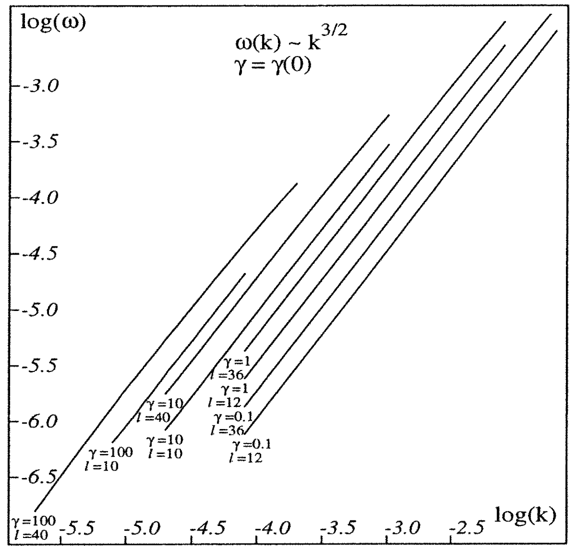

Figure 2: Spectrum of boundary excitations.
The low-momentum spectrum $\omega(k)$ of boundary excitations. The canal width l is measured in the coherence length units; $\gamma \simeq \rho c_t^2/\rho_{\rm gas} c_{\rm B}^2$.

such as the two-fluid hydrodynamics [22], should be used to describe the kinetics of damping.

4 Discussion

In our opinion, the obtained dispersion relation (see Fig. 2) in the framework of the proposed oversimplified model gives a possibility to conclude that the boundary interactions can play a key role in the superfluidity, namely, by reducing substantially the critical velocity of the superflow. If that is the case, it is reasonable to generalize our model to make it more realistic.

Recall that we have neglected the "smooth" part $\mathcal{U}$ of the gas-wall interactions. The models with repulsive $\mathcal{U}$ seems to be qualitatively similar to the one under consideration. We believe that they also lead to the zero *semiclassical* critical velocity, since nothing can apparently prevent the energy transfer from the gas flow to the walls (in this approximation). It is important to note that even if the higher quantum corrections yield nonzero critical velocity, the latter will be of much smaller value than the (semiclassical) Bogoliubov-Landau velocity found by the numerical simulations based on the nonlinear Schrödinger equation [27].

The situation might be different if the "smooth" part $\mathcal{U}$ of the interaction between the gas and wall atoms were attractive. For example, we can take the Hamiltonian of gas-wall interaction in the Deformation Potential form

$$\hat{H}_{\text{int}} = \int \sigma(\mathbf{r},\mathbf{r}')\, \boldsymbol{\nabla}\hat{\mathbf{q}}(\mathbf{r})\, \hat{\psi}^{\dagger}\hat{\psi}(\mathbf{r}')\, d\mathbf{r}d\mathbf{r}', \tag{28}$$

where $\mathbf{r}$ and $\mathbf{r}'$ change in the wall area ($|y| > l$) and in the channel area ($|y'| < l$) respectively, and the function $\sigma(\mathbf{r},\mathbf{r}')$ describes atom-lattice interaction. (At least two new parameters, which control the "smooth" part of gas-wall interaction, have to appear in the model with $\mathcal{U} \neq 0$: one, some characteristic value of energy, and, two, a length scale). Then the equations for the classical parts of $\hat{\psi}$ and $\hat{q}$, namely $\phi(y)$ and q_{0_y}, (see Eqs. (16), (17)), can be written as follows:

$$\left(\mu + \frac{\hbar^2}{2m}\partial_y^2 - \nu\phi^2(y)\right)\phi(y) = \left(\int \sigma(\mathbf{r}',\mathbf{r})\partial_{y'}Q(y')\, d\mathbf{r}'\right)\phi(y), \tag{29}$$

$$-c_l^2\, \partial_y^2 Q(y) = \rho^{-1}\partial_y \int \sigma(\mathbf{r},\mathbf{r}')\, \phi^2(y')\, d\mathbf{r}'. \tag{30}$$

After the exclusion of $Q(y)$ from Eq. (29), it can be rewritten in the following form:

$$\left(-\frac{\hbar^2}{2m}\partial_y^2 + \Lambda\, U(\mathbf{r}) + \nu\phi^2(y) - \int U_{\text{eff}}(\mathbf{r},\mathbf{r}')\phi^2(y')d\mathbf{r}'\right)\phi(y) = \mu\phi(y) \tag{31}$$

where $U_{\text{eff}}(\mathbf{r},\mathbf{r}') = \int\sigma(\mathbf{r}'',\mathbf{r})\sigma(\mathbf{r}'',\mathbf{r}')d\mathbf{r}''/\rho c_l^2$, $U(\mathbf{r}) = \int\sigma(\mathbf{r}',\mathbf{r})d\mathbf{r}'$, and $\Lambda = \text{const}$ is defined from the boundary condition (9).

It is easy to see from the structure of (31) that the atom-lattice interaction $\sigma(\mathbf{r},\mathbf{r}')$ (when exceeding a certain magnitude) can induce an attraction between the gas atoms in the boundary region. Indeed, there can exist such a scale of $|\mathbf{r}-\mathbf{r}'|$

that $\nu\delta(\mathbf{r}-\mathbf{r}')-U_{\text{eff}}(\mathbf{r},\mathbf{r}') < 0$. In that case, one would expect essential changes in the spectrum of boundary excitations. The study of exciton superfluidity [15],[40] hints such a possibility.

This study reveals a mechanism of the exciton-exciton attraction induced by the lattice effects. The exciton gas falls into the soliton-like state ψ_0, q_0, when the exciton-phonon coupling constant exceeds a certain value, or, equivalently, the velocity v exceeds some critical value. The excitation spectrum in an exciton branch has a gap, and the Landau critical velocity, calculated for this type of excitations, is given by

$$v_c \simeq \hbar/mL, \tag{32}$$

where L denotes the characteristic size of the soliton. We can adapt the similar considerations in our model, replacing the exciton-phonon by the gas-wall interactions. The critical velocity can be given by a formula similar to (32) with $L \simeq l$, provided l is the only macroscopical length in the theory.

In this article we did not consider the influence of the long-range van der Waals forces between the walls and gas atoms on the stability of superfluid flow. Although the attractive part of these forces originates from interaction between the electron shells of the particles [41], it can be included in our model in the form of a static van der Waals potential appearing in the r.h.s. of Eq. (11). Such an external potential being localized near the boundaries can be stronger than the effective potential $\Lambda U(y)$ in (31) and therefore can change the properties of the condensate wave function, the boundary excitations, etc..

5 Conclusions

In conclusion, our simple model manifests one of the possible microscopic mechanisms for dissipation processes in the quantum Hamiltonian system of coupled Bose-gas and channel walls. We show that the dissipation can also be caused by creation of boundary excitations in this system. Although, in the semiclassical approximation, this process is not prohibited at any velocity of the moving repulsive Bose-gas, the higher quantum corrections to the self-energy part of the boundary excitations may be essential to obtain the nonzero value of v_c in the theory of the dilute Bose-gas with gas-wall interaction. On the other hand, more rigorous consideration in the framework of the semiclassical approximation should involve solutions of (8)-(12) with $\mathcal{U} \neq 0$, where the attractive part of gas-wall interaction is taken into account. Such solutions can be still represented in the form (21), (22), though equations (18), (19), (20) would become integral. A direct numerical study of the flow dynamics would be also useful.

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References

[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science **269**, 198 (1995).

[2] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995).

[3] J.-L. Lin and J. P. Wolfe, Phys. Rev. Lett. **71**, 1222 (1993).

[4] E. Fortin, S. Farad, A. Mysyrowicz, Phys. Rev. Lett **70**, 3951 (1993).

[5] L. V. Butov, A. Zrenner, G. Abstreiter, G. Bohm, and G. Weimann, Phys. Rev. Lett **73**, 304, (1994).

[6] See http://amo.phys.gasou.edu/bec.html/ for an up-to-day bibliography on Bose-Einstein condensation.

[7] *Bose-Einstein Condensation*, edited by A. Griffin, D. W. Snoke, and S. Stringari, (Cambridge, New York, 1995).

[8] K. Huang, in *Bose-Einstein Condensation*, edited by A. Griffin, D. W. Snoke, S. Stringari, (Cambridge, New York, 1995).

[9] D. S. Rokhar, Phys. Rev. Lett. **79**, 2164 (1997).

[10] E. Zaremba, A. Griffin, and T. Nikuni, cond-mat/9705134, submitted to Phys. Rev. Lett..

[11] E. Benson, E. Fortin, A. Mysyrowicz, Phys. Stat. Sol. B **191**, 345, (1995).

[12] A. Mysyrowicz, E. Benson, E. Fortin, Phys. Rev. Lett. **77**, 896 (1996)

[13] E. Hanamura, Sol. Stat. Comm. **91**, 889 (1994).

[14] A. E. Bulatov, S. G. Tichodeev, Phys. Rev. B **46**, 15058 (1992)

[15] I. Loutsenko, D. Roubtsov, Phys. Rev. Lett. **78**, 3011 (1997)

[16] G. M. Kavoulakis, G. Baym, and J. P. Wolfe, Phys. Rev. B **53**, 7227 (1996)

[17] A. Imamoḡlu, R. G. Ram, Phys. Lett. A **214**, 193 (1996)

[18] S. G. Tichodeev, Phys. Rev. Lett. **78**, 3225 (1997);
A. Mysyrowicz, *ibidem*, 3226 (1997).

[19] D. R. Tilley, J. Tilley, *Superfluidity and Superconductivity*, (A. Hilger, Bristol, 1990).

[20] K. Huang, *Statistical Mechanics*, (Wiley, New York, 1987).

[21] D. R. Allum, P. V. E. McClintock, A. Phillips, and R. M. Bowley, Phil. Trans. R. Soc. A **284**, 179 (1977).

[22] S. J. Putterman, *Superfluid Hydrodynamics*, (North Holland, Amsterdam, 1974).

[23] L. Onsager, Nuovo Cimento Suppl. **6**, 249 (1949).

[24] W. M. Van Alphen, J. F.Olijhoek, R. de Bruyn Ouboter, and K. W. Taconis, Physica **32**, 1901 (1966).

[25] V. L. Ginzburg and L. P. Pitaevskii, Sov. Phys. JETP **7**, 858 (1958);
E. P. Gross, Nuovo Cimento **20**, 454 (1961).

[26] C. A. Jones and P. H. Roberts, J. Phys. A **15**, 2599 (1982).

[27] T. Frich, Y. Pomeau, and S. Rica, Phys. Rev. Lett. **69**, 1644 (1992).

[28] N. N. Bogoliubov, J. Phys. USSR **11**, 23 (1947).

[29] P. G. De Gennes, Rev. Mod. Phys. **53**, 827 (1987).

[30] H. Lüth, *Surfaces and Interfaces of Solids*, (Springer-Verlag, 1993).

[31] N. H. Christ and T. D. Lee, Phys. Rev. D **12**, 1606 (1975).

[32] S. Giorgini, L. P. Pitaevskii, S. Stringari, Phys. Rev. A **54**, 4633 (1996);
A. Griffin, Phys. Rev. B **53**, 9341 (1996).

[33] Al. L. Fetter, Phys. Rev. A **53**, 4245 (1996).

[34] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity*, (Pergamon, Oxford, 1957).

[35] Al. L. Fetter and J. D. Waleska, *Quantum Theory of Many-Particle System*, (McGrav-Hill, New York, 1971).

[36] *Handbook of mathematical functions*, edited by M. Abramowitz and I. A. Stegun, (Dover Paris inc., New York, 1965).

[37] The critical idea was provided by I. Loutsenko.

[38] We solve a difference analog of this system by the Gauss method using a FORTRAN program. The calculational error does not exceed the one of the semiclassical approximation.

[39] K. R. Atkins, Can. Journ. Phys. **31**, 1165 (1953);
L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, (Pergamon, Oxford, 1959).

[40] M. F. Miglei, S. A. Moskalenko, and A. V. Lelyakov, Phys. Stat. Sol. **35**, 389 (1969);
V. M. Nandkumaran and K. P. Sinha, Zeit. für Phys. B **22**, 173 (1975).

[41] V. Celli, in *Surface Phonons*, edited by W. Kress and F. W. de Wette, (Springer-Verlag, 1991).

Conclusion

The main result of this thesis is the development of a microscopic approach aimed to understand the anomalous transport of excitons observed in cuprous oxide. As this effect is a critical behavior, the interpretation in terms of the Bose-Einstein condensation of excitons is very appealing. However, it was not clear what is exactly the Bose-condensate in the packet of moving excitons and phonons and why the repulsive excitons form a soliton-like moving state appearing in the models with the particle-particle attraction.

We propose a relatively simple theoretical model which describes the packet of moving excitons and phonons as an exciton-phonon "comet" with the Bose-Einstein core of the correlated excitons and phonons. The optically inactive excitons can be described by the macroscopic wave function in the same manner as it is done for atomic Bose-gases. However, the crystal vibrations are very important in order to understand why a kind of the "bright" excitonic soliton can be formed. Assuming that some of (non-thermal) phonons of the exciton-phonon packet form a coherent state we showed that the effective exciton-exciton interaction between excitons in the Bose-condensate becomes attractive in the direction of motion. In other words, the spatial properties of the exciton-phonon condensate are described by a system of the coupled nonlinear Schrödinger and nonlinear wave equations that supports the "bright" soliton solutions.

Within the mean field theory, we introduced the elementary excitations of the exciton-phonon condensate: the well-known u-v Bogoliubov transform has to be generalized to include the field operators of the displacement field of the crystal and its conjugated momentum. Thus, we have two branches of the excitations to take into account in estimating of the condensate depletion. To simplify calculations, each of these two branches can be divided into the excitations forming the tail of the exciton-phonon condensate and the excitations forming the "coma" of it. For example, to a first approximation, the tail of the exciton-phonon condensate consists of the uncoupled exciton- and phonon-like excitations emitted out of the moving condensate.

To sum up, the subsonic anomalous propagation of exciton-phonon packets can be described by soliton solutions of the Nonlinear Schrödinger equation if we assume the exciton-phonon condensate was formed inside the moving packet. Then, the theory predicts two critical velocities for propagation of the packet as well as strong nonlinear interaction between the moving excitonic packets, both containing the condensates. This is in a good qualitative agreement with experimental data.

To finish this section, I would like to mention some questions which have not been discussed in the thesis as well as some open problems.

It seems the language of many-particle physics is quite suitable in attempts to interpret current experiments on the creation of highly correlated excited states in semiconductors and semiconductor heterostructures. However, the well-known critical phenomena, such as Bose-Einstein Condensation, Superfluidity, Kosterlitz-Thouless transition in 2D, Dicke Superradiance, have to be revisited in order to apply them correctly to excitons or exciton-polaritons in semiconductors and semiconductor "sandwich" structures.

For instance, to develop a many-particle theory explaining nonlinear phenomena in semiconductor microcavities, one has to consider a model of several interacting fields, such as the excitonic field, photon field, and, probably, the phonon one [45].

However, many of the interesting phenomena are beyond the reach of the standard perturbative methods of Quantum Field Theory and equilibrium Quantum Statistical Mechanics. It is possible to apply advanced theoretical methods, such as a proper combination of exactly solvable models and nonequilibrium quantum physics, to address the key questions in such challenging situations.

In this thesis, we studied the anomalous transport of exciton packets (droplets) in three dimensional crystals assuming the excitons are Bose-correlated. The anomalous optical properties of the Bose- condensed excitons were not discussed. From a theoretical point of view, the possibility to convert a coherent excitonic field into a coherent photon field is one of the challenging and important topics in the modern Physics of Semiconductors. From this perspective, the question on whether the moving exciton-phonon packet contains a Bose-correlated core with a rigid macroscopic phase at nonzero temperature could be answered both theoretically and experimentally. (The process with $\mathbf{k}_{\mathrm{x},1}+\mathbf{k}_{\mathrm{x},2}=\mathbf{k}_{\mathrm{ph},1}+\mathbf{k}_{\mathrm{ph},2}\approx 0$ can be allowed. For the case of Cu_20, however, one has to deal with the optically active orthoexcitons, so that it is probably necessary to switch on the magnetic field to suppress ortho-para conversion.) Therefore, we can speculate on the possibility to convert the coherent exciton field into the coherent photon one by colliding two moving exciton-phonon packets. Such an experiment could reveal how rigid is the macroscopic phase of the Bose-core (if any). If one can prescribe a macroscopic phase φ_c to each Bose-core of the moving packets, their head-on collision could result in the many-photon production without loss of the "phase memory", $\varphi_{c,1}-\varphi_{c,2}$. For example, the jets of photons originated from the condensate interaction (the Bose-core collision) seem to be highly directional in space and with a low noise level, see Fig. 1. The language of the coherent states can be applied to describe them. This problem, however, needs careful analysis and is left for future.

Another possibility is to develop a controllable pump of excitons into a controllable trap obtained, for example, by applying an inhomogeneous strain to a semiconductor [46], see Fig.2. Then, the photoluminescence signal from the trapped excitons can show a critical behavior. In addition, study of the anomalous exciton transport in strong magnetic field is interesting because, roughly, the exciton-exciton interaction strength depends on the value of the magnetic field. In crossed magnetic and electric fields, it is possible to find new observable effects.

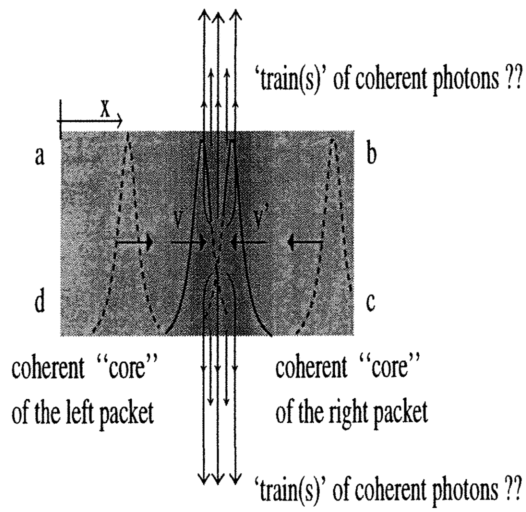

Figure 1: Head-on collision of packets.
After some amount of energy ($\delta E_{\rm left} = \delta E_{\rm right}$) has been pumped into the medium during a short time interval δt and absorbed near the left and right boundaries, two localized excited states are formed near the faces 'ad' and 'bc'. If there is a mechanism of the momentum transfer to the excited state, the droplets begin to move toward the opposite faces with the velocities $\langle v \rangle \approx \langle v' \rangle$. Such conditions can favor the appearance of a coherent boson-phonon state (an analog of Davydov soliton) inside both the left and right droplets. The collision of such exciton-phonon droplets seems to be analogous to the heavy nucleus collisions that result in the production of pion jets. In Cu_2O crystal, one can expect the production of two photon jets at *low* temperature $T < T_c$. As a result, coherence of the phonons in such jets can be checked experimentally.

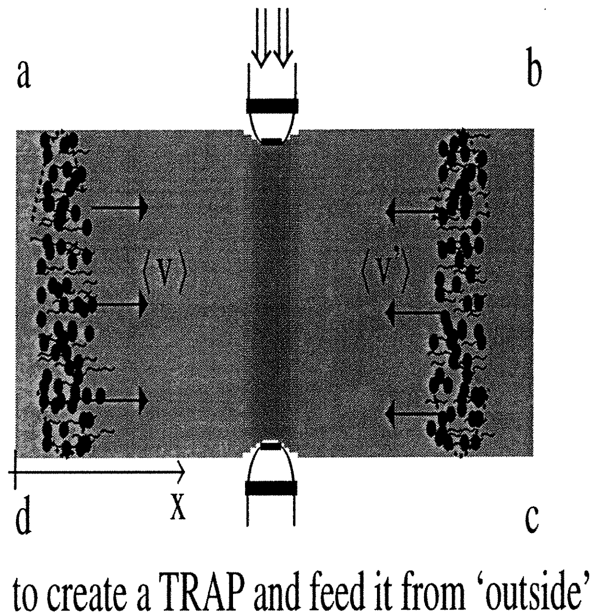

Figure 2: Trap for excitons inside a crystal.

Like the BEC of atomic gases, one can create a trap for excitons in a crystal. However, an exciton in such a trap has a finite life time, so that one cannot avoid strong fluctuations of any order parameter prescribed to the excitonic gas even though the conditions for the equilibrium Bose-Einstein condensation are met at the beginning. It is possible to suppress fluctuations of the phase of the order parameter if the average number of excitons in the trap will be kept constant. Then, we can expect the emitted photons to be coherent with the energy $\hbar\Omega = E(N_o) - E(N_o - 1)$.

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References

[1] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **79,** (1999) 463.

[2] J. J. Hopfield, Phys. Rev. **112,** (1958) 1555.

[3] J. L. Lin, J. P. Wolfe, Phys. Rev. Lett. **71,** (1993) 122;
L.V. Butov, A. I. Filin, Phys. Rev. B **58,** (1998) 1980;
T. Goto, M. Y. Shen et al., Phys. Rev. B **55,** (1997) 7609.
A. V. Larionov, V. B. Timofeev, J. Hvam, C.Soerensen, JETP Lett. **71,** (2000) 174.

[4] L. V. Keldysh and A. N. Kozlov, Sov. Phys. JETP **27,**, (1968) 521.

[5] S. A. Moskalenko, D. W. Snoke, *Bose-Einstein condensation of excitons and biexcitons: and coherent nonlinear optics with excitons*, (Cambridge University Press), 2000.

[6] Yu. Kagan, B. Svistunov, Sov. Phys. JETP **78,** (1994) 187.

[7] K. Huang, *Statistical Mechanics*, 2nd ed., (Wiley, New York, 1987).

[8] A. J. Leggett, Rev. Mod. Phys. **73,** (2001) 307.

[9] E. Fortin, S. Farad, and A. Mysyrowicz, Phys. Rev. Lett. **70,** (1993) 3951;
A. Mysyrowicz, E. Benson, and E. Fortin, Phys. Rev. Lett. **77,** (1996) 896;
E. Benson, E. Fortin et al., Europhys. Lett. **40,** (1997) 311;
E. Fortin, E. Benson, A. Mysyrowicz, Electrochem. Soc. Proceed. **98-25,** (1998) 1.

[10] H. Kondo, H. Mino et al., Phys. Rev. B **58,** (1998) 13835.

[11] T. Freixanet, B. Sermage et al., Phys. Rev. B **61,** (2000) 7233.

[12] G. B. Whitham, *Linear and Nonlinear Waves*, (Wiley Interscience, New York, 1974);
P. D. Weidman and T. Maxworthy, J. Fluid. Mech. **85,** (1978) 417.

[13] E. Benson, Ph.D. thesis, University of Ottawa, 1999.

[14] K. E. O'Hara, J. R. Gullingsrud, and J. P. Wolfe, Phys. Rev. B **60,** (1999) 10 872;
G. M. Kavoulakis, A. Mysyrowicz, Phys. Rev. B **61,** (2000) 16619.

[15] Y. Toyozawa, Progr. Theor. Phys. **20**, (1958) 53;
Y. Toyozawa, J. Phys. Chem. Solids **25**, (1964) 59.

[16] Yu. E. Lozovik, A. V. Poushnov, JETP **88**, 747 (1999); cond-mat/9803318.

[17] K. Johnsen and G. M. Kavoulakis, Phys. Rev. Lett. **86**, (2001) 858.

[18] V. A. Gergel', R. F. Kazarinov, and R. A. Suris, Zh. Eksp. Teor. Fiz. **53**, (1967) 544; *ibidem.* **54**, (1968) 298.

[19] E. Hanamura, Solid State Commun. **91**, (1994) 889.

[20] A. R. Vasconcellos, M. V. Mesquita, and R. Luzzi, Europhys. Lett. **49**, (2000) 637.

[21] N. N. Bogoliubov, *Lectures on Quantum Statistics*, Vol. 2, (Gordon and Breach, N.Y, 1970);
H. Shi and A. Griffin, Phys. Reports **304,** (1998) 1.

[22] Yu. Kagan and B. V. Svistunov, Phys. Rev. Lett. **79,** (1997) 3331.

[23] A. S. Davydov, *Solitons in Molecular Systems*, (Reidel, Dordrecht, 1984);
Davydov's Soliton Revisited, edited by P. L. Christiansen, A. C. Scott, NATO ASI Series B: Physics **243**, (Plenum Press, 1990).

[24] R. J. Glauber, Phys. Rev. **131,** (1963) 2766.

[25] J. Shumway and D. M. Ceperley, cond-mat/9907309.

[26] D. Pines and P. Noziéres, *Theory of Quantum Liquids*, (Addison-Wesley, New York, 1989), Vols. 1 and 2.

[27] Wei-Min Zhang, in *Quantum Field Theory. A Twentieth Century Profile*, ed. by A. N. Mitra, (Hindustan Book Agency, 2000).

[28] V. E. Zakharov, Sov. Phys. JETP **35** (1972), 908;
C. Sulem, P.-L. Sulem, *The nonlinear Schrödinger equation. Self-Focusing and Wave Collapse.*, (Springer-Verlag, New York Inc., 1999).

[29] S. Novikov, S. V. Manakov, L. P. Pitaevskii, V. E. Zakharov, *Theory of Solitons. The Inverse Scattering Method*, (Consultants Bureau, New York and London, 1984);
M. J. Ablowitz, P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, (Cambridge University Press, New York, 1991).

[30] D. Roubtsov, Y. Lépine, Phys. Stat. Sol. (B) 2002 and cond-mat/0008284 at http://xxx.lanl.gov/

[31] D. Hennig, G. P. Tsironis, Phys. Rep. **307**, 333 (1999).

[32] E.A. Bartnik, J.A. Tuszynski, and D. Sept, Phys. Lett. A **204**, (1995) 263.

[33] E. Fortin and M. Masse, will be published elsewhere.

[34] G. A. Kopelevich, S. G. Tikhodeev, and N. A. Gippius, JETP **82,** (1996) 1180;
S. G. Tikhodeev, N. A. Gippius, and G. A. Kopelevich, Phys. Stat. Sol. (a) **178,** (2000) 63.

[35] A. Griffin, Phys. Rev. B **53,** (1996) 9341.

[36] V. L. Ginzburg, Physics-Uspekhi, **167,** (1997) 429.

[37] S. L. Cornish, N. R. Claussen, J. L. Cornell, and C. E. Wieman, Phys. Rev. Lett. **85,** (2000) 1795.

[38] S. G. Tikhodeev, Phys. Rev. Lett. **78,** (1997) 3225, *ibid.* **84**, 3502 (2000).

[39] A. D. Jackson, G. M. Kavoulakis, cond-mat/0107103, submitted to Europhys. Lett.

[40] Ch. Josserand, Y. Pomeau, S. Rica, Phys. Rev. Lett. **75,** (1995) 3150.

[41] D. Mihalache, D. Mazilu, L.-C. Crasovan, B. A. Malomed, F. Lederer, Phys. Rev. E **61**, (2000) 7142.

[42] D. Kong, Phys. Lett. A **196**, (1995) 301;
B. Dey, A. Khare, C. N. Kumar, hep-th/9510054.

[43] Yu. E. Lozovik, S. A. Verzakov, M. Willander, cond-mat/9811037;
Yu. E. Lozovik, O. L. Berman, V. G. Tsvetus, Phys. Rev. B **59**, (1999) 5627.

[44] Ch. Josserand, Y. Pomeau, Nonlinearity **14,** (2001) 25.

[45] R. Huang, F. Tassone, and Y. Yamamoto, Phys. Rev. B **61,** (2000) R7854;
G. Khitrova, H. M. Gibbs, F. Jahnke, M. Kira, and S. W. Koch, Rev. Mod. Phys. **71,** (1999) 1591.

[46] D. W. Snoke, V. Negoita, Phys. Rev. B **61,** (2000) 2904.