Université de Montréal

SYMÉTRIES ET SINGULARITÉS DES ÉQUATIONS AUX VARIABLES DISCRÈTES

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Thèse présentée à la Faculté des études supérieures

en vue de l'obtention du grade de

Philosophiæ Doctor (Ph.D.)
en Physique

septembre 2001

Université de Montréal

Faculté des études supérieures

Cette thèse intitulée

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Thèse acceptée le:
18 décembre 2001

RÉSUMÉ

Cette thèse est consacrée à l'étude des symétries ainsi que de l'intégrabilité des équations aux variables discrètes indépendantes.

Dans les trois premiers chapitres nous introduisons une nouvelle méthode permettant de calculer les symétries de Lie ponctuelles des systèmes d'équations sur un réseau. Les transformations de Lie agissent à la fois sur l'équation discrète et sur le réseau lui-même. Ces transformations amènent les solutions du système en d'autres solutions. Le cas des équations à une seule variable discrète est d'abord traité, par la suite nous considérons le cas multidimensionnel.

Au chapitre 4, nous effectuons une classification par symétries des interactions possibles d'une chaîne moléculaire diatomique unidimensionnelle. Pour les interactions non-linéaires le groupe de Lie des transformations ponctuelles, amenant une solution en une autre solution et laissant le réseau invariant, est au plus de dimension 5. La classification des équations est faite en classes d'équivalence sous l'action de transformations permises.

La deuxième partie de la thèse concerne la question de l'intégrabilité. Au chapitre 5, nous étudions cette propriété pour les équations aux variables discrètes partielles bidimensionnelles. Pour ce faire, nous utilisons la méthode de la croissance de degré des itérations. Nous montrons que pour les équations non-intégrables, la croissance de degré est exponentielle alors que pour les équations intégrables, elle est polynômiale. Nous utilisons également cette méthode pour obtenir des équations non-autonomes à partir des équations autonomes étudiées. Finalement, nous montrons que la méthode de croissance de degré nous donne, en plus, l'information permettant de connaître la méthode d'intégration précise à utiliser.

Au chapitre 6, nous examinons dans quelle mesure la propriété de Painlevé est une condition nécessaire à l'intégrabilité des équations différentielles ordinaires non-linéaires. Des exemples, pour plusieurs systèmes linéarisables, nous montrent que ce n'est pas le cas. Dans le domaine discret, une étude tout à fait analogue est faite dans laquelle la propriété de Painlevé est substituée par la propriété du confinement des singularités. Des résultats similaires sont obtenus, c'est-à-dire que nous trouvons des systèmes discrets linéarisables dont les singularités ne sont pas confinées.

Finalement, en annexe, nous obtenons le nombre ainsi que la forme des invariants (opérateurs de Casimir généralisés) pour les algèbres de Lie triangulaires nilpotentes et résolubles.

MOTS CLEF : Équations aux différences, symétrie, intégrabilité, groupes de Lie, systèmes non-linéaires, Painlevé, entropie algébrique, algèbres de Lie triangulaires, opérateurs de Casimir.

ABSTRACT

This thesis is devoted to the study of symmetries and of the integrability of equations with discrete independent variables.

In the first three chapters, we introduce a new method which allows us to calculate Lie point symmetries for systems of equations on a lattice. The Lie transformations act simultaneously on the discrete equation and on the lattice itself. The transformations take solutions of the system into other solutions. The case of equations with only one discrete variable is treated first, then we consider the multidimensional case.

In Chapter 4, a symmetry classification of possible interactions in a unidimensional diatomic molecular chain is provided. For non-linear interactions, the group of Lie point transformations, leaving the lattice invariant and taking solutions into solutions, is at most 5-dimensional. The equations are classified into equivalence classes under the action of a group of "allowed" transformations.

The second part of the thesis is devoted to the question of integrability. In Chapter 5, we investigate this property for partial difference equations with two independent variables. We use newly developed techniques for studying the degree of the iterates. We show that for nonintegrable equations, the degree grows exponentially fast, for integrable lattice equations the degree growth is polynomial. The growth criterion is used in order to obtain integrable deautonomisations of the equations examined. Finally, we show that degree growth contains information that can be an indication as to the precise integration method to be used.

In the last Chapter, we examine whether the Painlevé property is a necessary condition for the integrability of nonlinear ordinary differential equations. We show that for a large class of linearisable systems this is not the case. In the

discrete domain, we investigate wheter the singularity confinement property is satisfied for the discrete analogues of the non-Painlevé continuous systems.

Finally, in the Appendix, we find the number and the form of the invariants (generalized Casimir operators) for nilpotent and solvable triangular Lie algebras.

KEY WORDS : Difference equations, symmetry, integrability, Lie groups, nonlinear systems, Painlevé, algebraic entropy, triangular Lie algebras, Casimir operators.

REMERCIEMENTS

Comme c'est habituellement la coutume, la première personne qu'un étudiant au doctorat devrait remercier est son directeur de recherche. Je suiverai ici cette loi implicite avec plaisir, non pas parce que c'est la coutume, mais bien parce que je suis d'une extrême reconnaissance envers mon directeur de thèse Pavel Winternitz. Celui-ci aura souvent éclairé mes travaux avec des remarques et des conseils judicieux qui m'auront permis de sortir de mon obscurité. Ses méthodes de travail et sa rigueur me seront une source d'inspiration tout au long de ma carrière. Par ailleurs, je lui suis reconnaissant de m'avoir permis de passer des séjours en France, en Italie et au Mexique qui auront été riches, autant du point de vue scientifique que humain. Finalement, je souhaite le remercier pour son soutien financier tout au long de mes études graduées.

En second lieu, je voudrais remercier Decio Levi qui m'a invité à deux reprises à Rome pour des séjours de un mois pour des collaborations scientifiques. Je voudrais le remercier sincèrement pour l'accueil chaleureux qu'il m'a alors offert et pour son soutien scientifique et financier.

Dans le même sens, je remercie Basile Grammaticos et Alfred Ramani de m'avoir si bien accueilli dans leur laboratoire, à Paris, pour une période de un mois. La méthode et l'environnement de travail «grammani» fut une expérience des plus enrichissante pour moi. Je les remercie sincèrement de m'avoir ouvert les portes et offert patiemment des explications sur la science intégrable.

Je remercie également mes compagnons au quotidien du CRM. Entre autres, mon autre collaborateur, et ami depuis mes debuts comme étudiant gradué, Stéphane Lafortune. Je voudrais aussi remercier Diego Clonda, Ervig Lapalme, Philippe St-Jean, Miguel Tremblay (bref tous les étudiants PhysNum) ainsi que

Vincent Lemaire pour leur appui ainsi que leur aide à la rédaction de la présente thèse.

J'aimerais remercier T. Seligman pour son invitation pour un séjour de un mois au «Centro Internacional de Ciencias», à Cuernavaca (Mexique), dans le cadre de l'année académique «Symmetries in Nature».

Mes remerciements vont aussi à l'Université de Rome III et l'INFN pour leur soutien financier lors de mes deux séjours à Rome ainsi qu'au Centre de Coopération Interuniversitaire Franco-Québécois (CCIFQ) qui m'a accordé une bourse me permettant de faire mon séjour à Paris. Je remercie, de plus, la FES de l'Université de Montréal pour les bourses accordées lors de mes études graduées.

Finalement, je remercie l'oxygène ambiant O_2 qui me permet de le respirer *consciemment* dans l'instant.

«Je préfère être le monde que de le maintenir.»

—*Wateufek*

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INTRODUCTION

Une des hypotèses fondamentales qui se trouve à la base de la description mathématique de la réalité physique est que l'espace-temps est continu. Les équations de mouvement physique sont alors modélisées par des équations différentielles. Ce que nos sens et l'extension de ceux-ci, par les instruments de mesure, nous disent, c'est que le monde physique semble continu et que nous n'avons aucune raison de croire que cette hypothèse de continuité a des chances de disparaître par l'amélioration de la précision des instruments de mesure. Tout ce que les instruments peuvent nous donner, c'est une limite supérieure de la «longueur» du réseau de l'espace-temps possiblement discret. Il existe en effet des spéculations sérieuses pour lesquelles l'espace-temps serait discret, à des longueurs au-delà de nos possibilités expérimentales **[1, 2, 3]**. Typiquement, ces longueurs sont de l'ordre de la longueur de Planck (10^{-32} cm). Si cela s'avérait vrai, cela voudrait dire que nous devrions utiliser les équations aux variables discrètes pour la description de mouvement de la physique. Les équations continues, celles qui nous sont familières, apparaîtraient alors comme des approximations ou des cas limites (en faisant tendre certains paramètres vers zéro) des équations fondamentales discrètes.

L'aspect spéculatif d'un espace-temps discret n'est cependant pas une condition *sine qua non* pour que les équations aux variables discrètes deviennent intéressantes. La motivation première provient du fait que les équations discrètes apparaissent de façon naturelle dans plusieurs domaines de la physique. On les retrouve, entre autres, dans la description des réseaux de spins en mécanique statistique, dans les réseaux cristallins, dans les chaînes moléculaires, dans la théorie des groupes quantiques, etc **[4, 5, 6]**. Une seconde motivation provient du fait que ce sont des objets intrinsèquement intéressant à étudier d'un point de vue purement mathématique. De là toute l'attention que l'on a portée aux systèmes discrets, particulièrement au cours de la dernière décennie, en ce qui concerne les symétries ainsi que l'intégrabilité de ces systèmes. Il existe toutefois un domaine où les systèmes discrets sont inévitables : l'analyse numérique. En effet, avec l'avènement des ordinateurs de plus en plus puissants, la simulation numérique des phénomènes en science est devenue chose courante dans le monde scientifique. Or, la simulation numérique est basée exclusivement sur des équations aux variables discrètes. D'ailleurs, les méthodes d'intégration numérique des équations différentielles sont sans doute les plus populaires ; suivant la capacité de la machine, elles permettent d'obtenir l'évolution d'un système physique avec plus ou moins de précision.

Mentionnons cependant que cette thèse se situe plutôt dans le cadre des méthodes analytiques pour les équations aux variables discrètes. Les méthodes analytiques, lorsqu'elles peuvent nous fournir des solutions exactes, présentent des avantages certains du point de vue de l'information qu'elles nous donnent : solutions compactes valides pour toutes conditions initiales, comportements solitoniques, comportements chaotiques, comportements asymptotiques, etc. Le problème qui se pose alors est de trouver les solutions d'un système d'équations (différentielles ou discrètes). Cette tâche peut s'avérer très ardue, particulièrement lorsque les systèmes sont non-linéraires.

En ce qui concerne les équations différentielles, la théorie des groupes de Lie est une méthode analytique puissante permettant la résolution d'équations différentielles. En fait, la théorie des groupes apparaît dès le début du 19^{e} siècle avec les travaux de Ruffini, Cauchy et surtout Galois sur la résolution des équations algébriques. Plus tard, le mathématicien norvégien Sophus Lie (1842–1899) cherche à réaliser pour les équations différentielles ce que Galois avait fait pour les équations algébriques. Ainsi, la théorie générale des groupes continus, que l'on appelle maintenant plus communément *groupes de Lie*, a fait son apparition.

Essentiellement, l'idée de cette méthode est d'obtenir le plus grand groupe de transformations locales agissant sur l'espace des variables indépendantes et dépendantes de notre équation différentielle et qui a la propriété de transformer une solution de notre équation en une autre solution. Ainsi, pour une équation différentielle (scalaire) de la forme

$$\begin{aligned}&E(x,u,u^{(1)},u^{(2)},\ldots,u^{(n)})=0,\\&x=(x_1,x_2,\ldots,x_p)\in\mathbb{R}^p,\ u\in\mathbb{R},\ \ p,n\in\mathbb{Z}^{>0},\end{aligned}$$

où $u^{(k)}$ dénote toutes les dérivées partielles d'ordre k de $u(x)$, on considère la variété M des $(p+1)$-tuples (x,u) où les x_i sont les variables indépendante et u est la variable dépendante de notre équation

$$M\subset X\times U,\qquad X\sim\mathbb{R}^p,\ U\sim\mathbb{R}.$$

On cherche un groupe continu G agissant sur l'espace M, dont l'action est donnée par

$$\begin{aligned}G:\quad M\quad&\longrightarrow M\\(x,u)\quad&\longmapsto(\tilde{x}(x,u),\tilde{u}(x,u))\end{aligned}$$

où $\tilde{x}(x,u)$ et $\tilde{u}(x,u)$ sont des fonctions localement lisses. On dit que G est un *groupe de symétrie* de l'équation différentielle $E=0$ s'il transforme des solutions de cette équation en d'autres solutions. Précisément, si $u=f(x)$ est une solution de l'équation différentielle, alors

$$E(\tilde{x},\tilde{u},\tilde{u}^{(1)},\tilde{u}^{(2)},\ldots,\tilde{u}^{(n)})=0,$$

c'est-à-dire $\tilde{u}=g\cdot f(x)=\tilde{f}(\tilde{x})$ est aussi une solution de l'équation (où $\tilde{f}(\tilde{x})$ est la transformée de $f(x)$ sous l'action de $g\in G$).

Ce qu'il faut noter, c'est que l'algèbre de Lie, associée au groupe de symétrie de notre équation différentielle, peut être obtenue de façon purement algorithmique **[7, 8, 9]**. Ainsi, une fois l'agèbre de Lie obtenue, souvent appelée *algèbre de* *symétrie*, on peut intégrer les champs de vecteurs de cette algèbre afin d'obtenir le groupe de symétrie.

Les motivations pour lesquelles on cherche le groupe de symétrie d'une équation différentielle sont multiples :

(1) Le groupe de symétrie, qui laisse l'ensemble des solutions de notre système invariant, peut être appliqué sur une solution connue pour générer de nouvelles solutions. Il est ainsi parfois possible d'obtenir une solution non-triviale à partir d'une solution triviale.

(2) Le groupe de symétrie peut être utilisé pour faire de la «réduction par symétrie». Ainsi, pour les équations aux dérivées ordinaires (EDO), cela nous permet de réduire l'ordre de l'équation. Pour les équations aux dérivées partielles (EDP), la réduction par symétrie réduit le nombre de variables indépendantes.

(3) Si deux équations sont équivalentes alors elles ont un groupe de symétrie isomorphe. Ainsi, l'isomorphisme des deux groupes peut être utilisé comme condition nécessaire pour que deux équations puissent être transformées l'une dans l'autre par une transformation ponctuelle. En particulier, on peut déterminer si une équation non-linéaire est linéarisable par son groupe de symétrie [**9**].

Etant donné la puissance de l'utilisation des groupes de Lie pour les équations différentielles, il est légitime de se demander si une telle méthode peut aussi s'appliquer pour les équations aux variables discrètes. Dans ce sens, Levi et Winternitz (et leurs collaborateurs) ont développé deux méthodes pour obtenir les groupes de symétrie des équations aux variables discrètes. La première méthode, appelée *la méthode intrinsèque* [**10**], nous donne les symétries de Lie ponctuelles pour les équations aux variables discrètes sur un réseau fixe et uniforme. Par la suite, *la méthode des équations différentielles* a été introduite [**11**]. Celle-ci constitue en quelque sorte le complément de la méthode intrinsèque puisqu'elle donne une classe de symétries généralisées en plus des symétries de Lie ponctuelles. Comme la méthode intrinsèque, la méthode des équations différentielles s'applique sur un réseau fixe et uniforme. Dans les deux cas, la méthode pour obtenir les symétries des équations aux variables discrètes est algorithmique et similaire au cas des équations différentielles, donc relativement simple. Par contre, quand on passe du cas continu au cas discret par une de ces deux méthodes, certaines symétries peuvent disparaître. Il est en effet difficile d'envisager une dilatation, par exemple, sur un réseau fixe et uniforme.

Pour les équations aux variables discrètes *linéaires* sur un réseau fixe et uniforme, une approche supplémentaire a été proposée **[12, 13]**. Dans cette approche, l'équation discrète est formulée en termes d'opérateurs linéaires qui formeront les éléments de l'algèbre obtenue. Ici, l'algèbre de symétrie de l'équation discrète et sa limite continue sont isomorphes. Cette méthode a été étendue pour certaines équations aux variables discrètes dites linéarisables **[14]** ou intégrables **[15, 16, 17, 18]**. Dans le cas linéaire comme dans le cas non-linéaire, la méthode donne des symétries généralisées puisqu'elles agissent sur plusieurs points dans le réseau.

Dans les méthodes présentées ci-dessus, permettant d'obtenir le groupe de symétries pour les équations aux variables discrètes, on considérait les équations ainsi que le réseau comme des objets donnés *a priori* afin de les étudier. Le but étant de résoudre les équations, les classifier et d'identifier les équations linéarisables ou intégrables. Cependant, une autre approche introduite par Dorodnitsyn et des collaborateurs **[19, 20, 21, 22, 23, 24]** considère l'équation discrète ainsi que le réseau sur laquelle elle est définit comme des objets auxiliaires. Le point de départ étant le groupe de symétrie associé à une équation différentielle (ou à une classe d'équations différentielles) plutôt que l'équation discrète elle-même et son réseau. Par la suite, on cherche à discrétiser l'équation différentielle originale de façon à préserver ses symétries. Cette approche est donc intéressante du point de vue de l'analyse numérique, lorsque l'on veut faire une discrétisation qui soit fidèle aux symétries du cas continu. Mentionnons également que c'est dans cette approche qu'a été introduite la notion de l'action du groupe de symétrie, agissant à la fois sur l'équation ainsi que sur le réseau lui-même.

Telle que mentionné ci-dessus, une des motivations pour laquelle nous cherchons le groupe de symétrie d'une équation différentielle concernait la réduction par symétrie [**7, 8, 9**]. En effet, la méthode de réduction par symétrie s'avère souvent le seul outil permettant d'obtenir des solutions analytiques explicites. Du point de vue physique, ceci correspond à imposer une symétrie au modèle. Cette symétrie ne doit cependant pas être quelconque mais se trouver au niveau des équations, c'est-à-dire dans le groupe de symétrie. Ainsi, mathématiquement, la méthode est une construction qui consiste à déterminer les solutions d'un système d'équations différentielles caractérisées par leur invariance sous un sous-groupe du groupe de symétrie du systéme. Pour une EDP, la méthode permet de réduire le nombre de variables impliquées dans l'équation originale. Par exemple, si on cherche les solutions invariantes sous les rotations pour l'équation de Laplace $\triangle u = 0$, ceci correspond à introduire la nouvelle variable $r = \sqrt{x^2 + y^2 + z^2}$ qui réduit notre équation de Laplace originale à $u_{rr} + \frac{2u_r}{r} = 0$ dont la solution est $u = \frac{c_1}{r} + c_2$.

Le groupe de symétrie nous permet donc de ramener une EDP non-linéaire, par des réductions par symétrie, à une EDO non-linéaire. Cependant, même si on a réussi à réduire le nombre de variables impliquées dans notre équation, la tâche qui consiste à déterminer les solutions de telles équations est souvent difficile puisqu'aucune méthode systématique d'intégration n'existe. Toutefois, une classe importante d'EDO, pour laquelle les solutions sont connues, a pu être déterminée par les mathématiciens dès la fin du 19^{e} siècle. On appelle cette classe *équations de type Painlevé.* Cette classification est faite en fonction du type de singularités pouvant apparaître dans les solutions des EDO dans le plan complexe.

Pour les EDO *linéaires*, il est connu que les seules singularités possibles surviennent uniquement lorsque les coefficients de l'équation sont eux-même singuliers. Ces singularités sont dites *fixes* puisqu'elles sont indépendantes des constantes d'intégration. Toutefois, les EDO non-linéaires peuvent, en plus des singularités fixes, avoir des singularités dépendantes des constantes d'intégration. Ces singularités sont dites *mobiles*. Par exemple, si on considère l'équation $w_z + w^2 = 0$ dont la solution est $w = (z - z_0)^{-1}$, on voit que la solution est singulière en $z = z_0$ (z_0 étant la constante d'intégration) même si aucune singularité ne se trouve dans l'équation.

Les mathématiciens se sont alors aperçus que les EDO ne possédant que des pôles comme singularités mobiles s'avéraient beaucoup plus intéressantes que les équations avec des points de branchement ou des singularités essentielles (on appelle ces singularités, autres qu'un pôle, *points critiques*). En effet, on a remarqué que les solutions explicites des EDO sans points critiques mobiles pouvaient être déterminer. Ainsi Fuchs montra, en 1884, que pour les équations du premier ordre de la forme $w_z = F(w, z)$, où F est rationnelle en w et analytique en z, que la seule équation sans point critique est celle de Riccati $w_z = a(z) + b(z)w + c(z)w^2$ **[25]**. Mentionnons d'ailleurs que cette équation peut être transformée en une équation linéaire homogène du deuxième ordre en posant $w = \frac{-u_z}{c(z)u}$. De façon plus générale, les EDO du type $w_z^n = F(w, z)$, sans point critique mobile, sont transformables en l'équation de Riccati ou intégrables en termes des fonctions elliptiques de Jacobi. Par la suite, Painlevé et Gambier firent un travail remarquable dans lequel ils montrèrent qu'il existe 50 types d'équations de la forme $w_{zz} = F(w_z, w, z)$, rationnelles pour w_z, w et analytiques pour z, sans point critique mobile **[26, 27]**. Parmis ces 50 équations, 6 définissent de nouvelles fonctions que l'on nomme *transcendants de Painlevé*. Les 44 autres équations sont intégrables en termes de ces 6 transcendants de Painlevé ou en termes de fonctions elliptiques, fonctions élémentaires ou réduites à des équations linéaires. Mais ce qui est important à savoir, c'est que dans tous les cas les solutions sont connues.

On appelle maintenant les équations ne possédant que des pôles comme singularités mobiles *équations à la propriété de Painlevé*. On considère, de façon générale, qu'une équation possédant la propriété de Painlevé est intégrable.

D'autres classifications d'EDO furent entreprises par la suite. Cosgrove et Scoufis ont, par exemple, traité les équations de la forme $w_{zz}^2 = F(w_z, w, z)$ **[28]**. Bureau avait précédemment considéré le cas plus général $w_{zz}^2 = G(w_z, w, z)w_{zz} + F(w_z, w, z)$, mais est arrivé à une classification partielle **[29]**. Enfin, les EDO du troisième ordre de la forme $w_{zzz} = F(w_{zz}, w_z, w, z)$ furent également classifiées, mais encore une fois la classifation est incomplète **[30]**.

Les classifications précédentes des équations possédant la propriété de Painlevé ont toutes été faites en des classes d'équivalence sous l'action des transformations de Möbius données par

$$W(Z) = \frac{\alpha(Z)w(z) + \beta(Z)}{\gamma(Z)w(z) + \delta(Z)}, \quad \alpha\delta - \beta\gamma = 1, \quad Z = \phi(z), \quad \phi_z \neq 0.$$

La propriété de Painlevé est en effet invariante sous ces transformations. Donc, si une équation pour $w(z)$ possède la propriété de Painlevé, la nouvelle équation en $W(Z)$, générée par la transformation de Möbius, aura aussi la propriété. Ainsi, toute équation du second ordre de la forme $w_{zz} = F(w_z, w, z)$, possédant la propriété de Painlevé, peut être transformée par les transformations de Möbius en une des 50 formes canoniques.

Un algorithme, appelé *test de Painlevé*, permet de savoir si notre EDO possède certaines conditions nécessaires à la propriété de Painlevé. L'idée du test consiste à écrire la solution générale de notre EDO en termes de série formelle de Laurent et de vérifier certaines conditions sur les coefficients et les exposants de la série **[31]**. L'algorithme n'est cependant pas suffisant puisque celui-ci détecte si l'EDO est libre de certains points critiques seulement (points de branchement algébriques ou logarithmiques) mais des singularités essentielles peuvent encore subsister.

Toutefois, en pratique, très peu d'équations passent le test sans avoir la propriété de Painlevé.

Tout comme il fut naturel de se demander si on pouvait utiliser la théorie des groupes de Lie pour les équations aux variables discrètes, étant donné la puissance de la méthode pour les équations différentielles, on peut également se poser la question à savoir s'il existe un critère d'intégrabilité lié aux singularités pour les équations aux variables discrètes. Nous allons ici nous attarder à deux extensions de l'analyse de Painlevé pour les équations discrètes.

Un de ces critères d'intégrabilité, introduit par Ramani, Grammaticos et Papageorgiou, appelé *le confinement des singularités*, a l'avantage d'être à la fois simple et efficace **[32]**. Ce critère a permis de trouver plusieurs équivalents discrets d'équations intégrables. La méthode est basée sur l'observation que les singularités qui apparaissent spontanément lors de l'itération d'un système discret ne se propagent pas indéfiniment lorsque le système est intégrable, mais disparaissent après quelques itérations. Illustrons la procédure pour l'équation suivante

$$u_{n+1} + u_{n-1} = \frac{a}{u_n} + \frac{1}{u_n^2}, \tag{1}$$

où a est une constante. On voit que la seule singularité possible apparaît si $u_n = 0$ pour un certain n. La conséquence de cette singularité se reflète alors dans le comportement des itérations subséquentes. Ainsi u_{n+1} devient infini, u_{n+2} devient nul et u_{n+3} possède une indétermination de la forme $\infty - \infty$. Afin de lever cette indétermination, on introduit une perturbation autour de 0, c'est-à-dire que l'on pose $u_n = \varepsilon$, et on calcule les itérations suivantes de u en analysant leurs comportements quand $\varepsilon \to 0$. On obtient alors

$$\begin{aligned}
u_{n+1} &= \tfrac{1}{\varepsilon^2} + \tfrac{a}{\varepsilon} - u_{n-1}, \\
u_{n+2} &= -\varepsilon + a\varepsilon^2 + \mathcal{O}(\varepsilon^3), \\
u_{n+3} &= u_{n-1} + 2(au_{n-1} + 1)\varepsilon + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Ainsi, lorsque ε tend vers 0, on retrouve $u_{n+1} \to \infty$ et $u_{n+2} \to 0$ mais l'indétermination sur u_{n+3} est levée et donne une valeur finie. En fait, on retrouve l'information initiale contenue dans u_{n-1}, avant l'apparition de la singularité en $u_n = 0$. Ainsi, la singularité ne se propage pas indéfiniment et est confinée selon la séquence $\{0,\ \infty,\ 0\}$. Le critère d'intégrabilité est donc respecté. Si on étudie maintenant le confinement des singularités pour l'équation

$$u_{n+1}u_{n-1} = \frac{(1-au_n)^3}{u_n(u_n-a)^3},$$

alors on peut voir qu'en introduisant une perturbation autour de a, c'est-à-dire $u_n = a + \varepsilon$, on obtient alors la séquence

$$\{\infty,\ 0,\ f_1,\ \infty,\ 0,\ f_2, \ldots\}$$

qui se répète indéfiniment, où f_i représente une valeur finie. La singularité n'est donc jamais confinée et le critère d'intégrabilité n'est pas vérifié.

Hietarinta et Viallet ont cependant démontré que le confinement des singularité n'est pas un caractère suffisant à l'intégrabilité des équations aux variables discrètes **[35]**. En effet, ceux-ci ont vérifié que l'équation

$$u_{n+1} + u_{n-1} = u_n + \frac{1}{u_n^2}, \tag{2}$$

est confinée, la séquence des singularités étant $\{0,\ \infty,\ \infty,\ 0\}$, mais que cette équation est pourvue d'un comportement chaotique. Cet exemple n'étant pas un cas isolé, un critère plus rigoureux fut alors requis. Hietarinta et Viallet ont alors proposé un nouveau critère basé sur les idées d'Arnold et Veselov (voir **[33, 34]**) sur la complexité d'une application **[35, 36]**. En effet, comme Veselov le mentionnais : «The *integrability* has an essential correlation with the *weak growth* of certain characteristics»[1]. Ainsi, les auteurs de **[35]** ont proposé de tester

[1]L'intégrabilité a une corrélation essentielle avec la croissance lente de certaines caractéristiques

directement le degré des itérations successives et d'introduire la notion *d'entropie algébrique*. L'entropie algébrique E étant définie comme

$$E = \lim_{n\to\infty} \frac{\log(d_n)}{n},$$

où d_n est le degré de la $n^{\text{ème}}$ itération de certaines données initiales sous l'action de l'application. Une équation non-intégrable présentera une croissance de degré exponentielle : une entropie algébrique différente de zéro indiquera donc la non-intégrabilité. Les applications dites intégrables auront, quant à elles, une entropie algébrique nulle associée à une croissance de degré plus lente qu'exponentielle (typiquement, la croissance est polynômiale).

Afin de mettre les idées au clair, considérons la méthode proposée par Grammaticos et Ramani pour l'étude de la croissance de degré. Cette méthode présente quelques différences à celle donnée dans **[35]**, mais rend l'utilisation de la croissance de degré plus simple. On commence par introduire des variables homogènes par l'intermédiaire d'un choix approprié de données initiales. Par exemple, pour une application à 3-points, on introduit typiquement $u_0 = r$, $u_1 = p/q$, où on choisit le poids de r comme 0 et les poids de p et q comme 1. On calcule alors le degré homogène dans p et q pour le numérateur et le dénominateur de u_n pour chaque itération. Afin de comprendre le mécanisme de la croissance de degré, considérons quelques unes des premières itérations pour l'équation (1). On obtient

$$\begin{aligned} u_2 &= \frac{q^2+apq-rp^2}{p^2}, \qquad & u_3 &= \frac{pP_4}{q(q^2+apq-rp^2)^2}, \\ u_4 &= \frac{(q^2+apq-rp^2)P_6}{P_4^2}, & u_5 &= \frac{P_4P_9}{qP_6^2}, \end{aligned}$$

où les P_k sont des polynômes homogènes de degré k. Si on calcule le degré des itérations successives pour le numérateur (ou le dénominateur, puisque le degré est le même), on trouve : 0, 1, 2, 5, 8, 13, 18, 25, 32, 41,... Ici, on voit que la croissance de degré est polynômiale : on obtient $d_{2m} = 2m^2$ et $d_{2m+1} =$

$2m^2 + 2m + 1$. Donc, l'entropie algébrique de cette application est nulle, ce qui est en accord avec son caractère intégrable **[37]**.

L'étude de l'équation (2) à l'aide de la croissance de degré nous donne, quant à elle, une croissance exponentielle, ce qui est en accord avec le caractère chaotique de cette équation. Dans ce cas, la séquence des degrés des itérations est donnée par $0,\ 1,\ 3,\ 8,\ 23,\ 61,\ 160,\ 421, \ldots$ avec le ratio asymptotique $(3+\sqrt{5})/2$.

Aux chapitres 1 à 3 de cette thèse, on introduit une nouvelle méthode pour obtenir les transformations de Lie ponctuelles des équations aux variables discrètes. Dans cette méthode, on se place du point de vue où l'équation ainsi que le réseau sont connus *a priori*. Ici, le réseau n'est pas nécessairement fixe et uniforme et l'action du groupe de symétrie peut agir à la fois sur l'équation et sur le réseau qui restent invariants sur l'ensemble solution. Au chapitre 1, on introduit le formalisme pour les équations à une seule variable discrète. On considère également les équations différentielles aux différences (une variable continue et une variable discrète). Plusieurs exemples sont traités en détail et on considère quelques cas de réduction par symétrie. Au chapitre 2, on revise quelque peu le formalisme présenté au chapitre 1 et on considère la méthode pour l'équation de la chaleur discrète, c'est-à-dire pour deux variables discrètes[2]. Au chapitre 3, on généralise la méthode pour les équations discrètes à plusieurs dimensions. On verra que, contrairement à ce que l'on pourrait croire, il s'agit d'une généralisation non-triviale. Le cas des équations différentielles aux différences (avec au moins deux variables discrètes) est aussi considéré. Plusieurs exemples sont présentés en détail. Au chapitre 4, on présente une classification des interactions possibles pour une chaîne moléculaire diatomique unidimensionnelle. Le formalisme utilisé dans ce chapitre, pour classifier les interactions selon leurs groupes de symétrie, est la méthode intrinsèque présentée ci-dessus. Par la suite, au chapitre 5, on étudie l'intégrabilité pour les équations à deux variables discrètes. Pour ce faire on utilise

[2]Article dans le cadre de l'atelier DI-CRM sur la physique mathématique, Prague, République Tchèque, 18–21 Juin, 2000.

la méthode de la croissance de degré, jusque-là utilisée uniquement pour les équations à une seule variable discrète. Finalement, au chapitre 6, nous examinons si la propriété de Painlevé est une condition nécessaire pour l'intégrabilité des EDO non-linéaires et montrons que ce n'est pas le cas pour une large classe de systèmes linéarisables. De la même façon, dans le domaine discret, nous étudions si la propriété du confinement des singularités est satisfaite pour les analogues discrets des systèmes linéarisables ne satisfaisant pas Painlevé. Enfin, mentionnons que l'annexe A de cette thèse traite des invariants (ou opérateurs de Casimir généralisés) pour les algèbres de Lie triangulaires nilpotentes et résolubles. La forme ainsi que le nombre des invariants sont obtenus et des cas spéciaux sont traités en détail.

Chapitre 1

SYMÉTRIES DE LIE PONCTUELLES POUR LES ÉQUATIONS AUX DIFFÉRENCES ET LEURS RÉSEAUX

Lie Point Symmetries of Difference Equations and Lattices

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Abstract

A method is presented for finding the Lie point symmetry transformations acting simultaneously on difference equations and lattices, while leaving the solution set of the corresponding difference scheme invariant. The method is applied to several examples. The found symmetry groups are used to obtain particular solutions of differential-difference equations.

1 Introduction

Lie groups have long been used to study differential equations. As a matter of fact, they originated in that context [1, 2]. They have been put to good use to solve differential equations, to classify them, and to establish properties of their solution spaces [3, ..., 8].

Applications of Lie group theory to discrete equations, like difference equations, differential-difference equations, or q-difference equations are much more recent [9, ..., 37].

Several different approaches are being pursued. One philosophy is to consider a given system of discrete equations on a given fixed lattice and to

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search for a group of transformations, taking solutions into solutions, while leaving the lattice invariant. Within this philosophy different approaches differ by the restrictions imposed on the transformations and by the methods used to find the symmetries. One thing that is clear is that within this philosophy it is necessary to generalize the concept of point symmetries for difference equations, if we wish to recover all point symmetries of a differential equation in the continuous limit [9, ..., 26].

A different philosophy is to consider a difference equation and a lattice as two relations involving a fixed number of points, in which we give the values of the independent and dependent variables say x_-, x, x_+ and u_-, u, u_+ respectively. The group transformations act on the equation and on the lattice. This philosophy was mainly developped by Dorodnitsyn and collaborators [27, ..., 33]. In this approach, the given object was a Lie group and its Lie algebra. Invariants of this Lie group, depending on x and u, calculated at a predetermined number of points were obtained. They were used to obtain invariant equations and lattices. The emphasis was on discretizing differential equations while preserving all of their point symmetries, or at least most of them.

The purpose of this article is to combine the two philosophies. More specifically, we will consider given equations on given lattices, but the lattice will also be given by some equation. We will then look for Lie point transformations, acting on both equations, and leaving the common solution sets of both equations invariant.

In Section 2 we develop the formalism necessary for calculating simultaneous symmetries of difference or differential-difference equations and lattices. Section 3 is devoted to examples of symmetries of purely difference equations, both linear and nonlinear ones. In Section 4 we also consider examples, this time of differential-difference equations. Some conclusions are drawn in the final Section 5.

2 Symmetries of differential-difference equations

2.1 The differential-difference scheme

In this article we shall only consider a restricted class of problems, for reasons of simplicity and clarity. However, the formalism involved can easily be extended to quite general systems of equations.

Thus we shall consider one scalar function $u(x,t)$ of two variables only. The variable t is continuous and varies in some interval $I \subset \mathbb{R}$. The variable x is also continuous and varies in some interval $\tilde{I} \subset \mathbb{R}$. However, x will be 'sampled' in a set of discrete points $\{\ldots, x_{n-2}\,, x_{n-1}\,, x_n\,, x_{n+1}\,, \ldots\}$. The points x_k are not necessarily equally spaced.

We shall study the symmetries of a pair of equations which we postulate to have the form

$$\begin{aligned} E = E\left(t, \{x_k\}_{k=n-n_1}^{n+n_2}, \{u_k\}_{k=n-n_1}^{n+n_2}, u_{n,t}, u_{n,tt}\right) &= 0 && (1)\\ \Omega = \Omega\left(t, \{x_k\}_{k=n-n_3}^{n+n_4}, \{u_k\}_{k=n-n_3}^{n+n_4}\right) &= 0 \quad n_i \geq 0\,. && (2) \end{aligned}$$

We have $k, n, n_i \in Z$, all n_i are finite. Equations (1) is a differential equation in t and a difference equation in x, since we define:

$$\begin{array}{rclrcl} x_n & \equiv & x & x_{n-1} & \equiv & x_n - h_-(x_n, t) \\ x_{n+1} & \equiv & x_n + h_+(x_n, t) & x_{n+2} & \equiv & x_n + h_+(x_n, t) + h_+(x_{n+1}, t) \ldots \\ u_n & \equiv & u(x_n, t) & u_{n+k} & \equiv & u(x_{n+k}, t)\,. \end{array} \tag{3}$$

At this stage we are not imposing any boundary conditions, so we assume that equations (1) and (2) can be shifted arbitrarily to the left and to the right. Thus, eq.(1) and (2) involve any n_1+n_2+1 or n_3+n_4+1 neighbouring points, respectively.

The fact that (1) involves only first and second derivatives and that there are no derivatives in (2) is also for simplicity only. The same goes for the fact that derivatives are evaluated at the reference point n only (i.e. we do not consider terms like $\partial u(x_{n+1}, t)/\partial t$).

In order to be able to consider eq.(1) and (2) as a difference scheme, we must be able to obtain x_{n+N}, u_{n+N} and also x_{n-M}, u_{n-M} ($N = \max(n_2, n_4)$, $M = \max(n_1, n_3)$). In other words, we impose two conditions:

$$\det\left(\frac{\partial(E,\Omega)}{\partial(x_{n+N}, u_{n+N})}\right) \not\equiv 0 \qquad \det\left(\frac{\partial(E,\Omega)}{\partial(x_{n-M}, u_{n-M})}\right) \not\equiv 0\,. \tag{4}$$

If necessary, when calculating (4) we shift one of the equations, (1) or (2), to the left or right, so that the same values $n+N$ and $n-M$ figure in both equations.

In general, we do not require that a continuous limit should exist. If it does, then eq.(1) should go into a differential equation in x and t and eq.(2) should go into the identity $0 = 0$. When taking the continuous limit it is convenient to introduce 'discrete derivatives', e.g.

$$u_{,x} = \frac{u_{n+1} - u_n}{x_{n+1} - x_n} \qquad u_{,\underline{x}} = \frac{u_n - u_{n-1}}{x_n - x_{n-1}} \qquad u_{,x\bar{x}} = 2\frac{u_{,x} - u_{,\underline{x}}}{x_{n+1} - x_{n-1}} \tag{5}$$

etc. In the continuous limit we have $h_+(x_k, t) \to 0$, $h_-(x_k, t) \to 0$, $x_{n+k} \to x$, $u_k \to u(x)$ and the discrete derivatives go to the continuous ones.

A solution of the system (1), (2) will have the form $x_n = \Phi(n, c_1, \dots, c_k)$, $u_n = f(x_n, c_1, \dots, c_k)$ where $c_1, \dots, c_k$ are constants needed to satisfy initial conditions and the functions Φ and f are such that (1) and (2) become identities.

As a clarifying example of eqs.(1) and (2), let us consider a three point purely difference scheme, namely

$$E = \frac{u_{n+1} - 2u_n + u_{n-1}}{(x_{n+1} - x_n)^2} - u_n = 0 \tag{6}$$

$$\Omega = x_{n+1} - 2x_n + x_{n-1} = 0\,. \tag{7}$$

The equation $\Omega = 0$ determining the lattice has constant coefficients and its solution is $x_n = hn + x_0$, where $h = h_+ = h_-$ and x_0 are constants. The equation $E = 0$ on this lattice also has constant coefficients (since we have $x_{n+1} - x_n = h$) and its general solution is

$$u(x_n) = c_1 K_+^{x_n} + c_2 K_-^{x_n} \qquad K_\pm = \left(\frac{2 + h^2 \pm h\sqrt{4 + h^2}}{2}\right)^{1/h}. \tag{8}$$

In the continuous limit we obtain $E = 0 \to u'' - u = 0$, $\Omega = 0 \to 0 = 0$, $u(x) = c_1 e^x + c_2 e^{-x}$. Eq. (7) happens to determine a regular (equally spaced) lattice. Below we shall see examples of other lattices.

2.2 Symmetries of differential-difference schemes

Let us consider a one-parameter group of local point transformations of the form

$$\tilde{x} = \Xi_\lambda(x,t,u) \quad \tilde{t} = \Gamma_\lambda(t) \quad \tilde{u}(\tilde{x},\tilde{t}) = \Phi_\lambda(x,t,u)\,. \tag{9}$$

We shall require that they leave the system of equations (1), (2) invariant on the solution set of this system. Since we are interested in continuous transformations (of discrete systems), we use an infinitesimal approach and write the transformations up to order λ as

$$\begin{aligned}
\tilde{x} &= x + \lambda\,\xi(x,t,u(x,t)) && (10)\\
\tilde{t} &= t + \lambda\,\tau(t) && (11)\\
\tilde{u}(\tilde{x},\tilde{t}) &= u(x,t) + \lambda\,\phi(x,t,u(x,t)) \qquad |\lambda| \ll 1. && (12)
\end{aligned}$$

This assumption is quite restrictive. Not only do we consider only point transformations, but we require that both t and $\tilde{t}$ are continuous. No dependence, explicit or implicit, on the discretely sampled variable x is allowed. Indeed, once the lattice equation is solved, we get a discrete set of points $\{x_n\}$and this would introduce discrete values $\tilde{t} = \tilde{t}_n$, which we do not allow. Moreover, the x-dependence of t, if allowed, remains unspecified, since the considered equations involve only time derivatives. This would lead to wrong results, i.e. infinite dimensional transformation groups that do **not** take solutions into solutions.

We must now prolong the action of the transformation (10) to the prolonged space. This space includes the derivatives $u_t(x,t), u_{tt}(x,t)$, the shifted points $x_\pm = x_{n\pm 1}, \ldots$ and the function at shifted points $u_\pm = u(x_\pm, t), \ldots$

It is convenient to express the invariance condition for the system (1), (2) in terms of a formalism involving vector fields and their prolongations. The vector field itself has the form

$$\hat{X} = \xi(x,t,u)\,\partial_x + \tau(t)\,\partial_t + \phi(x,t,u)\,\partial_u \tag{13}$$

with ξ, τ and ϕ the same as in eq.(10)–(12). The prolongation of the vector field (13) acting on the system (1), (2) is

$$\mathrm{pr}^{(M+N)}\,\hat{X} = \hat{X} + \sum_{k=n-M}^{n+N} \xi(x_k,t,u_k)\partial_{x_k} + \sum_{k=n-M}^{n+N} \phi^{(k)}\partial_{u_k} + \phi^t\partial_{u_t} + \phi^{tt}\partial_{u_{tt}} \tag{14}$$

with

$$\phi^{(k)} = \phi(x_k, t, u_k) \tag{15}$$

$$\phi^{t} = D_t\phi - (D_t\xi)\,u_x - (D_t\tau)\,u_t \tag{16}$$

$$\phi^{tt} = D_t\phi^t - (D_t\xi)\,u_{xt} - (D_t\tau)\,u_{tt}\,. \tag{17}$$

Thus the prolongation coefficients ϕ^t, ϕ^{tt} are the same as for differential equations, the coefficients $\phi^{(k)}$ are as in [10, ..., 27].

The requirement that the system (1), (2) be invariant under the considered one-parameter group translates into the requirement

$$\mathrm{pr}\hat{X}\, E\,|_{E=0\;\Omega=0} = 0 \qquad \mathrm{pr}\hat{X}\,\Omega\,|_{E=0\,,\Omega=0} = 0\,. \tag{18}$$

In eq.(18), once the equations (1), (2) are taken into account, all involved variables are to be considered as independent. Eq.(18) are thus the determining equations for the infinitesimal coefficients ξ, τ and ϕ.

For purely difference equations (u_t and u_{tt} absent in (1)) the procedure is the following

1. Extract u_{n+N} and x_{n+N} (or u_{n-M} and x_{n-M}) from the equations (1) and (2) and substitute into eq.(18). This provides us with two functional equations for ξ, τ and ϕ.

2. Assuming an analytical dependence of ξ, τ and ϕ on their own variables, we convert these two equations into differential equations by differentiating them with respect to appropriately chosen variables u_{n+k}, x_{n+k}. Use the fact that the coefficients ξ, τ and ϕ depend on x and u evaluated at one point only to simplify the equations. Differentiate sufficiently many times to obtain differential equations that we can integrate.

3. Solve the differential equations, substitute back into the two original functional equations and solve them.

For differential-difference equations, we solve for the highest derivative (in our case u_{tt}) and for either x_{n+N}, or u_{n+N} (or x_{n-M} or u_{n-M}) and substitute into eq.(18). In this case, the determining equation will be a polynomial expression in the derivatives of u with respect to t (in our case u_t only) and all their coefficients must vanish. For the remaining terms, which depend on shifted variables, we proceed as in the case of purely difference equations.

3 Examples of symmetries of difference equations

We shall give several examples of the calculation of symmetries acting on difference schemes. They will involve either three or four points on a lattice. Equations (1) and (2) simplify to

$$E(x, x_-, x_+, x_{++}, u, u_-, u_+, u_{++}) = 0 \tag{19}$$

$$\Omega(x, x_-, x_+, x_{++}, u, u_-, u_+, u_{++}) = 0 \tag{20}$$

for a four point scheme. A three point scheme is obtained if E and Ω are independent of x_{++} and u_{++}. Here $x = x_n$ is the reference point and $x_- = x_{n-1}$, $x_+ = x_{n+1}$, $x_{++} = x_{n+2}$ and similarly for u.

The prolongation (14) of the vector field simplifies to

$$\begin{aligned} \mathrm{pr}\hat{X} &= \xi(x,u)\partial_x + \phi(x,u)\partial_u + \xi(x_-,u_-)\partial_{x_-} + \xi(x_+,u_+)\partial_{x_+} + \phi(x_-,u_-)\partial_{u_-} \\ &\quad + \xi(x_{++},u_{++})\partial_{x_{++}} + \phi(x_+,u_+)\partial_{u_+} + \phi(x_{++},u_{++})\partial_{u_{++}}. \end{aligned} \tag{21}$$

(for three point schemes we drop the x_{++}, u_{++} terms).

A symmetry classification of three point schemes was provided in a recent article [35]. Here we solve a different problem. The equations and lattices are given and we determine their symmetries.

3.1 Polynomial nonlinearity on a uniform lattice

Let us consider the nonlinear ordinary differential equation

$$u_{xx} - u^N = 0 \qquad N \not\equiv 0, 1. \tag{22}$$

A straightforward calculation shows that for $N \not\equiv -3$ eq.(22) is invariant under a two-dimensional Lie group, the Lie algebra of which is spanned by

$$\hat{P} = \partial_x \qquad \hat{D} = (N-1)x\partial_x - 2u\partial_u. \tag{23}$$

For $N = -3$ the symmetry algebra is $sl(2, \mathbb{R})$ with a basis

$$\hat{P} = \partial_x \qquad \hat{D} = 2x\partial_x + u\partial_u \qquad \hat{C} = x^2\partial_x + xu\partial_u. \tag{24}$$

A natural way to discretize eq.(22) is to use a uniform lattice and put

$$E = \frac{u_+ - 2u + u_-}{(x_+ - x)^2} - u^N = 0 \tag{25}$$

$$\Omega = x_+ - 2x + x_- = 0\,. \tag{26}$$

Let us now apply the symmetry algorithm (18). The condition $prX\,\Omega = 0$ for $E = 0$, $\Omega = 0$ implies

$$\xi(2x - x_-, (x - x_-)^2 u^N + 2u - u_-) - 2\xi(x,u) + \xi(x_-, u_-) = 0\,. \tag{27}$$

Differentiating first by ∂_{u_-}, then by ∂_u we obtain

$$-\xi_{u_+}(2x - x_-, (x - x_-)^2 u^N + 2u - u_-) + \xi_{u_-}(x_-, u_-) = 0 \tag{28}$$

$$\left[N(x - x_-)^2\, u^{N-1} + 2\right]\, \xi_{u_+u_+}(2x - x_-, (x - x_-)^2 u^N + 2u - u_-) = 0\,. \tag{29}$$

Eq.(29) implies that ξ is linear in u

$$\xi(x,u) = a(x)u + b(x)\,. \tag{30}$$

Eq.(28) reduces to $a(x_+) = a(x)$, i.e. a is a constant. Substituing these results into eq.(27) we obtain

$$a\,[u_+ - 2u + u_-] + b(x_+) - 2b(x) + b(x_-) = 0\,. \tag{31}$$

This implies $a = 0$ and

$$b(x_+) - 2b(x) + b(x_-) = 0\,. \tag{32}$$

Differentiating successively with respect to x and x_- we find $b_{x_+x_+}(x_+) = 0$, i.e.

$$b(x) = b_1 x + b_0\,. \tag{33}$$

Thus, the invariance of eq.(26) implies $\xi = b_1 x + b_0$ with b_1, b_0 constants. The function $\phi(x,u)$ is restricted by the requirement $\mathrm{pr}X\,E = 0$ for $E = 0$, $\Omega = 0$. This invariance condition is given by

$$\begin{aligned}&\phi(2x-x_-,(x-x_-)^2u^N+2u-u_-)-2\phi(x,u)+\phi(x_-,u_-)\\&-(x-x_-)^2[N\phi(x,u)u^{N-1}+2b_1u^N]=0\,.\end{aligned}\tag{34}$$

We successively differentiate this equation with respect to u_- and u and we obtain

$$-\phi_{u_+}(x_+,u_+)+\phi_{u_-}(x_-,u_-)\ =\ 0 \tag{35}$$

$$\phi_{u_+u_+}(x_+,u_+)\ =\ 0\,. \tag{36}$$

These two equations require that $\phi=\phi_1u+\phi_0(x)$ with ϕ_1 a constant. Substituing back into eq.(34) we obtain the remaining determining equation

$$\begin{aligned}&\phi_0(2x-x_-)-2\phi_0(x)+\phi_0(x_-)-(x-x_-)^2[(N-1)\phi_1+2b_1]u^N\\&-N(x-x_-)^2\phi_0u^{N-1}=0\end{aligned}\tag{37}$$

Since we have $N\not\equiv 0,1$ eq.(37) implies $\phi_0(x)=0$ and $\phi_1(1-N)=2b_1$. Finally, we obtain the symmetry algebra of the difference system (25), (26). It is 2-dimensional and coincides with the algebra (23) of the differential equation (22), the continuous limit of eq.(25).

Notice that the case $N=-3$ is not distinguished from the generic case. As a matter of fact, no difference equation on a uniform lattice can be invariant under the $SL(2,\mathbb{R})$ group corresponding to the algebra (24). A basis for the difference invariants of this algebra in the space $\{x,x_-,x_+,u,u_-,u_+\}$ is

$$\rho_1=\frac{h_-u_+}{(h_++h_-)u}\qquad \rho_2=\frac{h_+u_-}{(h_++h_-)u}\qquad \rho_3=\frac{h_+h_-}{(h_++h_-)u^2} \tag{38}$$

where h_+ and h_- are defined as $h_+=x_+-x$, $h_-=x-x_-$, and no function of x, x_+ and x_- alone can be set equal to a constant. An $SL(2,\mathbb{R})$ invariant scheme must be constructed out of these invariants. For instance, an invariant scheme approximating eq.(22) for $N=-3$ is

$$\frac{h_-(u_+-u)-h_+(u-u_-)}{h_+h_-(h_++h_-)}\ =\ \frac{2h_+h_-}{(h_++h_-)^2}\frac{1}{u^3}\qquad h_-u_+=h_+u_-\,. \tag{39}$$

3.2 Discrete versions of linear second order equations

3.2.1 Discretization of $u_{xx} = u$

Consider the ordinary differential equation

$$u_{xx} = u. \tag{40}$$

Like every second order linear ODE, it is invariant under $SL(3,\mathbb{R})$ with the Lie algebra realized in this case by the vector fields

$$\begin{aligned} &\hat{X}_1 = \partial_x \ , \ \hat{X}_2 = u\partial_u \ \ \hat{X}_3 = \mathrm{e}^{x}\partial_u \ , \ \hat{X}_4 = \mathrm{e}^{-x}\partial_u \ \ \hat{X}_5 = \mathrm{e}^{2x}(\partial_x + u\partial_u) \\ &\hat{X}_6 = u\mathrm{e}^{x}(\partial_x + u\partial_u) \ \ \hat{X}_7 = \mathrm{e}^{-2x}(\partial_x - u\partial_u) \ \ \hat{X}_8 = u\mathrm{e}^{-x}(\partial_x - u\partial_u). \end{aligned} \tag{41}$$

A very straightford discretization of eq.(40) on a uniform lattice is

$$\frac{u_+ - 2u + u_-}{(x_+ - x)^2} = u \tag{42}$$

$$x_+ - 2x + x_- = 0. \tag{43}$$

Applying the same procedure to the system (42), (43) that was applied to the system (25), (26) (with $N \not\equiv 0, 1$), we again obtain a 2-dimensional symmetry algebra

$$\hat{P} = \partial_x \qquad \hat{D} = u\partial_u. \tag{44}$$

At first glance the absence of symmetries of the form $\phi(x)\partial_u$, representing the linear superposition principle, seems surprising. However, viewed as a system of two equations, the system (42), (43) is really nonlinear. Eq.(43) defines a uniform lattice with an arbitrary step $h = x_+ - x = x - x_-$, where the step h can be scaled by a dilatation of x.

An alternative approach to the system (42), (43) is to first integrate eq. (43) once, thus fixing the step on the x-axis. The system (42), (43) is then replaced by the equation

$$\frac{u_+ - 2u + u_-}{h^2} = u \tag{45}$$

where $h = x_+ - x = x - x_-$ is a fixed (non-scalable) constant. The symmetry algorithm described in Section 2 and applied in Section 3.1 yields a four-dimensional symmetry algebra

$$\hat{P} = \partial_x \qquad \hat{D} = u\partial_u \qquad \hat{S}_1 = K_+^x \partial_u \qquad \hat{S}_2 = K_-^x \partial_u \tag{46}$$

with $K_\pm$ as in eq.(8). The symmetries $\hat{S}_1$, $\hat{S}_2$ represent the linear superposition formula for the linear system (45).

We mention that eq.(40) (and any linear ODE) can be discretized in a manner that exactly preserves all of its solutions. To do this we must preserve a subalgebra of the symmetry algebra of the ODE, containing the elements corresponding to the linear superposition formula. In our case these are $\hat{X}_3$ and $\hat{X}_4$ of eq.(41). Let us consider the subalgebra $\{\hat{X}_1, \ldots, \hat{X}_6\}$. Its second order discrete prolongation allows no invariants. It does however allow an invariant manifold, namely

$$I = u\mathrm{e}^{-x}(\mathrm{e}^{-2x_+} - \mathrm{e}^{-2x_-}) + u_+\mathrm{e}^{-x_+}(\mathrm{e}^{-2x_-} - \mathrm{e}^{-2x}) + u_-\mathrm{e}^{-x_-}(\mathrm{e}^{-2x} - \mathrm{e}^{-2x_+}) = 0\,. \tag{47}$$

The expression

$$S = \frac{\mathrm{e}^{-2x} - \mathrm{e}^{-2x_-}}{\mathrm{e}^{-2x_+} - \mathrm{e}^{-2x}} \tag{48}$$

is an invariant on the manifold (47).

Indeed, we have

$$\begin{gathered} (\hat{X}_1 + 3\hat{X}_2)\, I = 0 \qquad \hat{X}_3\, I = \hat{X}_4\, I = \hat{X}_5\, I = \hat{X}_6\, I = 0 \qquad \hat{X}_2\, I = I \\ \hat{X}_i\, S = 0 \;\; (i = 1, \ldots, 5) \qquad \hat{X}_6\, S = \frac{2\, I}{(\mathrm{e}^{-2x_+} - \mathrm{e}^{-2x})^2} \end{gathered} \tag{49}$$

so that we have

$$\hat{X}_i\, I\,|_{I=0} = 0 \qquad \hat{X}_i\, S\,|_{I=0} = 0 \qquad i = 1, \ldots, 6. \tag{50}$$

A uniform lattice, to first order in h and an equation with (40) as its continuous limit, is obtained by putting

$$S = 1 \qquad \frac{\mathrm{e}^{3x}\, I}{2h^3} = 0. \tag{51}$$

Eq.(51), or $I = 0$, has $u = \mathrm{e}^x$ and $u = \mathrm{e}^{-x}$ as solutions and the general solution is

$$u = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{-x} \tag{52}$$

just as in the continuous case (40).

To check this, let us solve the system $S = 1$, $I = 0$ directly, with I and S given in eq. (47) and (48), respectively. We linearize $S = 1$ by a change of variables and obtain:

$$z = \mathrm{e}^{-2x} \qquad z_+ - 2z + z_- = 0. \tag{53}$$

The solution is:

$$z_n = c_3 n + c_4 \quad x_n = -\frac{1}{2}\ln(c_3 n + c_4) \tag{54}$$

so that the lattice in x is logarithmic (c_3 and c_4 are integration constants). On this lattice eq.(47) reduces to

$$2u\sqrt{c_3 n + c_4} - u_+\sqrt{c_3(n+1) + c_4} - u_-\sqrt{c_3(n-1) + c_4} = 0. \tag{55}$$

To solve this linear equation we put $u(x) = \mathrm{e}^x f(x)$ or, on the lattice

$$u(x_n) = \frac{1}{\sqrt{c_3 n + c_4}} f(x_n) \tag{56}$$

so that $f(x)$ satisfies

$$f(x_+) - 2f(x) + f(x_-) = 0. \tag{57}$$

We write the general solution of eq.(57) as

$$f(x_n) = f \circ x(n) = A\,n + B\ . \tag{58}$$

By rewriting A and B in terms of the new integration constants c_1 and c_2, i.e. by putting $A = c_2\,c_3$ and $B = c_2\,c_4 + c_1$, we obtain the general solution of the system (51) as

$$u = \frac{c_1}{\sqrt{c_3 n + c_4}} + c_2\sqrt{c_3 n + c_4} = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{-x} \tag{59}$$

in full agreement with eq.(52).

3.2.2 Discrete version of $u_{xx} = 1$

Let us consider the simplest 3 point difference scheme for the ODE $u_{xx} = 1$

$$\frac{u_+ - 2u + u_-}{(x_+ - x)^2} = 1 \quad x_+ - 2x + x_- = 0\,. \tag{60}$$

Applying the prolonged vector field to these equations and eliminating x_+ and u_+, we obtain two equations

$$\xi(2x - x_-, (x - x_-)^2 + 2u - u_-) - 2\xi(x,u) + \xi(x_-, u_-) = 0, \tag{61}$$

$$\begin{aligned} &\phi(2x - x_-, (x - x_-)^2 + 2u - u_-) - 2\phi(x,u) + \phi(x_-, u_-) = \\ &\quad 2(x - x_-)\left[\xi(2x - x_-, (x - x_-)^2 + 2u - u_-) - \xi(x,u)\right]\,. \end{aligned} \tag{62}$$

We first concentrate on eq.(61). Taking the second derivative with respect to u and u_- we find that ξ is linear in u. Substituing back into (61) and differentiating with respect to x and x_- we find

$$\xi(x,u) = \alpha(u - \frac{x^2}{2}) + \beta_1 x + \beta_0 \tag{63}$$

where α, β_1 and β_0 are constants. Substituing ξ into eq.(62) and solving for ϕ in a similar manner, we obtain:

$$\phi(x,u) = \alpha(xu - \frac{x^3}{2}) + c(u - \frac{x^2}{2}) + \beta_1 x^2 + \beta_2 x + \beta_3. \tag{64}$$

Finally, a basis for the symmetry algebra of the system (60) is

$$\begin{aligned} &\hat{X}_1 = \partial_x \quad \hat{X}_2 = \partial_u \quad \hat{X}_3 = x\partial_u \quad \hat{X}_4 = x\partial_x + x^2\partial_u \\ &\hat{X}_5 = (u - \tfrac{x^2}{2})\partial_u \quad \hat{X}_6 = (u - \tfrac{x^2}{2})\partial_x + (u - \tfrac{x^2}{2})x\partial_u. \end{aligned} \tag{65}$$

It is easy to check that this Lie algebra is isomorphic to the general affine Lie algebra $gaff(2,\mathbb{R})$. This is the symmetry algebra of the scheme [35]

$$w_+ - 2w + w_- = 0 \quad t_+ - 2t + t_- = 0. \tag{66}$$

Indeed the system (60) is transformed into (66) by putting

$$u = w + \frac{t^2}{2} \quad x = t. \tag{67}$$

3.3 Discrete versions of the equation $u_{xxx} = 0$

The symmetry algebra of the ODE $u_{xxx} = 0$ is 7-dimensional. A basis for this algebra is

$$\begin{gathered} \hat{X}_1 = \partial_x \quad \hat{X}_2 = \partial_u \quad \hat{X}_3 = x\partial_x \quad \hat{X}_4 = u\partial_u \quad \hat{X}_5 = x\partial_u \\ \hat{X}_6 = x^2\partial_u \ , \ \hat{X}_7 = x^2\partial_x + 2xu\partial_u . \end{gathered} \tag{68}$$

The generators $\hat{X}_2$, $\hat{X}_5$, $\hat{X}_6$ correspond to the linear superposition principle. We can add $u = c_2x^2 + c_1x + c_0$ to any solution and indeed, this itself is the general solution.

Let us now consider discretizations of this ODE.

3.3.1 Discretization on a uniform lattice

We consider the system

$$E = u_{++} - 3u_+ + 3u - u_- = 0 \tag{69}$$

$$\Omega_1 = x_+ - 2x + x_- = 0. \tag{70}$$

The lattice is uniform, since the general solution of (70) is $x_n = hn + x_0$ with h and x_0 constants. Eq.(70) must be shifted once to the right to obtain x_{++}.

The prolonged vector fields have the form (21). We apply the same method as in Section 3.2. to obtain the symmetry algebra of the system (69), (70). The result is a 6-dimensional Lie algebra generated by $\{\hat{X}_1 , \hat{X}_2 , \hat{X}_3 , \hat{X}_4 , \hat{X}_5 , \hat{X}_6\}$ of eq.(68). The system hence has exactly the same solutions as the ODE $u_{xxx} = 0$, however the lattice is not invariant under the projective transformations generated by $\hat{X}_7$.

3.3.2 Discretization on a four point lattice

We take the equation (69) on the lattice

$$\Omega_2 = x_{++} - 3x_+ + 3x - x_- = 0 . \tag{71}$$

The lattice given by equation (71) is not uniform but satisfies $x_n = L_2n^2 + L_1n + L_0$, where L_i are constants. We assume $L_2 \not\equiv 0$, otherwise the lattice is the same as for $\Omega_1 = 0$.

The symmetry algebra in this case is given by

$$\{\hat{X}_1\,,\,\hat{X}_2\,,\,\hat{X}_3\,,\,\hat{X}_4\,,\,\hat{X}_5\,,\,\hat{Y}=u\partial_x\} \tag{72}$$

with $\hat{X}_1,\ldots,\hat{X}_5$ as in eq.(68). Thus $\hat{X}_6$ of (68) is absent. This reflects the fact that $u=x^2$ is not an exact solution on the lattice $\Omega_2=0$. Indeed, if we take $L_2=1$ and $L_1=L_0=0$ in eq.(68) we have $u=n^4$ which would solve a fourth order equation, not however equation (69).

3.3.3 Discretization preserving the entire symmetry group

The third prolongation of the algebra (68) acts on an 8-dimensional space with coordinates $(x\,,\,x_+\,,\,x_{++}\,,\,x_-\,,\,u\,,\,u_+\,,\,u_{++}\,,\,u_-)$. If the 7 prolonged fields are linearly independent, they will allow only one invariant. This invariant can be calculated directly. It lies entirely in the subspace $\{x\,,\,x_+\,,\,x_{++}\,,\,x_-\}$ and is given by the anharmonic ratio of four points, namely

$$\frac{(x_{++}-x)(x_+-x_-)}{(x-x_-)(x_{++}-x_+)}=K\,. \tag{73}$$

This is the invariant of the projective action of $sl(2,\mathbb{R})$ on the real line $\mathbb{R}$, given by the ∂_x part of the subalgebra $\{\hat{X}_1\ ,\ \hat{X}_3\ ,\ \hat{X}_7\}$ of the algebra (68). Eq.(73) provides us with a lattice. The invariant equation is obtained by requiring that the third prolongation of $(\hat{X}_1,\ldots,\hat{X}_7)$ be linearly connected on some manifold. This manifold is given by the condition

$$\begin{aligned} I = \; & - \; (u_+-u)(x_{++}-x)(x-x_-)(x_{++}-x_-) \\ & + \; (u_{++}-u)(x_+-x)(x-x_-)(x_+-x_-) \\ & + \; (u-u_-)(x_+-x)(x_{++}-x)(x_{++}-x_+)=0\,. \end{aligned} \tag{74}$$

It is easy to check that I is indeed invariant, i.e.

$$\mathrm{pr}^{(3)}\hat{X}_i\,I\,|_{I=0}=0 \qquad i=1,\ldots,7\,. \tag{75}$$

Finally, a difference scheme, invariant under the group generated by the algebra (68), having $u_{xxx}=0$ as a continuous limit is given by

$$u_{,\underline{x}x\bar{x}}=\frac{6\,I}{(x_{++}-x_-)(x_{++}-x)(x_{++}-x_+)(x_+-x_-)(x_+-x)(x-x_-)}=0 \tag{76}$$

and eq.(71).

We define discrete derivatives as

$$\begin{aligned} u_{,x} &= \frac{u_+ - u}{x_+ - x} & u_{,\bar{x}} &= \frac{u_{++} - u_+}{x_{++} - x_+} & u_{,\underline{x}} &= \frac{u - u_-}{x - x_-} \\ u_{,x\bar{x}} &= 2\frac{u_{\bar{x}} - u_x}{x_{++} - x} & u_{,x\underline{x}} &= 2\frac{u_x - u_{\underline{x}}}{x_+ - x_-} \\ u_{,\underline{x}x\bar{x}} &= 3\frac{u_{x\bar{x}} - u_{x\underline{x}}}{x_{++} - x_-}. \end{aligned} \tag{77}$$

Any four solutions of a Riccati equation satisfy eq.(73) and we use this fact to solve this equation. Indeed, consider e.g. the Riccati equation

$$\dot{x} = Ax^2 + Bx + C \qquad B^2 - 4AC > 0 \tag{78}$$

where A, B and C are real constants and $A \not\equiv 0$. The general solution of eq.(78) is

$$x = \frac{x_1 + x_2\,\omega\,\mathrm{e}^{A(x_1 - x_2)t}}{1 - \omega\,\mathrm{e}^{A(x_1 - x_2)t}} \qquad x_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{79}$$

Let us take $\omega = n$, $x_1 = \alpha$, $x_2 = \beta$ and $\mathrm{e}^{A(x_1 - x_2)t} = \gamma$. A solution of eq.(79) is

$$x \equiv x(n) = \frac{\alpha n + \beta}{\gamma n + \delta} \qquad \alpha,\ \beta,\ \gamma,\ \delta = const.,\ \alpha\delta - \beta\gamma = 1. \tag{80}$$

Substituting into eq.(73) we find $K = 4$. The value $K = 4$ is also required to obtain the correct continuous limit. Indeed, putting $x_+ - x = \epsilon\sigma_1$, $x - x_- = \epsilon\sigma_2$, $x_{++} - x_+ = \epsilon\sigma_3$, $\sigma_i \in \mathbb{R}$ and $\epsilon \longrightarrow 0$ we have

$$\frac{\epsilon^2(\sigma_1 + \sigma_3)(\sigma_1 + \sigma_2)}{\epsilon^2\sigma_2\sigma_3} = K \tag{81}$$

and for $\sigma_1 = \sigma_2 = \sigma_3$ we have $K = 4$ and also $u_{,x} \longrightarrow u'$, $u_{,\bar{x}} \longrightarrow u'$, $u_{,\underline{x}} \longrightarrow u'$, $u_{,x\bar{x}} \longrightarrow u''$, $u_{,\underline{x}x} \longrightarrow u''$, $u_{,\underline{x}x\bar{x}} \longrightarrow u'''$, where the primes denote (continuous) derivatives.

Plots of $x(n)$ for lattices (70), (71) and (80) are shown on Figure 1,2 and 3, respectively. The expression (80) is singular for $\gamma = \delta/n$, so such values of γ are to be avoided.

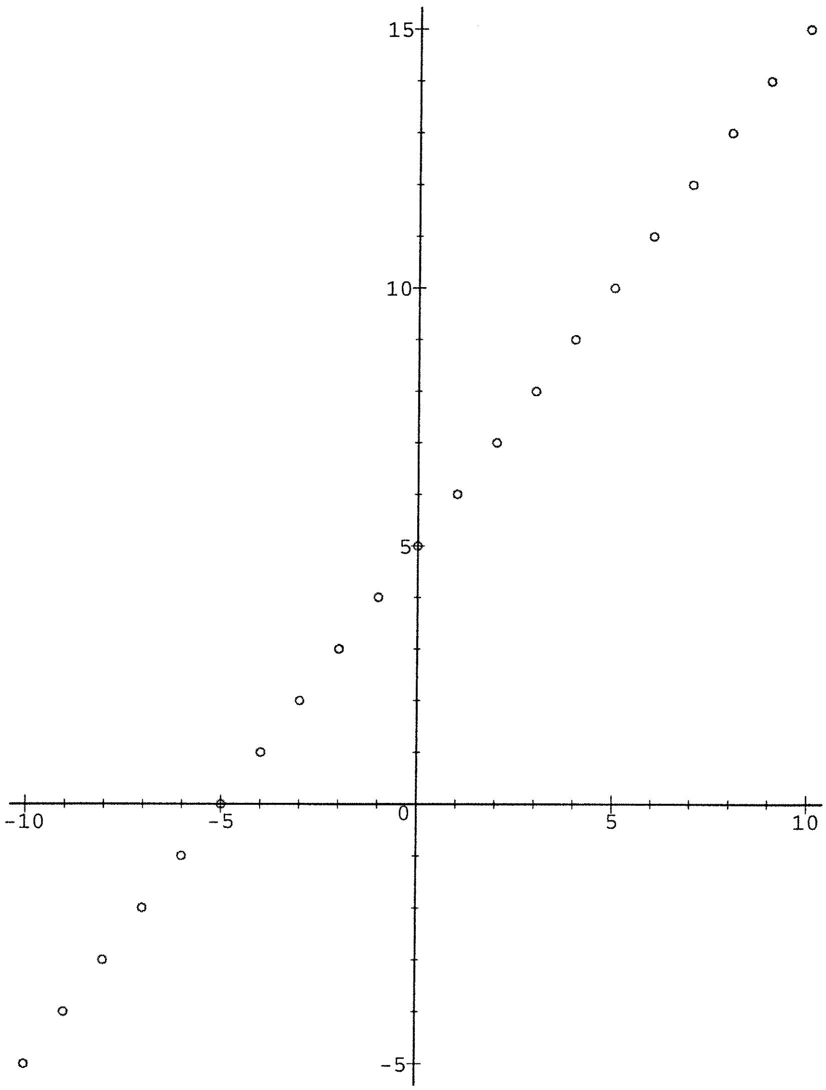

Figure 1: Variable x as a function of n for the lattice (70) $x_n = hn + x_0$ ($h = 1$, $x_0 = 5$)

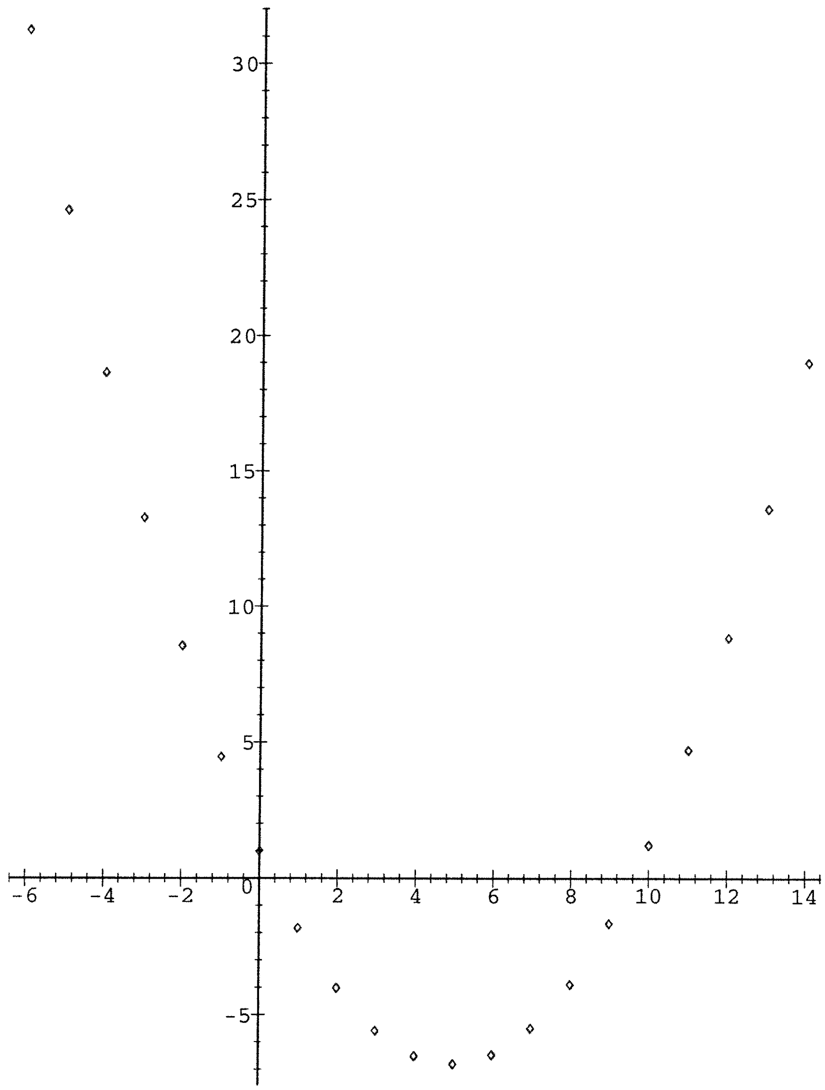

Figure 2: Variable x as a function of n for the lattice (71) $x_n = L_2 n^2 + L_1 n + L_0$ $(L_2 = 1/\sqrt{10},\ L_1 = -\pi,\ L_0 = 1)$

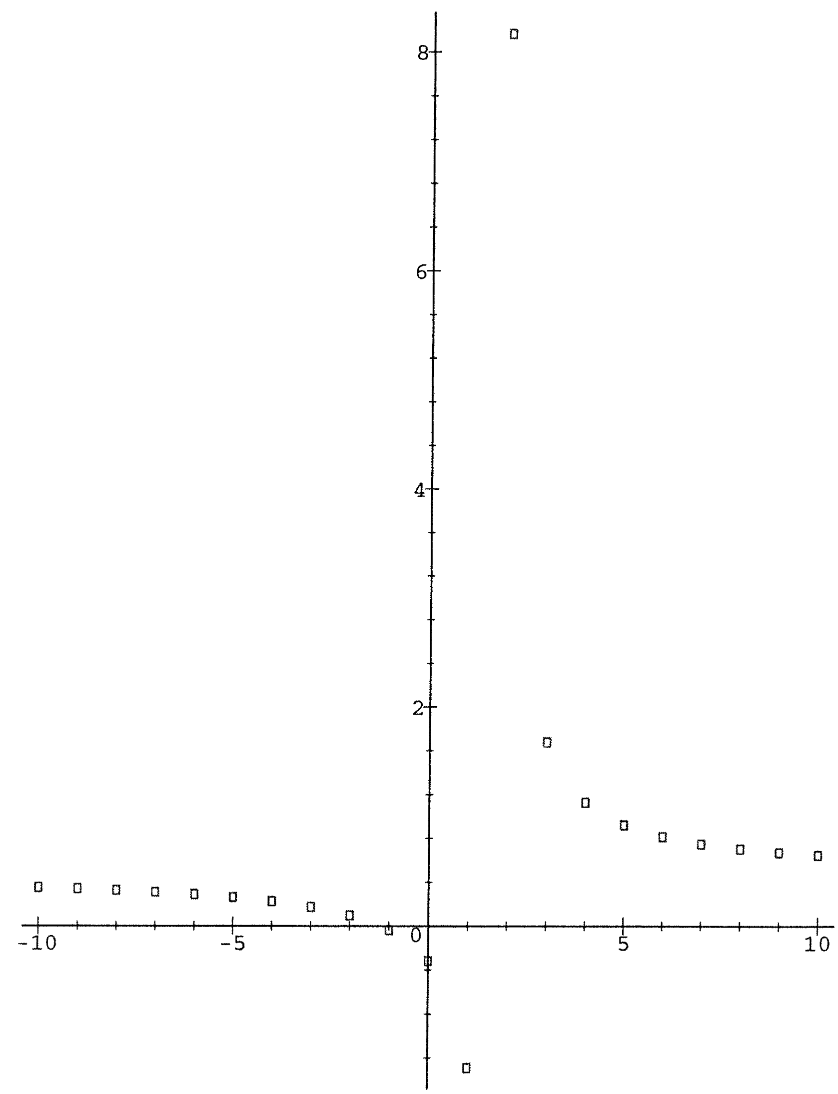

Figure 3: Variable x as a function of n for the lattice (80) $x_n = (\alpha n+\beta)(\gamma n+\delta)^{-1}$ ($\alpha = \sqrt{2}$, $\beta = -\sqrt{3}$, $\gamma = 3$, $\delta = -\sqrt{3}\pi$)

4 Examples for differential-difference equations

In this section we shall need the complete formalism of Section 2, in particular the vector field prolongation (14),...,(17).

4.1 Symmetries of the discrete Volterra equation

The discrete Volterra equation [17] on a uniform lattice is represented by the two equations

$$E \equiv u_t + u\frac{u_+ - u_-}{x_+ - x_-} = 0 \tag{82}$$

$$\Omega \equiv x_+ - 2x + x_- = 0 \tag{83}$$

where t is a continuous variable, $u = u(x,t)$ and $u_t = \partial u/\partial t$. The Volterra equation is integrable [17] but we make no use of that here.

The invariance condition for the lattice (83) is

$$\xi(2x - x_-, t, u_+) - 2\xi(x,t,u) + \xi(x_-, t, u_-) = 0\,. \tag{84}$$

Contrary to the cases in Section 3, the values u_+, u and u_- in eq.(84) are independent, since the equation $E = 0$ involves u_t (in addition to u_+, u and u_-). Differentiating eq.(84) with respect to e.g. u we obtain $\xi_u = 0$. Differentiating with respect to x_- and then x, we obtain $\xi_{x_+x_+}(x_+,t) = 0$. The function $\xi(x,t,u)$ hence reduces to

$$\xi = a(t)x + b(t) \tag{85}$$

with $a(t)$ and $b(t)$ so far arbitrary functions of t.

Invariance of the equation (82) implies:

$$\phi^t + \phi\frac{u_+ - u_-}{x_+ - x_-} + \frac{u}{x_+ - x}(\phi^{(+)} - \phi^{(-)}) - \frac{u(u_+ - u_-)}{(x_+ - x_-)^2}(\xi^{(+)} - \xi^{(-)})\,|_{E=\Omega=0} = 0\,. \tag{86}$$

The coefficients in the prolongation satisfy

$$\begin{aligned} \phi^t &= \phi_t + (\phi_u - \tau_t)u_t - \xi_t u_x - \xi_u u_t u_x - \tau_u u_t^2 & (87) \\ \phi^{(\pm)} &= \phi\left(x_\pm, t, u(x_\pm, t)\right). & (88) \end{aligned}$$

We substitute (85), (87) and (88) into eq.(86) and eliminate $u_t(x,t)$ and x_+ using the equations (82) and (83). The only term involving u_x is in ϕ^t. Its coefficient ξ_t must vanish and we find $\dot{a} = \dot{b} = 0$ in the expression (85).

The remaining determining equation is

$$\left\{\phi_t + [\phi - u(\phi_u - \tau_t - au)]\frac{u_+ - u_-}{x_+ - x_-} + \frac{u}{x_+ - x_-}\left[\phi(x_+, t, u(x_+, t)) - \phi(x_-, t, u(x_-, t))\right]\right\}_{x_+ = 2x - x_-} = 0\,. \qquad (89)$$

We differentiate twice with respect to u_+ and obtain $\phi_{u_+u_+} = 0$, so that we have $\phi(x,t,u) = \phi_1(x,t)u + \phi_0(x,t)$. Substituing back into eq.(89) we obtain the final result, namely

$$\xi = ax + b \qquad \tau = c_1 t + c_2 \qquad \phi = (a - c_1)u. \qquad (90)$$

Thus, the difference scheme (82), (83) which is the usual Volterra equation, is invariant under a 4-dimensional group of Lie point transformations. The symmetry algebra is spanned by

$$\hat{P}_0 = \partial_t \qquad \hat{P}_1 = \partial_x \qquad \hat{D}_0 = t\partial t - u\partial_u \qquad \hat{D}_1 = x\partial x + u\partial_u \qquad (91)$$

(two translations and two dilatations).

The continuous limit of the system (82), (83) is the Euler equation in $1+1$ dimensions

$$u_t + uu_x = 0\,. \qquad (92)$$

Its symmetry group is infinite-dimensional and can be obtained by standard techniques [3, ...,8] (though we have not found it given explicitely in the literature). Its symmetry algebra is spanned by

$$\begin{gathered} \hat{X}(\xi) = \xi(z,u)\partial_x \qquad \hat{T}(\tau) = \tau(z,t,u)\left(\partial_t + u\partial_x\right) \\ \hat{F}(\phi) = \phi(z,u)\left(t\partial_x + \partial_u\right) \qquad z = x - ut \end{gathered} \qquad (93)$$

where ξ, τ and ϕ are arbitrary functions of their arguments.

The Volterra equation (82) is certainly not a 'symmetry preserving' discretization of the Euler equation (92) on a uniform lattice. It only preserves the four-dimensional subalgebra (91) of the infinite-dimensional symmetry algebra (93). Let us mention here that eq.(82) is well known to be a bad numerical scheme for eq.(92).

4.2 A general nearest neighbour interaction equation

Let us consider the difference scheme

$$E \equiv u_{tt} - F(t, x_+, x, x_-, u_+, u, u_-) = 0 \tag{94}$$

$$\Omega \equiv x_+ - 2x + x_- = 0 \tag{95}$$

where F is an arbitrary smooth function satisfying

$$(F_{u_+}, F_{u_-}) \not\equiv (0,0)\,. \tag{96}$$

A symmetry analysis of a similar class of equations was recently performed for a fixed (non transformable) regular lattice [12]. More specifically, the assumption was $x_n = n$, $n \in Z$.

The prolongation formula for the vector field (13) is (14),...,(17). Applying it to eq.(95) we obtain that ξ has the form (85), just as for the Volterra equation. Applying the prolongation to the eq.(94) we obtain

$$\begin{aligned} &\phi^{tt} - \tau F_t - (ax+b)F_x - (ax_+ + b)F_{x_+} - (ax_- + b)F_{x_-} - \phi F_u \\ &-\phi^{(+)}F_{u_+} - \phi^{(-)}F_{u_-}\,|_{E=\Omega=0} = 0. \end{aligned} \tag{97}$$

We substitute the expression for ϕ^{tt}, $\phi^{(+)}$ and $\phi^{(-)}$ and set the coefficients of u_t^3, u_t^2, $u_t^2 u_x$, $u_t u_{xt}$, u_{xt}, u_t equal to zero, after eliminating u_{tt} and x_+, using equations (94), (95). The result is that for any interaction F satisfying condition (96), we have

$$\tau = \tau(t) \qquad \xi = ax + b \qquad \phi = \left[\frac{\dot{\tau}}{2} + \alpha(x)\right] u + B(x,t). \tag{98}$$

The as yet unspecified functions $\tau(t)$, $\alpha(x)$, $B(x,t)$ and constants a, b satisfy a remaining determining equation, namely

$$\Big\{\tfrac{1}{2}\tau_{ttt}u + B_{tt} - (\tfrac{3}{2}\tau_t - \alpha)F + \tau F_t - (ax+b)F_x - (ax_+ + b)F_{x+} \\ -(ax_- + b)F_{x_-} - \left[(\tfrac{1}{2}\tau_t + \alpha(x))u + B\right] F_u - \left[(\tfrac{1}{2}\tau_t + \alpha(x_+))u_+ + B(x_+,t)\right] F_{u_+} \\ - \left[(\tfrac{1}{2}\tau_t + \alpha(x_-))u_- + B(x_-,t)\right] F_{u_-}\Big\}_{x_+=2x-x_-} = 0 . \tag{99}$$

The results (98), (99) agree with those of Ref.[12], but are more general. The reason for the increase in generality is that here the lattice is not fixed *a priori* and hence the vector field (13) contains a term proportional to ∂_x.

To proceed further, we restrict the interaction F to have a specific form.

4.3 Equation with $F = (x_+ - x)^6(u_+ - 2u + u_-)^{-3}$

Let us consider a special case of the system (94), (95), namely

$$u_{tt} = \frac{(x_+ - x)^6}{(u_+ - 2u + u_-)^3} \tag{100}$$

$$x_+ - 2x + x_- = 0. \tag{101}$$

We substitute F of eq.(100) into the determining equation (99) and clear the denominator. The dependence on u, u_+ and u_- is explicit and we obtain

$$\tau_{ttt} = 0 \quad B_{tt} = 0 \quad B(x_+,t) - 2B(x,t) + B(x_-,t) = 0 \\ \alpha(x)(x_+ - x) + 6(ax+b) - 6(ax_+ + b) + 3\alpha(x_+)(x_+ - x) = 0. \tag{102}$$

Analysing the system (102) in the usual manner, we obtain a 9-dimensional Lie algebra with basis

$$\hat{P}_0 = \partial_t \;\; \hat{P}_1 = \partial_x \;\; \hat{D}_1 = 2t\partial_t + u\partial_u \;\; \hat{D}_2 = 2x\partial_x + 3u\partial_u \\ \hat{C} = t^2\partial_t + tu\partial_u \;, \;\; \hat{W}_1 = \partial_u \;\; \hat{W}_2 = t\partial_u \;\; \hat{W}_3 = x\partial_u \;\; \hat{W}_4 = tx\partial_u . \tag{103}$$

A related system was studied earlier [12], namely

$$\ddot{u}_n(t) = \left[(\gamma_n - \gamma_{n-1})u_{n+1} + (\gamma_{n+1} - \gamma_{n-1})u_n + (\gamma_{n-1} - \gamma_n)u_{n+1}\right]^{-3} \qquad (104)$$

where γ_n is any function of n, satisfying $\gamma_{n+1} \not\equiv \gamma_n$. If we take $\gamma_n = n$ in eq.(104) and $x = n$ in (100), (101) the two systems coincide. The symmetry algebra found in Ref.[12] is the subalgebra $\{\hat{P}_0, \hat{D}_1, \hat{C}, \hat{W}_1, \hat{W}_2, \hat{W}_3, \hat{W}_4\}$ of the algebra (103). The elements $\hat{P}_1$ and $\hat{D}_2$ are absent, since the lattice is fixed. Shifts $n' = n + N$ are allowed, but are not infinitesimal.

The system (100), (101) has a continuous limit

$$u_{tt} = \frac{1}{u_{xx}^3}. \qquad (105)$$

The symmetry algebra of eq.(105) coincides with (103), i.e. the system (100), (101) is a symmetry preserving discretization of eq.(105). We emphasize that eq.(100) was obtained as part of a classification of difference equations [12], not in any connection with the PDE (105).

4.4 Equation without a continuous limit

Let us now consider another special case of the system (94), (95), namely

$$u_{tt} = \frac{1}{(u_+ - 2u + u_-)^3} \qquad x_+ - 2x + x_- = 0. \qquad (106)$$

Substituing for F into eq.(99) and proceeding as in Section 4.3. we again obtain a 9-dimensional symmetry algebra. It differs from that given in eq.(103) only in that D_2 is replaced by $\tilde{D}_2 = x\partial x$. For $h = x_+ - x$ satisfying $h \to 0$ we find u_{tt} finite, but $(u_+ - 2u + u_-)^{-3} \to \infty$, so the limit $h \to 0$ does not exist.

5 Conclusions

The main questions to be addressed in a program aiming at using Lie group theory to solve difference equations are: (i) How does one define the symmetries? (ii) How does one calculate the symmetries? (iii) What does one do with the symmetries?

In this article we define the symmetries as in eq.(9), that is we consider only Lie point transformations that act simultaneously in a difference equation (1) and lattice equation (2). The fact that the lattice also transforms is in the spirit of Dorodnitsyn's approach to discretizing differential equations. In most symmetry studies of difference equations [9, ..., 26] the lattice is fixed and nontransformable, e.g. given by the equation $x = n$, $n \in Z$. For nontransforming lattices we need to go beyond point symmetries to catch transformations of interest[17].

Once the class of symmetries that we wish to consider is defined, the matter of calculating them becomes purely technical. We proposed an algorithm for calculating symmetries in Section 2 (see eq.(13),...,(18)) and applied it in Section 3 and 4. Symmetry algorithms for fixed lattices were presented elsewhere [10, ..., 14].

Equations (100) and (104) provide good examples of different approaches. The symmetry algebra (103) of the system (100), (101) happens to coincide with the symmetry algebra of the continuous limit (105). The symmetry algebra of the related equation (104) was calculated elsewhere [12]. It is a 7-dimensional subalgebra of the algebra (103), obtained by dropping $\hat{P}_1$ and $\hat{D}_2$. It was obtained by the 'intrinsic method' [11]. The symmetry algebra of eq.(104) can also be obtained from that of the system (100), (101) by taking a specific solution $x = n$ of eq.(101) and reducing the algebra (103) to the one that preserves this solution.

As far as applications of symmetries are concerned, they are the same for differential equations and difference ones, in particular, symmetry reduction.

First, consider translationally invariant solutions, i.e. solutions invariant under the subgroup generated by $\hat{X} = \hat{P}_0 - v\hat{P}_1$ with v constant and $\hat{P}_0$, $\hat{P}_1$ as in eq.(103). We find that the solution, the differential-difference equations (DΔE) (100), (101) and the PDE (106) reduce to

$$u(x,t) = G(\eta) \qquad \eta = x + vt \tag{107}$$

$$v^2 G_{\eta\eta}[G(\eta + h) - 2G(\eta) + G(\eta - h)]^3 = h^6 \tag{108}$$

$$v^2 G_{\eta\eta}^4 = 1 \tag{109}$$

respectively. Surprisingly, the difference equation (108) and the ODE (109) have exactly the same solution for all values of the spacing h, namely

$$G = \pm\frac{1}{2\sqrt{v}}\eta^2 + A\eta + B \qquad v \not\equiv 0 \tag{110}$$

where A and B are integration constants. Thus, the system (100), (101) is not only a symmetry preserving discretization. It also preserves translationally invariant solutions.

As a second example, consider solutions invariant under dilations generated by $\hat{D}_1$ of eq.(103).The reduction formula, reduced DΔE and reduced PDE are

$$u(x,t) = t^{1/2}G(x) \tag{111}$$

$$G(x)\,[G(x+h) - 2G(x) + G(x-h)]^3 = -4h^6 \tag{112}$$

$$G\,G_{xx}^3 = -4 \tag{113}$$

respectively. A particular solution of eq.(113) is $G(x) = 4(-3)^{-3/4}(x-x_0)^{3/2}$. This is not an exact solution of eq.(112), but the solution of (112) and (113) coincide to order h^2, rather than just h.

As a final example of symmetry reduction, consider the subgroup corresponding to $\hat{D}_2 - 3\hat{D}_1$ of eq.(103). The reduction formulas are

$$u(x,t) = G(\eta) \qquad \eta = x^3 t \tag{114}$$

$$G_{\eta\eta} = \frac{(\eta_+^{1/3} - \eta^{1/3})^6}{\eta^2[G(\eta_+) - 2G(\eta) + G(\eta_-)]^3} \qquad \eta_+^{1/3} - 2\eta^{1/3} + \eta_-^{1/3} = 0 \tag{115}$$

$$27\eta^3 G_{\eta\eta}[3\eta G_{\eta\eta} + 2G_\eta]^3 = 1 \tag{116}$$

While we are not able to solve the ODE (116), nor the difference scheme (115), we see that in both cases we get a reduction of the number of independent variables. We mention that this last reduction would not be obtained on a fixed lattice.

Let us sum up the situation with this particular approach to symmetries of difference equations.

1. Lie point symmetries acting simultaneously on given equations and lattices can be calculated using the reasonably simple algorithm presented in this article.

2. Symmetries can be used to perform symmetry reduction for DΔE.

Work is in progress on other applications of symmetries of discrete equations, in particular solving ordinary difference equations.

Acknowledgments

The authors thank V. Dorodnitsyn, R. Kozlov and S. Lafortune for stimulating discussions. The research of PW is partially supported by grants from NSERC of Canada and FCAR du Québec. The research reported here is also partly supported by the NATO grant CRG960717 and a Cultural Agreement Università Roma Tre–Université de Montréal. ST and PW thank the Università Roma Tre for hospitality, DL similarly thanks the Centre de Recherches Mathématiques, Université de Montréal.

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Chapitre 2

SYMÉTRIES CONTINUES DES ÉQUATIONS SUR UN RÉSEAU

Cze. J. Phys. **51** *No 4 (2001) 349–356*

Continuous Symmetries of Equations on Lattices

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Abstract

A method is presented for calculating the Lie point symmetries of difference equations with one, or several, independent variables. The equations are given on a priori specified lattices. The Lie transformations act on the lattice, as well as on the equation. The transformations take solutions into solutions and can be used to perform symmetry reduction.

1 Introduction

The theory of Lie groups and Lie algebras started out as a theory of transformations of solutions of differential equations [1, 2]. They are used to solve differential equations, to classify equations and solutions, to establish properties of their solution spces [3].

Applications of Lie groups, and of the fundamentally continuous transformations that they represent, to discrete equations, are much more recent. A concerted effort is presently being made to adapt Lie theory to difference equations, differential-difference equations and q-difference ones.

Two different philosophies are being pursued. In one, the discrete equation is a priori given; as is the lattice on which it is realized. The aim is to

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find the group of transformations, taking solutions into solutions and then to apply the group to solve, or at least to simplify the equation. Typically, in this approach the group acts on the equation, but not on lattice [4, ..., 9].

A different philosophy is to start from a differential equation and its known symmetry group. The aim is to discretize the equation while preserving its point symmetries. Thus, the symmetries are a priori given, one looks for a lattice and a difference equation on this lattice. The group acts on the equation and on the lattice (also given by an equation, or system of equations) [10, 11, 12].

The purpose of this article is to combine the two approaches (see also Ref.[13, 14]). We will consider given difference equations on given lattices. However, the lattice will also be given by an equation, or a system of equations. We will then construct Lie point transformations, acting on the difference equations and the lattice and leaving the solution set of this difference scheme invariant.

2 Symmetries of difference schemes in one dimension

2.1 General formalism

Let us consider a scheme involving one scalar function $u(x)$ of one scalar variable x. A difference scheme will consist of two equations

$$\begin{gathered} E_a\left(\{x_{n+k}\}_{k=-N_1}^{N_2}, \{u_{n+k}\}_{k=-N_1}^{N_2}\right) = 0 \\ N_1\,,\ N_2 \in \mathbb{Z}^{\geq 0}, \qquad a = 1,2 \end{gathered} \tag{1}$$

involving N_1+N_2+1 points. The equations must be such that given N_1+N_2 neighbouring values of $\{x_k, u_k\}$, we can calculate the values of $\{x, u\}$ at one further point, to the left or right of the given set. We assume that the lattice is infinite in both directions and that eq.(1) can be shifted arbitrarily to the right and to the left.

A solution of the difference scheme is a pair of expressions

$$\begin{aligned} x_n &= \Phi(n, c_1, \ldots, c_N) \\ u_n &= \Psi(n, c_1, \ldots, c_N)\ , \quad N \equiv 2(N_1 + N_2), \end{aligned} \tag{2}$$

where $c_1, \ldots, c_N$ are integration constants. The points $\{\ldots, x_{n-1}, x_n, x_{n+1}, \ldots\}$ are not necessarily evenly spaced and the value of x_{n+1} may depend not only on the previous values of x_k but also on the solutions of u_k at the previous points.

We choose some point $\{x, u\} = \{x_n, u_n\}$ as the origin in the space $X \times U \subset \mathbb{R}^2$ of independent and dependent variables and consider a group G of local point tranformations, acting in a neighbourhood of the point $\{x, u\}$. The transformations will be generated by a Lie algebra of vector fields of the form

$$\hat{X} = \xi(x, u)\, \partial_x + \phi(x, u)\, \partial_u. \tag{3}$$

When dealing with differential equations, we must prolong these vector fields to identify their action on derivatives $u_x, u_{xx}, \ldots$ (up to the order of the equation under study). When dealing with the difference scheme (1) we must prolong the action of the vector field to all points figuring in eq.(1). The finite group transformations act on the entire $\{x, u\}$ space simultaneously (at least locally). The prolongation of the vector field (3) can hence be given as

$$\mathrm{pr}\, \hat{X} = \sum_{k=-N_1}^{N_2} \left[\xi(x_{n+k}, u_{n+k}) \partial_{x_{n+k}} + \phi(x_{n+k}, u_{n+k}) \partial_{u_{n+k}} \right], \tag{4}$$

where the summation is over all points figuring in eq.(1). We now require that the scheme (1) be invariant, under the group G, on the solutions of eq.(1). In infinitesimal terms that means that the vector field (3) must satisfy

$$\mathrm{pr}\hat{X}\, E_a \,|_{E_i=0} = 0, \qquad \forall\ a, i. \tag{5}$$

Equation (5) provides the determining equations for the coefficients $\xi(x, u)$ and $\phi(x, u)$. The actual algorithm for calculating the symmetry algebra is as follows.

1. Use eq.(1) to express x_k and u_k for some value of k (usually the highest or the lowest one) in terms of the other ones, e.g.

$$\begin{gathered} x_{n+N_2} = R(x_k, u_k), \qquad u_{n+N_2} = S(x_k, u_k) \\ n - N_1 \le k \le n + N_2 - 1. \end{gathered} \tag{6}$$

   Substitute (6) into (5). We obtain two functional relations for $\xi(x, u)$ and $\phi(x, u)$. Since eq.(1) has already been used, these equations must hold for any $N_1 + N_2$ given neighbouring points. These equations are

$$\sum_{k=-N_1}^{N_2-1}\left[\xi_{n+k}\frac{\partial E_a}{\partial x_{n+k}}+\phi_{n+k}\frac{\partial E_a}{\partial u_{n+k}}\right]+\xi_{n+N_2}(R,S)\frac{\partial E_a}{\partial x_{n+N_2}} \qquad (7)$$
$$+\phi_{n+N_2}(R,S)\frac{\partial E_a}{\partial u_{n+N_2}}=0, \qquad a=1,2$$

where we use the notation

$$\xi_{n+k}\equiv\xi(x_{n+k},u_{n+k}), \qquad \phi_{n+k}\equiv\phi(x_{n+k},u_{n+k}),$$

and R, S are functions of (x_k,u_k), $k=n-N_1,\dots,n+N_2-1$, defined in eq.(6).

2. Assume that the dependence of ξ and ϕ on their arguments is analytic. Convert eq.(7) into a system of differential equations for ξ and ϕ by differentiating with respect to the variables x_k and u_k with $n-N_1\leq k\leq n+N_2-1$. The obtained equations will be first order linear differential equations, involving fewer terms (since e.g. a given x_k can figure only in ξ_k , ϕ_k , ξ_{n+N_2} , ϕ_{n+N_2}, via R and S and explicitly via E_1 and E_2). Further differentiations may produce one term equations that we can solve.

3. Substitute the general solutions back into eq.(7) and solve for ξ and ϕ.

If a continuous limit exists, then the pair of equations (1) goes into one single differential equations, generally speaking of order N_1+N_2. In other words, in the continuous limit, the two functions E_1 and E_2 are no longer linearly independent.

Let us mention that instead of considering the variables $\{x_k,u_k\}$ in different points, we can use a different basis, in which we introduce discrete derivatives and spacings between points. Thus, for instance for 3 points we can put

$$\{x_-\,,\,x\,,\,x_+\,,\,u_-\,,\,u\,,\,u_+\}\longleftrightarrow \qquad (8)$$
$$\left\{x\,,\,h_+=x_+-x\,,\,h_-=x-x_-\,,\,u\,,\,u_x=\tfrac{u_+-u}{x_+-x}\,,\,u_{x\bar{x}}=\tfrac{u_+-2u+u_-}{(x_+-x)^2}\right\},$$

having defined $x\equiv x_n$, $x_\pm\equiv x_{n\pm1}$. The continuous limit is particularly clear in the second basis.

2.2 Example

Let us consider the nonlinear ordinary differential equation

$$u_{xx} - u^N = 0, \qquad N \not\equiv 0, 1. \tag{9}$$

A straightforward calculation shows that for $N \not\equiv -3$ eq.(9) is invariant under a two-dimensional Lie group, the Lie algebra of which is spanned by

$$\hat{P} = \partial_x, \qquad \hat{D} = (N-1)x\partial_x - 2u\partial_u\,. \tag{10}$$

For $N = -3$ the symmetry algebra is $sl(2, \mathbb{R})$ with a basis

$$\hat{P} = \partial_x, \qquad \hat{D} = 2x\partial_x + u\partial_u, \qquad \hat{C} = x^2\partial_x + xu\partial_u. \tag{11}$$

A natural way to discretize eq.(9) is to use a uniform lattice and put

$$E_1 = \frac{u_+ - 2u + u_-}{(x_+ - x)^2} - u^N = 0, \tag{12}$$

$$E_2 = x_+ - 2x + x_- = 0. \tag{13}$$

Let us now apply the symmetry condition (5). First, the condition $pr\hat{X}\, E_2$ for $E_1 = E_2 = 0$ implies

$$\xi(2x - x_-, (x-x_-)^2 u^N + 2u - u_-) - 2\xi(x,u) + \xi(x_-, u_-) = 0\,. \tag{14}$$

Differentiating first by ∂_{u_-}, then by ∂_u we obtain

$$-\xi_{u_+}(2x - x_-, (x-x_-)^2 u^N + 2u - u_-) + \xi_{u_-}(x_-, u_-) = 0, \tag{15}$$

$$\left[N(x-x_-)^2\, u^{N-1} + 2\right]\, \xi_{u_+u_+}(2x - x_-, (x-x_-)^2 u^N + 2u - u_-) = 0\,. \tag{16}$$

Eq.(16) implies that ξ is linear in u

$$\xi(x,u) = a(x)u + b(x)\,. \tag{17}$$

Eq.(15) reduces to $a(x_+) = a(x)$, i.e. a is a constant. Substituing these results into eq.(14) we obtain

$$a\,[u_+ - 2u + u_-] + b(x_+) - 2b(x) + b(x_-) = 0\,. \tag{18}$$

This implies $a = 0$ and

$$b(x_+) - 2b(x) + b(x_-) = 0\,. \tag{19}$$

Differentiating successively with respect to x and x_- we find $b_{x_+x_+}(x_+) = 0$, i.e.

$$b(x) = b_1 x + b_0\,. \tag{20}$$

Thus, the invariance of eq.(13) implies $\xi = b_1 x + b_0$ with b_1, b_0 constants. The function $\phi(x,u)$ is restricted by the requirement $pr\hat{X}\,E_1 = 0$ for $E_1 = 0$, $E_2 = 0$. This invariance condition is given by

$$\begin{aligned} &\phi(2x - x_-, (x-x_-)^2 u^N + 2u - u_-) - 2\phi(x,u) + \phi(x_-,u_-) \\ &-(x-x_-)^2[N\phi(x,u)u^{N-1} + 2b_1 u^N] = 0\,. \end{aligned} \tag{21}$$

We successively differentiate this equation with respect to u_- and u and obtain

$$-\phi_{u_+}(x_+,u_+) + \phi_{u_-}(x_-,u_-) = 0, \tag{22}$$

$$\phi_{u_+u_+}(x_+,u_+) = 0. \tag{23}$$

These two equations require that $\phi = \phi_1 u + \phi_0(x)$ with ϕ_1 a constant. Substituing back into eq.(21) we obtain the remaining determining equation

$$\begin{aligned} &\phi_0(2x - x_-) - 2\phi_0(x) + \phi_0(x_-) - (x-x_-)^2[(N-1)\phi_1 + 2b_1]u^N \\ &-N(x-x_-)^2\phi_0 u^{N-1} = 0. \end{aligned} \tag{24}$$

Since we have $N \not\equiv 0,1$ eq.(24) implies $\phi_0(x) = 0$ and $\phi_1(1-N) = 2b_1$. Finally, we obtain the symmetry algebra of the difference system (12), (13). It is 2-dimensional and coincides with the algebra (10) of the differential equation (9), the continuous limit of eq.(12).

Notice that the case $N = -3$ is not distinguished from the generic case. As a matter of fact, no difference equation on a uniform lattice can be invariant under the $SL(2,\Re)$ group corresponding to the algebra (11). A basis for the difference invariants of this algebra in the space $\{x, x_-, x_+, u, u_-, u_+\}$ is

$$\rho_1 = \frac{h_- u_+}{(h_+ + h_-)u}, \qquad \rho_2 = \frac{h_+ u_-}{(h_+ + h_-)u}, \qquad \rho_3 = \frac{h_+ h_-}{(h_+ + h_-)u^2}, \tag{25}$$

where h_+ and h_- are defined as $h_+ = x_+ - x$, $h_- = x - x_-$. Hence, no function of x, x_+ and x_- alone can be set equal to a constant. An $SL(2,\mathbb{R})$ invariant scheme must be constructed out of these invariants. For instance, an invariant scheme approximating eq.(9) for $N = -3$ is

$$\frac{h_-(u_+ - u) - h_+(u - u_-)}{h_+ h_-} = \frac{2h_+ h_-}{(h_+ + h_-)} \frac{1}{u^3}, \qquad h_- u_+ = h_+ u_-. \tag{26}$$

3 Symmetries of multidimensional difference schemes

3.1 General formalism

For simplicity we restrict here to the case of two independent variables. The generalization to n variables is immediate.

As in the case of one variable, we consider a difference scheme to be a system of relations between N points in the space $\{x, t, u\}$ enabling us to calculate x, t and u in an $N - th$ point, if they are given in $N - 1$ points. In general a lattice in two dimensions is given by a system of 4 equations, allowing us to calculate both x and y in new points and to move in two linearly independent directions in the plane, from which we can calculate x and t at any point of the lattice.

We find it convenient to label points on the lattice by two subscripts: $(x_{m,n}, t_{m,n})$ with (in principle) $-\infty < m < \infty$, $-\infty < n < \infty$. We put $u(x_{m,n}, t_{m,n}) \equiv u_{m,n}$. The difference scheme has the form

$$E_a\left(x_{m+k,n+l}, t_{m+k,n+l}, u_{n+k,n+l}\,;\, -M_1 \leq k \leq M_2\,,\, -N_1 \leq l \leq N_2\right) = 0$$
$$a = 1, \ldots, 5\,. \tag{27}$$

We look for a Lie algebra of vector fields of the form

$$\hat{X} \equiv \hat{X}_{m,n} = \xi_{m,n}\,\partial_{x_{m,n}} + \tau_{m,n}\,\partial_{t_{m,n}} + \phi_{m,n}\,\partial_{u_{m,n}}, \tag{28}$$

(with e.g. $\xi_{m,n} \equiv \xi(x_{m,n}, t_{m,n}, u_{m,n})$) that will generate Lie point symmetry transformations, i.e. transformations taking solutions into solutions. The symmetry condition is again given by eq.(5). The prolongation of $\hat{X}$ is again calculated by shifting ξ, τ and ϕ to all points of the lattice involved in eq.(27)

$$\mathrm{pr}\,\hat{X} = \sum_{m,n} \hat{X}_{m,n}. \tag{29}$$

Equation (5) provides a system of functional equations for just three functions $\xi(x,t,u), \tau(x,t,u)$ and $\phi(x,t,u)$. As in the case of one independent variable, we convert them into differential equations and solve them to obtain the symmetry algebra and ultimately the symmetry group.

3.2 Example. The discrete heat equation

The continuous heat equation

$$u_t = u_{xx} \tag{30}$$

is invariant under an infinite dimensional pseudogroup. Factoring out the infinite dimensional pseudogroup corresponding to the linear superposition principle, we are left with a 6 dimensional group. A basis for its Lie algebra is

$$\begin{gathered} \hat{P}_0 = \partial_t,\ \hat{P}_1 = \partial_x,\ \hat{D} = x\partial_x + 2t\partial_t,\ \hat{W} = u\partial_u, \\ \hat{B} = t\partial_x - \tfrac{1}{2}xu\partial_u,\ \hat{K} = t^2\partial_t + xt\partial_x - \tfrac{1}{4}(x^2+2t)u\partial_u, \end{gathered} \tag{31}$$

corresponding to time and space translations, dilation, multiplication of the solution by a constant, Galilei boosts and expansions, respectively.

Let us now consider a discrete heat equation and lattice:

$$\frac{u_{m,n+1} - u_{m,n}}{t_{m,n+1} - t_{m,n}} = \frac{u_{m+1,n} - 2u_{m,n} + u_{m+1,n}}{(x_{m+1,n} - x_{m,n})^2}, \tag{32}$$

$$x_{m+1,n} - 2x_{m,n} + x_{m-1,n} \;=\; 0, \tag{33}$$

$$
\begin{aligned}
x_{m,n+1} - x_{m,n} &= 0, && (34)\\
t_{m+1,n} - t_{m,n} &= 0, && (35)\\
t_{m,n+1} - t_{m,n} - c(x_{m,n} - x_{m-1,n})^2 &= 0, \quad c \neq 0. && (36)
\end{aligned}
$$

The lattice equations (33),...,(36) can be solved to give

$$x_{m,n} = Am + x_0, \qquad t_{m,n} = cA^2 n + t_0. \tag{37}$$

Thus the lattice is regular (equally spaced), orthogonal and the lattice spacing in the x and t directions are related (by the constant c).

Using the lattice equation (36) we simplify eq.(32) to

$$u_{m,n+1} - u_{m,n} = c(u_{m+1,n} - 2u_{m,n} + u_{m-1,n}). \tag{38}$$

Applying $pr\,\hat{X}$ of eq.(29) to eq.(34) and (35), using the original equations and differentiating with respect to $u_{m+1,n}$ we find $\xi = \xi(x,t)$, $\tau = \tau(x,t)$. Differentiating the same relations with respect to $x_{m-1,n}$ we find that actually we have $\xi = \xi(x)$, $\tau = \tau(t)$. Acting on (33) and (36) we finally obtain

$$\xi = Ax + B, \qquad \tau = 2At + C, \tag{39}$$

where A, B and C are constants.

Finally, let us apply $pr\,\hat{X}$ to eq.(38). Using eq.(33),...,(38) we obtain

$$
\begin{aligned}
&\phi\left(x_{m,n}, t_{m,n} + c(x_{m,n} - x_{m-1,n})^2, u_{m,n} + c(u_{m+1,n} - 2u_{m,n} + u_{m-1,n})\right)\\
&+(2c-1)\phi(x_{m,n}, t_{m,n}, u_{m,n}) = c\left[\phi(2x_{m,n} - x_{m-1,n}, t_{m,n}, u_{m+1,n})\right.\\
&\left.+\phi(x_{m-1,n}, t_{m,n}, u_{m-1,n})\right].
\end{aligned} \tag{40}
$$

Taking the second derivative with respect to $u_{m-1,n}$ and $u_{m+1,n}$ of eq.(40) we obtain $\phi = R(x,t)u + S(x,t)$. Substituing back into eq.(40) and equating coefficients of $u_{m,n}, u_{m+1,n}, u_{m-1,n}$ and 1, we obtain 3 equations for $R_{m,n}$ and one for $S_{m,n}$. The function $S_{m,n}$ must satisfy eq.(38) and is an expression of the linear superposition principle. The three equations for $R_{m,n} = R(x_{m,n}, t_{m,n})$ imply $R = const.$

Finally, the symmetry algebra of the discrete heat equation (32) on the lattice (33),...,36) is given by

$$\hat{P}_0 = \partial_t,\ \hat{P}_1 = \partial_x,\ \hat{D} = x\partial_x + 2t\partial_t,\ \hat{W} = u\partial_u,$$
$$\hat{S} = S(x,t)\partial_u, \tag{41}$$

where $S(x,t)$ is any solution of the system (32),...,(36).

In other words, this particular discretization of the heat equation (30) has preserved a four dimensional subalgebra of the algebra (31). The entire symmetry group could also be preserved, but the discretization would be much more complicated [12].

4 Conclusion

We have presented a reasonably simple algorithm for calculating the Lie point symmetries of arbitrary difference schemes. The determining equations are linear functional equations for the coefficients of the vector fields that realize the symmetry algebra. They can be converted into systems of linear differential equations and solved. The transformations considered are point ones; the finite transformations are obtained by integrating the vector fields.

Applications of the symmetries to obtain solutions and perform symmetry reduction will be presented elsewhere, both for linear and nonlinear difference equations.

Acknowledgements

This article was written while S.T. and P.W. were visiting the Dipartimento di Fisica, Università di Roma Tre. They thank the Dipartimento, the INFN and the Agreement Università di Roma Tre – Université de Montréal for their support. The research of P.W. was partly supported by a research grant from NSERC of Canada.

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Chapitre 3

SYMÉTRIES DE LIE POUR LES ÉQUATIONS AUX DIFFÉRENCES MULTIDIMENSIONNELLES

Lie Symmetries of Multidimensional Difference Equations

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August, 2001

Abstract

A method is presented for calculating the Lie point symmetries of a scalar difference equation on a two-dimensional lattice. The symmetry transformations act on the equations and on the lattice. They take solutions into solutions and can be used to perform symmetry reduction. The method generalizes one presented in a recent publication for the case of ordinary difference equations. In turn, it can easily be generalized to difference systems involving an arbitrary number of dependent and independent variables.

1 Introduction

A recent article [1] was devoted to Lie point symmetries, acting on ordinary difference equations and lattices, while leaving their set of solutions invariant. The purpose of this article is to extend the previously obtained methods and results to the case of partial difference equations, i.e. equations involving more than one independent variable.

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Algebraic techniques, making use of Lie groups and Lie algebras, have proved themselves to be extremely useful in the theory of differential equations [2].

When applying similar algebraic methods to difference equations, several decisions have to be made.

The first decision is a conceptual one. One can consider difference equations and lattices as given objects to be studied. The aim then is to provide tools for solving these equations, simplifying the equations, classifying equations and their solutions, and identifying integrable, or linearizable difference equations [1, 3, ..., 26]. Alternatively, one can consider difference equations and the lattices on which they are defined, to be auxiliary objects. They are introduced in order to study solutions of differential equations, numerically or otherwise. The question to be asked in this is: how does one discretize a differential equation, while preserving its symmetry properties [27, 29, 30, 28, 31].

In this article we take the first point of view: the equation and the lattice are *a priori* given. The next decision to be made is a technical one: which aspect of symmetry to pursue. For differential equations one can look for point symmetries, or generalized ones. When restricting to point symmetries, and constructing the Lie algebra of the symmetry group, one can use vector fields acting on dependent and independent variables. Alternatively and equivalently, one can use evolutionary vector fields, acting only on dependent variables. For difference equations, these two approaches are in general not equivalent and may lead to different results, both of them correct and useful.

Several aspects of symmetry for discrete equations were pursued in earlier articles by two of the present authors (D.L. and P.W.) and collaborators. The "intrinsic method" which provides, in an algorithmic way, all purely point symmetries of a given differential - difference equation on a given uniform fixed lattice was introduced in [4]. This was complemented by the "differential equations method" in [5]. In addition to point symmetries the differential equation method provides a class of generalized symmetries. It was pointed out that in many cases the two methods provide the same result, i.e. all symmetries are point ones. The two methods were successfully applied to many specific problems [5, 7, 12, 13, 16]. The advantage of these two approaches are their simplicity, their algorithmic character, and their close analogy to symmetries of differential equations. Their disadvantage is that many interesting symmetries, like rotations among discrete variables, are lost in this approach.

A complementary approach was first developed for linear difference equations [19, 8], again given on fixed uniform lattices. It was formulated in terms of linear difference operators, commuting with the linear operator defining the original difference equation. This approach provides a large number of symmetries and the symmetry algebras of the discrete equations and their continuous limits are actually isomorphic. The symmetries of the difference equations are not point ones: they act at many points of the lattice. They do however provide flows that commute with the flow determined by the original equation and can thus be used to obtain solutions.

This aspect of commuting flows has been adapted to nonlinear difference and differential-difference equations [9, 10, 11, 15, 17]. The equations are defined on a fixed and uniform lattice. Generalized symmetries are considered together with point ones and some of the generalized symmetries reduce to point ones in the continuous limit. The methods for finding these generalized symmetries rely on either linearizability, as in the case of the discrete Burgers equation [9], or on integrability (the existence of a Lax pair) as in the case of the Toda hierarchy [10, 11, 15], or the discrete nonlinear Schrödinger equation [17, 18].

This symmetry approach is powerful whenever it is applicable. Together with point and generalized symmetries it provides Bäcklund transformations as a composition of infinitely many higher symmetry transformations. This aspect has been explored in detail for the Toda lattice [15]. We emphasize that Bäcklund transformations for difference equations, just as for differential ones, are not obtained directly as Lie symmetries (not even as generalized ones).

Each of the above methods has its own merits and will be further developed in the future.

In this article we take the same point of view as in our recent article [1]. We consider point symmetries only and use the formalism of vector fields acting on all variables, dependent and independent ones. In [1] we considered only one discretely changing variable. The lattice was not fixed. Instead it was given by a further difference equation. Point symmetries act on the entire difference system: the equation and the lattice. The lattice is not necessarily uniform and we explored the effect of choosing different types of lattices. The idea of using transforming lattices is due to Dorodnitsyn and coworkers [27, 29, 30, 28, 31]. We differ from them in one crucial aspect. They start from a given symmetry group and construct invariant difference schemes for a given group. We, on the other hand, start from a given difference scheme and find its Lie point symmetry group. Previously this was done for the case of one independent variable. In this article we generalize to the multidimensional case. The generalization is by no means trivial. The lattice is given by N^2 equations, where N is the number of independent variables, all of them varying discretely. Transformations of continuously varying independent variables, if present, are also taken into account.

We stress that the approach of this article complements those of previous ones. The results of [4] and [5] are obtained if we chose a special form of the lattice (e.g. $x_{m+1} - x_m = h$ in the case of one independent variable, where h is a fixed, nontransforming constant). We purpusely avoid any use of integrability. Like Lie theory for differential equations, this approach is applicable to arbitrary differential systems, integrable or not.

A general formalism for determining the symmetry algebra is presented in Section 2. It generalizes the algorithm presented earlier [1] for ordinary difference equations to the case of several independent variables. In Section 3 we apply the algorithm to a discrete linear heat equation which we consider on several different lattices, each providing its own symmetries. Section 4 is devoted to difference equations on lattices that are invariant under Lorentz transformations. In Section 5 we discuss two different discrete Burgers equations, one linearizable, the other not. The lattices are the same in both cases, the symmetry algebras turn out to be different. Section 6 treats symmetries of differential-difference equations, i.e. equations involving both discrete and continuous variables. Some conclusions are drawn in the final Section 7.

2 General symmetry formalism

2.1 The difference scheme

For clarity and brevity, let us consider one scalar equation for a continuous function of two (continuous) variables: $u = u(x,t)$. A lattice will be a set of points P_i, lying in the plane $\mathbb{R}^2$ and stretching in all directions with no boundaries. The points P_i in $\mathbb{R}^2$ will be labeled by two discrete labels $P_{m,n}$. The Cartesian coordinates of the point $P_{m,n}$ will be $(x_{m,n}, t_{m,n})$ with $-\infty < m < \infty$, $-\infty < n < \infty$ (we are of course not obliged to use Cartesian coordinates). The value of the dependent variable in the point $P_{m,n}$ will be denoted $u_{m,n} = u(x_{m,n}, t_{m,n})$.

A difference scheme will be a set of equations relating the values of

$\{x, t, u\}$ in a finite number of points. We start with one 'reference point' $P_{m,n}$ and define a finite number of points $P_{m+i,n+j}$ in the neighborhood of $P_{m,n}$. They must lie on two different curves, intersecting in $P_{m,n}$. Thus, the difference scheme will have the form

$$\begin{gathered} E_a\Big(\{x_{m+i,n+j}, t_{m+i,n+j}, u_{m+i,n+j}\} \Big) = 0 \qquad 1 \le a \le 5 \\ -i_1 \le i \le i_2 \qquad -j_1 \le j \le j_2 \qquad i_1, i_2, j_1, j_2 \in \mathbb{Z}^{\ge 0}. \end{gathered} \tag{1}$$

The situation is illustrated on Figure 1. It corresponds to a lattice determined by 6 points. Our convention is that x increases as m grows, t increases as n grows (i.e. $x_{m+1,n} - x_{m,n} \equiv h_1 > 0$, $t_{m,n+1} - t_{m,n} \equiv h_2 > 0$). The scheme on Figure 1 could be used e.g. to approximate a differential equation of third order in x, second in t.

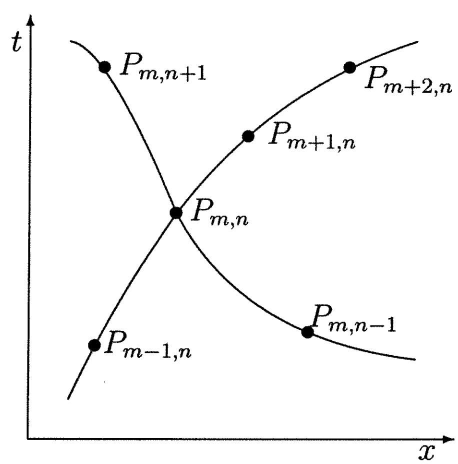

Figure 1: Points on a lattice

Of the above five equations in (1), four determine the lattice, one the difference equation. If a continuous limit exists, it is a partial differential equation in two variables. The four equations determining the lattice will reduce to identities (like $0 = 0$).

The system (1) must satisfy certain independence criteria. Starting from the reference point $P_{m,n}$ and a given number of neighboring points, it must be possible to calculate the values of $\{x, t, u\}$ in all points. This requires a minimum of five equations: to be able to calculate the (x, t) in two directions and u in all points. For instance, to be able to move upward and to the right along the curves passing through $P_{m,n}$ (with either m, or n fixed) we impose a condition on the Jacobian

$$|J| = \left| \frac{\partial(E_1, E_2, E_3, E_4, E_5)}{\partial(x_{m+i_2,n}, t_{m+i_2,n}, x_{m,n+j_2}, t_{m,n+j_2}, u_{m+i_2,n+j_2})} \right| \neq 0. \tag{2}$$

As an example of difference scheme, let us consider the simplest and most standard lattice, namely a uniformly spaced orthogonal lattice and a difference equation approximating the linear heat equation on this lattice. Equations (1) in this case are:

$$x_{m+1,n} - x_{m,n} = h_1 \qquad t_{m+1,n} - t_{m,n} = 0 \tag{3}$$

$$x_{m,n+1} - x_{m,n} = 0 \qquad t_{m,n+1} - t_{m,n} = h_2 \tag{4}$$

$$\frac{u_{m,n+1} - u_{m,n}}{h_2} = \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{(h_1)^2} \tag{5}$$

where h_1 and h_2 are constants.

The example is simple and the lattice and the lattice equations can be solved explicitly to give

$$x_{m,n} = h_1 m + x_0 \qquad t_{m,n} = h_2 n + t_0. \tag{6}$$

The usual choice is $x_0 = t_0 = 0$, $h_1 = h_2 = 1$ and then x is simply identified with m, t with n. We need the more complicated two index notation to describe arbitrary lattices and to formulate the symmetry algorithm (see below).

The example suffices to bring out several points:

1. Four equations are needed to describe the lattice.

2. Four points are needed for equations of second order in x, first in t. Only three figure in the lattice equation, namely $P_{m+1,n}$, $P_{m,n}$ and $P_{m,n+1}$. To get the fourth point, $P_{m-1,n}$, we shift m down by one unit in equations (3-5).

3. The independence condition (2) is needed to be able to solve for $x_{m+1,n}$, $t_{m+1,n}$, $x_{m,n+1}$, $t_{m,n+1}$ and $u_{m,n+1}$.

2.2 Symmetries of the difference scheme

We are interested in point transformations of the type

$$\tilde{x} = F_\lambda(x,t,u) \quad \tilde{t} = G_\lambda(x,t,u) \quad \tilde{u} = H_\lambda(x,t,u) \tag{7}$$

where λ is a group parameter, such that when (x,t,u) satisfy the system (1) then $(\tilde{x},\tilde{t},\tilde{u})$ satisfy the same system. The transformation acts on the entire space (x,t,u), at least locally, i.e. in some neighborhood of the reference point $P_{m,n}$, including all points $P_{m+i,n+j}$ figuring in equation (1). That means that the same functions F, G and H determine the transformation of all points. The transformations (7) are generated by the vector field

$$\hat{X} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u. \tag{8}$$

We wish to find the symmetry algebra of the system (1), that is the Lie algebra of the local symmetry group of local point transformations. To do this we must prolong the action of the vector field $\hat{X}$ from the reference point $(x_{m,n}, t_{m,n}, u_{m,n})$ to all points figuring in the system (1). Since the transformations are given by the same functions F, G and H at all points, the prolongation of the vector field (8) is obtained simply by evaluating the functions ξ, τ and ϕ at the corresponding points.

In order words, we can write

$$\begin{aligned} \mathrm{pr}\,\hat{X} = \textstyle\sum_{m,n} \Big[&\xi(x_{m,n}, t_{m,n}, u_{m,n})\partial_{x_{m,n}} + \tau(x_{m,n}, t_{m,n}, u_{m,n})\partial_{t_{m,n}} \\ &+\phi(x_{m,n}, t_{m,n}, u_{m,n})\partial_{u_{m,n}}\Big], \end{aligned} \tag{9}$$

where the summation is over all points figuring in the system (1). The invariance requirement is formulated in terms of the prolonged vector field as

$$\mathrm{pr}\hat{X}\, E_a\,|_{E_b=0} \qquad 1 \le a, b \le 5. \tag{10}$$

Just as in the case of ordinary difference equations, we can turn equation (10) into an algorithm for determining the symmetries, i.e. the coefficients in vector field (8).

The procedure is as follows:

1. Use the original equations (1) and the Jacobian condition (2) to express five independent quantities in terms of the other ones, e.g.

$$\begin{gathered} v_1 = x_{m+i_2,n} \qquad v_2 = t_{m+i_2,n} \qquad v_3 = x_{m,n+j_2} \\ v_4 = t_{m,n+j_2} \qquad v_5 = u_{m+i_2,n+j_2} \end{gathered} \tag{11}$$

as

$$\begin{gathered} v_a = v_a(x_{m+i,n+j}, t_{m+i,n+j}, u_{m+i,n+j}) \\ -i_1 \leq i \leq i_2 - 1 \qquad -j_1 \leq j \leq j_2 - 1. \end{gathered} \tag{12}$$

2. Write the five equations (10) explicitly and replace the quantities v_a using equation (12). We obtain five functional equations for the functions ξ, τ and ϕ, evaluated at different point of the lattice. Once the functions v_a are substituted into these equations, each value of $x_{i,k}, t_{i,k}$ and $u_{i,k}$ is independent. Moreover, it can only figure via the corresponding $\xi_{i,k}, \tau_{i,k}$ and $\phi_{i,k}$ (with the same values of i and k), via the functions v_a, or explicitly via the functions E_a.

3. Assume that the dependence of ξ, τ and ϕ on their variables is analytic. Convert the obtained functional equations into a system of differential equations by differentiating with respect to the variables $x_{i,k}, t_{i,k}$ and $u_{i,k}$. This provides an overdetermined system of linear partial differential equations which we must solve.

4. The solutions of the differential equations must be substituted back into the functional ones and these in turn must be solved.

The above algorithm provides us with the function $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$ figuring in equation (8). The finite transformations of the (local) Lie symmetry group are obtained in the usual manner, by integrating the vector field (8):

$$\begin{gathered} \frac{d\tilde{x}}{d\lambda} = \xi(\tilde{x},\tilde{t},\tilde{u}) \quad \frac{d\tilde{t}}{d\lambda} = \tau(\tilde{x},\tilde{t},\tilde{u}) \quad \frac{d\tilde{u}}{d\lambda} = \phi(\tilde{x},\tilde{t},\tilde{u}) \\ \tilde{x}\,|_{\lambda=0} = x \qquad \tilde{t}\,|_{\lambda=0} = t \qquad \tilde{u}\,|_{\lambda=0} = u. \end{gathered} \tag{13}$$

3 Discrete heat equation

The heat equation in one-dimension

$$u_t = u_{xx} \tag{14}$$

is invariant under a six-dimensional Lie group, corresponding to translations in x and t, dilations, Galilei transformations, multiplication of u by a constant and expansions. It is also invariant under an infinite dimensional pseudo-group, corresponding to the linear superposition principle.

Symmetries of the discrete heat equation have been studied, using different methods and imposing different restrictions on the symmetries [8, 19, 27, 28].

Here we will use the discrete heat equation to illustrate the methods of Section 2 and to show the influence of the choice of the lattice.

3.1 Fixed rectangular lattice

The discrete heat equation and a fixed lattice were given in equation (5) and (3), (4), respectively. Applying the operator (9) to the lattice, we obtain

$$\xi(x_{m+1,n}, t_{m+1,n}, u_{m+1,n}) = \xi(x_{m,n}, t_{m,n}, u_{m,n}) \tag{15}$$

$$\xi(x_{m,n+1}, t_{m,n+1}, u_{m,n+1}) = \xi(x_{m,n}, t_{m,n}, u_{m,n}). \tag{16}$$

The values $u_{m+1,n}, u_{m,n+1}, u_{m,n}$ are not related by equation (5) (since it also contains $u_{m-1,n}$). Hence if we differentiate equations (15), (16), e.g. with respect to $u_{m,n}$, we find that ξ is independent of u. We have $t_{m+1,n} = t_{m,n}$ so equation (15) implies that ξ does not depend on x. Similarly, equation (16) implies that ξ does not depend on t. Hence ξ is constant. Similarly, we obtain that $\tau(x, t, u)$ is also constant. Applying the prolongation $\mathrm{pr}\hat{X}$ to equation (5) we obtain the functional equation

$$\phi_{m,n+1} - \phi_{m,n} = \frac{h_2}{(h_1)^2}(\phi_{m+1,n} - 2\phi_{m,n} + \phi_{m-1,n}) \tag{17}$$

with e.g. $\phi_{m,n} \equiv \phi(x_{m,n}, t_{m,n}, u_{m,n})$.

In $\phi_{m,n+1}$ we replace $u_{m,n+1}$, using equation (5). We then differentiate with respect to $u_{m+1,n}$ and again with respect to $u_{m-1,n}$. We obtain

$$\phi_{m,n} = A(x_{m,n}, t_{m,n})u_{m,n} + B(x_{m,n}, t_{m,n}). \tag{18}$$

Substituting (18) into equation (17), using (5) again and setting the coefficient of $u_{m+1,n}, u_{m-1,n}, u_{m,n}$ and 1 equal to zero separately we find that

A must be constant and B must be a solution of equation (5). Thus, the symmetry algebra of the heat equation on the lattice (3), (4) is given by

$$\hat{P}_0 = \partial_t \quad \hat{P}_1 = \partial_x \quad \hat{W} = u\partial_u \quad \hat{S} = S(x,t)\partial_u \tag{19}$$

with S a solution of the equation itself. Thus, the only symmetries are those due to the fact that the equation is linear and autonomous.

3.2 Lattices invariant under dilations

There are at least two ways of making the discrete heat equation invariant under dilations.

A) Five point lattice

We replace the system of equations (3), (4) and (5) by

$$x_{m+1,n} - 2x_{m,n} + x_{m-1,n} = 0 \qquad x_{m,n+1} - x_{m,n} = 0 \tag{20}$$

$$t_{m+1,n} - t_{m,n} = 0 \qquad t_{m,n+1} - 2t_{m,n} + t_{m,n-1} = 0 \tag{21}$$

$$\frac{u_{m,n+1} - u_{m,n}}{t_{m,n+1} - t_{m,n}} = \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{(x_{m+1,n} - x_{m,n})^2}. \tag{22}$$

Applying pr$\hat{X}$ of equation (8) to (20) and substituting for $x_{m+1,n}, t_{m+1,n}, t_{m,n+1}$ and $x_{m,n+1}$ from the equations (20), (21) we obtain

$$\begin{aligned} &\xi(2x_{m,n} - x_{m-1,n}, t_{m,n}, u_{m+1,n}) - 2\xi(x_{m,n}, t_{m,n}, u_{m,n}) \\ &+\xi(x_{m-1,n}, t_{m-1,n}, u_{m-1,n}) = 0 \end{aligned} \tag{23}$$

$$\xi(x_{m,n}, 2t_{m,n} - t_{m,n-1}, u_{m,n+1}) = \xi(x_{m,n}, t_{m,n}, u_{m,n}). \tag{24}$$

Since $u_{m,n+1}$ and $u_{m,n}$ are independent a differentiation of (24) with respects to say $u_{m-1,n}$ (contained on the left hand side via $u_{m,n+1}$) implies that ξ does not depend on u. Differentiating (24) with respect to $t_{m,n-1}$ we find that ξ cannot depend on t either. Putting $\xi = \xi(x)$ into equation (23) and taking the second derivative with respect to $x_{m-1,n}$ and $x_{m,n}$, we obtain that ξ is linear in x. Similarly, invariance of equation (21) restricts the form of

$\tau(x,t,u)$. Finally the lattice (20), (21) is invariant under the transformation generated by $\hat{X}$ with

$$\xi = \alpha x + \beta \qquad \tau = \gamma t + \delta. \tag{25}$$

Now let us apply pr$\hat{X}$ to equation (22). We obtain

$$\frac{\phi_{m,n+1} - \phi_{m,n}}{t_{m,n+1} - t_{m,n}} = \frac{\phi_{m+1,n} - 2\phi_{m,n} + \phi_{m-1,n}}{(x_{m+1,n} - x_{m,n})^2} - (2\alpha - \gamma)\frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{(x_{m+1,n} - x_{m,n})^2}. \tag{26}$$

Taking the second derivative $\partial_{u_{m+1,n}}\partial_{u_{m-1,n}}$ of equation (26) after using the equation (22) to eliminate $u_{m,n+1}$, we find $\phi_{m,n} = A_{m,n}(x,t)u_{m,n} + B_{m,n}(x,t)$. Substituting back into equation (26) we obtain $A_{m,n} = A = const.$, and see that $B_{m,n}(x,t)$ must satisfy the original difference system. Moreover, we obtain the restriction $\gamma = 2\alpha$.

Finally, on the lattice (20), (21) the heat equation (22) has a symmetry algebra generated by the operators (19) and the additional dilation operator

$$\hat{D} = x\,\partial_x + 2t\,\partial_t. \tag{27}$$

We mention that the lattice equations (20), (21) can be solved to give $x = am + b$, $t = cn + d$. At first glance this seems to coincide with the lattice (6). The difference is that in equation (6) h_1 and h_2 are fixed constants. Here a, b, c and d are integration constants that can be chosen arbitrarily. In particular, they can be dilated. Hence the additional dilational symmetry.

B) A four point lattice

We only need four points to write the discrete heat equation, so it makes sense to write a four point lattice. Let us define the lattice by the equations

$$x_{m+1,n} - 2x_{m,n} + x_{m-1,n} = 0 \qquad x_{m,n+1} - x_{m,n} = 0 \tag{28}$$

$$t_{m+1,n} - t_{m,n} = 0 \qquad t_{m,n+1} - t_{m,n} - c(x_{m+1,n} - x_{m,n})^2 = 0. \tag{29}$$

On this lattice the discrete heat equation (22) simplifies to

$$u_{m,n+1} - u_{m,n} = c(u_{m+1,n} - 2u_{m,n} + u_{m-1,n}). \tag{30}$$

Applying the same method as above, we find that invariance of the lattice implies $\xi = Ax + B$, $\tau = 2At + C$. Invariance of equation (30) then implies

$\phi = Du + S(x,t)$ where A, B, C and D are constants and $S(x,t)$ solves the discrete heat equation. Thus, the discrete heat equation on the four point lattice (28), (29) is invariant under the same group as on the five point lattice (20), (21).

3.3 Exponential lattice

Let us now consider a lattice that is neither equally spaced, nor orthogonal, given by the equations

$$x_{m+1,n} - 2x_{m,n} + x_{m,n-1} = 0 \quad x_{m,n+1} = (1+c)\,x_{m,n} \tag{31}$$

$$t_{m,n+1} - t_{m,n} = h \qquad t_{m+1,n} - t_{m,n} = 0 \tag{32}$$

with $c \neq 0, -1$. These equations can be solved and explicitly the lattice is

$$t = hn + t_0 \qquad x = (1+c)^n\,(\alpha m + \beta) \tag{33}$$

where t_0, α and β are integration constants. Thus while t grows by constant increments, x grows with increments which vary exponentially with time (see Figure 2). Numerically this type of lattice may be useful if we can solve the equation asymptotically for large values of t and are interested in the small t behavior.

The heat equation on lattice (31), (31) can be written as

$$\frac{u_{m,n+1} - u_{m,n}}{h} = \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{(x_{m+1,n} - x_{m,n})^2}. \tag{34}$$

Applying the symmetry algorithm to the lattice equations (31), (32) we find that the symmetry algebra is restricted to

$$\hat{X} = \left[a\,x + b\,(1+c)^{t/h}\right]\,\partial_x + \tau_0\,\partial_t + \phi(x,t,u)\,\partial_u, \tag{35}$$

where a, b and τ_0 are arbitrary constants (whereas c and h are constants determining the lattice). Invariance of the equation (34) implies $a = 0$ in (35) and restricts $\phi(x,t,u)$ to reflect linearity of the equation and nothing more. The resulting symmetry algebra has a basis consisting of

$$\hat{P}_1 = (1+c)^{t/h}\,\partial_x \qquad \hat{P}_0 = \partial_t \qquad \hat{W} = u\,\partial_u \qquad \hat{S} = S(x,t)\,\partial_u \tag{36}$$

where $S(x,t)$ satisfies the heat equation. We see that the system is no longer invariant under space translations, or rather, that these 'translations' become time dependent and thus simulate a transformation to a moving frame.

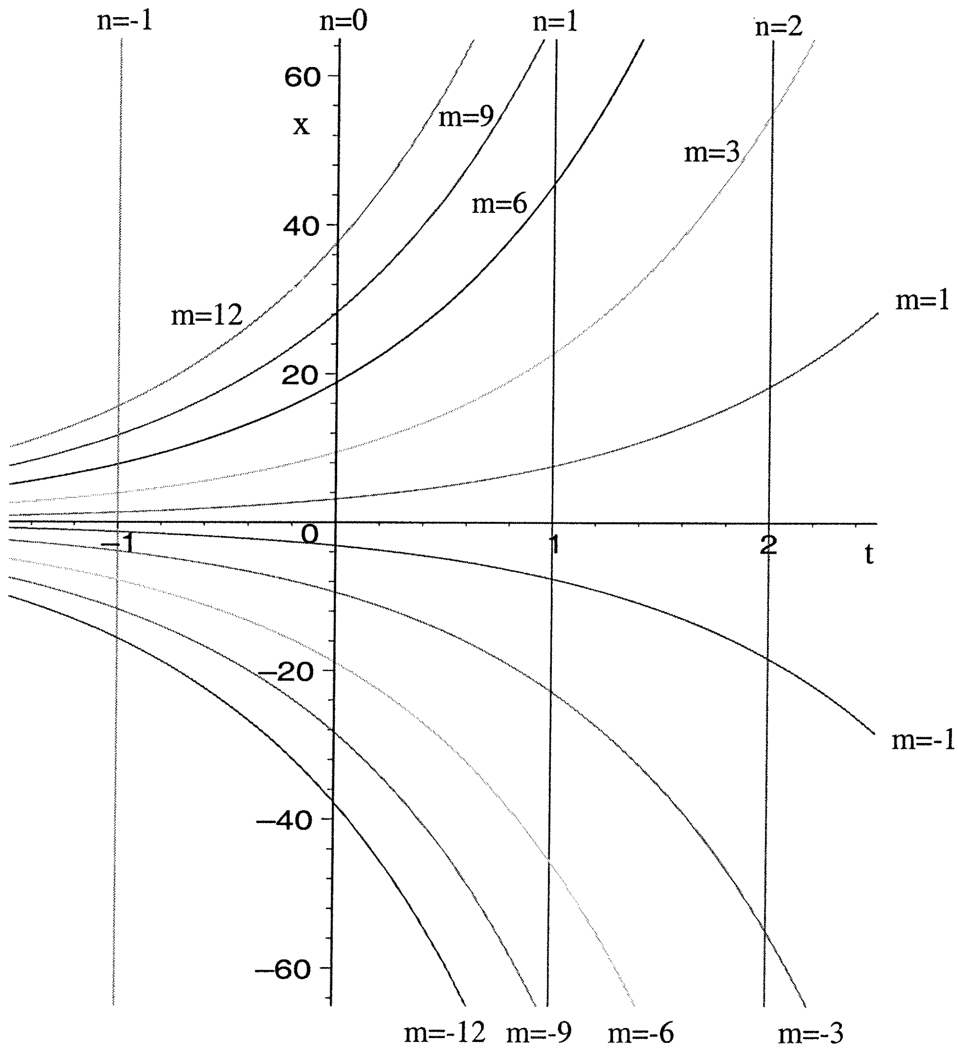

Figure 2: Variables (x, t) as functions of m and n for the lattice equations (31), (32). The parameters and the integration constants are, respectively, $c = \sqrt{2}, h = 1$ and $\alpha = \pi, \beta = 0, t_0 = 0$.

3.4 Galilei invariant lattice

Let us now consider the following difference scheme

$$\frac{u_{m,n+1} - u_{m,n}}{\tau_2} = \tau_2^2 \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{\zeta^2} \tag{37}$$

$$t_{m+1,n} - t_{m,n} = \tau_1 \qquad t_{m,n+1} - t_{m,n} = \tau_2 \tag{38}$$

$$x_{m+1,n} - 2x_{m,n} + x_{m-1,n} = 0 \tag{39}$$

$$(x_{m+1,n} - x_{m,n})\tau_2 - (x_{m,n+1} - x_{m,n})\tau_1 = \zeta \tag{40}$$

where τ_1, τ_2 and ζ are fixed constants.

The lattice equations can be solved and we obtain

$$t_{m,n} = \tau_1 \, m + \tau_2 \, n + t_0 \qquad x_{m,n} = \sigma\tau_1 \, m + \left(\frac{\sigma\tau_1\tau_2 - \zeta}{\tau_1}\right) n + x_0 \tag{41}$$

where σ, t_0 and x_0 are integration constants. The corresponding lattice is equally spaced and in general, nonorthogonal (see Figure 3). Indeed, the coordinate curves, corresponding to $m = const$ and $n = const$, respectively, are

$$\begin{aligned} x - x_0 &= \sigma (t - t_0) - \frac{\zeta}{\tau_1} n \\ x - x_0 &= \frac{\sigma\tau_1\tau_2 - \zeta}{\tau_1\tau_2} (t - t_0) + \frac{\zeta}{\tau_2} m. \end{aligned} \tag{42}$$

These are two families of straight lines, orthogonal only in the special case $(\sigma^2 + 1)\tau_1\tau_2 = \sigma\zeta$. If we choose

$$\sigma\tau_1\tau_2 - \zeta = 0 \tag{43}$$

then the second family of coordinate lines in equation (42) is parallel to the x axis.

Invariance of equation (38) implies that in the vector field we have $\tau(x, t, u) = \alpha = const$. From the invariance of equation (39) we obtain $\xi = A(t)\, x + B(t)$ with

$$A(t_{m+1,n}) = A(t_{m,n}) \qquad B(t_{m+1,n}) - 2B(t_{m,n}) + B(t_{m-1,n}) = 0. \tag{44}$$

Finally, invariance of equation (40) implies $A(t) = 0$ and $B(t) = \beta t + \gamma$ where β and γ are constants. Now let us apply the prolonged vector field to

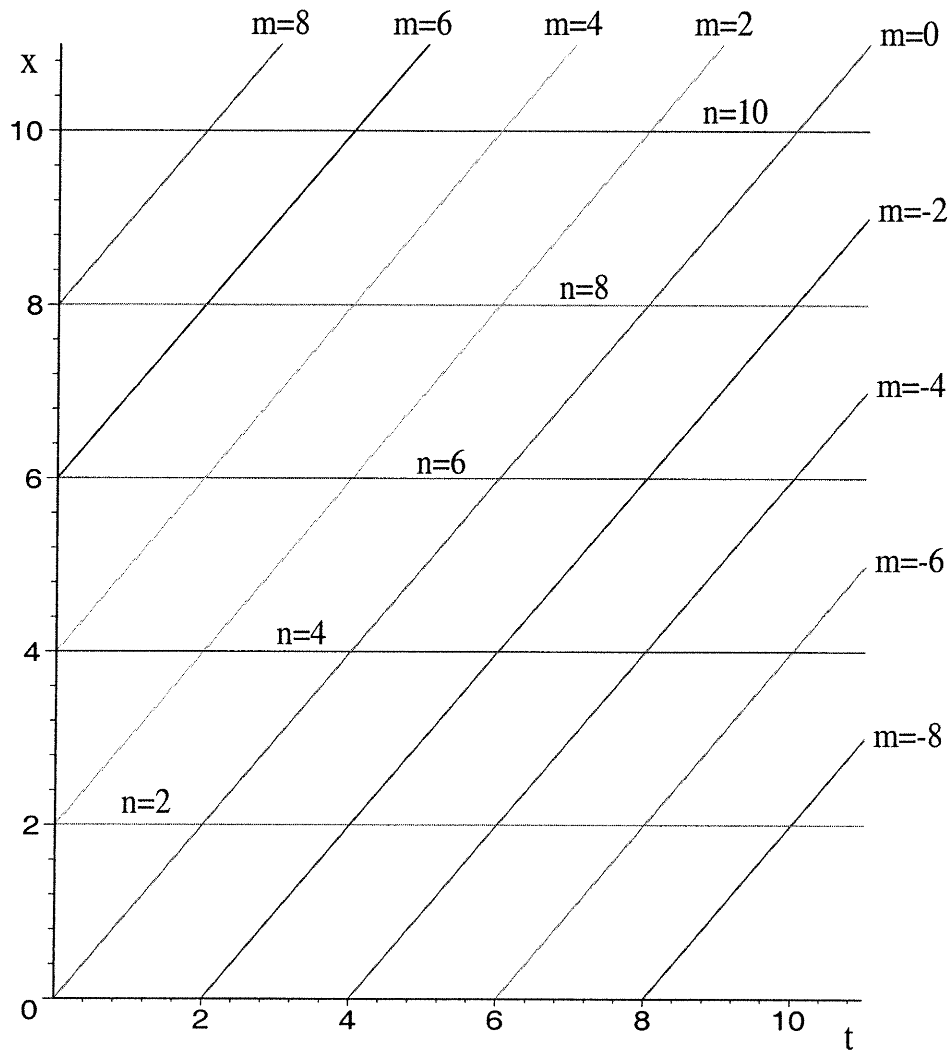

Figure 3: Variables (x, t) as functions of m and n for the lattice equations (38), (39), (40). The parameters and the integration constants are, respectively, $\tau_1 = 1, \tau_2 = 2, \zeta = 2$ and $\sigma = 1, x_0 = 0, t_0 = 0$.

equation (37). We obtain $\phi = Ru + S(x,t)$ where $S(x,t)$ satisfies the system (37),...,(40). The symmetry algebra is given by

$$\hat{P}_0 = \partial_t \quad \hat{P}_1 = \partial_x \quad \hat{B} = t\,\partial_x \quad \hat{W} = u\,\partial_u \quad \hat{S} = S(x,t)\,\partial_u. \tag{45}$$

Thus, the system is Galilei invariant with Galilei transformation generated by the operator $\hat{B}$.

Let us now consider the continuous limit of the system (37),...,(40). We use the solution (41) of the lattice equations (38), (39), (40) and for simplicity restrict the constants by imposing equation (43). We have, from equation (41), (43)

$$\begin{aligned} t_{m,n+1} &= t_{m,n} + \tau_2 \qquad x_{m,n+1} = x_{m,n} \\ x_{m\pm1,n} &= x_{m,n} \pm \sigma\tau_1 \quad t_{m\pm1,n} = t_{m,n} \pm \tau_1. \end{aligned} \tag{46}$$

The continuous limit is obtained by pushing $\tau_1 \ll 1$, $\tau_2 \ll 1$, $\zeta \ll 1$ and expanding both sides of equation (37) into a Taylor series, keeping only the lowest order terms. The LHS of equation (37) gives

$$\begin{aligned} \frac{u_{m,n+1} - u_{m,n}}{\tau_2} &= \frac{u(x_{m,n}, t_{m,n} + \tau_2) - u(x_{m,n}, t_{m,n})}{\tau_2} \\ &= u_t + \mathcal{O}(\tau_2) \end{aligned}$$

and the RHS is given by

$$\begin{aligned} &\left(\tfrac{\tau_2}{\zeta}\right)^2 (u_{m+1,n} - 2u_{m,n} + u_{m-1,n}) \\ &= \left(\tfrac{\tau_2}{\zeta}\right)^2 [u(x_{m,n} + \sigma\tau_1, t_{m,n} + \tau_1) - 2u(x_{m,n}, t_{m,n}) + u(x_{m,n} - \sigma\tau_1, t_{m,n} - \tau_1)] \\ &= u_{xx} + \tfrac{2}{\sigma}u_{x,t} + \tfrac{1}{\sigma^2}u_{tt} + \mathcal{O}(\tau_1). \end{aligned}$$

The continuous limit of the system (37),..., (40) is

$$u_t = u_{xx} + \frac{2}{\sigma}u_{x,t} + \frac{1}{\sigma^2}u_{tt} \quad \sigma \neq 0. \tag{47}$$

The symmetry algebra of this equation, for any value of σ, is isomorphic to that of the heat equation. In addition to the pseudo-group of the superposition principle, we have

$$
\begin{gathered}
\hat{P}_0 = \partial_t \qquad \hat{D} = x\,\partial_x + 2t\,\partial_t - \tfrac{1}{2}u\,\partial_u - cx\,\partial_t \\
\hat{K} = tx\,\partial_x + t^2\,\partial_t - \tfrac{1}{2}(t + \tfrac{1}{2}x^2)u\,\partial_u - c(x^2\,\partial_x + xt\,\partial_t - \tfrac{1}{2}xu\,\partial_u) \\
\hat{P}_1 = \partial_x + c\,\partial_t \qquad \hat{W} = u\,\partial_u \\
\hat{B} = t\,\partial_x - \tfrac{1}{2}xu\,\partial_u - c(x\,\partial_x - 2t\,\partial_t) - c^2x\,\partial_t \qquad c \equiv 1/\sigma .
\end{gathered}
\tag{48}
$$

The fact that the commutation relations do not depend on c suggest that equation (47) could be transformed into the heat equation. This is indeed the case and it suffices to put

$$
\begin{gathered}
u(x,t) = \mathrm{e}^{\frac{c[(2+c^2)x+ct]}{4(1+c^2)^2}}\, w(\alpha, \beta) \\
\alpha = x + ct \qquad \beta = (1 + c^2)(t - cx)
\end{gathered}
\tag{49}
$$

to obtain

$$w_\beta = w_{\alpha\alpha}. \tag{50}$$

Notice that while the difference equation (37) on the lattice (38), (39), (40) is Galilei invariant, this invariance is realized in a different manner, than for the continuous limit (47). To see this, compare the operator $\hat{B}$ of equation (45) with that of equation (48).

4 Lorentz invariant equations

The partial differential equation

$$u_{xy} = f(u) \tag{51}$$

is invariant under the inhomogeneous Lorentz group, with its Lie algebra realized as

$$\hat{X}_1 = \partial_x \qquad \hat{X}_2 = \partial_y \qquad \hat{L} = y\partial_x - x\partial_y \tag{52}$$

(for any function $f(u)$). In equation (51) x and y are 'light cone' coordinates. In the continuous case we can return to the usual space-time coordinates $z = x + y$, $t = x - y$, in which we have

$$u_{zz} - u_{tt} = f(u) \tag{53}$$

instead of equation (51) and the Lorentz group is generated by

$$\hat{P}_0 = \partial_t \qquad \hat{P}_1 = \partial_z \qquad \hat{L} = t\partial_z + z\partial_t. \tag{54}$$

Let us now consider a discrete system, namely

$$\frac{u_{m+1,n+1} - u_{m,n+1} - u_{m+1,n} + u_{m,n}}{(x_{m+1,n} - x_{m,n})(y_{m,n+1} - y_{m,n})} = f(u_{m,n}) \tag{55}$$

$$x_{m+1,n} - 2x_{m,n} + x_{m-1,n} = 0 \qquad x_{m,n+1} - x_{m,n} = 0 \tag{56}$$

$$y_{m,n+1} - 2y_{m,n} + y_{m,n-1} = 0 \qquad y_{m+1,n} - y_{m,n} = 0. \tag{57}$$

Applying the operator pr$\hat{X}$ (with t replaced by y) of equation (9) to equations (56), (57) we obtain

$$\xi = Ax + C \qquad \eta = By + D. \tag{58}$$

Requesting the invariance of equation (55) we find that ϕ must be linear

$$\phi = \alpha(x, y)u + \beta(x, y). \tag{59}$$

The remaining determining equations yield $\alpha = \alpha_0 = const.$ and

$$(A + B)\frac{\partial f}{\partial u_{m,n}} + (\alpha_0 u_{m,n} + \beta(x, y))\frac{\partial^2 f}{\partial u_{m,n}^2} = 0. \tag{60}$$

Thus, for any function $f = f(u)$ we obtain the symmetries (52), just as in the continuous case (they correspond to $B = -A$, $\alpha_0 = \beta = 0$). As in the continuous case, the symmetry algebra can be larger for special choices of the function $f(u)$. Let us analyze these cases.

a) Nonlinear interaction

We have $f'' \neq 0$, hence $\beta = \beta_0 = const.$ The function must then satisfy

$$(A + B - \alpha_0)f + (\alpha_0 u + \beta)f' = 0. \tag{61}$$

For $\alpha_0 \neq 0$ we take

$$f = u^p \qquad p \neq 0, 1 \tag{62}$$

(we have dropped some inessential constants). The system (55), (56), (57) is, in this case, invariant under a four-dimensional group generated by the algebra (52), complemented by dilation

$$\hat{D} = x\partial_x + y\partial_y + \frac{2}{1-p}u\partial_u. \tag{63}$$

For $\alpha_0 = 0$, $\beta \neq 0$ we have

$$f = \mathrm{e}^u. \tag{64}$$

The algebra is again four-dimensional with the additional dilation

$$\hat{D} = x\partial_x + y\partial_y - 2\partial_u. \tag{65}$$

b) Linear interaction $f(u) = u$

The only elements of the Lie algebra additional to (52) are

$$\hat{D} = u\partial_u \qquad \hat{S}(\beta) = \beta\partial_u \tag{66}$$

where β satisfies the system(55), (56), (57) with $f(u) = u$. The presence of $\hat{D}$ and $\hat{S}(\beta)$ is just a consequence of linearity.

c) Constant interaction $f(u) = 1$

The additional elements of the Lie algebra are again a consequence of linearity, namely

$$\hat{L} = x\partial_x + y\partial_y + 2u\partial_u \qquad \hat{S} = [S_1(x) + S_2(y)]\partial_u \tag{67}$$

where $S_1(x)$ and $S_2(y)$ are arbitrary (because $S_1(x) + S_2(y)$ is the general solution of equation (55) with $f(u) = 0$ on the lattice (56), (57)).

To find a discretization of equation (53), invariant under the group corresponding to (54) is more difficult and we will not go into that here.

As stressed in the Introduction, the methods of this article can be applied to any difference system, but they provide only point symmetries. We could treat the integrable discrete Liouville and Sine-Gordon equations of Faddeev [32], or Hirota [33], but would not otain the generalized symmetries that are of interest. The correct formalism to use for these equations is that of Ref. [11].

5 Discrete Burgers equation

The continuous Burgers equation is written as

$$u_t = u_{xx} + 2uu_x, \tag{68}$$

or in potential form as

$$v_t = v_{xx} + v_x^2 \qquad u \equiv v_x. \tag{69}$$

We shall determine the symmetry groups of two different discrete Burgers equations, both on the same lattice. The lattice is one of those used above for the heat equation, namely the four point lattice (28), (29). Each of the four lattice equations involves at most three points. Hence, for any difference equation on this lattice, involving all four points, the symmetry algebra will be realized by vector fields of the form (8) with

$$\xi = Ax + B \qquad \tau = 2At + D \tag{70}$$

where A, B and D are constants (see section 3.2B).

5.1 Nonintegrable discrete potential Burgers equation

An absolutely straightforward discretization of equation (69) on the lattice (28), (29) is

$$\frac{u_{m,n+1} - u_{m,n}}{t_{m,n+1} - t_{m,n}} = \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{(x_{m+1,n} - x_{m,n})^2} + \left(\frac{u_{m+1,n} - u_{m,n}}{x_{m+1,n} - x_{m,n}}\right)^2. \tag{71}$$

Applying the usual symmetry algorithm, we find a four-dimensional symmetry algebra

$$\hat{P}_1 = \partial_x \qquad \hat{P}_0 = \partial_t \qquad \hat{D} = x\,\partial_x + 2t\,\partial_t \qquad \hat{W} = \partial_u. \tag{72}$$

5.2 A linearizable discrete Burgers equation

A different discrete Burgers equation was proposed recently [9]. It is linearizable by a discrete version of the Cole-Hopf transformation. Using the notation of this article, we write the linearizable equation as

$$u_{m,n+1} = u_{m,n} + c\,\frac{(1 + h_x u_{m,n})[u_{m+2,n} - 2u_{m+1,n} + u_{m,n} + h_x u_{m+1,n}(u_{m+2,n} - u_{m,n})}{1 + ch_x[u_{m+1,n} - u_{m,n} + h_x u_{m,n} u_{m+1,n}]} \tag{73}$$

$$h_x \equiv x_{m+1,n} - x_{m,n} \quad h_t \equiv t_{m,n+1} - t_{m,n} = c h_x^2$$

$$t_{m+1,n} - t_{m,n} = 0 \qquad x_{m,n+1} - x_{m,n} = 0.$$

In equation (73) c is a constant, but h_x is a variable, subject to dilations. The determining equation is obtained in the usual manner. It involves the function $\phi_{m,n}$ at all points figuring in equation (73), and also the constant A of equation (70). The equation is too long to be included here, but is straightforward to obtain. The variable that we choose to eliminate using equation (73) is $u_{m,n+1}$. Differentiating twice with respect to $u_{m+2,n}$ we obtain

$$\frac{\partial^2 \phi_{m,n+1}}{\partial u_{m,n+1}^2} \frac{\partial u_{m,n+1}}{\partial u_{m+2,n}} = \frac{\partial^2 \phi_{m+2,n}}{\partial u_{m+2,n}^2}. \tag{74}$$

We differentiate (74) with respect to $u_{m,n}$ and then, separately, with respect to $u_{m-1,n}$. We obtain two equations that are compatible for $c(1+c)^2 h_x (1 + h_x u_{m,n}) = 0$. Otherwise they imply that ϕ is linear in u: $\phi = \alpha(x,t)\, u + \beta(x,t)$. We have $c \neq 0$, $h_x \neq 0$, but the case $c = -1$ must be considered separately. We first introduce the expression for ϕ into the determining equation and obtain, after a lengthy computation (using MAPLE): $\alpha = -A$, $\beta = 0$. For $c = -1$ we proceed differently, but got the same result. Finally, the Lie point symmetry algebra of the system (73), (28), (29) has the basis

$$\hat{P}_0 = \partial_t \quad \hat{P}_1 = \partial_x \quad \hat{D} = x\,\partial_x + 2t\,\partial_t - u\,\partial_u. \tag{75}$$

This result should be compared with the symmetry algebra of equation (73) on a fixed constant lattice, found earlier [9]. The symmetry algebra found there was five-dimensional. It was inherited from the heat equation, via the discrete Cole-Hopf transformation. It was realized in a 'discrete evolutionary formalism' by flows, commuting with the flow given by the Burgers equation. The symmetries found there were higher symmetries, and cannot be realized in terms of the vector fields of the form considered in this article.

6 Symmetries of differential-difference equations

6.1 General comments

Symmetries of differential-difference equations were discussed in our previous article [1]. Here we shall put them into the context of partial difference equations and consider a further example. As in the case of multiple discrete variables, we will consistently consider the action of vector fields at points in the space of independent and dependent variables. To do this we introduce a discrete independent variable n (or several such variables) and a continuous independent variable α (or a vector variable $\vec{\alpha}$). A point in the space of independent variables will be $P_{n,\alpha}$, its coordinates $\{x_{n,\alpha}, z_{n,\alpha}\}$ where both x and z can be vectors. The form of the lattice is specified by some relations between $x_{n,\alpha}, z_{n,\alpha}$ and $u_{n,\alpha} \equiv u(x_{n,\alpha}, z_{n,\alpha})$.

We shall not present the general formalism here, but restrict to the case of one discretely varying variable $z \equiv z_n$, $-\infty < n < \infty$ and either one continuous (time) variable (t), or two continuous variables (x, y).

For instance, a uniform lattice that is time independent can be given by the relations

$$\begin{aligned} z_{n+1,\alpha} - 2z_{n,\alpha} + z_{n-1,\alpha} &= 0 && (76) \\ z_{n,\alpha} - z_{n,\alpha'} &= 0 && (77) \\ t_{n+1,\alpha} - t_{n,\alpha} &= 0. && (78) \end{aligned}$$

where α' is a different value of the continuous variable α.

Conditions (77), (78) are rather natural. They state that time is the same at each point of the lattice and that the lattice does not evolve in time. They are however not obligatory. Similarly, equation (76) is not obligatory. The solution of equations (76), ..., (78) is of course trivial, namely

$$z_n = h\,n + z_0 \qquad t = t(\alpha) \tag{79}$$

and we can identify t and α ($t = \alpha$, h and z_0 are constants).

The prolongation of a vector field acting on a differential-difference scheme on the lattice (76),..., (78) will have the form

$$\begin{aligned}\mathrm{pr}\,\hat{X} = \;& \textstyle\sum_n \Big[\tau(z_{n,\alpha}, t_{n,\alpha}, u_{n,\alpha})\,\partial_{t_{n,\alpha}} + \zeta(z_{n,\alpha}, t_{n,\alpha}, u_{n,\alpha})\,\partial_{z_{n,\alpha}} \\ & +\phi(z_{n,\alpha}, t_{n,\alpha}, u_{n,\alpha})\,\partial_{u_{n,\alpha}}\Big] + \dots\end{aligned} \tag{80}$$

where the dots signify terms acting on time derivatives of u. Since $u_{n,\alpha}$, $u_{n,\alpha'}$ and $u_{n+1,\alpha}$ are all independent, equations (77) and (78) imply

$$\zeta = \zeta(z_n) \qquad \tau = \tau(t). \tag{81}$$

On any lattice satisfying equation (77), (78) we can simplify notation and write

$$\hat{X} = \zeta(z)\,\partial_z + \tau(t)\,\partial_t + \phi(z,t,u)\,\partial_u. \tag{82}$$

Similarly for an equation with one discretely varying independent variable z and two continuous ones (x, y) one can impose

$$z_{n+1,\alpha_1,\alpha_2} - 2z_{n,\alpha_1,\alpha_2} + z_{n-1,\alpha_1,\alpha_2} = 0 \tag{83}$$

$$\begin{aligned} z_{n,\alpha_1',\alpha_2} - z_{n,\alpha_1,\alpha_2} &= 0 \\ z_{n,\alpha_1,\alpha_2'} - z_{n,\alpha_1,\alpha_2} &= 0 \end{aligned} \tag{84}$$

$$\begin{aligned} x_{n+1,\alpha_1,\alpha_2} - x_{n,\alpha_1,\alpha_2} &= 0 \\ y_{n+1,\alpha_1,\alpha_2} - y_{n,\alpha_1,\alpha_2} &= 0. \end{aligned} \tag{85}$$

Invariance of the conditions (84) and (85) then implies that the vector fields realizing the symmetry algebra have the form

$$\hat{X} = \zeta(z)\,\partial_z + \xi(x,y)\,\partial_x + \eta(x,y)\,\partial_y + \phi(z,x,y,u)\,\partial_u. \tag{86}$$

We can again simplify notation identifying $x = \alpha_1$, $y = \alpha_2$ and solving (83) to give $z_n = h\,n + z_0$ (h and z_0 constant).

6.2 Examples

We shall consider here just one example that brings out the role of the lattice equations very clearly. The example is Toda field theory, or the two-dimensional Toda lattice [34, 35, 13]. It is given by the equation

$$u_{n,xy} = \mathrm{e}^{u_{n-1}-u_n} - \mathrm{e}^{u_n - u_{n+1}} \tag{87}$$

with $u_n \equiv u(z_n, x, y)$.

On the lattice (83),..., (85) we start with equation (86) and have

$$\mathrm{pr}\,\hat{X} = \xi(x,y)\,\partial_x + \eta(x,y)\,\partial_y + \sum_{k=-1}^{1} \zeta_{n+k}(z)\,\partial_{z_{n+k}} + \sum_{k=-1}^{1} \phi_{n+k}\,\partial_{u_{n+k}} + \phi_n^{xy}\,\partial_{u_{n,xy}} \tag{88}$$

where ϕ_n^{xy} is calculated in the same way as for differential equations [2].

Applying (88) to equations (83) and (87) we find

$$\xi = \xi(x) \qquad \eta = \eta(y) \qquad \zeta_n = A\,z_n + B \qquad \phi_n = \beta_n(x, y, z_n) \tag{89}$$

and we still have two equation to solve, namely

$$\beta_{n+1} - \beta_n + \xi_x + \eta_y = 0 \tag{90}$$

$$\beta_{n,xy} = 0. \tag{91}$$

On the lattice (83),..., (85) z_{n+1} and z_n are independent. Hence we can differentiate (90) with respect to z_{n+1} and find that β_{n+1} is independent of z_{n+1} and hence of n. We thus find a symmetry algebra generated by

$$\begin{gathered} \hat{P}_1 = \partial_x \qquad \hat{P}_2 = \partial_y \qquad \hat{L} = x\,\partial_y - y\,\partial_x \qquad \hat{S} = \partial_z \qquad \hat{D} = z\,\partial_z \\ \hat{U}(k) = k(x)\,\partial_u \qquad \hat{V}(h) = h(y)\,\partial_u \end{gathered} \tag{92}$$

where $k(x)$ and $h(y)$ are arbitrary smooth functions. Notice that $\hat{S}$ and $\hat{D}$ act only on the lattice and $\hat{U}(k)$ and $\hat{V}(h)$ generate gauge transformations, acting only on the dependent variables.

If we change the lattice to a fixed, nontransforming one, i.e. replace (83) by

$$z_{n+1,\alpha_1,\alpha_2} - z_{n,\alpha_1,\alpha_2} = h \tag{93}$$

$h = const$, the situation changes dramatically. We loose the dilation $\hat{D}$ of equation (92), however z_{n+1} and z_n are now related by equation (93). The solution of equation (90), (91) in this case is

$$\beta_n = \frac{z}{h}(\xi_x + \eta_y) + k(x) + h(y). \tag{94}$$

On this fixed lattice the Toda field equations are conformally invariant and the invariance algebra is spanned by

$$\begin{gathered} \hat{X}(f) = f(x)\,\partial_x + \tfrac{z}{h} f'(x)\,\partial_u \qquad \hat{Y}(g) = g(y)\,\partial_y + \tfrac{z}{h} g'(y)\,\partial_u \\ \hat{U}(k) = k(x)\,\partial_u \qquad \hat{V}(h) = h(y)\,\partial_u \qquad \hat{S} = \partial_z. \end{gathered} \tag{95}$$

We see that giving more freedom to the lattice (three points z_{n+1}, z_n, z_{n-1} instead of two) may lead to a reduction of the symmetry group, rather than to an enhancement. For the Toda field theory the reduction is a drastic one: the two arbitrary functions $f(x)$ and $g(y)$ reduce to $f = ax + b$, $g = -ay + d$, respectively (and only the element $\hat{D}$ is added to the symmetry algebra).

7 Conclusions and future outlook

The main conclusion is that we have presented an algorithm for determining the Lie point symmetry group of a difference system, i.e. a difference equation and the lattice it is defined on. The algorithm provides us with all Lie point symmetries of the system. In Ref. [1] we considered only one discretely varying independent variable. In this article we concentrated on the case of two such variables. The case of an arbitrary number of dependent and independent variables is completely analogous though it obviously involves more cumbersome notations and lengthier calculations. The problem of finding the symmetry group is reduced to solving linear functional equations. In turn, these are converted into an overdetermined system of linear partial difference equations, just as in the case of differential equations. The fact that the determining equations are linear, even if the the studied equations are nonlinear, is due to the infinitesimal approach.

The symmetry algorithm can be computerized, just as it has been for differential equations.

In previous articles (other than Ref. [1]) we considered only one discretely varying variable and a fixed (nontransforming) lattice [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. The coefficients in the vector fields, realizing the symmetry algebra, depended on variables evaluated at more than one point of the lattice, possibly infinitely many ones. Thus, one obtained generalized symmetries together with point ones. For integrable equations, including linear and linearizable ones, the symmetry structure can be quite rich [8, 9, 10, 11, 15, 17, 18]. In the continuous limit some of the generalized symmetries reduce to point ones [11, 17, 18] and the structure of the symmetry algebra changes.

A detailed comparison of various symmetry methods is postponed to a future article. Applications of Lie point symmetries, as well as generalized symmetries, to the solution of difference equations, will be given elsewhere.

Acknowledgments

The work reported in this article was performed while S.T. and P.W. were visiting the Dipartimento di Fisica of the Università di Roma Tre in the framework of the Université de Montréal – Università Roma Tre interchange agreement. They thank this University and INFN for hospitality and support. The research of P.W. is partly supported by grants from NSERC of Canada and FCAR of Québec.

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Chapitre 4

CLASSIFICATION PAR SYMÉTRIES DES CHAîNES MOLÉCULAIRES DIATOMIQUES

Symmetry Classification of Diatomic Molecular Chains

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Abstract

A symmetry classification of possible interactions in a diatomic molecular chain is provided. For nonlinear interactions the group of Lie point transformations, leaving the lattice invariant and taking solutions into solutions, is at most five-dimensional. An example is considered in which subgroups of the symmetry group are used to reduce the dynamical differential-difference equations to purely difference ones.

1 Introduction

The purpose of this article is to analyze possible interactions in a long one-dimensional molecule consisting of two types of atoms. The model we consider is a very general one, described by the equations

$$\begin{aligned} E_1 &\equiv \ddot{x}_n - F_n(\xi_n, t) - G_n(\eta_{n-1}, t) = 0, \\ E_2 &\equiv \ddot{y}_n - K_n(\xi_n, t) - P_n(\eta_n, t) = 0, \end{aligned} \tag{1.1}$$

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where the overdots denote time derivatives and x_n, y_n can be interpreted as the displacement of the n-th atom of type X or Y, respectively, from their equilibrium positions. We define

$$\xi_n \equiv y_n - x_n, \qquad \eta_n \equiv x_{n+1} - y_n \tag{1.2}$$

and t is time. The functions F_n, G_n, K_n and P_n are as yet unspecified smooth functions. Indeed, our aim is to classify such systems according to the Lie point symmetries that they allow, that is, to classify these functions F_n, G_n, K_n and P_n.

The assumptions built into the model are:

1. The atoms of type X and Y alternate along a fixed uniform one-dimensional chain with positions labeled by the integers n (see Figure 1).

2. Only nearest neighbor interactions are considered, i.e. the atom X_n interacts only with Y_{n-1} and Y_n and Y_n interacts only with X_n and X_{n+1} (see Figure 1).

3. The system is invariant under a uniform translation of all atoms in the molecule and also under a Galilei transformations of the chain.

4. The systems is strongly coupled, i.e. we assume

$$\frac{\partial F_n}{\partial \xi_n} \neq 0, \quad \frac{\partial G_n}{\partial \eta_{n-1}} \neq 0, \quad \frac{\partial K_n}{\partial \xi_n} \neq 0, \quad \frac{\partial P_n}{\partial \eta_n} \neq 0. \tag{1.3}$$

5. In the bulk of the article we assume that the interactions are nonlinear, i.e. at least one of the four functions F_n, G_n, K_n or P_n depends nonlinearly on the argument ξ or η, respectively. The linear case will be treated separately.

6. A discrete symmetry is built into the model. Indeed, the two equations (1.1) are permuted by the transformation

$$\begin{gathered} x_n \longrightarrow y_n, \qquad y_n \longrightarrow x_{n+1}, \\ F_n \longrightarrow P_n, \quad G_n \longrightarrow K_n, \quad K_{n-1} \longrightarrow G_n, \quad P_{n-1} \longrightarrow F_n. \end{gathered} \tag{1.4}$$

Figure 1. Interactions between atoms of type X and Y along a molecular chain.

Models of this type have many applications in classical mechanics, in molecular physics, or mathematical biology [1, 2, 3]. In applications, the form of the functions in eq.(1.1) are usually *a priori* fixed.

The formalism used in this article is the one called "intrinsic method" in earlier articles [4, 5]. It has already been applied to monoatomic molecular chains [6] and to a model with two species, or two types of atoms, distributed along a double chain [7].

In this approach the dependent variables x and y depend on one discrete variable n and one continuous variable t. Symmetry transformations, taking solutions into solutions, act on the variables x, y and t, not however on the lattice variable n. The Lie algebra of the symmetry group is realized by vector fields of the form

$$\hat{X} = \tau(x_n, y_n, t)\partial_t + \phi_n(x_n, y_n, t)\partial_{x_n} + \psi_n(x_n, y_n, t)\partial_{y_n}. \tag{1.5}$$

The functions τ, ϕ_n and ψ_n are determined from the requirement that the second prolongation of the vector field $\hat{X}$ should annihilate equations (1.1) on their solution surface. Explicitly we have [4, 5, 6, 7]

$$\begin{aligned} \mathrm{pr}^{(2)}\hat{X} &= \tau(t, x_n, y_n)\partial_t + \sum_{k=n-1}^{n+1} \phi_k(t, x_k, y_k)\partial_{x_k} \\ &+ \sum_{k=n-1}^{n+1} \psi_k(t, x_k, y_k)\partial_{y_k} + \phi_n^{tt}\partial_{\ddot{x}_n} + \psi_n^{tt}\partial_{\ddot{y}_n} \end{aligned} \tag{1.6}$$

with

$$\begin{aligned} \phi_n^{tt} &= D_t^2\phi_n - (D_t^2\tau)\,\dot{x}_n - 2(D_t\tau)\,\ddot{x}_n, \\ \psi_n^{tt} &= D_t^2\psi_n - (D_t^2\tau)\,\dot{y}_n - 2(D_t\tau)\,\ddot{y}_n \end{aligned} \tag{1.7}$$

(D_t is the total time derivative). In eq.(1.6) we have spelled out only those terms which act on eq.(1.1).

The use of this formalism is not obligatory. Indeed, the group transformations can also act on the lattice [8, 9, 10, 11] and generalized symmetries can be very useful [12]. In this article we restrict ourselves to the intrinsic formalism, described above.

The present article is organized as follows. In Section 2 we establish the general form of the vector fields (1.5) that realize the symmetry algebra of eq.(1.1). We also derive the determining equations for the symmetries and introduce a "group of allowed transformations". Allowed transformations take equations of the type (1.1) into other equations of the same type. They can change the functions F_n, G_n, K_n and P_n into other functions of the same arguments. As in previous articles, we classify equations into symmetry classes under the action of allowed transformations [6, 7, 13, 14]. We also establish that equations (1.1) are invariant under a two-dimensional Abelian group for any functions $F_n, \ldots, P_n$. Section 3 is devoted to Abelian symmetry algebras. We denote them $A_{j,k}$ where A means Abelian, j denotes the dimension and $k = 1, 2, 3, \ldots$ enumerates algebras of the same dimension. For each interaction we list only the maximal symmetry algebra. Section 4 is devoted to nilpotent symmetry algebras, denoted by $N_{j,k}$ with the same conventions as in Section 3. In Section 5 we find all solvable symmetry algebras with non-Abelian nilradicals ($SN_{j,k}$). In Section 6 those with Abelian nilradicals ($SA_{j,k}$). All nonsolvable symmetry algebras are listed in Section 7 ($NS_{j,k}$). In Sections 3 to 7 we consider only nonlinear interactions. Symmetries of the linear case are discussed in Section 8. Conclusions and some applications of the symmetries are summed up in the final Section 9.

2 Determining equations and allowed transformations

The algorithm for finding the symmetry algebra of eq.(1.1) is

$$\mathrm{pr}\hat{X}\, E_a \,|_{E_b=0} = 0, \qquad a = 1, 2, \quad b = 1, 2. \tag{2.1}$$

The coefficients of all terms of the type $\dot{x}_n^p\, \dot{y}_n^q$ must vanish independently and we find that the vector field (1.5) must actually have the form

$$\hat{X} = \tau(t)\partial_t + \left[\left(a + \frac{\dot{\tau}(t)}{2}\right)x_n + \lambda_n(t)\right]\partial_{x_n} + \left[\left(a + \frac{\dot{\tau}(t)}{2}\right)y_n + \mu_n(t)\right]\partial_{y_n}, \tag{2.2}$$

where a is a constant and $\lambda_n(t), \mu_n(t)$ and $\tau(t)$ are functions of the indicated variables. This form (2.2) is valid for any interactions F_n, G_n, K_n and P_n in eq.(1.1). Moreover, we have

$$\tau = \tau_0 + \tau_1 t + \tau_2 t^2, \tag{2.3}$$

where τ_0, τ_1 and τ_2 are constants.

The constants a, τ_i and the functions $\lambda_n(t)$ and $\mu_n(t)$ are subject to two further determining equations that involve the interaction functions explicitly. They are

$$\begin{aligned} &\ddot{\lambda}_n + \left(a - \tfrac{3}{2}\dot{\tau}\right)(F_n + G_n) + \left[\lambda_n - \mu_n - \left(a + \tfrac{\dot{\tau}}{2}\right)\xi_n\right]F_{n,\xi_n} \\ &+ \left[\mu_{n-1} - \lambda_n - \left(a + \tfrac{\dot{\tau}}{2}\right)\eta_{n-1}\right]G_{n,\eta_{n-1}} - \tau(F_{n,t} + G_{n,t}) = 0, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\ddot{\mu}_n + \left(a - \tfrac{3}{2}\dot{\tau}\right)(K_n + P_n) + \left[\lambda_n - \mu_n - \left(a + \tfrac{\dot{\tau}}{2}\right)\xi_n\right]K_{n,\xi_n} \\ &+ \left[\mu_n - \lambda_{n+1} - \left(a + \tfrac{\dot{\tau}}{2}\right)\eta_n\right]P_{n,\eta_n} - \tau(K_{n,t} + P_{n,t}) = 0. \end{aligned} \tag{2.5}$$

Our task is to perform a complete analysis of eq.(2.4) and (2.5). Conceptually, this is very similar to the problem considered in Ref. 7. However, the functions figuring in eq.(1.1) are less general than those of Ref. 7, hence the computations are simpler.

We shall classify the equations of type (1.1) into equivalence classes under the action of a group of "allowed transformations". These are transformations of the form

$$x_n = \Phi_n(\tilde{x}_n, \tilde{y}_n, \tilde{t}), \qquad y_n = \Psi_n(\tilde{x}_n, \tilde{y}_n, \tilde{t}), \qquad t = T(\tilde{t}) \tag{2.6}$$

that transform equations (1.1) into equations of the same form, but do not preserve the functions on the right hand side of eq.(1.1). The requirement that no first derivatives should appear and that the transformed functions

$\tilde{F}_n$ and $\tilde{K}_n$ should depend only on $\tilde{\xi}_n$ and $\tilde{t}$, $\tilde{G}_n$ and $\tilde{P}_n$ only on $\tilde{t}$ and $\tilde{\eta}_{n-1}$ or $\tilde{\eta}_n$, respectively, implies that the transformations actually have the form

$$\begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix} = q\,\dot{\tilde{t}}^{-1/2} \begin{pmatrix} \tilde{x}_n(\tilde{t}) \\ \tilde{y}_n(\tilde{t}) \end{pmatrix} + \begin{pmatrix} \alpha_n(t) \\ \beta_n(t) \end{pmatrix}, \tag{2.7}$$

$$\tilde{t} = \frac{c_1\,t + c_2}{c_3\,t + c_4}, \qquad c_1c_4 - c_2c_3 = 1, \qquad q \neq 0, \tag{2.8}$$

where $q, c_1, \ldots, c_4$ are constants and α_n and β_n are arbitrary functions of n and t.

The transformed system is

$$\begin{aligned} \ddot{\tilde{x}}_n(\tilde{t}) &= \tilde{F}_n(\tilde{\xi}_n, \tilde{t}) + \tilde{G}_n(\tilde{\eta}_{n-1}, \tilde{t}), \\ \ddot{\tilde{y}}_n(\tilde{t}) &= \tilde{K}_n(\tilde{\xi}_n, \tilde{t}) + \tilde{P}_n(\tilde{\eta}_n, \tilde{t}), \end{aligned} \tag{2.9}$$

with

$$\begin{pmatrix} \tilde{F}_n + \tilde{G}_n \\ \tilde{K}_n + \tilde{P}_n \end{pmatrix} = \frac{\dot{\tilde{t}}^{-3/2}}{q} \left[\begin{pmatrix} F_n(\xi_n, t) + G_n(\eta_{n-1}, t) \\ K_n(\xi_n, t) + P_n(\eta_n, t) \end{pmatrix} - \begin{pmatrix} \ddot{\alpha}_n(t) \\ \ddot{\beta}_n(t) \end{pmatrix} \right], \tag{2.10}$$

where

$$\xi_n = y_n - x_n = q\,\dot{\tilde{t}}^{-1/2}(\tilde{x}_n - \tilde{y}_n) + \alpha_n(t) - \beta_n(t), \tag{2.11}$$

$$\eta_n = x_{n+1} - y_n = q\,\dot{\tilde{t}}^{-1/2}(\tilde{x}_{n+1} - \tilde{y}_n) + \alpha_{n+1}(t) - \beta_n(t), \tag{2.12}$$

$$t = \frac{c_4\,\tilde{t} - c_2}{-c_3\,\tilde{t} + c_1}. \tag{2.13}$$

The vector field $\hat{X}$ of eq.(2.2) is transformed into a similar field with

$$\tilde{\tau}(\tilde{t}) = \tau(t(\tilde{t}))\,\dot{\tilde{t}}, \qquad \tilde{a} = a, \tag{2.14}$$

$$\begin{pmatrix} \tilde{\lambda}_n(\tilde{t}) \\ \tilde{\mu}_n(\tilde{t}) \end{pmatrix} = \frac{\dot{\tilde{t}}^{1/2}}{q} \left[(a + \frac{\dot{\tau}}{2}) \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} - \tau \begin{pmatrix} \dot{\alpha}_n \\ \dot{\beta}_n \end{pmatrix} + \begin{pmatrix} \lambda_n \\ \mu_n \end{pmatrix} \right]. \tag{2.15}$$

The transformed functions and constants must satisfy the same determining equations (2.4) and (2.5).

As mentioned in the Introduction, translational and Galilei invariance are built into the model. That is easy to check. Indeed $\lambda_n = \mu_n = 1$, $a = 0$, $\tau(t) = 0$ and $\lambda_n = \mu_n = t$, $a = 0$, $\tau(t) = 0$ are solutions of eq.(2.4) and (2.5) for F_n, G_n, K_n and P_n arbitrary. No other symmetries exist, unless some constraints on the interactions are imposed.

We shall use the allowed transformations to simplify the vector fields that occur. In particular the coefficient $\tau(t)$ of a given vector field can be transformed into one of the following expressions: 0, 1, t or t^2+1.

Our strategy will be to first find all Abelian symmetry algebras, then all nilpotent (non-Abelian) ones. Once these are known, we can determine all solvable ones, having the corresponding Abelian, or nilpotent ones as nilradicals [15]. Finally, all nonsolvable symmetry algebras will be determined, making use of their Levi decomposition [15].

Any symmetry algebra will contain the algebra

$$A_{2,1}: \quad \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \qquad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}), \tag{2.16}$$

as a subalgebra. Allowed transformations leave the algebra (2.16) invariant. Any further element of the symmetry algebra can be transformed into one of the following ones

$$\hat{Y}_1 = \partial_t + a(x_n\,\partial_{x_n} + y_n\,\partial_{y_n}), \qquad a = 0, 1, \tag{2.17}$$

$$\hat{Y}_2 = t\,\partial_t + (a + \frac{1}{2})(x_n\,\partial_{x_n} + y_n\,\partial_{y_n}), \tag{2.18}$$

$$\hat{Y}_3 = (t^2+1)\,\partial_t + (a+t)(x_n\,\partial_{x_n} + y_n\,\partial_{y_n}), \tag{2.19}$$

$$\hat{Y}_4 = \lambda_n(t)(\partial_{x_n} + \partial_{y_n}), \qquad \ddot{\lambda}_n \neq 0, \quad \lambda_{n+1} \neq \lambda_n, \tag{2.20}$$

$$\hat{Y}_5 = \lambda_n(t)\,\partial_{x_n} + \lambda_{n+1}(t)\,\partial_{y_n}, \qquad \ddot{\lambda}_n \neq 0, \quad \lambda_{n+1} \neq \lambda_n. \tag{2.21}$$

The interactions that allow these additional terms can easily be determined from equations (2.4) and (2.5). Once this is done, we determine whether the considered interactions allows further symmetries. For each interaction, we shall only list the maximal symmetry algebra allowed, not lower-dimensional subalgebras.

3 Abelian symmetry algebras

The lowest dimensional maximal symmetry algebra is $A_{2,1}$ of eq.(2.16), present for any interactions in eq.(1.1). This algebra can be enlarged into a higher dimensional Abelian algebra by adding elements of the type (2.20) or (2.21). The determining equations for a nonlinear system allow at most four commuting symmetry generators. Moreover, the three-dimensional symmetry algebras are never maximal.

Finally, we obtain two different four-dimensional Abelian symmetry algebras together with the interactions that allow them. They are

$$
\begin{aligned}
A_{4,1} \quad & \hat{X}_1 = \lambda_{1,n}(t)(\partial_{x_n} + \partial_{y_n}), \qquad \hat{X}_2 = \lambda_{2,n}(t)(\partial_{x_n} + \partial_{y_n}), \\
& \hat{X}_3 = \partial_{x_n} + \partial_{y_n}, \qquad \hat{X}_4 = t(\partial_{x_n} + \partial_{y_n}), \\
& F_n = F_n(\xi_n, t), \qquad G_n = \frac{\ddot{\lambda}_{1,n}}{\lambda_{1,n} - \lambda_{1,n-1}}\eta_{n-1}, \\
& K_n = K_n(\xi_n, t), \qquad P_n = \frac{\ddot{\lambda}_{1,n}}{\lambda_{1,n+1} - \lambda_{1,n}}\eta_n, \\
& \ddot{\lambda}_{1,n} \neq 0, \qquad \ddot{\lambda}_{2,n} \neq 0, \qquad \lambda_{1,n+1} \neq \lambda_{1,n}, \\
& \frac{\ddot{\lambda}_{2,n}}{\ddot{\lambda}_{1,n}} = \frac{\lambda_{2,n} - \lambda_{2,n-1}}{\lambda_{1,n} - \lambda_{1,n-1}} = \frac{\lambda_{2,n+1} - \lambda_{2,n}}{\lambda_{1,n+1} - \lambda_{1,n}}.
\end{aligned}
$$

$$
\begin{aligned}
A_{4,2} \quad & \hat{X}_1 = \lambda_{1,n}(t)\partial_{x_n} + \lambda_{1,n+1}(t)\partial_{y_n}, \qquad \hat{X}_2 = \lambda_{2,n}(t)\partial_{x_n} + \lambda_{2,n+1}(t)\partial_{y_n}, \\
& \hat{X}_3 = \partial_{x_n} + \partial_{y_n}, \qquad \hat{X}_4 = t(\partial_{x_n} + \partial_{y_n}), \\
& F_n = \frac{\ddot{\lambda}_{1,n}}{\lambda_{1,n+1} - \lambda_{1,n}}\xi_n, \qquad G_n = G_n(\eta_{n-1}, t), \\
& K_n = \frac{\ddot{\lambda}_{1,n+1}}{\lambda_{1,n+1} - \lambda_{1,n}}\xi_n, \qquad P_n = P_n(\eta_n, t), \\
& \ddot{\lambda}_{1,n} \neq 0, \qquad \ddot{\lambda}_{2,n} \neq 0, \qquad \lambda_{1,n+1} \neq \lambda_{1,n}, \\
& \frac{\ddot{\lambda}_{2,n}}{\ddot{\lambda}_{1,n}} = \frac{\lambda_{2,n} - \lambda_{2,n-1}}{\lambda_{1,n} - \lambda_{1,n-1}} = \frac{\lambda_{2,n+1} - \lambda_{2,n}}{\lambda_{1,n+1} - \lambda_{1,n}}.
\end{aligned}
$$

The algebras $A_{4,1}$ and $A_{4,2}$ are actually related by the discrete symmetry

(1.4). Algebra $A_{4,1}$ is transformed into $A_{4,2}$ by the substitutions

$$\begin{gathered} F_n(\xi_n) \longrightarrow P_n(\eta_n), \qquad G_n(\eta_{n-1}) \longrightarrow K_n(\xi_n), \\ K_{n-1}(\xi_{n-1}) \longrightarrow G_n(\eta_{n-1}), \qquad P_{n-1}(\eta_{n-1}) \longrightarrow F_n(\xi_n), \\ \sigma_n(t)\,\partial_{x_n} \longrightarrow \sigma_{n+1}(t)\,\partial_{y_n}, \qquad \sigma_n(t)\,\partial_{y_n} \longrightarrow \sigma_n(t)\,\partial_{x_n}. \end{gathered} \tag{3.1}$$

The functions $\lambda_{1,n}(t)$ and $\lambda_{2,n}(t)$ in algebras $A_{4,1}$, $A_{4,2}$ satisfy the equations

$$\frac{\ddot{\lambda}_{2,n}}{\ddot{\lambda}_{1,n}} = \frac{\lambda_{2,n} - \lambda_{2,n-1}}{\lambda_{1,n} - \lambda_{1,n-1}} = \frac{\lambda_{2,n+1} - \lambda_{2,n}}{\lambda_{1,n+1} - \lambda_{1,n}}. \tag{3.2}$$

These equations can be solved and we obtain

$$\begin{gathered} \lambda_{1,n} = f(t)\lambda_{2,n} + g(t), \qquad \lambda_{2,n} = \frac{\gamma_n}{\dot{f}(t)^{1/2}} - \frac{1}{2\,\dot{f}(t)^{1/2}} \int_{t_0}^{t} \frac{\ddot{g}(s)}{\dot{f}(s)^{1/2}} \mathrm{d}s, \\ \dot{f}(t) \neq 0, \qquad \gamma_{n+1} \neq \gamma_n, \end{gathered} \tag{3.3}$$

where $f(t)$, $g(t)$ are arbitrary smooth functions of t and γ_n is an arbitrary function of n.

Notice that the quantities $\lambda_{1,n}(t)$ and $\lambda_{2,n}(t)$ (or $f(t)$, $g(t)$ and γ_n) figure explicitly in the interaction functions G_n and P_n of $A_{4,1}$, or respectively in F_n and K_n of $A_{4,2}$. The two algebras are thus indeed four-dimensional and completely specified.

4 Nilpotent non-Abelian symmetry algebras

Nilpotent Lie algebras exist for all dimensions $\dim L \geq 3$. For $\dim L = 3$ only one type exists, namely the Heisenberg algebra. It has a two-dimensional Abelian ideal. Maximality requires that this ideal be the algebra $A_{2,1}$ of eq.(2.16). The Heisenberg algebra is obtained by adding the operator $\hat{T} = \partial_t$. We then calculate the interaction allowing this symmetry algebra, and obtain

$$\begin{aligned} N_{3,1} \quad & \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \qquad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}), \qquad \hat{T} = \partial_t, \\ & F_n = f_n(\xi_n), \qquad G_n = g_n(\eta_{n-1}), \\ & K_n = k_n(\xi_n), \qquad P_n = p_n(\eta_n). \end{aligned}$$

We mention that this algebra is invariant under the substitution (3.1).

Every nilpotent non-Abelian Lie algebra contains the Heisenberg algebra as a subalgebra. We can hence proceed by adding further operators to $N_{3,1}$. Moreover, they can only be added to the Abelian ideal. The determining equations (2.4), (2.5) allow us to add at most two operators. Maximality requires that we add precisely two. We thus obtain two mutually isomorphic five-dimensional nilpotent Lie algebras with four-dimensional Abelian ideals, namely

$$
\begin{aligned}
N_{5,1} \quad & \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}), \quad \hat{T} = \partial_t, \\
& \hat{X}_3 = (\sigma_n + t^2)(\partial_{x_n} + \partial_{y_n}), \quad \hat{X}_4 = (\sigma_n t + \tfrac{t^3}{3})(\partial_{x_n} + \partial_{y_n}), \\
& F_n = f_n(\xi_n), \quad G_n = \frac{2}{\sigma_n - \sigma_{n-1}} \eta_{n-1}, \\
& K_n = k_n(\xi_n), \quad P_n = \frac{2}{\sigma_{n+1} - \sigma_n} \eta_n, \quad \sigma_{n+1} \neq \sigma_n,
\end{aligned}
$$

where σ_n is an arbitrary function of n.

The second algebra $N_{5,2}$ is obtained from $N_{5,1}$ by the substitution (3.1). We mention that the interactions allowing the symmetry algebra $N_{5,1}$ are special cases of those allowing the Abelian algebra $A_{4,1}$. Similarly for $N_{5,2}$ and $A_{4,2}$.

5 Solvable nonnilpotent symmetry algebras with non-Abelian nilradicals

A solvable Lie algebra L always has a uniquely defined maximal nilpotent ideal, the nilradical $NR(L)$ [15]. If a solvable symmetry algebra of the system (1.1) has a non-Abelian nilradical, it must be $N_{3,1}$, $N_{5,1}$ or $N_{5,2}$ of Section 4, or a four-dimensional subalgebra of $N_{5,1}$ or $N_{5,2}$.

The determining equations (2.4) and (2.5) do not allow any extension of the four and five-dimensional nilpotent symmetry algebras to solvable ones.

The Heisenberg algebra $N_{3,1}$, on the other hand, leads to three different four-dimensional solvable symmetry algebras. The Lie algebras are given by four basis elements, $\hat{X}_1, \hat{X}_2$ and $\hat{T}$ of $N_{3,1}$ and an additional operator $\hat{Y}$. Below we list these elements $\hat{Y}$ together with the invariant interactions that allow the corresponding symmetry groups. In each case we present a matrix A defining the action of $\hat{Y}$ on the nilradical $N_{3,1}$ (i.e. $[\hat{X}_i, \hat{Y}] = A_{i,1}\hat{X}_1 + A_{i,2}\hat{T} + A_{i,3}\hat{X}_2$).

$$SN_{4,1} \quad \hat{Y} = t\partial_t + (a+\tfrac{1}{2})(x_n\partial_{x_n} + y_n\partial_{y_n}),$$
$$F_n = (\xi_n)^{\frac{2a-3}{2a+1}} f_n, \qquad G_n = (\eta_{n-1})^{\frac{2a-3}{2a+1}} g_n,$$
$$K_n = (\xi_n)^{\frac{2a-3}{2a+1}} k_n, \qquad P_n = (\eta_n)^{\frac{2a-3}{2a+1}} p_n,$$
$$A = \mathrm{diag}\,(a+\tfrac{1}{2}\,,\,1\,,\,a-\tfrac{1}{2}), \qquad a \neq -\tfrac{1}{2}\,,\,\tfrac{3}{2}.$$

$$SN_{4,2} \quad \hat{Y} = t\partial_t + (2x_n + t^2)\partial_{x_n} + (2y_n + t^2)\partial_{y_n},$$
$$F_n = f_n + \tfrac{1}{2}\ln(\xi_n), \qquad G_n = \tfrac{1}{2}\ln(\eta_{n-1}),$$
$$K_n = k_n + \tfrac{1}{2}\ln(\xi_n), \qquad P_n = \tfrac{1}{2}\ln(\eta_n),$$
$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$SN_{4,3} \quad \hat{Y} = t\partial_t + \sigma_{1,n}\partial_{x_n} + \sigma_{2,n}\partial_{y_n},$$
$$F_n = f_n \exp\left(\frac{2\xi_n}{\sigma_{1,n} - \sigma_{2,n}}\right), \qquad G_n = g_n \exp\left(\frac{-2\eta_{n-1}}{\sigma_{1,n} - \sigma_{2,n-1}}\right),$$
$$K_n = k_n \exp\left(\frac{2\xi_n}{\sigma_{1,n} - \sigma_{2,n}}\right), \qquad P_n = p_n \exp\left(\frac{-2\eta_n}{\sigma_{1,n+1} - \sigma_{2,n}}\right),$$
$$A = \mathrm{diag}\,(0\,,\,1\,,\,-1), \qquad \sigma_{1,n} \neq \sigma_{2,n}, \qquad \sigma_{1,n+1} \neq \sigma_{2,n}.$$

The quantities $f_n, g_n, p_n, k_n, \sigma_{1,n}$ and $\sigma_{2,n}$ depend on n alone.

The transformation (3.1) does not lead to any new algebras or interactions. In the case of the algebra $SN_{4,3}$ we may have $\sigma_{2,n+1} = \sigma_{2,n}$. Then σ_2 can be transformed into $\sigma_{2,n} = \sigma = 0$. Similarly, for $\sigma_{2,n+1} \neq \sigma_{2,n}$, but $\sigma_{1,n+1} = \sigma_{1,n} \equiv \sigma$, we can transform σ_1 into $\sigma_1 = \sigma = 0$.

6 Solvable nonnilpotent symmetry algebras with Abelian nilradicals

A large number of symmetry algebras of the system (1.1) is of this type. To identify and classify them, we use several known results on the structure of solvable Lie algebras [15].

1. The nilradical $NR(L)$ is unique and its dimension satisfies
$$\dim\ NR(L) \geq \frac{1}{2}\ \dim\ L. \tag{6.1}$$

2. Any solvable Lie algebra L can be written as the algebraic sum of the nilradical $NR(L)$ and a complementary linear space F, i.e. $L = F\dot{+}NR(L)$.

3. The derived algebra is contained in its nilradical: $[L, L] \subseteq NR(L)$.

4. For an Abelian nilradical $\{\hat{X}_1, \ldots, \hat{X}_n\}$, the commutation relations can be written as
$$[\hat{X}_i, \hat{Y}_k] = (A_k)_{ij}\hat{X}_j, \qquad [A_k, A_\ell] = 0, \qquad [\hat{Y}_i, \hat{Y}_k] = c^\ell_{ik}\hat{X}_\ell, \qquad [\hat{X}_i, \hat{X}_k] = 0, \tag{6.2}$$
where the elements $\hat{Y}_k$ are the nonnilpotent elements (outside the nilradical). The matrices A_k commute and are linearly nilindependent (i.e. no nontrivial linear combination of them is a nilpotent matrix). If only one element $\hat{Y}$ outside the nilradical exists, the nonnilpotent matrix A can be taken in Jordan canonical form.

In our case we can add that the Abelian nilradical must be one of the algebras found in Section 3. In principle, the nilradical could be a three-dimensional subalgebra of $A_{4,1}$, or $A_{4,2}$, containing $A_{2,1}$ as a subalgebra. However, it turns out that all choices of this type lead to symmetry algebras that are not maximal for the interactions that they allow.

The following solvable symmetry algebras occur.

6.1 dim $NR(L) = 2$

The only two-dimensional nilradical that leads to solvable Lie algebras that are maximal for the obtained interaction is $A_{2,1}$. The solvable Lie algebras are always three dimensional. A basis for them consists of $\hat{X}_1$ and $\hat{X}_2$ of eq.(2.16) and an additional element $\hat{Y}$. In each case we give the element $\hat{Y}$ and the matrix A representing the action of $\hat{Y}$ on the nilradical.

$$SA_{3,1} \quad \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}), \quad \hat{Y} = \partial_t + x_n\partial_{x_n} + y_n\partial_{y_n},$$

$$F_n = \xi_n f_n(\omega_n), \quad G_n = \eta_{n-1} g_n(\zeta_{n-1}),$$

$$K_n = \xi_n k_n(\omega_n), \quad P_n = \eta_n p_n(\zeta_n),$$

$$\omega_n = \xi_n \mathrm{e}^{-t}, \quad \zeta_n = \eta_n \mathrm{e}^{-t}, \quad A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

$$SA_{3,2} \quad \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}), \quad \hat{Y} = t\partial_t + (a + \tfrac{1}{2})(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = t^{-2}\xi_n f_n(\omega_n), \quad G_n = t^{-2}\eta_{n-1} g_n(\zeta_{n-1}),$$

$$K_n = t^{-2}\xi_n k_n(\omega_n), \quad P_n = t^{-2}\eta_n p_n(\zeta_n),$$

$$\omega_n = \xi_n t^{-(a+\frac{1}{2})}, \quad \zeta_n = \eta_n t^{-(a+\frac{1}{2})}, \quad A = \mathrm{diag}\,(a - \tfrac{1}{2}, \, a + \tfrac{1}{2}).$$

$$SA_{3,3} \quad \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}), \quad \hat{Y} = (t^2+1)\partial_t + (a+t)(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = (t^2+1)^{-2}\xi_n f_n(\omega_n), \quad G_n = (t^2+1)^{-2}\eta_{n-1} g_n(\zeta_{n-1}),$$

$$K_n = (t^2+1)^{-2}\xi_n k_n(\omega_n), \quad P_n = (t^2+1)^{-2}\eta_n p_n(\zeta_n),$$

$$\omega_n = \xi_n (t^2+1)^{-1/2} \exp[-a \arctan(t)], \quad \zeta_n = \eta_n (t^2+1)^{-1/2} \exp[-a \arctan(t)]$$

$$A = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}.$$

These three algebras are nonisomorphic (since the corresponding matrices A are not mutually conjugate). Each of these three cases is self conjugate under the substitution (3.1).

6.2 dim $NR(L) = 4$

The nilradical could be three-dimensional, however the obtained solvable Lie algebra is never maximal. We only need to deal with four-dimensional Abelian ideals of the form $A_{4,1}$ and $A_{4,2}$. An extension to a solvable Lie algebra is only possible for special cases of the functions $\lambda_{1,n}(t)$ and $\lambda_{2,n}(t)$ figuring in the vector fields and interactions. Below we list all inequivalent extensions of $A_{4,1}$. There are precisely nine of them. The corresponding extensions of $A_{4,2}$ are obtained by the substitution (3.1). The action of $\hat{Y}$ on $\{\hat{X}_1, \ldots, \hat{X}_4\}$ is represented by the matrix A.

$SA_{5,1}$ $\hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),$

$$\hat{X}_3 = \sigma_n \mathrm{e}^t(\partial_{x_n} + \partial_{y_n}), \quad \hat{X}_4 = \sigma_n \mathrm{e}^{-t}(\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = \partial_t + a(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = \xi_n f_n(\omega_n), \quad G_n = \frac{\sigma_n}{\sigma_n - \sigma_{n-1}}\eta_{n-1},$$

$$K_n = \xi_n k_n(\omega_n), \quad P_n = \frac{\sigma_n}{\sigma_{n+1} - \sigma_n}\eta_n,$$

$$\omega_n = \xi_n \mathrm{e}^{-at}, \quad \sigma_{n+1} \neq \sigma_n, \quad A = \begin{pmatrix} a-1 & 0 & 0 & 0 \\ 0 & a+1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & -1 & a \end{pmatrix}.$$

$SA_{5,2}$ $\hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),$

$$\hat{X}_3 = \sigma_n \cos(t)(\partial_{x_n} + \partial_{y_n}), \quad \hat{X}_4 = \sigma_n \sin(t)(\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = \partial_t + a(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = \xi_n f_n(\omega_n), \quad G_n = \frac{-\sigma_n}{\sigma_n - \sigma_{n-1}}\eta_{n-1},$$

$$K_n = \xi_n k_n(\omega_n), \quad P_n = \frac{-\sigma_n}{\sigma_{n+1} - \sigma_n}\eta_n,$$

$$\omega_n = \xi_n \mathrm{e}^{-at}, \quad \sigma_{n+1} \neq \sigma_n, \quad A = \begin{pmatrix} a & 1 & 0 & 0 \\ -1 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & -1 & a \end{pmatrix}.$$

$SA_{5,3}$ $\hat{X}_1 = \partial_{x_n} + \partial_{y_n}$, $\hat{X}_2 = t(\partial_{x_n} + \partial_{y_n})$,

$$\hat{X}_3 = (\sigma_n + t^2)(\partial_{x_n} + \partial_{y_n}), \quad \hat{X}_4 = (\sigma_n t + \tfrac{t^3}{3})(\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = \partial_t + a(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = \xi_n f_n(\omega_n), \quad G_n = \frac{2\eta_{n-1}}{\sigma_n - \sigma_{n-1}},$$

$$K_n = \xi_n k_n(\omega_n), \quad P_n = \frac{2\eta_n}{\sigma_{n+1} - \sigma_n},$$

$$\omega_n = \xi_n e^{-at}, \quad \sigma_{n+1} \neq \sigma_n, \quad A = \begin{pmatrix} a & 0 & 0 & -2 \\ -1 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & -1 & a \end{pmatrix}.$$

$SA_{5,4}$ $\hat{X}_1 = \partial_{x_n} + \partial_{y_n}$, $\hat{X}_2 = t(\partial_{x_n} + \partial_{y_n})$,

$$\hat{X}_3 = \sigma_n t^{\alpha}(\partial_{x_n} + \partial_{y_n}), \quad \hat{X}_4 = \sigma_n t^{1-\alpha}(\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = t\partial_t + (a + \tfrac{1}{2})(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = t^{-2}\xi_n f_n(\omega_n), \quad G_n = \alpha(\alpha - 1)t^{-2}\frac{\sigma_n}{\sigma_n - \sigma_{n-1}}\eta_{n-1},$$

$$K_n = t^{-2}\xi_n k_n(\omega_n), \quad P_n = \alpha(\alpha - 1)t^{-2}\frac{\sigma_n}{\sigma_{n+1} - \sigma_n}\eta_n,$$

$$\omega_n = \xi_n t^{-(a+\frac{1}{2})}, \quad \sigma_{n+1} \neq \sigma_n, \quad \alpha \neq 0, 1\ ,$$
$$A = \mathrm{diag}\,(a - \alpha + \tfrac{1}{2}\,,\ a + \alpha - \tfrac{1}{2}\,,\ a + \tfrac{1}{2}\,,\ a - \tfrac{1}{2}).$$

$SA_{5,5}$ $\hat{X}_1 = \partial_{x_n} + \partial_{y_n}$, $\hat{X}_2 = t(\partial_{x_n} + \partial_{y_n})$,

$$\hat{X}_3 = \sigma_n t^{1/2}\ln(t)(\partial_{x_n} + \partial_{y_n}), \quad \hat{X}_4 = \sigma_n t^{1/2}(\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = t\partial_t + (a + \tfrac{1}{2})(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = t^{-2}\xi_n f_n(\omega_n), \quad G_n = -\tfrac{1}{4}t^{-2}\frac{\sigma_n}{\sigma_n - \sigma_{n-1}}\eta_{n-1},$$

$$K_n = t^{-2}\xi_n k_n(\omega_n), \quad P_n = -\tfrac{1}{4}t^{-2}\frac{\sigma_n}{\sigma_{n+1} - \sigma_n}\eta_n,$$

$$\omega_n = \xi_n t^{-(a+\frac{1}{2})}, \quad \sigma_{n+1} \neq \sigma_n, \quad A = \begin{pmatrix} a & -1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a + \frac{1}{2} & 0 \\ 0 & 0 & 0 & a - \frac{1}{2} \end{pmatrix}.$$

$SA_{5,6}$ $\hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),$

$$\hat{X}_3 = \sigma_n t^{1/2} \cos[\ln(t)](\partial_{x_n} + \partial_{y_n}), \quad \hat{X}_4 = \sigma_n t^{1/2} \sin[\ln(t)](\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = t\partial_t + (a + \tfrac{1}{2})(x_n \partial_{x_n} + y_n \partial_{y_n}),$$

$$F_n = t^{-2}\xi_n f_n(\omega_n), \quad G_n = -\tfrac{5}{4}t^{-2}\frac{\sigma_n}{\sigma_n - \sigma_{n-1}}\eta_{n-1},$$

$$K_n = t^{-2}\xi_n k_n(\omega_n), \quad P_n = -\tfrac{5}{4}t^{-2}\frac{\sigma_n}{\sigma_{n+1} - \sigma_n}\eta_n,$$

$$\omega_n = \xi_n t^{-(a+\frac{1}{2})}, \quad \sigma_{n+1} \neq \sigma_n, \quad A = \begin{pmatrix} a & 1 & 0 & 0 \\ -1 & a & 0 & 0 \\ 0 & 0 & a+\frac{1}{2} & 0 \\ 0 & 0 & 0 & a-\frac{1}{2} \end{pmatrix}.$$

$SA_{5,7}$ $\hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),$

$$\hat{X}_3 = [\sigma_n - \ln(t)](\partial_{x_n} + \partial_{y_n}), \quad \hat{X}_4 = t[\sigma_n + \ln(t)](\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = t\partial_t + (a + \tfrac{1}{2})(x_n \partial_{x_n} + y_n \partial_{y_n}),$$

$$F_n = t^{-2}\xi_n f_n(\omega_n), \quad G_n = -t^{-2}\frac{\eta_{n-1}}{\sigma_n - \sigma_{n-1}},$$

$$K_n = t^{-2}\xi_n k_n(\omega_n), \quad P_n = -t^{-2}\frac{\eta_n}{\sigma_{n+1} - \sigma_n},$$

$$\omega_n = \xi_n t^{-(a+\frac{1}{2})}, \quad \sigma_{n+1} \neq \sigma_n, \quad A = \begin{pmatrix} a+\frac{1}{2} & 0 & 1 & 0 \\ 0 & a-\frac{1}{2} & 0 & -1 \\ 0 & 0 & a+\frac{1}{2} & 0 \\ 0 & 0 & 0 & a-\frac{1}{2} \end{pmatrix}.$$

$SA_{5,8}$ $\hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),$

$$\hat{X}_3 = \lambda_{1,n}(t)(\partial_{x_n} + \partial_{y_n}), \quad \lambda_{1,n} = \sigma_n(t^2+1)^{1/2}\exp[\alpha \arctan(t)],$$

$$\hat{X}_4 = \lambda_{2,n}(t)(\partial_{x_n} + \partial_{y_n}), \quad \lambda_{2,n} = \sigma_n(t^2+1)^{1/2}\exp[-\alpha \arctan(t)],$$

$$\hat{Y} = (t^2+1)\partial_t + (a+t)(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = (t^2+1)^{-2}\xi_n f_n(\omega_n), \quad G_n = (\alpha^2+1)(t^2+1)^{-2}\frac{\sigma_n}{\sigma_n - \sigma_{n-1}}\eta_{n-1},$$

$$K_n = (t^2+1)^{-2}\xi_n k_n(\omega_n), \quad P_n = (\alpha^2+1)(t^2+1)^{-2}\frac{\sigma_n}{\sigma_{n+1} - \sigma_n}\eta_n,$$

$$\omega_n = \xi_n(t^2+1)^{-1/2}\exp[-a\arctan(t)], \quad \sigma_{n+1} \neq \sigma_n, \quad \alpha \neq 0,$$

$$A = \begin{pmatrix} a-\alpha & 0 & 0 & 0 \\ 0 & a+\alpha & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & -1 & a \end{pmatrix}.$$

$SA_{5,9}$ $\hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),$

$$\hat{X}_3 = \sigma_n(t^2+1)^{1/2}(\partial_{x_n} + \partial_{y_n}), \quad \hat{X}_4 = \sigma_n(t^2+1)^{1/2}\arctan(t)(\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = (t^2+1)\partial_t + (a+t)(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = (t^2+1)^{-2}\xi_n f_n(\omega_n), \quad G_n = (t^2+1)^{-2}\frac{\sigma_n}{\sigma_n - \sigma_{n-1}}\eta_{n-1},$$

$$K_n = (t^2+1)^{-2}\xi_n k_n(\omega_n), \quad P_n = (t^2+1)^{-2}\frac{\sigma_n}{\sigma_{n+1} - \sigma_n}\eta_n,$$

$$\omega_n = \xi_n(t^2+1)^{-1/2}\exp[-a\arctan(t)], \quad \sigma_{n+1} \neq \sigma_n, \quad \alpha \neq 0,$$

$$A = \begin{pmatrix} a & 0 & 0 & 0 \\ -1 & a & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & -1 & a \end{pmatrix}.$$

In all cases the interaction terms G_n and P_n are specified, whereas F_n and K_n each involve an arbitrary function of one variable ω_n. The time dependence of the variable ω_n and the functions F_n and K_n depends on the form of the generator $\hat{Y}$.

After the substitution (3.1) we have altogether 18 five-dimensional Lie algebras. No further symmetry generators can be added, at least in the nonlinear case studied so far.

7 Nonsolvable symmetry algebras

Any finite dimensional Lie algebra L that is not solvable is either semisimple, or has a nontrivial and unique Levi decomposition

$$L = S \rhd R, \tag{7.1}$$

where S is semisimple and R is the radical, i.e. the maximal solvable ideal. The only semisimple Lie algebra that can be realized in terms of the vector fields (2.2) is actually simple, namely $sl(2,\mathbb{R})$. Up to allowed transformations the realization is unique (and given below by the operators $\hat{Y}_1$, $\hat{Y}_2$ and $\hat{Y}_3$). The determining equations (2.4) and (2.5) can be used to obtain the interaction invariant under the corresponding group $SL(2,\mathbb{R})$. Equations (1.1) will then be invariant under a five-dimensional group that contains the subalgebra $A_{2,1}$. We have:

$$\begin{aligned}
NS_{5,1}\quad & \hat{Y}_1 = \partial_t, \quad \hat{Y}_2 = t\partial_t + \tfrac{1}{2}(x_n\partial_{x_n} + y_n\partial_{y_n}), \quad \hat{Y}_3 = t^2\partial_t + t(x_n\partial_{x_n} + y_n\partial_{y_n}),\\
& \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),\\
& F_n = \xi_n^{-3} f_n, \quad G_n = \eta_{n-1}^{-3} g_n,\\
& K_n = \xi_n^{-3} k_n, \quad P_n = \eta_n^{-3} p_n.
\end{aligned}$$

The Lie algebra $NS_{5,1}$ is isomorphic to the special affine algebra $saff(2,\mathbb{R})$. This is the only maximal nonsolvable symmetry algebra that occurs.

This completes our analysis of possible symmetries of the system (1.1) with nonlinear interactions.

8 Symmetries of linear interactions

In Sections 3–7 we have excluded the case of linear interactions. Let us turn to this case now. We specify equations (1.1) to be

$$\begin{aligned}
\ddot{x}_n &= A_n(t)\,\xi_n + B_n(t)\,\eta_{n-1} + U_n(t),\\
\ddot{y}_n &= C_n(t)\,\xi_n + D_n(t)\,\eta_n + V_n(t)\,.
\end{aligned} \tag{8.1}$$

The system is still strongly coupled, i.e. the functions A_n, B_n, C_n, D_n are all nonzero. The determining equations reduce to

$$\ddot{\lambda}_n - (\mu_n - \lambda_n)A_n - (\lambda_n - \mu_{n-1})B_n + (a - \frac{3}{2}\dot{\tau})U_n - \tau\dot{U}_n = 0, \tag{8.2}$$

$$\ddot{\mu}_n - (\mu_n - \lambda_n)C_n - (\lambda_{n+1} - \mu_n)D_n + (a - \frac{3}{2}\dot{\tau})V_n - \tau\dot{V}_n = 0, \tag{8.3}$$

$$2\dot{\tau}A_n + \tau\dot{A}_n = 0, \tag{8.4}$$

$$2\dot{\tau}B_n + \tau\dot{B}_n = 0, \tag{8.5}$$

$$2\dot{\tau}C_n + \tau\dot{C}_n = 0, \tag{8.6}$$

$$2\dot{\tau}D_n + \tau\dot{D}_n = 0, \tag{8.7}$$

$$\tau = \tau_0 + \tau_1 + \tau_2 t^2, \tag{8.8}$$

since the coefficients of ξ_n, η_n, η_{n-1} and 1 vanish separately.

For $A_n(t), \ldots, D_n(t)$ generic, we obtain $\tau = 0$ and then only equations (8.2) and (8.3) (with $\tau = 0$) survive. These equations can be solved in the generic case and we obtain two types of symmetries, both just a consequence of linearity.

1. We take $a = 0$ and denote $(\lambda_{h,n}, \mu_{h,n})$ the general solution of the homogeneous equations, i.e. eq.(8.1) with $U_n = V_n = 0$. The vector field is

$$\hat{X}_h = \lambda_{h,n}(t)\,\partial_{x_n} + \mu_{h,n}(t)\,\partial_{y_n}. \tag{8.9}$$

2. For $a \neq 0$ we choose $a = -1$ and denote some chosen particular solution of the inhomogeneous system (8.1) $(\lambda_{p,n}, \mu_{p,n})$. The vector field is

$$\hat{X}_p = [x_n - \lambda_{p,n}(t)]\partial_{x_n} + [y_n - \mu_{p,n}(t)]\partial_{y_n}\,. \tag{8.10}$$

In particular, if we have $U_n = V_n = 0$, then we take $\lambda_{p,n} = \mu_{p,n} = 0$ in eq.(8.10).

The symmetry (8.9) only means that we can add any solution of the homogeneous equations to a solution of eq.(8.1). The symmetry (8.10) corresponds to the fact that a solution of the homogeneous system can be multiplied by a constant.

Let us now assume that a further symmetry generator exists. It is of the form (2.2) with $\tau(t)$ as in eq.(8.8). Allowed transformations can be used to transform τ into one of four cases. Let us consider them separately.

a) $\tau = 0$

No symmetries beyond the generic ones are obtained.

b) $\tau = 1$

Using allowed transformations we simplify the additional vector field into

$$\hat{T} = \partial_t + a(x_n\,\partial_{x_n} + y_n\,\partial_{y_n}). \tag{8.11}$$

The determining equations restrict the time dependence of the coefficients in eq.(8.1) and the system reduces to

$$\begin{aligned} \ddot{x}_n &= f_n\,\xi_n + g_n\,\eta_{n-1} + u_n \mathrm{e}^{at}, \\ \ddot{y}_n &= k_n\,\xi_n + p_n\,\eta_n + v_n \mathrm{e}^{at}. \end{aligned} \tag{8.12}$$

c) $\tau = t$

The additional vector field and invariant equations are reduced to

$$\hat{D} = t\,\partial_t + (a + \frac{1}{2})(x_n\,\partial_{x_n} + y_n\,\partial_{y_n}), \tag{8.13}$$

$$\begin{aligned} \ddot{x}_n &= \frac{f_n}{t^2}\xi_n + \frac{g_n}{t^2}\,\eta_{n-1} + u_n t^{a-\frac{3}{2}}, \\ \ddot{y}_n &= \frac{k_n}{t^2}\,\xi_n + \frac{p_n}{t^2}\,\eta_n + v_n t^{a-\frac{3}{2}}. \end{aligned} \tag{8.14}$$

d) $\tau = t^2 + 1$

The additional vector field and invariant equations are

$$\hat{C} = (t^2+1)\partial_t + (a+t)(x_n\partial_{x_n} + y_n\partial_{y_n}), \tag{8.15}$$

$$\begin{aligned} \ddot{x}_n &= \frac{f_n}{(t^2+1)^2}\xi_n + \frac{g_n}{(t^2+1)^2}\eta_{n-1} + \frac{u_n}{(t^2+1)^{3/2}}\exp[a\arctan(t)], \\ \ddot{y}_n &= \frac{k_n}{(t^2+1)^2}\xi_n + \frac{p_n}{(t^2+1)^2}\eta_n + \frac{v_n}{(t^2+1)^{3/2}}\exp[a\arctan(t)]. \end{aligned} \tag{8.16}$$

In all cases f_n, g_n, k_n, p_n, u_n and v_n are independent of t. No further symmetries exist for any of the interactions (8.12), (8.14) or (8.16).

9 Conclusions

Let us sum up the results obtained above.

For nonlinear interactions the symmetry algebra is at most five-dimensional. The following cases occur.

1. The nonsolvable algebra $NS_{5,1}$ of Section 7. The dependence of the right hand side of eq.(1.1) on ξ_n and η_n is completely specified by an inverse cube relation. The dependence on the discrete variable n remains arbitrary. The interactions are time independent.

2. The solvable Lie algebras with Abelian nilradicals $SA_{5,1}, \ldots, SA_{5,9}$ (and $SA_{5,10}, \ldots, SA_{5,18}$ by the substitution (3.1)) of Section 6.2. The interactions are all "semilinear". By this we mean that the dependence on one variable η_n is specified to be linear, whereas the dependence on ξ_n remains arbitrary (and vice versa for $SA_{5,10}, \ldots, SA_{5,18}$). The time dependence of the nonlinear terms in the interaction depends crucially on the form of the nonnilpotent element $\hat{Y}$.

   Any attempt to enlarge the symmetry algebra by further elements leads to linear interactions.

3. The nilpotent five-dimensional Lie algebras $N_{5,1}$ and the related algebra $N_{5,2}$ of Section 4. For $N_{5,1}$ the interaction is again semilinear with

G_n and P_n linear in η_{n-1} and η_n, respectively, and F_n and K_n arbitrary functions of ξ_n (and vice versa for $N_{5,2}$). The interaction is time independent.

4. Four-dimensional maximal symmetry algebras are either Abelian, or solvable with the Heisenberg algebra as a nilradical. For $A_{4,1}$ and $A_{4,2}$ the interaction is semilinear with an arbitrary time dependence in the nonlinear terms. For $SN_{4,1}$, $SN_{4,2}$ and $SN_{4,3}$ the dependence on ξ_n and η_n is completely specified as being monomial, logarithmic or exponential, respectively. There is no time dependence.

5. A three-dimensional maximal symmetry algebra is either nilpotent, or solvable with an Abelian nilradical. For $N_{3,1}$, the Heisenberg algebra, the interaction is time independent, otherwise arbitrary. The model, studied by Campa *et al* [1], namely

$$\begin{aligned} F_n(\xi_n) &= \tfrac{1}{M_1}(k_1\xi_n + \varepsilon\beta_1\xi_n^2) \quad , \quad K_n(\xi_n) = -\tfrac{M_1}{M_2}F_n(\xi_n), \\ G_n(\eta_{n-1}) &= -\tfrac{1}{M_1}(k_2\eta_{n-1} + \varepsilon\beta_2\eta_{n-1}^2) \quad , \quad P_n(\eta_n) = -\tfrac{M_1}{M_2}G_{n+1}(\eta_{n+1}), \end{aligned}$$

is of this type. For $SA_{3,1}$, $SA_{3,2}$ and $SA_{3,3}$ the interactions involve four arbitrary functions of one variable. The interaction is entirely specified by the element $\hat{Y}$.

6. As mentioned above, the general interaction in eq.(1.1) is invariant under the group of global translations and Galilei transformations, corresponding to the algebra $A_{2,1}$ of eq.(2.16).

The symmetries found in this article can be used to perform symmetry reduction on one hand, and to obtain new solutions from known ones, on the other.

Let us look at the example of algebra $NS_{5,1}$. The system (1.1) in this case reduces to

$$\ddot{x}_n = \frac{f_n}{\xi_n^3} + \frac{g_n}{\eta_{n-1}^3}, \qquad \ddot{y}_n = \frac{k_n}{\xi_n^3} + \frac{p_n}{\eta_n^3}. \tag{9.1}$$

The algebra $sl(2,\mathbb{R})$ has three inequivalent one-dimensional subalgebras, namely $\hat{Y}_1$, $\hat{Y}_2$ and $\hat{Y}_3 + \hat{Y}_1$. Each of them can be used to reduce the system (9.1) to a system of two difference equations. Let us look at the three individual cases separately.

Subalgebra $\hat{Y}_1$

This algebra leads to stationary solutions. We have

$$x_n = x_{n,0}, \qquad y_n = y_{n,0} \tag{9.2}$$

and hence

$$\xi_{n,0} = \left(-\frac{f_n}{g_n}\right)^{1/3} \eta_{n-1,0} = \left(-\frac{k_n}{p_n}\right)^{1/3} \eta_{n,0}. \tag{9.3}$$

Subalgebra $\hat{Y}_2$

The reduction formulas in this case are

$$x_n = x_{n,0}\sqrt{t}, \qquad y_n = y_{n,0}\sqrt{t} \tag{9.4}$$

and the recursion relations are

$$-\frac{x_{n,0}}{4} = \frac{f_n}{\xi^3_{n,0}} + \frac{g_n}{\eta^3_{n-1,0}}, \qquad -\frac{y_{n,0}}{4} = \frac{k_n}{\xi^3_{n,0}} + \frac{p_n}{\eta^3_{n,0}}. \tag{9.5}$$

Subalgebra $\hat{Y}_3 + \hat{Y}_1$

We put

$$x_n = x_{n,0}\sqrt{t^2+1}, \qquad y_n = y_{n,0}\sqrt{t^2+1} \tag{9.6}$$

and obtain the recursion relations

$$x_{n,0} = \frac{f_n}{\xi^3_{n,0}} + \frac{g_n}{\eta^3_{n-1,0}}, \qquad y_{n,0} = \frac{k_n}{\xi^3_{n,0}} + \frac{p_n}{\eta^3_{n,0}}. \tag{9.7}$$

In all three cases we can express ξ_n in terms of η_n and obtain a two term recursion relation for η_n. These can be solved, but we will not go into the details here.

Acknowledgments

We thank professor D. Levi for interesting discussions. The research of S.L. was successively supported by a FCAR Doctoral Scholarship and by an NSERC Postdoctoral Fellowship. The research of P.W. was partly supported by NSERC of Canada and FCAR du Québec. The work reported here was started while all three authors were visiting the CIF in Cuernavaca, Mexico. We thank Professor T. Seligman for his hospitality there. The final version was written while P.W. was visiting the Faculty of Nuclear Sciences and Physical Engineering of the Czech Technical University in Prague. He thanks Professors M. Havlicek and J. Tolar for their hospitality.

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Chapitre 5

LA CROISSANCE DE DEGRÉ POUR LES ÉQUATIONS DISCRÈTES INTÉGRABLES BIDIMENSIONNELLES

Integrable Lattice Equations and their Growth Properties

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Abstract

In this paper we investigate the integrability of two-dimensional partial difference equations using the newly developed techniques of study of the degree of the iterates. We show that while for generic, nonintegrable equations, the degree grows exponentially fast, for integrable lattice equations the degree growth is polynomial. The growth criterion is used in order to obtain the integrable deautonomisations of the equations examined. In the case of linearisable lattice equations we show that the degree growth is slower than in the case of equations integrable through Inverse Scattering Transform techniques.

The study of integrability of nonlinear evolution equations has spurred the development of efficient tools for its detection. The ARS [1] conjecture was formulated originally for partial differential equations and related integrability to the Painlevé property. In the discrete domain the singularity confinement [2] property was discovered while studying the lattice KdV equation and the singularities that can appear spontaneously during the evolution. The singularity confinement has been a most useful discrete integrability criterion in the sense that it is a necessary condition for lattice equations to be integrable by Inverse Scattering Transform (IST) methods. However, it has turned out [3] that singularity confinement is not sufficient for integrability and thus its use as an integrability detector must be subject to particular caution.

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Another property of integrable discrete systems, namely the growth of the degree of the iterates [4], has, in the long run, proven to be a reliable integrability detector. The main idea goes back to Arnold [5] and Veselov [6]. As Veselov summarized it: "integrability has an essential correlation with the weak growth of certain characteristics". The characteristic quantity which can be easily obtained and computed for a rational mapping is the degree of the numerators or denominators of (the irreducible forms of) the iterates of some initial condition. (In order to obtain the degree one must introduce homogeneous coordinates and compute the homogeneity degree). Those ideas were refined by Viallet and collaborators [7, 8], leading to the introduction of the notion of algebraic entropy. The latter is defined as $E = \lim_{n\to\infty} \log(d_n)/n$ where d_n is the degree of the n-th iterate. A generic, nonintegrable, mapping leads to exponential growth of the degrees of the iterates and thus has a nonzero algebraic entropy, while an integrable mapping has zero algebraic entropy. As we have shown in a previous work [9], this is too crude an estimate. The degree growth contains information that can be an indication as to the precise integration method to be used and thus should be studied in detail. (At this point, we must stress that, as was already pointed out in [8], the degrees of the iterates are *not* invariant under transformation of the variables. However the degree *growth* is invariant and characterises the system at hand).

In previous works of ours we have applied the techniques of degree growth to the study of one-dimensional mappings [9, 10]. A first important conclusion of these studies was the confirmation of the singularity confinement results [11] on the derivation of discrete Painlevé equations. We have shown that, when singularity confinement is used for the deautonomisation of an integrable autonomous mapping, the condition obtained is identical to the one found by requiring nonexponential growth of the degrees of the iterates. (The terms "degrees of the iterates" in the above sentence and in the rest of the paper must be understood as the common homogeneity degree of the numerators and denominators of their irreducible forms, obtained through the introduction of the homogeneous coordinates). This not only confirms the results previously obtained for discrete Painlevé equations, but also suggests a dual strategy for the study of discrete integrability based on the combined use of singularity confinement and study of degree growth. The second result [9] was that mappings which are linearisable are associated to a degree growth slower than the ones integrable through IST techniques. Thus, the detailed study of the degree is not only an indication of integrability but also of the integration method.

In this paper, we apply the techniques of degree growth to two-dimensional partial difference equations. We shall show that the main conclusions from the study of one-dimensional mappings carry over to the two-dimensional case in a rather straightforward way.

Let us start with the examination of the equation that serves as a paradigm in all integrability studies, namely KdV, the discrete form of which is [12, 13]:

$$X_{n+1}^{m+1} = X_n^m + \frac{1}{X_{n+1}^m} - \frac{1}{X_n^{m+1}}. \tag{1}$$

(Incidentally, this is precisely the equation we have studied in [2], while investigating the singularity confinement property.) The study of the degree growth of the iterates in the case of a 2-dimensional lattice is substantially more difficult than that of the 1-dimensional case. It is thus very important to make the right choices from the outset. Here are the initial conditions we choose: on the line $m = 0$ we take X_n^0 of the form $X_n^0 = p_n/q$ while on the line $n = 0$ we choose $X_0^m = r_m/q$ (with $r_0 = p_0$). We assign to q and the p's, r's the same degree of homogeneity. Then we compute the iterates of X using (1) and calculate the degree of homogeneity in p, q, r at the various points of the lattice. Here is what we find:

$$\begin{array}{ccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
 & 1 & 7 & 19 & 31 & 41 & 51 & \cdots \\
 & 1 & 5 & 13 & 19 & 25 & 31 & \cdots \\
 & 1 & 3 & 5 & 7 & 9 & 11 & \cdots \\
\uparrow m & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & \to n & & & & & & &
\end{array}$$

At this point we must indicate how the analytical expression for the degree can be obtained. First we compute several points on the lattice which allow us to have a good guess at how the degree behaves. In the particular case of a 2-dimensional discrete equation relating four points on an elementary square like (1), and with the present choice of initial conditions (and given our experience on 1-dimensional mappings) we can reasonably surmise that the dominant behaviour of the degree will be of the form $d_n^m \propto mn$. Moreover the subdominant terms must be symmetric in m, n and at most linear. With those indications it is possible to "guess" the expression $d_n^m = 4mn - 2\max(m,n) + 1$ (for $mn \not\equiv 0$) and subsequently calculate some more points in order to check its validity. This procedure will be used throughout this paper.

So the lattice KdV equation leads, quite expectedly, to a polynomial growth in the degrees of the iterates. Let us now turn to the more interesting question of deautonomisation. The form (1) of KdV is not very convenient and thus we shall study its potential form [14]:

$$x_{n+1}^{m+1} = x_n^m + \frac{z_n^m}{x_n^{m+1} - x_{n+1}^m}. \tag{2}$$

(The name 'potential' is given here in analogy to the continuous case: the dependent variable x of equation (2) is related to the dependent variable X of equation (1) through $x_n^{m+1} - x_{n+1}^m = X_n^m$ and (1) is recovered exactly if z_n^m=1). The deautonomisation we are referring to consists in finding an explicit m, n dependence of z_n^m which is compatible with integrability. Let us first compute the degrees of the iterates for constant z:

$$\begin{array}{cccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & 1 & 4 & 7 & 10 & 13 & 16 & \cdots \\
 & 1 & 3 & 5 & 7 & 9 & 11 & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\uparrow m & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & \rightarrow n & & & & & &
\end{array}$$

The degree d_n^m is given simply by $d_n^m = mn + 1$. Assuming a generic (m, n) dependence for z we obtain the following successive degrees:

$$
\begin{array}{cccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & 1 & 4 & 10 & 20 & 35 & 56 & \cdots \\
 & 1 & 3 & 6 & 10 & 15 & 21 & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\uparrow m & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & \rightarrow n & & & & & &
\end{array}
$$

We remark readily that the degrees form a Pascal triangle i.e. they are identical to the binomial coefficients, leading to an exponential growth at least on a strip along the diagonal. The way to obtain an integrable deautonomisation is to require that the degrees obtained in the autonomous and nonautonomous cases be identical. The first constraint can be obtained by reducing the degree of x_2^2 from 6 to 5. As a matter of fact, starting from the initial conditions $x_n^0 = p_n/q$, $x_0^m = r_m/q$ (with $r_0 = p_0$) we obtain $x_1^1 = (p_1p_0 - p_0r_1 - z_0^0q^2)/(q(p_1 - r_1))$, $x_1^2 = Q_3/(qQ_2)$ where Q_k is a polynomial of degree k, and a similar expression for x_2^1. Computing x_2^2 we find $x_2^2 = Q_6/(q(p_1 - r_1)Q_4)$. It is impossible for q to divide Q_6 for generic initial conditions. However, requiring $(p_1 - r_1)$ to be a factor of Q_6 we find the constraint $z_1^1 - z_0^1 - z_1^0 + z_0^0 = 0$. The relation of this result to singularity confinement is quite easy to perceive. The singularity corresponding to $q = 0$ is indeed a fixed singularity: it exists for *all* (n, m)'s where either n or m are equal to zero. On the other hand the singularity related to $p_1 - r_1 = 0$ appears only at a certain iteration and is thus movable. The fact that with the proper choice of z_n^m the denominator factors out, is precisely what one expects for the singularity to be confined.

Requiring that z satisfy

$$z_{n+1}^{m+1} - z_n^{m+1} - z_{n+1}^m + z_n^m = 0 \tag{3}$$

suffices to reduce the degrees of all higher x's to those of the autonomous case. The solution of (3) is $z_n^m = f(n) + g(m)$ where f, g are two arbitrary functions. This form of z_n^m is precisely the one obtained in the analysis of convergence acceleration algorithms [15] using singularity confinement. The integrability of the nonautonomous form of (2) (and its relation to cylindrical KdV) has been discussed by Nagai and Satsuma [16] in the framework of the bilinear formalism.

We must point out here that the kind of initial conditions we choose, while influencing the specific degrees obtained, do not modify the conclusions on the type of growth. Let us illustrate this by choosing for (2) a staircase type of initial conditions where $x_n^{-n} = p_n/q$, $x_n^{1-n} = r_n/q$ with the same convention as to the degrees of q and the p's, r's (but without the now unnecessary constraint $p_0 = r_0$). We find the degrees:

$$\begin{array}{llllllll}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & 1 & 2 & 4 & 7 & 11 & 16 & \cdots \\
 & 1 & 1 & 2 & 4 & 7 & 11 & \cdots \\
 & & 1 & 1 & 2 & 4 & 7 & \cdots \\
 & & & \underline{1} & 1 & 2 & 4 & \cdots \\
\uparrow m & & & & 1 & 1 & 2 & \cdots \\
 & \rightarrow n & & & & & &
\end{array}$$

where the underlined 1 corresponds to the origin. The growth is again quadratic and depends only on the sum $N = n + m$ of the coordinates: $d_n^m = 1 + N(N-1)/2$.

Two more well-known discrete equations can be treated along the same lines. In the case of the lattice mKdV [14]:

$$x_{n+1}^{m+1} = x_n^m \frac{x_n^{m+1} - z_n^m x_{n+1}^m}{z_n^m x_n^{m+1} - x_{n+1}^m} \tag{4}$$

we obtain for constant z the same degree growth, $d_n^m = mn + 1$, as for the potential lattice KdV. If we assume now a generic z we find the degrees:

$$
\begin{array}{lccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & 1 & 4 & 13 & 32 & 65 & & \cdots \\
 & 1 & 3 & 7 & 13 & 21 & 31 & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\uparrow m & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & \multicolumn{2}{l}{\longrightarrow n} & & & & &
\end{array}
$$

The degrees obey the recursion $d_{n+1}^{m+1} = d_n^{m+1} + d_{n+1}^m + d_n^m - 1$ leading to an exponential growth with asymptotic ration $(1+\sqrt{2})$. Requiring the degree of x_2^2 to be 5 instead of 7 we find the condition

$$z_{n+1}^{m+1} z_n^m - z_n^{m+1} z_{n+1}^m = 0 \tag{5}$$

with solution $z_n^m = f(n)g(m)$. This condition is sufficient for the degrees of the nonautonomous case to coincide with those of the autonomous one. It is also precisely the one obtained in [15] using the singularity confinement condition. We believe that the Nagai-Satsuma approach [16] for the construction of double Casorati determinant solutions can be extended to the case of the nonautonomous lattice modified-KdV.

The discrete sine-Gordon equation [17, 18]:

$$x_{n+1}^{m+1} x_n^m = \frac{1 + z_n^m x_n^{m+1} x_{n+1}^m}{x_n^{m+1} x_{n+1}^m + z_n^m} \tag{6}$$

in the autonomous case where z is a constant leads to the degree pattern:

$$
\begin{array}{lccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & 1 & 5 & 9 & 13 & 16 & 19 & \cdots \\
 & 1 & 4 & 7 & 9 & 11 & 13 & \cdots \\
 & 1 & 3 & 4 & 5 & 6 & 7 & \cdots \\
\uparrow m & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & \multicolumn{2}{l}{\longrightarrow n} & & & & &
\end{array}
$$

It can be represented by $d_n^m = mn + \min(m,n) + 1$. In the nonautonomous case of generic z we obtain the sequence of degrees:

$$
\begin{array}{ccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & 1 & 5 & 19 & 49 & 96 & & \cdots \\
 & 1 & 4 & 11 & 19 & 29 & 41 & \cdots \\
 & 1 & 3 & 4 & 5 & 6 & 7 & \cdots \\
\uparrow m & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & \rightarrow n & & & & & &
\end{array}
$$

obeying the relation $d_{n+1}^{m+1} = d_n^{m+1} + d_{n+1}^m + d_n^m - (1-\delta_n^m)$ leading again to exponential growth. The condition for a growth identical to that of the autonomous case is the same as (5). Thus equation (6) introduces a nonautonomous extension of the lattice sine-Gordon equation. (We intend to return to a study of its properties in some future work). We must point out that in the continuous limit, this nonautonomous form goes over to $w_{x,t} = f(x)g(t)\sin w$. This explicit x and t dependence can be absorbed through a redefinition of the independent variables leading to the standard, autonomous, sine-Gordon, but no such gauge exists in the discrete case.

We now turn to two discrete equations which are particular in the sense that they are not integrable through IST techniques but rather through direct linearisation. The first is the discrete Liouville equation [19]:

$$x_{n+1}^{m+1} x_n^m = x_n^{m+1} x_{n+1}^m + z_n^m. \tag{7}$$

If we assume that z is a constant we obtain the following degree pattern:

$$
\begin{array}{ccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & 1 & 4 & 5 & 6 & 7 & 8 & \cdots \\
 & 1 & 3 & 4 & 5 & 6 & 7 & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\uparrow m & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & \rightarrow n & & & & & &
\end{array}
$$

By inspection we find $d_n^m = m + n$. This result is not at all astonishing. As we have shown in [9], the degree growth of linearisable mappings is slower than that of the IST integrable ones. The same feature appears again here. The deautonomisation of (7) can proceed along the same lines as previously. For generic z_n^m, the degrees are organised in a Pascal triangle and thus the growth is exponential. The condition for the growth to be identical to that of the autonomous case is again (5) and thus $z_n^m = f(n)g(m)$. However, this nonautonomous extension is trivial: it can be absorbed through a simple gauge transformation. Indeed, putting $x = \phi X$ where $\phi = \alpha(n)\beta(m)$ with $f(n) = \alpha(n)\alpha(n+1)$, $g(m) = \beta(m)\beta(m+1)$ we can reduce equation (7) to one where $z \equiv 1$.

Finally, we analyse the discrete Burgers equation [19]:

$$x_n^{m+1} = x_n^m \frac{1 + z_n^m x_{n+1}^m}{1 + z_n^m x_n^m}. \tag{8}$$

When z is a constant we find $d_n^m = m + 1$. (Notice that contrary to all the previous examples, in the case of Burgers equation m and n do not play the same role and thus a d_n^m that is not symmetric in m, n is not surprising). For a generic z_n^m, we find $d_n^m = 2^m$, a manifestly exponential growth. The condition for the degree to grow like $m + 1$ is just

$$z_{n+1}^m - z_n^m = 0 \tag{9}$$

i.e. $z_n^m = g(m)$. This leads to a nonautonomous extension of the lattice Burgers equation. Moreover this extension cannot be removed by a gauge. On the other hand, this extension is perfectly compatible with linearisability. Indeed, putting $x_n^m = X_{n+1}^m / X_n^m$ we can reduce it to the linear equation:

$$X_n^{m+1} = f(m)(X_n^m + g(m) X_{n+1}^m) \tag{10}$$

where f is arbitrary and can be taken equal to unity. This nonautonomous extension is just a special case of the more general discrete Burgers:

$$x_n^{m+1} = x_n^m \frac{\alpha_n^m + \beta_n^m x_{n+1}^m}{1 + \gamma_n^m x_n^m} \tag{11}$$

which can be linearised through $x_n^m = \phi_n^m X_{n+1}^m / X_n^m$ to $X_n^{m+1} = \psi_n^m (X_n^m + \gamma_n^m \phi_n^m X_{n+1}^m)$ provided $\beta_n^m = \alpha_n^m \gamma_{n+1}^m$ and α, ϕ and ψ are related through $\alpha_n^m \psi_n^m \phi_n^m = \psi_{n+1}^m \phi_n^{m+1}$. We must point out here that the continuous Burgers equation also does possess a nonautonomous extension. It is straightforward to show that if ϕ is a solution of the equation $\phi_t = \phi^2\phi_{xx}$ then the nonautonomous Burgers $u_t = \phi^2 u_{xx} + 2\phi u u_x$ can be linearised to $v_t = \phi^2 v_{xx}$ through the Cole-Hopf transformation $u = \phi v_x/v$.

In this paper, we have applied the method of the slow degree growth to the study of the integrability of partial difference equations. Our study has focused on well-known integrable lattice equations for which we have tried to provide nonautonomous forms. We have shown that using degree-growth methods it is possible to obtain integrable nonautonomous forms for most of the equations studied, and confirmed results previously obtained through the singularity confinement method. In the case of linearisable lattice equations, our results are the logical generalisation of the ones obtained for 1-dimensional mappings: the linearisable mappings have a degree growth that is slower than the one of the IST-integrable discrete equations. Our estimate of the degree growth was based on the direct computation of the degree for successive iterations and obtaining a fit of some analytical expression confirmed by subsequent iterations. It would be interesting, of course, to provide a rigorous proof of the degree growth following, for instance, the methods of [20]. However, this has not yet been carried through even for one-dimensional, nonautonomous mappings that are integrable through spectral methods. On the other hand, the proof of the degree growth for the cases where the equations are linearisable looks more tractable and we intend to address this question for both the one-and two-dimensional cases in some future work.

The fact that we were able, through the adequate choice of initial data, to perform these calculations without being overwhelmed by their size is an indication of the usefulness of our approach. The study of degree growth, perhaps coupled with singularity confinement in the dual strategy we sketched in [10], can be a precious tool for the detection of integrability of multidimensional discrete systems. The interest of this method is not only that it can be used as a detector of new integrable lattice systems but also that it can furnish an indication as to the precise method of their integration.

Acknowledgments

S. Tremblay acknowledges a scholarship from Centre de Coopération Interuniversitaire Franco-Québécois.

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Chapitre 6

SYSTÈMES INTÉGRABLES CONTINUS (DISCRETS) SANS LA PROPRIÉTÉ DE PAINLEVÉ (LE CONFINEMENT DES SINGULARITÉS)

J. Phys. A **33** *No 15 (2000) 3045–3052*

Integrable Systems without the Painlevé Property

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Abstract

We examine whether the Painlevé property is a necessary condition for the integrability of nonlinear ordinary differential equations. We show that for a large class of linearisable systems this is not the case. In the discrete domain, we investigate whether the singularity confinement property is satisfied for the discrete analogues of the non-Painlevé continuous linearisable systems. We find that while these discrete systems are themselves linearisable, they possess nonconfined singularities.

1 Introduction

For over a century, the Painlevé property [1] has been the cornerstone of integrability. The reason Painlevé introduced this property, which later was called after him, was a question that was open at the time, and of particular interest: 'Is it possible to define (new) functions from the solutions of nonlinear differential equations?'

In some sense, this amounted to introducing the analogue of special functions into the nonlinear domain. The study of linear equations had shown where the difficulties lied [2]. In particular, one had to deal with the multivaluedness that could appear as a consequence of the singularities of the coefficients of the equation which, for linear equations, are the only possible

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singularities of the solutions. The extension of these ideas to the nonlinear domain appeared hopeless since the location of bad singularities could now depend on the initial conditions. Then Painlevé made a leap of faith by requesting that all critical (i.e. multivalued) movable (i.e. initial condition dependent) singularities be absent.

The ordinary differential equations (ODE's) without critical movable singularities are said to possess the Painlevé property. Their solutions define functions which in some cases (the Painlevé transcendents) cannot be expressed in terms of known functions. The precise way to integrate (i.e. to construct the solutions of) the ODE's with the Painlevé property can be very complicated [3] but the important fact is that this can in principle be done. Thus, the property came to be synonymous to integrability. At this point it must be made clear that the integrability we are talking about, related to the Painlevé property, is of a special kind often referred to as 'algebraic integrability' [4]. It is, for instance, the kind of integrability that characterises systems integrable in terms of Inverse Scattering Transform (IST) techniques [5]. However, in common practice, many other 'brands' of integrability do exist [6]. Integrability through quadratures, like that encountered in the case of finite-dimensional Hamiltonian systems, is of (relatively) frequent occurence, and is not identical to algebraic integrability. Linearisability, i.e. the reduction of the system to a system of linear equations through local transformation, is a further, different, type.

In this paper, we shall examine the relation of these kinds of integrability to the Painlevé property, focusing on linearisable systems. In the second part of the paper, we shall examine the discrete analogues of these notions. In this case, the role of the Painlevé property is played by singularity confinement [7]. The latter is believed to be a necessary condition for integrability (but unlike the Painlevé property it has turned out not to be sufficient as well [8]). We shall show that in both continuous and discrete settings, linearisable systems integrable through linearisation can exist without the Painlevé property.

2 Integrable continuous systems and the Painlevé property.

A first instance of integrability without the Painlevé property was the derivation of the integrable system described by the Hamiltonian [9]:

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + y^5 + y^3x^2 + \frac{3}{16}yx^4 \tag{2.1}$$

which has the second (besides the energy) constant of motion

$$C = -yp_x^2 + xp_xp_y + \frac{1}{2}y^4x^2 + \frac{3}{8}y^2x^4 + \frac{1}{32}x^6. \tag{2.2}$$

There are movable singularities where near some singular point t_0, one has $y \approx \alpha(t-t_0)^{-2/3}$, $x \approx \beta(t-t_0)^{-1/3}$ with $\alpha^3 = -2/9$, β arbitrary. Taking the cube of the variables is not sufficient to regularise them, however. Indeed, a detailed analysis of complex-time singularities shows that their expansions contain *all* powers of $(t-t_0)^{1/3}$. The fact that some multivaluedness was compatible with integrability led to the introduction of the notion of "weak Painlevé" property. However, it was soon realised [10] that (2.1) was a member of a vaster family of integrable Hamiltonian systems associated to the potential $V = (F(\rho+y) + G(\rho-y))/\rho$ where $\rho = \sqrt{x^2+y^2}$. Since the two functions F and G are free, one can easily show that the singularities of the solutions of the equations of motion can be arbitrary. The Hamiltonians of this family are integrable through quadratures and, in fact, the associated Hamilton-Jacobi equations are separable. This leads to the conclusion that this type of integrability is not necessarily related to the Painlevé property. (As a matter of fact, the same conclusion could have been reached if we had simply considered one-dimensional Hamiltonian systems). One may justifiably argue that in the case of Hamiltonian systems the term integrability is to be understood as Liouville integrability which is not the one we refer to in relation to the Painlevé property. Still, Liouville integrability, and the dynamical symmetries to which it is associated, may be of utmost importance for physical applications and a systematic method for the detection would have been most welcome.

We turn now to a second case of integrability where the necessary character of the Painlevé property can be critically examined: that of linearisable systems. The term linearisable is used here to denote systems that can be reduced to linear equations through a local variable transformation. The first family of such systems are the projective ones [11]. Starting from the linear system for $(N+1)$ variables:

$$X'_\mu = \sum_{\nu=0}^{N} A_{\mu\nu} X_\nu \qquad \mu = 0, 1, \ldots, N \tag{2.3}$$

and introducing the quantities $x_\mu = X_\mu / X_0$ we obtain the projective Riccati system:

$$x'_\mu = a_\mu + \sum_{\nu=1}^{N} b_{\mu\nu} x_\nu + x_\mu \sum_{\nu=1}^{N} c_{\mu\nu} x_\nu \qquad \mu = 1, \ldots, N \tag{2.4}$$

where a_μ, $b_{\mu\nu}$ and $c_{\mu\nu}$ are given in terms of $A_{\mu\nu}$. As we have shown in [12] this system can be rewritten as a single N-th order differential equation. For N=1, this is just the Riccati equation for x_1:

$$x'_1 = a_1 + b_{11} x_1 + c_{11} x_1^2 \tag{2.5}$$

For N=2, the system can be reduced to the equation VI of the Painlevé/Gambier classification [2]

$$\frac{d^2 w}{dz^2} = -3w \frac{dw}{dz} - w^3 + q(z)\left(\frac{dw}{dz} + w^2\right) \tag{2.6}$$

for z some function of the independent variable of (2.4) and w a homographic function of x_1 with some specific functions of z as coefficients. Because of the underlying linearisation, the projective Riccati systems possess the Painlevé property by construction. Indeed, the X_μ have no movable singularities at all, and the only movable singularities of the x_μ are poles coming from the zeros of X_0.

However, there exists another kind of linearisability for which the Painlevé property need not be satisfied. Let us discuss the best-known second order case. One of the equations of the Painlevé/Gambier classification, bearing the number XXVII, is the equation proposed by Gambier [13]:

$$x'' = \frac{n-1}{n}\frac{x'^2}{x} + \left(fx + \phi - \frac{n-2}{nx}\right) x' - \frac{nf^2}{(n+2)^2} x^3 + \frac{n(f' - f\phi)}{n+2} x^2 + \psi x - \phi - \frac{1}{nx} \tag{2.7}$$

where f, ϕ and ψ are definite rational functions of two arbitrary analytic functions and of their derivatives [2]. As Gambier has shown, equation (2.7) can be written as a system of two Riccati equations in cascade:

$$y' = -y^2 + \phi y + \frac{2f}{n(n+2)} + \frac{\psi}{n} \tag{2.8a}$$

$$x' = \frac{nf}{n+2}x^2 + nyx + 1. \tag{2.8b}$$

Gambier has shown that unless the parameter n appearing in (2.7) and (2.8) is integer, the equation does not possess the Painlevé property. (We must point out that this is a first necessary condition and, in general, not a sufficient one: further constraints on the coefficients are needed in order to ensure the Painlevé property). On the other hand, the integration of the two Riccati equations in cascade can always be performed, through reduction to linear second order equations. Thus, although the solution of (2.8) does not in general lead to a well-defined function as solution of (2.7), it can still be obtained in cascade.

Once the Painlevé property is deemed unnecessary for the linearisation of the Gambier system, it is straightforward to extend the latter to the form:

$$y' = \alpha y^2 + \beta y + \gamma \tag{2.9a}$$

$$x' = a(y,t)x^2 + b(y,t)x + c(y,t) \tag{2.9b}$$

where α, β and γ are arbitrary functions of t while a, b and c are arbitrary functions of y and t. The integration in cascade of (2.9) can be obtained as previously. As a matter of fact, an extension like (2.9) gives the handle to the $(N+1)$-variables generalisation of the Gambier system:

$$x_0' = a_0(t)x_0^2 + b_0(t)x_0 + c_0(t) \tag{2.10}$$

$$x_\mu' = a_\mu(x_0,\ldots,x_{\mu-1},t)x_\mu^2 + b_\mu(x_0,\ldots,x_{\mu-1},t)x_\mu + c_\mu(x_0,\ldots,x_{\mu-1},t)$$
$$\mu = 1,\ldots,N$$

where a_μ, b_μ and c_μ are arbitrary functions of their arguments. Again, system (2.10) does not possess, generically, the Painlevé property while it can be linearised and integrated in cascade.

Untill now, we have presented rather straightforward generalisations of integrable systems which violate the Painlevé property while preserving their linearisability. We shall close this section by introducing a new (at least to our knowledge) method of linearisation which again leads to integrable systems not possessing the Painlevé property. Let us describe our general approach. The idea is the following: we start from a linear second order equation in the form:

$$\frac{\alpha x'' + \beta x' + \gamma x + \delta}{\epsilon x'' + \zeta x' + \eta x + \theta} = K \tag{2.11}$$

where $\alpha, \beta, \dots, \theta$ are functions of t with K a constant, and a *nonlinear* second order equation of the form:

$$f(x'', x', x) = M \tag{2.12}$$

where f is a (possibly inhomogeneous) polynomial of degree two in x together with its derivatives, but linear in x'', and with M a constant. We then ask that the derivatives of both equations with respect to the independent variable, i.e. the resulting third order equations, be identical up to an overall factor. This is a novel linearisation approach. The explicit integration procedure is the following. We start from equation (2.12) with *given* M and initial conditions x_0, x_0' for some value t_0 of the independent variable t. We use (2.12) to compute x_0'' at t_0. Having these values, we can use (2.11) to compute the value of K. Since the latter is assumed to be a constant, we can integrate the linear equation (2.11) for all values of t. Since this solution will satisfy the third order equation mentioned above, it will also be a solution of (2.12).

In order to illustrate this approach, we derive one equation that can be integrated through this linearisation. Our starting assumption is that (2.12) contains a term $x''x'$. The more general term $x''(x' + cx + d)$ can always be reduced to this form, i.e. c=d=0 through a rescaling and translation of x. It is then straightforward to obtain the full expression in the homogeneous subcase $\delta = \theta = 0$. We thus find:

$$\frac{tx'' + (at - 1/2)x' + btx}{x'' + ax' + bx} = K \tag{2.13}$$

for the linear equation, and

$$x''x' + 2ax'^2 + 3bx'x + (2ab - b')x^2 = M \tag{2.14}$$

for the nonlinear one, with $b = a^2 - a'/2$ and a satisfying the equation

$$a''' = 6a''a + 7a'^2 - 16a'a^2 + 4a^4 \tag{2.15}$$

which is equation XII in the Chazy classification [14]. Its general solution can be obtained by putting $a = -u'/2u$. Equation (2.15) reduces to

$$u^{(IV)}u - u'''u' + u''^2/2 = 0 \tag{2.16}$$

which implies $u^{(V)}=0$, so u is a quartic polynomial in the independent variable t with one constraint on its coefficients, because of (2.16). Given a and the corresponding b, equation (2.14) is integrable by linearisation through (2.13). On the other hand, (2.14) violates the Painlevé property. Solving it for x'', we find a term proportional to x^2/x' (or, for that matter, to $1/x'$) which is incompatible with it.

More cases like the one above could have been derived but this is not necessary in order to prove our point. Integrability through linearisation does not require the Painlevé property. On the other hand we do not know of any systematic way to detect linearisability for a given differential system.

3 Discrete integrable systems

In the case of discrete systems, a difficulty appears from the outset in the sense that the discrete analogue of the Painlevé property is not well established. One of the properties that characterises discrete integrable systems is that of singularity confinement [7]. While analysing a host of integrable mappings it was observed that whenever a singularity appeared at some iteration, due to the particular initial conditions, it disappeared after some further iterations. Thus, confinement would have been an excellent candidate for the role of the discrete analogue of the Painlevé property were it not for the fact that it is not sufficient. There exist mappings which have only confined singularities and which are *not* integrable [8]. Another property which has been proposed as an indicator of integrability in (rational) mappings is that of the degree growth of the iterates [15].

Let us illustrate what we mean by degree in a specific example. We consider a three-point mapping of the form $\overline{x} = f(x, \underline{x}; n)$ where f is rational in $x, \underline{x}$. (The 'bar' notation, which will be used throughout this section, is a shorthand for the up- and down-shifts in n i.e. $\overline{x} \equiv x(n+1)$, $x \equiv x(n)$, $\underline{x} \equiv x(n-1)$). Starting from some initial conditions x_0, x_1 we introduce homogeneous variables through $x_0 = p$, $x_1 = q/r$ and compute the homogeneity degree of the iterates of the mapping in q, r, to which we assign the same degree 1, while p is assigned the degree 0. Other choices do exist but the result does not depend on the particular choice. While the degrees obtained do depend on it, the *growth* of the degree does not. Thus for a generic, nonintegrable, mapping the degree growth of the iterates is exponential [16,17]. On the contrary, for integrable mappings, the growth is just polynomial. Moreover, a detailed analysis of discrete Painlevé equations [18] and linearisable mappings [19] has shown that the latter have even slower growth properties (which can be used not only as a detector of integrability but as an indicator of the integration method). In what follows, we shall examine the results of the application of the two methods to integrable discrete systems.

The first case we shall analyse are projective mappings [11]. In perfect analogy to the continuous case one can introduce the discrete projective Riccati equations. The starting point is a linear system for $(N+1)$ variables:

$$\overline{X}_\mu = \sum_{\nu=0}^{N} A_{\mu\nu} X_\nu \qquad \mu = 0, 1, \ldots, N \tag{3.1}$$

Introducing again $x_\mu = X_\mu / X_0$, we obtain:

$$\overline{x}_\mu = \frac{A_{\mu 0} + \sum_{\nu=1}^{N} A_{\mu\nu} x_\nu}{A_{00} + \sum_{\nu=1}^{N} A_{0\nu} x_\nu} \qquad \mu = 1, \ldots, N \tag{3.2}$$

In fact we have shown [12] that this system can always be rewritten as a $N{+}1$-point mapping in terms of a single object. Clearly the case $N{=}1$ is just a homographic (discrete Riccati) mapping for x_1. For $N{=}2$ we finally find [20,21]:

$$\overline{w} = \alpha + \frac{\beta}{w} + \frac{1}{w\underline{w}} \tag{3.3}$$

for a quantity w which is obtained from x_1, say, through some homography and α, β are given in tems of the $A_{\mu\nu}$. Because of the underlying linearisation, any singularity appearing in the projective Riccati system is confined in one step. Moreover, the study of the degree of the iterates [19] shows that there is no growth at all: the degree is constant. Thus both criteria are satisfied in this case.

We turn now to the more interesting case of the Gambier mapping [22]. The latter is, in perfect analogy to the continuous case, a system of two (discrete) Riccati equations in cascade:

$$\overline{y} = \frac{\alpha y + \beta}{\gamma y + \delta} \tag{3.4a}$$

$$\overline{x} = \frac{ayx + bx + cy + d}{fyx + gx + hy + k} \tag{3.4b}$$

where α, ..., δ and a, ..., k are all functions of the independent discrete variable n. In [22] it was shown that system (3.4) is not confining unless the coefficients entering in the equation satisfy certain conditions. On the other hand the same argument presented in the continuous case can be transposed here: the integration of the two Riccati equations in cascade can always be performed, through reduction to linear second order mappings. The study of the degree growth of the iterates of (3.4) was performed in [19] where it was found that the growth is always linear, *independently* of the condition we referred to above.

This result leads naturally to the following generalisation of the discrete Gambier system, the singularities of which are, in general, not confined:

$$\overline{y} = \frac{\alpha y + \beta}{\gamma y + \delta} \tag{3.5a}$$

$$\overline{x} = \frac{a(y)x + b(y)}{c(y)x + d(y)} \tag{3.5b}$$

where a, ..., d are polynomials in y the coefficients of which may depend on the independent variable n. The study of the degree growth of the iterates of (3.5) is straightforward. We find that the degree growth of x is linear. Again, system (3.5) can be integrated in cascade. On the other hand, (3.5) cannot be written as a three-point mapping for x. Indeed, if we eliminate y, $\overline{y}$ between (3.5a) (3.5b) and the upshift of the latter, we obtain an equation relating x, $\overline{x}$ and $\overline{\overline{x}}$ which is polynomial in all three variables, generically *not* linear in $\overline{\overline{x}}$. This does not define a mapping but rather a correspondence which in general leads to exponential proliferation of the number of images and preimages. This correspondence is not integrable but this is not in contradiction with the integrability of (3.5). The two systems are not equivalent.

An $(N+1)$-variables extension of the Gambier system can be easily produced. We have:

$$\overline{x}_0 = \frac{\alpha x_0 + \beta}{\gamma x_0 + \delta} \tag{3.6}$$

$$\overline{x}_\mu = \frac{a_\mu(x_0,\ldots,x_{\mu-1})x_\mu + b_\mu(x_0,\ldots,x_{\mu-1})}{c_\mu(x_0,\ldots,x_{\mu-1})x_\mu + d_\mu(x_0,\ldots,x_{\mu-1})} \qquad \mu = 1,\ldots,N.$$

Again, the degree growth of (3.6) can be computed leading to a linear growth and, once more, the singularities of (3.6) do not confine in general.

Thus, several linearisable systems can be found for which the singularity confinement gives more restricted predictions than the degree growth. We shall comment on this point in the conclusion.

A last point concerns the discrete analogues of the linearisable systems we have presented at the end of section 2. The procedure can be transposed to a discrete setting in a pretty straightforward way. We have a linear equation

$$\frac{\alpha\overline{x} + \beta + \gamma\underline{x} + \delta}{\epsilon\overline{x} + \zeta + \eta\underline{x} + \theta} = K \tag{3.7}$$

where $\alpha,\ldots,\theta$ are all functions of n with K a constant, and a nonlinear mapping

$$f(\underline{x}, x, \overline{x}; n) = M \tag{3.8}$$

where f is globally polynomial of degree two in all the x's but not more than linear separately in each of $\underline{x}$ and $\overline{x}$. Writing that the l.h.s. of (3.7) is the same as that of its upshift we get an equation relating $\underline{x}$, x, $\overline{x}$ and $\overline{\overline{x}}$. For appropriate choices of $\alpha,\ldots,\theta$ this four point equation can be identical (up to unimportant factors) to the four-point equation obtained from (3.8) by writing $f(\underline{x}, x, \overline{x}; n) = f(x, \overline{x}, \overline{\overline{x}}; n{+}1)$. The integration method is quite similar to that described in the continuous case. Given M, and starting with $\underline{x}, x$ at some n, one gets $\overline{x}$ from (3.8). Implementing (3.7) this fixes the value of K. From then on, one integrates the linear equation (3.7) for all n. Since the four-point equation is always satisfied, this means that f computed at any n has a constant value, which is just M, so (3.8) is satisfied.

Several mappings derived in [23] as special limits of discrete Painlevé equations can be linearised in this way. For instance the nonlinear equation:

$$\left(\frac{\overline{x} + x - a}{\overline{z}} - \frac{x}{\zeta}\right)\left(\frac{\underline{x} + x - a}{z} - \frac{x}{\zeta}\right) - \frac{x^2}{\zeta^2} = M \tag{3.9}$$

with a a constant, where z and ζ are defined from a single arbitrary function g of n through $z = \overline{g}{+}\underline{g}$, $\zeta = \overline{g}{+}g$, can be solved through the linear equation:

$$\frac{A\overline{x} + B(x-a) + \overline{A}\underline{x}}{z\overline{x} + (\overline{z}+z)(x-a) + \overline{z}\underline{x}} = K \tag{3.10}$$

where $A = g^2(\overline{g} + \underline{g})$ and $B = -(\overline{g} + g)\overline{\overline{g}}\underline{g} - (\overline{\overline{g}} + \underline{g})\overline{g}g$. Mapping (3.9) is generically non-confining unless g is a constant.

4 Conclusion

In this paper we have adressed the question of integrability which does not necessitate the Painlevé property. We have found that for a large class of integrable, *linearisable* systems, the Painlevé property is not a prerequisite for integrability. The second-order Gambier system is the prototype of such a linearisable equation. Once we find that it can be linearised in the absence of the Painlevé property, it is straightforward to generalise the Gambier system and to extend it to N variables (violating the Painlevé property but preserving integrability through linearisability).

Having dispensed of the Painlevé property, it is possible to propose a new method of linearisation where the derivative of a nonlinear system coincides with that of a linear one. The usefulness of this method has been illustrated through the derivation of a linearisable system which does not satisfy the Painlevé criterion.

At this point, we must stress that the Painlevé property can still be considered as a necessary condition for integrability provided we qualify the latter. The integrability with which the Painlevé property is associated, often referred to as algebraic integrability, corresponds to the integration through IST methods. This is for instance the case of the transcendental Painlevé equations (or most of the integrable partial differential equations). For these cases, the Painlevé property is necessary and we believe, sufficient. What this paper shows is that for the simpler case of linearisability, the Painlevé property is superfluous.

In the case of discrete systems the situation is more complicated. It would appear that what would play the role of the Painlevé property is singularity confinement. (The caveat is that the latter was shown not to be a sufficient condition). Again, it turned out that singularity confinement is necessary for integrability through IST methods, as for instance in the integration of the discrete Painlevé equations through isomonodromy techniques. However for integrability through linearisation, singularity confinement is too restrictive just like the Painlevé property. The study of the degree growth, on the other hand, shows that this criterion is more suitable for the detection of integrability in a larger sense: it identifies all linearisable systems as integrable with no restrictions whatsoever. This is at variance with the continuous case where no linearisability criterion seems to exist (at least none has been found to date). Moreover, the detailed information on the degree growth is a most useful indication of the precise integration procedure. Thus, although it is not clear whether the degree growth is the discrete equivalent to the Painlevé property it can be a most reliable integrability detector.

Acknowledgments

S. Tremblay acknowledges a scholarship from Centre de Coopération Interuniversitaire Franco-Québécois.

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CONCLUSION

Les premiers chapitres de cette thèse concernaient la symétrie des équations aux variables discrètes. D'une part, aux chapitres 1 à 3, nous avons présenté une nouvelle méthode permettant d'obtenir les transformations de Lie ponctuelles, agissant de façon simultanée sur l'équation aux variables discrètes ainsi que sur le réseau. La méthode est purement algorithmique, tout comme celle pour obtenir le groupe de symétrie des équations différentielles [**7, 8, 9**]. La transformation du réseau sous ces transformations est empruntée à l'idée de Dorodnitsyn. Cependant, son approche concerne la discrétisation d'équations différentielles qui permet de conserver certaines symétries alors que, dans notre approche, on considère les équations aux variables discrètes comme des objets à étudier en eux-mêmes. Mentionnons que dans la plupart des méthodes proposées pour étudier les symétries des équations aux variables discrètes, le réseau est fixe et non transformable, c'est-à-dire donné habituellement par l'équation $x = n$, $n \in \mathbb{Z}$. Il pourrait donc être intéressant d'exploiter, dans un travail futur, l'aspect «transformation du réseau» et de regarder comment certaines symétries sont présentes pour un réseau donné, et si de nouvelles symétries supplémentaires apparaissent (ou disparaissent) pour un second réseau. D'autre part, la réduction par symétrie est un aspect qui a déjà été considéré très partiellement dans le chapitre 1 et pour lequel il serait intéressant de pousser l'investigation un peu plus loin.

Au chapitre 4, nous avons classifié un système d'équations différentielles aux différences représentant une chaîne unidimensionnelle à deux atomes. La classification a été faite en classes d'équivalence (sous des transformations permises) selon les interactions possibles admettant un groupe de transformations de Lie ponctuelles. Pour ce faire, on a utilisé la méthode intrinsèque [**10**]. On a ainsi obtenu une classification complète pour les interactions non-linéaires et linéaires. Pour les interactions non-linéaires, le groupe de symétrie des transformations de Lie ponctuelles, laissant le réseau invariant et amenant une solution en une autre solution, est au plus 5-dimensionnelle. Une investigation future serait de faire (possiblement) le même type de classification, mais avec la méthode présentée aux chapitres 1 à 3, plutôt qu'en utilisant la méthode intrinsèque.

La méthode de la croissance de degré lente est appliquée au chapitre 5 pour l'étude de l'intégrabilité des équations discrètes partielles. Notre étude s'est concentrée sur des équations discrètes intégrables pour lesquelles nous avons cherché des formes non-autonomes. Nous avons montré qu'en utilisant la croissance de degré, il est possible d'obtenir des systèmes intégrables non-autonomes. De plus, la croissance de degré nous donne plus d'informations que simplement nous dire qu'une équation non-intégrable aura une croissance de degré exponentielle et donc une entropie algébrique différente de zéro ou qu'une équation intégrable a une entropie algébrique nulle. En effet, tel que montré dans **[38]**, la croissance de degré peut nous donner une indication de la méthode à employer pour intégrer l'équation. En effet, dans le cas des équations aux variables discrètes linéarisables, la croissance de degré est plus lente que pour les équations intégrables par la méthode de diffusion inverse. Ce résultat est la généralisation naturelle de celui obtenu pour les équations à une seule variable discrète **[38, 39]**. La combinaison de l'utilisation de la croissance de degré avec le confinement des singularités semble donc être un duo très intéressant pour la détection de l'intégrabilité de systèmes discrets multidimensionnels **[39]**.

Enfin, au chapitre 6, nous avons posé la question : «Quels sont les systèmes intégrables continus qui ne possèdent pas la propriété de Painlevé ?». Nous avons trouvé que pour une large classe de systèmes linéarisables la propriété de Painlevé n'est pas un prérequis à l'intégrabilité. En effet, le système de Gambier de deuxième ordre **[40]** est un exemple d'un tel système. Nous nous sommes par la suite posé la même question, mais cette fois-ci pour le domaine discret : «Quels sont les systèmes intégrables discrets qui ne possèdent pas la propriété du confinement des singularités ?». Ici, nous avons substitué le rôle joué précédemment (dans le cas continu) par la propriété de Painlevé, par le confinement des singularités. De façon analogue au cas continu, nous trouvons que certains systèmes linéarisables existent sans que les singularités soient confinées (par exemple, le cas Gambier discret [**41**]).

Annexe A

INVARIANTS DES ALGÈBRES DE LIE TRIANGULAIRES NILPOTENTES ET RÉSOLUBLES

Invariants of the Nilpotent and Solvable Triangular Lie Algebras

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June, 2001

Abstract

Invariants of the coadjoint representation of two classes of Lie algebras are calculated. The first class consists of the nilpotent Lie algebras $T(M)$, isomorphic to the algebras of upper triangular $M \times M$ matrices. The Lie algebra $T(M)$ is shown to have $[M/2]$ functionally independent invariants. They can all be chosen to be polynomials and they are presented explicitly. The second class consists of the solvable Lie algebras $L(M, f)$ with $T(M)$ as their nilradical and f additional linearly nilindependent elements. Some general results on the invariants of $L(M, f)$ are given and the cases $M = 4$ for all f and $f = 1$, or $f = M - 1$ for all M are treated in detail.

1 Introduction

The purpose of this paper is to present some results on the invariants of two classes of Lie algebras, over the field of complex or real numbers ($K = \mathbb{C}$ or $\mathbb{R}$). The first class are the finite triangular nilpotent Lie algebras $T(M)$ of dimension $M(M-1)/2$. By triangular nilpotent Lie algebra, we mean the

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nilpotent Lie algebra isomorphic to the Lie algebra of strictly upper triangular $M \times M$ matrices. The second class of algebras studied below are the finite solvable triangular Lie algebras $L(M, f)$ that have $T(M)$ as their nilradicals (maximal nilpotent ideals) and contain f further nonnilpotent elements. For the algebras $L(M, f)$, use will be made of a recent article [1] in which we obtained a classification of such Lie algebras and presented the general form of the commutation relations.

In physics, invariant operators of the symmetry group of a physical system and its subgroups provide quantum numbers. Indeed, the eigenvalues of the invariant operators of the entire symmetry group will be the quantum numbers, characterizing the system as such (e.g., the particle mass and spin in the case of the Poincaré group). The invariant operators of subgroups will then characterize states of the system (its energy, linear or angular momentum, etc.) [2].

In other applications, invariant operators of dynamical groups provide mass formulas [3, 4], energy spectra [5, 6] and in general characterize specific properties of physical systems.

Let us stress here that in this context the concept of an invariant need not mean a Casimir operator. Indeed, the problem of finding invariants will be reduced to that of solving a certain set of linear first order partial differential equations [7, 8]. These may have polynomial solutions, giving rise to Casimir operators. They may also have rational solutions, giving rise to rational invariants. Finally, the equations may have more general solutions, including transcendental functions of various types, leading to general invariants.

Casimir operators are polynomials in the enveloping algebra of a Lie algebra that commute with all elements of the Lie algebra. In other words, a Casimir operator of a Lie algebra is an element of the centre of the enveloping algebra. For a Lie algebra L, the Casimir operators can be calculated directly. Namely, we impose that a general polynomial in the enveloping algebra commutes with all basis elements X_i of L. However, more efficiently, they can be calculated as invariants of the coadjoint representation of the corresponding Lie algebra [9, 10].

The Casimir operators of semisimple Lie algebras are well known. Their number p is equal to the rank of the considered Lie algebra [11, ... ,17]. Moreover, for semisimple Lie algebra, all invariants of the coadjoint representation can be expressed as functions of p homogeneous polynomials.

For solvable Lie algebras, the situation is less clear. Neither the specific type of functions, nor the number of functionally independent invariants is known.

One method, for calculating the polynomial and other invariants for arbitrary Lie algebras, is an infinitesimal one. This method has been presented in [7] and applied to low dimensional Lie algebras [18, 19], to subalgebras of the Poincaré Lie algebra [20] and to solvable Lie algebras with Heisenberg or Abelian nilradicals [21, 22].

From a mathematical point of view, in the representation theory of solvable Lie algebras, polynomial and non-polynomial invariants in the coadjoint representation appear on the same footing: they characterize irreducible representations. Casimir operators in the enveloping algebra correspond to polynomial invariants. The functions of the infinitesimal operators, corresponding to the non-polynomial invariants, will be called 'generalized Casimir operators'. In the study of the integrability of classical Hamiltonian systems, integrals of motion do not have to be polynomials in the dynamical variables [23, 24].

In Section 2 we formulate the problem of calculating the invariants of the coadjoint representation. Section 3 is devoted to the nilpotent algebras $T(M)$. We calculate the invariants explicitly. There are $[M/2]$ functionally independent invariants, all of them polynomials. In Section 4 we calculate the invariants of the solvable Lie algebras $L(M, f)$. We first treat the case $M = 4$ in detail, then present results and conjectures for $L(M, M-1)$ and $L(M, 1)$.

2 General results and formulation of the problem

Let us consider a N-dimensional Lie algebra given by the basis $\{Y_1, \ldots, Y_N\}$ and the commutation relations

$$[Y_i, Y_j] = \sum_{k=1}^{N} C_{ij}^k Y_k \qquad 1 \leq i, j, k \leq N. \tag{2.1}$$

In order to calculate the invariants of the Lie algebra L, we shall work on the dual of L. We consider smooth functions $F: (y_1, \ldots, y_N) \to K$ where the variables y_i are ordinary (commuting) variables on the space L^*, dual of L, and K is the field of complex or real numbers ($K = \mathbb{C}$ or $\mathbb{R}$). The generators Y_i are given in the coadjoint representation by the differential operators

$$\hat{Y}_i = \sum_{j,k} C_{ij}^k \, y_k \frac{\partial}{\partial y_j} \,. \tag{2.2}$$

We can verify easily that the differential operators $\hat{Y}_i$ satisfy the commutation relations (2.1).

The function F will be an invariant of the coadjoint representation of L if it satisfies the linear first order partial differential equations

$$\hat{Y}_i \cdot F = 0 \qquad i = 1, \ldots, N \tag{2.3}$$

which, one hopes, can be solved by standard methods.

Our aim is to find a complete set of functionally independent solutions to equation (2.3), for nilpotent and solvable triangular Lie algebras. If the solutions are polynomials, we obtain Casimir operators by replacing the variables y_i by the generators Y_i and symmetrizing, whenever necessary. The number of independent solutions n_I, i.e. the number of functionally independent invariants, is equal to

$$n_I = N - rank(M) \tag{2.4}$$

where M is the antisymmetric matrix with elements

$$M_{ij} = \sum_{k=1}^{N} C_{ij}^k \, y_k \tag{2.5}$$

(see Ref.[7]).

3 Invariants of nilpotent triangular Lie algebras

3.1 Structure of the nilpotent Lie algebra $T(M)$ and its realization by differential operators

Let us consider the finite triangular Lie algebra $T(M)$ over the field K of complex or real numbers. A basis for this algebra is

$$\begin{gathered}\{N_{ik} \mid 1 \leq i < k \leq M\} \\ (N_{ik})_{ab} = \delta_{i,a}\ \delta_{k,b} \qquad \dim\ T(M) = \tfrac{1}{2}M(M-1) \equiv r\end{gathered} \tag{3.1}$$

with $M > 3$. The Lie algebra $T(2)$ is trivial and $T(3)$ is isomorphic to the Heisenberg Lie algebra $H(1)$. The dimension $M = 3$ is the only case for which there is an isomorphism between the triangular and the Heisenberg Lie algebras.

The commutation relations of $T(M)$ are given by

$$[N_{ik}, N_{ab}] = \delta_{k,a}N_{ib} - \delta_{b,i}N_{ak}. \tag{3.2}$$

This basis can be represented by the standard basis of the strictly upper triangular $M \times M$ matrices.

The differential operators $\hat{N}_{ik}$ realizing the coadjoint representation of $T(M)$, are

$$\hat{N}_{ik} = \sum_{b=k+1}^{M} n_{ib}\frac{\partial}{\partial n_{kb}} - \sum_{a=1}^{i-1} n_{ak}\frac{\partial}{\partial n_{ai}}. \tag{3.3}$$

Note that $\hat{N}_{1M} \equiv 0$ in (3.3), since N_{1M} commutes with all the elements of $T(M)$.

We shall realize the coadjoint representation of $T(M)$ in a space of differentiable functions of r variables, i.e.

$$F = F(n_{12}, n_{23}, \ldots, n_{(M-1)M}, n_{13}, n_{24}, \ldots, n_{(M-2)M}, \ldots, n_{1M}). \tag{3.4}$$

The function F will be an invariant of the coadjoint representation of $T(M)$, if it satisfies the linear first order partial differential equations

$$\hat{N}_{ik} \cdot F = 0 \qquad 1 \leq i < k \leq M. \tag{3.5}$$

3.2 Definitions and results

Let us consider the set of strictly upper triangular $M \times M$ matrices $Q = Q(M)$ over the field K i.e.

$$Q_{ik} = \begin{cases} n_{ik} & for \quad k-i \geq 1 \\ \\ 0 & otherwise. \end{cases} \tag{3.6}$$

We define the determinant $Z_\mu = Z_\mu(M)$ constructed from the $\mu \times \mu$ right upper corner sub-matrix of the matrix Q, i.e.

$$Z_\mu = \begin{vmatrix} n_{1(M-\mu+1)} & n_{1(M-\mu+2)} & \cdots & n_{1M} \\ n_{2(M-\mu+1)} & n_{2(M-\mu+2)} & \cdots & n_{2M} \\ \vdots & \vdots & & \vdots \\ n_{\mu(M-\mu+1)} & n_{\mu(M-\mu+2)} & \cdots & n_{\mu M} \end{vmatrix} \qquad 1 \leq \mu \leq \left[\frac{M}{2}\right] \tag{3.7}$$

where we shall use the standard notation $[x]$ for the entire part of a positive number. In particular,

$$p = \left[\frac{M}{2}\right] = \begin{cases} \frac{M}{2} & for \quad M = 2p \\ \\ \frac{M-1}{2} & for \quad M = 2p+1. \end{cases} \tag{3.8}$$

Theorem 1 *The triangular Lie algebra $T(M)$ defined by equations* (3.1) *and* (3.2) *has exactly* $[M/2]$ *functionally independent invariants. A basis of invariants is given by*

$$I_\mu = Z_\mu \qquad \mu = 1, \ldots, \left[\frac{M}{2}\right] \tag{3.9}$$

where Z_μ is the determinant function given by eq.(3.7).

Proof. Let us first consider the cases M odd, i.e. $M = 2p+1$ for $p = 2, 3, \ldots$

We begin by applying the set of $p(p+2)$ differential operators of eq.(3.3), given by

$$\begin{matrix} \hat{N}_{1(p+1)} & \hat{N}_{1(p+2)} & \cdots & \hat{N}_{1M} \\ \hat{N}_{2(p+1)} & \hat{N}_{2(p+2)} & \cdots & \hat{N}_{2M} \\ \vdots & \vdots & & \vdots \\ \hat{N}_{p(p+1)} & \hat{N}_{p(p+2)} & \cdots & \hat{N}_{pM} \\ 0 & \hat{N}_{(p+1)(p+2)} & \cdots & \hat{N}_{(p+1)M} \end{matrix} \tag{3.10}$$

on the functions (3.4). The action of all these operators eliminates the dependence on the $p(p+1)$ variables n_{ik} for

$$1 \leq i \leq p \qquad i+1 \leq k \leq p+1 \tag{3.11}$$

and

$$p+1 \leq i \leq M-1 \qquad i+1 \leq k \leq M\,. \tag{3.12}$$

The p^2 remaining variables are

$$\begin{array}{cccc} n_{1(p+2)} & n_{1(p+3)} & \cdots & n_{1M} \\ n_{2(p+2)} & n_{2(p+3)} & \cdots & n_{2M} \\ \vdots & \vdots & & \vdots \\ n_{p(p+2)} & n_{p(p+3)} & \cdots & n_{pM} \end{array} \tag{3.13}$$

and the $p(p-1)$ remaining differential operators $\hat{N}_{ik}$ of eq.(3.3) are given by

$$\hat{N}_{ik} = \sum_{b=p+2}^{M} n_{ib} \frac{\partial}{\partial n_{kb}} \qquad 1 \leq i \leq p-1 \quad i+1 \leq k \leq p \tag{3.14}$$

$$\hat{N}_{ik} = -\sum_{a=1}^{p} n_{ak} \frac{\partial}{\partial n_{ai}} \qquad p+2 \leq i \leq M-1 \quad i+1 \leq k \leq M. \tag{3.15}$$

These differential operators are linearly independent. Therefore, the number of invariants for $T(2p+1)$ is p, i.e. the difference between the number of remaining variables and the number of remaining independent differential operators.

At this stage of the proof it is sufficient to verify that the remaining differential operators (3.14) and (3.15) annihilate determinants $Z_1, \ldots, Z_p$, i.e. $\hat{N}_{ik} \cdot Z_\alpha = 0$ for $\alpha = 1, \ldots, p$.

Let us first consider the set of differential operators (3.14). A given differential operator $\hat{N}_{ik}$ (i and k fixed) of (3.14) annihilates the determinant $Z_1, \ldots, Z_{k-1}$, since the variables n_{kb} $(p+2 \leq b \leq M)$ do not figure in these determinants. It is therefore sufficient to look how $\hat{N}_{ik}$ acts on $Z_k, \ldots, Z_p$.

The determinant Z_β $\beta \in \{k, k+1, \ldots, p\}$ can be expanded in terms of its k^{th} row

$$Z_\beta = \sum_{b=2p+2-\beta}^{M} n_{kb}\, C_{kb}^{(\beta)} \tag{3.16}$$

where $C_{kb}^{(\beta)}$ is the cofactor of the $\beta \times \beta$ square matrix associated with the determinant Z_β. Hence, the differential operator $\hat{N}_{ik}$ applied on these determinants gives

$$\hat{N}_{ik} \cdot Z_\beta = \sum_{b=2p+2-\beta}^{M} n_{ib}\, C_{kb}^{(\beta)}. \tag{3.17}$$

The right hand side of eq.(3.17) vanishes, since it corresponds to the expansion of determinant in terms of the cofactors of a different row. This gives the determinant of a matrix with two identical rows, hence zero.

The procedure is very similar for the set of differential operators (3.15). An operator $\hat{N}_{ik}$ of this set annihilates the determinants $Z_1, \ldots, Z_{M-i}$ since the operator acts only on variables not figuring in the determinants. Let us consider the action of N_{ik} in (3.15) for the determinants Z_γ, where $\gamma \in \{(M-i+1), (M-i+2), \ldots, p\}$.

We can write the determinants Z_γ as

$$Z_\gamma = \sum_{a=1}^{\gamma} n_{ai}\, C_{ai}^{(\gamma)} \tag{3.18}$$

and the action of the differential operators $\hat{N}_{ik}$ on these determinants is given by

$$\hat{N}_{ik} \cdot Z_\gamma = -\sum_{a=1}^{\gamma} n_{ak}\, C_{ai}^{(\gamma)}. \tag{3.19}$$

Hence, we obtain a determinant with two identical columns. More precisely, the action of differential operator $\hat{N}_{ik}$ in (3.15) on determinants (3.18) is the following: the column n_{ai} in determinants Z_γ is replaced by the column $-n_{ak}$, for $1 \leq a \leq \gamma$. Therefore, by the property of determinants, this action annihilates Z_γ.

The proof for the even case is very similar to the odd case and we omit it. □

4 Invariants of the solvable triangular Lie algebras

4.1 Structure of the solvable triangular Lie algebra $L(M, f)$

In this section we sum up the main results of Ref.[1] to make this article self-contained.

Let us extend the algebra $T(M)$ to an indecomposable solvable Lie algebra $L(M, f)$ of dimension $d = \frac{1}{2}M(M-1) + f$ having $T(M)$ as its nilradical. In other words, we add f further linearly nilindependent elements to $T(M)$. Let us denote them $\{X^1, \ldots, X^f\}$.

Definition 1 • *A set of elements $\{X^\alpha\}$ of a Lie algebra L is linearly nilindependent if no nontrivial linear combination of them is a nilpotent element.*

• *A set of matrices $\{A^\alpha\}_{\alpha=1,\ldots,n}$ is linearly nilindependent if no nontrivial linear combination of them is a nilpotent matrix, i.e. if*

$$\left(\sum_{i=1}^{n} c_i \, A^i \right)^k = 0 \tag{4.1}$$

for some $k \in \mathbb{Z}^+$, implies $c_i = 0 \ \forall i$.

The results on the structure of the Lie algebras $L(M, f)$ that we have obtained in [1] can be summed up as follows.

Each Lie algebra $L(M, f)$ can be transformed to a canonical basis $\{X^\alpha, N_{ik}\}$, $\alpha = 1, \ldots, f$, $1 \le i < k \le M$ with commutation relations (3.2) and

$$[X^\alpha, N_{ik}] = \sum_{p<q} A^\alpha_{ik\,,pq} \, N_{pq} \tag{4.2}$$

$$[X^\alpha, X^\beta] = \sigma^{\alpha\beta} N_{1M} \tag{4.3}$$

$$1 \le \alpha, \beta \le f \qquad A^\alpha_{ik\,,pq}, \ \sigma^{\alpha\beta} \in K.$$

The commutation relations (4.2) can be rewritten as

$$[X^\alpha, N] = A^\alpha N$$
$$N \equiv (N_{12}\ N_{23} \ldots N_{(M-1)M}\ N_{13} \ldots N_{(M-2)M} \ldots N_{1M})^T \tag{4.4}$$
$$A^\alpha \in K^{r\times r} \qquad N \in K^{r\times 1}$$

where the superscript T indicates transposition. We mention that the vector N introduces an order in lines (columns) of the matrices A^α, where each line (column) is represented by two numbers. The matrices $A^\alpha = \{A^\alpha_{ik\,,pq}\}$ have the following canonical form.

(i) They are upper triangular.

(ii) The only off-diagonal matrix elements that do not vanish identically and cannot be annulled by a redefinition of the elements X^α are:

$$A^\alpha_{12\,,2M} \qquad A^\alpha_{j(j+1)\,,1M}\ (2 \le j \le M-2) \qquad A^\alpha_{(M-1)M\,,1(M-1)}. \tag{4.5}$$

(iii) The diagonal elements $a^\alpha_{i(i+1)}$, $1 \le i \le M-1$ are free. The other diagonal elements satisfy

$$a^\alpha_{ik} = \sum_{p=i}^{k-1} a^\alpha_{p(p+1)} \qquad k > i+1 \tag{4.6}$$

where we have introduced the compact notation $A^\alpha_{ik\,,ik} \equiv a^\alpha_{ik}$.

The canonical forms of the characteristic matrices A^α and the constants $\sigma^{\alpha\beta}$ satisfy the following conditions:

1. The set of matrices A^α have the form specified above and are linearly nilindependent. For $f \ge 2$ they all commute, i.e.

$$[A^\alpha, A^\beta] = 0. \tag{4.7}$$

2. All constants $\sigma^{\alpha\beta}$ vanish unless we have $a^\gamma_{1M} = 0$ for $\gamma = 1, \ldots, f$ simultaneously for all γ.

3. The remaining off-diagonal elements $A^\alpha_{ik\,,ab}$ also vanish, unless the diagonal elements satisfy $a^\beta_{ik} = a^\beta_{ab}$ for $\beta = 1, \ldots, f$ simultaneously for all β.

4. The maximal number of non-nilpotent elements is $f_{max} = M - 1$ and in this case the non-nilpotent elements always commute, i.e.

$$\left[X^\alpha, X^\beta\right] = 0. \tag{4.8}$$

Furthermore, the characteristic matrices A^α are explicitly given by the diagonal matrices

$$a^\alpha_{ik} = \sum_{p=i}^{k-1} \delta_{\alpha,p} \qquad 1 \le i < k \le M \qquad 1 \le \alpha \le M-1. \tag{4.9}$$

5. For $f = 1$ the matrix A has at most $M-2$ off-diagonal elements that can be normalized to $+1$ for $K = \mathbb{C}$ and to $+1$, or -1 for $K = \mathbb{R}$.

4.2 Differential operators and the system of equations

Using the preceding results, we can construct (as in Section 2) the differential operators realizing a basis for the coadjoint representation of the Lie algebras $L(M, f)$:

$$\begin{aligned}
\hat{N}_{ik} &= \sum_{b=k+1}^{M} n_{ib}\frac{\partial}{\partial n_{kb}} - \sum_{a=1}^{i-1} n_{ak}\frac{\partial}{\partial n_{ai}} - \sum_{\alpha=1}^{f}\left(a^\alpha_{ik} n_{ik} + \Gamma^\alpha_{ik}\right)\frac{\partial}{\partial x^\alpha} && (4.10)\\
\hat{X}^\alpha &= \sum_{i<k}\left(a^\alpha_{ik} n_{ik} + \Gamma^\alpha_{ik}\right)\frac{\partial}{\partial n_{ik}} + \sum_{\beta=1}^{f}\left(\sigma^{\alpha\beta} n_{1N}\right)\frac{\partial}{\partial x^\beta}. && (4.11)
\end{aligned}$$

We have introduced the notation

$$\begin{aligned}
\Gamma^\alpha_{12} &\equiv A^\alpha_{12,2M}\, n_{2M} \\
\Gamma^\alpha_{j(j+1)} &\equiv A^\alpha_{j(j+1),1M}\, n_{1M} \qquad j = 2, 3, \ldots, M-2 \\
\Gamma^\alpha_{(M-1)M} &\equiv A^\alpha_{(M-1)M,1(M-1)}\, n_{1(M-1)} \\
\Gamma^\alpha_{lm} &\equiv 0 \quad m - l \ge 2.
\end{aligned} \tag{4.12}$$

In the generic case the differential operators (4.11) will not contain the second summation since $\sigma^{\alpha\beta} = 0$ unless $a^\gamma_{1M} = 0$ for $\gamma = 1, \ldots, f$.

Equation (2.3) determining the invariants in our case amounts to the system of equations

$$\hat{N}_{ik} \cdot F(n_{12}, n_{23}, \ldots, n_{1M}, x^1, \ldots, x^f) = 0 \qquad 1 \leq i < k \leq M \quad (4.13)$$

$$\hat{X}^\alpha \cdot F(n_{12}, n_{23}, \ldots, n_{1M}, x^1, \ldots, x^f) = 0 \qquad \alpha = 1, \ldots, f. \quad (4.14)$$

It is useful to construct linear combinations of these operators that involve only x derivatives. These linear combinations are not elements of the Lie algebra $L(M, f)$, since they have variable coefficients. This is permitted since we are now treating equations (4.13) and (4.14) simply as a system of linear partial differential equations.

Let us associate a differential operator $\hat{Z}_\mu$ with each invariant Z_μ of the nilpotent Lie algebra $T(M)$ (see eq.(3.7)). For each Z_μ we take a sum of μ determinants of the form (3.7) and in each of them we replace one column of scalars by a column of operators $\hat{N}_{ik}$. For examples, we have

$$\hat{Z}_1 = \hat{N}_{1M} \qquad \hat{Z}_2 = \begin{vmatrix} \hat{N}_{1(M-1)} & n_{1M} \\ \hat{N}_{2(M-1)} & n_{2M} \end{vmatrix} + \begin{vmatrix} n_{1(M-1)} & \hat{N}_{1M} \\ n_{2(M-1)} & \hat{N}_{2M} \end{vmatrix} \quad (4.15)$$

and in general, we have the formula

$$\hat{Z}_\mu = \sum_{j=1}^{\mu} \begin{vmatrix} n_{1(M-\mu+1)} & n_{1(M-\mu+2)} & \cdots & \hat{N}_{1(M-\mu+j)} & \cdots & n_{1M} \\ n_{2(M-\mu+1)} & n_{2(M-\mu+2)} & \cdots & \hat{N}_{2(M-\mu+j)} & \cdots & n_{2M} \\ \vdots & \vdots & & \vdots & & \vdots \\ n_{\mu(M-\mu+1)} & n_{\mu(M-\mu+2)} & \cdots & \hat{N}_{\mu(M-\mu+j)} & \cdots & n_{\mu M} \end{vmatrix} \qquad 1 \leq \mu \leq \left[\frac{M}{2}\right] . \quad (4.16)$$

It is a straightforward calculation to prove that we have

$$\hat{Z}_\mu = \sum_{\alpha=1}^{f} f_\alpha(n_{ik}) \frac{\partial}{\partial x^\alpha} \quad (4.17)$$

i.e. that all the n_{ik} derivatives drop out. For example, when the stucture matrices A^α are diagonal we obtain the formula

$$\hat{Z}_j = -Z_j \left(\sum_{\alpha=1}^{f} \sum_{\mu=1}^{j} a^\alpha_{\mu(M-\mu+1)} \frac{\partial}{\partial x^\alpha} \right) . \quad (4.18)$$

Remark: For non-diagonal matrices A^α, this formula is generic for odd M. However, for even M, off-diagonal terms will appear.

We can construct $[M/2]$ such operators; at most f of them are linearly independent.

4.3 Examples: Invariants of $L(4, f)$

Let us now illustrate the procedure to obtain the functionally independent invariants for the solvable Lie algebras $L(4, f)$, $f = 1, 2$ or 3. For each algebra $L(4, f)$ we will state results concerning the form and the number of invariants. For each Lemma, the strategy that we will adopt to prove it is the following.

We will separate the proof in two parts:

(A) We find the invariants depending only on the variables n_{ab} , $1 \leq a < b \leq 4$.

(B) We find the invariants which are dependent on variables n_{ik} **and** x^α , $\alpha = 1, \ldots, f$.

In each of these cases, we will apply the differential operators $\hat{N}_{ik}$ and $\hat{X}^\alpha$ of the coadjoint representation of $L(4, f)$, on the functions $F = F(\{n_{ab}\}, \{x^\alpha\})$. However in the case (A), since we postulate that the functions F only depend on the variables n_{ab}, the differential operators $\hat{N}_{ik}$ will be the same as the operators of the nilpotent Lie algebra $T(4)$ (the x derivatives do not act on F). Therefore, by using the results of Theorem 1, we will only have to apply the differential operators X^α on functions of the type

$$F = F(Z_1, Z_2) \tag{4.19}$$

where $Z_1 = n_{14}$ and $Z_2 = n_{13}\, n_{24} - n_{23}\, n_{14}$.

In the case (B), we will begin by imposing

$$\hat{Z}_j \cdot F(n_{ab}\, ,\, x^\alpha) = 0 \qquad j = 1, 2 \tag{4.20}$$

such that the dependence on the x^α variables is preserved in F. Then we will apply all the differential operators (4.10) and (4.11) of the coadjoint representation of $L(4, f)$.

4.3.1 The Lie algebras $L(4,1)$

The characteristic matrix A of these Lie algebras $L(4,1)$ has the form [1]

$$A = \begin{pmatrix} a_{12} & 0 & 0 & 0 & \lambda_1 & 0 \\ & a_{23} & 0 & 0 & 0 & \lambda_2 \\ & & a_{34} & \lambda_3 & 0 & 0 \\ & & & a_{13} & 0 & 0 \\ & & & & a_{24} & 0 \\ & & & & & a_{14} \end{pmatrix} \tag{4.21}$$

where we have at most 2 non-zero off-diagonal elements λ_i and by eq.(4.6) a_{13}, a_{24} and a_{14} are determined in terms of a_{12}, a_{23} and a_{34}.

Lemma 1 *A solvable triangular Lie algebra of the type* $L(4,1)$ *has either* 3 *invariants, or* 1 *invariant.*

1) *Three invariants exist iff the conditions*

$$a_{14} = a_{23} = \lambda_2 = 0 \tag{4.22}$$

are satisfied. In this case the algebra can be characterized by $a_{12} = -a_{34} = 1$, $a_{23} = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$ *in characteristic matrix* (4.21). *A basis for the invariants is:*

$$I_1 = Z_1 \tag{4.23}$$
$$I_2 = Z_2 \tag{4.24}$$
$$I_3 = (n_{12}\, n_{24} + n_{13}\, n_{34}) + n_{14}\, x. \tag{4.25}$$

Otherwise there exists precisely one invariant. Two types of Lie algebras occur.

2) $(a_{12} + a_{34}, a_{23}) \not\equiv (0,0)$ *and* $\lambda_2 = 0$ *in matrix* (4.21). *The invariant is:*

$$I = \frac{(Z_2)^{a_{14}}}{(Z_1)^{a_{14}+a_{23}}}. \tag{4.26}$$

3) $a_{12} + a_{34} = 0$, $\lambda_2 = 1$, a_{23} *is a free parameter in matrix* (4.21) *and the invariant is:*

$$I = a_{23}\frac{Z_2}{(Z_1)^2} - \ln Z_1. \tag{4.27}$$

Proof.

(A) We impose that the differential operator $\hat{X}$ of eq.(4.11) should annihilates the functions of type $F = F(Z_1, Z_2)$, i.e.

$$\begin{aligned}
\hat{X} \cdot F &= \left[(a_{12}\, n_{12} + \lambda_1\, n_{24}) \frac{\partial}{\partial n_{12}} + (a_{23}\, n_{23} + \lambda_2\, n_{14}) \frac{\partial}{\partial n_{23}} \right. \\
&\quad + (a_{34}\, n_{34} + \lambda_3\, n_{13}) \frac{\partial}{\partial n_{34}} + a_{13}\, n_{13} \frac{\partial}{\partial n_{13}} + a_{24}\, n_{24} \frac{\partial}{\partial n_{24}} \\
&\quad \left. + a_{14}\, n_{14} \frac{\partial}{\partial n_{14}} \right] F \\
&= a_{14}\, Z_1 \frac{\partial F}{\partial Z_1} + \left[(a_{14} + a_{23})\, Z_2 - \lambda_2\, (Z_1)^2 \right] \frac{\partial F}{\partial Z_2} = 0.
\end{aligned} \tag{4.28}$$

We first note that if we have $a_{14} = a_{23} = \lambda_2 = 0$, i.e. conditions (4.22) which implies $a_{12} + a_{34} = 0$ from eq.(4.6), then both Z_1 and Z_2 are invariants. Also, the matrix A can, with no loss of generality [1], be diagonalized and set equal to

$$A = \mathrm{diag}(1 \quad 0 \quad -1 \quad 1 \quad -1 \quad 0). \tag{4.29}$$

In all other cases eq.(4.28) implies that just one invariant of this type exists. We obtain it using the method of characteristics.

Two cases arise:

(i) $\lambda_2 = 0$: The invariant is then given by (4.26), with $(a_{12} + a_{34}, a_{23}) \not\equiv (0, 0)$.

(ii) $\lambda_2 \not\equiv 0$: From our previous article [1], we know that in this case we can normalize λ_2 to 1 and we necessarily have $a_{23} = a_{14}$, which implies $a_{12} + a_{34} = 0$. Hence, we obtain the invariant (4.27), where a_{23} is a free parameter.

(B) In this case we impose $\hat{Z}_j \cdot F = 0$ ($j = 1, 2$) for functions of the type $F = F(n_{12}, n_{23}, n_{34}, n_{13}, n_{24}, n_{14}, x)$ and the differential operators $\hat{Z}_j$ are given by

$$\hat{Z}_1 \equiv \hat{N}_{14} = -a_{14}\, Z_1 \frac{\partial}{\partial x} \tag{4.30}$$

$$\begin{aligned}
\hat{Z}_2 &\equiv n_{13}\, \hat{N}_{24} - n_{23}\, \hat{N}_{14} + n_{24}\, \hat{N}_{13} - n_{14}\, \hat{N}_{23} \\
&= \left[-(a_{14} + a_{23})\, Z_2 + \lambda_2\, (Z_1)^2 \right] \frac{\partial}{\partial x}.
\end{aligned} \tag{4.31}$$

Hence the required dependence on x will survive only if we have $a_{14} = a_{23} = \lambda_2 = 0$. This coincides with eq.(4.22), the condition for Z_1 and Z_2 to be invariant. Furthermore, we can normalize a_{12} to 1 and cancel λ_1 and λ_3 by transformations [1].

We now apply all the differential operators of the coadjoint representation of $L(4,1)$ and the final result is that we obtain two invariants (4.23) and (4.24) independent of x and one invariant (4.25) depending on x. □

4.3.2 The Lie algebras $L(4,2)$

The Lie algebras $L(4,2)$ have the following characteristic matrices [1]:

$$A^1 = \begin{pmatrix} a_{12} & & & & & \\ & a_{23} & & & & \\ & & a_{34} & & & \\ & & & a_{13} & & \\ & & & & a_{24} & \\ & & & & & a_{14} \end{pmatrix} \quad A^2 = \begin{pmatrix} b_{12} & 0 & 0 & 0 & \lambda_1 & 0 \\ & b_{23} & 0 & 0 & 0 & \lambda_2 \\ & & b_{34} & \lambda_3 & 0 & 0 \\ & & & b_{13} & 0 & 0 \\ & & & & b_{24} & 0 \\ & & & & & b_{14} \end{pmatrix} \tag{4.32}$$

where we have at most one off-diagonal element in A^2 and a_{ik}, b_{ik} satisfy the eq.(4.6). Furthermore, the coefficient σ^{12} in eq.(4.3) is in the generic case zero (i.e. the two non-nilpotent elements commute). However, for the particular case $a_{14} = 0 = b_{14}$, we can have $\sigma^{12} \not\equiv 0$ in eq.(4.3).

Lemma 2 *A solvable triangular Lie algebra of the type $L(4,2)$ has either 2 invariants or none. Two invariants exist iff the conditions*

$$b_{23}\,(a_{12} + a_{34}) - a_{23}\,(b_{12} + b_{34}) = 0 \tag{4.33}$$
$$a_{14}\,\lambda_2 = 0 \tag{4.34}$$

are satisfied simultaneously. They lead to the following algebras and invariants.

1) $a_{12} = -a_{34} = b_{23} = \lambda_2 = 1$ *and* $a_{23} = b_{12} = b_{34} = \lambda_1 = \lambda_3 = \sigma^{12} = 0$ *in matrices* (4.32) *and a basis for the invariants is:*

$$I_1 = \frac{Z_2}{(Z_1)^2} + \ln Z_1 \tag{4.35}$$

$$I_2 = \frac{n_{12}\, n_{24} + n_{13}\, n_{34}}{n_{14}} + x^1. \tag{4.36}$$

2a) $a_{12} = -a_{34} = b_{23} = 1$, $a_{23} = b_{12} = \lambda_1 = \lambda_2 = \lambda_3 = \sigma^{12} = 0$ *and* b_{34} *a free parameter in matrices* (4.32),

2b) $a_{12} = b_{34} = 1$ *and* $a_{23} = a_{34} = b_{12} = b_{23} = \lambda_1 = \lambda_2 = \lambda_3 = \sigma^{12} = 0$ *in matrices* (4.32)

In both cases we have the invariants:

$$I_1 = \frac{(Z_2)^{a_{14}}}{(Z_1)^{a_{14}+a_{23}}} \tag{4.37}$$

$$I_2 = (a_{34}b_{13} - b_{34}a_{13})\left(\frac{n_{12}n_{24} + n_{13}n_{34}}{n_{14}}\right) + a_{14}x^2 - b_{14}x^1 \tag{4.38}$$

3) $a_{12} = -a_{34} = b_{23} = -b_{34} = 1$ *and* $a_{23} = b_{12} = \lambda_1 = \lambda_2 = \lambda_3 = 0$ *in matrices* (4.32) *and the invariants are:*

$$I_1 = Z_1 \tag{4.39}$$

$$I_2 = n_{12}\, n_{24} + n_{13}\, n_{34} + Z_1\, x^1 + \sigma^{12}\, (Z_1)^2 \ln Z_2\,. \tag{4.40}$$

Otherwise, there is no invariant.

Proof.
(A) We first apply differential operators $\hat{X}_1$ and $\hat{X}_2$ on functions of type $F = F(Z_1, Z_2)$. We obtain a system of two linear partial differential equations given by

$$\begin{pmatrix} \hat{X}^1 \cdot F \\ \hat{X}^2 \cdot F \end{pmatrix} = \begin{pmatrix} a_{14}\, Z_1 & (a_{14} + a_{23})\, Z_2 \\ b_{14}\, Z_1 & (b_{14} + b_{23})\, Z_2 - \lambda_2\, (Z_1)^2 \end{pmatrix} \begin{pmatrix} \dfrac{\partial F}{\partial Z_1} \\ \dfrac{\partial F}{\partial Z_2} \end{pmatrix} = 0. \tag{4.41}$$

The rank of the 2×2 matrix in eq.(4.41) cannot be zero, since then matrices A^1 and A^2 would not be linearly nilindependent. Also, if the rank is 2 there is no invariant that depends only on Z_1 and Z_2. However, solution exist if the rank of the matrix is 1 for all values of Z_1 and Z_2. This gives conditions (4.33) and (4.34).

Let us now assume that the condition (4.33) is respected. We consider the diagonal and the non-diagonal cases separately.

(i) $\lambda_2 = 0$: In this case, we obtain the invariant (4.26) for $(a_{12} + a_{34}, a_{23}) \not\equiv (0, 0)$.

(ii) $\lambda_2 \not\equiv 0$, $a_{14} = 0$: Since λ_2 is non-zero in A^2, we necessarily have $b_{23} = b_{14}$, i.e. $b_{12} + b_{34} = 0$ which gives the condition $a_{23}\, b_{23} = 0$ by (4.33). Two cases are possible under these condition.

One case gives the invariant (4.35) for $a_{12} = -a_{34} = b_{23} = \lambda_2 = 1$ and $a_{23} = b_{12} = b_{34} = \lambda_1 = \lambda_3 = 0$.

In the other case, we simply obtain the invariant $I = Z_1$ for the Lie algebra characterized by $a_{23} = b_{12} = -b_{34} = \lambda_2 = 1$, $b_{23} = \lambda_1 = \lambda_3 = 0$ and $a_{34} = -(a_{12} + 1)$ (with a_{12} a free parameter).

Remark. The case $a_{23} = 0 = b_{23}$ gives two *nildependent* matrices A^1, A^2 and is therefore not considered.

(B) In this case, we begin by applying the differential operators $\hat{Z}_1$, $\hat{Z}_2$ on functions of type $F = F(n_{12}, n_{23}, n_{34}, n_{13}, n_{24}, n_{14}, x^1, x^2)$, i.e.

$$\begin{pmatrix} \hat{Z}_1 \cdot F \\ \hat{Z}_2 \cdot F \end{pmatrix} = \begin{pmatrix} -a_{14}\, Z_1 & -b_{14}\, Z_1 \\ (a_{23} + a_{14})\, Z_2 & (b_{23} + b_{14})\, Z_2 - \lambda_2\, (Z_1)^2 \end{pmatrix} \begin{pmatrix} \dfrac{\partial F}{\partial x^1} \\ \dfrac{\partial F}{\partial x^2} \end{pmatrix} = 0\,. \tag{4.42}$$

The dependence on x^1 and x^2 can exist only if the determinant of the 2×2 matrix in (4.42) is zero. This again imposes the conditions (4.33) and (4.34).

Let us again assume that the condition (4.33) is satisfied. We separate the problem into three distinct cases.

(i) $(a_{14}, b_{14}) \not\equiv (0,0)$, $\lambda_2 \not\equiv 0$: The condition $\lambda_2 \not\equiv 0$ implies two consequences. First we have from (4.34) that $a_{14} = 0$ and therefore $b_{14} \not\equiv 0$. Second, we necessarily have $b_{23} = b_{14}$ which implies from (4.34) that $b_{23}\, a_{23} = 0$

In this case, the invariants are (4.35) and (4.36) and the Lie algebra $L(4,2)$ satisfies $a_{12} = -a_{34} = b_{23} = \lambda_2 = 1$, $a_{23} = b_{12} = b_{34} = \lambda_1 = \lambda_3 = \sigma^{12} = 0$.

(ii) $(a_{14}, b_{14}) \not\equiv (0,0)$, $\lambda_2 = 0$: In this case, two triangular solvable Lie algebras are associated with the invariants (4.37) and (4.38). One Lie algebra is characterized by the parameters $a_{12} = -a_{34} = b_{23} = 1$, $a_{23} = b_{12} = \lambda_1 = \lambda_2 = \lambda_3 = \sigma^{12} = 0$ and b_{34} a free parameter. The other Lie algebra is characterized by $a_{12} = b_{34} = 1$ and $a_{23} = a_{34} = b_{12} = b_{23} = \lambda_1 = \lambda_2 = \lambda_3 = \sigma^{12} = 0$.

(iii) $(a_{14}, b_{14}) = (0,0)$: In this case, we see that conditions (4.33) and (4.34) are automatically respected. Also, we can have a non-zero σ^{12} in eq.(4.11).

Since $a_{14} = 0 = b_{14}$, we can substitute a_{34} by $-(a_{12} + a_{23})$ and b_{34} by $-(b_{12}+b_{23})$ in the characteristic matrices (4.32). However, by imposing the commutativity (4.7) and the nilindependence of the matrices A^1 and A^2, we obtain $a_{12} = -a_{34} = b_{23} = -b_{34} = 1$ and $a_{23} = b_{12} = \lambda_1 = \lambda_2 = \lambda_3 = 0$. Hence, we obtain the two invariants (4.39) and (4.40). □

4.3.3 The Lie algebra $L(4,3)$

For the Lie algebra $L(4,3)$, we have diagonal characteristic matrices given by

$$\begin{aligned} A^1 &= \mathrm{diag}\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \; A^2 = \mathrm{diag}\begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \\ A^3 &= \mathrm{diag}\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}. \end{aligned} \tag{4.43}$$

Furthermore, the non-nilpotent elements commute, i.e. $\sigma^{\alpha\beta} = 0, \;\alpha, \beta = 1,2,3$ (see equations (4.3) and (4.8))

Lemma 3 *The triangular solvable Lie algebra $L(4,3)$ has precisely* 1 *invariant given by*

$$I = \frac{n_{12}\, n_{24} + n_{13}\, n_{34}}{n_{14}} + (x^1 - x^3)\,. \tag{4.44}$$

Proof.

(A) In this case, it is easy to demonstrate that after we have applied the differential operator $\hat{X}^1$ on functions of type $F = F(Z_1, Z_2)$, we obtain the quotient of Z_2 over Z_1. However, when we apply operator $\hat{X}^2$ on functions $\tilde{F} = \tilde{F}(I)$ with $I = Z_2/Z_1$, we obtain

$$\begin{aligned} 0 &= \hat{X}^2 \cdot \tilde{F} = \left(n_{23}\, \frac{\partial}{\partial n_{23}} + n_{13}\, \frac{\partial}{\partial n_{13}} + n_{24}\, \frac{\partial}{\partial n_{24}} + n_{14}\, \frac{\partial}{\partial n_{14}} \right) \tilde{F} \\ &= I \frac{\partial}{\partial I} \tilde{F}\,. \end{aligned} \tag{4.45}$$

Therefore, there is no invariant in this case.

(B) We first impose that the differential operators $\hat{Z}_1$ and $\hat{Z}_2$ annihilate the functions of type $F = F(n_{12}, n_{23}, n_{34}, n_{13}, n_{24}, n_{14}, x^1, x^2, x^3)$, where

$$\hat{Z}_1 = -Z_1 \left(\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3} \right) \tag{4.46}$$

$$\hat{Z}_2 = -Z_2 \left(\frac{\partial}{\partial x^1} + 2 \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3} \right). \tag{4.47}$$

Since the Lie algebra $L(4,3)$ has no parameters, these conditions are not on the parameters of the algebra (as before) but on the x dependence of the invariant. Hence, the new functions on which we will apply all the differential operators of the coadjoint representation of $L(4,3)$ are of the type $\tilde{F} = \tilde{F}(n_{12}, n_{23}, n_{34}, n_{13}, n_{24}, n_{14}, x^1 - x^3)$. We then obtain the invariant (4.44) by imposing that the operators of the coadjoint representation of $L(4,3)$ annihilate $\tilde{F}$. □

4.4 General results

Proposition 1 *The triangular solvable Lie algebra $L(M, M-1)$ has precisely $\left[\frac{M-1}{2}\right]$ functionally independent invariants. A basis is given by*

$$I_\mu = \frac{(-1)^{\mu+1}}{Z_\mu}\left(\sum_{\rho=1}^{M-2\mu} W_\rho^{(\mu)}\right) + (x^\mu - x^{M-\mu}) \tag{4.48}$$

for $\mu = 1, \ldots, \left[\frac{M-1}{2}\right]$. The function Z_μ is the determinant given by eq.(3.7) and $W_\rho^{(\mu)}$ is also a determinant function given by the determinant of the $(\mu+1)\times(\mu+1)$ matrix:

$$W_\rho^{(\mu)} = \begin{vmatrix} n_{1(\rho+\mu)} & n_{1(M-\mu+1)} & n_{1(M-\mu+2)} & \cdots & n_{1M} \\ n_{2(\rho+\mu)} & n_{2(M-\mu+1)} & n_{2(M-\mu+2)} & \cdots & n_{2M} \\ \vdots & \vdots & \vdots & & \vdots \\ n_{\mu(\rho+\mu)} & n_{\mu(M-\mu+1)} & n_{\mu(M-\mu+2)} & \cdots & n_{\mu M} \\ 0 & n_{(\rho+\mu)(M-\mu+1)} & n_{(\rho+\mu)(M-\mu+2)} & \cdots & n_{(\rho+\mu)M} \end{vmatrix}. \tag{4.49}$$

Proposition 2 *A diagonal solvable Lie algebra of the type $L(M,1)$ has $\left[\frac{M}{2}\right] \pm 1$ functionally independent invariants.*

1) $\left[\frac{M}{2}\right]+1$ *invariants exist iff the conditions*

$$a_{i(i+1)} + a_{(M-i)(M-i+1)} = 0 \qquad i = 1, \ldots, \left[\frac{M}{2}\right] \tag{4.50}$$

are satisfied. A basis is given by $[M/2]$ invariants independent of x and one invariant depending on x:

$$I_\mu = Z_\mu \qquad \mu = 1, \ldots, \left[\frac{M}{2}\right] \tag{4.51}$$

$$I_{\left[\frac{M}{2}\right]+1} = \sum_{\mu=1}^{[(M-1)/2]} \sum_{\rho=1}^{M-2\mu} \frac{(-1)^{\mu+1}}{Z_\mu} a_{\mu(\mu+1)} W_\rho^{(\mu)} + x \tag{4.52}$$

where the function Z_μ and $W_\rho^{(\mu)}$ are determinant functions given by the equations (3.7) and (4.49), respectively.

2) *Otherwise there exist precisely* $\left[\frac{M}{2}\right] - 1$ *invariants, all independent of* x. *A basis is given by*

$$I_\mu = \frac{(Z_{\mu+1})^\alpha}{(Z_1)^\beta} \qquad \mu = 1, \ldots, \left[\frac{M}{2}\right] - 1 \tag{4.53}$$

with

$$\frac{\alpha}{\beta} = \frac{a_{1M}}{\sum\limits_{k=1}^{\mu+1} a_{k(M+1-k)}} \tag{4.54}$$

where the function Z_μ *is the determinant function given by eq.*(3.7).

By *diagonal* solvable Lie algebra of the type $L(M, 1)$ in Proposition 2, we mean that the characteristic matrix A of (4.2) is diagonal.

Propositions 1 and 2 each contains two types of information on the invariants: They give the form of the invariant functions and the number of functionally independent invariants. It is an easy calculation to prove that the functions I_μ of Proposition 1 and Proposition 2 are annihilated by the coadjoint representation (4.10), (4.11) of the Lie algebras $L(M, M-1)$ and $L(M, 1)$, respectively. However, it is much more difficult to establish the number of functionally independent invariants for Proposition 1 and Proposition 2. The difficulty is to prove that no further invariants exists. One way of doing that is to calculate the rank of the antisymmetric matrix $S = S(L(M, M-1))$ and $S = S(L(M, 1))$ of the commutation relations for the corresponding Lie algebra. The number of invariants is then given by the difference between the dimension of the solvable Lie algebra and the rank of the matrix S (see eq.(2.4)).

For the Lie algebra $L(M, M-1)$ of dimension $\frac{1}{2}(M-1)(M+2)$, S is the antisymmetric matrix given by the elements

$$\begin{gathered} S = \{[N_{ik}, N_{ab}] \quad [N_{ik}, X^\alpha]\} \\ 1 \leq i < k \leq M \qquad 1 \leq a < b \leq M \qquad \alpha = 1, \ldots, M-1 \end{gathered} \tag{4.55}$$

and for the Lie algebra $L(M, 1)$ of dimension $\frac{1}{2}(M^2 - M + 2)$, the matrix S is given by the elements

$$S = \{[N_{ik}, N_{ab}] \quad [N_{ik}, X]\}$$
$$1 \le i < k \le M \qquad 1 \le a < b \le M. \tag{4.56}$$

For example, the antisymmetric matrix S of the 7-dimensional Lie algebra $L(4,1)$ is given by

$$S = \begin{pmatrix} 0 & N_{13} & 0 & 0 & N_{14} & 0 & -a_{12}N_{12} \\ -N_{13} & 0 & N_{24} & 0 & 0 & 0 & -a_{23}N_{23} \\ 0 & -N_{24} & 0 & -N_{14} & 0 & 0 & -a_{34}N_{34} \\ 0 & 0 & N_{14} & 0 & 0 & 0 & -a_{13}N_{13} \\ -N_{14} & 0 & 0 & 0 & 0 & 0 & -a_{24}N_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_{14}N_{14} \\ a_{12}N_{12} & a_{23}N_{23} & a_{34}N_{34} & a_{13}N_{13} & a_{24}N_{24} & a_{14}N_{14} & 0 \end{pmatrix} \tag{4.57}$$

where the parameters a_{13}, a_{24} and a_{14} are given in terms of a_{12}, a_{23} and a_{34} by the relation (4.6). Hence, it is easy to calculate that

$$rank(S) = \begin{cases} 4 & for\ a_{14} = a_{23} = 0 \\ 6 & otherwise \end{cases} \tag{4.58}$$

giving, respectively, three and one invariants (in accordance with Proposition 2 and Lemma 1).

We have calculated the ranks of the matrices $S(L(M, M-1))$ and $S(L(M,1))$ for $M \le 13$ and $M \le 8$, respectively, using the symbolic package MAPLE. We conjecture that Proposition 1 and 2 hold for all M.

5 Conclusions

The problem of finding all invariants of the coadjoint representation of the triangular nilpotent algebras $T(M)$ is solved completely by Theorem 1. A basis for the invariants consists of polynomials and provides Casimir operators in the enveloping algebra of $T(M)$.

The situation with the solvable triangular Lie algebras $L(M,f)$ is more complicated. We have provided guidelines for calculating the invariants for all values of M, but presented comprehensive results only for $M = 4$. We have also presented conjectures concerning the invariants of $L(M, M-1)$ and

$L(M,1)$ for all values of M (and verified them for a large range of values of M).

The results for $M=4$ show that all invariants are polynomial only in special cases. In general, rational, irrational and logarithmic type invariants must be allowed in any basis of invariants.

Acknowledgments

The research of P.W. was supported in part by research grants from NSERC of Canada and FCAR du Québec.

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