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Symétries et intégrabilité des équations aux différences finies

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SOMMAIRE

La présente thèse porte sur l'étude des symétries et des propriétés d'intégrabilité des équations aux différences finies.

Dans le chapitre 1, le groupe de symétrie ponctuelle d'un système couplé à deux équations différentielles aux différences est étudié. On montre que dans certains cas, la dimension du groupe peut être infinie. Les équations peuvent décrire l'interaction de deux longues chaînes moléculaires, chacune étant composée d'atomes d'un même type.

Dans le chapitre 2, une classe de théories de champs avec interaction exponentielle est introduite. L'interaction dépend de deux matrices de “couplage” et est suffisamment générale pour inclure toutes les théories de champs de Toda existant dans la littérature. Les symétries de Lie ponctuelles sont obtenues pour les cas où l'on a un nombre fini, infini ou semi-infini de champs. Une attention spéciale est accordée à la présence de l'invariance conforme.

Dans le chapitre 3, nous procédons à la classification et à l'étude d'équations linéarisables. Nous examinons tout d'abord l'équation de Gambier continue qui contient, comme réductions, toutes les équations de deuxième ordre intégrables par linéarisation. Nous introduisons par la suite la forme discrète de cette équation et obtenons les conditions d'intégrabilité à l'aide du confinement des singularités. Nous étudions aussi les différentes réductions du cas discret. De plus, nous obtenons des transformations de Schlesinger pour les équations de Gambier discrète et continue. Dans la dernière partie du chapitre, nous étudions une famille

d'équations discrètes du deuxième ordre incluant des équations résolubles par linéarisation. Plusieurs cas intégrables sont obtenus. Dans le cas discret, l'étude de l'intégrabilité est faite à l'aide du confinement des singularités.

Dans le chapitre 4, nous étudions un autre critère d'intégrabilité: l'entropie algébrique. Nous montrons que les résultats obtenus avec ce critère pour les équations linéarisables sont les mêmes que ceux obtenus avec le confinement des singularités. Nous obtenons de plus une méthode algorithmique pour la détection de la linéarisabilité.

Le chapitre 5 est consacré à l'étude d'équations du troisième ordre. Nous obtenons des équations intégrables par des couplages d'équations du premier et du deuxième ordre. Les équations continues sont étudiées à l'aide de l'analyse de Painlevé et le confinement des singularités est utilisé dans le cas discret.

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INTRODUCTION

Pour un système Hamiltonien de dimension finie n, la notion d'intégrabilité a une définition précise depuis le dix-neuvième siècle. En effet, au sens de Liouville, un tel système est dit intégrable s'il possède n constantes du mouvement en involution et fonctionnellement indépendantes. Pour les systèmes de dimension infinie (correspondant à des équations aux dérivées partielles (EDP)), la notion d'intégrabilité a été introduite et étudiée durant les 30 dernières années principalement par Kruskal qui en a été l'instigateur et par beaucoup d'autres qui ont suivi.

C'est en 1844 que J. Scott Russell rapporte la première observation de ce que l'on appelle aujourd'hui un soliton [**1**]. Le phénomène qu'il observe dans le canal d'Edimbourg-Glasgow est caractérisé par le fait qu'il s'agit d'une onde solitaire possédant une grande stabilité. Il lui donne le nom de "grande onde de translation" ("great wave of translation"). Par la suite, Russell consacrecra la majeure partie de sa vie professionnelle à l'étude expérimentale des propriétés des solitons.

Ce n'est cependant qu'en 1895 que Korteweg et de Vries découvrent l'équation qui porte désormais leur nom [**2**]. On sait aujourd'hui que l'équation de Korteweg-de Vries (KdV) possède une infinité de solutions solitoniques et elle est considérée aujourd'hui comme le prototype d'une EDP intégrable. Elle possède toutes les propriétés que l'on attribue généralement à l'intégrabilité pour les systèmes de dimension infinie.

En 1955, Fermi, Pasta et Ulam travaillaient à Los Alamos sur un modèle numérique de phonons anharmoniques [3]. Il se trouve que ce système est étroitement lié à une approximation discrète de l'équation de KdV. Ils observèrent avec surprise qu'il n'y avait pas équipartition de l'énergie entre les différents modes.

Voyant les résultats des travaux de Fermi, Pasta et Ulam, Zabusky et Kruskal décidèrent, en 1965, de considérer le problème suivant

$$u_t + uu_x + \delta^2 u_{xxx} = 0 \tag{1}$$

avec une condition frontière périodique

$$u(x,0) = \cos \pi x, \quad 0 \leq x \leq 2$$

et u, u_x, u_{xx} périodiques sur $[0,2]$ pour tout t [4]. L'équation (1) est l'équation de KdV. Leurs études numériques de ce problème les amenèrent à la découverte des solitons qui sont la conséquence d'un équilibre fragile entre la dispersion et la non linéarité. Ils découvrirent aussi que deux de ces ondes solitaires préservent leur forme à travers une interaction non linéaire. Ce sont eux qui inventèrent le terme de soliton.

Cette découverte a mené, dans les 30 dernières années, à une intense étude des systèmes de dimension infinie possédant des propriétés analogues à celles de l'équation de KdV et que l'on désigne par le nom de systèmes intégrables. Cependant, le terme d'intégrabilité pour les EDP doit cependant encore aujourd'hui être précisé. Chacun donne sa définition ou apporte ses nuances à cette notion. Par ce terme, on veut généralement désigner, parmi les systèmes aux dérivées partielles, ceux qui possèdent une "classe riche" de solutions "suffisamment" globales. On admet habituellement qu'une EDP non linéaire est intégrable si elle possède des solutions à n solitons pour n'importe quel n. Les équations linéaires ou linéarisables (i.e. qui peuvent être amenées à une équation linéaire par une transformation) sont considérées comme intégrables *a priori*.

La définition précise d'un soliton implique les valeurs propres discrètes d'un problème de diffusion. Cependant, dans le cas où l'on a une seule variable d'espace et une variable de temps, on peut donner une définition plus simple et plus intuitive d'un soliton: il s'agit d'une solution d'une équation (ou d'un système) aux dérivées partielles qui (i) représente une onde de forme permanente dans le temps; (ii) est localisée, i.e. qui décroît vers zéro ou approche une constante à l'infini; (iii) peut interagir avec d'autres solitons et garder son identité.

Dans le cas des équations différentielles ordinaires, on considère comme intégrables les équations dont la solution générale s'écrit en terme d'un nombre fini de fonctions "acceptables". Ici encore, le terme d'intégrabilité est flou puisque l'ensemble des fonctions acceptables n'est pas bien défini. On considère cependant de façon générale qu'une équation possédant la propriété de Painlevé est intégrable. On dit qu'une équation possède la propriété de Painlevé si les singularités "mobiles" de sa solution générale dans le plan complexe ne sont pas des points de branchement [**5**]. Une singularité est dite mobile si sa position dans le plan complexe dépend des conditions initiales choisies.

Par exemple, considérons

$$\dot{y} + y^2 = 0 \tag{2}$$

dont la solution est

$$y = \frac{1}{t - t_0}.$$

Donc, l'équation (2) possède la propriété de Painlevé puisque la seule singularité mobile de sa solution générale n'est qu'un pôle d'ordre 1 à $t = t_0$. Prenons maintenant l'équation

$$\dot{y} + \frac{1}{3}y^4 = 0 \tag{3}$$

dont la solution générale est donnée par:

$$y = (t - t_0)^{-1/3}$$

qui a un point de branchement mobile à $t = t_0$. Donc, l'équation (3) n'a pas la propriété de Painlevé. Mais notons ici que par la transformation $z = y^3$, l'équation (3) devient l'équation (2) qui possède la propriété de Painlevé.

L'analyse de Painlevé consiste en l'application d'un algorithme appelé test de Painlevé [**6**]. Ce test nous permet de trouver la solution générale de l'équation donnée sous forme de série formelle de Laurent si elle peut s'exprimer ainsi. Ce test ne nous donne cependant qu'une condition nécessaire à la présence de la propriété de Painlevé puisque, par exemple, elle ne permet pas de détecter la présence d'une singularité essentielle multiforme. En pratique cependant, il y a très peu d'équation qui passent le test sans avoir la propriété de Painlevé.

Un sujet intimement lié à l'intégrabilité est l'étude des symétries des équations différentielles. Par exemple, il a été remarqué [**7**] que la présence d'une algèbre de symétrie possédant une structure de Kac-Moody-Virasoro est typique pour les EDP intégrables à trois variables (une de temps et deux d'espace).

L'origine de l'étude des symétries des systèmes d'équations différentielles nous fait retourner à la fin du dernier siècle. C'est en effet à cette époque que Sofus Lie introduit la notion de groupe continu (connue aujourd'hui sous le nom de groupe de Lie) et étudie ses applications aux équations différentielles [**8**].

Définissons brièvement ce qu'on entend par groupe de symétrie d'une EDP. Soit une EDP générale de la forme suivante

$$\begin{gathered} E(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(n)}) = 0, \\ x \in \mathbb{R}^p,\ u \in \mathbb{R},\ \ p, n \in \mathbb{N}, \end{gathered} \tag{4}$$

où $u^{(k)}$ dénote toutes les dérivées partielles d'ordre k de u. Il s'agit simplement d'une équation aux dérivées partielles pour la fonction u de x. Un groupe de symétrie est un groupe continu agissant sur l'espace formé de la variable dépendante u et la variable indépendante x et qui transforme une solution de l'équation (4) en une autre solution. Il laisse donc invariant l'ensemble des solutions de l'équation (4).

La théorie des groupes de Lie, utilisée avec l'analyse de Painlevé, nous donne une méthode très efficace pour trouver les solutions exactes d'EDP **[9, 10]**. En effet, le groupe de symétrie d'une EDP nous permet de la ramener, par des réductions par symétrie, à des équations différentielles ordinaires. L'analyse de Painlevé nous permet ensuite de détecter, parmi ces dernières, des candidats à l'intégrabilité. La dernière étape consiste en l'intégration des équations ainsi sélectionnées. De plus, le groupe de symétrie nous permet de générer de nouvelles solutions à partir de celles déjà connues. Les symétries peuvent aussi être utilisées pour établir des isomorphismes entre différentes équations.

Tout comme les EDP, les équations aux différences finies (EDF) sont très importantes en physique. Elles apparaissent naturellement lorsque des systèmes physiques discrets sont étudiés. Par exemple, dans les réseaux de spins en mécanique statistique, les réseaux cristallins, les chaînes moléculaires, etc... Les EDF sont aussi utiles pour l'étude numérique de phénomènes continus. Ainsi, ces dernières années, beaucoup de travaux ont porté sur l'étude des symétries et de l'intégrabilité des équations aux différences finies **[11, 12, 13]**.

Pour ce qui est de l'intégrabilité, plusieurs travaux proposent des extensions de l'analyse de Painlevé aux EDF. Nous allons nous intéresser ici au "confinement des singularités" **[14, 15]** qui est un critère d'intégrabilité qui a l'avantage d'être à la fois simple et efficace. En effet, depuis la découverte du confinement des singularités en 1991, plusieurs équivalents discrets d'équations intégrables importantes ont été trouvées grâce à ce critère (voir par exemple [**18**]).

On dit qu'une équation possède cette propriété si les singularités de ses solutions ne se propagent pas indéfiniment. Illustrons ceci par deux exemples. Considérons l'équation

$$x_{n+1} + x_n + x_{n-1} = a + \frac{b}{x_n},$$

où a et b sont des constantes non nulles. On voit ici que la seule singularité possible se produit si $x_n = 0$ pour un certain n. Ensuite, x_{n+1} est infini et x_{n+2} est de la forme indéterminée $\infty - \infty$. La méthode utilisée pour lever cette indétermination est la suivante. On introduit une perturbation autour de 0: $x_n = \epsilon$ (x_{n-1} est considéré non-nul). On calcule les valeurs subséquentes de x et on regarde leur comportement lorsque ϵ tend vers 0. Ainsi, pour x_{n+1}, on a

$$x_{n+1} = \frac{b}{\epsilon} + a - x_{n-1} + \mathcal{O}(\epsilon),$$

et donc $x_{n+1} \to \infty$ lorsque $\epsilon \to 0$. C'est notre singularité. Par la suite, on a

$$x_{n+2} = -\frac{b}{\epsilon} + x_{n-1} + \mathcal{O}(\epsilon),$$
$$x_{n+3} = -\epsilon + \mathcal{O}(\epsilon^2).$$

On trouve ensuite que $x_{n+4} \to x_{n-1}$ lorsque $\epsilon \to 0$. Donc, la singularité a disparu et elle ne se propage pas. Notre critère d'intégrabilité est respecté. Voyons maintenant un exemple où la singularité n'est pas confinée. Considérons l'équation

$$x_{n+1}x_{n-1} = \frac{(1 - ax_n)^3}{x_n(x_n - a)^3}.$$

Si $x_0 = a + \epsilon$, on trouve ensuite $x_1 \sim 1/\epsilon^3$, $x_2 \sim \epsilon^3$ et x_3, quant à elle, possède la valeur finie $1/a^5$. Plus loin cependant, on trouve $x_4 \sim 1/\epsilon^3$, $x_5 \sim \epsilon^3$ et x_6 a encore une valeur finie, $1/a^{11}$. La séquence (∞, 0, finie) se répète indéfiniment et donc la singularité n'est pas confinée.

Tout récemment, un nouveau critère d'intégrabilité a été introduit: l'entropie algébrique **[21, 22]**. Ce critère est basé sur les idées d'Arnold et de Veselov sur la complexité d'une application **[23, 24]**. Plusieurs exemples présentant un comportement chaotique sont maintenant connus où l'entropie algébrique détecte la non intégrabilité mais où le critère de confinement des singularités échoue **[21]**. En effet, le test du confinement est satisfait malgré le comportement chaotique. Ceci démontre que le confinement ne peut pas être considéré comme un critère suffisant à l'intégrabilité. De plus, puisque l'entropie algébrique est plus précise, toute étude de l'intégrabilité devra désormais être confirmée par une analyse de l'entropie.

Pour ce qui est des symétries, plusieurs approches, dont le but est d'appliquer la théorie des groupes aux équations aux différences finies, ont fait leur apparition dans les dernières années **[16, 17, 19, 20]**. Entre autres, en 1991, Winternitz et Levi ont introduit la méthode dite approche "équation différentielle" **[16]**. Cette méthode a deux grands avantages qui la distinguent d'autres méthodes: elle s'applique de façon complètement algorithmique et, de plus, on peut l'utiliser dans tous les cas d'équations linéaires ou non. Son désavantage est qu'elle ne concerne que les transformations laissant la variable discrète invariante.

Ma thèse se place dans le contexte d'une classification des systèmes intégrables discrets à une variable de deuxième ordre. Pour ce qui est des symétries, une étude systématique du lien entre le groupe de symétrie et l'intégrabilité n'a toujours pas été faite. De plus, il n'est pas encore possible d'utiliser les symétries pour l'étude des EDF de façon aussi efficace que pour les EDP.

La présente thèse est divisée en deux grandes parties. Premièrement, nous étudions et classifions des équations discrètes à l'aide de la théorie des groupes. Le but d'une telle classification est de pouvoir éventuellement établir un lien entre les symétries et la propriété d'intégrabilité. Deuxièment, nous classifions certaines familles d'équations intégrables discrètes.

Ces deux grandes parties sont incluses dans le programme de recherche dont le but est de pouvoir utiliser la théorie des groupes de Lie et l'analyse de l'intégrabilité pour les EDF de façon aussi efficace que pour les EDP. Ainsi, le chapitre 1 est consacré à la classification par symétries d'un système différentiel et aux différences. Ce système se retrouve dans les domaines de la biophysique, la physique moléculaire et la mécanique classique **[25]**. Dans le chapitre 2, nous étudions des généralisations de l'équation de Toda sous le point de vue des groupes de Lie **[26]**. Dans le chapitre 3, nous procédons à la classification et à l'étude d'équations discrètes intégrables par linéarisation **[27, 28, 29]**. Dans le chapitre 4, nous étudions les mêmes équations à l'aide de l'entropie algébrique **[30]**. À l'aide de couplages d'équations intégrables du premier et du deuxième ordre, nous obtenons une classe d'équations intégrables du troisième ordre dans le chapitre 5 **[31]**. Finalement, nous discutons des résultats obtenus dans l'ensemble de la thèse et tirons des conclusions générales.

Chapitre 1

CLASSIFICATION PAR SYMÉTRIES D'UN SYSTÈME DYNAMIQUE DISCRET IMPLIQUANT DEUX ESPÈCES

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Symmetries of discrete dynamical systems involving two species

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The Lie point symmetries of a coupled system of two nonlinear differential-difference equations are investigated. It is shown that in special cases the symmetry group can be infinite dimensional, in other cases up to ten dimensional. The equations can describe the interaction of two long molecular chains, each involving one type of atoms. © *1999 American Institute of Physics.* [S0022-2488(99)03206-5]

I. INTRODUCTION

Our purpose in this article is to perform a symmetry analysis of a system of two coupled differential-difference equations of the form

$$
\begin{aligned}
E_1 &= \ddot{u}_n - F_n(t, u_{n-1}, u_n, u_{n+1}, v_{n-1}, v_n, v_{n+1}) = 0, \\
E_2 &= \ddot{v}_n - G_n(t, u_{n-1}, u_n, u_{n+1}, v_{n-1}, v_n, v_{n+1}) = 0.
\end{aligned}
\tag{1.1}
$$

The overdots denote time derivatives. The discrete variable n plays the role of a space variable; it labels positions along a one-dimensional lattice. The functions F_n and G_n represent interactions, e.g., between different atoms along a double chain of molecules (see Fig. 1). The functions F_n and G_n are *a priori* unspecified; our aim is to classify equations of the type (1.1) according to the Lie point symmetries that they allow. The interactions in such a model depend on up to six neighboring particles. For instance, we can interpret u_n and v_n as deviations from equilibrium positions of two different types of atoms, say type U and type V. The accelerations $\ddot{u}_n$ and $\ddot{v}_n$ depend on the deviations u and v of both types of atoms at the neighboring sites $n-1$, n, and $n+1$. We do not restrict to two-body forces, nor do we impose translational invariance for the chain. We do, however, assume there is no dissipation, i.e., system (1.1) does not involve first derivatives with respect to time.

Such differential-difference equations typically arise when modeling phenomena in molecular physics, biophysics, or simply coupled oscillations in classical mechanics.[1–3]

A recent article[4] was devoted to a similar problem, but was concerned with a single species, i.e., one dependent variable $u_n(t)$. The approach adopted here is similar to that of Ref. 4. Thus, we shall consider only symmetries acting on the continuous variables t, u_n, and v_n. Transformations of the discrete variable n must then be studied separately.

Several different treatments of Lie symmetries of difference and differential-difference equations exist in the literature.[4–13] The one adopted in this article is that of Refs. 4–6. It has been

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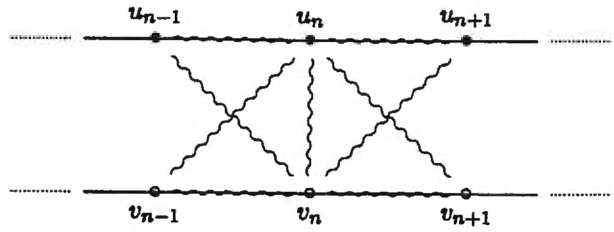

FIG. 1. Double molecular chain with two types of atoms.

called the "intrinsic method," makes use of a Lie algebraic approach, and is entirely algorithmic. The Lie algebra of the symmetry group, the "symmetry algebra" for short, is realized by vector fields of the form

$$\hat{X}=\tau(t,u_n,v_n)\partial_t+\phi_n(t,u_n,v_n)\partial_{u_n}+\psi_n(t,u_n,v_n)\partial_{v_n}. \tag{1.2}$$

The algorithm for finding the functions τ, ϕ_n, and ψ_n in (1.2) is to construct the appropriate prolongation $\mathrm{pr}\,\hat{X}$ of $\hat{X}$ (see Refs. 4–6 and Sec. II) and to impose that it should annihilate the studied system of equations on their solution set,

$$\mathrm{pr}\,\hat{X}E_1|_{E_1=E_2=0}=0, \quad \mathrm{pr}\,\hat{X}E_2|_{E_1=E_2=0}=0. \tag{1.3}$$

Our first step is to find and classify all interactions (F_n,G_n) for which the system (1.1) allows at least a one-dimensional symmetry algebra. The next step is to specify the interactions further and to find all those that allow a higher-dimensional, possibly infinite-dimensional, symmetry algebra.

As in previous articles,[4,14] our classification will be up to conjugacy under a group of "allowed transformations." These are fiber preserving locally invertible point transformations,

$$u_n=\Omega_n(\tilde{u}_n,\tilde{v}_n,\tilde{t}), \quad v_n=\Gamma_n(\tilde{u}_n,\tilde{v}_n,\tilde{t}), \quad t=t(\tilde{t}), \tag{1.4}$$

which preserve the form of Eqs. (1.1), but not necessarily the functions F_n and G_n (they go into new functions $\tilde{F}_n$ and $\tilde{G}_n$ of the new arguments).

Throughout the article we assume that both F_n and G_n depend on at least one of the quantities $u_{n-1},u_{n+1},v_{n-1},v_{n+1}$, so that nearest neighbors are genuinely involved. In the bulk of the article the interaction is assumed to be nonlinear.

In Sec. II we formulate the problem, establish the general form of the elements of the symmetry algebra, and present the determining equations for the symmetries. We also derive the "allowed transformations" under which we classify the interactions and their symmetries. Section III is devoted to a classification of interactions F_n, G_n, allowing at least a one-dimensional symmetry algebra. Ten classes of such interactions exist, each involving two arbitrary functions of six variables. In Sec. IV we study higher-dimensional symmetry algebras and introduce an important restriction. We first prove that four equivalence classes of symmetry algebras isomorphic to $\mathrm{sl}(2,\mathbb{R})$ exist. Then we restrict to just one of them, $\mathrm{sl}(2,\mathbb{R})_1$ generating a gauge group acting only on the fields u_n and v_n (in a global, coordinate-independent manner). We describe all symmetry algebras, containing the chosen $\mathrm{sl}(2,\mathbb{R})$ as a subalgebra. In Sec. V we obtain the invariant interactions for all algebras containing $\mathrm{sl}(2,\mathbb{R})_1$. The results are summed up and discussed in Sec. VI, where we also outline future work to be done.

II. FORMULATION OF THE PROBLEM

To find the Lie point symmetries of the system (1.1), we write the second prolongation of the vector field (1.2) in the form[4–6]

$$\mathrm{pr}^{(2)}\hat{X}=\tau(t,u_n,v_n)\partial_t+\sum_{k=n-1}^{n+1}\phi_k(t,u_n,v_n)\partial_{u_k}+\sum_{k=n-1}^{n+1}\psi_k(t,u_n,v_n)\partial_{v_k}+\phi_n^{tt}\partial_{\ddot{u}_n}+\psi_n^{tt}\partial_{\ddot{v}_n}, \tag{2.1}$$

with

$$\begin{aligned}\phi_n^{tt}&=D_t^2\phi_n-(D_t^2\tau)\dot{u}_n-2(D_t\tau)\ddot{u}_n,\\ \psi_n^{tt}&=D_t^2\psi_n-(D_t^2\tau)\dot{v}_n-2(D_t\tau)\ddot{v}_n,\end{aligned} \tag{2.2}$$

where D_t is the total time derivative. The determining equations for the symmetries are obtained by requiring that Eq. (1.3) be satisfied. The obtained equations will involve terms like $\dot{u}^k$, $\dot{v}^k$, and $\dot{u}^k\dot{v}^l$. The coefficients of each linearly independent term must vanish and this provides 16 linear differential equations that are easy to solve and do not involve the interaction functions F_n, G_n. The result is that an element $\hat{X}$ of the symmetry algebra must have the form

$$\hat{X}=\tau(t)\partial_t+\left[\left(\frac{\dot{\tau}}{2}+a_n\right)u_n+b_nv_n+\lambda_n(t)\right]\partial_{u_n}+\left[c_nu_n+\left(\frac{\dot{\tau}}{2}+d_n\right)v_n+\mu_n(t)\right]\partial_{v_n}, \tag{2.3}$$

where the overdots denote time derivatives. The functions $\tau(t)$, $\lambda_n(t)$, $\mu_n(t)$, a_n, b_n, c_n, and d_n satisfy the two remaining determining equations, namely,

$$\begin{aligned}&\frac{\dddot{\tau}}{2}u_n+\ddot{\lambda}_n+\left(a_n-\frac{3}{2}\dot{\tau}\right)F_n+b_nG_n-\tau F_{n,t}-\sum_{k=n-1}^{n+1}F_{n,u_k}\left[\left(\frac{\dot{\tau}}{2}+a_k\right)u_k+b_kv_k+\lambda_k(t)\right]\\&\quad-\sum_{k=n-1}^{n+1}F_{n,v_k}\left[\left(\frac{\dot{\tau}}{2}+d_k\right)v_k+c_ku_k+\mu_k(t)\right]=0,\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\frac{\dddot{\tau}}{2}v_n+\ddot{\mu}_n+\left(d_n-\frac{3}{2}\dot{\tau}\right)G_n+c_nF_n-\tau G_{n,t}-\sum_{k=n-1}^{n+1}G_{n,u_k}\left[\left(\frac{\dot{\tau}}{2}+a_k\right)u_k+b_kv_k+\lambda_k(t)\right]\\&\quad-\sum_{k=n-1}^{n+1}G_{n,v_k}\left[\left(\frac{\dot{\tau}}{2}+d_k\right)v_k+c_ku_k+\mu_k(t)\right]=0.\end{aligned} \tag{2.5}$$

In Eqs. (2.3), (2.4), and (2.5) the quantities a_n, b_n, c_n, and d_n are independent of t. To proceed further, one could specify the interaction functions F_n and G_n. Instead, we shall assume that at least one symmetry generator (2.3) exists and make use of allowed transformations to simplify this vector. The second step is to find interactions F_n and G_n compatible with such a symmetry.

Substituting (1.4) into Eq. (1.1) and requiring that the form of these two equations be preserved, we find that the allowed transformations are quite restricted, namely,

$$\begin{pmatrix}u_n(t)\\v_n(t)\end{pmatrix}=\begin{pmatrix}Q_n & R_n\\S_n & T_n\end{pmatrix}\dot{\tilde{t}}^{-1/2}\begin{pmatrix}\tilde{u}_n(\tilde{t})\\\tilde{v}_n(\tilde{t})\end{pmatrix}+\begin{pmatrix}\alpha_n(t)\\\beta_n(t)\end{pmatrix},\quad \tilde{t}=\tilde{t}(t),\quad \frac{d\tilde{t}}{dt}\neq 0. \tag{2.6}$$

The entries Q_n, R_n, S_n, and T_n are independent of t; $\tilde{t}(t)$ is an arbitrary locally invertible function of t; α_n, β_n are arbitrary functions of n and t, and the matrix

$$M_n=\begin{pmatrix}Q_n & R_n\\S_n & T_n\end{pmatrix},\quad \det M_n\neq 0, \tag{2.7}$$

is nonsingular.

It will be convenient to use a shorthand notation for the vector field X_n of Eq. (2.3), namely,

$$\left\{\tau(t), A_n, \begin{pmatrix}\lambda_n(t)\\ \mu_n(t)\end{pmatrix}\right\}, \quad A_n = \begin{pmatrix}a_n & b_n\\ c_n & d_n\end{pmatrix}. \tag{2.8}$$

If we perform an allowed transformation (2.6), then Eq. (1.1) goes into an equation of the same form, with F_n and G_n replaced by

$$\begin{pmatrix}\tilde F_n\\ \tilde G_n\end{pmatrix} = \dot{\tilde t}^{-3/2} M_n^{-1}\left[\begin{pmatrix}F_n\\ G_n\end{pmatrix} - \begin{pmatrix}\ddot\alpha_n\\ \ddot\beta_n\end{pmatrix}\right] + \left(\frac{1}{2}\dddot{\tilde t}\,\dot{\tilde t}^{-3} - \frac{3}{4}\ddot{\tilde t}^{2}\dot{\tilde t}^{-4}\right)\begin{pmatrix}\tilde u_n\\ \tilde v_n\end{pmatrix}, \tag{2.9}$$

where $\tilde F_n$ and $\tilde G_n$ are functions of the new variables.

The vector field characterized by the triplet (2.3) goes into a new one of the same form,

$$\left\{\tilde\tau(\tilde t), \tilde A_n, \begin{pmatrix}\tilde\lambda_n(\tilde t)\\ \tilde\mu_n(\tilde t)\end{pmatrix}\right\}, \tag{2.10}$$

with

$$\tilde\tau(\tilde t) = \tau(t(\tilde t))\dot{\tilde t},$$

$$\tilde A_n = M_n^{-1} A_n M_n,$$

$$\begin{pmatrix}\tilde\lambda_n(\tilde t)\\ \tilde\mu_n(\tilde t)\end{pmatrix} = M_n^{-1}\dot{\tilde t}^{1/2}\left[\left(A_n + \frac{\dot\tau}{2}\right)\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix} - \tau\begin{pmatrix}\dot\alpha_n\\ \dot\beta_n\end{pmatrix} + \begin{pmatrix}\lambda_n\\ \mu_n\end{pmatrix}\right].$$

We shall use the allowed transformations to simplify the vector field, rather than the equation itself.

III. SYSTEMS WITH ONE-DIMENSIONAL SYMMETRY GROUPS

Let us now assume that the system (1.1) has at least a one-dimensional symmetry group, generated by a vector field of the type (2.3). Using allowed transformations (2.6), we take $\hat X$ into one of ten inequivalent classes.

Indeed, for $\tau \neq 0$ we can choose the function $\tilde t(t)$ so as to transform $\tau(t)$ into $\tau = 1$, the functions $\alpha_n(t)$ and $\beta_n(t)$ so as to annul $\lambda_n(t)$, and $\mu_n(t)$ and the matrix M_n so as to take A_n into its canonical Jordan form.

For $\tau = 0$ the standardized form of $\hat X$ depends on the rank of the matrix A_n. For rank $A_n = 2$, we can again transform λ_n and μ_n into $\lambda_n = \mu_n = 0$ and take A_n into one of three canonical forms. For rank $A_n = 1$, only one of the functions λ_n or μ_n can be annulled. We choose it to be $\lambda_n(t) = 0$. Then A_n can be taken into one of the two standard matrices of rank 1 in $\mathbb{R}^{2\times 2}$. For rank $A_n = 0$ both $\lambda_n(t)$ and $\mu_n(t)$ survive.

We thus obtain ten mutually inequivalent one-dimensional symmetry algebras, listed below. The statement now is that any single vector field $\hat X$ of the form (2.3) can be transformed by an allowed transformation into precisely one of these vector fields.

The next step is to determine the interactions for which a one-dimensional symmetry group exists. To do this, we run through the canonical vector fields just obtained, substitute the corresponding $\tau(=1 \text{ or } 0)$, A_n, $\lambda_n(t)$, and $\mu_n(t)$ into Eqs. (2.4) and (2.5), and solve these equations for F_n and G_n.

Following this procedure, we arrive at the following list of interactions and their one-dimensional symmetry algebras:

$A_{1,1}$
$$\begin{aligned}
&\hat{X}=\partial_t+a_n u_n\partial_{u_n}+d_n v_n\partial_{v_n},\\
&F_n=e^{a_n t}f_n(\xi_k,\eta_k),\\
&G_n=e^{d_n t}g_n(\xi_k,\eta_k),\\
&\xi_k=u_k e^{-a_k t},\quad \eta_k=v_k e^{-d_k t},\\
&k=n-1,n,n+1;
\end{aligned}$$

$A_{1,2}$
$$\begin{aligned}
&\hat{X}=\partial_t+(a_n u_n+v_n)\partial_{u_n}+a_n v_n\partial_{v_n},\\
&F_n=e^{a_n t}[f_n(\xi_k,\eta_k)+t g_n(\chi_k,\eta_k)],\\
&G_n=e^{a_n t}g_n(\xi_k,\eta_k),\\
&\xi_k=(u_k-t v_k)e^{-a_k t},\quad \eta_k=v_k e^{-a_k t},\\
&k=n-1,n,n+1;
\end{aligned}$$

$A_{1,3}$
$$\begin{aligned}
&\hat{X}=\partial_t+(a_n u_n+b_n v_n)\partial_{u_n}+(-b_n u_n+a_n v_n)\partial_{v_n},\quad b_n>0,\\
&\begin{pmatrix}F_n\\ G_n\end{pmatrix}=e^{a_n t}\begin{pmatrix}\cos b_n t & \sin b_n t\\ -\sin b_n t & \cos b_n t\end{pmatrix}\begin{pmatrix}f_n(\xi_k,\eta_k)\\ g_n(\xi_k,\eta_k)\end{pmatrix},\\
&\xi_k=r_k e^{-a_k t},\quad \eta_k=\theta_k+b_k t,\\
&u_k=r_k\cos\theta_k,\quad v_k=r_k\sin\theta_k,\\
&k=n-1,n,n+1;
\end{aligned}$$

$A_{1,4}$
$$\begin{aligned}
&\hat{X}=a_n u_n\partial_{u_n}+d_n v_n\partial_{v_n},\quad |a_n|\geqslant|d_n|,\\
&F_n=u_n f_n(\xi_\alpha,\eta_k,t),\\
&G_n=v_n g_n(\xi_\alpha,\eta_k,t),\\
&\xi_\alpha=u_\alpha^{a_n}u_n^{-a_\alpha},\quad \eta_k=v_k^{a_n}u_n^{-d_k},\\
&k=n-1,n,n+1,\quad \alpha=n-1,n+1;
\end{aligned}$$

$A_{1,5}$
$$\begin{aligned}
&\hat{X}=(a_n u_n+v_n)\partial_{u_n}+a_n v_n\partial_{v_n},\quad a_n\neq 0,\\
&F_n=v_n f_n(\eta_\alpha,\xi_k,t)+v_n\ln(v_n)g_n(\eta_\alpha,\xi_k,t),\\
&G_n=a_n v_n g_n(\eta_\alpha,\xi_k,t),\\
&\xi_k=a_k\frac{u_k}{v_k}-\ln(v_k),\quad \eta_\alpha=v_\alpha^{a_n}v_n^{-a_\alpha},\\
&k=n-1,n,n+1,\quad \alpha=n-1,n+1;
\end{aligned}$$

$A_{1,6}$
$$\begin{aligned}
&\hat{X}=v_n\partial_{u_n},\\
&F_n=f_n(v_k,\xi_\alpha,t)+u_n g_n(v_k,\xi_\alpha,t),\\
&G_n=v_n g_n(v_k,\xi_\alpha,t),\\
&\xi_\alpha=-v_\alpha u_n+v_n u_\alpha,\\
&k=n-1,n,n+1,\quad \alpha=n-1,n+1;
\end{aligned}$$

$A_{1,7}$
$$\begin{aligned}
&\hat{X}=(a_n u_n+b_n v_n)\partial_{u_n}+(-b_n u_n+a_n v_n)\partial_{v_n},\quad b_n>0,\\
&\begin{pmatrix}F_n\\ G_n\end{pmatrix}=e^{-(a_n/b_n)\theta_n}\begin{pmatrix}\cos\theta_n & -\sin\theta_n\\ \sin\theta_n & \cos\theta_n\end{pmatrix}\begin{pmatrix}f_n(\xi_k,\eta_\alpha,t)\\ g_n(\xi_k,\eta_\alpha,t)\end{pmatrix},\\
&\xi_k=r_k^{b_n}e^{a_k\theta_n},\quad \eta_\alpha=b_n\theta_\alpha-b_\alpha\theta_n\\
&u_k=r_k\cos\theta_k,\quad v_k=r_k\sin\theta_k,\\
&k=n-1,n,n+1,\quad \alpha=n-1,n+1;
\end{aligned}$$

$A_{1,8}$
$$\begin{aligned}
&\hat{X}=a_n u_n\partial_{u_n}+\mu_n(t)\partial_{v_n},\quad \mu_n\neq 0,\\
&F_n=u_n f_n(\eta_\alpha,\xi_k,t),\\
&G_n=\frac{\dot{\mu}_n}{\mu_n}v_n+g_n(\eta_\alpha,\xi_k,t),\\
&\eta_\alpha=\mu_n v_\alpha-\mu_\alpha v_n,\quad \xi_k=u_k e^{-a_k v_n/\mu_n},\\
&k=n-1,n,n+1,\quad \alpha=n-1,n+1;
\end{aligned}$$

$A_{1,9}$

$$\hat{X}=v_n\partial_{u_n}+\mu_n(t)\partial_{v_n},\quad \mu_n\neq 0,$$
$$F_n=\frac{1}{2}\frac{\ddot{\mu}_n}{\mu_n^2}v_n^2+v_ng_n(\eta_\alpha,\eta_n,\xi_\alpha,t)+f_n(\eta_\alpha,\eta_n,\xi_\alpha,t),$$
$$G_n=\frac{\ddot{\mu}_n}{\mu_n}v_n+\mu_ng_n(\eta_\alpha,\eta_n,\xi_\alpha,t),$$
$$\eta_\alpha=\mu_n^2u_\alpha+\tfrac{1}{2}\mu_\alpha v_n^2-\mu_nv_nv_\alpha,\quad \xi_\alpha=\mu_\alpha v_n-\mu_nv_\alpha,$$
$$\eta_n=\mu_nu_n-\tfrac{1}{2}v_n^2,\quad \alpha=n-1,n+1;$$

$A_{1,10}$

$$\hat{X}=\lambda_n(t)\partial_{u_n}+\mu_n(t)\partial_{v_n},\quad \lambda_n,\ \mu_n\neq 0,$$
$$F_n=\frac{\ddot{\lambda}_n}{\lambda_n}u_n+f_n(\eta_k,\xi_\alpha,t),$$
$$G_n=\frac{\ddot{\mu}_n}{\mu_n}u_n+g_n(\eta_k,\xi_\alpha,t),$$
$$\xi_\alpha=\lambda_nu_\alpha-\lambda_\alpha u_n,\quad \eta_k=\mu_ku_n-\lambda_nv_k,$$
$$k=n-1,n,n+1,\quad \alpha=n-1,n+1.$$

We mention that the variables ξ_k and η_k are to be taken exactly as above. For instance, ξ_{n+1} is not an upshift of ξ_n.

The above results are summed up quite simply. Namely, the existence of a one-dimensional symmetry algebra restricts the interaction terms F_n and G_n to two arbitrary functions of six variables, rather than the original seven variables. The algebras $A_{1,1}$, $A_{1,2}$ and $A_{1,3}$ involve time translations. Hence, the time dependence in these cases is restricted: F_n and G_n depend on time explicitly and via invariant variables ξ_k and η_k that, in turn, depend explicitly on t. The algebras $A_{1,4},\ldots,A_{1,10}$ correspond to gauge transformations: the group transformations act on dependent variables only. The time variable figures in the arbitrary functions, as does the discrete independent variable n.

IV. HIGHER-DIMENSIONAL SYMMETRY ALGEBRAS

A. General strategy

The commutator of two symmetry operators (2.3) is an operator $X_3=[X_1,X_2]$ of the same form, satisfying

$$\tau_3=\tau_1\dot{\tau}_2-\tau_2\dot{\tau}_1,\quad A_{n,3}=-[A_{n,1},A_{n,2}],$$
$$\begin{pmatrix}\lambda_{n,3}\\ \mu_{n,3}\end{pmatrix}=\tau_1\begin{pmatrix}\dot{\lambda}_{n,2}\\ \dot{\mu}_{n,2}\end{pmatrix}-\tau_2\begin{pmatrix}\dot{\lambda}_{n,1}\\ \dot{\mu}_{n,1}\end{pmatrix}-\left(A_{n,1}+\frac{\dot{\tau}_1}{2}\right)\begin{pmatrix}\lambda_{n,2}\\ \mu_{n,2}\end{pmatrix}+\left(A_{n,2}+\frac{\dot{\tau}_2}{2}\right)\begin{pmatrix}\lambda_{n,1}\\ \mu_{n,1}\end{pmatrix}. \tag{4.1}$$

To obtain a finite-dimensional Lie algebra of symmetry operators, we see that the "differential components" $\tau_i(t)\partial_t$ must form a Lie algebra L_d, the "matrix components" $A_{n,i}$ must also form a Lie algebra L_m, homomorphic to L_d. Moreover, Eq. (4.1) shows that the "functional components" $(\lambda_{n,i}(t),\mu_{n,i}(t))$ must satisfy certain cohomology conditions.

The algebra of diffeomorphisms of $\mathbb{R}^1$, $\{\tau(t)\partial_t\}$ has only three mutually nondiffeomorphic finite-dimensional subalgebras, namely sl(2,R) and its subalgebras, realized, e.g., as

$$\{\partial_t,t\partial_t,t^2\partial_t\},\quad \{\partial_t,t\partial_t\},\quad \text{and } \{\partial_t\}, \tag{4.2}$$

respectively.

For n fixed, the matrices A_n generate the Lie algebra of gl(2,R). However, since the dependence on n is arbitrary, an unlimited number of copies of gl(2,R) and its subalgebras is available.

We shall not perform a complete classification of possible symmetry algebras here. Instead, we shall first concentrate on sl(2,R) symmetry algebras and show that, up to allowed transformations, four different sl(2,R) symmetry algebras can be constructed. We then consider just one of these four and study its extensions to higher-dimensional Lie algebras.

B. Equivalence classes of sl(2,R) symmetry algebras

Since sl(2,R) is a simple Lie algebra, it has no ideals. Hence, a homomorphism between sl(2,R) algebras is either an isomorphism, or one of the algebras is mapped onto zero. Correspondingly, we have three possibilities to explore: we shall call them $\mathrm{sl}(2,\mathbb{R})_d$, $\mathrm{sl}(2,\mathbb{R})_m$, and $\mathrm{sl}(2,\mathbb{R})_c$ (where d stands for "differential," m for "matrix," and c for "combined").

1. The algebra $\mathrm{sl}(2,\mathbb{R})_d$

We have *a priori*

$$X_1=\partial_t+\lambda_n(t)\partial_{u_n}+\mu_n(t)\partial_{v_n},$$

$$X_2=t\partial_t+(\tfrac{1}{2}u_n+\rho_n(t))\partial_{u_n}+(\tfrac{1}{2}v_n+\sigma_n(t))\partial_{v_n}, \tag{4.3}$$

$$X_3=t^2\partial_t+(tu_n+\omega_n(t))\partial_{u_n}+(tv_n+\kappa_n(t))\partial_{v_n}.$$

Using allowed transformations we transform $\lambda_n\to 0$, $\mu_n\to 0$. The commutation relation $[X_1,X_2]=X_1$ then requires $\dot\rho_n=\dot\sigma_n=0$. A further allowed transformation (2.6) with $\tilde{t}(t)=t$, $M_n=I$, and (α_n,β_n) constant will not change X_1, but take $\rho_n\to 0$, $\sigma_n\to 0$ (while leaving $\lambda_n=\mu_n=0$). The commutation relations $[X_2,X_3]=X_3$ and $[X_1,X_3]=2X_2$ then imply $\omega_n=\kappa_n=0$.

2. The algebra $\mathrm{sl}(2,\mathbb{R})_m$

A priori we have

$$X_1=b_nv_n\partial_{u_n}+\lambda_n(t)\partial_{u_n}+\mu_n(t)\partial_{v_n},$$

$$X_2=a_n(u_n\partial_{u_n}-v_n\partial_{v_n})+\rho_n(t)\partial_{u_n}+\sigma_n(t)\partial_{v_n}, \tag{4.4}$$

$$X_3=c_nu_n\partial_{v_n}+\omega_n(t)\partial_{u_n}+\kappa_n(t)\partial_{v_n}.$$

The structure constants cannot depend on n, so the commutation relations imply

$$a_n=a, \quad b_nc_n=bc. \tag{4.5}$$

Given that the product b_nc_n does not depend on n, we can use an allowed transformation to take $b_n\to b$, $c_n\to c$. A further allowed transformation will take $\rho_n\to 0$, $\sigma_n\to 0$. The commutation relations then imply $\lambda_n=\mu_n=0$ and $\omega_n=\kappa_n=0$.

3. The combined algebra $\mathrm{sl}(2,\mathbb{R})_c$

In view of the above results, we can write a "combined" algebra as

$$X_1=\partial_t+\alpha v_n\partial_{u_n}+\xi_n\partial_{u_n}+\eta_n\partial_{v_n}, \quad \alpha\neq 0,$$

$$X_2=t\partial_t+[(\tfrac{1}{2}+\beta)u_n+\lambda_n]\partial_{u_n}+[(\tfrac{1}{2}-\beta)v_n+\mu_n]\partial_{v_n}, \tag{4.6}$$

$$X_3=t^2\partial_t+(tu_n+\rho_n)\partial_{u_n}+(\gamma u_n+tv_n+\sigma_n)\partial_{v_n}.$$

We use allowed transformations to set $\alpha=1$, $\xi_n=\eta_n=0$. The commutation relations then determine $\beta=\frac{1}{2}$, $\gamma=-1$. The functions $\lambda_n(t)$, $\mu_n(t)$, $\rho_n(t)$, and $\sigma_n(t)$ are greatly restricted by the commutation relations. As a matter of fact, we either have $\lambda_n=\mu_n=\rho_n=\sigma_n=0$, or we can use allowed transformations to obtain $\lambda_n=t$, $\mu_n=1$, $\rho_n=2t^2$, $\sigma_n=2t$.

We arrive at the following result.

Theorem 1: *Precisely four classes of* sl(2,R) *algebras can be realized by vector fields of the form (2.3). Any such* sl(2,R) *algebra can be taken by an allowed transformation (2.6) into precisely one of the following algebras:*

$$\mathrm{sl}(2,\mathbb{R})_1: \quad \begin{aligned} X_1&=v_n\partial_{u_n},\\ X_2&=\tfrac{1}{2}(u_n\partial_{u_n}-v_n\partial_{v_n}),\\ X_3&=u_n\partial_{v_n}, \end{aligned} \tag{4.7}$$

$$\mathrm{sl}(2,\mathbb{R})_2: \quad \begin{aligned} X_1&=\partial_t,\\ X_2&=t\partial_t+\tfrac{1}{2}(u_n\partial_{u_n}+v_n\partial_{v_n}),\\ X_3&=t^2\partial_t+t(u_n\partial_{u_n}+v_n\partial_{v_n}), \end{aligned} \tag{4.8}$$

$$\mathrm{sl}(2,\mathbb{R})_3: \quad \begin{aligned} X_1&=\partial_t+v_n\partial_{u_n},\\ X_2&=t\partial_t+u_n\partial_{u_n},\\ X_3&=t^2\partial_t+tu_n\partial_{u_n}+(tv_n-u_n)\partial_{v_n}, \end{aligned} \tag{4.9}$$

$$\mathrm{sl}(2,\mathbb{R})_4: \quad \begin{aligned} X_1&=\partial_t+v_n\partial_{u_n}\\ X_2&=t\partial_t+(u_n+t)\partial_{u_n}+\partial_{v_n}\\ X_3&=t^2\partial_t+(tu_n+2t^2)\partial_{u_n}+(tv_n-u_n+2t)\partial_{v_n}. \end{aligned} \tag{4.10}$$

C. Indecomposable Lie algebras containing sl(2,R)$_1$

A Lie algebra L is called indecomposable if it cannot be written as a direct sum, $L=L_1\oplus L_2$. A Lie algebra over $\mathbb{R}$ containing sl(2,R) is either simple or it allows a nontrivial Levi decomposition,[15]

$$L=S\rhd R, \tag{4.11}$$

where S is a semisimple Lie algebra and R is the radical, that is, the maximal solvable ideal of L.

It follows from the results of Sec. IV A that the only simple Lie algebras that can be constructed from operators of the form (2.3) are the four sl(2,R) algebras obtained in Sec. IV B. We can hence concentrate on Lie algebras of the form (4.11).

The algebra S is either sl(2,R)$_1$ itself, or the direct sum of sl(2,R)$_1$ with one or more other sl(2,R) algebras.

Requiring that a symmetry operator Y should commute with all elements of sl(2,R)$_1$, we find that Y must have the form

$$Y_0=\tau\partial_t+(\tfrac{1}{2}\dot{\tau}+a_n)(u_n\partial_{u_n}+v_n\partial_{v_n}). \tag{4.12}$$

It is hence possible to construct precisely one semisimple Lie algebra properly containing sl(2,R)$_1$, namely, the direct sum sl(2,R)$_1\oplus$sl(2,R)$_2$ with sl(2,R)$_2$ defined in Eq. (4.8).

Let us introduce some notations for vector fields, to be used below. We put

$$V(a_n)=a_n(u_n\partial_{u_n}+v_n\partial_{v_n}), \tag{4.13}$$

$$T(a_n)=\partial_t+a_n(u_n\partial_{u_n}+v_n\partial_{v_n}), \tag{4.14}$$

$$D(a_n)=t\partial_t+(\tfrac{1}{2}+a_n)(u_n\partial_{u_n}+v_n\partial_{v_n}), \tag{4.15}$$

$$P(a_n)=t^2\partial_t+(t+a_n)(u_n\partial_{u_n}+v_n\partial_{v_n}), \tag{4.16}$$

$$R(a_n)=(t^2+1)\partial_t+(t+a_n)(u_n\partial_{u_n}+v_n\partial_{v_n}), \tag{4.17}$$

$$Y_u(\lambda_n)=\lambda_n(t)\partial_{u_n}, \quad Y_v(\lambda_n)=\lambda_n(t)\partial_{v_n}. \tag{4.18}$$

In all cases we have $\dot{a}_n=0$, but $\lambda_n(t)$ can be a function of t. Both a_n and $\lambda_n(t)$ can be functions of n.

Let us consider $S=\mathrm{sl}(2,\mathbb{R})_1$ and $S=\mathrm{sl}(2,\mathbb{R})_1\oplus\mathrm{sl}(2,\mathbb{R})_2$ in Eq. (4.11) separately.

1. $S=\mathrm{sl}(2,\mathbb{R})_1$

The considered Lie algebras will have a basis $\{X_1,X_2,X_3,Y_1,\ldots,Y_n\}$ with X_i given in Eq. (4.7). The basis elements $\{Y_1,\ldots,Y_n\}$ span the radical R. The algebra S acts on R according to some linear, not necessarily irreducible, finite-dimensional representation.

We start with the Cartan subalgebra $\{X_2\}$ of sl(2,R). It can be represented by a diagonal matrix in any finite-dimensional representation. Consider $Y\in R$. We have

$$[X_2,Y]=pY, \tag{4.19}$$

with Y as in Eq. (2.3). Equation (4.19) implies

$$\begin{gathered} p\tau=0, \\ p\left(\frac{\dot{\tau}}{2}+a_n\right)=0, \quad \left(p+\frac{1}{2}\right)\lambda_n=0, \quad (p+1)b_n=0, \\ p\left(\frac{\dot{\tau}}{2}+d_n\right)=0, \quad \left(p-\frac{1}{2}\right)\mu_n=0, \quad (p-1)c_n=0. \end{gathered} \tag{4.20}$$

For $p=0$ we obtain an operator that commutes not only with X_2, but with all of $\mathrm{sl}(2,\mathbb{R})_1$, namely, Y_0 of Eq. (4.12). This is a singlet representation of sl(2,R).

For $p=1$, or $p=-1$, Eq. (4.19) forces Y to be an element of $\mathrm{sl}(2,\mathbb{R})_1$, in other words, no such $Y\in R$ exists.

For $p=\pm\frac{1}{2}$ we obtain $Y_1=\lambda_n(t)\partial_{u_n}$ and $Y_2=\mu_n(t)\partial_{v_n}$, respectively. Acting with X_1 and X_3 on these operators, we find that the only representation of $\mathrm{sl}(2,\mathbb{R})_1$ that can be realized is a doublet one, namely $\{Y_u(\lambda_n),Y_v(\lambda_n)\}$ of Eq. (4.18), with $\lambda_n(t)$ an arbitrary function of n and t. The indecomposable Lie algebra $\{X_1,X_2,X_3,Y_u(\lambda_n),Y_v(\lambda_n)\}$ is isomorphic to the special affine Lie algebra saff(2,R).

All further indecomposable symmetry algebras containing $\mathrm{sl}(2,\mathbb{R})_1$ must be extensions of saff(2,R). The objects that we can add to saff(2,R) are either sl(2,R) doublets or singlets. Let us run through all possibilities.

(1) We can add an arbitrary number k of doublets of the form (4.18), where the k functions $\{\lambda_n^1(t),\lambda_n^2(t),\ldots,\lambda_n^k(t)\}$ must be linearly independent. However, we shall see in Sec. V that the presence of three such pairs forces the functions F_n and G_n in Eq. (1.1) to be linear. Moreover, even two such pairs are compatible with a nonlinear interaction only if they are of the form (or transformable into)

$$\lambda_n^1(t)=1, \quad \lambda_n^2(t)=t. \tag{4.21}$$

(2) We can add a singlet of the form (4.12). If we have $\tau=0$, then the commutation relations $[Y_0,Y_u]$ and $[Y_0,Y_v]$ imply $a_n=a_{n+1}$ and we can set $a_n=1$. We obtain an affine Lie algebra gaff(2,R)$_1$ consisting of saff(2,R) and $V(1)$ of Eq. (4.13).
If we have $\tau\neq 0$ in Eq. (4.12) and only one operator of this type, then we can use allowed transformations to take $\tau(t)$ into $\tau(t)=1$. The commutation relations $[Y_0,Y_u]$ and $[Y_0,Y_v]$ then imply

$$\lambda_n(t)=R_n e^{(a_n+k)t}, \quad \dot{R}_n=0.$$

For $k=0$, the algebra is decomposable. For $k\neq 0$ we can use allowed transformations to put $k=-1$ and $R_n=1$. We obtain a second algebra isomorphic to gaff(2,R), but not conjugate to the previous one. We have

$$\mathrm{gaff}(2,\mathrm{R})_2\sim\{X_1,X_2,X_3,Y_u(e^{(a_n-1)t}),Y_v(e^{(a_n-1)t}),T(a_n)\}. \tag{4.22}$$

In the special case of $a_n=a_{n+1}$ in Eq. (4.22), a further extension is possible. We transform $\lambda=e^{(a-1)t}$ into $\lambda=1$; then $T(a_n)$ goes into $D(b_n)$ with $b_n=b_{n+1}\equiv b\neq -\frac{1}{2}$, since for $b=-\frac{1}{2}$ the algebra is decomposable.

(3) We can add two singlets of the form (4.12). If they commute, they must be $\{V(1),T(0)\}$. The obtained algebra is decomposable. If they do not commute, they must form a two-dimensional Lie algebra, namely, $\{T(0),D(a),a_n=a_{n+1}\equiv a\}$. This implies $\lambda_n(t)\sim 1$, i.e., the entire radical is $\{Y_u(1),Y_v(1),T(0),D(a)\}$ with $a\neq\frac{1}{2}$ (the case $a=\frac{1}{2}$ corresponds to a decomposable algebra).

(4) If we add three singlets, the only case corresponds to the radical $\{Y_u(1),Y_v(1),V(1),T(0),D(0)\}$. There will then be no invariant interaction (see below).

(5) Let us consider the special case of two doublets of the form (4.18), namely,

$$Y_u(1)=\partial_{u_n}, \quad Y_v(1)=\partial_{v_n}, \quad Y_u(t)=t\partial_{u_n}, \quad Y_v(t)=t\partial_{v_n}. \tag{4.23}$$

This algebra can be extended by a further element, namely,

$$Z=(\tau_0+\tau_1 t+\rho_2 t^2)\partial_t+(\tfrac{1}{2}\tau_1+\tau_2 t+a)(u_n\partial_{u_n}+v_n\partial_{v_n}),$$
$$a_n=a_{n+1}\equiv a, \tag{4.24}$$

where τ_0, τ_1, and τ_2 are constants. By allowed transformations we can take Z into one of the four operators $V(1)$, $T(a)$, $D(a)$, or $R(a)$ of (4.13), (4.14), (4.15), and (4.17), respectively.

(6) We can add a two-dimensional algebra to (4.23), namely,

$$\{T(0),D(a)\}, \quad \{T(0),V(1)\}, \quad \{V(1),D(0)\}, \text{ or } \{V(1),R(0)\}.$$

(7) We can add only one three-dimensional algebra to (4.23), namely,

$$\{T(0), \quad D(0), \quad V(1)\}.$$

This completes the list of indecomposable symmetry algebras of the form (4.11) with $S=\mathrm{sl}(2,\mathrm{R})_1$.

2. $S=\mathrm{sl}(2,\mathrm{R})_1\oplus\mathrm{sl}(2,\mathrm{R})_2$

The algebra S is itself decomposable. It gives rise to precisely two indecomposable symmetry algebras. First, we have the one obtained by adding the Abelian ideal (4.23) to $\mathrm{sl}(2,\mathrm{R})_1\oplus\mathrm{sl}(2,\mathrm{R})_2$. Second, we get an 11-dimensional algebra by adding $V(1)$ to the first case.

D. Decomposable Lie algebras containing sl(2,R)$_1$

All decomposable Lie algebras L_D can be obtained from the indecomposable L_I ones, by adding their centralizers,

$$L_D = L_I \oplus C, \quad [C, L_I] = 0. \tag{4.25}$$

The centralizer C must commute with all elements of $\mathrm{sl}(2,\mathbb{R})_1$ and hence all of its elements will have the form of Y_0 of Eq. (4.12).

Let us consider the individual indecomposable algebras L_I.

1. $L_I = \mathrm{sl}(2,\mathbb{R})_1$

The centralizer C can be Abelian. Then we have the following possibilities: $C = \{V(a_{i,n}), i = 1,\ldots,k\}$ or $C = \{V(a_{i,n}), T(b_n), i = 1,\ldots,k\}$. The quantities $a_{1,n},\ldots,a_{k,n}$ must form a set of k linearly independent functions of n. If the centralizer is non-Abelian, then we have either $C \sim \mathrm{sl}(2,\mathbb{R})_2$ or $C = \{T(0), D(a)\}$. Both of these centralizers can be further extended by adding $V(a_{i,n})$, $i = 1,\ldots,k$, (with $a_{1,n},\ldots,a_{k,n}$ linearly independent).

2. $L_I = \mathrm{saff}(2,\mathbb{R})$

We must require Y_0 of Eq. (4.12) to commute with $Y_u(\lambda_n)$ and $Y_v(\lambda_n)$ of Eq. (4.18). We obtain

$$\lambda_n(\tfrac{1}{2}\dot{\tau} + a_n) - \tau\dot{\lambda}_n = 0. \tag{4.26}$$

For $\tau = 0$, Eq. (4.26) implies $\lambda_n a_n = 0$, and this is not allowed. For $\tau \neq 0$ we take $\tau \to 1$ by an allowed transformation, and Eq. (4.26) then implies $\lambda_n(t) = \gamma_n e^{a_n t}$. A further allowed transformation will take $\gamma_n \to 1$. We obtain the decomposable Lie algebra $\mathrm{saff}(2,\mathbb{R}) \oplus T(a_n)$. In the special case $a_n = a_{n+1}$ we transform $\lambda_n(t) \to 1$ and obtain a larger centralizer, namely, $\{T(0), D(-\tfrac{1}{2})\}$.

3. $L_I = \mathrm{gaff}(2,\mathbb{R})_1$

A nontrivial centralizer exists only if we have $\lambda_n(t) = e^{a_n t}$ in $\mathrm{saff}(2,\mathbb{R})$. In the case $a_n \neq 0$, the centralizer is $C = \{T(a_n)\}$. If $a_n = 0$ the centralizer is $C = \{T(0), D(-\tfrac{1}{2})\}$.

4. $L_I = \mathrm{gaff}(2,\mathbb{R})_2$

The centralizer is $C = \{T(a_n) - V(1)\}$. This algebra corresponds to the first one obtained in the case $L_I = \mathrm{gaff}(2,\mathbb{R})_1$ above.

E. Summary of possible symmetry algebras containing $\mathrm{sl}(2,\mathbb{R})_1$

The classification of possible symmetry algebras can now be summed up rather simply. In addition to $\mathrm{sl}(2,\mathbb{R})_1$ of Eq. (4.7), we have a further algebra L_C (the "complementary" algebra). The structure of each symmetry algebra is

$$L = \mathrm{sl}(2,\mathbb{R})_1 \dotplus L_C, \quad [\mathrm{sl}(2,\mathbb{R})_1, L_C] \subseteq L_C, \quad [L_C, L_C] \subseteq L_C. \tag{4.27}$$

The symbol $\dotplus$ denotes a direct sum of vector spaces. Moreover, Eq. (4.27) shows that L is either a direct sum or a semidirect one. The algebra L_C is also a representation space for $\mathrm{sl}(2,\mathbb{R})_1$. Irreducible representations in this case can be of dimension 1 or 2. All higher-dimensional representations are completely reducible into sums of one- and two-dimensional representations.

For further use it is convenient to split the symmetry algebras into four series, according to the structure of the Lie algebra L_C. In all cases L contains $\mathrm{sl}(2,\mathbb{R})_1$. We shall just specify L_C.

1. Series A

L_C is solvable and each element is a $\mathrm{sl}(2,\mathbb{R})_1$ singlet. There exist three different infinite-dimensional Lie algebras of this type:

$$A_1.\ \{V(a_{k,n})\}, \tag{4.28}$$

$$A_2.\ \{T(b_n), V(a_{k,n})\}, \tag{4.29}$$

$$A_3.\ \{T(0),D(b_n),V(a_{k,n})\}. \tag{4.30}$$

In each case we have $k=1,2,\ldots$, and the expressions a_k must be linearly independent functions of n. Taking $1\leqslant k\leqslant N$ for some finite N, we obtain finite-dimensional subalgebras.

2. Series B

L_C is solvable and contains precisely one sl(2,R)$_1$ doublet,

$$B_1=\{Y_u(\lambda_n),Y_v(\lambda_n)\}. \tag{4.31}$$

This is the indecomposable algebra saff(2,R) [B_1 together with sl(2,R)$_1$]. We have $\dim L=5$,

$$B_2=\{Y_u(\lambda_n),Y_v(\lambda_n),V(1)\}. \tag{4.32}$$

B_2 corresponds to the indecomposable algebra gaff(2,R)$_1$ with $\dim L=6$,

$$B_3=\{Y_u(e^{(a_n-1)t}),Y_v(e^{(a_n-1)t}),T(a_n)\}. \tag{4.33}$$

B_3 corresponds to the Lie algebra gaff(2,R)$_2$, isomorphic but not conjugate to B_2,

$$B_4=\{Y_u(e^{a_nt}),Y_v(e^{a_nt}),T(a_n)\}. \tag{4.34}$$

This algebra is saff(2,R)$\oplus T(a_n)$,

$$B_5=\{Y_u(1),Y_v(1),T(0),D(a)\}. \tag{4.35}$$

The algebra B_5 is indecomposable (except if $a=-\frac{1}{2}$),

$$B_6=\{Y_u(e^{(a_n-1)t}),Y_v(e^{(a_n-1)t}),T(a_n),V(1)\}. \tag{4.36}$$

The algebra B_6 is decomposable,

$$B_7=\{Y_u(1),Y_v(1),T(0),D(0),V(1)\}. \tag{4.37}$$

The algebra B_7 is indecomposable.

3. Series C

L_C contains two sl(2,R) doublets. The doublets could be characterized by any two functions $\lambda_{1,n}(t)$ and $\lambda_{2,n}(t)$. However, we shall only be interested in the case $\lambda_1=1$, $\lambda_2=t$. The others do not lead to invariant interactions. Similarly, we do not need algebras containing three or more doublets. In all cases the algebra L_C contains the elements (4.23). For $\dim L_C\geqslant 5$ it contains further elements. We have

$$C_1=\{Y_u(1),Y_v(1),Y_u(t),Y_v(t)\}. \tag{4.38}$$

Further, we just list the additional elements,

$$C_2.\ \{T(a)\},\quad a=0 \text{ or } 1, \tag{4.39}$$

$$C_3.\ \{D(a)\}, \tag{4.40}$$

$$C_4.\ \{R(a)\}, \tag{4.41}$$

$$C_5.\ \{V(1)\}, \tag{4.42}$$

$$C_6.\ \{T(0),D(a)\}. \tag{4.43}$$

In all cases above, a does not depend on $n(a_{n+1}=a_n)$,

$$C_7.\ \{V(1),T(0)\}, \tag{4.44}$$

$$C_8.\ \{V(1),D(0)\}, \tag{4.45}$$

$$C_9.\ \{V(1),R(0)\}, \tag{4.46}$$

$$C_{10}.\ \{T(0),D(0),P(0)\}\sim \mathrm{sl}(2,\mathbb{R})_2, \tag{4.47}$$

$$C_{11}.\ \{T(0),D(0),V(1)\}, \tag{4.48}$$

$$C_{12}.\ \{T(0),D(0),P(0),V(1)\}. \tag{4.49}$$

4. Series D

L_C contains $\mathrm{sl}(2,\mathbb{R})_2$ and (possibly) further elements, namely,

$$D_1.\ \text{None}, \tag{4.50}$$

$$D_2.\ \{V(a_n)\}, \tag{4.51}$$

$$D_3.\ \{V(a_{1,n}),V(a_{2,n})\}, \tag{4.52}$$

$$D_4.\ \{Y_u(1),Y_v(1),Y_u(t),Y_v(t)\}, \tag{4.53}$$

$$D_5.\ \{Y_u(1),Y_v(1),Y_u(t),Y_v(t),V(1)\} \tag{4.54}$$

(D_4 coincides with C_{10} and D_5 with C_{12}).

V. THE INVARIANT INTERACTIONS

A. General procedure and interactions invariant under SL(2,R)$_1$

In this section we shall find all interaction functions, invariant under symmetry groups, containing $\mathrm{SL}(2,\mathbb{R})_1$. We make use of the subalgebra classification provided in Sec. IV.

We first establish the form of the interaction, invariant under $\mathrm{SL}(2,\mathbb{R})_1$ itself. To do this we set $\tau(t)=\lambda_n(t)=\mu_n(t)=0$ in the determining equations (2.4) and (2.5) and consider the equations obtained for $a_n=-d_n=1$, $b_n=c_n=0$, then $b_n=1$, $a_n=-d_n=c_n=0$, and, finally, $c_n=1$, $a_n=-d_n=b_n=0$. The general solution of the obtained system of six equations can be written in the following form:

$$F_n=u_{n+1}f_n+u_ng_n, \quad G_n=v_{n+1}f_n+v_ng_n, \tag{5.1}$$

where f_n and g_n are functions of four variables each, namely,

$$t,\ \xi_n=u_{n+1}v_{n-1}-u_{n-1}v_{n+1}, \quad \xi_\alpha=u_\alpha v_n-u_nv_\alpha, \quad \alpha=n\pm 1. \tag{5.2}$$

Note that ξ_n, ξ_{n+1}, and ξ_{n-1} are as given in Eq. (5.2). They are not upshifts or downshifts of each other.

We shall proceed further by dimension of the symmetry algebra and by its structure. Thus, we can successively add sl(2,R) singlets of the form (4.12) or doublets of the form (4.18). We continue adding symmetry elements until the interaction is completely specified, i.e., it involves no further arbitrary functions. We then solve the "inverse problem." That is, we substitute the functions F_n and G_n back into the determining equations and solve for the symmetries. This provides a verification of previous calculations. More important, this procedure will find the largest symmetry algebra allowed by any given interaction.

Obviously, all invariant interactions will have the form (5.1). It is the functions f_n and g_n that will be further refined, and their dependence on the variables ξ_k and t will be restricted.

For future convenience we write down two further forms of the $\mathrm{SL}(2,\mathbb{R})_1$ invariant interaction functions, equivalent to (5.1). The first is

$$F_n=u_{n+1}\frac{\xi_{n-1}}{\xi_n}h_n+u_nk_n, \quad G_n=v_{n+1}\frac{\xi_{n-1}}{\xi_n}h_n+v_nk_n, \tag{5.3}$$

where h_n and k_n are arbitrary functions of the variables (5.2). The second convenient form is

$$\begin{aligned} F_n&=(\lambda_{n-1}u_{n+1}-\lambda_{n+1}u_{n-1})\phi_n+(\lambda_{n+1}u_n-\lambda_nu_{n+1})\psi_n+\frac{\ddot{\lambda}_n}{\lambda_{n+1}}u_{n+1}, \\ G_n&=(\lambda_{n-1}v_{n+1}-\lambda_{n+1}v_{n-1})\phi_n+(\lambda_{n+1}v_n-\lambda_nv_{n+1})\psi_n+\frac{\ddot{\lambda}_n}{\lambda_{n+1}}v_{n+1}, \end{aligned} \tag{5.4}$$

where $\lambda_n(t)$ is some arbitrary function of n and t and ϕ_n and ψ_n depend in an unspecified manner on the variables (5.2).

B. Interactions invariant under four-dimensional symmetry groups

As was shown in Sec. IV, two types of four-dimensional symmetry algebras containing $\mathrm{sl}(2,\mathbb{R})_1$ can exist. Both are decomposable according to the pattern $4=3+1$. Here and below we shall always list the operators that we can add to $\mathrm{sl}(2,\mathbb{R})_1$.

1. $V(a_n)=a_n(u_n\partial_{u_n}+v_n\partial_{v_n})$

The invariant interactions will have the form (5.3), but h_n and k_n will depend on three variables only.

(i) $a_{n-1}+a_{n+1}\neq 0$. The variables are

$$t, \quad \eta_\alpha=(\xi_\alpha)^{a_{n-1}+a_{n+1}}(\xi_n)^{-a_n-a_\alpha}, \quad \alpha=n\pm 1. \tag{5.5}$$

(ii) $a_{n-1}+a_{n+1}=0$. The variables are

$$t, \quad \xi_n, \quad \eta=(\xi_{n+1})^{a_{n+1}-a_n}(\xi_{n-1})^{a_{n+1}+a_n}. \tag{5.6}$$

2. $T(b_n)=\partial_t+b_n(u_n\partial_{u_n}+v_n\partial_{v_n})$

The invariant interaction will again have the form (5.3), however, in this case h_n and k_n are arbitrary functions of the three variables,

$$\zeta_n=\xi_ne^{-(b_{n-1}+b_{n+1})t}, \quad \zeta_\alpha=\xi_\alpha e^{-(b_n+b_\alpha)t}, \quad \alpha=n\pm 1. \tag{5.7}$$

We see that adding further singlets of the type $V(a_n)$ will restrict the variables in the functions h_n and k_n, not, however, the general form of Eq. (5.3).

C. Five-dimensional symmetry groups

From the results of Sec. IV, we know that three decomposable and one indecomposable symmetry algebras of dimension 5 can exist. Let us run through all four possibilities.

1. Decomposition 5=3+1+1

a. $V(a_{i,n})=a_{i,n}(u_n\partial_{u_n}+v_n\partial_{v_n})$ $i=1,2$, $a_{2,n}\neq\lambda a_{1,n}$. The interaction is of the form (5.3). The functions h_n and k_n depend on two variables each, namely, time t and

$$\eta=(\xi_{n-1})^A(\xi_{n+1})^B(\xi_n)^C, \tag{5.8}$$

$$\begin{aligned}A&=a_{1,n}(a_{2,n+1}+a_{2,n-1})+a_{1,n+1}(a_{2,n-1}-a_{2,n})-a_{1,n-1}(a_{2,n+1}+a_{2,n}),\\ B&=-a_{1,n}(a_{2,n+1}+a_{2,n-1})+a_{1,n+1}(a_{2,n-1}+a_{2,n})-a_{1,n-1}(a_{2,n+1}-a_{2,n}),\\ C&=a_{1,n}(a_{2,n+1}-a_{2,n-1})-a_{1,n+1}(a_{2,n-1}+a_{2,n})+a_{1,n-1}(a_{2,n+1}+a_{2,n}).\end{aligned}\tag{5.9}$$

Note that the variable η always exists since the condition $A=B=C=0$ (and hence η=const) only occurs for $a_{1,n-1}a_{2,n}-a_{1,n}a_{2,n-1}=0$, which implies $a_{2,n}=\lambda a_{1,n}$, λ=const, and this is not allowed.

b. $V(a_n)=a_n(u_n\partial_{u_n}+v_n\partial_{v_n})$, $T(b_n)=\partial_t+b_n(u_n\partial_{u_n}+v_n\partial_{v_n})$. The invariant interaction is as in Eq. (5.3) with h_n and k_n functions of two variables each. Namely, the following.

(i) $a_{n+1}+a_{n-1}\neq 0$:

$$\rho_\alpha=(\zeta_\alpha)^{a_{n+1}+a_{n-1}}(\zeta_n)^{-a_\alpha-a_n},\quad \alpha=n\pm 1,\tag{5.10}$$

with ζ_α, ζ_n as in Eq. (5.7).

(ii) $a_{n+1}+a_{n-1}=0$:

$$\rho_n=\zeta_n,\quad \sigma_n=(\zeta_{n-1})^{a_{n+1}+a_n}(\zeta_{n+1})^{a_{n+1}-a_n}.\tag{5.11}$$

2. Decomposition 5=3+2

a. $T(0)=\partial_t$, $D(b_n)=t\partial_t+(\frac{1}{2}+b_n)(u_n\partial_{u_n}+v_n\partial_{v_n})$. We impose $b_n\neq-\frac{1}{2}$; otherwise we have no invariant interaction. We must distinguish two subcases here.

(1) $b_{n+1}+b_{n-1}+1\neq 0$. The interaction as in Eq. (5.3), with

$$h_n=(\xi_n)^{-2/(b_{n+1}+b_{n-1}+1)}p_n,\quad k_n=(\xi_n)^{-2/(b_{n+1}+b_{n-1}+1)}q_n,\tag{5.12}$$

where p_n and q_n depend on two variables, namely,

$$\chi_\alpha=(\xi_\alpha)^{b_{n+1}+b_{n-1}+1}(\xi_n)^{-b_n-b_\alpha-1},\quad \alpha=n\pm 1.\tag{5.13}$$

(2) $b_{n+1}+b_{n-1}+1=0$, $b_{n+1}+b_n+1\neq 0$:

$$h_n=(\xi_{n+1})^{-2/(b_{n+1}+b_n+1)}p_n,\quad k_n=(\xi_{n+1})^{-2/(b_{n+1}+b_n+1)}q_n,\tag{5.14}$$

where p_n and q_n depend on

$$\chi_n=(\xi_{n-1})^{b_{n+1}+b_n+1}(\xi_{n+1})^{-b_{n-1}-b_n-1},\quad \xi_n.\tag{5.15}$$

Note that for $b_{n+1}+b_{n-1}+1=0$, $b_{n+1}+b_n+1=0$, we have $b_n=-\frac{1}{2}$, and there is no invariant interaction.

3. Indecomposable Lie algebra

$$Y_u(\lambda_n)=\lambda_n(t)\partial_{u_n},\quad Y_v(\lambda_n)=\lambda_n(t)\partial_{v_n}.\tag{5.16}$$

The invariant interaction is as in Eq. (5.4), but the functions ϕ_n and ψ_n depend on only two variables, namely,

$$t,\quad \omega=\lambda_{n-1}\xi_{n+1}-\lambda_n\xi_n-\lambda_{n+1}\xi_{n-1}.\tag{5.17}$$

D. Six-dimensional symmetry groups

1. Decomposition 6=3+1+1+1

a. $V(a_{i,n})=a_{i,n}(u_n\partial_{u_n}+v_n\partial_{v_n})$, $i=1,2,3$. The invariant interaction is as in Eq. (5.3), but h_n and k_n are functions of t only. We see that the coefficients $a_{i,n}$ do not figure in the interaction.

Hence, we can add an arbitrary number of vector fields $V(a_{i,n})$, $i \in \mathbb{Z}$ to the symmetry algebra. In other words, the symmetry algebra for the interaction (5.3) with h_n and k_n depending on t alone is infinite dimensional.

b. $V(a_{i,n}) = a_{i,n}(u_n\partial_{u_n} + v_n\partial_{v_n})$, $i=1,2$, $T(b_n) = \partial_t + b_n(u_n\partial_{u_n} + v_n\partial_{v_n})$. The invariant interaction is as in Eq. (5.3), but h_n and k_n depend on one variable only, namely,

$$\omega = \eta e^{-2t|M|}, \quad M = \begin{pmatrix} b_{n-1} & b_n & b_{n+1} \\ a_{1,n-1} & a_{1,n} & a_{1,n+1} \\ a_{2,n-1} & a_{2,n} & a_{2,n+1} \end{pmatrix}, \tag{5.18}$$

with η as in Eq. (5.8).

2. Decomposition 6=3+2+1

a. $V(a_n) = a_n(u_n\partial_{u_n} + v_n\partial_{v_n})$, $T(0) = \partial_t$, $D(c_n) = t\partial_t + (\frac{1}{2} + c_n)(u_n\partial_{u_n} + v_n\partial_{v_n})$. We start from Eq. (5.3). The presence of $T(0) = \partial_t$ implies that h_n and k_n do not depend on t. We first notice that if we have

$$\gamma_n = c_n + \tfrac{1}{2} = 0 \quad \text{or} \quad \gamma_n = c_n + \tfrac{1}{2} = \lambda a_n, \tag{5.19}$$

then we must have $h_n = k_n = 0$ (no invariant interaction). In all other cases, invariance under $V(a_n)$ and $D(c_n)$ implies

$$\begin{gathered} h_n = (\xi_n)^{\mu}(\xi_{n+1})^{\nu}(\xi_{n-1})^{\rho} p_n(\omega), \quad k_n = (\xi_n)^{\mu}(\xi_{n+1})^{\nu}(\xi_{n-1})^{\rho} q_n(\omega), \\ \omega = (\xi_{n-1})^{A}(\xi_{n+1})^{B}(\xi_n)^{C}, \end{gathered} \tag{5.20}$$

with A, B, and C as in Eq. (5.9), with the substitutions

$$a_{1,n} \to c_n + \tfrac{1}{2}, \quad a_{2,n} \to a_n .$$

The constants μ, ν, and ρ in Eq. (5.20) satisfy

$$\begin{gathered} (a_{n+1} + a_{n-1})\mu + (a_{n+1} + a_n)\nu + (a_{n-1} + a_n)\rho = 0, \\ (\gamma_{n+1} + \gamma_{n-1})\mu + (\gamma_{n+1} + \gamma_n)\nu + (\gamma_{n-1} + \gamma_n)\rho = -2. \end{gathered} \tag{5.21}$$

Thus, for $C \neq 0$ we can put

$$\mu = 0, \quad \nu = 2\frac{a_n + a_{n-1}}{C}, \quad \rho = -2\frac{a_n + a_{n+1}}{C}.$$

For $C=0$, $A \neq 0$,

$$\mu = 2\frac{a_n + a_{n+1}}{A}, \quad \nu = -2\frac{a_{n+1} + a_{n-1}}{A}, \quad \rho = 0.$$

For $C = A = 0$, $B \neq 0$,

$$\mu = -2\frac{a_{n-1} + a_n}{B}, \quad \nu = 0, \quad \rho = 2\frac{a_{n+1} + a_{n-1}}{B}.$$

The case $A = B = C = 0$ corresponds to Eq. (5.19) and hence to the absence of an invariant interaction.

3. Decomposition 6=3+3

a. $\mathrm{sl}(2,\mathbb{R})_1 \oplus \mathrm{sl}(2,\mathbb{R})_2$. The algebra $\mathrm{sl}(2,\mathbb{R})_2$ is as in Eq. (4.8) and the invariant interaction is

$$F_n = \frac{1}{(\xi_n)^2}\left[u_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n(\chi_{n+1},\chi_{n-1}) + u_n q_n(\chi_{n+1},\chi_{n-1})\right],$$

$$G_n = \frac{1}{(\xi_n)^2}\left[v_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n(\chi_{n+1},\chi_{n-1}) + v_n q_n(\chi_{n+1},\chi_{n-1})\right], \tag{5.22}$$

$$\chi_{n+1} = \frac{\xi_{n+1}}{\xi_n}, \quad \chi_{n-1} = \frac{\xi_{n-1}}{\xi_n}.$$

4. Decomposition 6=5+1

a. $\mathrm{saff}(2) \oplus A_1$. We have

$$Y_u(e^{a_n t}) = e^{a_n t}\partial_{u_n}, \quad Y_v(e^{a_n t}) = e^{a_n t}\partial_{v_n}, \quad T(a_n) = \partial_t + a_n(u_n\partial_{u_n} + v_n\partial_{v_n}).$$

The invariant interaction will be as in Eq. (5.4) with $\lambda_n = e^{a_n t}$. The functions ϕ_n and ψ_n will satisfy

$$\phi_n = e^{(a_n - a_{n-1} - a_{n+1})t}K_n(\omega), \quad \psi_n = e^{-a_{n+1}t}L_n(\omega),$$
$$\omega = e^{-(a_n + a_{n+1})t}\xi_{n+1} - e^{-(a_{n+1}+a_{n-1})t}\xi_n - e^{-(a_{n-1}+a_n)t}\xi_{n-1}. \tag{5.23}$$

5. Indecomposable symmetry algebras

It was shown in Sec. IV that two inequivalent gaff(2) symmetry algebras exist.

a. $\mathrm{gaff}(2,\mathbb{R})_1$.

$$Y_u(\lambda_n) = \lambda_n(t)\partial_{u_n}, \quad Y_v(\lambda_n) = \lambda_n(t)\partial_{v_n}, \quad V(1) = u_n\partial_{u_n} + v_n\partial_{v_n}.$$

The interaction is as in Eq. (5.4), however, ϕ_n and ψ_n depend only on t. This means that the equations are linear and, moreover, the equations (1.1) for u_n and v_n are decoupled.

b. $\mathrm{gaff}(2,\mathbb{R})_2$. The algebra is as in Eq. (4.22) [or (4.33)], the interaction as in Eq. (5.4) with $\lambda_n(t) = e^{(a_n - 1)t}$. The functions ϕ_n and ψ_n satisfy

$$\phi_n = e^{-(a_{n+1} + a_{n-1} - a_n - 1)t}K_n(\omega), \quad \psi_n = e^{(-a_{n+1}+1)t}L_n(\omega), \tag{5.24}$$

with ω as in Eq. (5.23).

E. Seven-dimensional symmetry groups

1. Decomposition 7=3+1+1+1+1

We exclude the case

$$V(a_{i,n}) = a_{i,n}(u_n\partial_{u_n} + v_n\partial_{v_n}), \quad i = 1,\ldots,4,$$

since the only invariant interaction is (5.3) with h_n and k_n functions of t. We already know that the symmetry algebra is infinite dimensional.

a. $V(a_{i,n}) = a_{i,n}(u_n\partial_{u_n} + v_n\partial_{v_n})$, $i = 1,2,3$, $T(b_n) = \partial_t + b_n(u_n\partial_{u_n} + v_n\partial_{v_n})$. The interaction is as in Eq. (5.3) with h_n and k_n constants (depending on n). The algebra is actually infinite dimensional: we can take any number of operators $V(a_{i,n})$.

2. Decomposition 7=3+2+1+1

a. $V(a_{i,n})=a_{i,n}(u_n\partial_{u_n}+v_n\partial_{v_n})$, $i=1,2$, $T(0)=\partial_t$, $D(c_n)=t\partial_t+(\frac{1}{2}+c_n)(u_n\partial_{u_n}+v_n\partial_{v_n})$. We put $\gamma_n=c_n+\frac{1}{2}$. An invariant interaction exists if and only if we have

$$\Delta=\det\begin{pmatrix}\gamma_n & \gamma_{n+1} & \gamma_{n-1}\\ a_{1,n} & a_{1,n+1} & a_{1,n-1}\\ a_{2,n} & a_{2,n+1} & a_{2,n-1}\end{pmatrix}\neq 0. \tag{5.25}$$

The invariant interaction is that of Eq. (5.3), with

$$h_n=\eta^k p_n,\quad k_n=\eta^k q_n,\quad k=-\frac{2}{\Delta}. \tag{5.26}$$

The variable η is as in Eq. (5.8); p_n and q_n are constants.

3. Decomposition 7=3+3+1

a. $\mathrm{sl}(2,\mathbb{R})_1\oplus\mathrm{sl}(2,\mathbb{R})_2\oplus A_1$. We have $A_1=\{V(a_n)\}$. The invariant interaction can be obtained from Eq. (5.22). The additional invariance implied by the presence of $V(a_n)$ restricts p_n and q_n to

$$p_n=\left(\frac{\xi_{n+1}}{\xi_n}\right)^{2(a_{n+1}+a_{n-1})/(a_n-a_{n-1})} r_n(\omega),$$

$$q_n=\left(\frac{\xi_{n+1}}{\xi_n}\right)^{2(a_{n+1}+a_{n-1})/(a_n-a_{n-1})} s_n(\omega), \tag{5.27}$$

$$\omega=(\xi_{n+1})^{a_{n+1}-a_n}(\xi_{n-1})^{a_n-a_{n-1}}(\xi_n)^{a_{n-1}-a_{n+1}},$$

and we must impose $a_n\neq a_{n-1}$ (otherwise we have $F_n=G_n=0$).

4. Decomposition 7=6+1

The algebra $\mathrm{gaff}(2,\mathbb{R})_1$ does not allow any nonlinear interactions. Let us consider $\mathrm{gaff}(2,\mathbb{R})_2$ of Eq. (4.22).

a. $\mathrm{gaff}(2,\mathbb{R})_2\oplus\{U=u_n\partial_{u_n}+v_n\partial_{v_n}\}$. The interaction is as in Eq. (5.4), with ϕ_n and ψ_n as in Eq. (5.24). Invariance under the dilations corresponding to U implies that ϕ_n and ψ_n do not depend on ω. Hence, the interaction is linear and decoupled.

5. Indecomposable Lie algebras

a. $Y_u(\lambda_n)=\lambda_n(t)\partial_{u_n}$, $Y_v(\lambda_n)=\lambda_n(t)\partial_{v_n}$, $Y_u(\mu_n)=\mu_n(t)\partial_{u_n}$, $Y_v(\mu_n)=\mu_n(t)\partial_{v_n}$. We start from Eq. (5.4) with ϕ_n and ψ_n functions of t and ω as in Eq. (5.17). If ϕ_n and ψ_n do not depend on ω, the interaction is already linear and decoupled. Hence, ω must be invariant under the transformations corresponding to $Y_u(\mu_n)$ and $Y_v(\mu_n)$. This implies that λ_n and μ_n are independent of n. Further, invariance implies

$$\frac{\ddot{\lambda}_n}{\lambda_n}=\frac{\ddot{\mu}_n}{\mu_n}=\bar{k}, \tag{5.28}$$

with $\bar{k}=\mathrm{const}$. Equation (5.28) allows solutions,

$$\begin{pmatrix}\lambda_n\\ \mu_n\end{pmatrix}=\begin{pmatrix}\sin kt\\ \cos kt\end{pmatrix},\quad \begin{pmatrix}\sinh kt\\ \cosh kt\end{pmatrix},\quad \begin{pmatrix}1\\ t\end{pmatrix}. \tag{5.29}$$

These solutions are all equivalent under allowed transformations. We choose $\lambda_n=1$, $\mu_n=t$, i.e.,

$$Y_u(1)=\partial_{u_n},\quad Y_v(1)=\partial_{v_n},\quad Y_u(t)=t\partial_{u_n},\quad Y_v(t)=t\partial_{v_n}. \tag{5.30}$$

The invariant interaction is

$$\begin{aligned} F_n&=(u_{n+1}-u_{n-1})\phi_n(\omega,t)+(u_n-u_{n+1})\psi_n(\omega,t),\\ G_n&=(v_{n+1}-v_{n-1})\phi_n(\omega,t)+(v_n-v_{n+1})\psi_n(\omega,t), \end{aligned} \tag{5.31}$$

with

$$\omega=\xi_{n+1}-\xi_{n-1}-\xi_n. \tag{5.32}$$

b. $Y_u(1)=\partial_{u_n}$, $Y_v(1)=\partial_{v_n}$, $T(0)=\partial_t$, $D(b)=t\partial_t+(\frac{1}{2}+b)(u_n\partial_{u_n}+v_n\partial_{v_n})$, $b\neq-\frac{1}{2}$, $b=\text{const}$. The invariant interaction is as in Eq. (5.31), with

$$\phi_n=k_n\omega^{-2/(2b+1)},\quad \psi_n=p_n\omega^{-2/(2b+1)}, \tag{5.33}$$

with k_n and p_n constants, ω as in Eq. (5.32). For $b=-\frac{1}{2}$ there is no invariant interaction. For $b\neq-\frac{1}{2}$ the symmetry algebra is actually larger and includes $Y_u(t)=t\partial_{u_n}$ and $Y_v(t)=t\partial_{v_n}$.

F. Symmetry groups of dimensions 8, 9, and 10

By now, all invariant interactions have been specified up to arbitrary constants (depending on n), except those involving symmetry algebras containing the subalgebra $\mathrm{sl}(2,\mathbb{R})_1\oplus\mathrm{sl}(2,\mathbb{R})_2$, or the subalgebra $\{Y_u(1),Y_v(1),Y_u(t),Y_v(t)\}$ of Eq. (5.30). Let us consider the remaining nonlinear interactions.

1. $\mathrm{sl}(2,\mathbb{R})_1\oplus\mathrm{sl}(2,\mathbb{R})_2\oplus\{V(a_{1,n})\}\oplus\{V(a_{2,n})\}$

The invariant interaction is obtained from Eq. (5.27) by specifying $r_n(\omega)$ and $s_n(\omega)$ to be specific powers of ω. The result is

$$\begin{aligned} F_n&=\xi_n^{-2}\left[u_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n+u_nq_n\right](\xi_{n-1})^{-2A/D}(\xi_{n+1})^{-2B/D}(\xi_n)^{2[(A+B)/D]},\\ G_n&=\xi_n^{-2}\left[v_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n+v_nq_n\right](\xi_{n-1})^{-2A/D}(\xi_{n+1})^{-2B/D}(\xi_n)^{2[(A+B)/D]}. \end{aligned} \tag{5.34}$$

Here p_n and q_n are constants, A and B are as in Eq. (5.9), and

$$D=a_{1,n}(a_{2,n+1}-a_{2,n-1})+a_{1,n+1}(a_{2,n-1}-a_{2,n})+a_{1,n-1}(a_{2,n}-a_{2,n+1}). \tag{5.35}$$

We assume $D\neq0$; otherwise there is no invariant interaction. In particular, we have $a_{1,n}\neq a_{1,n+1}$, $a_{2,n}\neq a_{2,n+1}$.

2. Algebras containing $(Y_u(1),Y_v(1),Y_u(t),Y_v(t))$ of (5.30) plus one additional operator Z

The interaction is as in Eq. (5.31) with a restriction on ϕ_n and ψ_n.

(i) $Z=T(a)=\partial_t+a(u_n\partial_{u_n}+v_n\partial_{v_n})$, $a\equiv a_n=a_{n+1}$,

$$\phi_n=\phi_n(\eta),\quad \psi_n=\psi_n(\eta),\quad \eta=\omega e^{-2at}. \tag{5.36}$$

(ii) $Z=D(a)=t\partial_t+(\frac{1}{2}+a)(u_n\partial_{u_n}+v_n\partial_{v_n})$, $a\equiv a_n=a_{n+1}$,

$$\phi_n=\frac{1}{t^2}r_n(\eta),\quad \psi_n=\frac{1}{t^2}s_n(\eta),\quad \eta=\omega t^{-(2a+1)}. \tag{5.37}$$

(iii) $Z=R(b)=(t^2+1)\partial_t+(t+b)(u_n\partial_{u_n}+v_n\partial_{v_n}),\ b\equiv b_n=b_{n+1},$

$$\phi_n=\frac{1}{(t^2+1)^2}r_n(\eta),\quad \psi_n=\frac{1}{(t^2+1)^2}s_n(\eta),$$
$$\eta=\frac{\omega}{1+t^2}e^{-2b\arctan t},\tag{5.38}$$

with ω as in Eq. (5.32) in all cases.

(iv) $Z=V(1)$. Then ϕ_n and ψ_n depend only on t and the interaction is linear.

We can add two operators to those of Eq. (5.30)

$$T(0)=\partial_t,\quad D(b)=t\partial_t+(\tfrac{1}{2}+b)(u_n\partial_{u_n}+v_n\partial_{v_n}).$$

The invariant interaction coincides with that of Eq. (5.33).

Finally, the interaction (5.31) is invariant under a ten-dimensional symmetry algebra of the form

$$(\mathrm{sl}(2,\mathbb{R})_1\oplus \mathrm{sl}(2,\mathbb{R})_2)\rhd\{Y_u(1),Y_v(1),Y_u(t),Y_v(t)\},$$

for

$$\phi_n=k_n\omega^{-2},\quad \psi_n=p_n\omega^{-2},\tag{5.39}$$

i.e., $b=0$ in Eq. (5.33).

VI. SUMMARY AND CONCLUSIONS

Let us first sum up the results on invariant interactions and the corresponding symmetry algebras. We shall follow the summary of possible symmetry algebras outlined in Sec. IV E. The results are presented in the following tables.

Table I. The Series A of symmetry algebras. The algebra L_C of Eq. (4.27) consists entirely of $\mathrm{sl}(2,\mathbb{R})_1$ singlets. In the first column of Table I we list the symmetry algebras. The number in brackets [e.g., $A_1(3)$] denotes the dimension of the symmetry algebra. The notation for basis elements in column 2 are as in Eqs. (4.13)–(4.18). Note that if the functions h_n and k_n in the interaction (5.3) depend only on t or are constants, then the symmetry algebra is infinite dimensional, although the interaction is nonlinear.

The case $A_3(7)$ corresponds to an algebra L with $\dim L=7$ and the interaction is completely specified [see (5.3), (5.25)–(5.26)]. In other cases the functions h_n and k_n depend on one, two, or three variables involving u_k and v_k.

Table II. The Series B of symmetry algebras. The symmetry algebras are either five or six dimensional. The interactions are as in Eq. (5.4) and involve two arbitrary functions, ϕ_n and ψ_n. A B-type symmetry allows ϕ_n and ψ_n to depend on just one variable involving u_k and v_k. Any extension of the B-type algebras will restrict $\lambda_n(t)$ to be $\lambda_n=1$ and will involve a further pair with $\lambda_n=t$. This takes us into the series C of symmetry algebras.

The algebras B_2, B_6, and B_7 of Eqs. (4.32), (4.36), and (4.37) lead to linear interactions. Any interaction invariant with respect to B_5 will be invariant under a larger group, corresponding to a Lie algebra in the series C. We do not include linear interactions in the tables and we list interactions together with their *maximal* symmetry algebras.

Table III. The Series C of symmetry algebras. The interaction will be as in Eq. (5.31), involving a variable ω as in Eq. (5.32). The algebras $C_5(8),C_7(9),C_8(9),C_9(9),C_{11}(10),C_{12}(11)$, absent in the table, lead to a linear interaction.

TABLE I. Series A of symmetry algebras. The interaction has the form (5.3).

No.	L_C	Restrictions on h_n and k_n	Variables and comments
$A_1(3)$	$\cdots$	$\cdots$	$t,\xi_{n+1},\xi_{n-1},\xi_n$ (5.2)
$A_1(4)$	$V(a_n)$	$\cdots$	$\begin{cases} t,\eta_{n+1},\eta_{n-1} \ (5.5) \\ t,\xi_n,\eta \ (5.6)\end{cases}$
$A_1(5)$	$V(a_{1,n}),V(a_{2,n})$	$\cdots$	t, η (5.8)
$A_1(\infty)$	$V(a_{i,n}), i\in Z^>$	$\cdots$	t
$A_2(4)$	$T(b_n)$	$\cdots$	$\zeta_{n+1},\zeta_{n-1},\zeta_n$ (5.7)
$A_2(5)$	$T(b_n),V(a_n)$	$\cdots$	$\begin{cases} \rho_{n-1},\rho_{n+1} \ (5.10) \\ \rho_n,\sigma_n \ (5.11)\end{cases}$
$A_2(6)$	$T(b_n),V(a_{1,n}),V(a_{2,n})$	$\cdots$	η (5.18)
$A_2(\infty)$	$T(b_n),V(a_{k,n}),k\in Z^>$	h_n,k_n constants	None
$A_3(5)$	$T(0),D(b_n)$	(5.12) or (5.14)	(5.13) or (5.15)
$A_3(6)$	$T(0),D(c_n),V(a_n)$	(5.20)	ω (5.20)
$A_3(7)$	$T(0),D(c_n),V(a_{1,n})V(a_{2,n})$	(5.26)	None

For $C_6(9)$ and $C_{10}(10)$ the interactions are specified up to constants (that can depend on n). In all other cases, the arbitrary functions depend on one variable, involving u_k and v_k.

Table IV. The Series D of symmetry algebras. There are three such algebras of dimension 6, 7, and 8, respectively. They all lead to nontrivial invariant interactions of the form (5.22). For $D_3(8)$, the interaction is completely specified. We do not list $D_4(10)$ in Table IV since it coincides with $C_{10}(10)$ of Table III. The algebra $D_5(11)$ corresponds to a linear interaction.

For each interaction we have verified that the given symmetry algebra is the maximal one.

A few words about the interpretation of the invariant interactions. From Eq. (5.1) and the variables (5.2) we see that invariance under $\mathrm{sl}(2,\mathbb{R})_1$ imposes very strong restrictions.

(1) In particular, if the interaction is linear and $\mathrm{sl}(2,\mathbb{R})_1$ invariant, we must have

$$F_n=\sum_{k=n-1}^{n+1} A_k(t)u_k, \quad G_n=\sum_{k=n-1}^{n+1} A_k(t)v_k, \tag{6.1}$$

i.e., the equations (1.1) for u_k and v_k decouple (into identical equations for u_n and v_n separately).

(2) If the interaction terms F_n and G_n in Eq. (5.1) are nonlinear, they always involve many-body forces. That is, they cannot be written as sums of terms of the type $h_n(u_n,v_n)$ or $h_n(u_n,v_{n+1})$, etc. Indeed, each invariant variable $\xi_n,\xi_{n+1},\xi_{n-1}$ itself involves four of the original variables u_i,v_i simultaneously. This many-body character becomes more pronounced when the invariance algebra is larger.

(3) The operators $V(a_n)$ correspond to site-depending dilations,

TABLE II. Series B of symmetry algebras. The algebra includes one pair $Y_u(\lambda_n)$, $Y_v(\lambda_n)$. The interaction has the form (5.4).

No.	Restrictions on λ_n, additional Elements of L_C	Restrictions on ϕ_n and ψ_n	Variables and comments
$B_1(5)$	$\cdots$	$\cdots$	t, ω as in (5.17)
$B_4(6)$	$\lambda_n=e^{a_n t},T(a_n)$	(5.23)	ω (5.23)
$B_3(6)$	$\lambda_n=e^{(a_n-1)t},T(a_n)$	(5.24)	ω (5.23)

TABLE III. Series C symmetry algebras. The algebras contain $sl(2,R)_1, Y_u(1), Y_v(1), Y_u(t), Y_v(t)$, and possibly additional elements. The interaction is as in Eq. (5.31).

No.	Additional elements	Conditions on ϕ_n and ψ_n	Variables
$C_1(7)$	-	...	ω, t (5.32)
$C_2(8)$	$T(a)$	...	$\eta=\omega e^{-2at}$
$C_3(8)$	$D(a)$	$\phi_n=t^{-2}r_n(\eta)$, $\psi_n=t^{-2}s_n(\eta)$	$\eta=\omega t^{-(2a+1)}$
$C_4(8)$	$R(b)$	$\phi_n=(t^2+1)^{-2}r_n(\eta)$, $\psi_n=(t^2+1)^{-2}s_n(\eta)$	$\eta=\omega(t^2+1)^{-1} e^{-2b \arctan t}$
$C_6(9)$	$T(0),D(a)$	$\phi_n=k_n\omega^{-2/(2a+1)}$, $\psi_n=p_n\omega^{-2(2a+1)}$ k_n, p_n constants, $2a+1\neq 0$	None
$C_{10}(10)$	$T(0),D(0),P(0)$	$\phi_n=k_n\omega^{-2}$, $\psi_n=p_n\omega^{-2}$	None

$$\tilde{u}_n=e^{\epsilon a_n}u_n, \quad \tilde{v}_n=e^{\epsilon a_n}v_n. \tag{6.2}$$

Invariance under two such one-dimensional symmetry groups, generated by $\{V(a_{1,n}),V(a_{2,n})\}$, where $a_{1,n}$ and $a_{2,n}$ are two linearly independent functions of n, introduces the symmetry variable

$$\omega_D\equiv(\xi_{n-1})^A(\xi_{n+1})^B(\xi_n)^C, \tag{6.3}$$

as in Eq. (5.8). Here all six variables are coupled together.

(4) The pair of operators $Y_u(\lambda_n),Y_v(\lambda_n)$ induces site-dependent (and time-dependent) shifts of the dependent variables,

$$\tilde{u}_n=u_n+\epsilon\lambda_n(t), \quad \tilde{v}_n=v_n+\epsilon\lambda_n(t). \tag{6.4}$$

The corresponding invariant variable again involves all six variables [see Eq. (5.17)],

$$\omega_T\equiv\lambda_{n-1}\xi_{n+1}-\lambda_n\xi_n-\lambda_{n+1}\xi_{n-1}. \tag{6.5}$$

A special case of the variable ω_T is obtained setting $\lambda_n=\lambda_{n-1}=\lambda_{n+1}=1$. This is the case of Eq. (5.32), where

$$\omega=\omega_S=\xi_{n+1}-\xi_n-\xi_{n-1} \tag{6.6}$$

is invariant with respect to two such translations:

$$\tilde{u}_n=u_n+\epsilon_1+\epsilon_2 t, \quad \tilde{v}_n=v_n+\epsilon_1+\epsilon_2 t \tag{6.7}$$

(ϵ_1 and ϵ_2 are group parameters and hence constants).

A continuation of this study is in progress. It involves several aspects.

The first is a study of the integrability properties of the equations that are completely specified by their symmetries. These are, first of all, those with infinite-dimensional symmetry groups, namely

$$\ddot{u}_n=u_{n+1}\frac{\xi_{n-1}}{\xi_n}h_n+u_nk_n, \quad \ddot{v}_n=v_{n+1}\frac{\xi_{n-1}}{\xi_n}h_n+v_nk_n, \tag{6.8}$$

TABLE IV. Series D of symmetry algebras. The algebra contains $sl(2,R)_1\oplus sl(2,R)_2$. The interaction has the form (5.22).

No.	Additional elements in L_C	Conditions on p_n and q_n	Variables
$D_1(6)$	...	...	χ_{n+1}, χ_{n-1} as in (5.22)
$D_2(7)$	$V(a_n)$	(5.27)	η as in (5.27)
$D_3(8)$	$V(a_{1,n})$, $V(a_{2,n})$	(5.34)	...

with h_n and k_n functions of t or constants [see $A_1(\infty)$ and $A_2(\infty)$ in Table I].

Completely specified equations with finite-dimensional symmetry algebras L are the following ones.

(i)

$$\ddot{u}_n=\left(u_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n+u_nq_n\right)\omega_D^{-2/\Delta}, \quad \ddot{v}_n=\left(v_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n+v_nq_n\right)\omega_D^{-2/\Delta}, \tag{6.9}$$

with ω_D as in Eq. (6.3), Δ as in Eq. (5.25). This is case $A_3(7)$ of Table I.

(ii)

$$\ddot{u}_n=[(u_{n+1}-u_{n-1})p_n+(u_n-u_{n+1})q_n]\omega_S^{-2/(2a+1)},$$

$$\ddot{v}_n=[(v_{n+1}-v_{n-1})p_n+(v_n-v_{n+1})q_n]\omega_S^{-2/(2a+1)}, \tag{6.10}$$

with ω_S as in Eq. (6.6), $p_n, q_n, a\neq-\frac{1}{2}$const. This is case $C_6(9)$ of Table III.

(iii) For $a=0$, Eq. (6.10) is invariant under a ten-dimensional symmetry algebra, namely $C_{10}(10)$ of Table III.

(iv)

$$\ddot{u}_n=(\xi_{n-1})^{-2A/k}(\xi_{n+1})^{-2B/D}(\xi_n)^{[2(A+B-D)/D]}\left[u_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n+u_nq_n\right],$$

$$\ddot{v}_n=(\xi_{n-1})^{-2A/D}(\xi_{n+1})^{-2B/D}(\xi_n)^{[2(A+B-D)/D]}\left[v_{n+1}\frac{\xi_{n-1}}{\xi_n}p_n+v_nq_n\right],$$

with p_n and q_n depending only on n. The constants A and B are given in Eq. (5.9), D in Eq. (5.35).

A further task is to complete the classification, that is, to treat the cases of other sl(2,R) algebras and also of solvable symmetry algebras.

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Chapitre 2

SYMÉTRIES PONCTUELLES DE THÉORIES DE CHAMPS DE TODA GÉNÉRALISÉES

L'article *Point Symmetries of Generalized Toda Field Theories* a été rédigé par Stéphane Lafortune, Pavel Winternitz et Luigi Martina et a été soumis à *Journal of Physics A*.

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Point Symmetries of Generalized Toda Field Theories

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Abstract *A class of two-dimensional field theories with exponential interactions is introduced. The interaction depends on two "coupling" matrices and is sufficiently general to include all Toda field theories existing in the literature. Lie point symmetries of these theories are found for an infinite, semi-infinite and finite number of fields. Special attention is accorded to conformal invariance and its breaking.*

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1 Introduction

The purpose of this article is to investigate the Lie point symmetries of a large class of "generalized Toda field theories". The class is characterized by the equation

$$u_{n,xy} = F_n, \qquad F_n = \sum_{m=n-n_1}^{n+n_2} K_{nm} \exp\left(\sum_{l=m-n_3}^{m+n_4} H_{ml} u_l\right) \tag{1.1}$$

where K and H are some real constant matrices and $n_1, \ldots, n_4$ are some finite non-negative integers. The range of n may be infinite, semi-infinite or finite, hence the matrices K and H may also be infinite, semi-infinite, or finite.

If the range of n is finite, K and H may be rectangular, not necessarily square. We assume that all the rows in H are different, that H contains no zero rows and K no zero columns. In all the cases we assume that the range of the interaction on the right hand side of eq. (1.1) is finite, hence the finite summation limits in both sums. "Generalized Toda lattices" are obtained from eq. (1.1) by symmetry reduction, using translational invariance, i.e. restricting to solutions of the form $u_n(x, y) = w_n(t)$ where $t = x + \lambda y$.

Toda lattices and their generalisations, Toda field theories, represent one of the most interesting, rich and fruitful developments in the realm of completely integrable systems. The original Toda lattice was introduced by M. Toda [1, 2] who found analytical solitons and periodic solutions in a discrete lattice with an exponential potential involving nearest neighbour interactions. It was also found that the Toda lattice admits a Lax representation and all the usual attributes of integrability [3, 4]. The Toda lattice was generalized to integrable lattices related to the root systems of simple Lie algebras [5] - [8]. The considered lattices can be finite, infinite, semi-infinite, or periodic.

The attractive features of Toda lattices have been generalized to two space dimensions in several different ways [9] - [19].

All of them can be recovered from eq. (1.1) by specifying the matrices K and H. Thus, the Mikhailov-Fordy-Gibbons field theories [9, 10] (for infinitely many fields)

$$u_{n,xy} = e^{u_{n-1}-u_n} - e^{u_n - u_{n+1}} \tag{1.2}$$

are obtained by putting $H_{nn-1} - H_{nn} = 1$, $K_{nn} = -K_{nn+1} = 1$ and all other components to zero. A class of Toda field theories

$$u_{n,xy} = \sum_{m=n-n_1}^{n+n_2} K_{nm} e^{u_m}, \tag{1.3}$$

studied by Leznov and Saveliev [12, 13], Olive, Turok and others [14] - [17] (usually for a finite number of fields u_n) are obtained by setting $H = I$ and taking K to be the Cartan matrix of a semisimple Lie algebra (or an affine one).

A further class of Toda field theories, also studied by Leznov and Saveliev [13, 14], by Bilal and Gervais [17], and Babelon and Bonora [18] (for a finite number of fields) can be written as

$$u_{n,xy} = \exp \sum_{l=m-n_3}^{m+n_4} H_{nl} u_l \tag{1.4}$$

and is obtained by taking $K = I$ and H as a Cartan matrix.

In this article we will be interested in point symmetries of the system (1.1), rather than in questions of integrability, or explicit solutions. The symmetries we are interested in will include conformal invariance, whenever it is present, and gauge invariance, not however higher, or generalized symmetries, be they local, or not.

In Section 2 we consider infinite Toda field theories, i.e. take $-\infty < n < \infty$. In this case eq. (1.1) can be viewed as a differential-difference equation. Continuous Lie symmetries of such equations have been studied using several different approaches [20] - [29]. We shall follow that of Ref. [20] - [24], using both the "intrinsic method" and the "differential equation method" [21].

In Section 3 we turn to finite Toda field theories, when we have $1 \leq n \leq N < \infty$ in eq. (1.1). Eq. (1.1) in this case represents a system of N differential equations and its point symmetries can be obtained in a standard manner [30, 31]. We first obtain general results, then specify the matrices H and K in several different ways.

Section 4 is devoted to semi-infinite Toda field theories, i.e. $0 \leq n < \infty$. Again we first obtain general results, then specify the matrices H and K, inforcing the cut-off at $n = 0$ in several different ways.

Some conclusions are drawn in Section 5.

2 Symmetries of Generalized $\infty-$Toda Field Theories

2.1 *General Results*

Let us consider eq. (1.1) with n in the range $-\infty < n < \infty$. We follow the "differential equation method" described in Ref.[21] and look for transformations of the form

$$\tilde{\vec{x}} = \Lambda_g(\vec{x}, \{u_k\}), \quad \tilde{u}_n = \Omega_g(\vec{x}, n, \{u_k\}), \quad \tilde{n} = n, \tag{2.1}$$

where we have used the notation $\vec{x} \equiv (x, y)$, $\tilde{\vec{x}} \equiv (\tilde{x}, \tilde{y})$, taking solutions of eq. (1.1) into solutions. The notation $\{u_k\}$ indicates that the new variables can depend on all the fields $\{u_k\}_{k\in\mathbf{Z}}$.

The Lie group transformation (2.1) is generated by a Lie algebra of vector fields of the form

$$\hat{v} = \xi(x, y, \{u_k\})\partial_x + \eta(x, y, \{u_k\})\partial_y + \sum_{j=-\infty}^{\infty} \phi_j(x, y, \{u_k\})\partial_{u_j}. \tag{2.2}$$

The prolongation of this vector field is constructed in the same manner as for differential equations [30, 31] (albeit an infinite system of them). For a general equation of the form

$$E_n = u_{n,xy} - F_n(x, y, \{u_k\}) = 0, \tag{2.3}$$

we require

$$pr^{(2)}\hat{v}E_n \mid_{E_n=0} = 0. \tag{2.4}$$

It was shown quite generally [21] that for eq (2.3) with F_n any sufficiently smooth function depending on at least one function u_k, $k \neq n$, the vector field (2.2) satisfying eq. (2.4) will have the form

$$\xi = \xi(x), \quad \eta = \eta(y), \quad \phi_n = \sum_{k=-\infty}^{\infty} A_{nk}u_k + B_n(x, y), \tag{2.5}$$

where $A = \{A_{n\alpha}\}$ is a constant (infinite) matrix. The functions in eq. (2.5) must satisfy a remaining determining equation, namely

$$B_{n,xy} - (\xi_x + \eta_y)F_n + \sum_{\alpha=-\infty}^{\infty} A_{n\alpha}F_\alpha - \xi F_{n,x} - \eta F_{n,y}$$
$$- \sum_{\alpha=-\infty}^{\infty} \left(\sum_{\beta=-\infty}^{\infty} A_{\alpha\beta}u_\beta + B_\alpha \right) F_{n,u_\alpha} = 0, \tag{2.6}$$

where F_{n,u_α} is the derivative of F_n with respect to the variable u_α.

Let us now specify the function F_n to be a sum of exponentials as in eq. (1.1). There are three types of terms in eq. (2.6): those independent of u_n, linear in u_n times exponentials and pure exponentials. Each type of term must vanish separately. Since H has no zero rows we get the determining equations

$$B_{n,xy} = 0, \tag{2.7}$$

$$\sum_{\alpha=-\infty}^{\infty} A_{\alpha m}F_{n,u_\alpha} = 0, \tag{2.8}$$

$$-(\xi_x + \eta_y)F_n + \sum_{\alpha=-\infty}^{\infty} A_{n\alpha}F_\alpha - \sum_{\alpha=-\infty}^{\infty} B_\alpha F_{n,u_\alpha} = 0. \tag{2.9}$$

Eq. (2.8) can be rewritten as

$$\sum_{\alpha\beta} K_{n\beta}H_{\beta\alpha}A_{\alpha m} \exp\left(\sum_{\gamma} H_{\beta\gamma}u_\gamma \right) = 0. \tag{2.10}$$

All exponentials in eq. (2.10) are linearly independent (since all rows in H are different), so the equation must hold for each β separately and the exponentials can be dropped. Moreover, the factor $K_{n\beta}$ can be dropped (since K has no zero column). We find that eq. (2.8) in this case implies an equation for the matrix A, namely

$$\sum_{\alpha=-\infty}^{\infty} H_{n\alpha}A_{\alpha m} = 0, \tag{2.11}$$

or in matrix form $HA = 0$ (however, the matrices are infinite).

Let us now turn to eq. (2.9) and make use of the finite range of the interaction F_n in eq. (1.1). We have

$$\frac{\partial F_n}{\partial u_k} = 0, \quad n + n_u < k \text{ or } k < n - n_d \tag{2.12}$$

for some non-negative integers n_u and n_d. In eq. (2.9) all exponentials, obtained after substituing for F_n from eq. (1.1), are linearly independent. This allows us to split eq. (2.9) into two types of equations. These are obtained as coefficients of $\exp\left(\sum_l H_{ml}u_l\right)$, with $m \in [n-n_1, n+n_2]$ and with m outside this interval, respectively. Thus we have:

$$-K_{nm}\left[(\xi_x + \eta_y) + \sum_{\alpha=m-n_3}^{m+n_4} B_\alpha H_{m\alpha}\right] + \sum_{\rho=m-n_1}^{m+n_2} A_{n\rho}K_{\rho m} = 0, \tag{2.13}$$

$$m \in [n-n_1, n+n_2],$$

$$\sum_{\rho=m-n_1}^{m+n_2} A_{n\rho}K_{\rho m} = 0, \quad m \notin [n-n_1, n+n_2]. \tag{2.14}$$

We shall show that eq. (2.14) actually holds for all values of m so that eq. (2.13) can be simplified. To do this, we view eq. (2.11) as a difference equation for $A_{\alpha m}$. To make this explicit we restrict H and K to be band matrices, with finite bands of constant width

$$H_{nm} = H_{n,n+\sigma} = \begin{cases} h_\sigma(n) & \sigma \in [p_1, p_2] \\ 0 & \sigma \notin [p_1, p_2] \end{cases}, \quad h_{p_1}(n) \neq 0, \quad h_{p_2}(n) \neq 0. \tag{2.15}$$

Similarly

$$K_{nm} = K_{m+\sigma,m} = \begin{cases} k_\sigma(m) & \sigma \in [q_1, q_2] \\ 0 & \sigma \notin [q_1, q_2] \end{cases}, \quad k_{q_1}(m) \neq 0, \quad k_{q_2}(m) \neq 0. \tag{2.16}$$

In these notations we see that eq. (2.11) is a linear difference equation for $A_{\sigma m}$ with $p_1 - p_2 + 1$ terms

$$\sum_{\sigma=p_1}^{p_2} h_\sigma(n) A_{\sigma+n,m} = 0. \tag{2.17}$$

Equation (2.17) determines the dependence of A_{nm} on n. Indeed the linear difference equation

$$\sum_{\sigma=p_1}^{p_2} h_\sigma(n)\ \psi_{\sigma+n} = 0 \tag{2.18}$$

has $p_2 - p_1$ linearly independent solutions, a basis of which we denote by $\{\psi^j_n,\ j = 1, 2, \ldots, p_2 - p_1\}$. Thus, we have

$$A_{nm} = \sum_{j=1}^{p_2-p_1} \psi^j_n C_{jm}, \tag{2.19}$$

where C_{jm} are constants to be determined by the remaining determining equations (2.13) and (2.14). In order to analyze them, let us define the quantities

$$Q_{nm} = \sum_{\sigma=m-n_1}^{m+n_2} A_{n\sigma} K_{\sigma m}.$$

From eq. (2.14) we have $Q_{nm} = 0$ for m "sufficiently far away" from n. But, by using the expansion (2.19), we get

$$Q_{nm} = \sum_{j=1}^{p_2-p_1} \psi^j_n \sum_{\sigma=m-n_1}^{m+n_2} C_{j\sigma} K_{\sigma m}$$

which, because of the linear independency of the ψ^j_n, implies

$$\sum_{\sigma=m-n_1}^{m+n_2} C_{j\sigma} K_{\sigma m} = 0 \tag{2.20}$$

for all values of m, since this relation does not depend on n and the index m is no longer tied to n. In other words, if $Q_{nm} = 0$ holds for certain values of n and m, as in eq. (2.14), then that equation must hold for all values. As in the case of eq. (2.17), we introduce a solution basis $\{\phi^l_m, l = 1, \ldots, q_2 - q_1\}$ for the equation

$$\sum_{\sigma=q_1}^{q_2} k_\sigma(m)\ \phi_{\sigma+m} = 0. \tag{2.21}$$

The general solution of eq. (2.20) now is

$$C_{jm} = \sum_{l=1}^{q_2-q_1} q_{jl}\, \phi^l_m,$$

where q_{jl} are arbitrary constants. The expression (2.19) for A_{nm} is replaced by

$$A_{nm} = \sum_{j=1}^{p_2-p_1} \sum_{l=1}^{q_2-q_1} q_{jl}\, \psi_n^j \phi_m^l . \tag{2.22}$$

A further consequence is that the last term in eq. (2.13) can be dropped. Then, using the general solution for eq. (2.7)

$$B_n(x,y) = \beta_n(x) + \gamma_n(y) ,$$

we separate the x from the y dependence in eq. (2.13) and reduce it to two inhomogeneous difference equations for $\beta_n(x)$ and $\gamma_n(y)$. The general solutions of which are

$$\beta_n(x) = \sum_{j=1}^{p_2-p_1} r_j(x)\, \psi_n^j - b_n \xi_x(x), \qquad \gamma_n(x) = \sum_{j=1}^{p_2-p_1} s_j(y)\; \psi_n^j - b_n \eta_y(y), \tag{2.23}$$

where b_n is an arbitrarily chosen solution of the inhomogeneous difference equation

$$\sum_{\sigma=p_1}^{p_2} h_\sigma(n)\; b_{\sigma+n} = 1. \tag{2.24}$$

Furthermore, in eq. (2.23) the functions $r_j(x)$ and $s_j(y)$ are arbitrarily chosen. Finally, we obtain the following theorem.

Theorem 1 *Consider all the generalized Toda theories of the form (1.1) for infinitely many fields $u_n(x,y)$, where the coupling matrices H and K satisfy eqs. (2.15) and (2.16). Their Lie point symmetry algebra is infinite-dimensional and a basis for it is given by the following vector fields:*

$$\hat{X}(\xi) = \xi(x)\partial_x - \xi_x(x) \sum_{n=-\infty}^{\infty} b_n \partial_{u_n}, \quad \hat{Y}(\eta) = \eta(y)\partial_y - \eta_y(y) \sum_{n=-\infty}^{\infty} b_n \partial_{u_n}, \tag{2.25}$$

$$\hat{U}_j(r_j) = r_j(x) \sum_{n=-\infty}^{\infty} \psi_n^j \partial_{u_n}, \quad \hat{V}_j(s_j) = s_j(y) \sum_{n=-\infty}^{\infty} \psi_n^j \partial_{u_n} \quad (j = 1, \ldots, p_2 - p_1), \tag{2.26}$$

$$\hat{Z}_{jl} = \left(\sum_{m=-\infty}^{\infty} \phi_m^l u_m \right) \left(\sum_{n=-\infty}^{\infty} \psi_n^j \partial_{u_n} \right) \quad (j = 1, \ldots, p_2 - p_1;\ l = 1, \ldots, q_2 - q_1). \tag{2.27}$$

The functions $\xi(x)$, $\eta(y)$, $r_j(x)$ and $s_j(y)$ are arbitrary, all the other quantities are determined by solving the linear difference eqs. (2.18), (2.21) and (2.24).

As far as interpretation is concerned, we see that the generalized $\infty-$ Toda lattice (1.1) is always conformally invariant, since the vector fields (2.25) generate arbitrary reparametrizations of x and y, accompanied by appropriate transformations of the fields u_n. More specifically, the conformal transformations leaving eq. (1.1) invariant are

$$\begin{aligned} &\tilde{x} = \psi(x,\lambda), \quad \tilde{y} = \chi(y,\lambda), \\ &\tilde{u}_n(\tilde{x},\tilde{y}) = u_n(x,y) - b_n \ln\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\frac{\mathrm{d}\chi}{\mathrm{d}y}\right), \end{aligned} \tag{2.28}$$

where $\psi(x,\lambda)$ and $\chi(y,\lambda)$ are arbitrary functions of x and y, related to $\xi(x)$ and $\eta(y)$ by the relations

$$\begin{aligned} &\tilde{x} = \psi(x,\lambda) = T^{-1}(\lambda + T(x)), \\ &\tilde{y} = \chi(y,\lambda) = S^{-1}(\lambda + S(y)), \end{aligned} \tag{2.29}$$

with

$$T(x) = \int_0^x \frac{\mathrm{d}s}{\xi(s)}, \qquad S(y) = \int_0^y \frac{\mathrm{d}t}{\eta(t)}. \tag{2.30}$$

The vector fields $\hat{U}_j(r)$ and $\hat{V}_j(s)$ generate gauge transformations: certain functions obtained by integrating the vector fields can be added to any solution. Formally, the operators $\hat{Z}_{jl}$ generate linear transformations among components of solutions. However, the sums are over infinite range, so convergence problems may arise. Moreover, we have

$$\partial_{xy}\left(\sum_m \phi^l_m u_m\right) = 0 \tag{2.31}$$

as a consequence of eq. (2.21). In other words, if the equation (2.21) admits non trivial solutions, than one can always perform a linear transformation among the u_n's, in such a way $q_2 - q_1$ new fields $v_l = \sum_m \phi^l_m u_m$, satisfying the wave equation $\partial_x\partial_y v_l = 0$, are replaced in the Toda system.

As stated in Theorem 1, the problem of finding all symmetries of eq. (1.1) reduces to solving the recursion relations (2.18), (2.21) and (2.24). In general, this may not be possible analytically in closed form. Well developed techniques exist for solving homogeneous and inhomogeneous difference equations with constant coefficients [32, 33]. This is the case that occurs for all generalized Toda field theories that we found in the literature: $h_\sigma(n)$ and $k_\sigma(m)$ do not depend on n and m, respectively. The nonzero commutation relations for the symmetry algebra of the generalized $\infty-$Toda theory (1.1) are:

$$\begin{gathered}
\left[\hat{X}(\xi_1),\,\hat{X}(\xi_2)\right]=\hat{X}(\xi_1\,\xi_{2,x}-\xi_{1,x}\,\xi_2),\quad \left[\hat{Y}(\eta_1),\hat{Y}(\eta_2)\right]=\hat{Y}(\eta_1\,\eta_{2,y}-\eta_{1,y}\,\eta_2),\\
\left[\hat{X}(\xi),\,\hat{U}_j(r)\right]=\hat{U}_j(\xi r_x),\quad \left[\hat{Y}(\eta),\,\hat{V}_j(s)\right]=\hat{V}_j(\eta\, s_y),\\
\left[\hat{X}(\xi),\,\hat{Z}_{jl}\right]=-\hat{U}_j\left(\xi_x\sum_n b_n\phi_n^l\right),\quad \left[\hat{Y}(\eta),\,\hat{Z}_{jl}\right]=-\hat{V}_j\left(\eta_y\sum_n b_n\phi_n^l\right),\\
\left[\hat{U}_a(r),\hat{Z}_{jl}\right]=\hat{U}_j\left(r\sum_m \phi_m^l\psi_m^a\right),\quad \left[\hat{V}_a(s),\hat{Z}_{jl}\right]=\hat{V}_j\left(s\sum_m \phi_m^l\psi_m^a\right),\\
\left[\hat{Z}_{ab},\hat{Z}_{cd}\right]=\left(\sum_m \phi_m^d\psi_m^a\right)\hat{Z}_{cb}-\left(\sum_m \phi_m^b\psi_m^c\right)\hat{Z}_{ad}.
\end{gathered}\tag{2.32}$$

The algebra of vector fields $\hat{Z}_{jl}$ is finite dimensional (its dimension is $d=(p_2-p_1)\times(q_2-q_1)$). However, its isomorphism class cannot be determined without specifying the functions ϕ_m^l and ψ_n^j, i.e. the matrices H and K in (1.1). In all examples in the literature, we have either $d=1$, or $d=0$. It is however easy to invent examples in which $\left\{\hat{Z}_{jl}\right\}$ is simple, semisimple, solvable, or whatever we postulate a priori.

The overall structure of the obtained Lie algebra is

$$\left(\left\{\hat{X}\right\}\oplus\left\{\hat{Y}\right\}\right)\ni\left(\left\{\hat{Z}\right\}\ni\left(\hat{U}\oplus\hat{V}\right)\right).\tag{2.33}$$

If $\left\{\hat{Z}\right\}$ is solvable, then (2.33) amounts to a Levi decomposition, since both $\left\{\hat{X}\right\}$ and $\left\{\hat{Y}\right\}$ are centerless Virasoro algebras and hence simple. We recall that the Levi theorem does not hold for infinite-dimensional Lie algebras and a Levi decomposition does not necessarily exist.

Let us sum up the general results obtained so far for the symmetries of the generalized $\infty-$Toda field theories (1.1) under the constraints imposed in Theorem 1.

1. The theory is always conformally invariant, since the inhomogeneous equation (2.24) always has a solution.
2. The theory allows gauge transformations $\hat{U}$ and $\hat{V}$ if $p_2 - p_1 \geq 1$.
3. The transformations of type $\hat{Z}$ exist if $(p_2 - p_1)(q_2 - q_1) \geq 1$.

2.2 *Special cases*

1. The Mikhailov-Fordy-Gibbons two dimensional ∞-Toda system (1.2)

We have

$$h_{-1}(n) = -h_0(n) = 1, \quad \text{and } k_{-1}(n) = -k_0(n) = -1, \tag{2.34}$$

so $p_2 - p_1 = q_2 - q_1 = 1$. From eqs. (2.18) and (2.21) we have

$$\psi_m = \phi_m = 1.$$

Equations (2.23) and (2.24) in this case imply

$$\beta_n = \beta(x) + n\xi_x, \quad \gamma_n = \gamma(y) + n\eta_y.$$

From Theorem 1 we now obtain all symmetries of eq (1.2), namely

$$\begin{aligned} \hat{X}(\xi) = \xi(x)\partial_x + \xi_x \sum_{n=-\infty}^{\infty} n\partial_{u_n},\ \hat{Y}(\eta) &= \eta(y)\partial_y + \eta_y \sum_{n=-\infty}^{\infty} n\partial_{u_n}, \\ \hat{U} = \beta(x) \sum_{n=-\infty}^{\infty} \partial_{u_n}, \qquad \hat{V} &= \gamma(y) \sum_{n=-\infty}^{\infty} \partial_{u_n}, \\ \hat{Z} = \left(\sum_{m=-\infty}^{\infty} u_m\right)\left(\sum_{n=-\infty}^{\infty} \partial_{u_n}\right). \end{aligned} \tag{2.35}$$

The generators $\hat{X}, \hat{Y}, \hat{U}$ and $\hat{V}$ were obtained in ref.[21] using the so called "intrinsic method". The generator $\hat{Z}$ was not obtained there and cannot be obtained by the intrinsic method.

2. The Toda field theory (1.3)

We take $H = I$. Then equations (2.18), (2.21) and (2.24) in this case imply

$$\beta_m = -\xi_x, \quad , \gamma_m = -\eta_y, \quad A_{nm} = 0.$$

The theory is only conformally invariant

$$\hat{X}(\xi) = \xi(x)\partial_x - \xi_x \sum_n \partial_{u_n}, \quad \hat{Y}(\eta) = \eta(y)\partial_y - \eta_y \sum_n \partial_{u_n} \tag{2.36}$$

and no further symmetries are obtained.

3. The Toda field theories (1.4)

We take $K = I$ and relation (2.21) implies

$$A_{nm} = 0.$$

The remaining equations (2.24) cannot be solved explicitely for general $h_\sigma(m)$, but as said above, we can easily deal with in the constant coefficients case. As an example, let us restrict to the case when H is the A_∞ Cartan matrix (This is the A_N Cartan matrix for $N \to \infty$, where the limit is taken symmetrically from a fixed, but not extremal, vertex in the corresponding Dynkin diagram). Thus we have

$$h_{-1} = h_{+1} = -1, \quad h_0 = 2, \tag{2.37}$$

the solutions (2.23) become

$$\beta_n = \frac{n^2}{2}\,\xi_x + n\,r_2(x) + r_1(x), \quad \gamma_n = \frac{n^2}{2}\,\eta_y + n\,s_2(y) + s_1(y). \tag{2.38}$$

The symmetry algebra is

$$\begin{aligned}
\hat{X}(\xi) &= \xi(x)\partial_x + \frac{1}{2}\xi_x \sum_{n=-\infty}^{\infty} n^2 \partial_{u_n}, \quad \hat{Y}(\eta) = \eta(y)\partial_y + \frac{1}{2}\eta_y \sum_{n=-\infty}^{\infty} n^2 \partial_{u_n}, \\
\hat{U}_1(r_1) &= r_1(x) \sum_{n=-\infty}^{\infty} \partial_{u_n}, \qquad \hat{V}_1(s_1) = s_1(y) \sum_{n=-\infty}^{\infty} \partial_{u_n}, \\
\hat{U}_2(r_2) &= r_2(x) \sum_{n=-\infty}^{\infty} n\,\partial_{u_n}, \qquad \hat{V}_2(s_2) = s_2(y) \sum_{n=-\infty}^{\infty} n\partial_{u_n},
\end{aligned} \tag{2.39}$$

where $\xi(x)$, $\eta(y)$, $r_1(x)$, $s_1(y)$, $r_2(x)$ and $s_2(y)$ are arbitrary smooth functions.

3 Symmetries of Finite Generalized Toda Field Theories

3.1 *General Results*

In this case we have a system of N partial differential equations in N fields $u_n(x,y)$, namely

$$u_{n,xy} = F_n, \qquad F_n = \sum_{m=1}^{M} K_{nm} \exp\left(\sum_{l=1}^{N} H_{ml} u_l\right) \quad (1 \le n \le N). \tag{3.1}$$

The "coupling constant" matrices H and K satisfy $H \in \mathbb{R}^{M\times N}$ and $K \in \mathbb{R}^{N\times M}$. The system (3.1) could arise in a quite general field theory with Lagrangian

$$\mathcal{L} = \frac{1}{2} \sum_{m,n=1}^{N} \kappa_{mn} \partial_x u_m \partial_y u_n - \sum_{m=1}^{M} c_m \exp\left(\sum_{l=1}^{N} H_{ml} u_l\right) \qquad (c_m \neq 0), \tag{3.2}$$

with

$$K = L^{-1} H^T C, \qquad L = \frac{\kappa + \kappa^T}{2}, \qquad C = \operatorname{diag}(c_1, \ldots, c_N). \tag{3.3}$$

Some general considerations concerning the system (3.1) are in order.

First, if either K, or H (or both) allow an inverse, or at least a left inverse, then this system can be simplified. Indeed, let K^{-1} exist. We put $u_n = \sum_m K_{nm} \rho_m$ and obtain

$$\rho_{m,xy} = \exp\left(\sum_{l=1}^{M} (HK)_{ml} \rho_l\right), \quad 1 \le m \le M. \tag{3.4}$$

Conversely, let H^{-1} exist and put $w_j = \sum_l H_{jl} u_l$, we obtain

$$w_{m,xy} = \sum_{j=1}^{M} (HK)_{mj} e^{w_j}, \quad 1 \le m \le M. \tag{3.5}$$

In other words, one of the matrices H, or K can be normalized to I_M, if it is left invertible.

The second comment is that the system (3.1) with $K = I$ admits Lie-Bäcklund transformations, and in this sense is completely integrable, if the matrix H is a Cartan, or a generalized Cartan matrix [19].

We mention that in the case of the infinite Toda field theories the matrices H and K in general have nontrivial kernels, are hence not invertible and we cannot normalize them.

Let us now turn to the Lie point symmetries of the system (3.1). We write a general element of the symmetry algebra in the form (2.2) (with the sum in the range $1 \leq n \leq N$), apply its prolongation to eq. (3.1) as in eq. (2.4). From the determining equations we find that for any F_n in eq. (3.1), in complete analogy with the ∞-Toda theory, a general element of the symmetry algebra will have the form (2.5), the summation being from 1 to N.

Two determining equations remain and they depend on the specific form of F_n in eq. (3.1). Making use of the fact that all the exponentials are linearly independent (no two rows in H coincide) and that the matrix K has no zero column, we reduce the remaining determining equations to two matrix relations

$$HA = 0, \tag{3.6}$$

$$\left[\left(A - \left(\xi_x + \eta_y\right) I\right) K\right]_{nm} = K_{nm} \left(HB\right)_m \quad (1 \leq n \leq N, \;\; 1 \leq m \leq M)\,. \tag{3.7}$$

We multiply eq. (3.7) by H from the left and use (3.6) to obtain

$$-\left(\xi_x + \eta_y\right) (H\,K)_{km} = (H\,K)_{km} \left(HB\right)_m \quad \forall k, m. \tag{3.8}$$

If the matrix HK has no zero column, then we obtain

$$HB = -\left(\xi_x + \eta_y\right) \bar{\mathbf{1}}_M, \tag{3.9}$$

where $\bar{\mathbf{1}}_M = (1, \ldots, 1)^T \in \mathbb{R}^M$, and from eq. (3.7)

$$AK = 0. \tag{3.10}$$

Thus, matrix A must satisfy the same two homogeneous equations (3.6) and (3.10) as in the infinite case. Furthermore, if $\bar{\mathbf{1}}_M$ is in the image of H, then we define $\mathbf{b}_N \in \mathbb{R}^N$ to be an arbitrarily chosen (but specified) solution of the inhomogeneous equation

$$H\mathbf{b}_N = \bar{\mathbf{1}}_M. \tag{3.11}$$

The results of these considerations can be summed up as follows

Theorem 2 *Consider the generalized Toda field theories (3.1) with a finite number of fields N. Assume that all rows in H are different and that the matrix HK has no zero column. Then 3 types of symmetries can occur and they depend on the properties of the fundamental spaces of the matrices H and K. The symmetries are of the same form as in Theorem 1, except that all summations range from 1 to N. However, if $\bar{\mathbf{1}}_M \in Im(H)$, then ξ and η are arbitrary functions of x and y, respectively, and the theory is conformally invariant. The quantities b_n are the components of the vector $\mathbf{b}_N$, itself an arbitrary solution of eq. (3.11). Otherwise, if $\bar{\mathbf{1}}_M \notin Im(H)$, the theory is invariant only under the Poincaré group, generated by*

$$\hat{P}_1 = \partial_x, \quad \hat{P}_2 = \partial_y, \quad \hat{L} = x\partial_x - y\partial_y. \tag{3.12}$$

Gauge transformations exist only if H is not invertible. Analogously to the formulas (2.26), r_j and s_j are arbitrary functions and the vectors ψ^j span $Ker\,(H)$. Finally, the vectors ϕ^l span the left kernel of K. If this space is not zero, then $\dim\left(Ker\left(K^T\right)\right) \times \dim\left(Ker\,(H)\right)$ *symmetries of the form (2.27) are admitted.*

From Theorem 2, contrary to the case of infinitely many fields, conformal invariance is not a priori guaranteed, but it imposes restrictions on the image of H. Gauge symmetries exist only if the matrix H has a nonzero kernel.

3.2 *Special cases*

1. **The Mikhailov-Fordy-Gibbons Toda theory and generalizations**
   Consider the field equation

$$\mathbf{U}_{xy} = \frac{\mu^2}{\beta} \sum_{i=1}^{N} \frac{\alpha_i}{\alpha_i^2} \exp\left(\beta \alpha_i \cdot \mathbf{U}\right), \tag{3.13}$$

where $\mathbf{U} = (u_1, \ldots, u_N)$ is an N-ple of real fields and $(\alpha_1, \ldots, \alpha_N)$ denote the simple roots of a classical simple finite Lie algebra. Equations (3.13) above take the form (1.2) for all n satisfying $N_0 \le n \le N-1$. For $n = N$ we obtain

$$u_{N,xy} = \exp\left(u_{N-1} - u_N\right). \tag{3.14}$$

The equations for $1 \le n < N_0$ are different for each Cartan series. The number N_0 is equal to 2 for A_N, B_N, C_N, and 3 for D_N.

For the A_N algebra we have

$$u_{1,xy} = -\exp\left(u_1 - u_2\right). \tag{3.15}$$

Conformal and gauge transformations are exactly the same as given in eq. (2.35) (except that the summations are from 1 to N).

For the B_N algebra we have

$$u_{1,xy} = \exp\left(-u_1\right) - \exp\left(u_1 - u_2\right). \tag{3.16}$$

Conformal transformations are as in eq. (2.35) (with the same comment about the summations) and there is no gauge invariance.

For the C_N algebra we have

$$u_{1,xy} = -\exp\left(u_1 - u_2\right) + 2\exp\left(-2u_1\right). \tag{3.17}$$

The only symmetry is conformal invariance, generated by

$$\begin{aligned}
\hat{X}(\xi) &= \xi(x)\partial_x + \xi_x \sum_{n=1}^{N} \left(n - \frac{1}{2}\right) \partial_{u_n}, \\
\hat{Y}(\eta) &= \eta(y)\partial_y + \eta_y \sum_{n=1}^{N} \left(n - \frac{1}{2}\right) \partial_{u_n}
\end{aligned} \tag{3.18}$$

Finally, for the D_N algebra we have

$$\begin{aligned}
u_{1,xy} &= \exp\left(-u_1 - u_2\right) - \exp\left(u_1 - u_2\right), \\
u_{2,xy} &= \exp\left(-u_1 - u_2\right) + \exp\left(u_1 - u_2\right) - \exp\left(u_2 - u_3\right).
\end{aligned} \tag{3.19}$$

Again, the only symmetry is conformal invariance, in this case generated by

$$\begin{aligned}
\hat{X}(\xi) &= \xi(x)\partial_x + \xi_x \sum_{n=1}^{N} (n-1)\,\partial_{u_n}, \\
\hat{Y}(\eta) &= \eta(y)\partial_y + \eta_y \sum_{n=1}^{N} (n-1)\,\partial_{u_n}.
\end{aligned} \tag{3.20}$$

We mention that the infinite system (1.2) can also be reduced to the finite one by imposing periodicity $u_{N+1} = u_1$. In this case $\mathbf{\bar{1}}_N$ is not contained in $Im(H)$ and there is no conformal invariance. Thus, the symmetry is given by the two dimensional Poincaré algebra (3.12) and by the gauge generators given in (2.35).

2. The Toda field theory (1.3)

The symmetries are the same in the finite case as in the infinite one, namely the conformal transformations generated by (2.36) (for any finite matrix k).

3. The finite Toda theories (1.4)

Since the Cartan matrix H is invertible, this theory is equivalent to that described by eq. (1.3) in the sense of eqs. (3.4) and (3.5). Hence this theory is always and only conformally invariant. However, the generators of the vector fields take a slightly different form, which we report for a subsequent discussion.

For the A_N algebra the generators are given by

$$\hat{W} = \xi(x)\partial_x + \eta(y)\partial_y + \frac{1}{2}(\xi_x + \eta_y)\sum_{n=1}^{N} n(n-N-1)\,\partial_{u_n}. \tag{3.21}$$

For the B_N algebra, the symmetry generator is given by

$$\hat{W} = \xi(x)\partial_x + \eta(y)\partial_y - \frac{1}{4}(\xi_x + \eta_y)\times \tag{3.22}$$

$$\left\{N(N+1)\,\partial_{u_1} + 2\sum_{n=2}^{N}\left[N(N+1) - n(n-1)\right]\partial_{u_n}\right\}. \tag{3.23}$$

For the C_N algebra, the symmetry generator is given by

$$\hat{W} = \xi(x)\partial_x + \eta(y)\partial_y + \frac{1}{2}(\xi_x + \eta_y)\sum_{n=1}^{N}\left[n(n-2) - N^2 + 1\right]\partial_{u_n}. \tag{3.24}$$

Finally, for the D_N algebra ($N \geq 4$), one has

$$\hat{W} = \xi(x)\partial_x + \eta(y)\partial_y - \frac{1}{4}(\xi_x + \eta_y)\times \tag{3.25}$$

$$\left\{N(N-1)(\partial_{u_1} + \partial_{u_2}) + 2\sum_{n=3}^{N}\left[N(N-1) - (n-2)(n-1)\right]\partial_{u_n}\right\} \tag{3.26}$$

4 Symmetries of Generalized Semi-Infinite Toda Field Theories

4.1 *General Results*

Let us now restrict the range of the discrete variable n to be $1 \leq n < \infty$. Both the equations (1.1) of the generalized Toda field theories, and their symmetries will be modified. The matrices H and K will no longer be pure band matrices but will have the form

$$H = \begin{pmatrix} H_{1,1} & \dots & \dots & \dots & H_{1,N} & & & & & & \\ \dots & \dots & \dots & \dots & \dots & & & & & & \\ H_{M,1} & \dots & \dots & \dots & H_{M,N} & & & & & & \\ & H_{M+1,M+1+p_1} & & & \dots & \dots & \dots & \dots & H_{M+1,M+1+p_2} & & \\ & & & & H_{M+2,M+2+p_1} & \dots & \dots & \dots & \dots & \dots & H_{M+2,M+2+p_2} \\ & & & & & \ddots & \ddots & \ddots & & \ddots & \ddots \end{pmatrix}, \tag{4.1}$$

where $M+p_1 \leq N \leq M+p_2$ and the void entries are equal to zero. Similarly, the matrix K takes the form

$$K = \begin{pmatrix} K_{1,1} & \dots & K_{1,N'} & & & \\ \dots & \dots & \dots & K_{N'+1+q_1,N'+1} & & \\ K_{M',1} & \dots & K_{M',N'} & \dots & K_{N'+2+q_1,N'+2} & \\ & & & \dots & \dots & \ddots \\ & & & K_{N'+1+q_2,N'+1} & \dots & \ddots \\ & & & & K_{N'+2+q_2,N'+2} & \ddots \\ & & & & & \ddots \end{pmatrix} \tag{4.2}$$

where $N' + q_1 \leq M' \leq N' + q_2$. Although one could easily construct non trivial models, which do not fit in the given scheme, they seem quite artificial and, moreover, all the cases which we found in the literature satisfy the above restrictions.

We denote by $\tilde{H}$ and $\tilde{K}$ respectively, the $M \times N$ and $M' \times N'$ matrices, which can be extracted by taking the first M rows and the first N columns from H and, in turn, the first M' rows and the first N' columns from K.

The symmetry algebra of the semi-infinite Toda field theory equation can either be obtained directly, *ab initio*, or we can obtain it from the infinite case of Section 2, by adding appropriate boundary conditions and requiring that they be invariant. As above, the functions $\xi(x)$, $\eta(y)$, A_{mn} and $B_n(x,y)$ must satisfy the remaining determining equations (2.7) - (2.9). Following the same reasoning as in the finite case (see Section 3), we obtain the analogs of all the relations (3.6) - (3.10), where now all the labels and summations range from 1 to ∞ (i.e. we take $N \to \infty$ in all formulas). The key equation of the discussion is eq. (3.9) and its associated homogeneous system. Here, we separate the problem into the finite subsystems

$$\begin{aligned} \tilde{H}\,\tilde{\mathbf{B}} &= 0, && (4.3)\\ \tilde{H}\,\tilde{\mathbf{B}} &= -(\xi_x+\eta_y)\,\bar{\mathbf{1}}_M, && (4.4)\end{aligned}$$

where $\tilde{\mathbf{B}} = (B_1, \ldots, B_N)$, and a difference linear equation, which we can put again in the form (2.18), or (2.24) respectively, for $n \geq M+1$. The eq. (4.3) has $Ker\left(\tilde{H}\right)$ as its solution space. On the other hand, the difference equation (2.18) has a (p_2-p_1)-dimensional solution space, the elements of which have the form

$$B_n = \sum_{j=1}^{p_2-p_1} \alpha_j \psi_n^j, \qquad n \geq M+1+p_1, \tag{4.5}$$

in terms of the basis $\{\psi_n^j\}$. Moreover, the difference eq. (2.18) has only the zero solution in the case $p_1 = p_2$. But, because of the imposed restrictions on the form of H, in such a case the components of the vector $\tilde{\mathbf{B}}$ are decoupled from the remaining $(B_{N+1}, \ldots)$. This means that the semi-infinite homogeneous linear system $HB = 0$ has zero-dimensional kernel only if both the finite system (4.3) and the homogeneous difference eq. (2.18) do.

Assuming now that $p_1 < p_2$ and, moreover, that $M + p_1 + 1 \leq N$, the components $(B_{M+1+p_1}, \ldots, B_N)$ have to satisfy both the finite linear eq. (4.3) and the difference eq. (2.18). Substituting the representation (4.5) into (4.3), we get $N - \dim\left(Ker\left(\tilde{H}\right)\right)$ constraints on the $\{\alpha_i\}_{i=1,\ldots,p_2-p_1}$. Thus, if it results that

$$M - N + p_2 + \dim\left(Ker\left(\tilde{H}\right)\right) = n_0 > 0, \tag{4.6}$$

then the semi-infinite homogeneous system $HB = 0$ admits a n_0-dimensional kernel, spanned by the set of linearly independent functions $\{\chi_n^j\}_{j=1,\ldots,n_0}$.

The above result implies that, if the constraint (4.6) holds, then the semi-infinite Toda model defined (4.1) and (4.2) possesses a symmetry group of gauge transformations, generated by the $2 \times n_0$ vector fields

$$\hat{U}_j(r_j) = r_j(x) \sum_{n=1}^{\infty} \chi_n^j \partial_{u_n}, \quad \hat{V}_j(s_j) = s_j(y) \sum_{n=1}^{\infty} \chi_n^j \partial_{u_n} \quad (j = 1, \ldots, p_2 - p_1). \tag{4.7}$$

As in the finite case, a semi-infinite theory is conformally invariant if the inhomogeneous eq. (3.9) (for semi-infinite matrices) has a solution. Thus, now we must require that the vector $\bar{\mathbf{1}} = (1, 1, \ldots)$ be contained in $Im(H)$. But, as outlined above, the problem is reduced to finding a solution of the eq. (4.4) and of the difference eq. (2.24). The former equation is solved if

$$\bar{\mathbf{1}}_M \in Im\left(\tilde{H}\right). \tag{4.8}$$

For the difference eq. (2.24) a solution always exists as seen in Sec. 2. Hence the structure of the matrix H shown in (4.1) garantees that a solution of the total inhomogeneous system always exists, once eq. (4.8) is satisfied ici. In conclusion, the condition (4.8) is not only necessary, but also sufficient to ensure the conformal invariance of the given Toda theories.

Finally, an analysis similar to the study of the gauge invariance can be performed for the $\hat{Z}$-type transformations, which exist if a common solution of the two semi-infinite homogeneous systems

$$HA = 0, \qquad AK = 0 \tag{4.9}$$

can be found. Thus, we are lead to the following theorem

Theorem 3 *Consider the semi-infinite Toda field theory (1.1), with H and K given by (4.1) and (4.2), respectively, and with all rows of H different. Moreover, let HK have no zero columns. Then, the symmetry algebra depends on the fundamental spaces of the finite dimensional submatrices $\tilde{H}$ and $\tilde{K}$, on the solutions of the difference eqs. (2.18) and (2.24) for $n \geq M+1$ and, finally, on the solutions of the difference eq. (2.21) for $m \geq N'+1$.*

The theory is conformally invariant if the condition (4.8) holds. The corresponding generators take the form (2.25). Otherwise, if (4.8) does not hold, the symmetry reduces to the Poincaré group generated by (3.12).

A gauge transformation group, involving $2n_0$ arbitrary functions of one variable, exists if the relation (4.6) holds. The algebra generators take the form (4.7). Finally, $\hat{Z}$-type gauge transformations exist if not only (4.6) *holds, but also the supplementary condition*

$$N' - M' + q_2 + \dim\left(Ker\left(\tilde{K}^T\right)\right) = m_0 > 0 \qquad (4.10)$$

is satisfied. In such a case they form a Lie algebra of dimension $m_0 \times n_0$.

4.2 *Special cases*

Now let us consider the same three examples as in the previous Sections.

1. Mikhailov-Fordy-Gibbons field theories

All examples of Section 3.2 can be generalized to the semi-infinite case, simply allowing N to go to ∞ for each classical Lie algebra. The equations labeled by $1 \leq n \leq N_0$ are explicitly given by (3.15), (3.16), (3.17) and (3.19), respectively. Moreover, for $i \geq N_0$ the equations are the same as in the infinite case, i.e. eq. (1.2).

For the $A_{\infty +}$ algebra (We use this notation in order to distinguish this semi-infinite model from the previously introduced A_∞ infinite one) we have $M = N = M' = N' = 0$ and hence the symmetries are exactly the same as in the infinite and in the finite cases (see eq. (2.35)), where the summations are over the appropriate range.

For the B_∞ algebra one has $\tilde{H} = -\tilde{K} = (-1)$, then also $M = N = M' = N' = 1$, as one can see from (3.16). Theorem 3 allows to establish that there are no gauge transformations of any kind and the generators of the conformal transformations are the same as given in (2.35).

From eq. (3.17) one sees that $\tilde{H} = -\tilde{K} = (-2)$ for the C_∞ algebra, then $M = N = M' = N' = 1$. Thus, Theorem 3 establishes that only the conformal invariance is admitted. Its generators have the same form as in eq. (3.18), where the summation is over the positive integers.

Finally, for the D_∞ algebra one has

$$\tilde{H} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} = -\tilde{K}^T.$$

Theorem 3 implies that only conformal transformations leave the system invariant and their generators are obtained by taking the limit $N \to \infty$ in the formulas (3.20).

2. The semi-infinite Toda field theory (1.3)

The discussion is very simple. Indeed, since H is the indentity matrix, there are no gauge transformations. Moreover, the generators of the conformal transformations in the infinite, semi-infinite and finite cases take always the same form (2.36), where the summations are over the appropriate range.

3. The semi-infinite Toda field theories (1.4)

As opposed to the finite case, the matrix H is no longer invertible, so now these theories are not equivalent to the ones given by (1.3).

First, we observe that, since K is the identity matrix, there are no $\hat{Z}$-type transformations. For any classical Lie algebra, extended to $N \to \infty$, the recursive part of the systems, i.e. the equations labeled by $n \geq N_0$ as defined in Sec. 3.2, are always the same as in the infinite case discussed in Sec. 2.2.3. The solution of the corresponding difference equations for B_n $(n \geq N_0)$, that is (2.18) and (2.24), are the same as in (2.38) and the generators are as in (2.39). However, for $1 \leq n < N_0$ the equations provide constraints of the form (4.3) and (4.4). The application of the Theorem 3 implies

1) All the semi-infinite systems (1.4) are conformally invariant.

2) All the semi-infinite systems (1.4) have $n_0 = 1$, as defined in (4.6), hence a gauge transformation algebra of the form (4.7) exists, with $j = 1$.

In the $A_{\infty+}$ case the $\hat{X}$ and $\hat{Y}$ conformal symmetries survive as in eq. (2.39), and so do $\hat{U}_2$ and $\hat{V}_2$ do. However the symmetries $\hat{U}_1$ and $\hat{V}_1$ are no longer present.

In the B_∞ case the generators $\hat{X}$, $\hat{Y}$ and $\hat{U}_2$, $\hat{V}_2$ combine together to give the new conformal symmetry generators

$$\hat{X} = \xi\left(x\right)\partial_x + \frac{1}{2}\xi_x \sum_{n=1}^{\infty} n\left(n-1\right)\partial_{u_n}, \qquad \hat{Y} = \eta\left(y\right)\partial_y + \frac{1}{2}\eta_y \sum_{n=1}^{\infty} n\left(n-1\right)\partial_{u_n}. \tag{4.11}$$

The remaining gauge invariance is generated by

$$\hat{U}\left(r\right) = r\left(x\right)\left[\partial_{u_1} + 2\sum_{n=2}^{\infty}\partial_{u_n}\right], \qquad \hat{V}\left(s\right) = s\left(y\right)\left[\partial_{u_1} + 2\sum_{n=2}^{\infty}\partial_{u_n}\right]. \tag{4.12}$$

For the C_∞ algebra the symmetry algebra is

$$\begin{aligned}
\hat{X} &= \xi(x)\,\partial_x + \frac{1}{2}\xi_x \sum_{n=1}^{\infty} n(n-2)\,\partial_{u_n}, \\
\hat{Y} &= \eta(y)\,\partial_y + \frac{1}{2}\eta_y \sum_{n=1}^{\infty} n(n-2)\,\partial_{u_n}, \\
\hat{U}(r) &= r(x)\sum_{n=1}^{\infty}\partial_{u_n}, \quad \hat{V}(s) = s(y)\sum_{n=1}^{\infty}\partial_{u_n}.
\end{aligned} \tag{4.13}$$

Finally, for the D_∞ algebra one has

$$\begin{aligned}
\hat{X} &= \xi(x)\,\partial_x + \frac{1}{2}\xi_x \sum_{n=1}^{\infty} (n-1)(n-2)\,\partial_{u_n}, \\
\hat{Y} &= \eta(y)\,\partial_y + \frac{1}{2}\eta_y \sum_{n=1}^{\infty} (n-1)(n-2)\,\partial_{u_n}, \\
\hat{U}(r) &= r(x)\left[\partial_{u_1} + \partial_{u_2} + 2\sum_{n=3}^{\infty}\partial_{u_n}\right], \\
\hat{V}(s) &= s(y)\left[\partial_{u_1} + \partial_{u_2} + 2\sum_{n=3}^{\infty}\partial_{u_n}\right].
\end{aligned} \tag{4.14}$$

The formulas for the semi-infinite models (1.4) are consistent with those obtained in the finite case in Sec. 3.2.3. The generators of the conformal invariance, in each case, are simply obtainable by dropping all terms involving N. Conversely, the terms proportional to a power of N provide us with the gauge invariance generators in the semi-infinite extensions. In this limit, the functions $r = \xi_x$ and $s = \eta_y$ must be considered as new linearly independent functions.

5 Conclusions

We have introduced the generalized Toda system (1.1) and investigated its Lie point symmetry group. It turned out that in the infinite case ($-\infty < n < \infty$)

these systems are always invariant under an infinite dimensional group of conformal transformations. It is also gauge invariant, if a certain homogeneous linear difference equation (i.e. eq. (2.18)) has non trivial solutions. Further gauge transformations exist if another linear homogeneous difference equation (i.e. eq. (2.21)) also has nontrivial solutions.

If we restrict the range of n to $1 \leq n < \infty$, in some cases the symmetry group remains the same, or is reduced to a subgroup of the original symmetry group. However, in other cases (see (4.12) and (4.14)) the symmetry group does not coincide with a Lie subgroup.

In the finite case, with $1 \leq n \leq N$, the symmetry group remains the same as in the semi-infinite case, or it is reduced further.

In some situations (see Theorem 2 and 3) the infinite dimensional conformal symmetry group is reduced to the Poincaré group in two dimensions (see eq. (3.12)).

These results were obtained directly, that is by analyzing the determining equations for the symmetries for all types of systems: infinite, semi-infinite and finite. The question to which we plan to devote a separate article is the application of the infinite generalized Toda systems. In particular we will establish the degree to which the symmetries of the semi-infinite and finite Toda systems are "inherited" from those of the infinite systems. In other words we plan to discuss symmetry breaking by boundary or periodicity conditions of the infinite chains.

One of the surprising results obtained in the present work is that the class of the conformally invariant Toda field theories is much larger than the class of the completely integrable models. Indeed, the existence of a Lax pair imposes severe algebraic restrictions on the matrices H and K (see for instance [19]).

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Chapitre 3

CLASSIFICATION ET ÉTUDE DE SYSTÈMES DISCRETS LINÉARISABLES

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PHYSICA A

The Gambier mapping, revisited

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Abstract

We examine critically the Gambier equation and show that it is the generic linearisable equation containing, as reductions, all the second-order equations which are integrable through linearisation. We then introduce the general discrete form of this equation, the Gambier mapping, and present conditions for its integrability. Finally, we obtain the reductions of the Gambier mapping, identify their integrable forms and compute their continuous limits. © 1998 Elsevier Science B.V. All rights reserved

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1. Introduction

The classification of the integrable second-order differential equations, based on their singularity properties, resulted to four classes [1]:

- equations that are simple derivatives of integrable first-order equations,
- equations that are autonomous (i.e. they do not have any explicit dependence on the independent variable) and which are integrable in terms of elliptic functions,
- equations which are nonautonomous but in which the independent variable appears in some simple form (linearly or at most quadratically) and which define the $\mathbb{P}$ transcendents,

and finally,

- equations which are nonautonomous and in which the independent variable enters through some free functions. These equations are solved by linearisation, i.e. they can be converted to a linear differential system.

Prominent among this last class is the Gambier equation [2]. This equation is, in fact, the generic equation of the linearisable class, in the sense that all the others

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can be obtained as its special limits. The essential feature of the Gambier equation is that it describes the coupling of two Riccati equations in cascade (i.e. the solution of the first Riccati equation appears in the coefficients of the second one). Thus, the Gambier equation is best written as

$$y' = -y^2 + by + c\,, \tag{1.1a}$$

$$x' = ax^2 + nyx + \sigma\,, \tag{1.1b}$$

where a, b and c are functions of the independent variable, σ is a constant which can be scaled to 1 unless it happens to be 0 and n is an integer. The precise form of the coupling is indicated by the singularity analysis which, moreover gives constraints on the coefficients a, b and c. In fact, out of these three functions only two (in general) are free. Eliminating y between Eqs. (1.1a) and (1.1b) one can write the Gambier equation as a second-order ODE:

$$x'' = \frac{n-1}{n}\frac{x'^2}{x} + a\frac{n+2}{n}xx' + bx' - \frac{n-2}{n}\frac{x'}{x}\sigma - \frac{a^2}{n}x^3 + (a' - ab)x^2$$
$$+ \left(cn - \frac{2a\sigma}{n}\right)x - b\sigma - \frac{\sigma^2}{nx}\,. \tag{1.2}$$

An important remark is in order at this point. The equations of the Painlevé/Gambier classification are usually given in canonical form, which means that all possible transformations of the dependent and the *independent* variables have been used in order to simplify their form. This does not seem to be done in the case of the Gambier equation. Indeed, as we will show in the next section, a suitable transformation of the dependent and the independent variables allows us to put $b = 0$. Thus, the Gambier equation contains only *two* functions, which, moreover, are constrained by the integrability requirement.

The discretisation of the Gambier system leads naturally to what we have called the Gambier mapping. In Ref. [3] we have proposed such a discretisation which we have studied using the discrete analog of the singularity analysis, namely the property of singularity confinement. In this paper we propose to reexamine the discrete form of the Gambier equation and determine its most general expression. Once this form is established we can proceed to the study of its particular, limiting, forms and propose expressions for the remaining linearisable discrete equations. For the sake of completeness we calculate, in the next section, the various limits of the Gambier equation in the continuous case.

2. The Gambier equation and its various limits

The canonical list of second-order equations with the Painlevé property is still an unsettled question. The simplest way out of the dilemma is to adopt the attitude of Gambier [2], who has presented a minimal list of 24 equations which contain, in principle all the basic equations. The remaining ones can be obtained through what in modern parlance would be called Miura transformations. Among the equations of the Gambier list some belong to the linearisable family. Here they are

$$(\mathrm{G5})\quad x''=-3xx'-x^3+q(x'+x^2)\,,$$

$$(\mathrm{G13})\quad x''=\frac{x'^2}{x}+q\frac{x'}{x}-q'+rxx'+r'x^2\,,$$

$$(\mathrm{G14})\quad x''=\left(1-\frac{1}{n}\right)\frac{x'^2}{x}+qxx'-\frac{nq^2}{(n+2)^2}x^3+\frac{nq'}{n+2}x^2\,,$$

$$(\mathrm{G15})\quad x''=\left(1-\frac{1}{n}\right)\frac{x'^2}{x}+f_n(q,r)xx'+\phi_n(q,r)x'-\frac{n-2}{n}\frac{x'}{x}-\frac{nf_n^2}{(n+2)^2}x^3$$
$$+\frac{n(f_n'-f_n\phi_n)}{n+2}x^2+\psi_n(q,r)x-\phi_n-\frac{1}{nx}\,. \tag{2.1}$$

To this list one must, in principle, add the equation

$$(\mathrm{G6})\quad x''=-2xx'+qx'+q'x$$

which is nothing but the derivative of the Riccati equation. It is easy to show that the Gambier equation (G15) contains all the previous ones: it is in some sense the general linearisable equation. Instead of using (G15), which corresponds to $\sigma=1$ in Eq. (1.2), we will work with Eq. (1.2) itself where one can directly see the relation to the coupled Riccati's.

First, we start with Eq. (1.2) for $\sigma=1$, and reduce it to its canonical form. For this we introduce the following transformation of the independent variable t to a new variable T through $dT/dt=g$ where g is defined by $(1/g)(dg/dt)=bn/(n-2)$ and simultaneously $X=gx$. This leads to an equation of the form (1.2) with $b=0$, which must be considered its canonical form (similarly (G15) is canonical for $\phi=0$). Moreover the Painlevé property requirement introduces one further relation between a and c (or, equivalently, between f and ψ).

Eq. (G14) is the easiest to obtain: it suffices to take $\sigma=0$. The canonical form corresponds to $b=c=0$. Indeed, in addition to the independent variable transformation which allows to put $b=0$, when $\sigma=0$ we have an additional gauge freedom which allows us to put $c=0$.

$$x''=\frac{n-1}{n}\frac{x'^2}{x}+a\frac{n+2}{n}xx'-\frac{a^2}{n}x^3+a'x^2 \tag{2.3}$$

(with $b=c=0$ Eq. (1.1a) leads to $y=1/(z-z_0)$ and Eq. (1.1b) for $\sigma=0$ is reduced to a linear ODE for $1/x$).

Eq. (G13) requires that we take the limit $n\to\infty$ on Eq. (1.2). The result is (where $d=\lim_{n\to\infty}cn$):

$$x''=\frac{x'^2}{x}+axx'+bx'-\frac{x'}{x}\sigma+(a'-ab)x^2+dx-b\sigma\,. \tag{2.4}$$

Eq. (2.4) is (G13) in non-canonical form. In order to reduce it to the standard expression we take $b=0$ and introduce a gauge $x \to \rho x$ such that $d = \rho''/\rho - \rho'^2/\rho^2$. The equation reduces then to

$$x'' = \frac{x'^2}{x} + q\frac{x'}{x} - q' + rx'x + r'x^2 \tag{2.5}$$

which is just (G13). What does the limit $n \to \infty$ really means in the level of the coupled Riccati's? Since n goes to infinity y must go to zero for the equation to remain meaningful and thus the quadratic term in Eq. (1.1a) disappears. The canonical form corresponds to $b=0$ and a new function is introduced through $d \equiv nc$. Finally, if we divide Eq. (1.1b) by x and take the derivative a term ny' appears, which from Eq. (1.1a) is equal to d. Thus, Eq. (G13) is, in fact, nothing but a derivative of a Riccati after we have divided by the dependent variable.

Finally, in order to obtain (G5) we start by taking $n=1$ which makes the x'^2/x term vanish. Integrability implies $\sigma = 0$ and we choose $a=-1$, $c=0$. This leads to equation

$$x'' = -3xx' - x^3 + b(x' + x^2) \tag{2.6}$$

which is (G5) in canonical form. Finally, it seems that (G6) is not in any sense related to the Gambier equation (G15).

3. The discrete analog of the Gambier equation, revisited

The discretisation of the Gambier equation is based on the idea of two Riccati equations in cascade. The discrete form of the first is simply

$$\bar{y} = \frac{ay+b}{y+1}, \tag{3.1}$$

where $y \equiv y_n$ and $\bar{y} \equiv y_{n+1}$. The denominator of Eq. (3.1) can be generically brought to this form through a scaling of y and a division by an overall factor. The second equation which contains the coupling can be discretised in several, not necessarily equivalent, ways. In Ref. [3] we have proposed the discretisation

$$\bar{x} = \frac{fxy + \sigma}{1 - gx}. \tag{3.2}$$

A different approach could be based on the direct discretisation of Eq. (1.1b) in the form

$$\bar{x} - x = -fx\bar{x} + (gx + h\bar{x})y + k. \tag{3.3}$$

In what follows we shall not choose a priori a particular form. We shall rather start (in the spirit of Ref. [4]) from a generic coupling of the form

$$\alpha x\bar{x}y + \beta x\bar{x} + \gamma\bar{x}y + \delta\bar{x} + \varepsilon xy + \zeta x + \eta y + \theta = 0. \tag{3.4}$$

Implementing a homographic transformation on x and y we can generically bring Eq. (3.4) under the form

$$x\bar{x} + \gamma\bar{x}y - \varepsilon xy - \theta = 0 \tag{3.5}$$

(the sign changes were introduced for future convenience). A choice of different transformations can bring Eq. (3.4) to the form Eq. (3.3) while Eq. (3.2) can be obtained through a special choice of the parameters of Eq. (3.4). Note that Eq. (3.4) contains an 'additive' type coupling $x\bar{x} + \delta\bar{x} + \zeta x + \eta y + \theta = 0$ for special values of its parameters, but the generic form of Eq. (3.5) is that of a 'multiplicative' coupling where γ, ε do not vanish. Solving Eq. (3.5) for $\bar{x}$ we obtain the second equation of the discrete Gambier system in the form

$$\bar{x} = \frac{\varepsilon xy + \theta}{x + \gamma y}\,. \tag{3.6}$$

Clearly, a scaling freedom remains in Eq. (3.6). We can use it in order to bring it to the final form

$$\bar{x} = \frac{xy/d + c^2}{x + dy}\,. \tag{3.7}$$

Eliminating y and $\bar{y}$ from Eq. (3.1) and Eq. (3.7) and its upshift, we can obtain a three-point mapping for x alone but the analysis is clearer if we deal with both y and x.

The main tool for the investigation of the integrability of the Gambier mapping will be the singularity confinement criterion [5]. A first remark before implementing the singularity confinement algorithm is that the singularities of a Riccati mapping are automatically confined. Indeed, if we start from $\bar{x} = (\alpha x + \beta)/(\gamma x + \delta)$ and assume that at some step $x = -\delta/\gamma$, we find that $\bar{x}$ diverges but $\bar{\bar{x}}$ and all subsequent x's are finite. Thus, the intrinsic singularities of Eq. (3.6) do not play any role. However, the singularities due to y (obtained from Eq. (3.1)) may cause problems at the level of Eq. (3.6). Whenever y takes a value that corresponds to either of the two roots $\pm c$ of the equation

$$y^2 - c^2 = 0\,, \tag{3.8}$$

we obtain $\bar{x} = \pm c/d$ irrespective of the value of x and thus the variable x loses a degree of freedom. On the other hand, once we enter a singularity there is no way to exit it unless y assumes again a special value after a certain number of steps. Thus, if we enter the singularity through, say $y = c$ we can exit it through $y = -c$ after N steps. However, if y were to take the value c again some steps after taking it for the first time, then it would take it periodically and the singularity would be periodic. This is contrary to the requirement that the singularity be movable: a periodic singularity (with fixed period) is 'fixed' in our terminology.

The first singularity condition can thus be obtained in the following way. We assume that at some step y assumes the value c solution of the condition (3.8). This value of

$y_0 = c_0$ evolves under the action of the Riccati and we obtain after N steps, y_N. We require that

$$y_N = -c_N\,, \tag{3.9}$$

i.e. the second root of Eq. (3.8). It is thus straightforward to write the first confinement conditions for the first few values of N. We have, for instance,

$$\begin{aligned} N=1:&\ \frac{ac+b}{c+1} + \bar{c} = 0\,,\\ N=2:&\ \frac{\bar{a}(ac+b)+\bar{b}(c+1)}{ac+b+c+1} + \bar{\bar{c}} = 0 \end{aligned} \tag{3.10}$$

and so on. The equivalent of this requirement in the continuous case is that the resonance be integer. We see here that the discrete condition is much more complicated and while one can easily compute the first few instances no general expression can be given. Once y passes through the second special value $-c$, there is a possibility for x to recover its lost degree of freedom through an indeterminate form 0/0. This is the confinement condition. In full generality (and somewhat abstract form) it reads

$$x_N + d_N y_N = 0 \tag{3.11}$$

or, using Eq. (3.9),

$$x_N = d_N c_N \tag{3.12}$$

where x_N is the Nth iterate of x through Eq. (3.6). We have, for example,

$$\begin{aligned} N=1:&\ \frac{c}{d} = \bar{d}\,\bar{c}\,,\\ N=2:&\ \frac{1}{\bar{\bar{d}}}\,\frac{(ac+b)c + d\bar{d}\bar{c}^2(c+1)}{c(c+1)+d\bar{d}(ac+b)} = \bar{\bar{d}}\bar{\bar{c}}\,. \end{aligned} \tag{3.13}$$

The two confinement conditions put constraints on the coefficients a, b, c and d just as the Painlevé requirement restricts the parameters in the continuous case. The better approach is to start with given a, c and use Eq. (3.10) to solve for b. The second condition becomes then an equation for d. For $N=1$ we find explicitly $d\bar{d} = c/\bar{c}$. For $N=2$ the equation for b is linear and the one for $d\bar{d}$ is just a homographic mapping with coefficients depending on a, b, c.

One important question that remains to be addressed is that of the continuous limit of the Gambier mapping. We start from the system

$$\bar{y} = \frac{ay+b}{y+1} \tag{3.18a}$$

$$\bar{x} = \frac{xy/d + c^2}{x+dy} \tag{3.18b}$$

and introduce the following expansions for the parameters:

$$\begin{aligned} a &= 1 + \left(\alpha + \frac{\gamma'}{\gamma} - \frac{2h}{n}\right)\varepsilon\,, \\ b &= \gamma\varepsilon^2\,, \\ c &= \frac{n\gamma}{2}\varepsilon^2\,, \\ d &= 1 + \delta\varepsilon\,, \end{aligned} \tag{3.19}$$

and for the dependent variables

$$y = \frac{n\gamma}{nY + h}\varepsilon\,, \tag{3.20a}$$

$$x = \frac{n\gamma(fX - 1)}{2(fX + 1)}\varepsilon^2\,, \tag{3.20b}$$

where $f = \delta + \gamma'/(2\gamma)$ and $h = f'/f$. We obtain at the limit $\varepsilon \to 0$ the two Riccati's:

$$X' = -f^2X^2 + nXY + 1\,, \tag{3.21a}$$

$$Y' = -Y^2 - \alpha Y + \gamma - \frac{\alpha h}{n} + \frac{h^2}{n^2} - \frac{h'}{n}\,, \tag{3.21b}$$

where the coefficient of the coupling term $n = 2c/b$ is a priori a function but with hindsight we have ignored its derivatives. While the continuous limit takes quite expectedly the form of two Riccati's in cascade we have still to show that they are indeed of the Gambier form and, in particular, that the coefficient of the coupling term n is in fact an integer and equal to N.

The key to this proof is the first confinement condition. Let us start with $y_0 = c$. Given the dependence of c on ε (3.19), we have

$$y_0 = \frac{n\gamma}{2}\varepsilon^2\,. \tag{3.22}$$

In order to do away with the ε^2 factor we introduce the auxiliary quantity ψ through $y = \varepsilon^2\psi$ and rewrite Eq. (3.22) as

$$\psi_0 = \frac{n\gamma}{2}\,. \tag{3.23}$$

The confinement condition is

$$\psi_N = -\frac{n\gamma}{2}\,. \tag{3.24}$$

Let us now compute ψ_N using the discrete Riccati (3.18) at lowest order in ε. Substituting the expressions (3.19) of a, b we have at lowest order:

$$\bar{\psi} = \psi + \gamma\,. \tag{3.25}$$

Thus, $\psi_N = \psi + N\gamma$ and substituting the values of ψ_0 and ψ_N we find $n = -N$. Thus, the coupling coefficient does indeed go over to the integer N which is the number of steps required for confinement. (Had we started from $y_0=-c$ we would have obtained $n=N$. The fact that $\pm c$ play different roles is due to the fact that the discrete Riccati (3.18) is not symmetric with respect to the upward–downward evolution).

We must remark here that the above continuous limit is incompatible with $N=1$. Indeed, for $N=1$ condition (3.13) implies $2\gamma\delta+\gamma'=0$ which would make Eqs. (3.20a) and (3.20b) meaningless. This is related to the fact that in the continuous case $n=1$ is never integrable for $\sigma=1$ while the $n=-1$ case is never integrable when the x equation is nonlinear.

4. Nongeneric forms of the Gambier mapping

An exhaustive study of all the nongeneric cases of the Gambier mapping is a task that lies beyond the scope of this work. In principle one has to go back to the system in Eqs. (3.1)–(3.4) and, following the steps of the derivation of Eq. (3.5), identify all instances where some transformation cannot be applied. The bulk of the resulting calculations makes this problem hardly tractable and we prefer, in what follows, to limit somewhat our scope.

We start thus with the Gambier mapping in its reduced form [Eqs. (3.1)–(3.7)] and consider the cases where the coefficients that have been assumed to be nonvanishing, do vanish. We are thus led to the systems given below. The equation for x assumes one of the following forms:

$$\bar{x}=\frac{xy/d+c^2}{x+dy}, \tag{4.1}$$

$$\bar{x}=\frac{xy+c^2}{x}, \tag{4.2}$$

$$\bar{x}=\frac{xy+c^2}{y}, \tag{4.3}$$

while that for y is given by

$$\bar{y}=\frac{ay+b}{y+1}, \tag{4.4}$$

$$\bar{y}=\frac{ay+1}{y}, \tag{4.5}$$

$$\bar{y}=y+b; \tag{4.6}$$

all the other cases obtained from Eqs. (3.1)–(3.7) can be brought to one of the above using homographic transformations on x and y. Next, we shall investigate the singularity confinement property of the system consisting of one of the Eqs. (4.1)–(4.3) coupled to one of the Eqs. (4.4)–(4.6). There exist, in principle, 9 possible couplings, the one of Eqs. (4.1) and (4.4) being the full discrete Gambier system studied in the previous section.

In order to investigate the coupling Eqs. (4.1)–(4.5) we apply the singularity confinement method. The principle is the same as for the full Gambier case: we enter a singularity when y passes through the value c. In order to confine this singularity we require that, after N steps, y pass through $-c$ and, moreover, x assume an indeterminate form 0/0. The condition for y to be equal to $-c$ can be worked out for the first few values of N:

$$N=1:\ \frac{ac+1}{c}=-\bar{c}\,,$$

$$N=2:\ \frac{\bar{a}(ac+1)+c}{ac+1}=-\bar{\bar{c}}\,. \tag{4.7}$$

The corresponding conditions for the denominator of x to vanish (which, in view of Eq. (4.7), entails the vanishing of the numerator) read:

$$N=1:\ \frac{c}{d}=\bar{c}\bar{d}\,,$$

$$N=2:\ \frac{1}{\bar{\bar{d}}}\frac{c(ac+1)+d\bar{d}\bar{c}^2c}{c^2+d\bar{d}(ac+1)}=\bar{\bar{c}}\bar{\bar{d}}\,. \tag{4.8}$$

In order to obtain the continuous limit of this system we start with the equation for y. The only continuous limit of Eq. (4.5) is obtained for $y=i+\varepsilon Y$. However, this is *incompatible* with the integrability condition where y assumes the values c and $-c$ (after N steps). Thus, the system of Eqs. (4.1)–(4.5) although integrable as a discrete system does not possess an *integrable* continuous limit.

Next we consider the coupling Eqs. (4.1)–(4.6) which can be treated just as the previous case. The first few conditions for the singularity to be confined are

$$N=1:\ c+b=-\bar{c}\,,$$

$$N=2:\ c+b+\bar{b}=-\bar{\bar{c}}\,, \tag{4.9}$$

combined with

$$N=1:\ \frac{c}{d}=\bar{c}\bar{d}\,,$$

$$N=2:\ \frac{1}{\bar{\bar{d}}}\frac{c(c+b)+\bar{c}^2d\bar{d}}{c+d\bar{d}(c+b)}=\bar{\bar{c}}\bar{\bar{d}}\,. \tag{4.10}$$

In this case the continuous limit is obtained through: $x=\varepsilon X$, $c=\varepsilon\gamma$, $d=1+\varepsilon\delta$ and $y=Y$. From the constraint (4.9) on b,c we find that at lowest order we have $b=-2c/N$. The continuous limit is then straightforward, but one must also verify the second integrability condition (4.10). It turns out that the resulting form is noncanonical. In order to bring it under canonical form a further transformation is needed on X: $X=\gamma(1-\delta W)/(1+\delta W)$ and, moreover, we must take $\gamma'=0$. In the case $N=1$ (where we have from Eq. (4.10): $2\delta+\gamma'/\gamma=0$), the canonical form can be recovered through a simpler transformation, $X=-\gamma+1/W$, leading to the linear equation: $W''+W'\gamma'/\gamma+W(\gamma'/\gamma)'=0$ which for $\gamma'=0$ reduces to just $W''=0$.

We turn now to the case of the mapping (4.2) coupled to any of the three homographic for y (4.4)–(4.6). A general remark is in order here. The mapping (4.2) has as only singularity $y=\infty$, i.e. $\bar{x}$ is defined independently of the value of x only when $y=\infty$. Once y in mappings (4.4) or (4.5) hits this special value, Eq. (4.2) loses one degree of freedom and cannot recover it because y cannot become infinite again (unless the mapping for y is periodic which we have excluded from the outset). Thus the combination of Eq. (4.2) with either of Eq. (4.4) or Eq. (4.5) is never integrable. On the contrary Eq. (4.2) coupled to Eq. (4.6) is always integrable because the latter, being linear, can never lead to $y=\infty$. In this case we find at the continuous limit equation (G6). Indeed, putting $x=1+\varepsilon X$, $y=2+\varepsilon^2 Y$ and $b=\beta\varepsilon^3$, $c^2=-1+\gamma\varepsilon^2$, we obtain $X'=-X^2+Y+\gamma$, $Y'=\beta$. Eliminating Y leads to $X''=-2XX'+\beta+\gamma'$ which can be brought to the canonical form (G6) through a simple translation.

Analogous arguments do apply to the case of the mapping (4.3). The singularity of this mapping occurs only if $y=0$. Again, the argument of y taking twice the value being possible only if Eqs. (4.4)–(4.6) are periodic, precludes the integrability of Eq. (4.3) coupled to any of these three. However, there exists a case where y *cannot* vanish. This is the case of Eq. (4.4) for $b=0$. This is the only integrable case of mapping (4.3) coupled to Eq. (4.4). However it is a trivial one. By transforming $y\to 1/y$ both mappings become linear.

5. Conclusions

In this paper we have examined the Gambier equation in both its continuous and discrete forms. For the continuous Gambier system we have shown that it is the generic second-order differential linearizable system: the other second-order linearizable ODEs can be obtained as special limits of the Gambier equation. In the discrete case we have obtained the Gambier mapping starting from the most general discrete Riccati in cascade system (instead of introducing an *ad hoc* parametrisation as we did in Ref. [3]). This most general form has made possible the interpretation of the number of steps necessary for confinement. In the particular case of the Gambier mapping this integer coincides with the one appearing in the coupling term of the ODEs obtained in the continuous limit and which is equal to the resonance of the Painlevé expansion.

This remark raises two important issues. The first is whether there exists a systematic relation between the Painlevé resonance and the number of steps for confinement, i.e. the length of the singularity pattern. We believe that the answer is, in general, negative, despite some tempting results like the Gambier system. The second remark is even more crucial. Since the Gambier system confines for any number of steps N, the limit $N\to\infty$ does in principle exist. We can thus wonder what is the meaning of confinement that requires an infinite number of steps. How can one distinguish the $N=\infty$ confining case from a nonconfining one? Although we cannot offer a rigorous statement, we can present some elements of an answer based on our experience with integrable discrete systems. In a nonconfining system the analysis of the singularity shows that there is *no* possibility for confinement ever. In many cases one can even formulate this in rigorous terms and prove the impossibility of confinement. In the case of a confining mapping the analysis indicates that the possibility of confinement does exist but is simply delayed (and pushed to infinity at the limit). More complicated situations may exist, those, among others, involving the discrete derivatives of homographic mappings. Clearly, at this level the refinement, the notion of confinement itself becomes quite delicate.

The reduced cases of the discrete Gambier system have been only cursorily studied in this work. The particular case where the Gambier mapping reduces, for $N=1$, to the two-dimensional projective system was not contained in the forms studied here. In order to obtain it one must go back to the initial complete form of the discrete Gambier system and perform the appropriate reductions there. This question is under active investigation [6].

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Schlesinger Transformations for Linearizable Equations

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Abstract. We introduce the Schlesinger transformations of the Gambier equation. The latter can be written, in both the continuous and discrete cases, as a system of two coupled Riccati equations in cascade involving an integer parameter n. In the continuous case, the parameter appears explicitly in the equation, while in the discrete case it corresponds to the number of steps for singularity confinement. Two Schlesinger transformations are obtained relating the solutions for some value n to that corresponding to either $n+1$ or $n+2$.

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1. Introduction

The existence of Schlesinger transformations is one of the very special properties of Painlevé equations [1]. These transformations are a particular kind of auto-Bäcklund transformations [2]. The latter relate a solution of a given equation to a solution of the same equation but corresponding to a different set of parameters. Schlesinger transformations do just that but the changes of parameters correspond to integer or half-integer shifts in the monodromy exponents. Both continuous and discrete (whether difference or multiplicative) Painlevé equations have been shown to possess Schlesinger transformations [3]. For the discrete case (and in particular for q-Painlevés) the relation of Schlesinger transformations to monodromy exponents is not quite clear and their derivation requires both experience and intuition, in particular in the choice of the proper parameters. With this minor *caveat* we hasten to say that we do possess a systematic approach to the construction of Schlesinger transformations. It is based on the bilinear formalism [4] which can be used to construct Miura transformations [5], the iteration of which can lead to auto-Bäcklunds.

The logical conclusion of the above introduction is that Schlesinger transformations should exist only for Painlevé equations. This is almost true with one excep-

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tion. As we shall report in this Letter, one equation exists which, without being a Painlevé, does possess Schlesinger transformations. This equation is known under the name of Gambier (who first derived it): it is the most general second-order ODE of linearisable type [6]. What makes possible the existence of Schlesinger transformations for this equation is the fact that its general expression involves an arbitrary integer n. It turns out that we can relate the solution of the equation for some value of n to that corresponding to $n+2$. These transformations are, for the linearizable case, the analogues of the Schlesinger transformations. Moreover, as we shall show, the same procedure can be followed in the discrete case, i.e. for the Gambier mapping [7, 8].

In Section 2, we shall review some basic facts about the Gambier equation. The Schlesinger transformations will be given in Section 3, while Section 4 is devoted to the study of the discrete case.

2. The Gambier Equation

The Gambier equation is given as a system of two Riccati equations in cascade. This means that we start with a first Riccati for some variable y

$$y' = -y^2 + by + c \tag{2.1}$$

and then couple its solution to a second Riccati by making the coefficients of the latter depend explicitly on y:

$$x' = ax^2 + nxy + \sigma. \tag{2.2}$$

The precise form of the coupling introduced in (2.2) is due to integrability requirements. In fact, the application of singularity analysis shows that the Gambier system cannot be integrable unless the coefficient of the xy term in (2.2) is an integer n. This is not the only integrability requirement. Depending on the value of n one can find constraints on the a, b, c, σ (where the latter is traditionally taken to be constant 1 or 0) which are necessary for integrability.

The common lore [9] is that two of the functions a, b, c are free. This turns out not to be the case. The reason for this is that system (2.1)–(2.2) is not exactly canonical, i.e. we have not used all possible transformations in order to reduce its form. We introduce a change of independent variable from t to T through $\mathrm{d}t = g\,\mathrm{d}T$, where g is given by

$$\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}t} = b\frac{n}{2-n},$$

a gauge through $x = gX$ and also

$$Y = gy - \frac{1}{n}\frac{\mathrm{d}g}{\mathrm{d}t}.$$

The net result is that system (2.1)–(2.2) reduces to one where $b = 0$, while σ remains equal to 0 or 1. It is clear from the equations above that n must be different from 2. On the other hand, when $n = 2$ the integrability condition, if $\sigma = 1$, is precisely $b = 0$. So we can always take $b = 0$. (As a matter of fact, in the case $\sigma = 0$ an additional gauge freedom allows us to take both b and c to zero for all n, even for $n = 2$.) Thus, the Gambier system can be written in full generality

$$y' = -y^2 + c, \tag{2.3a}$$

$$x' = ax^2 + nxy + \sigma. \tag{2.3b}$$

One further remark is in order here. The system (2.3) retains its form under the transformation $x \to 1/x$. In this case, $n \to -n$ and σ and $-a$ are exchanged. Thus, in some cases it will be interesting to consider a Gambier system where σ is not constant but rather a function of t. Still, it is possible to show that we can always reduce this case to one where $\sigma = 1$, while preserving the form of (2.3a), i.e. $b = 0$. To this end, we introduce the change of variables

$$\mathrm{d}t = h\,\mathrm{d}T, \qquad x = gX \quad \text{and} \quad Y = hy - \frac{1}{2}\frac{\mathrm{d}h}{\mathrm{d}t}$$

with $h = \sigma^{2/(n-2)}$, $g = \sigma^{n/(n-2)}$. With these transformations, system (2.3) reduces to one with $\sigma = 1$ and $b = 0$. (In the special case $n = 2$, with $b = 0$ integrability implies $\sigma =$ constant, whereupon its value can always be reduced to 1.)

3. Schlesinger Transformations for the Gambier Equations

The theory of auto-Bäcklund transformations of Painlevé equations is well established. As was shown in [2], the general form of auto-Bäcklund transformations for most Painlevé equations is of the form

$$\tilde{x} = \frac{\alpha x' + \beta x^2 + \gamma x + \delta}{\kappa x' + \zeta x^2 + \eta x + \theta}. \tag{3.1}$$

In the case of the Gambier equation considered as a coupled system of two Riccatis, it is more convenient to look for an auto-Bäcklund of the form:

$$\tilde{x} = \frac{\alpha xy + \beta x + \gamma y + \delta}{(\zeta y + \eta)(\theta x + \kappa)}. \tag{3.2}$$

with a factorized denominator, with hindsight from the discrete case. We require that the equation satisfied by $\tilde{x}$ does not comprise terms that are nonlinear in y. We examine first the case $\zeta \neq 0$ and easily reach the conclusion that no solution exists. So we take $\zeta = 0$, $\eta = 1$ which implies that α and γ do not both vanish (otherwise

(3.2) would have been independent of y). We find in this case $\alpha = 0$ and, thus, the general form of the auto-Bäcklund can be written as

$$\tilde{x} = \frac{\beta x + \gamma y + \delta}{\theta x + \kappa}. \tag{3.3}$$

From (3.3), we can obtain the two possible forms of the Gambier system auto-Bäcklund:

$$\tilde{x} = \beta x + \gamma y + \delta, \tag{3.4}$$

$$\tilde{x} = \frac{\beta x + \gamma y + \delta}{x + \kappa}. \tag{3.5}$$

As we shall see in what follows, both forms lead to Schlesinger transformations.

Let us first work form (3.4). Our approach is straightforward. We assume (3.4) and require that $\tilde{x}$ satisfies an equation of the form (2.3b), while y is always the same solution of (2.3a). The calculation is easily performed leading to

$$\tilde{x} = \gamma y + \frac{a\gamma}{n+1} x + \frac{\gamma'}{n}, \tag{3.6}$$

where γ satisfies

$$\frac{\gamma'}{\gamma} = \frac{n}{n+2} \frac{a'}{a}. \tag{3.7}$$

Here we have assumed $a \neq 0$, otherwise $\tilde{x}$ does not depend on x and (3.6) does not define a Schlesinger. The parameters of the equation satisfied by $\tilde{x}$ are given (in obvious notations) by

$$\tilde{n} + n + 2 = 0, \tag{3.8a}$$

$$\tilde{a} = \frac{n+1}{\gamma}, \tag{3.8b}$$

and

$$\tilde{\sigma} = \gamma \left(c + \frac{a\sigma}{n+1} + \frac{1}{n+2} \frac{a''}{a} - \frac{n+3}{(n+2)^2} \frac{a'^2}{a^2} \right). \tag{3.8c}$$

Thus, (3.6) is indeed a Schlesinger transformation, since it takes us from a Gambier system with parameter n to one with parameter $\tilde{n} = -n-2$. It suffices now to invert $\tilde{x}$ in order to obtain an equation with parameter $N = n + 2$. Expressions (3.6) and (3.8) can be written in a more symmetric way:

$$\tilde{a}\tilde{x} - ax = (n+1) \left(y - \frac{a'}{\tilde{n}a} \right) \tag{3.9}$$

and

$$\tilde{n}+1=-(n+1), \qquad \tilde{n}\frac{\tilde{a}'}{\tilde{a}}=n\frac{a'}{a}, \tag{3.10}$$

$$\tilde{a}\tilde{\sigma}-a\sigma=(n+1)\left(c-\frac{1}{\tilde{n}}\left(\frac{a'}{a}\right)'+\frac{1}{\tilde{n}^2}\frac{a'^2}{a^2}\right).$$

The inverse transformation can be easily obtained if we introduce $\tilde{\gamma}$ such that $a\tilde{\gamma}=-(n+1)=-\tilde{a}\gamma$. We thus find that

$$x=y\tilde{\gamma}+\frac{\tilde{a}\tilde{\gamma}}{\tilde{n}+1}\tilde{x}+\frac{\tilde{\gamma}'}{\tilde{n}} \tag{3.11}$$

and the relations (3.10) are still valid.

Iterating the Schlesinger transformations, one can construct integrable Gambier systems for higher n's and obtain by construction the functions which appear in them. However, it may happen that when we implement the Schlesinger, we find $\tilde{\sigma}=0$. If we invert x, we get a system with $N=-\tilde{n}=n+2$ but $A=0$ for which one cannot iterate the Schlesinger further.

Let us give an example of the application of this Schlesinger transformation. Let us start from $n=0$, in which case we find $\tilde{n}=-2$ and, after inversion, $N=2$. For $n=0$ we start from $a=-1$ and $\sigma=0$ or 1 (always possible through the appropriate changes of variable). This leads to $\tilde{a}=-1, \tilde{\sigma}=-c+\sigma$ and the Schlesinger reads $\tilde{x}=-y+x$. Next, we invert $\tilde{x}$ and have $X=1/(x-y)$. We thus find that the Schlesinger takes us from

$$y'=-y^2+c, \qquad x'=-x^2+\sigma \tag{3.12}$$

to the system

$$y'=-y^2+c, \qquad X'=AX^2+2Xy+\Sigma, \tag{3.13}$$

with $A=c-\sigma$, $\Sigma=1$. In the particular case $n=2$, a change of variable exists which allows us to put $A=-1$ (unless $A=0$), without introducing b in the equation for y, while keeping $\Sigma=1$ and changing only the value of c. Thus, the generic case of the Gambier equation for $n=2$ can be written with $A=-1$. Eliminating y between the two equations, we find

$$x''=\frac{x'^2}{2x}-2xx'-\frac{x^3}{2}-\frac{1}{2x}+(2c+1)x. \tag{3.14}$$

This is the generic form of the $n=2$ Gambier equation and it contains just one free function. The nongeneric cases corresponding to $A=0$ and $\sigma=0$ or 1 can be constructed in an analogous way.

We now turn to the second Schlesinger transformation corresponding to the form (3.5). As we shall show, a Schlesinger transformation of this form does indeed exist and corresponds to changes in n with $\Delta n = 1$. Let us start from the basic equations (2.3). Next we ask that $\tilde{x}$ defined by (3.5) satisfies a system like (2.3). We thus find that $\kappa = -x_0$ and γ must be given by

$$\frac{\gamma'}{\gamma} = y_0 + \frac{2ax_0}{n+1}, \tag{3.15}$$

where y_0 is a solution of the Riccati (2.3a) and x_0 a solution of (2.3b), obtained with y replaced by y_0. We introduce the quantities

$$\tilde{x}_0 = \frac{a\gamma}{n+1}, \qquad \tilde{a} = -\frac{nx_0}{\gamma}.$$

In this case, (3.15) becomes

$$\frac{\gamma'}{\gamma} = y_0 + \frac{2\tilde{a}\tilde{x}_0}{\tilde{n}+1} = y_0 + \frac{2x_0\tilde{x}_0}{\gamma}, \tag{3.16}$$

where

$$\tilde{n} + n + 1 = 0. \tag{3.17}$$

Thus, we have starting from a generic solution x, y of (2.3) for some n, the Schlesinger

$$\tilde{x} = \tilde{x}_0 + \frac{\gamma(y - y_0)}{x - x_0}, \tag{3.18}$$

where $\tilde{x}$ is indeed a solution of (2.3) for $\tilde{n} = -n - 1$ for the *same* y

$$\tilde{x}' = -\tilde{a}\tilde{x}^2 + \tilde{n}\tilde{x}y + \tilde{\sigma}, \tag{3.19}$$

where $\tilde{a}$ has been defined as $-(nx_0/\gamma)$ and

$$\tilde{\sigma} = \frac{\gamma}{n+1}\left(a' + a^2 x_0 \frac{n+2}{n+1} + a y_0 (n+2)\right). \tag{3.20}$$

Note that $\tilde{x}_0$ is a solution of the same equation with y replaced by y_0. As in the previous case, if we invert $\tilde{x}$, we obtain an equation corresponding to $N = n + 1$.

As an application of the $\Delta n = 1$ Schlesinger, we are going to construct the $n = 1$ equation starting from the $n = 0$ case, i.e. system (2.3) with $n = 0$. From (3.18), the Schlesinger reads

$$\tilde{x} = \gamma\left(a + \frac{y - y_0}{x - x_0}\right), \tag{3.21}$$

where γ satisfies the differential equation

$$\frac{\gamma'}{\gamma} = y_0 + 2ax_0, \tag{3.22}$$

where y_0 is a particular of (2.3b) and x_0 is a solution of (2.3a) with $n = 0$. Then $\tilde{x}$ satisfies a Gambier equation in which $n = 1$, $\tilde{\sigma} = \gamma(a' + 2a^2x_0 + 2ay_0)$ and $\tilde{a} = 0$. To obtain the $n = 1$ Gambier equation, we define $X = 1/\tilde{x}$ and we arrive at the following system:

$$y' = -y^2 + c, \qquad X' = AX^2 + Xy, \tag{3.23}$$

where $A = -\tilde{\sigma}$. The system (3.23) is the generic $n = 1$ case since, in this case, the condition for (2.3) to be integrable is $\sigma = 0$.

It is worth pointing out here that the Schlesinger transformation corresponding to $\Delta n = 2$ was known to Gambier himself. As a matter of fact, when faced with the problem of determining the functions appearing in his equation so as to satisfy the integrability requirement, Gambier proposed a recursive method which is essentially the Schlesinger $\Delta n = 2$. On the other hand, the Schlesinger $\Delta n = 1$ is quite new and we have first discovered it in the discrete case, whereupon we looked for (and found) its continuous analogue.

4. The Gambier Mapping

The Gambier equation has been already examined in [7, 8] and its discrete equivalent has been proposed there. These constructions of the Gambier mapping were *ad hoc* ones in the sense that we assumed a form and implemented the singularity confinement discrete integrability criterion in order to obtain the integrability conditions. In what follows, we shall use a slightly different approach based on the singularity structure.

Our starting point is the discrete equivalent of the system (2.3). We have thus one equation which is the discrete analogue of the Riccati, i.e. a homographic mapping for y and another homographic mapping for x, the coefficients of which depend linearly on y. Our derivation will be based on the study of singularities of the system. The general homographic equation for y involves three free parameters, but since we have the freedom of choice of a homographic transformation on y, we can always reduce it to $y =$ constant (i.e. $\bar{y} = y$). However, in the system under study, our aim is to study the singularities of x induced by *special* values of y. One could choose the singularity to enter at point n_0 if the value of y has some special value depending on n_0, say, $f(n_0)$. This would introduce one function in the homographic mapping, However, what is even more convenient is to decide what the special value of y is, say $y = 0$ for all n, at the price of the loss of part of the homographic freedom. Then the special value 0 will occur for some n depending on the particular solution. Of course, if we allow the full homographic freedom, we are back to the starting point, i.e. with three free functions. However, we have decided to have only one free function and thus we simplify the mapping by choosing its form so that it presents the pattern $\{-1, \infty, 1\}$. This fixes two of the functions and the result is

$$\bar{y} = \frac{y+c}{y+1}, \tag{4.1}$$

where c is a function of n and we use the notations $y = y(n)$, $\bar{y} = y(n+1)$.

Next, we turn to the equation for x. This equation is homographic in x. However, we require that when y takes the value 0, the resulting value of x is ∞. Thus, the denominator must be proportional to y, and since we can freely translate x, we can reduce its form to just xy. The remaining overall gauge factor is chosen so as to put the coefficient of xy of the numerator to unity resulting in the following mapping:

$$\bar{x} = \frac{x(y-r) + q(y-s)}{xy}. \tag{4.2}$$

The system (4.1)–(4.2) is a discrete form of the Gambier system. In order to study the confinement of the singularity induced by $y = 0$, we introduce the auxiliary quantity ψ_N which is the value of y if the Nth downshift of y had been zero. Thus

$$\psi_0 = 0, \qquad \psi_1 = \underline{c}, \qquad \psi_2 = \frac{\underline{\underline{c}} + \underline{c}}{\underline{\underline{c}} + 1}, \qquad \text{etc.}$$

The confinement requirement is that after N steps x becomes 0 in such a way as to lead to 0/0 at the next step. Thus, the mapping (4.2) has in fact the form

$$\bar{x} = \frac{x(y-r) + q(y-\psi_N)}{xy}. \tag{4.3}$$

Thus, when at some step N, we have $y = \psi_N$ and $x = 0$. In view of (4.3), $\bar{x}$ will then be indeterminate of the form 0/0. However, it turns out that, in fact, this value is well-determined and finite. Let us take a closer look at the conditions for confinement. The generic patterns for x and y are

$$y\colon \left\{ \begin{array}{cccccc} 0 & \overline{\psi_1} & \overline{\overline{\psi_2}} & \cdots & \overline{\overline{\overline{\psi_N}}} & \end{array} \right\}$$

$$x\colon \left\{ \begin{array}{cccccc} \text{free} & \infty & \dfrac{\overline{\psi_1} - \bar{r}}{\overline{\psi_1}} & \cdots & 0 & \text{finite} \end{array} \right\}.$$

At $N = 1$, it is clearly impossible to confine ourselves to a form of (4.3) since we do not have enough steps. In this case, the only integrable form of the x-equation is a linear one. The first genuinely confining case of the form (4.3) is $N = 2$. From the requirement $\overline{\overline{x}} = 0$, we have $r = \psi_1$ and q free: this is indeed the only integrability condition. For higher N's, we can similarly obtain the confinement condition which takes the form of an equation for r in terms of q.

At this point it is natural to ask whether the mapping (4.1)–(4.3) does indeed correspond to the Gambier equation (2.3). In order to do this, we construct its continuous limit. We first introduce

$$c = \varepsilon^2 D, \qquad y = \frac{\varepsilon D}{Y + H}, \tag{4.4}$$

with $H \approx D'/(2D)$, and obtain the continuous limit of (4.1) for $\varepsilon \to 0$. As expected, we find

$$Y' = -Y^2 + C, \tag{4.5}$$

i.e. Equation (2.3a), where

$$C = D - \frac{D''}{2D} + \frac{3}{4}\frac{D'^2}{D^2}.$$

Using (4.4) and (4.1), we can also compute ψ_N and we find, at lowest order,

$$\psi_N = \varepsilon^2 \Psi_N \quad \text{with} \quad \Psi_N \approx N\left(D - \varepsilon\frac{N+1}{2}D'\right) + \varepsilon^2 \Phi_N, \tag{4.6}$$

where Φ_N is an explicit function of D depending on N.

Next, we turn to the equation for x and introduce

$$r = \varepsilon^2 R, \qquad x \approx \frac{1}{2} + \frac{\varepsilon}{2X} - \varepsilon\frac{RD'}{4D^2}, \qquad q \approx -\tfrac{1}{4} + \varepsilon^2 Q \tag{4.7}$$

and for the continuous limit of the form (2.3b) to exist in canonical form (i.e. $b = 0, \sigma = 1$), we find that we must have

$$R \approx \frac{ND}{2} - \varepsilon(N+2)\frac{ND'}{8}, \tag{4.8}$$

This leads to the equation for x:

$$X' = AX^2 + NXY + 1, \tag{4.9}$$

with

$$A = \frac{N}{4}\left(\frac{N}{4+1}\right)\frac{D'^2}{D^2} - \frac{ND''}{4D} - 4Q.$$

Moreover, the confinement constraint implies a differential relation between D and Q which depends on N. We can verify explicitly in the first few cases that this is indeed the integrability constraint obtained in the continuous case. For instance, for $N = 2$, just imposing (4.8) in order to have the canonical form $b = 0, \sigma = 1$, is sufficient for integrability.

Once the singularity pattern of the Gambier mapping is established, we can use it in order to construct the Schlesinger transformation. Let us first look for a transformation that corresponds to $\Delta N = 2$. The idea is that, given the N-steps singularity pattern of the equation for x, we introduce a variable w with $N + 2$ singularity steps where we enter the singularity one step before x and exit it one step later. The general form of the Schlesinger transformation, which defines w, is

$$w = X\frac{y - \psi_{N+1}}{y}, \tag{4.10}$$

where X is homographic in x. The presence of the y and $y - \psi_{N+1}$ terms is clear: they ensure that w becomes infinite one step before x and vanishes one step after x. Next we turn to the determination of X. Since X is homographic in x, we can rewrite (4.10) as

$$w = \frac{\alpha x + \beta}{y}\frac{y - \psi_{N+1}}{\gamma x + \delta}. \tag{4.11}$$

Our requirment is that w becomes infinite when $y = 0$ for every value of x. This statement must be qualified. The numerator $\alpha x + \beta$ *will* vanish for some x (namely $x = -\beta/\alpha$) so this value of x must be the only one which should *not* occur in the confined singularity. Indeed, there is a unique value of x where, instead of being confined, the singularity extends to infinity in *both* directions of the independent variable n, while the only nonsingular values of the dependent variable occur in a finite range. The value of x such that $\bar{x}$ is finite and free even though y is zero, is such that the numerator $-xr - q\psi_N$ of $\bar{x}$ vanishes. For this value of x, the values of the dependent variable are fixed for $n \leqslant 0$ and $n \geqslant N + 1$ and the value can be considered as 'forbidden'. Thus, $\alpha x + \beta = xr + q\psi_N$ up to a multiplicative constant. Similarly, when $y = \psi_{N+1}$, w must vanish. Thus, $\gamma x + \delta$ must not be zero except for the unique value of x that does not occur in the confined singularity. Note that $y = \psi_{N+1}$ means $\underline{y} = \underline{\psi}_N$ and the only value of x that comes from a nonzero $\underline{x}$ in that case is $x = (\underline{\psi}_N - \underline{r})/\underline{\psi}_N$. In that case the values of the dependent variable are fixed for $n \geqslant 0$ and $n \leqslant -N - 1$. This value of x being 'forbidden', $\gamma x + \delta$ must be proportional to $\underline{\psi}_N x - (\underline{\psi}_N - \underline{r})$. We now have the first form of the Schlesinger:

$$w = \frac{xr + q\psi_N}{y}\frac{y - \psi_{N+1}}{\underline{\psi}_N x + \underline{r} - \underline{\psi}_N}, \tag{4.12}$$

where the proportionality constant has been taken as being equal to 1 (but any other value would have been equally acceptable). Here w effectively depends on x unless

$r(\underline{r} - \underline{\psi}_N) = q\underline{\psi}_N \psi_N$. But, in this case, the mapping (4.3) is in fact linear in the variable $\xi = (x - 1 + \overline{r}/\overline{\psi}_N)^{-1}$. This case is the analog of the case $a = 0$ in the continuous case where the Schlesinger does not exist.

Let us give an application of the Schlesinger transformation by obtaining the $N = 2$ equation starting from $N = 0$. We have always the equation for y which reads

$$\bar{y} = \frac{y + c}{y + 1} \tag{4.13}$$

and $\psi_0 = 0$, $\psi_1 = \underline{c}$. For $N = 0$ the equation for x reads

$$\bar{x} = \frac{x(y - r) + qy}{xy} = \frac{x + q}{x}, \tag{4.14}$$

since, for integrability, $r = 0$ and indeed $N = 0$ means that the x equation does *not* depend on y. We introduce the Schlesinger

$$w = x\frac{y - \underline{c}}{y} \tag{4.15}$$

(where we first write (4.12) for arbitrary r and since r factors, we take the limit $r \to 0$ afterwards). Using (4.14) and (4.15) to eliminate x, we obtain the equation for w:

$$\bar{w} = (1 - c)\frac{yw + q(y - \underline{c})}{(y + c)w}. \tag{4.16}$$

This equation is of the form (4.3) but not quite canonical. We can transform it to canonical form simply by introducing $\bar{y}$ instead of y because, indeed, w is infinite one step before x, so $w = \infty$ means $\bar{x} = \infty$, i.e. $\bar{y} = 0$. We thus obtain

$$\bar{w} = \frac{w(\bar{y} - c) + q(1 + \underline{c})(\bar{y} - \bar{\psi}_2)}{\bar{y}w} \tag{4.17}$$

with $\bar{\psi}_2 = (c + \underline{c})/(1 + \underline{c})$ which, coupled to (4.13), is indeed a $N = 2$ Gambier mapping.

As we pointed out above, the $N = 1$ case is not included in the parametrization (4.1)–(4.3): the x-mapping must be linear in order to ensure integrability. Thus, we are led to study the linear case separately. For an arbitrary N, the general form of the linear x-mapping can be obtained using confinement arguments in a way similar to what we did for the generic, nonlinear case. We obtain

$$\bar{x} = \frac{x(y - \psi_N) + g}{y}, \tag{4.18}$$

where g is free. The Schlesinger transformation is again given by

$$w = X\frac{y - \psi_{N+1}}{y} \tag{4.19}$$

and arguments similar to those of the nonlinear case allow us to determine the form of the homographic object X leading to

$$w = \frac{x\psi_N - g}{y}\frac{y - \psi_{N+1}}{x\underline{\psi}_N - \underline{g}}. \tag{4.20}$$

Thus, we can perform a Schlesinger in the linear case. This is not in disagreement with the continuous case. It is, in fact, the analog of the case where $\sigma = 0$ but $a \neq 0$ (which is linear in $1/x$) for which the Schlesinger *can* be performed. The analog of the case $\sigma = 0$ and $a = 0$ is the situation when $g = k\psi_N$ with constant k in which case the mapping rewrites $\bar{\xi} = \xi(y - \psi_N)/y$ with $\xi = x - k$. Then w does not depend on ξ (or x) and (4.20) does not define a Schlesinger in analogy to the case $r(\underline{r} - \underline{\psi}_N) = q\underline{\psi}_N\psi_N$ in the nonlinear case.

Using this form of the Schlesinger transformation we can, for example, construct the $N = 3$ case starting from the $N = 1$ case. In the case $N = 1$, the mapping for x is given by

$$\bar{x} = \frac{(y - \underline{c})x + g}{y}. \tag{4.21}$$

Using Equation (4.20), we introduce the Schlesinger

$$w = \frac{x\underline{c} - g}{y}\frac{(\underline{c} + 1)y - \underline{c} - \underline{\underline{c}}}{x\underline{\underline{c}} - \underline{g}}. \tag{4.22}$$

In order to simplify the final expression, we define p with $g = \underline{c}p$. We then have the following equation for w:

$$\bar{w} = \frac{c(c-1)(\underline{\underline{c}}w((\bar{p} - \underline{p})y + \underline{c}(\underline{p} - p)) + \underline{c}(p - \bar{p})(y(\underline{\underline{c}} + 1) - (\underline{\underline{c}} + \underline{c})))}{(p - \underline{p})c\underline{\underline{c}}w(y + c)} \tag{4.23}$$

To give this equation in the same parametrization as (4.3), we first write it in terms of $\bar{y}$ and then use a multiplicative gauge $\omega = \phi w$ to put to unity the coefficient of ωy in the numerator of $\bar{\omega}$. We than have

$$\bar{\omega} = \frac{\omega(\bar{y} - r) + q(\bar{y} - \bar{\psi}_3)}{\omega\bar{y}}, \tag{4.24}$$

where

$$q = \frac{(p - \bar{p})(\underline{p} - \underline{\underline{p}})(2\underline{\underline{c}} + \underline{c} + 1)}{(\underline{c}(p - \underline{p}) + \bar{p} - \underline{p})(\underline{\underline{c}}(\underline{p} - \underline{\underline{p}}) + p - \underline{\underline{p}})},$$

$$r = \frac{\underline{c}(p - \underline{p}) + c(\bar{p} - \underline{p})}{\underline{c}(p - \underline{p}) + \bar{p} - \underline{p}}, \tag{4.25}$$

$$\bar{\psi}_3 = \frac{\underline{\underline{c}}c + c + \underline{c} + \underline{\underline{c}}}{2\underline{\underline{c}} + \underline{c} + 1}.$$

Finally, we examine the possibility of the existence of a $\Delta N = 1$ Schlesinger. In this case, the structure of the transformation will be obtained by asking that the $N + 1$ case enter the singularity one step before the N case but exit at the same point. The general structure is thus

$$w = \frac{rx + q\psi_N}{y} \frac{y - \eta}{x - \xi}, \tag{4.26}$$

where η and ξ must be determined. We do this by requiring that the equation for w contains no coefficients nonlinear in y. As a result, we find that η must satisfy Equation (4.1) for y:

$$\bar{\eta} = \frac{\eta + c}{\eta + 1} \tag{4.27}$$

and ξ Equation (4.3) for x with η instead of y:

$$\bar{\xi} = \frac{\xi(\eta - r) + q(\eta - \psi_N)}{\xi\eta}. \tag{4.28}$$

We remark here on the perfect parallel to the continuous case (and as we pointed out, the discrete case led the investigation back to the continuous one). Let us point our here that the w obtained through (4.26) does not lead to $w = 0$ at the exit of the singularity (i.e. when $x = 0$, $y = \psi_N$) and a translation is needed. In principle, one has to define a new variable

$$\omega = w - w(x = 0, y = \psi_N) = w + \frac{q}{\xi}(\psi_N - \eta).$$

We are now going to study the particular case where we construct $N = 1$ starting from $N = 0$. As in the continuous case, our starting point is the $N = 0$ nonlinear equation. Thus, we are starting with the decoupled Equation (4.14) for x. We write the Schlesinger as

$$w = \frac{y - \eta}{y} \frac{x}{x - \xi}. \tag{4.29}$$

Once more, we find that η must satisfy (4.27) while ξ satisfies $\bar{\xi} = 1 + q/\xi$, i.e. the same equation as x. Using (4.27), we can easily obtain the equation for w

corresponding to $N = 1$. We write this equation in terms of $\bar{y}$ in order to enter the singularity with $\bar{y} = 0$. The expression of $\bar{w}$ is indeed linear in w and reads

$$\bar{w} = \frac{(q+\xi)w(c-\bar{y}) + q(\bar{y}(\eta+1) - c - \eta)}{\bar{y}q(\eta+1)}. \tag{4.30}$$

In order to cast it in the parametrization of (4.21), we introduce a shift in the w:

$$\omega = w + \omega_0. \tag{4.31}$$

We require that in the numerator of the equation for ω, $\bar{y}$ appears only as a product with ω. We find that ω_0 must satisfy

$$\bar{\omega}_0 = -\frac{\omega_0(q+\xi) + q(\eta+1)}{q(\eta+1)} \tag{4.32}$$

and we obtain the following equation for ω:

$$\bar{\omega} = \frac{(q+\xi)\omega(c-\bar{y}) - c\omega_0(q+\xi) - q(c+\eta)}{\bar{y}q(\eta+1)}, \tag{4.33}$$

which is in the form of (4.21) up to a multiplicative factor in ω. We note that ω_0 plays the role of $\tilde{x}_0$ in Equation (3.18) in the continuous case. Indeed, ω_0 satisfy (4.33) for $y = \eta$.

Finally, we derive the $\Delta N = 1$ Schlesinger for the case of a linear mapping (4.18). We start from

$$w = \frac{x\psi_N - g}{y}\frac{y-\eta}{x-\xi} \tag{4.34}$$

and again require for w an equation with coefficients linear in y. We find that η must again be a solution of the equation for y, i.e. it must satisfy (4.27) and, moreover, ξ is a solution of (4.18) with $y = \eta$:

$$\bar{\xi} = \frac{\xi(\eta - \psi_N) + g}{\eta}. \tag{4.35}$$

Thus, the list of the Schlesinger transformations of the Gambier mapping is complete.

5. Conclusion

In this letter, we have shown that the Gambier system possesses Schlesinger transformations just like Painlevé equations. This is a most interesting result given that the Gambier system is C-integrable (in the terminology of Calogero [10]), i.e. integrable through linearization, and not S-integrable.

We discovered that both the continuous and the discrete systems possess two kinds of Schlesinger transformations: one that allows changes of N by two units and one where the changes of N are by one unit. In the discrete case, our approach was based entirely on the singularity confinement approach. We have shown that a study of the singularities allows us to determine the form of the Gambier mapping and, at the same time, its Schlesinger transformations. This is one more argument in favour of the singularity analysis approach to the study of discrete systems.

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PHYSICA A

Again, linearizable mappings

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Abstract

We examine a family of three-point mappings that include mappings solvable through linearization. The different origins of mappings of this type are examined: projective equations and Gambier systems. The integrable cases are obtained through the application of the singularity confinement criterion and are explicitly integrated. © 1998 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

Integrability is a term far too general. Although the idea is simple and related to the integration of a differential system, integrability assumes various forms. Here we shall attempt neither a rigorous definition nor an exhaustive description of all the disguises of integrability. In order to fix the ideas we shall just present three most common types of integrability, which suffice in order to explain the properties of the majority of integrable systems [1]. These three types are:

- Reduction to quadrature through the existence of the adequate number of integrals of motion.
- Reduction to linear differential systems through a set of local transformations.
- Integration through IST techniques. This last case is mediated by the existence of a Lax pair (a linear system the compatibility of which is the nonlinear equation under integration) which allows the reduction of the nonlinear equation to a linear integrodifferential one. The above notions can be extended mutadis mutandis to the domain of discrete systems.

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This paper will focus on the second type of integrability, usually referred to as linearizability. The prototype of the linearizable equations is the Riccati. In differential form this equation writes

$$w' = \alpha w^2 + \beta w + \gamma \tag{1.1}$$

and the transformation $w = P/Q$ (Cole–Hopf) reduces it to the linear system:

$$P' = \beta P + \gamma Q, \qquad Q' = -\alpha P. \tag{1.2}$$

Similarly, the discrete Riccati equation, which assumes the form of a homographic mapping

$$\overline{x} = \frac{\alpha x + \beta}{\gamma x + \delta}, \tag{1.3}$$

where x stands for x_n, $\overline{x}$ for x_{n+1} (and, of course, $\underline{x}$ for x_{n-1}), can be linearized through a Cole–Hopf transformation. Putting $x = P/Q$ we obtain readily

$$\overline{P} = \alpha P + \beta Q, \qquad \overline{Q} = \gamma P + \delta Q. \tag{1.4}$$

The extension of the Riccati to higher orders can be and has been obtained [2]. The simplest linearizable system at N dimensions is the projective Riccati which assumes the form

$$w'_\mu = a_\mu + \sum_\nu b_{\mu\nu} w_\nu + w_\mu \sum_\nu c_\nu w_\nu \quad \text{with } \mu = 1, \ldots, N. \tag{1.5}$$

In two dimensions the projective Riccati system can be cast into a second-order equation without loss of generality,

$$w'' = -3ww' - w^3 + q(t)(w' + w^2). \tag{1.6}$$

The discrete analog of the projective Riccati does exist and is studied in detail in Ref. [3]. The corresponding form is

$$\overline{x}_\mu = \frac{a_\mu + x_\mu + \sum_\nu b_{\mu\nu} x_\nu}{1 - \sum_\nu c_\nu x_\nu}. \tag{1.7}$$

Again in two dimensions (and only in this case) the discrete projective Riccati can be written as a second-order mapping for a variable without any simplifying assumptions. One is thus led to a mapping of the form

$$\alpha \overline{x} x \underline{x} + \beta \overline{x} x + \gamma x \underline{x} + \delta \overline{x} \underline{x} + \varepsilon x + \zeta \overline{x} + \eta \underline{x} + \theta = 0 \tag{1.8}$$

which was first introduced in Ref. [4]. The coefficients $\alpha, \beta, \ldots, \theta$ are not totally free. Although the linearizability constraints have been obtained in Ref. [4], the study of mappings of the form (1.8) was not complete. In the present work we intend to present its exhaustive study from the point of view of integrability in general.

Another point must be brought to attention here. In the continuous case the study of second-order equations has revealed the relation of the linearizable Eq. (1.6) to the

Gambier equation [5]. The latter is obtained as a system of two coupled Riccati in cascade

$$y' = -y^2 + qy + c, \qquad w' = aw^2 + nyw + \sigma\,, \tag{1.9}$$

where n is an integer. The linearizable equation (1.6) is obtained from Eq. (1.9) for $n=1$ and $a=-1$, $c=0$ and $\sigma=0$. The discrete analog of the Gambier mapping was introduced in Ref. [6] and in full generality in Ref. [7]. However, the relation of the linearizable mapping to the discrete Gambier system has not been studied in full detail. In what follows we shall fill this gap by presenting the reduction of the Gambier mapping to the linearizable one.

In the next section we shall analyse Eq. (1.8) and all its reduced forms and isolate the integrable ones through the use of the singularity confinement criterion. The integration of all integrable cases will be given in detail and, in particular, the linearization through a projective system. Section 3 is devoted to the study of the reduction of a Gambier-type (coupled Riccati) system to a linearizable one together with the investigation of the integrable cases and their integration.

2. Linearizable mappings as projective systems

In Ref. [4] we have introduced a projective system as a way to linearize a second-order mapping. (The general theory of discrete projective systems has been recently presented in Ref. [3]). In this previous work we have focused on a three-point mapping that can be obtained from a 3×3 projective system. As a matter of fact, this is the only case where the projective equation can be converted to a single, one-component mapping without any simplifying assumptions. The main idea was to consider the system

$$\begin{pmatrix} \overline{u} \\ \overline{v} \\ \overline{w} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{2.1}$$

and conversely

$$\begin{pmatrix} \underline{u} \\ \underline{v} \\ \underline{w} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \tag{2.2}$$

where the matrix M is obviously related to the matrix P through $\overline{M} = P^{-1}$. Introducing the variable $x = u/v$ and the auxiliary $y = w/v$ we can rewrite Eqs. (2.1) and (2.2) as

$$\overline{x} = \frac{p_{11}x + p_{12} + p_{13}y}{p_{21}x + p_{22} + p_{23}y}\,, \tag{2.3}$$

$$\underline{x} = \frac{m_{11}x + m_{12} + m_{13}y}{m_{21}x + m_{22} + m_{23}y}\,. \tag{2.4}$$

(Since m_{3i}, p_{3i} do not appear in Eqs. (2.3) and (2.4) and we can simplify M, P by taking $m_{33} = p_{33} = 1$ and $m_{31} = p_{31} = m_{32} = p_{32} = 0$.) Finally, eliminating y between Eqs. (2.3) and (2.4) we obtain the mapping

$$\alpha\overline{x}x\underline{x} + \beta\overline{x}x + \gamma x\underline{x} + \delta\overline{x}\underline{x} + \varepsilon x + \zeta\overline{x} + \eta\underline{x} + \theta = 0\,, \tag{2.5}$$

where $\alpha, \beta, \ldots, \theta$ are related to m, p's.

Eq. (2.5) will be the starting point of the present study. Our question will be when is an equation of this form integrable? (Clearly, the relation to the projective system works only for a particular choice of the parameters.) In order to investigate the integrability of Eq. (2.5) we shall use the singularity confinement approach that was introduced in Ref. [8]. The first question is what are the singularities of Eq. (2.5)? Given the form of Eq. (2.5) it is clear that a diverging x does not lead to any difficulty. However, another (subtler) difficulty arises whenever x_{n+1} is defined independently of x_{n-1}. In this case the mapping "loses one degree of freedom". Thus, the singularity condition is

$$\frac{\partial x_{n+1}}{\partial x_{n-1}} = 0\,,$$

which leads to

$$(\alpha x + \delta)(\varepsilon x + \theta) = (\beta x + \zeta)(\gamma x + \eta)\,. \tag{2.6}$$

Eq. (2.6) is the condition for the appearance of a singularity. Given the invariance of Eq. (2.5) under homographic transformations it is clear that one can use them in order to simplify Eq. (2.6). Several choices exist, but the one we shall make here is to choose the roots of Eq. (2.6) so as to be equal to 0 and ∞, unless of course Eq. (2.6) has two equal roots, in which case we bring them both to 0. Let us examine first the distinct root case. For the roots of Eq. (2.6) to be 0 and ∞ we must have

$$\alpha\varepsilon = \beta\gamma, \qquad \delta\theta = \zeta\eta\,. \tag{2.7}$$

The generic mapping of the form (2.5) has $\alpha\theta \neq 0$ and we can take $\alpha = \theta = 1$ (by the appropriate scaling of x and a division). We have thus

$$\overline{x}x\underline{x} + \beta\overline{x}x + \gamma x\underline{x} + \zeta\eta\overline{x}\underline{x} + \beta\gamma x + \zeta\overline{x} + \eta\underline{x} + 1 = 0\,. \tag{2.8}$$

Nongeneric cases do exist as well and we shall examine them in detail. Thus, let us first assume that $\alpha = 0$ in which case $\beta\gamma = 0$ and we can decide that $\beta = 0$ without loss of generality (β and γ are interchanged if one reverses the direction of the evolution). The mapping then becomes: $(\gamma\underline{x} + \varepsilon)x + (\zeta\overline{x} + 1)(\eta\underline{x} + 1) = 0$ and we must have $\zeta \neq 0$, lest the mapping become a two-point one, in which case a scaling of x can bring its value to 1. We have thus the general form

$$x(\gamma\underline{x} + \varepsilon) + (\overline{x} + 1)(\eta\underline{x} + 1) = 0\,. \tag{2.9}$$

Clearly, we cannot take both $\eta = 0$, $\gamma = 0$, neither $\varepsilon = 0$, $\gamma = 0$. So, in order to obtain the reduced forms of Eq. (2.9) we assume first $\eta = 0$ leading to

$$\overline{x} + 1 + x(\gamma\underline{x} + \varepsilon) = 0 \tag{2.10}$$

and then $\eta \neq 0$ with two possible combinations either $\gamma = 0$

$$(\overline{x}+1)(\eta\underline{x}+1)+\varepsilon x=0 \tag{2.11}$$

or $\varepsilon = 0$

$$(\overline{x}+1)(\eta\underline{x}+1)+\gamma x\underline{x}=0\,. \tag{2.12}$$

Going back to mapping (2.5) we can obtain a further reduced form by assuming $\alpha=\theta=0$, in which case $\beta\gamma=0$ and $\zeta\eta=0$. As previously, we can assume that $\beta=0$, but this fixes the direction of evolution so the two choices $\zeta=0$ and $\eta=0$ are distinct. We have thus two reduced mappings

$$\gamma x\underline{x}+\delta\overline{x}\underline{x}+\varepsilon x+\zeta\overline{x}=0 \tag{2.13}$$

and

$$\gamma x\underline{x}+\delta\overline{x}\underline{x}+\varepsilon x+\eta\underline{x}=0\,. \tag{2.14}$$

A further reduction can be obtained if we assume that $\beta=\gamma=0$, in which case we can freely choose $\zeta=0$ and find

$$\delta\overline{x}\underline{x}+\varepsilon x+\eta\underline{x}=0\,. \tag{2.15}$$

Mappings (2.8)–(2.15) are all that can be obtained from Eq. (2.5) with the assumptions of the distinct roots for Eq. (2.6) up to homographic transformations.

Next, we turn to the case where Eq. (2.6) possesses a double root equal to 0. The constraints in this case read

$$\begin{aligned}\delta\theta&=\zeta\eta\,,\\ \alpha\theta+\delta\varepsilon&=\beta\eta+\zeta\gamma\,.\end{aligned} \tag{2.16}$$

The generic case corresponds to $\theta=\varepsilon=1$, leading to

$$\alpha\overline{x}x\underline{x}+\beta\overline{x}x+\gamma x\underline{x}+\zeta\eta\overline{x}\underline{x}+x+\zeta\overline{x}+\eta\underline{x}+1=0 \tag{2.17}$$

with $\alpha=\beta\eta+\zeta\gamma-\delta$. Nongeneric cases also exist and we start by considering $\theta=0$. This leads to $\zeta\eta=0$ and we choose $\zeta=0$. One further constraint must be satisfied in this case $\delta\varepsilon=\beta\eta$. Assuming $\varepsilon=1$, we have $\delta=\beta\eta$ and obtain

$$\alpha\overline{x}x\underline{x}+\beta\overline{x}x+\gamma x\underline{x}+\beta\eta\overline{x}\underline{x}+x+\eta\underline{x}=0\,. \tag{2.18}$$

If we take $\varepsilon=0$ then two choices exist: $\eta=0$ or $\beta=0$. The first leads to

$$\alpha\overline{x}x\underline{x}+\beta\overline{x}x+\gamma x\underline{x}+\delta\overline{x}\underline{x}=0\,, \tag{2.19}$$

while the second gives

$$\alpha\overline{x}x\underline{x}+\gamma x\underline{x}+\delta\overline{x}\underline{x}+\eta\underline{x}=0\,. \tag{2.20}$$

We can immediately point out that mappings (2.19) and (2.20) are trivial. The first reduces to an affine one under the transformations $x \to 1/x$, while in the second the term $\underline{x}$ is a common factor and the mapping is, in fact, a two-point one. Mappings (2.17)–(2.20) complete the reductions of Eq. (2.5) in the case of a double root of Eq. (2.6).

In order to investigate the integrability of the mappings obtained above we shall apply the singularity confinement criterion as previously explained. We start with the generic mapping (2.8). Here the singularities are by construction 0 and ∞. Following the results of Ref. [4] we require confinement in just one step. This leads to the condition $\beta=\zeta=0$. We obtain thus the mapping:

$$\overline{x}x\underline{x}+\gamma x\underline{x}+\eta\underline{x}+1=0 \tag{2.21}$$

or, solving for $\overline{x}$:

$$\overline{x}=-\gamma+\frac{\eta\underline{x}+1}{x\underline{x}}, \tag{2.22}$$

where γ and η are free. We expect Eq. (2.22) to be integrable. This is indeed the case: we can show that Eq. (2.22) can be obtained from the projective system (2.1) and (2.2) provided we take

$$P=\begin{pmatrix} -(\gamma q+1) & q\overline{q} & 1 \\ q & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.23}$$

and $M=\underline{P}^{-1}$ provided q is some solution of the equation $\overline{q}q\underline{q}+q\underline{q}\eta+\underline{q}\,\underline{\gamma}+1=0$. Then we can use the projective system (2.1) and (2.2) in order to construct the general solutions of Eq. (2.21). (Let us point out here that the equation for q is really different from Eq. (2.21) due to the presence of $\underline{\gamma}$ rather than γ.)

We turn now to the remaining mappings starting with Eq. (2.9). For the application of the singularity confinement criterion we must find the values of x for which the mapping loses one degree of freedom. For mapping (2.9) this happens (by construction) when x is 0 or ∞, but another source of singularity exists, when $\gamma\underline{x}+\varepsilon=0$. The investigation of the singularity $x=0$ leads to the confinement condition $\varepsilon=\eta=1$. Proceeding now to the examination of the singularity starting from $\gamma\underline{x}+1=0$ we find that there is no way to confine it: the sequence is $\{-1/\gamma, x, -1, f, \infty, \infty, \ldots\}$ where f is some finite value. We conclude that Eq. (2.9) is not integrable.

Next, we examine mapping (2.10):

$$\overline{x}=-1-x(\gamma\underline{x}+\varepsilon)\,. \tag{2.24}$$

If $x=0$ we find $\overline{x}=-1$ and all the subsequent values are independent of the initial data: there is no way for mapping (2.24) to recover the lost degree of freedom and thus, we conclude that this mapping is not integrable. Next, we turn to Eq. (2.11):

$$\overline{x}=-1-\frac{\varepsilon x}{\eta\underline{x}+1} \tag{2.25}$$

and again start with $x=0$. We obtain $\overline{x}=-1$, $\overline{\overline{x}}=-1+\overline{\varepsilon}$ and then $\overline{\overline{\overline{x}}}$ may become indeterminate, in the form $0/0$, provided $\varepsilon=\eta=1$. A careful analysis of this case shows that the singularity is periodic. Translating x we find a mapping of the form

$$\overline{x}\underline{x}+x-1=0\,, \tag{2.26}$$

which must be integrable. It turns out that Eq. (2.26) is indeed integrable. Its integral can be readily obtained:

$$K=\frac{x}{\underline{x}}+\frac{\underline{x}}{x}+\frac{1}{x\underline{x}}-x-\underline{x}-2\left(\frac{1}{x}+\frac{1}{\underline{x}}\right) \tag{2.27}$$

and Eq. (2.27) can be integrated in terms of the elliptic functions. Mapping (2.26) is a member of the QRT family of integrable mappings [9] and can be easily shown to be the only member of this family included in the parametrization (2.5). Mapping (2.12):

$$\overline{x}=-1-\frac{\gamma x\underline{x}}{\eta\underline{x}+1}\,, \tag{2.28}$$

has two kinds of singularities induced either by $x=0$ or $\eta\underline{x}+1=0$. The latter leads to an unconfined sequence $\overline{x}=\infty$, $\overline{\overline{x}}=\infty$, etc. and thus we conclude that Eq. (2.28) is not integrable. We turn now to mapping Eq. (2.13):

$$\overline{x}=-x\frac{\gamma x\underline{x}+\varepsilon}{\delta\underline{x}+\zeta}\,. \tag{2.29}$$

A singularity starting with $\delta\underline{x}+\zeta=0$ leads to an unconfined sequence $\overline{x}=\infty$, $\overline{\overline{x}}=\infty,\ldots$ unless $\gamma=0$. However, even if $\gamma=0$ we obtain a sequence ∞, ∞, finite, $0,0,\ldots$ again without recovering the lost degree of freedom. Thus, this mapping too is not integrable. Next, we consider mapping (2.14). If $\delta\neq 0$ we can put $\delta=1$ by division (if $\delta=0$ the mapping becomes a two-point one)

$$\overline{x}=-(\gamma x+\eta)-\frac{\varepsilon x}{\underline{x}}\,. \tag{2.30}$$

The singularity of Eq. (2.30) occurs when $x=0$ and we obtain successively $\overline{x}=-\eta$ and $\overline{\overline{x}}=\infty$. The condition for the subsequent x's to be finite is just $\overline{\varepsilon}=\overline{\gamma}\underline{\eta}$ or $\varepsilon=0$. The latter is trivial because the mapping becomes a two-point one and we consider only the first condition. We take $\eta\neq 0$, in which case a scaling can bring its value to $\eta=1$, resulting to $\varepsilon=\gamma$. Thus, based on singularity confinement the mapping

$$\overline{x}(\gamma x+\underline{x}+1)+\gamma x=0 \tag{2.31}$$

should be integrable. This is indeed the case. Its integration can be obtained in a straightforward way. We find as the integral of Eq. (2.31) the quantity

$$\frac{(\overline{x}+1)(x+1)}{a\overline{x}}=K\,, \tag{2.32}$$

where a is related to γ through $\gamma=-\underline{a}/a$. Thus, Eq. (2.31) is the discrete derivative of a homographic mapping. Once Eq. (2.32) is obtained the complete integration of Eq. (2.31) is elementary.

Finally, to make a long story short, we can just give without detailed proof the result that none of the mappings (2.15), (2.17) and (2.18) satisfies the singularity confinement criterion. Thus, we expect all of them to be nonintegrable.

Before closing this section let us present the continuous limits of the two linearizable equations we have identified above. For Eq. (2.21) we put $x=-1+vw$, $\gamma=3+v^2p$, $\eta=\gamma+v^3q$ and we obtain at the limit $v\to 0$ the equation

$$w''=3ww'-w^3+pw+q\,. \tag{2.33}$$

This is Eq. (6) in the Painlevé/Gambier classification [10] (in noncanonical form) and precisely the one that can be obtained from a $N=2$ projective Riccati system. For Eq. (2.31) we put $x=1+vw$, $\gamma=-1-v^3q$ and obtain, at the limit $v\to 0$

$$w''=ww'+2q\,. \tag{2.34}$$

This is Eq. (5) of the Painlevé/Gambier list [10] and the one resulting from the derivative of a Riccati equation. Its canonical form can be obtained by a translation and scaling of w.

3. Linearizable mappings from the Gambier systems

The Gambier equation is obtained as a system of two Riccati equations in cascade. (Cascade in this context means that the first equation contains only one variable, while the second one contains both. To solve the system one is thus led to integrate the two equations in that order, substituting the solution of the first one into the second one.) In a discrete context the Gambier system was studied in detail (first in Ref. [6] and in full detail in Ref. [7]). The approach is the same as in the continuous case. One introduces two discrete Riccati equations in cascade:

$$y=\frac{b\underline{y}+c}{a\underline{y}+d}\,, \tag{3.1}$$

$$\overline{x}=\frac{\varepsilon xy+\zeta x+\eta y+\theta}{\alpha xy+\beta x+\gamma y+\delta} \tag{3.2}$$

with the obvious assumption that $ac-db\neq 0$. Eliminating y between Eqs. (3.1) and (3.2) one obtains a single 3-point mapping for x. In this section we shall investigate the cases of the Gambier mapping that contain the linearizable mapping. To this end we shall first reduce the Gambier equation to the form Eq. (2.5). In practice, this means that the terms quadratic in x must be absent. The conditions for the mapping to be linear in all x, $\overline{x}$, $\underline{x}$ read:

$$\begin{aligned}
&\underline{\beta}(a\zeta+b\varepsilon)=\underline{\alpha}(c\varepsilon+d\zeta)\,,\\
&\underline{\delta}(a\zeta+b\varepsilon)=\underline{\gamma}(c\varepsilon+d\zeta)\,,\\
&\underline{\alpha}(c\alpha+d\beta)=\underline{\beta}(a\beta+b\alpha)\,,\\
&\underline{\gamma}(c\alpha+d\beta)=\underline{\delta}(a\beta+b\alpha)\,.
\end{aligned} \tag{3.3}$$

The solution of system (3.3) is long and tedious. We shall not enter into all the details but work out only the generic case and for the remaining cases give just the results. The generic case is based on the assumption that none of the terms appearing in Eq. (3.3) vanishes. By dividing the first two (or the last two) equations we obtain $\alpha\delta=\beta\gamma$. We then form the product of the second and third equation, use the previously derived relation and expanding we find $\alpha\zeta=\beta\varepsilon$. It suffices then to substitute into any of the four equations in order to get a final relation. The nongeneric cases are obtained by assuming that specific terms do vanish. Five cases can be distinguished, finally:

(1) $\zeta=\dfrac{\varepsilon\beta}{\alpha}, \qquad \beta=\dfrac{\alpha\delta}{\gamma}, \qquad c=\dfrac{a\delta\underline{\delta}+b\underline{\delta}\gamma-d\delta\underline{\gamma}}{\gamma\underline{\gamma}}$,

(2) $\alpha=\beta=0, \qquad \underline{\delta}=\dfrac{\underline{\gamma}(c\varepsilon+d\zeta)}{a\zeta+b\varepsilon}$,

(3) $\alpha=\beta=\gamma=0, \qquad a\zeta+b\varepsilon=0$,

(4) $\alpha=\gamma=\varepsilon=a=0$,

(5) $\delta=\gamma=0, \qquad \zeta=\dfrac{\varepsilon\beta}{\alpha}, \qquad c=\dfrac{\underline{\beta}(a\beta+b\alpha)-d\beta\underline{\alpha}}{\alpha\underline{\alpha}}$.

Once these basic conditions are obtained we still have full homographic freedom which will allow us to reduce further Eqs. (3.1) and (3.2). Let us work out in detail the case corresponding to constraint (1) above. First, we can translate y so as to bring δ to zero. The direct consequence of $\delta=0$ is $\zeta=\beta=c=0$. We can further translate x so as to get $\gamma=0$ and by division put $b=1$, $\alpha=1$. Finally, by scaling x we can take $\varepsilon=1$, unless $\varepsilon=0$ and by scaling y we can put $\theta=1$. In the case $\varepsilon=0$ we can use the scaling of x to put $\eta=1$ (unless, of course, $\eta=0$) and then put $\theta=1$. As a result of these transformations case (1) can be reduced to three cases:

$$y=\frac{\underline{y}}{a\underline{y}+d}, \tag{3.4}$$

combined with one of the three equation for x below:

$$\overline{x}=\frac{xy+\eta y+1}{xy} \tag{3.5}$$

or

$$\overline{x}=\frac{y+1}{xy} \tag{3.6}$$

or

$$\overline{x}=\frac{1}{xy}. \tag{3.7}$$

Concerning the remaining cases we just give the final results. Case (2) leads to the following mapping:

$$\overline{y}=\frac{b\underline{y}-1}{a\underline{y}+1}, \tag{3.8}$$

$$\overline{x}=\frac{xy+x+\theta}{y}, \tag{3.9}$$

which is the main type and also to three simpler cases

$$\overline{y}=\frac{b\underline{y}}{a\underline{y}+1}, \tag{3.10}$$

$$\overline{x}=\frac{xy+1}{y} \tag{3.11}$$

and also

$$\overline{y}=\frac{b\underline{y}+c}{\underline{y}} \tag{3.12}$$

coupled to either

$$\overline{x}=\frac{x+y}{y} \tag{3.13}$$

or

$$\overline{x}=\frac{x}{y}. \tag{3.14}$$

Case (3) leads to cases identical to some of the ones identified in case (2) plus one trivial case corresponding to a purely linear mapping $\overline{y}=b\underline{y}+c$, $\overline{x}=x+y$.

Case (4) yields the mapping:

$$\overline{y}=b\underline{y}+c, \tag{3.15}$$

$$\overline{x}=\frac{\zeta x+y}{x}. \tag{3.16}$$

Finally, case (5) is identical to case (1). We can readily remark that the nongeneric mapping (3.4) or (3.10) is, in fact, linear for $1/y$. Thus, system (3.10) and (3.11) can be readily discarded as being linear. Moreover, Eqs. (3.15) and (3.16) are a subcase of Eqs. (3.4) and (3.5) (once y is inverted). We can thus summarize the cases to be studied in detail in the following list (where we have inverted y in Eq. (3.4)). We obtain just three cases, where $\sigma=0$ or 1.

(I)

$$\overline{y}=a+d\underline{y}, \qquad x\overline{x}=\sigma x+y \tag{3.17}$$

(obtained from case (1) up to a translation of y).

(II)

$$\overline{y}=b+\frac{c}{\underline{y}}, \qquad \overline{x}=\frac{x}{y}+\sigma \tag{3.18}$$

(obtained from the simplified case (2)) and

(III)

$$\overline{y}=\frac{b\underline{y}-1}{a\underline{y}+1}, \qquad \overline{x}=\frac{xy+x+\sigma\theta}{y} \tag{3.19}$$

(obtained from the main type of case (2)). The cases with $\sigma=0$ are trivially integrable: once y is given the integration for x is obtained through a multiplicative equation of the form $\overline{x}x=F(n)$ or $\overline{x}/x=F(n)$, which can be linearized by just taking the logarithm of both sides.

In order to investigate the integrability of the $\sigma=1$ cases we shall apply the singularity confinement criterion. In the case of mapping (3.17) a singularity appears when $y=0$ in which case $\overline{x}$ does not depend on the value of x. No way exists for the mapping to retrieve this lost degree of freedom through some indeterminate form. Thus, Eq. (3.17) for $\sigma=1$ is not integrable. We consider now mapping (3.18) for $\sigma=1$:

$$y=b+\frac{c}{\underline{y}}, \tag{3.20a}$$

$$\overline{x}=\frac{x}{y}+1. \tag{3.20b}$$

The sequence of singular values of (3.20a) is $y=0$, $\overline{y}=\infty$ and finite values for $\overline{\overline{y}}$ and beyond. The fact that $y=0$ induces a divergence on $\overline{x}$, and at leading order we have $\overline{x} \propto x/y$. At the next step we must take into account that $\overline{y} \propto \overline{c}/y$ and thus the computation of $\overline{\overline{x}}$ leads to $\overline{\overline{x}}=1+x/\overline{c}$ that is a finite value depending on the initial condition x. Thus, the singularity is confined and mapping (3.20) is expected to be integrable. Finally, we examine mapping (3.19) for $\sigma=1$. The only singularity sequence is $\underline{y}=1/b$, $y=0$, $\overline{y}=-1$, etc. Starting with a regular $\underline{x}$ we find a finite value for x, while $\overline{x}$ diverges. However, a close examination of the singularity shows that $\overline{\overline{x}}$ is finite and the singularity is confined without any constraint. (Moreover, the special value $a=1$, which leads to a more complicated singularity pattern for y, also has confined singularities for x.) Thus, mapping (3.19) is expected to be integrable in general. Before proceeding further let us transform Eq. (3.19) to a form close to that of Eq. (3.20). It is possible to show that by translating x, $x \to x-\theta$, and a homographic transformation on y, $y \to y/(1-y)$, we can bring Eq. (3.19) to the form

$$y=A+\frac{B}{\underline{y}}, \tag{3.21a}$$

$$\overline{x}=\frac{x}{y}+\rho, \tag{3.21b}$$

where $A=(b+1)/(a+b)$, $B=-1/(a+b)$ and $\rho=\overline{\theta}-\theta$.

We turn now to the integration of mappings (3.20) and (3.21). Instead of restricting ourselves to the particular forms of Eqs. (3.20) and (3.21), we shall generalize our setting a little since this will lead to interesting results. Let us start with the more general Gambier system:

$$y = a + \frac{b}{\underline{y}}, \tag{3.22a}$$

$$\overline{x} = \frac{x}{y} + \frac{py+q}{ry+s}. \tag{3.22b}$$

The application of singularity confinement follows exactly the same steps as for Eqs. (3.20) and (3.21) and shows that Eq. (3.22) satisfies this integrability requirement. Thus Eq. (3.22) is expected to be integrable. Next, we can ask for which values of p, q, r, s is the 3-point mapping for x of the linearizable form (2.5). It turns out that this is possible only if the equation for x is of the form

$$\overline{x} = \frac{x}{y} + \frac{\theta - \phi y}{y}. \tag{3.23}$$

If $\phi = \overline{\theta}$ then it is possible to bring them both to zero (by translation of x), and the mapping becomes trivial. Otherwise it is possible, through a translation, to bring either ϕ or θ to zero and scale the remaining one to unity. Thus, the only interesting forms of Eq. (3.23) are

$$\overline{x} = \frac{x+1}{y} \tag{3.24}$$

or (with $\xi = x + 1$)

$$\overline{\xi} = \frac{\xi}{y} + 1. \tag{3.25}$$

One introduces the transformation $x = -(1 + \omega X)^{-1}$ (or equivalently $\xi = \omega X(1+\omega X)^{-1}$) where $\overline{\omega}\omega\underline{\omega} = -b$ and reduces the 3-point mapping resulting from the elimination of y between Eq. (3.22a) and (3.24) to

$$\overline{X}X\underline{X} + \underline{X}X\frac{a+1}{\overline{\omega}} + \underline{X}\,\underline{\omega}\left(1 - \frac{a}{b}\right) + 1 = 0. \tag{3.26}$$

We remark that Eq. (3.26) is the generic linearizable mapping presented in its fully reduced form in Eq. (2.21). Thus, the Gambier mapping leads again to the linearizable one obtained from the projective discrete Riccati system is perfect analogy to the continuous case.

4. Conclusions

In the previous sections we have investigated three-point mappings that are integrable through linearization. Our analysis was guided by the analogy with the continuous situation and results of ours on $N=3$ projective systems and the Gambier equation. We have presented an exhaustive analysis of the linearizable mapping and identified all its integrable forms. We thus have shown that the discrete equation

$$\overline{X}X\underline{X}+\alpha\underline{X}X+\beta\underline{X}+1=0 \tag{4.1}$$

can be reduced to a linear 3×3 system. The same equation, with the transformation $X=c(1+1/x)$ can be reduced to the Gambier system:

$$\begin{aligned} y&=a+\frac{b}{\underline{y}}\,,\\ \overline{x}&=\frac{x+1}{y} \end{aligned} \tag{4.2}$$

(where a, b, c are related to the α, β), which provides a different method for its solution. Nongeneric cases of the general linearizable mapping were also identified and, in particular, the equation:

$$\overline{x}(\gamma x+\underline{x}+1)+\gamma x=0\,, \tag{4.3}$$

that is the discrete derivative of a homographic mapping and constitute thus the discretization of Eq. (5) on the Painlevé/Gambier classification. The extension of the present results to higher-order (third and beyond) systems could be most interesting. We intend to return to this question in some future work.

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Chapitre 4

ENTROPIE ALGÉBRIQUE ET LINÉARISABILITÉ POUR LES SYSTÈMES DISCRETS

L'article *Linearisable Mappings and the Low-Growth Criterion* a été rédigé par Alfred Ramani, Basile Grammaticos, Stéphane Lafortune et Yasuhiro Ohta et a été soumis à *Letters in Mathematical Physics.*

LINEARISABLE MAPPINGS AND THE LOW-GROWTH CRITERION

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Abstract

We examine a family of discrete second-order systems which are integrable through reduction to a linear system. These systems were previously identified using the singularity confinement criterion. Here we analyse them using the more stringent criterion of nonexponential growth of the degrees of the iterates. We show that the linearisable mappings are characterised by a very special degree growth. The ones linearisable by reduction to projective systems exhibit zero growth, i.e. they behave like linear systems, while the remaining ones (derivatives of Riccati, Gambier mapping) lead to linear growth. This feature may well serve as a detector of integrability through linearisation.

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Integrability of discrete systems is a concept that can be understood on the basis of our experience on integrable continuous systems. The progress accomplished in the domain of discrete systems this last decade has made possible the identification of the possible types of integrability. The parallel with continuous systems is almost perfect. Three main types of integrable discrete systems seem to exist [1]:

a) Systems which possess a sufficient number of constants of motion. The QRT family of mappings [2] is a nice example of such a system.

b) Systems which can be reduced to linear mappings. They will be examined in detail in this paper.

c) Systems which can be obtained as the compatibility condition for some linear system i.e. systems that possess a Lax pair. Nice examples of such systems are the discrete Painlevé equations [3]. Given the Lax pair one can reduce the integration of the nonlinear mapping to the solution of an isomonodromy problem.

It is clear that the integration of a given integrable discrete system may proceed along any of the lines sketched above. One can, for example, perform one first integration using a constant of motion whereupon the system becomes linearisable and so on.

The very existence of integrable mappings (and their relative rarity) made their detection particularly interesting. Integrability detectors must, of course, be based on the properties which are characteristic of integrability. In this spirit we have proposed the singularity confinement property [4] based on the observation that a singularity spontaneously appearing in an integrable mapping disappears after some iterations: it is "confined" in the sense that it does not propagate *ad infinitum*. The singularity confinement criterion is a necessary one for integrability but, as we have already remarked in [1], it is not sufficient. This was explained in ample details by Hietarinta and Viallet [5] who have proposed the notion of algebraic entropy as a stronger criterion which could well be sufficient. This criterion is based on the ideas of Arnold [6] and Veselov [7] on the growth of the degrees of the iterates of some initial data under the action of the mapping. The main argument is that a generic, nonintegrable mapping has an exponential degree growth, while integrability is associated with low growth, typically polynomial. Although the degree itself is not invariant under coordinate changes, the type of growth, as pointed out by Bellon and Viallet [8], is invariant. The authors of [5] and [8] have introduced the notion of algebraic entropy defined as $E = \lim_{n\to\infty} (\log d_n)/n$, where d_n is the degree of the nth iterate. Generic, nonintegrable mappings have nonzero algebraic entropy. The conjecture is that integrability, associated to polynomial growth, leads to zero algebraic entropy. In [9] we have examined the results on discrete Painlevé equations based on the singularity confinement criterion in the light of the low-growth approach. Our main finding was that singularity confinement is sufficient in order to deautonomize a given integrable autonomous mapping. This result led to the proposal of a dual approach for the study of discrete integrability based on the successive applications on the singularity confinement and low-growth criteria, the latter being implemented only after the first is used to simplify the problem down to tractable proportions.

The aim of this paper is to examine this particular class of mappings which are linearisable and study their growth properties. Most of these systems were obtained using the singularity confinement criterion and thus a study of the growth of the degree of the iterates would be an interesting complementary information. Moreover, as we will show, the linearisable systems do possess particular growth properties which set them apart from the other integrable discrete systems.

The first mapping we are going to treat is a two-point mapping of the form $x_{n+1} = f(x_n, n)$ where f is rational in x_n and analytical in n. In [1] we have shown that for all f's of the form $\sum_i \frac{\alpha_i}{(x+\beta_i)^{\nu_i}}$ the singularity confinement requirement is satisfied. However all those mappings cannot be integrable: the discrete Riccati, $x_{n+1} = \alpha + \frac{\lambda}{x_n+\beta}$, is the only expected integrable one. Our argument in [1], for the rejection of these confining but nonintegrable cases, was based on the proliferation of the preimages of a given point. If we solve the mapping for x_n in terms of x_{n+1} we do not find a uniquely defined x_n and, iterating, the number of x_{n-k} grows exponentially. In what follows we shall analyse this two-point mapping in the light of the algebraic entropy approach. We start from the simplest case which we expect to be nonintegrable,

$$x_{n+1} = \alpha + \frac{\lambda}{x_n + \beta} + \frac{\mu}{x_n + \gamma}. \tag{1}$$

The initial condition we are going to iterate is $x_0 = p/q$ and the degree we calculate is the homogeneous degree in p and q of the numerator (or the denominator) of the iterate. We obtain readily the following degree sequence $d_n = 1, 2, 4, 8, 16, \ldots$ i.e. $d_n = 2^n$. Thus the algebraic entropy of the mapping is $\log(2) > 0$, an indication that the mapping cannot be integrable. In the present case it was quite easy to guess an analytical expression for the degree. What we do in general in order to obtain a closed-form expression for the degrees of the iterates, is to compute a sufficient number of them. Then we establish heuristically an expression of the degree, compute the next few ones and check that they agree with the analytical expression prediction. Now we ask how can one curb the growth and make it nonexponential. It turns out that the only possibilities are $\lambda\mu = 0$ or $\beta = \gamma$. In either case mapping (1) becomes a homography. The degree in this case is simply $d_n = 1$ for all n. This is an interesting result, clearly due to the fact that the homographic mapping is linearisable through a simple Cole-Hopf transformation.

The second mapping we shall examine is one due to Bellon and collaborators [10]

$$
\begin{aligned}
x_{n+1} &= \frac{x_n + y_n - 2x_n y_n^2}{y_n(x_n - y_n)}, \\
y_{n+1} &= \frac{x_n + y_n - 2x_n^2 y_n}{x_n(y_n - x_n)}.
\end{aligned} \tag{2}
$$

The degree growth in this case is studied starting from $x_0 = r$, $y_0 = p/q$ and again we calculate the homogeneous degree of the iterate in p and q, i.e. we set the degree of r to zero. (Other choices could have been possible but the conclusion would not depend on these details.) We obtain the degrees $d_{x_n} = 0, 2, 2, 4, 4, 6, 6, \ldots$ and $d_{y_n} = 1, 1, 3, 3, 5, 5, \ldots$ i.e. a linear degree-growth. This is in perfect agreement with the integrable character of the mapping. As was shown in [11] it does satisfy the unique preimage requirement and possesses a constant of motion $k = \frac{1 - x_n y_n}{y_n - x_n}$, the use of which reduces it to a homographic mapping for x_n or y_n.

The third mapping we are going to study is the one proposed in [1]

$$
\begin{aligned}
x_{n+1} &= \frac{x_n(x_n - y_n - a)}{x_n^2 - y_n}, \\
y_{n+1} &= \frac{(x_n - y_n)(x_n - y_n - a)}{x_n^2 - y_n}
\end{aligned} \tag{3}
$$

where a was taken constant. We start by assuming that a is an arbitrary function of n and compute the growth of the degree. We find $d_{x_n} = 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots$ and $d_{y_n} = 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots$ i.e. again a linear growth. This is an indication that (3) is integrable for arbitrary a_n and indeed it is. Dividing the two equations we obtain $y_{n+1}/x_{n+1} = 1 - y_n/x_n$ i.e. $y_n/x_n = 1/2 + k(-1)^n$ whereupon (3) is reduced to a homographic mapping for x. Thus in this case the degree-growth has succesfully predicted integrability.

A picture starts emerging at this point. While in our study of discrete Painlevé equations and the QRT mapping we found quadratic growth of the degree of the iterate, linearisable second-order mappings seem to lead to slower growth. In order to investigate this property in detail we shall analyse the three-point mapping we have studied in [12,13] from the point of view of integrability in general and linearisability in particular. The generic mapping studied in [13] was one trilinear in x_n, x_{n+1}, x_{n-1}. Several cases were considered. Our starting point is the mapping,

$$
x_{n+1}x_n x_{n-1} + \beta x_n x_{n+1} + \zeta\eta x_{n+1}x_{n-1} + \gamma x_n x_{n-1} + \beta\gamma x_n + \eta x_{n-1} + \zeta x_{n+1} + 1 = 0. \tag{4}
$$

We start with the initial conditions $x_0 = r$, $x_1 = p/q$ and compute the homogeneous degree in p, q at every n. We find $d_n = 0, 1, 1, 2, 3, 5, 8, 13, \ldots$ i.e. a Fibonacci sequence $d_{n+1} =$

$d_n + d_{n-1}$ leading to exponential growth of d_n with asymptotic ratio $\frac{1+\sqrt{5}}{2}$. Thus mapping (4) is not expected to be integrable in general. However, as shown in [13] integrable subcases do exist. We start by requiring that the degree growth be less rapid and as a drastic decrease in the degree we demand that $d_3 = 1$ instead of 2. We find that this is possible when either $\beta = \zeta = 0$ in which case the mapping reduces to:

$$x_{n+1} = -\gamma - \frac{\eta}{x_n} - \frac{1}{x_n x_{n-1}} \tag{5}$$

or $\gamma = \eta = 0$, giving a mapping identical to (5) after $x \to 1/x$. In this case the degree is $d_n = 1$ for $n > 0$. Equation (5) is the generic linearisable three-point mapping, written in canonical form. Its linearisation can be obtained in terms of a projective system [13] i.e. a system of three linear equations, a fact which explains the constancy of the degree.

The trilinear three-point mapping possesses also many nongeneric subcases, some of which are integrable. The first nongeneric case writes:

$$x_n(\gamma x_{n-1} + \epsilon) + (x_{n+1} + 1)(\eta x_{n-1} + 1) = 0. \tag{6}$$

The degrees of the iterates of mapping (6) form again a Fibonacci sequence even in the case $\epsilon = 0$ or $\eta = 0$. The only case that presents a slightly different behaviour is the case $\gamma = 0$:

$$(x_{n+1} + 1)(\eta x_{n-1} + 1) + \epsilon x_n = 0. \tag{7}$$

In the generic case the degree of the iterate behaves like d_n =0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49,... satisfying the recursion relation $d_{n+1} = d_{n-1} + d_{n-2}$ leading to an exponential growth with asymptotic ratio $\left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}$. Although the mapping is generically nonintegrable it does possess integrable subcases. Requiring for example that $d_4 = 1$ we obtain the constraint $\epsilon = \eta = 1$ and the mapping becomes periodic with period 5. If we require $d_5 = 1$, we obtain $\epsilon = -\eta_{n+1}(\eta_n - 1)$ and $\eta_{n+1}\eta_n\eta_{n-1} - \eta_{n+1}\eta_n + \eta_{n+1} - 1 = 0$, leading again to a periodic mapping with period 8. In these cases, the degree of the iterates exhibits, of course, a periodic behaviour. A more interesting result is obtained if we require $d_9 < 7$. We find that the condition $\eta = 1$ and ϵ an arbitrary constant leads to a nonexponential degree growth $d_n = 0, 1, 1, 1, 2, 2, 3, 4, 5, 6, 7, 9, 10, 12, 14, 15, 18, 20, 22, 25, 27, 30, 33, 36, 39, 42, 46, 49, \ldots$. Although the detailed behaviour of d_n is pretty complicated one can see that the growth is quadratic: we have, for example, $d_{4m+1} = m(m+1)$ for $m > 0$. Thus this mapping is expected to be integrable and indeed, it is a member of the QRT family. Its constant of motion is given by

$$K = y_{n+1} + y_n - \epsilon\left(\frac{y_{n+1}}{y_n} + \frac{y_n}{y_{n+1}}\right) + \epsilon(\epsilon+1)\left(\frac{1}{y_n} + \frac{1}{y_{n+1}}\right) - \frac{\epsilon^2}{y_n y_{n+1}}$$

where $y_k = x_k + 1$. The second nongeneric case is:

$$\gamma x_n x_{n-1} + \delta x_{n+1} x_{n-1} + \epsilon x_n + \zeta x_{n+1} = 0. \tag{8}$$

A study of the degree-growth leads always to exponential growth with asymptotic ratio $\frac{1+\sqrt{5}}{2}$, except when $\gamma = 0$ in which case the degrees obey the recurrence $d_{n+1} = d_{n-1} + d_{n-2}$. No integrable subcases are expected for mapping (8). The last nongeneric case we shall examine is

$$\gamma x_n x_{n-1} + x_{n+1} x_{n-1} + \epsilon x_n + \eta x_{n-1} = 0. \tag{9}$$

Again the degree sequence is a Fibonacci one except when $\gamma = 0$ or $\eta = 0$, in which case we have the recursion $d_{n+1} = d_{n-1} + d_{n-2}$, or when $\epsilon_n = \gamma_n \eta_{n-2}$. In the latter case the degree-growth follows the pattern $d_n = 0, 1, 1, 2, 2, 3, 3, \ldots$ i.e. a linear growth. Thus we expect this case to be integrable. This is precisely what we found in [13]. Assuming $\eta \neq 0$ we can scale it to $\eta = 1$, and thus $\epsilon = \gamma$. The mapping can then be integrated to the homography $(x_{n-1} + 1)(x_n + 1) = kax_{n-1}$ where k is an integration constant and a is related to γ through $\gamma_n = -a_{n+1}/a_n$. Thus in this case mapping (9) is a discrete derivative of a homographic mapping.

This leads us naturally to the consideration of the generic three-point mapping that can be considered as the discrete derivative of a (discrete) Riccati equation. Let us start from the general homographic mapping which we can write as

$$Ax_n x_{n+1} + Bx_n + Cx_{n+1} + D = 0. \tag{10}$$

where A, B, C, D are linear in some constant quantity κ. In order to take the discrete derivative we extract the constant κ and rewrite (10) as:

$$\kappa = \frac{\alpha x_n x_{n+1} + \beta x_n + \gamma x_{n+1} + \delta}{\epsilon x_n x_{n+1} + \zeta x_n + \eta x_n up + \theta}. \tag{11}$$

Using the fact that κ is a constant, it is now easy to obtain the discrete derivative by downshifting (11) and subtracting it form (11) above. Instead of examining this most general case we concentrate on the forms proposed in [14]. They correspond to the reduction of (11) to the two cases:

$$\kappa = x_{n+1} + a + \frac{b}{x_n} \tag{12}$$

$$\kappa = \frac{x_{n+1}(x_n + a)}{x_n + b} \tag{13}$$

Next we compute the discrete derivatives of (12) and (13). We find:

$$x_{n+1} = x_n + a_{n-1} - a_n - \frac{b_n}{x_n} + \frac{b_{n-1}}{x_{n-1}} \tag{14}$$

and

$$x_{n+1} = x_n \frac{x_{n-1} + a_{n-1}}{x_n + a_n} \frac{x_n + b_n}{x_{n-1} + b_{n-1}} \tag{15}$$

The study of the degree of growth of (14) and (15) can be performed in a straightforward way. For both mappings we find the sequence $d_n = 0, 1, 2, 3, 4, 5, 6, \ldots$ i.e. a linear growth just as in the cases of mappings (2), (3) and the integrable subcases of (9). If we substitute b_{n-1} by c_{n-1} in the last term of the rhs of (14) or the denominator of (15) we find $d_n = 0, 1, 2, 4, 8, 16, \ldots$ i.e. $d_n = 2^n$ for $n > 0$ unless $c = b$. Investigating all the possible ways to curb the growth we find for both (14) and (15) that $c = 0$ is also a possibility to bring d_3 down to 3. However a detailed analysis of this case shows that for $c = 0$ we have $d_n = 0, 1, 2, 3, 5, 8, 13, 21, \ldots$ i.e. a Fibonacci sequence with slower, but still exponential, growth (i.e. ratio $\frac{1+\sqrt{5}}{2}$ instead of 2).

One more family of linearisable discrete systems has been studied in detail in [15] and [16]. They are what we called the Gambier mappings which constitute the discretisation of the continuous Gambier equation [17]. The latter is a system of two Riccati's in cascade. In the discrete case the Gambier system is written as two homographic mappings which we write in canonical form as:

$$y_{n+1} = \frac{a_n y_n + b_n}{y_n + 1} \tag{16a}$$

$$x_{n+1} = \frac{x_n y_n / d_n + c_n^2}{x_n + d_n y_n} \tag{16b}$$

Eliminating y we can also write the discrete Gambier system as a single three-point mapping for x. The study of the degree growth of (16) is straightforward. We start from $x_0 = r$, $y_0 = p/q$ and compute the homogeneous in p, q degree of (16a) and (16b). Since (16a) is a Riccati its degree does not grow i.e. we have $d_{y_n} = 1$. Given the structure of (16b) we have $d_{x_{n+1}} = d_{x_n} + d_{y_n}$ and thus $d_{x_n} = n$. What is interesting here is that the Gambier mapping exhibits a linear degree-growth independently of the precise values of a, b, c, d. The fact that it can be reduced to Riccati's in cascade is enough to guarantee its integrability. On the other hand, if we had asked, (as we have done in [15]) for the possibility to express the solution as an infinite product of matrices,even across singularities, this would have led to constraints on the parameters (which were given in detail in [16]).

In this work we have examined a class of integrable discrete systems (mainly three-point mappings) from the point of view of the degree-growth of the iterates of some initial data. Our study was motivated from the recent works connecting slow-growth and integrability. Our present analysis confirms our previous findings based on the singularity confinement necessary discrete integrability criterion. But what is more important is that a relation between the details of integrability and the degree-growth seems to emerge. In this work we have found two main types of degree-growth: zero and linear growth. Zero growth is associated to systems which are linearisable through a reduction to a projective system. Linear growth is characteristic of systems which can be reduced to linear ones although at the price of some more complicated transformations, usually through the existence of some constant of motion or, as in the case of the Gambier mapping, through the solutions of linear equations in cascade. On the other hand, in our study on discrete Painlevé equations and the QRT mapping we found that quadratic growth was the rule. These results are, of course, characteristic of three-point (second-order) mappings and we do not expect the details concerning the precise exponents to carry over to higher-order mappings. Still, we expect the pattern detected here, namely that linearisable mappings lead to slower growth than the nonlinearisable integrable ones, to persist. It could be used for the classification of integrable discrete systems and be a valuable indication as to the precise method of their integration. We intend to return to this point in some future work.

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Chapitre 5

CONSTRUCTION DE SYSTÈMES INTÉGRABLES DU TROISIÈME ORDRE SELON L'APPROCHE DE GAMBIER

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Constructing integrable third-order systems: the Gambier approach

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Abstract. We present a systematic construction of integrable third-order systems based on the coupling of an integrable second-order equation and a Riccati equation. This approach is an extension of the Gambier method that led to the equation that bears his name. Our study is carried through for both continuous and discrete systems. In both cases the investigation is based on the study of the singularities of the system (the Painlevé method for ordinary differential equations and the singularity confinement method for mappings).

1. Introduction

The investigation of the integrability of second-order differential equations has been one of the most important enterprises in the history of integrable systems. Initiated by Painlevé [1] and completed by Gambier [2], it established the importance of singularity analysis as an integrability criterion. Following the spirit of Painlevé, the property that bears his name (absence of movable critical singularities) is synonymous with integrability, since it allows the definition of a function from the solution of an ordinary differential equation (ODE). (In contrast, an equation, the solution of which is explicitly given through quadratures but presents multivaluedness, is not integrable in Painlevé's point of view.)

The results of the Painlevé–Gambier investigations are of capital importance since they showed the existence of new transcendents, known since then under the name of Painlevé. Overshadowed by this momentous discovery, the work of Gambier on linearizable systems did not receive the attention it deserved. The recent discovery of integrable *discrete* systems has led naturally to a critical examination of the work of the 19th century masters. In particular, we have shown that it is possible to find discrete forms not only for the Painlevé equations, but, in fact, for every single equation in the Painlevé–Gambier list. The equation #XXVII of the list of 50 canonical equations [3], which we decided to call the Gambier equation, was of course among them. Its discretization necessitated a thorough understanding of the Gambier approach.

The main idea of Gambier (we are aware that the historical truth may be different) was to construct an integrable second-order equation by suitably coupling two integrable first-order ones. The latter were well known: at first order the only integrable (in the sense of having the Painlevé property) ODEs are either linear or of Riccati type. The Gambier equation is precisely the coupling of two Riccati in cascade (and it contains as a subcase the

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coupling involving one or even two linear equations). From the point of view of singularity analysis this coupling of two integrable equations is not harmless. Each of the equations has singlevalued movable singularities. However, the singularities induced on the second equation by the singularities of the solution of the first one (and which would thus look superficially as fixed) may lead to multivaluedness. This feature makes the application of singularity analysis mandatory. Its implementation leads to the (algebraically) integrable forms of the Gambier equation.

In perfect analogy to the continuous case, we have introduced in [4] the Gambier mapping. The latter is a system of two coupled homographic mappings (which play the role of the discrete Riccati) in cascade. The integrable forms were obtained through the application of the discrete integrability criterion that we have proposed under the name of singularity confinement.

In this work we shall address the question of the construction of integrable third-order systems in the spirit of Gambier. Namely we shall start with a second-order integrable equation and couple it with a Riccati (or a linear) first-order (also integrable) equation. This enterprise may easily assume staggering proportions. While at second order one had only two first-order building blocks at one's disposal, at third order there are minimally 24 equations (the Gambier list) to be coupled to the two first-order integrable ones. The situation is even more overwhelming in the discrete case since it is well known that each continuous equation of the Gambier list may possess several discrete avatars. In order to limit the scope of our investigation we shall consider coupled systems where the dependent variable enters only in a polynomial way. This leads naturally to the coupling of a Painlevé ($\mathbb{P}$) I or II to a Riccati.

Historically the coupling of a $\mathbb{P}$ equation with a Riccati was first considered by Chazy [5]. He examined an additive coupling of Painlevé I (P_I) with a Riccati. Starting with P_I in the form

$$w'' = 6w^2 + z$$

he introduced a Riccati:

$$y' = \alpha y^2 + \beta y + \lambda w + \gamma \tag{1.1}$$

where $\alpha, \beta, \lambda, \gamma$ are functions of z. This coupling is additive as opposed to that introduced by Gambier which is multiplicative and assumes the form

$$y' = \alpha y^2 + (\beta + \lambda w)y + \gamma.$$

(In the case of the Gambier coupling w is the solution of a Riccati equation.) Since the singularities of P_I are double poles ($6/(z - z_0)^2$), the only coupling that is compatible with integrability is the additive one. Assuming that $\alpha \neq 0$, we can put $\beta = 0$ by a simple translation of y and Chazy found that the only cases where the leading singularity does not induce multivaluedness were when equation (1.1) assumed the form

$$y' = \frac{1 - k^2}{4} y^2 + w + \gamma \tag{1.2}$$

where k is an integer not multiple of 6. Thus five cases had to be examined as $k = 6m + n$ with $n = 1, \ldots, 5$. Chazy found the following necessary integrability constraints:

$$\begin{array}{ll} n = 2 & \gamma = 0 \\ n = 3 & \gamma' = 0 \\ n = 4 & \gamma'' = \mu\gamma^2 + \nu z \\ n = 5 & \gamma''' = \mu\gamma\gamma' + \nu \end{array}$$

where μ and ν are specific numerical constants. It turns out that for $k = n$ they are also sufficient. For $k = 6m + 1$ the first condition appears at $k = 7$. In this case the constraint reads:

$$y^{(5)} = 48yy''' + 120y'y'' - \tfrac{2304}{5}y'y^2 - 24zy' - 48y.$$

This equation has the $\mathbb{P}$ property and is thus expected to be integrable. Still it is interesting to point out that this equation is more difficult to solve than the one we started with, which is of third order.

Chazy offers only a rapid comment concerning the case $k \geqslant 8$. In fact, the constraints obtained are necessary, but not sufficient for higher k's. We have examined the first few cases beyond $k = 7$ using the same method as Chazy, namely singularity analysis (but unlike Chazy our approach has profited from the existence of computer algebra tools). It turned out that none of the cases we examined satisfied the $\mathbb{P}$ criterion. So, although this is not a proof in a strict sense, we can suppose that no integrable cases exist beyond the five identified by Chazy.

2. Coupling of integrable second-order ODEs with a Riccati

As we have explained in the introduction we shall not attempt an exhaustive treatment of all 24 [2] (or 50 [3], or more [6]) second-order equations of the Painlevé/Gambier list with a Riccati. Instead we shall limit ourselves to the simplest case, namely equations where the dependent variable enters in a polynomial way (instead of rational). This limits the research to just three generic equations: $\mathrm{P_I}$ (already examined by Chazy), Painlevé II ($\mathrm{P_{II}}$) and the linearizable G5 (number 5 of the Gambier list) equation. Both $\mathrm{P_{II}}$ and G5 have dominant singularities that are single poles i.e. $w \sim 1/(z-z_0)$. Thus the adequate coupling is through a multiplicative Riccati. An additive coupling would lead to logarithmic singularities in the Riccati and thus to multivaluedness incompatible with integrability.

2.1. Coupling P_{II} with a Riccati

We start with the canonical form of $\mathrm{P_{II}}$, namely,

$$w'' = 2w^3 + zw + \mu \tag{2.1}$$

where μ is a constant, and consider the following multiplicative coupling:

$$y' = \alpha y^2 + (nw + \beta)y + \gamma. \tag{2.2}$$

where α, β, γ and, *a priori* n, are functions of z. A gauge transformation on y can be used in order to put β to zero. Next, we proceed to determine α, γ through the application of singularity analysis so as to ensure the $\mathbb{P}$ property for the system. Equation (2.1) has of course the $\mathbb{P}$ property and the expansion of its solution around a singularity is

$$w = \frac{\sigma}{z - z_0} + \cdots + a_4(z - z_0)^4 + \cdots \tag{2.3}$$

where $\sigma^2 = 1$ and a_4 is a free parameter (the second one besides z_0). The coupling of w with y must not lead to multivaluedness. Thus the coefficient n of the coupling must be an integer. This is only the first condition and, by far, not sufficient. In order to proceed further we expand y around the singularity z_0 and assume that y either has a pole at the same location or is regular. Substituting the expansion form (2.3) we can compute the terms of the series of y and obtain the compatibility conditions for the absence of logarithmic terms in the expansion of y. We find thus, the following condition for $n = 1$:

$$\gamma = \alpha = 0. \tag{2.4}$$

Let us point out that $\alpha = \gamma = 0$ works for every value of n: w is simply related to the logarithmic derivative of y. We have furthermore, for $n = 2$,

$$\gamma' = \alpha' = 0. \tag{2.5}$$

For $n = 3$ we obtain

$$\frac{\gamma}{\alpha} = -\left(\frac{2\alpha'' + \alpha z}{\alpha^3}\right) \qquad \text{and} \qquad \frac{\alpha}{\gamma} = -\left(\frac{2\gamma'' + \gamma z}{\gamma^3}\right). \tag{2.6}$$

Eliminating γ and integrating once, we find

$$\alpha\alpha''' - 3\alpha'\alpha'' - \alpha'\alpha z - k\alpha^2 = 0 \tag{2.7}$$

and putting $\phi = \frac{\alpha'}{\alpha}$ we find

$$\phi'' = 2\phi^3 + z\phi + k. \tag{2.8}$$

Thus the logarithmic derivative of α satisfies precisely P_{II} (with a free constant k). For $n = 4$ we find a more complicated condition:

$$3\alpha\gamma\gamma' + \alpha'\gamma^2 + 3\alpha(1 + \mu) + 3\gamma'z + 9\gamma''' = 0 \tag{2.9a}$$

$$3\gamma\alpha\alpha' + \gamma'\alpha^2 + 3\gamma(1 - \mu) + 3\alpha'z + 9\alpha''' = 0. \tag{2.9b}$$

Putting $\alpha^2 = \phi'$ we can integrate (2.9b) (multiplied by α) for the quantity $\alpha^3\gamma$ and thus obtain γ. Then (2.9a) gives a sixth-order homogeneous equation for ϕ and putting $u = \phi'/\phi$ leads to a fifth-order equation. This equation passes the $\mathbb{P}$ test and is thus presumably integrable but its integration is a more complicated task than the equation we started with, which is only of third order.

For $n = 5$ we obtain again as a first condition $\alpha = \gamma = 0$ which as we explained is sufficient. For $n = 6$ a first condition (as in the $n = 2$ case) is $\alpha' = \gamma' = 0$. However, a second condition appears. In fact, for $n > 4$ the free parameter of the expression (2.3) a_4 starts appearing in the compatibility condition which must be identically satisfied. Thus for $n = 6$ we find the second condition either $\alpha = 0$ and $\mu = -\frac{7}{6}$ or $\gamma = 0$ and $\mu = \frac{7}{6}$. Thus this coupling works only for some particular case of P_{II} with a specific μ. For $n \geqslant 7$ we have not been able to find any integrable case, besides the trivial $\alpha = \gamma = 0$ one. In some cases it is even possible to prove the incompatibility of the constraints. We surmise that the multiplicative coupling of P_{II} with a Riccati does not possess any integrable case besides those listed above.

2.2. *Coupling the linearizable G5 with a Riccati*

The canonical form of the linearizable equation, G5 in the Gambier list is,

$$w'' = -3w'w - w^3 + q(z)(w' + w^2). \tag{2.10}$$

The Cole–Hopf transformation $w = u'/u$ reduces (2.10) to a linear equation $u''' = q(z)u''$. The function $q(z)$ is completely free. Given this fact one can make two different couplings. The first is the 'standard' one where the solution of (2.10) for given $q(z)$ is injected into a Riccati:

$$y' = \alpha y^2 + nwy + \gamma. \tag{2.11}$$

The condition for the $\mathbb{P}$ property for $n < 0$ turns out to be $\alpha = 0$, while for $n > 0$ it is $\gamma = 0$. In both cases (2.10) becomes a linear equation (either for y or for $1/y$) and the remaining free function (γ or α) does not produce multivaluedness.

The second case of coupling is when $q(z)$ is itself proportional to the solution of a Riccati. Thus the coupled system now becomes

$$w' = \alpha w^2 + \beta w + \gamma \tag{2.12a}$$

$$y'' = -3yy' - y^3 + nw(y' + y^2). \tag{2.12b}$$

Only the case $\alpha \neq 0$ needs to be considered: when $\alpha = 0$ equation (2.12a) is linear and thus its solutions do not have any movable singularities. Since $\alpha \neq 0$ we can take $\alpha = 1$. As previously the coupling enters through nw with integer n since the singularity of (2.12a) is a simple pole. For $n < 0$ the system has always the $\mathbb{P}$ property and thus β and γ are free. In contrast, for $n > 0$ we have stringent integrability conditions. For $n = 1, 2$ there is no solution for β, γ leading to the $\mathbb{P}$ property for the system. For $n = 3$ we find as the only solution $\beta = \gamma = 0$. For $n = 4$ we obtain the condition,

$$\gamma = -\tfrac{11}{4}\beta^2 + \beta' \tag{2.13a}$$

and

$$\beta'' = 12\beta\beta' - 16\beta^3. \tag{2.13b}$$

Putting $\beta = -\phi'/4\phi$, (2.13b) reduces to $\phi''' = 0$ and we thus have elementary expressions for β and γ.

For $n \geqslant 5$ we can obtain the two compatibility conditions in the form of a higher order nonlinear system for β, γ. It turns out that for the first few cases studied this system has the weak $\mathbb{P}$ property [7]. We have not tried to integrate these systems since their integration is more difficult than the problem we started with.

3. Coupling of a second-order mapping with a discrete Riccati equation

Constructing integrable discrete systems in the same spirit as Gambier is quite straightforward once the basic ingredients are available. What is needed is a detailed knowledge of the forms of the equations to be coupled and a reliable integrability detector. The second-order mappings which play the role of the $\mathbb{P}$ equations in the discrete domain have been the object of numerous detailed studies and we are now in possession of discrete forms of all the equations of the Painlevé/Gambier classification. The discrete integrability detector is based on the singularity confinement that we discovered in [8] and which has turned out to be of the upmost reliability.

The coupling we are going to consider is a homographic mapping (discrete Riccati) for the variable y:

$$\overline{y} = \frac{(\alpha x + \beta)y + (\eta x + \theta)}{(\epsilon x + \zeta)y + (\gamma x + \delta)} \tag{3.1}$$

(where $\overline{y}$ stands for y_{n+1}, y for y_n and $\alpha, \beta, \ldots, \theta$ depend in general on n) the coefficients of which depend linearly on x, the solution of the discrete $\mathrm{P_I}$ or $\mathrm{P_{II}}$. (We shall not present here the coupling of the discrete analogue of the linearizable equation to a Riccati. As a matter of fact this equation is the simplest non-trivial member of the hierarchy of projective Riccati systems, the discretization of which was presented in full generality in [9].) The mapping (3.1) can be simplified and brought under canonical form through the application of homographic transformations on y. The generic form of the result is,

$$\overline{y} = \frac{(\alpha x + \beta)y + 1}{y + (\gamma x + \delta)}. \tag{3.2}$$

Non-generic cases do exist as well, and foremost among those is the linear relation,

$$\overline{y}(\gamma x + \delta) - y(\alpha x + \beta) - 1 = 0. \tag{3.3}$$

In what follows we shall examine in detail the coupling of (3.2) and (3.3) with either discrete Painlevé I (d-$\mathrm{P_I}$) or discrete Painlevé II (d-$\mathrm{P_{II}}$) (under various forms).

How does one apply the singularity confinement criterion to a mapping such as (3.2) when x is given by some discrete equation like d-$\mathrm{P_I}$ or d-$\mathrm{P_{II}}$? The first step consists of determining the singularities of (3.2). As we have explained in [10] the singularity manifests itself by the fact that $\overline{y}$ is independent of y. (We say in this case that y 'forgets the initial condition' or 'loses one degree of freedom'.) The condition for $\overline{y}$ to be independent of y is just

$$(\gamma x + \delta)(\alpha x + \beta) = 1. \tag{3.4}$$

This quadratic equation has two roots which we will denote by X_1, X_2: they can be easily related to α, β, γ, δ. The confinement condition is for y to recover the lost degree of freedom. This can be done if y assumes an indeterminate form 0/0. This means that x at this stage must again satisfy (3.4) and moreover be such that the denominator (or, equivalently, the numerator) vanishes.

Let us assume now that for some n we have $x_n = X_1$. The confinement requirement is that k steps later $x_{n+k} = X_2$. (We must point out here that we require that x_{n+k} be equal to X_2 and not to X_1 again. The latter would mean that X_1 is a singularity occurring periodically. Such singularities are not really movable, i.e. their position cannot be freely adjusted by choosing the appropriate initial conditions. Our conjecture is that they do not play any role in integrability, just as the fixed singularities in the continuous case.) Starting from $x_n = X_1$ and some initial datum x_{n-1}, we can iterate the mapping for x and obtain x_{n+k} as a complicated function of x_{n-1} and X_1. Since x_{n+k} depends on the free parameter x_{n-1} there is no hope for x_{n+k} to be equal to X_2 if X_1 is a generic point for the mapping of x. The only possibility is that both X_1 and X_2 be special values. What the special values of this equation are depends on its details, but clearly in the case of the discrete $\mathbb{P}$'s we shall examine here, these values can only be those related to the singularities. To be more specific, let us examine d-$\mathrm{P_{II}}$:

$$\overline{x} + \underline{x} = \frac{zx + a}{1 - x^2}. \tag{3.5}$$

The only special values of x are those related to the singularity $x_n = \pm 1$, $x_{n+1} = \infty$, $x_{n+2} = \mp 1$ while $\ldots, x_{n-2}, x_{n-1}$ and $x_{n+3}, x_{n+4}, \ldots$ are finite. This means that the two roots of (3.4) must be two of $\{+1, \infty, -1\}$ and moreover that confinement must occur in two steps. The precise implementation of singularity confinement requires that the denominator of (3.2) at $n+2$ vanishes (and because of (3.4) this ensures that the numerator vanishes as well). Moreover, we must make sure that the lost degree of freedom (i.e. the dependence on y) is indeed recovered through the indeterminate form.

3.1. Coupling various d-P_I's with a discrete Riccati

In this section we are going to analyse the coupling of four different forms of d-$\mathrm{P_I}$ to the homographic mapping (3.2) and to a linear equation (3.3). The d-$\mathrm{P_I}$'s we are going to consider are the following (presented below together with their singularity pattern):

$$\overline{x} + \underline{x} = \frac{z_n}{x} + \frac{1}{x^2} \qquad \{0, \infty, 0\} \tag{3.6}$$

$$\overline{x} + x + \underline{x} = \frac{z_n}{x} + 1 \qquad \{0, \infty, \infty, 0\} \tag{3.7}$$

$$\overline{x} + \underline{x} = \frac{z_n}{x} + 1 \qquad \{0, \infty, 1, \infty, 0\} \tag{3.8}$$

$$\overline{x}\underline{x} = -\frac{q_n}{x^2} + \frac{1}{x} \qquad \{q, 0, \infty, 0, q\} \tag{3.9}$$

with $z_n = an + b$ and $q_n = q_0\lambda^n$ (a, b, q_0, λ are constants). (More forms of the discrete $\mathrm{P_I}$ [11] are known but we shall restrict our analysis of the possible couplings to just these simplest forms.)

All the singularity patterns above have as a common characteristic that one can enter the singularity through 0 and exit it again through 0. This means that the condition (3.4) can have 0 as a double pole. This results in the following conditions:

$$\begin{aligned} &\beta\delta = 1 \\ &\alpha\delta + \beta\gamma = 0 \end{aligned} \tag{3.10}$$

and since neither α nor γ can vanish (lest the x^2 term disappear) we have $\delta = 1/\beta$ and $\gamma = -\alpha/\beta^2$. One can, of course, consider the case where one (or two) of the roots of (3.4) are equal to ∞: after all ∞ is part of the special values of the singularity pattern. It has turned out that except for the case (3.8) the consideration of these cases does not lead to any interesting result. (Let us point out that the value 1 appearing in the singularity pattern of (3.8) should not be considered as a special value: it may well occur outside any singularity pattern). Thus the first discrete Riccati we are going to consider is of the form:

$$\overline{y} = \frac{y(\alpha x + \beta) + 1}{y - (\alpha x - \beta)/\beta^2}. \tag{3.11}$$

In all the cases considered, the first confinement condition, namely that y (at a suitable n) assumes the form 0/0 does not suffice in order to reintroduce the dependence in the initial conditions. It is thus necessary to proceed to the next order and introduce one further constraint (which turns out to be sufficient). Let us work out in detail the case of the $\mathrm{d\text{-}P_I}$ (3.6). Starting with $\underline{x} = 0$ we obtain $y = \underline{\beta}$ i.e. independent of the value of $\underline{y}$. For $x = \infty$ we obtain $\overline{y} = -\underline{\beta}\beta^2$ and finally at the next step, $\overline{x} = 0$, we ask that the numerator and denominator of $\overline{\overline{y}}$ vanish. This leads to the first condition

$$\underline{\beta}\beta^2\overline{\beta} = 1. \tag{3.12}$$

Implementing this constraint leads to a second confinement condition that reads: $\underline{\alpha}/\underline{\beta} = \overline{\alpha}/\overline{\beta}$. This means that $\alpha = c\beta$ where c is a constant with an even–odd dependence. The solution of the constraint (3.12) is straightforward. Taking the logarithm of both members and calling $b = \log\beta$ we find the linear equation

$$\underline{b} + 2b + \overline{b} = 0 \tag{3.13}$$

with solution $b = (p + qn)(-1)^n$. Simple solutions to (3.12) can be obtained from this last solution. On the other hand just by inspection we can obtain solutions to (3.12) where β is constant: $\beta = \pm 1, \pm \mathrm{i}$.

The case of the 'standard' $\mathrm{d\text{-}P_I}$ (3.7) can be treated along similar lines. The first confinement condition reads

$$\underline{\beta}\beta^2\overline{\beta}^2\overline{\overline{\beta}} = -1 \tag{3.14}$$

while the second becomes too complicated to be exactly solved. We prefer to proceed using one particular solution of (3.14) corresponding to constant β's, for example $\beta = \mathrm{i}$. This leads to a second confinement condition $\underline{\alpha}/\underline{z} = \overline{\overline{\alpha}}/\overline{\overline{z}}$. Thus $\alpha = cz$ where c is a constant with ternary freedom ($\overline{\overline{c}} = \underline{c}$). If we implement $\beta = \mathrm{e}^{\pm \mathrm{i}\pi/6}$ and define $\chi = -\alpha z$, we obtain, as a second confinement condition, the equation

$$\overline{\chi} + \chi + \underline{\chi} = 3\beta^2 \frac{z}{\chi} + c \tag{3.15}$$

where c is a constant of integration. Thus, after considering the coupling with a d-P$_{\mathrm{I}}$ we obtain a d-P$_{\mathrm{I}}$ of the same type as one of the confinement conditions. This is in perfect parallel to the continuous case of Chazy (coupling (1.2) with $n = 4$) where we find another P$_{\mathrm{I}}$ as the integrability condition for a coupling between a Riccati and a P$_{\mathrm{I}}$.

The case (3.8) leads to still more complicated equations. One way to simplify them is to choose β satisfying:

$$\underline{\beta}\beta^2\overline{\beta} = 1 \tag{3.16}$$

which is sufficient (but not necessary) to satisfy the first confinement condition. We can then implement the solutions $\beta = \mathrm{i}$ and $\beta = 1$. If $\beta = \mathrm{i}$, the second confinement condition is $\alpha = c/z$ (where c is a constant with quaternary freedom $\underline{\underline{c}} = \overline{\overline{c}}$). If $\beta = 1$, we define $\chi = -\alpha z$ and we obtain, as the second confinement condition, the following equation:

$$\overline{\chi} + \underline{\chi} = -\frac{4z}{\chi} + c \tag{3.17}$$

where c is a constant with binary freedom. So, again here, we find a d-P$_{\mathrm{I}}$ of the same type as the one we started with as a confinement condition.

For the case (3.8), it is also possible to consider a coupling where the condition (3.4) has 0 and ∞ as roots. This means that $\alpha = 0$ (we could also choose $\gamma = 0$ but these two cases are equivalent under the homographic transformation $w \to 1/w$) and $\delta = 1/\beta$. The first confinement condition then is $\gamma = -\delta$. We define $\chi = \overline{\beta}\beta$ and the second integrability condition reads,

$$\overline{\chi} + \underline{\chi} = -\frac{\underline{z} + c}{\chi} + 1 \tag{3.18}$$

where c is a constant of integration. Thus again we get a d-P$_{\mathrm{I}}$ of the same type as the one we started with. Finally we can also consider the case where the condition (3.4) has ∞ as a double root. We then must have $\alpha = \beta = 0$. The first confinement condition is $\delta = -\gamma$ and we obtain the following relation for γ:

$$\overline{\gamma}\gamma = \frac{1}{-\underline{z} + k} \tag{3.19}$$

(where k is a constant of integration) which can be solved in an elementary way for γ.

In the case of q-P$_{\mathrm{I}}$ (3.9) the full singularity pattern is one where we enter the singularity at q and exit it at q after four steps. However the complete study of this singularity pattern turns out to be intractable. Thus we shall limit ourselves here to the case where we enter the singularity through 0 and exit it through 0 after two steps. In this case the first confinement condition is just (3.12). Once this is implemented the second condition reads $\overline{\alpha}\underline{\beta} = \lambda\underline{\alpha}\overline{\beta}$ which means $\alpha = c\beta\mu^n$ where c is a constant with binary freedom and $\mu^2 = \lambda$.

Let us now turn to the case of the coupling of d-P$_{\mathrm{I}}$ with a linear equation (3.3). For the special values of d-P$_{\mathrm{I}}$ 0 and ∞, only three couplings have to be considered:

$$\overline{y} = \alpha y + \frac{1}{\gamma x} \tag{3.20a}$$

$$\overline{y} = \alpha x y + \frac{1}{\delta} \tag{3.20b}$$

$$\overline{y} = \frac{\beta y + 1}{\gamma x}. \tag{3.20c}$$

It turns out that in every case examined the second (3.20*b*) and third (3.20*c*) are always incompatible with confinement. The only remaining candidate is thus the coupling of the form (3.20*a*). By the appropriate gauge of y we can bring it to the form,

$$\overline{y} - y = \frac{1}{\gamma x}. \tag{3.21}$$

Let us work out in detail the case of (3.6). A detailed analysis of the singularity pattern shows that if $\underline{x}$ vanishes as does ϵ then x diverges like $1/\epsilon^2$ and $\overline{x}$ vanishes like $-\epsilon$. We compute the corresponding y's and find, at leading order, $y \sim 1/\underline{\gamma}\epsilon$, $\overline{y} \sim 1/\underline{\gamma}\epsilon$ and the condition for $\overline{\overline{y}}$ to be finite is $1/\underline{\gamma} - 1/\overline{\gamma} = 0$, i.e. γ must be a constant with binary freedom (i.e. even–odd dependence). The analysis of the remaining cases proceeds along similar lines. For (3.7) we have the pattern $\{\epsilon, \underline{z}/\epsilon, -\underline{z}/\epsilon, -\epsilon\overline{\overline{\overline{z}}}/\underline{z}\}$ and the condition for $\overline{\overline{\overline{y}}}$ to be finite is $\overline{\overline{\overline{\gamma}}}\,\overline{\overline{\overline{z}}} = \underline{\gamma}\underline{z}$, i.e. $\gamma = k/z$ where k is a constant with ternary freedom. The case (3.8) is related to the pattern $\{\epsilon, \underline{\underline{z}}/\epsilon, 1, -\underline{\underline{z}}/\epsilon, \overline{\overline{\overline{z}}}\epsilon/\underline{\underline{z}}\}$ leading to the confinement condition $\underline{\underline{\gamma}}\,\underline{\underline{z}} = \overline{\overline{\gamma}}\,\overline{\overline{z}}$, i.e. $\gamma = k/z$ where k is a constant with quaternary freedom. Finally the case (3.9) is related to the pattern $\{\underline{\underline{q}} + \epsilon, a\epsilon, -\lambda/(a^2\epsilon^2), -\epsilon a/\lambda, \overline{\overline{q}}\}$ (where a is a free constant). Again we concentrate on the singularity induced by $x = 0$ and which confines when $x = 0$ again. This results in the condition $\overline{\gamma} = \lambda\underline{\gamma}$ which means $\gamma = k\mu^n$ where k is a constant with binary freedom and $\mu^2 = \lambda$.

3.2. *Coupling discrete P_{II}'s with a discrete Riccati*

In this section we shall examine the coupling of two different discrete forms of P_{II} with a Riccati: a difference one (which is the 'standard' d-P_{II})

$$\overline{x} + \underline{x} = \frac{zx + \mu}{1 - x^2} \tag{3.22}$$

where z is linear in the discrete variable n and μ is a constant, and one of q-type:

$$\overline{x}\underline{x} = \frac{p(x - q)}{x(x - 1)} \tag{3.23}$$

where $q = q_0\lambda^n$ and $p = p_0\lambda^n$.

Let us start with d-P_{II} (3.22). The singularity pattern of this equation is $\{\pm 1, \infty, \mp 1\}$. This means that the singularity condition (3.4) must have ± 1 as roots (the case when one root is ∞ does not lead to anything interesting). As a result we have that δ and γ are given by $\delta = -\beta/(\alpha^2 - \beta^2)$, $\gamma = \alpha/(\alpha^2 - \beta^2)$. The pattern $\{1, \infty, -1\}$ leads to a confinement condition: $\gamma = (\underline{\alpha} + \underline{\beta})\alpha(\overline{\alpha} - \overline{\beta})$ while the second pattern $\{-1, \infty, 1\}$ leads to $\gamma = (\underline{\alpha} - \underline{\beta})\alpha(\overline{\alpha} + \overline{\beta})$. Equating the two expressions for γ we find $\underline{\alpha}\overline{\beta} = \overline{\alpha}\underline{\beta}$ i.e. $\beta = k\alpha$ where k is a constant with binary freedom which we will ignore from now on. Expressing γ in two possible ways we obtain finally for α the equation

$$\underline{\alpha}\alpha^2\overline{\alpha} = \frac{1}{(1 - k^2)^2}. \tag{3.24}$$

This equation can be solved by linearization simply by taking the logarithm of both sides.

The q-$\mathrm{P_{II}}$ has also two singularity patterns $\{q, 0, \infty, 1\}$ and $\{1, \infty, 0, q\}$. Requiring 0 and 1 to be roots of (3.4) gives the following expressions for γ, δ: $\delta = 1/\beta$ and $\gamma = -\frac{\alpha}{\beta(\alpha+\beta)}$. Next we obtain the confinement conditions for the two patterns of singularities:

$$\underline{\alpha}\alpha\beta\overline{\beta} + \alpha\underline{\beta}\beta\overline{\beta} + \underline{\alpha}\beta^2\overline{\beta} + \underline{\beta}\beta^2\overline{\beta} - 1 = 0 \tag{3.25a}$$

$$\alpha\overline{\alpha}\underline{\beta}\beta + \overline{\alpha}\underline{\beta}\beta^2 + \alpha\underline{\beta}\beta\overline{\beta} + \underline{\beta}\beta^2\overline{\beta} - 1 = 0. \tag{3.25b}$$

Subtracting these two equations we obtain $\overline{\alpha}\underline{\beta} = \underline{\alpha}\overline{\beta}$ i.e. $\beta = k\alpha$ where k is a constant with binary freedom which we again ignore. Substituting back to (3.25) we obtain the final condition:

$$\underline{\alpha}\alpha^2\overline{\alpha} = \frac{1}{k^2(k+1)^2} \tag{3.26}$$

which can be integrated through linearization as explained above. The case where (3.3) has ∞ and q as roots is equivalent to the one treated above by a homographic transformation.

We now consider the case where (3.4) has 0 and q as roots which imposes the relations $\delta = 1/\beta$ and $\gamma = \frac{-\alpha}{\beta(q\alpha+\beta)}$. As a first condition we then find that β is a constant with binary freedom. We ignore this freedom and consider β as a constant and we obtain the following relation for α: $\alpha = -\frac{(\beta^2+1)}{\beta q}$. Finally, the case where (3.4) has q and 1 as roots has been studied but the resulting equations are far too complicated to be of any use. There is no other possible coupling of the form (3.2) with the q-$\mathrm{P_{II}}$ (3.23).

Let us now turn to the case of a linear coupling given by equation (3.3). In the case of d-$\mathrm{P_{II}}$ (3.22) we require that the only singularities of the coupling terms $(\alpha x + \beta)/(\gamma x + \delta)$ be the two singularities ± 1. This leads to a coupling of the form:

$$\overline{y} = \frac{\alpha(x \pm 1)y + 1}{x \mp 1} \tag{3.27}$$

where one of the parameters (e.g. γ) has been put to 1 through the appropriate gauge of y. Computing the successive y's we find that the condition for having a finite $\overline{\overline{\overline{y}}}$, depending on the initial condition y, is just $\overline{\alpha}\alpha = 1$. This means that all even α's are constant while all odd ones are equal to the inverse of this constant.

For q-$\mathrm{P_{II}}$ (3.23), in the case where (3.4) has 0 and 1 as roots, we have two possible couplings:

$$\overline{y} = \frac{\alpha x y + 1}{x - 1} \tag{3.28a}$$

and

$$\overline{y} = \frac{\alpha(x-1)y + 1}{x}. \tag{3.28b}$$

It turns out that in both cases the confinement condition is the same as in the case of d-$\mathrm{P_{II}}$ namely $\overline{\alpha}\alpha = 1$. When the roots are 0 and q, the possible couplings are:

$$\overline{y} = \frac{\alpha x y + 1}{x - q} \tag{3.29a}$$

and

$$\overline{y} = \frac{\alpha(x-q)y + 1}{x}. \tag{3.29b}$$

The condition for integrability in the two cases is $\alpha = 1/\lambda$. Two other couplings are possible when the roots of (3.4) are q and ∞:

$$\overline{y} = \frac{\alpha y + 1}{x - q} \tag{3.30a}$$

and

$$\overline{y} = \alpha(x - q)y + 1. \tag{3.30b}$$

The integrability condition for (3.30a) is $\alpha = \underline{q}$ and for (3.30b), it is $\alpha = 1/q$. Finally if (3.4) has 1 and q as roots, the possible couplings are

$$\overline{y} = \frac{\alpha(x - 1)y + 1}{x - q} \tag{3.31a}$$

and

$$\overline{y} = \frac{\alpha(x - q)y + 1}{x - 1}. \tag{3.31b}$$

In the case of (3.31a), the integrability condition is

$$\overline{\alpha}\alpha\underline{\alpha}(-\overline{\overline{q}} + 1) + \overline{\alpha}(-\overline{q}q + \overline{\overline{q}} + \underline{q} - 1) + q^2 - \overline{q} = 0 \tag{3.32}$$

and in the case of (3.31b), the condition reads

$$\overline{\alpha}\alpha\underline{\alpha}(\overline{q}^2 - q) + \overline{\alpha}\alpha(-\overline{q}q + \overline{\overline{q}} + \underline{q} - 1) - \underline{q} + 1 = 0. \tag{3.33}$$

Equations (3.32) and (3.33) are integrable and they belong to the family of linearizable equations [10].

One last remark is necessary at this point since we have seen that almost all the equations we obtained contain terms with binary, ternary or quaternary freedom. The presence of these terms indicates that our systems must be augmented by adding more components. This will not alter the order of the resulting equation: it just increases the number of its parameters. The continuous limit is, of course, affected by this choice.

4. Conclusion

In this work we have presented a systematic approach for the construction of integrable third-order systems through the coupling of a second-order equation to a Riccati or a linear first-order equation. Thus we have extended the Gambier approach (first used in his derivation of the second-order ODE that bears his name) to higher order systems. We have applied this coupling method to both continuous and discrete systems (given that we have already presented in [4] the discrete equivalent of the Gambier equation).

One point remains to be discussed. It is often argued that, since the Riccati is a linearizable equation, the coupling of the Riccati to another of the same kind or to an integrable second order is always integrable. The (naive) argument is the following: first solve the first equation, substitute the solution into the second and solve it by linearizing it. The argument about singularities is usually swept aside by the statement that one is interested only in solutions on the real-time axis. However the situation is not that simple. What integrability consists of is a global description of the solutions of the equations. The argument about solutions on the real-time axis is not acceptable since it offers just a local description of the solution of the equation. A global representation of the solution of a linear equation (and, thus, also of a Riccati) involves path integrals winding over the complex-time plane. Thus the study of movable singularities is crucial and the $\mathbb{P}$ property a necessary condition for integrability of the systems.

How do these arguments carry over to the discrete setting? One must go back to the way difference equations are formally solved. Given a linear difference (or $q-$) equation, we can express the solution as an infinite product of matrices, the elements of which depend on the coefficients of the equation. A singularity appears whenever one of the matrices is singular. In this case the solution of the linear difference equation cannot be defined for every n. However it is in general possible to choose the coefficients of the equation so as to avoid these singularities. In the case of a coupling the coefficients depend on the solutions of some other equation. Thus there is no way to control the singularities (which depend on the initial conditions of the first equation). As a consequence the solution of the second equation cannot be defined everywhere unless the confinement property is satisfied. Thus, again, despite the linearizability of the discrete Riccati, whenever we talk about a global description of the solution of the coupled system, the application of the adequate integrability criterion is mandatory.

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CONCLUSION

Dans cette thèse, nous avons tout d'abord utilisé la théorie des groupes de Lie afin de classifier un système différentiel aux différences. Ce système peut décrire des phénomènes en biophysique, en physique moléculaire et en mécanique classique. Un des résultats les plus intéressants de ce travail est l'identification des cas où l'algèbre de symétrie est de dimension infinie. Une suite logique à ce travail est l'étude dese liens entre la présence d'une algèbre de dimension infinie et l'intégrabilité.

Dans le deuxième chapitre, nous avons étudié, à l'aide de la théorie des groupes de Lie, des généralisations de l'équation de Toda. À notre grande surprise, nous avons pu identifier plusieurs cas présentant une algèbre de symétrie conforme de dimension infinie qui ne sont pas complètement intégrables.

Dans le troisième chapitre, nous avons classifié et étudié une famille importante d'équations discrètes linéarisables. Ce travail s'inscrit dans le cadre du projet général de classifier les équations discrètes à une variable de deuxième ordre. Une telle classification donnera aux physiciens un puissant outil pour l'étude des phénomènes discrets.

Par la suite, nous avons étudié les équations linéarisables du point de vue de l'entropie algébrique. Nous avons montré que les résultats obtenus avec cette approche sont les mêmes que ceux obtenus à l'aide du confinement des singularités.

Finalement, nous avons obtenu de grandes classes de systèmes intégrables du troisième ordre, autant dans le cas continu que dans le cas discret. L'étude des systèmes intégrables du troisième ordre est loin d'être terminée, pourtant une classification complète de ces systèmes sera un outil très utile aux physiciens.

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RÉSUMÉ

Symétries et intégrabilité des équations aux différences finies

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Résumé

La théorie des groupes de Lie joue un rôle très important dans l'étude des équations différentielles. Par exemple, le groupe de symétrie d'un système d'équations différentielles nous permet de construire une famille de solutions exactes à partir d'une solution déjà connue. On peut aussi classifier les équations selon leurs symétries et ainsi établir des liens entre des équations qui, *a priori*, n'en ont pas. De plus, l'analyse de Painlevé est une technique mathématique nous permettant d'étudier l'intégrabilité des équations différentielles.

Utilisées ensemble, l'étude du groupe de symétrie et l'analyse de Painlevé nous donnent un outil puissant pour trouver des solutions exactes de différents systèmes d'équations différentielles apparaissant en physique. Ces solutions sont déterminées à l'aide du processus de réduction par symétrie.

Tout comme les équations différentielles, les équations aux différences finies (EDF) sont souvent utilisées en physique. Elles peuvent décrire des phénomènes apparaissant dans des chaînes moléculaires unidimensionelles (A.D.N.) ou dans des réseaux cristallins. De même, elles apparaissent dans la théorie des groupes quantiques. Il est donc nécessaire de développer un formalisme nous permettant d'étudier les symétries et l'intégrabilité des équations aux différences finies tout comme on le fait présentement pour le cas continu.

Dans ma thèse, les symétries sont utilisées dans un premier temps pour la classification d'un système d'équations différentielles aux différences finies. Ce système se retrouve entre autres dans les domaines de la physique moléculaire, de la biophysique et de la mécanique classique. Un des résultats les plus intéressants obtenus dans cette thèse concerne l'existence de certains systèmes possédant un groupe de symétrie de dimension infinie. Mes travaux sur ce sujet sont la base d'un projet entamé récemment sur les liens entre les symétries et l'intégrabilité d'une équation aux différences finies. Nous étudions aussi des systèmes de Toda généralisés du point de vue de ses symétries. Les systèmes de Toda font partie des équations les plus importantes et les plus étudiées en physique mathématique moderne. Un des résultats intéressants que nous avons obtenu est l'identification de cas n'étant pas complètement intégrables mais possédant un groupe de symétrie conforme.

Pour ce qui est de l'intégrabilité, la présente thèse porte principalement sur des équations dites linéarisables, i.e. des équations qui sont équivalentes à un système linéaire. La principale méthode utilisée est le "confinement des singularités". Ce travail s'insère dans le vaste projet de recherche dont le but est de classifier toutes les équations discrètes intégrables à une variable. Nous classifions de grandes familles de systèmes linéarisables. Finalement, nous utilisons l'équation de Riccati afin d'obtenir des équations du troisième ordre intégrables.

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