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Études sur la gravitation : théories alternatives en 2+1 et 3+1 dimensions

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Sommaire

La présente thèse comprend quatre articles portant sur la gravitation, dont trois ont été publiés dans des revues scientifiques et un a été soumis pour publication. Les quatre articles se divisent naturellement en deux parties. Le Chapitre 1 comprend deux articles concernant la gravité conforme, une théorie de la gravité en 3+1 dimensions, tandis que le Chapitre 2 comprend deux articles sur la gravité en 2+1 dimensions avec masse topologique.

La gravité conforme est une théorie métrique de la gravitation qui est covariante comme la gravité d'Einstein, mais qui nécessite aussi que l'action demeure invariante sous les transformations conformes de la métrique. La solution sphériquement symétrique dans le vide (l'analogue de la solution de Schwarzschild pour la gravité d'Einstein) a été obtenue par Mannheim et Kazanas (MK) en 1989. La caractéristique principale de la solution est un terme de la métrique qui croît linéairement avec le rayon. On peut ensuite utiliser ce terme linéaire, lequel joue un rôle important aux échelles galactiques, pour imiter les effets de la matière sombre présumée, qui existerait en abondance aux échelles galactiques

Dans les premier et deuxième articles, nous avons évalué la viabilité de la solution de MK. Dans le premier article, nous avons calculé la déflexion de la lumière due à une source sphériquement symétrique et obtenu un résultat remarquablement simple : en plus de la déflexion d'Einstein habituelle, nous obtenons une déflexion qui croît linéairement (valide pour petites déflexions). Les données expérimentales portant sur la déflexion de la lumière dans les galaxies et les amas de galaxies requièrent qu'un paramètre libre γ soit de signe négatif, ce qui est contraire au signe de γ obtenu par MK dans leur analyse des courbes de

rotation galactique (lesquelles concernent le mouvement des particules massives non-relativistes). Dans ces deux analyses, aucune matière sombre n'a été incluse. Nous expliquons clairement dans le premier article les raisons de la divergence des deux résultats.

Dans le deuxième article, nous étudions en détail les propriétés géométriques et la structure causale de la solution de MK. Nous avons démontré que le terme linéaire dans la métrique contribue à une singularité à $r = 0$ et que l'espace-temps possède également une singularité conique à $r = 0$. À l'infini, l'espace-temps n'est pas plat mais plutôt conformément plat. En utilisant les diagrammes (conformes) de Penrose, nous démontrons que certains espaces-temps, qui correspondent à certains paramètres de la métrique, ne permettent pas à la lumière de se propager d'un infini à l'autre. Dans le cas des espaces-temps qui permettent une telle propagation, nous obtenons les trajectoires et la déflexion de la lumière (sans faire l'approximation que la déflexion est petite).

Afin de simplifier l'étude de la gravité, on fait souvent appel à un modèle en 2+1 dimensions. Cependant, en 2+1 dimensions, l'analogue de la gravité d'Einstein devient trivial. L'espace-temps à l'extérieur des sources est plat et les particules au repos n'interagissent pas. L'espace-temps de dimensions impaires nous permet d'inclure un terme gravitationnel de Chern-Simons qui donne une masse au graviton (une masse topologique). Les solutions dans le vide de cette gravité avec masse topologique (TMG) ont été trouvées par Vuorio qui a obtenu deux solutions distinctes : un secteur plat et un secteur cosmologique. La théorie linéarisée avec sources ponctuelles a été étudiée par Deser, qui a trouvé que le terme topologique induit un spin. Dans les troisième et quatrième articles, nous considérons la masse et le spin localisés dans la théorie non-linéaire. Nous obtenons des solutions pour les deux secteurs de Vuorio. Dans le troisième article, nous démontrons que les sources de fonction delta, bien que permises dans la théorie linéarisée, ne sont pas compatibles avec les équations non linéaires. Nous démontrons particulièrement qu'un résultat obtenu antérieurement par Clément est erroné et nous en donnons les raisons. Enfin, dans le quatrième article, nous étudions la façon d'inclure des sources ponctuelles dans le secteur cosmologique de Vuorio et discutons de la solution dans les cas où les observateurs sont proches ou éloignés de la source.

Summary

This thesis contains four articles on gravitation of which three have been published in scientific journals and one has been submitted for publication. The four articles divide naturally into two parts. Chapter 1 contains two articles on conformal gravity, a 3+1 dimensional theory of gravity and chapter 2 contains two articles on 2+1 dimensional topologically massive gravity.

Conformal gravity is a metric theory of gravitation which is generally covariant like Einstein gravity but also requires the action to be invariant under conformal transformations of the metric. The spherically symmetric vacuum solution to this theory (the analog to the Schwarzschild solution in Einstein gravity) was obtained by Mannheim and Kazanas in 1989. The novel feature of the solution is a term in the metric that grows linearly with the radius. This linear term, which plays a significant role on large distance scales, can then be used to imitate the effects of the dark matter that is presumed to exist in abundance on galactic scales. In the first and second article, we test the viability of the vacuum solution of conformal gravity. In the first article we calculate the deflection of light due to a spherically symmetric source and obtain a remarkably simple result: besides the usual Einstein deflection, we obtain an extra deflection that increases linearly as the "impact parameter" increases (valid for small deflections). The experimental data on the deflection of light in galaxies and clusters of galaxies require that a free parameter γ be of negative sign and this is the opposite sign for γ obtained by Mannheim and Kazanas by fitting galactic rotation curves (which concern non-relativistic massive particles). In both analyses no dark mattter has been included. We explain clearly in our first article the reasons for the discepancy between the two results. In the second article, we study in detail the geometrical

properties and the causal structure of the vacuum solution to conformal gravity. It is shown that the linear term in the metric contributes a singularity at $r = 0$ and that the space-time also possesses a conical singularity at $r = 0$. At infinity the space-time is not flat but conformal to flat. Through the use of Penrose (conformal) diagrams, we demonstrate that certain space-times, corresponding to certain parameters in the metric, do not allow light to scatter from infinity to infinity. For those which do, we obtain the trajectories and the deflection of light for arbitrary large values of r_0 (the radius of closest approach).

To simplify the study of gravity, it is common to study a 2+1 dimensional model. However, in 2+1 dimensions, the analog to Einstein gravity is trivial. The space-time outside sources is flat and particles at rest do not interact. The odd dimensional space-time allows one to include a gravitational Chern-Simons term called a topological mass term because it gives mass to the graviton. The solutions to this topologically massive gravity (TMG) in the vacuum were found by Vuorio and consisted of two distinct solutions: a flat sector and a cosmological sector. The linearized theory with point sources was treated by Deser where he obtained that the topological term induces a spin. In article three and four, we consider localized mass and spin sources in the full non-linear theory. We obtain solutions for both of Vuorio's sectors. In article 3 we demonstrate that delta function sources, while allowed in the linearized theory are not consistent with the full non-linear field equations. In particular we show that a result previously obtained by Clément was incorrect and point out the reasons for his error. In article four, we consider how to embed point sources in Vuorio's cosmological sector and discuss the metric solution for the case when observers are near and far away from the source.

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Je dédie la présente thèse à ma femme Ethel.

Les cieux racontent la gloire de Dieu,
Et l'étendue céleste annonce l'oeuvre de ses mains.

Psaume 19:1

The heavens declare the glory of God;
the skies proclaim the work of his hands.

Psalm 19:1

This most beautiful system of the sun, planets, and
comets, could only proceed from the counsel and
dominion of an intelligent and powerful Being.

Isaac Newton in
Philosophiae Naturalis Principia Mathematica

Introduction

L'histoire de la gravitation est une histoire fascinante qui est très pertinente à la présente thèse. Au milieu du XVIIe siècle, Isaac Newton (1642-1727) fit une prodigieuse découverte : il expliqua les mouvements planétaires de notre système solaire en émettant l'hypothèse que tous les corps s'attirent les uns les autres selon une force proportionnelle à l'inverse du carré de leur distance. Il démontra que divers phénomènes, tels que les marées, la chute d'un objet sur la Terre ou la trajectoire d'une planète autour du soleil, obéissaient tous à une loi gravitationnelle universelle. Plus tard, au milieu du XIXe siècle, John Adams (1819-1892) et Jean Le Verrier (1811-1877) ont chacun de leur côté utilisé la théorie de Newton pour expliquer les irrégularités de l'orbite d'Uranus et ont prédit l'existence et la position de la planète Neptune [1]. Au lieu de changer la théorie de Newton, ils ont postulé l'existence d'une autre planéte dans le système solaire, laquelle n'avait pas encore été observée. La découverte de Neptune quelques années plus tard à sa position prédite a donc été une solide confirmation de la théorie de Newton. Il demeurait cependant une petite divergence entre la théorie et l'expérience : la précession du périhélie de Mercure. Le Verrier et Simon Newcomb (1835-1909) ont tenté de résoudre ce problème en ajoutant de la matière à certains endroits : cette tentative s'est révélée un échec. En 1896, H.H. Seeliger a construit un modèle complexe où la matière était placée sur des ellipsoïdes proches du soleil. Il est intéressant de constater que ce modèle complexe de Seeliger a pu convaincre Newcomb et le reste de la communauté scientifique qu'il n'était pas nécessaire de modifier la loi de la gravitation. Le modèle de Seeliger est complètement faux et il a fallu le génie d'Einstein pour proposer et élaborer une théorie alternative à celle de Newton. Nous voyons donc qu'historiquement, les scientifiques ont tenté de résoudre les divergences entre théorie et expérience concernant la théorie de Newton de deux façons : d'une part, en supposant la présence de matière et, d'autre part, en modifiant la loi de la gravitation. Or dans un cas, nous avons vu qu'il était correct de présupposer la présence de matière et que dans l'autre, il est approprié de modifier la loi. Nous reviendrons sur ce point lorsque nous aborderons la question de la matière sombre.

Einstein n'a pas essayé de modifier la loi de la gravitation de Newton afin d'expliquer la précession du périhélie de Mercure et nous ne savons pas si cette anomalie a motivé ses recherches. Il était plutôt intéressé à mettre au point une théorie relativiste de la gravitation, c'est-à-dire une théorie de la gravitation en accord avec la relativité restreinte. La première étape d'Einstein sur le chemin de sa théorie fut la présentation du principe d'équivalence en 1907, qui postule qu'à chaque point dans un champ gravitationnel arbitraire il est possible de choisir des coordonées localement inertielles où les lois de la nature sont celles de la relativité restreinte. En utilisant ce principe, Einstein a voulu calculer la déflexion de la lumière par le soleil, mais il obtenait un résultat égal à la moitié de la déflexion réelle. Il n'avait pas encore compris la structure du champ gravitationnel. Il a par la suite conclu que le champ gravitationnel était un tenseur métrique à 10 composantes de la géométrie espace-temps pseudo-riemannienne. Selon Einstein, la gravitation se manifestait par la courbure de l'espace-temps quadri-dimensionnel! Ce concept constituait bien sûr une divergence radicale par rapport à la loi gravitationnelle de Newton. Afin d'incorporer le principe d'équivalence aux équations du champ gravitationnel, Einstein a établi des équations covariantes, c'est-à-dire invariantes sous transformations générales de coordonnées. En langage plus moderne, cela revient à dire que l'action (comme dans "action minimale") de la gravitation et de la matière devraient être des scalaires. Cela étant bien sûr insuffisant pour obtenir les équations d'Einstein, deux autres hypothèses ont été émises : d'une part, l'action gravitationnelle ne devrait pas excédée deux dérivées de la métrique et, d'autre part, les équations doivent se réduire aux équations de

Newton dans la limite où le champ est faible et les particules se déplacent à des vitesses non-relativistes. Avec ces deux hypothèses, l'action se réduit simplement au scalaire R de Ricci (le coefficient étant de $1/16\pi G$). Dans l'action, une constante Λ (appelée "constante cosmologique") ne va pas à l'encontre de la première hypothèse, mais ne respecte pas la phénoménologie newtonienne; elle est donc considérée égale à zéro ou extrêmement petite. L'important ici est que le principe d'équivalence n'est pas le seul à déterminer les équations d'Einstein, les deux autres hypothèses étant en effet cruciales. Nous reviendrons ultérieurement sur ce point. La théorie d'Einstein a été un grand succès : elle a pu expliquer l'anomalie du périhélie de Mercure, prédire correctement la déflexion de la lumière par le soleil ainsi que le retard des échos radars traversant le soleil.

Il existe toutefois aujourd'hui une divergence entre théorie et expérience : la masse lumineuse des galaxies et leur masse dynamique newtonienne inférée par les courbes de rotation des galaxies ne sont pas égaux. La vitesse des étoiles et des gaz entourant la périphérie des galaxies spirales ne diminue pas en fonction de la distance, mais demeure constante jusqu'à des distances qui dépassent la région lumineuse! Les courbes de rotation plates sont typiques de la plupart des galaxies spirales [5]. Près du centre, les courbes suivent assez bien les prédictions newtoniennes, mais si les distances à partir du centre sont plus grandes, les vitesses ne diminuent pas et restent constantes. Cette divergence s'explique généralement par le fait que la galaxie visible est enveloppée dans un halo sombre plus vaste, autrement dit, la majorité de la masse des galaxies est composée d'une matière non lumineuse distribuée (généralement sphériquement) à l'intérieur et à l'extérieur de la région lumineuse de la galaxie. Quant à la nature, à la distribution et à la quantité de cette matière sombre, elles demeurent mystérieusement inconnues. La matière sombre baryonique n'est plus considérée comme une des composantes principales possibles, mais la constitution de la matière non baryonique demeure inconnue [6]. L'hypothèse de la matière sombre n'est pas toujours très convaincante. Les modèles constitués d'un disque lumineux et d'un halo sombre étendu ont généralement trois paramètres ajustables qui peuvent toujours être adaptés pour retrouver les courbes de rotation. Dans nos rappels historiques, nous avons mentionné que Seeliger, en 1896, avait construit un modèle complexe de distribution de la matière pour essayer d'expliquer la précession du périhélie de Mercure. Sommes-nous en train de faire quelque chose de semblable? De nombreux auteurs sont de cet avis et ont par conséquent proposé des théories alternatives de la gravitation. C'est le cas de Milgrom, avec sa dynamique newtonienne modifiée (MOND, *Modified Newtonian Dynamics*) [2], de Bekenstein et Sanders, avec leur théorie tenseur-scalaire [4], et de Mannheim et Kazanas, avec leur théorie sur la gravité conforme [3]. La théorie qui a été le mieux vérifiée empiriquement est celle de Milgrom [5]. Selon la MOND, à des accélérations faibles typiques des orbites aux échelles galactiques, les lois newtoniennes sont modifiées. Cette théorie a pu remarquablement bien expliquer les courbes de rotation galactique en l'absence de matière sombre. Malheureusement, la MOND est une théorie non-relativiste qui ne peut en tant que telle faire aucune prédiction quant aux phénomènes relativistes telles la déflexion de la lumière, la cosmologie, etc. Pour ce qui est de la théorie scalaire-tenseur, elle n'est pas viable, car elle ne prédit pas une déflexion suffisante de la lumière [4].

Les deux articles figurant au Chapitre 1, l'article 1 et l'article 2, portent sur la gravité conforme (Weyl) en 3+1 dimensions. Il s'agit d'une théorie métrique de la gravitation qui est covariante comme celle d'Einstein, mais qui requiert également que l'action demeure invariante dans les transformations conformes de la métrique : $g_{\mu\nu}(x) \rightarrow \Omega^2(x) g_{\mu\nu}(x)$. Avec ces propriétés, l'action la plus simple pouvant être construite à partir de la métrique est le tenseur de Weyl au carré, $I = -\alpha \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa}$ où α est une constante de couplage sans dimension. Les équations de mouvement sont $2\alpha\, W_{\mu\nu} = (1/3) R_{;\mu;\nu} - R_{\mu\nu}{}^{;n}{}_{;n} + (1/6) g_{\mu\nu} R^{;n}{}_{;n} + 2R^{n\lambda} R_{\mu n\nu\lambda} - (2/3) R R_{\mu\nu} - (1/2) g_{\mu\nu} (R^{n\lambda} R_{n\lambda} - 1/3\, R^2) = (-1/2)\, T_{\mu\nu}$. Cette théorie est par conséquent une théorie à plus haute dérivée et a suscité beaucoup d'intérêt comme théorie quantique. Contrairement à la théorie d'Einstein qui est non renormalisable et comprend une constante de couplage dimensionnelle G, la théorie de la gravité conforme est renormalisable de façon perturbative [7, 8] grâce à l'absence d'échelle, la constante de couplage étant sans dimension. Il a par ailleurs été démontré que cette théorie était asymptotiquement libre [9, 10]. C'est dans les années 1980 que les recherches sur la gravité conforme classique ont vraiment commencé, grâce au travail de R. Riegert. Riegert a pu démontrer que le théorème de Birkhoff était valide pour la gravité conforme [11] et a démontré l'unicité et la stabilité des solutions symétriques sphériques. Riegert a également résolu la théorie linéarisée [12]. Il a conclu qu'en plus de la particule sans masse de spin 2, il existe une particule sans masse de spin 1 et une particule fantôme sans masse de spin 2. Vers la fin des années 1980, Mannheim et Kazanas (MK) [3] ont proposé la gravité conforme comme alternative classique à la théorie d'Einstein. En 1989, ils ont obtenu la solution statique sphériquement symétrique dans le vide (cette solution est analogue à la solution de Schwarzschild de la Relativité générale). La caractéristique principale de cette solution réside dans la présence d'un terme γr linéaire dans la métrique : en plus du terme "newtonien" $1/r$, il existe un terme proportionnel à r qui domine lorsque les distances deviennent élevées. Dans le but de comparer la théorie et les observations sur les échelles galactiques, là où le terme linéaire peut avoir un rôle significatif, MK ont comparé leur théorie aux courbes de rotation galactique et ont supposé l'absence de matière sombre : dans leur théorie, le terme linéaire imite les effets de la matière sombre présumée. Pour une métrique spécifique (obtenue en fixant le facteur conforme), ils ont pu obtenir une valeur du paramètre γ ainsi que certaines courbes raisonables.

Dans le premier article [13] de la présente thèse, nous avons évalué la viabilité de la solution de MK en calculant la déflexion de la lumière causée par une source sphériquement symétrique. Le résultat est remarquablement simple : en plus de la déflexion d'Einstein habituelle, nous obtenons une déflexion supplémentaire qui croît linéairement (valide pour des petites déflexions). Les données expérimentales portant sur la déflexion de la lumière dans les galaxies et les amas de galaxies requièrent que le paramètre γ soit de signe négatif (en supposant l'absence de matière sombre). Mannheim et Kazanas ont obtenu un signe opposé pour γ dans leur analyse des courbes de rotation des galaxies. Autrement dit, un γ négatif attire la lumière mais repousse les particules massives non relativistes, et vice-versa. Afin de comprendre la cause de cette différence, le potentiel effectif pour la théorie a été obtenu et utilisé pour redériver le résultat de la déflexion de la lumière. Nous discutons également la raison pour laquelle l'analyse des courbes de rotation galactique dans une théorie conformément invariante est problématique.

Dans le deuxième article, nous étudions en détail les propriétés géométriques ainsi que la structure causale de la solution de MK. Nous avons démontré que le terme linéaire dans la métrique contribue à une singularité à $r = 0$, chose qui était autrefois passée inaperçue. L'espace-temps possède aussi une singularité conique à $r = 0$, laquelle est causée par une petite constante dans la métrique. Très loin de la source, le tenseur de Riemann ne tend pas vers zéro, donc, pour des distances élevées, on ne retrouve pas l'espace-temps de Minkowski comme c'est le cas dans la solution de Schwarzschild. Cependant, le tenseur de Weyl révèle que l'espace-temps est conformément plat lorsque r est suffisamment grand. La structure causale des espaces-temps conformément plats devient transparente avec des coordonnées où la métrique a la forme suivante : facteur conforme fois la métrique de Minkowski. À l'aide des diagrammes (conformes) de Penrose, nous démontrons que certains espaces-temps, reliés à certains paramètres dans la métrique, ne permettent pas à la lumière d'aller d'un infini à l'autre. Quant aux espaces-temps qui permettent un tel déplacement, nous obtenons les trajectoires de la lumière et calculons la déflexion de la lumière pour des valeurs élevées de r_0 (le rayon le plus petit de la trajectoire).

L'un des plus grands problèmes de la physique théorique est d'obtenir une théorie quantique de la gravitation. La théorie gravitationnelle d'Einstein n'est pas renormalisable. Les deux grands piliers de la physique théorique, la mécanique quantique et la relativité générale, semblent pour le moment incompatibles. Au début des années 1980, à cause de la difficulté à comprendre la gravitation en quatre dimensions, les scientifiques se sont tournés vers un modèle plus simple en 2+1 dimensions, et ce en espérant apporter plus de clarté à la théorie quadri-dimensionnelle. Les équations d'Einstein en 2+1 dimensions sont triviales dans le vide et il n'existe aucun mode de propagation. La disparition du tenseur de Ricci implique la disparition du tenseur de Riemann et par conséquent, l'espace-temps est plat. Jackiw et coll. [14] ont démontré que si une masse m et un spin J localisés sont inclus, l'espace-temps devient coniquement plat avec une structure temps hélicoïdale; en d'autres termes, la masse produit un cône dont l'angle est proportionnel à la masse m et le spin contribue au temps hélicoïdal (avec un saut de $8\pi J$ tous les 2π). Comme nous l'avons déjà mentionné, il n'existe pas de mode de propagation dans la gravité d'Einstein en 2+1 dimensions et deux particules au repos n'interagissent pas. Les espaces-temps de dimensions impaires nous permettent toutefois d'inclure un terme avec masse topologique dans la théorie [15]. La théorie s'appelle alors "gravité en 2+1 dimensions avec masse topologique" (TMG, *Topologically Massive Gravity*). Le terme supplémentaire est un tenseur de deuxième rang $C^{\mu\nu}$ appelé tenseur de Weyl tridimensionnel et il a les propriétés suivantes : il disparaît si et seulement si la métrique tridimensionnelle est conformément plate, est conservée de manière covariante et possède au moins une dérivée de plus que le tenseur de Ricci $R^{\mu\nu}$ (ainsi, la constante de proportionnalité entre lui et le tenseur d'Einstein a des dimensions de masse inverse). Ce terme avec masse topologique correspond à l'analogue gravitationnel du terme de masse topologique de Chern-Simons qui existe dans les théories tridimensionnelles de jauge. La théorie en trois dimensions est significativement modifiée par le terme de masse topologique. Les modes de propagation deviennent possibles et la solution aux équations d'Einstein dans le vide ainsi qu'avec des sources ponctuelles acquièrent de nouvelles caractéristiques. Vuorio [16] a obtenu les solutions de TMG dans le vide et a obtenu deux secteurs différents : un secteur plat et un secteur cosmologique de courbure constante négative. Des sources ponctuelles ont ensuite été introduites dans la théorie linéarisée par Deser [17] et le résultat suivant a été obtenu : le terme topologique induit un spin de m/μ (où m est la masse de la source et μ, la masse topologique). Dans sa linéarisation de l'espace-temps plat, Deser n'a pu atteindre le secteur cosmologique de Vuorio.

Dans le troisième article, nous avons déterminé les solutions stationnaires à la théorie non-linéaire avec une masse et un spin localisés [18]. Nous avons plus particulièrement démontré qu'un résultat antérieur [19] était incorrect et nous avons établi que dans la théorie non linéaire, les sources de fonction delta n'étaient pas compatibles avec les équations de TMG. Il a aussi été démontré que le terme de masse topologique induit un spin et que le spin total est compris dans un intervalle de valeurs dépendant de la structure de la source. Dans la limite où la masse et le spin sont petits, nous retrouvons le résultat de la théorie linéarisée, à savoir que le spin induit est m/μ. Nos solutions représentent les sources dans le secteur plat de Vuorio et c'est pourquoi il est possible de retrouver le résultat linéarisé dans la limite appropriée. Cela signifie que les théories linéarisée et non linéarisée s'appliquent dans le secteur plat, ce qui n'est pas le cas du secteur cosmologique. En fait, seule la théorie non linéaire peut atteindre le secteur cosmologiqe de Vuorio.

Dans le quatrième article [20], nous avons incorporé la masse et le spin dans le secteur cosmologique. Cet exercice est très pertinent, car l'univers dans lequel nous vivons comprend des sources présentes dans une cosmologie et non dans un espace-temps plat. Notre résultat démontre que le spin associé à la cosmologie interagit avec la particule par un terme de couplage spin-spin. Cette interaction se traduit par un angle de cône supérieur par un facteur de trois à celui de l'espace plat. Dans une région proche des sources, mais extérieure à ces dernières, la métrique ressemble à la métrique coniquement plate avec spin, déjà obtenue dans le secteur plat [17, 18]. Cependant, près de la source, l'angle du cône reste trois fois plus grand que dans le secteur plat, ce qui implique que la cosmologie a influencé les propriétés locales de l'espace-temps. Loin de la source, nous retrouvons l'espace-temps cosmologique homogène de Vuorio.

Le terme gravitationnel de Chern-Simons est inversement proportionnel à la masse topologique. Il est donc raisonnable de s'attendre à ce qu'il tende vers zéro à la limite où la masse topologique tend vers l'infini, et de retrouver la gravité d'Einstein en 2+1 dimensions. Tel est le cas dans le secteur plat de la TMG où on retrouve les solutions d'Einstein obtenues par Jackiw et coll. [14]. Toutefois, dans le secteur cosmologique, on ne retrouve pas les solutions de la gravité d'Einstein à cette limite. Certaines composantes du terme de Chern-Simons ne tendent pas vers zéro lorsque la masse topologique tend vers l'infini : le numérateur et le dénominateur tendent tous deux vers l'infini et le résultat est différent de zéro. Le secteur cosmologique est donc un secteur détaché et aucune limite ne permet un contact avec la gravité d'Einstein en 2+1 dimensions.

Chapitre 1

Gravité Conforme (Weyl)

1.1 Article 1 : Classical tests for Weyl gravity: deflection of light and time delay

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Abstract

Weyl gravity has been advanced in the recent past as an alternative to General Relativity (GR). The theory has had some success in fitting galactic rotation curves without the need for copious amounts of dark matter. To check the viability of Weyl gravity, we propose two additional classical tests of the theory: the deflection of light and time delay in the exterior of a static spherically symmetric source. The result for the deflection of light is remarkably simple: besides the usual positive (attractive) Einstein deflection of $4GM/r_0$ we obtain an extra deflection term of $-\gamma r_0$ where γ is a constant and r_0 is the radius of closest approach. With a negative γ, the extra term can increase the deflection on large distance scales (galactic or greater) and therefore imitate the effect of dark matter. Notably, the negative sign required for γ is opposite

to the sign of γ used to fit galactic rotation curves. The experimental constraints show explicitly that the magnitude of γ is of the order of the inverse Hubble length something already noted as an interesting numerical coincidence in the fitting of galactic rotation curves [9].

I. Introduction

The higher-derivative conformally invariant Weyl action, the integral of the square of the Weyl tensor, has attracted much interest as a candidate action for quantum gravity . Unlike GR, the lack of scale in the theory probably implies that it is pertubatively renormalizable [1, 2]. The theory is also asymptotically free [3, 4].

Weyl gravity, as a classical theory, has attracted less attention because GR has been so remarkably successful at large distances i.e. on solar system scales, and therefore there seems no pressing need to study a higher-derivative alternative classical theory. However, GR may not be free of difficulties either theoretical or experimental. At present, it is faced with one long-standing problem: the notorious cosmological constant problem [5] whose solution is not yet in sight. There may however be an experimental problem with GR: the so-called dark matter problem. The clearest evidence for the existence of large amounts of dark matter comes from the flat rotation curves of galaxies, velocities of galaxies in clusters and the deflection of light from galaxies and clusters [6] (for short, we will call these observations "galactic phenomenology"). From this evidence, there is a consensus in the astrophysical community that most of the mass of galaxies (and of our universe) consists of non-luminous matter. However, the nature of this dark matter is still unknown and is one of the great unsolved problems in astrophysics. At first it was thought that it may be faint stars or other forms of baryonic matter i.e. the so-called massice compact halo objects (MACHOS). However, it is safe to say that observations have obtained much fewer events than required for an explanation of the galactic phenomenology with a dark halo dominated by MACHOS [7] (though there is still the possibility that future experiments might show otherwise). One is then left to consider non-baryonic forms of dark matter such as massive neutrinos, axions and WIMPS i.e. the weakly interacting massive particles as predicted for example by supersymmetric theories. The direct experimental observation of such non-baryonic candidates is of date singularly lacking (though many experiments are currently under development) [15]. Hence, to date, the nature of the dark matter that is thought to comprise most of the mass of our universe is still elusive. Is it possible that the copious amounts of dark matter we are searching for is simply not there? We believe it is reasonable at this juncture to consider such a possibility.

As far as we know, the deviation of galactic rotation curves from the Newtonian expectation occurs at distances way beyond the solar-system scale [14]. In other words, it is a galactic scale phenomena. Newton's gravity theory, which GR recovers in the non-relativistic weak gravity limit, was originally formulated to explain solar-system phenomenology and it may be incorrect to extrapolate this theory to galactic scales. It has therefore been suggested by a handful of authors [8, 9, 14] that there may not be large amounts of dark matter after all and that the "galactic phenomenology" may be signaling a breakdown of Newtonian gravity (and hence GR) on galactic scales.

Some authors have therefore proposed alternative classical theories of gravity. Most notably there is Milgrom's MOND program [8], Mannheim and Kazanas' Weyl (conformal) gravity program [9] and Bekenstein and Sander's scalar-tensor gravity theory [13]. In MOND, Newtonian dynamics are modified at low accelerations typical of orbits on galactic scales. It has had success in fitting galactic rotation curves without the need for dark matter [8, 14]. MOND, however, is a non-relativistic theory and therefore cannot make any predictions on relativistic phenomena such as the deflection of light, cosmology, etc. In the scalar-tensor theory, it has been shown that the bending of light cannot exceed that which is predicted by GR [13], in conflict with the observations i.e. the observed bending is actually even greater than that predicted by GR. On aesthetic grounds, conformal gravity is more appealing than other alternative theories because it is based on a local invariance principle i.e. conformal invariance of the metric. Weyl gravity encompasses the largest symmetry group which keep the light cones invariant i.e. the 15 parameter conformal group. It has already been stressed in the past that unlike Weyl gravity and gauge theories, GR is not based on an invariance principle. The Principle of General Covariance, which follows from the Principle of Equivalence, is not an invariance principle. It describes how physical systems behave in a given arbitrary gravitational field but it does not tell us much about the gravitational field itself beyond restricting the gravitational action to a scalar. The lack of an invariance principle is partly the reason why guesswork is inevitable in the derivation of Einstein's gravitational field equations (see [17] for details). In contrast, the Weyl action is unique due to its conformal invariance. Besides its aesthetic appeal, Weyl gravity has many other attractive features not the least being that it is renormalizable owing to its lack of length scale. Since the early days of GR, it has been known that the vacuum GR equations $R_{\mu\nu} = 0$ are also vacuum solutions of the Weyl theory. One therefore expects the Schwarzschild metric to be one possible solution to the spherically symmetric Weyl vacuum equations. More recently, Weyl gravity has attracted some interest because it has had reasonable success in fitting galactic rotation curves without recourse to any dark matter [10].

The principal reason that Weyl gravity has not received general acceptance is because some solutions of the classical theory are expected to have no lower energy bound and therefore exhibit instabilities [20] i.e. runaway solutions common to higher-derivative theories. For example, there may exist some Weyl vacuum solutions other than $R_{\mu\nu} = 0$ which are not desirable. Though it has been shown that the Einstein-Hilbert action plus higher-derivative terms has a well posed initial value problem [16] this has yet to be shown for the pure fourth order Weyl gravity. Fortunately, however, the static spherically symmetric vacuum solutions [9], the analog to the Schwarzschild metric, has been found to be stable and to make important corrections to the Schwarzschild metric at large distances i.e. it contains a linear potential that plays a non-trivial role on galactic scales. It therefore becomes compelling and interesting to compare Weyl gravity to GR in their classical predictions.

II. Geodesic Equations

Weyl gravity is a theory that is invariant under the conformal transformation $g_{\mu\nu}(x) \rightarrow \Omega^2(x) g_{\mu\nu}(x)$ where $\Omega^2(x)$ is a finite, non-vanishing, continuous real function. The metric exterior to a static spherically symmetric source (i.e. the analog of the Schwarzschild solution in GR) has already been obtained in Weyl gravity by Mannheim and Kazanas [9]. For a metric in the standard form

$$d\tau^2 = B(r)\,dt^2 - A(r)\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) \tag{1.1}$$

they obtain the vacuum solutions

$$B(r) = A^{-1}(r) = 1 - \frac{2\beta}{r} + \gamma\,r - kr^2 \tag{1.2}$$

where β,γ and k are constants. The authors note that with $\beta = GM$, the Schwarzschild metric can be recovered on a certain distance scale (say the solar system) provided γ and k are small enough. The linear γ term would then be significant only on larger distance scales (say galactic or greater) and hence would deviate from Schwarzschild only on those scales. The constant k, which should be taken negative, can then be made even smaller so that the kr^2 term becomes significant only on cosmological scales (in fact, it has been shown [9] that k is proportional to the cosmological scalar curvature). It should be noted that the solution (1.2) is not unique. The Weyl gravitational field equations are conformally invariant so that any metric which is related to the standard metric (1.1) by a conformal factor $\Omega^2(r)$ is also a valid solution. This is in contrast to GR where the Schwarzschild solution is the unique vacuum solution for a spherically symmetric source. Two metrics that differ by a conformal factor of course have different curvatures. Remarkably, however, the geodesic equations for light are conformally invariant. Massive particles, on the other hand, have geodesics that depend on the conformal factor (though it is conceivable to envisage some spontaneous conformal symmetry breaking mechanism which gives rise to conformally covariant massive geodesics. e.g. see [11]. We do not entertain conformal symmetry breaking in this paper).

The geodesic equations along the equatorial plane ($\theta = \pi/2$) for a metric of the form (1.1) are [17]

$$r^2\frac{d\varphi}{dt} = J\,B(r) \tag{1.3}$$

$$\frac{A(r)}{B^2(r)}\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B(r)} = -E \tag{1.4}$$

$$d\tau^2 = E\,B^2(r)\,dt^2 \tag{1.5}$$

where E and J are constants with $E = 0$ for null geodesics (photons) and $E > 0$ for massive particles. The above geodesic equations are only conformally invariant for photons and therefore two classical tests can be carried out unambiguously: the deflection of light and the time delay of radar echos.

III. Deflection of Light

The geodesic equations (1.3)-(1.5) enable one to express the angle φ as a function of r

$$\varphi(r) = \int \frac{A^{1/2}(r)}{r^2\left(\frac{1}{J^2\,B(r)} - \frac{E}{J^2} - \frac{1}{r^2}\right)^{1/2}}\,dr. \tag{1.6}$$

where the functions $A(r)$ and $B(r)$ are given by (1.2). To do a scattering experiment, the light is taken to approach the source from infinity. Unlike the Schwarzschild solution where the metric is Minkowskian at large distances from the source i.e. $B(r)$ and $A(r) \to 1$ as $r \to \infty$, $B(r)$ given by the solution (1.2) diverges as $r \to \infty$ and we do not recover Minkowski space at large distances. However, this is not a problem. At large r it has been shown that the metric is conformal to a Robertson Walker metric with three space curvature $K = -k - \gamma^2/4$ [9]. Hence, at large r the photon is simply moving in a "straight" line in this background geometry (i.e. with $B(r)$ given by (1.2) and $\varphi(r)$ given by (1.6), it is easy to see that $d\varphi/dr \to 0$ as $r \to \infty$). The photon then deviates from this "straight" line path as it approaches the source.

We now substitute the appropriate quantities in Eq.(1.6). For the photon we set $E = 0$. At the point of closest approach $r = r_0$, we have that $dr/d\varphi = 0$ and using equations (1.5) one obtains $(1/J^2) = B(r_0)/r_0^2$. From the solutions (1.2) we know that $A^{1/2}(r) = B^{-1/2}(r)$. The deflection of the photon as it moves from infinity to r_0 and off to infinity can be expressed as

$$\Delta\varphi = 2\int_{r_0}^{\infty}\left(\frac{B(r_0)}{r_0^2} - \frac{B(r)}{r^2}\right)^{-1/2}\frac{dr}{r^2} \;-\pi \tag{1.7}$$

where π is the change in the angle φ for straight line motion and is therefore subtracted out. We now calculate the integral in (1.7) using $B(r) = 1 - \frac{2\beta}{r} + \gamma\, r - kr^2$. This yields

$$\int_{r_0}^{\infty}\left(\left(1 - \frac{2\beta}{r_0} + \gamma r_0\right)\frac{r^4}{r_0^2} - \gamma r^3 - r^2 + 2\beta\, r\right)^{-1/2} dr \tag{1.8}$$

The above integral, being the inverse of the square root of a fourth-degree polynomial, can be expressed in terms of elliptic integrals. However, this is not very illuminating.

It will prove more instructive to evaluate the integral after expanding the integrand in some small parameters. Note that the constant k, important on cosmological scales, has cancelled out and does not appear in the integral (1.8). The deflection of light is insensitive to the cosmology of the theory and in general would not be affected by a spherically symmetric Hubble flow. On the other hand, the motion of massive particles on galactic or greater scales is affected by the Hubble flow [10, 19]. Hence, the bending of light is highly appropriate for testing Weyl gravity.

We now evaluate the integral (1.8). It can be rewritten in the form

$$\int_{r_0}^{\infty} \left(\frac{1}{r_0^2} - \frac{1}{r^2}\right)^{-1/2} \left\{1 - 2\beta\left(\frac{1}{r_0} + \frac{1}{r} - \frac{1}{r + r_0}\right) + \frac{\gamma r_0}{1 + r_0/r}\right\}^{-1/2} \frac{dr}{r^2}. \tag{1.9}$$

After making the substitution $\sin\theta = r_0/r$ the integral becomes

$$\int_0^{\pi/2} \left[1 - \frac{2\beta}{r_0}\left(1 + \sin\theta - \frac{\sin\theta}{1 + \sin\theta}\right) + \frac{\gamma r_0}{(1 + \sin\theta)}\right]^{-1/2} d\theta \tag{1.10}$$

For any realistic situation, such as the bending of light from the sun, galaxies or cluster of galaxies the deflection is of the order of arc seconds and therefore the parameters β/r_0 and γr_0, which measure the deviation from straight line motion in Eq. (1.10), must be much less than one. We will therefore expand the integrand to first order in the small parameters β/r_0 and γr_0. One obtains

$$\int_0^{\pi/2} \left[1 + \frac{\beta}{r_0}\left(1 + \sin\theta - \frac{\sin\theta}{1 + \sin\theta}\right) - \frac{\gamma r_0}{2(1 + \sin\theta)}\right] d\theta = \frac{\pi}{2} + \frac{2\beta}{r_0} - \frac{\gamma\, r_0}{2} \tag{1.11}$$

The deflection, given by (1.7), is therefore

$$\Delta\varphi = \frac{4\beta}{r_0} - \gamma\, r_0 \tag{1.12}$$

a simple modification of the standard "Einstein" result of $4GM/r_0$ (where $\beta = GM$). The constant γ must be small enough such that the extra term $-\gamma\, r_0$ is negligible compared to $4GM/r_0$ on solar distance scales. The linear γ term, however, can begin to make important contributions on larger distance scales where discrepencies between experiment and theory presently exist i.e. the "Einstein" deflection due to the luminous matter in galaxies or clusters of galaxies is less than the observed deflection. Of course, these discrepencies are usually taken as evidence for the existence of large amounts of dark matter in the halos of galaxies. If the extra term $-\gamma\, r_0$ is to ever replace or imitate this dark matter on large distance scales it would have to be positive (i.e.

attractive), implying that γ must be negative. The sign of γ used to fit galactic rotation curves [10] however, is positive (the reason why the sign of γ is different for null and non-relativistic massive geodesics is discussed in the next section on potentials). Therefore there is a glaring incomaptibility between these two analyses. This means that Weyl gravity does not seem to solve the dark matter problem, although this does not signal any inconsistency of Weyl gravity itself. In addition, the mechanism of conformal symmetry breaking is not well understood and it must be addressed in more detail before considering massive geodesics or just mass in general. The analysis of the deflection of light is more reliable since it is completely independent of any such conformal symmetry breaking mechanism.

IV. The Potential in Weyl Gravity

In General Relativity, the Schwarzschild geodesic equations can be viewed as "Newtonian" equations of motion with a potential (see [18]). In Weyl gravity, a potential can also be extracted from the vacuum equations and for this purpose it is convenient to define a new "time" coordinate p such that $dp = B(r)dt$. The vacuum equations (1.3)-(1.5) in these new coordinates are

$$r^2\frac{d\varphi}{dp} = J \tag{1.13}$$

$$\frac{1}{2}\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{2r^2}B(r) - \frac{1}{2} = \frac{-E\,B(r)}{2} \tag{1.14}$$

$$d\tau^2 = E\,dp^2. \tag{1.15}$$

Let $B(r) \equiv 1 + 2\phi(r)$ where ϕ is not necessarily a weak field. Equation (1.14) becomes

$$\frac{1}{2}\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{2r^2} + \phi\left(\frac{J^2}{r^2} + E\right) = \frac{1-E}{2}. \tag{1.16}$$

The above geodesic equation together with eq.(1.13) can be viewed as a particle having energy per unit mass $(1-E)/2$ and angular momentum J moving in ordinary mechanics with a potential

$$V(r) = \phi\left(\frac{J^2}{r^2} + E\right). \tag{1.17}$$

The derivative of the potential is

$$V'(r) = \frac{\beta}{r^2}\left(\frac{3J^2}{r^2} + E\right) + \frac{\gamma}{2}\left(E - \frac{J^2}{r^2}\right) - krE. \tag{1.18}$$

where $\phi(r) = -\beta/r + \gamma r/2 - kr^2/2$ was used. There are three terms in Eq. (1.18): a β, γ and k term respectively. The k term vanishes for null geodesics in agreement with our results on the deflection of light. For massive geodesics the k term is non-zero but is negligible unless one is considering cosmological scales. Hence, this term will be ignored. The factor $3J^2/r^2 + E$ in front of the β term is always positive since $E \geq 0$. Therefore, the β term is attractive for both massive and null geodesics (which is the case in GR). On the other hand, the factor $E - J^2/r^2$ in front of the γ term, can be positive or negative depending on the physical situation. For a non-relativistic particle moving in a weak field, which is the case of galactic rotation curves, we obtain $E \approx 1$, $J^2/r^2 \ll 1$, and therefore the factor $E - J^2/r^2$ is positive. For light, E is zero and the factor is negative. The potential (1.17) is different for non-relativistic particles and light: the γr term in ϕ contributes a linear potential for non-relativistic particles but an inverse r potential for light. Their corresponding derivatives therefore have opposite sign and this explains why γ obtained through galactic rotation curves has the opposite sign to that obtained in the deflection of light.

Of course, a negative γ term is not reserved to null geodesics only. Any massive particle which is sufficiently relativistic will also have this property. For example consider a particle moving in a weak field ϕ with a negligible "radial velocity" dr/dp. One obtains from eq. (1.16) that $J^2/r^2 \approx 1 - E - 2\phi$ and therefore $E - J^2/r^2 \approx 2E + 2\phi - 1$. It follows that if a particle is sufficiently relativistic such that $E < 1/2 - \phi \approx 1/2$ then we obtain a negative γ term.

We can actually reproduce the deflection of light result Eq.(1.12) in a most straightforward way using the potential Eq.(1.17). For null geodesics($E = 0$) the potential is given by

$$V_{null}(r) = \frac{-\beta\, J^2}{r^3} + \frac{\gamma\, J^2}{2\, r} + \frac{-k\, J^2}{2}. \tag{1.19}$$

The deflection by a potential $V(r)$ is obtained by integrating along the straight line path the gradient of $V(r)$(in the $\perp$ direction i.e. in the direction of r_0). As long as the deflection is very small, integrating along the straight line path instead of the curved path gives the same results. The deflection is given by

$$\Delta\varphi = \int_{-\infty}^{\infty} \nabla_{\perp} V(r)\, dZ. \tag{1.20}$$

where Z is the distance along the straight line path i.e. $r^2 = Z^2 + r_0^2$. In the potential

V_{null}, the γ term is an inverse r potential. This is the reason why its contribution to the deflection of light Eq.(1.20) is finite and comes with a relative negative sign. If V_{null} had contained a linear potential, the integral for the deflection would diverge, implying that no scattering states could exist.

Using $J^2 = r_0^2/B(r_0)$ given in section III and V_{null} as the potential, the deflection Eq. (1.20) yields

$$\Delta\varphi = \frac{4\beta}{r_0} - \gamma\, r_0 \tag{1.21}$$

where only first order terms in β/r_0 and $\gamma\, r_0$ were kept. The deflection of light result Eq.(1.12) is therefore reproduced in a straightforward fashion that allows one to trace clearly the origin of the negative sign in $-\gamma r_0$.

V. The Weyl Radius

The geometry of a typical lens system is shown in Fig. 1, below. A light ray from a source S is deflected by an angle α at the lens and reaches an observer at O. The angle between the optic axis and the true position of the source is β and the angle between the optic axis and the image I is θ. The angular diameter distances between observer and lens, lens and source, and observer and source are d_{ol}, d_{ls} and d_{os} respectively. For a spherically symmetric lens, image formation is governed by the one dimensional lens equation

$$\beta = \theta - \alpha(d_{ls}/d_{os}). \tag{1.22}$$

A source is imaged as a ring if the source , the lens and the observer lie on a "straight" line(i.e. $\beta = 0$). For an Einstein deflection angle of $\alpha = 4GM/r_0$, the radius of the ring is called the Einstein radius and is given by

$$\theta_E = \left(\frac{4GM}{D}\right)^{1/2} \tag{1.23}$$

where $D \equiv \dfrac{d_{ol}\, d_{os}}{d_{ls}}$ and M is the mass of the lens enclosed in the Einstein radius. For a Weyl deflection angle given by Eq.(1.12), the radius of the ring, which we will call the "Weyl" radius, can be readily calculated and yields

$$\theta_w = \left(\frac{4GM}{D + \gamma\,(d_{ol})^2}\right)^{1/2}. \tag{1.24}$$

The above result for the Weyl radius will be used later to obtain an estimate for the constant γ. If the source, lens and observer are not alligned in a "straight" line(i.e. $\beta \neq 0$) then instead of a ring one obtains two images, one inside and one outside the Weyl ring. Using the Weyl deflection angle Eq.(1.12) and the definitions for the Einstein and Weyl radius, the lens equation (1.22) gives

$$\beta = (1 + n_\gamma)\,\theta - \frac{4GM}{D\,\theta} \tag{1.25}$$

where $n_\gamma \equiv \gamma d_{ol}^2/D$. The two solutions to the above equation are

$$\theta_\pm = \frac{1}{2(1+n_\gamma)}\left(\beta \pm \sqrt{\beta^2 + 4\theta_w^2(1+n_\gamma)^2}\right). \tag{1.26}$$

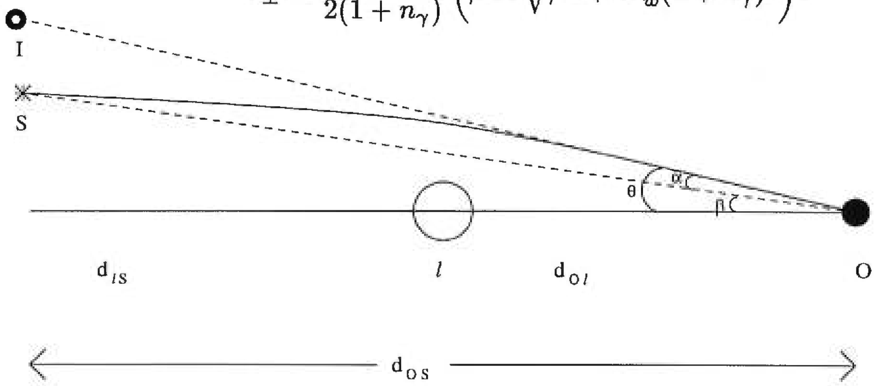

Figure 1. Light from the source S bends at the lens l and arrives at the observor O who then sees the image I.

VI. Circular Orbits in Equilibrium

In the Schwarzschild metric, it is known that photons do not have circular orbits with stable equilibrium but have one unstable equilibrium at the radius $r = 3GM$. We now determine the radii of equilibrium for photons in the Weyl vacuum solution (1.2). The geodesic equation of interest is Eq. (1.4) where we substitute $E = 0$ for photons and set dr/dt to zero at the radius of orbit $r = R$. Equation (1.4) becomes

$$\frac{J^2}{R^2} - \frac{1}{B(R)} = 0. \tag{1.27}$$

For equilibrium, the derivative of the LHS of (1.27) at $r = R$ must vanish and we obtain

$$\frac{-2J^2}{R^3} + \frac{B'(R)}{B^2(R)} = 0 \tag{1.28}$$

With J^2 given by (1.27) and $B(r)$ given by (1.2), equation (1.28) becomes

$$\gamma R^2 + 2R - 6GM = 0 \tag{1.29}$$

where $\beta = GM$ was used. Note that the constant k has again cancelled out. The two solutions to equation (1.29) are

$$R \simeq 3GM \text{ and } R \simeq -2/\gamma \tag{1.30}$$

where it has been assumed that $|\beta\gamma| << 1$. We see that besides the $R = 3GM$ solution a second equilibrium exists at $R = -2/\gamma$ if γ is negative. By differentiating equation (1.29) we see that this second equilibrium is a stable one while the first is an unstable one as in the Schwarzschild case. This stable equilibrium provides us with a natural length scale i.e. a scale which determines the "region of influence" of a particular localized source in contrast to the background or global aspects. A length scale of this sort is probably necessary if we ever want to develop a concept of "energy of an isolated system" in Weyl gravity. In the Scwarzschild case, the metric tends towards Minkowski space in the limit $r \to \infty$ and a Gauss's law formulation of total energy of an "isolated" system is possible. In the Weyl case we obtain a metric conformal to a Robertson-Walker spacetime in the limit $r \to \infty$. We therefore need a natural cut-off radius at which the influence of the specific source in question ceases and the global aspects take over. Indeed, we have shown that the constant k, which is proportional to the cosmological curvature, plays no role in determining the radius of stable equilibrium and lends support to the idea that the stable radius is determined by the localized source. Hence, from the arguments above, a negative γ is desirable.

VII. Time Delay

We now calculate the time taken by a photon on a return trip between any two points in a gravitational field produced by a central mass. We expect modifications to the standard GR result when the radius of closest approach to the central mass is on the order of galactic scales. The equation governing the time evolution of orbits is Eq. (1.4), with $E = 0$ for light. At the point of closest approach $r = r_0$, $dr/dt = 0$ so that Eq. (1.4) gives $J^2 = r_0^2/B(r_0)$. The time for light to travel from r_0 to r_1, given by Eq. (1.4), is

$$t = \int_{r_0}^{r_1} \left(\frac{A(r)/B(r)}{1 - \dfrac{B(r)\, r_0^2}{B(r_0)\, r^2}} \right)^{1/2} dr \tag{1.31}$$

We evaluate the above integral with $A(r)$ and $B(r)$ given by Eq. (1.2). This yields

$$t = \int_{r_0}^{r_1} \frac{r(1-2\beta/r+\gamma\, r - k\, r^2)^{-1}(1-2\beta/r_0+\gamma\, r_0 - k\, r_0^2)^{1/2}}{\sqrt{r^2-r_0^2}\left[1-\dfrac{2\beta}{r_0}\left(1+\dfrac{r_0^2}{r(r+r_0)}\right)+\gamma r_0\left(\dfrac{r}{r+r_0}\right)\right]^{1/2}}\, dr \tag{1.32}$$

We can expand the above integral to first order in the parameters β/r, $\gamma\, r$ and $k\, r^2$ which are much less than 1 within the usual limits of integration. To first order in the parameters, the integral (1.32) yields

$$t \simeq \int_{r_0}^{r_1} \frac{r}{\sqrt{r^2-r_0^2}}\left[(1-\frac{1}{2}kr_0^2)+\frac{2\beta}{r}+\frac{\beta\, r_0}{r(r+r_0)}-\gamma\, r+\frac{\gamma\, r_0^2}{2(r+r_0)}+k\, r^2\right] dr \tag{1.33}$$

There are six elementary integrals to evaluate above. The result is

$$\begin{aligned} t &\simeq \sqrt{r_1^2-r_0^2}+2\beta\ln\left(\frac{r_1+\sqrt{r_1^2-r_0^2}}{r_0}\right)+\beta\sqrt{\frac{r_1-r_0}{r_1+r_0}} \\ &- \frac{\gamma}{2}\left(\frac{r_1^3-r_0^3}{\sqrt{r_1^2-r_0^2}}\right)+\frac{k}{6}(2r_1^2+r_0^2)\sqrt{r_1^2-r_0^2}. \end{aligned} \tag{1.34}$$

The leading term is identified as the time for light to travel in a straight line in Minkowski space (where $\beta = \gamma = k = 0$) and we recognize the β terms as the standard "Shapiro" time delay. The γ and k terms evidently produce a modification of the time delay. We see that the effect of the γ term is to increase the time delay if γ is negative and to decrease it if γ is positive.

VIII. Constraints on γ from Experiments

A. Solar Gravitational Deflection

In solar experiments, the sun can be treated as a point mass and no parametrized lens model is required. To date, the best measurements on the deflection of light from the sun were obtained using radio-interferometric methods and verified Einstein's prediction to within 1 %. The measured deflection at the solar limb was 1.761 ± 0.016 arc sec [21] compared to Einstein's prediction of $4GM_{\odot}/R_{\odot} = 1.75$ arc sec. Using the Weyl deflection angle Eq.(1.12) these measurements constrain the constant γ to the range 3.45x $10^{-19} cm^{-1} \geq \gamma \geq$ -1.87x $10^{-18} cm^{-1}$. Clearly, the solar gravitational deflection experiments constrain strongly the order of magnitude of γ but leave open the possibility for a positive or negative γ.

B. Signal retardation by solar gravity

The results of the Viking Relativity Experiment published in 1979 [22] confirmed the "Shapiro" time delay on solar system scales to an accuracy of 0.1%. For example, a ray that leaves the earth, grazes the sun, reaches Mars and comes back would have a time delay of $248 \pm 0.25 \mu s$ where the $248 \mu s$ is the exact prediction of the "Shapiro" time delay and the uncertainty $\pm 0.25 \mu s$ can be used to constrain γ. At superior conjunction, the radius of the sun to the Earth , r_e, and to Mars, r_m, are much greater than the radius of the sun $R_{\odot}$ so that r_0 can be neglected in the factor in front of γ in Eq.(1.34). We therefore have $-\gamma(r_e^2 + r_m^2) = \pm 0.25 \text{x} 10^{-6} s$. This constrains γ to the range $|\gamma| \leq 1.02$ x$10^{-23} cm^{-1}$. This is roughly five orders of magnitude better than the constraint on γ from solar deflection experiments but does not allow us to draw any conclusions on the sign of γ.

C. Deflection of light by galaxies and clusters

One should expect measurements on the deflection of light by galaxies and clusters to determine the most accurate value for γ because it is on those scales where the γ term plays a significant role. However, the interpretation of the experimental data on those scales is more difficult than in the solar system because galaxies and clusters have unknown matter distributions and in general cannot be assumed to be either point masses or spherically symmetric. A parametrized lens model is therefore required for each case of gravitational lensing. Nonetheless, it has been pointed out [23, 24] that though a spherically symmetric lens model is an idealization it is a good first approximation and is extremely useful in obtaining the same order of magnitude results as the more realistic case. We will therefore be able to assume spherical symmetry to obtain a value for γ, with much confidence that the order of magnitude is correct. To constrain γ beyond "order of magnitude" accuracy, a detailed lens model for each case is required and would take us beyond the scope of the present paper.

One type of lensing phenomena observed in clusters are the large arcs. In a spherically symmetric lens, the radius of the tangentially oriented large arcs, θ_{arc}, is a good estimate of the radius which occurs at $\beta = 0$ in the lens equation Eq. (1.22). In GR, the radius of the arc is therefore interpreted as the Einstein radius where M is the total mass(M_{total}) i.e. the sum of the luminous and presumed dark matter. In the context of Weyl gravity, the *same* arc is to be interpreted as the Weyl radius, with M representing only the luminous matter(M_L) and γ a constant to be determined. Using equations (1.23) and (1.24) for the Einstein and Weyl radius respectively and equating them for the same observed arc, one obtains

$$\gamma = \left(\frac{d_{os}}{d_{ls}\, d_{ol}}\right)\left(\frac{M_L}{M_{total}} - 1\right). \tag{1.35}$$

In experiments one measures the redshifts Z_l and Z_s in the spectrum of the light reaching us from the lens(i.e. the cluster) and source respectively. We define a dimensionless quantity(often called the angular size distance) $y \equiv (1+Z)d/L_{H_0}$ where $L_{H_0} \equiv c/H_0$ is the Hubble length and d are the angular diameter distances that appear in Eq.(1.35)(we have reinserted the speed of light c for clarity) The values of y are related to redshifts by [25]

$$y_{ox} = \begin{cases} \dfrac{Z_x(1+Z_x/2)}{1+Z_x} & \text{for } \Omega = 0 \\ 2 - \dfrac{2}{\sqrt{1+Z_x}} & \text{for } \Omega = 1 \end{cases} \tag{1.36}$$

$$y_{ls} = \begin{cases} y_{os}(1+y_{ol}^2)^{1/2} - y_{ol}(1+y_{os}^2)^{1/2} & \text{for } \Omega = 0 \\ y_{os} - y_{ol} & \text{for } \Omega = 1 \end{cases} \tag{1.37}$$

where x represents either the lens l or the source s and Ω is the cosmological density parameter(we have taken the two extremes $\Omega = 0$ and $\Omega = 1$ as examples. In the end we will see that γ is insensitive to this parameter). Equation (1.35) can be rewritten in terms of angular size distances y and yields

$$\gamma = \frac{1}{L_{H_0}}\left(\frac{y_{os}}{y_{ls}\, y_{ol}}\right)\left(\frac{M_L}{M_{total}} - 1\right) \tag{1.38}$$

where $L_{H_0} \equiv c/H_0$ is the Hubble length(we have reinserted the speed of light c). To obtain a value for γ reliable data on the redshifts Z_l and Z_s is required as well as values for the mass-to-light ratios of clusters derived from gravitational lensing. Fortunately, such data exists for many large arcs in clusters. Before looking at the data it is important to note that the mass-to-light ratio is large for a typical cluster(>100) and therefore $M_L/M_{total} << 1$. It follows that the factor $(M_L/M_{total} - 1)$ in Eq.(1.38) will not vary greatly from cluster to cluster. Data is shown below for different clusters with the value of γ calculated in each case.

Cluster	Z_l	Z_s	M_L/M_{total}	$\left.\frac{y_{os}}{y_{ls}\,y_{ol}}\right\|_{\Omega=1,0}$	$\gamma\|_{\Omega=1,0}$
A370	0.375	0.724	$\approx 1/200$	6.877, 7.765	-6.83, -7.72
A2390	0.231	0.913	$\approx 1/120$	7.885, 7.308	-7.25, -7.82
Cl2244-02	0.331	0.83	$< 1/100$	7.68, 6.87	-7.60, -6.80

where γ is in units of the inverse Hubble length, $1/L_{H_0}$ (which is equal to $(H_0/100)\times 1.08\times 10^{-28} cm^{-1}$). As can be seen from the data, the sign of γ is negative and its order of magnitude remains constant. Interestingly enough, Mannheim and Kazanas [9] had already noted, within the context of galactic rotation curves, that γ is of the order of the inverse Hubble length(though no use of Hubble's constant was used to determine γ in their calculations). On the other hand, the lensing data coupled with Eq.(1.38) show explicitly that γ is related to Hubble's constant. In conclusion, we obtain the same order of magnitude for γ as in galactic rotation curves but with opposite sign. It is certainly worth investigating further this anomaly.

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1.2 Article 2 : A study of the spherically symmetric vacuum solution to Conformal Weyl Gravity

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Abstract

We study in detail the static spherically symmetric vacuum solution to conformal (Weyl) gravity. One of the more striking aspects of the solution is the presence of a physical singularity at $r = 0$ due to the linear term in the metric. Using Penrose diagrams we determine which conformally flat space-times allow light to scatter from infinity and calculate the trajectories and deflection of light for these space-times.

I. Introduction

Conformal (Weyl) gravity has been advanced in the recent past as an alternative classical theory to Einstein gravity[1, 2]. Conformal gravity is a metric theory that insists on conformal invariance of the action, and as such encompasses the largest symmetry group which keep the light cones invariant i.e. the 15 parameter conformal group (which includes the 10 parameter Poincaré group). Interest in conformal gravity grew in the early 90's after the metric exterior to a static spherically symmetric source was obtained[1] and had the possibility of explaining galactic rotation curves without recourse to dark matter. The deflection of light in the theory was later calculated[3] and it was discovered that at large distances, if the source attracted light it would repel non-relativistic massive particles and vice-versa. It was recognized, however, that this discrepency between null and massive geodesics did not pose a serious problem for the theory. Unlike the Schwarzschild solution, the metric in conformal gravity is not unique and is determined up to an arbitrary conformal factor. The fitting of galactic rotation curves requires one to fix in advance the conformal factor because massive geodesics are not conformally invariant. In contrast, null geodesics are conformally invariant and therefore the deflection of light result is independent of any conformal factor. In short, we can reliably extract only the light cones from the solutions to conformal gravity because only these are conformally invariant.

Conformal gravity is generally covariant and is invariant under the conformal transformation $g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x)$ where $\Omega^2(x)$ is a finite, non-vanishing, continuous real function. Its action is the Weyl tensor squared. For a metric in the standard form

$$d\tau^2 = B(r)\, dt^2 - A(r)\, dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{1.39}$$

the static spherically symmetric vacuum solutions are[1]

$$B(r) = A^{-1}(r) = 1 - \frac{(2 - 3\gamma\beta)\beta}{r} - 3\beta\gamma + \gamma\, r - kr^2 \tag{1.40}$$

where β,γ and k are constants. Constraints from phenomenology imply that γ, k and $\gamma\beta << 1$ (see [1, 2, 3]). The constant $\gamma\beta$ is usually negligible but for the purposes of the next section we include it here. In subsequent sections it will be dropped.

Our goal in this paper is to study in detail the metric (1.39) with solution Eq.(1.40) for different choices of the constants γ and k. This leads us to the study of conformally flat space-times. After making suitable coordinate transformations to obtain the metrics in a form which is manifestly conformally flat, we analyze their causal structure using Penrose diagrams. One can then identify from these diagrams which space-times allow light to approach the source from infinity. The trajectories and deflection of light for these metrics are then obtained for arbitrary large values of r_0 (the point of closest approach).

II. Curvature and related tensors

Curvature scalars are invariant under coordinate transformations and therefore are useful for detecting physical singularities. In contrast, the metric (which is coordinate dependent) may have a coordinate singularity which is not a physical singularity (the classic example is the coordinate singularity at the Schwarzschild radius $r = 2m$ which is not a physical singularity but a horizon). The metric under study is

$$(1-3\gamma\beta-\frac{\beta(2-3\gamma\beta)}{r}+\gamma r-kr^2)dt^2-\frac{dr^2}{1-3\gamma\beta-\beta(2-3\gamma\beta)/r+\gamma r-kr^2}-r^2d\Omega^2 \tag{1.41}$$

The Ricci scalar, $R \equiv R_{\mu\nu}g^{\mu\nu}$, for the general metric (1.39) with $B(r) = A^{-1}(r)$ can readily be calculated and yields $R = B'' + 4B'/r + 2B/r^2 - 2/r^2$ where a prime denotes differentition with respect to r. Note that each term that is part of $B(r)$ contributes separately to the Ricci scalar i.e. there are no cross terms. Each term in the Ricci scalar can therefore be traced to one term in $B(r)$. The Ricci scalar for the metric (1.41) i.e. for the solution (1.40), is equal to

$$R = 6\gamma/r - 6\gamma\beta/r^2 - 12k. \tag{1.42}$$

The Ricci scalar clearly has a physical singularity at $r = 0$ due to the γr term and the constant $3\beta\gamma$ term in the metric. The singularity due to the constant term is a conical singularity. To illustrate this, consider the metric (1.41) with only the constant term present i.e. the r dependent terms are dropped because they do not contribute to the quantity $6\gamma\beta/r^2$ in the Ricci scalar. The metric $(1-3\gamma\beta)dt^2 - 1/(1-3\gamma\beta)dr^2 - r^2(d\theta^2 - sin^2\theta d\varphi^2)$ has a conical singularity because the ratio of the area of a sphere at coordinate radius r to the proper radius squared $r^2/(1-3\gamma\beta)$ is the constant $4\pi(1-3\gamma\beta) \neq 4\pi$. Correspondingly, the deflection of light is given by the angular defect in the scattering two plane, $3\pi\gamma\beta$ in the limit $\gamma\beta << 1$.

We now turn our attention to the singularity due to the γr term and analyze the metric obtained by setting $\beta = k = 0$ in the metric (1.41) i.e. $ds^2 = (1+\gamma r)dt^2 - 1/(1+\gamma r)dr^2 - r^2 d\Omega^2$. At first glance, it seems that no singularity exists as r decreases to zero. One seems to recover Minkowski space-time! A singularity at $r = 0$ is made apparent by rewritting the metric for small r i.e. $\gamma r << 1$. The metric then takes the form

$$ds^2 = (dt^2 - dr^2 - r^2 d\Omega^2) + \gamma r(dt^2 + dr^2). \tag{1.43}$$

The metric has therefore been decomposed into Minkowski space-time plus a small term. One can write the metric (1.43) as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $|h_{\mu\nu}| << 1$. Although neither term is singular, the derivatives of $h_{\mu\nu}$ are singular at $r = 0$. The connection and the Riemann tensor are constructed out of these derivatives and the inverse metric $g^{\mu\nu}$ (which is not singular) subsequently giving rise to physical singularities at $r = 0$ in the space-time.

As r tends to infinity the curvature scalar tends to the constant $-12k$. The metric (1.41) therefore describes a space-time where the region far from the source i.e. the background, is not flat but of constant curvature (it will be shown that the background is nonetheless conformal to flat). We now calculate the Riemann tensor, $R_{\mu\nu\sigma\tau}$ for the metric (1.41). Its non-vanishing components are

$$\begin{aligned}
R_{r\theta r\theta} &= R_{r\phi r\phi} = \frac{\beta(2-3\gamma\beta)}{2r^3} + \frac{\gamma}{2r} - k\,, \quad R_{rtrt} = \frac{\beta(2-3\gamma\beta)}{r^3} + k \\
R_{\theta\phi\theta\phi} &= \frac{-\beta(2-3\gamma\beta)}{r^3} - \frac{3\gamma\beta}{r^2} + \frac{\gamma}{r} - k\,, \\
R_{\theta t\theta t} &= R_{\phi t\phi t} = \frac{-\beta(2-3\gamma\beta)}{2r^3} - \frac{\gamma}{2r} + k. \qquad (1.44)
\end{aligned}$$

where other non-vanishing components related to the above by anti-symmetry are not shown. Clearly, the Riemann tensor and the scalar constructed from it i.e. the Riemann tensor squared, diverge at $r = 0$. Every term in the metric except for the kr^2 term contributes to a physical singularity. Note that the components of the Riemann tensor do not vanish as r tends to infinity but approach the constant $\pm k$. As previously mentioned, the space-time is not flat at infinity. The kr^2 term contributes a constant to the Riemann tensor. In contrast, the terms containing β or γ are responsible for the physical singularity at $r = 0$ and their contribution to the Riemann tensor decreases gradually as one moves away from the singularity. These features of the β, γ and k terms lead us to the following interpretation: the β and γ terms represent sources and the k term represents the background in which these sources are embedded.

Let us now compute the Weyl tensor $C_{\mu\nu\sigma\tau}$ for the metric (1.41). This tensor is useful because the requirement that a space-time be conformal to flat is for the components of the Weyl tensor to vanish. The components of the Weyl tensor are

$$\begin{aligned}
C_{r\theta r\theta} &= C_{r\phi r\phi} = \frac{\beta(2-3\gamma\beta)}{2r^3} + \frac{\gamma\beta}{2r^2} \\
C_{rtrt} &= \frac{\beta(2-3\gamma\beta)}{r^3} + \frac{\gamma\beta}{r^2},;\ C_{\theta\phi\theta\phi} = \frac{-\beta(2-3\gamma\beta)}{r^3} - \frac{\gamma\beta}{r^2}; \\
C_{\theta t\theta t} &= C_{\phi t\phi t} = \frac{-\beta(2-3\gamma\beta)}{2r^3} - \frac{\gamma\beta}{2r^2}. \qquad (1.45)
\end{aligned}$$

where the components related to the above by antisymmetry are not shown. The Weyl tensor is zero when $\beta = 0$ and therefore the metric (1.41) without β terms is conformal to flat. We see that the β terms break the conformal flatness. One also observes that the Weyl tensor tends to zero as r approaches infinity. Hence, the space-time is asymptotically conformal to flat. In the large r limit when terms containing β become negligible, the metric (1.41) reduces to

$$ds^2 = (1 + \gamma r - kr^2)dt^2 - 1/(1 + \gamma r - kr^2)dr^2 - r^2 d\Omega^2. \tag{1.46}$$

The above metric describes the large r behaviour of the original metric (1.41) and is conformal to flat because the Weyl tensor is zero. In particular, we would like to investigate how the γ and k terms affect the causal structure of the space-time at large radii and whether light has scattering states. We therefore analyze the metric Eq.(1.46) in detail in the next sections.

III. Conformally Flat Space-Times

In the original r, t coordinates, the components of the metric (1.46) change sign whenever $1 + \gamma r - kr^2 = 0$. It is then no longer convenient to use these coordinates to analyze the causal structure. Our task in this section will be to rewrite the conformally flat metric (1.46) in coordinates where the conformal flatness is manifest i.e. in a form where the metric is a conformal factor times the Minkowski metric. The effort spent in obtaining the new coordinates is rewarded by having the metric in a form that has the same causal structure as that of Minkowski space-time i.e. null geodesics do not depend on the conformal factor and therefore the light cones are drawn at 45^0 to the horizontal axis as in Minkowski space-time. These diagrams depicting the causal structure of the space-time are called Penrose (conformal) diagrams. There are constraints on the new coordinates that naturally occur when one transforms from the old to the new coordinates. In short, the causal structure of the conformally flat space-time can be analysed in the new coordinates as a *patch* in Minkowski space-time.

We now perform the coordinate transformation from the r, t coordinates to a new set of coordinates ρ, τ where the metric (1.46) is written in a form which is manifestly conformal to flat. We write

$$ds^2 = (1 + \gamma r - kr^2)dt^2 - 1/(1 + \gamma r - kr^2)dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

$$= \quad \Omega^2(\rho,\tau)\left[d\tau^2 - d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right] \tag{1.47}$$

where τ and ρ are the new coordinates and $\Omega(\rho,\tau)$ is the conformal factor. We therefore have the following two relations:

$$r = \rho\,\Omega \tag{1.48}$$

$$\Omega^2(d\tau^2 - d\rho^2) = (1+\gamma r - kr^2)dt^2 - 1/(1+\gamma r - kr^2)dr^2. \tag{1.49}$$

The coordinates r and t are now functions of both ρ and τ so that $dr = r'd\rho + \dot{r}d\tau$ and $dt = t'd\rho + \dot{t}d\tau$ where a prime and dot on r and t represent partial derivatives with respect to ρ and τ respectively. Equations (1.48) and (1.49) lead to the following three partial differential equations

$$(1+\gamma r - kr^2)t'\dot{t} - \frac{r'\dot{r}}{1+\gamma r - kr^2} \quad = \quad 0 \tag{1.50}$$

$$(1+\gamma r - kr^2)\dot{t}^2 - \frac{\dot{r}^2}{1+\gamma r - kr^2} \quad = \quad \frac{r^2}{\rho^2} \tag{1.51}$$

$$(1+\gamma r - kr^2)t'^2 - \frac{r'^2}{1+\gamma r - kr^2} \quad = \quad -\frac{r^2}{\rho^2} \tag{1.52}$$

We can eliminate t from the above three equations to obtain two partial differential equations for r:

$$\frac{r^2}{\rho^2(\dot{r}+r')} = f(\tau - \rho) \tag{1.53}$$

$$r'^2 - \dot{r}^2 = \frac{r^2(1+\gamma r - kr^2)}{\rho^2} \tag{1.54}$$

where $f(\tau-\rho)$ is an arbitrary function of $\tau-\rho$. To solve the above two equations for r it is convenient to introduce two new coordinates u and v related to τ and ρ by

$$u = \tau - \rho \quad ; \quad v = \tau + \rho\,. \tag{1.55}$$

In u, v coordinates Eq.(1.53) reduces to

$$\frac{2r^2}{(v-u)^2\,\partial r/\partial v} = f(u). \tag{1.56}$$

The solution to the above equation is

$$r = \frac{f(u)(v-u)}{2 + h(u)(v-u)} \tag{1.57}$$

where $h(u)$ is an arbitrary function of u. Substituting Eq. (1.56) into Eq. (1.54) one obtains

$$-2\int \frac{dr}{1+\gamma r - kr^2} = \int f(u)du = g(u) + p(v) \tag{1.58}$$

where $dg(u)/du = f(u)$ and $p(v)$ is an arbitrary function of v. The solution to the above equation depends on whether the polynomial $1 + \gamma r - kr^2$ has roots or not. If the polynomial has roots the integral of $1/(1 + \gamma r - kr^2)$ is given by

$$\frac{-1}{k(r_+ - r_-)} \ln \left| \frac{r - r_+}{r - r_-} \right| ; \quad k > -\frac{\gamma^2}{4} \tag{1.59}$$

where the two roots r_+ and r_-, which can have negative values, are given by

$$r_\pm = \frac{\gamma}{2k} \pm \sqrt{\frac{\gamma^2}{4k^2} + \frac{1}{k}}. \tag{1.60}$$

If the polynomial has no roots the integral of $1/(1 + \gamma r - kr^2)$ is given by

$$\left(-k - \gamma^2/4\right)^{-1/2} \arctan\left(\frac{-kr + \gamma/2}{\sqrt{-k - \gamma^2/4}}\right) ; \quad k < -\gamma^2/4. \tag{1.61}$$

We now solve Eq. (1.58) separately for each of the two cases i.e. case 1: polynomial has roots and case 2: polynomial has no roots.

Case 1: Roots at $r_\pm$

Substituting (1.59) for the integral in Eq.(1.58) one obtains

$$r = \frac{r_+ \pm r_- e^{k(r_+ - r_-)(g(u)+p(v))/2}}{1 \pm e^{k(r_+ - r_-)(g(u)+p(v))/2}} \tag{1.62}$$

where the negative sign corresponds to the region where $\infty > r > r_+$ and $0 < r < r_-$ whereas the plus sign corresponds to the region where $r_+ > r > r_-$. We now equate r in Eq.(1.57) to r in Eq. (1.62). Note that $f(u)$ in Eq.(1.57) is $g'(u) \equiv dg(u)/du$. One obtains the following equality

$$\frac{g'(u)e^{-k(r_+ - r_-)(g(u)+p(v))/2}}{r_+ e^{-k(r_+ - r_-)(g(u)+p(v))/2} \pm r_-} \pm \frac{g'(u)e^{k(r_+ - r_-)(g(u)+p(v))/2}}{r_+ \pm r_- e^{k(r_+ - r_-)(g(u)+p(v))/2}} = \frac{2}{v-u} + h(u) \tag{1.63}$$

After integrating the above equation and performing some algebraic manipulations we obtain

$$\ln\left(r_+ e^{-k(r_+ - r_-)(g(u)+p(v))/4} \pm r_- e^{k(r_+ - r_-)(g(u)+p(v))/4}\right) = \ln(v-u) + S(u) + T(v) \tag{1.64}$$

where $S(u)$(related to $h(u)$) and $T(v)$ are arbitrary functions of u and v repectively. After exponentiating both sides one can reduce Eq.(1.64) to

$$r_+ e^{-k(r_+ - r_-)P(v)/2} \pm r_- e^{k(r_+ - r_-)g(u)/2} = (v-u)A(u)B(v). \tag{1.65}$$

The functions $A(u), B(v), g(u)$ and $p(v)$ are arbitrary functions of u and v and we can therefore write the above equation as

$$(v-u)A(u)B(v) = N(v) + M(u) \tag{1.66}$$

where all the functions above are arbitrary functions of u and v. In terms of the functions $M(u)$ and $N(v)$ the coordinate r given by Eq.(1.62) is

$$r = \frac{r_+ r_- (M(u) + N(v))}{r_- N(v) + r_+ M(u)} \tag{1.67}$$

where the above is valid for the entire region $\infty > r > 0$. Fortunately, equation (1.66) can be solved exactly for $M(u)$ and $N(v)$. We arrive also at Eq.(1.66) in case 2 and therefore postpone finding the solution to Eq.(1.66) until case 2 is completed.

Case 2: Polynomial has no roots

We proceed in a fashion similar to case 1. Substituting (1.61) for the integral in Eq.(1.58) one obtains

$$r = \frac{1}{kc}\tan\left[(g(u)+p(v))/2c\right] + \frac{\gamma}{2k} \tag{1.68}$$

where $c \equiv -1/\sqrt{-k-\gamma^2/4}$. We now equate r in Eq.(1.57) to r in Eq.(1.68). After integration one obtains

$$\ln\left[\cos\left((g(u)+p(v))/2c\right)\gamma\, c/2 + \sin\left((g(u)+p(v))/2c\right)\right] = \ln(v-u) + S(u) + T(v) \tag{1.69}$$

where $S(u)$ and $T(v)$ are arbitrary functions. After algebraic manipulations one obtains

$$\tan(g(u)/2c) + \frac{\sin(p(v)/2c) + \cos(p(v)/2c)\,\gamma\, c/2}{\sin(p(v)/2c)\,\gamma\, c/2 + \cos(p(v)/2c)} = (v-u)A(u)B(v). \tag{1.70}$$

We therefore obtain the same equation as in case 1 i.e.

$$(v-u)A(u)B(v) = N(v) + M(u)$$

In terms of the functions $M(u)$ and $N(v)$ the coordinate r given by Eq.(1.68) is

$$r = \frac{-c(M(u)+N(v)}{1 + (M(u)+N(v))\gamma\, c/2 - M(u)N(v)}. \tag{1.71}$$

Though we arrive at the same equation (1.66), the coordinate r in case 1 and case 2 are not the same.

We now solve Eq.(1.66) and discuss its physical significance. The right hand side of the equation does not contain any mixed terms of u and v and therefore the mixed terms on the left hand side must vanish. We write $A(u)$ as

$$A(u) = A_0 + a(u') \tag{1.72}$$

where $A_0 = A(u_0)$ is a constant and $a(u')$ is a function of $u' \equiv u - u_0$ which vanishes at $u' = 0$. Similarly

$$B(v) = B_0 + b(v'). \tag{1.73}$$

With $A(u)$ and $B(v)$ given above, the left hand side of Eq.(1.66) yields

$$(v' - u' + C_0)\,(A_0 B_0 + A_0 b(v') + B_0 a(u') + a(u')b(v')) \tag{1.74}$$

where $C_0 = v_0 - u_0$ is a constant. The mixed terms must vanish and we obtain the following equation

$$v'a(u')B_0 + v'a(u')b(v') - u'b(v')A_0 - u'a(u')b(v') + C_0 a(u')b(v') = 0. \tag{1.75}$$

We isolate one solution to the above equation which is readily obtained

$$a(u') = b(v') = 0. \tag{1.76}$$

This solution will be useful for later analysis. The general solution to Eq. (1.75) is obtained by separating the variables i.e.

$$b(v') = \frac{-v'B_0}{C_0 + v' - u'\,(1 + A_0/a(u'))}. \tag{1.77}$$

The function $b(v')$ is a function of v' only and therefore $u'\,(1 + A_0/a(u'))$ must be a constant(call it D). We therefore obtain the following solutions

$$a(u') = \frac{-u'A_0}{D + u'}; \quad b(v') = \frac{-v'B_0}{C_0 + D + v'}. \tag{1.78}$$

One recovers solution Eq.(1.76) in the limit $D \to \infty$. The solutions to $(v-u)\,A(u)\,B(v) = M(u) + N(v)$ are therefore

$$A(u) = A_0,\ B(v) = B_0,\ M(u) = -A_0 B_0 u,\ N(v) = A_0 B_0 v \tag{1.79}$$

and

$$A(u) = \frac{A}{B+u},\ B(v) = \frac{C}{B+v},\ M(u) = \frac{-AC\,u}{B(B+u)},\ N(v) = \frac{AC\,v}{B(B+v)} \tag{1.80}$$

where solutions (1.76) and (1.78) were substituted into equations (1.72) and (1.73) and the quantities A, B and C are constants related to the constants A_0, B_0, C_0 and D. Note that the solutions to $M(u)$ and $N(v)$ are determined only up to a constant i.e. one can add a constant +k to M(u) and a constant -k to N(v). With the above solutions we can finally obtain the coordinate r. For case 1, r is given by Eq.(1.67) and yields

$$r = \begin{cases} \dfrac{r_+ r_-(v-u)}{vr_- - ur_+} & \text{for solution (1.79)} \\ \dfrac{Br_+ r_-(v-u)}{vr_-(B+u) - ur_+(B+v)} & \text{for solution (1.80)} \end{cases} \tag{1.81}$$

For case 2, r is given by Eq.(1.71) and yields

$$r = \begin{cases} \dfrac{-cA_0B_0(v-u)}{1-\gamma c A_0 B_0(v-u)/2 + (A_0B_0)^2 uv} & \text{for solution (1.79)} \\ \dfrac{-cAC(v-u)}{(B+u)(B+v) - \gamma c AC(v-u)/2 + (AC/B)^2 uv} & \text{for solution (1.80)} \end{cases} \tag{1.82}$$

It is worth mentioning that solutions to Eq.(1.66) are related to each other by the following coordinate transformation

$$u \to \frac{Au}{B+u} \quad v \to \frac{Av}{B+v}. \tag{1.83}$$

where A and B are arbitrary constants. These transformations form a sub-group of the special conformal transformations that leave the light-cones invariant i.e. leave the equation

$$ds^2 = d\tau^2 - d\rho^2 - \rho^2 d\Omega^2 = dudv - \left(\frac{v-u}{2}\right)^2 d\Omega^2 = 0 \tag{1.84}$$

invariant (where u and v are related to τ and ρ via Eq.(1.55)). The transformations form only a sub-group of the 15 parameter conformal group because two coordinates are not involved in the transformation. Hence, equations (1.66) and (1.84) contain the same symmetries. There are two more transformations that leave equations (1.66) and (1.84) invariant. These are

$$\text{space inversion: } u \to v\,, \quad v \to u \quad i.e. \quad \rho \to -\rho \tag{1.85}$$

$$\text{time reversal: } u \to -v\,, \quad v \to -u \quad i.e. \quad \tau \to -\tau. \tag{1.86}$$

Of course, these can be combined with transformations (1.83).

IV. Penrose Diagrams

The causal structure of the conformally flat metric Eq.(1.46) will now be analyzed for different choices of γ and k in the "conformally flat coordinates" u and v (or τ and ρ). The possible choices of γ and k are the following

$$\begin{aligned} &a)\ k > 0 :\ r_+ > 0,\ r_- < 0 \\ &b)\ -\gamma^2/4 < k < 0 \text{ and } \gamma < 0 :\ r_+ > 0,\ r_- > 0 \\ &c)\ -\gamma^2/4 < k < 0 \text{ and } \gamma > 0 :\ r_+ < 0,\ r_- < 0 \\ &d)\ k < -\gamma^2/4 :\ \text{no roots} \end{aligned} \tag{1.87}$$

Altogether there are four cases to consider and a Penrose diagram has been drawn for each showing the axes of both the u, v and ρ, τ coordinates (see figure 2). The causal analysis proceeds as in Minkowski space-time except that only a patch of the u, v (or ρ, τ) coordinates are allowed. This is due to the condition that in the original r, t coordinates the radius r must be positive. In every diagram the singularity at $r = 0$ is shown in bold as a vertical dashed line occuring at $u = v$ or $\rho = 0$. At the point $u = v = 0$, we draw a circle to show that r is indeterminate at that point i.e. it is not zero and hence not a singularity. The value of r at this point actually depends on the slope with which a line crosses it. This is the reason that different values of r can cross this point. For each case the line at $r = \infty$ is represented by a dashed line. The region where r is positive and runs from the singularity at $r = 0$ to the dashed line at $r = \infty$ is shown by an arc (there is a second arc that is shown that represents an identical patch but with time running the opposite direction). The lines with arrows represent null geodesics i.e. the light cones. For diagrams a), b) and c) which represent the case with roots at $r_\pm$, we use the simpler solution of Eq.(1.81) to obtain the proper patch. One obtains the following features for all three diagrams: r_+ is a 45^0 line at $u = 0$ and r_- is a -45^0 line at $v = 0$, lines of constant r are simply straight lines that go through the origin and as can be seen from Eq.(1.81) the radius approaches r_+ as $v \to \infty$ (this is not shown on the diagrams to avoid clutter). In diagram d), which represents the case with no roots, we use the simpler solution of Eq.(1.82) to obtain the correct patch. In contrast to the first three diagrams, lines of constant r do not go through the origin and are hyperbolas. Let us now look at the causal structure for all

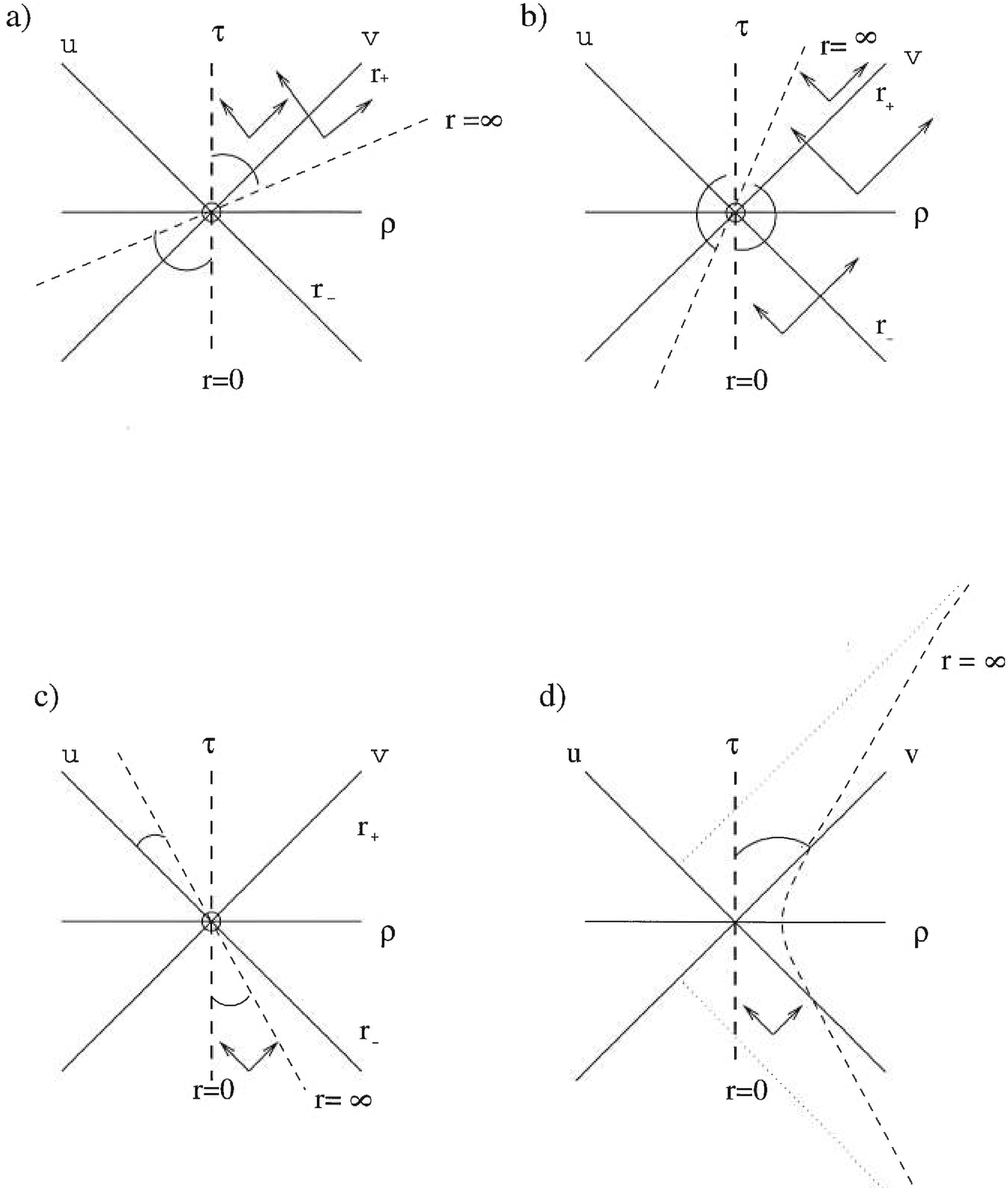

Figure 2. Penrose diagrams for four different space-times: a) horizon at r_+ b) horizon at r_- and r_+ c) roots $r_\pm$ are negative, no horizon d) no roots, no horizon

four diagrams. In diagram a), the case of $k > 0$, r_+ is a horizon because light between $r = 0$ and $r = r_+$ either ends at the singulatity or at r_+ (i.e. light cannot escape beyond r_+). Also there exists no point from which light can reach infinity. Hence, there does not exist scattering states for the space-time with $k > 0$. In diagram b), both r_- and r_+ act as horizons. Light at a radius greater than r_- cannot penetrate the r_- line i.e. light is pushed out towards greater values of r. Light between r_+ and infinity is trapped between these two values and therefore there are no scattering states in this space-time either. In diagram c), there are no horizons i.e. both r_+ and r_- are in the negative r region which is not permitted. In this space-time there is clearly scattering allowed as one can reach infinity from any radius. In diagram d), there are no horizons and we see again that light can reach infinity from any radius. Hence, scattering states can also exist for case d).

V. Scattering of Light

We saw in the previous section that light has scattering states only if $k < -\gamma^2/4$ or if $0 > k > -\gamma^2/4$ with $\gamma > 0$ (note that no scattering states exist for a positive value of k). We can therefore calculate the deflection of light for the two cases above. The deflection of light has already been calculated for the original metric (1.41). The result obtained is [3]

$$\frac{4\beta}{r_0} - \gamma r_0 \tag{1.88}$$

where r_0 is the point of closest approach. The calculation was done with the approximation that both terms in Eq.(1.88) are much smaller than one. At large r_0, the γr_0 term does not remain small and the approximation is no longer valid. It is therefore worthwhile to perform a separate calculation for the γ term alone. We can either set $\beta = 0$ or let r_0 be large enough so that the β term can be neglected. Light will therefore be scattering in a conformally flat space-time. The angle φ as a function of r for the metric (1.46) is given by (see [3])

$$\varphi(r) = \int \left[1 + \frac{\gamma r_0}{1 + \sin\theta}\right]^{-1/2} d\theta \tag{1.89}$$

where $\sin\theta = r_0/r$. We therefore obtain the condition that

$$1 + \frac{\gamma r_0}{1 + \sin\theta} \geq 0. \tag{1.90}$$

The above condition is automatically satisfied if γ is positive and this implies that for positive γ light will reach infinity($\theta = 0$) for any value of r_0. If γ is negative, then condition (1.90) implies that

$$r_0 \leq \frac{1+\sin\theta}{|\gamma|} \quad ; \quad \gamma \text{ negative} . \tag{1.91}$$

With the above condition, light can reach infinity($\theta = 0$) only if $r_0 \leq 1/|\gamma|$. If r_0 is in the range $2/|\gamma| \geq r_0 > 1/|\gamma|$ then light is moves in a closed orbit i.e. a bound state. Let us now calculate the integral Eq.(1.89). This yields

$$\varphi(r) = \arcsin\left(\frac{r_0/r + \gamma r_0/2}{1+\gamma r_0/2}\right) \tag{1.92}$$

The deflection from infinity to r_0 and back to infinity is

$$\begin{aligned} \Delta\varphi &= 2\left(\varphi(r_0) - \varphi(\infty)\right) - \pi \\ &= -2\arcsin\left(\frac{\gamma r_0}{2+\gamma r_0}\right). \end{aligned} \tag{1.93}$$

The above result for the scattering is valid for negative γ as long as the point of closest approach $r_0 \leq 1/|\gamma|$. It is always valid for positive γ. For $\gamma r_0 << 1$ the deflection is equal to $-\gamma r_0$ in agreement with Eq.(1.88). The deflection Eq.(1.93) is repulsive for a positive γ and ranges from 0 at $r_0 = 0$ to $-\pi$ at $r_0 \to \infty$. Therefore light "bounces off" sharply at large r_0 and there is little chance of observing it. For a negative γ the deflection is attractive and ranges from 0 at $r_0 = 0$ to π at $r_0 = 1/|\gamma|$. Let us now obtain the shape of the orbits. One can obtain r as a function of φ from Eq.(1.88). This yields

$$r = \frac{-2/\gamma}{1 - \left(\frac{2+\gamma r_0}{\gamma r_0}\right)\sin\varphi}. \tag{1.94}$$

This is of course the equation for a conic section in polar coordinates with eccentricity

$$e = \left|\frac{2+\gamma r_0}{\gamma r_0}\right|. \tag{1.95}$$

The shapes are determined by the value of e. The orbits we obtain are

$$\text{positive } \gamma : \text{ hyperbola } (e > 1) \tag{1.96}$$

$$\text{negative } \gamma : \begin{cases} r_0 < \frac{1}{|\gamma|}, \text{ hyperbola } (e > 1) \\ r_0 = \frac{1}{|\gamma|}, \text{ parabola } (e = 1) \\ \frac{2}{|\gamma|} > r_0 > \frac{1}{|\gamma|}, \text{ ellipse } (0 < e < 1) \\ r_0 = \frac{2}{|\gamma|}, \text{ circle } (e = 0) \end{cases} . \tag{1.97}$$

For a positive γ, the shapes of all orbits are hyperbolas and these describe scattering states. For a negative γ, the shapes of the orbits depend on the value of r_0 and bound states as well as scattering states can exist. The ellipse (see figure 3) has a minimum value of r which occurs at $\varphi = \pi/2$ and is $r_{min} = r_0$. It has a maximum value of r which occurs at $\varphi = -\pi/2$ and is $r_{max} = r_0/(|\gamma|r_0 - 1)$. The semi-latus rectum L, defined to be the point which occurs at an angle of $\pm\pi/2$ away from r_{min} i.e. at $\varphi = 0$ or π, is $L = 2/|\gamma|$. Note that L is independent of r_0. For the negative γ case, what is observable lies in the region where the trajectory is a hyperbola i.e. for typical values of γ, its inverse is typically of the order of the Hubble length [1, 3] and light in the bound states remain at large distances. It is also important to realize that on these huge scales, one must begin to consider the effects of other sources and the solution for one isolated source is no longer realistic.

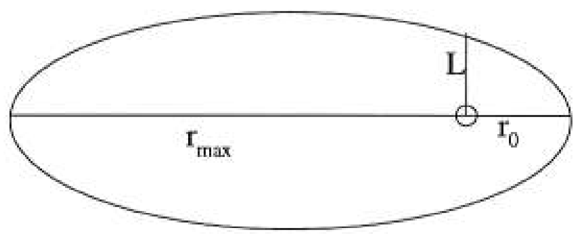

Figure 3. An ellipse showing focal point, r_{min}, r_{max} and the semi-latus rectum L equal to $2/|\gamma|$ which is independent of r_0.

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Chapitre 2

Gravité en 2+1 Dimensions avec Masse Topologique

2.1 Article 3 : Localized Mass and Spin in 2+1 Dimensional Topologically Massive Gravity

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Abstract

Stationary solutions to the full non-linear topologically massive gravity (TMG) are obtained for localized sources of mass m and spin σ. Our results show that the topological term induces spin and that the total spin J (which is the spin observed by an asymptotic observer) ranges from 0 to $\sigma + \frac{m}{\mu}\left(\frac{4\pi+m}{4\pi+2m}\right)$ depending on the structure of the spin source (here μ is the topological mass). We find that it is inconsistent to consider actual delta function mass and spin sources. In the point-like limit, however, we find no condition constraining m and σ contrary to a previous analysis [2].

I. Introduction

A few years ago Deser [1] obtained solutions to linearized TMG with point mass

m and point spin σ as source. Among other things, he found that the topological term induced a spin m/μ. In our work we consider the full non-linear theory with a highly localized (approaching a point) mass source m and spin source σ. Like the linear theory, we obtain an induced spin due to the presence of the topological term but it has a range of values (the value depends on the structure of the spin source σ). For the case where the total spin is at its maximum value, we obtain an induced spin of $\frac{m}{\mu}\left(\frac{4\pi+m}{4\pi+2m}\right)$. As m gets smaller this result approaches m/μ which is the value obtained in the linear theory.

Clément [2] has also obtained solutions to the full non-linear theory with delta function mass m and delta function spin σ as sources. His results showed that the mass and spin must be constrained by the condition $m + \mu\sigma = 0$. We argue, however, that the use of a delta function spin source, while allowed in the linear theory [1], is not allowed in the full theory and renders the condition $m + \mu\sigma = 0$ invalid.

An alternative approach exists [5] using dreibeins and the spin connection, where delta function spin and mass sources can be treated directly. This formulation facilitates the inclusion of torsion and one finds non-zero torsion localized at the position of the source. We will not pursue the possibility of allowing for torsion below. Existence of torsion is an additional structure on the manifold; one has to specify the equation governing the torsion field. We restrict our work to metrical theories of gravitation, where the torsion is zero and the metric is covariantly conserved.

II. Field Equations

We begin our work by writing down the well known field equations for TMG with energy-momentum tensor $T_{\mu\nu}$ as source. The Einstein field equations including a topological mass term is given by [1, 2] (in units where $8\pi G = 1$)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{1}{\mu}C_{\mu\nu} = -\kappa^2 T_{\mu\nu} \tag{2.1}$$

where $R_{\mu\nu}$ is the Ricci tensor , $R \equiv R_{\mu\nu}g^{\mu\nu}$ is the curvature scalar, and $C_{\mu\nu}$ is the 3 dimensional Weyl (Cotton) tensor defined by

$$C_{\mu\nu} = \frac{1}{2}(detg_{\delta\sigma})^{-1/2}\left(\epsilon_{\mu}^{\alpha\beta}D_\alpha R_{\nu\beta} + \epsilon_{\nu}^{\alpha\beta}D_\alpha R_{\mu\beta}\right).$$

Note that the trace of $C_{\mu\nu}$ is identically zero. The reason for the negative sign in front of $T_{\mu\nu}$ in (2.1) has been discussed by Deser [1]. We simplify the field equations (2.1) by choosing a rotationally symmetric, stationary metric. The most general form for such a metric is given by [3]

$$\begin{aligned} ds^2 &= n^2\left(dt + \omega_i\, dx^i\right)^2 + h_{ij}\, dx^i dx^j \qquad i = 1,2 \\ &= n^2(\, dt + \psi(r)\, d\theta\,)^2 + h_{ij}\, dx^i dx^j \end{aligned} \tag{2.2}$$

with $det\; g_{\mu\nu} = n^2\; det\; h_{ij}$. We find that, for the present purpose, it suffices to consider the case with $n = 1$ (i.e. we show that the metric with $n = 1$ does support solutions with point mass and point spin). In two space dimensions any metric is conformally flat so that we can express h_{ij} as

$$h_{ij} = -e^{\phi(r)}\delta_{ij}\,. \tag{2.3}$$

The negative sign in (2.3) corresponds to Minkowski signature. The functions $\psi(r)$ and $\phi(r)$ completely determine the metric. We are particularly interested in the asymptotic behaviour of $\psi(r)$ since it is proportional to the total spin J (see [4]).

The scalar twist $\rho(r)$ is defined by [3],

$$\begin{aligned} \rho &\equiv \frac{n}{\sqrt{det\, h_{ij}}}\epsilon^{ij}\partial_i\omega_j \\ &= e^{-\phi(r)}\left(\frac{\psi'(r)}{r} + 2\pi\,\psi(0)\delta^2(\vec{r})\right) \end{aligned} \tag{2.4}$$

where $\psi'(r) \equiv \dfrac{d\psi}{dr}$. Then the Ricci tensor and curvature scalar are given by

$$\begin{aligned} R_{00} &= \frac{1}{2}\rho^2 \\ R_0^j &= \frac{\epsilon^{jk}e^{-\phi}}{2}\,\partial_k(\rho) \\ R^{ij} &= \frac{1}{2}\hat{R}h^{ij} - \frac{h^{ij}\rho^2}{2} \\ R &= \hat{R} - \frac{\rho^2}{2} \end{aligned} \tag{2.5}$$

and the Weyl (Cotton) tensor is given by

$$\begin{aligned} C_{00} &= \rho^3 - \frac{1}{2}\left(\hat{\nabla}^2\rho + \hat{R}\,\rho\right) \\ C_0^j &= \frac{\epsilon^{jk}e^{-\phi}}{2}\partial_k\left(\frac{3}{2}\rho^2 - \frac{1}{2}\hat{R}\right) \\ C^{ij} &= h^{ij}\left(\frac{-3}{8}\rho^3 + \frac{1}{2}\hat{\nabla}^2\rho + \frac{1}{4}\rho\,\hat{R}\right) - \frac{1}{2}\hat{\nabla}^i\hat{\nabla}^j\,\rho \end{aligned} \tag{2.6}$$

All quantities in (2.5) and (2.6) are defined with respect to the spatial metric h_{ij}. Indices i, j, k are raised and lowered by h_{ij}, $\hat{\nabla}$ stands for covariant differentiation with respect to h_{ij} and $\hat{R}$ is the two dimensional curvature scalar given by

$$\hat{R} = e^{-\phi}\nabla^2\phi \text{ where } \nabla^2 \text{ is the flat laplacian.} \tag{2.7}$$

We now solve the field equations (2.1) for each component using (2.5), (2.6) and (2.7). Without loss of generality we set $\kappa = 1$.

The (0,0), (0,j) and (i,j) component equations are respectively

$$\frac{3}{4}\rho^2 - \frac{1}{2}e^{-\phi}\nabla^2\phi + \frac{1}{\mu}\rho^3 - \frac{1}{2\mu}\hat{\nabla}^2\rho - \frac{1}{2\mu}\,\rho\, e^{-\phi}\nabla^2\phi = -T_{00} \tag{2.8}$$

$$\frac{\epsilon^{jk}e^{-\phi}}{2}\partial_k\left(\rho + \frac{3}{2\mu}\rho^2 - \frac{1}{2\mu}e^{-\phi}\nabla^2\phi\right) = -T_0^j \tag{2.9}$$

$$-e^{-\phi}\delta^{ij}\left(-\frac{\rho^2}{4} - \frac{1}{2\mu}\rho^3 + \frac{1}{2\mu}\hat{\nabla}^2\rho + \frac{1}{4\mu}\,\rho\, e^{-\phi}\nabla^2\phi\right) - \frac{1}{2\mu}\hat{\nabla}^i\hat{\nabla}^j\,\rho = -T^{ij}\,. \tag{2.10}$$

Note that the scalar twist ρ appearing in the above equations cannot contain a delta function or else quantities like ρ^2 and ρ^3 in those equations would be ill defined. This implies that $\psi(0)$ appearing in the definition of ρ i.e. (2.4) must be zero. Therefore ρ reduces to

$$\rho = e^{-\phi}\,\frac{\psi'(r)}{r}. \tag{2.11}$$

We approach the problem of localized sources, not by actually specifying $T_{\mu\nu}$ but by examining the metric dependent side of the field equations (2.8-2.10), and drawing conclusions on the scalar twist ρ and the function ϕ if $T_{\mu\nu}$ were localized. Hence $T_{\mu\nu}$ is only defined via the field equations (2.8-2.10), and therefore automatically covariantly conserved (i.e. since the metric dependent side of the field equations obey the Bianchi identities).

III. Solving the Field Equations

We consider a localized spin source σ by allowing the scalar twist $\rho(r)$ be a rapidly decreasing function of r. The total spin J, which is conserved and invariant under general coordinate transformations, is given by (see [4])

$$J \equiv 2\pi \lim_{r\to\infty} \psi(r) = 2\pi \int_0^\infty \rho e^{\phi} r dr\,, \tag{2.12}$$

where (2.11) was used. The scalar twist ρ can therefore be interpreted as the total spin density. The mass m is defined as the total energy when the total spin is zero (i.e. $\rho = 0$) and when the topological term is absent (i.e. $\mu \to \infty$). It is therefore given by

$$m = \int \frac{1}{2}\left(e^{-\phi}\nabla^2\phi\right) e^{\phi}\, d^2r = \int \frac{1}{2}\nabla^2\phi\, d^2r\,. \tag{2.13}$$

Here $\nabla^2\phi$ can be arbitrarily localized. The above equation for m leads to the condition that

$$r\,\phi'\Big|_0^\infty = \frac{m}{\pi} \quad . \tag{2.14}$$

For reasons given in section IV we do not allow $\nabla^2\phi$ to be a delta function (i.e. we exclude the possibility that $\phi(r) \propto \ln r$ as $r \to 0$). Therefore

$$\lim_{r\to 0}(r\,\phi') = 0 \tag{2.15}$$

i.e. if $\lim_{r\to 0}(r\,\phi') = k$ where $k \neq 0$, then $\phi(r) \propto k \ln r$ as $r \to 0$.

We now want to find an expression for the total spin J. We first integrate the T_0^j equation (2.9) and find that

$$\int \epsilon^{ij}\, x^i \left(-T_0^j\right) e^{2\phi}\, d^2r = \frac{-1}{2}\int \left(\rho + \frac{3}{2\mu}\rho^2 - \frac{1}{2\mu}e^{-\phi}\nabla^2\phi\right)' e^{\phi}\, r\, d^2r\,. \tag{2.16}$$

The integral of the third term can be readily evaluated using (2.14) and (2.15) and gives $-(m/\mu)(1 + m/4\pi)$. As we will show in section IV, this result is different from that obtained using naive manipulations with delta function sources. For the first two terms we take $\rho(r)$ to be a rapidly decreasing function of r with the condition that $\lim_{r\to\infty}\rho(r) r^2 e^{\phi} = \lim_{r\to\infty}\rho(r)^2 r^2 e^{\phi} = 0$. The integrals of these two terms are well defined for any regular mass distribution $\nabla^2\phi$, but depend on the actual profile and there will be small corrections for any well localized mass. However, when evaluated in the point mass limit the result is

$$(2\pi + m)\int_0^\infty \left(\rho + \frac{3}{2\mu}\rho^2\right) e^{\phi} r dr\,. \tag{2.17}$$

Then using (2.12) for the total spin, one can rewrite (2.16) as

$$J = \sigma + \frac{m}{\mu}\left(\frac{4\pi + m}{4\pi + 2m}\right) - 2\pi\int_0^\infty \frac{3}{2\mu}\rho^2 e^{\phi} r dr\,. \tag{2.18}$$

where

$$\sigma \equiv \frac{2\pi}{2\pi + m} \int \epsilon^{ij}\, x^i \left(-T_0^j\right) e^{2\phi}\, d^2r. \tag{2.19}$$

Here σ is identified as the spin source (or bare spin) i.e. it is equal to the total spin when the topological term is absent ($\mu \to \infty$). Clearly, as m approaches zero equation (2.19) reduces to the usual definition of spin in the linear theory [1]. The m appearing in the denominator of the spin source σ arises because the spatial metric describes a cone with a negative angular defect m and therefore the angle ranges from 0 to $2\pi + m$ instead of 2π (see [1]).

The last two terms on the right hand side of (2.18) can be regarded as the induced spin, which has a dependence on $\rho(r)$. The total spin J can therefore range from 0 to its maximum value of $\sigma + \frac{m}{\mu}\left(\frac{4\pi+m}{4\pi+2m}\right)$. The induced spin reduces to the linearized result m/μ in the limit where m and ρ are small.

We do not obtain any specific equation relating the mass m to the spin σ as in ref.[2] where the condition $m + \mu\sigma = 0$ is obtained. For the specific case $\rho = 0$ (which implies $J = 0$) we obtain $m\left(\frac{4\pi+m}{4\pi+2m}\right) + \mu\sigma = 0$ in disagreement with the result $m + \mu\sigma = 0$ of Ortiz [6]. We now discuss the reasons for this discrepency.

IV. The Point Like Limit

We now give an analysis of the use of delta functions for sources. A delta function spin source for T_0^j was used in the work of Clément [2]

$$T_0^j = \frac{-1}{2}\sigma e^{-\phi}\epsilon^{jk}\partial_k\left(e^{-\phi}\,\delta^2(\vec{r})\right) \tag{2.20}$$

where σ is equivalent to the spin source defined in (2.19). Then (2.9) becomes

$$\partial_k\left(\rho + \frac{3}{2\mu}\rho^2 - \frac{1}{2\mu}e^{-\phi}\nabla^2\phi\right) = \sigma\partial_k\left(e^{-\phi}\,\delta^2(\vec{r})\right). \tag{2.21}$$

The ρ and ρ^2 terms on the left hand side of (2.21) cannot match the delta function on the right hand side (see comment above (2.11)) and for the purposes of our forthcoming argument will simply be dropped. The term $e^{-\phi}\nabla^2\phi$, with ϕ proportional to $\ln r$, cannot match the delta function because the non-linearity in (2.21) imposes the products of derivatives of $\ln r$ which are ill defined at the origin. Consider the contraction of (2.21) with x^k. This implies

$$-\frac{1}{2\mu}e^{-\phi}\left(x^k\partial_k\left(\nabla^2\phi\right) - r\phi'\nabla^2\phi\right) = \sigma e^{-\phi}\left(x^k\partial_k\delta^2(\vec{r}) - r\phi'\delta^2(\vec{r})\right) \tag{2.22}$$

which requires, for $\phi = -(\mu\sigma/\pi)\ln r$,

$$r\phi'\nabla^2\phi = -2\mu\sigma\, r\phi'\delta^2(\vec{r}) \tag{2.23}$$

where the first term in (2.22) is properly matched and the $e^{-\phi}$ is canceled with the corresponding e^{ϕ} in any volume element. Equation (2.23) is not sensible for the imposed choice $\phi \propto \ln r$. We cannot simply cancel the $r\phi'$ from both sides since it is ill defined exactly at the point where the delta function has all its weight. It is not consistent to take $r\phi'$ to be identically constant while taking $\nabla^2\phi \equiv (r\phi')'/r$ to be a delta function. The product $r\phi'\nabla^2\phi$ is clearly ill defined for $\phi \propto \ln r$. The right hand side of (2.23) is also ill defined for $\phi \propto \ln r$ because $r\phi'$ does not lie in the space of functions on which $\delta^2(\vec{r})$ acts.

A reasonable way to handle the product $r\phi'\nabla^2\phi$ is to use the equivalence between a delta function $\delta^2(\vec{r})$ and the limit of a sequence of functions $\delta_n(r)$ (see [7]) where

$$\int \delta_n(r)\, d^2r = 1 \;\; \forall\, n \quad \text{and} \quad \lim_{n\to\infty}\int \delta_n(r) f(r)\, d^2r = f(0). \tag{2.24}$$

Here $\delta_n(r)$ can be any smooth function, like a gaussian, that peaks as $n \to \infty$. Then to satisfy (2.13) we let

$$\nabla^2\phi_n(r) = 2m\delta_n(r)\,. \tag{2.25}$$

Clearly (2.14)

$$r\,\phi_n'\Big|_0^\infty = \frac{m}{\pi} \tag{2.26}$$

is valid for each n. Since $\nabla^2\phi_n(r)$ is a smooth function for every n, it follows that $\phi_n(r)$ for any given n cannot be proportional to $\ln r$ as $r \to 0$ (since $\nabla^2 \ln r$ is not smooth at r=0). Hence $\lim_{r\to 0}(r\phi_n') = 0$. Then the moment

$$\begin{aligned} M &\equiv \frac{-1}{2}\int x^k\partial_k\left(\frac{-1}{2\mu}e^{-\phi}\nabla^2\phi\right)e^{\phi}\, d^2r \\ &= -\frac{1}{4\mu}\int\left(2 + r\phi'\right)\nabla^2\phi\, d^2r \end{aligned} \tag{2.27}$$

which was used to calculate the third term of (2.16) yields

$$\begin{aligned} M &= \frac{-\pi}{2\mu} \lim_{n\to\infty} \int_0^\infty \frac{1}{r} \left(r\phi_n^{'}\right)^{'} \left(2 + r\phi_n^{'}\right) r\, dr \\ &= \frac{-\pi}{2\mu} \lim_{n\to\infty} \left(2 \left(r\phi_n^{'}\right)\Big|_0^\infty + \frac{1}{2} \left(r\phi_n^{'}\right)^2 \Big|_0^\infty \right) \\ &= \frac{-\pi}{2\mu} \lim_{n\to\infty} \left(2\frac{m}{\pi} + \frac{1}{2} \left(\frac{m}{\pi}\right)^2 \right) \\ &= -\frac{m}{\mu} \left(1 + \frac{m}{4\pi}\right) \end{aligned} \tag{2.28}$$

where the limits of $r\phi_n^{'}$ i.e.

$$\lim_{r\to\infty} r\phi_n^{'} = \frac{m}{\pi} \text{ and } \lim_{r\to 0} r\phi_n^{'} = 0 \tag{2.29}$$

were used (also note that the result (2.28) is valid for any n and is independent of the limit $n \to \infty$). The right hand side of (2.23) would be equal to zero if the functions $\phi_n(r)$ were used, showing clearly the impossibility of satisfying equation (2.23) (and similarly (2.21)). A naive attempt to satisfy this equation by having $\phi = (m/\pi)\ln r$, $r\phi^{'} = m/\pi$ and $\nabla^2\phi = 2m\delta^2(\vec{r})$, which is not a consistent treatment of $r\phi^{'}$, leads to the incorrect conclusion that $m + \mu\sigma = 0$ (see [2]) and to a wrong value for the moment M, namely

$$M = -\frac{m}{\mu}\left(1 + \frac{m}{2\pi}\right). \tag{2.30}$$

The factor $\left(\frac{4\pi+m}{4\pi+2m}\right)$ in (2.18) was obtained using the result (2.28) and would not appear if the result (2.30) were used. This is why for the specific case $\rho = 0$ we obtain the condition $m\left(\frac{4\pi+m}{4\pi+2m}\right) + \mu\sigma = 0$ and not the condition $m + \mu\sigma = 0$ of Ortiz [6].

We have shown that the correct procedure in the non-linear theory is to take the source to be a function $\delta_n(r)$ that peaks as $n \to \infty$ instead of starting directly with a delta function. In the linearized theory [1] such a procedure is not necessary because products like $e^{-\phi}\nabla^2\phi$ do not appear in the equations and one can begin directly with a delta function source. These products actually reflect the non-linearity of the field equations and this is why the factor $\left(\frac{4\pi+m}{4\pi+2m}\right)$ in (2.18) does not appear in the linearized theory.

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2.2 Article 4 : Cosmological sector for localized mass and spin in 2+1 dimensional topologically massive gravity

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Abstract

The cosmological sector to the full non-linear topologically massive gravity (TMG) is obtained for localized sources of mass m and spin σ besides the asymptotically flat sector previously obtained [7]. In a small region near but outside the sources, the metric resembles the spinning conical flat metric but we find that the mass m creates a negative deficit angle of $3m$ as opposed to m. Furthermore, it is not possible to recover the results of pure Einstein gravity in the limit $\mu \to \infty$ unlike the flat sector.

I. Introduction

Gravity in 2+1 dimensions has attracted much interest in the last few years. In supergravity theory, a mechanism in 2+1 dimensions originally proposed by Witten [11], has been implemented as a possible solution to the cosmological constant problem [9]. The authors demonstrated that although unbroken supersymmetry prohibits the possibility of a cosmological constant in the vacuum, massive states, more precisely massive soliton states, cause the space-time to be conically flat and the generators of the supersymmetry, which cause the usual spectrum doubling, fail to exist. That work was done in the context of ordinary Einstein gravity (appropriately supersymmetrized). It is well known that Einstein gravity is trivial in 2+1 dimensions i.e there is no propagating graviton and matter free regions are flat. However, 2+1 dimensions allows one to include the parity violating gravitational Chern-Simons term [1, 10] and the gravitational field now becomes a non-trivial propagating massive field. Such an inclusion is in fact obligatory for a theory containing fermions and parity violation; it is automatically induced via 1 loop quantum corrections with strength $N/2$ where N is the number of species of fermions [12]. The results of [9] rested on the fact that exterior to the soliton the space-time was conically flat. It becomes a compelling and interesting exercise to discern what would the soliton and the exterior space-time represent for the case of topologically massive gravity (TMG).

Many years ago, Vuorio [6] obtained two solutions to TMG in vacuum ($T_{\mu\nu} = 0$): a trivial flat space and a cosmological solution i.e. a homogenous space-time with constant curvature scalars. Localized sources can now be embedded in either of these two backgrounds. In the case of the flat background, solutions with localized mass m and spin σ have been obtained in the linearized [1] and in the non-linear theory [7] and gives rise in both cases to an asymptotic spinning conical space-time. As mentioned in [1], the cosmological solutions are not accessible in the linearized case and one is required to employ the full non-linear theory. Cosmological solutions have already been obtained in the non-linear theory using delta function sources [2]. It has been shown that delta function sources can be accomodated with torsion at the source [8] but leads to inconsistencies in the torsion-free TMG theory (see [7]). An alternative and more consistent approach is to use non-singular sources that are arbitrarily localized.

We demonstrate that non-singular localized mass and spin sources in the full non-linear TMG theory support a cosmological sector. In this sector, the spin source dominates the helical time structure of the metric at short distances outside the source and the mass creates a negative deficit angle of $3m$ instead of the value m found in the flat sector. Far from the sources, the metric resumes the form of the cosmological homogenous space-time found by Vuorio. The cosmological sector is a disjoint sector in that it is not possibe to recover pure Einstein gravity in the limit $\mu \to \infty$ where one would expect the topological term to vanish.

II. Field Equations

We begin our work by writing down the well known field equations for TMG with energy-momentum tensor $T_{\mu\nu}$. The Einstein field equations including a topological mass term is given by [1, 2] (in units where $8\pi G = 1$)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{\mu} C_{\mu\nu} = -\kappa^2 T_{\mu\nu} \tag{2.31}$$

where $R_{\mu\nu}$ is the Ricci tensor , $R \equiv R_{\mu\nu} g^{\mu\nu}$ is the curvature scalar, and $C_{\mu\nu}$ is the 3 dimensional Weyl-Cotton tensor. We simplify the field equations (2.31) by choosing a rotationally symmetric, stationary metric. The most general form for such a metric is given by [3]

$$ds^2 = q^2 (\, dt + \psi(r)\, d\theta\,)^2 - e^{\phi(r)} \left(dr^2 + r^2\, d\theta^2 \right) \qquad i = 1, 2 \tag{2.32}$$

To recover Vuorio's solution in the region exterior to the source we set $q = 1$ as in his work [6] (we will also show that the metric with $q = 1$ does support solutions with localized mass and spin). The functions $\psi(r)$ and $\phi(r)$ completely determine the metric. The scalar twist $\rho(r)$ is given by [7]

$$\rho = e^{-\phi} \, \frac{\psi'(r)}{r}. \tag{2.33}$$

where $\psi'(r) \equiv \dfrac{d\psi(r)}{dr}$. To avoid a delta function ρ, $\psi(r)$ is set to zero at $r = 0$ (see [7] for more details). The (0,0), (0,j) and (i,j) component field equations (2.31) are respectively [7]

$$\frac{3}{4}\rho^2 - \frac{1}{2} e^{-\phi} \nabla^2 \phi + \frac{1}{\mu}\rho^3 - \frac{1}{2\mu} \hat{\nabla}^2 \rho - \frac{1}{2\mu}\, \rho\, e^{-\phi} \nabla^2 \phi = -T_{00} \tag{2.34}$$

$$\frac{\epsilon^{jk} e^{-\phi}}{2} \partial_k \left(\rho + \frac{3}{2\mu}\rho^2 - \frac{1}{2\mu} e^{-\phi} \nabla^2 \phi \right) = -T_0^j \tag{2.35}$$

$$-e^{-\phi}\delta^{ij} \left(-\frac{\rho^2}{4} - \frac{1}{2\mu}\rho^3 + \frac{1}{2\mu}\hat{\nabla}^2 \rho + \frac{1}{4\mu}\, \rho\, e^{-\phi} \nabla^2 \phi \right) - \frac{1}{2\mu} \hat{\nabla}^i \hat{\nabla}^j \rho = -T^{ij} . \tag{2.36}$$

As in [7], we approach the problem of localized sources not by actually specifying $T_{\mu\nu}$ but by examining the metric dependent side of the field equations and drawing conclusions on the scalar twist ρ and the function ϕ if $T_{\mu\nu}$ were localized.

III. Solving the Field Equations: Cosmological Sector

When solving the field equations in the vacuum i.e. $T_{\mu\nu} = 0$, Vuorio obtains that the scalar twist $\rho(r)$ is a constant that can have two possible values (corresponding to two metrics): one is given by $\rho(r) = 0$ and $e^{-\phi}\nabla^2\phi = 0$ which describes a flat metric (Minkowski space) and the second is $\rho(r) = -2\mu/3$ and $e^{-\phi}\nabla^2\phi = 2\mu^2/9$ which describes a cosmological space-time. To include sources of mass m and spin σ in this cosmological space, we introduce two functions $M(r)$ and $S(r)$ such that

$$e^{-\phi}\nabla^2\phi = 2\mu^2/9 + M(r) \quad \text{and} \quad \rho(r) = \frac{-2\mu}{3} + S(r) \tag{2.37}$$

where $M(r)$ and $S(r)$ are zero outside the sources. We will now see that $M(r)$ is proportional to the mass density and $S(r)$ to the spin source density.

When substituting ρ above into the T_{0j} "spin" equation (2.35) one observes that $\rho + 3\rho^2/2\mu = -S + 3S^2/2\mu$. As expected the constant part of $\rho(r)$ i.e. $-2\mu/3$, has disappeared and does not contribute to the spin source. Clearly, $\rho(r)$ has been replaced by $-S(r)$ in Eq. (2.35). In flat space, $\rho(r)$ was the total conserved spin source density (see [7]) and we see that in the cosmological case, it is $-S(r)$ that takes on that role.

The mass m is defined as the total energy (i.e. the volume integral of T_{00}) when the spin source density $S(r)$ is zero. It will be localized in a region from $r = 0$ to $r = \epsilon$ so that $M(r)$ and T_{00} are zero for $r > \epsilon$. Substituting the quantities in Eq. (2.37) into the T_{00} equation (2.34), one obtains

$$\begin{aligned} m &= \frac{1}{6}\int M(r) e^{\phi}\, d^2r \\ &= \frac{\pi}{3}\int_0^{\epsilon} \left(\nabla^2\phi - \frac{2\mu^2}{9}e^{\phi}\right) r\, dr \\ &= \frac{\pi}{3}\, r\,\phi'\Big|_0^{\epsilon} - \frac{2\pi\mu^2}{27}\int_0^{\epsilon} e^{\phi}\, r\, dr. \end{aligned} \tag{2.38}$$

There are two terms to evaluate above. In the first term, the limit $r \to \epsilon$ can be obtained by matching the exterior to the interior solution at $r = \epsilon$. Outside the mass source ($r > \epsilon$), $e^{-\phi}\nabla^2\phi = 2\mu^2/9$ and the most general solution is given by [6]

$$\phi(r) = 2\ln\left(\frac{6n\, r^{n-1}}{\mu\, r_0^n\left[1 - \left(\dfrac{r}{r_0}\right)^{2n}\right]}\right) \quad \text{for } r > \epsilon \tag{2.39}$$

where n and r_0 are arbitrary constants and $r < r_0$. Note that $\phi(r)$ is invariant under the transformation $n \to -n$ so that positive and negative values for n are both valid (nonetheless, this invariance will have no direct physical consequences and for the sake of clarity and without loss of generality we can assume n is positive). With ϕ given above, one obtains

$$r\phi' = 2(n-1) + \frac{4n(r/r_0)^{2n}}{[1-(r/r_0)^{2n}]} \quad \text{for } r > \epsilon \tag{2.40}$$

and

$$\lim_{r\to\epsilon}(r\phi') = 2(n-1) \tag{2.41}$$

where ϵ is sufficiently small so that terms proportional to ϵ are negligible and have been dropped in the above limit. For reasons given in the introduction, we do not allow delta function sources and therefore we exclude the possibility that $\phi(r) \propto \ln r$ as $r \to 0$. Therefore

$$\lim_{r\to 0}(r\phi') = 0 \tag{2.42}$$

i.e. if $\lim_{r\to 0}(r\phi') = k$ where $k \neq 0$, then $\phi(r) \propto k \ln r$ as $r \to 0$. With the above limits, Eqs. (2.41) and (2.42), the first term in Eq. (2.38) is

$$r\phi'\Big|_0^\epsilon = 2(n-1). \tag{2.43}$$

The mass m is positive and therefore $n > 1$. To evaluate the second term in Eq. (2.38), the behaviour of $\phi(r)$ in the source region needs to be known. In the exterior, $\phi(r)$ is negative and decreases as $r \to \epsilon$ from the right i.e. behaves as $\ln r$ as $r \to \epsilon$. For an elementary localized particle the mass density $M(r)$ (and hence $\nabla^2\phi$) should be positive in the source region and this implies that $\phi(r)$ must continue to decrease from $r = \epsilon$ to $r = 0$. It follows that the quantity e^ϕ is of the order ϵ (or smaller) in the source region and that the second term in Eq. (2.38) can be neglected for ϵ sufficiently small. Substituting Eq. (2.43) into Eq. (2.38) one obtains

$$r\phi'\Big|_0^\epsilon = \frac{3m}{\pi} \quad \text{and} \quad n = 1 + \frac{3m}{2\pi} \tag{2.44}$$

We now turn to spin. The spin source $S(r)$ will be localized in a small region from $r = 0$ to r_s where $r_0 \gg r_s \gg \epsilon$. By integrating the T_0^j equation (2.35)with $\rho(r)$ given by Eq. (2.37) one obtains

$$\int \epsilon^{ij}\, x^i \left(-T_0^j\right) e^{2\phi}\, d^2r = -\pi \int_0^{r_s} \left(-S(r) + \frac{3}{2\mu}S^2(r) - \frac{1}{2\mu}\, e^{-\phi}\nabla^2\phi\right)' e^\phi\, r^2\, dr\,. \tag{2.45}$$

The integral of the third term can be readily evaluated using Eqs. (2.40) and (2.42) and yields $-(3m/\mu)(1 + 3m/4\pi)$ (where the second term in Eq. (2.40), proportional to $(r_s/r_0)^{2n}$, is negligible and has been dropped). For the integral of the first two terms we take $S(r)$ to be a rapidly decreasing function of r with the condition that $\lim_{r\to r_s} S(r)r^2e^{\phi} = \lim_{r\to r_s} S^2(r)r^2e^{\phi} = 0$. These integrals are well defined for any regular mass distribution $\nabla^2\phi$, but depend on the actual profile and there will only be small differences for any two well localized mass distributions. However, in the point mass limit the result is

$$(2\pi + 3m)\int_0^{r_s}\left(-S + \frac{3}{2\mu}S^2\right)e^{\phi}r dr\,. \tag{2.46}$$

Equation (2.45) can now be expressed as

$$J_S \equiv 2\pi\int_0^{r_s} -S(r)e^{\phi}\,r\,dr = \sigma + \frac{3m}{\mu}\left(\frac{4\pi+3m}{4\pi+6m}\right) - 2\pi\int_0^{r_s}\frac{3}{2\mu}S^2e^{\phi}r dr\,. \tag{2.47}$$

where

$$\sigma \equiv \frac{2\pi}{2\pi+3m}\int \epsilon^{ij}\,x^i\left(-T_0^j\right)e^{2\phi}\,d^2r. \tag{2.48}$$

Here σ is identified as the bare spin i.e. it is equal to the spin source J_S when the topological term is absent ($\mu \to \infty$). The other two terms in Eq. (2.47) represent the induced spin (see [1, 7]).

We now proceed to find the metric Eq. (2.32). The function $\psi(r)$, given by Eq. (2.33) is

$$\begin{aligned}\psi(r) &= \int_0^r \rho(r)e^{\phi}rdr \\ &= \int_0^r S(r)e^{\phi}rdr - \frac{2\mu}{3}\int_0^r e^{\phi}rdr \\ &= \frac{-J_S}{2\pi} - \frac{12n\,(r/r_0)^{2n}}{\mu\,[1-(r/r_0)^{2n}]} \qquad \text{for } r > r_s\end{aligned} \tag{2.49}$$

where we see that there are two distinct contributions to $\psi(r)$: one from the spin source density $S(r)$ and one from the "background" spin density $-2\mu/3$. In Eq. (2.49), the expression for ϕ exterior to the source Eq. (2.39), was used to evaluate the second integral. For $r < r_s$, $\psi(r)$ and hence the metric, are well behaved but depend intimately on the distribution of the mass and spin sources. Also, as mentioned in section II, $\psi(r)$ is zero at $r = 0$ and the metric is therefore nonsingular.

As in Vuorio's work [6] we define new variables

$$\tilde{\theta} = n\theta, \quad \sinh x = \frac{2(r/r_0)^n}{1-(r/r_0)^{2n}} \qquad \text{where } n = 1 + \frac{3m}{2\pi} \tag{2.50}$$

so that the metric, in terms of these new variables (dropping the tilde), is given by

$$ds^2 = \left[dt - \left(\frac{J_S}{(2\pi + 3m)} + \frac{6}{\mu}(\cosh x - 1)\right) d\theta\right]^2 - \frac{9}{\mu^2}\left(dx^2 + \sinh^2 x d\theta^2\right). \qquad (2.51)$$

This metric is similar to Vuorio's "cosmological" metric [6] but differs from it in two ways: the constant n, which appears in the redefinition of θ, is not equal to 1 as in Vuorio's case and our metric has a non-zero spin J_S.

Since n is not 1, our metric has a deficit angle i.e. with $n = 1 + 3m/2\pi$ the new angle θ runs from 0 to $2\pi + 3m$ instead of 2π. Locally, this represents a conical space with negative angular defect of $3m$. In the case of the flat background the deficit angle was simply the mass m [1, 7]. The extra factor of three in the cosmological case is due to the spin-spin coupling term $\rho\, e^{-\phi}\nabla^2\phi/2\mu$ appearing in the T_{00} equation (2.34). This term contributes to the mass i.e. it is equal to $-e^{-\phi}\nabla^2\phi/3$ when m is being defined, that is when $S(r) = 0$ and ρ is $-2\mu/3$. The coupling is therefore between a "background" spin $\rho = -2\mu/3$ and an "induced" spin $e^{-\phi}\nabla^2\phi/2\mu$. In the flat case, the spin-spin term makes no contribution to the mass because there is no background spin. It is important to note that the spin-spin term $-e^{-\phi}\nabla^2\phi/3$ is independent of μ and therefore it is impossible to make it vanish in the non-topological limit $\mu \to \infty$. Hence, the cosmological sector is a disjoint sector and one cannot recover pure Einstein gravity in the appropriate limit.

To see the effect of the spin J_S, note that in a region close to the source where x is small, the term with $\cosh x - 1$ in the metric is negligible compared to J_S and $\sinh^2 x \approx x^2$. In the neighbourhood of the source the metric is

$$ds^2 = \left[dt - \frac{J_S}{(2\pi + 3m)}\, d\theta\right]^2 - \frac{9}{\mu^2}\left(dx^2 + x^2 d\theta^2\right) \qquad (2.52)$$

which is a spinning conical space with the spin J_S governing the helical- time structure of the metric(see [5, 6] for a discussion on helical- time structure). Far from the sources, the metric (2.51) of course describes the same space-time as Vuorio's i.e. a homogenous space-time with constant curvature scalars.

In conclusion, we have shown that there exists a distinct cosmologicial sector for the exterior space-time to localized spin and mass sources in 2+1 dimensional topologically massive gravity in addition to the usual flat, conical solution. This is in contradistinction to ordinary Einstein gravity in 2+1 dimensions and also in 3+1 dimensions which admit only a unique exterior space-time. The ramifications of the existence of this sector should be investigated for the work of [9], for example.

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Conclusion

Dans la première partie de la présente thèse, nous avons étudié la gravité conforme, théorie proposée comme alternative à la gravité d'Einstein par Mannheim et Kazanas(MK). Notre recherche a démontré que pour la métrique choisie par MK (obtenue en fixant le facteur conforme), il n'est pas possible d'expliquer à la fois la déflexion de la lumière et les courbes de rotation plates des galaxies. Les solutions à la gravité conforme dans le vide, contrairement à la gravité d'Einstein, ne sont pas uniques et toute métrique se rapportant par un facteur conforme à celle qui a été choisie par MK est valide. Cela pose un problème sérieux quant à l'analyse des courbes de rotation galactique, car une géodésique massive n'est pas conformément invariante, ce qui nous oblige à fixer le facteur conforme afin de concorder avec les expériences. Par-contre, le résultat de la déflexion de la lumière est conformément invariant. La divergence entre la géodésique massive et la géodésique nulle n'est pas un problème de gravité conforme, mais plutôt une application erronée de la théorie. Pour le moment, la théorie ne permet pas d'avoir des géodésiques massives ou une masse dans le tenseur énergie-moment (ceci brise la symétrie conforme). Phénoménologiquement, nous obtenons un résultat dont l'ordre de grandeur de γ demeure constant. Bien sûr, cela suppose l'absence de matière sombre dans les amas de galaxies et si des grandes quantités de cette matière venaient à être découvertes un jour, cela annulerait ce résultat. En laissant de côté la question de la matière sombre, le défi absolu de la théorie serait de savoir s'il est possible de réaliser un mécanisme qui casserait la symétrie dynamique conforme pour permettre à la masse d'être inclue sans aucune difficulté. Sans ce mécanisme de brisure, la gravité conforme ne peut vraiment être une alternative viable à la gravité d'Einstein.

Indépendamment de la gravité conforme, le problème de la matière sombre subsiste. Mais l'auteur de la présente thèse ne trouve toujours pas l'hypothèse de la matière sombre assez convaincante. Les courbes de rotation galactique dans plusieurs galaxies spirales suivent un schéma simple. En fait, Milgrom a obtenu une loi empirique

(MOND) avec seulement un paramètre libre, et a eu un succès remarquable à expliquer les courbes de rotation galactique sans inclure de la matière sombre [2]. Ce modèle semble difficile à concilier avec l'hypothèse de la matière sombre. Il est par ailleurs important de se souvenir que jusqu'à aujourd'hui, la matière non baryonique, prédite en grande quantité dans les galaxies et amas de galaxies, n'a toujours pas été directement confirmée expérimentalement.

La gravité en 2+1 dimensions avec masse topologique, sujet de la deuxième partie de la thèse, n'est évidemment pas censé donner des résultats phénoménologiques. Nous voyons que l'analogue en 2+1 dimensions de la gravité d'Einstein n'a aucune ressemblance avec le cas en quatre dimensions. En 2+1 dimensions, l'espace-temps est plat à l'extérieur des sources! La gravité d'Einstein est dynamique qu'à partir de quatre dimensions, ce qui diffère de la théorie de Newton selon laquelle les particules peuvent interagir en trois dimensions. La théorie en 2+1 dimensions, vu ses dimensions impaires, nous permet d'ajouter un terme de Chern-Simons (qui n'existe pas en quatre dimensions).

Le terme de Chern-Simons modifie significativement la théorie en 2+1 dimensions. Il existe un espace-temps cosmologique dans le vide et nous avons démontré que des sources peuvent être inclues dans cet espace-temps. Nos résultats montrent que le spin associé à la cosmologie interagit avec la particule et augmente l'angle conique comparativement au secteur plat; en d'autres termes, la cosmologie a des effets sur les propriétés locales de l'espace-temps. Il a été démontré que la théorie non-linéaire, contrairement à la théorie linéarisée, ne permet pas des sources de fonction delta. C'est ce que nous avons prouvé à l'aide de calculs mathématiques détaillés. Le fait que les fonctions delta ne sont pas permises dans la théorie non-linéaire ne représente pas qu'un détail mathématique. En effet, la mise en application de telles sources de fonction delta mène à des conclusions physiques erronées, comme cela a été démontré dans notre troisième article.

Un projet pertinent pour le futur serait d'obtenir une théorie relativiste en accord avec la théorie de Milgrom (MOND). Dans une telle théorie on imposerait bien sûr le principe d'équivalence et les équations seraient covariantes. Cependant, dans la limite où le champ est faible et les particules se deplacent avec des vitesses non-relativistes, la théorie se réduirait à MOND et non à la loi gravitationelle de Newton. Il n'est pas clair si cela est possible en gardant seulement deux dérivées ou moins de la métrique.

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