Respecting Improvement in Markets with Indivisible Goods[*]

Lars Ehlers[†]

September 17, 2023

Abstract

We study markets with indivisible goods where monetary compensations are fixed (or are not possible). Each individual is endowed with an object and a preference relation over all objects. Respect for improvement means that when the ranking of an agent's endowment improves in some other agent's preference (while keeping other preferences unchanged), then this agent weakly benefits from it. As a main result we show that on the strict domain *individual rationality*, *strategy-proofness*, and *non-bossiness* imply *respecting improvement*. As a consequence we obtain that top trading with fixed-tie breaking and random tie-breaking, respectively, satisfy *respecting improvement* on the weak domain. We further show that trading cycles rules with fixed tie-breaking satisfy *respecting improvement*. Finally, we put our results in the contexts of generalized matching problems, roommate problems and school choice.

JEL Classification: C78, D63, D70.

Keywords: indivisible goods, respecting improvement, top trading cycles, indifferences, tie-breaking.

1 Introduction

We consider exchange markets where any agent owns one indivisible good and monetary transfers are fixed (or absent, i.e. they are set equal to zero). Such markets arise in numerous applications ranging from entry-level medical markets to school choice, on-campus housing and kidney exchange. In all these markets any agent possesses a preference relation over the possible allotments to receive. Preferences are private information and need to be elicited, and a centralized clearinghouse (a mechanism) assigns the indivisible goods among the agents for any reported profile.

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[*] The author is grateful to Florian Brandl for early inputs, conversations on this project, detailed comments and suggestions, and to the SSHRC (Canada) for financial support under Insight Grant 435-2023-0129.

[†] Département de Sciences Économiques and CIREQ, Université de Montréal, Montréal, QC H3C 3J7, Canada. Email: lars.ehlers@umontreal.ca.

As a basic property we study *respecting improvement*: this means that for a given profile, if the endowment of a fixed agent improves in the ranking of some agent while keeping preferences over other indivisible goods and other agents' preferences unchanged, then the fixed agent shall be weakly better off as the result of the improvement. This requirement is natural as endowments are goods and any agent weakly benefits when his endowment is ranked more desirable. It prevents agents from trying to lower the ranking of their endowment in other agents' preferences.

The most important applications described above are matching markets which are either two-sided or one-sided. For two-sided markets, like for example for school choice, there are two sides, students and schools, who are matched to each other. For two-sided markets with strict preferences, *respecting improvement* is well understood for the deferred-acceptance (DA-)algorithm. On the one hand, Balinski and Sönmez (1999) show that the student-proposing DA-mechanism respects improvements for students, and Kominers and Sönmez (2016) generalize this finding to matching with slot-specific priorities.[1] Recently, Hirata, Kasuya and Okumura (2023) show the equivalence of *respecting improvement* and *strategy-proofness* for stable mechanisms under a weak independence condition. On the other hand, Hatfield, Kojima and Narita (2016) show that for schools with several seats there exist no stable mechanisms which respect improvements for schools, and the same is true for mechanisms which are efficient for students. Note that for two-sided markets respecting improvement is not at all studied on the weak domain.

For one-sided markets such as on-campus housing or kidney exchange (e.g. Roth, Sönmez and Ünver (2004)), any agent owns one house (or a donor with a kidney) and desires at most one house. When preferences are strict, the top-trading-cycles (TTC-)mechanism is the most important one for applications. The study of *respecting improvement* is pioneered for those markets by Biro et al. (2023). We provide an in-depth study of deterministic mechanisms for one-sided markets on both the strict and the weak domain. Our first main result shows an implication of *respecting improvement* via three basic properties on the domain of strict preferences. Namely, *individual rationality*, *strategy-proofness* and *non-bossiness* imply *respecting improvement*. There is a large class of rules satisfying *individual rationality*, *strategy-proofness* and *non-bossiness* on the strict domain. As the TTC-mechanism satisfies these three properties, we immediately obtain that the TTC-mechanism satisfies *respecting improvement* (which is also shown by Biro et al. (2023)). Now one might wonder whether replacing *strategy-proofness* with *respecting improvement* in Ma (1994)'s characterization of the TTC-mechanism (by *individual rationality*, *strategy-proofness* and *efficiency*) continues to be true. We show that this is not the case even when we add *non-bossiness*, i.e. there are rules other than the TTC-mechanism satisfying *individual rationality*, *efficiency*, *respecting improvement* and *non-bossiness* on the strict domain. Hence, in contrast to two-sided markets and stable mechanisms, *strategy-proofness* and *respecting improvement* are not equivalent under *individual rationality*, *efficiency* and *non-bossiness*.

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[1] Kominers (2020) applied this result to obtain alternative direct proofs comparative statics of the student-proposing DA-mechanism (like for instance, Crawford (1991)).

For one-sided markets with weak preferences, one needs to obtain strict preferences via tie-breaking in order to apply the TTC-mechanism (as it is defined only for strict preferences). One way is to break ties in a fixed manner, i.e. to use the same tie-breaker for any reported preference profile. Now from our first main result it follows directly that the TTC-mechanism with fixed tie-breaking satisfies *respecting improvement*. This is the first deterministic mechanism (i.e. selecting always a unique assignment) satisfying *respecting improvement* on the weak domain which is in contrast to the multi-valued solutions studied in Biro et al. (2023). Instead of breaking ties in a fixed manner, in applications ties might be broken randomly and random mechanisms emerge. Our second main result considers three versions of tie-breaking: (i) uniformly (or multiple-tie-breaking (MTB)), (ii) identically (or single-tie-breaking (STB)) or (iii) locally (or local tie-breaking (LTB)). We show that on the weak domain the TTC-mechanism with any of these versions of tie-breaking satisfies *respecting improvement* (where (iii) was shown by Biro et al. (2023)).

Finally, for the weak domain we consider mechanisms satisfying *individual rationality*, *efficiency* and *strategy-proofness*. Again by Ma (1994) these properties characterize the TTC-mechanism on the strict domain (and they imply *respecting improvement*). On the weak domain, we show that this is no longer the case: on the one hand *individual rationality*, *efficiency* and *strategy-proofness* do not imply *respecting improvement*; and on the other hand there exist rules satisfying these four properties. For these results we show most importantly top cycles rules with fixed tie-breaking from Jaramillo and Manjunath (2012) to satisfy *respecting improvement*.

The paper is organized as follows. Section 2 introduces housing markets, basic properties and the TTC-mechanism. Section 3 defines *respecting improvement*, states our first main result (where *respecting improvement* is implied by three properties) and shows that Ma (1994)'s characterization does not hold when we replace *strategy-proofness* with *respecting improvement* and add *non-bossiness*. Section 4 applies our first main result to top-trading with fixed tie-breaking and random tie-breaking on the weak domain. Section 5 considers rules satisfying the properties in Ma's characterization and *respecting improvement* on the weak domain. Section 6 discusses our results in the contexts of generalized matching problems, roommate problems and school choice.

2 Housing markets

Let $N = \{1, \ldots, n\}$ denote the finite set of potential agents. Agent i's object (or endowment or house) is denoted by i. Given $i \in N$, agent i is equipped with a (weak) preference relation R_i on N. Any such relation is reflexive, complete and transitive (but not necessarily strict). Let P_i denote the strict relation associated with R_i and I_i the indifference relation associated with R_i. We assume that agent i is never indifferent between his endowment i and any other object, i.e. for all $j \in N \backslash \{i\}$, we have jP_ii or iP_ij.[2] This means agent i is able to distinguish his endowment from

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[2] Indifferences with the endowment are often ruled out, see for example Sönmez (1999) and Klaus, Klijn and Walzl (2010).

any other object. Under this restriction any agent strictly prefers to keep his own object to any object with the same characteristics (because i likes to avoid moving). In Section 6 we describe how to adjust our main results when indifferences with the endowment are allowed.

Let $\mathcal{R}_i^N$ denote the set of all such preference relations of agent i on N. We use the convention to write $R_i : jk \ldots$ if jP_ikP_il for all $l \in N\backslash\{j,k\}$ and $R_i : [jk] \ldots$ if jI_ikP_il for all $l \in N\backslash\{j,k\}$. Given $S \subseteq N$, let $R_i|_S$ denote the restriction of R_i to S. A relation $R_i \in \mathcal{R}_i^N$ is strict when for all $j,k \in N$ such that $j \neq k$, we have either jP_ik or kP_ij. Let $\mathcal{P}_i^N$ denote the set of all strict preference relations of agent i on N. Given $R_i \in \mathcal{P}_i^N$, let $top(R_i)$ denote the most preferred object under R_i. Let $\mathcal{R}^N = \times_{i \in N}\mathcal{R}_i^N$ denote the set of all weak preference profiles, respectively, the weak domain, and let $\mathcal{P}^N = \times_{i \in N}\mathcal{P}_i^N$ denote the set of all strict preference profiles, respectively, the strict domain. Given $R \in \mathcal{R}^N$ and $i \in N$, let $R_{-i} = (R_j)_{j \in N\backslash\{i\}}$ denote the profile R without i's preference R_i.

Since agent i's endowment is denoted by i, an economy is for short a profile $R \in \mathcal{R}^N$. An allocation for N is a mapping $\mu : N \to N$ such that for all $i,j \in N$ with $i \neq j$, $\mu(i) \neq \mu(j)$. Under any allocation each agent is receiving some object and no two agents receive the same object. Here $\mu(i)$ denotes the object received by agent i. Let $\mathcal{A}_N$ denote the set of all allocations for N. A(n allocation) rule (or mechanism) is a mapping

$$\varphi : \mathcal{R}^N \longrightarrow \mathcal{A}_N.$$

Given $R \in \mathcal{R}^N$, let $\varphi(R)$ denote the allocation chosen by φ for profile R and $\varphi_i(R)$ denote the object assigned to i by φ for R. We will be interested in the following properties.

Individual rationality means that each individual should always weakly prefer the assigned object to his endowment. If a rule is not *individually rational*, then agents are not necessarily willing to reallocate their endowments.

Individual Rationality: For all $R \in \mathcal{R}^N$, we have $\varphi_i(R)R_ii$ for all $i \in N$.

Strategy-proofness means that no individual can manipulate the rule to his advantage by misreporting his preference. This incentive-compatibility condition ensures that agents report truthfully and the allocation chosen by the rule is based on true preferences.

Strategy-Proofness: For all $R \in \mathcal{R}^N$, all $i \in N$, and all $R_i' \in \mathcal{R}_i^N$, we have $\varphi_i(R)R_i\varphi_i(R_i', R_{-i})$.

Weak efficiency means that it is impossible to make all agents strictly better off through some other allocation. Note that this requirement is very weak.

Weak Efficiency: For all $R \in \mathcal{R}^N$, there exists no $\mu \in \mathcal{A}_N$ such that $\mu(i)P_i\varphi_i(R)$ for all $i \in N$.

(Strong) Efficiency means that it is impossible to make some agent strictly better off while all agents are weakly better off through some other allocation.

Efficiency: For all $R \in \mathcal{R}^N$, there exists no $\mu \in \mathcal{A}_N$ such that $\mu(j) P_j \varphi_j(R)$ for some $j \in N$ and $\mu(i) R_i \varphi_i(R)$ for all $i \in N$.

Non-bossiness means that no individual can change the allocation without changing his assigned object. This property prevents collusion among agents because if a rule violates this condition, then an agent may be bribed by others in order to change the allocation without changing his assigned object.

Non-Bossiness: For all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}_i^N$, if $\varphi_i(R) = \varphi_i(R'_i, R_{-i})$, then $\varphi(R) = \varphi(R'_i, R_{-i})$.[3]

Even though all properties above are described for the weak domain, sometimes we only consider them for rules defined on the strict domain. Below we define Gale's top-trading-cycles algorithm for this domain. Informally, the algorithm works as follows: any profile, each agent points to his most preferred object (or agent). Because N is finite, there must be at least one (top) cycle and for each top cycle agents trade their endowments (as specified by the cycle). The objects of these trading cycles are deleted from the preferences of the remaining agents and we apply the same procedure to the remaining agents and their preferences restricted to the remaining objects (or agents), and so on.

Top-trading-cycles (TTC-)rule

Let $R \in \mathcal{P}^N$ and set $N_0 = N$.

Step 1. Each $i \in N$ points to his most preferred object $top(R_i)$ in N_0. Then there exists at least one cycle $i_1 - i_2 - \cdots - i_t$ (where $top(R_{i_l}) = i_{l+1}$ for $l \in \{1, \ldots, t-1\}$ and $top(R_{i_t}) = i_1$) and for any such cycle we set $f_{i_l}(R) = top(R_{i_l})$ for all $l \in \{1, \ldots, t\}$. Let C_1 denote the set of agents assigned in Step 1 and $N_1 = N_0 \backslash C_1$.

Step k+1. Each $i \in N_k$ points to his most preferred object $top(R_i|_{N_k})$ in N_k. Then there exists at least one cycle $i_1 - i_2 - \cdots - i_t$ (where $top(R_{i_l}|_{N_k}) = i_{l+1}$ for $l \in \{1, \ldots, t-1\}$ and $top(R_{i_t}|_{N_k}) = i_1$)

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[3] Note that under this condition, if an agent changes his report from R_i to R'_i and is assigned the same object, then independently of his true preference relation agent i is indifferent between the two reports R_i and R'_i (given R_{-i}) and all other agents are indifferent independently of their true preferences if the same allocation is chosen under R and (R'_i, R_{-i}).

and for any such cycle we set $f_{i_l}(R) = top(R_{i_l}|_{N_k})$ for all $l \in \{1, \ldots, t\}$. Let C_{k+1} denote the set of agents assigned in Step $k+1$ and $N_{k+1} = N_k \backslash C_{k+1}$.

Stop. $N_k = \emptyset$.

For any strict R, let $f(R)$ denote the outcome of Gale's top trading cycles algorithm,[4] and f denotes the top-trading-cycles (TTC-)rule. Roth (1982) showed *strategy-proofness* and from Bird (1984) and Pápai (2000) follows *non-bossiness* of the TTC-rule.

Ma (1994) showed that on the strict domain the TTC-rule is the unique one satisfying *individual rationality*, *efficiency* and *strategy-proofness*.[5]

3 Respecting improvement on the strict domain

The main property requires that for any profile, for a fixed agent, if the endowment of the fixed agent improves in the ranking of some other agent (while keeping unchanged the preference over other goods and other agents' preferences), then the fixed agent is weakly better off after the improvement of his endowment. This prevents agents from trying to lower their endowment in other agents' preferences.

Definition 1. *Let* $i \in N$ *and* $R_i, R'_i \in \mathcal{R}_i^N$. *Then*

(I) R'_i *is a (local) k-improvement of* R_i *if (i)* $k \in N\backslash\{i\}$, *(ii)* $R_i|_{N\backslash\{k\}} = R'_i|_{N\backslash\{k\}}$, *and (iii) for some* $j \in N\backslash\{k\}$ *we have either* ($[jP_ik$ *and* $kR'_ij]$ *or* $[kI_ij$ *and* $kP'_ij]$) *and for all* $l \in N\backslash\{j,k\}$, $[lP_ik$ *implies* $lR'_ik]$ *and* $[lP_ij$ *implies* $lP'_ik]$; *and*

(II) R'_i *is a global k-improvement of* R_i *if (i)* $k \in N\backslash\{i\}$, *(ii)* $R_i|_{N\backslash\{k\}} = R'_i|_{N\backslash\{k\}}$, *and (iii) for some* $j \in N\backslash\{k\}$ *we have either* ($[jP_ik$ *and* $kR'_ij]$ *or* $[kI_ij$ *and* $kP'_ij]$).

In the first part of Definition 1 we consider minimal improvements of an object k in an agent's ranking. For instance, if $R_i : 6[12]k\ldots$, then $R'_i : 6[12k]\ldots$ is the only (local) k-improvement of R_i whereas $R''_i : 6k[12]\ldots$ is a global k-improvement of R_i but not a (local) k-improvement of R_i. Furthermore, it still allows for pairwise reversals of adjacent alternatives in strict preferences.

Respecting Improvement: For all $i, k \in N$, all $R \in \mathcal{R}^N$ and all k-improvements R'_i of R_i, we have $\varphi_k(R'_i, R_{-i})R_k\varphi_k(R)$.

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[4] Note that the outcome is independent of the choice of cycles.

[5] Svensson (1999) provides an elegant and short proof of Ma (1994)'s characterization. Ekici (2023) replaces *efficiency* by pair-efficiency (whereby no two agents should benefit by swapping their assigned object) in this characterization whereas Lemma 1 in Ehlers (2014) replaces *efficiency* with *weak efficiency* and consistency. Miyagawa (2002) showed that *individual rationality*, *strategy-proofness*, *non-bossiness* and anonymity characterize the TTC-rule and the no-trade rule whereby all agents always keep their endowments.

$$\begin{tikzcd}
\begin{array}{c|c} R_i & R_k \\ \hline \vdots & \vdots \\ \underline{a} & \underline{c} \\ k & \vdots \\ \vdots & d \end{array} \arrow[r] \arrow[d] & \begin{array}{c|c} R_i & R'_k \\ \hline \vdots & \vdots \\ \underline{a} & \underline{c} \\ k & k \\ \vdots & \vdots \end{array} \arrow[d] \\
\begin{array}{c|c} R'_i & R_k \\ \hline \vdots & \vdots \\ \underline{k} & c \\ a & \vdots \\ \vdots & \underline{d} \end{array} \arrow[r] & \begin{array}{c|c} R'_i & R'_k \\ \hline \vdots & \vdots \\ k & c \\ \underline{a} & \underline{k} \\ \vdots & \vdots \end{array}
\end{tikzcd}$$

Table 1: Proof of Theorem 1: underlined houses are allocated to those agents at the corresponding profiles (where $R_{-i,k}$ remains fixed throughout).

Global Respecting Improvement: For all $i, k \in N$, all $R \in \mathcal{R}^N$ and all global k-improvements R'_i of R_i, we have $\varphi_k(R'_i, R_{-i}) R_k \varphi_k(R)$.

It is clear that the global version of *respecting improvement* is obtained from successive application of the above local version. Furthermore, we may require *respecting improvement* just on the strict domain.

Our first theorem shows that on the strict domain *respecting improvement* is implied by three basic properties.

Theorem 1. *On the strict domain, if a rule satisfies individual rationality, strategy-proofness and non-bossiness, then it satisfies respecting improvement.*

Proof. Let $\varphi : \mathcal{P}^N \to \mathcal{A}_N$ be a rule satisfying *individual rationality*, *strategy-proofness* and *non-bossiness*. Suppose φ violates *respecting improvement*. Then there exist $i, k \in N$, $R \in \mathcal{P}^N$ and a k-improvement R'_i of R_i such that $c = \varphi_k(R) P_k \varphi_k(R'_i, R_{-i}) = d$. Obviously, $c \neq d$.

If $\varphi_i(R) \neq k \neq \varphi_i(R'_i, R_{-i})$, then by $R_i|_{N\setminus\{k\}} = R'_i|_{N\setminus\{k\}}$ and *strategy-proofness*, $\varphi_i(R) = \varphi_i(R'_i, R_{-i})$ which together with $c \neq d$ yields a contradiction to *non-bossiness*. The same argument yields a contradiction if $\varphi_i(R) = k = \varphi_i(R'_i, R_{-i})$. If $\varphi_i(R) = k \neq \varphi_i(R'_i, R_{-i})$, then by *strategy-proofness* and R_i being strict, $k P_i \varphi_i(R'_i, R_{-i})$, which is then a contradiction to *strategy-proofness* as R'_i is a k-improvement of R_i and thus, $\varphi_i(R) = k P'_i \varphi_i(R'_i, R_{-i})$. Hence, $\varphi_i(R) = a \neq k = \varphi_i(R'_i, R_{-i})$ and by *strategy-proofness* we have $R'_i = R_i^{a \leftrightarrow k}$,[6] i.e. R'_i is obtained by the pairwise switch of a and k (and a plays the role of j in (iii) of Definition 1).

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[6] If $R_i : 1 \dots n$, then $R_i^{m \leftrightarrow m+q} : 1 \dots m-1 m+q m+1 \dots m+q-1 m m+q+1 \dots n$.

We illustrate in Table 1 the remainder of the proof. Let $R'_k \in \mathcal{P}_k^N$ be such that $R'_k|_{N\setminus\{k\}} = R_k|_{N\setminus\{k\}}$ and $\{l \in N : lP'_k k\} = \{l \in N : lR_k c\}$. Now by $cP_k d$, *strategy-proofness* and *individual rationality* we have $\varphi_k(R'_k, R_{-k}) = c$ and $\varphi_k(R'_i, R'_k, R_{-i,k}) = k$. Furthermore, by *non-bossiness* and $\varphi_k(R) = c$ we have $\varphi(R) = \varphi(R'_k, R_{-k})$ and $\varphi_i(R) = \varphi_i(R'_k, R_{-k}) = a \neq k$. But then from $\varphi_k(R'_i, R'_k, R_{-i,k}) = k$ we have $\varphi_i(R'_i, R'_k, R_{-i,k}) \neq k$. As $R_i|_{N\setminus\{k\}} = R'_i|_{N\setminus\{k\}}$ and $\varphi_i(R'_k, R_{-k}) \neq k \neq \varphi_i(R'_i, R'_k, R_{-i,k})$, *strategy-proofness* implies $\varphi_i(R'_k, R_{-k}) = \varphi_i(R'_i, R'_k, R_{-i,k})$. Now this is a contradiction to *non-bossiness* as $\varphi_k(R'_k, R_{-k}) = c \neq k = \varphi_k(R'_i, R'_k, R_{-i,k})$. □

There is a rich class of rules satisfying *individual rationality*, *strategy-proofness*, and *non-bossiness* on the strict domain. For instance, fix an allocation $\mu \in A_N$ and for all $R \in \mathcal{P}^N$, (i) μ is *individually rational* under R, then $\varphi^\mu(R) = \mu$, and (ii) otherwise $\varphi_i^\mu(R) = i$ for all $i \in N$. Then φ^μ satisfies *individual rationality*, *strategy-proofness*, and *non-bossiness* (and *respecting improvement* by Theorem 1).

Another type of rule is the following, denoted by ϕ^1 where agent 1 trades in a pairwise exchange his house for his most preferred house subject to *individual rationality*. Formally, for all $R \in \mathcal{P}^N$, let $N_1(R) = \{i \in N : 1P_i i\}$, and (i) if $N_1(R) \neq \emptyset$, then set $\phi_1^1(R) = k$ for $k \in N_1(R)$ such that $kR_1 h$ for all $h \in N_1(R)$, $\phi_k^1(R) = 1$, and $\phi_i^1(R) = i$ for all $i \in N\setminus\{1, k\}$, and (ii) otherwise $N_1(R) = \emptyset$ and set $\phi_i^1(R) = i$ for all $i \in N$. Then ϕ^1 satisfies *individual rationality*, *strategy-proofness*, *non-bossiness* and *weak efficiency* (and *respecting improvement* by Theorem 1).

Of course, by Ma (1994)'s characterization, on the strict domain there is only one rule which satisfies *individual rationality*, *strategy-proofness*, *non-bossiness* and *efficiency*, namely the TTC-rule. Note that the TTC-rule does not Pareto dominate ϕ^1, i.e. the TTC-rule does not Pareto dominate any rule satisfying *individual rationality*, *strategy-proofness* and *non-bossiness*.

Below we provide a rule satisfying *individual rationality*, *efficiency*, *non-bossiness* and *respecting improvement* on the strict domain. This implies that we cannot replace *strategy-proofness* with *respecting improvement* in Ma (1994)'s characterization of the TTC-rule on the strict domain.[7]

Example 1. *Let $N = \{1, 2, 3\}$. We define $g : \mathcal{P}^N \to \mathcal{A}_N$ as follows. For all $R \in \mathcal{P}^N$, (i) if $R_1 : 231$, $R_2 : 1\ldots$ and $R_3 : 2\ldots$, then $g(R) = (3, 1, 2)$ and (ii) otherwise $g(R) = f(R)$ (where f denotes the TTC-rule). Note that in (i) we have $g(R) = (3, 1, 2) \neq (2, 1, 3) = f(R)$, i.e. g is not the TTC-rule. It is immediate that g satisfies* individual rationality *and* efficiency*.*

We show that g satisfies non-bossiness*. Let $i \in N$, $R, R' \in \mathcal{P}^N$ be such that $R' = (R'_i, R_{-i})$ (where $R'_i \neq R_i$) and $g_i(R) = g_i(R')$. If either R and R' belong to (i) or R and R' belong to (ii), then* non-bossiness *is obvious. Thus, let R belong to (i) and R' belong to (ii). We consider three cases.*

If $i = 1$, then $g_1(R) = 3 = g_1(R')$, $3P'_1 1$ and by $R'_1 \neq R_1$, we must have $R'_1 : 3\ldots$. Thus, $f(R') = (3, 1, 2) = g(R') = g(R)$, the desired conclusion.

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[7] I thank Florian Brandl for early input on this.

If $i = 2$, then $g_2(R) = 1 = g_2(R')$, $1P'_2 2$ and as R' is not in (i), we must have $R'_2 : 3 \ldots$ But then $f(R') = (1,3,2) = g(R')$ implying $g_2(R) \neq g_2(R')$, a contradiction.

If $i = 3$, then $g_3(R) = 2 = g_3(R')$, $2P'_3 3$ and as R' is not in (i), we must have $R'_3 : 1 \ldots$ But then $f(R') = (2,1,3) = g(R')$ implying $g_3(R) \neq g_3(R')$, a contradiction.

We show that g satisfies respecting improvement*. Let $i, k \in N$ (with $i \neq k$) and $R, R' \in \mathcal{P}^N$ be such that $R' = (R'_i, R_{-i})$ where R'_i is a k-improvement of R_i. If either R and R' belong to (i) or R and R' belong to (ii), then* respecting improvement *is obvious. Thus, let R belong to (i) and R' belong to (ii) (and by the same arguments this also establishes* respecting improvement *when R' belong to (i) and R belongs to (ii)). We consider three cases.*

If $i = 1$, then $g_1(R) = 3$ and as R'_1 is a k-improvement of R_1, we must have $k = 3$ and $R'_1 : 3 \ldots$ Thus, $f(R') = (3,1,2) = g(R') = g(R)$, the desired conclusion.

If $i = 2$, then $g_2(R) = 1$ and as R'_2 is a k-improvement of R'_2 and R' is in (ii), we must have $k = 3$ and $R'_2 : 3 \ldots$ But then $f(R') = (1,3,2) = g(R')$ implying $g_3(R') R_3 g_3(R')$, the desired conclusion.

If $i = 3$, then $g_3(R) = 2$ and as R'_3 is a k-improvement of R_3 and R' is in (ii), we must have $k = 1$ and $R'_3 : 1 \ldots$ But then $f(R') = (2,1,3) = g(R')$ implying $g_1(R') \neq g_1(R)$, the desired conclusion.

Now by Example 1, under *individual rationality* and *efficiency*, *strategy-proofness* and *respecting improvement* are not equivalent as the above rule violates *strategy-proofness* but satisfies *respecting improvement*. This is in contrast to two-sided markets as Hirata, Kasuya and Okumara (2023) have shown this equivalence for stable rules in two-sided markets, and stability implies *individual rationality* and *efficiency*.[8]

Even though Theorem 1 holds only for the strict domain, it has numerous implications for the weak domain which we examine in detail next.

4 Top trading with fixed tie-breaking

For the strict domain, the most important rule is Gale's top trading cycles algorithm. For the weak domain, this rule is extended by breaking ties in a fixed way and then applying Gale's top trading cycles algorithm to the obtained strict profile, as done below.

Given $R_i, R'_i \in \mathcal{R}_i^N$, we say R'_i is a strict transformation of R_i if (i) R'_i is strict and (ii) for all $j, k \in N$, we have $jP_ik \Rightarrow jP'_ik$. Given $R, R' \in \mathcal{R}^N$, we say R' is a strict transformation of R if for all $i \in N$, R'_i is a strict transformation of R_i. Let $ST(R)$ denote the set of all strict transformations of R. Of course, for strict R we have $ST(R) = \{R\}$.

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[8] They also need a weak independence condition which can be ignored when agents are assigned exactly one object (with fixed contractual terms).

Note that in $ST(R)$ ties are broken arbitrarily. We will be interested in the case where ties are broken in a fixed manner. Let $\succ_i \in \mathcal{P}_i^N$ denote a fixed tie-breaker. The choice of the tie-breaker is arbitrary but it will be used independently of the preference profile to break ties in any weak preference relation of i in order to obtain a strict preference relation. Given $R_i \in \mathcal{R}_i^N$, let $\succ_i (R_i)$ denote the strict transformation $R'_i \in \mathcal{P}_i^N$ of R_i such that for all $j, k \in N$ with $j \neq k$, if jI_ik and $j \succ_i k$, then jP'_ik.

Let $\succ = (\succ_i)_{i \in N}$. For any $R \in \mathcal{R}^N$, let $\succ (R) = (\succ_i (R_i))_{i \in N}$. Obviously, $\succ (R) \in ST(R)$ and if $R \in \mathcal{P}^N$, then $\succ (R) = R$.

Given a profile of fixed tie-breakers $\succ$ and all $R \in \mathcal{R}^N$, let $f^{\succ}(R)$ denote Gale's top trading cycles algorithm with fixed tie-breaking $\succ$:

$$f^{\succ}(R) = f(\succ (R)).$$

In other words, in any weak preference profile ties are broken according to $\succ$ and then Gale's top trading cycles algorithm is applied to the resulting strict profile. Top trading (rule) with fixed tie-breaking $f^{\succ}$ satisfies *weak efficiency* but violates *efficiency* (see for instance, Ehlers (2014)).

The following is an immediate consequence of Theorem 1 (where (i) is Theorem 1 of Biro et al. (2023)).

Corollary 1. *(i) On the strict domain, the TTC-rule satisfies respecting improvement.*

(ii) On the weak domain, top-trading with fixed tie-breaking satisfies respecting improvement.

Proof. Since on the strict domain the TTC-rule satisfies *individual rationality*, *strategy-proofness* and *non-bossiness*, we obtain (i) from Theorem 1.

In showing (ii), let $\succ$ be a profile of tie-breakers. Note that $f^{\succ}$ satisfies *individual rationality*, *strategy-proofness* and *non-bossiness* (see for instance, Theorem 1 in Ehlers (2014)). Suppose that $f^{\succ}$ violates *respecting improvement*. Then there exist $i, k \in N$, $R \in \mathcal{R}^N$ and a k-improvement R'_i of R_i such that $f^{\succ}_k(R)P_kf^{\succ}_k(R'_i, R_{-i})$. As tie-breaking respects strict preferences we obtain

$$f^{\succ}_k(R) = f_k(\succ(R)) \succ_k(R_k) f_k(\succ(R'_i, R_{-i})) = f^{\succ}_k(R'_i, R_{-i}),$$

where the preference is strict. Note that $\succ(R'_i)$ is obtained from $\succ(R_i)$ by (possibly a sequence of) k-improvements of R_i. But then the TTC-rule f violates the global version of *respecting improvement*, which is a contradiction to (i) as the local version of *respecting improvement* implies the global version of *respecting improvement*. □

Part (ii) of Corollary 1 establishes the first full class of (deterministic) rules satisfying *respecting improvement* on the weak domain.

Instead of fixed tie-breaking, we consider three versions of random tie-breaking (which then induce random assignments and random assignment rules). One is the global version of multiple tie-breaking (MTB) by attaching equal probability to each profile of tie-breakers $\succ$: for any profile $R \in \mathcal{R}^N$ we set

$$MTB(R) = \frac{1}{(n!)^n} \sum_{\succ \in \mathcal{P}^N} f^{\succ}(R).$$

Note that here we implicitly add probabilities for any allocation which is the output of several tie-breakers. Another one is the global version of single tie-breaking (STB) by attaching equal probability to each profile of tie-breakers $\succ$ where ties are broken identically across agents: for any profile $R \in \mathcal{R}^N$ we set

$$STB(R) = \frac{1}{n!} \sum_{\succ \in \mathcal{P}^N : \succ_1 = \cdots = \succ_n} f^{\succ}(R).$$

Another one is the local version of single tie-breaking (STB) by attaching equal probability to each profile of tie-breakers $\succ$ where ties are broken randomly across agents: for any profile $R \in \mathcal{R}^N$, for R_i, let $R_i : [H_1][H_2] \cdots [H_m]$ denote the indifference classes of R_i. Then there are

$$\Pi_{l=1}^{m} \frac{1}{|H_l|!} \equiv |R_i|$$

ways to break the ties in R_i. We also write $P_i \subseteq \succ_i$ if for all $j, l \in N$, jP_il implies $j \succ_i l$. For profile R, let $|R| = \Pi_{i=1}^{n} |R_i|$. We also write $P \subseteq \succ$ if $P_i \subseteq \succ_i$ for all $i \in N$. Then

$$LTB(R) = \frac{1}{|R|} \sum_{\succ \in \mathcal{P}^N : P \subseteq \succ} f^{\succ}(R).$$

We show that all three versions of tie-breaking satisfy *respecting improvement* (where the local one is shown in Theorem 2 of Biro et al. (2023)). Here we use R_i^{sd} to denote first-order stochastic dominance among random assignments.[9]

Theorem 2. *Let $R, R' \in \mathcal{R}^N$ be such that R' is a k-improvement of R.*

(i) $MTB_k(R') R_k^{sd} MTB_k(R)$.

(ii) $STB_k(R') R_k^{sd} STB_k(R)$.

(iii) $LTB_k(R') R_k^{sd} LTB_k(R)$.

Proof. Note that (i) and (ii) follow immediately as for any profile of tie-breakers $\succ \in \mathcal{P}^N$ from (ii) of Corollary 1 we have $f_k^{\succ}(R') R_k f_k^{\succ}(R)$.

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[9] A random assignment $\tilde{\mu}$ is a probability distribution over $\mathcal{A}_N$. Then $\Pr\{\tilde{\mu} = \mu\}$ denotes the probability of μ under $\tilde{\mu}$. Given $R_i \in \mathcal{R}_i^N$ and two random assignments $\tilde{\mu}$ and $\tilde{\nu}$, we have $\tilde{\mu}(i) R_i^{sd} \tilde{\nu}(i)$ if for all $k \in N$ we have $\sum_{\mu \in \mathcal{A}_N : \mu(i) R_i k} \Pr\{\tilde{\mu} = \mu\} \geq \sum_{\mu \in \mathcal{A}_N : \mu(i) R_i k} \Pr\{\tilde{\nu} = \mu\}$.

In showing (iii), it suffices to consider minimal k-improvement R'_i of R_i and $R' = (R'_i, R_{-i})$.

In the first case, suppose jI_ik for some $j \neq k$. Let $H_j = \{l \in N : lI_ij\}$. Then $k \in H_j$, $H'_j = H_j \backslash \{k\}$ and for all $l \in N \backslash H_j$, lP'_ik or kP'_il. Suppose that R_{-i} is strict and $R_i|_{N \backslash H_j}$ is strict, i.e. R_i only contains the non-singleton indifference class H_j. Now for any tie-breaking $\succ$ we have $f^{\succ}_k(R')R_kf^{\succ}_k(R)$. Note that any tie-breaking of H_j in R_i corresponds to $|H_j|$ times breaking the ties of $H_j \backslash \{k\}$ in R'_i identically (and vice versa, i.e. any tie-breaking of $H_j \backslash \{k\}$ corresponds to $|H_j|$ breaking the ties in H_j). This observation implies $LTB_k(R')R^{sd}_kLTB_k(R)$. Now a similar observation implies the desired conclusion when R_i contains more ties or/and R_{-i} contains ties.

In the second case, suppose that for all $l \neq k$, lP_ik or kP_il, and jI'_ik for some $j \neq k$. Let $H'_j = \{l \in N : lI'_ij\}$. Then $k \in H'_j$ and $H_j = H'_j \backslash \{k\}$. Suppose that R_{-i} is strict and $R_i|_{N \backslash H_j}$ is strict, i.e. R_i only contains the non-singleton indifference class H_j. Then the same arguments as above yield the desired conclusion. □

We relate Theorem 1 to the core and to the set of competitive allocations in a housing market. Given $R \in \mathcal{R}^N$, $T \subseteq N$, and $\mu \in \mathcal{A}_N$, if for some $\bar{\mu} \in \mathcal{A}_N$, (i) $\bar{\mu}(T) = T$, (ii) for all $i \in T$, $\bar{\mu}(i)R_i\mu(i)$, and (iii) for some $j \in T$, $\bar{\mu}(j)P_j\mu(j)$, then we say that *coalition T blocks μ under R*. The (strong) core of R, denoted by $C(R)$, consists of all allocations which are not blocked by any coalition.

Given a preference profile $R \in \mathcal{R}^N$, an allocation is called *competitive* when there exist prices for all $i \in N$ such that each agent maximizes his preference on the set of affordable objects and the price of his assignment at μ equals the price of his initial endowment. We consider the solution that chooses for each preference profile its set of competitive allocations, called the *competitive solution*. Formally, given $R \in \mathcal{R}^N$, $\mu \in \mathcal{A}_N$ is a competitive allocation if there exists a price vector $(p(i))_{i \in N} \in \mathbb{R}^N$ such that for all $i \in N$, (i) $p(\mu(i)) = p(i)$; and (ii) for all $j \in N$, if $p(j) \leq p(i)$, then $\mu(i)R_ij$. Now a rule is *competitive* if it chooses for any profile a competitive allocation.

When preferences are strict, the core contains exactly one allocation, which is the unique competitive allocation (Roth and Postlewaite, 1977). When preferences are weak, the core is always a subset of the competitive solution (Wako, 1984). Moreover, for each profile, the set of competitive allocations is obtained by computing the cores of all its strict transformations (Shapley and Scarf, 1974).[10]

Hence, top trading with fixed tie-breaking chooses for each profile a competitive allocation. This supports the market-based approach even when preferences are weak, also because competitive allocations always exist whereas the core may be empty. Let *comp*(R) denote the set of all competitive allocations for profile R. It is known that

$$comp(R) = \cup_{\succ \in \mathcal{P}^N} \{f^{\succ}(R)\}.$$

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[10] Ehlers (2004) shows that the minimal monotonic extension of the core is the correspondence choosing for each profile its set of competitive allocations. Klaus, Klijn and Walzl (2010) established on the domain $\mathcal{R}^N$ that the set of competitive allocations is the unique von Neumann-Morgenstern farsightedly stable set based on weak dominance.

We obtain Corollary 1 in Biro et al. (2023) as an immediate consequence from (ii) of Corollary 1 as we explain below.

Corollary 2 (Biro et al. (2023, Corollary 1))**.** *On the weak domain we have for all* $i, k \in N$*, all* $R \in \mathcal{R}^N$ *and all* k*-improvements* R'_i *of* R_i*,*

(i) There exists $\mu' \in comp(R'_i, R_{-i})$ *such that* $\mu'(k) R_k \mu(k)$ *for all* $\mu \in comp(R)$*.*

(ii) There exists $\mu \in comp(R)$ *such that* $\mu'(k) R_k \mu(k)$ *for all* $\mu' \in comp(R'_i, R_{-i})$*.*

Proof. In showing (i), let $\bar{\mu} \in comp(R)$ be such that $\bar{\mu}(k) R_k \mu(k)$ for all $\mu \in comp(R)$. Then there exists a profile $\succ$ of tie-breakers such that $f^{\succ}(R) = \bar{\mu}$. As $f^{\succ}$ satisfies *respecting improvement*, we have $f^{\succ}_k(R'_i, R_{-i}) R_k f^{\succ}_k(R)$. By setting $\mu' = f^{\succ}(R'_i, R_{-i})$ we obtain the desired conclusion.

In showing (ii), let $\bar{\mu}' \in comp(R'_i, R_{-i})$ be such that $\mu'(k) R_k \bar{\mu}'(k)$ for all $\mu' \in comp(R'_i, R_{-i})$. Then there exists a profile $\succ$ of tie-breakers such that $f^{\succ}(R'_i, R_{-i}) = \bar{\mu}'$. As $f^{\succ}$ satisfies *respecting improvement*, we have $f^{\succ}_k(R'_i, R_{-i}) R_k f^{\succ}_k(R)$. By setting $\mu = f^{\succ}(R)$ we obtain the desired conclusion. □

5 Respecting improvement and efficiency on the weak domain

On the one hand, on the strict domain the TTC-rule is characterized by *individual rationality*, *efficiency*, *strategy-proofness*, *non-bossiness* and *respecting improvement*, and on the other hand, on the weak domain top-trading with fixed tie-breaking satisfies *individual rationality*, *weak efficiency*, *strategy-proofness*, *non-bossiness*[11] and *respecting improvement* but violates *efficiency*. In achieving *efficiency*, we need to search for rules other than top-trading with fixed tie-breaking. It is known that on the weak domain *individual rationality*, *efficiency*, *strategy-proofness* and *non-bossiness* are incompatible (Jaramillo and Manjunath, 2012, Proposition 2). Hence, in achieving *individual rationality*, *efficiency* and *strategy-proofness*, we need to drop *non-bossiness*.

For the weak domain, we show the following: on the one hand, the three basic properties in Ma's characterization do not imply *respecting improvement* (which is in contrast to the strict domain); and on the other hand, there exists a full class of rules satisfying the three basic properties in Ma's characterization and *respecting improvement*. In showing the latter, we show that the class of top cycles rules with fixed tie-breaking from Jaramillo and Manjunath (2012) satisfies *respecting improvement* (and we refer to their paper for the detailed description of those rules). Of course, by the discussion above these rules must violate *non-bossiness*.[12]

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[11] Theorem 1 in Ehlers (2014) characterizes top-trading with fixed tie-breaking in terms of welfare with these properties and consistency.

[12] Note that there other ways to define *non-bossiness* on the weak domain as discussed by Bogomolnaia, Deb and Ehlers (2005). For instance, in the definition of *non-bossiness* one could require $\varphi_i(R) I_i \varphi_i(R'_i, R_{-i})$ implies $\varphi_l(R) I_l \varphi_l(R'_i, R_{-i})$ for all $l \in N$. It is easily seen that this notion is incompatible with *efficiency* as for $N = \{1, 2\}$, $R_1 : 12$ and $R_2 : 12$ either $(1, 2)$ or $(2, 1)$ is chosen, say $\varphi(R) = (1, 2)$. Then for $R'_1 : [12]$ we have by

Theorem 3. *(i) On the strict domain, individual rationality, efficiency and strategy-proofness imply respecting improvement.*

(ii) On the weak domain, individual rationality, efficiency and strategy-proofness do not imply respecting improvement.

(iii) On the weak domain, there exist rules satisfying individual rationality, efficiency, strategy-proofness and respecting improvement. In particular, top cycles rules with fixed tie-breaking from Jaramillo and Manjunath (2012) satisfy respecting improvement.

Proof. Note that by Ma (1994), the TTC-rule is characterized by *individual rationality*, *efficiency* and *strategy-proofness*. Since the TTC-rule satisfies *non-bossiness*, Theorem 1 yields (i).

Below we use the same notation as in Jaramillo and Manjunath (2012) and denote by $TC^{\succ}$ the top cycles rule with priority $\succ$ (where we drop the endowment vector as any agent i owns object i). By Propositions 2 and 5 in Jaramillo and Manjunath (2012), $TC^{\succ}$ satisfies *individual rationality*, *efficiency* and *strategy-proofness*.

In showing (ii), let $N = \{1, 2, 3, 4\}$, $\succ$: 1234 and $\succ'$: 4321. Let $g : \mathcal{R}^N \to \mathcal{A}_N$ be defined as follows (where we exclude indifferences with the endowment): for all $R \in \mathcal{R}^N$, if $R_1 : 12\ldots$, then $g(R) = TC^{\succ'}(R)$ and otherwise $g(R) = TC^{\succ}(R)$. As $TC^{\succ}$ and $TC^{\succ'}$ satisfy *individual rationality* and *efficiency*, g satisfies *individual rationality* and *efficiency*. Furthermore, g satisfies *strategy-proofness* as $g_1(R) = 1$ if $R_1 : 1\ldots$ and g satisfies *individual rationality* for agent 1 (and $TC^{\succ}$ and $TC^{\succ'}$ satisfy *strategy-proofness*).

However, g violates *respecting improvement*. Let $R_1 : 1234$, $R_2 : 42\ldots$, $R_3 : 43\ldots$ and $R_4 : [23]41$. Then $g(R) = TC^{\succ'}(R) = (1, 2, 4, 3)$ (as $3 \succ' 2$). For $R'_1 : 1324$ we have $g(R'_1, R_{-1}) = TC^{\succ}(R'_1, R_{-1}) = (1, 4, 3, 2)$ (as $2 \succ 3$). Hence, $g_2(R'_1, R_{-1}) P_2 g_2(R)$ and R_1 is a 2-improvement over R'_1 yielding a violation to *respecting improvement* (and at the same time this is a violation of *non-bossiness*).[13]

In showing (iii), without loss of generality, let $1 \succ 2 \succ \cdots \succ n$. Suppose that $TC^{\succ}$ violates *respecting improvement*. Then as $TC^{\succ}$ satisfies *individual rationality* and *strategy-proofness*, then there exist $i, k \in N$ (with $i \neq k$) and $R, R' = (R'_i, R_{-i}) \in \mathcal{R}^N$ such that R'_i is a k-improvement of R_i, $TC^{\succ}_k(R) \neq k = TC^{\succ}_k(R')$ and $R_k : TC^{\succ}_k(R)k\ldots$.[14] Note that it suffices to consider minimal

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efficiency, $\varphi(R'_1, R_2) = (2, 1)$, which is now a contradiction to this notion of *non-bossiness* as $\varphi_1(R) I'_1 \varphi_1(R'_1, R_2)$ but $\varphi_2(R'_1, R_2) P_2 \varphi_2(R)$.

[13] Note that g can be easily modified when $R_1 : 1\ldots$ by allowing for more "complicated choices" of the priority. This confirms that a characterization of the rules satisfying *individual rationality*, *efficiency* and *strategy-proofness* seems to be out of reach (see also Saban and Sethuraman (2013)). One could also simply set the priority $\succ$ equal to R_1 in those cases, it follows then from the same arguments below that this rule satisfies *respecting improvement*. Hence, again a characterization of the rules satisfying *individual rationality*, *efficiency*, *strategy-proofness* and *respecting improvement* seems to be out of reach.

[14] Let φ be a rule satisfying *individual rationality* and *strategy-proofness*. If φ violates *respecting improvement*, then there exist $i, k \in N$, a profile R and k-improvement R'_i of R_i such that $\varphi_k(R) P_k \varphi_k(R'_i, R_{-i})$. By *individual rationality* and *strategy-proofness*, we may suppose that $R_k \in \mathcal{P}^N_k$, $R_k : \varphi_k(R)k\ldots$ and $\varphi_k(R'_i, R_{-i}) = k$.

k-improvements of R_i where either k is ranked in R'_i just above all houses indifferent with k under R_i (and k is not indifferent with any other house under R'_i) or k is not indifferent with any other house under R_i and then indifferent under R'_i with the houses ranked just above k under R_i.

By $TC^{\succ}_k(R') = k \neq TC^{\succ}_i(R')$, we have the following six mutually exclusive cases.

1. If $TC^{\succ}_i(R)P'_ik$, then by *strategy-proofness*, both $TC^{\succ}_i(R)I_iTC^{\succ}_i(R')$ and $TC^{\succ}_i(R)I'_iTC^{\succ}_i(R')$ and in the execution of $TC^{\succ}(R)$ and $TC^{\succ}(R')$ agent i does not point to k which implies $TC^{\succ}(R) = TC^{\succ}(R')$ (and *respecting improvement* is satisfied).

2. If $kP_iTC^{\succ}_i(R)$, then by *strategy-proofness*, both $TC^{\succ}_i(R)I_iTC^{\succ}_i(R')$ and $TC^{\succ}_i(R)I'_iTC^{\succ}_i(R')$ and in the execution of $TC^{\succ}(R)$ and $TC^{\succ}(R')$ agent k departs before agent i and the pointing of the agents in $N\backslash\{i\}$ remains identical in $TC^{\succ}(R)$ and $TC^{\succ}(R')$ until agent k departs. This implies $TC^{\succ}_k(R) = TC^{\succ}_k(R')$ (and *respecting improvement* is satisfied).

3. If $k = TC^{\succ}_i(R)$ and lI_ik for some $l \neq k$, then R'_i is a local push-up of R_i at k and by Lemma 2 (Invariance) of Jaramillo and Manjunath (2012) we obtain $TC^{\succ}_i(R') = k$. Thus, by $R_k : TC^{\succ}_k(R)k \ldots$ and *individual rationality* we obtain $TC^{\succ}_k(R') = TC^{\succ}_k(R)$ (and *respecting improvement* is satisfied).

4. If $k = TC^{\succ}_i(R)$ and for all $l \neq k$ either lP_ik or kP_il, then by *individual rationality* we have kP_ii and for R'_i we have lI'_ik for some $l \neq k, i$. But then by *strategy-proofness* we have $TC^{\succ}_i(R') = k$ and by $R_k : TC^{\succ}_k(R)k \ldots$ and *individual rationality* we obtain $TC^{\succ}_k(R') = TC^{\succ}_k(R)$ (and *respecting improvement* is satisfied).

5. If $TC^{\succ}_i(R) \neq k \neq TC^{\succ}_i(R')$ and $kI_iTC^{\succ}_i(R)$, then for R'_i we have for all $l \neq k$ either lP'_ik of kP'_il. Note that $kP'_iTC^{\succ}_i(R')$ as R'_i is a k-improvement of R_i. By *strategy-proofness*, $TC^{\succ}_i(R') \neq k$ and as R'_i is a k-improvement of R_i, we have $TC^{\succ}_i(R)I_iTC^{\succ}_i(R')$ and $TC^{\succ}_i(R)I'_iTC^{\succ}_i(R')$. But then in the execution of $TC^{\succ}(R')$ agent i cannot depart before agent k (as agent k remains unsatisfied (by $TC^{\succ}_k(R') = k$) and i continues to point to agent k). Now when considering the execution of $TC^{\succ}(R)$, at some Stage t agent k must belong to a trading cycle $j_1 - \cdots - j_s$. As agent k is unsatisfied, the pointing of all agents other than i remains unchanged in this trading cycle under R'. If i does not belong to the trading cycle, then this trading cycle is also executed under R'. If i belongs to the trading cycle, then under R' agent i points to agent k (as k is unsatisfied and for all $l \neq k$ either lP'_ik or kP'_il) and a shorter trading cycle is executed. Thus, $TC^{\succ}_k(R') \neq k$ and $TC^{\succ}_k(R') = TC^{\succ}_k(R)$ (and *respecting improvement* is satisfied).

6. If $TC^{\succ}_i(R) \neq k \neq TC^{\succ}_i(R')$ and $kI'_iTC^{\succ}_i(R')$, then as R'_i is a k-improvement of R_i we obtain $TC^{\succ}_i(R')P_ik$. By *strategy-proofness*, $TC^{\succ}_i(R) \neq k$ and as R'_i is a k-improvement of R_i, we have $TC^{\succ}_i(R)I_iTC^{\succ}_i(R')$ and $TC^{\succ}_i(R)I'_iTC^{\succ}_i(R')$. Now when considering the execution of $TC^{\succ}(R)$, at some Stage t agent k must belong to a trading cycle $j_1 - \cdots - j_s$. As agent k is

unsatisfied, the pointing of all agents other than i remains unchanged in this trading cycle under R'. If i does not belong to the trading cycle, then this trading cycle is also executed under R'. If i belongs to the trading cycle, then under R' either i points to the same agent, say j_l (and both j_l is unsatisfied and $j_l \succ k$) or i points to agent k (if j_l is satisfied or both j_l is unsatisfied and $k \succ j_l$) and a shorter trading cycle is executed. In both case we have $TC_k^{\succ}(R') \neq k$ and $TC_k^{\succ}(R') = TC_k^{\succ}(R)$ (and *respecting improvement* is satisfied). ☐

Next we discuss replacing *respecting improvement* with *non-bossiness* in Theorem 3: now (i) continues to hold, i.e. on the strict domain, *individual rationality*, *efficiency* and *strategy-proofness* imply *non-bossiness* (as the TTC-rule is characterized by these three properties and satisfies *non-bossiness*); furthermore, the incompatibilty of *individual rationality*, *efficiency*, *strategy-proofness* and *non-bossiness* on the weak domain implies that (ii) continues to hold but not (iii) when *respecting improvement* is replaced with *non-bossiness*. Now one might wonder whether on the weak domain there exist rules satisfying *individual rationality*, *efficiency*, *respecting improvement* and *non-bossiness*, Of course, once we replace *efficiency* with *weak efficiency* we obtain a compatibility as top trading with fixed tie-breaking satisfies all properties (and is even characterized by Ehlers (2014, Theorem 1) when consistency is added).

We show the following in the Appendix.

Proposition 1. *For three agents and the weak domain,*

(i) on the one hand individual rationality, efficiency, non-bossiness and strategy-proofness are incompatible whereas

(ii) on the other hand individual rationality, efficiency, non-bossiness and respecting improvement are compatible.

Here for the last part we construct explicitly a three agent rule. We also show that any such rule cannot coincide with the TTC-rule on the strict domain, i.e. it is not possible to extend the TTC-rule from the strict domain to the weak domain while maintaining the properties in (ii) of Proposition 1. This again establishes for the weak domain the non-equivalence of *strategy-proofness* and *respecting improvement* under *individual rationality*, *efficiency* and *non-bossiness*.

Finally we show that Theorem 3 implies that for any profile and any local k-improvement, if the cores are non-empty under both profiles, then the core satisfies *respecting improvement* which is an extension of (i) of Corollary 1 from the strict domain to the weak domain (and this was shown in Theorem 3 by Biro et al. (2023)).

Given $R \in \mathcal{R}^N$ and $\mu, \nu \in \mathcal{A}_N$, μ strongly blocks ν if there exists $\emptyset \neq S \subseteq N$ such that $\mu(S) = S$ and $\mu(i) P_i \nu(i)$ for all $i \in S$. Then the weak core of R, denoted by $WC(R)$, consists of all allocations which are not strongly blocked.

Corollary 3 (Biro et al. (2023, Theorem 3))**.** *Let $R, R' \in \mathcal{R}^N$ be such that R' is a k-improvement over R. If $C(R) \neq \emptyset \neq C(R')$, then $\mu'(k) R_k \mu(k)$ for all $\mu \in C(R)$ and all $\mu' \in C(R')$.*

Proof. First, we show that $TC^{\succ}(R) \in C(R)$ whenever $C(R) \neq \emptyset$. Suppose not. Then there is a weak blocking coalition S and $\mu \in \mathcal{A}_N$ such that $\mu(S) = S$, $\mu(i) R_i TC_i^{\succ}(R)$ for all $i \in S$ and $\mu(j) P_j TC_j^{\succ}(R)$ for some $j \in S$. Without loss of generality, we may suppose that S consists of one trading cycle $j_1 - \cdots - j_s$. By Jaramillo and Manjunath (2012, Proposition 4), $TC^{\succ}(R) \in WC(R)$. Thus, there exists $l \in \{1, \ldots, s\}$ such that $TC_{j_l}^{\succ}(R) I_{j_l} j_{l+1}$ and $j_{l+2} P_{j_{l+1}} TC_{j_{l+1}}^{\succ}(R)$. Then j_l cannot depart before j_{l+1} as j_{l+1} is unsatisfied. Similarly all agents before j_{l+1} in the cycle cannot depart before. Furthermore, j_{l+1} does not depart before j_{l+2}, and so on. This implies that no agent in S departs before any other agent in S, which is a contradiction.[15]

Second, by (iii) of Theorem 3 we obtain $TC_k^{\succ}(R') R_k TC_k^{\succ}(R)$. By the above and $C(R) \neq \emptyset \neq C(R')$, we have $TC^{\succ}(R) \in C(R)$ and $TC^{\succ}(R') \in C(R')$. Now by Wako (1991) we have $\mu'(k) I'_k TC_k^{\succ}(R')$ for all $\mu' \in SC(R')$ and $\mu(k) I_k TC_k^{\succ}(R)$ for all $\mu \in C(R)$, which yields the desired conclusion. ☐

6 Discussion

6.1 Generalized matching problems

First, it is easily seen that our first main result remains true for generalized matching problems introduced by Sönmez (1996).

Recall that N denotes the (finite) set of agents. Given $i \in N$, let $S_i \subseteq N$ denote agent i's set of possible assignments. This set includes his endowment, i.e., $i \in S_i$. The preference relation of each $i \in N$ is a reflexive, complete, and transitive binary relation R_i over S_i. For convention, we use the notation $\bar{\mathcal{R}}_i$ to denote all such preference relations, and the weak domain by $\bar{\mathcal{R}}^N$ (and similar for the strict domain $\bar{\mathcal{P}}_i$ and $\bar{\mathcal{P}}^N$).

Now a *matching* is a bijection $\mu : N \to N$ such that each agent's assignment belongs to his set of possible assignments, i.e. for all $i \in N$, $\mu(i) \in S_i$. Let $\mathcal{M}$ denote the set of all matchings. Let μ^I denote the initial matching such that for all $i \in N$, $\mu^I(i) = i$. We specify a subset $\mathcal{M}^f$ of $\mathcal{M}$ as the set of *feasible matchings*. We always require that $\mu^I \in \mathcal{M}^f$ and for all $i \in N$, $S_i = \{\mu(i) : \mu \in \mathcal{M}^f\}$. Since N, $(S_i)_{i \in N}$, and $\mathcal{M}^f$ remain fixed, a *generalized matching problem* is simply a preference profile $R \in \bar{\mathcal{R}}^N$.

All the definitions from housing markets extend in a straightforward way to generalized matching problems. Furthermore, the proof of our first main result continues to remain true.

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[15] Alcalde-Unzu and Molis (2011, Theorem 4) have shown this for their class of rules satisfying *individual rationality*, *efficiency* and *strategy-proofness*. Alcalde-Unzu and Molis (2011, Theorem 1), Jaramillo and Manjunath (2012, Proposition 4) and Saban and Sethuraman (2013, Theorem 2) have shown that their rules always select weak core allocations. Ma (1994, Theorem 2) has shown this for the core and any correspondence where any selection from it satisfies *individual rationality*, *efficiency* and *strategy-proofness*.

Theorem 4 (Generalized matching problems). *On the strict domain, if a rule* $\varphi : \bar{\mathcal{P}}^N \to \mathcal{M}^f$ *satisfies individual rationality, strategy-proofness and non-bossiness, then* φ *satisfies respecting improvement.*

This means for generalized matching problems we obtain the same conclusions as above, i.e. on the strict domain any rule satisfying *individual rationality*, *strategy-proofness* and *non-bossiness* satisfies *respecting improvement*, and any such rule can be extended to the weak domain via fixed tie-breaking (while maintaining all these properties). Furthermore, if indifferences with the endowment are allowed, then we may break ties always in favor of the endowment, i.e. for any fixed tie-breaker $\succ_i$ we have $i \succ_i j$ for all $j \neq i$, and Corollary 1 and Theorem 2 carry over. When considering *efficiency*, we already know from Theorem 1 in Sönmez (1999) that except for housing markets the class of rules satisfying *individual rationality*, *efficiency* and *strategy-proofness* is often empty.

The elegance of generalized matching problems is that they cover the three most important matching problems for applications: (i) housing markets (Shapley and Scarf, 1974) by setting $S_i = N$ for all $i \in N$ and $\mathcal{M}^f = \mathcal{A}_N$; (ii) roommate markets (Gale and Shapley, 1962) by setting $S_i = N$ for all $i \in N$ and $\mu \in \mathcal{M}^f \Leftrightarrow [\mu \in \mathcal{A}_N$ and $\mu(\mu(i)) = i$ for all $i \in N]$; (iii) marriage markets (Gale and Shapley, 1962) by setting $N = M \cup W$ with $M \cap W = \emptyset$, $S_m = W \cup \{m\}$ for all $m \in M$, $S_w = M \cup \{w\}$ for all $w \in W$, and $\mu \in \mathcal{M}^f \Leftrightarrow [\mu \in \mathcal{A}_N, \mu(\mu(i)) = i$ for all $i \in N$, $\mu(m) \in W \cup \{m\}$ for all $m \in M$, and $\mu(w) \in M \cup \{w\}$ for all $w \in W$.

Housing markets were discussed in detail above. The other two applications are considered below.

6.2 Roommate problems

For roommate problems, any two agents need to be put into one (dormitory) room. Below we will not provide a detailed analysis of roommate problems as this is beyond the scope of the paper but highlight some important differences for three agents. More precisely, we show that *individual rationality*, *efficiency* and *strategy-proofness* are incompatible[16] but there exist rules satisfying *individual rationality*, *efficiency*, *non-bossiness* and *respecting improvement*.

Example 2. *Let* $N = \{1, 2, 3\}$ *and consider the strict domain* $\bar{\mathcal{P}}^N$ *and* $\mathcal{M}^f$ *as described above.*

On the one hand, let $\varphi : \bar{\mathcal{P}}^N \to \mathcal{M}^f$ *satisfy individual rationality, efficiency and strategy-proofness. Let* $R_1 : 231$, $R_2 : 312$ *and* $R_3 : 123$. *By efficiency,* $\varphi(R)$ *must pair two agents in one room, say* $\varphi(R) = (2, 1, 3)$. *Let* $R'_3 : 231$. *By strategy-proofness and efficiency,* $\varphi(R'_3, R_{-3}) =$

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[16] We do not claim that we are the first ones to show this for three agents. For four or more agents this follows from Theorem 1 and Corollary 5 in Sönmez (1999) but not for three agents as then the core is either empty or contains exactly one allocation (as otherwise it contains two allocations where two agents are put into one room, and the agent who is put into a room under both allocations has a strict prference over these two allocations, which means that one allocation blocks the other one).

$(2,1,3)$. *Let* $R_2' : 321$. *By individual rationality and efficiency,* $\varphi(R_1, R_2', R_3') = (1,3,2)$, *which is a contradiction to strategy-proofness as* $\varphi_2(R_1, R_2', R_3') = 3P_21 = \varphi(R_3', R_{-3}) = (2,1,3)$.

Hence, individual rationality, efficiency and strategy-proofness are incompatible for roommate problems with three agents.

On the other hand, we modify the rule ϕ^1 *defined after Theorem 1 for roommate problems to* $\bar{\phi}^1$ *as follows: for all* $R \in \bar{\mathcal{P}}^N$, *(i) if* $R_2 : 321$ *and* $R_3 : 231$, *then* $\bar{\phi}^1(R) = (1,3,2)$ *and (ii) otherwise* $\bar{\phi}^1(R) = \phi^1(R)$. *Then it is straightforward that* $\bar{\phi}^1$ *satisfies individual rationality, efficiency, non-bossiness and respecting improvement.*

Hence, individual rationality, efficiency, non-bossiness and respecting improvement are compatible for roommate problems with three agents.

For three agents, the above example shows on the strict domain an important difference between housing markets and roommate problems: for housing markets *individual rationality*, *efficiency*, *strategy-proofness* and *respecting improvement* characterize the TTC-rule whereas for roommate problems the four properties are incompatible. Furthermore, for both housing markets and roommate problems there exist other rules than the TTC-rule satisfying *individual rationality*, *efficiency* and *respecting improvement*. We summarize our findings of the above example below.

Proposition 2 (Roommate problems). *For roommate problems with three agents, on the strict domain*

(i) individual rationality, efficiency and strategy-proofness are incompatible and

(ii) individual rationality, efficiency, non-bossiness and respecting improvement are compatible.

6.3 School choice

Even though our first main result covers only housing markets, it can be applied to the TTC-mechanism in school choice. The main difference to a marriage market is that schools have non-unit capacities and are treated as "passive" whereas students correspond to the "active" agents.

We briefly describe the model. Let N denote the set of agents (or students) and O denote the set of finite set of objects (or schools). Each $i \in N$ has strict preference relation R_i over $O \cup \{i\}$ which we denote by $\hat{\mathcal{P}}_i$ and the set of profiles by $\hat{\mathcal{P}}^N$. Each $o \in O$ has a capacity $q_o \in \mathbb{N}$ and a priority order $\succ_o$ over N (and we set $\succ = (\succ_o)_{o \in O}$). An assignment is a function $\mu : N \to O \cup N$ such that for all $i \in N$, $\mu(i) \in O \cup \{i\}$, and for all $o \in O$, $|\mu^{-1}(o)| \leq q_o$. Let $\hat{\mathcal{M}}$ denote the set of all assignments. An assignment μ is stable for profile R if μ is individually rational and there exist no $i \in N$ and $o \in O$ such that $oP_i\mu(i)$ and either $|\mu^{-1}(o)| < q_o$ or $i \succ_o j$ for some $j \in \mu^{-1}(o)$. A rule (or mechanism) is a function $\varphi : \hat{\mathcal{P}}^N \to \hat{\mathcal{M}}$.

Except for *respecting improvement* all properties are defined for rules in the same way as for housing markets. Note that here for (weak) efficiency we only consider agents' welfare and ignore objects' priorities. A rule is stable if it chooses a stable assignment for any profile.

As a first observation, there is no hope to apply Theorem 4 to stable rules as Kojima (2010) has shown the incompatibility of stability and *non-bossiness*.

Next for *respecting improvement*, there are two versions: one is with respect to improving an agent's ranking in object priorities and the other one is with respect to improving an object's ranking in agent preferences. For stable rules the first one has been studied in detail by Balinski and Sönmez (1999), Kominers and Sönmez (2016) and most recently by Hirata, Kasuya and Okumura (2023) while the second one leads to impossibilities as shown by Hatfield, Kojima and Narita (2016).

Even though here agents have no endowments, for any object implicitly the highest priority agent has the claim (or right) to be assigned this object. As objects have non-unit capacities, any agent who is among the q_o highest priority agents, has the claim to o. Given profile R and assignment μ, μ respects claims if there exist no $i \in N$ and $o \in O$ such that $oP_i\mu(i)$ and i has a claim to o, i.e. $|\{k \in N : k \succ_o i\}| \leq q_o - 1$ (and similarly, respecting claims is defined for rules).

Definition 2. *Let N be a set of agents, $i \in N$, $o \in O$ and $R_i, R'_i \in \hat{\mathcal{P}}_i$. Then R'_i is a (local) o-improvement of R_i if (i) i does not have a claim for o, (ii) $R_i|_{O\backslash\{o\}} = R'_i|_{O\backslash\{o\}}$, (iii) for some $o' \in O\backslash\{o\}$ we have $o'P_io$ and oR'_io', and (iv) for all $o'' \in O\backslash\{o, o'\}$, [$o''P_io'$ implies $o''P'_io'$] and [$o'P'_io''$ implies $o'P_io''$].*

In Definition 2 we consider minimal improvements of an object k in an agent's ranking only on the strict domain.

Weak Respecting Improvement: For all $i, k \in N$, all $R \in \hat{\mathcal{P}}^N$ and all o-improvements $R'_i \in \hat{\mathcal{P}}_i$ of R_i such that k has a claim for o we have $\varphi_k(R'_i, R_{-i})R_k\varphi_k(R)$.

Note that weak respecting improvement is distinct from the notions described above for two-sided matching markets. Our proposed notion is in between those as it considers students' welfare and at the same time improvement is only possible on the students' side in terms of their preferences (while keeping schools' priorities fixed).

For the following we just replicate the proof of Theorem 1.

Theorem 5. *For school choice, if a rule satisfies individual rationality, strategy-proofness, non-bossiness and respecting claims, then it satisfies weak respecting improvement.*

Abdulkadiroğlu and Sönmez (2003) have proposed the following mechanism for school choice.

Top-trading-cycles (TTC-)mechanism

Let $R \in \hat{\mathcal{P}}^N$ and set $N_0 = N$, $O_0 = O$, and $q^0 = (q^0_o)_{o\in O} = (q_o)_{o\in O}$.

Step 1. Each $i \in N$ points to his most preferred object $top(R_i)$ in O and each object $o \in O$ points to the highest priority agent $top(\succ_o)$ in N. Then there exists at least one cycle and fix one, say

$i_1 - o_1 - i_2 - o_2 - \cdots - i_t - o_t$ (where $top(R_{i_l}) = o_l$ and $top(\succ_{o_l}) = i_{l+1}$ for $l \in \{1, \ldots, t-1\}$, $top(R_{i_t}) = o_t$ and $top(\succ_{o_t}) = i_1$). Then we set $g_{i_l}(R) = top(R_{i_l})$ for all $l \in \{1, \ldots, t\}$. Let $C_1 = \{i_1, \ldots, i_t\}$ denote the set of agents assigned in the fixed cycle, $N_1 = N_0 \backslash C_1$, $q_o^1 = q_o^0 - 1$ if $o \in \{o_1, \ldots, o_l\}$ and otherwise $q_o^1 = q_o^0$, $E_1 = \{o \in O : q_o^1 = 0\}$ denote the set of objects which have exhausted their capacity, and $O_1 = O \backslash E_1$.

Step k+1. Each $i \in N_k$ points to his most preferred object $top(R_i|_{O_k})$ in O_k and each object $o \in O_k$ points to the highest priority agent $top(\succ_o |_{N_k})$ in N_k. Then there exists at least one cycle and fix one, say $i_1 - o_1 - i_2 - o_2 - \cdots - i_t - o_t$ (where $top(R_{i_l}|_{O_k}) = o_l$ and $top(\succ_{o_l} |_{N_k}) = i_{l+1}$ for $l \in \{1, \ldots, t-1\}$, $top(R_{i_t}|_{O_k}) = o_t$ and $top(\succ_{o_t} |_{O_k}) = i_1$). Then we set $g_{i_l}(R) = top(R_{i_l})$ for all $l \in \{1, \ldots, t\}$. Let $C_{k+1} = \{i_1, \ldots, i_t\}$ denote the set of agents assigned in the fixed cycle, $N_{k+1} = N_k \backslash C_{k+1}$, $q_o^{k+1} = q_o^k - 1$ if $o \in \{o_1, \ldots, o_t\}$ and otherwise $q_o^{k+1} = q_o^k$, $E_{k+1} = \{o \in O : q_o^{k+1} = 0\}$ denote the set of which have exhausted their capacity, and $O_{k+1} = O_k \backslash E_{k+1}$.

Stop. $N_k = \emptyset$ or $O_k = \emptyset$.

Let g denote the TTC-mechanism. This mechanism belongs to the class of "hierarchical exchange rules" (Pápai, 2000) which satisfy *strategy-proofness* and *non-bossiness*. As the TTC-mechanism satisfies *individual rationality* and respecting claims, the most important application of Theorem 5 is the following.[17]

Corollary 4. *For school choice, the TTC-mechanism satisfies weak respecting improvement.*

Furthermore, one can verify that the above remains true on the subdomain of the strict domain where all schools are preferred to being unmatched (which corresponds to $\mu(i)$). Note that the TTC-mechanism is efficient for students and satisfies weak respecting improvement whereas the "student-proposing" deferred acceptance mechanism is inefficient for students (but satisfies *weak efficiency* for students).

Appendix

Throughout, let $N = \{1, 2, 3\}$. Note that there are exactly two exchange cycles of length three, namely $\nu = (3, 1, 2)$ and $\eta = (2, 3, 1)$.

First, for completeness, we show the incompatibility of *individual rationality*, *efficiency*, *non-bossiness* and *strategy-proofness*: let φ be a rule satisfying these properties and consider profile R where $R_1 : [23]1$, $R_2 : 132$ and $R_3 : 123$. Then by *individual rationality* and *efficiency*, $\varphi(R) \in \{\nu, \eta\}$, say $\varphi(R) = \nu$. Then for $R'_1 : 321$ we obtain from *strategy-proofness*, $\varphi_1(R'_1, R_{-1}) = \varphi_1(R)$,

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[17] Without going into details, for house allocation with existing tenants the mechanism proposed by Abdulkadiroğlu and Sönmez (1999) also belongs to this class and respects claims for existing tenants, meaning that weak respecting improvements is again satisfied.

and from *non-bossiness*, $\varphi(R_1', R_{-1}) = \varphi(R)$. As $(R_1', R_{-1}) \in \mathcal{P}^N$ and $f(R_1', R_{-1}) = (3, 2, 1) \neq \nu$, φ does not coincide with TTC-rule on the strict domain, which is a contradiction to Ma (1994)'s characterization of the TTC-rule on the strict domain.

Second, we show that if φ satisfies *individual rationality*, *efficiency*, *non-bossiness* and *respecting improvement*, then φ cannot coincide with the TTC-rule on the strict domain. Consider the same profile R as above and suppose $\varphi(R) = \nu$. We show again $\varphi(R) = \varphi(R_1', R_{-1}) = \nu \neq f(R_1', R_{-1})$. By *efficiency*, we have $\varphi_1(R_1', R_{-1}) \neq 1$. If $\varphi_1(R_1', R_{-1}) = 3$, then this follows from *non-bossiness*. Thus, suppose $\varphi_1(R_1', R_{-1}) = 2$ which implies by *efficiency*, $\varphi(R_1', R_{-1}) = (2, 3, 1)$. Now for R_3' : $[12]3$ we obtain from *efficiency*, $\varphi(R_1', R_2, R_3') = (3, 1, 2)$. But then $\varphi_1(R_1', R_2, R_3') P_1' \varphi_1(R_1', R_{-1})$ even though R_3 is a 1-improvement over R_3', which is a contradiction to *respecting improvement*. Hence, we must have $\varphi(R_1', R_{-1}) \neq f(R_1', R_{-1})$.

Third, we show the compatibility of *individual rationality*, *efficiency*, *non-bossiness* and *respecting improvement* via the rule $\phi : \mathcal{R}^N \to \mathcal{A}_N$ defined as follows.

Let $R \in \mathcal{R}^N$. We have the following cases:

(1) if $R_i : i \ldots$ for some $i \in N$, then there exists a unique individually rational and efficient allocation which coincides with $\phi(R)$;

(2) if $\neg(1)$ and there exists an efficient μ such that $\mu(i) R_i j$ for all $i, j \in N$, then $\mu \in \{\nu, \eta\}$ and we set $\phi(R) = \nu$ if ν is efficient and otherwise $\phi(R) = \eta$;

(3) if $\neg(2)$ and $R \notin \mathcal{P}^N$, then for some $i \in N$ we have $R_i : [jk]i$ and either (3.1) $R_j : k \ldots$, $R_k : j \ldots$ and we set $\phi_i(R) = i$, $\phi_j(R) = k$ and $\phi_k(R) = j$ or (3.2) $R_j : i \ldots$, $R_k : i \ldots$ and we set $\phi(R) = \nu$ if ν is individually rational and otherwise we have $l P_l \nu(l)$ for some $l \in N$ and we set $\phi_l(R) = l$ (and this determines the choice of $\phi(R)$ by efficiency); or

(4) if $\neg(3)$, then $R \in \mathcal{P}^N$ and there exists $i \in N$ such that $R_j : i \ldots$ and $R_k : i \ldots$, and (4.1) we set $\phi(R) = \nu$ if $\nu(i) = top(R_i)$ and ν is individually rational and (4.2) otherwise in $\phi(R)$ we make a pairwise exchange among i and $top(R_i)$ and the other agent keeps his endowment.

The rule ϕ discriminates among ν and η as it selects η only in (2) when η is the unique efficient allocation implying that all agents receive one of their most preferred objects under η. Furthermore, note that in (4) always agent i receives his top object and at least one other agent receives his top object.

It is obvious that ϕ satisfies *individual rationality* and *efficiency*. In order to verify *non-bossiness*, note that for any profile R which contains indifferences, breaking ties in favor of the assigned object keeps the chosen allocation unchanged. Hence, for *non-bossiness*, it suffices to consider the restriction of ϕ to the strict domain.

For *non-bossiness*, without loss of generality, we consider agent 1, and let $R \in \mathcal{P}^N$ and $R_1' \in \mathcal{P}_1^N$ be such that $\phi_1(R_1', R_{-1}) = \phi_1(R)$. If $\phi_1(R) = 1$, then *individual rationality* and *efficiency* imply

$\phi(R) = \phi(R'_1, R_{-1})$. If both $\phi(R)$ and $\phi(R'_1, R_{-1})$ are exchange cycles of length three, then by $\nu(i) \neq \eta(i)$ for all $i \in N$, we obtain $\phi(R) = \phi(R'_1, R_{-1})$ from $\phi_1(R'_1, R_{-1}) = \phi_1(R)$.

Thus, $\phi(R)$ or $\phi(R'_1, R_{-1})$ is an exchange cycle of length two. Without loss of generality, let $\phi(R) = (2, 1, 3)$. If *non-bossiness* is violated, then $\phi(R'_1, R_{-1}) = (2, 3, 1)$. But then from *individual rationality* we obtain $2P_1 1$, $2P'_1 1$ and $1P_3 3$, and by *efficiency* of $\phi(R)$, $R_2 : 132$. Note that under R two agents rank the same object at the top as otherwise there is a unique efficient allocation where each agent receives his most preferred object. If $R_3 : 1\ldots$, then as $\phi(R) = (2, 1, 3)$, by (4.2) we must have $R_1 : 2\ldots$ and $R'_1 : 321$ (as for $R'_1 : 2\ldots$ again (4.2) would apply), but then this is impossible as either $R_3 : 123$ would imply by (4.1) $\phi(R'_1, R_{-1}) = (3, 1, 2)$ or $R_3 : 132$ would imply by (4.2) $\phi(R'_1, R_{-1}) = (3, 2, 1)$, a contradiction to $\phi_1(R'_1, R_{-1}) = 2$. Thus, we must have $R_3 : 213$ and $R_1 : 2\ldots$, which would imply by either by (4.1) $\phi(R) = (3, 1, 2)$ or by (4.2) $\phi(R) = (1, 3, 2)$, a contradiction to $\phi_1(R) = 2$.

For *respecting improvement*, without loss of generality, we consider agent 1 and $k \neq 1$, and let $R \in \mathcal{P}^N$ and $R'_1 \in \mathcal{P}^N_1$ be a k-improvement of R_1. If $\phi_1(R'_1, R_{-1}) = \phi_1(R)$, then by *non-bossiness*, $\phi(R'_1, R_{-1}) = \phi(R)$ and *respecting improvement* is satisfied. Thus, let $\phi_1(R'_1, R_{-1}) \neq \phi_1(R)$. We consider the same cases as in the definition of ϕ.

First, suppose that R is in (1). Then $R_i : i\ldots$ for some $i \in N$. If $i \neq 1$, then *respecting improvement* is obvious. Suppose that $top(R_2) \neq 2$, $top(R_3) \neq 3$, $R_1 : 1\ldots$ and $R'_1 : k1\ldots$. If (R'_1, R_{-1}) belongs to (2), again *respecting improvement* is obvious. Thus, (R'_1, R_{-1}) must belong to (4) (as otherwise (2) applies). If $top(R_2) = k$ or $top(R_3) = k$, then by (4.1) or (4.2) we must have $\phi_k(R'_1, R_{-1}) = top(R_k) R_k \phi_k(R)$ and *respecting improvement* is satisfied. Otherwise $top(R_2) = top(R_3) = 1$, either by (4.2) agents 1 and k make a pairwise exchange or by (4.1) agent k is weakly better off as $\phi_k(R'_1, R_{-1}) \in \{1, \phi_1(R)\}$.

Second, suppose that R is in (2). But then $\phi_1(R) \neq 1$. If $\phi_1(R) = k$, then as R'_1 is a k-improvement over R_1 we must have $R_1 : [23]1$ and $top(R'_1) = k$ which then implies by *non-bossiness*, $\phi(R) = \phi(R'_1, R_{-1})$ and *respecting improvement* is satisfied. If $\phi_1(R) \neq k$, then either [$R'_1 : [23]1$, (R'_1, R_{-1}) is in (2) and *respecting improvement* is satisfied] or [$R_1 : [23]1$, $top(R'_1) = k$ and as above *respecting improvement* is satisfied].

Third, suppose that R is in (3). Then $R_i \notin \mathcal{P}^N_i$ for some $i \in N$. If $i = 1$, then either [$R_1 : [23]1$, $R_2 : 3\ldots$, $R_3 : 2\ldots$, and $R'_1 : k\ldots$, which implies by (4.1) and (4.2) that agent k is weakly better off (and the other agent j is weakly worse off)] or [$R_1 : [23]1$, $R_2 : 1\ldots$, $R_3 : 1\ldots$ and $R'_1 : k\ldots$, which implies by (4) $\phi_1(R'_1, R_{-1}) = k$ and if *respecting improvement* is violated, $\phi_k(R'_1, R_{-1}) = j$ and $\phi_j(R'_1, R_{-1}) = 1$, which means $\phi(R'_1, R_{-1}) = \nu = (3, 1, 2)$ and $\phi(R) = (2, 3, 1) = \eta$, a contradiction as in (4.1) ν must be then chosen for R (and the other agent j is weakly worse off)].

Fourth, suppose that R is in (4). Then $R \in \mathcal{P}^N$. If $(R'_1, R_{-1}) \notin \mathcal{P}^N$, then we must have $R'_1 : [23]1$ and R_1 is a j-improvement over R_1, and the above for (3) established that then k is weakly worse off. Thus, $(R'_1, R_{-1}) \in \mathcal{P}^N$ and $R'_1 : top(R_1)k1$. If $top(R_j) = top(R_i)$ it follows by (4) that $\phi_k(R) = \phi_k(R'_1, R_{-1}) = top(R_k)$ and *respecting improvement* is satisfied. Otherwise $top(R_2) = top(R_3) = 1$ and as for the last case for (3) *respecting improvement* follows.

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