BILEVEL KNAPSACK PROBLEMS

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1 Introduction

Bilevel knapsack problems (BKPs) extend to two-stage sequential games the classical knapsack problem (KP). In a BKP, a player called the leader fixes the value of their variables x and then, a player called the follower, observes the leader's decision and reacts optimally by solving a knapsack problem. Hence, we have the following formulation:

$$
\begin{aligned}
(BKP) \quad & \min_{x} && f(x,y) && \text{(1a)}\\
& \text{subject to} && (x,y) \in X && \text{(1b)}\\
& && \text{where } y_1,\dots,y_n \text{ solves the follower's problem} &&\\
& (KP(x)) && \max_{y \in Y} \sum_{i \in N(x)} p_i(x) y_i \quad \text{s.t.} \sum_{i \in N(x)} w_i(x) y_i \leq W(x). &&
\end{aligned}
$$

The reason why BKPs are a game is the follower's KP parametrization on the leader's decision variables x. Indeed, BKPs are a special case of bilevel programs.

The BKPs proposed in the literature differ accordingly with

- the definition of the upper level (1a)-(1b), *i.e.*, leader's variables, objective and constraints;
- the consideration of a continuous or binary KP for the follower, i.e., $Y = [0,1]^{N(x)}$ or $Y = \{0,1\}^{N(x)}$, respectively;
- the effect of the leader's decision on the follower's problem which can be on the set of items $N(x)$ available, the knapsack capacity $W(x)$, or profit $p_i(x)$ and weight $w_i(x)$ of each item $i \in N(x)$.

Remark that for a BKP to be well defined, it must be clarified the follower's action when $KP(x)$ has multiple optimal solutions for a fixed x. Typically, the bilevel programming literature uses the *pessimistic* and *optimistic* versions of these problems. Under the pessimistic case, it is assumed that the follower picks among their optimal solutions the one damaging the most the leader's objective. In the optimistic case, the follower picks among their optimal solutions the one benefiting the most the leader's objective.

Applications of BKPs have been describe in corporate strategy [1], revenue management [2] and telecommunications [3], to name few. Another motivation to investigate BKPs is methodological since they form simple to formulate bilevel programs whose understanding can potentially be leveraged to tackle more general problems.

Standard algorithmic ideas for solving the KP, such as dynamic programming and the critical item computation, reveal to be very useful for solving its bilevel versions.

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2 Models

In the BKP variants reviewed next, when the follower solves a continuous KP, the problem is designated by *continuous* BKP, otherwise the word *continuous* is omitted. Besides linear BKPs, non-linear versions are also described. Unless stated, all parameters in the presented models are non-negative integers.

Continuous BKP The *continuous knapsack problem with interdiction constraints* introduced by Carvalho et al. [4] is as follows:

$$
\begin{aligned}
(cBKP) \quad \min_{x \in [0,1]^n} \quad & \sum_{i=1}^{n} p_i y_i \\
\text{subject to} \quad & \sum_{i=1}^{n} v_i x_i \leq V && \text{(2a)} \\
& \text{where } y_1, \ldots, y_n \text{ solves the follower's problem} \\
& \max_{y \in [0,1]^n} \sum_{i=1}^{n} p_i y_i \quad \text{s.t.} \quad \sum_{i=1}^{n} w_i y_i \leq W \quad \text{and} \\
& \qquad y_i \leq 1 - x_i \quad \text{for } 1 \leq i \leq n. && \text{(2b)}
\end{aligned}
$$

In simple words, the leader aims to minimize the profit of the follower's continuous KP by interdicting x_i of item i, constraints (2b), and subject to their own knapsack constraint (2a). Note that constraints (2b) of cBKP can be equivalently modeled within the BKP formulation by setting $p_i(x) = p_i(1 - x_i)$ and $w_i(x) = w_i(1 - x_i)$ for each item i. For cBKP, it is unnecessary to distinguish between its pessimistic and optimistic versions since both the leader and the follower have the same objective function.

Binary BKP Dempe and Richter [5] proposed a BKP where the leader controls a single variable deciding the follower's knapsack capacity

$$
\begin{aligned}
(DR) \quad \max_{x \in \mathbb{R}} \quad & Ax + \sum_{i=1}^{n} a_i y_i \\
\text{subject to} \quad & V' \leq x \leq V \\
& \text{where } y_1, \ldots, y_n \text{ solves the follower's problem} \\
& \max_{y \in \{0,1\}^n} \sum_{i=1}^{n} p_i y_i \quad \text{s.t.} \quad \sum_{i=1}^{n} w_i y_i \leq x,
\end{aligned}
$$

with parameter A taking any integer value. Since x is continuous, DR may fail to have a solution given that the set of (bilevel) feasible solutions can be non-closed. Nevertheless, if A is non-positive, Dempe and Richter [5] proved that there is always an optimal solution. Moreover, Brotcorne et al. [2] showed that whenever DR has an optimal solution, it is also optimal for DR with x restricted to integer values. It is important to remark that these results hold for the *pessimist* and *optimistic* versions of DR. In [6], DR is considered but the interference of the leader's choice of capacity x for the follower is uncertain; when this uncertainty is characterized by a finite set and x is restricted to integer values, it is shown that the problem is equivalent to a two-stage stochastic program.

In the BKP formulated by Mansi et al. [3], the leader and the follower share the knapsack capacity

$$
\begin{aligned}
(MACH) \quad \max_{x \in \{0,1\}^m} \quad & \sum_{i=1}^{m} a_i x_i + \sum_{i=1}^{n} b_i y_i \\
& \text{where } y_1, \ldots, y_n \text{ solves the follower's problem} \\
& \max_{y \in \{0,1\}^n} \sum_{i=1}^{n} p_i y_i \quad \text{s.t.} \quad \sum_{i=1}^{n} w_i y_i \leq W - \sum_{i=1}^{m} v_i x_i.
\end{aligned}
$$

As in DR, the leader in MACH affects the follower's capacity but instead of controlling a single continuous variable, the leader controls m binary variables. Brotcorne et al. [7] consider a more general version of MACH, where the upper level can have linear constraints.

DeNegre [1] proposed the binary version of cBKP

$$
\begin{aligned}
(DN) \quad & \min_{x\in\{0,1\}^n} && \sum_{i=1}^n p_i y_i \\
& \text{subject to} && \sum_{i=1}^n v_i x_i \le V \\
& && \text{where } y_1,\dots,y_n \text{ solves the follower's problem} \\
& && \max_{y\in\{0,1\}^n} \sum_{i=1}^n p_i y_i \quad \text{s.t. } \sum_{i=1}^n w_i y_i \le W \quad \text{and} \\
& && \qquad\qquad\qquad\qquad y_i \le 1 - x_i \quad \text{for } 1 \le i \le n.
\end{aligned}
$$

In fact, it must be noted that DN chronologically precedes cBKP. In this bilevel, the leader interdicts items from being taken by the follower and, analogously to its continuous version, there is no need to discern between the pessimistic and optimistic cases.

Non-linear BKP Chen and Zhang [8] propose a non-linear BKP where the same sets of items are available to both leader and follower but they may have different knapsack capacities. In their formulation, the same item can be selected by both players, potentially resulting in a change of the profits:

$$
\begin{aligned}
(CZ) \quad & \max_{x\in\{0,1\}^n} && \sum_{i=1}^n p_i(x_i + y_i) + 2\sum_{i=1}^n a_i x_i y_i \\
& \text{subject to} && \sum_{i=1}^n w_i x_i \le V \\
& && \text{where } y_1,\dots,y_n \text{ solves the follower's problem} \\
& && \max_{y\in\{0,1\}^n} \sum_{i=1}^n p_i y_i + \sum_{i=1}^n a_i x_i y_i \quad \text{s.t. } \sum_{i=1}^n w_i y_i \le W,
\end{aligned}
$$

where the parameters a_i are not restricted to non-negative values. Note that the leader optimizes over the total profit of both players while the follower only maximizes their own profit.

Another non-linear BKP is formulated by Pferschy et al. [9], the *Stackelberg knapsack problem with weight selection*

$$
\begin{aligned}
(SKPW) \quad & \max_{x\in\mathbb{R}^{|L|}} \sum_{i\in L} x_i y_i \\
& \qquad \text{where } y_1,\dots,y_{|L|+|F|} \text{ solves the follower's problem} \\
& \qquad \max_{y\in\{0,1\}^{|L|+|F|}} \sum_{i\in L\cup F} p_i y_i \quad \text{s.t. } \sum_{i\in L} x_i y_i + \sum_{i\in F} w_i y_i \le W.
\end{aligned}
$$

Here, the leader decides the weights of a subset L of the follower's items while maximizing the profit achieved by the items in L selected by the follower. Hence, the challenge for the leader is in finding a good balance between making the items in L attractive for the follower (*i.e.*, lowering their weight) and optimizing the profit of items in L.

In the same vein as SKPW, Pferschy et al. [10] propose the *Stackelberg knapsack problem with profit selection*

$$
\begin{aligned}
(SKPP) \quad & \max_{x\in\mathbb{R}^{|L|}} \sum_{i\in L} (p_i - x_i) y_i \\
& \qquad \text{where } y_1,\dots,y_{|L|+|F|} \text{ solves the follower's problem} \\
& \qquad \max_{y\in\{0,1\}^{|L|+|F|}} \sum_{i\in L} x_i y_i + \sum_{i\in F} p_i y_i \quad \text{s.t. } \sum_{i\in L\cup F} w_i y_i \le W.
\end{aligned}
$$

In this model, the leader decides the profit of the items in L. Again, the leader must balance between increasing the profits in L, which incentives the follower to pick them, and the loss in their objective due to increasing the follower's profits.

3 Methods

Despite linear bilevel programs being NP-hard [11], cBKP can be solved in polynomial time [4, 12]. The key ingredient of the algorithms tackling it relates to guessing the *critical item* [13] for the follower's knapsack when the leader selects the optimal interdiction. This is because Dantzig's famous result [14] provides a direct way to determine the optimal solution for the follower.

Definition 1 *If the items of KP are sorted such that* $\frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \ldots \geq \frac{p_n}{w_n}$ *holds, the item* c *defined by* $c = \min\{j : \sum_{i=1}^{j} w_i > W\}$ *is called the critical item.*

Theorem 1 (Dantzig [14]) *If the items of KP are sorted such that* $\frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \ldots \geq \frac{p_n}{w_n}$ *holds, then an optimal solution for the continuous KP is to fully pack the items from 1 to* $c-1$*, where* c *is the critical item, and pack* $\frac{W - \sum_{i=1}^{c-1} w_i}{w_c}$ *of item* c*.*

Pferschy et al. [9] showed that when the follower's variables in SKPW are relaxed, it can also be solved in polynomial solvable. On the other hand, for binary bilevel knapsack problems the associated computational complexity changes drastically.

Theorem 2 (Caprara et al. [15]) *The decision versions of DR, MACH and DN are* Σ_2^p*-complete under the pessimistic and optimistic cases.*

The result above implies the impossibility of formulating these BKPs as integer programs of polynomial size, unless the entire polynomial hierarchy collapses to the first level. The same result holds for SKPP.

Theorem 3 (Pferschy et al. [10]) *The decision version of SKPP is* Σ_2^p*-complete under the pessimistic and optimistic cases.*

Surprisingly, DN was the first Σ_2^p-hard problem for which a polynomial time approximation scheme was presented [15]. For the remaining Σ_2^p-hard bilevel knapsack problems mentioned above, there is no polynomial time approximation schemes, except if $P = NP$ [15, 10].

Given the landscape of computational complexity provided above, we now move to discuss algorithmic methodologies. Brotcorne et al. [2] provided a pseudo polynomial time dynamic programming approach with worst-case time complexity $\theta(nV)$ for both pessimistic and pessimistic cases of DR. For a more general version of MACH at the upper-level, Brotcorne et al. [7] give a one-level pseudo polynomial time reduction by using dynamic programming in the follower's KP. Succinctly, their approach identifies the follower's optimal solutions accordingly with the available follower's knapsack capacity and uses it to build a single-level formulation.

The fact that cBKP can be efficiently solved does not enlighten us on how to solve its binary version, DN. Indeed, in integer bilevel programming, the relaxation of binary requirements does not generally leads to a relaxed problem. Concretely, an optimal solution of cBKP is not necessarily a lower bound of DN. Caprara et al. [16] proposed the first tailored approach for DN. In short, first, the follower's knapsack is relaxed to its continuous version and strong duality is used to achieve a single-level mixed integer linear problem (MILP); second, the obtained MILP is iteratively solved with additional cuts until optimality is proven. It is worth noting that this method first step has been successfully used within an heuristic devised by Fischetti et al. [17] for general interdiction games. A branch-and-cut algorithm for general interdiction games is presented in [18] and shown to outperform the method in [16] for the *hard* instances. An extremely significant advance on the practical efficiency of solving DN was achieved by the approach designed by Croce and Scatamacchia [19]. Again, the critical item plays an important role in the proposed algorithm as well as the concept of *core* of a knapsack used by state-of-the-art algorithms tackling KP [20, 21].

In [8], the authors propose approximation algorithms for two cases of the parameters a_i for CZ: *(i)* the competitive version where the profit of an item selected by both players results in a profit decrease ($a_i < 0$), and *(ii)* the beneficial version where the profit of an item selected by both players leads to a profit increase ($a_i > 0$). Qiu and Kern [22] improve the approximation algorithms for these two problem versions and prove that the approximation ratios are tight.

Pferschy et al. [9, 10] motivate that in practice the follower can have limited computational power, preventing them to solve KP (a NP-hard problem). Nevertheless, even assuming different (polynomial time) algorithms for the follower, the leader's problem in SKPW and SKPP can still fail to have a polynomial time approximation algorithm with a constant approximation ratio.

4 Conclusions

Although bilevel knapsack problems are simple to describe, they already encompass the dynamics and challenges of general bilevel programs. Thus, BKPs have been an exciting line of research due to their potential for methodological insights. In particular, the described variants relate to important classes of problems in optimization. For instance, the variant DN belongs to the class of interdiction games and it can also be seen as a robust optimization problem. Another example is SKPW which relates to pricing problem.

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