Université de Montréal

Distribution of reflection points of periodic billiard trajectories in a strictly convex table

par

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Résumé

Ce mémoire de maîtrise porte sur les billards mathématiques et la distribution des points de réflexion des trajectoires périodiques d'une table de billard strictement convexe. Un billard mathématique est un système dynamique généré par le mouvement libre d'une particule à l'intérieur d'un domaine dont la frontière est parfaitement réfléchissante. Une question d'intérêt particulier dans l'étude des billards mathématiques est celle de ses trajectoires périodiques. Nous considérons le cas des billards planaires strictement convexes. Il est connu que les points de réflexion des trajectoires périodiques de période n faisant un tour de table sont équidistribués par rapport à une mesure naturelle sur la frontière. Nous montrons ce résultat par une méthode nouvelle et relativement élémentaire utilisant la théorie de Lazuktin [**12**]. Dans le premier chapitre, nous donnons une description précise de la dynamique des billards et une brève introduction à la théorie de Lazuktin, aux applications de torsion et aux caustiques. Dans les chapitres 2 à 4, nous développons chacun des concepts précédents et expliquons comment ceux-ci sont liés aux billards. Le chapitre 5 est consacré à la preuve de notre résultat principal, divisée en deux parties. Nous concluons en donnant une annexe sur la théorie de la mesure.

Mots clés : billard, orbites périodiques, distribution uniforme, caustiques, application de torsion, revêtements universels.

Abstract

This master's thesis is concerned with mathematical billiards and distribution of reflection points of periodic trajectories of a strictly convex billiard table. A mathematical billiard is a dynamical system generated by the free motion of a particle inside of a domain with a perfectly reflecting boundary. A question of particular interest in the study of mathematical billiards is that of its periodic trajectories. We consider the case of planar strictly convex billiards. It is known that the reflection points of periodic trajectories of period n making one turn around the table are equidistributed with respect to a natural measure on the boundary. We show this result by a new and relatively elementary method using Lazuktin's theory [**12**]. In the first chapter, we give a precise description of billiard dynamics and a brief introduction of Lazuktin's theory, twist mappings and caustics. In Chapter 2 to 4, we elaborate each of the previous concepts and explain how they are related to billiards. Chapter 5 is dedicated to the proof of our main result, divided into two parts. We conclude by giving an appendix about measure theory.

Keywords: billiard, periodic orbits, uniform distribution, caustics, twist maps, universal covering.

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Introduction

A mathematical billiard is a dynamical model generated by the free motion of a particle (billiard ball) inside a domain (billiard table) with a perfectly reflecting boundary. The billiard ball is a point mass which moves along a straight line with a constant speed and is subject to elastic collision with the boundary. The velocity (unit) vector is decomposed into the normal and tangential component to the reflection point. Upon reflection, the normal component changes its sign, whereas the tangential component remains the same. In dimension two, this is the famous reflection law in optics: *the angle of incidence equals the angle of reflection*. However, the billiard dynamics are not defined if the ball hits the corners of the table.

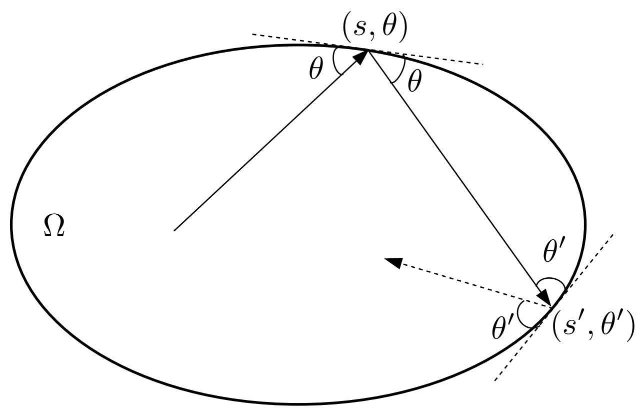

Fig. 0.1. Billiard trajectory according to reflection law

The billiard model was first introduced by Poincaré and Birkhoff, who found its relation with models in physics while studying dynamical systems [**2**]. Birkhoff mostly investigated billiard trajectories, and in particular those that are periodic (which will be explained later). For a beautiful introduction to theory of mathematical billiards we refer to the book [**20**] of Tabachnikov. A recent research article of particular relevance to this thesis is [**1**] by Bialy, Mironov and Tabachnikov about wire billiard. The authors found a special property about the set of reflection points of periodic orbits in a strictly convex billiard table. Marvizi and Melrose investigated the same problem using interpolating Hamiltonians [**13**]. Lazutkin, on the other hand, studied a different geometric aspect of the billiard dynamics: the existence of caustics inside a billiard table [**12**]. Nowadays, the subject of billiards is studied by many mathematicians, and it is a very active field of research.

We now describe the billiard dynamic mathematically. Let M be the space of units vectors $(x,\mathbf{v})$ in our domain Ω, where x is the foot point on $\partial\Omega$ and $\mathbf{v}$ is the inward velocity vector. We can then define the *billiard ball map* T on M that sends $(x,\mathbf{v})$ to $(x',\mathbf{v}')$, where x' is the foot point upon reflection and $\mathbf{v}'$ is the velocity vector reflected off the boundary.

Let us now introduce coordinates on the phase space M. Let $\gamma : [0,l] \to \mathbb{R}^2$ be arclength parametrization of $\partial\Omega$ with l the its length. Then x can be associated with the arclength parameter s. We then define θ to be the angle between the velocity vector $\mathbf{v}$ and the tangent at $\gamma(s)$. We only consider the angle from 0 to π no matter its orientation. Therefore, M can be associated with $[0,l] \times (0,\pi)$ and the billiard ball map becomes

$$\begin{aligned} T : [0,l] \times (0,\pi) &\to [0,l] \times (0,\pi) \\ (s,\theta) \quad &\mapsto \quad (s',\theta'). \end{aligned}$$

Let $\kappa(s)$ be the curvature of $\partial\Omega$ at $\gamma(s)$. By considering the neighborhood of the line $\theta = 0$, Lazutkin introduced a special change of coordinates (x,y) with

$$\begin{aligned} x &= C_1 \int_0^s \kappa^{2/3}(s)\, ds \\ y &= C_2 \kappa^{-1/3}(s) \sin\frac{\theta}{2} \end{aligned}$$

where

$$C_1 = \left(\int_{\partial\Omega} \kappa^{2/3}\, ds\right)^{-1}, C_2 = 4C_1.$$

He then proved in [**12**] that under this change of coordinates, the billiard ball map takes the form

$$T(x,y) = (x + y + O(y^3), y + O(y^4)).$$

The billiard orbits of particular interest are periodic orbits, which in the simplest terms, are the vertices of a n-gon if the period is n. Let $X = \{x_k\}_{k\in\mathbb{Z}} = \{(s_k,\theta_k)\}_{k\in\mathbb{Z}}$ be an orbit such that there exists $n \geq 2$ such that $x_{k+n} = x_k$ for all $k \in \mathbb{Z}$. The smallest n that satisfies the last relation is called the period of X. The winding number m, on the other hand, is the number of turns that our trajectory makes around the table before coming back to the initial position. We can then completely describe the periodic orbits by its winding number m and its period n. Hence, we define such periodic orbits to be *periodic orbit of type* (m,n).

Results

Billiards are related to many others domains of mathematics, such as dynamical systems, symplectic geometry, KAM theory, spectral theory, and so on. However, many results are restricted to strictly convex domains. In this master's thesis, we would like to investigate the distribution of the set of reflection points (also called the impact points) of the periodic billiard trajectories in a strictly convex region in $\mathbb{R}^2$. In fact, the main theorem below states that the set of reflection points of periodic orbit of type $(1,n)$ is uniformly distributed as the period n goes to infinity.

Theorem 0.0.1 (Bialy, Mironov, Tabachnikov [**1**])**.** *In a strictly convex billiard table with curvature $\kappa > 0$, the distribution of reflection points of Birkhoff periodic orbits of type $(1,n)$ is uniform with respect to the measure $\kappa^{2/3}\,dx$ as n tends to ∞. In other words, for an interval $(a,b) \subset \partial\Omega$, we have*

$$\frac{\#\{x_n^j \in I\}}{n} \to \frac{\int_I \kappa^{2/3}\,dx}{\int_{\partial\Omega} \kappa^{2/3}\,dx},$$

where $x_n^1,\dots,x_n^n$ are the reflection points.

The proof in [**1**] relies on a complicated method, called interpolating Hamiltonians, due to Melrose (see [**13**]). Our goal is to obtain a new proof using Lazutkin's coordinates.

Structure of the thesis

In Chapter 1, we present the precise description of the billiard system, and various concepts associated to the proof of the main theorem. We begin by defining the billiard ball map and some of its properties. We then describe the periodic orbits and we show its existence for strictly convex billiard table. We then introduce the concept of caustics, twist mapping and Lazutkin's coordinates by giving essential definitions and results. We conclude this chapter by presenting briefly the ideas of proof of the main theorem using those concepts.

Chapter 2 is dedicated to describe Lazutkin's theory that will be used in the proof of our main theorem. We first introduce the concept of caustics and some quantities such as the rotation number and normalized number associated to it. Those numbers will be shown to be irrational and cannot be well approximated by rational numbers, in contrast with those of periodic orbits that are rational. This fact suggests that periodic orbit are "squeezed" between discontinuous family of caustics. Secondly, we describe in details Lazutkin's change of coordinates from the (s,θ) coordinates and how it implies that the billiard ball will take a nice form under those coordinates. The main theorem in this chapter shows that these coordinates can be expressed in terms of the rotation number and normalized parameter up to some degree of errors. We finish by proving the reflection points of periodic orbit of type $(1,n)$ are uniformly distributed in the round circle.

Chapter 3 presents the theory of twist mapping and shows that our billiard ball map is a twist map. To understand the definition of a twist map, we need to know that of the circle map and its lift. The first part explains who one can interpret the billiard dynamics using the concept of the lift of a circle map. The rotation number of a circle map is then introduced. Its existence and some important properties are proven by using the concept of a quasi-morphism. We then define the twist maps and show that the billiard ball map is a twist map, so we can talk about rotation number of the billiard map in our proof.

Chapter 4 continues the discussion of the caustics introduced in Chapter 2. We rigorously define caustics and invariant circles associated to them using the concept of an envelope. We first show that we can recover the curve by knowing its caustic. The latter is defined as the envelope of the invariant circle constituting of the family of rays at the fixed angle of points of the curve. Birkhoff showed that caustics of the billiard table can be expressed as a graph of a continuous function, and they lie inside the table. We then state and prove the mirror equation, famous in geometrical optics. We conclude by using this equation to prove Mather's result about the non-existence of convex caustics in convex table with a point of zero curvature.

Chapter 5 elaborates the proof of our main result about the uniform distribution of reflection points of periodic orbit of type (1,n) in a strictly convex billiard table. Using Lazutkin's coordinates, it suffices to show that the coordinate y along such a periodic orbit is bounded by $\frac{C}{n}$, for some real constant C, to show the equidistribution. To show the first part, we bound y along a caustic in terms of the normalized parameter η which represents also the angle. To do so, we show that a periodic orbit of type (1,n) can be found within two caustics, so its rotation number is bounded by that of the caustics, which are described by the normalized parameter η. To show the second part on how this implies equidistribution, we show that any reflection points x_j can be written as $x_0 + \frac{j}{n} + O(\frac{1}{n^2})$, where x_0 is the initial point. We then prove that for an interval of the boundary and for n sufficiently large, the difference of the points of the form x_j and those of the form $x_0 + \frac{j}{n} + O(\frac{1}{n^2})$ does not exceed 2.

This thesis concludes with Appendix A about measure theory, which we use to prove that irrational rotation numbers corresponding to the caustics cannot be well-approximated by rational numbers. The general ideas and motivation to measure theory are firstly introduced. The algebra of sets, which are structures about sets, are then presented to talk about measure on a half-ring. We finally define the outer measure in order to define the Lebesgue measure on a set, and some essential properties associated to it.

Chapter 1

Billiard System

A mathematical billiard is a dynamical model (which means a mathematical model which describes a moving object) generated by the free motion of a particle (billiard ball) inside of a domain (billiard table) with a perfectly reflecting boundary. More precisely, a billiard table Ω is usually a subset of $\mathbb{R}^2$ with a piecewise smooth boundary $\partial\Omega$. The billiard ball is a point mass which moves along a straight line with a constant speed and is subject to elastic collision with the boundary. The latter means that the energy and the norm of momentum are preserved after each collision. The velocity (unit) vector is decomposed into the normal and tangential component to the reflection point. Upon reflection, the normal component changes its sign, whereas the tangential component remains the same. In dimension two, this is the famous reflection law in optics: *the angle of incidence equals the angle of reflection*. However, the reflection is assumed to occur only at smooth points of the boundary, which means the billiard dynamics is not defined if the ball hits the corners of the table.

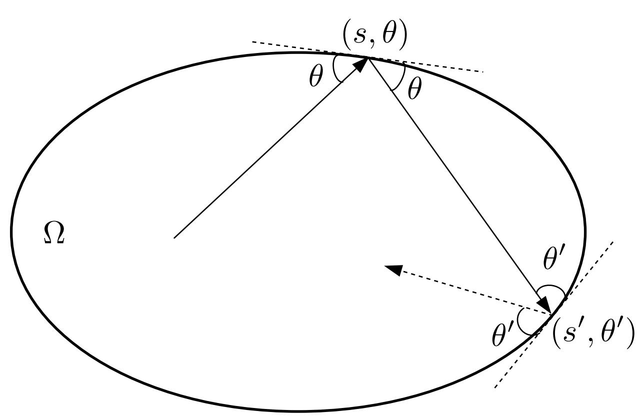

Fig. 1.1. Billiard trajectory according to reflection law

1.1. Billiard ball map

Let M be the space of pairs (x,v), where x is a point on $\partial\Omega$ and v is a unit tangent vector to $\mathbb{R}^2$ at x, which points inwards into Ω. This space M is called the *phase space* of the billiard system. We can then define the *billiard ball map* $T : M \to M$ from M to M as follows: $T(x,v) = (x',v')$, where $x' \in \partial\Omega$ is the first reflection point of the billiard dynamics starting with (x,v), and v' is the velocity vector reflected off the boundary. The table Ω is required to be convex in order to make T continuous, since otherwise there would exist periodic orbits touching the boundary. (See Figure 1.2)

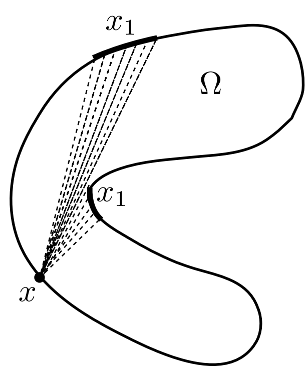

Fig. 1.2. Discontinuity of T in non-convex domain

Let us now introduce coordinates on the phase space M. Each unit vector can be described by its foot point's position on the boundary and the angle that it makes with the direction tangent to $\partial\Omega$ at this point. Let $\gamma : [0,l] \to \mathbb{R}^2$ be arclength parametrization of $\partial\Omega$ with l its length. Then x can be associated with the arclength parameter s. We then define θ to be the angle between the velocity vector $\mathbf{v}$ and the tangent vector $\dot{\gamma}(s)$ of γ at the value s of the arclength parameter. See Figure 1.1. We orient the boundary anticlockwise starting from $\theta = 0$ so a trajectory starting from $\theta = \pi$ is in the opposite orientation. In this way, we only consider the angle from 0 to π no matter its orientation. Therefore, M is isomorphic to with $[0,l] \times (0,\pi)$ and the billiard ball map becomes

$$\begin{aligned} T : [0,l] \times (0,\pi) &\to [0,l] \times (0,\pi) \\ (s,\theta) \quad &\mapsto \quad (s_1,\theta_1). \end{aligned}$$

An important property of the billiard ball map is that it preserves an area form on M.

Theorem 1.1.1. *The area form $\alpha = \sin\theta \, d\theta \wedge ds$ is invariant under the billiard map T.*

Proof. Since $\theta \in (0,\pi)$, then $\sin\theta > 0$, and therefore this 2-form is an area form. We wish to prove that $\sin\theta_1 \, d\theta_1 \wedge ds_1 = \sin\theta \, d\theta \wedge ds$. We consider the distance function $f(s,s_1)$ between $\gamma(s)$ and $\gamma(s_1)$. The gradient of this distance ∇f is the unit vector from $\gamma(s)$ to $\gamma(s_1)$. To see this, we denote $\gamma(s) = (x,y)$ and $\gamma(s_1) = (x_1, y_1)$ and we compute $(\partial f/\partial x, \partial f/\partial y)$. We have $\nabla f = \frac{1}{|\gamma(s)\gamma(s_1)|}(x_1 - x, y_1 - y)$, which is the unit direction vector from $\gamma(s)$ to $\gamma(s_1)$. Therefore, $\partial f/\partial s_1 = \nabla f \cdot \dot{\gamma}(s_1)$ is the length of the projection of ∇f on the tangent vector $\dot{\gamma}(s_1)$ at $\gamma(s_1)$, and hence $\partial f/\partial s_1 = \cos\theta_1$. Similarly, $\partial f/\partial s = -\cos\theta$. Taking the total (exterior) derivative of f, we have

$$df = \frac{\partial f}{\partial s} ds + \frac{\partial f}{\partial s_1}\, ds_1 = -\cos\theta ds + \cos\theta_1\, ds_1.$$

By the property of exterior derivative, we have $d^2 f = 0$ and hence

$$0 = d^2 f = \sin\theta\, d\theta \wedge ds - \sin\theta_1\, d\theta_1 \wedge ds_1,$$

and the result follows. □

One application of this area form is the computation of the area of the phase space. Let l be the length of γ, which is the parametrisation of $\partial\Omega$. The area of the phase space M is equal to

$$\int_0^l \int_0^\pi \sin\theta\, d\theta ds = \int_0^l -\cos\theta\Big|_0^\pi ds = \int_0^l 2\, ds = 2l.$$

Now, instead of considering the phase space M, one can consider the space of oriented lines N where each line is characterised by its direction (angle ϕ) and its signed distance from the origin orthogonal to the line (see Figure 1.3). The space N is hence a cylinder with coordinates (ϕ, p). The area form of N is $A = d\phi \wedge dp$.

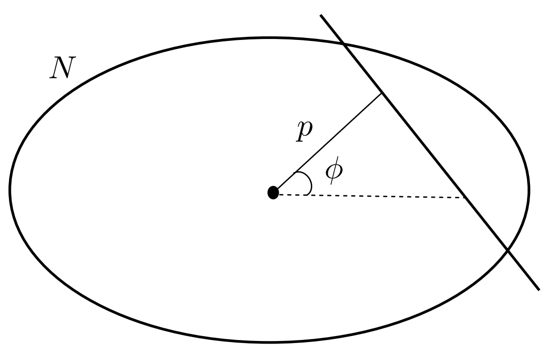

Fig. 1.3. Space of oriented lines

We wish to show the correspondence of these two spaces. Let $\Phi : M \to N$ be the map that associates to each unit vector (x,v) an oriented line passing through it. Since A is assumed to be convex, we have one-to-one correspondence. We show the following relation:

$$\Phi^\star(A) = \alpha$$

Proof. Let $\psi(s)$ be the direction of the tangent line to γ at $\gamma(s)$, and $\gamma = (\gamma_1,\gamma_2)$. See Figure 1.4. We can then write the coordinates (ϕ,p) as

$$\phi = \theta + \psi(s), \quad p = \gamma \times (\cos\phi, \sin\phi).$$

Computing the derivative gives

$$\begin{aligned}
d\phi &= d\theta + \psi' ds \\
dp &= (\gamma_1 \sin\psi - \gamma_2 \cos\psi)' = (\gamma_1' \sin\psi - \gamma_2' \cos\psi)ds + (\gamma_1 \cos\psi + \gamma_2 \sin\psi)d\phi
\end{aligned}$$

By the properties of the wedge product, we have $d\phi \wedge dp = (\gamma_1' \sin\psi - \gamma_2' \cos\psi)\ d\theta \wedge ds$.

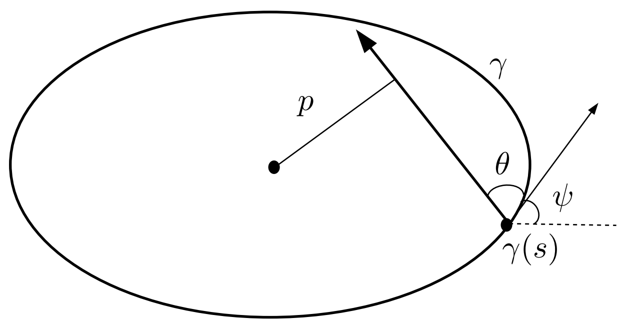

Fig. 1.4. Correspondence between the two area forms

Since $(\gamma_1',\gamma_2') = (\cos\psi, \sin\psi)$, then

$$\begin{aligned}
\cos\psi \sin\phi - \sin\psi \cos\phi &= \cos(\phi-\theta)\sin\phi - \sin(\phi-\theta)\cos\phi \\
&= (\cos\phi\cos\theta + \sin\phi\sin\theta)\sin\phi - (\sin\phi\cos\theta - \cos\phi\sin\theta)\cos\phi \\
&= (\sin^2\phi + \cos^2\phi)\sin\theta = \sin\theta.
\end{aligned}$$

Hence, $d\phi \wedge dp = \sin\theta\ d\theta \wedge ds$, which concludes the proof. $\square$

To sum up, for a convex billiard table, the billiard ball map T can be viewed as map of the space of oriented lines which intersect the table and this map preserves the area form Ω.

1.2. Periodic orbits

Among the billiard orbits, those of particular interest are periodic orbits, which in the simplest terms, are the vertices of a n-gon if the period is n. Let $X := \{x_k\}_{k\in\mathbb{Z}} = \{(s_k,\theta_k)\}_{k\in\mathbb{Z}}$ be an orbit such that there exists $n \geq 2$ such that $x_{k+n} = x_k$ for all $k \in \mathbb{Z}$. The smallest n that satisfies the last relation is called the *period* of X.

However, periodic orbits of same period can be topologically very different. A topological invariant that allows us to distinguish among these periods is the *rotation number*.

See [9]. For X an periodic orbit of period n, consider the n-tuple $(s_1,\dots,s_n)$. For all $1 \leq k \leq n$, there exists a quantity $\lambda_k \in [0,l]$ such that $s_{k+1} = s_k + \lambda_k$. We can see it as the oriented arclength distance between two reflection points. Since X is periodic, we have $\lambda_1 + \cdots + \lambda_k \in l\mathbb{Z}$ and more precisely this sum is between l and $(n-1)l$. We define the *winding number* of the orbit to be $m := \frac{\lambda_1+\cdots+\lambda_k}{n}$, which represents the number of turns that our trajectory makes around the table before coming back to (s_1,θ_1). The rotation number of X will then be defined as $\rho(X) := \frac{m}{n}$. Note that we do not distinguish between opposite orientations of the trajectory, and since changing the orientation replaces the winding number m by $n-m$, then the rotation number $\rho(X)$ will take values in $(0,\frac{1}{2}] \subset \mathbb{Q}$.

In order to consider periodic orbits, we first have to show the existence of those orbits by proving the Birkhoff's theorem. We only need to prove the existence of one periodic orbit (i.e. of maximum length) for later propose.

Theorem 1.2.1 (Birkhoff)**.** *Let n be the number of reflection points of a strictly convex billiard table and m its winding number. For every $n \geq 2$ and $m \leq \lfloor (n-1)/2 \rfloor$, there exists two geometrically distinct n-periodic trajectories with winding number m.*

Remark 1.2.2. *Such periodic orbits are also called Birkhoff periodic orbits of type (m,n).*

Proof. ([8], [20]) We consider the space $C_{m,n}$ of potential periodic orbits that make m turns in counterclockwise direction around the table before coming back to the initial point. In other words, we can see them as n-gons inscribed into the table with vertices $x_1,x_2,\dots,x_n$ on the boundary. Let s and s' the arc-length corresponding to the points p and p' on the boundary and define $H(s,s') := -d(s,s')$ to be the negative of the Euclidean distance between p and p'. We now define the perimeter length function (an action functional)

$$L_{m,n}(x_1,\dots,x_n) = -(H(s_1,s_2) + H(s_2,s_3) + \cdots + H(s_n,s_1)).$$

Let us first show a lemma.

Lemma 1.2.3. *If θ' is the angle between the segment joining p and p' with the tangent at p' and θ is the angle between the segment joining p and p' with the tangent at p, then*

$$\frac{\partial}{\partial s'} H(s,s') = -\cos\theta',$$
$$\frac{\partial}{\partial s} H(s,s') = \cos\theta.$$

Proof. Let γ be a unit-speed parametrization of the boundary $\partial\Omega$. Then

$$\begin{aligned}\frac{\partial}{\partial s'}H(s,s') = -\frac{d}{dt}d(p,\gamma(t)) &= -\frac{d}{dt}\sqrt{\langle \gamma(t)-p,\gamma(t)-p\rangle} \\ &= -\frac{1}{2\sqrt{\langle \gamma(t)-p,\gamma(t)-p\rangle}}\cdot 2\langle \gamma'(t),\gamma(t)-p\rangle \\ &= -\frac{\langle \gamma'(t),\gamma(t)-p\rangle}{\|\gamma(t)-p\|}\end{aligned}$$

For $t = s'$, we have

$$\frac{\partial}{\partial s'}H(s,s') = -\frac{\langle \gamma'(s'),\gamma(s')-p\rangle}{\|\gamma(s')-p\|} = -\frac{\|\gamma'(s')\|\,\|\gamma(s')-p\|\cos\theta'}{\|\gamma(s')-p\|} = -\cos\theta'$$

since γ is unit-speed so $\|\gamma'(s')\| = 1$. The second equation is proved in the same way. $\square$

Using this lemma, we can show that three successive reflections points x_{i-1},x_i,x_{i+1} form an orbit segment if and only if the partial derivative of $L_{m,n}$ with respect to x_i vanishes. In fact, three points x_{-1},x_0,x_1 are on the same orbit if the angle between the segment $\overline{x_{-1}x_0}$ and the tangent at x_0 is equal to the angle between the segment $\overline{x_0x_1}$ and the tangent at x_0. Hence,

$$\frac{\partial}{\partial s}H(s_{-1},s) + \frac{\partial}{\partial s}H(s,s_1) = 0 \quad \text{at } s = s_0.$$

Therefore, x_0 is a critical point of the functional $s \mapsto H(s_{-1},s) + H(s,s_1)$. Iterating this process, we have that critical points of $L_{m,n}$ are exactly the configurations of Birkhoff periodic orbits. And to show there are two distinct Birkhoff periodic orbits of type (m,n), it suffices to show that there are two critical points for the perimeter length function.

We first show that the configuration corresponding to the maximum of $L_{m,n}$ is a Birkhoff orbit. Since $C_{m,n}$ is not compact, we consider instead its closure $\overline{C}_{m,n}$. This closure is obtained by adding configurations where successive points may coincide, which corresponds to allowing zero side lengths. These configurations are the degenerate polygons with fewer than n sides of positive length. We know that $L_{m,n}$ attains the maximum on $\overline{C}_{m,n}$ since it's compact, and we want to show that this maximum is attained in the interior $C_{m,n}$ of $\overline{C}_{m,n}$, that is, not on the degenerate polygons. The triangle inequality shows that the perimeter increases when we increase the number of sides (see Figure 1.5). Hence, the maximum can not be achieved on k-gons with $k < n$ and it is a critical point of $L_{m,n}$ by Fermat's Theorem. Therefore, we found a a periodic orbit of type (m,n) having the maximum length for L. $\square$

Remark 1.2.4. *We can use the minimax (mountain pass) principle to show the existence of another critical point for $L_{m,n}$. As the name suggests, it tells that to pass from one peak of the mountain to another, we have to pass through a saddle or a mountain pass (see Figure 1.6). Note that a cyclic permutation of the configuration, i.e. $(x_2,\cdots,x_n,x_1)$, is also a maximum of $L_{m,n}$. We join these two maxima by a smooth path $\gamma(t) = (\gamma_1(t),\ldots,\gamma_n(t))$,*

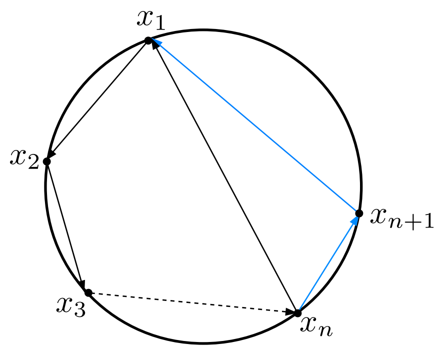

Fig. 1.5. Increasing perimeter of n-gon

$0 \leq t \leq 1$, such that $\gamma_i(t)$ stays between x_i and x_{i+1} for all $i = 1, \ldots, n$. The perimeter length function is either constant or attains a minimum value strictly less than common value at x_i and x_{i+1} on such paths. Taking the maximum over these possible paths, and by differentiating the function $L_{m,n}$ on it we show that this corresponds to a critical point different from the first one.

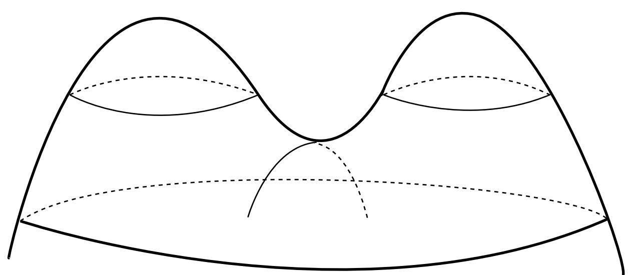

Fig. 1.6. Mountain pass type critical point

1.3. Caustics

An object in geometrical optics that appears to be extremely useful for our billiard problem is the caustic. In the simplest terms, a caustic is a curve inside the billiard table such that if a segment of the billiard trajectory is tangent to it, it will remain tangent after reflection. (See Figure 1.7) To make this definition more precise, we need to define the notion of an invariant circle.

Definition 1.3.1. *An invariant circle Γ for a map ϕ is a ϕ-invariant set in its domain of definition. In particular, an invariant circle for the billiard ball map T is a simple closed T-invariant curve inside the phase space M making one turn around M.*

If an invariant circle Γ is a smooth curve, then Γ can be described as a smooth one-parameter family of oriented lines intersecting the billiard table. A caustic is said to be associated to a invariant circle if the family of tangent lines (which can be also called rays) defining it constitutes an invariant circle of ϕ. This is in fact the case with convex caustics.

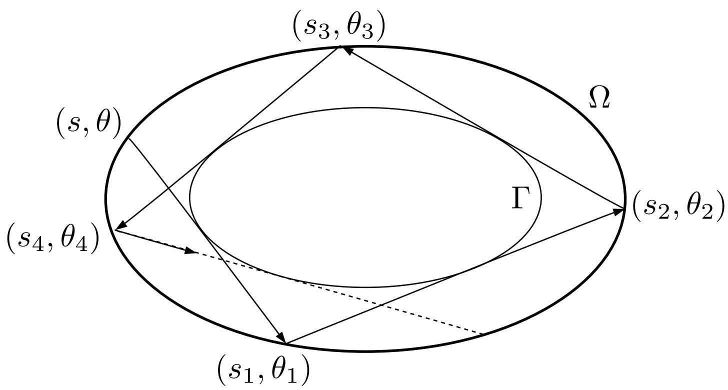

Fig. 1.7. Caustic Γ inside the domain Ω

Therefore, the caustic is the envelope of rays defining it, which is a smooth curve tangent to every ray in the family. Another way to see it is to consider the intersection of all left (or right) half-planes to these rays, and the boundary of this region is a caustic.

A natural question to ask is whether such caustics exist in our billiard table. Mather showed in his paper (see [**14**]) that in general, for an arbitrary convex domain, continuous family of caustics does not exist. However, if the boundary of the table is sufficiently smooth (with 553 continuous derivatives, but this number is later reduced to 6!) and its curvature is everywhere positive, there exists a discontinuous family of caustics near the boundary of the table.

Since we are working with strictly convex billiard table with convex boundary, we can assume the existence of caustics inside the table. One useful fact about the caustics is given by Birkhoff's theorem (see [**2**]).

Theorem 1.3.2. *(Birkhoff) In the standard coordinates* (s,θ) *in the phase space* M*, the caustic is a graph* $\theta = f(s)$ *of a continuous function* f*.*

As we will see in Chapter 5, this result is the heart of our proof together with twist mapping in Section 1.4 and Lazutkin's coordinates in Section 1.5.

1.4. Twist mapping

An important and interesting fact is that the billiard ball map is a twist map, as we will prove in this section. For a more detailed description of the twist mapping, see Chapter 3.

We begin by defining the twist map in the most general case.

Definition 1.4.1 (Twist map, [**8**]). *A diffeomorphism* $\phi : C \to C$ *of the open cylinder* $C = S^1 \times (-1,1)$ *is said to be a* ***twist map*** *if*

(1) it preserves orientation, and there exists $\varepsilon > 0$ such that

$$(x,y) \in S^1 \times (-1,\varepsilon - 1) \Longrightarrow \phi(x,y) \in S^1 \times (-1,0);$$

(2) $\frac{\partial}{\partial y}\Phi_1(x,y) > 0$, where $\Phi = (\Phi_1,\Phi_2)$ is a lift of ϕ to the strip $S = \mathbb{R} \times (0,1)$;

(3) ϕ extends to a homeomorphism $\overline{\phi}$ of the closed cylinder $S^1 \times [-1,1]$, which preserves the boundary components, i.e.

$$\overline{\phi}(S^1 \times \{\pm 1\}) = S^1 \times \{\pm 1\}.$$

*The map ϕ is said to be a **differentiable twist map** if for $\varepsilon > 0$, there exists $\delta > 0$ such that $\frac{\partial}{\partial y}\Phi_1(x,y) > \delta$ on $C_\varepsilon := S^1 \times (\varepsilon - 1,1 - \varepsilon)$.*

Below we prove that the billiard ball map $T : C \to C$ of the open cylinder $S^1 \times (-1,1)$ is a twist map. We will prove a more complete result in Chapter 3.

Proof. We can reparametrize the boundary $\partial\Omega$ or normalize the arclength parameter s by dividing by the length of the boundary l to have $x \in [0,1]$. By taking $y = -\cos\theta$, we have $y \in (-1,1)$. Therefore, the billiard ball map T is a mapping from the open cylinder C to C.

To see that T is a diffeomorphism, it is firstly invertible, since by reflection law, we can find the previous reflection point of any given point on the boundary. It is also smooth by setting $r = -\cos\theta$ and applying the Implicit Function Theorem [**8**] to the map

$$0 = F(s,s',r,r') := \begin{pmatrix} \frac{\partial}{\partial s'}H(s,s') - r' \\ \frac{\partial}{\partial s}H(s,s') + r \end{pmatrix},$$

so that $\frac{\partial}{\partial s'}H(s,s') = r'$ and $\frac{\partial}{\partial s}H(s,s') = -r$ as in Lemma 1.2.3. The function H is called the generating function of the billiard ball map T. Hence, T and T^{-1} are both smooth.

We now verify the three conditions of a twist map. Increasing the angle θ will increase y, and therefore will increase the distance x between s and s' (see Figure 1.8). We have $\frac{\partial}{\partial y}\Phi_1(x,y) > 0$, and the condition (2) is verified.

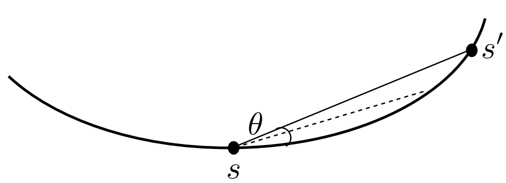

Fig. 1.8. Increasing θ increases s'

The condition (3) is easy to see, since we can set $\theta = 0$ (so $y = -1$) if $x' = x$ and $\theta = \pi$ (so $y = 1$) if $x' = x + 1$. In other words, making an angle of 0 means that the billiard ball did not move upon reflection, and making and angle of π means that it made a whole turn around the table and came back to the initial position. Therefore, T can be extended to $\overline{T}$

of the closed cylinder.

Finally, we verify condition (1). By definition, a differentiable map $f : O \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to preserve orientation if $|Jf| > 0$. We use the coordinates (s,r) as defined previously, so the billiard ball map becomes $T(s,r) = (S(s,r), R(s,r))$. We set $\tilde{H}(s,r) := H(s, S(s,r))$ where H is the generating function of T. Partially deriving $\tilde{H}(s,r)$ with respect to s and r gives

$$\frac{\partial \tilde{H}}{\partial s} = \frac{\partial H}{\partial s} + \frac{\partial H}{\partial s'}\frac{\partial S}{\partial s} = -r + R\frac{\partial S}{\partial s}$$
$$\frac{\partial \tilde{H}}{\partial r} = \frac{\partial H}{\partial s'}\frac{\partial S}{\partial r} = R\frac{\partial S}{\partial r}$$

Calculating $\partial^2 \tilde{H}/\partial s \partial r$ in two different ways (by Clairaut's Theorem) gives

$$-1 + \frac{\partial R}{\partial r}\frac{\partial S}{\partial s} + R\frac{\partial^2 S}{\partial r \partial s} = \frac{\partial^2 \tilde{H}}{\partial r \partial s} = \frac{\partial^2 \tilde{H}}{\partial s \partial r} = \frac{\partial R}{\partial s}\frac{\partial S}{\partial r} + R\frac{\partial^2 S}{\partial s \partial r},$$

which can be simplified to

$$1 = \frac{\partial R}{\partial s}\frac{\partial S}{\partial r} - \frac{\partial R}{\partial r}\frac{\partial S}{\partial s}.$$

The right-hand side of the last equation is the determinant of the Jacobian of T. Hence by definition, T preserves orientation.

Lastly, we show that T preserves the boundary components. More precisely, the y-coordinate being in the interval $(-1,0)$ after reflection means that the angle θ_1 that the trajectory makes with the boundary at the next reflection point is between 0 and $\frac{\pi}{2}$. Let $\gamma(s)$ be an arclength parametrization of the boundary and let s_0 be the initial position and s_1 be the position upon reflection. We draw a line between s_0 and s_1 and we let $v = (\cos\theta, \sin\theta)$ be the unit direction vector of that line. Finally, let θ_0 be the angle between v and $\dot{\gamma}(s_0)$, and θ_1 be the angle between v and $\dot{\gamma}(s_1)$ (see Figure 1.9).

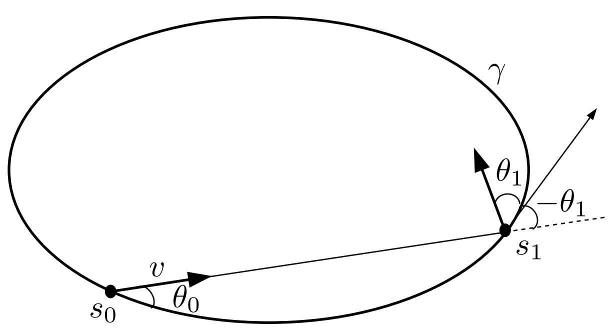

Fig. 1.9. Billiard trajectory with direction vector

We want to prove that if $|\angle(v,\dot{\gamma}(s_0))|$ is sufficiently small, then $|\angle(v,\dot{\gamma}(s_1))| < \frac{\pi}{2}$. This can be shown by a suitable application of the inverse function theorem. However, we choose a more geometric approach, which yields stronger estimates. We first show a lemma whose proof we defer to the end of the section.

Lemma 1.4.2. *Let* Γ *be the boundary of a strictly convex domain in* $\mathbb{R}^2$*. There exists* $R > 0$ *such that for all* $x \in \Gamma$ *there exists a circle of radius* R*, tangent to* Γ *at* x *and containing* Γ*.*

We then prove that $|\theta_1 - \theta_0| \leq \dfrac{|s_1 - s_0|}{R}$. We first show that $\frac{d}{ds}\angle(v,\dot\gamma(s)) = \frac{d\theta}{ds} = \kappa_s$, where κ_s is the signed curvature (see [**18**]). We can rewrite $\dot\gamma(s)$ as $(\cos\theta, \sin\theta)$, so that

$$\ddot\gamma(s) = \theta'(-\sin\theta, \cos\theta).$$

We know that the unit normal vector is $\mathbf{n} = (-\sin\theta, \cos\theta)$. Then

$$\ddot\gamma(s) = \kappa_s \mathbf{n} \Longrightarrow \kappa_s = \theta' = \frac{d\theta}{ds}.$$

Therefore by Lemma 1.4.2,

$$|\theta_1 - \theta_0| = \int_{s_0}^{s_1} \kappa_s(s)ds \leq \int_{s_0}^{s_1} \frac{1}{R} ds = \frac{|s_1 - s_0|}{R}. \tag{1.4.1}$$

Finally, we claim that $|s_1 - s_0|$ is small if $\angle(v,\dot\gamma(s_0))$ is small. Indeed, by Lemma 1.4.2, we have a circle of radius R tangent to Γ at s_0 and containing Γ. Let $\tilde{s_1}$ be the intersection of the extension of the segment from s_0 to s_1 with the circle. We denote by $d(\cdot,\cdot)$ the euclidean distance between two points. We have

$$d(s_0,s_1) \leq d(s_0,\tilde{s_1}) \leq R \cdot \theta_0.$$

Denoting by $d_{\text{arc}}(\cdot,\cdot)$ the arclength between two points, and using the Crofton formula (see [**20**]), we obtain

$$d_{\text{arc}}(s_0,s_1) \leq d_{\text{arc}}(s_0,s_1) + d(s_0,s_1) \leq R \cdot \theta_0 + d(s_0,\tilde{s_1}) \leq 2R \cdot \theta_0.$$

Therefore, $\theta_0 = \angle(v,\dot\gamma(s_0))$ is small implies that $|s_1 - s_0|$ is small.

By (1.4.1), $|\theta_1 - \theta_0|$ is small. Hence, θ_1 is small if θ_0 is small, concluding the proof. □

We now prove the lemma.

Proof. The proof of this lemma is given by Blaschke's rolling ball theorem [**3**], which says that we can choose R to be $1/\kappa_{\min}$, where κ is the curvature of Γ. Let $\gamma : [0,l(\Gamma)] \to \Gamma$ be an arclength parametrization of Γ and s be the arclength parameter. Let $\gamma(s)$ be a point of the boundary. We define $p(\theta)$ to be the distance of the line segment starting at $\gamma(s)$ until the intersection with Γ by making an angle of θ with the tangent at $\gamma(s)$. We also define $q(\theta)$ to be the distance of the line segment starting at $\gamma(s)$ until the intersection with the circle of radius $\kappa_{\min}^{-1}$ tangent to the boundary.

We want to show that $p(\theta) \leq q(\theta)$ for all $\theta \in [0,2\pi]$. This is equivalent to show that $p'(\theta) \leq q'(\theta)$.

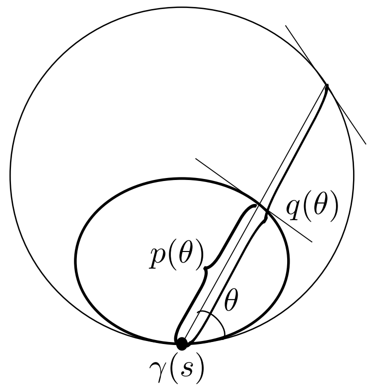

Fig. 1.10. Convex boundary inside the circle

We have that $p(\theta) = \gamma(s(\theta)) \cdot e^{i\theta}$. The angle θ is such that $\gamma'(s(\theta)) = ie^{i\theta} = e^{i\theta + \frac{\pi}{2}}$. By calculation,

$$p'(\theta) = \underbrace{\gamma'(s(\theta))\frac{ds}{d\theta} \cdot e^{i\theta}}_{=0} + \gamma(s(\theta)) \cdot e^{i\left(\theta+\frac{\pi}{2}\right)} = \gamma(s(\theta)) \cdot e^{i\left(\theta+\frac{\pi}{2}\right)},$$

since $\gamma(s)$ is perpendicular to $e^{i\theta}$. By differentiating again, we have

$$p''(\theta) = \gamma'(s(\theta))\frac{ds}{d\theta} \cdot e^{i(\theta+\frac{\pi}{2})} - \gamma(s(\theta) \cdot e^{i\theta} = \gamma'(s(\theta))\frac{ds}{d\theta} \cdot e^{i(\theta+\frac{\pi}{2})} - p'(\theta). \tag{1.4.2}$$

On the other hand, $\gamma'(s) = e^{i*\theta+\frac{\pi}{2})}$, so

$$\gamma''(s) = -e^{i\theta} \cdot \frac{d\theta}{ds} \quad \text{and} \quad \gamma''(s) = i \cdot \gamma'(s) \cdot \kappa(s) = e^{i\theta} \cdot \kappa(s)$$

by the definition of the curvature κ. Combining those two equations, we get

$$\frac{d\theta}{ds} = \kappa(s) \Longrightarrow \frac{ds}{d\theta} = \kappa^{-1}(s).$$

Therefore, (1.4.1) becomes

$$p''(\theta) + p'(\theta) = \kappa^{-1}(\theta).$$

By solving this differential equation of second order for $0 \le \theta \le \pi$, we obtain

$$p(\theta) = \int_0^\theta \kappa_1^{-1}(\phi)\sin(\theta - \phi)d\phi.$$

Similarly, we can find

$$q(\theta) = \int_0^\theta \kappa_2^{-1}\sin(\theta - \phi)d\phi.$$

In the case where $-\pi \le \theta \le 0$, we have

$$p(\theta) = -\int_\theta^0 \kappa_1^{-1}(\phi)\underbrace{\sin(\theta-\phi)}_{\le 0}d\phi = \int_0^\theta \kappa_2^{-1}\sin(\theta-\phi)d\phi$$
$$q(\theta) = -\int_\theta^0 \kappa_2^{-1}\sin(\theta-\phi)d\phi = \int_0^\theta \kappa_2^{-1}\sin(\theta-\phi)d\phi.$$

Since κ_2 is the minimal curvature, we have $p(\theta) \le q(\theta)$. □

1.5. Lazutkin's coordinates

An essential element in the proof of our theorem is the change of coordinates in order to make the billiard ball map having a nice form. In this section, we will see how this is done with the Lazutkin's coordinates. We refer the reader to Chapter 2 and [**12**] for a more detailed description of the Lazutkin's theory.

We have seen that the phase space M of the billiard dynamics can be associated with the arclength coordinate s and the angle coordinate θ. Under these coordinates, the billiard ball map T becomes

$$T(s,\theta) = (s_1,\theta_1),$$

where (s_1,θ_1) are the coordinates of the reflection point. Lazutkin showed that s_1 and θ_1 can be written explicitly in terms of s and θ as follow:

$$\begin{aligned} s_1 &= s + \alpha_1(s)\theta + \alpha_2(s)\theta^2 + \alpha_3(s)\theta^3 + F(s,\theta)\theta^4 \\ \theta_1 &= \theta + \beta_2(s)\theta^2 + \beta_3(s)\theta^3 + G(s,\theta)\theta^4 \end{aligned} \tag{1.5.1}$$

with

$$\begin{aligned} \alpha_1(s) = 2\rho(s),\ \alpha_2(s) = \frac{4}{3}\rho(s)\rho'(s),\ \alpha_3(s) = \frac{2}{3}\rho^2(s)\rho''(s) + \frac{4}{9}\rho(s)\rho'(s) \\ \beta_2(s) = -\frac{2}{3}\rho'(s),\ \beta_3(s) = -\frac{2}{3}\rho(s)\rho''(s) + \frac{4}{3}\rho'^2(s). \end{aligned} \tag{1.5.2}$$

The functions $F(s,\theta)$ and $G(s,\theta)$ are continuous and have three less derivatives than the radius of the curvature $\rho(s)$.

Let $\kappa(s)$ be the curvature of $\partial\Omega$ at $\gamma(s)$. We consider the neighborhood of the line $\theta = 0$, which is the region close to the boundary $\partial\Omega$. Lazutkin introduced a special change of coordinates (x,y) with

$$\begin{aligned} x &= C_1 \int_0^s \kappa^{2/3}(s)\ ds \\ y &= C_2 \kappa^{-1/3}(s) \sin\frac{\theta}{2} \end{aligned} \tag{1.5.3}$$

where

$$C_1 = \left(\int_{\partial\Omega} \kappa^{2/3}\, ds\right)^{-1}, C_2 = 4C_1.$$

The constant C_1 is chosen to make the period of x unity, so that $x \in S^1$. The choice of the coordinates x and y is due to diffraction theory and the invariant measure having the form $dm = \text{const}|y|dxdy$. The equations (1.5.1), (1.5.2) and (1.5.3) imply that the billiard ball map takes the form

$$T(x,y) = (x + y + O(y^3), y + O(y^4)),$$

which will make the proof of our main theorem about the uniform distribution of the reflection points much more easier.

In particular, near $\{y = 0\}$, this map is reduced to small perturbation of $(x,y) \mapsto (x + y,y)$. Lazutkin proved using this and KAM theory that for sufficiently smooth boundary $\partial\Omega$ there exists a set of caustics with positive measure accumulating on $\partial\Omega$.

1.6. Main result

In this thesis, we restrict the domains of our billiard table to be strictly convex, and we study the distribution of the set of reflection of the periodic billiard orbit in these domains. We have the following theorem.

Theorem 1.6.1 (Bialy, Mironov, Tabachnikov [**1**]). *In a strictly convex billiard table with curvature $\kappa > 0$, the distribution of reflection points of Birkhoff periodic orbits of type $(1,n)$ is uniform with respect to the measure $\kappa^{2/3}\, dx$ as n tends to ∞. In other words, for an interval $(a,b) \subset \partial\Omega$, we have*

$$\frac{\#\{x_n^j \in I\}}{n} \to \frac{\int_I \kappa^{2/3}\, dx}{\int_{\partial\Omega} \kappa^{2/3}\, dx},$$

where $x_n^1, \ldots, x_n^n$ are the reflection points.

This result has been proven by Melrose in his paper [**13**] using the theory of interpolating Hamiltonians. However, we found a new proof using Lazutkin's coordinates that will be elaborated in Chapter 5. The idea of this proof is to prove that the Lazutkin's coordinate y is bounded by C/n along the $(1,n)$ Birkhoff periodic orbit γ_n for some constant $C \in \mathbb{R}$. To do this, we bound y along a caustic in term of angle η defined by Lazutkin. We then "sandwich" γ_n by two caustics with $\eta \sim \frac{1}{n}$.

Once we have this bound, we prove the uniform distribution by showing that the j-th reflection x_j can be written as $x_0 + \frac{j}{n} + O(\frac{1}{n^2})$ where x_0 is the initial point of the billiard orbit. We then show that given an interval (a,b), for sufficiently large n, the difference between the number of x_j and the number of $x_0 + \frac{j}{n}$ in this interval is small (≤ 2).

Chapter 2

Lazutkin's Theory

As we can see in the introduction, the core idea of the method of proof of our main result is to use Lazutkin's coordinates. In this chapter, we will give the detailed theory behind those "magical" coordinates. We first briefly introduce the concept of caustics and quantities associated to it. Then, we discuss about Lazutkin's change of coordinates and the billiard ball map expressed in those coordinates. Finally, we show the uniform distribution of billiards's reflection points in round circle. We refer the reader to [**12**] for an elaborated discussion about this theory.

2.1. Caustics problem

When studying the billiard dynamics, one can be interested about the caustic of the domain (billiard table). A caustic Γ of a planar convex domain Ω is a smooth closed convex curve such that when a tangent is drawn to Γ at any point, it will remain tangent to Γ after reflection according to the reflection rule.

A natural question that we can ask about caustics is whether such object exists for a given convex domain. Mather proved in [**14**] that in general such caustic does not exist, unless the boundary $\partial\Omega$ is sufficiently smooth with its curvature being bounded by positive constant. In this case, there exists a discontinuous family of caustics in a small neighborhood of $\partial\Omega$.

The caustic is characterised by some quantities that are naturally associated with. First of all, let M be a point on the caustic. The tangent drawn to M will intersect a point on the boundary $\partial\Omega$, and we denote it by N. The reflection of the tangent line will be tangent to another point on the caustic, and we denote it by M_1.

Remark 2.1.1. *We define the number Q to be*

$$Q = |NM| + |NM_1| - \widehat{MM_1}$$

where $\widehat{MM_1}$ is by convention the arc-length of the convex upward arc. Since the quantity Q is independent of the point N, we can see that each caustic is characterised by an unique Q.

Then, we notice that the map $M \mapsto M_1$ is a diffeomorphism of Γ onto itself. For circle homeomorphisms, Poincaré introduced a topological invariant, the rotation number $\eta = \eta(\Gamma)$, associated to each caustic Γ [**16**]. We will discuss this number in detail in Chapter 3. For now, we only consider caustics with irrational rotation number.

Finally, if the map $M \mapsto M_1$ is sufficiently smooth, we can see it as a map $\xi \mapsto \xi + \eta$, where ξ depends on the arclength. If one turn is made on the caustic, the map becomes $\xi \mapsto \xi + 1$. Such parameter, introduced by Denjoy [**4**], is called the normalized parameter.

An important property of Lazutkin's caustics is that they lie near the boundary $\partial\Omega$, and their rotation numbers are small and cannot be well-approximated by rational numbers. This is suggested by the following result of Lazutkin, which we will use below. For the reader's convenience we present its proof.

Proposition 2.1.2. *Let $\sigma \geq 3$ be an integer, and $A < 1$ be a positive number. Set*

$$E := \{\eta \in \mathbb{R} : |n\eta - m| \geq A|m|n^{-\sigma+\frac{1}{2}} \ \forall m \in \mathbb{Z}, n \in \mathbb{N}\}.$$

For α small, say $\alpha < 1$, let $E(\alpha) := E \cap [0,\alpha]$. Then its Lebesgue measure has a lower bound

$$\mu(E(\alpha)) > \alpha - c_1\alpha^{\sigma-\frac{1}{2}},$$

where c_1 only depends on σ and A.

Proof. Notice that $E(\alpha) = [0,\alpha]\backslash E(\alpha)^c$. By properties of Lebesgue measure (see Appendix A), we have

$$\mu(E(\alpha)) = \mu([0,\alpha]) - \mu(E(\alpha)^c) = \alpha - \mu(E(\alpha)^c).$$

We only need to show that $\mu(E(\alpha)^c) < c_1\alpha^{\sigma-\frac{1}{2}}$. By definition,

$$E(\alpha)^c = \{\eta \in [0,\alpha] : \exists m \in \mathbb{Z}, m \in \mathbb{N} \text{ s.t. } |n\eta - m| < A|m|n^{-\sigma+\frac{1}{2}}\}$$

We fix m,n and we denote by

$$E_{m,n}(\alpha)^c = \{\eta \in [0,\alpha] : |n\eta - m| < A|m|n^{-\sigma+\frac{1}{2}}\}.$$

Notice that $E(\alpha)^c \subset \bigcup_{m,n} E_{m,n}(\alpha)^c$, so $\mu(E(\alpha)^c) \leq \sum_{m,n} \mu(E_{m,n}(\alpha)^c)$. Note that m has to be positive, since otherwise dividing $|n\eta - m| < \frac{A|m|}{n^{\sigma-1/2}}$ by $|m|$ yields $|\frac{n}{m}\eta - 1| < \frac{A}{n^{\sigma-1/2}}$ and $|\frac{n}{m}\eta - 1| > 1$, but $\frac{A}{n^{\sigma-1/2}} < A < 1$.

We show three claims. We first show that $\mu(E_{m,n}(\alpha)^c) \leq \frac{2A|m|}{n^{\sigma+\frac{1}{2}}}$. For $\eta \in E_{m,n}(\alpha)^c$, we have

$$\begin{aligned} &-\frac{A|m|}{n^{\sigma-1/2}} < n\eta - m < \frac{A|m|}{n^{\sigma-1/2}} \\ \iff\ &-\frac{A|m|}{n^{\sigma+1/2}} + \frac{m}{n} < \eta < \frac{A|m|}{n^{\sigma+1/2}} + \frac{m}{n}. \end{aligned}$$

Therefore, $\mu(E_{m,n}(\alpha)^c) \leq \mu((-\frac{A|m|}{n^{\sigma+1/2}} + \frac{m}{n}, \frac{A|m|}{n^{\sigma+1/2}} + \frac{m}{n})) = 2\frac{A|m|}{n^{\sigma+1/2}}$.

We then show that if $n < \frac{1-A}{\alpha}$ and $\eta \leq \alpha$, then $\mu(E_{m,n}(\alpha)^c) = 0$. We have

$$\Big|\frac{n}{m}\eta - 1\Big| = 1 - \frac{n}{m}\eta \geq 1 - \frac{(1-A)}{\alpha \cdot m}\alpha = A,$$

and this is only true for $E_{m,n}(\alpha)$. Hence, $\mu(E_{m,n}(\alpha)^c) = \mu(\varnothing) = 0$.

Lastly, we show that if for some constant $c > 0$ we have $m > cn\alpha$, then $\mu(E_{m,n}(\alpha)^c) = 0$. We know that $\eta \in E_{m,n}(\alpha)^c$ satisfies $\eta < \frac{A|m|}{n^{\sigma+1/2}} + \frac{m}{n} \leq \alpha$. In particular, $\frac{m}{n} \leq c\alpha$ for some positive constant c, so $m \leq cn\alpha$. However, if $m > cn\alpha$, then all the η would be in $E(\alpha)$, and $\mu(E_{m,n}(\alpha)^c) = \mu(\varnothing) = 0$.

With these observations in mind, we are ready to show that $\mu(E(\alpha)^c) < c_1\alpha^{\sigma-1/2}$.

First off,

$$\mu(E(\alpha)^c) \leq \sum_{m,n} \mu(E_{m,n}(\alpha)^c) \leq 2A \sum_{n \geq (1-A)/\alpha} \frac{1}{n^{\sigma+1/2}} \sum_{m \leq cn\alpha} m.$$

Now for some constant $C = \tilde{C}c^2/2$, we have that

$$\sum_{m \leq cn\alpha} m \leq C \int_0^{cn\alpha} m = \tilde{C}\frac{c^2n^2\alpha^2}{2} = Cn^2\alpha^2.$$

We therefore find that

$$2A \sum_{n \geq (1-A)/\alpha} \frac{1}{n^{\sigma+1/2}} \sum_{m \leq cn\alpha} m \leq 2AC\alpha^2 \sum_{n \geq (1-A)/\alpha} \frac{1}{n^{\sigma-2+1/2}}.$$

For some constant C_1, we have that

$$\sum_{n \geq (1-A)/\alpha} \frac{1}{n^{\sigma-2+1/2}} \leq C_1 \int_{(1-A)/\alpha}^{\infty} \frac{1}{x^{\sigma-2+1/2}} = C_2\frac{1}{((1-A)/\alpha)^{\sigma-3+1/2}} = C_3\alpha^{\sigma-3+1/2}$$

for some constant C_3. Then the result follows by letting $c_1 = 2ACC_3$. □

This proposition will allow us to prove in Chapter 5 the fact that rotation numbers of periodic orbits are between those of the caustics. The proof comes from Lazutkin [**12**], who showed in his Theorem 1 the existence of a discontinuous family of caustics. In fact, "caustic tube", which are the spaces where caustics can exist, are in the cap between periodic orbits with sufficiently small rational rotation number.

2.2. Lazutkin's coordinates

Let us recall the billiard system in (s,θ) coordinates. Let s be the arc-length parameter on the boundary $\partial\Omega$ of the billiard table. This corresponds to the position of the billiard ball on the boundary. To give a motion to the ball, we attach an unit inward vector $\vec{t}$ to the point s and we let θ to be the angle between this vector and the tangent vector $\dot{\gamma}(s)$ at s.

The phase space M of the billiard dynamics can be seen as a restriction of the unit tangent bundle of $\mathbb{R}^2$ onto $\partial\Omega$.

We define the billiard ball map by the diffeomorphism $T : M \to M$ as follows. The ball moves along a straight line in the direction of $\vec{t}$ starting from s until it hits the boundary. Upon collision, the trajectory is reflected according to the rule "the angle of incidence equals the angle of reflection". The position of the collision will be denoted by s_1 and the direction vector upon reflection, $\vec{t_1}$. The angle between $\vec{t_1}$ and $\dot{\gamma}(s_1)$ is henceforth θ_1. Hence, (s_1,θ_1) is the image of (s,θ) under the billiard map T.

When $\theta = 0$ or $\theta = \pi$, the point (s,θ) is a stationary point of the billiard map T. The set of all stationary points forms two stationary curves, namely $\{\theta = 0\}$ and $\{\theta = \pi\}$, which are simply our boundary curve depending on its orientation.

Consider the stationary curve $\{\theta = 0\}$ and the map T is a small neighborhood of this curve. Lazutkin stated that we can make calculations to find an expression for (s_1,θ_1):

$$\begin{aligned} s_1 &= s + \alpha_1(s)\theta + \alpha_2(s)\theta^2 + \alpha_3(s)\theta^3 + F(s,\theta)\theta^4 \\ \theta_1 &= \theta + \beta_2(s)\theta^2 + \beta_3(s)\theta^3 + G(s,\theta)\theta^4 \end{aligned} \tag{2.2.1}$$

where

$$\begin{aligned} \alpha_1(s) = 2\rho(s),\ \alpha_2(s) = \frac{4}{3}\rho(s)\rho'(s),\ \alpha_3(s) = \frac{2}{3}\rho^2(s)\rho''(s) + \frac{4}{9}\rho(s)\rho'(s) \\ \beta_2(s) = -\frac{2}{3}\rho'(s),\ \beta_3(s) = -\frac{2}{3}\rho(s)\rho''(s) + \frac{4}{3}\rho'^2(s). \end{aligned} \tag{2.2.2}$$

The functions $F(s,\theta)$ and $G(s,\theta)$ are continuous and have three less derivatives than the radius of curvature $\rho(s)$.

Let $\kappa(s)$ be the curvature of $\partial\Omega$ at $\gamma(s)$. Lazutkin then made a change of coordinates based on certain formulas in diffraction theory to make the billiard map taking a nice form. He introduced the coordinate system (x,y) as follows.

$$\begin{aligned} x &= C_1 \int_0^s \kappa^{2/3}(s)\ ds \\ y &= C_2 \kappa^{-1/3}(s) \sin\frac{\theta}{2} \end{aligned} \tag{2.2.3}$$

where

$$C_1 = \left(\int_{\partial\Omega} \kappa^{2/3}\ ds \right)^{-1},\ C_2 = 4C_1.$$

The choice of the constant C_1 is to make the period of x unity. In those coordinates, the billiard ball map becomes

$$T_L(x,y) = (x + y + O(y^3), y + O(y^4)). \tag{2.2.4}$$

If the coefficient of y^3 and y^4 is 0, the billiard map becomes $T_L(x,y) = (x + y,y)$, and has the continuous family of invariant curves y =const. Such a system is called an integrable system (see Chapter 4 for more details). Therefore, we can see the original map as a small perturbation of $T_L(x,y) = (x + y,y)$. KAM theory suggests that invariant curves will only deform by a little under this perturbation, meaning that those curves with irrational rotation number that are poorly approximated by rational numbers are stable under this perturbation.

Let us consider a more general billiard map than (2.2.4) using Lazutkin's coordinates.

$$T_L(x,y) = (x + y + O(y^{m+1}),y + O(y^{m+2})),$$

where $m \geq 1$ is a natural number, and the coefficient of y^{m+1} and y^{m+2} that we will denote by $f(x,y)$ and $g(x,y)$ respectively are functions of the class C^l, 1-periodic in x and specified in the strip $|y| < \alpha$. The rotation numbers of caustics near $\{y = 0\}$ will run through the set $E \cap [-b^\star, b^\star]$, with $b^\star$ a small enough positive constant. Let s be the desired number of continuous derivatives for the family of invariant curves associated to the caustics. Then the number of continuous derivatives of $f(x,y)$ and $g(x,y)$ must satisfy

$$l > (3\sigma + 6)\left\{(2s + 1)(m + 1) + \frac{4(s + 1)^3}{2s + 1}\right\}$$

with l being the smallest natural number satisfying this inequality. Finally, let $c_0 > 0$ be a real number such that

$$|D_x^{\rho_1} D_y^{\rho_2} f| + |D_x^{\rho_1} D_y^{\rho_2} g| \leq c_0, \quad 0 \leq \rho_1 + \rho_2 \leq l.$$

Lazutkin proved in [**12**] the following theorem that allows us to estimate x and y using the rotation number η and the normalized parameter ξ. We only give a simplified version of this theorem here. For the complete version of this theorem, see Section 1 of [**12**].

Theorem 2.2.1. *There exists a $b^\star \in E(\alpha)$, depending only on c_0, s, σ and A, and functions $p(\xi,\eta)$ and $q(\xi,\eta)$ of the class C^s that are specified in the strip $|\eta| \leq b^\star$, 1-periodic in η, and satisfy the condition that the formulas*

$$x = \xi + \eta^m p(\xi,\eta),$$
$$y = \eta + \eta^{m+1} q(\xi,\eta)$$

specify a diffeomorphic mapping of the strip $|\eta| \leq b^\star$ onto the strip $|y| \leq \alpha$.

2.3. Equidistribution in round circle

In this section, we will use Lazutkin's coordinates to show an easier case of our main theorem, that of the uniform distribution of the reflection points in the round circle S^1.

Let γ_n be a Birkhoff periodic orbit of type $(1,n)$. We denote by N_n the number of reflection points of γ_n, which is equal to n. We denote by $N_n(I)$ the number of reflection points in an interval $I = (a,b) \subset S^1$. Let $T : S^1 \times [0,\delta) \to S^1 \times [0,\delta)$ be the billiard map in a circle with

$$T_L(x,y) = (x + y,y),$$

where $\delta > 0$ is sufficiently small so the angle parameter y is small. By basic geometry, we can show that is the angle between the inward vector at a point and the tangent at the same point is θ, then the angle after reflection remains θ and the arc-length between those two points is 2θ. That justifies the form that the billiard map takes, i.e. without perturbation. Our goal is to prove that the reflection points on a circle are uniformly distributed.

Proposition 2.3.1. *The reflection points of periodic billiard orbit in a round circle are uniformly distributed, that is, for any interval $I \subset S^1$,*

$$\frac{N_n(I)}{N_n} \to |I|.$$

Proof. By Lazultkin's coordinates, we have

$$x + ny = x + 1 \Longrightarrow ny = 1 \Longrightarrow y = \frac{1}{n}.$$

If $(k-1)y \leq |I| < ky$, then for $f(I,y) \in \{0,1\}$,

$$N_n(I) = k - 1 + f(I,y).$$

Hence, $\frac{N_n(I)}{n} = \frac{k}{n} + O(\frac{1}{n})$ and $|I| = \frac{k}{n} + O(\frac{1}{n})$. Hence, $\frac{k}{n} \to |I|$, so $\frac{N_n(I)}{n} \to |I|$. $\square$

To see how this result is translated in terms of the arc length s, we know that the curvature of a circle is the inverse of its radius r, so $\kappa = \frac{1}{r}$. Lazutkin's coordinate x becomes

$$x = \frac{\int_0^s \kappa^{2/3}\, ds}{\int_{\partial\Omega} \kappa^{2/3}\, ds} = \frac{\int_0^s r^{-2/3}\, ds}{\int_{\partial\Omega} r^{-2/3}\, ds} = \frac{s}{2\pi r}.$$

Let $A : a \leq \frac{s}{2\pi r} \leq b$ be the arc corresponding to I. So

$$|A| = 2\pi r(b-a) = 2\pi r|I| \iff |I| = \frac{|A|}{2\pi r}.$$

Then in terms of s, $\frac{N_n(I)}{n} \to \frac{|A|}{2\pi r}$.

Chapter 3

Twist Maps

In this chapter, we will discuss a special type of map called a twist map and its lift to a strip, and we will show that our billiard ball map is in fact a twist map. The reader can refer to [**8**] for a thorough discussion about this topic.

3.1. Circle map and its lift

We shall first introduce the concept of a lift of a circle map to the strip.

A diffeomorphism from S^1 to S^1 is called a **circle map**. It has interesting properties that diffeomorphism of $\mathbb{R}$ does not necessarly have, such as the existence of periodic points of any period. The circle map is said to be orientation-perserving if the order of the points on the circle is perserved after mapping. In this section, we will only consider orientation-preserving circle map, since the orientation-reversing circle map is somehow similar.

The dynamics of the circle is often better understood if we consider its lift to the strip $\mathbb{R}$. We first define the map $\pi : \mathbb{R} \to S^1$ by the following:

$$\pi(x) = \exp(2\pi i x) = \cos(2\pi x) + i \sin(2\pi x).$$

It is not hard to see that infinitely many points x map to the same point $\pi(x) \in S^1$ and they are all integer translates of each other. We can now define the lift of the circle map.

Definition 3.1.1 ([**5**]). *A map $F : \mathbb{R} \to \mathbb{R}$ is called a **lift** of a circle map $f : S^1 \to S^1$ if*

$$\pi \circ F = f \circ \pi.$$

One can interpret this definition as the process of "unrolling" the circle up to an helix (see Figure 3.1). A point on the circle is lifted to the corresponding position on the strip after mapping. If the point comes back to that position on the circle after some iterations, it will instead go to the next layer on the strip.

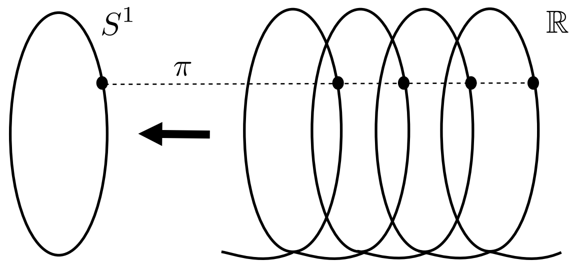

Fig. 3.1. The lift of a circle map

Example 3.1.2. *Let $\phi_\omega(\theta) = \theta + 2\pi\omega$ be the translation by the angle $2\pi\omega$. We can prove that the map $\Phi_{\omega,k}(x) = x + \omega + k$ is a lift of ϕ_ω for each integer k. We first compute $\pi \circ \Phi$.*

$$\begin{aligned}(\pi \circ \Phi)(x) = \pi(x + \omega + k) &= \exp(2\pi i(x + \omega + k)) \\ &= \exp(2\pi i(x + \omega)) \underbrace{\exp(2\pi i k)}_{=1\ \forall k \in \mathbb{Z}} \\ &= \exp(2\pi i x) \exp(2\pi i \omega)\end{aligned}$$

Then we compute $\phi \circ \pi$.

$$(\phi \circ \pi)(\theta) = \phi(\pi(\theta)) = \phi(\exp(2\pi i\theta)) = \exp(2\pi i\theta) + 2\pi\omega.$$

The two quantities $\exp(2\pi i x)\exp(2\pi i\omega)$ and $\exp(2\pi i\theta) + 2\pi\omega$ are the same, since multiplying by $\exp(2\pi\omega)$ rotates the angle by $2\pi\omega$ and translating the angle by $2\pi\omega$ is equivalent.

Below we state some properties of the lift of a circle map.

Proposition 3.1.3. *There are infinitely many lifts for a circle map. Indeed, the lift is unique up to an additive integer constant.*

Proof. Let F, $\tilde{F}$ be two lifts of $f : S^1 \to S^1$. Then by definition we have

$$\begin{aligned}(\pi \circ F)(x) = (f \circ \pi)(x) &= (\pi \circ \tilde{F})(x) \\ \iff\ & \pi(F(x)) = \pi(\tilde{F}(x)) \\ \iff\ & \exp(2\pi i F(x)) = \exp(2\pi i \tilde{F}(x)) \\ \iff\ & F(x) = \tilde{F}(x) + k, \quad \text{where } k \in \mathbb{Z}.\end{aligned}$$

Conversely, we show that if $F(x)$ is a lift, then $F(x) + k$ is also a lift.

$$\pi(F(x) + k)) = \exp(2\pi i F(x) + k) = \exp(2\pi i F(x))\exp(2\pi i k) = \pi(F(x)) = f(\pi(x)).$$

Thus, two lifts of f only differ by an integer constant. $\square$

Definition 3.1.4. *The quantity $\deg(f) := F(x+1) - F(x)$ is called the **degree** of f.*

Definition 3.1.5 ([8]). *Suppose f is invertible. If $\deg(f) = 1$, we say that f is orientation-preserving; if $\deg(f) = -1$, we say that f is orientation-reversing.*

Proposition 3.1.6. *If F is a lift of an orientation-preserving map f, then $F'(x) > 0$ and $F(x+1) = F(x) + 1$.*

Proof. Since f is orientation-preserving, $F(x)$ increases when x increases, so $F'(x) > 0$. By definition, $\deg(f) = 1$, and

$$1 = F(x+1) - F(x) \Longrightarrow F(x+1) = F(x) + 1.$$

A direct consequence of the last equality is that $F(x+1) - (x+1) = F(x) - x$, so that $F(x) - Id$ is a periodic function of period 1. □

We now give a different interpretation of the billiard dynamics using the concept of the lift. Let $\phi : C \to C$ of the open cylinder $C := S^1 \times (-1,1)$ be the billiard ball map. We can construct its lift Φ in the following way. Given a coordinate $(x,y) \in S := \mathbb{R} \times (-1,1)$ on the strip, we find the corresponding point $(s,y) \in C$ on the cylinder by projecting x modulo 1. We then find the coordinate of the point (s',y') after reflection by taking the image of (s,y) under the billiard ball map. We now choose the smallest $x' > x$ such that $x' \equiv s'$ mod 1. By letting $y \to 0$ after fixing s, we obtain a continuous map Φ, which is a lift of ϕ.

3.2. Rotation number

A fundamental topological invariant associated to a circle map is its rotation number. It measures the extent to which the points are rotated by each iteration. More precisely, it gives the proportion of points that make one turn after mapping, and therefore, this number is between 0 and 1. We will start be introducing a quantity that leads to the definition of the rotation number.

Let $f : S^1 \to S^1$ be an orientation-preserving diffeomorphism and F be its lift. We define

$$\rho(F) := \lim_{|n|\to\infty} \frac{F^n(x) - x}{n}. \tag{3.2.1}$$

We first show that this quantity does not depend on the choice of x. We will show in Proposition 3.2.4 below that the limit exists. From Proposition 3.1.6, if $|x - y| < 1$ (without loss of generality assume that $y > x$), then $y < x + 1$, and $F(y) < F(x+1) = F(x) + 1$. Similarly, $y > x - 1$, and $F(y) > F(x) - 1$. Therefore, $|F(y) - F(x)| < 1$. By induction, we have $|F^n(x) - F^n(y)| < 1$. Thus,

$$\left| \frac{F^n(x) - x}{n} - \frac{F^n(y) - y}{n} \right| \le \frac{1}{n}(|F^n(x) - F^n(y)| + |x - y|) \le \frac{2}{n}.$$

Taking the limit when $n \to \infty$, the difference between the two quantities is 0, so that ρ is independent of x. Note that for general x,y, there exists $y_0 = y + m$ with m being an integer, such that $\dfrac{F^n(y_0) - y_0}{n} = \dfrac{F^n(y) - y}{n}$ and $|x - y_0| < 1$.

However, ρ does depends on the choice of the lift. Let us consider the lift of the translation mentioned previously.

Example 3.2.1. *Let $\phi_\omega(\theta) = \theta + 2\pi\omega$ be translation and $\Phi_{\omega,k}(x) = x + \omega + k$ be a lift of ϕ_ω. We compute the number defined in (3.2.1).*

$$\rho(\Phi) = \lim_{|n|\to\infty} \frac{x + n\omega + nk - x}{n} = \omega + k.$$

We can see that different lifts will produce different values of ρ, but they all differ by an integer constant.

In fact, this is true for any lift of a circle map f. Let F_1 and F_2 be two lifts of f. By Proposition 3.1.3, there exists an integer k such that $F_2(x) = F_1(x) + k$. After one iteration,

$$F_2^2(x) = F_2(F_2(x)) = F_2(F_1(x) + k) = F_1(F_1(x) + k) + k = (F_1^2(x) + k) + k = F_1^2(x) + 2k.$$

It follows that $F_2^n(x) = F_1^n(x) + nk$, so that

$$\begin{aligned}
\rho(F_2^n) &= \lim_{|n|\to\infty} \frac{F_2^n(x) - x}{n} \\
&= \lim_{|n|\to\infty} \frac{F_1^n(x) + nk - x}{n} \\
&= \rho(F_1^n) + \lim_{|n|\to\infty} \frac{nk}{n} \\
&= \rho(F_1^n) + k.
\end{aligned}$$

Therefore, to remove the dependence of the lift, one may only consider the fractional part of the value of ρ, which brings us to the definition of the **rotation number**.

Definition 3.2.2. *The rotation number $\rho(f) \in [0,1)$ of a circle map f is obtained by taking the projection modulo 1 of $\rho(F)$ on the circle S^1, where F is any lift of f.*

To make sure that the rotation number exists, we need to prove the existence of the number defined in (3.2.1). Let us first introduce the concept of a quasi-morphism.

Definition 3.2.3. *Let G be a group. A function $\mu : G \to \mathbb{R}$ is called a **quasi-morphism** if there exists a constant $C \geq 0$ such that*

$$|\mu(gh) - \mu(g) - \mu(h)| \leq C \quad \forall g,h \in G.$$

*A quasimorphism is said to be **homogeneous** if*

$$\mu(g^k) = k \cdot \mu(g) \quad \forall g \in G, k \in \mathbb{Z}.$$

We call the *defect* of μ the smallest constant C in the definition above (See [**17**]). A homomorphism is a quasi-morphism with defect 0.

Let $\widetilde{\mathrm{Diff}}_0(S^1)$ be the universal cover of the group of orientation-preserving diffeomorphisms of the circle S^1. It can be explicitly written as

$$\widetilde{\mathrm{Diff}}_0(S^1) = \{F \in C^\infty(\mathbb{R}) : F \text{ is strictly increasing, } F(x+1) = F(x) + 1 \ \forall x \in \mathbb{R}\}.$$

In other words, it is the group of all lifts of orientation-preserving circle diffeomorphisms. We define $\mu : \widetilde{\mathrm{Diff}}_0(S^1) \to \mathbb{R}$ by $\mu(F) = F(0)$. We first prove that μ is a quasi-morphism.

Proof. We denote the floor and ceiling function by $\lfloor \cdot \rfloor$ and by $\lceil \cdot \rceil$ respectively. We have

$$\mu(FG) = F(G(0)) \leq F(\lceil G(0) \rceil) = F(0) + \lceil G(0) \rceil \leq F(0) + G(0) + 1.$$

Similarly,

$$\mu(FG) = F(G(0)) \geq F(\lfloor G(0) \rfloor) = F(0) + \lfloor G(0) \rfloor \geq F(0) + G(0) - 1.$$

Combining those two equations, we get

$$|\mu(FG) - \mu(F) - \mu(G)| \leq 1,$$

so that we found the defect $C = 1$. $\square$

We need a proposition to show that the rotation number exists.

Proposition 3.2.4. *Let G be group and $\mu : G \to \mathbb{R}$ be a quasi-morphism with defect C. Set*

$$\overline{\mu}(g) = \lim_{n\to\infty} \frac{\mu(g^n)}{n}.$$

Then

(1) The limit exists for all $g \in G$.

(2) $|\overline{\mu} - \mu| \leq C$, hence $\overline{\mu}$ is a quasi-morphism.

(3) $\overline{\mu}$ is homogeneous.

Proof. (1) Let $g,h \in G$. Since μ is a quasi-morphism, we have $|\mu(gh) - \mu(g) - \mu(h)| \leq C$. Taking $h = g$,

$$-C \leq \mu(g^2) - 2\mu(g) \leq C \iff |\mu(g^2)| \leq C + 2\mu(g).$$

Using the same argument, we have

$$\begin{aligned} &-C \leq \mu(g^3) - \mu(g^2) - \mu(g) \leq C \\ \iff\ &\mu(g) + \mu(g^2) - C \leq \mu(g^3) \leq C + \mu(g) + \mu(g^2) \\ \iff\ &3\mu(g) - 2C \leq \mu(g^3) \leq 2C + 3\mu(g) \end{aligned}$$

Repeating this argument, we have for all $n \in \mathbb{N}$

$$n\mu(g) - (n-1)C \le \mu(g^n) \le n\mu(g) + (n-1)C. \tag{3.2.2}$$

Now set $a_n := \mu(g^n) + (n+1)C$. By the inequality above, $a_n \ge n\mu(g) + 2C \ge 0$. Hence,

$$a_{m+n} = \mu(g^{n+m}) + (m+n+1)C \le \mu(g^n) + \mu(g^m) + C + (m+n+1)C = a_m + a_n,$$

so a_n is a non-negative subadditive sequence. By Fekete's lemma, the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists. Therefore, the limit

$$\lim_{n\to\infty} \frac{\mu(g^n)}{n} = \lim_{n\to\infty} \frac{a_n - (n+1)C}{n}$$

also exists, finishing the proof.

(2) Let $g \in G$. By (3.2.2), we have

$$\begin{aligned} |\overline{\mu}(g) - \mu(g)| &= |\lim_{n\to\infty} \frac{\mu(g^n)}{n} - \mu(g)| \\ &= \lim_{n\to\infty} \frac{|\mu(g^n) - n\mu(g)|}{n} \\ &\le \lim_{n\to\infty} \frac{(n-1)C}{n} = C. \end{aligned}$$

and this implies that $\overline{\mu}$ is a quasi-morphism.

(3) By definition of $\overline{\mu}$, $\overline{\mu}(g^k) = \left(\lim_{n\to\infty} \frac{\mu(g^{nk})}{nk}\right) \cdot k = \overline{\mu}(g) \cdot k$, hence $\overline{\mu}$ is homogeneous. $\square$

Since the $\rho(F)$ does not depend on the choice of x, we have

$$\lim_{n\to\infty} \frac{F^n(x) - x}{n} = \lim_{n\to\infty} \frac{F^n(0)}{n} = \lim_{n\to\infty} \frac{\mu(F^n)}{n}.$$

By Proposition 3.2.4., the last quantity exists and therefore we have proved the existence of rotation number.

An important property of the rotation number $\rho(f)$ is that of its invariance under conjugacy.

Proposition 3.2.5. *Let $\tilde{G} = \widetilde{\mathrm{Diff}}_0(S^1)$ be the universal cover of the group of diffeomorphism of the circle S^1 and let $\rho : \tilde{G} \to \mathbb{R}$ be the rotation number map. Then for all orientation-preserving f,$g \in \tilde{G}$,*

$$\rho(fgf^{-1}) = \rho(g).$$

Proof. Let f,$g \in \overline{G}$. We first find an expression for $(fgf^{-1})^k$. Note that

$$(fgf^{-1})^k = \underbrace{(fgf^{-1}) \cdot (fgf^{-1}) \cdots (fgf^{-1})}_{k \text{ times}} = fg^kf^{-1}$$

Since ρ is a homogeneous quasi-morphism, $\rho(f^{-1}) = -\rho(f)$, and

$$\begin{aligned}|\rho(fg^kf^{-1}) - \rho(g^k)| &= |\rho(fg^kf^{-1}) - \rho(fg^k) - \rho(f^{-1}) + \rho(fg^k) + \rho(f^{-1}) - \rho(g^k)| \\ &\leq |\rho(fg^kf^{-1}) - \rho(fg^k) - \rho(f^{-1})| + |\rho((g^k) - \rho(f) - \rho(g^k)| \\ &\leq 2C_\rho.\end{aligned}$$

By homogeneity and by previous inequality,

$$|\rho((fgf^{-1})^k) - \rho(g^k)| = k|\rho(fgf^{-1}) - \rho(g)| \leq 2C_\rho \Longrightarrow |\rho(fgf^{-1}) - \rho(g)| \leq \frac{2C_\rho}{k}$$

for all $k \in \mathbb{N}$. Then $|\rho(fgf^{-1}) - \rho(g)| = 0$ which implies that $\rho(fgf^{-1}) = \rho(g)$. $\square$

Lastly, we prove a property in dynamical system about rotation number that will be useful to understand the proof of the main result of this thesis.

Proposition 3.2.6. *$\rho(f)$ is irrational if and only if f has no periodic point.*

Proof. We first prove the contrapositive: If f has a periodic point, then $\rho(f)$ is rational. If f has a q-periodic point, then there exists some $p \in \mathbb{N}$ such that $F^q(x) = x + p$ where F is a lift of f. Then

$$\begin{aligned}\rho(F) = \lim_{n\to\infty} \frac{F^{nq}(x) - x}{nq} &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{F^{iq}(x) - F^{q(i-1)}(x)}{nq} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{F^q(F^{(q(i-1)}(x)) - F^{q(i-1)}(x)}{nq} \\ &= \lim_{n\to\infty} \frac{np}{nq} = \frac{p}{q} \in \mathbb{Q}.\end{aligned}$$

Conversely, suppose by contradiction that $\rho(f)$ is rational. Since ρ is an homogeneous quasi-morphism, we have $\rho(F^m) = m\rho(F)$ for any lift F. If $\rho(F) = \frac{p}{q}$, then $\rho(f^q) = [\rho(F^q)] = [q \cdot \rho(F)] = [q\frac{p}{q}] = [p] = 0$, where $[x]$ denotes the equivalence class of x in S^1. Therefore, we can assume that $\rho(f) = 0$.

By hypothesis, f has no fixed point, the lift F does not have fixed points either, so without loss of generality one can assume that $F(x) > x$ for all $x \in \mathbb{R}$. We have two cases:

(1) $F^n(0) < 1$ for all $n \in \mathbb{N}$. Since F is strictly increasing, then $F^n(0)$ converges, say, to p. Then

$$F(p) = F(\lim_{n\to\infty}(F^n(0)) = \lim_{n\to\infty} F(F^n(0)) = \lim_{n\to\infty} F^{n+1}(0) = p.$$

Contradiction to the fact that f has no periodic point.

(2) There exists $k \in \mathbb{N}$ such that $F^k(0) > 1$. This implies that $F^{mk}(0) > m$, so

$$\rho(F^{mk}) = mk\rho(F) > m \iff \rho(F) > \frac{1}{k}$$

which is a contradiction to $\rho(f) = 0$.

Those two cases concludes the proof. $\square$

3.3. Definition of twist map

Now that we saw the concept of the lift of a circle map and that of the rotation number, we are ready to properly introduce the twist map, which is commonly studied in the field of dynamical systems and which plays an essential role in the proof of our main theorem.

Roughly speaking, a twist map is a map of the open cylinder that preserves geometrical aspects of the dynamics and is strictly increasing in the first component when we increase its second component while keeping the first one fixed. Let's recall the definition.

Definition 3.3.1 ([**8**]). *A diffeomorphism $\phi : C \to C$ of the open cylinder $C = S^1 \times (-1,1)$ is said to be a **twist map** if*

(1) it preserves orientation, and there exists $\varepsilon > 0$ such that

$$(x,y) \in S^1 \times (-1,\varepsilon - 1) \Longrightarrow \phi(x,y) \in S^1 \times (-1,0);$$

(2) $\frac{\partial}{\partial y}\Phi_1(x,y) > 0$, where $\Phi = (\Phi_1,\Phi_2)$ is a lift of ϕ to the strip $S = \mathbb{R} \times (0,1)$;

(3) ϕ extends to a homeomorphism $\overline{\phi}$ of the closed cylinder $S^1 \times [-1,1]$, which preserves the boundary components, i.e.

$$\overline{\phi}(S^1 \times \{\pm 1\}) = S^1 \times \{\pm 1\}.$$

*The map ϕ is said to be a **differentiable twist map** if for $\varepsilon > 0$, there exists $\delta > 0$ such that $\frac{\partial}{\partial y}\Phi_1(x,y) > \delta$ on a compact subset $C_\varepsilon := S^1 \times (\varepsilon - 1, 1 - \varepsilon)$.*

The second condition can be easily illustrated by Figure 3.2, where for $x_1 = x_2$ and $y_2 > y_1$, we have $\Phi_1(x_2,y_2) = x_2 > x_1 = \Phi_1(x_1,y_1)$.

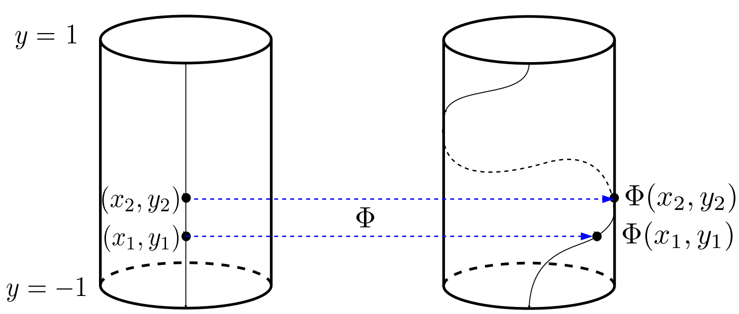

Fig. 3.2. Twist map with increasing x as y increases

The last condition is not necessary [**8**] and may not always be the case if such map does not extend continuously to closed cylinder (see [**10**]). However, it allows us to talk about the rotation number of ϕ on the "boundary" circles. We denote by ρ_- and ρ_+ the rotation numbers of $\overline{\phi}$ restricted to the bottom circle $S^1 \times \{-1\}$ and the top circle $S^1 \times \{1\}$ respectively. We should notice that the interval $[\rho_-,\rho_+]$ will be changed up to integers if we choose another lift. This interval is called the **twist interval** of the twist map.

3.4. Billiard ball map as twist map

The reason why twist map is so crucial for this thesis is because the billiard ball map is in fact a twist map, as we shall show in the following proposition.

Proposition 3.4.1. *The billiard map $\phi : C \to C$ of the open cylinder $C = S^1 \times (-1,1)$ is an area-preserving twist map with the additional property that a lift Φ satisfies $\Phi_1(x,y) \xrightarrow[y\to -1]{} x$ and $\Phi_1(x,y) \xrightarrow[y\to 1]{} x+1$. Thus the twist interval for a billiard map is $[0,1]$.*

Proof. The first part has been proven in Section 1.4. Since $y = -\cos\theta$, we have

$$\lim_{y\to -1} \Phi_1(x,y) = \lim_{\theta\to 0} \Phi_1(x,y) = x.$$

Similarly, $\lim_{y\to 1} \Phi_1(x,y) = \lim_{\theta\to \pi} \Phi_1(x,y) = x+1$ since by making a whole turn for the lift means that the distance in x increases by 1. Thus, we deduce that on the "bottom" circle $S^1 \times \{-1\}$

$$\rho(\Phi) = \lim_{n\to\infty} \frac{\Phi_1(x,y) - x}{n} = \lim_{n\to\infty} \frac{x-x}{n} = 0.$$

Similarly, $\rho(\Phi)$ restricted to the "top" circle is 1. Hence, the twist interval is $[0,1]$. $\square$

Although we introduce twist map to study billiard map, there are general examples of twist map that could be discussed. We give one example below. For more examples, see [**10**].

Example 3.4.2 (Integrable twist map)**.** *A twist map is said to be* ***integrable*** *if it has the form*

$$f(x,y) = (x + g(y),y).$$

From this form, we can see that all the circles $S^1 \times \{y\}$ stay invariant but are rotated by the angle given by the function $g(y)$. In particular, if $g(y)$ is rational, then the circle associated is an invariant circle with rotation number $g(y)$, since

$$\rho_g(f) = \lim_{n\to\infty} \frac{F_1^n(x,y) - x}{n} = \lim_{n\to\infty} \frac{x + ng(y) - x}{n} = g(y).$$

From this fact, one can deduce that there exists infinitely many family of periodic orbits separated by circles with irrational rotation number. Later on, those circles will be associated to caustics. Thus, the twist interval of this map is $(\lim_{y\to -1} g(y), \lim_{y\to 1} g(y))$.

As an obvious example, the billiard map in a circle is an integrable twist map with $g(y) = 2y$. However, with Lazutkin's coordinates, the billiard map in a general strictly convex domain is not integrable, although it is up to certain perturbations.

We end our discussion by talking briefly about Birkhoff periodic orbits defined in Chapter 1 in the context of twist map.

Definition 3.4.3 ([**8**]). *Let $\phi : C \to C$ be a twist map and Φ be its lift. A point $w \in C$ is said to be a Birkhoff periodic point of type (p,q) and its orbit a Birkhoff periodic orbit of type (p,q) if for a lift $z \in S$ of w there exists a sequence $\{(x_n,y_n)\}_{n\in\mathbb{Z}}$ in S such that*

(1) $(x_0,y_0) = z$,

(2) $x_{n+1} > x_n$ $(n \in \mathbb{N})$,

(3) $(x_{n+q},y_{n+q}) = (x_n + 1,y_n)$,

(4) $(x_{n+p},y_{n+p}) = \Phi(x_n,y_n)$.

We notice that in order to make sense the last condition, the sequence $\{(x_n,y_n)\}_{n\in\mathbb{Z}}$ has to be ordered geometrically instead once we project Φ on the circle S^1 (see Figure 3.3).

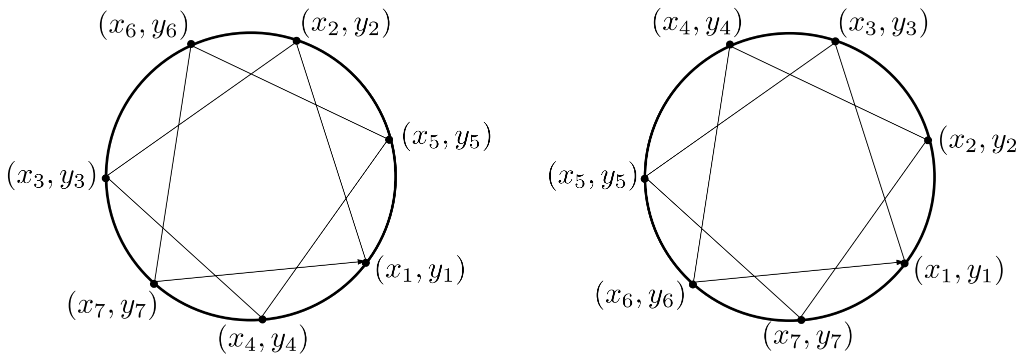

Fig. 3.3. Geometric ordering (left) and dynamical ordering (right)

Chapter 4

Caustics

We introduced the concept of a caustic in Chapter 3 while introducing Lazutkin's theory. In this chapter, we will give more details about this important object in geometrical optics and some properties of it. We refer the reader to [**20**], [**12**], [**7**], and [**8**] for a thorough discussion about this topic.

4.1. Definition of caustics

A caustic, in its simplest definition, is a curve inside a planar billiard table such that if a segment of a billiard trajectory is tangent to it, then it remains tangent after each reflection.

A natural question rises: if we know that γ is a caustic of Γ, can we recover Γ from γ? The answer is yes, and it is due to string construction. We surround the caustic γ with a non-stretchable string of length greater than that of γ, we pull it tight at a point, and we turn around the caustic to form a new curve (see Fig 4.1).

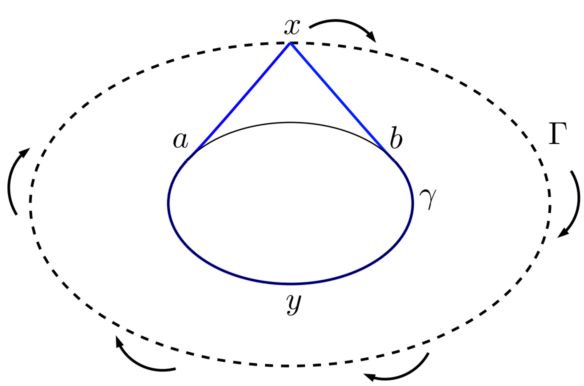

Fig. 4.1. Recovering the curve from its caustic

We shall now prove its existence mathematically. We prove the following theorem.

Theorem 4.1.1. *Billiard inside* Γ *has* γ *as its caustic.*

Proof. Fix a reference point $y \in \gamma$. For a point $x \in \Gamma$, let $f(x)$ and $g(x)$ be the distance from x to y from the left and from the right respectively. Then Γ is a level curve of the function $f + g$. Let θ_1 and θ_2 be the angles made by ax and bx with Γ respectively. Our goal is to show that $\theta_1 = \theta_2$.

Lemma 4.1.2. *$\nabla f(x)$ is a unit vector in the direction of ax.*

Proof. We parametrize γ by arc-length s with $y = \gamma(0)$. We consider the level curve $f = c$ passing through x and we show that it is orthogonal to ax, so that the gradient is parallel to ax. The point x can be written as $x = \gamma(s) + (c - s)\gamma'(s)$, where $a = \gamma(s)$ and $(c - s)$ is the distance between a and x. Its derivative is $x' = (c-s)\gamma''(s)$. Since $\gamma(s)$ is the arc-length parametrization of the caustic, the vectors $\gamma'(s)$ and $\gamma''(s)$ are orthogonal, so x' is orthogonal to ax. Finally, the directional derivative of f in the direction ax has unit norm. □

By this lemma, $\nabla(f + g)$ bissects angle axb, and therefore ax and bx makes equal angle. □

However, caustics inside a billiard table might not always be convex. For instance, caustics might have cusps. Consider the following example of an astroid, given by the equation

$$|x|^{2/3} + |y|^{2/3} = 1.$$

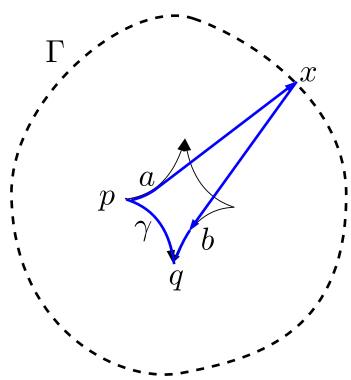

Fig. 4.2. String construction for caustic with cusps

A modification of the string construction will still work. We have the closed path $xapqbx$. Its length is the algebraic sum of the length of each arc with orientation (positive if it follows the path and negative otherwise). In this case, the length of the arc pq will be negative. Let Γ be the locus of the points x such that the string $xapqbx$ has a constant length. We claim that γ is a caustic of the billiard inside Γ.

The proof will be essentially the same as that for Theorem 4.1.1. Fix a reference point $y \in \gamma$ (say at halfway on the arc pq). Let $f(x)$ and $g(x)$ be the respective distance between x and y from the left and from the right. Hence, Γ is a level curve of $f + g$. It suffices to show that the angles made by ax and bx with Γ are equal.

Lemma 4.1.2 still applies up to some modification of the constant $(c-s)$. It follows that $\nabla(f+g)$ bisects the angle axb, and therefore ax and bx makes equal angle with Γ.

We now define the caustic in a more precise way using the concept of invariant circle. Let the phase space M be the set of unit tangent vectors in $\mathbb{R}^2$ with footpoints on the table Ω. We identity M with the space of oriented lines N that intersect the boundary $\partial\Omega$. We will give an alternative definition of invariant circle.

Definition 4.1.3. [**7**] *A curve $\Gamma \subset M$ is an **invariant circle** if it is isotopic to a boundary component of M, and for the billiard ball map $\phi : M \to M$ we have $\phi(\Gamma) = \Gamma$.*

We distinguish different invariant circles by the angle each oriented line (which can also be called ray) makes with the boundary. For instance, the invariant circle $\theta = \theta_0$ is the family of rays intersecting the $\partial\Omega$ with an angle of θ_0.

Definition 4.1.4. *Let X be a family of rays. The **envelope** of this family is a curve that intersects each ray once and tangent to the rays (see Figure 4.3).*

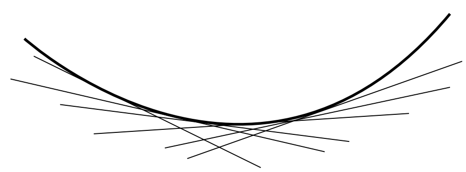

Fig. 4.3. An envelope of a family of rays

Therefore, a caustic can be defined as the envelope of an invariant circle $\theta = \theta_0$. A caustic is said to be associated to a invariant circle if the family of rays defining it constitutes an invariant circle of the billiard map ϕ. Another way to see it is to consider the intersection of all left half-planes to these rays, and the boundary of this region is a caustic.

4.2. Properties of caustics

In this section, we discuss some of the properties of the caustics. First off, in order to talk about such geometric object, we need to ensure its existence. Lazutkin showed in his paper [**12**] that in general, for an arbitrary convex domain, continuous family of caustics does not exist. However, if the boundary of the table is sufficiently smooth with 553 continuous derivatives (this number is later reduced to 6!) and the curvature is everywhere positive, there exists a continuous family of caustics near the boundary. Mather established a more quantitative result in [**15**], compatible with Lazutkin's result, saying that if the minimal curvature of the table is sufficiently small, then the convex caustics are located in a neighbourhood of the boundary.

Now that caustics exist with sufficiently smooth boundary, one can ask if caustics can have points outside of $\partial\Omega$. Before answering the question, we introduce a very useful fact about the caustics given by Birkhoff's Theorem. See [**10**] for the proof the the following theorem.

Theorem 4.2.1. *(Birkhoff) In the standard coordinates (s,θ) in the phase space M, the caustic is a graph $\theta = f(s)$ of a continuous function f.*

The consequence of this theorem will answer our question.

Lemma 4.2.2. *Let γ be a caustic associated to an invariant circle of the billiard ball map inside a convex boundary Γ. The γ lies inside Γ.*

Proof. Denote the invariant curve by δ. Since δ is a graph $\theta = f(s)$, we can write the billiard ball map T restricted to δ as

$$T(s,f(s)) = (g(s), f(g(s)).$$

Here, g is an increasing function since the billiard map preserves orientation. Let $s_1 = s + \varepsilon$ to be a very close point. Then the straight line $(\Gamma(s)\Gamma(g(s))$ and $(\Gamma(s_1)\Gamma(g(s_1))$ intersect inside Γ (See Figure 4.4). Letting $\varepsilon \to 0$, we obtain the result. $\square$

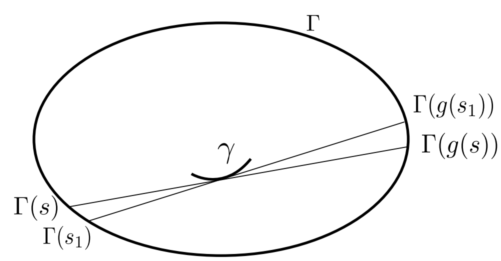

Fig. 4.4. Caustic inside the table

Let us now introduce a very useful formula in geometrical optics: the mirror equation.

Theorem 4.2.3 (Mirror equation)**.** *Let Γ be a reflecting curve. Let A and B two points inside the curve such that a ray passing through A reflects by passing through B. Denote by X the reflection point, and a and b the respective length of AX and BX. Let k be the curvature of Γ at X and α the angle between the segment AX and the tangent to Γ at X.(See figure 4.5) We have*

$$\frac{1}{a} + \frac{1}{b} = \frac{2k}{\sin\alpha}.$$

Proof. We parameterize the curve Γ by arc-length s such that $\Gamma(0) = X$. Consider the length function

$$f(s) = |\Gamma(s) - A| + |\Gamma(s) - B|.$$

Since the ray AX reflects to XB, then the length $|AX| + |XB|$ is a minimum of f, so $f'(0) = 0$. Since infinitesimally close rays from A reflects through B, we can also deduct that

$f''(0) = 0$.

We now differentiate the length $|\Gamma(s) - A|$ with respect to s.

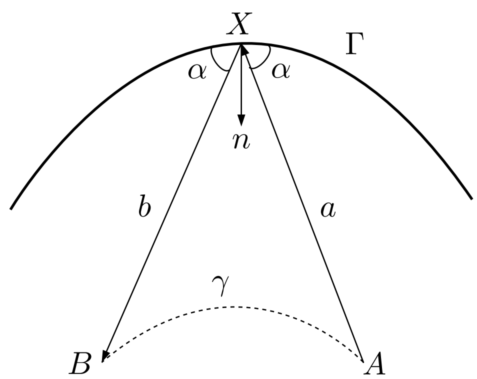

Fig. 4.5. Mirror equation

$$\begin{aligned}
a' = |\Gamma(s) - A|' &= \Big([(\Gamma(s) - A) \cdot (\Gamma(s) - A)]^{1/2}\Big)' \\
&= \frac{1}{2[(\Gamma(s) - A) \cdot (\Gamma(s) - A)]^{1/2}}[(\Gamma(s) - A) \cdot (\Gamma(s) - A)]' \\
&= \frac{1}{2|\Gamma(s) - A|}(\Gamma'(s) \cdot (\Gamma(s) - A) + (\Gamma(s) - A) \cdot \Gamma'(s)) \\
&= \frac{(\Gamma(s) - A) \cdot \Gamma'(s)}{a} = \cos\alpha
\end{aligned}$$

since $(\Gamma(s) - A) \cdot \Gamma'(s)$ is the norm of the projection of $(\Gamma(s) - A)$ onto $\Gamma'(s)$. Similarly, we can show that $|\Gamma(s) - B|' = -\cos\alpha$. We note that $\Gamma''(s) = kn$ where n is the normal inward vector at X and k is the curvature at X.

We now differentiate $|\Gamma(s) - A|'$ by using the quotient rule.

$$\begin{aligned}
|\Gamma(s) - A|'' &= \frac{[(\Gamma(s) - A) \cdot \Gamma'(s)]'a - [(\Gamma(s) - A) \cdot \Gamma'(s)]a'}{a^2} \\
&= \frac{\Gamma'(s) \cdot \Gamma'(s)}{a} + \frac{(\Gamma(s) - A) \cdot \Gamma''(s)}{a} - \frac{[(\Gamma(s) - A) \cdot \Gamma'(s)]^2}{a^3} \\
&= \frac{1}{a} - k\sin\alpha - \frac{\cos^2\alpha}{a} = \frac{\sin^2\alpha}{a} - k\sin\alpha
\end{aligned}$$

Likewise, we can show that $|\Gamma(s) - B|'' = \frac{\sin^2\alpha}{b} - k\sin\alpha$. Since $f''(0) = 0$, we have

$$0 = \frac{\sin^2\alpha}{a} + \frac{\sin^2\alpha}{b} - 2k\sin\alpha \iff \frac{1}{a} + \frac{1}{b} = \frac{2k}{\sin\alpha}.$$

$\square$

This formula implies the following theorem about non-existence of caustics discovered by J. Mather [**14**].

Theorem 4.2.4. *A convex C^2 billiard table with a point of zero curvature has no convex caustics.*

PROOF. We prove it by contradiction by supposing the existence of such a caustic. We consider two rays tangent to it with the intersection point $p \in \Gamma$, where the curvature at this points is zero. If a and b are the distance from p to the point of tangency, then by the mirror equation, we have

$$\frac{1}{a} + \frac{1}{b} = \frac{2k}{\sin \alpha}.$$

The left-hand side of this equation is strictly positive, implying that $k \neq 0$, which is a contradiction to our initial hypothesis. $\square$

Chapter 5

Proof of the Main Result

5.1. Distribution in strictly convex table

We recall the main theorem that we would like to prove in this thesis:

Theorem 5.1.1 (Bialy, Mironov, Tabachnikov)**.** *In a strictly convex billiard table with curvature $\kappa > 0$, the distribution of reflection points of Birkhoff periodic orbits of type $(1,n)$ is uniform with respect to the measure $\kappa^{2/3}\,dx$ as n tends to ∞.*

The guiding idea is to consider the billiard ball map using Lazutkin's coordinates (x,y) defined in Chapter 2

$$f_L(x,y) = (x + y + O(y^3), y + O(y^4))$$

and notice the key fact that we can control y by giving it an upper bound: $y \leq \frac{C}{n}$ along the $(1,n)$ orbit for some real constant C.

The outline of the proof will be as follows:

Step 1 Proof that $y \leq \frac{C}{n}$ implies the equidistribution in two steps:
- (a) prove that $x_j = x_0 + \frac{j}{n} + O(\frac{1}{n^2})$
- (b) prove that for an interval $(a,b) \subset \partial\Omega$ and for n sufficiently large, we have

$$|\#\{j \mid x_j \in (a,b)\} - \#\{j \mid x_0 + \tfrac{j}{n} \in (a,b)\}| \leq 2$$

Step 2 Prove that $y \leq \frac{C}{n}$ along the orbit γ_n
- (a) bound y along a caustic in terms of η (angle)
- (b) "sandwich" γ_n between two caustics with $\eta \sim \frac{1}{n}$

Proof. (Step 1) We first prove that for all $1 \leq j \leq n$, $x_j = x_0 + \frac{j}{n} + O(\frac{1}{n^2})$ if $y \leq \frac{C}{n}$. We want to prove that $|x_j - x_0 - \frac{j}{n}| \leq \frac{C}{n^2}$ for some $C \in \mathbb{R}$.
Let z_{j-1} be such that $x_j = x_{j-1} + y_{j-1} + z_{j-1}$. It represents the distance between the j-th reflection point and its approximation.

By recurrence, we have

$$\begin{aligned} x_j &= z_{j-1} + y_{j-1} + z_{j-2} + y_{j-2} + x_{j-2} \\ &= \sum_{i=0}^{j-1} z_i + \sum_{i=0}^{j-1} y_i + x_0. \end{aligned}$$

We know by Lazutkin's coordinates and $y \leq \frac{C}{n}$ that $|z_j| = O(y^3) = O(\frac{1}{n^3})$. Let $w_j = y_{j+1} - y_j$ be the difference between the angle of the j-th reflection point and the angle of its iteration. Similarly, $|w_j| = O(y^4) = O(\frac{1}{n^4})$. Then,

$$\begin{aligned} \left|x_j - x_0 - \frac{j}{n}\right| &= \left|\sum_{i=0}^{j-1} z_j + \sum_{i=0}^{j-1} y_j - j\, y_0 + j\, y_0 + \frac{j}{n}\right| \\ &\leq \left|\sum_{i=0}^{j-1} z_j\right| + \left|\sum_{i=0}^{j-1} y_j - j\, y_0\right| + \left|j\, y_0 - \frac{j}{n}\right| \\ &\leq O\left(n \cdot \frac{1}{n^3}\right) + O\left(n^2 \cdot \frac{1}{n^4}\right) + \left|j\left(y_0 - \frac{1}{n}\right)\right| \end{aligned}$$

To bound $y_0 - \frac{1}{n}$, we have $x_n = x_0 + ny_0 + O(y^3)$ from the Lazutkin's coordinates. Then

$$\begin{aligned} x_n &= x_0 + ny_0 + O\left(n \cdot \frac{1}{n^3}\right) + O\left(n \cdot \frac{1}{n^4}\right) \\ \iff x_n - x_0 &= ny_0 + O\left(\frac{1}{n^2}\right) \\ \iff 1 &= ny_0 + O\left(\frac{1}{n^2}\right) \\ \implies y_0 &= \frac{1}{n} + O\left(\frac{1}{n^3}\right) \end{aligned}$$

Hence

$$\left|x_j - x_0 - \frac{j}{n}\right| = O\left(\frac{1}{n^2}\right) + O\left(\frac{1}{n^2}\right) + O\left(n \cdot \frac{1}{n^3}\right) = O\left(\frac{1}{n^2}\right).$$

□

PROOF. (Step 1) We then prove that for an interval $(a,b) \subset \partial\Omega$ and for n sufficiently large, we have

$$|\#\{j \mid x_j \in (a,b)\} - \#\{j \mid x_0 + \tfrac{j}{n} \in (a,b)\}| \leq 2.$$

We have two cases:
<u>Case 1</u>: $x_j \in (a,b)$ and $x_0 + \frac{j}{n} \notin (a,b)$. Since $x_j = x_0 + \frac{j}{n} + w_j$ where $|w_j| \leq \frac{C}{n^2}$, we have

$$d\left(x_0 + \frac{j}{n}, a\right) \leq \frac{C}{n^2} \quad \text{and} \quad d\left(b, x_0 + \frac{j}{n}\right) \leq \frac{C}{n^2}.$$

Thus $\min\{d\left(x_0+\frac{j}{n},a\right), d\left(b,x_0+\frac{j}{n}\right)\} \le \frac{C}{n^2}$. For n sufficiently large,

$$\frac{C}{n^2} = \frac{C}{n}\cdot\frac{1}{n} \le \frac{1}{2}\cdot\frac{1}{n} = \frac{1}{2n} \quad \text{if } n \ge 2C.$$

However, for any $j' \ne j$, we have

$$d\left(x_0+\frac{j}{n},x_0+\frac{j'}{n}\right) = \left|\left(x_0+\frac{j}{n}\right)-\left(x_0+\frac{j'}{n}\right)\right| = \left|\frac{j-j'}{n}\right| \ge \frac{1}{n}$$

Hence, $\#\{j \mid d\left(x_0+\frac{j}{n},a\right) \le \frac{C}{n^2}\} \le 1$ and $\#\{j \mid d\left(b,x_0+\frac{j}{n}\right) \le \frac{C}{n^2}\} \le 1$

$$\Longrightarrow |\#\{j \mid x_j \in (a,b)\} - \#\{j \mid x_0+\tfrac{j}{n} \in (a,b)\}| \le 2.$$

<u>Case 2</u>: $x_0+\frac{j}{n} \in (a,b)$ but $x_j \notin (a,b)$. Since $|x_j - x_0 - \frac{j}{n}| \le \frac{C}{n^2}$, then

$$d\left(x_j,a\right) \le \frac{C}{n^2} \quad \text{and} \quad d\left(b,x_j\right) \le \frac{C}{n^2}.$$

Similarly, for n sufficiently large, we have $\frac{C}{n^2} \le \frac{1}{2n}$ so

$$d\left(x_j,a\right) + d\left(b,x_j\right) \le \frac{1}{n}.$$

However, for any $j' \ne j$, we have

$$\begin{aligned}
d\left(x_j,x_{j'}\right) = |x_j - x_j'| &= \left|x_0+\frac{j}{n}+w_j-\left(x_0+\frac{j'}{n}+w_{j'}\right)\right| \\
&= \left|\frac{j-j'}{n}-(w_{j'}-w_j)\right| \\
&\ge \left|\left|\frac{j-j'}{n}\right| - |w_{j'}-w_j|\right| \qquad \text{(by Triangle inequality)} \\
&\ge \left|\frac{1}{n}-\frac{C}{n^2}\right| \\
&> \left|\frac{1}{n}-\frac{2}{n}\right| \quad \text{if} \quad \frac{C}{n} < 2 \iff n > \frac{C}{2} \\
&= \frac{1}{n}
\end{aligned}$$

Therefore, $|\#\{j \mid x_j \in (a,b)\} - \#\{j \mid x_0+\frac{j}{n} \in (a,b)\}| \le 2$. $\square$

In other words, the difference between the number of reflection points of the form x_j and the number of points of the form $x_0+\frac{j}{n}$ is relatively small as n increases, so the difference z_j won't matter too much as n goes to infinity. Since the distribution of the reflection points of the second form is uniform, that the distribution of x_j is also uniform with respect to the natural measure on the boundary.

Before proving the step 2, we need a lemma about the rotation numbers.

Lemma 5.1.2. *There exists a sequence of caustics with rotation numbers $\eta_{i_0}, \eta_{i_0+1}, \ldots$ such that $\forall n \geq i_0$, we have*

$$\eta_n > \frac{1}{n} > \eta_{n+1}.$$

Proof. By Lazutkin's paper [**12**], the rotation number of caustics are described by

$$E = \{\eta \in \mathbb{R} \mid |n\eta - m| \geq A|m|n^{-\eta+1/2}, \forall n \in \mathbb{N}, \forall m \in \mathbb{Z}\}$$

for $\sigma \geq 3$ and $A < 1$. It is enough to prove that $E \cap (\frac{1}{n+1}, \frac{1}{n}) \neq \varnothing$. It is therefore sufficient to prove that $m(E \cap (\frac{1}{n+1}, \frac{1}{n})) > 0$.

We write $E \cap (\frac{1}{n+1}, \frac{1}{n})$ as

$$\left(E \cap \left[0, \frac{1}{n+1}\right]\right) \cup \left(E \cap \left(\frac{1}{n+1}, \frac{1}{n}\right)\right) = E \cap \left[0, \frac{1}{n}\right).$$

Since it's a disjoint union, we have

$$m\left(E \cap \left[0, \frac{1}{n+1}\right]\right) + m\left(E \cap \left(\frac{1}{n+1}, \frac{1}{n}\right)\right) = m\left(E \cap \left[0, \frac{1}{n}\right)\right).$$

Lazutkin showed in [**12**] that $m(E \cap [0, \alpha]) > \alpha - c_1\alpha^{\sigma-1/2}$ where c_1 depends on σ and A. Letting $\alpha = \frac{1}{n}$, we have

$$m\left(E \cap \left[0, \frac{1}{n+1}\right]\right) + m\left(E \cap \left(\frac{1}{n+1}, \frac{1}{n}\right)\right) > \frac{1}{n} - c_1\left(\frac{1}{n}\right)^{\sigma-1/2}$$

Hence,

$$\begin{aligned}
m\left(E \cap \left(\frac{1}{n+1}, \frac{1}{n}\right)\right) &= \frac{1}{n} - c_1\left(\frac{1}{n}\right)^{\sigma-1/2} - m\left(E \cap \left[0, \frac{1}{n+1}\right]\right) \\
&\geq \frac{1}{n} - c_1\left(\frac{1}{n}\right)^{\sigma-1/2} - m\left(\left[0, \frac{1}{n+1}\right]\right) \\
&= \frac{1}{n} - c_1\left(\frac{1}{n}\right)^{\sigma-1/2} - \frac{1}{n+1} \\
&= \frac{1}{n(n+1)} - c_1\left(\frac{1}{n}\right)^{\sigma-1/2} > 0
\end{aligned}$$

for all n sufficiently large, since $\sigma - \frac{1}{2} \geq \frac{5}{2} > 2$. $\square$

Our goal is to prove that if z is a periodic orbit between Γ_{η_k} and $\Gamma_{\eta_{k+1}}$, where Γ_{η_k} and $\Gamma_{\eta_{k+1}}$ are caustics of rotation number η_k and η_{k+1} respectively, then the rotation number of z is between η_k and η_{k+1}. We will instead prove a more general statement.

Proposition 5.1.3. *Let Γ_α and Γ_β two Lazutkin caustics with rotation numbers $\alpha < \beta$. If z is a periodic point between Γ_α and Γ_β, then its rotation number q satisfies $\alpha < q < \beta$.*

The key idea of the proof is to use the following theorem which says that every Lazutkin caustic is the graph of a Lipschitz function.

Theorem 5.1.4. *(Katok & Hasselblatt [**8**]) If U is an open invariant set of a differentiable twist map ϕ that contains a neighborhood of the "bottom" $S^1 \times \{-1\}$ and has connected boundary, then the boundary of U is the graph of a Lipschitz function.*

Proof. (Proposition 5.1.3) By Theorem 5.1.4, the caustics Γ_α and Γ_β are graphs of a Lipschitz function. We have that $0 < \alpha < \beta < 1$. Let Φ be a lift of a billiard map ϕ.

Let $\Gamma = \Gamma_\beta = \text{graph}(\gamma)$ be the upper caustic. We denote by $(x^i,y^i) = \Phi^i(x^0,y^0)$ the i-th iteration of the initial periodic point $(x^0,y^0) = (x,y)$.

For a point (x^i,y^i), let (x^i,y^i_+) be the corresponding point of (x^i,y^i) on the graph Γ that has the same x^i (see 5.1).

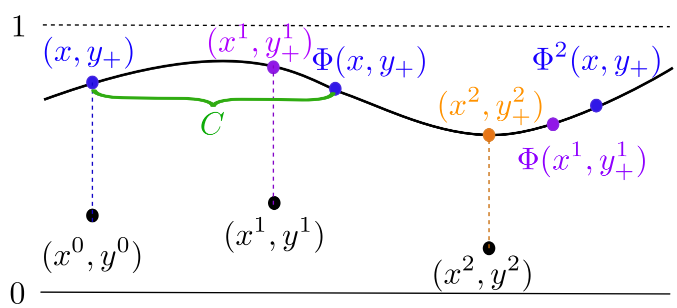

Fig. 5.1. Lift of the billiard map to caustics

We have $x^1 \in (x,\Phi(x,y_+)_1)$. Set $C(I) := \text{graph}(\gamma_{|_I})$. Then

$$(x^1,y^1_+) \in C((x,y_+),\Phi(x,y_+))$$
$$\Phi((x^1,y^1_+)) \in \Phi(C) = C(\Phi(x,y_+),\Phi^2(x,y_+))$$

We show the following lemma:

Lemma 5.1.5. *The distance in x between (x^k,y^k) and (x,y) is smaller than the distance between (x,y_+) and its k-th iteration for all $k \in \mathbb{N}$, i.e.*

$$(\Phi^k(x,y_+) - (x,y_+))_1 > x^k - x$$

Since $(x,y_+) = x$, then we only have to show that $\Phi^k(x,y_+) > x^k$.

Proof. We show it by induction on k.

In the case $k = 2$, since $\Phi(x^1,y^1_+) \in C(\Phi(x,y_+),\Phi^2(x,y_+))$, then $\Phi(x^1,y^1_+)_1 < \Phi^2(x,y_+)_1$ by inclusion. Also, by twist condition, $x^2 = \Phi(x^1,y^1)_1 < \Phi(x^1,y^1_+)_1$. Therefore,

$$x^2 < \Phi(x^1,y^1_+)_1 < \Phi^2(x,y_+)_1.$$

Now suppose it holds for $k = s$, that is, $x^s < \Phi^s(x,y_+)_1$. We show this inequality also holds for $k = s+1$. Since $(x^s,y^s_+) \in C(\Phi^{s-1}(x,y_+),\Phi^s(x,y_+))$ by hypothesis, then

$$\Phi(x^s,y^s_+) \in \Phi(C) = (\Phi^s(x,y_+),\Phi^{s+1}(x,y_+))$$

by inclusion. By twist condition, we also have $x^{s+1} = \Phi(x^s,y^s)_1 < \Phi(x^s,y^s_+)_1$. Hence,

$$x^{s+1} < \Phi(x^s,y^s_+)_1 < \Phi^{s+1}(x,y_+)_1.$$

By induction, the lemma is true for all $k \in \mathbb{N}$. □

We are ready to show that $\rho(\Phi) < \beta$. By definition of rotation number,

$$\begin{aligned} q = \rho(\Phi) &= \lim_{|n|\to\infty} \frac{(\Phi^n(x,y) - (x,y))_1}{n} \\ &= \lim_{|n|\to\infty} \frac{x^n - x}{n} \\ \text{(by Lemma 5.1.5)} \quad &< \lim_{|n|\to\infty} \frac{(\Phi^n(x,y_+) - (x,y_+))_1}{n} = \beta \end{aligned}$$

since caustics are preserved by Φ. Similarly,

$$q = \lim_{|n|\to\infty} \frac{x^n - x}{n} > \lim_{|n|\to\infty} \frac{(\Phi^n(x,y_-) - (x,y_-))_1}{n} = \alpha$$

where (x,y_-) is the corresponding point of (x,y) on the graph of the lower caustic . We can now conclude that $\alpha < \rho(\Phi) \leq \beta$. □

Proof. (Step 2) Since z is a $(1,n)$ periodic points, its rotation number is $q = \frac{1}{n}$. By Lemma 6.0.2. and Proposition 6.0.3.,

$$\eta_{k+1} \leq \frac{1}{n} \leq \eta_k.$$

By Theorem 2 in Lazutkin's paper, $y = \eta + O(\eta^{m+1})$. Therefore, $y = O(\frac{1}{n}) \leq \frac{C}{n}$ for some $C \in \mathbb{R}$ along the orbit γ_n. □

5.2. Density in convex billiards

One might ask what happens if the billiard table is convex, but not strictly convex. This means that we include the possibilities that at some points the curvature κ vanishes. Does the main result still hold?

By Theorem 4.2.4, our approach using Lazutkin caustics would fail, since such caustics would not exist. However, it remains true in such case that those reflection points are dense in the convex billiard table, as stated in the following theorem.

Theorem 5.2.1. *In a strictly convex billiard table B with convex boundary ∂B, the set of reflection points of periodic trajectories is dense in ∂B. In other words, every open subset of the boundary contains reflection points of periodic billiard trajectories.*

Proof. In the case where the boundary is strictly convex, meaning that the curvature at all points $x \in \partial B$ is strictly positive, the result follows from Theorem 5.1.1.

In the convex boundary case, where ∂B can have points x with $\kappa(x) = 0$, we consider two functions: the Euclidean distance function defined by

$$\begin{aligned} d : \partial B \times \partial B &\longrightarrow \mathbb{R}_{\geq 0} \\ (x,y) \quad &\mapsto \quad |x - y| \end{aligned}$$

and the arc-length distance function defined by

$$\rho : \partial B \times \partial B \longrightarrow \mathbb{R}_{\geq 0}$$

$$\rho(x,y) = d|_{\partial B}(x,y) = \begin{cases} \min\{\text{length}(l_1), \text{length}(l_2)\} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

with $l_1 \cap l_2 = \{x,y\}$ and $l_1 \cup l_2 = \partial B$. Notice that d and ρ are continuous.

Consider $X_\varepsilon \subset \partial B \times \partial B$ such that $X_\varepsilon := \{\rho \geq \varepsilon\}$. We claim that X_ε is compact. In fact, the product space $\partial B \times \partial B$ is compact since ∂B is. Also, X_ε is closed since $X_\varepsilon^c = \{\rho < \varepsilon\}$ is open. Therefore, X_ε is compact because it's a closed subset of a compact space.

Set $\lambda := \rho - d$. We show that $\lambda(x,y) > 0$ if $x \neq y$. Since B is strictly convex, then for any $x,y \in \partial B$, the open line segment $(x,y) \subset \mathring{B}$. We know that the geodesic (x,y) is the shortest distance between x and y, so in particular it's shorter than the arc between x and y on ∂B. Thus, $d(x,y) < \rho(x,y)$, and $\lambda(x,y) = \rho(x,y) - d(x,y) > 0$.

By compacity of X_ε and continuity of λ, $\inf_{X_\varepsilon} \lambda$ is attained at $(x,y) \in X_\varepsilon$ with $x \neq y$. We have that $0 < \lambda(x,y) = \min_{X_\varepsilon} \lambda$. Hence, $\delta(\varepsilon) := \min_{X_\varepsilon} \lambda$ is a strictly positive function.

Now let us introduce a function $\mathcal{L}(\gamma)$ that gives the length of a particular periodic orbit. There exists a sequence $(x_1, \ldots, x_n)$ such that

$$\mathcal{L}(x_1, \ldots, x_n) \xrightarrow{n \to \infty} L = \text{length}(\partial B)$$

since ∂B is smooth, in particular rectifiable: it has finite length. This means that

$$L = \sup_{0 = s_1 < \cdots < s_n = 1} \mathcal{L}(\gamma(s_1), \ldots, \gamma(s_n)).$$

If there is an interval $I \subset \partial B$ with $\partial I = \{x,y\}$ such that none of the points $x_1, \dots, x_n$ belongs to I (see 5.2), then

$$\mathcal{L}(x_1, \dots, x_n) \leq L - \lambda(x,y) \leq L - \delta(\underbrace{\rho(x,y)}_{=\varepsilon}) < L.$$

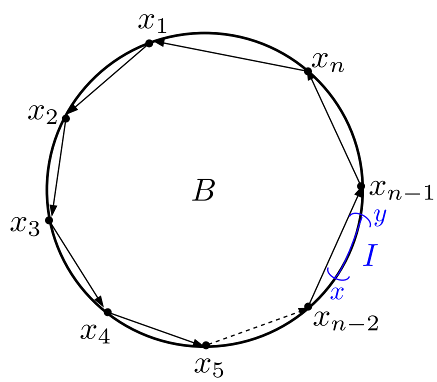

Fig. 5.2. Non-existence of such interval

Thus, $\mathcal{L}(x_1, \dots, x_n) \not\to L$, which is a contradiction, concluding the proof of the theorem. $\square$

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Appendix A

Measure Theory

In this appendix, we introduce the concept of measure theory and we give some useful results of it. For more details, we refer the reader to [**19**],[**6**] and [**21**].

A.1. Motivation to measure theory

The area of a regular shape such as rectangle or triangle is known by its formula. What about the area of a shape bounded by a closed curve in general?

In case where the shape is bounded by two curves who are graphs of some integrable functions, we can use integral calculus to compute its area. If not, we have not seen until now tools to compute the area of such shape. One idea would be to approximate the area by fitting as many rectangles as we can inside the boundary in a way that those rectangles do not overlap on each other (see Figure A.1). The area of the shape will be approximately the sum of area of those rectangles.

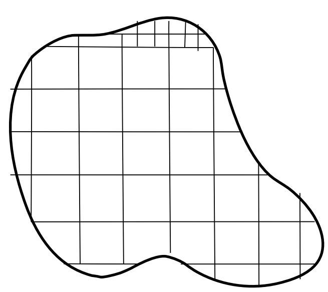

Fig. A.1. Fitting rectangles inside an irregular shape

One can generalize this idea to approximate the "area" of higher dimensional objects. For instance, in $\mathbb{R}^3$, we can use cubes to approximate the volume of any irregular solid. In $\mathbb{R}^4$, we can use 4D-rectangles $[a_1,b_1] \times [a_2,b_2] \times [a_3,b_3] \times [a_4,b_4]$ to approximate the hyper-volume of an 4D object, and so on. We will use the word "measure" to generalize the notion of length, area and volume, and we call those hyper objects subsets of $\mathbb{R}^d$ with $d \in \mathbb{N}$. In those terms, we wish to calculate the measure of a subset of $\mathbb{R}^d$ by approximating it by union of other sets with simple geometry such as rectangles that covers the set in question. This marks the beginning of measure theory.

Before getting precisely into measures of sets in the following sections, let us first introduce some notation and preliminary results.

We mostly work with compact sets in $\mathbb{R}^d$, which are sets that are bounded and closed. Those sets have the Heine-Borel covering property: any covering of compact set by a collection of open sets contains a finite subcovering. More formally,

> *Assume K is compact, $K \subset \bigcup_{\lambda\in\Lambda} \mathcal{O}_\lambda$, and each $\mathcal{O}_\lambda$ is open. Then there are finitely many of the open sets, $\mathcal{O}_{\lambda_1}, \mathcal{O}_{\lambda_1}, \dots, \mathcal{O}_{\lambda_n}$ such that $K \subset \bigcup_{i=1}^{n} \mathcal{O}_{\lambda_i}$.*

A rectangle R in $\mathbb{R}^d$ is the product of d one-dimensional intervals

$$R = [a_1,b_1] \times [a_2,b_2] \times \cdots \times [a_d,b_d]$$

where $a_i < b_i$ for all $i \in \{1,2,\dots,d\}$. We denote its volume by $|R|$, and it is given by the product of length of each side

$$|R| = (b_1 - a_1)(b_2 - a_2)\cdots(b_d - a_d).$$

We say that the union of rectangles are **almost disjoint** if the interior of each rectangle are disjoint to each other. Since the main idea is to cover sets by union of rectangles, we state two theorems to describe the structure of open sets in terms of rectangles. We first begin by open sets in $\mathbb{R}$.

Theorem A.1.1. *Every open subset $\mathcal{O}$ of $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.*

Proof. We first describe the form of the open intervals. Fix $x \in \mathcal{O}$ and let I_x be the largest open interval containing x that is contained in $\mathcal{O}$. Set

$$a_x = \inf\{a < x : (a,x) \subset \mathcal{O}\} \quad \text{and} \quad b_x = \sup\{b > x : (x,b) \subset \mathcal{O}\}.$$

We must have $a_x < x < b_x$. We now let $I_x = (a_x,b_x)$. By construction, $x \in I_x \subset \mathcal{O}$. Hence,

$$\mathcal{O} = \bigcup_{x\in\mathcal{O}} I_x.$$

We then show that distinct open intervals are disjoints. Suppose by contradiction that there exist two sets containing x that is contained in $\mathcal{O}$, namely I_x and I_y. Their union $I_x \cup I_y$ is also contained in $\mathcal{O}$ and contains x. Since I_x is maximal, we have $(I_x \cup I_y) \subset I_x$. Similarly, $(I_x \cup I_y) \subset I_y$. This happens only if $I_x = I_y$.

Finally, we show that this family of open intervals is countable. Each open I_x contains a rational number. Since distinct I_x are disjoint, they contain different rational numbers. $\square$

Remark A.1.2. *If $\mathcal{O}$ is open and $\mathcal{O} = \bigcup_{j=1}^{\infty} I_j$, where I_j are disjoint, then the measure of $\mathcal{O}$ is naturally equal to $\sum_{j=1}^{\infty} |I_j|$, where $|I_j|$ denotes the measure of I_j.*

Lastly, we state an analogue of this theorem in $\mathbb{R}^d$. The reader can refer to [**19**] for its proof.

Theorem A.1.3. *Every open subset $\mathcal{O}$ of $\mathbb{R}^d$, $d \geq 1$, can be written as a countable collection of almost disjoint closed cubes.*

A.2. Algebras of sets

Before getting into the measure of a set, we first get a closer look on some structure about sets that will be used when discussing about the notion of measure. We call those structures the algebras of sets, and we refer the reader to [**11**] more details.

Let X be a set. We denote by $P(X)$ the collection of all subsets of X or the power set of X.

Definition A.2.1 ([**11**]). *The **ring of subsets** in X is a nonempty family $R \subset \mathcal{P}(X)$ that is closed under the operators of union, intersection and difference.*

On one hand, we can refine the rings of subsets by giving those sets some special properties.

Definition A.2.2 (Refinements of Rings). *Let R be a ring of subsets.*

(1) An algebra of sets is a ring $R \subset \mathcal{P}(X)$ that contains the whole set X as an element;

(2) A $\sigma-$ring is a ring R closed under the operation of countable union.

(3) A $\sigma-$algebra is a ring R that is both an algebra and a σ-ring.

On the other hand, we can relax the rings of subsets by making some assumptions less restrictive.

Definition A.2.3 (Relaxation of Rings). *A **half-ring** is a family $S \subset \mathcal{P}(X)$ that is closed under intersection, and if $A, B \subset S$, then there exist $C_1, \ldots, C_n \in S$ such that $A \backslash B = C_1 \cup C_2 \cup \cdots \cup C_n$.*

Example A.2.4. *The collection S of all half-open intervals of the form* [a,b) *is a half-ring, but not a ring since it is not closed under the operation of intersection.*

Finally, let X be a topological space, and $U \subset \mathcal{P}(X)$ the family of all open subsets of X. There is a minimal $\sigma-$ring among those containing U, denoted by $R_\sigma(U)$, and elements of that minimal $\sigma-$ring are called the **Borel subsets** of X.

Example A.2.5. *The set of all rational points (which are open subsets in $\mathbb{R}$) of the segment* [0,1] *is a Borel set on the real line $\mathbb{R}$.*

A.3. Measurable sets and Lebesgue measure

As mentioned in the introduction, the measure of a set is the generalization of the volume of the set. We will now give a precise definition of it on the half-ring.

Definition A.3.1. *A **measure** on a half-ring $S \subset \mathcal{P}(X)$ is a non-negative function on S having the additivity property*

$$\mu(A \sqcup B) = \mu(A) + \mu(B)$$

where $\sqcup$ is the disjoint union of two sets.

*Furthermore, a measure μ is said to be **countably additive** (or σ-additive) if it has the property*

$$\mu(\bigsqcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$$

Example A.3.2. *Let S be a half-ring of the open interval of the form $[a,b) \subset \mathbb{R}$. It makes sense that its measure is its length. Indeed, if we define $\mu([a,b)) = b-a$, then μ is a measure on S.*

The following theorem tells us that we can extend a measure on the half-ring to an unique measure on the ring itself.

Theorem A.3.3. *Every measure μ on a half-ring S extends uniquely to the measure μ on the ring $R(S)$. If the original measure is countably additive, its extension will also be countably additive.*

The reader can refer to [**11**] for a proof of this theorem.

Finally, an important property of the measure is that of the countable **monotonicity**: if $A, A_k \in S$ and $A \subset \bigcup A_k$, then $\mu(A) \leq \sum \mu(A_k)$.

Remark A.3.4. *Countable additive measure extends not only to the ring $R(S)$ or to the sigma-ring ($R_\sigma(s)$), but also to a bigger collection called measurable sets.*

We will now define the measure of a set,

Definition A.3.5. *Let X be a set, and $S \subset \mathcal{P}(X)$ be a half-ring, and μ be a σ-additive measure on S.*

*(1) For any $A \in \mathcal{P}(X)$ we define the **outer measure** $\mu^*(A)$ by*

$$\mu^*(A) = \inf \sum_{k=1}^{\infty} \mu(A_k), \quad A \subset \bigcup_{k=1}^{\infty} A_k, \quad A_k \subset S.$$

*(2) $A \in \mathcal{P}(X)$ is said to be **Lebesgue measurable** with respect to μ if for any $\varepsilon > 0$ there is a set $B \in R(S)$ such that $\mu^*(A \triangle B) < \varepsilon$.*

The operation $A \triangle B$ is called the symmetric difference, defined as $A \triangle B = (A \cup B) \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A)$. In other words, a set A is measurable if it can be approximated as precise as we wish by sets B in the ring $R(S)$. We lastly state the Lebesgue's theorem, for which the proof can be found in [**11**].

Theorem A.3.6. *The collection $L(S,\mu)$ of measurable sets forms a σ-algebra, and μ^* is a σ-additive measure on it.*