# On the constrained efficiency of strategy-proof random assignment*

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**Abstract**

We study random assignment of indivisible objects among a set of agents with strict preferences. Random Serial Dictatorship is known to be only ex-post efficient and there exist mechanisms which Pareto-dominate it ex-ante. However, we show that there is no mechanism that is likewise (i) strategy-proof and (ii) boundedly invariant, and that Pareto-dominates Random Serial Dictatorship. Moreover, the same holds for all mechanisms that are ex post efficient, strategy-proof and boundedly invariant: no such mechanism is dominated by any other mechanism that is likewise strategy-proof and boundedly invariant.

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*Keywords:* random assignment, strategy-proofness, ex post efficiency, bounded invariance.

# 1 Introduction

Consider the problem of assigning indivisible objects among a set of agents – each agent is to receive at most one and we assume they have strict preferences over the set of objects. Further, while objects' characteristics may include a fixed monetary payment, there are no additional transfers. Problems like this arise in many real-life applications such as on-campus housing (where rents are fixed), organ allocation, school choice with ties in applicants' priorities, etc.. Whenever several agents would like to consume the same object, the indivisible

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nature of objects, together with the absence of any compensating transfers, will render any deterministic assignment unfair. For that reason, both theorists and policy makers have implemented of random assignments in such contexts.

Since agents' preferences are private information, the design of random assignment mechanisms has to take into account agents' incentives to reveal their preferences. Moreover, eliciting preferences over all possible lotteries is often difficult in practice, so that assignments should instead be based on agents rankings of objects alone – for example, school choice programs will typically ask applicants to provide a list of schools, ranked from most- to least-preferred.

Strategy-proofness makes truthful reporting a dominant strategy and thus should ensure that agents truthfully reveal their ordinal preferences over objects for any underlying utility representation of preferences. Unfortunately, the literature on random assignment mechanisms contains numerous impossibility results as soon as strategy-proofness and equal-treatment-of-equals, as a minimal fairness requirement, are married with different ex-ante notions of efficiency.[1] We will consider throughout ex-post efficiency and constrained ex-ante efficiency in a class of mechanisms satisfying certain properties. Furthermore, for the applications given above, indivisible objects can be seen as goods where any agent desires, or needs to consume, exactly one object and finds all objects acceptable. We will refer to this as the no-disposal domain.

A mechanism frequently used in real life is the random serial dictatorship (RSD) mechanism. It works as follows. An order of the agents is drawn uniformly at random and then agents pick their most preferred objects among the remaining objects according to the selected order. RSD satisfies many desirable criteria: (0) equal treatment of equals as any two agents with the same preference obtain identical random assignments, (1) ex-post efficiency as it selects only assignments which cannot be Pareto improved, and (2) strategy-proofness as no agent profits from misreporting his true preference. Furthermore, it satisfies the condition of (3) bounded invariance saying that the chosen random assignment of any object $x$ depends only on agents' preferences over objects that are weakly preferred to $x$ – hence changing the reported ordering of less preferred objects should not affect the probability with which agents are assigned an object. According to our knowledge no real-life mechanism used in practice and virtually no mechanism considered in the theoretical literature violates bounded invariance. For example, Probabilistic Serial, Deferred- and Immediate Acceptance, as well as the Top-Trading-Cycles Mechanism all satisfy bounded invariance.

Our main result establishes that any mechanism satisfying properties (1)-(3) is not Pareto

[1] Throughout 'ex-ante' is to be understood as before realizing the final deterministic assignment; this corresponds to the term 'interim' used in mechanism design outside of the literature on random assignments.

dominated (in terms of first-order stochastic dominance) by any strategy-proof and bounded invariant mechanism. As immediate corollary we obtain that the RSD mechanism is not Pareto dominated by any mechanism satisfying strategy-proofness and bounded invariance. It is important to stress that our main result applies to any mechanism and is not exclusive to RSD. For instance, in applications one might take into account affirmative action constraints with respect to minorities or disadvantaged groups by not choosing certain orders of agents (where majorities or advantaged groups come first in the order) and apply a weighted version of RSD. Any such mechanism satisfies (1)-(3) and is not Pareto dominated by any strategy-proof and bounded invariant mechanism. This answers the longstanding question whether RSD is constrained efficient in the class of strategy-proof mechanisms (while admittedly we impose bounded invariance in addition), and provides a positive partial answer on the no-disposal domain to the long-standing open question by Zhou (1990) whether RSD is undominated in the class of mechanisms satisfying (0)-(2) – our result does not impose (0) but instead imposes (3). This is the first affirmative result for RSD in connection with ex-post efficiency and strategy-proofness.

We connect our main result to previous literature. Numerous contributions establish the impossibility of strategy-proofness, envy-freeness and ex-ante efficiency. Zhou (1990) showed that in the cardinal framework there exists no mechanism satisfying equal treatment of equals, strategy-proofness and ex-ante efficiency (where the latter postulates always to choose a random assignment which is not Pareto dominated in terms of expected utility by any other one). In the ordinal framework, Bogomolnaia and Moulin (2001) show the impossibility when envy-freeness is weakened to equal-treatment-of-equals. Nesterov (2017) shows that the impossibility persists when ex-ante efficiency is weakened to ex-post efficiency (while maintaining envy-freeness).[2] Shende and Purohit (2020) show that strategy-proofness and envy-freeness are incompatible with unanimity (which they refer to as contention-free efficiency). When strengthening equal-treatment-of-equals to envy-freeness, Basteck and Ehlers (2023) showed that a strategy-proof mechanism can be unanimous and hence ex-post efficient with probability of at most $\frac{2}{n}$ (where $n$ is the number of agents). In other words, for any strategy-proof and envy-free mechanism there exist profiles where ex-post efficient assignments are chosen with probability of at most $\frac{2}{n}$ (and inefficient assignments are chosen with probability of at least $1-\frac{2}{n}$). This finding strengthens significantly the impossibility of strategy-proofness, envy-freeness and ex-post efficiency and provides an exact upper bound for ex-post unanimity.

Furthermore, one might allow agents to rank certain objects unacceptable, which we

[2]Zhang (2019) proves a strong group-manipulability result, imposing ex-post efficiency and auxiliary fairness axioms that are by themselves weaker than envy-freeness.

refer to the full domain. If agents may rank objects unacceptable and possibly receive no object, notions of efficiency have to take into account the set of (un)assigned objects: a deterministic assignment is non-wasteful if no agent prefers an unassigned object to her assignment. As a stronger requirement, ex-ante non-wastefulness demands that if an agent prefers an object over another and is assigned the less-preferred with positive probability, then the more-preferred object must be assigned with probability one. On the full domain there is no relation between ex-post efficiency and ex-ante non-wastefulness, and RSD might be ex-ante wasteful. Erdil (2014) established that there are mechanisms Pareto-dominating RSD which are less ex-ante wasteful, which is a negative answer on the full domain to the open question by Zhou (1990). In particular, the mechanism constructed in the proof of Proposition 3 by Erdil (2014) satisfies equal treatment of equals and strategy-proofness but violates bounded invariance. It is unknown which mechanisms Pareto dominate RSD and at the same time are not Pareto dominated by any other strategy-proof mechanism. Our main result implies that any strategy-proof and bounded invariant mechanism, which dominates RSD on the full domain, must coincide with RSD on the no-disposal domain. In other words, then Pareto improvements over RSD are only possible for profiles where objects are classified unacceptable in a "certain" way. For the full domain we show the following object-by-object domination lemma: if one mechanism sd-dominates another mechanism, then there is a profile where the first mechanism assigns each object to each agent with weakly greater probability than the second one, with strict inequality holding for some agent and some object. Furthermore, Martini (2016) shows that there is no mechanism satisfying strategy-proofness, equal-treatment-of-equals and ex-ante non-wastefulness, i.e. another principal impossibility result on the full domain.

Finally, for the no-disposal domain, Bogomolnaia and Moulin (2001) introduced the probabilistic serial (PS) mechanism and show that it is envy-free and ex-ante efficient (hence necessarily violates strategy-proofness). Bogomolnaia and Heo (2012) provide an axiomatic characterization of the PS mechanism via ex-ante efficiency, envy-freeness and bounded invariance. Hashimoto et al. (2014) weakened bounded invariance to weak invariance in this characterization, a property which is satisfied by any strategy-proof mechanism.

The paper is organized as follows. Section 2 introduces random assignments, their properties and several popular mechanisms. Section 3 states our main result pertaining to the constrained efficiency of any mechanism satisfying ex-post efficiency, strategy-proofness and bounded invariance. Section 4 contains the proof of our main result. Section 5 allows agents to rank objects unacceptable and the waste of objects, and states the object-by-object domination lemma. Section 6 concludes.

## 2 Model

Let $N = \{1, \ldots, n\}$ denote the set of agents and $O = \{o_1, \ldots, o_n\}$ denote the finite set of objects. Throughout we suppose $|N| = |O| \geq 3$. Each agent $i$ has strict preferences over $O \cup \{i\}$ where $i$ stands for being unassigned; let $R_i$ denote the corresponding linear order[3] and write $P_i$ for its asymmetric part (where $xP_iy$ is defined by $xR_iy$ and $x \neq y$). Let $\mathcal{W}^i$ denote the set of all strict preferences of agent $i$ over $O \cup \{i\}$. Let $\mathcal{W}^N = \times_{i \in N} \mathcal{W}^i$ denote the set of all preference profiles $R = (R_1, \ldots, R_n)$. Let $\mathcal{R}^i$ denote the set of all strict preferences of agent $i$ over $O \cup \{i\}$ such that $oR_ii$ for all $o \in O$, i.e., where all objects are acceptable. We denote this domain by $\mathcal{R}^N = \times_{i \in N} \mathcal{R}^i$ and refer to it as the no-disposal domain, as no agent would ever dispose of any assigned object. We call $\mathcal{W}^N$ the full domain.

An assignment is a mapping $\mu : N \to O \cup N$ such that[4] $\mu_i \in O \cup \{i\}$ for all $i \in N$ and $\mu_i \neq \mu_j$ for all $i \neq j$. Let $\mathcal{M}$ denote the set of all assignments.

An assignment $\mu$ is efficient under $R$ if there exists no $\mu' \in \mathcal{M}$ such that $\mu'_i R_i \mu_i$ for all $i \in N$ and $\mu'_j P_j \mu_j$ for some $j \in N$. Let $\mathcal{PO}(R)$ denote the set of all efficient assignments under $R$. If $R$ is a unanimous profile (where any two agents rank different objects first), then the unique efficient assignment allocates each agent his most preferred object.

An assignment $\mu$ is weakly efficient under $R$ if there exists no $\mu' \in \mathcal{M}$ such that $\mu'_i P_i \mu_i$ for all $i \in N$. Let $\mathcal{WPO}(R)$ denote the set of all weakly efficient assignments under $R$.

An assignment $\mu$ is non-wasteful under $R$ if for all $i \in N$ and all $x \in O \cup \{i\}$, $xR_i\mu_i$ implies there exists $j \in N$ with $\mu_j = x$. Note that this implies $\mu_i R_i i$. Let $\mathcal{NW}(R)$ denote the set of all non-wasteful assignments under $R$.

For any profile $R$, we have $\mathcal{PO}(R) \subseteq \mathcal{NW}(R)$, and there is no relation between non-wastefulness and weak efficiency.

Let $\Delta(\mathcal{M})$ denote the set of all probability distributions over $\mathcal{M}$. Given $p \in \Delta(\mathcal{M})$, let $p_{ia}$ denote the associated probability of $i$ being assigned $a$. Let $supp(p)$ denote the support of $p$. Then (i) $p$ is ex-post efficient under $R$ if $supp(p) \subseteq \mathcal{PO}(R)$, (ii) $p$ is ex-post weakly efficient under $R$ if $supp(p) \subseteq \mathcal{WPO}(R)$, and (iii) $p$ is ex-post non-wasteful under $R$ if $supp(p) \subseteq \mathcal{NW}(R)$.

For all $i \in N$, all $R_i \in \mathcal{W}^i$ and all $x \in O \cup \{i\}$, let $B(x, R_i) = \{y \in O \cup \{i\} : yR_ix\}$. Then given any $p, q \in \Delta(\mathcal{M})$, $p_i$ stochastically $R_i$-dominates $q_i$ if for all $x \in O \cup \{i\}$,

$$\sum_{y \in B(x,R_i)} p_{iy} \geq \sum_{y \in B(x,R_i)} q_{iy}.$$

[3]Thus, $R_i$ is (i) complete, (ii) transitive and (iii) antisymmetric ($xR_iy$ and $yR_ix$ implies $x = y$).
[4]We will use throughout the convention to write $\mu_i$ instead of $\mu(i)$ for any $i \in N$.

A random assignment $p$ stochastically $R$-dominate another random assignment $q$ if $p_i$ $R_i$-dominates $q_i$ for all $i \in N$. A random assignment is sd-efficient if there is no random assignment $q \neq p$ that stochastically $R$-dominates it.[5] Given two random assignments $p$ and $q$, we say that $p$ and $q$ are *equivalent* if $p_i = q_i$ for all $i \in N$.

A mechanism is a mapping $f : \mathcal{R}^N \to \Delta(\mathcal{M})$. Then $f(R)$ denotes the random assignment chosen for $R$, and $f_{ia}(R)$ denotes the probability of agent $i$ being assigned object $a$. For $i \in N$, $f_i(R)$ denotes the tuple of assignment probabilities $(f_{ia}(R))_{a \in O}$, and for $a \in O$, $f_a(R)$ is defined accordingly as the tupple of probabilities with which $a$ is assigned to the various agents. Then $f$ is *sd-efficient* if for all $R \in \mathcal{R}^N$, $f(R)$ is sd-efficient under $R$. Similarly, we define ex-post (weak) efficiency and ex-post non-wastefulness for a mechanism.

Then $f$ is *strategy-proof* if for all $R \in \mathcal{R}^N$, all $i \in N$ and all $R'_i \in \mathcal{R}^i$, $f_i(R)$ stochastically $R_i$-dominates $f_i(R'_i, R_{-i})$. Note that for any ordinal mechanism (where an agent only submits his ordinal ranking), strategy-proofness is equivalent to the requirement that for any von Neumann-Morgenstern utility presentation of his true ordinal ranking, submitting the true ordinal ranking maximizes his expected utility. Most real-life mechansims only elicit this ordinal information (instead of von Neumann-Morgenstern utilities).

Furthermore, $f$ is *envy-free* if for all $R \in \mathcal{R}^N$ and all $i \in N$, $f_i(R)$ stochastically $R_i$-dominates $f_j(R)$ (where in $f_j(R)$ the outside option $j$ is replaced by $i$). If $f(R)$ attaches probability one to assignment $\mu$, then this is equivalent to $\mu_i R_i \mu_j$ for all $i, j \in N$. Finally, $f$ is *symmetric* (respectively, treats equals equally) if for all $R \in \mathcal{R}^N$ and all $i, j \in N$, $R_i = R_j$ implies $f_{io}(R) = f_{jo}(R)$ for all $o \in O$.

Note that most properties are defined in terms of an agent's random assignment. For a given set of properties, we say that a mechanism $f$ is *unique in terms of probability shares*, if for any other mechanism $\phi$ satisfying this set of properties, $f(R)$ and $\phi(R)$ are equivalent for any profile $R$.

Below we introduce some of the well-known mechanisms on the no-disposal domain.

The uniform assignment (UA) mechanism[6] randomizes uniformly over all $|N|!$ deterministic non-wasteful assignments (irrespective of agents preferences). Hence for individual object assignment probabilities we have: for all $R \in \mathcal{R}^N$, $UA_{io}(R) = \frac{1}{n}$ for all $i \in N$ and $o \in O$.

A strict priority ranking over $N$ is denoted by $\succ$. Let $\mathcal{L}$ denote the set of all strict priority rankings. Given $\succ \in \mathcal{L}$, let $f^{\succ}$ denote the (deterministic) serial dictatorship mechanism where agents are assigned their most-preferred among all available objects in order of their

[5] Bogomolnaia and Moulin (2001) refer to this as 'ordinal efficiency'. It implies Pareto-efficiency with respect to expected utilities for some von Neumann-Morgenstern-representations of agents' ordinal preferences over objects McLennan (2002).

[6] Chambers (2004) characterizes UA via consistency.

priority.[7] Then the random serial dictatorship (RSD) mechanism is defined by $RSD(R) = \frac{1}{n!}\sum_{\succ \in \mathcal{L}} f^{\succ}(R)$ for all $R \in \mathcal{R}^N$.[8]

We omit the formal definition of the probabilistic serial (PS) mechanism[9] and provide an intuitive formulation instead: each agent starts eating with uniform speed from his most-preferred object; once an object is exhausted, each agent eats with uniform speed from his most-preferred among the remaining objects, and so on until all objects are exhausted. The assignment probabilities of any agent in PS are simply the shares of objects the agent has eaten during this process.[10]

# 3 Main Result

Before we state our main result, we introduce two more definitions.

Given $i \in N$, $R_i \in \mathcal{R}^i$ and $x \in O$, let $R_i(x) = R_i|B(x, R_i)$ denote the restriction of $R_i$ to the weak upper contour set of $x$. Now a mechanism $f$ satisfies *bounded invariance (BI)* if for all $R \in \mathcal{R}^N$, all $i \in N$, all $R'_i \in \mathcal{R}^i$ and all $x \in O$, if $R'_i(x) = R_i(x)$, then $f_x(R) = f_x(R'_i, R_{-i})$.

In other words, if agent $i$'s preference above object $x$ remains unchanged, then the random assignment of $x$ remains unchanged.

Given two mechanisms $f$ and $g$, $f$ sd-dominates $g$ (denoted by $f \succeq^{sd} g$) if for any profile $R$ the random assignment $f(R)$ stochastically $R$-dominates the random assignment $g(R)$, and for some profile $\bar{R}$ and $i \in N$ we have $f_i(\bar{R}) \neq g_i(\bar{R})$.

**Theorem 1** *On the no-disposal domain, if mechanism $g$ satisfies ex-post efficiency, bounded invariance and strategy-proofness, then there exists no bounded invariant and strategy-proof mechanism $f$ which sd-dominates $g$.*

RSD satisfies ex-post efficiency, bounded invariance and strategy-proofness, and by Theorem 1 RSD cannot be sd-dominated by another mechanism satisfying bounded invariance and strategy-proofness. The same is true for weighted versions of RSD, i.e. where we

[7]For any $R \in \underline{\mathcal{R}}^N$ and $i_1 \succ i_2 \succ \cdots \succ i_n$, $i_1$ receives his most $R_{i_1}$-preferred object in $O$ (denoted by $f^{\succ}_{i_1}(R)$), and for $l = 2, \ldots, n$, $i_l$ receives his most $R_{i_l}$-preferred object in $O \backslash \{f^{\succ}_{i_l}(R), \ldots, f^{\succ}_{i_{l-l}}(R)\}$ (denoted by $f^{\succ}_{i_l}(R)$).

[8]Pycia and Troyan (2023a) construct a random mechanism distinct from RSD but where for any profile $R$ the chosen random assignment is equivalent to $RSD(R)$.

[9]For that, we refer the reader to Bogomolnaia and Moulin (2001); Bogomolnaia (2015) offers an alternative definition of PS, and Katta and Sethuraman (2006) extend PS to the domain where indifferences are allowed.

[10]Note that the PS-mechanism pins down individuals' object assignment probabilities directly, rather than a random assignment per se, i.e., a convex combination of deterministic assignments. Nonetheless, corresponding random assignments exists as any bistochastic matrix $(p_{ia})_{i \in N, a \in O}$ can be decomposed as a convex combination of permutation matrices by the Birkhoff-von Neumann Theorem Birkhoff (1946).

attach different weights to different orders of agents and apply SD. Such weights could take into account minorities/majorities and (dis)advanted groups. Furthermore, in Theorem 1 ex-post efficiency cannot be weakened to ex-post weak efficiency. For instance, the Random-Dictatorship-cum-Equal-Division[11] by Basteck and Ehlers (2023) satisfies ex-post weak efficiency, bounded invariance and strategy-proofness, but is sd-dominated by RSD.

Several questions remain. First, does Theorem 1 remain unchanged when we drop bounded invariance from the second mechanism, i.e. we keep bounded invariance just for the first mechanism? Second, do we obtain a characterization of RSD by adding symmetry to the mechanism $g$ (i.e. RSD is the only symmetric, strategy-proofness and ex-post-eff mechanism which is undominated)? Third, is RSD characterized by ex-post efficiency, bounded invariance, strategy-proofness and symmetry? Obviously, an affirmative answer to the third question implies the affirmative answer to the second question, but not the reverse.

We provide an outline of the proof of Theorem 1. As a basic step we show that for any efficient deterministic assignment, any agent must rank his allocated object weakly above some non-top ranked object. Then for a fixed object, say $z$, we count for any profile and for any agent the number of non-top ranked objects below $z$, and lexicographically minimize with respect to those numbers. If $g$ sd-dominates $f$, then the set of profiles where $f$ and $g$ differ is non-empty. Now in this set we choose a profile where object $z$ is ranked as low as possible with respect to the minimization outlined above and show that the random assignment of $z$ must coincide for $f$ and $g$. Remaining in the set of profiles where $f$ and $g$ differ and $z$ is ranked as low as possible, we take another object, say $y$, choose a profile where $y$ is ranked as low as possible and show that the random assignment of $y$ (and $z$) is identical for $f$ and $g$. Iterating we eventually exhaust the set of objects and obtain that $f$ and $g$ coincide, which implies that the set of profiles where $f$ and $g$ differ was empty yielding the final contradiction.

# 4 Proof

First, we show a basic consequence of efficiency. Given $R_i \in \mathcal{R}^i$, let $top(R_i)$ denote the top ranked object in $O$ by $R_i$. For profile $R$, let $top(R) = \cup_{i \in N}\{top(R_i)\}$ denote the set of top ranked objects under $R$.

**Lemma 1** *Let $R \in \mathcal{R}^N$ and $\mu \in \mathcal{PO}(R)$. Then for all $i \in N$ there exists $x \in O \backslash top(R)$ such that $\mu_i R_i x$.*

[11] We omit the formal definition and refer to Basteck and Ehlers (2023). Informally, the mechanism works as follows: any agent $i$ is chosen with probability $\frac{1}{n}$, then agent $i$ picks his most preferred object and the remaining objects are assigned uniformly among the other agents.

**Proof.** Suppose not, i.e. say for agent 1 we have $xP_1\mu_1$ for all $x \in O\backslash top(R)$. Then $O \neq top(R)$ as otherwise $R$ is a unanimous profile and the unique efficient assignment gives each agent his most preferred object. Thus, $O\backslash top(R) \neq \emptyset$ and $top(R_1)P_1xP_1\mu_1$ for all $x \in O\backslash top(R)$. Now $\mu_1 \in top(R)$, and say $top(R_2) = \mu_1$. By efficiency, $\mu_2 \notin O\backslash top(R)$ and $\mu_2 \in top(R)$, say $top(R_3) = \mu_2$, and so on. At some point we find a (improvement) cycle, a contradiction to efficiency of $\mu$. □

Suppose that mechanism $g$ satisfies ex-post efficiency, bounded invariance and strategy-proofness, and there a bounded invariant and strategy-proof mechanism $f$ such that $f \succeq^{sd} g$. By Lemma 1, if $g$ is ex-post efficient, then for all $R \in \mathcal{R}^N$, all $i \in N$ and all $x \in top(R)$ such that $yP_ix$ for all $y \in O\backslash top(R)$, we have $g_{ix}(R) = 0$. Furthermore, by $f \succeq^{sd} g$ we then also have $f_{ix}(R) = 0$.

We introduce additional notation. If $f \succeq^{sd} g$, then let

$$\mathcal{R}^{\neq} = \{R \in \mathcal{R}^N : f_i(R) \neq g_i(R) \text{ for some } i \in N\}$$

denote the set of profiles where $f$ and $g$ differ. In the set $\mathcal{R}^{\neq}$ we identify a minimal profile as follows for a given object.

Note that for any $R \in \mathcal{R}^{\neq}$ we have $O\backslash top(R) \neq \emptyset$ as otherwise $O = top(R)$, $R$ is a unanimous profile, and by ex-post efficiency of $g$ and $f \succeq^{sd} g$ both $f$ and $g$ attach probability one to the unique assignment where each agent received his top ranked object.

Let $\mathbb{N} = \{0, 1, \ldots\}$ denote the set of natural numbers including zero. Let $\mathbb{N}^n_{\geq}$ denote the set of all vectors $v \in \mathbb{N}^n$ such that $v_1 \geq v_2 \geq \cdots \geq v_n$, i.e. the coordinates of $v$ are arranged in non-increasing order. Let $\precsim$ denote the lexicographical ordering on $\mathbb{N}^n_{\geq}$: for all $v, w \in \mathbb{N}^n_{\geq}$, $v \precsim w$ means either $v = w$ or there is $1 \leq t \leq n$, such that $v_i = w_i$ for every $1 \leq i < t$ and $v_t < w_t$. We write $v \prec w$ if $v \precsim w$ and $w \not\precsim v$.

Given $z \in O$ and $R$, consider for each $i \in I$ the number of objects in $O\backslash top(R)$ ranked below $z$. Furthermore, for any $z \in O$ and $R_i$, let

$$L(z, R_i) = \{y \in O : zP_iy\}$$

denote the strict lower contour set at $z$ of $R_i$. Note that this set excludes $z$.

We are going to build a vector with all these numbers in non-increasing order from highest to lowest. Formally, let[12]

$$\theta(z,R)\in\mathbb{N}^n_{\geq} \quad \text{, where} \quad \{\theta_i(z,R):1\leq i\leq n\}=\{|L(z,R_i)\cap(O\backslash top(R))|:i\in N\} \text{ and } \theta_1(z,R)\geq\theta_2(z,R)\geq\cdots\geq\theta_n(z,R).$$

We say that $z$ is ranked lower in $R'$ than in $R''$, if, starting with the highest number in each case, that number is lower for $R'$ than for $R''$ (and vice versa). If the two are equal, move to the second highest number for each profile and iterate. Hence, we lexicographically minimize $\theta(z,R)$, but we start from the highest number (first trying to lower $z$ further for those agents that already rank it rather high). Let

$$\mathcal{R}^{\neq}_z=\{R\in\mathcal{R}^{\neq}: \text{ there exists no } \bar{R}\in\mathcal{R}^{\neq} \text{ such that } \theta(z,\bar{R})\prec\theta(z,R).\}.$$

We show in two lemmas that for any profile in $\mathcal{R}^{\neq}_z$ the random assignment of $z$ is identical under $f$ and $g$.

**Lemma 2** *Let $z\in O$ and $R\in\mathcal{R}^{\neq}_z$. If there is some $j\in I$ such that $f_{jz}(R)>g_{jz}(R)$, then we partition $I$ as follows: $I=I_1\cup I_2$ where for all $i\in I_1$, $z$ is ranked least relative to $O\backslash top(R)$, i.e., $L(z,R_i)\cap(O\backslash top(R))=\emptyset$, while for all $i\in I_2$ (where $j\in I_2$), $L(z,R_j)\cap(O\backslash top(R))\subseteq L(z,R_i)\cap(O\backslash top(R))$ (i.e., $i$'s lower contour set of $z$ at $R$ contains all objects in $O\backslash top(R)$ that are contained in $j$'s lower contour set).*

**Proof.** First note that for $j$, $L(z,R_j)\cap(O\backslash top(R))\neq\emptyset$ (as $j$ has more of $z$ under $f$ and hence needs to have less of some lower ranked object for $f$ to dominate $g$ – hence by Lemma 1, $L(z,R_j)\cap(O\backslash top(R))$ is non-empty). Now take any $a\in L(z,R_j)\cap(O\backslash top(R))$ and move it up to just below $z$. Since $R$ was minimal, we cannot push it past $z$, so we have $f_{jz}(R)>g_{jz}(R)$ and $f_{ja}(R)<g_{ja}(R)$. Any agent $i\neq j$ who does not rank $z$ least relative to $O\backslash top(R)$, i.e., for whom $L(z,R_j)\cap(O\backslash top(R))\neq\emptyset$, must also rank $a$ below $z$: otherwise, they could move $z$ to the bottom of $O\backslash top(R)$ while, by BI, we still have $f_{ja}(R)<g_{ja}(R)$ – contradicting the minimality of $R$. Since $a\in L(z,R_j)$ was chosen arbitrarily, this proves the lemma. □

**Lemma 3** *Let $z\in O$ and $R^*\in\mathcal{R}^{\neq}_z$. Then $f_{iz}(R^*)=g_{iz}(R^*)$ for all $i\in I$.*

**Proof.** Suppose not. Then there exists $j\in I$ with $f_{jz}(R^*)>g_{jz}(R^*)$ and the strict lower contour set $L(z,R^*_j)$ must intersect with $O\backslash top(R^*)$ as otherwise we have $L(z,R^*_j)\subseteq top(R^*)$

[12]Here identical numbers appear multiple times, i.e. we could have $\{2,2,2,1,1,0,\ldots\}$.

and by Lemma 1 and $f \succeq^{sd} g$, $g_{jo}(R^*) = 0 = f_{jo}(R^*)$ for all $o \in L(z, R_j^*)$ (which is then a contradiction to $f \succeq^{sd} g$ and $f_{jz}(R^*) > g_{jz}(R^*)$).

Consider the partition $\{I_1, I_2\}$ as in Lemma 2 (where $j \in I_2$). Let $top(R_{I_1}^*)$ denote the set of top objects of agents in $I_1$. If $top(R_{I_1}^*) \cap L(z, R_j^*) \neq \emptyset$, take any $i \in I_1$ such that $top(R_i^*) = x \in L(z, R_j^*)$. Then move $x$ in $R_j^*$ just below $z$. Denoting the obtained profile by $R$, we have $f_{jz}(R) > g_{jz}(R)$.

If $f_{jx}(R) < g_{jx}(R)$, let $i$ push $\{z\} \cup (L(z, R_j^*) \cap (O \backslash top(R)))$ to the bottom of $R_i$ in the same order as $R_j$. Same for all other $i'$ in $I_1$ who rank $x$ first. Then by BI, we still have $f_{jx}(R) < g_{jx}(R)$. Towards a contradiction, if $f_{jz}(R) \leq g_{jz}(R)$, then there is some higher ranked object $y$ such that $f_{jy}(R) > g_{jy}(R)$. Moreover, $yP_hz$ for all $h$ in $I_1$ that rank $x$ first as well as for $h = j$ – hence, we can push $z$ to the bottom for all $R_h$, arrive at a profile where $f$ and $g$ differ in the assignment probabilities of $y$ yet $z$ is ranked lower relative to objects $O \backslash top(R)$ than at our initial profile. Since this contradicts the minimality of our initial profile, we conclude that $f_{jz}(R) > g_{jz}(R)$.

If instead we have $f_{jx}(R) \geq g_{jx}(R)$, swap $x$ and $z$ in the ranking of $j$: if we now have $f_{jx}(R) > g_{jx}(R)$ then we can push down $z$ down for $j$, below all other $O \backslash top(R)$ – by SP this preserves $f_{jx}(R) > g_{jx}(R)$ and by BI we may push down again $z$ for any $i \in I_1$ with $top(R_i) = x$ (as $xP_i(O \backslash top(R)) \cap L(z, R_j)$), contradicting minimality of $R^*$. Therefore, we know that, after having swapped $x$ and $z$, we must have $f_{jx}(R) \leq g_{jx}(R)$ and thus $f_{jz}(R) > g_{jz}(R)$.

Thus, independently of whether we are able to push $x$ past $z$ or not, we still have $f_{jz}(R) > g_{jz}(R)$. Note that if we are able to push $x$ above $z$, then after the swap of $x$ and $z$ we have $f_{jx}(R) \leq g_{jx}(R)$ and the preferences of agents in $I_1$ are unchanged.

If there is any other $x'$ that is a top object of some agent in $I_1$ and also in $L(z, R_j)$, we proceed as before and move $x'$ in $R_j$ just below $z$. By BI, we have $f_{jz}(R) > g_{jz}(R)$.

If $f_{jx'}(R) < g_{jx'}(R)$ do the same as we did above with $x$: let all $i \in I_1$ who rank $x'$ first push $\{z\} \cup (L(z, R_j^*) \cap O \backslash top(R))$ to the bottom of $R_i$ in the same order as $R_j$. Then by BI we still have $f_{jx'}(R) < g_{jx'}(R)$ and moreover $f_{jz}(R) > g_{jz}(R)$ as otherwise there would be a higher ranked object in the ranking of $j$ as well as all other agents $i \in I_1$ for whom we have so far constructed new preferences – hence we could move $z$ down in their ranking and arrive at a contradiction to the minimality or $R^*$.

If instead we have $f_{jx'}(R) \geq g_{jx'}(R)$, swap $x'$ and $z$ in the ranking of $j$: if we now have $f_{jx'}(R) > g_{jx'}(R)$ then we can push down $z$ to the bottom for $j$ as well as for all other $i \in I_1$ (i.e., those were we had already raised $z$ in the previous steps as they must prefer $x'$ to $z$) – by BI this preserves $f_{jx'}(R) > g_{jx'}(R)$, contradicting minimality of $R^*$. Hence, instead we know that, after having swapped $x'$ and $z$, we must have $f_{jx'}(R) \leq g_{jx'}(R)$ and thus

$f_{jz}(R) > g_{jz}(R)$.

Thus, independently of whether we are able to push $x'$ past $z$ or not, we still have $f_{jz}(R) > g_{jz}(R)$.

Repeat the last step until there are no more objects in $top(R^*_{I_1}) \cap L(z, R^*_j)$ that we have not considered. Then we have arrived at a profile, refer to it once more as $R$, where $top(R_i) = top(R^*_i)$ for all $i \in I$, $f_{jz}(R) > g_{jz}(R)$ and $I_1$ has been partitioned into two subsets: $I'_1$ includes all agents $i \in I_1$ for whom $L(z, R_i) \cap (O \backslash top(R^*)) = \emptyset$, $R_i = R^*_i$, and whose top ranked objects are ranked above $z$ by $j$ – in $R^*$ but also in $R$ since $j$'s lower contour set has only gotten smaller. Second, $I''_1$ includes all agents $i \in I_1$ whose lower contour set $L(z, R_i)$ consists of all objects $(O \backslash top(R^*)) \cap L(z, R^*_j)$, ranked in the same order as by $R^*_j$ and $R_j$ ($j's$ ranking over these objects was not altered moving from $R^*$ to $R$). Compared to $R^*$, $j$'s lower contour set at $z$ has gotten smaller, while the ranking of other agents $k \in I_2$ is unchanged.

By Lemma 2 as well as the preceding construction, we still have

$$L(z, R_i) \cap (O \backslash top(R)) \subseteq L(z, R_j) \cap (O \backslash top(R)) \subseteq L(z, R_k) \cap (O \backslash top(R))$$

for all $i \in I''_1$ and all $k \in I_2$. Now, for all $k \in I_2$ (including $j$) change the order of objects in the lower contour set $L(z, R_k)$ as follows: (i) objects that are in $L(z, R_k) \backslash L(z, R_j)$ are ranked immediately below $z$ (beyond that, their order does not matter), (ii) objects that are also in $(O \backslash top(R)) \cap L(z, R_j)$ are ranked next, in the same order as by $R_j$, (iii) last, all objects in $L(z, R_k) \cap L(z, R_j) \cap top(R)$ are ranked below (beyond that, their order does not matter). By BI, we still have $f_{jz}(R) > g_{jz}(R)$. By Lemma 1 and $f \succeq^{sd} g$, we have $f_{kx}(R) = 0 = g_{kx}(R)$ for all $k \in I_2$ and all $x \in L(z, R_k) \cap L(z, R_j) \cap top(R)$.

Hence, we now have all agents in $I''_1 \cup I_2$ ranking objects $L(z, R_j) \cap (O \backslash top(R))$ adjacent and in the same order as $R_j$, and below that only objects in $top(R)$. Since $f_{jz}(R) > g_{jz}(R)$, there is some $y$, ranked below $z$ by $R_j$, such that $f_{jy}(R) < g_{jy}(R)$ – and thus some $i \in I$ with $f_{iy}(R) > g_{iy}(R)$. Moreover by Lemma 1 we have $y \in L(z, R_j) \cap (O \backslash top(R))$.

If $i \in I''_1 \cup I_2$ then there is $y'$ with $yR_iy'$, such that $f_{iy'}(R) < g_{iy'}(R)$ – and thus some $i' \in I$ with $f_{i'y'}(R) > g_{i'y'}(R)$. Moreover, by Lemma 1 and $y \in O \backslash top(R)$, we have $y' \in L(y, R_j) \cap (O \backslash top(R))$. Hence, $y'$ is ranked lower than $y$ according to $R_j$.

If $i' \in I''_1 \cup I_2$ then there is $y''$ with $y'R_{i'}y''$, such that $f_{i'y''}(R) < g_{i'y''}(R)$ – and thus some $i'' \in I$ with $f_{i''y''}(R) > g_{i''y''}(R)$, and so on.

Since $L(z, R_j) \cap (O \backslash top(R))$ is finite and we move down (according to $R_j$) in each iteration, eventually we have exhausted the set. Hence, eventually, there is some $y^* \in L(z, R_j) \cap (O \backslash top(R))$ and $i^* \in I'_1$ such that $f_{i^*y^*}(R) > g_{i^*y^*}(R)$.

Note that $R_{i^*} = R^*_{i^*}$. But then $i^*$ can exchange the positions of $y^*$ and $top(R_{i^*})$ in $R_{i^*}$, reduce the number of non-top objects ranked below $z$ for $j$ (as well as all $i \in I_1''$) and it weakly decreases for all in $I_2\backslash\{j\}$. Hence, $z$ was not minimal in initial $R^*$. $\square$

Now we continue in the same way: having established that $f$ and $g$ agree on the assignment probabilities of $z$ for all profiles $R \in \mathcal{R}_z^{\neq}$, we then take $y \neq z$ and consider those profiles in $\mathcal{R}_z^{\neq}$ where $y$ is ranked as low as possible relative to $(O\backslash\{z\})\backslash top(R)$. Repeating the same arguments in Lemma 3 where $y$ plays the role of $z$ we get again $f_y(R) = g_y(R)$ and $f_z(R) = g_z(R)$. Reiterating, we thereby prove the theorem.

Formally, let $O = \{z_1, \ldots, z_n\}$ and define $Z_t = \{z_1, \ldots, z_t\}$ as well as $O_{-t} = O\backslash Z_t$, for any $1 \leq t < n$. Moreover, let

$$\theta(z_{t+1}, R) \in \mathbb{N}^n_{\geq} \quad \text{where} \quad \{\theta_i(z_{t+1}, R) : 1 \leq i \leq n\} = \{|L(z_{t+1}, R_i) \cap (O_{-t}\backslash top(R))| : i \in N\}\}$$

be the vector of ranks that $z_{t+1}$ occupies in agents' preferences (ordered in non-increasing fashion and allowing for multiple identical entries). We rewrite $\mathcal{R}_{z_1}^{\neq}$ as $\mathcal{R}_1^{\neq}$ and define

$$\mathcal{R}_{t+1}^{\neq} = \{R \in \mathcal{R}_t^{\neq} : \text{there exists no } \bar{R} \in \mathcal{R}_t^{\neq} \text{ such that } \theta(z_{t+1}, \bar{R}) \prec \theta(z_{t+1}, R)\},$$

where $\theta(z_{t+1}, \bar{R})$ and $\theta(z_{t+1}, R)$ ordered by lexicographic minimization. Hence, $\mathcal{R}_{z_{t+1}}^{\neq}$ contains all profiles where $z_{t+1}$ is ranked as low as possible, provided that (i) $f$ and $g$ still differ in the assignment probabilities of *some* object, and that (ii) all objects $z \in Z_t$ are likewise ranked as low as possible (with rank-minimization of $z_m$ taking precedence over the rank-minimization of $z_{m'}$ for any $m < m' \leq t$).

As a first observation, we show that if we rank an object $z_{t+1}$ as low as possible in $\mathcal{R}_t^{\neq}$, then $z_{t+1}$ cannot be ranked at the top of any agent $j$'s ranking for whom $f_j$ and $g_j$ differ.

**Lemma 4** *Let $0 \leq t < n$, $z_{t+1} \in O$ and $R \in \mathcal{R}_{t+1}^{\neq}$. If there is some $j \in I$ such that $f_{jz_{t+1}}(R) > g_{jz_{t+1}}(R)$, then we partition $I$ as follows: $I = I_1 \cup I_2$ where for all $i \in I_1$, $z_{t+1}$ is ranked least relative to $O_{-t}\backslash top(R)$, i.e., $L(z_{t+1}, R_j) \cap (O_{-t}\backslash top(R)) = \emptyset$, while for all $i \in I_2$ (where $j \in I_2$), $L(z_{t+1}, R_j) \cap (O_{-t}\backslash top(R)) \subseteq L(z_{t+1}, R_i) \cap (O_{-t}\backslash top(R))$ (i.e., $i$'s lower contour set of $z_{t+1}$ at $R$ contains all objects in $O_{-t}\backslash top(R)$ that are contained in $j$'s lower contour set).*

**Lemma 5** *Let $0 \leq t < n$, $z_{t+1} \in O$ and $R \in \mathcal{R}_{t+1}^{\neq}$. Then $f_{iz_{t+1}}(R) = g_{iz_{t+1}}(R)$ for all $i \in I$.*

**Proof of Lemma 4 and 5.**

For $t = 0$, this is established by Lemma 2 and 3, which serve as the basis for the following induction. For the induction step, assume we have established both statements for all

$0 \leq t < k \leq n$ (induction hypothesis). It remains to show that both hold for $t = k$.

*Induction step for Lemma 4.* First note that for $j$, $L(z_{k+1}, R_j) \cap (O_{-k} \backslash top(R)) \neq \emptyset$ (as $j$ has more of $z_{k+1}$ under $f$ and hence needs to have less of some lower ranked object for $f$ to dominate $g$ – hence by Lemma 1, and our induction hypothesis, $L(z_k, R_j) \cap (O_{-k} \backslash top(R))$ is non-empty). Now take any $a \in L(z_{k+1}, R_j) \cap (O_{-k} \backslash top(R))$ and move it up to just below $z_{k+1}$. Since $R$ was minimal, we cannot push it past $z_{k+1}$, so we have $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$ and $f_{ja}(R) < g_{ja}(R)$. Any agent $i \neq j$ who does not rank $z_{k+1}$ least relative to $O_{-k} \backslash top(R)$ must also rank $a$ below $z_{k+1}$: otherwise, they could move $z_{k+1}$ to the bottom of all $O_{-k} \backslash top(R)$ while, by BI, we still have $f_{ja}(R) < g_{ja}(R)$ – contradicting the minimality of $R$. Since $a \in L(z_{k+1}, R_j)$ was chosen arbitrarily, this completes the induction step for lemma 4.

*Induction step for Lemma 5.* Suppose the statement is not true for $t = k$. Then there exists $R^* \in \mathcal{R}^{\neq}_{k+1}$ and $j \in I$ with $f_{jz_{k+1}}(R^*) > g_{jz_{k+1}}(R^*)$. Without loss of generality, we may assume that all objects in $Z_k$ are ranked at the bottom of $R^*_j$ such that $z'_m R^*_j z_m$ for $m < m' \leq k$: Otherwise we can begin by moving $z_1$ to the bottom of $j$'s preference list in single, pairwise swaps. Since these transformations keep the profile in $\mathcal{R}^{\neq}_k \subseteq \mathcal{R}^{\neq}_1$ we have $f_{jz_1}(R) = g_{jz_1}(R)$ both before and after the swap and hence, by SP, $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$ (where $R$ denotes an arbitrary profile in the sequence starting at $R^*$). Repeating this for each $m$ with $1 < m \leq k$ establishes the claim.

For most of the subsequent modifications of preference profiles, up until the very end, we will leave all objects $Z_k \subseteq L(z_{k+1}, R^*_j)$ at the bottom of $j$'s preference in the order just described.

Second, note that the strict lower contour set $L(z_{k+1}, R^*_j)$ must intersect with $O_{-k} \backslash top(R^*)$: otherwise we could move to $R$ by pushing all $o \in L(z_{k+1}, R^*_j) \cap Z_k$ up against $z_{k+1}$, leaving only objects $o \in L(z_{k+1}, R^*_j) \cap top(R^*)$ at the bottom of $R_j$, so that by induction hypothesis and Lemma 1 we have the same assignment probabilities $f_{jo}(R) = g_{jo}(R)$ for all $o \in L(z_{k+1}, R_j)$. Moreover, by SP, we still have $f_{jz_{k+1}}(R^*) > g_{jz_{k+1}}(R^*)$. Together that contradicts the standing assumption that $f$ sd-dominates $g$.

Consider the partition $\{I_1, I_2\}$ as in Lemma 4 (where $j \in I_2$) – by the induction hypothesis and the induction step for Lemma 4 above, this exists for $t = k$. Let $top(R^*_{I_1})$ denote the set of objects most-preferred by some agent in $I_1$. If $top(R^*_{I_1}) \cap L(z_{k+1}, R^*_j) \cap O_{-k} \neq \emptyset$, take any $i \in I_1$ such that $top(R^*_i) = x \in L(z_{k+1}, R^*_j) \cap O_{-k}$. Then move $x$ in $R^*_j$ just below $z_{k+1}$. Denoting the obtained profile by $R$, we have $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$ (by SP).

If $f_{jx}(R) < g_{jx}(R)$, then for all $i \in I_1$ who rank $x \in L(z_{k+1}, R^*_j) \cap O_{-k}$ at the top, we can let $i$ push all objects in $(L(z_{k+1}, R^*_j) \cap (O_{-k} \backslash top(R))) \cup Z_k$ to the bottom in the same order as $R_j$: $\{z_{k+1}\} \cup (L(z_{k+1}, R_j^*) \cap (O_{-k} \backslash top(R)))$ is ranked first, then $Z_k$ at the very bottom (with $z_{m'}$ ranked above $z_m$ for any $m < m' \leq k$). Same for all other $i'$ in $I_1$ who rank $x$ first. Then by BI, we still have $f_{jx}(R) < g_{jx}(R)$. Towards a contradiction, if $f_{jz_{k+1}}(R) \leq g_{jz_{k+1}}(R)$ then there is some higher ranked object $y$ such that $f_{jy}(R) > g_{jy}(R)$. Moreover, $yP_h z_{k+1}$ for all $h \in I_1 \cup \{j\}$ – hence, we can push $z_{k+1}$ below $L(z_{k+1}, R_j^*) \cap (O_{-k} \backslash top(R))$ for all $R_h$, arrive at a profile where $f$ and $g$ differ in the assignment probabilities of $y$ yet $z_{k+1}$ is ranked lower relative to objects $O_{-k} \backslash top(R)$ than at our initial profile. Since this contradicts the minimality of our initial profile, we conclude that $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$. Moreover, since all objects in $Z_k$ are ranked as low as possible for all $i$ for whom we have constructed new preferences, we still have $R \in \mathcal{R}_k^{\neq}$ (though not necessarily $R \in \mathcal{R}_{k+1}^{\neq}$ since $z_{k+1}$ is not ranked as low as possible).

If instead we have $f_{jx}(R) \geq g_{jx}(R)$, then swap $x$ and $z_{k+1}$ in the ranking of $j$: if we now have $f_{jx}(R) > g_{jx}(R)$ then we can push down $z_{k+1}$ down for $j$, below all other $O_{-t} \backslash top(R)$ (but above $Z_t$) – by SP this preserves $f_{jx}(R) > g_{jx}(R)$, contradicting minimality of $R^*$. So instead we know that, after having swapped $x$ and $z_{k+1}$, we must have $f_{jx}(R) \leq g_{jx}(R)$ and thus $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$. Again, since the swap does not raise any object in $Z_k$ in the ranking of any agent, we have $R \in \mathcal{R}_k^{\neq}$. So independently of whether we are able to push $x$ past $z_{k+1}$ or not, we still have $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$ and $R \in \mathcal{R}_k^{\neq}$.

If there is any other $x' \in O_{-k}$ that is a top object of some agent in $I_1$ and also in $L(z_{k+1}, R_j)$, move $x'$ in $R_j$ just below $z_{k+1}$. By SP, we have $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$.

If $f_{jx'}(R) < g_{jx'}(R)$, then do the same as we did above with $x$: let all $i \in I_1$, who rank $x'$ first, push all objects in $(L(z_{k+1}, R_j^*) \cap (O_{-k} \backslash top(R))) \cup Z_k$ to the bottom in the same order as $R_j$ – first $\{z_{k+1}\} \cup (L(z_{k+1}, R_j^*) \cap (O_{-k} \backslash top(R)))$, then $Z_k$. By BI, we still have $f_{jx}(R) < g_{jx}(R)$ and moreover we also have $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$ as otherwise there would be some $y$ with $f_{jy}(R) > g_{jy}$ that is ranked above $z_{k+1}$ by $j$ and all $i \in I_1$ for whom we have so far constructed new preferences. Hence, by pushing $z_{k+1}$ down further below $L(z_{k+1}, R_j^*) \cap (O_{-k} \backslash top(R))$ in the ranking of $j$ and all $i \in I_1$ for whom we have so far constructed new preferences leads to a contradiction to minimality of $R^*$.

If instead we have $f_{jx'}(R) \geq g_{jx'}(R)$, swap $x'$ and $z_{k+1}$ in the ranking of $j$: if we now have $f_{jx'}(R) > g_{jx'}(R)$ then we can push down $z_{k+1}$ to the bottom of all $O_{-k} \backslash top(R)$ (but above $Z_k$) for $j$ as well as for all other $i \in I_1$ for whom we have so far constructed new preferences – by BI this preserves $f_{jx'}(R) > g_{jx'}(R)$, contradicting minimality of $R^*$. So instead we know that, after having swapped $x'$ and $z_{k+1}$, we must have $f_{jx'}(R) \leq g_{jx'}(R)$ and thus $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$. So independently of whether we are able to push $x'$ past $z_{k+1}$ or not, we still have $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$.

Note that neither of the two manipulations increases the rank of objects in $Z_k$. Hence, as a result, we have another profile $R$ where $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$ and $R \in \mathcal{R}_k^{\neq}$.

Repeat the last step until there are no more objects in $O_{-k} \cap top(R^*_{I_1}) \cap L(z_{k+1}, R^*_j)$ that we have not considered. Then we have arrived at a profile, refer to it once more as $R$, where $top(R_i) = top(R^*_i)$ for all $i \in I$, $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$ and $I_1$ has been partitioned into two subsets: $I'_1 \subseteq I_1$ includes all agents $i \in I_1$ that rank $z_{k+1}$ least relative to $O_{-k} \backslash top(R^*)$ and for which $R_i = R^*_i$ and whose top ranked objects are either in $Z_t$ or ranked above $z_{k+1}$ by $j$ – in $R^*_j$ but also in $R_j$ since $j$'s lower contour set has only gotten smaller. Second, $I''_1$ includes all agents $i \in I_1$ whose lower contour set $L(z_{k+1}, R_i)$ consists of all objects $(L(z_{k+1}, R^*_j) \cap (O_{-k} \backslash top(R))) \cup Z_k$, ranked in the same order as by $R^*_j$ and $R_j$ ($j's$ ranking over these objects was not altered moving from $R^*$ to $R$). Compared to the initial profile $R^*$, $j$'s lower contour set at $z_{k+1}$ may have gotten smaller, while the preference order of other agents in $I_2$ is unchanged.

By Lemma 4 as well as the preceding construction, we still have

$$L(z_{k+1}, R_i) \cap (O_{-k} \backslash top(R)) \subseteq L(z_{k+1}, R_j) \cap (O_{-k} \backslash top(R)) \subseteq L(z_{k+1}, R_h) \cap (O_{-k} \backslash top(R))$$

for all $i \in I''_1$ and all $h \in I_2$. Now, for all $h \in I_2$ (other than $j$) change the order of objects in the lower contour set $L(z_{k+1}, R_h)$ as follows: (i) objects that are in $L(z_{k+1}, R_h) \backslash L(z_{k+1}, R_j)$ are ranked immediately below $z_{k+1}$ (beyond that, their order does not matter), (ii) objects that are in $L(z_{k+1}, R_h) \cap L(z_{k+1}, R_j)$ are ranked next, in the same order as by $R_j$ (hence, with objects in $L(z_{k+1}, R_h) \cap L(z_{k+1}, R_j) \cap Z_k$ at the bottom). By BI, we still have $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$. Moreover, since objects in $Z_k$ have only seen their rank lowered, we still have $R \in \mathcal{R}_k^{\neq}$.

In a final reshuffling, let all agents in $I''_1 \cup I_2$ push all objects in $L(z_{k+1}, R_j) \cap top(R)$ to the bottom. Again, we will call the new profile $R$. Since top ranked objects do not matter in the rank-minimization of objects $Z_k$, we still have $R \in \mathcal{R}_k^{\neq}$, as well as $f_{jz_{k+1}}(R) > g_{jz_{k+1}}(R)$ (by BI). Hence there is some $y$, ranked below $z_{k+1}$ by $R_j$, such that $f_{jy}(R) < g_{jy}(R)$ – and thus some $i \in I$ with $f_{iy}(R) > g_{iy}(R)$. Moreover, by Lemma 1 and our induction hypothesis, we have $y \in L(z_{k+1}, R_j) \cap (O_{-k} \backslash top(R))$.

If $i \in I''_1 \cup I_2$ then there is $y'$ with $y R_i y'$, such that $f_{iy'}(R) < g_{iy'}(R)$ – and thus some $i' \in I$ with $f_{i'y'}(R) > g_{i'y'}(R)$. Moreover, by Lemma 1 and our induction hypothesis, we have $y' \in L(y, R_j) \cap (O_{-k} \backslash top(R))$. Hence, $y'$ is ranked lower than $y$ according to $R_j$.

If $i' \in I''_1 \cup I_2$ then there is $y''$ with $y' R_{i'} y''$, such that $f_{i'y''}(R) < g_{i'y''}(R)$ – and thus some $i'' \in I$ with $f_{i''y''}(R) > g_{i''y''}(R)$, and so on.

Since $L(z_{k+1}, R_j) \cap (O_{-k} \backslash top(R))$ is finite and we move down (according to $R_j$) in each iteration, eventually we have exhausted the set. So eventually, there is some $y^* \in$

$L(z_{k+1}, R_j) \cap (O_{-k} \backslash top(R))$ and $i^* \in I'_1$ such that $f_{i^*y^*}(R) > g_{i^*y^*}(R)$.

If $top(R_{i^*}) \in O_{-k}$, let $i^*$ exchange $y^*$ with his top ranked object in $R_{i^*}$. since, as a result of our changes to the initial preference profile $R^*$, we know that $top(R_{i*})$ is not included in $j$'s lower contour set of $z_{k+1}$, this reduces the number of non-top objects ranked below $z_{k+1}$ for $j$ (as well as all $i \in I''_1$) and it weakly decreases it for all in $I_2 \backslash \{j\}$. Hence $z_{k+1}$ was not minimal in the initial $R^*$.

Finally, consider $top(R_{i^*}) = z_m \in Z_k$ (i.e., $m \leq k$). Since $f_{i^*y^*}(R) > g_{i^*y^*}(R)$ there must be some lower ranked $\hat{y}$ such that $f_{i^*\hat{y}}(R) < g_{i^*\hat{y}}(R)$. But then, consider the strict upper countour set of $\hat{y}$, i.e. $U(\hat{y}, R^*_i) = \{o \in O : oP^*_i\hat{y}\}$. Push all elements in $U(\hat{y}, R_{i^*}) \cap Z_k$ to just above $\hat{y}$ to arrive at $R'$. This preserves $f_{i^*\hat{y}}(R') < g_{i^*\hat{y}}(R')$ (by SP). Moreover, since we have pushed these objects below $y^*$ and $y^* \in O_{-k} \backslash top(R)$, we have reduced their rank. But that contradicts $R \in \mathcal{R}^{\neq}_k$ – which concludes the proof. □

# 5 Unacceptable Objects

So far, our results were confined to the no-disposal domain $\mathcal{R}^N$, where every agent finds all objects acceptable. For the full domain $\mathcal{W}^N$ being unassigned is not necessarily ranked at the bottom of an agent's preference. As this represents the outside option, we want to avoid any agent to be assigned to an unacceptable object.

Given profile $R$, the random assignment $p$ is individually rational if for all $i \in N$ and $x \in O$ such that $iP_ix$ we have $p_{ix} = 0$.[13] Obviously, for the no-disposal domain any random assignment is individually rational.

Given two random assignments $p$ and $q$, we say that $p$ object-by-object dominates $q$ if for all $i \in N$ and all $o \in O$ we have $p_{io} \geq q_{io}$ (with strict inequality holding for some $j \in N$ and some $x \in O$). Object-by-object domination makes no reference to a preference profile. Now given profile $R$ and two individually rational random assignments $p$ and $q$, if $p$ object-by-object dominates $q$, then $p$ stochastically $R$-dominates $q$ and $p_i \neq q_i$ for some $i \in N$. Note that for this assertion we do not need to know the exact preferences over acceptable objects. On the full domain we show that if one strategy-proof mechanism sd-dominates another strategy-proof mechanism, then we can find a profile where object-by-object dominance holds.

**Lemma 6** ***(Object-by-object domination lemma)*** *Let $f$ and $g$ be two individually rational and strategy-proof mechanisms on the full domain. If $f$ sd-dominates $g$, then there*

[13]For individual rationality its ex-ante and ex-post notions are equivalent.

*exists a profile $\hat{R}$ where $f(\hat{R})$ object-by-object dominates $g(\hat{R})$.*

**Proof.** As $f \succeq^{sd} g$, there exists $R \in \mathcal{W}^N$, $i \in N$ and $x \in O$ such that $f_{ix}(R) > g_{ix}(R)$. Without loss of generality, let $x$ be the $R_i$-most preferred object where $f_i(R)$ and $g_i(R)$ differ. Let $R'_i$ be such that $\{o \in O : oP'_i i\} = B(x, R_i)$ and $R'_i|B(x, R_i) = R_i|B(x, R_i)$. Let $R' = (R'_i, R_{-i})$. By our choice of $x$ and both strategy-proofness and individual rationality of $f$ and $g$, we obtain $f_{ix}(R') = f_{ix}(R) > g_{ix}(R) = g_{ix}(R')$ and $f_{io}(R') = g_{io}(R')$ for all $o \in O\backslash\{x\}$. If $f(R')$ object-by-object dominates $g(R')$, then we are done. Otherwise there exist $j \in N\backslash\{i\}$ and $y \in O$ such that $f_{jy}(R') < g_{jy}(R')$. By individual rationality, $yP_j j$. As $f_j(R')$ sd-dominates $g_j(R')$, there exists $z \in O$ such that $zP_j y$ and $f_{jz}(R') > g_{jz}(R')$. But then we repeat the same procedure for $j$ as above and obtain the same conclusions for $R'$, $j$ and $z$. As at each iteration the set of acceptable objects of some agent shrinks and the set of objects is finite, eventually we find $\hat{R}$ such that $f(\hat{R})$ object-by-object dominates $g(\hat{R})$.□

The power of the object-by-object domination lemma is that for sd-dominance among strategy-proof mechanisms on the full domain, we can always find a profile where one mechanism unambiguously dominates the other mechanism, i.e., knowing only which objects are considered acceptable to agents. Lemma 6 does not hold for the no-disposal domain as, for example, UA is strategy-proof and sd-dominated by RSD, but there exists no $R \in \mathcal{R}^N$ such that $RSD(R)$ object-by-object dominates $UA(R)$.

Erdil (2014) was the first one to take into account possible ex-ante waste of objects on the full domain. Given profile $R$ and random assignment $p$, $p$ is ex-ante non-wasteful if there exists no $i \in N$ and $x, y \in O \cup \{i\}$ such that $xP_i y$, $p_{iy} > 0$ and $\sum_{j\in N} p_{jx} < 1$. In other words, if agent $i$ prefers $x$ to $y$ and is assigned with positive probability to $y$, then object $x$ is assigned with probability one. Obviously, on the no-disposal domain ex-post efficiency implies ex-ante non-wastefulness as all objects are assigned with probability one. A weaker version is ex-ante weak non-wastefulness of a random assignment $p$ where there exists no $i \in N$ and $x \in O$ such that $xP_i i$, $p_{ii} > 0$ and $\sum_{j\in N} p_{jx} < 1$. In other words, if agent $i$ remains unassigned with positive probability, then all objects, which $i$ finds acceptable, are assigned with probability one.

The size of any random assignment $p$ is the total amount of objects allocated by $p$, i.e. $|p| = \sum_{o\in O} \sum_{i\in N} p_{io}$. Now given two mechanisms $f$ and $g$, $f$ is of greater size than $g$ if $|f(R)| \geq |g(R)|$ for all $R \in \mathcal{W}^N$ and $|f(\hat{R})| > |g(\hat{R})|$ for some $\hat{R}$.

Erdil (2014, Proposition 1) showed that no strategy-proof and ex-ante non-wasteful is sd-dominated by another strategy-proof mechanism.[14] Of course, this is not true for the

[14]Clearly any deterministic mechanism, which is non-wasteful and strategy-proof (e.g. serial dictatorship mechanisms), is not sd-dominated by another strategy-proof mechanism.

no-disposal domain as UA is ex-ante non-wasteful and strategy-proof, but sd-dominated by RSD. By the same reasoning, Erdil (2014, Proposition 2) does not hold on the no-disposal domain as he shows that if strategy-proof mechanism $f$ sd-dominates another strategy-proof mechanism $g$, then $f$ must be of greater size than $g$.

The object-by-object domination lemma allows for alternative proofs of the important Propositions 1 and 2 in Erdil (2014) (where in (1) below ex-ante weak non-wastefulness suffices).

**Corollary 1** *Let $f$ and $g$ be two individually rational and strategy-proof mechanisms on the full domain.*

*(1) If $g$ is ex-ante weakly non-wasteful, then $f$ cannot sd-dominate $g$.*

*(2) If $f$ sd-dominates $g$, then $f$ is of greater size than $g$.*

**Proof.** In showing (1), suppose $f \succeq^{sd} g$. By Lemma 6 there exists $\hat{R}$ such that $f(\hat{R})$ object-by-object dominates $g(\hat{R})$. Then for some $i \in N$ and $x \in O$ we have $f_{ix}(\hat{R}) > g_{ix}(\hat{R})$. Since $f_{jx}(\hat{R}) \geq g_{jx}(\hat{R})$ for all $j \in N$, we obtain $1 \geq \sum_{j \in N} f_{jx}(\hat{R}) > \sum_{j \in N} g_{jx}(\hat{R})$, i.e. object $x$ is not assigned with probability one under $g(\hat{R})$. Since $f_{io}(\hat{R}) \geq g_{io}(\hat{R})$ for all $o \in O$, we obtain $f_{ii}(\hat{R}) < g_{ii}(\hat{R})$. As $x \hat{P}_i i$ and $\sum_{j \in N} g_{jx}(\hat{R}) < 1$, this implies that $g(\hat{R})$ is ex-ante weakly wasteful, a contradiction.

In showing (2), from $f \succeq^{sd} g$ and Lemma 6 there exists $\hat{R}$ such that $f(\hat{R})$ object-by-object dominates $g(\hat{R})$ which implies $|f(\hat{R})| > |g(\hat{R})|$. As for any $R \in \mathcal{W}^N$, $f(R)$ sd-dominates $g(R)$ and both $f(R)$ and $g(R)$ are individually rational, we obtain $|f(R)| \geq |g(R)|$. Hence, $f$ is of greater size than $g$. $\square$

Most importantly, Erdil (2014, Proposition 3) showed that RSD is sd-dominated by a strategy-proof mechanism with less ex-ante waste. Unfortunately, Erdil's mechanism cannot be adapted to satisfy bounded invariance (while continuing to sd-dominate RSD) in a straightforward manner.[15] Erdil (2014, Proposition 4) showed that any strategy-proof mechanism, which sd-dominates RSD, must be ex-ante wasteful. The problems of (i) determining the minimal waste in the class of strategy-proof mechanisms which sd-dominate RSD and (ii) whether there exists a strategy-proof and bounded invariant mechanism, which

[15] Without going into details and using the same notation, consider the construction on p.158-159 or Erdil (2014) (where all agents other than agent 1 always receive the same random assignment under $g$ and RSD): let $R_1 : ba$; then $R_1 \in \mathcal{R}_1^b$ and $g(R)$ gives $\epsilon$-more for $a$ to 1 (compared to RSD); now let $Q_2 : cab$ and by BI, $g(Q_2, R_{-2})$ would have to continue to give $\epsilon$-more of $a$ to 1 (compared to RSD) and give the same of $b$ to 1 under $g$ and RSD; and then for $Q_1 : ab$ we obtain by SP that $g(Q_1, Q_2, R_3, R_4)$ has to give more to $a$ and $b$ to 1 than under RSD, but this is a contradiction to $g \succeq^{sd} RSD$ and $|RSD(Q_1, Q_2, R_3, R_4)| = 1$.

sd-dominates RSD on the full domain remain open. Lemma 6 might offer a tool for resolving those questions.

# 6 Conclusion

Instead of reporting ordinal preferences, one might ask agents to report cardinal utility functions. Then agents compare random assignments via their expected utilities. We implicitly assume ordinality of mechanisms, i.e. the chosen random assignment is the same across cardinal utility profiles which induce identical ordinal preferences. For applications ordinality is a natural requirement as it facilitates reporting, given that agents might be unable to determine their exact utilities but are able to compare individual objects. Of course, allowing cardinal reports but imposing ordinality gives us the same result. In particular, in such contexts RSD is not dominated by any mechanism satisfying ordinality, strategy-proofness and bounded invariance. This is a positive answer on the no-disposal domain to the open question by Zhou (1990), and who showed that in the cardinal framework there exists no mechanism satisfying equal treatment of equals, strategy-proofness and ex-ante efficiency. The latter postulates always to choose a random assignment which is not Pareto dominated in terms of expected utility by any other one. It is clear that in the cardinal context the properties of ordinality, equal treatment of equals and ex-ante efficiency are incompatible: as a simple example, let $N = \{1, 2, 3\}$, $O = \{a, b, c\}$, $u_1 = (u_{1a}, u_{1b}, u_{1c}) = (1, 1 - \epsilon, 0) = u_2$ and $u_3 = (1, \epsilon, 0)$ where $\epsilon > 0$ is small; when all agents have utility function $u_3$ equal treatment of equals requires each agent to obtain $a$ with probability one third, and similarly, when all agents have utility function $u_1$ equal treatment of equals requires each agent to obtain $c$ with probability one third; now ordinality requires for the profile $(u_1, u_2, u_3)$ that each agent obtains any object with probability $\frac{1}{3}$, which is dominated in terms of expected utility by assigning agent 3 object $b$ with probability one, and assigning agents 1 and 2 objects $a$ and $c$ each with probability one half.

The last example shows the disrelation of Zhou's result and the impossibility results in the ordinal framework with respect to efficiency, equity and strategy-proofness. Ordinality, sd-efficiency and envy-freeness are compatible as PS satisfies all those properties. As soon as strategy-proofness is added, we obtain an incompatibility, which is robust when weakening sd-efficiency to ex-post efficiency, unanimity and respectively, to ex-ante non-wastefulness on the full domain, or envy-freeness to equal treatment of equals.

The question whether RSD is characterized by ex-post efficiency, equal treatment of equals and strategy-proofness remains open. A partial answer has been recently found by

Brandt et al. (2023) for five agents via computational methods.[16] However, an affirmative answer to this question remains elusive for an arbitrary number of agents. Pycia and Troyan (2023b) recently showed that RSD is characterized by symmetry, efficiency, and obvious strategyproofness among all mechanisms that, roughly speaking, can be represented as a symmetrization of an extensive-form game where in each stage, one agent is allowed to pick one house from a subset of the remaining houses or "pass" on this opportunity. For the assignment of one object and the full domain, Ehlers (2002) characterized the uniform random dictatorship mechanism with ex-post efficiency, envy-freeness and strategy-proofness. Another strand of the literature studies large markets. Here one may make markets large in two different ways: either by keeping the set of object types fixed and adding copies to match an increasing number of agents, or by considering economies with a large number of distinct agents and distinct objects. First, when we add object copies, Liu and Pycia (2016) have shown in their Theorem 2 that any two symmetric and "regular"[17] mechanisms, which are asymptotically strategy-proof and asymptotically efficient, coincide asymptotically, i.e., they choose the same allocations in the limit. For instance, this implies asymptotic coincidence of RSD[18] and PS (which was first shown by Che and Kojima (2010)), and that RSD and, respectively, PS satisfy ex-post efficiency and asymptotically both strategy-proofness and envy-freeness.[19] In some sense, then it does not matter in the large whether we choose RSD or PS (or any other mechanism satisfying the above three properties). However, continued discussions in real-life markets show the importance of the choice of the random assignment mechanism to be implemented. As we have shown, RSD cannot be improved in an unambiguous way while maintaining our two basic properties.

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[16] While this is an interesting approach, it crucially hinges on the computational power of computers which will most likely not resolve the question for an arbitrary number of agents and objects.

[17] Loosely speaking, this means that agents cannot change to "too much" the random assignments of other agents (in terms of probability shares) as the market becomes large.

[18] RSD is regular, provided the number of copies for each object type grows at the same rate as the number of agents, e.g., in replica economies.

[19] When we consider economies with a large number of distinct agents and distinct objects, Manea (2009) has shown that RSD is sd-efficient with probability zero, and hence RSD and PS diverge with probability one.

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