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Approaches to Boyd's Conjectures and Their Applications

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Abstract

In this dissertation, we consider four cases of Boyd's conjectures for the Mahler measure of polynomials. The first case involves a polynomial defining a genus 1 curve, two other cases cover genus 2 curves, and the final case deals with a genus 3 curve.

For the case of the genus 1 curve, we study an identity conjectured by Boyd and proven by Boyd and Rodriguez-Villegas. We find an expression of the Mahler measure given by a linear combination of some values of the Bloch-Wigner dilogarithm. Combining this with the result proven by Boyd and Rodriguez-Villegas, we can establish some identities among different values of the Bloch-Wigner dilogarithm.

For the problems related to the genus 2 curves, we use the elliptic regulator to recover some identities between Mahler measures involving certain families of genus 2 curves that were conjectured by Boyd and proven by Bertin and Zudilin by differentiating the parameter in the Mahler measure formulas and using hypergeometric identities.

For the case involving the genus 3 curve, we use the elliptic regulator to prove an entirely new identity between the Mahler measures of a genus 3 polynomial family and of a genus 1 polynomial family that was initially conjectured by Liu and Qin.

Since our proofs for the cases of genus 2 and 3 curves involve the regulator, they yield light into the relation of the Mahler measures of the genus 2 or 3 families with special values of the L-functions associated to the genus 1 families.

Keywords: Mahler measure, L-function of elliptic curve, regulator, Bloch-Wigner dilogarithm, volume of a hyperbolic 3-manifold.

Résumé

Dans cette thèse, nous considérons quatre cas de conjectures de Boyd pour la mesure de Mahler de polynômes. Le premier cas concerne un polynôme associé à une courbe de genre 1, deux autres cas couvrent des courbes de genre 2, et le dernier cas traite d'une courbe de genre 3.

Pour le cas de la courbe de genre 1, nous étudions une identité conjecturée par Boyd et prouvée par Boyd et Rodriguez-Villegas. On trouve un expression de la mesure de Mahler donnée par une combinaison linéaire de certaines valeurs du dilogarithme de Bloch-Wigner. En combinant cela avec le résultat prouvé par Boyd et Rodriguez-Villegas, nous pouvons établir certaines identités entre différentes valeurs du dilogarithme de Bloch-Wigner.

Pour les problèmes liés aux courbes de genre 2, nous utilisons le régulateur elliptique pour récupérer des identités entre les mesures de Mahler des certaines familles de courbes de genre 2 qui ont été conjecturées par Boyd et prouvées par Bertin et Zudilin en différenciant le paramètre des formules de la mesure de Mahler et en utilisant des identités hypergéométriques.

Pour le cas impliquant la courbe de genre 3, nous utilisons le régulateur elliptique pour prouver une identité entièrement nouvelle entre les mesures de Mahler d'une famille polynomiale de genre 3 et d'une famille polynomiale de genre 1 qui a été initialement conjecturée par Liu et Qin.

Comme nos preuves pour les cas des courbes des genres 2 et 3 impliquent le régulateur, elles éclairent la relation des mesures de Mahler des familles des genres 2 ou 3 avec des valeurs spéciales des fonctions L associées aux familles de genre 1.

Mots clés: mesure de Mahler, fonction L de courbe elliptique, régulateur, dilogarithme de Bloch-Wigner, volume d'une variété hyperbolique 3D.

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Chapter 1

Introduction: Mahler Measure and Boyd's Conjectures

1.1 Boyd's Conjectures

The *logarithmic Mahler measure* of a nonzero rational function $P \in \mathbb{C}(x_1, \dots, x_n)^\times$ is defined by

$$\begin{aligned} \mathrm{m}(P) &= \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n})| \, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(z_1, \cdots, z_n)| \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_n}{z_n}, \end{aligned} \tag{1.1}$$

where $\mathbb{T}^n$ is the unit torus given by

$$\mathbb{T}^n = \{(z_1, \cdots, z_n) \in \mathbb{C}^n \mid |z_1| = \cdots = |z_n| = 1\}.$$

The *Mahler measure* of P is then defined by

$$M(P) = e^{\mathrm{m}(P)}.$$

For a nonzero rational function $P \in \mathbb{C}(x_1, \dots, x_n)^\times$, the integral (1.1) converges, and furthermore, if P is a Laurent polynomial with integral coefficients, then $\mathrm{m}(P) \geq 0$ (see [13, Lemma 3.7, p. 57] in detail). For a one variable polynomial

$$P(x) = a_d \prod_{i=1}^{d} (x - \alpha_i) \in \mathbb{C}[x],$$

by Jensen's formula, the Mahler measure of P has an equivalent expression given by

$$M(P) = |a_d| \prod_{i=1}^{d} \max\{1, |\alpha_i|\}. \tag{1.2}$$

If a polynomial $P \in \mathbb{Z}[x]$, then from (1.2) its Mahler measure $M(P)$ is an algebraic number and $M(P) \geq 1$. A well-known conjecture, Lehmer's Mahler measure problem, claims that there exists a real number $r > 1$ such that for any polynomial $P \in \mathbb{Z}[x]$, the Mahler measure satisfies $M(P) > r$ or $M(P) = 1$. In addition, it is known that for a nonzero primitive $P \in \mathbb{Z}[x]$, $M(P) = 1$ if and only if all the zeros of P are roots of unity or 0 (see [13, Theorem 1.33, p. 28]). As $M(P)$ is actually an algebraic number, Lehmer's problem involves the distribution of algebraic numbers and is still an open question.

For the two-variable case, the Mahler measure is more complex. In favorable cases, one finds a relationship with special values of L-functions. For instance, consider the following formula due to Smyth (see [5] and [32])

$$\mathrm{m}(y + x + 1) = L'(\chi_{-3}, -1),$$

where $L(\chi, s)$ is the nontrivial Dirichlet L-function defined in Section 2.3 and χ_{-3} is the Dirichlet character of conductor 3 (see the notation χ_{-f} in Definition 2.3.2 and the conductor f in Definition 2.3.4). In 1997, Deninger [10] expressed the Mahler measures of certain Laurent polynomials without vanishing points on the unit torus in terms of a regulator from a higher K-group to a cohomology group. From this connection and the conjectural relationships between the regulator and special value of L-function (Bloch-Beilinson conjectures), Deninger predicted that under some conditions, the Mahler measure is related to an L-value. These connections of Mahler measure with special values of L-functions attract considerable research interest. Many formulas of this type of relationships have been conjectured and some have been proven.

In order to understand and generalize the relationship of Mahler measure with L-functions, Boyd [5] conducted a systematic study of certain families of two-variable reciprocal and tempered polynomials of the form (the definitions of reciprocal and tempered polynomials are given in Section 2.2):

$$P_k(x, y) = A(x)y^2 + B_k(x)y + C(x), \tag{1.3}$$

where k is a parameter. In Section 2.2 we will give a brief description of Boyd's polynomials.

When the curve $P_k(x,y)=0$ has genus 0 or 1, Boyd [5] found some numerical examples of the form:

$$\mathrm{m}(P_k) \stackrel{?}{=} \sum_j r_j d_{f_j}, \tag{1.4}$$

where $r_j \in \mathbb{Q}$ and

$$d_f = L'(\chi_{-f},-1) = \frac{f^{3/2}}{4\pi} L(\chi_{-f},2), \tag{1.5}$$

where $L(\chi_{-f},s)$ is the Dirichlet L-function with the quadratic Dirichlet character χ_{-f} of conductor f. We will use the question mark to indicate conjectural identities that have been verified numerically up to at least 20 decimal places. In this dissertation, we will be particularly interested in the following identity, conjectured by Boyd [5] and proven by Boyd and Rodriguez-Villegas [6]:

$$\mathrm{m}(Q) = \frac{1}{6} d_{15},$$

where

$$Q(x,y) = (x^2+x+1)y^2 + 3x(x+1)y + x(x^2+x+1).$$

Note that $Q(x,y)=0$ defines a genus 1 curve. We study this identity and find an expression for the Mahler measure given by a linear combination of special values of the Bloch-Wigner dilogarithm. Combining our findings with the result proven by Boyd and Rodriguez-Villegas, we can establish some identities among different values of the Bloch-Wigner dilogarithm. The results will be discussed in Chapter 3.

Consider a family of genus 1 curves $P_k(x,y)=0$, with

$$P_k(x,y) = (x+1)(y+1)(x+y) - kxy,$$

where k is an integer, and consider the elliptic curve E_k corresponding to the zero locus of $P_k(x,y)$, given by

$$E_k : Y^2 + (k-2)XY + kY = X^3. \tag{1.6}$$

Then, Boyd [5] found some numerical examples of the form:

$$\mathrm{m}(P_k) \stackrel{?}{=} r_k L'(E_k,0), \tag{1.7}$$

where r_k is a rational number and $L(E_k,s)$ is the L-function of the elliptic curve E_k.

k	r_k	N_k	Proven by
-4	2	36	Rodriguez-Villegas [34]
2	1/2	36	Rodriguez-Villegas [34]
-8	10	14	Mellit [25]
1	1	14	Mellit [25]
7	6	14	Mellit [25]
-2	3	20	Rogers–Zudilin [30]
4	2	20	Rogers–Zudilin [30]

Table 1.1: Proven cases corresponding to $g = 1$ of formula (1.7). Here N_k is the conductor of E_k.

Rodriguez-Villegas [34] studied these identities in the context of Beĭlinson's conjectures and was able to prove those related to elliptic curves of conductor 36, which have complex multiplication. Subsequently, other cases were proven involving conductors 14 and 20 (see Table 1.1).

Boyd [5] also investigated some families of curves of generic genus 2 such as

$$Q_k(x,y) = (x^2+x+1)y^2 + kx(x+1)y + x(x^2+x+1),$$

$$R_k(x,y) = (x^2+x+1)y^2+(x^4+kx^3+(2k-4)x^2+kx+1)y+x^2(x^2+x+1),$$

and

$$S_k(x,y) = y^2 + (x^4+kx^3+2kx^2+kx+1)y + x^4.$$

In Boyd's polynomials, the parameter k is a real number. Especially for the direct computation method, the Mahler measure is differentiated respect to the parameter $k \in \mathbb{R}$. For this reason, in this dissertation we focus on Boyd's polynomials for a real parameter k. We will see in Example 2.2.2 that $Q_k(x,y) = 0$ defines a curve of genus 2 for $k \in \mathbb{R}\setminus\{\pm 3, 0\}$, genus 1 for $k \in \{-3,3\}$ and genus 0 for $k = 0$. By Example 2.2.3, $R_k(x,y) = 0$ defines a curve of genus 2 for $k \in \mathbb{R}\setminus\{-1,2,5,6\}$, genus 1 for $k \in \{-1,2,6\}$ and genus 0 for $k = 5$. By Example 2.2.1, $S_k(x,y) = 0$ defines a curve of genus 2 for $k \in \mathbb{R}\setminus\{-1,0,4,8\}$, genus 1 for $k = \{-1,4\}$, genus 0 for $k = 8$ and has two irreducible components of genus 0 for $k = 0$. In Section 2.2, we will see that the curves corresponding to $Q_k(x,y) = 0$, $R_k(x,y) = 0$ and $S_k(x,y) = 0$ are hyperelliptic and by some quadratic changes of variables, these genus 2 curves can be seen to have factors corresponding to genus 1 curves, i.e., the Jacobians associated to the genus 2 curves have an elliptic curve factor. In particular, the Jacobian associated to $S_k(x,y) = 0$ is isogenous to a product of two elliptic curves, one of which is E_k given by (1.6). Boyd found numerical relations of the type

$$\mathrm{m}(S_k) \stackrel{?}{=} s_k L'(E_k, 0).$$

Similarly, the Jacobians associated to $Q_k(x,y) = 0$ and $R_k(x,y) = 0$ are isogenous to the products of two elliptic curves, with a common factor, and Boyd found that the corresponding Mahler measures are numerically related to the L-function of an elliptic curve isomorphic to the common factor.

The above findings led Boyd to conjecture relationships between the Mahler measures of P_k and S_k and between the Mahler measures of Q_k and R_k. These results were eventually proven by Bertin and Zudilin [1, 2] and are summarized as follows.

Theorem 1.1.1. *[1] For k taking real values, we have*

$$\mathrm{m}(S_k) = \begin{cases} 2\,\mathrm{m}(P_k) & 0 \le k \le 4, \\ \\ \mathrm{m}(P_k) & k \le -1. \end{cases}$$

Theorem 1.1.2. *[2] For real $k \ge 4$, we have*

$$\mathrm{m}(Q_k) = \mathrm{m}(R_{2+k}).$$

Bertin and Zudilin proved these results by studying the corresponding Mahler measures as functions on the parameter k and by differentiating respect to k. The equalities were then established by using several identities of hypergeometric functions.

The family $S_k(x,y)$ was also studied by Bosman in his thesis [4]. He considered the relationship with the regulator and proved exact formulas for $\mathrm{m}(S_k)$ in the cases $k \in \{-1, 2, 8\}$. The genus 1 case of $k = 4$ can be proven by techniques of modular unit parametrizations [1, 37].

Bosman [4] used the regulator to relate the Mahler measure of the family $S_k(x,y)$ to a combination of elliptic dilogarithms. A proof of Theorem 1.1.1 could be achieved by relating those elliptic dilogarithms to the ones corresponding to the Mahler measure of $P_k(x,y)$. This is essentially the strategy of Lalín and Wu [20], who provided further clarification by reproving Theorems 1.1.1 and 1.1.2 by using the regulator theory. The proof of Lalín and Wu will be presented in Chapter 4.

We also consider a problem involving a genus 3 curve.

Liu and Qin [23] extended Boyd's ideas (particularly allowing more general expressions for $B_k(x)$) to obtain many more conjectural families generically corresponding to genus 2 and genus 3 curves. The polynomials are still of general type (1.3), and they are still endowed with the automorphism $\sigma : (x, y) \to (1/x, 1/y)$, as Boyd's polynomials (see Section 2.2 in details). They have hyperelliptic models that are covers of at least one common genus 1 curve.

Lalín and Wu [21] proved the following result conjectured by Liu and Qin [22].

Theorem 1.1.3. *Let*

$$P_k(x, y) = y^2 + (x^6 + kx^5 - x^4 + (2 - 2k)x^3 - x^2 + kx + 1)y + x^6$$

and

$$Q_k(x, y) = xy^2 + (kx - 1)y - x^2 + x.$$

Then, for $k \geq 2$*,*

$$\mathrm{m}(Q_k) = \mathrm{m}(P_k).$$

The proof will be presented in Chapter 5.

The curve defined by $P_k(x, y) = 0$ has genus 3, except for $k = \pm 2$, when it has genus 2 (see Example 2.2.4), while the curve defined by $Q_k(x, y) = 0$ has genus 1 for all k.

More precisely, Liu and Qin conjectured that the common value of $\mathrm{m}(Q_k)$ and $\mathrm{m}(P_k)$ is given by $r_k L'(E_k, 0)$, where

$$E_k : Y^2 = X^3 + (k^2 - 4)X^2 - 8kX + 16.$$

The first few values for $1/r_k$ are given in Table 1.2. (Here we list $1/r_k$ because it has the tendency to be an integer.)

k	2	3	4	5	6	7	8	9	10
$1/r_k$	$-1/2$	-1	-2	-4	6	14	-18	36	52
N_k	37	79	197	469	997	1907	3349	5497	8549

Table 1.2: Numerical values of $1/r_k$ for the conjectural formulas $\mathrm{m}(Q_k) = \mathrm{m}(P_k) = r_k L'(E_k, 0)$ found by Liu and Qin [23]. N_k indicates the conductor of E_k.

For $k = 2$, $E_2 \colon Y^2 = X^3 - 16X + 16$ is the elliptic curve 37a1 in Cremona's classification given by

$$y^2 - y = x^3 - x.$$

To our knowledge, Theorem 1.1.3 is the first result shedding light on the Mahler measure of a genus 3 curve. Our method of proof is similar to the one employed in [20], establishing identities between the regulators, but the regulator of the genus 3 curve is more difficult to evaluate and we employ a few strategies to simplify it before comparing it to the regulator of the genus 1 curve. The major new idea for evaluating the regulator of the genus 3 curve is to use equation (1.11) as opposed to (1.10) to simplify the evaluation of the diamond operator on $(x_1) \diamond (y_1)$ (we will explain how this works in Section 1.2). This simple idea has potential for other cases. Another interesting feature of this example is that the regulators are supported in powers of a point of infinite order in the elliptic curve. The majority of the examples that have been proven so far have the regulators supported in torsion points.

1.2 Approaches to Boyd's Conjectures

Here, we briefly introduce some approaches for proving Boyd's conjectures and some techniques used in these methods.

1. Direct computation of $\mathrm{m}(P_k)$ for a family of polynomial P_k by studying the Mahler measures as functions on the parameter k and differentiating with respect to the parameter k in order to find relations among different values of k or between the Mahler measures of different families. This was the method employed by Bertin and Zudilin in [1] and [2] to prove Theorem 1.1.1 and Theorem 1.1.2.

   Of all the methods, direct computation is often the first approach to be considered. Sometimes the proof by the direct computation may be gratifyingly straightforward and it can be achieved just by applying complex analysis and properties of special functions. Recently, Ringeling and Zudilin [29] used this method to prove another identity conjectured by Lin and Qin [23, Eq (3.22)]. The identity proven by Ringeling and Zudilin looks quite complex and involves Mahler measures of three different families. But the proof is very concise.

2. For some cases where the curve $P(x, y) = 0$ has genus 0 or 1, the term $\frac{f^{3/2}}{4\pi} L(\chi_{-f}, 2)$ can be obtained from the hyperbolic volume of an oriented hyperbolic three-manifold (see Section 2.4 for details). Meanwhile, the Mahler measure may be expressed as $\mathrm{m}(P) = \sum_j r_j D(\alpha_j)$ with $r_j \in \mathbb{Q}$, where $D(\alpha_j)$ is the Bloch-Wigner dilogarithm defined in Section 2.4. In favorable cases, the Bloch-Wigner dilogarithm D can be expressed as a hyperbolic volume (see Section 2.4 for detail). If those volumes are consistent

between $\frac{f^{3/2}}{4\pi}L(\chi_{-f}, 2)$ and $\sum_j r_j D(\alpha_j)$, then we can conclude the relation between $m(P)$ and the L-value. This technique will be studied further in Chapter 3.

3. Boyd's conjectures can also be proven by using the *regulator map* on the second K-group $K_2(C)$ of an algebraic curve C. The theory for this method can be found in Section 2.5. Here, we briefly summarize this method as follows.

   Let $C\colon P(x, y) = 0$ be an algebraic curve over $\mathbb{Q}$, where P is a polynomial over $\mathbb{Q}$ of the form:

$$P(x, y) = P^*(x)\big(y - y_1(x)\big)\big(y - y_2(x)\big)$$

   with algebraic functions $y_i(x)$. Suppose that by a change of variables $x = x(X, Y)$ and $y = y(X, Y)$, we obtain the standard form $Y^2 = h(X^2)$ of a hyperelliptic curve, where $E\colon Y^2 = h(Z)$ is an elliptic curve. On the unit tours $\mathbb{T}^2$, if $[\{x \mid |x| = 1, |y_i(x)| \geq 1\}]$ can be seen as an element in the homology group $H_1(C, \mathbb{Z})$, then for the case where $|y_1(x)| \geq 1$ and $|y_2(x)| \leq 1$ (or $|y_1(x)| \leq 1$ and $|y_2(x)| \geq 1$) on $|x| = 1$, we can construct the *regulator map* (described in Section 2.5 in detail) r_E from the second K-group $K_2(E)$ of E to the cohomology group $H^1(E, \mathbb{R})$ with a relationship of the form:

$$\mathrm{m}(P) - \mathrm{m}(P^*) = -\frac{1}{2\pi} r_E(\{x, y\})[\gamma], \tag{1.8}$$

   where $\{x, y\}$ is the *Milnor symbol* (defined in Section 2.5.1) in $K_2\big(\mathbb{Q}(E)\big)$ of the function field $\mathbb{Q}(E)$ of E over $\mathbb{Q}$ with two rational functions $x(X, Y), y(X, Y) \in \mathbb{Q}(E)^\times$ and $[\gamma]$ is a generator of the homology group $H_1\big(E(\mathbb{C}), \mathbb{Z}\big)$. Meanwhile, one may find a function: $\hat{D}^E\colon E(\mathbb{C}) \to \mathbb{R}$ with a relationship of the form:

$$r_E(\{x, y\})[\gamma] = c\hat{D}^E\big((x) \diamond (y)\big), \tag{1.9}$$

   where c is a constant depending on the cycle $[\gamma]$ in $H_1\big(E(\mathbb{C}), \mathbb{Z}\big)$ and $\diamond$ denotes the *diamond operator* (described in Section 2.5.3 in detail). Formulas (1.8) and (1.9) provide tools for the computations and comparisons in the proof of Boyd's conjectures. For instance, consider two polynomials $P(x_1, y_1)$, $Q(x_2, y_2)$ over $\mathbb{Q}$. From certain changes of variables $(x_1, y_1) \mapsto (X_1, Y_1)$ and $(x_2, y_2) \mapsto (X_2, Y_2)$, we obtain two elliptic curves $E_1\colon Y_1^2 = h_1(Z_1)$ and $E_2\colon Y_2^2 = h_2(Z_2)$ with $Z_1 = X_1^2$ and $Z_2 = X_2^2$. By (1.8) and

(1.9),

$$\mathrm{m}(P) - \mathrm{m}(P^*) = -\frac{c_1}{2\pi}\hat{D}^{E_1}\big((x_1) \diamond (y_1)\big),$$
$$\mathrm{m}(Q) - \mathrm{m}(Q^*) = -\frac{c_2}{2\pi}\hat{D}^{E_2}\big((x_2) \diamond (y_2)\big).$$

If the curves E_1, E_2 are isomorphic and $(x_1) \diamond (y_1) = (x_2) \diamond (y_2)$, we have the following desired result:

$$\mathrm{m}(P) - \mathrm{m}(P^*) = \frac{c_1}{c_2}\big(\,\mathrm{m}(Q) - \mathrm{m}(Q^*)\big).$$

This is the idea of proving Boyd's conjectures by the regulator method. We will discuss how to find (1.8) and (1.9) in Chapter 2.

Lalín and Wu [20, 21] used the regulator approach to prove the identities in Theorem 1.1.1, Theorem 1.1.2 and Theorem 1.1.3. The proofs will be presented in Chapter 4 and Chapter 5.

4. Computations of Boyd's conjectures are always difficult. In many cases, we obtain an elliptic curve $E\colon Y^2 = h(Z)$ with $Z = X^2$ from the curve $P(x, y) = 0$ by the certain change of variables: $(x, y) \mapsto (X, Y)$, that is to say: our curve is a 2-cover of an elliptic curve. In such cases, we may not be able to compute $x \wedge y$ or $(x) \diamond (y)$ directly, where $x \wedge y$ is an element in $\bigwedge^2\big(\mathbb{C}(E)^\times\big)$ and $(x) \diamond (y)$ is the diamond operation of two divisors (x) and (y) with $x, y \in \mathbb{Q}(E)^\times$. In order to make the computations feasible, Bosman in his thesis [4] introduced two rational functions $a(Z, Y)$ and $b(Z, Y)$ such that

$$a(X^2, Y)x(X, Y) + b(X^2, Y)y(X, Y) = 1. \qquad (1.10)$$

Then, we perform the computations (wedge products or diamond operations) in terms of the functions $a(Z, Y)$ and $b(Z, Y)$. Sometimes, we are unable to get the appropriate functions $a(Z, Y)$ and $b(Z, Y)$ from (1.10). Then, we may get them from an equation of the form

$$a(X^2, Y)x(X, Y) + b(X^2, Y)\frac{y(X, Y)}{\big(x(X, Y)\big)^m} = 1 \qquad (1.11)$$

with some integer m. Notice that

$$x(X, Y) \wedge y(X, Y) = x(X, Y) \wedge \frac{y(X, Y)}{\big(x(X, Y)\big)^m}.$$

The functions $a(Z, Y)$ and $b(Z, Y)$ obtained from Eq (1.11) have the same effect as the ones from (1.10). This technique will be used for all our results in Chapter 3, Chapter 4 and Chapter 5.

Chapter 2

Some Background and Basic Results

In this chapter, we will introduce the necessary background that will be needed to attack Boyd's conjectures. We will explain how those techniques work.

2.1 Two Properties of Mahler Measure

Here, we introduce two properties for the Mahler measure that are at the basis of many computations.

Proposition 2.1.1 (Properties of the Mahler measure, [13, p. 51–52]). *The following properties for the logarithmic Mahler measure hold.*

1. *For any rational functions* $P, Q \in \mathbb{C}(x_1, \cdots, x_n)^\times$,

$$\mathrm{m}(PQ) = \mathrm{m}(P) + \mathrm{m}(Q).$$

2. *Let*

$$P(\mathbf{x}) = \sum c_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \in \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$$

be a Laurent polynomial, where $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_n^{m_n}$. *Let* A *be an* $n \times n$ *integral matrix with* $\det A \neq 0$. *Define* $P^{(A)}(\mathbf{x}) = \sum c_{\mathbf{m}} \mathbf{x}^{\mathbf{m}A}$. *Then,*

$$\mathrm{m}(P) = \mathrm{m}(P^{(A)}).$$

Proof. Property 1 follows directly from the definition of Mahler measure and the property of the logarithm: $\log |PQ| = \log |P| + \log |Q|$. We will prove Property 2.

Consider $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$. Notice that

$$A = \begin{bmatrix} a_{11} & \dots & a_{1j}/d_j & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj}/d_j & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & d_j & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix},$$

where $d_j = \gcd(a_{1j}, \dots, a_{nj})$. By LU-factorization and the Euclidean algorithm, the matrix A is a product of $n \times n$ integer matrices L, U and D with nonzero determinant, where L is a lower triangular matrix with 1's on the diagonal, U is a upper triangular matrix with 1's on the diagonal and D is a diagonal matrix. From composing the changes of variables, it suffices to prove that

$$\mathrm{m}(P) = \mathrm{m}(P^{(U)}), \quad \mathrm{m}(P) = \mathrm{m}(P^{(L)}), \quad \mathrm{m}(P) = \mathrm{m}(P^{(D)}).$$

Write $\mathbf{m} = (m_1, \dots, m_n)$ and $U = \begin{bmatrix} 1 & a_{12} & \dots & a_{1n} \\ 0 & 1 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \in \mathrm{GL}(n, \mathbb{Z})$.

Then,

$$\begin{aligned} \mathbf{m}U &= (m_1, \dots, m_n) \begin{bmatrix} 1 & a_{12} & \dots & a_{1n} \\ 0 & 1 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \\ &= (m_1, a_{12}m_1 + m_2, \dots, a_{1n}m_1 + \dots + a_{n-1,n}m_{n-1} + m_n), \end{aligned}$$

Then,

$$\begin{aligned} \mathrm{m}(P^{(U)}) &= \int_0^1 \cdots \int_0^1 \log |P^{(U)}(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n})| \, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \\ &= \int_0^1 \cdots \int_0^1 \log |\sum c_{\mathbf{m}} e^{(\theta_1 + \theta_2 a_{12} + \dots + \theta_n a_{1n})2\pi i m_1} \cdots e^{\theta_n 2\pi i m_n}| \, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n. \end{aligned}$$

Let $\varphi_1 = \theta_1 + \theta_2 a_{12} + \dots + \theta_n a_{1n}$, $\varphi_2 = \theta_2 + \theta_3 a_{23} + \dots + \theta_n a_{2n}$, ..., $\varphi_n = \theta_n$. Then, $\theta_n = \varphi_n$, $\theta_{n-1} = \varphi_{n-1} - \varphi_n a_{n-1,n}$, ..., $\theta_1 = \varphi_1 - (\varphi_2 - \dots - \varphi_n a_{2n})a_{12} - \dots - \varphi_n a_{1n}$. The Jacobian is then

$$\frac{\partial(\theta_1, \dots, \theta_n)}{\partial(\varphi_1, \dots, \varphi_n)} = 1.$$

We obtain that

$$\begin{aligned}
&\mathrm{m}(P^{(U)})\\
&= \int_0^1 \cdots \int_0^1 \log \left| \sum c_{\mathbf{m}} e^{(\theta_1+\theta_2 a_{12}+\cdots+\theta_n a_{1n})2\pi i m_1} \cdots e^{\theta_n 2\pi i m_n} \right| \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \\
&= \int_0^1 \cdots \int_{f_1(\varphi_2,\ldots,\varphi_n)}^{1+f_1(\varphi_2,\ldots,\varphi_n)} \log \left| \sum c_{\mathbf{m}} e^{2\pi i m_1 \varphi_1} \cdots e^{2\pi i m_n \varphi_n} \right| \frac{\partial(\theta_1,\ldots,\theta_n)}{\partial(\varphi_1,\ldots,\varphi_n)} \mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_n \\
&= \int_0^1 \cdots \int_{f_1(\varphi_2,\ldots,\varphi_n)}^{1+f_1(\varphi_2,\ldots,\varphi_n)} \log |P(e^{2\pi i \varphi_1},\ldots,e^{2\pi i \varphi_n})| \, \mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_n,
\end{aligned}$$

where

$$\begin{aligned}
&f_1(\varphi_2,\ldots,\varphi_n) = \theta_2 a_{12} + \ldots + \theta_n a_{1n} = (\varphi_2 - \ldots - \varphi_n a_{2n}) a_{12} + \ldots + \varphi_n a_{1n},\\
&\qquad \ldots\\
&f_{n-2}(\varphi_{n-1},\varphi_n) = \theta_{n-1} a_{n-2,n-1} + \theta_n a_{n-2,n}\\
&\qquad\qquad\qquad\quad = (\varphi_{n-1} - \varphi_n a_{n-1,n}) a_{n-2,n-1} + \varphi_n a_{n-2,n},\\
&f_{n-1}(\varphi_n) = \theta_n a_{n-2,n-1} = \varphi_n a_{n-2,n-1},\\
&f_{n-1} = 0.
\end{aligned}$$

Notice that for each φ_j, the function $P(\ldots,\varphi_j,\ldots) = P(e^{2\pi i\varphi_1},\ldots,e^{2\pi i\varphi_n})$ has period 1. Hence,

$$\int_{f_j(\varphi_{j+1},\ldots,\varphi_n)}^{1+f_j(\varphi_{j+1},\ldots,\varphi_n)} \log |P(e^{2\pi i\varphi_1},\ldots,e^{2\pi i\varphi_n})| \, \mathrm{d}\varphi_j = \int_0^1 \log |P(e^{2\pi i\varphi_1},\ldots,e^{2\pi i\varphi_n})| \, \mathrm{d}\varphi_j,$$

This gives the desired result:

$$\mathrm{m}(P^{(U)}) = \mathrm{m}(P).$$

Similarly, one can show that

$$\mathrm{m}(P^{(L)}) = \mathrm{m}(P).$$

Now, write $D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$ with integers $d_j \neq 0$. Then,

$$\mathbf{m}D = (m_1,\ldots,m_n) \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} = (m_1 d_1,\ldots,m_n d_n)\,.$$

and

$$\begin{aligned}
\mathrm{m}(P^{(D)}) &= \int_0^1 \cdots \int_0^1 \log \left| P\left(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}\right) \right| \,\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \\
&= \int_0^1 \cdots \int_0^1 \log |\sum c_{\mathbf{m}} e^{2\pi i m_1 d_1 \theta_1} \cdots e^{2\pi i m_n d_n \theta_n}| \,\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n.
\end{aligned}$$

Let $\eta_1 = d_1\theta_1, \ldots, \eta_n = d_n\theta_n$. Then,

$$\begin{aligned}
\mathrm{m}(P^{(D)}) &= \int_0^{d_n} \cdots \int_0^{d_1} \log |\sum c_{\mathbf{m}} e^{2\pi i m_1 \eta_1} \cdots e^{2\pi i m_n \eta_n}| \frac{\mathrm{d}\eta_1}{d_1} \cdots \frac{\mathrm{d}\eta_n}{d_n} \\
&= \int_0^{d_n} \cdots \int_0^{d_1} \log \left| P\left(e^{2\pi i\eta_1}, \ldots, e^{2\pi i\eta_n}\right) \right| \frac{\mathrm{d}\eta_1}{d_1} \cdots \frac{\mathrm{d}\eta_n}{d_n}.
\end{aligned}$$

Notice that $d_j \in \mathbb{Z}$ with $d_j \neq 0$ and for each j, the function $P(\ldots, \eta_j, \ldots) = P\left(e^{2\pi i\eta_1}, \ldots, e^{2\pi i\eta_n}\right)$ has period 1. Thus,

$$\begin{aligned}
\int_0^{d_j} \log \left| P\left(e^{2\pi i\eta_1}, \ldots, e^{2\pi i\eta_n}\right) \right| \frac{\mathrm{d}\eta_j}{d_j} &= d_j \int_0^1 \log \left| P\left(e^{2\pi i\eta_1}, \ldots, e^{2\pi i\eta_n}\right) \right| \frac{\mathrm{d}\eta_j}{d_j} \\
&= \int_0^1 \log \left| P\left(e^{2\pi i\eta_1}, \ldots, e^{2\pi i\eta_n}\right) \right| \mathrm{d}\eta_j.
\end{aligned}$$

This gives the desired result:

$$\mathrm{m}(P^{(D)}) = \mathrm{m}(P). \qquad \square$$

2.2 The Curves Appearing in Boyd's Conjectures

An *algebraic plane curve* over $\mathbb{C}$ is defined by

$$C = \{(x, y) \in \mathbb{C}^2 \mid P(x, y) = 0\},$$

where $P(x, y) \in \mathbb{C}[x, y]$ is a nonconstant polynomial with no repeated factors. A complex curve $C\colon P(x, y) = 0$ is often called an *affine* plane curve (for distinguishing them from *projective* curves defined below). An affine plane curve in $\mathbb{C}^2$ is never compact (see [18, p. 46]).

Note that we can identify $\mathbb{C}^2$ with the open subset $\{[x, y, z] \in \mathbb{P}^2 \mid z \neq 0\}$ of the projective plane $\mathbb{P}^2 = \{[x, y, z] \mid (x, y, z) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}\}$, where $[x, y, z]$ in $\mathbb{P}^2$ is the equivalence class of 3-tuples (x, y, z) in $\mathbb{C}^3 \setminus \{(0, 0, 0)\}$ under the equivalence relation $\sim$ defined by $(x, y, z) \sim (\lambda x, \lambda y, \lambda z)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Then, a plane curve C can be defined on the projective plane $\mathbb{P}^2$ by

$$C = \{[x, y, z] \in \mathbb{P}^2 \mid P(x, y, z) = 0\},$$

where $P \in \mathbb{C}[x, y, z]$ is a homogeneous polynomial. Every projective curve $C\colon P(x, y, z) = 0$ in $\mathbb{P}^2$ is compact and Hausdorff (see [18, Lemma 2.30, p. 42]). A projective curve $C\colon P(x, y, z) = 0$ in $\mathbb{P}^2$ may be obtained from an affine plane curve $P(x, y) = 0$ by homogenization of $P(x, y)$ to $P(x, y, z)$.

A smooth algebraic projective curve in $\mathbb{P}^2$ can be seen as a compact Riemann surface. A compact, connected, orientable surface without boundary is homeomorphic to a sphere with g handles. The number $g \geq 0$ is called the *genus* of the surface. For a nonsingular projective plane curve $C : P(x, y, z) = 0$ in $\mathbb{P}^2$, the genus of C can be obtained by (see [18, p. 85])

$$g = \frac{1}{2}(d-1)(d-2), \tag{2.1}$$

where d is the degree of the polynomial P. Inspired by the definition of genus of a surface, one can obtain an invariant g in the classification of algebraic curves (with singularities or without singularities). This invariant g is called the *genus of a curve*. For the rigorous definition of genus of a curve, we refer the reader to standard algebraic geometry or Riemann surface textbooks (e.g. [14, 17, 24]).

A curve of $g = 1$ with a rational point is an *elliptic curve*. It can be defined by the Weierstrass form (2.8).

A curve C of $g > 1$ with a double covering of $\mathbb{P}^1$, i.e., such that there exists a finite morphism $\pi\colon C \to \mathbb{P}^1$ of degree 2, is called *hyperelliptic*. A hyperelliptic curve can be written in the canonical form (see [24, Proposition 4.24, p. 294]) as

$$y^2 = f(x), \tag{2.2}$$

where $f \in \mathbb{C}[x]$ of degree d with d distinct roots. For the hyperelliptic curve (2.2), the genus is given by (see [24, Proposition 4.24, p. 294] or [4, Lemma 15, p. 35])

$$\begin{aligned} g &= \frac{d-1}{2}, \quad \text{if } d \text{ is odd}, \\ g &= \frac{d-2}{2}, \quad \text{if } d \text{ is even}. \end{aligned} \tag{2.3}$$

In [5], Boyd considered some specific algebraic plane curves $C : P(x, y) = 0$, where $P \in \mathbb{C}[x, y]$ of the form:

$$P(x, y) = A(x)y^2 + B(x)y + C(x) \tag{2.4}$$

with $A, B, C \in \mathbb{C}[x]$. In this section, we discuss some fundamentals of plane curves and give some precisions about the curves appearing in Boyd's conjectures.

Example 2.2.1 (cf. [4, Lemma 17, p. 43])**.** Consider the family

$$S_k(x, y) = y^2 + (x^4 + kx^3 + 2kx^2 + kx + 1)y + x^4.$$

In Section 4.1, the curve $S_k(x, y) = 0$ will be written in the form (2.2) by

$$Y_1^2 = h_1(X_1^2),$$

where

$$h_1(Z_1) = (k^2 + k)Z_1^3 + (-2k^2 + 5k + 4)Z_1^2 + (k^2 - 5k + 8)Z_1 - k + 4.$$

The discriminant of $h_1(X_1^2)$ is $2^{22}k^3(k - 8)^2(k - 4)(k + 1)$. By (2.3), the curve $Y_1^2 = h_1(X_1^2)$ has genus

$$g = \frac{6 - 2}{2} = 2$$

for $k \in \mathbb{C} \setminus \{-1, 0, 4, 8\}$. When $k = -1$, $h_1(X_1^2) = -3X_1^4 + 14X_1^2 + 5$. So, $Y_1^2 = h_1(X_1^2)$ is a curve of genus 1 for $k = -1$. When $k = 0$, $h_1(X_1^2) = 4(X_1^2+1)^2$. So, $Y_1^2 = h_1(X_1^2)$ has two irreducible components of genus 0 for $k = 0$. When $k = 4$, $h_1(X_1^2) = 4X_1^2(5X_1^4 - 2X_1^2 + 1)$. The genus can be obtained from the curve $Y_1^2 = 5X_1^4 - 2X_1^2 + 1$, which is a curve of genus 1. When $k = 8$, $h_1(X_1^2) = 4(3X_1^2 - 1)^2(2X_1^2 - 1)$. The genus can be obtained from the curve $Y_1^2 = 2X_1^2 - 1$, which is a curve of genus 0.

Example 2.2.2. Consider the family

$$Q_k(x, y) = (x^2 + x + 1)y^2 + kx(x + 1)y + x(x^2 + x + 1).$$

In Section 4.2, the curve $Q_k(x, y) = 0$ will be written in the form (2.2) by

$$Y_2^2 = h_2(X_2^2),$$

where

$$h_2(Z_2) = (k^2 - 9)Z_2^3 - (2k^2 - 3)Z_2^2 + (k^2 + 5)Z_2 + 1.$$

The discriminant of $h_2(X_2^2)$ is $2^{22}k^4(k - 3)(k + 3)(k^2 + 16)^2$. By (2.3), the curve $Y_2^2 = h_2(X_2^2)$ has genus

$$g = \frac{6 - 2}{2} = 2$$

for $k \in \mathbb{C}\setminus\{\pm 3, 0, \pm 4i\}$. When $k = \pm 3$, $h_2(X_2^2) = -15X_2^4 + 14X_2^2 + 1 = -(15X_2^2+1)(X_2^2-1)$. So, $Y_2^2 = h_2(X_2^2)$ is a curve of genus 1 for $k = \pm 3$. When $k = 0$, $h_2(X_2^2) = -(X_2-1)(X_2+1)(3X_2^2+1)^2$. So, $Y_2^2 = h_1(X_2^2)$ is a curve of genus 0 for $k = 0$.

Example 2.2.3. Consider the family

$$R_k(x,y) = (x^2+x+1)y^2 + (x^4+kx^3+(2k-4)x^2+kx+1)y \\ + x^2(x^2+x+1).$$

In Section 4.2, the curve $R_k(x,y) = 0$ can be written in the form (2.2) by

$$Y_3^2 = h_3(X_3^2),$$

where

$$h_3(Z_3) = (k^2-k-2)Z_3^3 + (-2k^2+11k-2)Z_3^2 \\ + (k^2-11k+26)Z_3 + k - 6.$$

The discriminant of $h_3(X_3^2)$ is $2^{22}(k-6)(k-5)^4(k-2)(k+1)(k^2-4k+20)$. By (2.3), the curve $Y_3^2 = h_3(X_3^2)$ has genus

$$g = \frac{6-2}{2} = 2$$

for $k \in \mathbb{C} \setminus \{-1, 2, 5, 6, 2 \pm 4i\}$. When $k = -1$, $h_3(X_3^2) = -15X_3^4 + 38X_3^2 - 7 = -(3X_3^2-7)(5X_3^2-1)$. When $k = 2$, $h_3(X_3^2) = 4(3X_3^4 + 2X_3^2 - 1) = 4(X_3^2+1)(3X_3^2-1)$. So, $Y_3^2 = h_3(X_3^2)$ is a curve of genus 1 for $k \in \{-1, 2\}$. When $k = 5$, $h_3(X_3^2) = (2X_3^2-1)(3X_3^2+1)^2$. The genus can be obtained from the curve $Y_3^2 = 2X_3^2 - 1$, which is a curve of genus 0. When $k = 6$, $h_3(X_3^2) = 4X_3^2(7X_3^4 - 2X_3^2 - 1)$. The genus can be obtained from the curve $Y_3^2 = 7X_3^4 - 2X_3^2 - 1$, which is a curve of genus 1.

Example 2.2.4. Consider the family

$$P_k(x,y) = y^2 + (x^6 + kx^5 - x^4 + (2-2k)x^3 - x^2 + kx + 1)y + x^6.$$

In Section 5.1, the curve $P_k(x,y) = 0$ can be written in the form (2.2) by

$$Y^2 = h(X^2),$$

where

$$h(u) = (k+2)u^4 + 4(k^2+3k+3)u^3 - 2(4k^2-3k-16)u^2 \\ + 4(k^2-5k+5)u + k - 2.$$

The discriminant of $h(X^2)$ is $2^{48}(k-2)(k+2)(k^4-k^3-8k^2+36k-11)^2$. By (2.3), the curve $Y^2 = h(X^2)$ has genus

$$g = \frac{8-2}{2} = 3$$

when $k \neq \pm 2$. For $k = 2$,

$$h(u) = 4u^4 + 52u^3 + 12u^2 - 4u = 4u(u^3 + 13u^2 + 3u - 1).$$

The genus of the curve $Y^2 = h(X^2)$ is equal to that of $Y^2 = X^6 + 13X^4 + 3X^2 - 1$. So, the curve has genus

$$g = \frac{6-2}{2} = 2$$

when $k = 2$. For $k = -2$,

$$h(u) = 4u^3 - 12u^2 + 76u - 4 = 4(u^3 - 3u^2 + 19u - 1).$$

So, the curve has genus

$$g = \frac{6-2}{2} = 2$$

when $k = -2$.

Furthermore, the polynomials (2.4) in Boyd's conjectures are reciprocal and tempered. We now explain what these conditions mean.

We say that the polynomial $P(x_1, \dots, x_n)$ is *reciprocal* if $P(x_1, \dots, x_n)$ equals $\pm P(1/x_1, \dots, 1/x_n)$ multiplied by a monomial.

Let

$$P(x,y) = \sum_{(m,n)\in J} a(m,n) x^m y^n \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}],$$

where J is the finite subset of $\mathbb{Z}^2$ with $a(m,n) \neq 0$ for all $(m,n) \in J$. Define the *Newton polygon* of P, denoted by $\mathcal{C}(P)$, to be the convex hull of J. For each side τ of $\mathcal{C}(P)$, let $\tau(j)$ be the lattice points (enumerated clockwise by $j = 0, 1, 2, \dots$) on the side τ. Let

$$P_\tau(t) = \sum_j a\big(\tau(j)\big) t^j \in \mathbb{C}[t].$$

The polynomial P_τ is said to be a *face polynomial* of P. A polynomial $P(x,y)$ is *tempered* if for every side τ of the Newton polygon $\mathcal{C}(P)$, the zeros of the face polynomials $P_\tau(t)$ are roots of unity.

Example 2.2.5. Consider the polynomial $Q_k(x,y)$ in Example 2.2.2:

$$\begin{aligned} Q_k(x,y) &= (x^2 + x + 1)y^2 + kx(x+1)y + x(x^2 + x + 1) \\ &= x^2y^2 + xy^2 + y^2 + kx^2y + kxy + x^3 + x^2 + x. \end{aligned}$$

The Newton polygon is the convex hull of J represented by

$$\begin{array}{cccc} 1 & 1 & 1 & \\ & k & k & \\ & 1 & 1 & 1 \end{array}.$$

The face polynomials of Q_k are given by

$$\begin{aligned} P_1(t) &= 1 + t + t^2, \\ P_2(t) &= 1 + t, \\ P_3(t) &= 1 + t + t^2, \\ P_4(t) &= 1 + t. \end{aligned}$$

We thus conclude that the polynomial $Q_k(x, y)$ is tempered.

Observing Boyd's polynomial (2.4), we have that

$$\begin{aligned} P_k(x, y) &= A(x)y^2 + B_k(x)y + C(x) \\ &= \frac{\big(2A(x)y + B_k(x)\big)^2 - \big(B_k^2(x) - 4A(x)C(x)\big)}{4A(x)}. \end{aligned}$$

The curve $P_k(x, y) = 0$ is given by

$$\hat{y}^2 = B_k^2(x) - 4A(x)C(x),$$

where

$$\hat{y} = 2A(x)y + B_k(x).$$

Let

$$D_k(x) = B_k^2(x) - 4A(x)C(x).$$

In order to apply the approach used in Chapters 3, 4 and 5, we need write the curves $P_k(x, y) = 0$ in the form (2.2). This may be realized by the change of variables

$$\begin{aligned} X &= \frac{x+1}{x-1}, \quad Y = \frac{2A(x)y + B_k(x)}{\gamma(x)(x-1)^m}, \\ x &= \frac{X+1}{X-1}, \quad y = \frac{\gamma\left(\frac{X+1}{X-1}\right)\left(\frac{2}{X-1}\right)^m Y - B_k\left(\frac{X+1}{X-1}\right)}{2A\left(\frac{X+1}{X-1}\right)} \end{aligned} \tag{2.5}$$

with some appropriate polynomial $\gamma(x)$ and integer m. Then, we obtain the curves:

$$Y^2 = \left(\frac{2A(x)y + B_k(x)}{\gamma(x)(x-1)^m}\right)^2 = \frac{B_k^2(x) - 4A(x)C(x)}{\gamma^2(x)(x-1)^{2m}} = \frac{(X-1)^{2m} D_k\left(\frac{X+1}{X-1}\right)}{2^{2m}\gamma^2\left(\frac{X+1}{X-1}\right)}.$$

From some specific curve $P_k(x,y)=0$, if

$$(X-1)^{2m}\,D_k\left(\frac{X+1}{X-1}\right)/\gamma^2\left(\frac{X+1}{X-1}\right)=\hat{h}(X^2)$$

with $\hat{h}\in\mathbb{C}[X]$, then we obtain a hyperelliptic curve in the form:

$$Y^2=h(X^2). \tag{2.6}$$

Note that for our cases in Chapter 4 and Chapter 5, D_k is a reciprocal polynomial such that

$$D_k\left(\frac{1}{x}\right)=\frac{D_k(x)}{x^d}$$

with even degree d. Write $D_k(x)=\sum_{j=0}^{d}c_jx^j$. Then $c_d=c_0, c_{d-1}=c_1,\ldots$. Let $2m=d$. Then,

$$\begin{aligned}(X-1)^{2m}D_k\left(\frac{X+1}{X-1}\right)&=(X-1)^d\left(c_d\left(\frac{X+1}{X-1}\right)^d+\cdots+c_0\left(\frac{X+1}{X-1}\right)\right)\\&=(X-1)^d\left(c_\ell\sum_{\ell=0}^{\frac{d}{2}-1}\left(\left(\frac{X+1}{X-1}\right)^{d-\ell}+\left(\frac{X+1}{X-1}\right)^{\ell}\right)+c_{\frac{d}{2}}\left(\frac{X+1}{X-1}\right)^{\frac{d}{2}}\right).\end{aligned}$$

Observe that for $\ell<d/2$,

$$\begin{aligned}&(X-1)^d\left(\left(\frac{X+1}{X-1}\right)^{d-\ell}+\left(\frac{X+1}{X-1}\right)^{\ell}\right)\\&=(X-1)^d\left(\frac{X+1}{X-1}\right)^{\ell}\left(\left(\frac{X+1}{X-1}\right)^{d-2\ell}+1\right)\\&=(X^2-1)^{\ell}\left((X+1)^{d-2\ell}+(X-1)^{d-2\ell}\right)=h_\ell(X^2)\end{aligned}$$

since d is even, and in addtion,

$$(X-1)^d\left(\frac{X+1}{X-1}\right)^{d/2}=((X+1)(X-1))^{\frac{d}{2}}=(X^2-1)^{\frac{d}{2}}.$$

We thus conclude that

$$(X-1)^{2m}D_k\left(\frac{X+1}{X-1}\right)=\hat{h}(X^2)$$

for some polynomial $\hat{h}$. We then obtain a hyperelliptic curve (2.6) by setting $\gamma(x)=1$.

Here we only deal with the families of the curves from Chapters 3, 4 and 5. All these curves can be rewritten in the form (2.6) by the change of variables (2.5). From (2.5), we see that the transformation $x \to \frac{1}{x}$, $y \to \frac{1}{y}$ may lead to $X \to -X$, $Y \to \pm Y$. In order to successfully prove the conjectures with our methods, we should have that

$$x \to \frac{1}{x} \text{ and } y \to \frac{1}{y} \quad \Longleftrightarrow \quad X \to -X \text{ and } Y \to Y.$$

2.3 A Review of L-Functions

Boyd's conjectures are some numerical identities that show a relationship between the Mahler measure of a given polynomial and the L-function of the corresponding elliptic curve. For this reason, we review some definitions and concepts of L-functions. The content of this section can be found in standard textbooks on elliptic curves (e.g. [19]).

2.3.1 Dirichlet Series and Euler Products

A series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ with $a_n, s \in \mathbb{C}$ is called a *Dirichlet series.*

Proposition 2.3.1 (cf. [19, Proposition 7.2, p. 192])**.** *Let* $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *be a Dirichlet series.*

1. *If the series is convergent for* $s = s_0$*, then it is convergent uniformly on compact sets for* $\operatorname{Re} s > \operatorname{Re} s_0$*, and the sum of the series is analytic in this region.*

2. *If the series is absolutely convergent for* $s = s_0$*, then it is uniformly and absolutely convergent for* $\operatorname{Re} s \geq \operatorname{Re} s_0$*.*

3. *If the series is convergent for* $s = s_0$*, then it is absolutely convergent for* $\operatorname{Re} s > \operatorname{Re} s_0 + 1$*.*

4. *If the series is convergent at some* s_0 *and sums to* 0 *in a right half plane, then all the coefficients are* 0*.*

Consider a formal product

$$\prod_{p \text{ prime}} (1 + a_p p^{-s} + \cdots + a_{p^m} p^{-ms} + \cdots). \tag{2.7}$$

If this product is expanded without regard to convergence, the result is the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where

$$a_1 = 1, \quad a_n = a_{p_1^{r_1}} \cdots a_{p_k^{r_k}} \text{ if } n = p_1^{r_1} \cdots p_k^{r_k}.$$

If an absolutely convergent Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ has the property that all its coefficients are *multiplicative*, i.e.,

$$a_1 = 1, \quad a_{mn} = a_m a_n \text{ whenever } \gcd(m, n) = 1,$$

then the Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ can be expressed as (2.7). In this case, the product (2.7) is called the *Euler product* of the Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$. In particular, if the coefficients are *strictly multiplicative*, i.e.,

$$a_1 = 1, \quad a_{mn} = a_m a_n \text{ for all } m \text{ and } n,$$

then we have $a_{p^m} = (a_p)^m$ for all m. Consequently, the Dirichlet series has a *first degree Euler product*:

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p \text{ prime}} (1 + a_p p^{-s} + \cdots + a_{p^m} p^{-ms} + \cdots) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{a_p}{p^s}}.$$

In Section 2.3.3, we consider the Euler product for the L-function of an elliptic curve. The expansion (2.7) is called a k^{th} *degree Euler product* if for each prime p, there is a polynomial $P_p(X) \in \mathbb{C}[X]$ of the degree $\leq k$ and zero constant term such that

$$1 + a_q X^{-s} + \cdots + a_{q^m} X^{-ms} + \cdots = \frac{1}{1 - P_p(X)}.$$

As $\mathbb{C}$ is a closed field, we may factor $1 - P_p(X)$ over $\mathbb{C}$ as

$$1 - P_p(X) = (1 - r_p^{(1)} X) \cdots (1 - r_p^{(k)} X).$$

The complex numbers $r_p^{(j)}$ are said to be the *reciprocal roots* of $1 - P_p(X)$.

The properties of Euler products can be found in [19, VII.2, p. 192–199].

2.3.2 Dirichlet L-Series

Definition 2.3.2 (Dirichlet character)**.** A *Dirichlet character modulo* d is a function $\chi\colon \mathbb{Z} \to \mathbb{C}$ that has the following properties.

1. For all $n \in \mathbb{Z}$, $\chi(n) = \chi(n + d)$.
2. For all $n \in \mathbb{Z}$, $\chi(n) = 0$ if and only if $\gcd(n, d) > 1$.

3. For all $m, n \in \mathbb{Z}$, $\chi(nm) = \chi(n)\chi(m)$.

We write χ to be χ_{-d} if the modulo d should be emphasized. The *principal* Dirichlet character modulo d is defined by

$$\chi^0(n) = \chi^0_{-d}(n) = \begin{cases} 1 & \text{if } \gcd(n,d) = 1, \\ 0 & \text{if } \gcd(n,d) > 1. \end{cases}$$

One can easily verify that χ^0 is a Dirichlet character modulo d.

From Definition 2.3.2, we can deduce the following properties for the Dirichlet character.

Proposition 2.3.3. *Let $\chi\colon \mathbb{Z} \to \mathbb{C}$ be a Dirichlet character modulo d. Then:*

1. $\chi(1) = 1$.
2. *If $m \equiv n \mod d$, then $\chi(m) = \chi(n)$.*
3. *Let ζ_n denote a primitive n^{th} root of unity. Let k be a positive integer such that* $\gcd(k,d) = 1$. *Then,*

$$\chi(k) = \zeta_{\phi(d)}^{t}$$

for some integer t which depends on k, ζ and χ, where $\phi\colon \mathbb{Z}_+ \to \mathbb{Z}_+ \cup \{0\}$ is the Euler's ϕ-function. It follows that $\chi\colon \mathbb{Z} \to S^1 \cup \{0\}$ given by

$$\chi(n) = \begin{cases} z \in S^1, & \text{if } \gcd(n,d) = 1, \\ 0, & \text{if } \gcd(n,d) > 1. \end{cases}$$

This implies that the number of Dirichlet characters for a given modulo d is finite.

Proof. Part 1 and part 2 are the direct corollaries of Definition 2.3.2. Part 3 follows from

$$\chi(k)^{\phi(d)} = \chi(k^{\phi(d)}) = \chi(1) = 1,$$

by Euler's theorem: $a^{\phi(b)} \equiv 1 \mod b$ if $\gcd(a,b) = 1$. □

Notice that the multiplicative group of integers modulo d is given by

$$(\mathbb{Z}/d\mathbb{Z})^\times = \{\bar{k} \in \mathbb{Z}/d\mathbb{Z} \mid \gcd(k,d) = 1\}.$$

Thus, the Dirichlet character χ_{-d} can be viewed as a group homomorphism

$$\chi\colon\ (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^\times, \qquad k \mapsto \zeta_{\phi(d)}^{t_k}.$$

So, the Dirichlet character modulo d is a character of $(\mathbb{Z}/d\mathbb{Z})^\times$ which has order $\phi(d)$.

Definition 2.3.4 (Conductor of associate characters. cf. [19, p. 213])**.** Two nonprincipal Dirichlet characters χ_{-d} and $\chi'_{-d'}$ are said to be *associate* if $\chi_{-d}(p) = \chi'_{-d'}(p)$ for all but finitely many primes. One can easily verify that the associate relation is an equivalence relation. The *conductor* of an equivalence class is the least d'' such that $\chi''_{-d''}$ is contained in this class.

Definition 2.3.5 (Primitive Dirichlet character. cf. [19, p. 213])**.** A Dirichlet character modulo d is *primitive* if its conductor is d.

Definition 2.3.6 (L-Series)**.** A *Dirichlet L-series* is a series on $\mathbb{C}$ given by

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where $s \in \mathbb{C}$ such that $\operatorname{Re} s > 1$ and χ is a Dirichlet character as in Definition 2.3.2.

In particular, the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

can be thought of the Dirichlet L-series corresponding to the principal character χ_0 modulo 1.

Remark 2.3.7. In Definition 2.3.6, since $\operatorname{Re} s > 1$ and $\chi(n) \in S^1 \cup \{0\}$, we see that the Dirichlet L-series is absolutely convergent. This implies that for a fixed Dirichlet character χ, the Dirichlet L-series $L(\chi, s)$ is an analytic function on the open subset $U = \{s \in \mathbb{C} \mid \operatorname{Re} s > 1\}$ of $\mathbb{C}$.

Proposition 2.3.8 (cf. [19, Proposition 7.10, p. 202])**.** *Fix an integer $d > 0$. Let χ be a Dirichlet character modulo d.*

1. *In the region $\operatorname{Re} s > 1$, the Dirichlet L-series $L(\chi, s)$ is given by a first degree Euler product:*

$$L(\chi, s) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{\chi(p)}{p^s}}.$$

2. *If χ is not principal, then the series $L(\chi, s)$ is convergent for the region $\operatorname{Re} s > 0$, and the sum is analytic in that region.*

3. *Let χ^0 be the principal Dirichlet character modulo d. In the region $\operatorname{Re} s > 0$, $L(\chi^0, s)$ has a unique pole at $s = 1$ and the pole is simple. Furthermore, $L(\chi^0, s)$ is given in terms of the Riemann zeta function $\zeta(s)$ by*

$$L(\chi^0, s) = \zeta(s) \prod_{\substack{p \text{ prime} \\ p|d}} \left(1 - \frac{1}{p^s}\right).$$

Remark 2.3.9. By analytic continuation, the Dirichlet L-series $L(\chi, s)$ can be extended to a meromorphic function on the complex plane $\mathbb{C}$. This extended function is called a *Dirichlet L-function* and also denoted by $L(\chi, s)$.

2.3.3 L-Function of an Elliptic Curve

A smooth projective plane curve of genus $g = 1$ with a specified point O is called an *elliptic curve*. The specified point O in the elliptic curve is called the base point, which represents the unique point at infinity. Every elliptic curve E defined over the field $\mathbb{C}$ can be written by the following Weierstrass equation (see [19, Theorem 10.3 (Néron), p. 292]):

$$E\colon y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3, \tag{2.8}$$

where $a_j \in \mathbb{C}$. The base point in the Weierstrass form (2.8) is given by $O = [0, 1, 0]$.

Now, consider an elliptic curve E over $\mathbb{Q}$ given by (2.8) with discriminant Δ, where, with some admissible change of variables, we can assume all coefficients $a_i \in \mathbb{Z}$ (see [19, Proposition 10.2, p. 292]). Let E_p be the reduction of E modulo p with a given prime p. The reduction E_p is given by a Weierstrass form written by (2.8) with the coefficients $a_i \in \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. The discriminant Δ_p of E_p is given by

$$\Delta_p \equiv \Delta \mod p.$$

This implies that E_p is nonsingular if and only if $p \nmid \Delta$. Denote

$$\hat{a}_p = p + 1 - \#E_p(\mathbb{F}_p),$$

where $E_p(\mathbb{F}_p)$ is the set of projective solutions of E_p.

Definition 2.3.10 (L-Function of an elliptic curve)**.** The L-function of E is the product of the local L factors, given by

$$L(E,s) = \prod_{p|\Delta} \left(\frac{1}{1-\hat{a}_p p^{-s}} \right) \prod_{p\nmid\Delta} \left(\frac{1}{1-\hat{a}_p p^{-s} + p^{1-2s}} \right).$$

By the Euler product expansion described in Section 2.3.1, we have that

$$L(E,s) = \sum_{n=1}^{\infty} \frac{\hat{a}_n}{n^s},$$

where

$$\begin{aligned}
&\hat{a}_1 = 1, \\
&\hat{a}_n = \hat{a}_p, \quad \text{when } n = p, \text{ for all prime } p, \\
&\hat{a}_{p^m} = \hat{a}_p \hat{a}_{p^{m-1}}, \quad \text{when } p \mid \Delta, \text{ for a prime } p, \\
&\hat{a}_{p^m} = \hat{a}_p \hat{a}_{p^{m-1}} - p\hat{a}_{p^{m-2}}, \quad \text{when } p \nmid \Delta, \text{ for a prime } p, \\
&\hat{a}_{mn} = \hat{a}_m \hat{a}_n, \quad \text{when } \gcd(m,n) = 1.
\end{aligned}$$

Theorem 2.3.11 (Hasse's Theorem, cf. [19, Theorem 10.5, p. 296])**.** *Let E be an elliptic curve defined over $\mathbb{Q}$ with integer coefficients. For each $p \nmid \Delta$, let E_p be the reduction of the curve E modulo p. Then, $|\hat{a}_p| = |p+1-\#E_p(\mathbb{F}_p)| < 2\sqrt{p}$.*

By Hasse's Theorem, the L-function $L(E,s) = \sum_{n=1}^{\infty} \frac{\hat{a}_n}{n^s}$ converges absolutely when $\mathrm{Re}(s) > 3/2$ (cf. [19, Corollary 10.6, p. 296]).

During the process of proving Fermat's Last Conjecture, Wiles and Taylor proved a modularity theorem for all semistable elliptic curves over $\mathbb{Q}$. Then, some mathematicians worked to extend and generalize Wiles' ideas. In 2001, Breuil, Conrad, Diamond, and Taylor completed the proof of the modularity theorem for all elliptic curves over $\mathbb{Q}$ (see the history in [31, p. 443]). Here, we present the following theorem as a corollary of the modularity theorem.

Theorem 2.3.12. *Let E be an elliptic curve over $\mathbb{Q}$. The L-series $L(E,s)$ has an analytic continuation to the whole complex plane.*

Remark 2.3.13 (A brief description of the modularity theorem)**.** Recall that a *cuspform f for the congruence subgroup $\Gamma_0(N)$ of weight* 2 is a modular form of weight 2 whose Fourier expansion is given by (see [11, p. 4, p 6 and p. 13] for the definitions of modular form, cuspform and $\Gamma_0(N)$):

$$f(\tau) = \sum_{n=1}^{\infty} a_n(f) q^{n/N}, \qquad a_n(f) \in \mathbb{C}, \quad q = e^{2\pi i \tau}.$$

There is a family of Hecke operators $\{T_\ell\}$ acting on the $\mathbb{C}$-vector space of cuspforms for $\Gamma_0(N)$ of weight 2. For a cuspform f, if f is an eigenvector for all T_ℓ, then f is called an *eigenform*. In addtion, f is *normalized* if $a_1(f) = 1$ (see [11, p. 196]). The L-function of a normalized eigenform f for $\Gamma_0(N)$ is defined by (see [11, p. 201])

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s}.$$

The modularity theorem consists of the following two statements.

Theorem 2.3.13-1: Let f be a normalized eigenform for $\Gamma_0(N)$ of weight 2. The L-function $L(f,s)$ has an analytic continuation to the whole complex plane $\mathbb{C}$.

Theorem 2.3.13-2: Let E be an elliptic curve over $\mathbb{Q}$ of conductor N_E. There is a normalized eigenform f for $\Gamma_0(N_E)$ of weight 2 such that

$$L(E,s) = L(f,s).$$

Theorem 2.3.12 follows.

2.4 The Dilogarithm

2.4.1 Bloch-Wigner Dilogarithm

In many cases, Mahler measure can be expressed as a finite sum

$$\pi \, \mathrm{m}(P) = \sum_i r_i D(z_i),$$

where $r_i \in \mathbb{Q}$, $z_i \in \mathbb{C}$ with $|z_i| = 1$, and $D(z)$ is the Bloch-Wigner dilogarithm of z (see Definition 2.4.2 given below). For example (see [6, p. 7]),

$$2\pi \, \mathrm{m}(x+y+1) = D(e^{\pi i/3}) - D(e^{-\pi i/3}) = 2\pi L'(\chi_{-3}, -1).$$

This gives us motivation to study the Bloch-Wigner dilogarithm in order to prove Boyd's conjectures. Here we introduce the definition, and some basic notions and properties of the Bloch-Wigner dilogarithm.

Recall the Taylor series for the logarithm:

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \text{for } |z| < 1.$$

Definition 2.4.1 (Polylogarithm)**.** We define the *polylogarithm* as

$$\mathrm{Li}_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}, \qquad \text{for } |z| < 1, \quad m = 1, 2, \ldots$$

The polylogarithm $\mathrm{Li}_m(z)$ is continuous and analytic on $|z| < 1$ for $m \geq 1$. For $m \geq 2$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}z} \mathrm{Li}_m(z) = \frac{1}{z} \mathrm{Li}_{m-1}(z) \quad \text{for } |z| < 1.$$

Then, the *dilogarithm* $\mathrm{Li}_2(z)$ can be given by (see [35, p. 5])

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-\tau)}{\tau} \,\mathrm{d}\tau. \tag{2.9}$$

This gives an analytic continuation to $\mathbb{C} \setminus [1, \infty)$. Recall that

$$\log(1-\tau) = \log|1-\tau| + i \arg(1-\tau).$$

For $\tau \in (1, \infty)$, $\arg(1-\tau) = -\pi$. So, when the integral path in (2.9) crosses $(1, \infty)$, the dilogarithm $\mathrm{Li}_2(z)$ will jump by $2\pi i \log|\tau_0|$, where τ_0 is the crossing point of the integral path passing the interval $(1, \infty)$. This implies that the function $\log|z| \arg(1-z) + \mathrm{Im}\,\mathrm{Li}_2(z)$ is continuous on $\mathbb{C}$. This leads to the following definition.

Definition 2.4.2 (Bloch-Wigner Dilogarithm)**.** For $z \in \mathbb{C} \cup \{\infty\}$, we define

$$D(z) = \log|z| \arg(1-z) + \mathrm{Im}\,\mathrm{Li}_2(z), \quad \text{for } z \in \mathbb{C},$$
$$D(\infty) = 0,$$

the *Bloch-Wigner dilogarithm* of z, where the dilogarithm $\mathrm{Li}_2(z)$ is given by (2.9).

Remark 2.4.3 (cf. [3, Corollary 6.1.2, p. 44])**.** The function $D\colon \mathbb{C} \cup \{\infty\} \to \mathbb{R}$ is well-defined.

Proposition 2.4.4 (Properties of the Bloch-Wigner Dilogarithm. [35, p. 10])**.** *Let $D(z)$ be the the Bloch-Wigner Dilogarithm of z. Then, we have some properties as follows.*

1. *The function D(z) is continuous on $\mathbb{C} \cup \{\infty\}$ and real analytic on $\mathbb{C} \setminus \{0, 1\}$.*
2. *For $z, w \in \mathbb{C} \cup \{\infty\}$, we have the five-term relation:*

$$D(z) + D(w) + D(1-zw) + D\left(\frac{1-z}{1-zw}\right) + D\left(\frac{1-w}{1-zw}\right) = 0.$$

3. *One has the following 6-fold symmetry:*

$$D(z) = D\left(1 - \frac{1}{z}\right) = D\left(\frac{1}{1-z}\right) = -D\left(\frac{1}{z}\right)$$
$$= -D(1-z) = -D\left(\frac{-z}{1-z}\right).$$

4.
$$D(\bar{z}) = -D(z), \quad \textit{in particular,} \quad D(z) = 0, \ \textit{for } z \in \mathbb{R},$$

5. *For $z \in \mathbb{C}$, we have that*

$$D(z) = \frac{1}{2}\left(D\left(\frac{z}{\bar{z}}\right) + D\left(\frac{1-1/z}{1-1/\bar{z}}\right) + D\left(\frac{1/(1-z)}{1/(1-\bar{z})}\right)\right).$$

Notice that $\left|\frac{z}{\bar{z}}\right| = \left|\frac{1-1/z}{1-1/\bar{z}}\right| = \left|\frac{1/(1-z)}{1/(1-\bar{z})}\right| = 1$. So, $D\left(\frac{z}{\bar{z}}\right) = D(e^{i\alpha})$, $D\left(\frac{1-1/z}{1-1/\bar{z}}\right) = D(e^{i\beta})$ and $D\left(\frac{1/(1-z)}{1/(1-\bar{z})}\right) = D(e^{i\gamma})$ for some $\alpha, \beta, \gamma \in \mathbb{R}$. Hence, the function $D\colon \mathbb{C} \to \mathbb{R}$ can be expressed as a sum of functions of a single real variable.

6.
$$D(z^n) = n \sum_{k=0}^{n-1} D(ze^{2\pi k i/n}).$$

Here we give an example which will be used in Chapter 3.

Example 2.4.5. We have that

$$2D\left(\frac{3+i\sqrt{15}}{3-i\sqrt{3}}\right) + 2D\left(\frac{3+i\sqrt{15}}{3+i\sqrt{3}}\right) = \frac{1}{3}D\left(-\omega^3\right) - 3D\left(-\omega\right),$$

where

$$\omega = \frac{-1+i\sqrt{15}}{4}.$$

Proof. By Proposition 2.4.4-5, we obtain that

$$2D\left(\frac{3+i\sqrt{15}}{3-i\sqrt{3}}\right) = D\left(\frac{(3+i\sqrt{15})(3+i\sqrt{3})}{(3-i\sqrt{15})(3-i\sqrt{3})}\right)$$
$$+ D\left(-\frac{3-i\sqrt{15}}{3+i\sqrt{15}}\right) + D\left(-\frac{3-i\sqrt{3}}{3+i\sqrt{3}}\right)$$

and

$$2D\left(\frac{3+i\sqrt{15}}{3+i\sqrt{3}}\right)=D\left(\frac{(3+i\sqrt{15})(3-i\sqrt{3})}{(3-i\sqrt{15})(3+i\sqrt{3})}\right)$$
$$+D\left(-\frac{3-i\sqrt{15}}{3+i\sqrt{15}}\right)+D\left(-\frac{3+i\sqrt{3}}{3-i\sqrt{3}}\right).$$

By Proposition 2.4.4-4,

$$D\left(-\frac{3-i\sqrt{3}}{3+i\sqrt{3}}\right)+D\left(-\frac{3+i\sqrt{3}}{3-i\sqrt{3}}\right)=0.$$

In addition,

$$\left|\frac{3+i\sqrt{15}}{3-i\sqrt{15}}\right|=\left|-\frac{3-i\sqrt{15}}{3+i\sqrt{15}}\right|=\left|\frac{3+i\sqrt{3}}{3-i\sqrt{3}}\right|=\left|\frac{3-i\sqrt{3}}{3+i\sqrt{3}}\right|=1.$$

We have that

$$\frac{3+i\sqrt{15}}{3-i\sqrt{15}}=\frac{-1+i\sqrt{15}}{4}=\omega,\quad -\frac{3-i\sqrt{15}}{3+i\sqrt{15}}=\frac{1+i\sqrt{15}}{4}=-\omega,$$
$$\frac{3+i\sqrt{3}}{3-i\sqrt{3}}=\frac{1+i\sqrt{3}}{2}=e^{i\frac{\pi}{3}},\quad \frac{3-i\sqrt{3}}{3+i\sqrt{3}}=\frac{1-i\sqrt{3}}{2}=e^{-i\frac{\pi}{3}}.$$

Then,

$$D\left(\frac{(3+i\sqrt{15})(3+i\sqrt{3})}{(3-i\sqrt{15})(3-i\sqrt{3})}\right)+D\left(\frac{(3+i\sqrt{15})(3-i\sqrt{3})}{(3-i\sqrt{15})(3+i\sqrt{3})}\right)$$
$$=D(\omega e^{i\frac{\pi}{3}})+D(\omega e^{-i\frac{\pi}{3}})$$
$$=\operatorname{Im}\operatorname{Li}_2\left(\omega\frac{1+i\sqrt{3}}{2}\right)+\operatorname{Im}\operatorname{Li}_2\left(\omega\frac{1-i\sqrt{3}}{2}\right).$$

We conclude that

$$2D\left(\frac{3+i\sqrt{15}}{3-i\sqrt{3}}\right)+2D\left(\frac{3+i\sqrt{15}}{3+i\sqrt{3}}\right)$$
$$=D\left(\frac{3+i\sqrt{15}}{3-i\sqrt{15}}e^{i\frac{\pi}{3}}\right)+D\left(\frac{3+i\sqrt{15}}{3-i\sqrt{15}}e^{-i\frac{\pi}{3}}\right)+2D\left(-\frac{3-i\sqrt{15}}{3+i\sqrt{15}}\right)$$
$$=D\left(\omega e^{i\frac{\pi}{3}}\right)+D\left(\omega e^{-i\frac{\pi}{3}}\right)+2D\left(-\bar{\omega}\right).$$

By Proposition 2.4.4-6, we have that

$$\begin{aligned}
\frac{1}{6}D(\omega^6) &= \sum_{k=0}^{5} D(\omega e^{2\pi k i/6}) \\
&= D(\omega) + D\left(\omega e^{i\frac{\pi}{3}}\right) + D\left(\omega e^{i\frac{2\pi}{3}}\right) + D\left(\omega e^{i\pi}\right) + D\left(\omega e^{i\frac{4\pi}{3}}\right) + D\left(\omega e^{i\frac{5\pi}{3}}\right) \\
&= D(\omega) + D\left(\omega e^{i\frac{\pi}{3}}\right) + D\left(\omega e^{-i\frac{\pi}{3}}\right) + D(-\omega) + D\left(\omega e^{i\frac{2\pi}{3}}\right) + D\left(\omega e^{i\frac{4\pi}{3}}\right),
\end{aligned}$$

and

$$\frac{1}{3}D(\omega^3) = \sum_{k=0}^{2} D(\omega e^{2\pi k i/3}) = D(\omega) + D\left(\omega e^{i\frac{2\pi}{3}}\right) + D\left(\omega e^{i\frac{4\pi}{3}}\right).$$

Thus,

$$\begin{aligned}
&2D\left(\frac{3+i\sqrt{15}}{3-i\sqrt{3}}\right) + 2D\left(\frac{3+i\sqrt{15}}{3+i\sqrt{3}}\right) \\
&= D\left(\omega e^{i\frac{\pi}{3}}\right) + D\left(\omega e^{-i\frac{\pi}{3}}\right) - 2D(-\omega) \\
&= \frac{1}{6}D(\omega^6) - D(-\omega) - D(\omega) - D\left(\omega e^{i\frac{2\pi}{3}}\right) - D\left(\omega e^{i\frac{4\pi}{3}}\right) - 2D(-\omega) \\
&= \frac{1}{6}D(\omega^6) - \frac{1}{3}D(\omega^3) - 3D(-\omega).
\end{aligned}$$

Again, by Proposition 2.4.4-6, we have that

$$\frac{1}{2}D(\omega^6) = \sum_{k=0}^{1} D(\omega^3 e^{2\pi k i/2}) = D\left(\omega^3\right) + D\left(-\omega^3\right).$$

Thus,

$$\begin{aligned}
&2D\left(\frac{3+i\sqrt{15}}{3-i\sqrt{3}}\right) + 2D\left(\frac{3+i\sqrt{15}}{3+i\sqrt{3}}\right) \\
&= \frac{1}{6}D(\omega^6) - \frac{1}{3}D(\omega^3) - 3D(-\omega) = \frac{1}{3}D\left(-\omega^3\right) - 3D(-\omega). \quad \square
\end{aligned}$$

2.4.2 The Elliptic Dilogarithm

Let $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ be a lattice in the complex plane $\mathbb{C}$. Let $\mathbb{C}/\Lambda$ be the quotient group given by

$$\mathbb{C}/\Lambda = \{z + \Lambda \mid z \in \mathbb{C}\}.$$

The group $\mathbb{C}/\Lambda$ is also a Riemann surface with the quotient topology induced by the projection $\mathbb{C} \to \mathbb{C}/\Lambda$ given by $z \mapsto z+\Lambda$ (see [14, p. 4]). Denote the group of K-rational points on an elliptic curve E over a number field K by $E(K)$. Then, we have an isomorphism of Riemann surfaces that is also a group isomorphism given by (see [31, Proposition 3.6, p. 170])

$$\mathbb{C}/\Lambda \to E(\mathbb{C}), \qquad z \mapsto [\wp(z), \wp'(z), 1],$$

where $\wp$ is the Weierstrass function relative to the lattice Λ, defined by the series

$$\wp(z) = \wp(z; \Lambda) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

and $\wp'$ is its derivative, given by

$$\wp'(z) = -2 \sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^3}.$$

Let $q = e^{2\pi i\tau}$, where $\tau \in \mathbb{H} = \{x + iy \mid x, y \in \mathbb{R} \text{ with } y > 0\}$. Let $\mathbb{C}^\times/q^{\mathbb{Z}}$ be the quotient group given by

$$\mathbb{C}^\times/q^{\mathbb{Z}} = \{zq^{\mathbb{Z}} \mid z \in \mathbb{C}^\times\}.$$

Now, let $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ with $\tau \in \mathbb{H}$. Define $\varphi\colon \mathbb{C}/\Lambda \to \mathbb{C}^\times/q^{\mathbb{Z}}$ by

$$\varphi(u + \Lambda) = e^{2\pi i u} q^{\mathbb{Z}}.$$

Notice that $e^u e^v = e^{u+v}$ for all $u, v \in \mathbb{C}$. So, φ is a group homomorphism. Since the function $z = e^u$ has the inverse image: $u = \log|z| + i \operatorname{Arg} z = \operatorname{Log} z$, we see that φ is a bijection. Thus, $\varphi\colon \mathbb{C}/\Lambda \to \mathbb{C}^\times/q^{\mathbb{Z}}$ is a group isomorphism. Then, we conclude that we have the following isomorphisms:

$$\begin{array}{ccccc} E(\mathbb{C}) & \stackrel{\cong}{\to} & \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) & \stackrel{\cong}{\to} & \mathbb{C}^\times/q^{\mathbb{Z}} \\ P = \big(\wp(u), \wp'(u)\big) & \mapsto & u \mod \Lambda & \mapsto & z = e^{2\pi i u}. \end{array} \tag{2.10}$$

From (2.10), every point on $\mathbb{C}$ of an elliptic curve E corresponds to an element in $\mathbb{C}^\times/q^{\mathbb{Z}}$. In this context, Bloch [3] gave following definition.

Definition 2.4.6 (Elliptic Dilogarithm. [3, p. 61])**.** The elliptic dilogarithm is a function D^E on an elliptic curve $E(\mathbb{C})$ defined by

$$D^E(P) = \sum_{n\in\mathbb{Z}} D(q^n z) = \sum_{n\in\mathbb{Z}} D\big(e^{2\pi i(u+n\tau)}\big), \tag{2.11}$$

where $q = e^{2\pi i\tau}$, τ, u and $z = e^{2\pi i u}$ are given in (2.10).

Remark 2.4.7 (cf. [3, Lemma 8.1.1, p. 61])**.** The expression in Definition 2.4.6 defines a continuous function $D^E\colon E \to \mathbb{R}$.

In Section 2.5.3, we will see the relationship (2.15) between the regulator and the elliptic dilogarithm D^E for an elliptic curve E. Our proof strategies in Chapters 4 and 5 are based on the relationship (2.15).

2.4.3 The Bloch Group

Let F be a field. Let $\bigwedge^2(F^\times)$ be the set of all formal linear combinations $z \wedge w$ with $z, w \in F^\times$ subject to the conditions

$$\begin{aligned} z \wedge z &= 0, \\ (z_1 z_2) \wedge w &= z_1 \wedge w + z_2 \wedge w, \quad z \wedge (w_1 w_2) = z \wedge w_1 + z \wedge w_2. \end{aligned}$$

Let $z, w \in F^\times$. Since $1 \wedge z = (1 \cdot 1) \wedge z = 1 \wedge z + 1 \wedge z$, we have that

$$1 \wedge z = 0.$$

Since

$$\begin{aligned} 0 &= (zw) \wedge (zw) = z \wedge (zw) + w \wedge (zw) \\ &= z \wedge z + z \wedge w + w \wedge z + w \wedge w = z \wedge w + w \wedge z, \end{aligned}$$

we have that

$$z \wedge w = -w \wedge z.$$

Since

$$0 = 1 \wedge w = \frac{z}{z} \wedge w = z \wedge w + \frac{1}{z} \wedge w,$$

we have that

$$\frac{1}{z} \wedge w = -z \wedge w.$$

Let $\mathcal{A}(F)$ be the additive abelian group of formal sums

$$n_1[z_1] + \cdots + n_m[z_m]$$

with $n_j \in \mathbb{Z}$ and $z_j \in F \setminus \{0, 1\}$ such that

$$\sum_{j=1}^{m} n_j z_j \wedge (1 - z_j) = 0.$$

Proposition 2.4.8. *Let F be an algebraic number field. For all $z, w \in F \setminus \{0,1\}$ with $zw \neq 1$, the elements*

$$[z] + \left[\frac{1}{z}\right], \quad [z] + [1-z], \quad [z] + [w] + [1-zw] + \left[\frac{1-z}{1-zw}\right] + \left[\frac{1-w}{1-zw}\right] \tag{2.12}$$

are in $\mathcal{A}(F)$.

Proof. We have that

$$\begin{aligned}
& z \wedge (1-z) + \left(\frac{1}{z} \wedge \left(1 - \frac{1}{z}\right)\right) \\
& = z \wedge (1-z) + \left(\frac{1}{z} \wedge \left(\frac{z-1}{z}\right)\right) \\
& = z \wedge (1-z) + 1 \wedge (z-1) - z \wedge (z-1) - 1 \wedge z + z \wedge z \\
& = 0.
\end{aligned}$$

We thus conclude that

$$[z] + \left[\frac{1}{z}\right] \in \mathcal{A}(F).$$

We also have that

$$z \wedge (1-z) + (1-z) \wedge \big(1 - (1-z)\big) = 0.$$

We thus conclude that

$$[z] + [1-z] \in \mathcal{A}(F).$$

Finally,

$$\begin{aligned}
& z \wedge (1-z) + w \wedge (1-w) + (1-zw) \wedge \big(1 - (1-zw)\big) \\
& \quad + \left(\frac{1-z}{1-zw}\right) \wedge \left(1 - \left(\frac{1-z}{1-zw}\right)\right) + \left(\frac{1-w}{1-zw}\right) \wedge \left(1 - \left(\frac{1-w}{1-zw}\right)\right) \\
& = z \wedge (1-z) + w \wedge (1-w) + (1-zw) \wedge \big(zw\big) \\
& \quad + \left(\frac{1-z}{1-zw}\right) \wedge \left(\frac{z(1-w)}{1-zw}\right) + \left(\frac{1-w}{1-zw}\right) \wedge \left(\frac{w(1-z)}{1-zw}\right) \\
& = z \wedge (1-z) + w \wedge (1-w) + (1-zw) \wedge z + (1-zw) \wedge w \\
& \quad + (1-z) \wedge z + (1-z) \wedge (1-w) - (1-z) \wedge (1-zw) - (1-zw) \wedge z \\
& \quad - (1-zw) \wedge (1-w) + (1-w) \wedge w + (1-w) \wedge (1-z) \\
& \quad - (1-w) \wedge (1-zw) - (1-zw) \wedge w - (1-zw) \wedge (1-z) \\
& = 0.
\end{aligned}$$

We thus conclude that

$$[z]+[w]+[1-zw]+\left[\frac{1-z}{1-zw}\right]+\left[\frac{1-w}{1-zw}\right]\in\mathcal{A}(F). \qquad \square$$

Let $\mathcal{C}(F)$ be the subgroup generated by the elements in (2.12). The *Bloch group* is defined by

$$\mathcal{B}(F)=\mathcal{A}(F)/\mathcal{C}(F).$$

From Proposition 2.4.4, the Bloch-Wigner function D can be extended to a linear map

$$\begin{aligned} D\colon \mathcal{B}(\mathbb{C}) &\to \mathbb{R}, \\ n_1[z_1]+\cdots+n_m[z_m] &\mapsto n_1D(z_1)+\cdots+n_mD(z_m). \end{aligned}$$

2.4.4 Volumes of Hyperbolic 3-manifolds

As we mentioned before, the Mahler measure can be expressed as a finite sum of special values of the Bloch-Wigner dilogarithm. In this section, we will see that in some cases the Bloch-Wigner dilogarithm can be expressed as the volume of a tetrahedron in a hyperbolic space. Meanwhile, in some special cases, the L-function corresponding to the curve is also related with the volume of a hyperbolic manifold. This gives a way to prove Boyd's conjectures by showing relations between certain hyperbolic volumes.

Let $\mathbb{H}^3=\mathbb{C}\times\mathbb{R}_+$ be the upper half-space model of the hyperbolic 3-space with the standard hyperbolic metric:

$$\mathrm{d}s=\frac{\sqrt{\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}t^2}}{t}, \qquad x+iy\in\mathbb{C}, \quad t\in\mathbb{R}_+.$$

With this metric, $\mathbb{H}^3$ is a complete metric space. For the upper half-space model, the volume element is given by

$$\mathrm{d}V=\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t}{t^3}.$$

The boundary of $\mathbb{H}^3$ is given by $\partial\mathbb{H}^3=\mathbb{C}\cup\{\infty\}$, where $\mathbb{C}\cup\{\infty\}$ is the Riemann sphere with the topology of one point compactification of $\mathbb{C}\cong\mathbb{R}^2$.

A *geodesic segment* connecting two points in the hyperbolic space is the shortest path between these two points obtained by the hyperbolic metric. A subset Y of a hyperbolic space is *convex* if the geodesic segment between each two points in Y is contained in Y. A *tetrahedron*

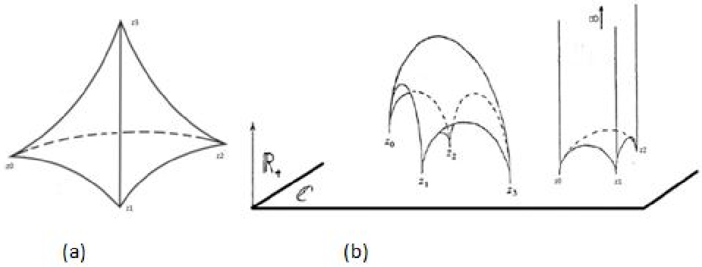

Figure 2.1: Hyperbolic tetrahedron. This figure can be found in [35] p.14.

T in $\overline{\mathbb{H}^3}$ is a convex polyhedron whose boundary consists of 4 vertices, 6 edges and 4 faces. An edge of T is the geodesic segment between two vertices and a face of T is the convex hyperbolic plane bounded by three edges. Figure 2.1 (a) illustrates a tetrahedron in $\overline{\mathbb{H}^3}$. An *ideal tetrahedron* is a tetrahedron T in $\overline{\mathbb{H}^3}$ such that all vertices of T are in $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$. Figure 2.1 (b) illustrates two ideal tetrahedra, one has four vertices in $\mathbb{C}$ and another has three vertices in $\mathbb{C}$ and another vertex is ∞.

Remark 2.4.9. For the rigid definitions of convex polyhedron and its sides, faces, edges, vertices, we refer the reader to standard hyperbolic geometry textbooks (e.g. [28]).

Theorem 2.4.10 (Lobachevsky Theorem [35, p. 13–14])**.** *The hyperbolic volume of an ideal tetrahedron T can be found by*

$$\mathrm{Vol}(T) = \tilde{D}(z_0, z_1, z_2, z_3),$$

where

$$\tilde{D}(z_0, z_1, z_2, z_3) = D\left(\frac{(z_0 - z_2)(z_1 - z_3)}{(z_0 - z_3)(z_1 - z_2)}\right)$$

is the Bloch-Wigner Dilogarithm of the cross-ratio of 4 elements $z_0, z_1, z_2, z_3 \in \mathbb{C} \cup \{\infty\}$*. Now, let T_z be the ideal tetrahedron with vertices* $0, 1, \infty, z$*. Then,*

$$\mathrm{Vol}(T_z) = \tilde{D}(0, 1, \infty, z) = D\left(\frac{1-z}{-z}\right) = D(z),$$

by Proposition 2.4.4-3.

Theorem 2.4.11 (Triangulation for 3-manifold. [35, p. 14–15])**.** *A complete oriented hyperbolic 3-manifold M with finite volume can be triangulated by ideal tetrahedra $n_1 T_{z_1}, \ldots, n_m T_{z_m}$ for some positive integers n_j such that*

$$\sum_{j=1}^{m} n_j z_j \wedge (1 - z_j) = 0,$$

where the vertices of T_{z_j} *are* $0, 1, \infty, z_j$. *For such triangulation,*

$$\sum_{j=1}^{m} n_j[z_j] \in \mathcal{B}(\mathbb{C}).$$

Then,

$$\mathrm{Vol}(M) = \sum_{j=1}^{m} n_j \mathrm{Vol}(T_{z_j}) = \sum_{j=1}^{m} n_j D(z_j).$$

If the Mahler measure $\mathrm{m}(P)$ for $P \in \mathbb{C}[x, y]$ is given by

$$\pi\,\mathrm{m}(P) = \sum_{j=1}^{m} r_j D(z_j) = \frac{1}{r}\sum_{j=1}^{m} n_j D(z_j)$$

for some $r_j, r \in \mathbb{Q}$ and $n_j \in \mathbb{Z}$ such that

$$\sum_{j=1}^{m} n_j[z_j] \in \mathcal{B}(\mathbb{C}),$$

then by Theorem 2.4.11,

$$\frac{1}{r}\sum_{j=1}^{m} n_j D(z_j) = \frac{1}{r}\mathrm{Vol}(M)$$

for some hyperbolic 3-manifold M. Consequently,

$$\pi\,\mathrm{m}(P) = \frac{1}{r}\mathrm{Vol}(M).$$

Let Γ be a group acting on a metric space X. The *orbit space* of the action is the set of Γ-orbits given by

$$X/\Gamma = \{\Gamma x \mid x \in X\}$$

with the quotient topology. Define a distance function $d_\Gamma\colon X/\Gamma \times X/\Gamma \to \mathbb{R}$ by

$$d_\Gamma(\Gamma x, \Gamma y) = \mathrm{dist}(\Gamma x, \Gamma y) = \inf\{\mathrm{dist}(u, v) \mid u \in \Gamma x, v \in \Gamma y\}.$$

Theorem 2.4.12 (cf. [28, Theorem 6.6.1, p. 232])**.** *Let* Γ *be a group of isometries of a metric space* X. *Then,* d_Γ *is a metric on* X/Γ *if and only if each* Γ*-orbit is a closed subset of* X.

Notice that every discrete group Γ of isometries of $X = \mathbb{E}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$ admits a *proper fundamental region* R, and all the proper fundamental regions for Γ have same volume (see [28, Theorem 6.7.2, p. 244]). This leads to the following definition.

Definition 2.4.13 (cf. [28, p. 245])**.** Let Γ be a discrete group of isometries of $X = \mathbb{E}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$. The *volume of* X/Γ is the volume of a proper fundamental region for Γ in X.

Remark 2.4.14. For the definition of *fundamental region* for a group Γ of isometries of a metric space X, we refer the reader to standard textbooks (e.g. [28, p. 233]). A fundamental region R for a discrete group Γ of isometries of $X = \mathbb{E}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$ is said to be *proper* if $\mathrm{Vol}(\partial R) = 0$ in X.

Let $\mathcal{O}_d$ be the ring of integers in the imaginary quadratic field $F = \mathbb{Q}(i\sqrt{d})$, where d is a square-free positive integer. Now, let $\Gamma = \mathrm{PSL}(2, \mathcal{O}_d)$. Then, the group Γ acts discretely on the hyperbolic space $\mathbb{H}^3$. This implies that each Γ-orbit is a closed subset of $\mathbb{H}^3$ (see [28, Corollary 1, p. 232]). Moreover, $\mathrm{PSL}(2, \mathbb{C})$ is the group of orientation preserving isometries of $\mathbb{H}^3$. It follows that $\mathbb{H}^3/\Gamma$ is a metric space and then an oriented hyperbolic 3-manifold (see [28, Theorem 8.1.3, p. 337–338]). For Boyd's conjectures, we are particularly interested in how to calculate the volume of $\mathbb{H}^3/\Gamma$ since some Mahler measures are related to $\mathrm{Vol}(\mathbb{H}^3/\Gamma)$. Here is an example which will be used in Chapter 3.

Example 2.4.15 (cf. [6, p. 14])**.** Let

$$Q_3(x, y) = (x^2 + x + 1)y^2 + 3x(x + 1)y + x(x^2 + x + 1).$$

Boyd's conjecture

$$\mathrm{m}\left(Q_3\right) \stackrel{?}{=} \frac{15^{3/2}}{24\pi} L(\chi_{-15}, 2)$$

for the Dirichlet character χ_{-15} of conductor 15 can be written as

$$\mathrm{m}\left(Q_3\right) \stackrel{?}{=} \frac{1}{\pi} \mathrm{Vol}\big(\mathbb{H}^3/\,\mathrm{PSL}(2, \mathcal{O}_{15})\big).$$

In [6], Boyd and Rodriguez-Villegas showed that for $c = \frac{1+i\sqrt{15}}{2}$,

$$\begin{aligned}
\pi \operatorname{m}\big(Q_3\big) &= \frac{1}{3}\left(3D\left(\frac{c+3}{4}\right) + 2D(c-1) + 2D(c+1) + 4D\left(\frac{c}{2}\right)\right) \\
&= \frac{1}{3}\Bigg(3D\left(\frac{7+i\sqrt{15}}{8}\right) + 2D\left(\frac{-1+i\sqrt{15}}{2}\right) \\
&\qquad + 2D\left(\frac{3+i\sqrt{15}}{2}\right) + 4D\left(\frac{1+i\sqrt{15}}{4}\right)\Bigg) \\
&= \frac{1}{3}\Bigg(3D\left(\frac{1+i\sqrt{15}}{2}\right) + 2D\left(\frac{-1+i\sqrt{15}}{2}\right) \\
&\qquad + 2D\left(\frac{3+i\sqrt{15}}{2}\right) + 4D\left(\frac{3+i\sqrt{15}}{6}\right)\Bigg) \\
&\qquad\qquad \left(\text{since } D(z) = D\left(\frac{1}{1-z}\right)\right) \\
&= \frac{1}{3}\left(3D(c) + 2D(c-1) + 2D(c+1) + 4D\left(\frac{c+1}{3}\right)\right).
\end{aligned}$$

Then, Boyd and Rodriguez-Villegas found that

$$\delta = 3[c] + 4[(c+1)/3] + 2[c+1] + 2[c-1] \in \mathcal{B}\big(\mathbb{Q}(i\sqrt{15})\big),$$

and that the hyperbolic manifold $\mathbb{H}^3/\operatorname{PSL}(2,\mathcal{O}_{15})$ can be triangulated into ideal tetrahedra with shape parameter 2δ. It follows that

$$\begin{aligned}
\pi \operatorname{m}\big(Q_3\big) &= \frac{1}{3}\left(3D(c) + 2D(c-1) + 2D(c+1) + 4D\left(\frac{c+1}{3}\right)\right) \\
&= \operatorname{Vol}\big(\mathbb{H}^3/\operatorname{PSL}(2,\mathcal{O}_{15})\big).
\end{aligned}$$

One has the following theorem.

Theorem 2.4.16 (Humbert Formula, cf. [33, Theorem 7.4.1] and [36, Section I.1])**.** *Let $\mathcal{O}_d$ be the ring of integers in the field $\mathbb{Q}(i\sqrt{d})$, where d is a square-free positive integer. Let $\Gamma = \operatorname{PSL}(2,\mathcal{O}_d)$. Then,*

$$\operatorname{Vol}(\mathbb{H}^3/\Gamma) = \frac{\hat{d}^{3/2}}{24}\zeta_{\mathbb{Q}(i\sqrt{d})}(2)/\zeta(2) = \frac{\hat{d}^{3/2}}{4\pi^2}\zeta_{\mathbb{Q}(i\sqrt{d})}(2),$$

where ζ is the Riemann zeta function, $\zeta_{\mathbb{Q}(i\sqrt{d})}$ is the Dedekind zeta function of the number field $\mathbb{Q}(i\sqrt{d})$ and

$$\hat{d} = \begin{cases} d & \text{if } d \equiv 3 (\operatorname{mod} 4), \\ 4d & \text{otherwise.} \end{cases}$$

This volume can be found by (see [33, p. 169])

$$\mathrm{Vol}(\mathbb{H}^3/\Gamma) = \frac{\hat{d}}{12} \sum_{k \bmod \hat{d}} \left(\frac{-\hat{d}}{k}\right) \Pi\left(\pi k/\hat{d}\right),$$

where $\Pi(\theta)$ *is the* Lobachevsky function *defined by*

$$\Pi(\theta) = -\int_0^\theta \log|2\sin t|\,\mathrm{d}t,$$

and $\left(\frac{-\hat{d}}{n}\right)$ *is the* Jacobi quadratic symbol *determined by the following properties.*

1. *If* $n = p_1 \cdots p_\ell$ *with primes* p_i*, then* $\left(\frac{-\hat{d}}{n}\right) = \left(\frac{-\hat{d}}{p_1}\right) \cdots \left(\frac{-\hat{d}}{p_\ell}\right)$.
2. $\left(\frac{-\hat{d}}{1}\right) = +1$ *and if* $p \mid \hat{d}$*, then* $\left(\frac{-\hat{d}}{p}\right) = 0$.
3. *If* p *is an odd prime, then*
$$\left(\frac{-\hat{d}}{p}\right) = \begin{cases} +1 & \text{if } -\hat{d} \equiv m^2 \bmod p \text{ for some } m, \\ -1 & \text{otherwise.} \end{cases}$$
4. $$\left(\frac{-\hat{d}}{2}\right) = \begin{cases} +1 & \text{if } -\hat{d} \equiv 1 \bmod 8, \\ -1 & \text{if } -\hat{d} \equiv 5 \bmod 8. \end{cases}$$

The Lobachevsky function has the Fourier series expansion:

$$\Pi(\theta) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n^2}.$$

For the Bloch-Wigner Dilogarithm $D(z)$, we also have the following expansion:

$$D(e^{i\theta}) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}, \quad \theta \in \mathbb{R}.$$

Thus, for $\theta \in \mathbb{R}$, we have

$$\Pi(\theta) = \frac{1}{2} D(e^{i2\theta}).$$

Then, Humbert's formula is given by

$$\mathrm{Vol}(\mathbb{H}^3/\Gamma) = \frac{\hat{d}}{24} \sum_{k \bmod \hat{d}} \left(\frac{-\hat{d}}{k}\right) D\left(e^{2\pi k i/\hat{d}}\right).$$

The following formula gives a relationship between the Dirichlet series and the Dedekind zeta function of the algebraic number field $\mathbb{Q}(i\sqrt{d})$.

Proposition 2.4.17 (Hecke's Formula, cf. [33, p. 168])**.** *We have that*

$$\zeta_{\mathbb{Q}(i\sqrt{d})}(s) = \zeta(s)L(\chi, s),$$

where ζ is the Riemann zeta function, $\zeta_{\mathbb{Q}(i\sqrt{d})}$ is the Dedekind zeta function of the number field $\mathbb{Q}(i\sqrt{d})$ and $L(\chi, s)$ is the Dirichlet series defined by

$$L(\chi, s) = \sum_{n=1}^{\infty} \left(\frac{-\hat{d}}{n}\right) n^{-s}.$$

Recall (6) that $d_f = \frac{f^{3/2}}{4\pi} L(\chi_{-f}, 2)$. In particular,

$$d_{15} = \frac{15^{3/2}}{4\pi} \zeta_{\mathbb{Q}(i\sqrt{15})}(2)/\zeta(2) = \frac{3 \times 15^{3/2}}{2\pi^3} \zeta_{\mathbb{Q}(i\sqrt{15})}(2).$$

Here we give an example which will be used in Chapter 3.

Example 2.4.18. For $d = 15 \equiv 3 (\mathrm{mod}\ 4)$, we have that $\hat{d} = d = 15$ and

$$\begin{aligned}
\frac{\pi d_{15}}{6} &= \frac{15^{3/2}}{24} \zeta_{\mathbb{Q}(i\sqrt{15})}(2)/\zeta(2) \\
&= \mathrm{Vol}\big(\mathbb{H}^3 / \mathrm{PSL}(2, \mathcal{O}_{15})\big) = \frac{15}{24} \sum_{k \bmod 15} \left(\frac{-15}{k}\right) D\left(e^{2\pi k i/15}\right) \\
&= \frac{5}{8} \Big(D\left(e^{2\pi i/15}\right) + D\left(e^{4\pi i/15}\right) + D\left(e^{8\pi i/15}\right) - D\left(e^{14\pi i/15}\right) \\
&\quad + D\left(e^{16\pi i/15}\right) - D\left(e^{22\pi i/15}\right) - D\left(e^{26\pi i/15}\right) - D\left(e^{28\pi i/15}\right) \Big) \\
&= \frac{5}{8} \Big(D\left(e^{2\pi i/15}\right) + D\left(e^{4\pi i/15}\right) + D\left(e^{8\pi i/15}\right) - D\left(e^{14\pi i/15}\right) \\
&\quad + D\left(e^{-14\pi i/15}\right) - D\left(e^{-8\pi i/15}\right) - D\left(e^{-4\pi i/15}\right) - D\left(e^{-2\pi i/15}\right) \Big) \\
&= \frac{5}{4} \Big(D\left(e^{2\pi i/15}\right) + D\left(e^{4\pi i/15}\right) + D\left(e^{8\pi i/15}\right) - D\left(e^{14\pi i/15}\right) \Big).
\end{aligned}$$

2.5 The Regulator

In algebraic K-theory, a *regulator* is a homomorphism from a K-group to a cohomology group. The idea of the regulator approach for proving Boyd's conjectures is described in Section 1.2. The regulator map used in Chapters 4 and 5 is defined on the second K-group $K_2(C)$ of a smooth curve C. Quillen [27] defined higher K-groups for schemes. This gives the definition of $K_2(C)$ since a smooth algebraic curve C is a scheme.

In Section 2.5.1, we introduce the definition of $K_2(F)$ for a field F given by Milnor [26] and a relationship between $K_2(C)$ and $K_2\big(F(C)\big)$, where $F(C)$ is the function field of the curve C over the number field F. In Section 2.5.2, we show how to define the *regulator map* that will be used for proving Boyd's conjectures. In Section 2.5.3, we explain how this approach works for our cases in Chapters 4 and 5.

2.5.1 The Second K-Group

First, we introduce the basic notion of the K_2-group of a field.

Definition 2.5.1 (Steinberg Symbol - [26, p. 94])**.** Let F be a field and let A be an abelian (additive) group. A *Steinberg symbol* $\{x,y\}_S$ on F is a bimultiplicative mapping $\sigma\colon F^\times \times F^\times \to A$: $(x,y) \mapsto \{x,y\}_S \in A$, subject to the following relations:

- Bilinearity: $\{xx',y\}_S = \{x,y\}_S + \{x',y\}_S$, $\{x,yy'\}_S = \{x,y\}_S + \{x,y'\}_S$;
- Steinberg identity: $\{x,1-x\}_S = 0$ for all $x \in F^\times \setminus \{1_F\}$.

By bilinearity, we have $\{1_F,y\}_S = \{1_F,y\}_S + \{1_F,y\}_S$ and $\{x,1_F\}_S = \{x,1_F\}_S + \{x,1_F\}_S$. This implies that

$$\{x,1_F\}_S = 0, \quad \{1_F,x\}_S = 0, \qquad \forall x \in F^\times.$$

Then, $0 = \{1_F,y\}_S = \{x^{-1}x,y\}_S = \{x^{-1},y\}_S + \{x,y\}_S$. We thus conclude that

$$\{x^{-1},y\}_S = -\{x,y\}_S, \qquad \{x,y^{-1}\}_S = -\{x,y\}_S.$$

Note that $-x(1_F - x^{-1}) = 1_F - x$. Hence, $(1_F - x)(1_F - x^{-1})^{-1} = -x$. We have that $\{x,1_F - x^{-1}\}_S = -\{x^{-1},1_F - x^{-1}\}_S = 0$. Thus, we obtain that

$$\begin{aligned}\{x,-x\}_S &= \{x,(1_F - x)(1_F - x^{-1})^{-1}\}_S \\ &= \{x,1_F - x\}_S - \{x,1_F - x^{-1}\}_S = 0, \quad \forall x \in F^\times.\end{aligned}$$

Now,

$$\begin{aligned}\{x,y\}_S &= \{x,y\}_S + \{x,-x\}_S - \{xy,-xy\}_S + \{y^{-1},-y^{-1}\}_S \\ &= \{x,-yx\}_S - \{xy,-xy\}_S + \{y^{-1},-y^{-1}\}_S \\ &= \{y^{-1},-xy\} + \{y^{-1},-y^{-1}\} \\ &= \{y^{-1},x\}_S = -\{y,x\}_S, \qquad \forall x,y \in F^\times.\end{aligned}$$

Let F be a field. Milnor [26] defined a canonical homomorphism $\phi\colon \mathrm{St}(F) \to \mathrm{GL}(F)$, where $\mathrm{St}(F)$ is the *Steinberg group* defined in [26, p. 39]. The kernel of ϕ is denoted by $K_2(F)$ (see [26, p. 39]). We usually call $K_2(F)$ the " Milnor's second K group of F".

Remark 2.5.2. The generators of Steinberg group are different from the Steinberg symbols. In fact, $K_2(F) = \ker\phi \subseteq \mathrm{St}(F)$, but the Steinberg symbols are defined in an abelian group A. We will not describe the definition of Milnor's second K group of F in detail, since we will exclusively work with Matsumoto theorem (Theorem 2.5.6).

Definition 2.5.3 (Milnor Symbol - [26, p. 67])**.** From the construction of $K_2(F)$, there is a canonical map $\psi\colon F^\times \otimes_{\mathbb{Z}} F^\times \to K_2(F)$ (see [26, p. 67]). For $x, y \in F^\times$, $\psi(x \otimes y)$ is called the *Milnor symbol*, denoted by $\{x, y\}$.

Proposition 2.5.4 (Properties of Milnor Symbol - [26, p. 74])**.** *We have that*

- *for* $x \in F^\times$, $\{x,-x\} = 0$*;*
- *for* $x, y \in F^\times$, $\{y,x\} = -\{x,y\}$*;*
- *for* $x \in F^\times \setminus \{1_F\}$, $\{x, 1_F - x\} = 0$ *.*

Theorem 2.5.5 (cf. [26, Corollary 9.13, p. 78])**.** *Let F be a field. Then the second K-group $K_2(F)$ is an abelian group that is generated by the set of Milnor symbols $\{x,y\}$ with $x, y \in F^\times$.*

Theorem 2.5.6 (Matsumoto Theorem - [26, Corollary 11.3, p. 94])**.** *Let F be a field and let A be an abelian (additive) group. Then, there is a unique homomorphism*

$$\begin{aligned} K_2(F) &\to A \\ \{x,y\} &\mapsto \{x,y\}_S \end{aligned}$$

for all $x, y \in F^\times$, where $\{x,y\}_S$ is a Steinberg symbol in A.

The Matsumoto theorem implies

$$K_2(F) \cong (F^\times \otimes_{\mathbb{Z}} F^\times)/\langle x \otimes (1-x) \colon x \in F^\times \setminus \{1\}\rangle.$$

This gives an alternative definition of $K_2(F)$. In particular, this gives the definition of $K_2\big(F(C)\big)$ as $F(C)$ is the function field of the curve C over F.

Recall that a complete nonsingular curve over an algebraically closed field F is an integral scheme of dimension 1 over F. Quillen [27] defined the K_n-group of X for a scheme X. We will not provide the formal definition of the second K-group $K_2(C)$ of a curve C in details. Note that our goal in this section is to deduce the relationships (1.8) and (1.9). For this purpose, we just give the definition of *tame symbol* from an exact sequence below.

Let C' be a smooth connected plane curve in $\mathbb{C}$ and let C be the compactification of C'. Then, $C' = C \setminus S$ with a finite subset S of C. Quillen showed a localization theorem for the K-groups (see [27, Theorem 5 and Corollary, p. 113]). Applying this localization theorem, we have following exact sequence (see [8, p. 86]):

$$\cdots \to K_2(C') \to K_2\big(\mathbb{C}(C)\big) \xrightarrow{\tau} \bigoplus_{P \in C'} \mathbb{C}^\times \to \cdots,$$

with the *tame symbol* τ whose P-component is given by

$$\tau_P\{f,g\} = (-1)^{\operatorname{ord}_P(f)\operatorname{ord}_P(g)} \frac{f^{\operatorname{ord}_P(g)}}{g^{\operatorname{ord}_P(f)}}\bigg|_P, \qquad f,g \in \mathbb{C}(C)^\times,$$

where $\operatorname{ord}_P(f)$ is the order of zero or pole of the function f at $P \in C$ (positive for a zero and negative for a pole). One can show that $\tau_P\{f,g\}$ is a Steinberg symbol (see Lemma 11.5, [26, p. 98]). We will denote it by $\{f,g\}_P = \tau_P\{f,g\}$.

2.5.2 The Regulator Map

Let C be a compact Riemann surface (i.e., a smooth algebraic curve). Denote the function field of C by $\mathbb{C}(C)$. For $f \in \mathbb{C}(C)$, denote the set of zeros and poles of f by S_f. Let $f,g \in \mathbb{C}(C)^\times$ and let $S = S_f \cup S_g$. Define a real analytic differential 1-form $\eta(f,g)$ on $C \setminus S$:

$$\eta(f,g) = \log|f|\,\mathrm{d}\arg g - \log|g|\,\mathrm{d}\arg f, \tag{2.13}$$

where d denotes the differential operator. Then, $\eta(f,g)$ is bilinear, and

$$\eta(f,g) = -\eta(g,f), \qquad \eta(f,f) = 0,$$
$$\eta(f_1 f_2, g) = \eta(f_1,g) + \eta(f_2,g).$$

Denote the $\mathbb{R}$-vector space of real analytic 1-forms on $C\backslash S$ by $\mathcal{M}(C\backslash S)$. Notice that $\mathcal{M}(C\setminus S)$ is a $\mathbb{Q}$-vector space. Then, by identifying $f\wedge g = (f\wedge g)\otimes 1$, one has following canonical map:

$$\begin{aligned}\big(\mathbb{C}(C)^\times \wedge \mathbb{C}(C)^\times\big) \otimes_{\mathbb{Z}} \mathbb{Q} &\to \mathcal{M}(C\setminus S),\\ f\wedge g &\mapsto \eta(f,g).\end{aligned}$$

Proposition 2.5.7. *We have that*

$$\mathrm{d}D(z) = \eta(z, 1-z),$$

where $D(z)$ is the Bloch-Wigner dilogarithm of z.

Proof. Recall that

$$\begin{aligned}D(z) &= \log|z|\arg(1-z) - \operatorname{Im}\int_0^z \frac{\log(1-\tau)}{\tau}\,\mathrm{d}\tau\\ &= \log|z|\arg(1-z) - \operatorname{Im}\int_0^z \log(1-\tau)\,\mathrm{d}\log\tau.\end{aligned}$$

Then,

$$\begin{aligned}\mathrm{d}D(z) &= (\,\mathrm{d}\log|z|)\arg(1-z) + \log|z|\,\mathrm{d}\arg(1-z) - \operatorname{Im}\left(\log(1-z)\,\mathrm{d}\log(z)\right)\\ &= (\,\mathrm{d}\log|z|)\arg(1-z) + \log|z|\,\mathrm{d}\arg(1-z)\\ &\qquad - \operatorname{Im}\big(\big(\log|1-z| + i\arg(1-z)\big)\big(\,\mathrm{d}\log|z| + i\,\mathrm{d}\arg z\big)\big)\\ &= (\,\mathrm{d}\log|z|)\arg(1-z) + \log|z|\,\mathrm{d}\arg(1-z)\\ &\qquad - \log|1-z|\,\mathrm{d}\arg z - \arg(1-z)\,\mathrm{d}\log|z|\\ &= \log|z|\,\mathrm{d}\arg(1-z) - \log|1-z|\,\mathrm{d}\arg z = \eta(z,1-z). \quad \square\end{aligned}$$

From $f = |f|e^{i\arg f}$, we have that $\log f = \log|f| + i\arg f$. Then,

$$\mathrm{d}\log f = \frac{\mathrm{d}f}{f} = \mathrm{d}\log|f| + i\,\mathrm{d}\arg f.$$

This gives

$$\begin{aligned}\mathrm{d}\arg f &= \operatorname{Im}\frac{\mathrm{d}f}{f},\\ \mathrm{d}\log|f| &= \operatorname{Re}\frac{\mathrm{d}f}{f} = \frac{1}{2}\left(\frac{\mathrm{d}f}{f} + \overline{\frac{\mathrm{d}f}{f}}\right).\end{aligned}$$

Then, we obtain that

$$\begin{aligned}\mathrm{d}\eta(f,g) &= \mathrm{d}\left(\log|f|\,\mathrm{d}\arg g - \log|g|\,\mathrm{d}\arg f\right)\\ &= \operatorname{Im}\left(\frac{1}{2}\left(\frac{\mathrm{d}f}{f} + \overline{\frac{\mathrm{d}f}{f}}\right)\wedge\frac{\mathrm{d}g}{g} - \frac{1}{2}\left(\frac{\mathrm{d}g}{g} + \overline{\frac{\mathrm{d}g}{g}}\right)\wedge\frac{\mathrm{d}f}{f}\right)\\ &= \operatorname{Im}\left(\frac{\mathrm{d}f}{f}\wedge\frac{\mathrm{d}g}{g}\right).\end{aligned}$$

In local coordinates, this says

$$\operatorname{Im}\left(\frac{f'(z)}{f(z)}\,\mathrm{d}z \wedge \frac{g'(z)}{g(z)}\,\mathrm{d}z\right) = 0,$$

since $\mathrm{d}z \wedge \mathrm{d}z = 0$. Hence, $\eta(f,g)$ is a closed 1-form. This implies that for a closed path γ in $C \setminus S$, the integral $\int_\gamma \eta(f,g)$ only depends on the homology class $[\gamma]$ of γ in the first homology group $H_1(C \setminus S, \mathbb{Z})$. This determines an element $r(f,g) \in H^1(C \setminus S, \mathbb{R})$ as a map $r(f,g)\colon H_1(C \setminus S, \mathbb{Z}) \to \mathbb{R}$, defined by

$$r(f,g)\big([\gamma]\big) = \int_\gamma \eta(f,g). \tag{2.14}$$

This induces a map $r\colon K_2\big(C\big) \to H^1(C,\mathbb{R})$ given by $\{f,g\} \mapsto r(f,g)$. This map is called the *regulator map.*

2.5.3 The Diamond Operation

Let C be an algebraic curve over $\mathbb{C}$. Denote the *group of divisors of* C by $\operatorname{Div}(C)$, i.e.,

$$\operatorname{Div}(C) = \left\{ \sum_{P \in C} n_P(P) \,\middle|\, n_P \in \mathbb{Z} \text{ and } n_P = 0 \text{ for all but finitely many } P \in C \right\}.$$

Each element in $\operatorname{Div}(C)$ is called a *divisor of* C.

Assume that the curve C is smooth. Let $f \in \mathbb{C}(C)^\times$. Define the *divisor of* f by

$$\operatorname{div}(f) = \sum_{P \in C} \operatorname{ord}_P(f)(P),$$

where $\operatorname{ord}_P(f)$ is the order of zero or pole of the regular function f at $P \in C$ (see [17, p. 15–16 and p. 130–131] or [31, p. 17–18] for the rigorous definition). Normally, we also use the notation:

$$(f) = \operatorname{div}(f).$$

For a smooth curve C, $\operatorname{ord}_P(f) \neq \infty$ for all $P \in C$ and there are only finitely many $P \in C$ such that $\operatorname{ord}_P(f) \neq 0$ (see [31, Proposition II.1.2, p. 18]). Hence, $\operatorname{div}(f)$ is a divisor of C, i.e., $\operatorname{div}(f) \in \operatorname{Div}(C)$.

Now, let E be a smooth elliptic curve given by a Weierstrass form. Then, the points of E satisfy a group law $+$. Consequently, if $P_1, P_2 \in E$, then $P_1 - P_2 \in E$.

Definition 2.5.8 (Diamond Operation)**.** Let

$$D_1 = \sum_{i=1}^{s} n_i(P_i), \quad D_2 = \sum_{j=1}^{t} m_j(Q_j) \quad \in \mathrm{Div}(E)$$

for a smooth elliptic curve E. Define the *diamond operation* as a function $\diamond\colon \mathrm{Div}(E) \times \mathrm{Div}(E) \to \mathrm{Div}(E)$ by

$$D_1 \diamond D_2 = \sum_{i=1}^{s} \sum_{j=1}^{t} n_i m_j (P_i - Q_j).$$

Notation 2.5.9 (Plus and negative parts of H_1)**.** Let C be a complete nonsingular curve over $\mathbb{Q}$. Denote the subgroup of $H_1\big(C(\mathbb{C}), \mathbb{Z}\big)$ consisting of the homology classes that are invariant (resp. anti-invariant) under the action of complex conjugation by

$$H_1\big(C(\mathbb{C}), \mathbb{Z}\big)^+ \qquad \big(\text{resp. } H_1\big(C(\mathbb{C}), \mathbb{Z}\big)^-\big).$$

Let E be an elliptic curve over $\mathbb{Q}$ and let $u \in K_2(E) \otimes \mathbb{Q}$. By Theorem 2.5.5, $u = \sum_i \{f_i, g_i\}$ for some rational functions f_i, g_i on E. Let

$$\eta(u) = \sum_i \eta(f_i, g_i).$$

Brunault [7] showed the following theorem.

Theorem 2.5.10 (cf. [7, Proposition 19, p. 555])**.** *For every $\gamma \in H_1\big(E(\mathbb{C}), \mathbb{Z}\big)$, we have*

$$\int_\gamma \eta(u) = -\left(\gamma_E^+ \bullet \gamma\right) D^E \left(\sum_i (f_i) \diamond (g_i) \right), \tag{2.15}$$

where $\bullet$ is the intersection product *on $H_1\big(E(\mathbb{C}), \mathbb{Z}\big)$, γ_E^+ is the generator of $H_1\big(E(\mathbb{C}), \mathbb{Z}\big)^+$ corresponding to a chosen orientation of $E(\mathbb{R})$ and D^E is the elliptic dilogarithm on $E(\mathbb{C})$ given in Definition 2.4.6.*

For an algebraic curve C, the intersection product $\gamma_1 \bullet \gamma_2$ on $H_1\big(C(\mathbb{C}), \mathbb{Z}\big)$ is computed from the number of intersection points of the closed paths γ_1 and γ_2 on C. The rigorous definition of *intersection product* can be found in some standard intersection theory textbooks (e.g. [12, 15, 16]).

Notice that in Boyd conjectures, we only consider polynomials $P(x, y)$ in $\mathbb{Q}[x, y]$ of degree 2 on y. Now, write

$$P(x, y) = \sum_{j=0}^{2} a_j(x) y^j = P^*(x)\big(y - y_1(x)\big)\big(y - y_2(x)\big), \tag{2.16}$$

where $y_1(x), y_2(x)$ are algebraic functions. For some specific cases, one may give an explicit formula where the Mahler measure of an elliptic curve E is determined by the diamond operations $(x) \diamond (y)$ with $x, y \in \mathbb{Q}(E)^\times$.

In fact, we have

$$\begin{aligned}
&\mathrm{m}(P) - \mathrm{m}(P^*) \\
&= \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log|y - y_1(x)| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} + \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log|y - y_2(x)| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}.
\end{aligned}$$

By Jensen's formula, $\int_0^1 \log|a - e^{2\pi i t}| \,\mathrm{d}t = \log\max\{1, |a|\}$, and we have

$$\frac{1}{2\pi i} \int_{|y|=1} \log|y - y_j(x)| \frac{\mathrm{d}y}{y} = \log\max\{1, |y_j(x)|\}.$$

Then,

$$\begin{aligned}
&\mathrm{m}(P) - \mathrm{m}(P^*) \\
&= \frac{1}{2\pi i} \int_{|x|=1, |y_1(x)| \geq 1} \log|y_1(x)| \frac{\mathrm{d}x}{x} + \frac{1}{2\pi i} \int_{|x|=1, |y_2(x)| \geq 1} \log|y_2(x)| \frac{\mathrm{d}x}{x}.
\end{aligned} \tag{2.17}$$

For $|x| = 1$, $x = e^{i \arg x}$ and then $\log x = i(\arg x + 2n\pi)$ for some integer n. Hence, for $|x| = 1$,

$$\mathrm{d}\arg x = \frac{\mathrm{d}x}{ix},$$

is well defined, independently of the branch of log. Then,

$$\begin{aligned}
\eta\big(x, y_j(x)\big) &= \log|x| \,\mathrm{d}\arg y_j(x) - \log|y_j(x)| \,\mathrm{d}\arg x \\
&= -\log|y_j(x)| \,\mathrm{d}\arg x = -\frac{1}{i} \log|y_j(x)| \frac{\mathrm{d}x}{x}.
\end{aligned}$$

Thus, (2.17) becomes

$$\begin{aligned}
&\mathrm{m}(P) - \mathrm{m}(P^*) \\
&\quad = -\frac{1}{2\pi} \int_{|x|=1, |y_1(x)| \geq 1} \eta(x, y_1) - \frac{1}{2\pi} \int_{|x|=1, |y_2(x)| \geq 1} \eta(x, y_2).
\end{aligned} \tag{2.18}$$

When the set $\{x \mid |x| = 1, |y_i(x)| \geq 1\}$ can be seen as a cycle in $H_1\big(E(\mathbb{C}), \mathbb{Z}\big)^-$, from (2.15), the Mahler measure of P may be determined by the diamond operations $(x) \diamond (y)$ for $x, y \in \mathbb{Q}(E)^\times$. In particular, when we encounter the case where $|y_1(x)| \geq 1$ and $|y_2(x)| \leq 1$ on $|x| = 1$, we have

$$\mathrm{m}(P) - \mathrm{m}(P^*) = \frac{c}{2\pi} D^E\big((x) \diamond (y)\big), \tag{2.19}$$

for $x, y \in \mathbb{Q}(E)^\times$, where c is a nonzero integer which only depends on the cycle $\{x \mid |x| = 1, |y_1(x)| \geq 1\}$ in $H_1\big(E(\mathbb{C}), \mathbb{Z}\big)^-$. The conditions to get (2.19) have to be examined on case by case.

Remark 2.5.11. Usually, the set $\{x \mid |x| = 1, |y_i(x)| \geq 1\}$ can only be seen as a cycle in $H_1\big(C(\mathbb{C}), \mathbb{Z}\big)$. By an appropriate transformation f of variables, we obtain an elliptic curve E from the curve $C \colon P(x, y) = 0$, and the cycle $[\gamma_E] \in H_1\big(E(\mathbb{C}), \mathbb{Z}\big)$ from $[\gamma_P] \in H_1\big(C(\mathbb{C}), \mathbb{Z}\big)$ by

$$[f(\gamma_P)] = a[\gamma_E],$$

with a constant a. Then we have

$$\int_{\gamma_P} \eta_C(x, y) = a \int_{\gamma_E} \eta_E\big(f_*\{x, y\}\big).$$

If, moreover, $[\gamma_E] \in H_1\big(E(\mathbb{C}), \mathbb{Z}\big)^-$, then we have the formula (2.19) by replacing c with ac.

Formula (2.19) provides a strategy for the proof of some specific Boyd's conjectures, in particular, for the cases in Chapters 4 and 5. More precisely, suppose that the Mahler measure of the polynomial Q can be expressed by

$$\mathrm{m}(Q) - \mathrm{m}(Q^*) = \frac{c'}{2\pi} D^{E'}\big((x') \diamond (y')\big),$$

with an elliptic curve E' and a nonzero integer c' (depending on the cycle) for $x', y' \in \mathbb{Q}(E')^\times$. If E and E' are isogenous with the isogeny $\varphi \colon E \to E'$, then for $P' \in E'(\mathbb{C})$, we have (see [7, Lemma 23, p. 557])

$$D^{E'}\big(P'\big) = d_\varphi \sum_{P \in \varphi^{-1}(P')} D^E\big(P\big),$$

where d_φ is the topological degree of $E(\mathbb{R})^0 \to E'(\mathbb{R})^0$ and the notation $(\cdot)^0$ means the connected component of the origin. If $\varphi \colon E \to E'$ is an isomorphism of two curves, then $d_\varphi = 1$. In addition, if $\varphi\big((x) \diamond (y)\big) = (x') \diamond (y')$, then we obtain the desired result:

$$\mathrm{m}(P) - \mathrm{m}(P^*) = \frac{c}{c'}\big(\mathrm{m}(Q) - \mathrm{m}(Q^*)\big). \tag{2.20}$$

For the case of Theorem 1.1.1, the elliptic curves E and E' are isomorphic. For the cases of Theorem 1.1.2 and Theorem 1.1.3, E and E' are same curve. We thus have following two steps for the regulator proofs of Boyd's identities in Chapters 4 and 5: (1) show the relationship between the regulators by computing the diamond operation at torsion points for both curves, and (2) show the relationship between the cycles to determine c/c'.

2.6 Methods for Genus 0 Curves

Consider an algebraic curve $C\colon P(x,y)=0$, where $P(x,y)\in\mathbb{Q}[x,y]$ given by (2.16). Notice that if we can write

$$x \wedge y_j = \sum_{j_k} \alpha_{j_k} z_{j_k} \wedge (1-z_{j_k}), \tag{2.21}$$

in $\mathbb{C}(C)^\times \wedge \mathbb{C}(C)^\times$, then

$$\mathrm{m}(P)-\mathrm{m}(P^*) = -\frac{1}{2\pi}\sum_{j=1}^{2}\sum_{j_k}\alpha_{j_k} D(z_{j_k})|_{\partial\{|x|=1,|y_j(x)|\geq 1\}}, \tag{2.22}$$

where $D(z)$ is the Bloch-Wigner dilogarithm of z, defined in Definition 2.4.2. For a curve of genus 0, the polynomial $P(x,y)$ can be parametrized. This gives solutions to (2.21). In order to write the form (2.21), we need the following lemma. (Its proof is given by a direct computation, see [9, Lemma 7, p. 10].)

Proposition 2.6.1 (cf. [9, Lemma 7, p. 10])**.** *Let $a,b,c,d\in\mathbb{C}$ and let t be a variable. Suppose that $ad-bc\neq 0$. Then, in $\left(\mathbb{C}(t)^\times \wedge \mathbb{C}(t)^\times\right)_{\mathbb{Q}}$, we have that*

$$\begin{aligned}
&(at+b)\wedge(ct+d)\\
&\quad= \frac{act+bc}{ad-bc}\wedge\frac{act+ad}{ad-bc} - (ad-bc)\wedge\frac{act+bc}{act+ad}\\
&\qquad - c\wedge(ct+d) - (at+b)\wedge a - c\wedge a.
\end{aligned}$$

Chapter 3

A Case of a Genus 1 Curve

In [5], Boyd showed numerically the following identity:

$$\mathrm{m}\left(Q_3\right) = \frac{d_{15}}{6}, \tag{3.1}$$

where d_{15} is given by (6) : $d_f = L'(\chi_{-f}, -1) = \frac{f^{3/2}}{4\pi} L(\chi_{-f}, 2)$ and Q_3 is given by setting $k = 3$ in the following family

$$Q_k(x, y) = (x^2 + x + 1)y^2 + kx(x + 1)y + x(x^2 + x + 1).$$

In Section 2.4.4, we discussed that

$$\frac{\pi d_{15}}{6} = \frac{15^{3/2}}{24} \zeta_{\mathbb{Q}(i\sqrt{15})}(2)/\zeta(2) = \mathrm{Vol}\big(\mathbb{H}^3 / \,\mathrm{PSL}(2, \mathcal{O}_{15})\big).$$

Thus, the conjectural formula (3.1) becomes

$$\pi\, \mathrm{m}\left(Q_3\right) = \mathrm{Vol}\big(\mathbb{H}^3 / \,\mathrm{PSL}(2, \mathcal{O}_{15})\big). \tag{3.2}$$

Recall Example 2.4.15. Boyd and Rodriguez-Villegas [6] have shown Boyd's conjecture (3.2):

$$\begin{aligned}
\mathrm{m}(Q_3) &= \frac{1}{3\pi} \left(3D(c) + 4D\left((c+1)/2\right) + 2D(c+1) + 2D(c-1)\right) \\
&= \frac{1}{3\pi} \left(3D\left((c+3)/4\right) + 4D\left(c/2\right) + 2D(c+1) + 2D(c-1)\right) \\
&= \frac{1}{\pi} \mathrm{Vol}\big(\mathbb{H}^3 / \,\mathrm{PSL}(2, \mathcal{O}_{15})\big) = \frac{d_{15}}{6}
\end{aligned}$$

where $c = \frac{1+i\sqrt{15}}{2}$.

The goal of this chapter is to provide other expressions for $\mathrm{m}\left(Q_3\right)$. We prove the following formula.

Theorem 3.0.2. *We have that*

$$\begin{aligned}\mathrm{m}\left(Q_3\right) &= \frac{2}{\pi}\left(D\left(\frac{3+i\sqrt{15}}{3-i\sqrt{3}}\right)+D\left(\frac{3+i\sqrt{15}}{3+i\sqrt{3}}\right)\right)\\ &= \frac{1}{3\pi}D\left(b\right)-\frac{3}{\pi}D\left(a\right),\end{aligned} \tag{3.3}$$

where $a=\frac{1-i\sqrt{15}}{4}\in\mathbb{Q}(i\sqrt{15})$ *and* $b=a^3\in\mathbb{Q}(i\sqrt{15})$.

Combining (3.3) with Examples 2.4.15 and 2.4.18, we establish the following identities.

$$\begin{aligned}\mathrm{m}\left(Q_3\right) &= \frac{d_{15}}{6}=\frac{15^{3/2}}{24\pi}L(\chi_{-15},2)\\ &= \frac{2}{\pi}\left(D\left(\frac{3+i\sqrt{15}}{3-i\sqrt{3}}\right)+D\left(\frac{3+i\sqrt{15}}{3+i\sqrt{3}}\right)\right)=\frac{1}{3\pi}D\left(a^3\right)-\frac{3}{\pi}D\left(a\right)\\ &= \frac{1}{\pi}D\left(\frac{7+i\sqrt{15}}{8}\right)+\frac{2}{3\pi}D\left(\frac{-1+i\sqrt{15}}{2}\right)\\ &\quad+\frac{2}{3\pi}D\left(\frac{3+i\sqrt{15}}{2}\right)+\frac{4}{3\pi}D\left(\frac{1+i\sqrt{15}}{4}\right)\\ &= \frac{5}{4\pi}\left(D\left(e^{2\pi i/15}\right)+D\left(e^{4\pi i/15}\right)+D\left(e^{8\pi i/15}\right)-D\left(e^{14\pi i/15}\right)\right).\end{aligned} \tag{3.4}$$

3.1 Changes of Variables

Equation $Q_k(x,y)=0$ gives generally a hyperelliptic curve of genus 2. In order to get the standard form $Y^2=h(X^2)$ of a hyperelliptic curve, we proceed to make changes of variables $x=x(X,Y), y=y(X,Y)$ as follows.

From $Q_k(x,y)=0$, we have that

$$\begin{aligned}0 &= (x^2+x+1)y^2+kx(x+1)y+x(x^2+x+1)\\ &= \frac{1}{4(x^2+x+1)}\Big(\big(2(x^2+x+1)y+kx(x+1)\big)^2-k^2x^2(x+1)^2\\ &\qquad +4x(x^2+x+1)^2\Big).\end{aligned}$$

This allows us to write

$$\hat{y}^2=k^2x^2(x+1)^2-4x(x^2+x+1)^2, \tag{3.5}$$

where

$$\hat{y} = 2(x^2 + x + 1)y + kx(x + 1).$$

Boyd made a change of variables (see [5, p. 68]):

$$x = \frac{X+1}{X-1}, \quad 2Y = (X-1)^3\hat{y}.$$

Then, we obtain the transformations:

$$\begin{aligned} y &= \frac{Y - kX(X^2-1)}{(X-1)(3X^2+1)}, \\ X &= \frac{x+1}{x-1}, \quad Y = \frac{(X-1)^3}{2}\hat{y} = \frac{8(x^2+x+1)y + 4kx(x+1)}{(x-1)^3}. \end{aligned}$$

Now, we can write the curve (3.5) as

$$Y^2 = k^2X^2(X^2-1)^2 - (X^2-1)(3X^2+1)^2 = h(X^2),$$

where

$$\begin{aligned} h(Z) &= k^2Z(Z-1)^2 - (Z-1)(3Z+1)^2 \\ &= (k^2-9)Z^3 - (2k^2-3)Z^2 + (k^2+5)Z + 1. \end{aligned}$$

Fix $k = 3$. We have a genus 0 curve:

$$Y^2 = -15Z^2 + 14Z + 1, \tag{3.6}$$

where $Z = X^2$, with the change of variables:

$$\begin{aligned} x &= \frac{X+1}{X-1}, \quad y = \frac{Y - 3X(X^2-1)}{(X-1)(3X^2+1)}, \\ X &= \frac{x+1}{x-1}, \quad Y = \frac{8(x^2+x+1)y + 12x(x+1)}{(x-1)^3}. \end{aligned}$$

Now, we want to write

$$x \wedge y = \left(\frac{X+1}{X-1}\right) \wedge \left(\frac{Y - 3X(X^2-1)}{(X-1)(3X^2+1)}\right) \tag{3.7}$$

in the form (2.21). The genus 0 curve (3.6) can be parametrized by $\big(Z(t), Y(t)\big)$. But $X^2 = Z$. So, the parametrization $X(t)$ will cause problems at branch points. Consequently, it is not evident how to make a direct computation $x \wedge y$ by the equation (3.7). In order to make the computation feasible, Bosman [4] introduced a method involving two rational functions $a(X^2, Y)$ and $b(X^2, Y)$ such that

$$a(X^2, Y)x(X, Y) + b(X^2, Y)y(X, Y) = 1. \tag{3.8}$$

These two functions a and b can be taken as

$$\begin{aligned} a(Z,Y) &= \frac{Y-3Z+3}{Y+3Z-3}, \\ b(Z,Y) &= \frac{-6Z-2}{Y+3Z-3}. \end{aligned}$$

For these two rational functions $a(X,Y)$ and $b(X,Y)$, we have that

$$\begin{aligned} &a(X^2,Y)x(X,Y) \wedge b(X^2,Y)y(X,Y) \\ &\quad = a(X^2,Y) \wedge b(X^2,Y) + a(X^2,Y) \wedge y(X,Y) \\ &\qquad + x(X,Y) \wedge b(X^2,Y) + x(X,Y) \wedge y(X,Y). \end{aligned} \tag{3.9}$$

Notice that the transformation of $x \to 1/x$ and $y \to 1/y$ leads to $X \to -X$ and $Y \to Y$. This yields

$$\begin{aligned} &a(X^2,Y)x(-X,Y) \wedge b(X^2,Y)y(-X,Y) \\ &\quad = a(X^2,Y)\frac{1}{x(X,Y)} \wedge b(X^2,Y)\frac{1}{y(X,Y)} \\ &\quad = a(X^2,Y) \wedge b(X^2,Y) - a(X^2,Y) \wedge y(X,Y) \\ &\qquad - x(X,Y) \wedge b(X^2,Y) + x(X,Y) \wedge y(X,Y). \end{aligned} \tag{3.10}$$

Adding Eq (3.10) to (3.9) gives

$$\begin{aligned} &a(X^2,Y)x(X,Y) \wedge b(X^2,Y)y(X,Y) + a(X^2,Y)x(-X,Y) \wedge b(X^2,Y)y(-X,Y) \\ &\quad = 2a(X^2,Y) \wedge b(X^2,Y) + 2x(X,Y) \wedge y(X,Y). \end{aligned} \tag{3.11}$$

By (3.8), we have that

$$\begin{aligned} &a(X^2,Y)x(X,Y) \wedge b(X^2,Y)y(X,Y) \\ &\qquad = a(X^2,Y)x(X,Y) \wedge \big(1 - a(X^2,Y)x(X,Y)\big) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &a(X^2,Y)x(-X,Y) \wedge b(X^2,Y)y(-X,Y) \\ &\qquad = a(X^2,Y)x(-X,Y) \wedge \big(1 - a(X^2,Y)x(-X,Y)\big). \end{aligned} \tag{3.13}$$

From (3.11) with (3.12) and (3.13), we obtain

$$\begin{aligned} &x(X,Y) \wedge y(X,Y) \\ &\quad = -a(X^2,Y) \wedge b(X^2,Y) + \frac{1}{2}\left(a(X^2,Y)x(X,Y) \wedge \big(1 - a(X^2,Y)x(X,Y)\big)\right) \\ &\qquad + \frac{1}{2}\left(a(X^2,Y)x(-X,Y) \wedge \big(1 - a(X^2,Y)x(-X,Y)\big)\right). \end{aligned} \tag{3.14}$$

Recall the arguments for (2.21) and (2.22). If $x \wedge y$ has a term $f \wedge (1-f)$ with $f \in \mathbb{C}(Q_3)$, then $\mathrm{m}(Q_3)$ has a term $D(f)|_{\partial\{|x|=1,|y(x)|\geq 1\}}$. So, for the term $a(X^2,Y)x(X,Y) \wedge \big(1 - a(X^2,Y)x(X,Y)\big)$, it integrates to

$$D\big(a(X^2,Y)x(X,Y)\big)_{\partial\{|x|=1,|y(x)|\geq 1\}},$$

and for the term $a(X^2,Y)x(-X,Y) \wedge \big(1 - a(X^2,Y)x(-X,Y)\big)$, it integrates to

$$D\big(a(X^2,Y)x(-X,Y)\big)_{\partial\{|x|=1,|y(x)|\geq 1\}}.$$

3.2 Parameterization

We are back to considering the curve $Y^2 = -15Z^2 + 14Z + 1$. We can rewrite the equation as a circle given by

$$\left(\frac{Y}{\sqrt{15}}\right)^2 + \left(Z - \frac{7}{15}\right)^2 = \left(\frac{8}{15}\right)^2.$$

Pick $(Z,Y) = (1,0)$ to be the base point. We can parametrize the curve by

$$\begin{aligned} Y &= \frac{-16t}{t^2+15}, \\ Z &= \frac{t^2-1}{t^2+15}. \end{aligned} \tag{3.15}$$

Then,

$$\begin{aligned} a(Z,Y) &= \frac{Y-3Z+3}{Y+3Z-3} = \frac{t-3}{t+3}, \\ b(Z,Y) &= \frac{-6Z-2}{Y+3Z-3} = \frac{t^2+3}{2(t+3)}. \end{aligned} \tag{3.16}$$

We obtain

$$\begin{aligned} &a(Z,Y) \wedge b(Z,Y) \\ &\quad = \frac{t-3}{t+3} \wedge \frac{t^2+3}{2(t+3)} = \frac{t-3}{t+3} \wedge \frac{(t+i\sqrt{3})(t-i\sqrt{3})}{2(t+3)} \\ &\quad = (t-3) \wedge (t+i\sqrt{3}) + (t-3) \wedge (t-i\sqrt{3}) - (t-3) \wedge (2t+6) \\ &\qquad - (t+3) \wedge (t+i\sqrt{3}) - (t+3) \wedge (t-i\sqrt{3}) + (t+3) \wedge 2, \end{aligned}$$

since $(t+3) \wedge (t+3) = 0$. For every term $f(t) \wedge g(t)$ in $a(Z,Y) \wedge b(Z,Y)$, we need an expression of the form (2.21): $f(t) \wedge g(t) = \sum_j \alpha_j h_j(t) \wedge$

$\big(1-h_j(t)\big)$. This can be realized by Proposition 2.6.1 as follows.

$$\begin{aligned}(t-3)\wedge(t+i\sqrt{3}) &= \frac{t-3}{i\sqrt{3}+3}\wedge\frac{t+i\sqrt{3}}{i\sqrt{3}+3} - (i\sqrt{3}+3)\wedge\frac{t-3}{t+i\sqrt{3}}\\ &\quad - 1\wedge(t+i\sqrt{3}) - (t-3)\wedge 1 - 1\wedge 1,\end{aligned}$$

$$\begin{aligned}(t-3)\wedge(t-i\sqrt{3}) &= \frac{t-3}{-i\sqrt{3}+3}\wedge\frac{t-i\sqrt{3}}{-i\sqrt{3}+3} - (-i\sqrt{3}+3)\wedge\frac{t-3}{t-i\sqrt{3}}\\ &\quad - 1\wedge(t-i\sqrt{3}) - (t-3)\wedge 1 - 1\wedge 1,\end{aligned}$$

$$\begin{aligned}(t-3)\wedge(2t+6) &= \frac{2t-6}{12}\wedge\frac{2t+6}{12} - 12\wedge\frac{2t-6}{2t+6}\\ &\quad - 2\wedge(2t+6) - (t-3)\wedge 1 - 2\wedge 1,\end{aligned}$$

$$\begin{aligned}(t+3)\wedge(t+i\sqrt{3}) &= \frac{t+3}{i\sqrt{3}-3}\wedge\frac{t+i\sqrt{3}}{i\sqrt{3}-3} - (i\sqrt{3}-3)\wedge\frac{t+3}{t+i\sqrt{3}}\\ &\quad - 1\wedge(t+i\sqrt{3}) - (t+3)\wedge 1 - 1\wedge 1,\end{aligned}$$

$$\begin{aligned}(t+3)\wedge(t-i\sqrt{3}) &= \frac{t+3}{-i\sqrt{3}-3}\wedge\frac{t-i\sqrt{3}}{-i\sqrt{3}-3} - (-i\sqrt{3}-3)\wedge\frac{t+3}{t-i\sqrt{3}}\\ &\quad - 1\wedge(t-i\sqrt{3}) - (t+3)\wedge 1 - 1\wedge 1.\end{aligned}$$

Combining the above equations, $a(Z,Y)\wedge b(Z,Y)$ can be expressed as a sum of the form: $\sum_j \alpha_j h_j(t)\wedge\big(1-h_j(t)\big)$ as follows.

$$\begin{aligned}&-a(Z,Y)\wedge b(Z,Y)\\ &\quad= -(t-3)\wedge(t+i\sqrt{3}) - (t-3)\wedge(t-i\sqrt{3}) + (t-3)\wedge(2t+6)\\ &\qquad + (t+3)\wedge(t+i\sqrt{3}) + (t+3)\wedge(t-i\sqrt{3}) - (t+3)\wedge 2\\ &\quad= -\frac{t-3}{i\sqrt{3}+3}\wedge\frac{t+i\sqrt{3}}{i\sqrt{3}+3} + (i\sqrt{3}+3)\wedge\frac{t-3}{t+i\sqrt{3}} + 1\wedge(t+i\sqrt{3}) + (t-3)\wedge 1\\ &\qquad - \frac{t-3}{-i\sqrt{3}+3}\wedge\frac{t-i\sqrt{3}}{-i\sqrt{3}+3} + (-i\sqrt{3}+3)\wedge\frac{t-3}{t-i\sqrt{3}} + 1\wedge(t-i\sqrt{3})\\ &\qquad + (t-3)\wedge 1 + \frac{2t-6}{12}\wedge\frac{2t+6}{12} - 12\wedge\frac{2t-6}{2t+6} - 2\wedge(2t+6) - (t-3)\wedge 1\\ &\qquad - 2\wedge 1 + \frac{t+3}{i\sqrt{3}-3}\wedge\frac{t+i\sqrt{3}}{i\sqrt{3}-3} - (i\sqrt{3}-3)\wedge\frac{t+3}{t+i\sqrt{3}} - 1\wedge(t+i\sqrt{3})\\ &\qquad - (t+3)\wedge 1 + \frac{t+3}{-i\sqrt{3}-3}\wedge\frac{t-i\sqrt{3}}{-i\sqrt{3}-3} - (-i\sqrt{3}-3)\wedge\frac{t+3}{t-i\sqrt{3}}\\ &\qquad - 1\wedge(t-i\sqrt{3}) - (t+3)\wedge 1 - (t+3)\wedge 2.\end{aligned}$$

Noting that $2\wedge(2t+6) = 2\wedge 2(t+3) = 2\wedge 2+2\wedge(t+3) = -(t+3)\wedge 2$, we have that $-2\wedge(2t+6)-(t+3)\wedge 2 = 0$. Then,

$$\begin{aligned}
&(i\sqrt{3}+3)\wedge\frac{t-3}{t+i\sqrt{3}} + (-i\sqrt{3}+3)\wedge\frac{t-3}{t-i\sqrt{3}} - 12\wedge\frac{t-3}{t+3} - 2\wedge(2t+6) \\
&\quad - (i\sqrt{3}-3)\wedge\frac{t+3}{t+i\sqrt{3}} - (-i\sqrt{3}-3)\wedge\frac{t+3}{t-i\sqrt{3}} - (t+3)\wedge 2 \\
&= -2(i\sqrt{3}+3)\wedge(t+i\sqrt{3}) + 2(i\sqrt{3}+3)\wedge(t-i\sqrt{3}) \\
&\quad - 12\wedge(t-i\sqrt{3}) + 12\wedge(t+i\sqrt{3}) \\
&\quad\quad - (-1)\wedge(t+3) + (-1)\wedge(t-i\sqrt{3}).
\end{aligned}$$

In addition, $f(t)\wedge 1 = \{f(t),1\} = 0$ for any function $f(t)$ by Definition 2.5.1. We eventually obtain that

$$\begin{aligned}
&-a(Z,Y)\wedge b(Z,Y) \\
&\quad = -\frac{t-3}{i\sqrt{3}+3}\wedge\frac{t+i\sqrt{3}}{i\sqrt{3}+3} - \frac{t-3}{-i\sqrt{3}+3}\wedge\frac{t-i\sqrt{3}}{-i\sqrt{3}+3} + \frac{t-3}{6}\wedge\frac{t+3}{6} \\
&\quad\quad + \frac{t+3}{i\sqrt{3}-3}\wedge\frac{t+i\sqrt{3}}{i\sqrt{3}-3} + \frac{t+3}{-i\sqrt{3}-3}\wedge\frac{t-i\sqrt{3}}{-i\sqrt{3}-3}.
\end{aligned}$$

Thus, by (3.14), $x(X,Y)\wedge y(X,Y)$ can be expressed as a sum of the form: $\sum_j \alpha_j h_j(t)\wedge\big(1-h_j(t)\big)$. Write

$$Q_3(x,y) = (x^2+x+1)\big(y-y_+(x)\big)\big(y-y_-(x)\big).$$

From (2.22), we obtain

$$\begin{aligned}
\mathrm{m}(Q_3) &= \mathrm{m}(Q_3) - \mathrm{m}(x^2+x+1) \\
&= -\frac{1}{2\pi}\Bigg(-D\left(\frac{t-3}{i\sqrt{3}+3}\right) - D\left(\frac{t-3}{-i\sqrt{3}+3}\right) + D\left(\frac{t-3}{6}\right) \\
&\qquad + D\left(\frac{t+3}{i\sqrt{3}-3}\right) + D\left(\frac{t+3}{-i\sqrt{3}-3}\right) \\
&\qquad + \frac{1}{2}D\left(a(X^2,Y)x(X,Y)\right) \\
&\qquad + \frac{1}{2}D\left(a(X^2,Y)x(-X,Y)\right)\Bigg)_{\partial\{|x|=1,|y(x)|\geq 1\}},
\end{aligned}$$

where $D(z)$ is the Bloch-Wigner dilogarithm of z. By (3.15),

$$x(t) = \frac{X+1}{X-1} = \frac{\sqrt{t^2-1}+\sqrt{t^2+15}}{\sqrt{t^2-1}-\sqrt{t^2+15}} = -\frac{t^2+7+\sqrt{(t^2-1)(t^2+15)}}{8}, \tag{3.17}$$

and by (3.16),

$$a(X^2, Y) = \frac{t-3}{t+3}. \tag{3.18}$$

From Formula (2.18), we need to determine the integration path $|x| = 1$ and $|y| \geq 1$ in the computation of the Mahler measure of Q_3. For $x(t)$ to be a complex number with $|x(t)| = 1$, we need $-15 \leq t^2 \leq 1$. Then, the path of integration can be described at the level of t by

$$t \in [-i\sqrt{15}, 0] \cup [0, 1] \cup [-1, 0] \cup [0, i\sqrt{15}],$$

where $[-i\sqrt{15}, 0]$ and $[0, i\sqrt{15}]$ are intervals in the imaginary axis, and the sign of t should be chosen so that $|y| \geq 1$. Now, set

$$\begin{aligned} f(t) := -\frac{1}{2\pi}\Bigg(& -D\left(\frac{t-3}{i\sqrt{3}+3}\right) - D\left(\frac{t-3}{-i\sqrt{3}+3}\right) + D\left(\frac{t-3}{6}\right) \\ & + D\left(\frac{t+3}{i\sqrt{3}-3}\right) + D\left(\frac{t+3}{-i\sqrt{3}-3}\right) \\ & + \frac{1}{2}D\left(a(X^2, Y)x(X, Y)\right) + \frac{1}{2}D\left(a(X^2, Y)x(-X, Y)\right)\Bigg). \end{aligned}$$

Since $x(X, Y) = 1/x(-X, Y)$ and $x(X, Y)\overline{x(X, Y)} = |x(X, Y)|^2 = 1$, for $t \in \mathbb{R}$ we have that

$$\overline{a(X^2, Y)x(X, Y)} = \frac{t-3}{t+3}\overline{x(X, Y)} = \frac{t-3}{t+3}\left(\frac{1}{x(X, Y)}\right) = a(X^2, Y)x(-X, Y).$$

By Proposition 2.4.4, $D(\bar{z}) = -D(z)$. Thus, we conclude that $f(t) = 0$ for a real t. Assume that t is purely imaginary. Then, $-t = \bar{t}$ and from $D(\bar{z}) = -D(z)$, we see that

$$\begin{aligned} f(t) = -\frac{1}{2\pi}\Bigg(& 2D\left(\frac{t+3}{i3+\sqrt{3}}\right) + 2D\left(\frac{t+3}{-i\sqrt{3}+3}\right) + D\left(\frac{t-3}{6}\right) \\ & + \frac{1}{2}D\left(a(X^2, Y)x(X, Y)\right) + \frac{1}{2}D\left(a(X^2, Y)x(-X, Y)\right)\Bigg). \end{aligned}$$

We first concentrate in $t = i\sqrt{15}$. This corresponds to $x = 1$ by (3.17). Then,

$$\begin{aligned} & \frac{1}{2}D\left(a(X^2, Y)x(X, Y)\right) + \frac{1}{2}D\left(a(X^2, Y)x(-X, Y)\right) \\ & = D\left(\frac{i\sqrt{15}-3}{i\sqrt{15}+3}\right) = D\left(\frac{1+i\sqrt{15}}{4}\right). \end{aligned}$$

Since

$$1-\frac{1}{\frac{1+i\sqrt{15}}{4}}=\frac{3+i\sqrt{15}}{6},$$

we have

$$D\left(\frac{3-i\sqrt{15}}{6}\right)+D\left(\frac{1+i\sqrt{15}}{4}\right)=0.$$

Then, we consider $t=-i\sqrt{15}$ and obtain similar result by complex conjugation. Finally, we conclude

$$\mathrm{m}\left(Q_3\right)=f(-i\sqrt{15})-f(i\sqrt{15})=\frac{2}{\pi}\left(D\left(\frac{3+i\sqrt{15}}{3-i\sqrt{3}}\right)+D\left(\frac{3+i\sqrt{15}}{3+i\sqrt{3}}\right)\right).$$

By Example 2.4.5,

$$\begin{aligned}\mathrm{m}\left(Q_3\right)&=\frac{2}{\pi}\left(D\left(\frac{3+i\sqrt{15}}{3-i\sqrt{3}}\right)+D\left(\frac{3+i\sqrt{15}}{3+i\sqrt{3}}\right)\right)=\frac{1}{3\pi}D\left(-\omega^3\right)-\frac{3}{\pi}D\left(-\omega\right)\\&=\frac{1}{3\pi}\left(D\left(b\right)-9D\left(a\right)\right),\end{aligned}\tag{3.19}$$

where

$$\omega=\frac{-1+i\sqrt{15}}{4},\quad a=-\omega\in\mathbb{Q}(i\sqrt{15}),\quad b=(-\omega)^3\in\mathbb{Q}(i\sqrt{15}).$$

Together with Examples 2.4.15 and 2.4.18, we obtain (3.4).

Remark 3.2.1. From (3.19), we can show that

$$\mathrm{m}\left(Q_3\right)=\frac{1}{3\pi}\left(D\left(b\right)-9D\left(a\right)\right)=rd_{15}$$

for some $r\in\mathbb{Q}$.

In fact, we see that

$$\begin{aligned}&b\wedge(1-b)-9a\wedge(1-a)=a^3\wedge(1-a^3)-9a\wedge(1-a)\\&=a^3\wedge(1-a)(1+a+a^2)-9a\wedge(1-a)\\&=3a\wedge(1-a)+3a\wedge(1+a+a^2)-9a\wedge(1-a)\\&=3a\wedge(1+a+a^2)-6a\wedge(1-a)=3a\wedge(1+a+a^2)-3a\wedge(1-a)^2\\&=3a\wedge\left(\frac{3-3i\sqrt{15}}{8}\right)-3a\wedge\left(\frac{-3+3i\sqrt{15}}{8}\right)\\&=3a\wedge\left(\frac{3-3i\sqrt{15}}{8}\right)+3a\wedge\left(-\frac{3-3i\sqrt{15}}{8}\right)^{-1}=3a\wedge(-1)\\&=3a\wedge1=0.\end{aligned}$$

Thus,

$$[b] - 9[a] \in \mathcal{B}\big(\mathbb{Q}(i\sqrt{15})\big).$$

We thus conclude that

$$\frac{d_{15}}{6} = \frac{1}{\pi}\mathrm{Vol}\big(\mathbb{H}^3/\,\mathrm{PSL}(2, \mathcal{O}_{15})\big) = \frac{r_1}{3\pi}\big(D(b) - 9D(a)\big) = r_1\,\mathrm{m}(Q_3)$$

for some $r_1 \in \mathbb{Q}$.

Chapter 4

Regulator Proofs for Boyd's Identities on Genus 2 Curves

The goal of this chapter is to provide a proof of Theorems 1.1.1 and 1.1.2 by using the regulator theory. Here we rewrite Theorems 1.1.1 and 1.1.2 as follows.

Theorem 1.1.1. For $k \in \mathbb{R}$, we have

$$\mathrm{m}(S_k) = \begin{cases} 2\,\mathrm{m}(P_k), & 0 \le k \le 4, \\ \mathrm{m}(P_k), & k \le -1, \end{cases}$$

where

$$\begin{aligned} P_k(x,y) &= (x+1)(y+1)(x+y) - kxy, \\ S_k(x,y) &= y^2 + \big(x^4 + kx^3 + 2kx^2 + kx + 1\big)y + x^4. \end{aligned} \tag{4.1}$$

Theorem 1.1.2. For real $k \ge 4$, we have

$$\mathrm{m}(Q_k) = \mathrm{m}(R_{2+k}),$$

where

$$\begin{aligned} Q_k(x,y) &= (x^2+x+1)y^2 + kx(x+1)y + x(x^2+x+1), \\ R_k(x,y) &= (x^2+x+1)y^2 + \big(x^4 + kx^3 + (2k-4)x^2 + kx + 1\big)y \\ &\quad + x^2(x^2+x+1). \end{aligned} \tag{4.2}$$

In the case of Theorem 1.1.1 we finish the work started by Bosman [4]. In the case of Theorem 1.1.2, the development is entirely new. Our proofs show the role of the regulator in these relationships, which is the key step to an eventual understanding of the conjectural relationship of each of these Mahler measures to their corresponding L-value.

As discussed in Section 2.5.3, for the cases in Chapters 4 and 5, we consider $P \in \mathbb{Q}[x, y]$ of degree 2 on y. Write $P(x, y) = P^*(x)\big(y - y_1(x)\big)\big(y - y_2(x)\big)$, we have (2.19):

$$\mathrm{m}(P) - \mathrm{m}(P^*) = \frac{c}{2\pi} D^E\big((x) \diamond (y)\big),$$

for $x, y \in \mathbb{Q}(E)^\times$, where c is a nonzero integer which only depends on the cycle $\{x \mid |x| = 1, |y_1(x)| \geq 1\}$ in $H_1\big(E(\mathbb{C}), \mathbb{Z}\big)^-$. This is the fundamental formula to prove Boyd's identity (2.20) by using the regulator theory. Here we introduce our proof strategy for Theorem 1.1.1 as follows.

1. From a change of variables $x = x(X, Y), y = y(X, Y)$ given by (4.3) for the curve $P_k(x, y) = 0$, we obtain a Weierstrass form $E_k \colon Y^2 = h(X)$. From a change of variables $x_1 = x_1(X_1, Y_1), y_1 = y_1(X_1, Y_1)$ given by (4.6) for the curve $S_k(x_1, y_1) = 0$, we obtain the curve $C_k \colon Y_1^2 = h_1(X_1^2)$ such that $E_k' \colon Y_1^2 = h_1(Z_1)$ is an elliptic curve, where $Z_1 = X_1^2$ and h_1 is a cubic polynomial given by (4.5. Note that if $k \in \mathbb{Q}$, then $x, y \in \mathbb{Q}(E_k)^\times$ and $x_1, y_1 \in \mathbb{Q}(C_k)^\times$. Suppose that $k \in \mathbb{Q}$. We have the isogeny

$$\begin{aligned} \varphi_k \colon E_k &\to E_k' \\ (X, Y) &\mapsto (Z_1, Y_1) \end{aligned}$$

given by (4.7). For $P' \in E'(\mathbb{C})$, we have that

$$D^{E_k'}\big(P'\big) = d_{\varphi_k} \sum_{P \in \varphi_k^{-1}(P')} D^{E_k}\big(P\big)$$

where d_{φ_k} is the topological degree of $E_k(\mathbb{R})^0 \to E_k'(\mathbb{R})^0$. From (4.7), we have the isogeny $\varphi_k' \colon E_k' \to E_k$ given by $\varphi_k'\big(Z_1, Y_1\big) = (X, Y)$. Thus, $\varphi_k \colon E_k \to E_k'$ is an isomorphism. Consequently, $d_{\varphi_k} = 1$.

2. We show that

$$\varphi_k^{-1}\big((x_1) \diamond (y_1)\big) = (x) \diamond (y), \quad (x_1) \diamond (y_1) = \varphi_k\big((x) \diamond (y)\big).$$

This may be written as

$$(x_1) \diamond (y_1) = (x) \diamond (y).$$

Since the variables of the Weierstrass form $Y_1^2 = h_1(Z_1)$ are (Z_1, Y_1) with $Z_1 = X_1^2$, we cannot compute $(x_1) \diamond (y_1)$ directly. Bosman [4] built two rational functions $a(Z_1, Y_1)$, $b(Z_1, Y_1)$ such that

$$a(X_1^2, Y_1)x_1(X_1, Y_1) + b(X_1^2, Y)y_1(X_1, Y_1) = 1.$$

Then, one can show that

$$\big(x_1(X_1, Y_1)\big) \diamond \big(y_1(X_1, Y_1)\big) = -\big(a(X_1^2, Y_1)\big) \diamond \big(b(X_1^2, Y_1)\big).$$

So, we can compute $\big(a(Z_1, Y_1)\big) \diamond \big(b(Z_1, Y_1)\big)$ instead of $\big(x_1(X_1, Y_1)\big) \diamond \big(y_1(X_1, Y_1)\big)$.

3. Let ω be the standard invariant differential of the elliptic curves E_k. We show that for $k \in \mathbb{R}$,

$$\begin{cases} 2\int_{|x|=1} \omega = \int_{\phi_*(|x_1|=1)} \omega & \text{for } 0 \leq k \leq 4, \\ \int_{|x|=1} \omega = \int_{\phi_*(|x_1|=1)} \omega & \text{for } k \leq -1. \end{cases}$$

This implies Theorem 1.1.1 for the case where $k \in \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, Theorem 1.1.1 holds for $k \in \mathbb{R}$.

We can prove Theorem 1.1.2 similarly.

The remainder of this chapter corresponds to Sections 3 and 4 of the article [20] by Matilde Lalín and Gang Wu: Regulator proofs for Boyd's identities on genus 2 curves. *Int. J. Number Theory*, 15(5):945-967, 2019 .

4.1 The Families from Theorem 1.1.1

4.1.1 The relationship between the regulators

Recall that the family $P_k(x, y)$ is given by

$$P_k(x, y) = (x + 1)(y + 1)(x + y) - kxy,$$

which is birational to the Deuring form

$$E_k : Y^2 + (k - 2)XY + kY = X^3.$$

The change of variables is given by

$$\begin{aligned} X(x, y) &= k\frac{x + y + 1}{x + y - k}, & x(X, Y) &= \frac{X - Y}{X - k}, \\ Y(x, y) &= k\frac{-kx + y + 1}{x + y - k}, & y(X, Y) &= \frac{Y + (k - 1)X + k}{X - k}. \end{aligned} \tag{4.3}$$

The torsion group of E_k for $k \in \mathbb{Q}$ has order 6, generated by $P = (k,k)$, with $2P = (0,0)$, $3P = (-1,-1)$, $4P = (0,-k)$, $5P = (k,-k^2)$.

Our first goal is to compute the diamond operation $(x) \diamond (y)$ in E_k. This will allow us to understand the differential form $\eta(x,y)$ that is involved in the computation of $\mathrm{m}(P_k)$. Thus, we proceed to compute the divisors (x) and (y).

$$\begin{aligned}
(x) &= ((P) + (2P) + (3P) - 3O) - ((P) + (5P) - 2O) \\
&= (2P) + (3P) - (5P) - O \\
(y) &= ((3P) + (4P) + (5P) - 3O) - ((P) + (5P) - 2O) \\
&= -(P) + (3P) + (4P) - O
\end{aligned}$$

The diamond operation yields

$$(x) \diamond (y) = -6(P) - 6(2P). \tag{4.4}$$

Now we proceed to compute the diamond operation $(x_1) \diamond (y_1)$. Recall that

$$S_k(x_1,y_1) = y_1^2 + (x_1^4 + kx_1^3 + 2kx_1^2 + kx_1 + 1)y_1 + x_1^4.$$

Bosman ([4], p. 47) considers the curve

$$C_k : Y_1^2 = h_1(X_1^2),$$

where

$$h_1(Z_1) = (k^2+k)Z_1^3 + (-2k^2+5k+4)Z_1^2 + (k^2-5k+8)Z_1 - k + 4. \tag{4.5}$$

We have an isomorphism

$$\begin{array}{ccc}
C_k & \overset{\psi_k}{\to} & \{S_k = 0\} \\
(X_1,Y_1) & \to & (x_1,y_1),
\end{array}$$

where ψ_k^{-1} and ψ_k are given by

$$\begin{aligned}
X_1(x_1,y_1) =& \frac{x_1+1}{x_1-1}, \\
Y_1(x_1,y_1) =& \frac{4(y_1^2 - x_1^4)}{y_1(x_1-1)^3(x_1+1)}, \\
x_1(X_1,Y_1) =& \frac{X_1+1}{X_1-1}, \\
y_1(X_1,Y_1) =& \frac{2X_1Y_1 - (2k+1)X_1^4 + (2k-6)X_1^2 - 1}{(X_1-1)^4},
\end{aligned} \tag{4.6}$$

respectively. We also have a map

$$\begin{array}{ccc} E_k & \stackrel{\phi_k}{\to} & C_k \\ (Z_1, Y_1) & \to & (X_1, Y_1) \end{array}$$

defined by $\phi_k(Z_1) = X_1^2$, $\phi_k(Y_1) = Y_1$. The map ϕ_k has no inverse since $\phi_k(Z_1) = X_1^2$. This implies that we can not compute the diamond operation between $(x_1(X_1, Y_1))$ and $(y_1(X_1, Y_1))$ directly. Bosman [4] resolved this problem by introducing the rational functions $a(Z_1, Y_1)$, $b(Z_1, Y_1)$ that we will explain as follows.

The relationship between the rational functions Z_1, Y_1 and X, Y in E_k is given by the following transformations.

$$\begin{aligned} Y_1(k^2 + k) =& 4(2Y + (k-2)X + k), \\ (k^2 + k)Z_1 - (k^2 - 3k) =& 4X, \end{aligned}$$

so that

$$\begin{aligned} Y =& \frac{k((k+1)Y_1 + (-k^2 + k + 2)Z_1 + (k^2 - 5k + 2))}{8}, \\ Y_1 =& \frac{4(2Y + (k-2)X + k)}{k^2 + k}, \\ Z_1 =& \frac{4X + k^2 - 3k}{k^2 + k}. \end{aligned} \tag{4.7}$$

Our goal is to compute

$$r_{C_k}(\{x_1(X_1, Y_1), y_1(X_1, Y_1)\})[\psi_k \circ \gamma],$$

where γ is the path in $\{S_k = 0\}$ defined by $|x_1| = 1, |y_1| \geq 1$ (for certain choice of a root y_1), that will be made precise later. In order to do this, we will consider the pushforward by ϕ_k to the regulator r_{E_k} in E_k.

Bosman does this by finding rational functions $a(Z_1, Y_1)$, $b(Z_1, Y_1)$ such that

$$a(X_1^2, Y_1)x_1(X_1, Y_1) + b(X_1^2, Y_1)y_1(X_1, Y_1) = 1.$$

Then, he proves the following result, which we reproduce here for completeness.

Lemma 4.1.1. *We have*

$$r_{C_k}(\{x_1(X_1, Y_1), y_1(X_1, Y_1)\})[\psi_k \circ \gamma] = -r_{E_k}(\{a(Z_1, Y_1), b(Z_1, Y_1)\})[\phi_k \circ \psi_k \circ \gamma].$$

Proof. It suffices to see the identity at the level of the diamond operator, namely, to prove that

$$(x_1(X_1, Y_1)) \diamond (y_1(X_1, Y_1)) \sim -(a(X_1^2, Y_1)) \diamond (b(X_1^2, Y_1)).$$

Because of the triviality of the Steimberg symbol, $(f) \diamond (1 - f) \sim 0$, and

$$\begin{aligned} 0 \sim &(a(X_1^2, Y_1)x_1(X_1, Y_1)) \diamond (b(X_1^2, Y_1)y_1(X_1, Y_1)) \\ \sim &(a(X_1^2, Y_1)) \diamond (b(X_1^2, Y_1)) + (x_1(X_1, Y_1)) \diamond (b(X_1^2, Y_1)) \\ &+ (a(X_1^2, Y_1)) \diamond (y_1(X_1, Y_1)) + (x_1(X_1, Y_1)) \diamond (y_1(X_1, Y_1)). \end{aligned}$$

Now consider the automorphism of $S_k(x_1, y_1) = 0$ given by $x_1 \to \frac{1}{x_1}$, $y_1 \to \frac{1}{y_1}$. We remark that $X_1 \to -X_1$ and $Y_1 \to Y_1$. Then

$$\begin{aligned} 0 \sim &(a(X_1^2, Y_1)x_1(-X_1, Y_1)) \diamond (b(X_1^2, Y_1)y_1(-X_1, Y_1)) \\ \sim &(a(X_1^2, Y_1)) \diamond (b(X_1^2, Y_1)) - (x_1(X_1, Y_1)) \diamond (b(X_1^2, Y_1)) \\ &- (a(X_1^2, Y_1)) \diamond (y_1(X_1, Y_1)) + (x_1(X_1, Y_1)) \diamond (y_1(X_1, Y_1)). \end{aligned}$$

Combining the above expressions, we obtain the result. $\square$

Following Bosman, we take

$$\begin{aligned} a(Z_1, Y_1) =&\frac{(-Z_1^2 - 6Z_1 - 1)Y_1 + (4k + 2)Z_1^3 + 14Z_1^2 + (-4k + 14)Z_1 + 2}{(Z_1 - 1)((-Z_1 - 1)Y_1 + (2k + 1)Z_1^2 + (-2k + 6)Z_1 + 1)} \\ =&\frac{\begin{pmatrix} 2X^2Y + 4k^2XY + (k^4 - 2k^3 - k^2)Y + (-3k - 4)X^3 \\ + (-k^3 + 2k)X^2 + (k^3 + 2k^2)X - k^3 \end{pmatrix}}{(X - k)((k^2 - k)Y + 2XY - (k + 3)X^2 + 2kX)} \end{aligned}$$

and

$$\begin{aligned} b(Z_1, Y_1) =&\frac{(Z_1 - 1)^2}{(-Z_1 - 1)Y_1 + (2k + 1)Z_1^2 + (-2k + 6)Z_1 + 1} \\ =&- \frac{(X - k)^2}{(k^2 - k)Y + 2XY - (k + 3)X^2 + 2kX}. \end{aligned}$$

We proceed to compute the diamond operation for $(a(X, Y))$ and

$(b(X,Y))$. Consider the following points on E_k.

$$\begin{aligned}
P =&(k,k),\\
U_\pm =&\left(\frac{k(-k\pm\sqrt{k^2-16k+32})}{8}, \frac{k^2(k-8\mp\sqrt{k^2-16k+32})}{16}\right),\\
V_\pm =&\left(\frac{-k^2+4k-3\pm(k+1)\sqrt{k^2-10k+9}}{8},\right.\\
&\left.\frac{k^3-7k^2-k-9\mp(k^2-2k-3)\sqrt{k^2-10k+9}}{16}\right),
\end{aligned}$$

where we also have that $U_+ + U_- = P$ and $V_+ + V_- = 2P$. Thus we write U for U_+, V for V_+, $P-U$ for U_-, and $2P-V$ for V_-.

One can check that

$$\begin{aligned}
(X-k) =&(P)+(5P)-2O,\\
((k^2-k)Y+2XY-(k+3)X^2+2kX) =&2(P)+(2P)+(V)+(2P-V)-5O,
\end{aligned}$$

and

$$\begin{aligned}
&(2X^2Y+4k^2XY+(k^4-2k^3-k^2)Y+(-3k-4)X^3+(-k^3+2k)X^2\\
&\qquad+(k^3+2k^2)X-k^3)\\
=&5(P)+(U)+(P-U)-7O.
\end{aligned}$$

In sum, this gives

$$\begin{aligned}
(a(Z_1,Y_1)) =&2(P)+(U)+(P-U)-(5P)-(2P)-(V)-(2P-V),\\
(b(Z_1,Y_1)) =&2(5P)-(2P)-(V)-(2P-V)+O.
\end{aligned}$$

By applying Lemma 4.1.1,

$$\begin{aligned}
-(x_1)\diamond(y_1) \sim&5(P)+3(2P)+(U)+(P-U)+3(P+U)+3(2P-U)+(V-U)\\
&+(2P-U-V)+(U+V-P)+(U-V+P)-(V)-(2P-V)\\
&-3(V+P)+3(V+3P). \qquad (4.8)
\end{aligned}$$

Now we record other divisors.

$$\begin{aligned}
(X+k) =&(V-P)+(P-V)-2O,\\
(kX+2Y+k^2) =&(5P)+(U)+(P-U)-3O,\\
(Y) =&3(2P)-3O.
\end{aligned}$$

The above relations imply

$$\begin{aligned}\left(\frac{X+k}{Y}\right) =&(V-P)+(P-V)+O-3(2P),\\ \left(\frac{kX+2Y+k^2}{Y}\right) =&(5P)+(U)+(P-U)-3(2P),\end{aligned}$$

and

$$\begin{aligned}0 \sim& \left(-\frac{k(X+k)}{2Y}\right) \diamond \left(\frac{kX+2Y+k^2}{2Y}\right)\\ =&(P)+3(2P)-(U)-(P-U)-3(P+U)-3(2P-U)-(V-U)\\ &-(2P-U-V)-(U+V-P)-(U-V+P)+(V)+(2P-V)\\ &+3(V+P)-3(V+3P).\end{aligned}$$

Combining the above equation with (4.8) we obtain

$$(x_1) \diamond (y_1) \sim -6(P)-6(2P).$$

By comparing with Eq. (4.4), we conclude,

$$(x_1) \diamond (y_1) \sim (x) \diamond (y).$$

4.1.2 The relationship between the cycles

We consider the integration cycle for the Mahler measure over P_k first. It is convenient to make the change of variables $x = x_0^2$ as well as $y_0 = y/x_0$. In this case we have

$$(x_0+x_0^{-1})y_0^2 + (x_0^2+(2-k)+x_0^{-2})y_0 + (x_0+x_0^{-1}) = 0.$$

This gives

$$\begin{aligned}y_{0\pm} =& \frac{-(x_0^2+(2-k)+x_0^{-2})}{2(x_0+x_0^{-1})}\\ &\pm \frac{\sqrt{(x_0^2-2x_0+2-k-2x_0^{-1}+x_0^{-2})(x_0^2+2x_0+2-k+2x_0^{-1}+x_0^{-2})}}{2(x_0+x_0^{-1})}.\end{aligned}$$

Now write $x_0 = e^{i\theta}$ with $0 \le \theta \le \pi$. We have

$$y_{0\pm} = \frac{(k-4\cos^2\theta) \pm \sqrt{(k-4\cos^2\theta)^2 - 16\cos^2\theta}}{4\cos\theta}.$$

Further taking $t = \cos^2\theta$, we have that the polynomial inside the square root is $16t^2 - 8(2+k)t + k^2$ which has roots $t = \frac{2+k\pm 2\sqrt{k+1}}{4}$.

When this polynomial takes negative values, both roots are complex conjugate of each other and both have absolute value 1. We are interested in the cases that the polynomial takes positive values and one of the roots has absolute value larger than 1.

When $0 \leq k \leq 4$ this polynomial takes positive values for $0 \leq t \leq \frac{2+k-2\sqrt{k+1}}{4}$. We can see that in this case $|y_{0+}| > 1$.

When $k \leq -1$, the polynomial inside the square root has no real roots and therefore it is positive for $0 \leq t \leq 1$. Both roots are then real. We see that $|y_{0-}| > 1$.

In order to characterize the homology class given by the integration set, we integrate respect to the standard invariant differential ω of the elliptic curves. Recall that

$$\omega = \frac{\mathrm{d}X}{2Y + (k-2)X + k}.$$

By looking at the transformations, we have

$$\mathrm{d}X = -\frac{k(k+1)(\,\mathrm{d}x + \,\mathrm{d}y)}{(x+y-k)^2}.$$

By differentiating P_k, we have,

$$(2(y+1)x + y^2 + (2-k)y + 1)\,\mathrm{d}x + (2(x+1)y + x^2 + (2-k)x + 1)\,\mathrm{d}y = 0.$$

Putting the above together, we obtain,

$$\begin{aligned}\mathrm{d}X =& \frac{k(k+1)(y-x)\,\mathrm{d}x}{(2(x+1)y + x^2 + (2-k)x + 1)(x+y-k)} \\ =& \frac{k(k+1)y(y-x)\,\mathrm{d}x}{(x+1)(y^2-x)(x+y-k)}.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}X}{2Y + (k-2)X + k} =& \frac{\mathrm{d}x}{2(x+1)y + x^2 + (2-k)x + 1} = \frac{y\,\mathrm{d}x}{(x+1)(y^2-x)} \\ =& \frac{2\,\mathrm{d}x_0}{(2(x_0 + x_0^{-1})y_0 + x_0^2 + (2-k) + x_0^{-2})x_0}.\end{aligned}$$

At this point, we either have to specify the choice of the root $y_{0\pm}$ or leave the sign in front of the square-root undetermined. Since all the Mahler measures are non negative, and the integration sets are connected, we can leave the sign to be determined later.

$$\omega = \pm \frac{2i\,\mathrm{d}\theta}{\sqrt{(k-4\cos^2\theta)^2 - 16\cos^2\theta}}.$$

Take $t = \cos^2\theta$, then $\frac{\mathrm{d}t}{\sqrt{t(1-t)}} = -2\,\mathrm{d}\theta$ and

$$\omega = \pm \frac{i\,\mathrm{d}t}{\sqrt{t(1-t)((k-4t)^2 - 16t)}}$$

In sum, we must consider, for $0 \le k \le 4$,

$$\int_{\varphi_*(|x|=1)} \omega = \pm \int_0^{\frac{2+k-2\sqrt{k+1}}{4}} \frac{2i\,\mathrm{d}t}{\sqrt{t(1-t)((k-4t)^2 - 16t)}}$$

and for $k \le -1$,

$$\int_{\varphi_*(|x|=1)} \omega = \pm \int_0^1 \frac{2i\,\mathrm{d}t}{\sqrt{t(1-t)((k-4t)^2 - 16t)}}.$$

In both cases, the extra factor 2 comes from changing $0 \le \theta \le \pi$ to $0 \le \theta \le \frac{\pi}{2}$.

Now we analyze the cycle for S_k. Make the change of variables $y_0 = y_1/x_1^2$. This gives

$$y_0^2 + (x_1^2 + kx_1 + 2k + kx_1^{-1} + x_1^{-2})y_0 + 1 = 0$$

and

$$\begin{aligned} y_{0\pm} =& \frac{-(x_1^2 + kx_1 + 2k + kx_1^{-1} + x^{-2})}{2} \\ &\pm \frac{\sqrt{(x_1 + 2 + x_1^{-1})(x_1 + k - 2 + x_1^{-1})(x_1^2 + kx_1 + 2(k+1) + kx_1^{-1} + x_1^{-2})}}{2}. \end{aligned}$$

By setting $x_1 = e^{i\theta}$ with $0 \le \theta \le 2\pi$, we have

$$\begin{aligned} y_{0\pm} =& -(2\cos^2\theta + k\cos\theta + (k-1)) \\ &\pm \sqrt{(\cos\theta + 1)(2\cos\theta + k - 2)(2\cos^2\theta + k\cos\theta + k)}. \end{aligned}$$

Taking $t = \cos\theta$, the polynomial inside the square root is $(t+1)(2t+k-2)(2t^2+kt+k)$. The roots for the quadratic factor are given by $\frac{-k\pm\sqrt{k^2-8k}}{4}$. As before, we are interested in the case when the polynomial inside the square-root takes positive values.

When $0 \le k \le 4$, the polynomial is positive for $\frac{2-k}{2} \le t \le 1$.

When $k \le -1$, the polynomial is positive for $\frac{-k-\sqrt{k^2-8k}}{4} \le t \le 1$.

In both cases, this leads to a root that has absolute value greater or equal to 1 and another that has absolute value less or equal to 1. As observed in the previous case, we do not have to determine the exact sign of this root as long as each integral is done over a fixed root.

On the other hand, we have,

$$\mathrm{d}X = \frac{k(k+1)\,\mathrm{d}Z_1}{4} = \frac{k(k+1)X_1\,\mathrm{d}X_1}{2} = -\frac{k(k+1)(x_1+1)\,\mathrm{d}x_1}{(x_1-1)^3}.$$

We also have

$$2Y + (k-2)X + k = \frac{k(k+1)Y_1}{4} = \frac{k(k+1)(y_1^2-x_1^4)}{y_1(x_1-1)^3(x_1+1)}.$$

Therefore,

$$\frac{\mathrm{d}X}{2Y+(k-2)X+k} = -\frac{(x_1+1)^2 y_1\,\mathrm{d}x_1}{(y_1^2-x_1^4)} = \frac{(x_1+1)^2 y_1\,\mathrm{d}x_1}{(x_1^4+kx_1^3+2kx_1^2+kx_1+1)y_1+2x_1^4}.$$

Let $y_0 = y_1/x_1^2$. Then

$$\frac{\mathrm{d}X}{2Y+(k-2)X+k} = \frac{(x_1+2+x_1^{-1})y_0\,\mathrm{d}x_1}{((x_1^2+kx_1+2k+kx_1^{-1}+x_1^{-2})y_0+2)x_1}.$$

Writing $x_1 = e^{i\theta}$ with $0 \le \theta \le 2\pi$, this leads to

$$\omega = \pm\frac{(1+\cos\theta)i\,\mathrm{d}\theta}{\sqrt{(\cos\theta+1)(2\cos\theta+k-2)(2\cos^2\theta+k\cos\theta+k)}}.$$

Take $t = \cos\theta$. Then $-\frac{\mathrm{d}t}{\sqrt{1-t^2}} = \mathrm{d}\theta$ and

$$\omega = \pm\frac{i\,\mathrm{d}t}{\sqrt{(1-t)(2t+k-2)(2t^2+kt+k)}}.$$

In sum, for $0 \le k \le 4$, we must consider

$$\int_{\varphi_*(|x_1|=1)} \omega = \pm\int_{\frac{2-k}{2}}^{1} \frac{2i\,\mathrm{d}t}{\sqrt{(1-t)(2t+k-2)(2t^2+kt+k)}}$$

and for $k \le -1$,

$$\int_{\varphi_*(|x_1|=1)} \omega = \pm \int_{\frac{-k-\sqrt{k^2-8k}}{4}}^{1} \frac{2i\,\mathrm{d}t}{\sqrt{(1-t)(2t+k-2)(2t^2+kt+k)}}.$$

The result is completed with the following statement which covers the necessary identities, except for the boundary cases, which can be deduced by continuity.

Lemma 4.1.2. *For $0 < k < 8$, we have*

$$2\int_0^{\frac{2+k-2\sqrt{k+1}}{4}} \frac{\mathrm{d}t}{\sqrt{t(1-t)((k-4t)^2-16t)}} = \int_{\frac{2-k}{2}}^{1} \frac{\mathrm{d}t}{\sqrt{(1-t)(2t+k-2)(2t^2+kt+k)}}. \tag{4.9}$$

For $k < -1$, we have

$$\int_0^1 \frac{\mathrm{d}t}{\sqrt{t(1-t)((k-4t)^2-16t)}} = \int_{\frac{-k-\sqrt{k^2-8k}}{4}}^{1} \frac{\mathrm{d}t}{\sqrt{(1-t)(2t+k-2)(2t^2+kt+k)}}. \tag{4.10}$$

Proof. First consider the change of variables $t = \frac{ks-k+2}{2}$. Then for $0 < k$,

$$\int_{\frac{2-k}{2}}^{1} \frac{\mathrm{d}t}{\sqrt{(1-t)(2t+k-2)(2t^2+kt+k)}} = \int_0^1 \frac{\mathrm{d}s}{\sqrt{s(1-s)(k^2s^2+k(4-k)s+4)}},$$

and for $k < -1$,

$$\int_{\frac{-k-\sqrt{k^2-8k}}{4}}^{1} \frac{\mathrm{d}t}{\sqrt{(1-t)(2t+k-2)(2t^2+kt+k)}}$$
$$= -\int_{\frac{k-4-\sqrt{k^2-8k}}{2k}}^{1} \frac{\mathrm{d}s}{\sqrt{s(1-s)(k^2s^2+k(4-k)s+4)}}.$$

The right-hand sides of the above equations are related to the formulas found by Rogers and Zudilin [30] and Bertin and Zudilin [1], and this allows us to use their changes of variables to manipulate those sides of the equations. However, the left-hand sides do not appear in those works. As we will eventually see, the connection between the two sides are given by periods in two isogenous elliptic curves.

Notice that we can modify the integration limits in the last integral by the involution $s = \frac{1-w}{1+kw}$, which gives for $k < -1$,

$$- \int_{\frac{k-4-\sqrt{k^2-8k}}{2k}}^{1} \frac{\mathrm{d}s}{\sqrt{s(1-s)(k^2s^2+k(4-k)s+4)}}$$
$$= \int_0^{\frac{k-4+\sqrt{k^2-8k}}{2k}} \frac{\mathrm{d}w}{\sqrt{w(1-w)(k^2w^2+k(4-k)w+4)}}.$$

In sum, we have to prove, for $0 < k < 8$,

$$2\int_0^{\frac{2+k-2\sqrt{k+1}}{4}} \frac{\mathrm{d}t}{\sqrt{t(1-t)((k-4t)^2-16t)}} = \int_0^1 \frac{\mathrm{d}s}{\sqrt{s(1-s)(k^2s^2+k(4-k)s+4)}} \tag{4.11}$$

and for $k < -1$,

$$\int_0^1 \frac{\mathrm{d}t}{\sqrt{t(1-t)((k-4t)^2-16t)}} = \int_0^{\frac{k-4+\sqrt{k^2-8k}}{2k}} \frac{\mathrm{d}w}{\sqrt{w(1-w)(k^2w^2+k(4-k)w+4)}}. \tag{4.12}$$

First we concentrate on Eq. (4.12). Consider the change $t = \frac{1}{1+\frac{4u}{k^2}}$. Then the left-hand side of Eq. (4.12) becomes

$$\int_0^1 \frac{\mathrm{d}t}{\sqrt{t(1-t)((k-4t)^2-16t)}} = \frac{1}{2}\int_0^{\infty} \frac{\mathrm{d}u}{\sqrt{u\left(u^2+2\left(\frac{k^2}{4}-k-2\right)u+\frac{k^3}{16}(k-8)\right)}}. \tag{4.13}$$

Consider $w = \frac{1}{1+v}$. The right-hand side of Eq. (4.12) becomes

$$\int_0^{\frac{k-4+\sqrt{k^2-8k}}{2k}} \frac{\mathrm{d}w}{\sqrt{w(1-w)(k^2w^2+k(4-k)w+4)}}$$
$$=\frac{1}{2}\int_{\frac{k^2-4k-8-k\sqrt{k^2-8k}}{8}}^{\infty} \frac{\mathrm{d}v}{\sqrt{v(v^2-\left(\frac{k^2}{4}-k-2\right)v+k+1)}}. \tag{4.14}$$

The integrals on the right-hand sides of Eqs. (4.13) and (4.14) correspond to the same periods in isogenous elliptic curves. We can use the standard isogeny of degree 2 for the Weierstrass form $y^2 = x(x^2+ax+b)$ to describe the change of variables between them. More precisely, $u = v - \left(\frac{k^2}{4} - k - 2\right) + \frac{k+1}{v}$ yields

$$\begin{aligned}
&\int_0^\infty \frac{\mathrm{d}u}{\sqrt{u\left(u^2 + 2\left(\frac{k^2}{4} - k - 2\right)u + \frac{k^3}{16}(k-8)\right)}}\\
=&\int_{\frac{k^2-4k-8-k\sqrt{k^2-8k}}{8}}^\infty \frac{\mathrm{d}v}{\sqrt{v\left(v^2 - \left(\frac{k^2}{4} - k - 2\right)v + k + 1\right)}}.
\end{aligned}$$

This concludes the proof of Eq. (4.12) and therefore of Eq. (4.10).

For (4.11), consider the following observation. If we set $\beta = -\frac{8}{k}$, then

$$\int \frac{\mathrm{d}t}{\sqrt{t(1-t)((k-4t)^2 - 16t)}} = \frac{|\beta|}{4} \int \frac{\mathrm{d}t}{\sqrt{t(1-t)(\beta^2t^2 + \beta(4-\beta)t + 4)}}.$$

Applying the above transformation to Eq. (4.12), we get, for $0 < \beta < 8$,

$$\begin{aligned}
&\frac{\beta}{4}\int_0^1 \frac{\mathrm{d}t}{\sqrt{t(1-t)(\beta^2t^2 + \beta(4-\beta)t + 4)}}\\
=&\frac{4}{\left|\frac{-8}{\beta}\right|}\int_0^{\frac{2+\beta-2\sqrt{\beta+1}}{4}} \frac{\mathrm{d}w}{\sqrt{w(1-w)((\beta-4w)^2 - 16w)}},
\end{aligned}$$

which implies Eqs. (4.11) and (4.9). □

4.2 The Families from Theorem 1.1.2

4.2.1 The relationship between the regulators

In this section we work with the families

$$Q_k(x_2, y_2) = (x_2^2 + x_2 + 1)y_2^2 + kx_2(x_2 + 1)y_2 + x_2(x_2^2 + x_2 + 1)$$

and

$$R_{k'}(x_3, y_3) = (x_3^2+x_3+1)y_3^2+(x_3^4+k'x_3^3+(2k'-4)x_3^2+k'x_3+1)y_3+x_3^2(x_3^2+x_3+1).$$

For $Q_k(x_2, y_2)$, Boyd [5] writes

$$Y_2^2 = h_2(X_2^2),$$

where

$$h_2(Z_2) = (k^2-9)Z_2^3 - (2k^2-3)Z_2^2 + (k^2+5)Z_2 + 1,$$

and

$$\begin{aligned}
X_2(x_2, y_2) =& \frac{x_2+1}{x_2-1}, \\
Y_2(x_2, y_2) =& \frac{4(2(x_2^2+x_2+1)y_2 + kx_2(x_2+1))}{(x_2-1)^3}, \\
x_2(X_2, Y_2) =& \frac{X_2+1}{X_2-1}, \\
y_2(X_2, Y_2) =& \frac{Y_2 - kX_2(X_2^2-1)}{(X_2-1)(3X_2^2+1)}.
\end{aligned}$$

By applying the transformation

$$Z = (k^2-9)(Z_2-1), \qquad W = (k^2-9)Y_2,$$

we obtain

$$F_k : W^2 = Z^3 + (k^2-24)Z^2 - 16(k^2-9)Z.$$

Our goal as before is to compute $(x_2) \diamond (y_2)$ in F_k. We will do this by applying Lemma 4.1.1. Thus we take

$$\begin{aligned}
a(Z_2, Y_2) =& \frac{Y_2 - k(Z_2-1)}{Y_2 + k(Z_2-1)} = \frac{W - kZ}{W + kZ}, \\
b(Z_2, Y_2) =& -\frac{2(3Z_2+1)}{Y_2 + k(Z_2-1)} = -\frac{2(3Z + 4(k^2-9))}{W + kZ},
\end{aligned}$$

and we can easily see that

$$a(X_2^2, Y_2)x_2(X_2, Y_2) + b(X_2^2, Y_2)y_2(X_2, Y_2) = 1.$$

Lemma 4.1.1 still applies with the same change $x_2 \to \frac{1}{x_2}$ and $y_2 \to \frac{1}{y_2}$ that leads to $X_2 \to -X_2$ and $Y_2 \to Y_2$.

We consider the following points of F_k (in Z, W coordinates).

$$\begin{aligned}
P =& (0, 0), \\
S_\pm =& (\pm 4k + 12, k(\pm 4k + 12)), \\
T =& \left(-\frac{4(k^2-9)}{3}, \frac{4i(k-3)k(k+3)}{3\sqrt{3}} \right),
\end{aligned}$$

where $S_+ + S_- = P$. Thus we rename S to be S_+ and $P-S$ to be S_-. Notice also that $2P = O$

We compute some divisors.

$$\begin{aligned}(W-kZ) =&(S)+(P-S)+(P)-3O,\\ (W+kZ) =&(-S)+(P+S)+(P)-3O,\\ (3Z+4(k^2-9)) =&(T)+(-T)-2O.\end{aligned}$$

This leads to

$$\begin{aligned}(a(Z_2,Y_2)) =&(S)+(P-S)-(-S)-(P+S),\\ (b(Z_2,Y_2)) =&(T)+(-T)+O-(-S)-(P+S)-(P),\end{aligned}$$

and

$$\begin{aligned}&(a(Z_2,Y_2))\diamond(b(Z_2,Y_2))\\ &= 2(S-T)+2(S+T)-2(P+S+T)-2(P+S-T)+4(S)-4(P+S).\end{aligned}$$

Finally, by Lemma 4.1.1, we conclude,

$$-(x_2)\diamond(y_2) \sim 2(S-T)+2(S+T)-2(P+S+T)-2(P+S-T)+4(S)-4(P+S). \tag{4.15}$$

We now consider the case of $R_{k'}(x_3,y_3)$. We have

$$Y_3^2 = h_3(X_3^2),$$

where

$$h_3(Z_3) = (k'^2-k'-2)Z_3^3+(-2k'^2+11k'-2)Z_3^2+(k'^2-11k'+26)Z_3+k'-6,$$

and

$$\begin{aligned}X_3(x_3,y_3) =&\frac{x_3+1}{x_3-1},\\ Y_3(x_3,y_3) =&\frac{4(2(x_3^2+x_3+1)y_3+x_3^4+k'x_3^3+(2k'-4)x_3^2+k'x_3+1)}{(x_3-1)^3(x_3+1)},\\ x_3(X_3,Y_3) =&\frac{X_3+1}{X_3-1},\\ y_3(X_3,Y_3) =&\frac{2X_3Y_3-(2k'-1)X_3^4+(2k'-10)X_3^2+1}{(X_3-1)^2(3X_3^2+1)}.\end{aligned}$$

By applying the transformation

$$Z = (k'^2-k'-2)Z_3-(k'^2-5k'-6), \quad W = (k'^2-k'-2)Y_3,$$

we obtain

$$W^2 = Z^3 + (k'^2 - 4k' - 20)Z^2 - 16(k'^2 - 4k' - 5)Z.$$

Notice that this is precisely $F_{k'-2}$.

We proceed to compute the diamond operation $(x_3) \diamond (y_3)$. Using the usual strategy of Lemma 4.1.1, we find

$$\begin{aligned}
a(Z_3, W_3) =& \frac{(Z_3+1)Y_3 - (2k'-1)Z_3^2 + (2k'-10)Z_3 + 1}{(Z_3-1)Y_3} \\
=& \frac{\left(\begin{array}{l} ZW + 2(k'^2 - 3k' - 4)W - (2k'-1)Z^2 \\ \quad - 2(k'^3 - 5k'^2 - 10k' - 4)Z + 16(k'^3 - 3k'^2 - 9k' - 5) \end{array}\right)}{W(Z - 4(k'+1))} \\
b(Z_3, W_3) =& -\frac{3Z_3+1}{Y_3} = -\frac{3Z + 4(k'^2 - 4k' - 5)}{W}
\end{aligned}$$

and one can easily see that

$$a(X_3^2, Y_3)x_3(X_3, Y_3) + b(X_3^2, Y_3)y_3(X_3, Y_3) = 1.$$

Lemma 4.1.1 still applies with the same change $x_3 \to \frac{1}{x_3}$ and $y_3 \to \frac{1}{y_3}$ that leads to $X_3 \to -X_3$ and $Y_3 \to Y_3$.

We consider the following points (in Z, W coordinates),

$$\begin{aligned}
P =& (0,0), \\
A =& \left(\frac{-(k'^2 - 4k' - 20) + \sqrt{k'^4 - 8k'^3 + 40k'^2 - 96k' + 80}}{2}, 0 \right), \\
A + P =& \left(\frac{-(k'^2 - 4k' - 20) - \sqrt{k'^4 - 8k'^3 + 40k'^2 - 96k' + 80}}{2}, 0 \right), \\
S =& (4(k'+1), 4(k'-2)(k'+1)), \\
2S =& (16, -16), \\
T =& \left(-\frac{4(k'-5)(k'+1)}{3}, \frac{4i(k'-5)(k'-2)(k'+1)}{3\sqrt{3}} \right), \\
P - S =& (4(5-k'), -4(k'-5)(k'-2)), \\
P - 2S =& ((5-k')(k'+1), (k'-5)(k'+1)).
\end{aligned}$$

Notice that the points P, S, T are the same that were previously considered in F_k. The formulas are different since they depend on the parameter k'.

We then obtain

$$\begin{aligned}(W) =&(P) + (A) + (A + P) - 3O,\\ (Z - 4(k' + 1)) =&(S) + (-S) - 2O,\\ (3Z + 4(k' - 5)(k' + 1)) =&(T) + (-T) - 2O,\end{aligned}$$

and

$$\begin{aligned}&(ZW + 2(k'^2 - 3k' - 4)W\\ &\qquad - (2k' - 1)Z^2 - 2(k'^3 - 5k'^2 - 10k' - 4)Z + 16(k'^3 - 3k'^2 - 9k' - 5))\\ &\ = 3(S) + (P - S) + (P - 2S) - 5O.\end{aligned}$$

This implies

$$\begin{aligned}(a(Z_3, Y_3)) =&2(S) + (P - S) + (P - 2S) - (P) - (A) - (A + P) - (-S),\\ (b(Z_3, Y_3)) =&(T) + (-T) + O - (P) - (A) - (A + P).\end{aligned}$$

Thus,

$$\begin{aligned}-(x_3) \diamond (y_3) =&(a(X_3^2, Y_3)) \diamond (b(X_3^2, Y_3))\\ =&3(S - T) + 3(S + T) + 4(S) - 4(P + S)\\ &- (P + S + T) - (P + S - T)\\ &+ (2S) - (P + 2S) - (P + 2S + T) + (P - 2S + T) - 2(S + A)\\ &+ (2S + A) - 2(S + A + P) + (2S + A + P).\end{aligned} \tag{4.16}$$

Now consider

$$(W - 3Z - 4(k' - 5)(k' + 1)) =(S) + (P + S) + (P - 2S) - 3O$$

and

$$\begin{aligned}\left(\frac{W - 3Z - 4(k' - 5)(k' + 1)}{W}\right) =&(S) + (P + S) + (P - 2S) - (P) - (A) - (A + P)\\ \left(\frac{3Z + 4(k' - 5)(k' + 1)}{W}\right) =&(T) + (-T) + O - (P) - (A) - (A + P).\end{aligned}$$

Combining the above divisors, we have

$$\left(\frac{W-3Z-4(k'-5)(k'+1)}{W}\right)\diamond\left(\frac{3Z+4(k'-5)(k'+1)}{W}\right)=(S-T)+(S+T)$$
$$+(P+S+T)+(P+S-T)+(P-2S+T)+(P-2S-T)+(2S)-(P+2S)$$
$$-2(S+A)-2(S+A+P)+(2S+A)+(2S+A+P).$$

By comparing with Eqs. (4.15) and (4.16), we get

$$(x_2)\diamond(y_2)\sim(x_3)\diamond(y_3).$$

4.2.2 The relationship between the cycles

We start by considering Q_k. It is convenient to make the change of variables $x_2 = x_0^2$ and $y_2 = y_0 x_0$. We then consider

$$(x_0^2+1+x_0^{-2})y_0^2+k(x_0+x_0^{-1})y_0+(x_0^2+1+x_0^{-2})=0.$$

In this case we have

$$\begin{aligned}y_{0\pm}=&\frac{-k(x_0+x_0^{-1})}{2(x_0^2+1+x_0^{-2})}\\&\pm\frac{\sqrt{-(2x_0^2-kx_0+2-kx_0^{-1}+2x_0^{-2})(2x_0^2+kx_0+2+kx_0^{-1}+2x_0^{-2})}}{2(x_0^2+1+x_0^{-2})}.\end{aligned}$$

Write $x_0 = e^{i\theta}$ with $0\le\theta\le\pi$. We have

$$y_{0\pm}=\frac{-k\cos\theta\pm\sqrt{k^2\cos^2\theta-(4\cos^2\theta-1)^2}}{4\cos^2\theta-1}.$$

If we take $t=\cos^2\theta$, the polynomial inside the square root is $-16t^2+(8+k^2)t-1$ and is positive when $k\ge 4$ for $\frac{8+k^2-k\sqrt{k^2+16}}{32}\le t\le 1$. As observed in the previous section, we do not have to determine which of the roots has absolute value greater or equal than 1.

We evaluate $\omega=\frac{\mathrm{d}Z}{2W}$. First we have

$$\mathrm{d}Z=-4(k^2-9)\frac{x_2+1}{(x_2-1)^3}\,\mathrm{d}x_2.$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}Z}{2W}&=-\frac{x_2+1}{2(2(x_2^2+x_2+1)y_2+kx_2(x_2+1))}\,\mathrm{d}x_2\\&=-\frac{x_0+x_0^{-1}}{2(x_0^2+1+x_0^{-2})y_0+k(x_0+x_0^{-1})}\frac{\mathrm{d}x_0}{x_0}.\end{aligned}$$

Writing $x_0 = e^{i\theta}$, this leads to

$$\omega = \pm \frac{\cos\theta i \,\mathrm{d}\theta}{\sqrt{k^2\cos^2\theta - (4\cos^2\theta - 1)^2}}.$$

Take $t = \cos^2\theta$ and $\mathrm{d}\theta = -\frac{\mathrm{d}t}{2\sqrt{t(1-t)}}$.

$$\omega = \pm \frac{i\,\mathrm{d}t}{2\sqrt{(1-t)(k^2t - (4t-1)^2)}}$$

We must consider for $k \geq 4$,

$$\int_{\varphi_*(|x_2|=1)} \omega = \pm \int_{\frac{8+k^2-k\sqrt{k^2+16}}{32}}^{1} \frac{i\,\mathrm{d}t}{\sqrt{(1-t)(k^2t-(4t-1)^2)}},$$

where the extra factor 2 comes from changing $0 \leq \theta \leq \pi$ to $0 \leq \theta \leq \frac{\pi}{2}$.

We consider $R_{k'}$. We write $y_3 = y_0 x_3$. Then we have

$$(x_3+1+x_3^{-1})y_0^2+(x_3^2+k'x_3+(2k'-4)+k'x_3^{-1}+x_3^{-2})y_0+(x_3+1+x_3^{-1}) = 0.$$

$$\begin{aligned} y_{0\pm} =& \frac{-(x_3^2+k'x_3+(2k'-4)+k'x_3^{-1}+x_3^{-2})}{2(x_3+1+x_3^{-1})} \\ &\pm \frac{\sqrt{\begin{aligned}&(x_3+2+x_3^{-1})(x_3+(k'-4)+x_3^{-1}) \\ &\quad\times\left(x_3^2+(k'+2)x_3+2(k'-1)+(k'+2)x_3^{-1}+x_3^{-2}\right)\end{aligned}}}{2(x_3+1+x_3^{-1})}. \end{aligned}$$

Write $x_3 = e^{i\theta}$ with $0 \leq \theta \leq 2\pi$,

$$\begin{aligned} y_{0\pm} =& \frac{-(4\cos^2\theta + 2k'\cos\theta + (2k'-6))}{2(1+2\cos\theta)} \\ &\pm \frac{\sqrt{2(1+\cos\theta)((k'-4)+2\cos\theta)(4\cos^2\theta+2(k'+2)\cos\theta+2(k'-2))}}{2(1+2\cos\theta)}. \end{aligned}$$

Write $t = \cos\theta$, the polynomial inside the square root is

$$2(1+t)((k'-4)+2t)(4t^2+2(k'+2)t+2(k'-2)).$$

For $k' \geq 6$, we must consider $\frac{-(k'+2)+\sqrt{k'^2-4k'+20}}{4} \leq t \leq 1$ for the polynomial inside the square root to be positive.

We evaluate $\omega = \frac{\mathrm{d}Z}{2W}$. We have

$$\mathrm{d}Z = -4(k'^2-k'-2)\frac{x_3+1}{(x_3-1)^3}\,\mathrm{d}x_3.$$

$$\frac{\mathrm{d}Z}{2W} = -\frac{x_3 + 2 + x_3^{-1}}{2(2(x_3 + 1 + x_3^{-1})y_0 + x_3^2 + k'x_3 + (2k' - 4) + k'x_3^{-1} + x_3^{-2})} \frac{\mathrm{d}x_3}{x_3}$$

$$\omega = \pm \frac{(1 + \cos\theta) i \,\mathrm{d}\theta}{\sqrt{2(1 + \cos\theta)((k' - 4) + 2\cos\theta)(4\cos^2\theta + 2(k' + 2)\cos\theta + 2(k' - 2))}}$$

Take $t = \cos\theta$, then

$$\omega = \pm \frac{i \,\mathrm{d}t}{2\sqrt{(1 - t)((k' - 4) + 2t)(2t^2 + (k' + 2)t + (k' - 2))}}.$$

Therefore, we must consider

$$\int_{\varphi_*(|x_3|=1)} \omega$$
$$= \pm \int_{\frac{-(k'+2)+\sqrt{k'^2-4k'+20}}{4}}^{1} \frac{i \,\mathrm{d}t}{\sqrt{(1 - t)((k' - 4) + 2t)(2t^2 + (k' + 2)t + (k' - 2))}},$$

where the extra factor 2 comes from changing $0 \le \theta \le 2\pi$ to $0 \le \theta \le \pi$.

Since we have $k = k' - 2$, we must prove for $k \ge 4$

$$\int_{\frac{8+k^2-k\sqrt{k^2+16}}{32}}^{1} \frac{\mathrm{d}t}{\sqrt{(1 - t)(k^2 t - (4t - 1)^2)}}$$
$$= \int_{\frac{-(k+4)+\sqrt{k^2+16}}{4}}^{1} \frac{\mathrm{d}s}{\sqrt{(1 - s)(2s + k - 2)(2s^2 + (k + 4)s + k)}}.$$

In fact, we can go from one side to the other by setting

$$t = \frac{(k + 1)s + k - 1}{2(2s + k - 2)}.$$

(This change of variables can be deduced from [2].)

Chapter 5

The Mahler Measure of A Genus 3 Family

The goal of this chapter is to provide a proof of Theorems 1.1.3 conjectured by Liu and Qin [22] by using the regulator theory. Here we rewrite Theorems 1.1.3 as follows.

Theorem 1.1.3. For a real $k \geq 2$, we have

$$\mathrm{m}(Q_k) = \mathrm{m}(P_k),$$

where

$$\begin{aligned} P_k(x,y) &= y^2 + \big(x^6 + kx^5 - x^4 + (2-2k)x^3 - x^2 + kx + 1\big)y + x^6, \\ Q_k(x,y) &= xy^2 + (kx-1)y - x^2 + x. \end{aligned} \tag{5.1}$$

Our proof strategy is similar to the one described at the beginning of Chapter 4. The major new idea for evaluating the regulator of the genus 3 curve is to use a modified version of Bosman's method (given by equation (5.4) as opposed to (5.3)) to simplify the evaluation of the diamond operator on $(x_1) \diamond (y_1)$.

The remainder of this chapter corresponds to Sections 3 and 4 of the article [21] by Matilde Lalín and Gang Wu: The Mahler measure of a genus 3 family. *Ramanujan J.*, 55(1):309-326, 2021 .

5.1 The regulator relationship

5.1.1 The genus 3 curve

We start by considering the regulator in

$$C_k : P_k(x_1, y_1) = 0,$$

where

$$P_k(x_1,y_1) = y_1^2 + (x_1^6 + kx_1^5 - x_1^4 + (2-2k)x_1^3 - x_1^2 + kx_1 + 1)y_1 + x_1^6.$$

A standard procedure to obtain a hyperelliptic equation from a polynomial of type (1.3) is to complete squares and write $(2A(x)y + B_k(x))^2 = B_k(x)^2 - 4A(x)C(x)$ and set $X = \frac{x+1}{x-1}$, $Y = \frac{2A(x)y+B_k(x)}{\delta(x,y)}$ for a conveniently chosen polynomial $\delta(x,y)$. In our case, the following birational transformation

$$\begin{aligned} X(x_1,y_1) =& \frac{x_1+1}{x_1-1}, \\ Y(x_1,y_1) =& \frac{8(2y_1 + (x_1^6 + kx_1^5 - x_1^4 + (2-2k)x_1^3 - x_1^2 + kx_1 + 1))}{(x_1^2-1)(x_1-1)^4}, \end{aligned}$$

leads to

$$\begin{aligned} Y^2 =& (k+2)X^8 + 4(k^2+3k+3)X^6 - 2(4k^2-3k-16)X^4 \\ &+ 4(k^2-5k+5)X^2 + k - 2 \\ =& (X^6 + (4k+5)X^4 + (11-4k)X^2 - 1)((k+2)X^2 + (2-k)). \end{aligned}$$

If we further set

$$\begin{aligned} Z =& \frac{4(X^2-1)}{(k+2)X^2 + (2-k)}, \\ W =& \frac{8Y}{((k+2)X^2 + (2-k))^2}, \end{aligned}$$

we obtain the family of elliptic curves

$$E_k : W^2 = Z^3 + (k^2-4)Z^2 - 8kZ + 16. \tag{5.2}$$

In sum, we have,

$$f : C_k \to C_k/\langle\sigma\rangle \cong E_k,$$

given by

$$\begin{aligned} Z(x_1,y_1) =& \frac{4x_1}{x_1^2 + kx_1 + 1}, \\ W(x_1,y_1) =& \frac{4(2y_1 + (x_1^6 + kx_1^5 - x_1^4 + (2-2k)x_1^3 - x_1^2 + kx_1 + 1))}{(x_1^2-1)(x_1^2+kx_1+1)^2} \\ =& \frac{4(y_1^2 - x_1^6)}{y_1(x_1^2-1)(x_1^2+kx_1+1)^2}. \end{aligned}$$

We can then write

$$\int_{\gamma_{P_k}} \eta_{C_k}(x_1, y_1) = \int_{f(\gamma_{P_k})} \eta_{E_k}(f_*(\{x_1, y_1\}),$$

where

$$\gamma_{P_k} = \{(x_1, y_1) \,:\, |x_1| = 1, |y_1| \geq 1\},$$

and our goal in this section is to evaluate $\eta_{E_k}(f_*(\{x_1, y_1\})$. For simplicity of notation, we will refer to $(x_1) \diamond (y_1)$ but we will think of x_1, y_1 as functions on Z, W. However, x_1, y_1 are rational functions on X, W and not on Z, W. Indeed, we have

$$\begin{aligned} x_1(X, W) =& \frac{X+1}{X-1}, \\ y_1(X, W) =& \frac{((2+k)X^2 + 2 - k)^2 XW - 2(X^6 + (8k+13)X^4 - (8k-19)X^2 - 1)}{2(X-1)^6}. \end{aligned}$$

In order to find their divisors in E_k, we follow a version of a modification of an idea of Bosman. First we recall Bosman's original result for completeness.

Lemma 5.1.1 ([4], Lemma 7 [20]). *Suppose that we have rational functions $a(Z, W), b(Z, W)$ such that*

$$a(X^2, W)x_1(X, W) + b(X^2, W)y_1(X, W) = 1. \qquad (5.3)$$

Then we have

$$r_{C_k}(\{x_1(X, W), y_1(X, W)\})([\gamma_{P_k}]) = -r_{E_k}(\{a(Z, W), b(Z, W)\})([f(\gamma_{P_k})]).$$

Proof. It suffices to see the identity at the level of the diamond operator, namely, to prove that

$$(x_1(X, W)) \diamond (y_1(X, W)) \sim -(a(X^2, W)) \diamond (b(X^2, W)).$$

Since the Steimberg symbol is trivial, $(f) \diamond (1 - f) = 0$, and

$$\begin{aligned} 0 \sim& (a(X^2, W)x_1(X, W)) \diamond (b(X^2, W)y_1(X, W)) \\ \sim& (a(X^2, W)) \diamond (b(X^2, W)) + (x_1(X, W)) \diamond (b(X^2, W)) \\ &+ (a(X^2, W)) \diamond (y_1(X, W)) + (x_1(X, W)) \diamond (y_1(X, W)). \end{aligned}$$

Now consider the automorphism σ of C_k given by $x_1 \to \frac{1}{x_1}$, $y_1 \to \frac{1}{y_1}$. We remark that $X \to -X$ and $W \to W$. Then

$$\begin{aligned} 0 \sim& (a(X^2, W)x_1(-X, W)) \diamond (b(X^2, W)y_1(-X, W)) \\ \sim& (a(X^2, W)) \diamond (b(X^2, W)) - (x_1(X, W)) \diamond (b(X^2, W)) \\ &- (a(X^2, W)) \diamond (y_1(X, W)) + (x_1(X, W)) \diamond (y_1(X, W)). \end{aligned}$$

We conclude by combining the above expressions. □

Instead of searching for an equation of the type (5.3), we will search for

$$a(X^2,W)x_1(X,W) + b(X^2,W)\frac{y_1(X,W)}{x_1(X,W)^3} = 1, \qquad (5.4)$$

where we have modified the y_1 component to $\frac{y_1}{x_1^3}$ in order to eliminate as much as possible the number of monomials with odd degree in X. Indeed, y_1 has the factor $(X-1)^6$ in the denominator, but this becomes $(X^2-1)^3$ in the denominator of $\frac{y_1}{x_1^3}$.

Lemma 5.1.2. *Let $a, b \in \mathbb{Q}(E)$ satisfying* (5.4). *We have*

$$r_{C_k}(\{x_1(X,W), y_1(X,W)\})([\gamma_{P_k}]) = -r_{E_k}(\{a(Z,W), b(Z,W)\})([f(\gamma_{P_k})]).$$

Proof. From Lemma 5.1.1 we have that

$$(a(Z,W)) \diamond (b(Z,W)) \sim -(x_1(X,W)) \diamond \left(\frac{y_1(X,W)}{x_1(X,W)^3}\right).$$

The result follows because

$$(x_1) \diamond \left(\frac{y_1}{x_1^3}\right) \sim (x_1) \diamond (y_1). \qquad \square$$

In our computations it will be convenient to introduce another variable to name the even powers of X:

$$Z_1 := X^2 = \frac{(k-2)Z-4}{(k+2)Z-4}.$$

For simplicity of notation, we also let

$$A(Z_1) = ((2+k)Z_1 + 2 - k)^2,$$
$$B(Z_1) = Z_1^3 + (8k+13)Z_1^2 - (8k-19)Z_1 - 1.$$

Then we write

$$y_1(X,W) = \frac{AXW - 2B}{2(X-1)^6}.$$

We have to solve

$$a\frac{(X+1)^2}{Z_1-1} + b\frac{AXW-2B}{2(Z_1-1)^3} = 1,$$

which leads to

$$\begin{cases} a(Z_1+1)(Z_1-1)^2 - bB &= (Z_1-1)^3, \\ 4a(Z_1-1)^2 + bAW &= 0, \end{cases}$$

and then

$$a(Z,W) = \frac{(Z_1-1)AW}{(Z_1+1)AW+4B} = \frac{4ZW}{2(4-kZ)W+Z^3+2(k^2-4)Z^2-16kZ+32},$$

$$b(Z,W) = -\frac{4(Z_1-1)^3}{(Z_1+1)AW+4B}$$

$$= -\frac{Z^3}{4(2(4-kZ)W+Z^3+2(k^2-4)Z^2-16kZ+32)}.$$

After ignoring constants, and grouping together terms, the diamond operation gives

$$\begin{aligned}(a)\diamond(b) =& 2(Z)\diamond(2(4-kZ)W+Z^3+2(k^2-4)Z^2-16kZ+32),\\ &-(W)\diamond(2(4-kZ)W+Z^3+2(k^2-4)Z^2-16kZ+32)+3(W)\diamond(Z).\end{aligned} \tag{5.5}$$

If we proceeded as usual (see, for example, [20]), we would compute the divisors $(Z),(W),(2(4-kZ)W+Z^3+2(k^2-4)Z^2-16kZ+32)$. With the exception of (Z), these divisors are supported in non-rational points and it is difficult to find relationships among them. Instead of directly computing the divisors, we consider some further manipulations. Notice from (5.2) that

$$2(4-kZ)W+Z^3+2(k^2-4)Z^2-16kZ+32 = 2(4-kZ+W)W-Z^3.$$

We consider the following trivial symbol (trivial because it is of the form $(g)\diamond(1-g)$):

$$\begin{aligned}0 \sim& \left(\frac{2(4-kZ+W)W}{Z^3}\right)\diamond\left(\frac{Z^3-2(4-kZ+W)W}{Z^3}\right)\\ \sim&(4-kZ+W)\diamond(2(4-kZ+W)W-Z^3)-3(4-kZ+W)\diamond(Z)\\ &+(W)\diamond(2(4-kZ+W)W-Z^3)-3(W)\diamond(Z)\\ &-3(Z)\diamond(2(4-kZ+W)W-Z^3).\end{aligned}$$

Combining with (5.5), we have,

$$\begin{aligned}(a)\diamond(b) =& 2(Z)\diamond(2(4-kZ+W)W-Z^3)-(W)\diamond(2(4-kZ+W)W-Z^3)\\ &+3(W)\diamond(Z)\\ \sim& 2(Z)\diamond(2(4-kZ+W)W-Z^3)+(4-kZ+W)\diamond(2(4-kZ+W)W-Z^3)\\ &-3(4-kZ+W)\diamond(Z)-3(Z)\diamond(2(4-kZ+W)W-Z^3)\\ \sim&\left(\frac{4-kZ+W}{Z}\right)\diamond(2(4-kZ+W)W-Z^3)-3(4-kZ+W)\diamond(Z).\end{aligned} \tag{5.6}$$

The advantage of working with (5.6) as opposed to (5.5) is that we do not have to consider the divisor (W) anymore. Instead, we will compute the divisor $(4 - kZ + W)$, supported on rational points.

The family E_k given by (5.2) has a point $P = (0, 4)$ of infinite order that satisfies $2P = (4, 4(k - 1))$, $3P = (4(1 - k), 4(k^2 - 3k + 1))$.

$$(Z) = (P) + (-P) - 2O, \tag{5.7}$$
$$(4 - kZ + W) = 2(-P) + (2P) - 3O, \tag{5.8}$$
$$(2(4 - kZ + W)W - Z^3) = 2(-P) + (U) + (P - U) + (V) + (P - V) - 6O, \tag{5.9}$$

for certain points U, V. We remark that

$$\left(\frac{4 - kZ + W}{Z}\right) = (-P) + (2P) - (P) - O,$$

and

$$\left(\frac{4 - kZ + W}{Z}\right) \diamond (2(4 - kZ + W)W - Z^3) = 10(P) - 8(2P) + 2(3P),$$

since the terms involving U cancel themselves, and the same applies for the terms involving V.

From (5.6), we obtain

$$\begin{aligned}(a) \diamond (b) \sim & 10(P) - 8(2P) + 2(3P) - 3(5(P) - 4(2P) + (3P)) \\ = & -(5(P) - 4(2P) + (3P)).\end{aligned}$$

Finally,

$$(x_1) \diamond (y_1) \sim 5(P) - 4(2P) + (3P).$$

For the elliptic curve E_k expressed by (5.2):

$$E_k : W^2 = Z^3 + (k^2 - 4)Z^2 - 8kZ + 16,$$

$E_k(\mathbb{R})$ is connected if and only if the polynomial $Z^3 + (k^2 - 4)Z^2 - 8kZ + 16$ has only one real root. Notice that this is a cubic polynomial. So, if $E_k(\mathbb{R})$ is not connected, then $E_k(\mathbb{R})$ has two connected components. By (2.15), we have an identity:

$$\int_{\gamma_{P_k}} \eta_{C_k}(x_1, y_1) = m_k c_k D^{E_k}(5(P) - 4(2P) + (3P)), \tag{5.10}$$

where c_k is a constant defined by

$$[f(\gamma_{P_k})] = c_k[\gamma_k],$$

$[\gamma_k]$ is a generator of $H_1(E_k, \mathbb{Z})^-$ and m_k is given by

$$m_k = \begin{cases} 1, & \text{if } E_k(\mathbb{R}) \text{ is connected,} \\ 2, & \text{if } E_k(\mathbb{R}) \text{ is not connected} \end{cases} \tag{5.11}$$

that obtained by the intersection product $\gamma_{E_k}^+ \bullet \gamma_k$.

Recall that our goal is to prove Theorem 1.1.3, and therefore it suffices to consider $k \geq 2$. One can see that the cubic polynomial $Z^3 + (k^2 - 4)Z^2 - 8kZ + 16$ has one real root when $k \geq 2$, and therefore $E_k(\mathbb{R})$ is connected. This implies that $m_k = 1$ for $k \geq 2$.

5.1.2 The genus 1 curve

We now consider the genus 1 family given by

$$Q_k(x_2, y_2) = x_2 y_2^2 + (kx_2 - 1)y_2 - x_2^2 + x_2.$$

The following birational transformation

$$\begin{aligned} Z(x_2, y_2) &= 4x_2, & x_2(Z, W) &= \frac{Z}{4}, \\ W(x_2, y_2) &= 4(2x_2y_2 + kx_2 - 1), & y_2(Z, W) &= \frac{W - kZ + 4}{2Z}, \end{aligned}$$

leads directly to the Weierstrass form (5.2)

$$W^2 = Z^3 + (k^2 - 4)Z^2 - 8kZ + 16.$$

We can compute the relevant divisors by using equations (5.7) and (5.8),

$$\begin{aligned} (x_2) &= (P) + (-P) - 2O, \\ (y_2) &= (-P) + (2P) - (P) - O, \end{aligned}$$

Then

$$(x_2) \diamond (y_2) = -5(P) + 4(2P) - (3P).$$

Again, by (2.15), we have an identity:

$$\int_{\gamma_{Q_k}} \eta_{E_k}(x_2, y_2) = -m_k d_k D^{E_k}(5(P) - 4(2P) + (3P)), \tag{5.12}$$

where d_k is a constant defined by

$$[\gamma_{Q_k}] = d_k[\gamma_k],$$

$[\gamma_k]$ is a generator of $H_1(E_k,\mathbb{Z})^-$ and m_k is the constant given by (5.11).

We remark that the right-hand side of equation (5.12) is the same as the left-hand side of (5.13):

$$5D^E(P) - 4D^E(2P) + D^E(3P) = -4\pi L'(E_2,0) \tag{5.13}$$

found by Zagier and Gangl [36] for $k = 2$. Combining this with (2.19), we get that $\mathrm{m}(Q_2)$ is $L'(E_2,0)$ multiplied by a rational number (that is expected to be $-1/2$).

5.2 The cycles of integration

In this section we consider the relationship between the cycles of integration. From (5.10) and (5.12), we understand the relationship between the regulators. It remains to compare the cycles of integration, namely, to find the relationship between c_k and d_k. It suffices to compare the integral of the holomorphic differential in E_k respect to each cycle. The strategy is to evaluate $\omega = \frac{\mathrm{d}Z}{W}$ both in terms of x_1, y_1 and x_2, y_2, integrate over $f(\gamma_{P_k})$ and γ_{Q_k} respectively, and compare both integrals.

In our calculations we ignore the sign in front, since the Mahler measure is always non-negative.

5.2.1 The genus 3 curve

Since $P_k(x,y)$ is reciprocal, the path $f(\gamma_{P_k})$ to be considered corresponds to a fixed choice of a root y_+ or y_-. We do not need to specify the choice, since working with the wrong root will only lead to the opposite sign in the integral of the holomorphic differential. We have,

$$\begin{aligned}\frac{\mathrm{d}Z}{W} &= -\frac{y_1(1-x_1^2)^2\,\mathrm{d}x_1}{y_1^2 - x_1^6}\\ &= \pm\frac{(1-x_1^2)^2\,\mathrm{d}x_1}{\sqrt{(x_1^6+kx_1^5-x_1^4+(2-2k)x_1^3-x_1^2+kx_1+1)^2-4x_1^6}}\\ &= \pm\frac{(1-x_1^2)\,\mathrm{d}x_1}{\sqrt{(x_1^2+kx_1+1)(x_1^6+kx_1^5-x_1^4+2(2-k)x_1^3-x_1^2+kx_1+1)}}.\end{aligned}$$

We see from the change of variables $x_1 \to \frac{1}{x_1}$ that the integral over $|x_1| = 1$ is purely imaginary.

By writing $x_1 = e^{i\theta}$, we have

$$\frac{\mathrm{d}Z}{W} = \pm\frac{\sin\theta\,\mathrm{d}\theta}{\sqrt{(2\cos\theta+k)(2\cos^3\theta+k\cos^2\theta-2\cos\theta+1-k)}}.$$

Setting $t = \cos\theta$,

$$\frac{\mathrm{d}Z}{W} = \pm \frac{\mathrm{d}t}{\sqrt{(2t+k)(2t^3+kt^2-2t+1-k)}}.$$

Now set $s = \frac{1}{2t+k}$,

$$\frac{\mathrm{d}Z}{W} = \pm \frac{\mathrm{d}s}{\sqrt{4s^3+(k^2-4)s^2-2ks+1}}.$$

For $k > 1$, the polynomial $p(s)$ inside the square root has one negative root θ_0 and two roots θ_1, θ_2 between 0 and 1 (where $\theta_2 = 1$ for $k = 1$). More precisely, assume that $k > 2$, then $p(-1) = (k+1)^2 - 8 > 0$, $p\left(\frac{1}{k+2}\right) = \frac{4}{(k+2)^3} > 0$, $p\left(\frac{1}{k}\right) = \frac{4(1-k)}{k^3} < 0$, $p\left(\frac{1}{k-2}\right) = \frac{4}{(k-2)^3} > 0$. In addition notice that $p(1) = (k-1)^2 > 0$. In conclusion, for $k > 2$, $p(s)$ has three real roots satisfying

$$\theta_0 < -1 < 0 < \frac{1}{k+2} < \theta_1 < \frac{1}{k} < \theta_2 < \min\left\{1, \frac{1}{k-2}\right\}.$$

And the above is also true for $k = 2$ by taking the limit.

Then we must integrate

$$\begin{aligned}\int_{\gamma_{P_k}} \omega(Z(x_1,y_1), W(x_1,y_1)) &= \pm\, i2\,\mathrm{im}\left(\int_{\frac{1}{k+2}}^{\frac{1}{k-2}} \frac{\mathrm{d}s}{\sqrt{4s^3+(k^2-4)s^2-2ks+1}}\right)\\ &= \pm\, 2\int_{\theta_1}^{\theta_2} \frac{\mathrm{d}s}{\sqrt{4s^3+(k^2-4)s^2-2ks+1}},\end{aligned} \tag{5.14}$$

where we have multiplied by 2 because the change of variable $t = \cos\theta$ implies that there are two values of x_1 yielding the same value of s.

5.2.2 The genus 1 curve

Since $Q_k(x_2, y_2)$ is not reciprocal, we must first verify that the integration path γ_{Q_k} is closed. First we prove that $Q_k(x_2, y_2) = 0$ does not intersect the unit torus $\{|x_2| = |y_2| = 1\}$ for $k \geq 2$. This means that $|y_{2,+}|$ and $|y_{2,-}|$ stay always > 1 or < 1 while $|x| = 1$. The case $k = 2$ will then follow by continuity. We start by making the change $x_3 = x_2/y_2$ and by writing the equation as

$$x_3 - (y_2 + k + y_2^{-1}) + x_3^{-1}y_2^{-1} = 0.$$

We look for a solution with $|x_3| = |y_2| = 1$. Assuming such solution exists, it must also verify that

$$x_3^{-1} - (y_2 + k + y_2^{-1}) + x_3 y_2 = 0.$$

By combining both equations, we obtain

$$x_3(1 - y_2) = x_3^{-1}(1 - y_2^{-1}),$$

and this implies that either $y_2 = 1$ or $x_3^2 y_2 = -1$. In the first case, we get $x_3 + x_3^{-1} = 2 + k$, which has no solution in $|x_3| = 1$ for $k > 0$. In the second case we get $x_3^{-2} - k + x_3^2 = 0$, which has no solution on $|x_3| = 1$ for $k > 2$ (for $k = 2$ the only solutions are $x_3 = \pm 1$).

This proves that the paths $\{|x_2| = 1, |y_{2+}| \geq 1\}$ and $\{|x_2| = 1, |y_{2-}| \geq 1\}$ are either closed or empty. Notice that $|y_{2+} y_{2-}| = |1 - x_2|$. Since $|1 - x_2| > 1$ for $x_2 = -1$, we conclude that at least one of these cycles is not empty, since we must have $|y_{2\pm}| > 1$ for a certain choice of the sign. On the other hand $|1 - x_2| < 1$ for $x_2 = 1$ and we conclude that at least one of these cycles is empty since we must have $|y_{2\pm}| < 1$ for a certain choice of the sign. Thus, we obtain exactly one nonempty cycle γ_{Q_k}.

We have,

$$\frac{\mathrm{d}Z}{W} = \frac{\mathrm{d}x_2}{2x_2 y_2 + kx_2 - 1} = \pm \frac{\mathrm{d}x_2}{\sqrt{4x_2^3 + (k^2 - 4)x_2^2 - 2kx_2 + 1}}.$$

Then we must integrate

$$\int_{\gamma_{Q_k}} \omega(Z(x_2, y_2), W(x_2, y_2)) = \pm \int_{|x_2|=1} \frac{\mathrm{d}x_2}{\sqrt{4x_2^3 + (k^2 - 4)x_2^2 - 2kx_2 + 1}}, \tag{5.15}$$

5.2.3 The end of the proof

To prove that $c_k = \pm d_k$, we must combine equations (5.14) and (5.15) and prove

$$\pm 2 \int_{\theta_1}^{\theta_2} \frac{\mathrm{d}s}{\sqrt{4s^3 + (k^2 - 4)s^2 - 2ks + 1}} = \int_{|x_2|=1} \frac{\mathrm{d}x_2}{\sqrt{4x_2^3 + (k^2 - 4)x_2^2 - 2kx_2 + 1}},$$

but this is true because $\theta_0 < -1 < 0 < \theta_1 < \theta_2 < 1$ and therefore, exactly θ_1 and θ_2 are in the interior of the cycle $|x_2| = 1$. Thus, integrating over $|x_2| = 1$ gives the complex period of E_k, which is twice the semi-period obtained by integrating between θ_1 and θ_2.

By combining equations (5.10) and (5.12), we conclude the proof of Theorem 1.1.3.

Chapter 6

Conclusion

This dissertation studies four cases of Boyd's conjectures, involving curves of genus 1, 2 and 3. The results have been presented in Chapters 3, 4 and 5.

Instead of direct computations of the Mahler measure, we used some alternative approaches to relate the Mahler measure to other objects or to the Mahler measure of a different polynomial. For the genus 1 curve in Chapter 3, we interpreted the Mahler measure as the volume of a hyperbolic 3-manifold. For the genus 2 and 3 curves in Chapters 4 and 5, we used the regulator method to prove identities between Mahler measures of different families of polynomials. These approaches work very well in our cases and we expect that these ideas can be applied to other polynomial families. Unfortunately, these approaches are far from omnipotent. We failed to prove other identities in several attempts with these techniques. Here are two identities conjectured by Boyd [5] that we cannot prove.

$$\begin{aligned} &\mathrm{m}(P) \stackrel{?}{=} \frac{d_{40}}{6} \\ &\text{with } P(x,y) = (x^2-x+1)(x^2+1)(y^2+1) + x(14x^2-32x+14)y, \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} &\mathrm{m}(P) \stackrel{?}{=} \frac{d_{120}}{36} \\ &\text{with } P(x,y) = (x^2+1)(x^2+x+1)(y+1)^2 - 24x^2y, \end{aligned} \tag{6.2}$$

where

$$d_f = \frac{f^{3/2}}{4\pi} L(\chi_{-f}, 2).$$

We summarize some reasons and factors that hinder our success as follows.

For all the cases we considered in this dissertation, we obtain an elliptic curve $Y^2 = h(Z)$ with $Z = X^2$ from the original curve $P(x, y) = 0$ by the changes of variables $(x, y) \mapsto (X, Y)$. Recall that for the genus 1 curve in Chapter 3, we need Eq (3.14) to calculate $x(X, Y) \wedge y(X, Y)$. In order to obtain Eq (3.14), the curve should have following involution:

$$\begin{cases} x \to 1/x \\ y \to 1/y \end{cases} \iff \begin{cases} X \to -X \\ Y \to Y \end{cases} . \tag{6.3}$$

Condition (6.3) is also key for the application of Lemmas 4.1.1 and 5.1.1 and is therefore essential for the successful proofs of the conjectures in Chapters 4 and 5.

In all of Boyd's conjectures, the polynomial $P(x, y)$ is reciprocal. From the reciprocity of the polynomial, the transformation $x \to \frac{1}{x}$, $y \to \frac{1}{y}$ may also lead to $X \to -X$, $Y \to Y$ or $X \to -X$, $Y \to -Y$. However, the latter case may be fatal for the success of the proof. For example, in the cases (6.1) and (6.2), the transformation $x \to \frac{1}{x}$, $y \to \frac{1}{y}$ leads to $X \to -X$, $Y \to -Y$. This prevents us from using Eq (3.14) to calculate $x(X, Y) \wedge y(X, Y)$.

Even if the condition (6.3) is met, the computations may also be very difficult. In most cases, the divisors of the rational functions under consideration are very complicated, or are supported in points that are difficult to understand. So we have to analyze case by case to see the feasibility of the approach of proof.

It would be very interesting to see if these methods can be extended to prove other similar results, particularly some of those conjectured by Liu and Qin [23]. One could consider identities between different genus 3 families or between a genus 3 family and a genus 2 family. Some concrete examples can be found in [23, Section 3.4 and 4.3].

It would also be interesting to find an example relating a high genus curve with a genus 1 curve, for a case where the Mahler measure of the genus 1 curve is actually proven. Unfortunately, for the case in Chapter 5, no connection with $L'(E, 0)$ has been proven for any value of $k > 2$.

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