Student-Optimal Interdistrict School Choice: District-Based versus School-Based Admissions*

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Abstract

Hafalir, Kojima and Yenmez (2022) introduce a model of interdistrict school choice: each district consists of a set of schools and the district's admission rule places applicants to the schools in the district. We show that any district's admission rule satisfying their assumptions is uniquely rationalized by a collection of schools' choice functions satisfying substitutability and acceptance. We then establish that all students weakly prefer the outcome of the cumulative offer process (COP) under the school-based admissions to the outcome under the district-based admissions. This has the implication that if students prefer the interdistrict outcome for the district-based admissions to the intradistrict outcome, then all students are weakly better off under the school-based admissions compared to either of these outcomes. Therefore, for student-optimal interdistrict school choice the introduction of district admission rules hurts students and it suffices to endow schools with usual choice priorities (if students' welfare is more important than districts' policy goals) and to (de)centralize district admissions by letting schools choose.

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1 Introduction

Until the 1990s children were assigned to public schools in the district where their parents live in. Rich families could choose the district where their children will go to school by moving to that district. Poor families did not have this choice. Since "better" public schools are often located in richer neighborhoods, children from poor families did not have access to those schools. This inequality of opportunity between rich and poor families triggered U.S. cities to establish centralized interdistrict school choice programs. Such a program offers students (or children) the possibility to be admitted at schools which are not located in the district in which they live.

Abdulkadiroğlu and Sönmez (2003) introduce the model of (public) school choice assigning students to schools on the basis of schools' priorities and students' (or parents') preferences. Only recently Hafalir, Kojima and Yenmez (2022) have provided a precise formulation of interdistrict school choice: each district consists of a set of (district) schools and a set of (home district) students. Students have strict preferences over schools and districts have admission rules to admit students from their district and outside the district.

The modelling of intradistrict school choice uses matching with contracts (Hatfield and Milgrom, 2005). Any contract consists of a student, a district and a school in this district. Students have strict preferences over contracts and schools have choice functions over sets of contracts. Any school's choice function is feasible in the sense that each student is associated with at most one chosen contract, the chosen set of students does not exceed its capacity, and only contracts associated with the school are chosen. The law of aggregate demand (LAD) says that from a bigger set of contracts weakly more contracts are chosen compared to the smaller set. Substitutability means that any chosen contract remains chosen when other contracts are removed. LAD and substitutability have proven to be essential for the cumulative offer process (COP) to be stable (in the sense that the outcome is not blocked by a student and a school) and strategy-proof in the sense that no student gains from misrepresenting his preferences. In addition, under these conditions COP is outcome equivalent to the student-proposing deferred-acceptance algorithm (SPDA) (first defined by Gale and Shapley (1962)).

In a recent contribution Hatfield and Kominers (2014) weakened substitutability to the requirement that any school's choice function has a completion which is LAD and substitutable. Here a completion of a choice function is another choice function which chooses either the same set as the original choice function or several contracts associated with the same student. Hafalir, Kojima and Yenmez (2022) endow each district with an admission rule which is feasible for the schools in the district, acceptant and has a completion satisfying LAD and substitutability. We show that then it is possible to deduce for each school its choice function satisfying acceptance and substitutability. In other words, any admission rule satisfying their assumptions can be rationalized via a choice function profile of the district's schools. The union of these choice functions is an expansion of any completion (which satisfies LAD and substitutability) of the district's admission rule. This in turn implies that all students are weakly better off under the deduced school admission priorities compared to the district admission priorities. For policy implications, this means for the district not to impose any admission rules if students' welfare is more important than districts' welfare, i.e. to decentralize admissions by letting schools choose their students. For the student-optimal interdistrict COP it suffices to endow schools with usual choice priorities satisfying acceptance and substitutability. Furthermore, this rationalization is canonical in the sense that two district admission rules with the same rationalization yield the same outcomes for the student-optimal interdistrict COP (when fixing the other districts' admission rules). Endowed with these results, we show that several policy goals for stable and strategy-proof mechanisms need to be imposed (implicitly) at the school level.

Any policy goal of the district is incorporated in its admission rule. Four main policy goals are formulated for the student-proposing deferred-acceptance mechanism (SPDA) by Hafalir, Kojima and Yenmez (2022): (i) individual rationality meaning that any student prefers his assigned school to his initial (walk-zone) school, (ii) balanced exchange meaning that any district sends outside and receives the same numbers of students, (iii) all students prefer the interdistrict outcome to the intradistrict outcome and (iv) diversity meaning that student bodies respect certain desegregation policies. All these policy goals are formulated in terms of the districts' admission rules. We show that when the policy goals (i), (iii) and (iv) hold for the districts' admission rules, then they also hold for the deduced schools' choice functions. Therefore, those policy goals are easier to achieve at the school level than at the district level.

The paper is organized as follows. Section 2 introduces the model and states our main results. Section 3 applies those results for district-based and school-based admissions for intradistrict and interdistrict school choice. Section 4 discusses further district-level policy goals and whether those have to be imposed (implicitly) at the school level or not. The Appendix contains all proofs omitted from the main text.

2 The Model

We follow Hafalir, Kojima and Yenmez (2022). There exist finite sets of students $\mathcal{S}$, districts $\mathcal{D}$ and schools $\mathcal{C}$. Each student s and school c has a home district denoted by $d(s)$ and

$d(c)$, respectively. For later purposes, let $\mathcal{C}_d$ denote set of all schools located in district d. We assume that $\mathcal{D}$ contains at least two districts (as otherwise there is only intradistrict school choice), $|\mathcal{C}_d| \geq 2$ for at least one district d (as otherwise each district contains only one school) and any district contains at least one student.

Each school c has a capacity of q_c. For each district d, let k_d denote the number of students whose home district is d. Each district d has sufficiently large capacity to accommodate all students in its district, i.e. $k_d \leq \sum_{c \in \mathcal{C}_d} q_c$, and each district's number of home students exceeds the capacity of any of its district schools, i.e. $k_d \geq q_c$ for all $c \in \mathcal{C}_d$.

A contract $x = (s, d, c)$ specifies a student, a district and a school c in this district, i.e. $d(c) = d$. For any contract x, let $s(x)$, $d(x)$ and $c(x)$ denote the student, district and school associated with x. Let $\mathcal{X} = \{(s, d, c) | d(c) = d\}$ denote the set of all contracts. For any $X \subseteq \mathcal{X}$, let X_s denote the set of all contracts in X associated with s, i.e. $X_s = \{x \in X | s(x) = s\}$, and $X_{\mathcal{S}} = \cup_{s \in \mathcal{S}} X_s$. Similarly, we define X_d and X_c for any district d and for any school c.

2.1 Admission Rules

Each district d has an admissions rule that is represented by a choice function Ch_d choosing from any set of contracts a subset of contracts associated with itself: given $X \subseteq \mathcal{X}$, we have $Ch_d(X) = Ch_d(X_d) \subseteq X_d$.

A matching is a set of contracts. A matching X is feasible for students if $|X_s| \leq 1$ for all $s \in \mathcal{S}$. A matching X is feasible if it is feasible for students and $|X_c| \leq q_c$ for all $c \in \mathcal{C}$.

The following properties are important for admission rules:

(i) Ch_d is feasible if it always chooses a feasible matching;

(ii) Ch_d is acceptant if for any matching X that is feasible for students, $x \in X_d \setminus Ch_d(X)$ implies $|\{y \in Ch_d(X) | c(y) = c(x)\}| = q_{c(x)}$ or $|Ch_d(X)| \geq k_d$;

(iii) Ch_d is substitutable if for all $x \in X \subseteq Y \subseteq \mathcal{X}$, $x \in Ch_d(Y)$ implies $x \in Ch_d(X)$;

(iv) Ch_d is LAD (law of aggregate demand) if for all $X \subseteq Y \subseteq \mathcal{X}$, $|Ch_d(X)| \leq |Ch_d(Y)|$;

(v) Ch_d has a completion if there exists another admissions rule Ch'_d such that for every matching X either $Ch'_d(X) = Ch_d(X)$ or $Ch'_d(X)$ is not feasible for students;[1]

(vi) Ch'_d is an expansion of Ch_d if for every matching X we have $Ch'_d(X) \supseteq Ch_d(X)$; and

[1] This concept has been introduced by Hatfield and Kominers (2014) for many-to-one matching. Yenmez (2018) proposed a more general approach for many-to-many matching.

(vii) Ch_d is strongly acceptant if for any matching X that is feasible for students, $x \in X_d \backslash Ch_d(X)$ implies $|\{y \in Ch_d(X)|c(y) = c(x)\}| = q_{c(x)}$.

Note that strong acceptance implies acceptance.

Analogously, a choice function Ch_c for school c chooses from any set of contracts a subset of contracts associated with itself: given $X \subseteq \mathcal{X}$, we have $Ch_c(X) = Ch_c(X_c) \subseteq X_c$. The choice function Ch_c is q_c-acceptant if for any $X \subseteq \mathcal{X}$ we have $|Ch_c(X)| = \min\{q_c, |X_c|\}$, i.e. Ch_c only rejects contracts associated with it if the capacity of school c is exhausted. We call Ch_c acceptant if Ch_c is q_c-acceptant.

Hafalir, Kojima and Yenmez (2022) assume that Ch_d is feasible and acceptant and has a completion satisfying substitutability and LAD. Our first main result determines for each admission rule a rationalization of it by schools' choice functions such that its completion is a subcorrespondence of the union of the schools' choice functions.

Proposition 1. *Let Ch_d be feasible and acceptant and have a completion satisfying substitutability and LAD. For any $c \in \mathcal{C}_d$ and any $X \subseteq \mathcal{X}$, we set*

$$Ch_d(X_c) \equiv Ch_c(X_c) = Ch_c(X). \tag{1}$$

Then the following holds:

(i) Ch_d induces in (1) a unique choice function profile $(Ch_c)_{c\in\mathcal{C}_d}$ such that each Ch_c satisfies substitutability and q_c-acceptance.

(ii) $\cup_{c\in\mathcal{C}_d} Ch_c$ is the unique strongly acceptant, substitutable and LAD expansion of any completion of Ch_d satisfying substitutability and LAD.

(iii) If $k_d = \sum_{c\in\mathcal{C}_d} q_c$, then $\cup_{c\in\mathcal{C}_d} Ch_c$ is a completion of Ch_d satisfying substitutability and LAD.

Proof. (i): Let Ch_d be feasible and acceptant, and let Ch'_d be a completion of Ch_d satisfying substitutability and LAD. Let $c \in \mathcal{C}_d$. For any $X \subseteq \mathcal{X}$, we have

$$Ch'_d(X_c) = Ch_d(X_c) = Ch_c(X_c) = Ch_c(X) \tag{2}$$

where the first equality follows from the fact that $Ch_d(X_c)$ and $Ch'_d(X_c)$ are feasible for students and Ch'_d is a completion of Ch_d and the remaining equalities from (1). Since Ch'_d is substitutable and LAD, Ch_c is substitutable and LAD. As $q_c \leq k_d$ the constructed Ch_c is q_c-acceptant: as Ch_d is acceptant we have $|Ch_d(X_c)| = \min\{q_c, |X_c|\}$ for any $X_c \subseteq \mathcal{X}_c$ and the claim follows from (2). Furthermore, by (2), Ch_c is uniquely defined by Ch_d.

(ii): We show $Ch'_d \subseteq \cup_{c\in\mathcal{C}_d} Ch_c$. Let $X \subseteq \mathcal{X}$. Then $Ch'_d(X) = Ch'_d(X_d)$. Note that $X_d = \cup_{c\in\mathcal{C}_d} X_c$. By substitutability of Ch'_d,

$$Ch'_d(X_d) \cap X_c \subseteq Ch'_d(X_c) = Ch_c(X_c) = Ch_c(X)$$

where the last two equalities follows from (2). Thus, $Ch'_d(X) \subseteq \cup_{c\in\mathcal{C}_d} Ch_c(X)$.

For any $c \in \mathcal{C}_d$, we have $q_c \leq k_d$. Now by (2) and acceptance of Ch_d, we obtain that for all $X \subseteq \mathcal{X}$, $x \in X_c \backslash Ch_c(X)$ implies $|Ch_c(X)| = q_c$. Hence, $\cup_{c\in\mathcal{C}_d} Ch_c$ is a feasible and strongly acceptant expansion of Ch'_d. Furthermore, by (i), $\cup_{c\in\mathcal{C}_d} Ch_c$ is substitutable and LAD.

For uniqueness, let Ch''_d be a strongly acceptant expansion of Ch'_d satisfying substitutability and LAD. Then by strong acceptance of Ch''_d, we have for any $X_c \subseteq \mathcal{X}_c$, $Ch''_d(X_c) = X_c$ (if $|X_c| \leq q_c$) or $|Ch''_d(X_c)| = q_c$. Note that the same holds for Ch_c by strong acceptance. Hence, by (2) and the fact that Ch''_d is an expansion of Ch'_d, we have for any $X \subseteq \mathcal{X}$ and any $c \in \mathcal{C}_d$, $Ch''_d(X_c) = Ch_c(X_c)$. Because both Ch''_d and $\cup_{c\in\mathcal{C}_d} Ch_c$ are strongly acceptant, we have for any $X \subseteq \mathcal{X}$, $Ch''_d(X) \cap X_c = X_c = Ch_c(X_c)$ (if $|X_c| \leq q_c$) or $|Ch''_d(X) \cap X_c| = q_c = |Ch_c(X_c)|$ (if $|X_c| > q_c$). By (2), we have in both cases $Ch''_d(X) \cap X_c = Ch_c(X_c)$. By substitutability and LAD of Ch''_d, we obtain

$$Ch''_d(X) = \cup_{c\in\mathcal{C}_d}(Ch''_d(X) \cap X_c) = \cup_{c\in\mathcal{C}_d} Ch_c(X_c) = \cup_{c\in\mathcal{C}_d} Ch_c(X),$$

which implies $Ch''_d = \cup_{c\in\mathcal{C}_d} Ch_c$, the desired conclusion.

(iii): Let $k_d = \sum_{c\in\mathcal{C}_d} q_c$. Let $X \subseteq \mathcal{X}$ be a matching. If $Ch'_d(X)$ is not feasible for students, then by $Ch'_d(X) \subseteq \cup_{c\in\mathcal{C}_d} Ch_c(X)$, $\cup_{c\in\mathcal{C}_d} Ch_c(X)$ is not feasible for students. If $Ch'_d(X)$ is feasible for students, then as Ch'_d is a completion of Ch_d, $Ch_d(X_d) = Ch_d(X) = Ch'_d(X) = Ch'_d(X_d)$. As Ch_d is feasible, $Ch_d(X_d)$ is a feasible matching and $|Ch'_d(X_d) \cap X_c| \leq q_c$ for all $c \in \mathcal{C}_d$. We distinguish two cases for any $c \in \mathcal{C}_d$.

If $|Ch'_d(X_d) \cap X_c| = q_c$, then

$$Ch'_d(X_d) \cap X_c = Ch'_d(X_c) = Ch_c(X_c) = Ch_c(X),$$

where the first equality follows from substitutability and LAD of Ch'_d and the fact that $|Ch'_d(X_c)| \leq q_c$ (as $Ch'_d(X_c)$ is necessarily feasible for students, $Ch'_d(X_c) = Ch_d(X_c)$ from Ch'_d being a completion of Ch_d and $|Ch_d(X_c)| \leq q_c$ from feasibility of Ch_d) and $\sum_{c\in\mathcal{C}_d} q_c = k_d$, and the last two equalities follow from the definition of Ch_c.

If $|Ch'_d(X_d) \cap X_c| < q_c$, then

$$Ch'_d(X_d) \cap X_c = Ch'_d(X_c) = Ch_c(X_c) = Ch_c(X),$$

where the first equality follows from substitutability and LAD of Ch'_d and the fact that $|Ch'_d(X_c)| < q_c$ (as $Ch'_d(X_c)$ is necessarily feasible for students and $Ch'_d(X_c) = Ch_d(X_c)$ from Ch'_d being a completion of Ch_d and $Ch_d(X_c) = X_c$ by acceptance of Ch_d and $k_d \geq q_c$), and the last two equalities follow from the definition of Ch_c.

Thus,

$$Ch_d(X) = Ch'_d(X) = \cup_{c \in \mathcal{C}_d}(Ch'_d(X_d) \cap X_c) = \cup_{c \in \mathcal{C}_d} Ch_c(X),$$

which is the desired conclusion. □

Proposition 1 (Part (ii)) implies that $\cup_{c \in \mathcal{C}_d} Ch_c$ is the "largest strongly acceptant choice function containing all completions satisfying substitutability and LAD" of any district admission rule (inducing the same schools' choice functions). From now on we will call the choice function profile $(Ch_c)_{c \in \mathcal{C}_d}$ **the rationalization of** Ch_d. Note that any Ch_d, which has a completion satisfying substitutability and LAD, has a unique rationalization, and the set of all rationalizations induces a partition of the set of district admission rules which have a completion satisfying substitutability and LAD.

The following is immediate from Proposition 1.

Corollary 1. *Let Ch_d be feasible and strongly acceptant. Then Ch_d has a completion satisfying substitutability and LAD if and only if there exists a choice function profile $(Ch_c)_{c \in \mathcal{C}_d}$ such that each Ch_c satisfies substitutability and acceptance and $\cup_{c \in \mathcal{C}_d} Ch_c$ is a completion of Ch_d.*

In other words, Ch_d has then a unique completion satisfying substitutability and LAD.

The following shows that in Proposition 1 (Part (iii)) the assumption $k_d = \sum_{c \in \mathcal{C}_d} q_c$ is necessary.[2]

Example 1. *There are four students, s_1, s_2, s_3, and s_4. Let district d have two schools, c and c', with sufficiently large capacity, say $q_c = q_{c'} = 4$. Let $k_d = 2$. The district's admission rule Ch_d is defined as follows: according to the order (s_1, d, c), (s_2, d, c), (s_3, d, c), (s_4, d, c), (s_1, d, c'), (s_2, d, c'), (s_3, d, c'), and (s_4, d, c') let Ch_d accept up to two contracts (while not choosing more than one contract with the same student).*

Then Ch_d has the following completion Ch'_d which satisfies both substitutability and LAD: according to the order (s_1, d, c), (s_2, d, c), (s_3, d, c), (s_4, d, c), (s_1, d, c'), (s_2, d, c'), (s_3, d, c'),

[2]I thank Fuhito Kojima for providing this example.

(s_4, d, c') let Ch'_d accept up to two contracts (while choosing possibly two contracts with the same student).

Then

$$Ch_c(\{(s_1, d, c), (s_2, d, c)\}) = Ch_d(\{(s_1, d, c), (s_2, d, c)\}) = \{(s_1, d, c), (s_2, d, c)\}$$

and

$$Ch_{c'}(\{(s_3, d, c'), (s_4, d, c')\}) = Ch_d(\{(s_3, d, c'), (s_4, d, c')\}) = \{(s_3, d, c'), (s_4, d, c')\},$$

but

$$Ch_d(\{(s_1, d, c), (s_2, d, c)\} \cup \{(s_3, d, c'), (s_4, d, c')\}) = Ch_d(\{(s_1, d, c), (s_2, d, c)\}),$$

which is different from

$$Ch_c(\{(s_1, d, c), (s_2, d, c)\}) \cup Ch_{c'}(\{(s_3, d, c'), (s_4, d, c')\}) = \{(s_1, d, c), (s_2, d, c), (s_3, d, c'), (s_4, d, c')\}.$$

Note that $\{(s_1, d, c), (s_2, d, c), (s_3, d, c'), (s_4, d, c')\}$ is a feasible matching.

2.2 Cumulative Offer Process

We consider a district admissions rule profile $(Ch_d)_{d \in \mathcal{D}}$ satisfying the assumptions in Proposition 1 and denote its rationalization by the choice function profile $((Ch_c)_{c \in \mathcal{C}_d})_{d \in \mathcal{D}} \equiv (Ch_c)_{c \in \mathcal{C}}$. We say that $(Ch_d)_{d \in \mathcal{D}}$ is feasible and acceptant if each Ch_d is feasible and acceptant. Similarly, we say $(Ch_c)_{c \in \mathcal{C}}$ is a rationalization (satisfying substitutability and LAD) of $(Ch_d)_{d \in \mathcal{D}}$ if for each $d \in \mathcal{D}$, $(Ch_c)_{c \in \mathcal{C}_d}$ is a rationalization (satisfying substitutability and LAD) of Ch_d.

Each student s has a strict preference relation P_s over $\mathcal{X}_s \cup \{\emptyset\}$ such that for all $x \in \mathcal{X}_s$, $xP_s\emptyset$. In other words, being unmatched is always the worst option (in public school choice). The corresponding weak preference relation is denoted by R_s, i.e. xR_sy implies xP_sy or $x = y$. Let $P = (P_s)_{s \in \mathcal{S}}$ denote the student preference profile. Since everything except students' preferences remains fixed, a problem is simply a profile P. Below we fix the profile P.

A matching X is stable if it is feasible, and for all $d \in \mathcal{D}$, $Ch_d(X_d) = X_d$ (individual rationality), and there exists no $x = (s, d, c) \notin X$ such that xP_sX_s and $x \in Ch_d(X \cup \{x\})$ (no blocking contract).[3] Furthermore, a matching X is individually rational for schools if

[3]Usually the second requirement is referred as "no blocking pair" but given our context we refer to it as "no blocking contract".

for all $d \in \mathcal{D}$ and all $c \in \mathcal{C}_d$, $Ch_d(X_c) = X_c$.

A mechanism ϕ chooses for any P a feasible matching, which we denote by $\phi(P)$. Then ϕ is stable if $\phi(P)$ is stable (under P). The mechanism ϕ is strategy-proof if for any P, any s and P'_s, we have $\phi_s(P) R_s \phi_s(P'_s, P_{-s})$.

The following mechanism plays the important role for stability and strategy-proofness.

Cumulative Offer Process (COP) based on $(Ch_d)_{d\in\mathcal{D}}$.

Step 1: For all $i \in \mathcal{S}$, let x_i^1 denote i's most preferred acceptable available contract. For all $d \in \mathcal{D}$, let $X_d^1 = \cup_{i\in\mathcal{S}}\{x_i^1\} \cap \mathcal{X}_d$ denote the set of contracts proposed to d. For all $d \in \mathcal{D}$, let $M_d^1 = Ch_d(X_d^1)$ denote the set of tentatively accepted contracts by d, and $M^1 = \cup_{d\in\mathcal{D}} M_d^1$.

Step k: For all $i \in \mathcal{S}\backslash M_{\mathcal{S}}^{k-1}$, let x_i^k denote i's most preferred acceptable available contract among the available ones i did not propose yet. For all $d \in \mathcal{D}$, let $X_d^k = X_d^{k-1} \cup (\cup_{i\in\mathcal{S}\backslash M_{\mathcal{S}}^{k-1}}\{x_i^k\} \cap \mathcal{X}_d)$ denote the set of contracts proposed to d until Step k. For all $d \in \mathcal{D}$, let $M_d^k = Ch_d(X_d^k)$ denote the set of tentatively accepted contracts by d, and $M^k = \cup_{d\in\mathcal{D}} M_d^k$.

Stop: If for some Step k, $\mathcal{S}\backslash M_{\mathcal{S}}^{k-1} = \emptyset$, then $f(P) \equiv M^{k-1}$ is the matching chosen by COP f for the problem P.

When districts' admission rules have completions satisfying substitutability and LAD, then by Hatfield and Kominers (2014) the COP based on $(Ch_d)_{d\in\mathcal{D}}$ is stable and strategy-proof. Furthermore, COP yields identical outcomes when applied either to the rationalization $(Ch_c)_{c\in\mathcal{C}}$ or to the district admissions rule profile $(\cup_{c\in\mathcal{C}_d} Ch_c)_{d\in\mathcal{D}}$.

Below we show that the COP based on the rationalization in Proposition 1 produces a feasible and acceptant matching which does not contain any blocking contract with respect to the district admission rules. Furthermore, all students are weakly better off when comparing the outcome of the COP based on the rationalization with the outcome of the COP based on the district admission rules. This rationalization is canonical for the outcome of COP in the following sense: obviously, if district admission rules have the same rationalization, then the outcomes of COP (based on the rationalization) coincide. In other words, to distinguish district admission rules in terms of their outcomes of the COP (based on the rationalization), it is sufficient to know their rationalizations. Furthermore, in our context, the student-proposing deferred acceptance mechanism (SPDA) used by Hafalir, Kojima and Yenmez (2022) is outcome equivalent to COP (as SPDA is stable and strategy-proof).

Proposition 2. *Let* $(Ch_d)_{d\in\mathcal{D}}$ *be feasible and acceptant and have completions satisfying substitutability and LAD. Let* $(Ch_c)_{c\in\mathcal{C}}$ *denote its rationalization satisfying substitutability and acceptance. Then the following holds:*

(i) Given profile P, let X denote the outcome of COP based on $(Ch_d)_{d\in\mathcal{D}}$ and Y denote the outcome of COP based on $(Ch_c)_{c\in\mathcal{C}}$. Then

(i.i) Y is a feasible and individually rational for schools matching, and Y contains no blocking contract with respect to $(Ch_d)_{d\in\mathcal{D}}$.

(i.ii) Y is student-optimal in the following sense: $Y_s R_s X_s$ for all $s \in S$.

(ii) COP based on $(Ch_c)_{c\in\mathcal{C}}$ is the unique student-optimal (with respect to (i.ii) for district admission rules with rationalization $(Ch_c)_{c\in\mathcal{C}}$) and strategy-proof mechanism.

Proof. For each $d \in \mathcal{D}$, let Ch_d be feasible and acceptant, and let Ch'_d be a completion of Ch_d satisfying substitutability and LAD.

(i.i): It is straightforward that Y is feasible. Furthermore, Y is individually rational for schools as for all $d \in \mathcal{D}$ and $c \in \mathcal{C}_d$, we have $Y_c = Ch_c(Y_c) = Ch_d(Y_c)$ (where the last equality follows from (1)). Suppose that Y contains a blocking contract $\bar{x} = (\bar{s}, \bar{d}, \bar{c})$ with respect to $(Ch_d)_{d\in\mathcal{D}}$. Then $\bar{x} P_{\bar{s}} Y_{\bar{s}}$ and $\bar{x} \in Ch_{\bar{d}}(Y_{\bar{d}} \cup \{\bar{x}\})$.

If $Ch'_{\bar{d}}(Y_{\bar{d}} \cup \{\bar{x}\}) = Ch_{\bar{d}}(Y_{\bar{d}} \cup \{\bar{x}\})$, then by $Ch'_{\bar{d}}(Y_{\bar{d}} \cup \{\bar{x}\}) \subseteq \cup_{c\in\mathcal{C}_{\bar{d}}} Ch_c(Y_{\bar{d}} \cup \{\bar{x}\})$ we have $\bar{x} \in Ch_{\bar{c}}(Y_{\bar{d}} \cup \{\bar{x}\})$ and Y contains a blocking contract under $(Ch_c)_{c\in\mathcal{C}}$, a contradiction.

If $Ch'_{\bar{d}}(Y_{\bar{d}} \cup \{\bar{x}\}) \neq Ch_{\bar{d}}(Y_{\bar{d}} \cup \{\bar{x}\})$, then $Ch'_{\bar{d}}(Y_{\bar{d}} \cup \{\bar{x}\})$ is not feasible for students. If $\bar{x} \notin Ch'_{\bar{d}}(Y_{\bar{d}} \cup \{\bar{x}\})$, then

$$Ch'_{\bar{d}}(Y_{\bar{d}} \cup \{\bar{x}\}) = Ch'_{\bar{d}}(Y_{\bar{d}}) = Ch_{\bar{d}}(Y_{\bar{d}})$$

where the first equality follows from $\bar{x} \notin Ch'_{\bar{d}}(Y_{\bar{d}} \cup \{\bar{x}\})$ together with substitutability and LAD of $Ch'_{\bar{d}}$, and the second one from the fact that $Y_{\bar{d}}$ is feasible for students and $Ch'_{\bar{d}}$ is a completion of $Ch_{\bar{d}}$. But now $Ch'_{\bar{d}}(Y_{\bar{d}} \cup \{\bar{x}\})$ is feasible for students and we must have $Ch'_{\bar{d}}(Y_{\bar{d}} \cup \{\bar{x}\}) = Ch_{\bar{d}}(Y_{\bar{d}} \cup \{\bar{x}\})$ as $Ch'_{\bar{d}}$ is a completion of $Ch_{\bar{d}}$, which is a contradiction. Thus, $\bar{x} \in Ch'_{\bar{d}}(Y_{\bar{d}} \cup \{\bar{x}\})$, and as above it follows $\bar{x} \in Ch_{\bar{c}}(Y_{\bar{d}} \cup \{\bar{x}\})$ and Y is contains a blocking contract under $(Ch_c)_{c\in\mathcal{C}}$, a contradiction.

(i.ii): We show (i.ii) in the Appendix using arguments from Chambers and Yenmez (2013).

(ii): As any Ch_c satisfies substitutability and acceptance, the outcome of COP is the student-optimal stable matching for the profile $(Ch_c)_{c\in\mathcal{C}}$. By Corollary 1, if Ch_d is strongly acceptant for any $d \in \mathcal{D}$, then $\cup_{c\in\mathcal{C}_d} Ch_c$ is the unique completion of Ch_d satisfying substitutability and LAD. Thus, if Ch_d is strongly acceptant, then the COP based on $(Ch_c)_{c\in\mathcal{C}}$ yields the unique student-optimal stable matching (as any Ch_c satisfies substitutuability and acceptance) by Hatfield and Milgrom (2005). Hence, the COP based on $(Ch_c)_{c\in\mathcal{C}}$ is the unique student-optimal mechanism with respect to (i.ii). Strategy-proofness of the COP based on $(Ch_c)_{c\in\mathcal{C}}$ follows also from Hatfield and Milgrom (2005). $\square$

Note that (i) and (ii) imply that all students weakly prefer the outcome of COP under $(Ch_c)_{c\in\mathcal{C}}$ to any matching which is stable for some district admissions rule profile with rationalization $(Ch_c)_{c\in\mathcal{C}}$. Given Proposition 2 and the rationalization of the district admission rule, we call the COP based on the rationalization **the student-optimal COP**.

Hafalir, Kojima and Yenmez (2022, Section 2.2.1) provide an example of a district's admission rule by letting schools in the district choose according to a serial dictatorship from any set of contracts (where any school is endowed with a choice function satisfying substitutability and LAD). Formally, if $\mathcal{C}_d = \{c_1, \ldots, c_n\}$, then for $X \subseteq \mathcal{X}$ the district admissions rule according to the serial dictatorship $c_1 - \cdots - c_n$ chooses

$$Ch_d(X) = Ch_{c_1}(X) \cup Ch_{c_2}(X \backslash Y_1) \cup \cdots \cup Ch_{c_n}(X \backslash Y_{n-1})$$

where Y_i for $i = 1, \ldots, n-1$ is the set of all contracts in X associated with students who have been chosen in $Ch_{c_1}(X) \cup Ch_{c_2}(X \backslash Y_1) \cup \cdots \cup Ch_{c_i}(X \backslash Y_{i-1})$. The rationalization of Ch_d is $(Ch_c)_{c\in\mathcal{C}_d}$, which is independent of the order of schools. Now Corollary 1 implies that the outcome of the student-optimal COP is invariant to the order according to which schools choose from any set of contracts (as their individual choice functions form a rationalization of the district's admission rule).

Endowed with Proposition 1 and 2, we show that for the main results of Hafalir, Kojima and Yenmez (2022) the admission goals at the district level must be imposed (implicitly) at the school level. Throughout we fix the admissions rule profile $(Ch_d)_{d\in\mathcal{D}}$ and its corresponding rationalization $(Ch_c)_{c\in\mathcal{C}}$.

3 District-Based versus School-Based Admissions

Without district admission rules, the original approach was to endow each of the district schools with a choice function satisfying substitutability and LAD, and then use them for the COP mechanism to accept and reject students. This is equivalent to the use of the union of the schools' choice functions as district admission rule. Conversely, any district admission rule as in Proposition 1 induces a profile of school choice functions satisfying substitutability and LAD such that any completion of the district admission rule is a subcorrespondence of the union of the schools' choice functions. Now the largest expansion is the union which gives the student-optimal district admission rule.

In order for interdistrict school choice to be implemented via COP, all students shall be weakly better off under interdistrict COP when compared to intradistrict COP. The following condition turns out to be necessary and sufficient at both the district level and the school level.

Definition 1. *The choice function Ch_c (and, respectively, Ch_d)* ***favors own students*** *if for any matching X that is feasible for students, $Ch_c(X) \supseteq Ch_c(\{x \in X | d(s(x)) = d(c)\})$ (and, respectively, $Ch_d(X) \supseteq Ch_d(\{x \in X | d(s(x)) = d\})$).*

Below we consider a district admissions rule profile $(Ch_d)_{d\in\mathcal{D}}$ and denote its rationalization by the choice function profile $((Ch_c)_{c\in\mathcal{C}_d})_{d\in\mathcal{D}} \equiv (Ch_c)_{c\in\mathcal{C}}$.

Theorem 1. *[Hafalir, Kojima and Yenmez, 2022, Theorem 2] The following are equivalent.*

(i) Every student weakly prefers the student-optimal COP outcome under interdistrict school choice to the COP outcome under intradistrict school choice for all preferences.

(ii) Ch_c favors own students for each $c \in \mathcal{C}$.

Furthermore, if the COP based on $(Ch_d)_{d\in\mathcal{D}}$ favors own students, then the student-optimal COP favors own students.

Proof. From the proof of Hafalir, Kojima and Yenmez (2022, Theorem 2) it follows (i)$\Leftrightarrow$(ii) by setting $k_d = \sum_{c\in\mathcal{C}_d} q_c$ for all $d \in \mathcal{D}$.

Let X be a matching which is feasible for students and $d \in \mathcal{D}$. Then for all $c \in \mathcal{C}_d$, $Ch_c(X_c) = Ch_c(X)$ is feasible for students and $\cup_{c\in\mathcal{C}_d} Ch_c(X)$ is feasible for students. Because $\cup_{c\in\mathcal{C}_d} Ch_c$ is an expansion of Ch'_d and Ch'_d is a completion of Ch_d, it follows

$$\cup_{c\in\mathcal{C}_d} Ch_c(X_c) = \cup_{c\in\mathcal{C}_d} Ch_c(X) \supseteq Ch'_d(X) = Ch_d(X) = \cup_{c\in\mathcal{C}_d}(Ch_d(X) \cap X_c).$$

Now if Ch_d favors own students, then

$$\cup_{c\in\mathcal{C}_d} Ch_c(\{x \in X_c | d(s(x)) = d(c)\}) = \cup_{c\in\mathcal{C}_d} Ch_d(\{x \in X_c | d(s(x)) = d(c)\}) \subseteq \cup_{c\in\mathcal{C}_d} Ch_d(X_c) = \cup_{c\in\mathcal{C}_d} Ch_c(X_c)$$

where the first equality follows from (1) together with the fact that $\{x \in X_c | d(s(x)) = d(c)\}$ is feasible for students (as X is feasible for students), the inclusion relation from Ch_d favoring own students (when applied to X_c), and the last equality follows again from (1). Hence, for each $c \in \mathcal{C}_d$, Ch_c favors own students. $\square$

Note that if the student-optimal COP favors own students, then this does not necessarily imply the COP based on $(Ch_d)_{d\in\mathcal{D}}$ to favor own students. Hence, favoring own students of the student-optimal COP is "easier to achieve" than for the COP based on $(Ch_d)_{d\in\mathcal{D}}$.

This allows us to obtain the main policy application for interdistrict admissions: if the district admissions rules favor own students, then all students weakly prefer the outcome of the student-optimal COP to the outcome of interdistrict COP (which in turn is weakly preferred to the outcome of intradictrict COP). In addition, any school's choice function is simply derived from the district admission rule.

Corollary 2. *Let $(Ch_d)_{d\in\mathcal{D}}$ favor own students. Given profile P, let X denote the outcome of COP under $(Ch_d)_{d\in\mathcal{D}}$, Y denote the outcome of COP under $(Ch_c)_{c\in\mathcal{C}}$, and Z denote the outcome of interdistrict COP under $(Ch_d)_{d\in\mathcal{D}}$[4]. Then we have for all $s \in \mathcal{S}$,*

$$Y_s R_s X_s R_s Z_s.$$

Note that Corollary 2 follows from (ii) of Proposition 2 together with Hafalir, Kojima and Yenmez (2022, Theorem 2).

4 Further Applications

4.1 Individual Rationality

Suppose that there exists a feasible initial matching $\tilde{X}$ such that every student has exactly one contract. Then for any student s, $\tilde{X}_s = \{(s,d,c)\}$ for some district d and school c, meaning c is the initial school of s and d is the home district of s.

Individual rationality of COP requires each student to be matched to a school which is weakly preferred to his initial school. The following condition is necessary and sufficient for COP to satisfy individual rationality. This requirement can be either imposed at the district level or at the school level.

Definition 2. *The choice function Ch_c (and, respectively, Ch_d)* ***respects the initial matching*** *if for any matching X that is feasible for students, $(s,d,c) \in \tilde{X} \cap X \Rightarrow (s,d,c) \in Ch_c(X)$ (and, respectively, $(s,d,c) \in Ch_d(X)$).*

Theorem 2. *[Hafalir, Kojima and Yenmez, 2022, Theorem 1] The following are equivalent.*

(i) The student-optimal COP satisfies individual rationality.

(ii) Ch_c respects the initial matching for each $c \in \mathcal{C}$.

Furthermore, if the COP based on $(Ch_d)_{d\in\mathcal{D}}$ satisfies individual rationality, then the student-optimal COP satisfies individual rationality.

[4]This means all students are applying only to schools located in their home district.

Proof. From the proof of Hafalir, Kojima and Yenmez (2022, Theorem 1) and Corollary 1 it follows (i)$\Leftrightarrow$(ii) under the assumption that $k_d = \sum_{c \in \mathcal{C}_d} q_c$ for all $d \in \mathcal{D}$.

Let X be a matching which is feasible for students and $d \in \mathcal{D}$. Then for all $c \in \mathcal{C}_d$, $Ch_c(X_c) = Ch_c(X)$ is feasible for students and $\cup_{c \in \mathcal{C}_d} Ch_c(X)$ is feasible for students. Because $\cup_{c \in \mathcal{C}_d} Ch_c$ is an expansion of Ch'_d and Ch'_d is a completion of Ch_d, it follows $\cup_{c \in \mathcal{C}_d} Ch_c(X) \supseteq Ch'_d(X) = Ch_d(X) = \cup_{c \in \mathcal{C}_d}(Ch_d(X) \cap X_c)$. Thus, for all $c \in \mathcal{C}_d$, $Ch_c(X) \supseteq Ch_d(X) \cap X_c$. Hence, if Ch_d respects the initial matching, then for all $c \in \mathcal{C}_d$, Ch_c respects the initial matching. $\square$

Note that individual rationality of the student-optimal COP does not imply individual rationality of the COP based on $(Ch_d)_{d \in \mathcal{D}}$. Hence, individual rationality of the student-optimal COP is "easier to achieve" than individual rationality of the COP based on $(Ch_d)_{d \in \mathcal{D}}$.

4.2 Balanced Exchange

The balanced-exchange policy means that in the outcome of COP the district receives the same number of students outside the district as the number of students who are sent elsewhere by the district.

Definition 3. *The choice function Ch_d is **rationed** if for any matching X that is feasible for students, $|Ch_d(X)| \leq k_d$.*

We show that the student-optimal COP satisfies the balanced-exchange policy if and only if the total number of seats in a district is equal to the number of students who live in the district. This means that the balanced-exchange policy is implicitly given by the numbers of seats available at the district's schools.

Theorem 3. *[Hafalir, Kojima and Yenmez, 2022, Theorem 3] The following are equivalent.*

(i) The student-optimal COP satisfies the balanced-exchange policy.

(ii) $k_d = \sum_{c \in \mathcal{C}_d} q_c$ for each $d \in \mathcal{D}$.

Proof. Note that for all $d \in \mathcal{D}$ and all $c \in \mathcal{C}_d$, $q_c \leq k_d$, which implies that Ch_c is q_c-acceptant (by Remark 1). Now from the proof of Hafalir, Kojima and Yenmez (2022, Theorem 2) it follows that (ii)$\Rightarrow$(i). It remains to show (i)$\Rightarrow$(ii). Let $d \in \mathcal{D}$. By assumption we have $k_d \leq \sum_{c \in \mathcal{C}_d} q_c$.

Suppose that any student s ranks contracts from $\mathcal{X}_d \cap \mathcal{X}_s$ above $(\mathcal{X} \setminus \mathcal{X}_d) \cap \mathcal{X}_s$. Let Y be the outcome of the student-optimal COP.

Suppose that $k_d < \sum_{c \in \mathcal{C}_d} q_c$. Then the schools in $\mathcal{C}_d$ are assigned more than k_d students (as there is at least one other district containing one or more students), i.e. $|Y_d| > k_d$ and $Y_d = \cup_{c \in \mathcal{C}_c} Ch_c(Y_c) = \cup_{c \in \mathcal{C}_c} Ch_c(Y)$. Now $\cup_{c \in \mathcal{C}_d} Ch_c$ is not rationed, a contradiction. $\square$

4.3 Diversity

Let $\mathcal{T}$ denote the set of types. Each student s has a type $\tau(s) \in \mathcal{T}$. Given type t, let $\mathcal{S}_t$ denote the set of type-t students and $k^t = |\mathcal{S}_t|$. Given $X \subseteq \mathcal{X}$, let $X_t = \{x \in X | \tau(s(x)) = t\}$ denote the set of contracts associated with type-t students.

Definition 4. *The choice function Ch_c (and, respectively, Ch_d) has a **school-level type-specific ceiling** of q_c^t at school c for type-t students if for any matching X that is feasible for students, $|Ch_c(X) \cap X_t| \leq q_c^t$ (and, respectively, $|Ch_d(X) \cap X_c \cap X_t| \leq q_c^t$).*

Now the following lemma uses similar arguments as the proof of Proposition 1.

Lemma 1. *The admissions rule Ch_d has a school-level type-specific ceiling of q_c^t at school c for type-t students if and only if the choice function Ch_c (from its rationalization) has a school-level type-specific ceiling of q_c^t at school c for type-t students.*

Proof. First, let Ch_d have a school-level type-specific ceiling of q_c^t at school c for type-t students. Then for any matching X that is feasible for students we have $|Ch_d(X) \cap X_c \cap X_t| \leq q_c^t$. As X is feasible for students and Ch'_d is a completion of Ch_d, we have $Ch'_d(X) = Ch_d(X)$ and $|Ch'_d(X) \cap X_c \cap X_t| \leq q_c^t$. By substitutability of Ch'_d, we have $Ch'_d(X) \cap X_c \subseteq Ch'_d(X_c)$. As X_c is feasible for students, again $Ch'_d(X_c) = Ch_d(X_c)$. As Ch_d has a school-level type-specific ceiling of q_c^t at school c for type-t students, we have $|Ch'_d(X_c) \cap X_t| \leq q_c^t$. By (1) in Proposition 1, we have $Ch_c(X) = Ch'_d(X_c)$ and Ch_c has a school-level type-specific ceiling of q_c^t at school c for type-t students.

Second, let Ch_c have a school-level type-specific ceiling of q_c^t at school c for type-t students. As X is feasible for students and Ch'_d is a completion of Ch_d, we have $Ch_d(X) = Ch'_d(X)$. By (ii) of Proposition 1 we have $Ch_d(X) \cap X_c \subseteq Ch_c(X)$, and Ch_d has a school-level type-specific ceiling of q_c^t at school c for type-t students. $\square$

Once school-level type specific ceilings are imposed admission rules often violate acceptance. The following weaker version of acceptance incorporates type-specific ceilings:

Ch_d is weakly acceptant if for any matching X that is feasible for students, $x \in X_d \backslash Ch_d(X)$ implies $|\{y \in Ch_d(X) | c(y) = c(x)\}| = q_{c(x)}$ or $|Ch_d(X)| \geq k_d$ or $|\{y \in Ch_d(X) | c(y) = c(x) \& \tau(s(y)) = \tau(s(x))\}| = q_{c(x)}^{\tau(s(x))}$.

Now note that the proof of Proposition 1 remains unchanged for weakly acceptant admission rules having a completion which is substitutable and LAD, and thus the induced rationalization is substitutable and LAD.

Definition 5. *The choice function profile $(Ch_c)_{c\in\mathcal{C}}$ (and, respectively, $(Ch_d)_{d\in\mathcal{D}}$) **accomodates unmatched students** if for any feasible matching X in which student s is unmatched, there exists $x = (s, d, c) \in \mathcal{X}$ such that $x \in Ch_c(X \cup \{x\})$ (and, respectively, $x \in Ch_d(X \cup \{x\})$).*

The following lemma follows from similar arguments as above using (ii) of Proposition 1.

Lemma 2. *(i) If Ch_d is weakly acceptant, then for all $c \in \mathcal{C}_d$, Ch_c is weakly acceptant.*

(ii) If the admissions rule profile $(Ch_d)_{d\in\mathcal{D}}$ accomodates unmatched students, then its rationalization $(Ch_c)_{c\in\mathcal{C}}$ accomodates unmatched students.

Proof. In showing (i), let X be a matching which is feasible for students. Then $Ch'_d(X) = Ch_d(X)$ (as Ch'_d is a completion of Ch_d) and by (ii) of Proposition 1 we have $Ch_d(X) \subseteq \cup_{c\in\mathcal{C}_d} Ch_c(X)$. Thus, $X_d \backslash Ch_d(X) \supseteq X_d \backslash (\cup_{c\in\mathcal{C}_d} Ch_c(X))$. Now for any $c \in \mathcal{C}_d$, $x \in X_c \backslash Ch_c(X)$ implies $x \in X_d \backslash Ch_d(X)$ and Ch_c is weakly acceptant.

In showing (ii), let X be a matching which is feasible for students in which student s is unmatched. Then there exists $x = (s, d, c')$ such that $x \in Ch_d(X \cup \{x\})$. Note that $X \cup \{x\}$ is a matching which is feasible for students. Then $Ch'_d(X \cup \{x\}) = Ch_d(X \cup \{x\})$ (as Ch'_d is a completion of Ch_d) and by (ii) of Proposition 1 we have $Ch_d(X \cup \{x\}) \subseteq \cup_{c\in\mathcal{C}_d} Ch_c(X \cup \{x\})$. Thus, $x \in Ch_{c'}(X \cup \{x\})$, the desired conclusion. □

Now note that our results imply the following: if the admission rule profile $(Ch_d)_{d\in\mathcal{D}}$ has school-level type-specific ceilings, is weakly acceptant, and accomodates unmatched students, then the choice function profile $(Ch_c)_{c\in\mathcal{C}}$ has school-level type-specific ceilings, is weakly acceptant, and accomodates unmatched students. Thus, those assumptions in Hafalir, Kojima and Yenmez (2022, Theorem 3) on the admissions rule profile imply the corresponding ones on the choice function profile.

A matching X is legitimate if it satisfies the following conditions: (i) $|X_d| = k_d$ for all $d \in \mathcal{D}$, (ii) $|X_t| = k^t$ for all $t \in \mathcal{T}$, (iii) $|X_c| \leq q_c$ for all $c \in \mathcal{C}$ and (iv) $|X_c \cap X_t| \leq q_c^t$ for all $t \in \mathcal{T}$ and all $c \in \mathcal{C}$.

The following result has important policy implications for school-level type specific ceiling constraints.

Theorem 4. *Suppose that for any $d \in \mathcal{D}$, $\cup_{c \in \mathcal{C}_d} Ch_c$ respects the initial matching, favors own students, is rationed, has school-level type-specific ceiling q_c^t, and for any c and t, suppose $q_c^t \geq |\tilde{X}_c \cap \tilde{X}_t|$, i.e. $\tilde{X}$ is feasible. Then*

(i) The outcome of the student-optimal COP is legitimate.

(ii) Every student weakly prefers the student-optimal COP outcome under interdistrict school choice to the student-optimal COP outcome under intradistrict school choice for all preferences.

Proof. For (i), note that if Ch_c respects the initial matching for all $c \in \mathcal{C}$, then $(Ch_c)_{c \in \mathcal{C}}$ accommodates unmatched students. Thus, under the outcome of the student-optimal COP, say Y, all students are matched. Since $(Ch_c)_{c \in \mathcal{C}}$ is rationed, by Theorem 3 we have $k_d = \sum_{c \in \mathcal{C}_d} q_c$ for all $d \in \mathcal{D}$. As own students are favored, we have $|Y_d| = k_d = \sum_{c \in \mathcal{C}_d} q_c$ for all $d \in \mathcal{D}$. As Ch_c has the school-level type-specific ceiling q_c^t and $\tilde{X}$ is feasible, we have $|Y_c \cap Y_t| \leq q_c^t$.

For (ii), the proof of (i.ii) of Proposition 2 remains unchanged as it is valid for any rationalization of Ch_d (with $d \in \mathcal{D}$) and the completion satisfies substitutability and LAD, respects the initial matching, favors own students and is rationed. □

Here (ii) is somewhat surprising given the contributions of Kojima (2013) and Hafalir, Yenmez and Yildirim (2013): they showed that relaxing school-level type-specific constraints does not necessarily benefit student for stable mechanisms. Here Theorem 4 can be seen as a special case of relaxing school-level type-specific constraints from $|\tilde{X}_c \cap \tilde{X}_t|$ to q_c^t.

Note also that in Theorem 4 we could have equivalently stated the conditions on the districts' admissions rules (by the above results).

In another application, Kamada and Kojima (2018) consider regional contrains for hospital-residents matching in Japan. It would be useful to know whether our approach is extendable to their model by allowing several layers of completions/ rationalizations for the COP mechanism.

APPENDIX: Comparative Statics

Below we follow Chambers and Yenmez (2013).[5] Let Ch'_d be a completion of Ch_d satisfying LAD and substitutability. Then by Aizerman and Malishevski (1981) there exists a finite set of orders $\{\succeq_e\}_{e \in E_d}$ such that each $\succeq_e$ is complete, transitive and antisymmetric on

[5]The contents of this paper were subsequently published in Chambers and Yenmez (2017, 2018a, 2018b).

$\mathcal{X} \cup \emptyset$ and for each $X \subseteq \mathcal{X}$ we have

$$Ch'_d(X) = \cup_{e \in E_d}\{\max_{X \cup \emptyset} \succeq_e\}.$$

We will use the convention and call each $e \in E_d$ a Ch'_d-position.

The following uses arguments of Chambers and Yenmez (2013, Theorem 2) with an important modification to apply COP to district $\hat{d}$ and the students who would like to sign a contract with $\hat{d}$.

Proposition 3. *Let $Ch' = (Ch'_d)_{d \in \mathcal{D}}$ be LAD and substitutable. Let $\hat{d} \in D$ and $Ch''_{\hat{d}}$ be an expansion of $Ch'_{\hat{d}}$ satisfying LAD and substitutability, and $Ch'' = (Ch''_{\hat{d}}, Ch'_{-\hat{d}})$. If μ is Ch'-stable, then there exists a Ch''-stable ν such that $\nu_i R_i \mu_i$ for all $i \in \mathcal{S}$.*

Proof. As μ is Ch'-stable, we have for all $d \in \mathcal{D}$, $Ch'_d(\mu_d) = \mu_d$. As $Ch''_{\hat{d}}$ is an expansion of $Ch'_{\hat{d}}$ we have $\mu_{\hat{d}} = Ch'_{\hat{d}}(\mu_{\hat{d}}) \subseteq Ch''_{\hat{d}}(\mu_{\hat{d}}) \subseteq \mu_{\hat{d}}$. Thus, $Ch''_{\hat{d}}(\mu_{\hat{d}}) = \mu_{\hat{d}}$ and μ is Ch''-individually rational. If μ is Ch''-stable, then we are done.

Otherwise, μ contains a blocking contract $x = (s, d, c)$. Obviously, we must have $d = \hat{d}$. Let $\hat{X} = \{x \in \mathcal{X}_{\hat{d}} | x R_{s(x)} \mu_{s(x)}\}$ denote the set of contracts with $\hat{d}$ which students would like to sign with district $\hat{d}$. Note that $\hat{X} \supseteq \mu_{\hat{d}}$ as μ is individually rational for students.

Now apply the COP to $Ch''_{\hat{d}}$ and $\hat{X}_{\mathcal{S}}$ where we restrict R_i to $R_i|\hat{X}_i$ for $i \in \hat{X}_{\mathcal{S}}$. Let $\mu^0_{\hat{d}}$ be its outcome and $\hat{Y}$ denote the proposals made. Note that $\hat{Y} \subseteq \hat{X}$ and $Ch''_{\hat{d}}(\hat{Y}) = \mu^0_{\hat{d}}$. We claim that any student who is matched to $\hat{d}$ under μ does not become unmatched under $\mu^0_{\hat{d}}$. Suppose to the contrary for $i \in (\mu_{\hat{d}})_{\mathcal{S}}$, then $\hat{X}_i \subseteq \hat{Y}_i$ and $i \notin (\mu^0_{\hat{d}})_{\mathcal{S}}$. As $Ch'_{\hat{d}}(\hat{X}) = \mu_{\hat{d}}$, we obtain by substitutability of $Ch'_{\hat{d}}$, $\mu_i \in Ch'_{\hat{d}}(\hat{Y})$. On the other hand, as $Ch''_{\hat{d}}$ is an expansion of $Ch'_{\hat{d}}$, we have $Ch'_{\hat{d}}(\hat{Y}) \subseteq Ch''_{\hat{d}}(\hat{Y})$, which is now a contradiction.

Define μ^0 as follows: for all $x \in \mu^0_{\hat{d}}$, $\mu^0_{s(x)} = x$, and for all $i \in \mathcal{S} \backslash \{s(x) | x \in \mu^0_{\hat{d}}\}$, $\mu^0_i = \mu_i$. Note that μ^0 is well defined as all students matched to $\hat{d}$ under μ remain matched to $\hat{d}$ under μ^0. Thus, μ^0 is feasible for students. Furthermore, note that there exists no blocking contract $x = (s, d, c)$ with $d \neq \hat{d}$ such that for the Ch'_d-position e we have $x \succ_e \max_\mu \succeq_e$ as otherwise μ would not have been Ch'-stable (by $x P_s \mu_s$).

Now as in Chambers and Yenmez (2013), starting from μ^0 we can use the district-proposing algorithm to find a stable matching:

Step 0. If there is no rejected position, then the algorithm stops with μ^0.

Step k. Each rejected position applies to the next best acceptable contract (if there is any left), otherwise the position remains unmatched. Any student who receives a new proposal chooses the best acceptable contract from the proposed one and the one she is tentatively holding. And so on. If there are no rejections, then Stop. Let μ' be the final outcome matching, Note that matched students never become unmatched, and any student always becomes better off. Also at some point all contracts are proposed and we must arrive at a stable matching: individual rationality is obvious, and there is no blocking contract as students are always weakly better off and there was none at μ. $\square$

Given profile P, let X denote the outcome of COP based on $(Ch_d)_{d\in\mathcal{D}}$ and Y denote the outcome of COP based on $(Ch_c)_{c\in\mathcal{C}}$. Note that X is also the outcome of COP based on $(Ch'_d)_{d\in\mathcal{D}}$ and Y is also the outcome of COP based on $(\cup_{c\in\mathcal{C}_d}Ch_c)_{d\in\mathcal{D}}$. Then X is $(Ch'_d)_{d\in\mathcal{D}}$-stable. As for all $d\in\mathcal{D}$, $\cup_{c\in\mathcal{C}_d}Ch_c$ is an expansion of Ch'_d satisfying LAD and substitutability, and Y is the student-optimal stable matching under P and $(Ch_c)_{c\in\mathcal{C}}$, repetitive applications of Proposition 3 (whereby we replace one-by-one Ch'_d by $\cup_{c\in\mathcal{C}_d}Ch_c$ in the profile $(Ch'_d)_{d\in\mathcal{D}}$) yield $Y_sR_sX_s$ for all $s\in\mathcal{S}$. Hence, (i.ii) of Proposition 2 is true.

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