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Asymptotics of the sloshing eigenvalues for a two-layer fluid

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Résumé

Dans ce mémoire de maîtrise, nous étudions l'asymptotique spectrale pour les problèmes d'oscillation de deux fluides incompressibles idéaux remplissant un volume arbitraire avec une surface supérieure ouverte ou fermée. Dans le premier chapitre, nous introduisons les notions de base de la géométrie spectrale, illustrons le problème de Steklov pour un fluide dans un volume arbitraire ainsi que les principaux résultats qui seront nécessaires pour comprendre et démontrer les énoncés du manuscrit. Nous dérivons également les principales relations et équations des petites oscillations d'un fluide incompressible idéal.

La deuxième partie présente les principaux résultats sur les petites oscillations de deux liquides à surface supérieure fermée, obtenus par Solomyak, Kopachevsky et leurs collaborateurs, qui justifient et vérifient la cohérence des résultats pour le problème considéré.

Finalement, nous traitons le problème avec une surface ouverte. Une question similaire a été abordée par Kuznetsov. Un canal rectangulaire rempli de deux liquides est un exemple révélateur vérifiant tous les principaux résultats de la recherche. Entre autres, nous avons trouvé un cas particulier intéressant dans lequel la famille de solutions correspondant au paramètre spectral disparaît. En conclusion, nous trouvons sur les conditions d'existence et l'unicité des solutions.

Mots clés : géométrie spectrale, petites oscillations de fluides, asymptotique spectrale.

Abstract

In this M.Sc. thesis, we investigate the spectral asymptotics for a problem describing oscillations of two ideal incompressible fluids filling an arbitrary volume with either open or closed upper surface. In the first chapter, we introduce the basic notions of spectral geometry and illustrate the Steklov problem for fluid in an arbitrary volume, as well as the main results needed to understand and prove the statements in the manuscript. We also derive the equations of small oscillations of an ideal incompressible fluid. The second part presents the main results on small oscillations of two liquids with a closed upper surface, obtained by Solomyak, Kopachevsky, and their collaborators that justify and verify the consistency of the findings for the problem under consideration. In the third chapter, we treat the problem with an open surface. A similar question was previously addressed by Kuznetsov. A rectangular channel filled with two liquids is a telling example that confirms all the main research results. Interestingly enough, we found a particular case in which the family of solutions corresponding to the spectral parameter disappears. In conclusion, we describe the condition of existence and the uniqueness of such solutions.

Keywords: spectral geometry, small fluid oscillations, spectral asymptotics

Contents

Résumé ... ii

Abstract ... iii

List of figures ... vi

Acknowledgments ... vii

Chapter 1. Introduction ... 1

1.1. Dirichlet and Neumann eigenvalue problems ... 2

1.2. Steklov problem ... 4

1.3. Sloshing problem ... 4

1.4. Basic equations of a two-layer fluid ... 6

Chapter 2. Main results ... 10

2.1. Overview ... 10

2.2. Operator equations ... 10

2.3. Discreteness of the spectrum ... 15

2.4. The Weyl law ... 15

2.5. Examples ... 16

2.5.1. Rectangular container ... 16

2.5.2. The case of an extremely heavy fluid ... 18

2.5.3. The study of the limiting case ... 18

2.6. Existence and uniqueness of the solution 19

2.7. General case 20

Chapter 3. Proofs 22

3.1. Operator equations 22

3.1.1. Operator equation for a three-layer fluid in a closed container 22

3.1.2. Operator equation for a two-layer fluid in an open domain 24

3.1.3. Proof of Theorem 2.2.1 24

3.2. Discreteness of the spectrum 25

3.2.1. Proof of Lemma 2.3.1 25

3.2.2. Proof of Lemma 2.3.2 26

3.2.3. Proof of Lemma 2.3.3 28

3.2.4. Proof of Lemma 2.3.4 30

3.3. The Weyl law 33

3.3.1. Proof of Theorem 2.4.1 and 2.4.2 33

3.4. Examples 34

3.4.1. Rectangular container 34

3.4.2. The case of the extremely heavy fluid 37

3.4.3. The study of the limiting case 38

3.5. Existence and uniqueness of the solution 41

3.5.1. Proof of Proposition 2.6.1 41

3.6. General case 43

3.6.1. Proof of Proposition 2.7.1 43

Conclusions 46

References 47

List of figures

1.1 Bounded domain in the Euclidean space 5

1.2 Container with two-layer fluid 6

1.3 Deviation of fluid at the interface between two media Ω_1 and Ω_2 8

2.1 Three-layer fluid in a closed container 11

2.2 Container with three-layer fluid 11

2.3 Domain Ω_1 - three-layer fluid 12

2.4 Domain Ω_3 - three-layer fluid 12

2.5 Domain Ω_2 - three-layer fluid 13

2.6 Rectangular symmetric container 16

2.7 Numerical results for a rectangular container 17

2.8 Rectangular container with two fluids 18

2.9 A supplementary problem domain 19

2.10 Arbitrary fluid domain 20

3.1 Multi-layer fluid in the closed container 33

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Chapter 1

Introduction

The sloshing problem [**9**] is an eigenvalue problem that describes small surface oscillations of a fluid in a container. It is a Steklov-type eigenvalue problem e.g. [**5**], [**2**] with the spectral parameter in the boundary conditions. There has been a lot of research on the sloshing problem for homogeneous fluids e.g. [**6**], [**9**]. While the case of multi-layer fluids has been less explored, some basic results on this subject have been obtained by Kopachevsky [**10**] and Karazeeva-Solomyak [**8**]. These authors considered the oscillations of a multi-layer fluid in a closed container, where they established the discreteness of the spectrum as well as Weyl's law for eigenvalue asymptotics. Exact solutions have also been calculated for some regions admitting separation of variables.

Recently, Kuznetsov [**11**] proposed to investigate the sloshing problem for a two-layer fluid in an open container. In particular, he obtained explicit formulas for the eigenvalue asymptotics in the case of rectangle. This article has served as a motivation for the present work. In particular, we show that the two-layer sloshing problem in an open container can be reduced to the case of a three-layer fluid in a closed container, by adding an auxiliary weightless layer of fluid on the top. Applying this idea, one can explore a similar problem considered in [**11**] using the tools developed in [**10**] and [**8**]. Also, we correct an inaccuracy in the formula obtained in [**11**] for the coefficient in the leading term in Weyl's law. Finally, we consider a limiting case when the ratio between the fluid densities is infinite (this can be viewed as a model for the situation when the bottom fluid is much heavier than the top one). In this case, the eigenvalue problem in an open container may fail to have a discrete spectrum. We provide some relevant examples illustrating this surprising phenomenon, and provide its justification in terms of an auxiliary Robin-Neumann boundary value problem with a negative Robin parameter.

1.1. Dirichlet and Neumann eigenvalue problems

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. The Laplace operator $\triangle : C^\infty(\Omega) \to C^\infty(\Omega)$ is defined as

$$\triangle u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2}, \tag{1.1.1}$$

where $u \in C^\infty(\Omega)$.

The study of eigenvalues and eigenfunctions of the Laplace operator under different boundary conditions is a central subject in spectral geometry. The Dirichlet eigenvalue problem is given by

$$\begin{cases} -\triangle u = \lambda u \text{ in } \Omega \\ u = 0 \text{ on } \mathrm{S} = \partial\Omega, \end{cases}$$

and the Neumann eigenvalue problem is given by

$$\begin{cases} -\triangle u = \lambda u \text{ in } \Omega \\ \frac{\partial u}{\partial n} = 0 \text{ on } \mathrm{S} = \partial\Omega. \end{cases}$$

Note that for the Neumann eigenvalue problem one needs to assume some regularity of the boundary, for example, that Ω is a Lipschitz domain.

One of the central questions is to determine spectral parameters $\lambda \in \mathbb{C}$ such that there exists a $u \neq 0$ satisfying the boundary value problems. The collection of these parameters is called the Dirichlet / Neumann spectrum of Ω. These problems have been studied extensively, both theoretically and from the viewpoint of practical applications (e.g. [**13, 15, 1**]).

Theorem 1.1.1. *The spectrum of the Dirichlet (Neumann) eigenvalue problem in Ω is discrete and forms an increasing sequence of real positive (non-negative) eigenvalues of finite multiplicity*

$$0(=\lambda_0) < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \nearrow +\infty.$$

The corresponding eigenfunctions form a complete orthogonal basis in $L_2(\Omega)$.

Define $N(\lambda)$ as the eigenvalue counting function,

$$N(\lambda) = \#\{\lambda_k : \lambda_k < \lambda\},$$

which is used to study the distribution of the eigenvalues. We denote by N_D and N_N the counting functions for the Dirichlet and Neumann eigenvalue problems, respectively. The following result holds for $N_D(\lambda)$.

Theorem 1.1.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Then*

$$\lim_{\lambda\to\infty} \frac{N_D(\lambda)}{\lambda^{d/2}} = (2d)^{-d}\,\omega_d vol(\Omega),$$

where ω_d is a volume of the unit ball in $\mathbb{R}^d$ and $vol(\Omega)$ is the Lebesgue measure of the domain Ω. The same result holds for $N_N(\lambda)$, provided $\partial\Omega$ is Lipschitz.

Consider the eigenvalue problem for the vibrations of a membrane without external forces. The equation of motion is

$$\rho\frac{\partial^2 u}{\partial t^2} = T_0\left(\frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2}\right),$$

where $u(x,t)$ is the deviation from the equilibrium position and T_0 is the tension of the membrane, $x \in \mathbb{R}^2$.

We look for a solution in the form

$$u(x,t) = \exp(i\omega t)U(x),$$

which leads to the equation

$$-\lambda U = c^2 \triangle U \text{ in } \Omega, \tag{1.1.2}$$

where $\lambda = \omega^2$ and $c^2 = \frac{T_0}{\rho}$.

Note that the boundary conditions are

$$\begin{cases} u = 0 & \text{on } \partial\Omega \text{ if the membrane is fixed (Dirichlet), or} \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \text{ if the membrane is free (Neumann).} \end{cases} \tag{1.1.3}$$

We now consider a rectangular membrane with sides of length a and b.

If the boundary is fixed the eigenvalues and the natural basis of eigenfunctions are given by:

$$u^D_{lm} = \sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}, \lambda^D_{lm} = \pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}\right), N_D(\lambda) \sim \frac{\lambda\pi ab}{16} \qquad (l,m = 1,2,\dots)$$

and if the membrane is free the eigenvalues and the natural basis of eigenfunctions are given by

$$u^N_{lm} = \cos\frac{l\pi x}{a}\cos\frac{m\pi y}{b}, \lambda^N_{lm} = \pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}\right), N_N(\lambda) \sim \frac{\lambda\pi ab}{16} \qquad (l,m = 0,1,2,\dots)$$

1.2. Steklov problem

Steklov problem is an eigenvalue problem where the spectral parameter appears in the boundary condition.

The Steklov eigenvalue problem was first introduced by Russian mathematician Vladimir Steklov [**14**]. Early applications considered are the small vibrations of the fluid, a membrane vibration where the whole mass is distributed along the boundary. This problem has been studied extensively (e.g. [**4, 5, 9**] etc.).

The Steklov problem in the bounded domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz boundary $M = \partial\Omega$ is given by

$$\begin{cases} \triangle u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \sigma u & \text{on M,} \end{cases}$$

where $\triangle$ is the Laplace operator and σ is a spectral parameter.

Theorem 1.2.1. *The spectrum of the Steklov problem is discrete and the eigenvalues form an increasing sequence* [**14**]

$$0 = \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n \leq \cdots \nearrow +\infty$$

and the eigenfunctions form an orthogonal basis in $L_2(M)$.

The next theorem states the Weyl law for the Steklov eigenvalue problem.

Theorem 1.2.2. *Let $\Omega \subset \mathbb{R}^d$ be a domain with a Lipschitz boundary $M = \partial\Omega$. Then*

$$N(\sigma) = \frac{\omega_{d-1} Vol\,(M)}{(2\pi)^{d-1}} \sigma^{d-1} + O\left(\sigma^{d-2}\right),$$

where ω_{d-1} is a volume of the unit ball in $\mathbb{R}^{d-1}$.

1.3. Sloshing problem

Sloshing is a small vibration of liquid in a container. Problems of sloshing were studied, for example, in [**9**]. In this paper, we consider the main properties of a two-layer fluid in an open container of the arbitrary shape and study the properties of the solutions. A sloshing

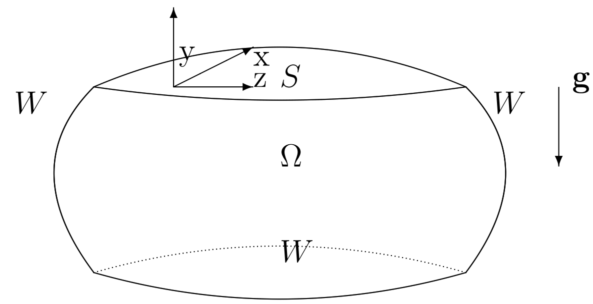

Fig. 1.1. Bounded domain in the Euclidean space

problem in a bounded domain $\Omega \subset \mathbb{R}^3$ and Lipschitz boundary $W \cup S = \partial\Omega$ is given by

$$\begin{cases} \triangle u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } W \\ \frac{\partial u}{\partial n} = \lambda u & \text{on } S. \end{cases} \tag{1.3.1}$$

The spectrum of the sloshing problem is discrete and the eigenfunctions form an orthogonal basis in $L_2(S)$ [**2**].

The boundary value problem (1.3.1) can be also stated in the planar case, where S is a line segment, Ω is a simply connected bounded planar domain, the shape of a container or a canal, and W is a boundary of the Ω called "wall". The following theorem [**2**] gives a formula on the asymptotics of sloshing eigenvalues.

Theorem 1.3.1. *Let Ω be a simply connected bounded Lipshitz planar domain with the sloshing surface $S = (A, B)$ of length L and wall W which are C^1 -regular near the corner points A and B. Let α and β be interior angles between W and S at the points A and B, resp., and assume $0 < \beta \leq \alpha < \pi/2$. Then the following asymptotic expansion holds as $k \to \infty$:*

$$\lambda_k L = \pi\left(k - \frac{1}{2}\right) - \frac{\pi^2}{8}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + r(k), \qquad \textit{where } r(k) = o(1)$$

If, moreover walls are straight near the corners then:

$$r(k) = O(k^{1-\frac{\pi}{2\alpha}}).$$

The counting function in the planar case is given by

$$N(\sigma) = \frac{vol(S)}{\pi}\sigma + O(1).$$

Asymptotic formulas for Steklov eigenvalues of curvilinear polygons in terms of their side lengths and angles are obtained in [**3**].

Theorem 1.3.2. *Let $P = P(\alpha, \mathbf{l})$ be a curvilinear polygon in $\mathbb{R}^2$ with n vertices $V_1,V_2,\dots,V_n$ number clock-wise, corresponding internal angles $0 < \alpha_j < \pi$ at V_j, and smooth sides I_j of length l_j joining V_{j-1} and V_j. Here $\alpha = (\alpha_1,\dots,\alpha_n) \in \prod^n$, where $\prod = (0,\pi)$, and $\mathbf{l} = (l_1,\dots,l_n) \in \mathbb{R}^n_+$.Let $\{\sigma_m\}$ denote the sequence of quasi-eigenvalues (defined in [**3**]) ordered increasingly with account of multiplicities. Then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0,\epsilon_0)$ the Steklov eigenvalues of P satisfy*

$$\lambda_m = \sigma_m + O(m^{-\epsilon}), \text{ as } m \to \infty.$$

1.4. Basic equations of a two-layer fluid

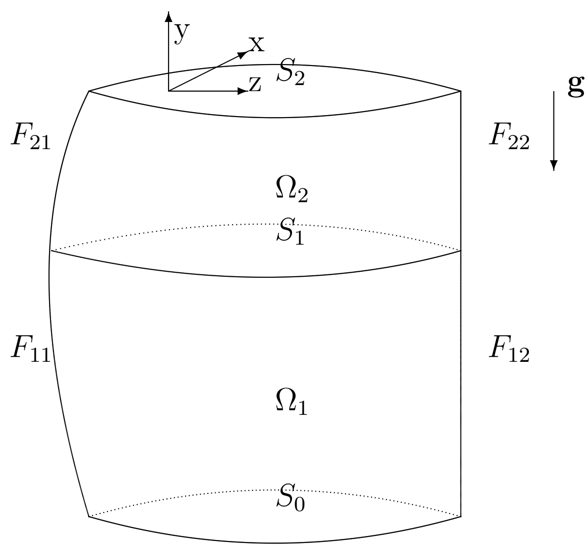

Fig. 1.2. Container with two-layer fluid

In this section, we describe the main eigenvalue problem of the small oscillations of a two-layer fluid in an open container.

We consider a container filled with two ideal incompressible fluids. Our goal is to deduce the equations of the small vibrations of a two-layer fluid in an open container, in the gravitational field, without taking into account the forces of surface tension.

The equations of motion of an ideal incompressible fluid can be written in the form of the Gromeka-Lamb [**7**]

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \frac{v^2}{2} + 2\omega \times \mathbf{v} = -\frac{1}{\rho}\nabla p - \nabla U, \tag{1.4.1}$$

where $\mathbf{v}$ is a velocity vector field of the particles of the fluid, $\omega = \frac{\mathbf{1}}{\mathbf{2}}\nabla \times \mathbf{v}$ is the vector field of the vorticity of the particle, ρ and p are respectively the density and the pressure at the point $x \in \mathbb{R}^3$ and U is a potential of the external forces.

When the motion is assumed to be infinitely small, we find, neglecting the square of the velocity and putting $\omega = 0$,

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho}\nabla p - \nabla U. \tag{1.4.2}$$

The continuity equation for a homogeneous, incompressible fluid is given by

$$\nabla \cdot \mathbf{v} = 0. \tag{1.4.3}$$

Note that this condition means that for any arbitrarily chosen volume with a closed surface there can be neither continuing accumulation of fluid within the volume nor continuing loss. It follows that the net flux of fluid across the surface must be zero. Let us introduce the velocity-potential u

$$\mathbf{v} = -\nabla u.$$

From (1.4.3) we obtain

$$\triangle u = 0. \tag{1.4.4}$$

Let $U = gy$, where g is a gravitational constant. After integrating (1.4.2) we have

$$\frac{p}{\rho} = \frac{\partial u}{\partial t} - gy + F(t), \tag{1.4.5}$$

where $F(t)$ is an arbitrary function of t. It is often convenient to incorporate this arbitrary function in the value $\frac{\partial u}{\partial t}$. This is permissible by (1.4.3), since, the velocity vector field is not affected.

We consider a surface between two domains Ω_1 and Ω_2 which are filled with different fluids in coordinates $\mathbf{r} = (x_1,x_2,y) \in \mathbb{R}^3$, where y denotes a vertical direction and the interface is defined by $y = 0$.

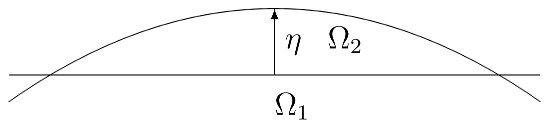

Fig. 1.3. Deviation of fluid at the interface between two media Ω_1 and Ω_2

If η denotes the elevation of the interface at the time t above the point $(x_1,x_2,0)$, we have equality of the pressures on both sides.

$$p_1 = \rho_1 \left(\frac{\partial u_1}{\partial t} - g\eta \right) = \rho_2 \left(\frac{\partial u_2}{\partial t} - g\eta \right) = p_2. \tag{1.4.6}$$

Taking into account that the time derivative of η determines the normal component of the velocity,

$$\frac{\partial \eta}{\partial t} = \mathbf{v}|_{y=0} = -\frac{\partial u}{\partial y}|_{y=0}. \tag{1.4.7}$$

Differentiating (1.4.6) with respect to t, we get the following interface condition

$$\rho_1 \left(\frac{\partial^2 u_1}{\partial t^2} - g\frac{\partial \eta}{\partial t} \right) = \rho_2 \left(\frac{\partial^2 u_2}{\partial t^2} - g\frac{\partial \eta}{\partial t} \right).$$

For steady-state oscillations, we define

$$u_1(t,x) = u_1(x)\exp(i\omega t)$$

and

$$u_2(t,x) = u_2(x)\exp(i\omega t).$$

The condition at the interface between two fluids is

$$\rho_1 \left(-\omega^2 u_1 + g\frac{\partial u_1}{\partial y} \right) = \rho_2 \left(-\omega^2 u_2 + g\frac{\partial u_2}{\partial y} \right)$$

or equivalently

$$\rho_1 \left(-\sigma u_1 + \frac{\partial u_1}{\partial y} \right) = \rho_2 \left(-\sigma u_2 + \frac{\partial u_2}{\partial y} \right), \tag{1.4.8}$$

where $\sigma = \frac{\omega^2}{g} > 0$ is the spectral parameter. It is the same at all interfaces. If the surface is open, the density of the upper layer is equal to zero, which yields

$$\rho_2 \left(\frac{\partial u_2}{\partial y} - \sigma u_2 \right) = 0. \tag{1.4.9}$$

We now have the equations describing the motion of ideal and incompressible two-layer fluids in a gravitational field in an open container,

$$\begin{cases} \triangle u_1 = 0 \text{ in } \Omega_1 \\ \triangle u_2 = 0 \text{ in } \Omega_2 \\ \rho_1 \left(\frac{\partial u_1}{\partial y} - \sigma u_1 \right) |_{S_1} = \rho_2 \left(\frac{\partial u_2}{\partial y} - \sigma u_2 \right) |_{S_1} \\ \left(\frac{\partial u_2}{\partial y} - \sigma u_2 \right) |_{S_2} = 0 \\ \frac{\partial u}{\partial n} |_{F_{ij}} = 0, \end{cases} \tag{1.4.10}$$

where u_1 and u_2 are the velocity-potentials of the fluids in domains Ω_1 and Ω_2, respectively, $\mathbf{n}$ is an outward normal to the boundary, and F_{ij} is defined as on the Figure.1.2.

Chapter 2

Main results

2.1. Overview

Small oscillations of a multi-layer fluid have previously been studied by Kopachevsky. In [**10**], Kopachevsky considered the problem of small oscillations of a multi-layer fluid, taking into account surface tension forces in the gravitational field, that is, when $m+1$ immiscible fluids (on the Figure 2.1 $m = 2$) with densities $\rho_1, \rho_2, ..., \rho_{m+1}$ fill a container completely, and occupy the equilibrium position in domains Ω_k, $k = 1,2,...,m+1$, σ_k, $k = 1,2,...$, are to be the coefficients of the surface tension on surface Γ_k between the domains Ω_k and Ω_{k+1}. On the basis of this work, the equation of the vibrations of immiscible liquids in a container has a real, positive, discrete spectrum $\{\lambda_k\}$ with the limit point $\lambda = +\infty$. The associated eigenfunctions form a complete and orthogonal system in the corresponding Hilbert space. Due to surface tension, in [**10**] surfaces Γ_k are supposed to have curvature, but in our work we do not take into consideration the surface tension. Consequently, the surfaces Γ_k are supposed to be straight. Based on the Kopachevsky result, the spectrum of a three-layer fluid in a closed domain is discrete. It also tells us that when for the domain Ω_{m+1} $\rho_{m+1} = 0$, then the spectrum is discrete as well.

2.2. Operator equations

We now construct operator equations for a three-layer liquid in a closed container and a two-layer fluid in an open container. The sloshing problem of a three-layer fluid in a closed

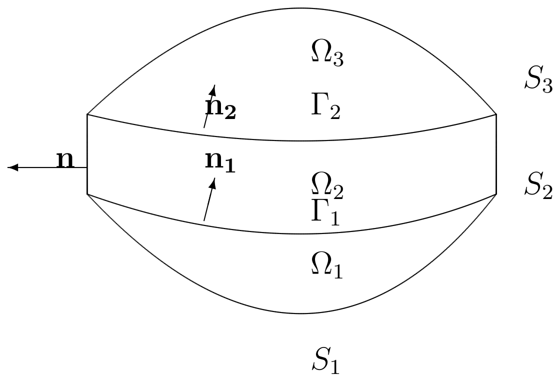

Fig. 2.1. Three-layer fluid in a closed container

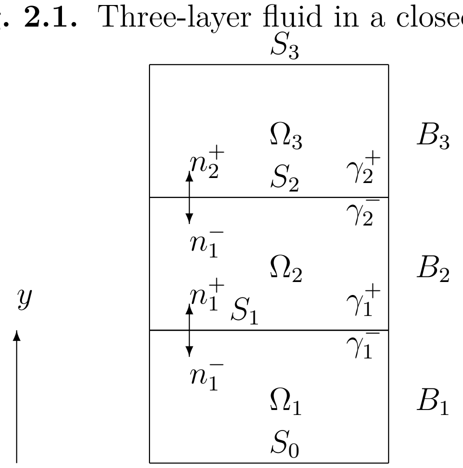

Fig. 2.2. Container with three-layer fluid

container is given by

$$\begin{cases} \triangle u_1 = 0 \qquad \text{in } \Omega_1 \\ \triangle u_2 = 0 \qquad \text{in } \Omega_2 \\ \triangle u_3 = 0 \qquad \text{in } \Omega_3 \\ \frac{\partial u_3}{\partial y}|_{S_3} = 0 \\ \rho_3 \left(\frac{\partial u_3}{\partial y} - \sigma u_3 \right) |_{S_2} = \rho_2 \left(\frac{\partial u_2}{\partial y} - \sigma u_2 \right) |_{S_2} \\ \frac{\partial u_3}{\partial y}|_{S_2} = \frac{\partial u_2}{\partial y}|_{S_2} \\ \rho_1 \left(\frac{\partial u_1}{\partial y} - \sigma u_1 \right) |_{S_1} = \rho_2 \left(\frac{\partial u_2}{\partial y} - \sigma u_2 \right) |_{S_1} \\ \frac{\partial u_1}{\partial y}|_{S_1} = \frac{\partial u_2}{\partial y}|_{S_1}, \frac{\partial u_2}{\partial y}|_{S_0} = 0 \\ \frac{\partial u_1}{\partial n}|_{B_1} = \frac{\partial u_2}{\partial n}|_{B_2} = \frac{\partial u_3}{\partial n}|_{B_3} = 0, \end{cases} \tag{2.2.1}$$

where u_i are velocity-potentials in domains Ω_i, ρ_i are the densities, $i \in \{1,2,3\}$.
Let us now consider the domain Ω_1.

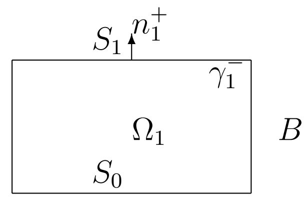

Fig. 2.3. Domain Ω_1 - three-layer fluid

We define the operator $(T_{\Omega_1})_1 : L_2'(S_1) \to H^1(\Omega_1)$, such that $(T_{\Omega_1})_1\phi_1 = u_1$, where $\phi_1 \in L_2'(S_1)$, and $u_1 \in H^1(\Omega_1)$, is a solution to the following boundary problem

$$\begin{cases} \triangle u_1 = 0 \text{ in } \Omega_1 \\ \frac{\partial u_1}{\partial n}|_{S_0} = 0 \\ \frac{\partial u_1}{\partial n}|_B = 0 \\ \frac{\partial u_1}{\partial n_1^+}|_{S_1} = \phi_1, \end{cases} \tag{2.2.2}$$

where $L_2'(S_1) = \{\phi \in L_2(S_1) : \int_{S_1} \phi dS = 0\}$, $L_2(S_1)$ is a space of the square-summable functions defined on S_1 and $H^1(\Omega_1)$ is a Sobolev space defined on Ω_1. Equation $\int_{S_1} \phi dS = 0$ in the boundary problem (2.2.2) means incompressibility, because ϕ defines the velocity of particles on S_1.
Now, we consider the domain Ω_3 and define the operator $(T_{\Omega_3})_2 : L_2'(S_2) \to H^1(\Omega_3)$, such

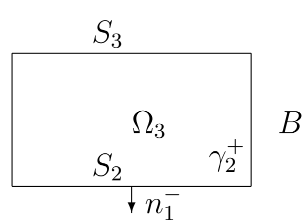

Fig. 2.4. Domain Ω_3 - three-layer fluid

that $(T_{\Omega_3})_2\phi_2 = u_3$ for $\phi_2 \in L_2'(S_2)$ and $u_3 \in H^1(\Omega_3)$ is the solution of the problem

$$\begin{cases} \triangle u_3 = 0 \qquad \text{in } \Omega_3 \\ \frac{\partial u_3}{\partial n}|_B = 0 \\ \frac{\partial u_3}{\partial n}|_{S_3} = 0 \\ \frac{\partial u_3}{\partial n_2^+}|_{S_2} = \phi_2. \end{cases} \tag{2.2.3}$$

We consider the domain Ω_2 and determine the function u_2 satisfied

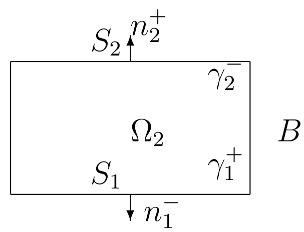

Fig. 2.5. Domain Ω_2 - three-layer fluid

$$\begin{cases} \triangle u_2 = 0 \qquad \text{in } \Omega_2 \\ \frac{\partial u_2}{\partial n}|_B = 0 \\ \frac{\partial u_2}{\partial n_1^+}|_{S_1} = \phi_1 \\ \frac{\partial u_2}{\partial n_2^+}|_{S_2} = \phi_2. \end{cases} \tag{2.2.4}$$

We define operators $(T_{\Omega_2})_1 : L_2'(S_1) \to H^1(\Omega_2)$ and $(T_{\Omega_2})_2 : L_2'(S_2) \to H^1(\Omega_2)$, such that $u_{21} = (T_{\Omega_2})_1 \phi_1, u_{22} = (T_{\Omega_2})_2 \phi_2$, where u_{21}, u_{22} satisfy

$$\begin{cases} \triangle u_{21} = 0 \qquad \text{in } \Omega_2 \\ \frac{\partial u_{21}}{\partial n}|_B = 0 \\ \frac{\partial u_{21}}{\partial n_1^+}|_{S_1} = \phi_1 \\ \frac{\partial u_{21}}{\partial n_2^+}|_{S_2} = 0, \end{cases} \tag{2.2.5}$$

$$\begin{cases} \triangle u_{22} = 0 \qquad \text{in } \Omega_2 \\ \frac{\partial u_{22}}{\partial n}|_B = 0 \\ \frac{\partial u_{22}}{\partial n_1^+}|_{S_1} = 0 \\ \frac{\partial u_{22}}{\partial n_2^+}|_{S_2} = \phi_2. \end{cases} \tag{2.2.6}$$

After decomposing the solution, $u_2 = u_{21} + u_{22}$, we arrive at

$$u_2 = (T_{\Omega_2})_1 \, \phi_1 + (T_{\Omega_2})_2 \, \phi_2. \tag{2.2.7}$$

Let us introduce the operator $(C^{\pm}_{ij})_k = \gamma^{\pm}_i (T_{\Omega_j})_k : L'_2(S_k) \to L'_2(S_i)$, where $\gamma^{\pm}_i : H^1(\Omega_j) \to L'_2(S_i)$ is a trace operator and $S_i \subset \partial\Omega$, S_k is a boundary where the normal derivative is defined. A sign of the γ_i is taken as on the Figure.2.2. The action of the operator $(C^{\pm}_{ij})_k$ will be defined in the subsection 3.1.1.

We denote $u \in L'_2(S_1) \bigoplus L'_2(S_2)$ and $u = (\phi_1, \phi_2)$. The operator equation for a three-layer fluid in a closed container including spectral parameter is now given by

$$Au = \sigma Bu, \tag{2.2.8}$$

where

$$A = \begin{pmatrix} \rho_1 - \rho_2 & 0 \\ 0 & \rho_2 - \rho_3 \end{pmatrix}, \tag{2.2.9}$$

and

$$B = \begin{pmatrix} \rho_1 (C^{-}_{11})_1 + \rho_2 (C^{+}_{12})_1 & \rho_2 (C^{+}_{12})_2 \\ \rho_2 (C^{-}_{22})_1 & \rho_2 (C^{-}_{22})_2 + \rho_3 (C^{+}_{23})_2 \end{pmatrix}. \tag{2.2.10}$$

For a two-layer fluid in an open container, the equation is given by

$$Au = \sigma Bu, \tag{2.2.11}$$

where

$$A = \begin{pmatrix} \rho_1 - \rho_2 & 0 \\ 0 & \rho_2 \end{pmatrix}, \tag{2.2.12}$$

and

$$B = \begin{pmatrix} \rho_1 (C^{-}_{11})_1 + \rho_2 (C^{+}_{12})_1 & \rho_2 (C^{+}_{12})_2 \\ \rho_2 (C^{-}_{22})_1 & \rho_2 (C^{-}_{22})_2 \end{pmatrix}. \tag{2.2.13}$$

Note that the operator equation for two-layer fluids in an open container follows directly from the operator equation of three-layer fluids in a closed container.

Theorem 2.2.1. *The spectrum of the operator equation of two-layer fluids in an open container coincides with the spectrum of the operator equation of three-layer fluids in the closed container, provided the top layer has density equal to zero.* [**10**]

This theorem allows us to apply all results obtained for multi-layer fluids in a closed container to open containers.

2.3. Discreteness of the spectrum

This section covers the properties of operators A and B for a three-layer fluids in closed container. According to Theorem 2.2.1, the spectral properties are the same for a two-layer fluid in an open container. We want to prove that the operator equation $Au = \lambda Bu$, where λ is a spectral parameter, has discrete spectrum. Let $L_2'(S_1) = \{u \in L_2(S_1) : \int_{S_1} u dS = 0\}$ and $L_2'(S_2) = \{u \in L_2(S_2) : \int_{S_2} u dS = 0\}$.

Lemma 2.3.1. *The operator A is positive definite in $H = L_2'(S_1) \bigoplus L_2'(S_2)$.*

Lemma 2.3.2. *The operator B is non-negative in $H = L_2'(S_1) \bigoplus L_2'(S_2)$.*

Lemma 2.3.3. *The operator B is self-adjoint in $H = L_2'(S_1) \bigoplus L_2'(S_2)$.*

Lemma 2.3.4. *The operator B is compact in $H = L_2'(S_1) \bigoplus L_2'(S_2)$.*

The results stated here will be proved in the section 3.2. We now immediately arrive at [**6**]

Theorem 2.3.5. *The operator equation $Au = \lambda Bu$ of the vibrations of a three-layer fluid in a closed container has a discrete spectrum*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \nearrow +\infty.$$

The eigenfunctions form a complete and orthogonal system in $H = L_2'(S_1) \bigoplus L_2'(S_2)$.

Theorem 2.3.6. *The operator equation $Au = \lambda Bu$ of the vibrations of a two-layer fluid in an open container has a discrete spectrum*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \nearrow +\infty.$$

The eigenfunctions form a complete and orthogonal system in $H = L_2'(S_1) \bigoplus L_2'(S_2)$.

2.4. The Weyl law

We obtain now the counting function for boundary value problem (1.4.10), which is for two-layer fluid in an open container.

Theorem 2.4.1. *The eigenvalue counting function of the sloshing problem for a two-layer fluid in an open bounded container in $\mathbb{R}^3$ satisfies the asymptotic relation*

$$N(\sigma) \sim \frac{1}{4\pi} \left[\int_{S_2} dS + \left(\frac{\rho + 1}{\rho - 1} \right)^2 \int_{S_1} dS \right] \sigma^2 \text{ as } \sigma \to \infty, \qquad (2.4.1)$$

where $\rho = \rho_1/\rho_2 > 1$. This result is a correction to the result obtained in [**11**] for the counting function of the spectrum for the sloshing of a two-layer fluid in a vertical cylinder. In the planar case the following result holds.

Theorem 2.4.2. *The eigenvalue counting function of the sloshing problem for a two-layer fluid in an open bounded container in* $\mathbb{R}^2$ *satisfies the asymptotic relation*

$$N(\sigma) \sim \frac{1}{\pi}\left[\int_{S_2} dS + \left(\frac{\rho+1}{\rho-1}\right)\int_{S_1} dS\right]\sigma \text{ as } \sigma \to \infty. \tag{2.4.2}$$

2.5. Examples

2.5.1. Rectangular container

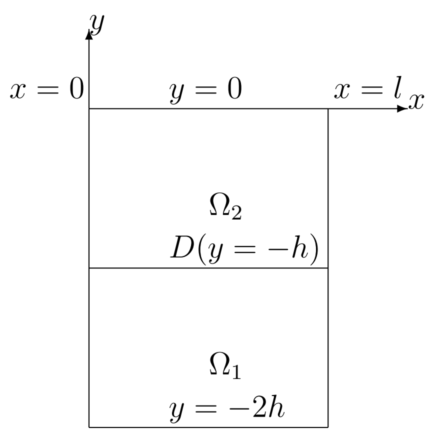

Fig. 2.6. Rectangular symmetric container

Let $\rho = \frac{\rho_1}{\rho_2} > 1$, where ρ_1 is a density of the fluid in Ω_1 and ρ_2 is a density of the fluid in Ω_2, $u \in H^1(\Omega_2)$ and $v \in H^1(\Omega_1)$ are the solutions. For a two-layer fluid in an open rectangular open container, the eigenvalues and eigenfunctions are

$$\begin{cases} \sigma_{1n} = \frac{\pi n}{l}\frac{\sqrt{2}\sqrt{-1+2\rho+\cosh\left(\frac{\pi n}{l}2h\right)\sinh\left(\frac{\pi n}{l}h\right)+\rho\sinh\left(\frac{\pi n}{l}2h\right)}}{\rho-1+(1+\rho)\cosh\left(\frac{\pi n}{l}2h\right)} \\ u_n(x,y) = \left(A_{1n}\cosh(\frac{\pi n}{l}y) + A_{2n}\sinh(\frac{\pi n}{l}y)\right)\cos(\frac{\pi n}{l}x) \\ v_n(x,y) = \left(A_{3n}\cosh(\frac{\pi n}{l}y) + A_{4n}\sinh(\frac{\pi n}{l}y)\right)\cos(\frac{\pi n}{l}x), \end{cases} \tag{2.5.1}$$

where A_{1n},A_{2n},A_{3n},A_{4n} are defined by

$$\begin{cases} A_{2n} = A_{1n}\frac{\sqrt{2}l^3\sqrt{\frac{n^6\left(-1+2\rho+\cosh(2h\frac{\pi n}{l})\right)}{l^6}}\sinh(h\frac{\pi n}{l})+n^3\rho\sinh(2h\frac{\pi n}{l})}{n^3\left(-1+\rho+(1+\rho)\cosh(2h\frac{\pi n}{l})\right)} \\ A_{3n} = -A_{1n}\cosh(2h\frac{\pi n}{l})\frac{-n^3+n^3\cosh(2h\frac{\pi n}{l})-\coth(h\frac{\pi n}{l})\sqrt{2}l^3\sqrt{\frac{n^6\left(-1+2\rho+\cosh(2h\frac{\pi n}{l})\right)}{l^6}}\sinh(h\frac{\pi n}{l})}{n^3\left(-1+\rho+(1+\rho)\cosh(2h\frac{\pi n}{l})\right)} \\ A_{4n} = A_{1n}\cosh(2h\frac{\pi n}{l})\frac{3n^3\sinh(h\frac{\pi n}{l})+2\cosh(h\frac{\pi n}{l})\sqrt{2}l^3\sqrt{\frac{n^6\left(-1+2\rho+\cosh(2h\frac{\pi n}{l})\right)}{l^6}}\sinh(h\frac{\pi n}{l})-n^3\sinh(3h\frac{\pi n}{l})}{n^3\left(-1+\rho+(1+\rho)\cosh(2h\frac{\pi n}{l})\right)}, \end{cases}$$

and

$$\begin{cases} \sigma_{2n} = -\frac{\pi n}{l}\frac{\sqrt{2}\sqrt{-1+2\rho+\cosh\left(\frac{\pi n}{l}2h\right)}\sinh\left(\frac{\pi n}{l}h\right)-\rho\sinh\left(\frac{\pi n}{l}2h\right)}{\rho-1+(1+\rho)\cosh\left(\frac{\pi n}{l}2h\right)} \\ u_n(x,y) = \left(B_{1n}\cosh(\frac{\pi n}{l}y)+B_{2n}\sinh(\frac{\pi n}{l}y)\right)\cos(\frac{\pi n}{l}x) \\ v_n(x,y) = \left(B_{3n}\cosh(\frac{\pi n}{l}y)+B_{4n}\sinh(\frac{\pi n}{l}y)\right)\cos(\frac{\pi n}{l}x), \end{cases} \tag{2.5.2}$$

where B_{1n},B_{2n},B_{3n},B_{4n} are defined by

$$\begin{cases} B_{2n} = B_{1n}\frac{-\sqrt{2}l^3\sqrt{\frac{n^6\left(-1+2\rho+\cosh(2h\frac{\pi n}{l})\right)}{l^6}}\sinh(h\frac{\pi n}{l})-n^3\rho\sinh(2h\frac{\pi n}{l})}{n^3\left(-1+\rho+(1+\rho)\cosh(2h\frac{\pi n}{l})\right)} \\ B_{3n} = -B_{1n}\cosh(2h\frac{\pi n}{l})\frac{-n^3+n^3\cosh(2h\frac{\pi n}{l})+\coth(h\frac{\pi n}{l})\sqrt{2}l^3\sqrt{\frac{n^6\left(-1+2\rho+\cosh(2h\frac{\pi n}{l})\right)}{l^6}}\sinh(h\frac{\pi n}{l})}{n^3\left(-1+\rho+(1+\rho)\cosh(2h\frac{\pi n}{l})\right)} \\ B_{4n} = -B_{1n}\cosh(2h\frac{\pi n}{l})\frac{-3n^3\sinh(h\frac{\pi n}{l})+2\cosh(h\frac{\pi n}{l})\sqrt{2}l^3\sqrt{\frac{n^6\left(-1+2\rho+\cosh(2h\frac{\pi n}{l})\right)}{l^6}}\sinh(h\frac{\pi n}{l})-n^3\sinh(3h\frac{\pi n}{l})}{n^3\left(-1+\rho+(1+\rho)\cosh(2h\frac{\pi n}{l})\right)}. \end{cases}$$

Here we again have that $\rho = \rho_1/\rho_2 > 1$.

For the rectangular domain considered above we will compare the counting function based on the eignevalues in theorem 2.4.2, for the planar case when $\rho = 1.2$, $l/2h = 0.5$.

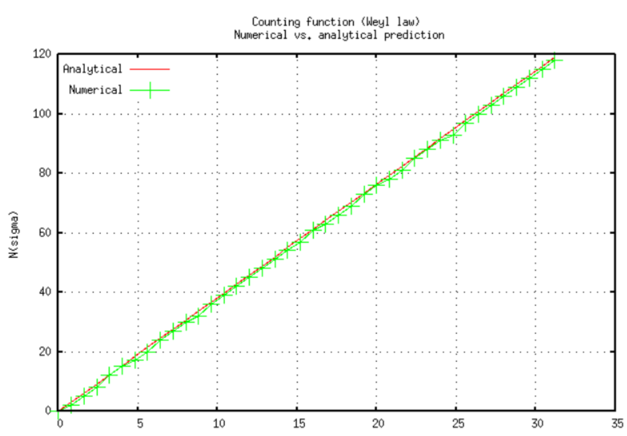

Fig. 2.7. Numerical results for a rectangular container

2.5.2. The case of an extremely heavy fluid

We now consider the situation when the bottom fluid is much heavier than the top one, which is similar to the condition when $\rho \to \infty$. In case when ρ is infinite, the eigenvalues and eigenfunctions are

$$\begin{cases} \sigma_{1n} = \sigma_{2n} = \frac{\pi n}{l} \tanh\left(\frac{\pi n}{l}\right) \\ u_n(x,y) = A_{1n} \frac{\cosh(y+h)\frac{\pi n}{l}}{\cosh(h\frac{\pi n}{l})} \cos(\frac{\pi n}{l} x) \\ v_n(x,y) = 0. \end{cases} \tag{2.5.3}$$

It is apparent that if $\frac{\pi n}{l} \to \infty$, then $\sigma_{1n} \sim \frac{\pi n}{l}$, where l is as on the Figure 2.6. The counting function gives

$$N(\sigma) \sim \frac{l}{\pi}\sigma. \tag{2.5.4}$$

Note that the counting function for this case is different than the result in Theorem 2.4.2 which predicts $\frac{2l}{\pi}\sigma$. This phenomenon is discussed in the following section.

2.5.3. The study of the limiting case

In this section, we consider the solution for a rectangular container as $\rho \to \infty$. Assume that d and h are arbitrary real numbers, such that $d > h$. Let ϕ_{1n} and ϕ_{2n} be the normal

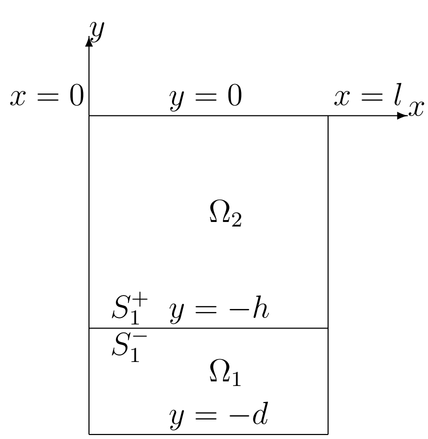

Fig. 2.8. Rectangular container with two fluids

derivatives defined on boundaries $y = -h$ and $y = 0$, respectively. Hence, the solution is

$$\begin{cases} \sigma_{1n} = \frac{\pi n}{l} \tanh\left(\frac{\pi n}{l}(d-h)\right) \\ \phi_{1n} = \phi_{2n} \frac{\sinh\left(\frac{\pi n}{l}(d-2h)\right)}{\sinh\left(\frac{\pi n}{l}(d-h)\right)} \\ d \neq 2h, \end{cases} \tag{2.5.5}$$

and

$$\begin{cases} \sigma_{2n} = \frac{\pi n}{l} \tanh\left(\frac{\pi n}{l} h\right) \\ \phi_{1n} = 0 \qquad \phi_{2n} \neq 0. \end{cases} \tag{2.5.6}$$

If $d = 2h$, then we obtain $\phi_{1n} = 0$ and $\phi_{2n} \neq 0$ for the first family of solutions. Hence, we get only one family of eigenfunctions that correspond to $\sigma_n = \frac{\pi n}{l} \tanh\left(\frac{\pi n}{l} h\right)$.

2.6. Existence and uniqueness of the solution

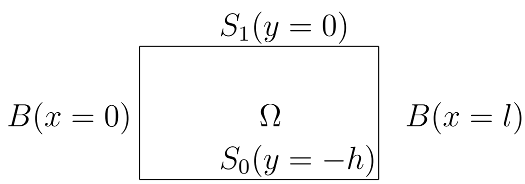

Fig. 2.9. A supplementary problem domain

Consider now an auxiliary problem

$$\begin{cases} \triangle u = 0 \qquad \text{in } \Omega \\ \frac{\partial u}{\partial n}|_B = 0 \\ \frac{\partial u}{\partial n} - \sigma u|_{S_1} = 0 \\ \frac{\partial u}{\partial n}|_{S_0} = \phi, \end{cases} \tag{2.6.1}$$

where $\sigma > 0$. We wish to solve the equations for function ϕ and parameter σ.

Proposition 2.6.1. *The boundary value problem (2.6.1) satisfies the following two properties*

- *if* $\sigma \notin \{\frac{\pi n}{l} \tan\left(\frac{\pi n}{l} h\right) : n = 1, \dots\}$, *then a solution exists and is unique.*
- *if* $\exists n : \sigma = \frac{\pi n}{l} \tan\left(\frac{\pi n}{l} h\right)$, *then a solution exists if and only if* $\left(\phi(x), \cos\left(\frac{\pi n}{l} x\right)\right)_{L_2(S_0)} = 0$.

2.7. General case

Finally, we consider the problem of the existence and uniqueness of the solution with a negative Robin parameter in general. Let Ω be an arbitrary domain with smooth boundary.

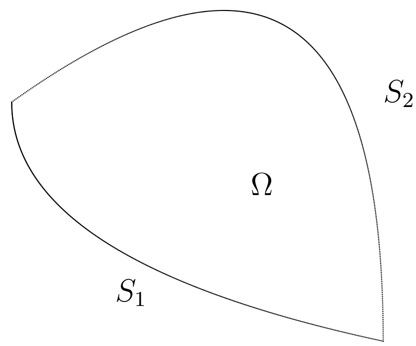

Fig. 2.10. Arbitrary fluid domain

$$\begin{cases} \triangle u = 0 \\ \frac{\partial u}{\partial n} - \sigma u|_{S_1} = 0 \\ \frac{\partial u}{\partial n}|_{S_2} = \phi, \end{cases} \tag{2.7.1}$$

where σ is a parameter. In addition to (2.7.1), we consider the auxiliary problem

$$\begin{cases} \triangle w = 0 \\ \frac{\partial w}{\partial n}|_{S_1} = 0 \\ \frac{\partial w}{\partial n}|_{S_2} = \phi \end{cases} \tag{2.7.2}$$

and the sloshing problem

$$\begin{cases} \triangle v = 0 \\ \frac{\partial v}{\partial n} - \lambda v|_{S_1} = 0 \\ \frac{\partial v}{\partial n}|_{S_2} = 0. \end{cases} \tag{2.7.3}$$

Proposition 2.7.1. *Let* $\{\lambda_s\}$ *and* v_s *be the eigenvalues and eigenfunctions of the sloshing problem (2.7.3). The boundary value problem of (2.7.1) possesses the following three properties*

- *if* $\sigma \notin \{\lambda_s\}$, *then a solution exists and is unique*
- *if* $\exists n : \sigma = \lambda_n$, *then a solution exists in case of* $\int_{S_2} v_s \phi = 0$
- *otherwise, a solution does not exist.*

Chapter 3

Proofs

In this chapter we explain the theorems and formulas obtained in the previous chapter.

3.1. Operator equations

3.1.1. Operator equation for a three-layer fluid in a closed container

In this section, we provide the construction of the equations in section 2.2. The boundary value problem (2.2.2) is a Neumann problem, which has a solution unique up to a constant term [**13**]. This means we may select u_1 such that

$$\int_{S_1} u_1 dS = 0. \tag{3.1.1}$$

Taking into the account the trace operator $\gamma_1^- : H^1(\Omega_1) \to L_2'(S_1)$, we get

$$u_1|_{S_1} = \gamma_1^-(T_{\Omega_1})_1\phi_1. \tag{3.1.2}$$

Finally, for Ω_1, we conclude

$$\begin{cases} \frac{\partial u_1}{\partial y}|_{S_1} = \phi_1 \\ u_1|_{S_1} = (C_{11}^-)_1\phi_1, \end{cases} \tag{3.1.3}$$

where $(C_{ij}^{\pm})_k$ were already defined.

Similarly, the boundary value problem (2.2.3) is a Neumann value problem where we again select u_3 such that

$$\int_{S_2} u_3 dS = 0. \tag{3.1.4}$$

Considering a trace operator $\gamma_2^+ : H^1(\Omega_3) \to L_2'(S_2)$, we get

$$u_3|_{S_2} = \gamma_2^+ (T_{\Omega_3})_2 \phi_2. \tag{3.1.5}$$

For the domain Ω_3, we obtain the following result

$$\begin{cases} \frac{\partial u_3}{\partial y}|_{S_2} = \phi_2 \\ -u_3|_{S_2} = (C_{23}^+)_2 \phi_2. \end{cases} \tag{3.1.6}$$

Based on the operators defined previously, we obtain

$$\begin{cases} -u_2|_{S_1} = \gamma_1^+ \left((T_{\Omega_2})_1 \phi_1 + (T_{\Omega_2})_2 \phi_2 \right) = \left(C_{12}^+ \right)_1 \phi_1 + \left(C_{12}^+ \right)_2 \phi_2 \\ u_2|_{S_2} = \gamma_2^- \left((T_{\Omega_2})_1 \phi_1 + (T_{\Omega_2})_2 \phi_2 \right) = \left(C_{22}^- \right)_1 \phi_1 + \left(C_{22}^- \right)_2 \phi_2. \end{cases} \tag{3.1.7}$$

Choose u_2 such, that

$$\begin{aligned} \int_{S_1} u_2 dS = 0, \\ \int_{S_2} u_2 dS = 0. \end{aligned} \tag{3.1.8}$$

Summarizing all above, we get

$$\begin{aligned} \frac{\partial u_1}{\partial y}|_{S_1} = \phi_1, \qquad \frac{\partial u_2}{\partial y}|_{S_1} = \phi_1, \\ \frac{\partial u_2}{\partial y}|_{S_2} = \phi_2, \qquad \frac{\partial u_3}{\partial y}|_{S_2} = \phi_2, \\ -u_3|_{S_2} = (C_{23}^+)_2 \phi_2, \qquad u_1|_{S_1} = (C_{11}^-)_1 \phi_1, \\ -u_2|_{S_1} = \left(C_{12}^+ \right)_1 \phi_1 + \left(C_{12}^+ \right)_2 \phi_2, \\ u_2|_{S_2} = \left(C_{22}^- \right)_1 \phi_1 + \left(C_{22}^- \right)_2 \phi_2. \end{aligned} \tag{3.1.9}$$

Using (2.2.1) and taking into account (3.1.9), we obtain

$$\begin{cases} \rho_1 \left(\frac{\partial u_1}{\partial y} - \sigma u_1 \right) |_{S_1} = \rho_1 \left(\phi_1 - \sigma (C_{11}^-)_1 \phi_1 \right) \\ \rho_2 \left(\frac{\partial u_2}{\partial y} - \sigma u_2 \right) |_{S_1} = \rho_2 \left(\phi_1 + \sigma \left(\left(C_{12}^+ \right)_1 \phi_1 + \left(C_{12}^+ \right)_2 \phi_2 \right) \right) \\ \rho_2 \left(\frac{\partial u_2}{\partial y} - \sigma u_2 \right) |_{S_2} = \rho_2 \left(\phi_2 - \sigma \left(\left(C_{22}^- \right)_1 \phi_1 + \left(C_{22}^+ \right)_2 \phi_2 \right) \right) \\ \rho_3 \left(\frac{\partial u_3}{\partial y} - \sigma u_3 \right) |_{S_2} = \rho_3 \left(\phi_2 + \sigma (C_{23}^+)_2 \phi_2 \right). \end{cases} \tag{3.1.10}$$

It follows that

$$\begin{cases} (\rho_1 - \rho_2) \phi_1 = \sigma \left(\rho_1 (C_{11}^-)_1 + \rho_2 (C_{12}^+)_1 \right) \phi_1 + \sigma \rho_2 (C_{12}^+)_2 \phi_2 \\ (\rho_2 - \rho_3) \phi_2 = \sigma \rho_2 (C_{22}^-)_1 \phi_1 + \sigma \left(\rho_2 (C_{22}^-)_2 + \rho_3 (C_{23}^+)_2 \right) \phi_2. \end{cases} \tag{3.1.11}$$

Equations (3.1.11) are the operator equations related to a three-layer fluid in a closed container. Let $u \in L_2'(S_1) \bigoplus L_2'(S_2)$ and $u = (\phi_1, \phi_2)^T$. For a closed container, the operator equation (3.1.11) can be written as

$$Au = \sigma Bu, \tag{3.1.12}$$

where

$$A = \begin{pmatrix} \rho_1 - \rho_2 & 0 \\ 0 & \rho_2 - \rho_3 \end{pmatrix}, \tag{3.1.13}$$

and

$$B = \begin{pmatrix} \rho_1(C_{11}^-)_1 + \rho_2(C_{12}^+)_1 & \rho_2(C_{12}^+)_2 \\ \rho_2(C_{22}^-)_1 & \rho_2(C_{22}^-)_2 + \rho_3(C_{23}^+)_2 \end{pmatrix}. \tag{3.1.14}$$

3.1.2. Operator equation for a two-layer fluid in an open domain

Using (1.4.10), we derive the following for a two-layer fluid in an open domain,

$$\begin{cases} \rho_1 \left(\frac{\partial u_1}{\partial y} - \sigma u_1\right)|_{S_1} = \rho_1 \left(\phi_1 - \sigma(C_{11}^-)_1\phi_1\right) \\ \rho_2 \left(\frac{\partial u_2}{\partial y} - \sigma u_2\right)|_{S_1} = \rho_2 \left(\phi_1 + \sigma \left(\left(C_{12}^+\right)_1 \phi_1 + \left(C_{12}^+\right)_2 \phi_2\right)\right) \\ \left(\frac{\partial u_2}{\partial y} - \sigma u_2\right)|_{S_2} = \left(\phi_2 - \sigma \left(\left(C_{22}^-\right)_1 \phi_1 + \left(C_{22}^+\right)_2 \phi_2\right)\right) = 0. \end{cases} \tag{3.1.15}$$

Hence

$$Au = \sigma Bu, \tag{3.1.16}$$

where

$$A = \begin{pmatrix} \rho_1 - \rho_2 & 0 \\ 0 & \rho_2 \end{pmatrix}, \tag{3.1.17}$$

and

$$B = \begin{pmatrix} \rho_1(C_{11}^-)_1 + \rho_2(C_{12}^+)_1 & \rho_2(C_{12}^+)_2 \\ \rho_2(C_{22}^-)_1 & \rho_2(C_{22}^-)_2 \end{pmatrix}. \tag{3.1.18}$$

3.1.3. Proof of Theorem 2.2.1

Note that this result agrees with the physical intuition and that follows by substituting $\rho_3 = 0$ in (2.2.1).

If we assume $\rho_3 = 0$ in (2.2.1), then the system splits into two systems

$$\begin{cases} \triangle u_3 = 0 \qquad \text{in } \Omega_3 \\ \frac{\partial u_3}{\partial y}|_{S_3} = 0 \\ \frac{\partial u_3}{\partial n}|_{\partial\Omega} = 0 \\ \frac{\partial u_3}{\partial y}|_{S_2} = \frac{\partial u_2}{\partial y}|_{S_2}, \end{cases} \tag{3.1.19}$$

and

$$\begin{cases} \triangle u_1 = 0 \qquad \text{in } \Omega_1 \\ \triangle u_2 = 0 \qquad \text{in } \Omega_2 \\ \rho_2 \left(\frac{\partial u_2}{\partial y} - \sigma u_2\right)|_{S_2} = 0 \\ \rho_1 \left(\frac{\partial u_1}{\partial y} - \sigma u_1\right)|_{S_1} = \rho_2 \left(\frac{\partial u_2}{\partial y} - \sigma u_2\right)|_{S_1} \\ \frac{\partial u_1}{\partial y}|_{S_1} = \frac{\partial u_2}{\partial y}|_{S_1} \\ \frac{\partial u_2}{\partial y}|_{S_0} = 0 \\ \frac{\partial u_1}{\partial n}|_{\partial\Omega} = \frac{\partial u_2}{\partial n}|_{\partial\Omega} = 0. \end{cases} \tag{3.1.20}$$

The second system coincides with (1.4.10). The first system does not contain any spectral parameter and is the Neumann problem with respect to function u_3 defined by the derivative of u_2 on S_2. This means that the spectrum of the operator equation of a three-layer fluid in a closed domain with the weightless layer at the top coincides with the spectrum of the operator equation of a two-layer fluid in an open domain.

3.2. Discreteness of the spectrum

3.2.1. Proof of Lemma 2.3.1

Let $\phi = (\phi_1,\phi_2) \in L_2'(S_1) \bigoplus L_2'(S_2)$ and A be the operator defined by (3.1.13). We have

$$(A\phi,\phi)_H = (\rho_1-\rho_2)(\phi_1,\phi_1)_{L_2'(S_1)}+(\rho_2-\rho_3)(\phi_2,\phi_2)_{L_2'(S_2)} \geq \gamma^2 \left((\phi_1,\phi_1)_{L_2'(S_1)} + (\phi_2,\phi_2)_{L_2'(S_2)}\right) = \gamma^2||\phi||_H^2,$$

where $H = L_2'(S_1) \bigoplus L_2'(S_2)$, $||\phi||_H^2 = \left((\phi_1,\phi_1)_{L_2'(S_1)} + (\phi_2,\phi_2)_{L_2'(S_2)}\right)$ and $\gamma^2 = min\{\rho_1 - \rho_2,\rho_2 - \rho_3\}$.

Therefore, we have shown

$$(A\phi,\phi)_H \geq \gamma^2||\phi||_H^2. \tag{3.2.1}$$

This proves Lemma 2.3.1.

3.2.2. Proof of Lemma 2.3.2

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary $\partial\Omega$ and $S \subset \partial\Omega$. Consider $W_2^1(\Omega)$, Sobolev space. This functional space is equipped with the norm [**6**] defined by

$$||u||^2_{W_2^1(\Omega)} = ||u||^2_{L_2(\Omega)} + |||\nabla u|||^2_{L_2(\Omega)} \tag{3.2.2}$$

for each $u \in W_2^1(\Omega)$.

According to [**13**, formula (9.13), p. 64], one can define an equivalent norm on $W_2^1(\Omega)$ by the formula

$$||u||_2^2 = \left(\int_S u|_S dS\right)^2 + |||\nabla u|||^2_{L_2(\Omega)}.$$

Hence, if we consider a subspace $\dot{W}_2^1(\Omega) \subset W_2^1(\Omega)$ defined by

$$\dot{W}_2^1(\Omega) = \{u \in W_2^1(\Omega) : \int_S u|_S dS = 0, S \subset \partial\Omega\}, \tag{3.2.3}$$

then the norm

$$||u||^2_{\dot{W}_2^1(\Omega)} = |||\nabla u|||^2_{L_2(\Omega)} \tag{3.2.4}$$

is equivalent to the usual Sobolev norm.

Let B be the operator defined by (3.1.14), we have

$$(B\phi,\phi)_H = \rho_1\left((C_{11}^-)_1\phi_1,\phi_1\right)_{L_2'(S_1)} + \rho_2\left(\left(C_{12}^+\right)_1\phi_1 + \left(C_{12}^+\right)_2\phi_2,\phi_1\right)_{L_2'(S_1)} + \\ \rho_2\left(\left(C_{22}^-\right)_1\phi_1 + \left(C_{22}^-\right)_2\phi_2,\phi_2\right)_{L_2'(S_2)} + \rho_3\left((C_{23}^+)_2\phi_2,\phi_2\right)_{L_2'(S_2)}. \tag{3.2.5}$$

Also, we have that

$$\left((C_{11}^-)_1\phi_1,\phi_1\right)_{L_2'(S_1)} = \int_{S_1}((C_{11}^-)_1\phi_1)\phi_1 dS = \int_{S_1}(u_1|_{S_1})\phi_1 dS = \tag{3.2.6}$$

$$\int_{S_1}(u_1|_{S_1})\frac{\partial u_1}{\partial n_1^+}dS = \int_{S_1}(u_1|_{S_1})\frac{\partial u_1}{\partial n}dS = \int_{S_1} u_1 n\cdot\nabla u_1 dS = \int_{\partial\Omega_1} u_1 n\cdot\nabla u_1 dS = \tag{3.2.7}$$

$$\int_{\partial\Omega_1} u_1 n\cdot\nabla u_1 dS = \int_{\Omega_1}\nabla\cdot(u_1\nabla u_1)dS = \int_{\Omega_1}\nabla u_1\cdot\nabla u_1 + u_1\triangle u_1)d\Omega = \tag{3.2.8}$$

$$\int_{\Omega_1}\nabla u_1\cdot\nabla u_1 d\Omega, \tag{3.2.9}$$

where $(C_{ij}^{\pm})_k = \gamma_i^{\pm}(T_{\Omega_j})_k : L_2'(S_k) \to L_2'(S_i)$, $\gamma_i^{\pm} : H^1(\Omega_j) \to L_2'(S_i)$ is a trace operator and $S_i \subset \partial\Omega$, $\partial\Omega$ is a boundary of the Ω. S_k is a boundary, where the normal derivative is defined.

Finally, based on the (3.2.4) and (3.2.3), we have that

$$\Big((C_{11}^-)_1\phi_1,\phi_1\Big)_{L_2'(S_1)} = ||u_1||^2_{\dot{W}_2^1(\Omega_1)}, \tag{3.2.10}$$

and

$$\Big((C_{23}^+)_2\phi_2,\phi_2\Big)_{L_2'(S_2)} = ||u_3||^2_{\dot{W}_2^1(\Omega_3)}. \tag{3.2.11}$$

Let us now calculate

$$\Big(\Big(C_{12}^+\Big)_1\phi_1 + \Big(C_{12}^+\Big)_2\phi_2,\phi_1\Big)_{L_2'(S_1)} = \int_{S_1}(-u_2)|_{S_1}\phi_1 dS =$$

$$\int_{S_1}(-u_2)|_{S_1}\frac{\partial u_2}{\partial n_1^+}dS = \int_{S_1}u_2|_{S_1}\left(\frac{\partial u_2}{\partial n}\right)dS = \int_{S_1}u_2 n\cdot\nabla u_2 dS.$$

Here we used the fact that $\frac{\partial u_2}{\partial n_1^+} = -\frac{\partial u_2}{\partial n}$ on the surface S_1 for the domain Ω_2, since we should use an outward normal. Summarizing, we get

$$\Big(\Big(C_{12}^+\Big)_1\phi_1 + \Big(C_{12}^+\Big)_2\phi_2,\phi_1\Big)_{L_2'(S_1)} = \int_{S_1}u_2 n\cdot\nabla u_2 dS. \tag{3.2.12}$$

According to the same calculations,

$$\Big(\Big(C_{22}^-\Big)_1\phi_1 + \Big(C_{22}^-\Big)_2\phi_2,\phi_2\Big)_{L_2'(S_2)} = \int_{S_2}u_2 n\cdot\nabla u_2 dS. \tag{3.2.13}$$

We now have

$$\Big(\Big(C_{12}^+\Big)_1\phi_1 + \Big(C_{12}^+\Big)_2\phi_2,\phi_1\Big)_{L_2'(S_1)} + \Big(\Big(C_{22}^-\Big)_1\phi_1 + (C_{22}^-)_2\phi_2,\phi_2\Big)_{L_2'(S_2)} =$$

$$\int_{S_1}u_2 n\cdot\nabla u_2 dS + \int_{S_2}u_2 n\cdot\nabla u_2 dS =$$

$$\int_{\partial\Omega_2}u_2 n\cdot\nabla u_2 dS = \int_{\Omega_2}\nabla u_2\cdot\nabla u_2 + u_2\triangle u_2)d\Omega = \int_{\Omega_2}\nabla u_2\cdot\nabla u_2 d\Omega = ||u_2||^2_{\dot{W}_2^1(\Omega_2)}.$$

Finally, we have that

$$\Big(\Big(C_{12}^+\Big)_1\phi_1 + \Big(C_{12}^+\Big)_2\phi_2,\phi_1\Big)_{L_2'(S_1)} + \Big(\Big(C_{22}^-\Big)_1\phi_1 + (C_{22}^-)_2\phi_2,\phi_2\Big)_{L_2'(S_2)} = ||u_2||^2_{\dot{W}_2^1(\Omega_2)}. \tag{3.2.14}$$

Therefore, we conclude

$$(B\phi,\phi)_H = \rho_1||u_1||^2_{\dot{W}_2^1(\Omega_1)} + \rho_2||u_2||^2_{\dot{W}_2^1(\Omega_2)} + \rho_3||u_3||^2_{\dot{W}_2^1(\Omega_3)} \geq 0. \tag{3.2.15}$$

This proves Lemma 2.3.2.

3.2.3. Proof of Lemma 2.3.3

We first calculate $(B\phi,\theta)_H$,

$$(B\phi,\theta)_H = \rho_1\left((C_{11}^-)_1\phi_1,\theta_1\right)_{L_2'(S_1)} + \rho_2\left(\left(C_{12}^+\right)_1\phi_1 + \left(C_{12}^+\right)_2\phi_2,\theta_1\right)_{L_2'(S_1)} +$$
$$\rho_2\left(\left(C_{22}^-\right)_1\phi_1 + \left(C_{22}^-\right)_2\phi_2,\theta_2\right)_{L_2'(S_2)} + \rho_3\left((C_{23}^+)_2\phi_2,\theta_2\right)_{L_2'(S_2)}.$$

Next, we have

$$\left((C_{11}^-)_1\phi_1,\theta_1\right)_{L_2'(S_1)} = \int_{S_1}((C_{11}^-)_1\phi_1)\theta_1 dS = \int_{S_1} u_1\theta_1 dS.$$

Let us introduce the function v, such that

$$\begin{cases} \triangle v = 0 \qquad \text{in } \Omega_1 \\ \frac{\partial v}{\partial n}|_{S_0} = 0 \\ \frac{\partial v}{\partial n}|_{B} = 0 \\ \int_{S_1} v dS = 0 \\ \frac{\partial v}{\partial n_1^+}|_{S_1} = \theta_1. \end{cases}$$

Hence,

$$\int_{S_1} u_1\frac{\partial v}{\partial n_1^+}dS = \int_{S_1} u_1\frac{\partial v}{\partial n}dS = \int_{\partial\Omega_1} u_1\frac{\partial v}{\partial n}dS = \int_{\partial\Omega_1} u_1 n\cdot\nabla v dS =$$
$$\int_{\Omega_1}\nabla\cdot(u_1\nabla v)d\Omega = \int_{\Omega_1}\left(\nabla u_1\cdot\nabla v + u_1\triangle v\right)d\Omega = \int_{\Omega_1}\left(\nabla u_1\cdot\nabla v + v\triangle u_1\right)d\Omega = \int_{\Omega_1}\nabla\cdot(v\nabla u_1)d\Omega =$$
$$\int_{\partial\Omega_1} vn\cdot\nabla u_1 dS = \int_{S_1} v\frac{\partial u_1}{\partial n}dS = \int_{S_1} v\frac{\partial u_1}{\partial n_1^+}dS = \int_{S_1} v\phi_1 dS.$$

Since $v|_{S_1} = (C_{11}^-)_1\theta_1$, we have that

$$\left((C_{11}^-)_1\phi_1,\theta_1\right)_{L_2'(S_1)} = \left((C_{11}^-)_1\theta_1,\phi_1\right)_{L_2'(S_1)} = \left(\phi_1,(C_{11}^-)_1\theta_1\right)_{L_2'(S_1)}. \tag{3.2.16}$$

Similarly,

$$\left((C_{23}^+)_2\phi_2,\theta_2\right)_{L_2'(S_2)} = \left((C_{23}^+)_2\theta_2,\phi_2\right)_{L_2'(S_2)}. \tag{3.2.17}$$

We continue with $\left(\left(C_{12}^+\right)_1\phi_1 + \left(C_{12}^+\right)_2\phi_2,\theta_1\right)_{L_2'(S_1)}$ and introduce v_{21} such that

$$\begin{cases} \triangle v_{21} = 0 \qquad \text{in } \Omega_2 \\ \frac{\partial v_{21}}{\partial n}|_{B} = 0 \\ \frac{\partial v_{21}}{\partial n_1^+}|_{S_1} = \theta_1 \\ \frac{\partial v_{21}}{\partial n_2^+}|_{S_2} = 0. \end{cases}$$

Hence,

$$\left(\left(C_{12}^{+}\right)_1 \phi_1 + \left(C_{12}^{+}\right)_2 \phi_2, \theta_1\right)_{L_2'(S_1)} = \int_{S_1} (-u_2)|_{S_1} \theta_1 dS =$$

$$\int_{S_1} (-u_2)|_{S_1} \frac{\partial v_{21}}{\partial n_1^{+}} dS = \int_{S_1} u_2|_{S_1} \frac{\partial v_{21}}{\partial n_1^{-}} dS,$$

where $(n_1^{+} = -n_1^{-})$ is the outward normal to Ω_2 on S_1,

$$\int_{S_1} u_2|_{S_1} n \cdot \nabla v_{21} dS = \int_{\partial\Omega_2} u_2|_{S_1} n \cdot \nabla v_{21} dS =$$

$$\int_{\Omega_2} \nabla u_2 \cdot \nabla v_{21} d\Omega = \int_{\partial\Omega_2} v_{21} \frac{\partial u_2}{\partial n} dS.$$

Finally, we get

$$\left(\left(C_{12}^{+}\right)_1 \phi_1 + \left(C_{12}^{+}\right)_2 \phi_2, \theta_1\right)_{L_2'(S_1)} = \int_{\partial\Omega_2} v_{21} \frac{\partial u_2}{\partial n} dS. \tag{3.2.18}$$

Following the same method, we get

$$\left(\left(C_{22}^{-}\right)_1 \phi_1 + \left(C_{22}^{-}\right)_2 \phi_2, \theta_2\right)_{L_2'(S_2)} = \int_{\partial\Omega_2} v_{22} \frac{\partial u_2}{\partial n} dS, \tag{3.2.19}$$

where

$$\begin{cases} \triangle v_{22} = 0 \qquad \text{in } \Omega_2 \\ \frac{\partial v_{22}}{\partial n}|_B = 0 \\ \frac{\partial v_{22}}{\partial n_1^{+}}|_{S_1} = 0 \\ \frac{\partial v_{22}}{\partial n_2^{+}}|_{S_2} = \theta_2. \end{cases}$$

Hence, we get

$$\int_{\partial\Omega_2} v_{21} \frac{\partial u_2}{\partial n} dS + \int_{\partial\Omega_2} v_{22} \frac{\partial u_2}{\partial n} dS = \int_{\partial\Omega_2} v_2 \frac{\partial u_2}{\partial n} dS, \tag{3.2.20}$$

where

$$\begin{cases} \triangle v_2 = 0 \qquad \text{in } \Omega_2 \\ \frac{\partial v}{\partial n}|_B = 0 \\ \frac{\partial v_2}{\partial n_1^{+}}|_{S_1} = \theta_1 \\ \frac{\partial v_2}{\partial n_2^{+}}|_{S_2} = \theta_2. \end{cases}$$

After setting

$$\begin{cases} -v_2|_{S_1} = \left(C_{12}^{+}\right)_1 \theta_1 + \left(C_{12}^{+}\right)_2 \theta_2 \\ v_2|_{S_2} = \left(C_{22}^{-}\right)_1 \theta_1 + \left(C_{22}^{-}\right)_2 \theta_2, \end{cases} \tag{3.2.21}$$

we have that

$$\int_{\partial\Omega_2} v_2 \frac{\partial u_2}{\partial n} dS = \int_{S_1} v_2 \frac{\partial u_2}{\partial n_1^-} dS + \int_{S_2} v_2 \frac{\partial u_2}{\partial n_1^+} dS = \int_{S_1} (-v_2) \frac{\partial u_2}{\partial n_1^+} dS + \int_{S_2} v_2 \frac{\partial u_2}{\partial n_1^+} dS =$$

$$\int_{S_1} (-v_2)\phi_1 dS + \int_{S_2} v_2\phi_2 dS = \int_{S_1} (-v_2)\phi_1 dS + \int_{S_2} v_2\phi_2 dS =$$

$$\int_{S_1} \left(\left(C_{12}^+\right)_1 \theta_1 + \left(C_{12}^+\right)_2 \theta_2 \right) \phi_1 dS + \int_{S_2} \left(\left(C_{22}^-\right)_1 \theta_1 + \left(C_{22}^-\right)_2 \theta_2 \right) \phi_2 dS.$$

Summarizing (3.2.18), (3.2.19) and (3.2.20), we have

$$\begin{gathered}\left((C_{12}^+)_1\phi_1 + (C_{12}^+)_2\phi_2, \theta_1\right)_{L_2'(S_1)} + \left((C_{22}^-)_1\phi_1 + (C_{22}^-)_2\phi_2, \theta_2\right)_{L_2'(S_2)} = \\ \left((C_{12}^+)_1\theta_1 + (C_{12}^+)_2\theta_2, \phi_1\right)_{L_2'(S_1)} + \left((C_{22}^-)_1\theta_1 + (C_{22}^-)_2\theta_2, \phi_2\right)_{L_2'(S_2)}.\end{gathered}$$

It follows that $(B\phi,\theta)_H = (B\theta,\phi)_H$, which proves Lemma 2.3.3.

3.2.4. Proof of Lemma 2.3.4

We prove that if $\phi_n \to \phi$ weakly converges, then $(B\phi_n,\phi_n) \to (B\phi,\phi)$. We have that

$$(B(\phi_n - \phi), \phi_n - \phi)_H = \rho_1 ||v_{1n}||^2_{\dot{W}_2^1(\Omega_1)} + \rho_2 ||v_{2n}||^2_{\dot{W}_2^1(\Omega_2)} + \rho_3 ||v_{3n}||^2_{\dot{W}_2^1(\Omega_3)},$$

where functions v_{1n}, v_{2n} and v_{3n} satisfy the equations

$$\begin{cases} \triangle v_{1n} = 0 \qquad \text{in } \Omega_1 \\ \frac{\partial v_{1n}}{\partial n}|_{S_0} = 0 \\ \frac{\partial v_{1n}}{\partial n}|_B = 0 \\ \frac{\partial v_{1n}}{\partial n_1^+}|_{S_1} = \phi_{1n} - \phi_1, \end{cases}$$

$$\begin{cases} \triangle v_{2n} = 0 \qquad \text{in } \Omega_2 \\ \frac{\partial v_{2n}}{\partial n}|_B = 0 \\ \frac{\partial v_{2n}}{\partial n_1^+}|_{S_1} = \phi_{1n} - \phi_1 \\ \frac{\partial v_{2n}}{\partial n_1^+}|_{S_2} = \phi_{2n} - \phi_2, \end{cases}$$

$$\begin{cases} \triangle v_{3n} = 0 \qquad \text{in } \Omega_3 \\ \frac{\partial v_{3n}}{\partial n}|_{S_3} = 0 \\ \frac{\partial v_{3n}}{\partial n}|_{B} = 0 \\ \frac{\partial v_{3n}}{\partial n_2^+}|_{S_2} = \phi_{2n} - \phi_2. \end{cases}$$

We consider the first equation and take an arbitrary function $\psi \in \dot{W}_2^1(\Omega_1)$. We have

$$0 = (\triangle v_{1n}, \psi)_{L_2'(\Omega_1)} = \int_{\Omega_1} \triangle v_{1n} \psi d\Omega = \int_{\Omega_1} \nabla \cdot (\nabla v_{1n}) \psi d\Omega =$$

$$\int_{\Omega_1} (\nabla \cdot (\nabla v_{1n} \psi) - \nabla v_{1n} \cdot \nabla \psi) d\Omega = \int_{\partial\Omega_1} \psi \frac{\partial v_{1n}}{\partial n} dS - \int_{\Omega_1} \nabla v_{1n} \cdot \nabla \psi d\Omega.$$

By the definition of v_{1n},

$$\int_{\partial\Omega_1} \psi \frac{\partial v_{1n}}{\partial n} dS - \int_{\Omega_1} \nabla v_{1n} \cdot \nabla \psi d\Omega = \int_{S_1} \psi \frac{\partial v_{1n}}{\partial n} dS - \int_{\Omega_1} \nabla v_{1n} \cdot \nabla \psi d\Omega. \tag{3.2.22}$$

We then obtain

$$(\phi_{1n} - \phi_1, \psi)_{L_2'(S_1)} = (v_{1n}, \psi)_{\dot{W}_2^1(\Omega_1)}. \tag{3.2.23}$$

For a fixed n, since ψ is an arbitrary function, we put $\psi = v_{1n}$. It follows from (3.2.23) that

$$(\phi_{1n} - \phi_1, v_{1n})_{L_2'(S_1)} = (v_{1n}, v_{1n})_{\dot{W}_2^1(\Omega_1)}. \tag{3.2.24}$$

Recall the operators $(T_{\Omega_1})_1 : L_2'(S_1) \to H^1(\Omega_1)$ and $\gamma_1^- : H^1(\Omega_1) \to L_2'(S_1)$ defined in sections 2.2 and 3.1. We have

$$(v_{1n}, \phi_{1n} - \phi_1)_{L_2'(S_1)} = (v_{1n}|_{S_1}, \phi_{1n} - \phi_1)_{L_2'(S_1)} = \Big((C_{11}^-)_1(\phi_{1n} - \phi_1), \phi_{1n} - \phi_1\Big)_{L_2'(S_1)}, \tag{3.2.25}$$

where we used the operator $(C_{11}^-)_1 = \gamma_1^-(T_{\Omega_1})_1$ defined in the section 2.2 and the relations $v_{1n}|_{S_1} = (C_{11}^-)_1(\phi_{1n} - \phi_1)$, $v_{1n}|_{S_1} \in L_2'(S_1)$ by the construction of the operators (see section 3.1). Based on the Cauchy-Schwarz inequality, we have

$$\left|\Big((C_{11}^-)_1(\phi_{1n} - \phi_1), \phi_{1n} - \phi_1\Big)_{L_2'(S_1)}\right| \le \left|\left|(C_{11}^-)_1(\phi_{1n} - \phi_1)\right|\right|_{L_2'(S_1)} ||\phi_{1n} - \phi_1||_{L_2'(S_1)} =$$
$$||v_{1n}||_{L_2'(S_1)} ||\phi_{1n} - \phi_1||_{L_2'(S_1)} \tag{3.2.26}$$

Since $||v_{1n}||^2_{\dot{W}_2^1(\Omega_1)} = (v_{1n}, v_{1n})_{\dot{W}_2^1(\Omega_1)} = (v_{1n}, \phi_{1n} - \phi)_{L_2'(S_1)}$, we have

$$||v_{1n}||^2_{\dot{W}_2^1(\Omega_1)} \le ||v_{1n}||_{L_2'(S_1)} ||\phi_{1n} - \phi_1||_{L_2'(S_1)}. \tag{3.2.27}$$

Since $v_{1n}|_{S_1} \in L_2'(S_1)$ and $v_{1n} \in W_2^1(\Omega)$, $v_{1n} \in \dot{W}_2^1(\Omega)$, where norm (3.2.4) is equivalent to the norm in $W_2^1(\Omega)$ and based on the Sobolev trace restriction theorem [**13**] we may write

$$||v_{1n}||_{L_2'(S_1)} \leq C||v_{1n}||_{\dot{W}_2^1(\Omega_1)},$$

where C is a positive constant. Therefore, from (3.2.27) it follows that

$$||v_{1n}||_{\dot{W}_2^1(\Omega_1)} \leq C||\phi_{1n} - \phi_1||_{L_2'(S_1)}.$$

Since $v_{1n} = (T_{\Omega_1})_1(\phi_{1n} - \phi_1)$ in Ω_1, based on the definition of the operator $(T_{\Omega_1})_1$ given in section 2.2, we have

$$||(T_{\Omega_1})_1(\phi_{1n} - \phi_1)||_{\dot{W}_2^1(\Omega_1)} \leq C||\phi_{1n} - \phi_1||_{L_2'(S_1)}. \tag{3.2.28}$$

Inequality (3.2.28) also holds for any arbitrary function $\kappa \in L_2'(S_1)$, because currently we only used the property of $\phi_{1n} - \phi_1$ to be in $L_2'(S_1)$. From which it follows that the operator $(T_{\Omega_1})_1$ is bounded. Based on the Sobolev trace restriction and Rellich-Kondrashov theorems [**13**], operator γ_1^- is compact [**10**]. Consequently, the operator $(C_{11}^-)_1 = \gamma_1^-(T_{\Omega_1})_1$ is compact [**10**] as a product of a bounded and a compact operator. It follows then [**6**] that

$$\Big((C_{11}^-)_1(\phi_{1n} - \phi_1), \phi_{1n} - \phi_1\Big)_{L_2'(S_1)} \to 0$$

as $\phi_{1n} \to \phi_1$ weakly converges. Based on the (3.2.24) and (3.2.25) we have

$$||v_{1n}||^2_{\dot{W}_2^1(\Omega_1)} \to 0 \tag{3.2.29}$$

as $\phi_{1n} \to \phi_1$ weakly converges. The same is valid for v_{2n} and v_{3n}. Finally,

$$(B(\phi_n - \phi), \phi_n - \phi)_H \to 0, \text{ as } \phi_n \to \phi \text{ weakly converges.} \tag{3.2.30}$$

By Lemma 2.3.2, B is non-negative, so it follows that B has a unique positive and self-adjoint square root $B^{\frac{1}{2}}$ [**12**] such that $B = B^{\frac{1}{2}} \cdot B^{\frac{1}{2}}$. Therefore, we have

$$(B(\phi_n-\phi), \phi_n-\phi)_H = (B^{\frac{1}{2}} \cdot B^{\frac{1}{2}}(\phi_n-\phi), \phi_n-\phi)_H = (B^{\frac{1}{2}}(\phi_n-\phi), B^{\frac{1}{2}}(\phi_n-\phi))_H = ||B^{\frac{1}{2}}(\phi_n-\phi)||^2_H. \tag{3.2.31}$$

Hence, for any sequence $\{\phi_n : \phi_n \in H\}$ that converges to $\phi \in H$ weakly, based on the (3.2.30) and (3.2.31), the sequence $\{B^{\frac{1}{2}}\phi_n\} \to B^{\frac{1}{2}}\phi$. It follows [**6**, p.13] that $B^{\frac{1}{2}}$ is a compact operator and, as a consequence, B is a compact operator as well, as a square of the compact operator. This completes the proof of Lemma 2.3.4.

3.3. The Weyl law

3.3.1. Proof of Theorem 2.4.1 and 2.4.2

In this section, we calculate the asymptotics for eigenvalues of problem 1.4.10 of a two-layer fluid in an open container. Let $\Omega \subset \mathbb{R}^{m+1}$ be divided S_1,$S_2, \ldots, S_p$ by the system of the smooth surfaces into subregions Ω_1,$\Omega_2, \ldots$,Ω_{p+1} as in the Figure 3.1. Let us consider the following problem [**8**],

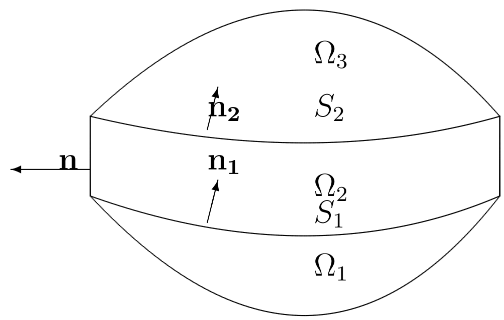

Fig. 3.1. Multi-layer fluid in the closed container

$$\begin{cases} \triangle u_k = 0 \text{ in } \Omega_k,\ k = 1, \ldots, p+1 \\ \frac{\partial u}{\partial n}_k = 0 \text{ on } \partial\Omega \bigcap \partial\Omega_k \\ \frac{\partial u_k}{\partial n_k} = \frac{\partial u_{k+1}}{\partial n_k} = \lambda^{-1} a_k^{-1} (\rho_k u_k - \rho_{k+1} u_{k+1}) \text{ on } S_k, \end{cases} \tag{3.3.1}$$

where a_k are positive functions, ρ_k are positive constants, n is a normal to $\partial\Omega$ and n_k is a normal to the surface S_k. Our goal is to determine the counting function for eignevalue problem (3.3.1). Based on the [**8**], we have

Theorem 3.3.1. *For the counting functions of the spectrum of the spectral problem, one has* $\lambda \to +0$ *the asymptotic equation*

$$N_\pm (\lambda; A, B) \sim (2\pi)^{-m} \lambda^{-m} \sum_{j=1}^{p} \int_{S_j} dS(x) \int_{\xi \perp n_j(x)} n_\pm^j (1; x, \xi) d\xi,$$

where $n^j_{\pm}(1;x,\xi)$ is a counting function for the spectrum of the supplementary problem constructed in [**8**].

If we compare problem (3.3.1) with (2.2.1), then we have $\sigma = \lambda^{-1}$. Therefore, the asymptotic formula for a three-layer fluid in a closed container in $\mathbb{R}^3 (m = 2)$ is

$$N(\sigma) \sim \frac{1}{4\pi}\left[\frac{(\rho_3+\rho_2)^2}{(\rho_3-\rho_2)^2}\int_{S_2} dS + \frac{(\rho_2+\rho_1)^2}{(\rho_2-\rho_1)^2}\int_{S_1} dS\right]\sigma^2 \text{ as } \sigma \to \infty. \qquad (3.3.2)$$

In the planar case [**8**], $m = 1$, we have that

$$N(\sigma) \sim \frac{1}{\pi}\left[\frac{(\rho_3+\rho_2)}{(\rho_3-\rho_2)}\int_{S_2} dS + \frac{(\rho_2+\rho_1)}{(\rho_2-\rho_1)}\int_{S_1} dS\right]\sigma \text{ as } \sigma \to \infty. \qquad (3.3.3)$$

From theorem 2.2.1 and theorem 2.4.1, theorem 2.4.2 immediately follows.

3.4. Examples

3.4.1. Rectangular container

We consider a sloshing problem for a two-layer fluid in an open rectangular container

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \\ \frac{\partial u}{\partial y} - \sigma u \mid_{y=0} = 0 \\ \left(\frac{\partial u}{\partial y} - \sigma u\right) \mid_{y=-h} = \rho \left(\frac{\partial v}{\partial y} - \sigma v\right) \mid_{y=-h} \\ \frac{\partial u}{\partial x}\big|_{x=0,l} = \frac{\partial v}{\partial x}\big|_{x=0,l} = 0 \\ \frac{\partial u}{\partial y}\big|_{y=-h} = \frac{\partial v}{\partial y}\big|_{y=-h} \\ \frac{\partial v}{\partial y}\big|_{y=-2h} = 0, \end{cases} \qquad (3.4.1)$$

where $\rho = \frac{\rho_1}{\rho_2} > 1$, $u \in H^1(\Omega_2)$ and $v \in H^1(\Omega_1)$.

The solutions u and v for the n^{th} harmonic are given by

$$\begin{cases} u_n(x,y) = \left(A_{1n}\cosh(\frac{\pi n}{l}y) + A_{2n}\sinh(\frac{\pi n}{l}y)\right)\cos(\frac{\pi n}{l}x) \\ v_n(x,y) = \left(A_{3n}\cosh(\frac{\pi n}{l}y) + A_{4n}\sinh(\frac{\pi n}{l}y)\right)\cos(\frac{\pi n}{l}x). \end{cases} \qquad (3.4.2)$$

For $y = 0, -h, -2h$, we have the following algebraic homogeneous equations relative to $A_{1n},...,A_{4n}$, including spectral parameter σ,

$$M[A_{1n}A_{2n}A_{3n}A_{4n}]^T = 0, \qquad (3.4.3)$$

where

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

$$M_{11} = \begin{bmatrix} -\sigma & \frac{\pi n}{l} \\ -\frac{\pi n}{l}\sinh\left(\frac{\pi n}{l}h\right) & \frac{\pi n}{l}\cosh\left(\frac{\pi n}{l}h\right) \end{bmatrix},$$

$$M_{12} = \begin{bmatrix} 0 & 0 \\ \frac{\pi n}{l}\sinh\left(\frac{\pi n}{l}h\right) & -\frac{\pi n}{l}\cosh\left(\frac{\pi n}{l}h\right) \end{bmatrix},$$

$$M_{21} = \begin{bmatrix} -\sigma\cosh\left(\frac{\pi n}{l}h\right) - \frac{\pi n}{l}\sinh\left(\frac{\pi n}{l}h\right) & \frac{\pi n}{l}\cosh\left(\frac{\pi n}{l}h\right) + \sigma\sinh\left(\frac{\pi n}{l}h\right) \\ 0 & 0 \end{bmatrix},$$

$$M_{22} = \begin{bmatrix} \rho\sigma\cosh\left(\frac{\pi n}{l}h\right) + \frac{\pi n}{l}\rho\sinh\left(\frac{\pi n}{l}h\right) & -\rho\sigma\sinh\left(\frac{\pi n}{l}h\right) - \frac{\pi n}{l}\rho\cosh\left(\frac{\pi n}{l}h\right) \\ -\frac{\pi n}{l}\sinh\left(\frac{\pi n}{l}2h\right) & \frac{\pi n}{l}\cosh\left(\frac{\pi n}{l}2h\right) \end{bmatrix}.$$

To obtain a nontrivial solution $[A_{1n}, A_{2n}, A_{3n}, A_{4n}]$, we set the determinant $det(M) = 0$. We have that $det(M) = a\sigma^2 + b\sigma + c = 0$, where

$$\begin{cases} a = -\left(\frac{\pi n}{l}\right)^2 \frac{1}{2}\left(\rho - 1 + (\rho+1)\cosh\left(\frac{\pi n}{l}2h\right)\right) \\ b = \left(\frac{\pi n}{l}\right)^3 \rho\sinh\left(\frac{\pi n}{l}2h\right) \\ c = -\left(\frac{\pi n}{l}\right)^4 (\rho-1)\sinh^2\left(\frac{\pi n}{l}h\right). \end{cases} \tag{3.4.4}$$

It follows that

$$\sigma_{1,2} = \frac{-b \mp \sqrt{D}}{2a}, \tag{3.4.5}$$

where $D = b^2 - 4ac$. We then have

$$\begin{cases} \sigma_{1n} = \frac{\pi n}{l}\frac{\sqrt{2}\sqrt{-1+2\rho+\cosh\left(\frac{\pi n}{l}2h\right)}\sinh\left(\frac{\pi n}{l}h\right)+\rho\sinh\left(\frac{\pi n}{l}2h\right)}{\rho-1+(1+\rho)\cosh\left(\frac{\pi n}{l}2h\right)} \\ \sigma_{2n} = -\frac{\pi n}{l}\frac{\sqrt{2}\sqrt{-1+2\rho+\cosh\left(\frac{\pi n}{l}2h\right)}\sinh\left(\frac{\pi n}{l}h\right)-\rho\sinh\left(\frac{\pi n}{l}2h\right)}{\rho-1+(1+\rho)\cosh\left(\frac{\pi n}{l}2h\right)} \end{cases}. \tag{3.4.6}$$

As $\frac{\pi n}{l} \to \infty$, we have that

$$\begin{cases} \sigma_{1n} \sim \frac{\pi n}{l} \\ \sigma_{2n} \sim \frac{\pi n}{l}\frac{\rho-1}{\rho+1} \end{cases}. \tag{3.4.7}$$

and counting function satisfies

$$N(\sigma) \sim \left(\left(\frac{\rho+1}{\rho-1}\right) + 1\right)\frac{vol(D)}{\pi}\sigma, \tag{3.4.8}$$

where $vol(D) = l$ is a Lebesgue measure of the interface. If

$$\sigma = \sigma_{1n} = \frac{\pi n}{l} \frac{\sqrt{2}\sqrt{-1 + 2\rho + \cosh\left(\frac{\pi n}{l} 2h\right)} \sinh\left(\frac{\pi n}{l} h\right) + \rho \sinh\left(\frac{\pi n}{l} 2h\right)}{\rho - 1 + (1 + \rho)\cosh\left(\frac{\pi n}{l} 2h\right)}, \tag{3.4.9}$$

then

$$\begin{cases} A_{2n} = A_{1n} \frac{\sqrt{2} l^3 \sqrt{\frac{n^6\left(-1+2\rho+\cosh(2h\frac{\pi n}{l})\right)}{l^6}} \sinh(h\frac{\pi n}{l}) + n^3 \rho \sinh(2h\frac{\pi n}{l})}{n^3\left(-1+\rho+(1+\rho)\cosh(2h\frac{\pi n}{l})\right)} \\ A_{3n} = -A_{1n}\cosh(2h\frac{\pi n}{l}) \frac{-n^3 + n^3\cosh(2h\frac{\pi n}{l}) - \coth(h\frac{\pi n}{l})\sqrt{2} l^3 \sqrt{\frac{n^6\left(-1+2\rho+\cosh(2h\frac{\pi n}{l})\right)}{l^6}} \sinh(h\frac{\pi n}{l})}{n^3\left(-1+\rho+(1+\rho)\cosh(2h\frac{\pi n}{l})\right)} \\ A_{4n} = A_{1n}\cosh(2h\frac{\pi n}{l}) \frac{3n^3\sinh(h\frac{\pi n}{l}) + 2\cosh(h\frac{\pi n}{l})\sqrt{2} l^3 \sqrt{\frac{n^6\left(-1+2\rho+\cosh(2h\frac{\pi n}{l})\right)}{l^6}} \sinh(h\frac{\pi n}{l}) - n^3\sinh(3h\frac{\pi n}{l})}{n^3\left(-1+\rho+(1+\rho)\cosh(2h\frac{\pi n}{l})\right)}. \end{cases} \tag{3.4.10}$$

Finally, we have that

$$\begin{cases} \sigma_{1n} = \frac{\pi n}{l} \frac{\sqrt{2}\sqrt{-1+2\rho+\cosh\left(\frac{\pi n}{l}2h\right)}\sinh\left(\frac{\pi n}{l}h\right)+\rho\sinh\left(\frac{\pi n}{l}2h\right)}{\rho-1+(1+\rho)\cosh\left(\frac{\pi n}{l}2h\right)} \\ u_n(x,y) = \left(A_{1n}\cosh(\frac{\pi n}{l}y) + A_{2n}\sinh(\frac{\pi n}{l}y)\right)\cos(\frac{\pi n}{l}x) \\ v_n(x,y) = \left(A_{3n}\cosh(\frac{\pi n}{l}y) + A_{4n}\sinh(\frac{\pi n}{l}y)\right)\cos(\frac{\pi n}{l}x), \end{cases} \tag{3.4.11}$$

where $A_{1n}, A_{2n}, A_{3n}, A_{4n}$ are defined by 3.4.10. If

$$\sigma = \sigma_{2n} = -\frac{\pi n}{l} \frac{\sqrt{2}\sqrt{-1 + 2\rho + \cosh\left(\frac{\pi n}{l} 2h\right)} \sinh\left(\frac{\pi n}{l} h\right) - \rho \sinh\left(\frac{\pi n}{l} 2h\right)}{\rho - 1 + (1 + \rho)\cosh\left(\frac{\pi n}{l} 2h\right)}, \tag{3.4.12}$$

then

$$\begin{cases} \sigma_{2n} = -\frac{\pi n}{l} \frac{\sqrt{2}\sqrt{-1+2\rho+\cosh\left(\frac{\pi n}{l}2h\right)}\sinh\left(\frac{\pi n}{l}h\right)-\rho\sinh\left(\frac{\pi n}{l}2h\right)}{\rho-1+(1+\rho)\cosh\left(\frac{\pi n}{l}2h\right)} \\ u_n(x,y) = \left(B_{1n}\cosh(\frac{\pi n}{l}y) + B_{2n}\sinh(\frac{\pi n}{l}y)\right)\cos(\frac{\pi n}{l}x) \\ v_n(x,y) = \left(B_{3n}\cosh(\frac{\pi n}{l}y) + B_{4n}\sinh(\frac{\pi n}{l}y)\right)\cos(\frac{\pi n}{l}x), \end{cases} \tag{3.4.13}$$

where

$$\begin{cases} B_{2n} = B_{1n} \frac{-\sqrt{2} l^3 \sqrt{\frac{n^6\left(-1+2\rho+\cosh(2h\frac{\pi n}{l})\right)}{l^6}} \sinh(h\frac{\pi n}{l}) - n^3 \rho \sinh(2h\frac{\pi n}{l})}{n^3\left(-1+\rho+(1+\rho)\cosh(2h\frac{\pi n}{l})\right)} \\ B_{3n} = -B_{1n}\cosh(2h\frac{\pi n}{l}) \frac{-n^3 + n^3\cosh(2h\frac{\pi n}{l}) + \coth(h\frac{\pi n}{l})\sqrt{2} l^3 \sqrt{\frac{n^6\left(-1+2\rho+\cosh(2h\frac{\pi n}{l})\right)}{l^6}} \sinh(h\frac{\pi n}{l})}{n^3\left(-1+\rho+(1+\rho)\cosh(2h\frac{\pi n}{l})\right)} \\ B_{4n} = -B_{1n}\cosh(2h\frac{\pi n}{l}) \frac{-3n^3\sinh(h\frac{\pi n}{l}) + 2\cosh(h\frac{\pi n}{l})\sqrt{2} l^3 \sqrt{\frac{n^6\left(-1+2\rho+\cosh(2h\frac{\pi n}{l})\right)}{l^6}} \sinh(h\frac{\pi n}{l}) - n^3\sinh(3h\frac{\pi n}{l})}{n^3\left(-1+\rho+(1+\rho)\cosh(2h\frac{\pi n}{l})\right)}. \end{cases} \tag{3.4.14}$$

3.4.2. The case of the extremely heavy fluid

If $\rho \to \infty$, then the sloshing problem for a two-layer fluid in an open rectangular container is

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \\ \frac{\partial u}{\partial y} - \sigma u \mid_{y=0} = 0 \\ \left(\frac{\partial v}{\partial y} - \sigma v\right) \mid_{y=-h} = 0 \\ \frac{\partial v}{\partial x}\big|_{x=0,l} = 0 \\ \frac{\partial u}{\partial y}\big|_{y=-h} = \frac{\partial v}{\partial y}\big|_{y=-h} \\ \frac{\partial v}{\partial y}\big|_{y=-2h} = 0, \end{cases} \tag{3.4.15}$$

where $u \in H^1(\Omega_2)$ and $v \in H^1(\Omega_1)$.
The solutions for u and v are given by

$$\begin{cases} u_n(x,y) = \left(A_{1n} \cosh(\frac{\pi n}{l} y) + A_{2n} \sinh(\frac{\pi n}{l} y)\right) \cos(\frac{\pi n}{l} x) \\ v_n(x,y) = \left(A_{3n} \cosh(\frac{\pi n}{l} y) + A_{4n} \sinh(\frac{\pi n}{l} y)\right) \cos(\frac{\pi n}{l} x). \end{cases}$$

The spectral parameter is determined by the boundary condition, that is, by the equations

$$M[A_{1n},A_{2n},A_{3n},A_{4n}]^T = 0, \tag{3.4.16}$$

where matrix M is defined by

$$M = \begin{bmatrix} -\sigma & \frac{\pi n}{l} & 0 & 0 \\ -\frac{\pi n}{l} \sinh\left(\frac{\pi n}{l} h\right) & \frac{\pi n}{l} \cosh\left(\frac{\pi n}{l} h\right) & \frac{\pi n}{l} \sinh\left(\frac{\pi n}{l} h\right) & -\frac{\pi n}{l} \cosh\left(\frac{\pi n}{l} h\right) \\ 0 & 0 & -\sigma \cosh\left(\frac{\pi n}{l} h\right) - \frac{\pi n}{l} \sinh\left(\frac{\pi n}{l} h\right) & \sigma \sinh\left(\frac{\pi n}{l} h\right) + \frac{\pi n}{l} \cosh\left(\frac{\pi n}{l} h\right) \\ 0 & 0 & -\frac{\pi n}{l} \sinh\left(\frac{\pi n}{l} 2h\right) & \frac{\pi n}{l} \cosh\left(\frac{\pi n}{l} 2h\right) \end{bmatrix}.$$

In order to have a nontrivial solution, we set the determinant $det(M) = 0$,

$$det(M) = a\sigma^2 + b\sigma + c = 0, \tag{3.4.17}$$

where

$$\begin{cases} a = \left(\frac{\pi n}{l}\right)^2 \cosh^2\left(\frac{\pi n}{l}h\right) \\ b = -\left(\frac{\pi n}{l}\right)^3 \sinh\left(\frac{\pi n}{l}2h\right) \\ c = \left(\frac{\pi n}{l}\right)^4 \sinh^2\left(\frac{\pi n}{l}h\right). \end{cases}$$

Hence, $\sigma_{1,2} = \frac{-b\mp\sqrt{D}}{2a}$, where $D = b^2 - 4ac$.

$$\begin{cases} \sigma_{1n} = \frac{\pi n}{l} \tanh\left(\frac{\pi n}{l}\right) \\ \sigma_{2n} = \frac{\pi n}{l} \tanh\left(\frac{\pi n}{l}\right) \end{cases} \tag{3.4.18}$$

If $\sigma = \sigma_{1n} = \sigma_{2n}$, then

$$\begin{cases} \sigma_{1n} = \sigma_{2n} = \frac{\pi n}{l} \tanh\left(\frac{\pi n}{l}\right) \\ u_n(x,y) = A_1 \frac{\cosh(y+h)\frac{\pi n}{l}}{\cosh(h\frac{\pi n}{l})} \cos(\frac{\pi n}{l}x) \\ v_n(x,y) = 0. \end{cases} \tag{3.4.19}$$

As $\frac{\pi n}{l} \to \infty$, we have that $\sigma_{1n} \sim \frac{\pi n}{l}$, and

$$N(\sigma) \sim \frac{vol(D)}{\pi}\sigma.$$

3.4.3. The study of the limiting case

The differential equations for this case are

$$\begin{cases} \triangle u_1 = 0 \\ \triangle u_2 = 0 \\ \left(\frac{\partial u_2}{\partial y} - \sigma u_2\right)|_{y=0} = 0 \\ \frac{\partial u_2}{\partial y}|_{y=-h} = \frac{\partial u_1}{\partial y} \\ \left(\frac{\partial u_1}{\partial y} - \sigma u_1\right)|_{y=-h} = 0 \\ \frac{\partial u_1}{\partial y} = 0|_{y=-d} \\ \frac{\partial u_i}{\partial n} = 0|_{\partial\Omega}, \end{cases} \tag{3.4.20}$$

where $u_1 \in H^1(\Omega_1)$ and $u_2 \in H^1(\Omega_2)$.

The solution for functions u_{1n} and u_{2n} corresponding to the n^{th} harmonic is

$$\begin{cases} u_{1n}(x,y) = (A_{1n}\cosh(\frac{\pi n}{l}y) + B_{1n}\sinh(\frac{\pi n}{l}y))\cos(\frac{\pi n}{l}x) \\ u_{2n}(x,y) = (A_{2n}\cosh(\frac{\pi n}{l}y) + B_{2n}\sinh(\frac{\pi n}{l}y))\cos(\frac{\pi n}{l}x), \end{cases} \tag{3.4.21}$$

from which it follows that

$$\frac{\partial u_{in}}{\partial y} = \frac{\pi n}{l}(A_{in}\sinh(\frac{\pi n}{l}y) + B_{in}\cosh(\frac{\pi n}{l}y))\cos(\frac{\pi n}{l}x). \tag{3.4.22}$$

We denote

$$\begin{cases} \phi_{1n} = \frac{\partial u_{1n}}{\partial y}|_{y=-h} = \frac{\partial u_{2n}}{\partial y}|_{y=-h} = \frac{\pi n}{l}(-A_{in}\sinh(\frac{\pi n}{l}h) + B_{in}\cosh(\frac{\pi n}{l}h))\cos(\frac{\pi n}{l}x) \\ \phi_{2n} = \frac{\partial u_{2n}}{\partial y}|_{y=0} = \frac{\pi n}{l}B_{2n}\cos(\frac{\pi n}{l}x), \end{cases} \tag{3.4.23}$$

where ϕ_{1n} and ϕ_{2n} correspond to a σ_n, because we are considering the solution which corresponds to the n^{th} harmonic. We note, that $\int_0^l \phi_{1/2n} dx = 0$, so $\phi_{1n} \in L_2'(S_1)$. Similarly, $\phi_{2n} \in L_2'(S_2)$, we have that

$$\begin{cases} u_{2n}|_{y=0} = A_{2n}\cos(\frac{\pi n}{l}x) \\ \phi_{2n} = \sigma_n A_{2n}\cos(\frac{\pi n}{l}x) \qquad \text{on } S_2. \end{cases} \tag{3.4.24}$$

On S_1, we have

$$u_{in}|_{y=-h} = (A_{in}\cosh(\frac{\pi n}{l}h) - B_{in}\sinh(\frac{\pi n}{l}h))\cos((\frac{\pi n}{l}x). \tag{3.4.25}$$

Hence, on the surface S_1, taking into account that $\phi_{1n} = \frac{\partial u_{1n}}{\partial y}|_{y=-h}$, we have

$$\phi_{1n} - \sigma_n(A_{1n}\cosh(\frac{\pi n}{l}h) - B_{1n}\sinh(\frac{\pi n}{l}h))\cos(\frac{\pi n}{l}x) = 0. \tag{3.4.26}$$

Finally, we have

$$\begin{cases} \phi_{1n} = \sigma_n(A_{1n}\cosh(\frac{\pi n}{l}h) - B_{1n}\sinh(\frac{\pi n}{l}h))\cos(\frac{\pi n}{l}x) \\ \phi_{2n} = \sigma_n A_{2n}\cos(\frac{\pi n}{l}x). \end{cases} \tag{3.4.27}$$

For $\frac{\partial u_{1n}}{\partial y}|_{y=-d} = 0$, we have

$$(-A_{1n}\sinh(\frac{\pi n}{l}d) + B_{1n}\cosh(\frac{\pi n}{l}d))\cos(\frac{\pi n}{l}x) = 0. \tag{3.4.28}$$

Hence,

$$B_{1n} = A_{1n}\tanh(\frac{\pi n}{l}d), \tag{3.4.29}$$

and $\phi_{1n} = \frac{\partial u_{1n}}{\partial y}|_{y=-h} = \frac{\pi n}{l}(-A_{1n}\sinh(\frac{\pi n}{l}h) + B_{1n}\cosh(\frac{\pi n}{l}h))\cos(\frac{\pi n}{l}x)$. It follows then

$$\begin{cases} A_{1n}\cos(\frac{\pi n}{l}x) = \phi_{1n}\frac{l}{\pi n}\frac{\cosh(\frac{\pi n}{l}d)}{\sinh(\frac{\pi n}{l}(d-h))} \\ B_{1n}\cos(\frac{\pi n}{l}x) = \phi_{1n}\frac{l}{\pi n}\frac{\sinh(\frac{\pi n}{l}d)}{\sinh(\frac{\pi n}{l}(d-h))}. \end{cases} \tag{3.4.30}$$

$\phi_{2n} = \frac{\partial u_{2n}}{\partial y}|_{y=0} = \frac{\pi n}{l}B_{2n}\cos(\frac{\pi n}{l}x)$, so

$$B_{2n}\cos(\frac{\pi n}{l}x) = \frac{l}{\pi n}\phi_2. \tag{3.4.31}$$

$\phi_{1n} = \frac{\partial u_{1n}}{\partial y}|_{y=-h} = \frac{\partial u_{2n}}{\partial y}|_{y=-h} = \frac{\pi n}{l}(-A_{2n}\sinh(\frac{\pi n}{l}h) + B_{2n}\cosh(\frac{\pi n}{l}h))\cos(\frac{\pi n}{l}x)$. It follows then

$$A_{2n}\cos(\frac{\pi n}{l}x) = (\phi_{2n}\cosh(\frac{\pi n}{l}h) - \phi_{1n})\frac{1}{\sinh(\frac{\pi n}{l}h)}. \tag{3.4.32}$$

Finally, we get

$$\begin{cases} A_{1n}\cos(\frac{\pi n}{l}x) = \phi_{1n}\frac{l}{\pi n}\frac{\cosh(\frac{\pi n}{l}d)}{\sinh(\frac{\pi n}{l}(d-h))} \\ B_{1n}\cos(\frac{\pi n}{l}x) = \phi_{1n}\frac{l}{\pi n}\frac{\sinh(\frac{\pi n}{l}d)}{\sinh(\frac{\pi n}{l}(d-h))} \\ A_{2n}\cos(\frac{\pi n}{l}x) = (\phi_{2n}\cosh(\frac{\pi n}{l}h) - \phi_1)\frac{1}{\sinh(\frac{\pi n}{l}h)}\frac{l}{\pi n} \\ B_{2n}\cos(\frac{\pi n}{l}x) = \frac{l}{\pi n}\phi_{2n}. \end{cases} \tag{3.4.33}$$

Based on (3.4.33), we have (3.4.27) as an operator equation $A\phi = \sigma_n B\phi$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.4.34}$$

and

$$B = \begin{pmatrix} \frac{l}{\pi n}\frac{\cosh(\frac{\pi n}{l}(d-h))}{\sinh(\frac{\pi n}{l}(d-h))} & 0 \\ -\frac{l}{\pi n}\frac{1}{\sinh(\frac{\pi n}{l}h)} & \frac{l}{\pi n}\frac{\cosh(\frac{\pi n}{l}h)}{\sinh(\frac{\pi n}{l}h)} \end{pmatrix}, \tag{3.4.35}$$

and $\phi = (\phi_{1n}, \phi_{2n})^T$. The eigenvalues are then

$$\begin{cases} \sigma_{1n} = \frac{\pi n}{l}\tanh\left(\frac{\pi n}{l}(d-h)\right) \\ \sigma_{2n} = \frac{\pi n}{l}\tanh\left(\frac{\pi n}{l}h\right). \end{cases} \tag{3.4.36}$$

If $\sigma_n = \sigma_{1n}$, we have that

$$\phi_{1n} = \phi_{2n}\frac{\sinh\left(\frac{\pi n}{l}(d-2h)\right)}{\sinh\left(\frac{\pi n}{l}(d-h)\right)} \tag{3.4.37}$$

If $\sigma_n = \sigma_{2n}$, we have that

$$\begin{cases} \phi_{1n} = 0 \\ \phi_{2n} \neq 0. \end{cases} \tag{3.4.38}$$

Finally,

$$\begin{cases} \sigma_{1n} = \frac{\pi n}{l} \tanh\left(\frac{\pi n}{l}(d-h)\right) \\ \phi_{1n} = \phi_2 \frac{\sinh\left(\frac{\pi n}{l}(d-2h)\right)}{\sinh\left(\frac{\pi n}{l}(d-h)\right)} \\ d \neq 2h, \end{cases} \tag{3.4.39}$$

$$\begin{cases} \sigma_{2n} = \frac{\pi n}{l} \tanh\left(\frac{\pi n}{l}h\right) \\ \phi_{1n} = 0 \qquad \phi_2 \neq 0. \end{cases} \tag{3.4.40}$$

3.5. Existence and uniqueness of the solution

3.5.1. Proof of Proposition 2.6.1

The tentative assumption for our solution is [**11**]

$$u(x,y) = \sum_{n=1} \left(A_n \sinh(\frac{\pi n}{l}y) + B_n \cosh(\frac{\pi n}{l}y)\right) \cos(\frac{\pi n}{l}x). \tag{3.5.1}$$

From this, it follows that

$$\begin{cases} \frac{\partial u}{\partial y}(x,y) = \sum_{n=1} \frac{\pi n}{l} \left(A_n \cosh(\frac{\pi n}{l}y) + B_n \sinh(\frac{\pi n}{l}y)\right) \cos(\frac{\pi n}{l}x) \\ u(x,0) = \sum_{n=1} \ B_n \cos(\frac{\pi n}{l}x) \\ \frac{\partial u}{\partial y}(x,0) = \sum_{n=1} \frac{\pi n}{l} A_n \cos(\frac{\pi n}{l}x). \end{cases} \tag{3.5.2}$$

The boundary condition on S_1 is

$$\frac{\partial u}{\partial n} - \sigma u|_{S_1} = \sum_{n=1} \left(\frac{\pi n}{l}A_n - \sigma B_n\right) \cos(\frac{\pi n}{l}x) = 0. \tag{3.5.3}$$

Hence, for each n, we have that

$$A_n = \frac{l}{\pi n}\sigma B_n \tag{3.5.4}$$

For $y = -h$, we have that

$$\frac{\partial u}{\partial y}(x,-h) = \sum_{n=1} \frac{\pi n}{l} \left(A_n \cosh(\frac{\pi n}{l}h) - B_n \sinh(\frac{\pi n}{l}h)\right) \cos(\frac{\pi n}{l}x) = \phi. \tag{3.5.5}$$

Let

$$\phi(x) = \sum_{n=1} \frac{2}{l}\phi_n \cos\left(\frac{\pi n}{l}x\right), \tag{3.5.6}$$

where $\phi_n =< \phi, \cos\left(\frac{\pi n}{l}x\right) >_{L_2([0,l])}$. Hence

$$\sum_{n=1} \frac{\pi n}{l}\left(A_n \cosh(\frac{\pi n}{l}h) - B_n \sinh(\frac{\pi n}{l}h)\right)\cos(\frac{\pi n}{l}x) = \sum_{n=1} \frac{2}{l}\phi_n \cos\left(\frac{\pi n}{l}x\right) \tag{3.5.7}$$

and

$$\frac{\pi n}{l}\left(A_n \cosh(\frac{\pi n}{l}h) - B_n \sinh(\frac{\pi n}{l}h)\right) = \frac{2}{l}\phi_n, \tag{3.5.8}$$

and when taking into account (3.5.4),

$$\cosh\left(\frac{\pi n}{l}h\right) B_n \left(\sigma - \frac{\pi n}{l}\tanh\left(\frac{\pi n}{l}h\right)\right) = \frac{2}{l}\phi_n. \tag{3.5.9}$$

This condition must be satisfied for an arbitrary positive integer n. If

$$\sigma \neq \frac{\pi n}{l}\tanh\left(\frac{\pi n}{l}h\right)$$

for each n, then we can calculate A_n and B_n to define the solution

$$\begin{cases} B_n = \frac{2}{l}\phi_n \frac{1}{\cosh\left(\frac{\pi n}{l}h\right)\left(\sigma - \frac{\pi n}{l}\tanh\left(\frac{\pi n}{l}h\right)\right)} \\ A_n = \frac{l}{\pi n}\sigma B_n. \end{cases} \tag{3.5.10}$$

If there exists an n such that $\sigma = \frac{\pi n}{l}\tanh\left(\frac{\pi n}{l}h\right)$, then for this particular n

$$\cosh\left(\frac{\pi n}{l}h\right) B_n \left(\sigma - \frac{\pi n}{l}\tanh\left(\frac{\pi n}{l}h\right)\right) = 0 = \frac{2}{l}\phi_n. \tag{3.5.11}$$

So if $\phi_n \neq 0$, a solution does not exist. In order to have a solution we must require that $\phi_n = 0$ and that B_n is arbitrary. We return to the spectral problem of sloshing in a rectangular container and consider the case when $d \neq 2h$. We change h in such way that m and n satisfy the following condition

$$\frac{\pi n}{l}\tanh\left(\frac{\pi n}{l}(d-h)\right) = \frac{\pi m}{l}\tanh\left(\frac{\pi m}{l}\right). \tag{3.5.12}$$

As $d \neq 2h$, it follows that $m \neq n$. For domain Ω_1, the solution

$$u_1 = \frac{A_1}{\cosh\left(\frac{\pi n}{l}d\right)}\cosh\left(\frac{\pi n}{l}(d+y)\right)\cos(\frac{\pi n}{l}x). \tag{3.5.13}$$

ϕ on S_1 satisfies

$$\phi = \frac{\partial u_1}{\partial y}|_{S_1(y=-h)} = \frac{\pi n}{l}\frac{A_1}{\cosh\left(\frac{\pi n}{l}d\right)}\sinh\left(\frac{\pi n}{l}(d-h)\right)\cos(\frac{\pi n}{l}x). \tag{3.5.14}$$

The solution which corresponds to $\frac{\pi m}{l} \tanh\left(\frac{\pi m h}{l}\right)$ is

$$v_m = \frac{C_1 \cosh\left(\frac{\pi m}{l}(h+y)\right)}{\cosh\left(h\frac{\pi m}{l}\right)} \cos(\frac{\pi m}{l} x). \tag{3.5.15}$$

So, we have that

$$< \phi, v_m >= \int_{S_1} \phi v_m = B \int_0^l \cos\left(\frac{\pi n}{l} x\right) \cos\left(\frac{\pi m}{l} x\right) = 0, \text{ as } m \neq n. \tag{3.5.16}$$

Consequently, there is always a solution for $d \neq 2h$ for a rectangular container. This cocludes the proof of Proposition 2.6.1.

3.6. General case

3.6.1. Proof of Proposition 2.7.1

We would like to understand the conditions for existence of a solution to the equations

$$\begin{cases} \triangle u = 0 \\ \frac{\partial u}{\partial n} - \sigma u|_{S_1} = 0 \\ \frac{\partial u}{\partial n}|_{S_2} = \phi, \end{cases} \tag{3.6.1}$$

where σ is a parameter. In addition to (3.6.1), we consider the following boundary problem

$$\begin{cases} \triangle \beta = 0 \\ \frac{\partial \beta}{\partial n}|_{S_1} = 0 \\ \frac{\partial \beta}{\partial n}|_{S_2} = \phi. \end{cases} \tag{3.6.2}$$

The solution to this problem exists if $\phi \in L_2'(S_2)$ [**13**].
Let us represent the solution as

$$u = \omega + \beta, \tag{3.6.3}$$

where the function β is known and defined by (3.6.2). We have that

$$\begin{cases} 0 = \triangle u = \triangle(\omega + \beta) = \triangle\omega + \triangle\beta = \triangle\omega \\ \triangle\omega = 0 \\ \frac{\partial u}{\partial n} - \sigma u|_{S_1} = 0 = \frac{\partial \omega}{\partial n} + \frac{\partial \beta}{\partial n} - \sigma\omega - \sigma\beta|_{S_1} = 0 \\ \frac{\partial \beta}{\partial n}|_{S_1} = 0 \\ \frac{\partial \omega}{\partial n} - \sigma\omega|_{S_1} = \sigma\beta|_{S_1}, \end{cases} \tag{3.6.4}$$

$$\begin{cases} \frac{\partial u}{\partial n}|_{S_2} = \phi = \frac{\partial \omega}{\partial n} + \frac{\partial \beta}{\partial n}|_{S_1} = \frac{\partial \omega}{\partial n} + \phi \\ \frac{\partial \omega}{\partial n}|_{S_2} = 0. \end{cases} \tag{3.6.5}$$

Finally, ω satisfies the system

$$\begin{cases} \triangle\omega = 0 \\ \frac{\partial \omega}{\partial n} - \sigma\omega|_{S_1} = \sigma\beta|_{S_1} \\ \frac{\partial \omega}{\partial n}|_{S_2} = 0, \end{cases} \tag{3.6.6}$$

where β is again defined from (3.6.2). We consider the following sloshing problem

$$\begin{cases} \triangle v = 0 \\ \frac{\partial v}{\partial n} - \lambda v|_{S_1} = 0 \\ \frac{\partial v}{\partial n}|_{S_2} = 0. \end{cases} \tag{3.6.7}$$

The spectrum is discrete $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \dots$, and the eigenfunctions $v_1, v_2, \dots$ form an orthogonal basis in $L_2(S_1)$. In fact, if $\lambda_i \neq \lambda_j$ and v_i,v_j are the corresponding eigenfunctions, we have that

$$0 = \int_\Omega v_i \triangle v_j d\Omega = \int_\Omega (\nabla \cdot (v_i \nabla v_j) - \nabla v_i \cdot \nabla v_j) d\Omega = 0. \tag{3.6.8}$$

Since

$$\int_\Omega \nabla\cdot(v_i \nabla v_j)\, d\Omega = \int_{\partial\Omega} v_i \frac{\partial v_j}{\partial n} dS = \int_{S_1} v_i \frac{\partial v_j}{\partial n} dS + \int_{S_2} v_i \frac{\partial v_j}{\partial n} dS = \int_{S_1} v_i(\lambda_j v_j) dS = \lambda_j \int_{S_1} v_i v_j dS,$$

we obtain that

$$\begin{cases} \lambda_j \int_{S_1} v_i v_j dS = \int_\Omega \nabla v_i \cdot \nabla v_j d\Omega \\ \lambda_i \int_{S_1} v_j v_i dS = \int_\Omega \nabla v_j \cdot \nabla v_i d\Omega. \end{cases} \tag{3.6.9}$$

Hence,

$$(\lambda_i - \lambda_j) \int_{S_1} v_j v_i dS = 0. \tag{3.6.10}$$

Assuming $\omega = a^s v_s$ and using the summation Einstein convention, we have

$$\begin{cases} \triangle\omega = \triangle a^s v_s = a^s \triangle v_s = a^s 0 = 0 \\ \frac{\partial \omega}{\partial n}|_{S_2} = a^s \frac{\partial v}{\partial n}_s|_{S_2} = 0. \end{cases} \tag{3.6.11}$$

We consider the conditions on the surface S_1

$$\begin{cases} \frac{\partial \omega}{\partial n} - \sigma\omega|_{S_1} = \sigma\beta|_{S_1} = a^s \frac{\partial v_s}{\partial n} - \sigma a^s v_s|_{S_1} = a^s(\lambda_s v_s) - \sigma a^s v_s|_{S_1} = a^s(\lambda_s - \sigma) v_s|_{S_1} \\ a^s(\lambda_s - \sigma) v_s|_{S_1} = \sigma\beta|_{S_1}. \end{cases} \tag{3.6.12}$$

Since the function β is known and defined by (2.7.2), it can be expanded in Fourier series on S_1

$$\beta|_{S_1} = b^k v_k|_{S_1}. \tag{3.6.13}$$

Consequently,

$$a^s(\lambda_s - \sigma) = \sigma b^s. \tag{3.6.14}$$

If $\sigma \notin \{\lambda_s\}$, then we can uniquely define

$$a^s = \frac{\sigma b^s}{\lambda_s - \sigma}. \tag{3.6.15}$$

So a solution exists and is unique. If there exists an s such as $\sigma = \lambda_s$, in order to have the solution, need b^s to satisfy $b^s = 0$. So

$$\int_{S_1} \beta v_k dS = 0. \tag{3.6.16}$$

We have

$$\begin{cases} 0 = \int_\Omega v_s \triangle\beta d\Omega = \int_\Omega \nabla \cdot (v_s \nabla\beta) - \nabla v_s \cdot \nabla\beta d\Omega = \\ \int_{\partial\Omega} v_s \frac{\partial\beta}{\partial n} dS - \int_\Omega \nabla v_s \cdot \nabla\beta d\Omega = \int_{S_2} v_s \phi dS - \int_\Omega \nabla v_s \cdot \nabla\beta d\Omega \\ 0 = \int_\Omega \beta \triangle v_s d\Omega = \int_\Omega \nabla \cdot (\beta \nabla v_s) - \nabla\beta \cdot \nabla v_s d\Omega = \\ \int_{\partial\Omega} \beta \frac{\partial v_s}{\partial n} dS - \int_\Omega \nabla\beta \cdot \nabla v_s d\Omega = \int_{S_1} \lambda_s v_s \beta dS - \int_\Omega \nabla v_s \cdot \nabla\beta d\Omega. \end{cases} \tag{3.6.17}$$

It follows then

$$\lambda_s \int_{S_1} v_s \beta dS = \int_{S_2} v_s \phi dS. \tag{3.6.18}$$

This completes the proof of Proposition 2.7.1.

Conclusions

In this paper, we studied the problem of small vibrations of a two-layer fluid in an open container. This question can be considered as an operator equation with a spectral parameter, for which the spectrum is discrete and the eigenfunctions form an orthogonal basis in the corresponding functional space. We have concluded that this problem is a special case of the multi-layer fluid in a closed domain. In parallel, we defined the conditions for the existence and uniqueness of the solution of problems with Robin conditions.

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