Relative Egalitarianism and Related Criteria

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Abstract

We reconsider the problem of aggregating individual preference orderings into a single social ordering when alternatives are lotteries and individual preferences are of the von Neumann-Morgenstern type. *Relative egalitarianism* ranks alternatives by applying the leximin ordering to the distributions of (0-1) normalized utilities they generate. We propose an axiomatic characterization of this aggregation rule and discuss related criteria.

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1. Introduction

The problem of aggregating individual preference orderings into a single social ordering admits nontrivial solutions if Arrow's (1963) axiom of Independence of Irrelevant Alternatives is suitably weakened. Several interesting aggregation rules are known in at least three contexts.

In the traditional voting model, Young and Levenglick (1978) define and axiomatize the so-called maximum likelihood rule suggested by Condorcet (1785). If there exists a "correct" ordering of the social alternatives which individual orderings reflect imperfectly, this rule chooses an ordering that has the highest probability of being the correct one. The axiomatization of this rule by Young and Levenglick relies on a weakening of Arrow's axiom known as Local Independence of Irrelevant Alternatives (Young, 1995).

When social alternatives are lotteries and individual preferences obey the von Neumann-Morgenstern axioms, Dhillon (1998) and Dhillon and Mertens (1999) advocate relative utilitarianism: order social alternatives according to the sum of (0-1) normalized utilities they generate for the individuals who are not completely indifferent between all alternatives.

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The axiomatizations by Dhillon and Mertens use several weak axioms that are implied by Arrow's axiom.

In economic models where alternatives are production and allocation decisions, the literature focuses on inequality-averse aggregation rules. These rules compare allocations according to the welfare enjoyed by the worst-off individual, where the appropriate measure of welfare is endogenously determined by the axioms imposed on the aggregation rule itself. Examples include Maniquet and Sprumont (2004, 2005) for public-good economies and Fleurbaey (2007) and Fleurbaey and Maniquet (2008) for private-good economies. The axiomatizations use variants of a weakening of Arrow's axiom known as Hansson Independence (Hansson, 1973)[1].

While the rules proposed in these three contexts share some fundamental properties –they rely on purely ordinal information, obey the Pareto principle and are anonymous–, they are based on different views of the social good. In particular, Dhillon and Mertens's proposal follows the utilitarian tradition while the work of Fleurbaey, Maniquet and Sprumont is rooted in the egalitarian conception of justice.

The purpose of the current paper is to reconsider the aggregation problem in Dhillon and Mertens's random framework from an egalitarian perspective. If there are just two pure alternatives, a and b, and society consists of just two individuals with opposite von Neumann-Morgenstern preferences over the lotteries between a and b, relative utilitarianism deems all lotteries equally good. Flipping a fair coin is not better than choosing, say, b. Of course, relative utilitarianism shares this view with standard utilitarianism (or any theory requiring society's preference to obey the independence axiom of expected utility theory). Utilitarians essentially contend that flipping the coin has no value because, eventually, either a or b will be selected anyway: see, for instance, Harsanyi (1975) and Broome (1984). Critics of utilitarianism, such as Diamond (1967) and Sen (1970), argue that it prevents society from valuing the process through which social decisions are made. They claim that a lottery between a and b is superior to b because it gives a chance to the supporter of b.

We push this argument one step further. In practice, lotteries are used as social decision devices primarily when serious indivisibilities are present and compensatory payments between individuals are ruled out (see Elster, 1989 for a discussion). Their raison d'être is to permit some form of compromise between decisions that seem too radical. In some social choice problems, however, compromises may exist which are not lotteries. In a problem where all individuals have strict preferences between a and b, let us define a compromise between these two alternatives to be any alternative strictly preferred to b (but not to a) by the supporters of a and strictly preferred to a (but not to b) by the supporters of b. Lotteries are compromises but compromises need not be lotteries. We are interested in aggregation rules which "value compromise" in general. The axiom we propose, Preference for Compromise, says that if all individuals have strict preferences between a and b and alternative a is deemed at least as good as b from the social viewpoint, then any compromise

[1] In private-good economies, Fleurbaey, Suzumura and Tadenuma (2005) study how much Arrow's axiom must be weakened to allow for Paretian and anonymous (or non-dictatorial) aggregation rules.

between a and b is at least as good as b.

We combine this axiom with a weakening of Arrow's independence condition which we call Independence of Inessential Expansions. This condition requires that society's preference over a set of alternatives be unaffected by the addition of new alternatives that no individual finds better than his most preferred alternative or worse than his least preferred one.

We set up our formal model in Section 2 and present our results in Sections 3 to 5. Theorem 1 states that Preference for Compromise and Independence of Inessential Expansions, along with the Pareto Principle and Anonymity, lead to what we call the relative maximin principle: society should prefer an alternative a to an alternative b whenever the lowest individual normalized utility generated by a is higher than the lowest individual normalized utility generated by b. Theorem 2 asserts that only the relative leximin rule, which uses the leximin ordering to compare vectors of normalized utilities, satisfies the four axioms of Theorem 1 and Separability, a condition demanding that the social ranking be independent of the preferences of "unconcerned" individuals. Next, we weaken Preference for Compromise to a condition named Preference for Lotteries which requires that randomizing between two alternatives be at least as good as choosing the worst of these two. Theorem 3 says that the rules satisfying the Pareto Principle, Anonymity, Preference for Lotteries, Independence of Inessential Expansions, and Separability compare alternatives by applying a strictly monotonic, symmetric, convex, and separable ordering to the vectors of normalized utilities they generate. Theorem 4 clarifies the role of Independence of Inessential Expansions by identifying the class of rules meeting that axiom and the basic condition of Pareto Indifference. Section 6 offers a discussion of our results and a comparison with related work. Proofs are gathered in Section 7.

2. Framework

Let A be an infinite reference set of *pure* (*social*) *alternatives* and let $\mathcal{A}$ denote the set of nonempty finite subsets of A. For each $X \in \mathcal{A}$, let $\Delta(X)$ be the set of lotteries on X, that is, $\Delta(X) = \left\{a \in [0,1]^X \mid \sum_{x \in X} a(x) = 1\right\}$. If $x \in X$, we abuse notation and also denote by x the lottery in $\Delta(X)$ assigning probability 1 to x. Let $\mathcal{R}(X)$ and $\mathcal{R}_*(X)$ denote respectively the set of all preference orderings and the subset of von Neumann-Morgenstern preference orderings over $\Delta(X)$. Write $\mathcal{R} = \cup_{X \in \mathcal{A}} \mathcal{R}(X)$ and $\mathcal{R}_* = \cup_{X \in \mathcal{A}} \mathcal{R}_*(X)$.

Let $N = \{1, ..., n\}$ be a fixed finite set of individuals. A (*social choice*) *problem* is a list (X, R) where $X \in \mathcal{A}$ and $R \in (\mathcal{R}_*(X))^N$. We call R a *preference profile*. The set of all problems is denoted by $\mathcal{P}$. An (*aggregation*) *rule* is a mapping $\mathbf{R} : \mathcal{P} \to \mathcal{R}$ such that $\mathbf{R}(X, R) \in \mathcal{R}(X)$ for every $(X, R) \in \mathcal{P}$.

We make five comments on the above framework.

1) As in Dhillon and Mertens (1999), the set X is meant to include the pure social alternatives that are both feasible and just. The term "just" is used here in the weak sense of "ethically acceptable" and could be approximated by "lawful". We call X the *set of acceptable pure alternatives* and $\Delta(X)$ the *set of acceptable alternatives*. When

there is no risk of confusion, we refer to either set as the "acceptable set". Defining X is a fundamental ethical issue which cannot be addressed without further knowledge of the nature of the alternatives in A. We discuss the issue in Section 6 in the context of a particular application to multi-stage social decision problems.

2) We interpret $\mathbf{R}(X, R)$ as the ordering over $\Delta(X)$ that should guide society's choices when individual preferences are given by the profile R. We refer to it as *society's preference*. Note that society's preference over $\Delta(X)$ is constrained to depend only upon individual preferences *over that set*. This is a serious restriction. But it is a natural one because A is a large unstructured set and individual preferences over lotteries involving arbitrary alternatives in A may therefore be difficult to elucidate. Moreover, since no structure is imposed on A, there is no natural reference point outside X which could help define the aggregation rule.

3) When the set of acceptable alternatives expands, the social preference over the originally acceptable alternatives is allowed to change: if $X \subseteq X'$ and the preference profile R' over $\Delta(X')$ coincides over $\Delta(X)$ with the profile R, $\mathbf{R}(X', R')$ need not coincide with $\mathbf{R}(X, R)$ on $\Delta(X)$[2].

4) Even though individual preferences are of the von Neumann-Morgenstern type, society's preference is not restricted to be of that type. As we explained in the Introduction, we believe that it should not be. See Section 6 for more discussion.

5) All individual von Neumann-Morgenstern preferences over the acceptable alternatives are admissible. This is in line with Arrow's universal domain assumption and guarantees that the aggregation rules we discuss are not restricted to a particular type of social choice problem. In applications, however, it may be desirable to impose restrictions on preferences. For instance, if X is the set of acceptable allocations in a private-good economy, it is natural to require that preferences be selfish. We believe that our results can be reformulated in such restricted contexts but such a reformulation is not straightforward. Our proofs do rely on the universal domain assumption.

3. A set of axioms implying the relative maximin principle

What we call the relative maximin principle says that society should prefer an alternative a to an alternative b whenever the lowest individual normalized utility generated by a is higher than the lowest individual normalized utility generated by b. This section proposes a set of axioms leading to that principle.

We begin with two familiar conditions. If $(X, R) \in \mathcal{P}$ and $i \in N$, let us write $R(i) = R_i$ and denote by P_i and I_i the strict preference and indifference relations associated with R_i. Likewise, $\mathbf{P}(X, R)$ and $\mathbf{I}(X, R)$ denote the strict social preference and indifference relations associated with $\mathbf{R}(X, R)$. Let $\Pi(N)$ be the set of permutations on N. If $\sigma \in \Pi(N)$,

[2] In Dhillon and Mertens (1999), the acceptable set (which they denote by A) is kept fixed. No axiom linking social preferences accross different acceptable sets is used to characterize relative utilitarianism. Yet, if the acceptable set is allowed to expand, the social preference recommended by relative utilitarianism over the originally acceptable alternatives may be –and often is– affected.

$\sigma R \in (\mathcal{R}_*(X))^N$ denotes the preference profile such that $(\sigma R)_{\sigma(i)} = R_i$ for all $i \in N$.

Pareto Principle. If $(X, R) \in \mathcal{P}$, $a, b \in \Delta(X)$, and aR_ib for all $i \in N$, then $a\mathbf{R}(X, R)b$. If, in addition, aP_jb for some $j \in N$, then $a\mathbf{P}(X, R)b$.

Anonymity. For all $(X, R) \in \mathcal{P}$ and $\sigma \in \Pi(N)$, $\mathbf{R}(X, R) = \mathbf{R}(X, \sigma R)$.

Next we state our central axiom. Preference for Compromise expresses the ethical judgement that it is desirable to compromise between two social alternatives over which individual preferences are antagonistic.

Preference for Compromise. Let $(X, R) \in \mathcal{P}$, $a, b, c \in \Delta(X)$, and $\varnothing \subsetneqq S \subsetneqq N$. If $a\mathbf{R}(X, R)b$, aR_icP_ib for all $i \in S$ and bR_jcP_ja for all $j \in N \setminus S$, then $c\mathbf{R}(X, R)b$.

The justification goes as follows. If a is deemed at least as good as b from the social viewpoint, choosing b would constitute an unfair treatment of the individuals supporting a against b. This is true even if society is exactly indifferent between a and b: the supporters of a are treated unfairly because society cannot invoke any *reason*[3] to select b rather than a. Therefore, if a third alternative c makes the supporters of a better off than at alternative b (where they are treated unfairly) while keeping the supporters of b better off than at a (where they could perhaps claim to be treated unfairly), society should prefer c to b. In other words: if switching from b to a is a social improvement, then switching from b to an alternative that is a "compromise" between b and a should also be.

Two remarks are in order. First, the above argument suggests that society's preference for compromise should be strict: if $\varnothing \subsetneqq S \subsetneqq N$, $a\mathbf{R}(X, R)b$, aR_icP_ib for all $i \in S$ and bR_jcP_ja for all $j \in N \setminus S$, then in fact $c\mathbf{P}(X, R)b$. While we strongly believe that this strict version of Preference for Compromise is desirable, we do not impose it. Section 4 shows how it follows from the weak version and other axioms. Second, the alternative c in the axiom is a compromise between a and b in the sense that aR_icP_ib for all $i \in S$ and bR_jcP_ja for all $j \in N \setminus S$. This does not imply that c is a lottery between a and b. Preference for Compromise therefore goes beyond the requirement that randomizing between two alternatives be at least as good as choosing the worst of the two. We will consider that alternative requirement in Section 5.

We now turn to our weakening of Arrow's independence axiom. As pointed out in Comment 3 in Section 2, the definition of an aggregation rule allows the social preference over a given subset of alternatives to vary with the set of acceptable alternatives. Such flexibility is necessary in order to construct "fair" aggregation rules. In particular, it is needed if society values compromise. To see this, suppose that the acceptable set consists of the lotteries between two pure alternatives, a and b. If society is composed of two individuals with opposite von Neumann-Morgenstern preferences over $\Delta(\{a, b\})$, very basic

[3] In our view, there is a fundamental difference between social and individual choices. While an individual needs no reason to select one of two alternatives between which he is indifferent because he decides for himself, society does need a reason because it does not decide for itself. Its choices matter exactly to the extent that they affect individuals. Even if society is indifferent between two alternatives, choosing remains a nontrivial issue as long as some members of society are not indifferent.

requirements (such as the usual Anonymity and Neutrality axioms) force social indifference between a and b. But if a third pure alternative, c, becomes acceptable and individual 1 strictly prefers a to b to c while 2 strictly prefers c to b to a, society should probably strictly prefer b to a in $\Delta(\{a, b, c\})$. The reason is that b may now be regarded as a compromise between the two agents while a has become a more extreme alternative.

In this example, adding the pure social alternative c to the set $\{a, b\}$ alters the preference aggregation problem in an essential way because it changes the worst possible outcome for individual 1 and the best outcome for 2. Independence of Inessential Expansions says that society's preferences over $\Delta(X)$ should be unaffected by the addition of new alternatives that leave the best and worst outcomes of all individuals unchanged –what we call an "inessential expansion" of the acceptable set. Given $(X, R) \in \mathcal{P}$ and $i \in N$, denote by $\overline{A}(X, R_i)$ and $\underline{A}(X, R_i)$ the sets of best and worst alternatives in $\Delta(X)$ according to R_i. Our formal requirement is the following.

Independence of Inessential Expansions. Let $(X, R), (X', R') \in \mathcal{P}$ be two problems such that $X \subseteq X'$ and R' coincides with R on $\Delta(X)$. If $\overline{a} R'_i a' R'_i \underline{a}$ for all $i \in N$ and all $\overline{a} \in \overline{A}(X, R_i), a' \in X', \underline{a} \in \underline{A}(X, R_i)$, then $\mathbf{R}(X', R')$ coincides with $\mathbf{R}(X, R)$ on $\Delta(X)$.

While we argued that the social preference should be allowed to vary when the individuals' best or worst alternatives change, it is not clear why it should remain unaffected by "inessential expansions" of the acceptable set. We do not think there are compelling ethical reasons to insist on this requirement. As is the case with other independence axioms (such as those proposed by Arrow, 1963, Hansson, 1973, or Young and Levenglick, 1978), the primary justification is practical convenience. An aggregation rule satisfying Independence of Inessential Expansions is relatively simple to implement in practice because the social ranking of alternatives is only affected by fairly radical changes in the environment.

The restriction to those expansions which do not change the best and worst outcomes of all individuals, though definitely somewhat *ad hoc,* is rather natural. Notice the similarity between our axiom and the Restricted Monotonicity condition often used in bargaining theory to characterize the Kalai-Smorodinsky (1975) solution. Restricted Monotonicity (see, e.g., Thomson, 1994) says that no individual should suffer from an expansion of the utility possibility set which does not affect the minimal and maximal utilities of anyone. Our axiom is a relative –in the ordinal aggregation context– of this cardinal choice axiom[4]. It retains the idea that "inessential" changes in the environment should not distort social decisions but it does not incorporate the solidarity concerns embedded in Restricted Monotonicity.

In order to state our first result, let us now give the formal definition of normalized utilities. Given $X \in \mathcal{A}$, we denote by R_0 the complete indifference relation on $\Delta(X)$. Given $(X, R) \in \mathcal{P}$ and $i \in N$, we let $u(., X, R_i) : \Delta(X) \to [0, 1]$ be the *normalized von Neumann-Morgenstern numerical representation* of R_i : if $R_i \neq R_0$, then $u(a, X, R_i) = \alpha \Leftrightarrow a I_i [\alpha \overline{a} + (1 - \alpha) \underline{a}]$ for any $\overline{a} \in \overline{A}(X, R_i)$ and $\underline{a} \in \underline{A}(X, R_i)$; if $R_i = R_0$, then $u(a, X, R_i) = 1$ for all $a \in \Delta(X)$.

[4]See de Clippel (2008) for an ordinal reformulation of Restricted Monotonicity.

Theorem 1. *If the aggregation rule* $\mathbf{R}$ *satisfies the Pareto Principle, Anonymity, Preference for Compromise, and Independence of Inessential Expansions, then for all* $(X, R) \in \mathcal{P}$ *and* $a, b \in \Delta(X)$, $\min_{i \in N} u(a, X, R_i) > \min_{i \in N} u(b, X, R_i) \Rightarrow a\mathbf{P}(X, R)b$.

The discussion following Theorem 2 shows that the axioms in Theorem 1 are independent.

It is important to recall that no utility information is available in our framework. An aggregation rule is a purely ordinal procedure transforming profiles of individual preference orderings into social orderings. The axioms in Theorem 1 thus perform a double task: 1) they select the (0-1) normalized von Neumann-Morgenstern representation of preferences as the adequate measure of individual welfare and 2) they force the use of the maximin criterion to compare welfare vectors. Very roughly, Independence of Inessential Expansions performs the first task while the other axioms take care of the second.

Pinning down the 0-1 normalization involves two main difficulties: a priori, the adequate numerical representation of an individual's preference need not be a von Neumann-Morgenstern utility function and it could vary with the preferences of the other individuals. The two examples blow illustrate these difficulties. They show that the Pareto Principle, Anonymity and Preference for Compromise allow for a complex variety of "welfare egalitarian" rules incompatible with the relative maximin principle.

Example 1. Given $(X, R) \in \mathcal{P}$ and $i \in N$, define $v(., X, R_i) : \Delta(X) \to [0, 1]$ by letting $v(a, X, R_i) = \frac{\mu(\{b \in \Delta(X) | aR_i b\})}{\mu(\Delta(X))}$ for all $a \in \Delta(X)$, where μ is the Lebesgue measure on $\mathbb{R}^{|X|-1}$. That is, an individual's utility from alternative a is measured by the proportion of acceptable alternatives that he does not consider better than a. Note that even though $v(., X, R_i)$ is not linear, it is a numerical representation of the von Neumann-Morgenstern preference R_i. For all $(X, R) \in \mathcal{P}$ and $a, b \in \Delta(X)$, let $a\mathbf{R}(X, R)b \Leftrightarrow (v(a, X, R_1), ..., v(a, X, R_n)) \succsim^L (v(b, X, R_1), ..., v(b, X, R_n))$, where $\succsim^L$ is the leximin ordering on $[0, 1]^N$. This aggregation rule satisfies all axioms in Theorem 1 except Independence of Inessential Expansions.

The difficulty exemplified by this rule is proper to our framework. In Dhillon and Mertens (1999), the assumption that society's preference obeys the independence axiom prevents the construction of comparable examples. The "utilitarian" ordering $a\mathbf{R}(X, R)b \Leftrightarrow \sum_{i \in N} v(a, X, R_i) \geq \sum_{i \in N} v(b, X, R_i)$ is generally not a von Neumann-Morgenstern preference over $\Delta(X)$.

Example 2. For each $(X, R) \in \mathcal{P}$, let $U(X, R)$ denote the normalized utility set generated by (X, R), that is, $U(X, R) = \{(u(a, X, R_1), ..., u(a, X, R_n) \mid a \in \Delta(X)\}$. Let $\alpha(X, R) = (\alpha_1(X, R), ..., \alpha_n(X, R))$ be the unique maximizer of $\prod_{i \in N} z_i$ over all $z \in U(X, R)$. For each $i \in N$, define the function $v_i(., X, R) : \Delta(X) \to [0, 1]$ by $v_i(a, X, R) = u(a, X, R_i)^{\alpha_i(X,R)}$ for all $a \in \Delta(X)$. This function is a numerical representation of R_i but it is one that changes with the preferences of the agents other than i. The rule $a\mathbf{R}(X, R)b \Leftrightarrow (v_1(a, X, R), ..., v_n(a, X, R_n)) \succsim^L (v_1(b, X, R), ..., v_n(b, X, R))$ for all $(X, R) \in \mathcal{P}$ and $a, b \in \Delta(X)$ satisfies all axioms in Theorem 1 except Independence of Inessential Expansions.

Observe that this aggregation rule satisfies the following weakening of Independence of

Inessential Expansions.

Independence of Redundant Alternatives. Let $(X, R), (X', R') \in \mathcal{P}$ be two problems such that $X \subseteq X'$ and R' coincides with R on $\Delta(X)$. If for all $a' \in X'$ there exists $a \in \Delta(X)$ such that $a' I'_i a$ for all $i \in N$, then $\mathbf{R}(X', R')$ coincides with $\mathbf{R}(X, R)$ on $\Delta(X)$.

This condition merely says that adding alternatives that are Pareto equivalent to some originally acceptable alternatives does not change the social ranking over the original acceptable set. This axiom is the translation in our framework of the property bearing the same name in Dhillon and Mertens (1999). It cannot replace Independence of Irrelevant Expansions in Theorem 1.

4. An axiomatization of the relative leximin rule

Theorem 1 is not a complete characterization result because it does not identify all the aggregation rules meeting the stated axioms. A prominent example of a rule satisfying all these axioms is the *relative leximin* aggregation rule $\mathbf{R}^L$ defined as follows: for all $(X, R) \in \mathcal{P}$ and $a, b \in \Delta(X)$, $a\mathbf{R}^L(X, R)b \Leftrightarrow (u(a, X, R_1), ..., u(a, X, R_n)) \succsim^L (u(b, X, R_1), ..., u(b, X, R_n))$, where $\succsim^L$ is again the leximin ordering on $[0, 1]^N$. When there are only two individuals, it turns out that no other rule satisfies the axioms in Theorem 1.

Corollary to Theorem 1. *Let $n = 2$. Then the aggregation rule $\mathbf{R}$ satisfies the Pareto Principle, Anonymity, Preference for Compromise, and Independence of Inessential Expansions if and only if $\mathbf{R} = \mathbf{R}^L$.*

This corollary does not extend beyond the two-individual case. For instance, it is straightforward to check that the lexicographic combination of relative maximin and relative utilitarianism, $\mathbf{R}^{MU}$, satisfies all the axioms in Theorem 1. This rule is defined by letting $a\mathbf{R}^{MU}(X, R)b$ if and only if (i) $\min_{i \in N} u(a, X, R_i) > \min_{i \in N} u(b, X, R_i)$ or (ii) $\min_{i \in N} u(a, X, R_i) = \min_{i \in N} u(b, X, R_i)$ and $\sum_{i \in N} u(a, X, R_i) \geq \sum_{i \in N} u(b, X, R_i)$.

A key property distinguishing $\mathbf{R}^L$ from $\mathbf{R}^{MU}$ is the following separability condition, adapted to our context from Fleming (1952).

Separability. Let $X \in \mathcal{A}$, $a, b \in \Delta(X)$, and $S \subseteq N$. Let $R, R' \in (\mathcal{R}_*(X))^N$ be such that $R_i = R'_i$ for all $i \in S$ and aI_jb and aI'_jb for all $j \in N \setminus S$. Then $a\mathbf{R}(X, R)b$ if and only if $a\mathbf{R}(X, R')b$.

Separability says that the social ranking of two alternatives should be independent of the preferences of all unconcerned individuals, namely those who are indifferent between the two alternatives in any case. This is a familiar condition in social choice theory: see d'Aspremont (1985) for references.

It is easy to see that $\mathbf{R}^L$ satisfies Separability while $\mathbf{R}^{MU}$ does not. In fact, relative leximin is the only rule satisfying Separability and the axioms in Theorem 1.

Theorem 2. *The aggregation rule $\mathbf{R}$ satisfies the Pareto Principle, Anonymity, Preference for Compromise, Independence of Inessential Expansions, and Separability if and only if $\mathbf{R} = \mathbf{R}^L$.*

We note that the relative leximin rule actually exhibits a "strict preference for compromise". Specifically, our proof of Theorem 2 establishes the following statement: if $\varnothing \subsetneqq S \subsetneqq N$, $a\mathbf{R}^L(X,R)b$, aR_icP_ib for all $i \in S$ and bR_jcP_ja for all $j \in N \setminus S$, then in fact $c\mathbf{P}^L(X,R)b$: the "compromise" alternative c is *strictly* preferred to b. In particular, when two individuals have opposite preferences over the lotteries between two alternatives, the relative leximin rule strictly prefers to toss a coin rather than choose either of the two alternatives.

A few comments are in order regarding the role of Separability. The axiom is used in our proof of Theorem 2 only to bridge the gap between relative maximin and relative leximin when $n \geq 3$ and it can be replaced with the following weaker requirement[5].

Weak Separability. Let $X \in \mathcal{A}$, $a, b \in \Delta(X)$, $R \in (\mathcal{R}_*(X))^N$, $i \in N$, and let (R_0, R_{-i}) denote the profile obtained from R by replacing R_i with the complete indifference relation R_0. If $a\mathbf{P}(R_0, R_{-i})b$ and aI_ib, then $a\mathbf{P}(R)b$.

It is clear, however, that Separability (or Weak Separability) is a powerful condition which does exclude some welfare egalitarian rules where the representation of an individual's preference depends upon others' preferences, such as the rule described in Example 2. But Separability does not disqualify all such rules, as the following example shows.

Example 3. Partition $\mathcal{R}_*$ into two sets $\mathcal{R}_*^1$, $\mathcal{R}_* \setminus \mathcal{R}_*^1$, each containing preferences differing from complete indifference. For instance, $\mathcal{R}_*^1$ could be the set of preferences with at most two indifference classes of pure alternatives. For each $(X,R) \in \mathcal{P}$ and $i \in N$, define $\beta_i(X,R) = |\{j \in N \setminus \{i\} \mid R_j \in \mathcal{R}_* \setminus \mathcal{R}_*^1\}|$ and define $w_i(.,X,R) : \Delta(X) \to [0,1]$ by letting $w_i(a,X,R) = u(a,X,R_i)^{2^{\beta_i(X,R)}}$ for all $a \in \Delta(X)$. Then let $a\mathbf{R}(X,R)b \Leftrightarrow (w_1(a,X,R),...,$ $w_n(a,X,R)) \succsim^L (w_1(b,X,R)\ ,...,\ w_n(b,\ X,R))$ for all $a,b \in \Delta(X)$. This aggregation rule satisfies all axioms in Theorem 2 except Independence of Inessential Expansions. To see why Separability is satisfied, consider two problems $(X,R),(X,R')$ where $R_i = R_i'$ for all $i \neq 1$, $R_1 \in \mathcal{R}_*^1$, and $R_1' \in \mathcal{R}_* \setminus \mathcal{R}_*^1$. Then, for all $i \neq 1$,

$$\begin{aligned} w_i(.,X,R') &= u(.,X,R_i)^{2^{\beta_i(X,R')}} \\ &= u(.,X,R_i)^{2^{\beta_i(X,R)+1}} \\ &= (u(.,X,R_i)^{2^{\beta_i(X,R)}})^2 \\ &= (w_i(.,X,R))^2, \end{aligned}$$

that is, the numerical representations of the preferences of all agents other than 1 are modified according to a *common* increasing transformation when 1's preference changes from R_1 to R_1'. Using the fact that the leximin ordering $\succsim^L$ is separable (in the usual sense defined just before Theorem 3), it follows easily that for all $a,b \in \Delta(X)$ such that aI_1b and $aI_1'b$, $a\mathbf{R}(X,R)b \Leftrightarrow a\mathbf{R}(X,R')b$.

We conclude this section by showing that the axioms in Theorem 2 are independent.

[5] Weak Separability is the translation in our framework of the strict part of the condition dubbed "consistency" in Dhillon and Mertens (1999).

1) The constant complete indifference rule $a\mathbf{I}(X,R)b$ for all $(X,R)\in\mathcal{P}$ and $a,b\in\Delta(X)$ satisfies all the axioms except the Pareto Principle. This rule does not satisfy the strict version of Preference for Compromise described just after Theorem 2. For an example satisfying that version along with Anonymity, Independence of Inessential Expansions, and Separability, consider the opposite of the relative leximin rule defined as follows: for all $(X,R)\in\mathcal{P}$ and $a,b\in\Delta(X)$, $a\mathbf{R}(X,R)b \Leftrightarrow (1-u(a,X,R_1),...,\ 1-u(a,X,R_n)) \succsim^L (1-u(b,X,R_1)\ ,...,\ 1-u(b,\ X,R_n))$.

2) An example of a rule violating only Anonymity is relative serial dictatorship: for all $(X,R)\in\mathcal{P}$ and $a,b\in\Delta(X)$, $a\mathbf{R}(X,R)b \Leftrightarrow (u(a,X,R_1),...,\ u(a,X,R_n)) \succsim^{(1,...,n)} (u(b,X,R_1)\ ,...,\ u(b,\ X,R_n))$, where $\succsim^{(1,...,n)}$ is the lexicographic ordering on $[0,1]^N$ corresponding to the natural ordering over N.

3) The rules in Examples 1 and 3 satisfy all axioms in Theorem 2 but Independence of Inessential Expansions. (The rule in Example 1 satisfies Separability because the numerical representation of an individual's preference does not depend on the preferences of the others.) Notice that both rules violate the weaker requirement of Independence of Redundant Alternatives. An interesting and difficult question is whether that axiom can replace Independence of Inessential Expansions in the statement of Theorem 2.

4) Relative utilitarianism $a\mathbf{R}(X,R)b \Leftrightarrow \sum_{i\in N} u(a,X,R_i) \geq \sum_{i\in N} u(b,X,R_i)$ for all $(X,R)\in\mathcal{P}$ and $a,b\in\Delta(X)$ violates only Preference for Compromise.

5) The lexicographic combination of relative maximin and relative utilitarianism $\mathbf{R}^{MU}$ violates only Separability.

5. Related aggregation rules

We call an aggregation rule "relative" if it compares alternatives according to the individual normalized utilities they generate. The purpose of this section is twofold. First, we study relative aggregation rules which "value compromise" to a lesser degree than what is required by Preference for Compromise. Second, we clarify the role of Independence of Inessential Expansions in obtaining characterizations of relative rules.

A natural alternative to Preference for Compromise is the requirement that randomizing between two alternatives be at least as good as choosing the worst of these two. More precisely, we consider the following axiom.

Preference for Lotteries. Let $(X,R)\in\mathcal{P}$, $a,b\in\Delta(X)$. If $a\mathbf{R}(X,R)b$, then $(\lambda a+(1-\lambda)b)\mathbf{R}(X,R)b$ for all $\lambda\in(0,1)$.

A variant of this axiom for a different aggregation exercise is used by Epstein and Segal (1992). Translated in our framework, Epstein and Segal's Randomization Preference axiom states that if $(X,R)\in\mathcal{P}$, $a,b\in\Delta(X)$, $a\mathbf{I}(X,R)b$, and a is not Pareto equivalent to b, then $(\frac{1}{2}a+\frac{1}{2}b)\mathbf{P}(X,R)b$. We discuss the differences between Epstein and Segal's work and ours in the next section.

Our first result in this section, Theorem 3, is a variant of Theorem 2. A few definitions are needed in order to state it. Let $\succsim$ be an ordering on $[0,1]^N$ and let $\sim$ and $\succ$ denote

the symmetric and asymmetric components of $\succsim$. Using the notation $\geq, >, \gg$ for vector inequalities, we say that $\succsim$ is *strictly monotonic* if, for all $v, w \in [0,1]^N$, $v > w$ implies $v \succ w$. Using the obvious notation, $\succsim$ is *symmetric* if, for all $v, w \in [0,1]^N$ and $\sigma \in \Pi(N)$, $v \succsim w$ if and only if $\sigma v \succsim \sigma w$. We say that $\succsim$ is *convex* if, for all $v, w \in [0,1]^N$ and $\lambda \in (0,1)$ $v \succsim w$ implies $\lambda v + (1-\lambda) w \succsim w$.[6] Finally, $\succsim$ is *separable* if, for all $S \subseteq N$ and $v, v', w, w' \in [0,1]^N$ such that $v_i = v'_i$ and $w_i = w'_i$ for all $i \in S$ and $v_j = w_j$ and $v'_j = w'_j$ for all $j \in N \setminus S$, we have $v \succsim w$ if and only if $v' \succsim w'$.

Theorem 3. *The aggregation rule* $\mathbf{R}$ *satisfies the Pareto Principle, Anonymity, Preference for Lotteries, Independence of Inessential Expansions, and Separability if and only if there exists a strictly monotonic, symmetric, convex and separable ordering* $\succsim$ *on* $[0,1]^N$ *such that for all* $(X,R) \in \mathcal{P}$ *and* $a, b \in \Delta(X)$, $a\mathbf{R}(X,R)b \Leftrightarrow (u(a,X,R_1), ..., u(a,X,R_n)) \succsim (u(b,X,R_1), ..., u(b,X,R_n))$.

Note an important difference with respect to Theorem 2: the aggregation rules in Theorem 3 need not satisfy the strict version of Preference for Lotteries requiring that if $(X,R) \in \mathcal{P}$, $a, b \in \Delta(X)$, $a\mathbf{R}(X,R)b$, and a is not Pareto equivalent to b, then $(\lambda a + (1-\lambda)b)\mathbf{P}(X,R)b$ for all $\lambda \in (0,1)$.

Theorem 3 suggests a natural conjecture: Pareto Indifference (whose definition will be recalled shortly) and Independence of Inessential Expansions might suffice to force society to compare alternatives by ranking the vectors of individual normalized utilities they generate according to a fixed ordering on $[0,1]^N$. The Pareto Principle, Anonymity, Preference for Lotteries, and Separability would then simply translate, respectively, into strict monotonicity, symmetry, convexity, and separability of that fixed ordering of utility vectors.

This conjecture is not quite correct. While an aggregation rule satisfying Pareto Indifference and Independence of Inessential Expansions must indeed be based solely on the individual normalized utilities, the criterion used to aggregate these utilities may in fact vary with the preference profile –albeit in a very restricted way. We conclude this section with a precise formulation of this assertion.

First, we recall the definition of Pareto Indifference and introduce one last piece of notation.

Pareto Indifference. If $(X,R) \in \mathcal{P}$, $a, b \in \Delta(X)$, and aI_ib for all $i \in N$, then $a\mathbf{I}(X,R)b$.

Given $(X,R) \in \mathcal{P}$, define $N_0(X,R) = \{i \in N \mid R_i = R_0\}$. For each $S \subseteq N$, let $\mathcal{P}_S = \{(X,R) \in \mathcal{P} \mid N_0(X,R) = S\}$. This is the set of problems where the individuals indifferent between all acceptable alternatives are the members of S.

Theorem 4. *The aggregation rule* $\mathbf{R}$ *satisfies Pareto Indifference and Independence of Inessential Expansions if and only if for each* $S \subseteq N$ *there exists an ordering* $\succsim_S$ *on* $[0,1]^N$ *such that for all* $(X,R) \in \mathcal{P}_S$ *and* $a, b \in \Delta(X)$, $a\mathbf{R}(X,R)b \Leftrightarrow (u(a,X,R_1), ..., u(a,X,R_n)) \succsim_S (u(b,X,R_1), ..., u(b,X,R_n))$.

[6] This slightly unusual statement is equivalent to the more standard definition requiring that for all $v, w, z \in [0,1]^N$ and $\lambda \in (0,1)$, $v \succsim z$ and $w \succsim z$ imply $\lambda v + (1-\lambda) w \succsim z$.

If, in addition to the axioms listed in Theorem 4, the aggregation rule also meets Separability, then the ordering used to aggregate normalized utility vectors must be fixed. It is then easy to show that the Pareto Principle, Anonymity, and Preference for Lotteries imply strict monotonicity, symmetry, and convexity of that ordering. Indeed, Section 6 proves Theorem 3 as a corollary to Theorem 4. Without Separability, however, the ordering used to compare utility vectors may vary with the set of completely indifferent individuals. In particular, Theorem 1 is not a direct corollary of Theorem 4.

6. Discussion

In this section we compare relative egalitarianism with related theories of social evaluation. We also address some of the criticisms often formulated against theories which, like relative egalitarianism, recommend social orderings violating the independence axiom of expected utility theory. In particular, we explain how relative egalitarianism may be applied in multi-stage social decision problems.

6.1. Relative egalitarianism and relative utilitarianism

Relative egalitarianism is a purely ordinal theory of social evaluation requiring no comparison of individual utilities. It may therefore be regarded as an "operational" egalitarian theory, just like relative utilitarianism is an operational version of classical utilitarianism. Society is dispensed with the delicate task of comparing individual utilities. Of course, it still must decide which feasible alternatives should be included in the acceptable set X over which individual preferences are to be aggregated. But this is the only ethical issue society must settle, and it is expressed in simple terms.

The primary advantage of relative egalitarianism over relative utilitarianism is that it embodies a concern for fairness. More specifically: it satisfies the strict version of Preference for Compromise. Obviously, that axiom reflects a value judgement about the social good. It is our view that promoting compromise and fairness is an important social goal.

A second important advantage, which relative egalitarianism shares with all aggregation rules comparing social alternatives by applying a strictly convex ordering to the vectors of relative utilities they generate, is its lack of ambiguity. Relative utilitarianism faces the following difficulty: if several *Pareto non-comparable* alternatives maximize the sum of normalized utilities over the acceptable set, which alternative should society choose? On the one hand, no tie-breaking device can be recommended without contradicting the theory. On the other hand, society is not an individual: having no free will or decision ability of its own, it cannot choose. Nor should it be allowed to choose. In a fully satisfactory normative theory, all social choices should follow unambiguously from the knowledge of individual preferences, unless of course all individuals themselves are indifferent. Except in that case, relative egalitarianism always delivers a unique best alternative.

Relative egalitarianism leads to social preferences that violate the independence axiom. Proponents of (relative or classical) utilitarianism criticize such social orderings on

two grounds. First, they argue that the implicit reason invoked for abandoning the independence axiom, namely, a concern for fairness, suggests that individuals themselves care about fairness: in that case, they claim, social alternatives should be redefined so as to encompass everything that matters to the individuals. The second criticism is that violations of the independence axiom lead to time-inconsistent choices: if society prefers to flip a coin rather than choose a or b, then it should also ignore the result of the flip and toss the coin again, indefinitely. A related difficulty is that the stochastic dominance principle may be violated.

Regarding the first criticism, it is important to keep in mind that relative egalitarianism does satisfy Pareto Indifference. Note that social indifference between two alternatives a and b may arise in two different cases: it may reflect unanimous individual indifference or result from the aggregation of opposite individual preferences. In the former case, relative egalitarianism deems any lottery between alternatives a and b equally good as either alternative. Only in the second case does society prefer a lottery. We see no logical inconsistency in having a social ordering embody collective values that complement individual preferences as long as these values do not contradict the Pareto Principle. In particular, a social concern for fairness is perfectly valid even if the individuals themselves are completely selfish[7]. Ultimately, collective values cannot be dispensed with anyway: relative egalitarianism and relative utilitarianism face the preliminary problem of delimiting the set of acceptable alternatives X. This, by the very specification of the model, cannot be done on the basis of individual preferences.

Regarding the second criticism, we claim that a proper definition of the set X in an explicit model of multi-stage social decisions avoids the time-inconsistency problem[8]. In the spirit of Hammond (1988, 1996), let us describe a multi-stage (social decision) problem as a list (T, π, R). The *decision tree* T has a finite set of nodes $V(T)$ which is partitioned into a set of (social) decision nodes $V^*(T)$ at which society must make a choice, a set of chance nodes $V^0(T)$ at which nature decides the course of events, and a set $Z(T)$ of terminal nodes at which all uncertainty is resolved. To fix ideas, we assume that the initial node of the tree, $v_0(T)$, is a decision node. The *probability function* π assigns to each chance node v a lottery $\pi(., v)$ over $V_{+1}(v)$, the set of nodes which immediately follow v. The *preference profile* R is a list of von Neumann-Morgenstern preferences over $\Delta(Z(T))$, the set of lotteries over the terminal nodes.

In order to apply relative egalitarianism, we first need to define the feasible alternatives

[7] The utilitarian argument seems to be based on the belief that a social concern for fairness necessarily contradicts Pareto Indifference if individuals themselves do not care about fairness. But this is incorrect: "not caring about fairness" is not a property of preferences which should be respected by an aggregation rule satisfying Pareto Indifference. Consider again two individuals with purely selfish opposite von Neumann-Morgenstern preferences over the lotteries between a and b. Neither individual cares about fairness, yet Pareto Indifference places no constraint at all on society's preference.

[8] For a completely different line of argument showing that a strict social preference for lotteries need not generate time-inconsistent decisions, see Epstein and Segal (1992). These authors point out that the source of the alleged time inconsistency is the implicit assumption that society's preference after the coin flip is the same as before. They show how a meaningful preference updating procedure avoids time-inconsistent choices.

at each decision node $v \in V^*(T)$ in a way that guarantees that individual preferences over such alternatives are well defined. Denote by T_v the subtree of T starting at node v. We consider as a feasible alternative at v any (social) strategy for the decision tree T_v. A *pure strategy* for T_v is a function s which assigns to each decision node $v' \in V^*(T_v)$ a node $s(v') \in V_{+1}(v')$. Denoting by $S(T_v)$ the set of such pure strategies, a *strategy* for T_v is an element σ_v of $\Delta(S(T_v))$. Since each strategy $\sigma_v \in \Delta(S(T_v))$ determines via the probability function π a unique lottery $a(\sigma_v) \in \Delta(Z(T))$, the individual preferences over $\Delta(Z(T))$ induce preferences over $\Delta(S(T_v))$ in an unambiguous way.

It should now be clear that not all feasible alternatives at v need be *acceptable*. This is the crucial difference between social and individual multi-stage decision problems. Which alternative is acceptable may depend on past choices. Indeed, we submit that if society chooses strategy σ at the initial node $v_0(T)$, the continuation of σ in T_v should be regarded as the *only* acceptable alternative at v. Our view is that the strategy σ constitutes a complete "contract" between society and its members. Any deviation from this fully specified plan of action is therefore ethically unacceptable. By choosing σ at node $v_0(T)$, society creates a right for individuals to insist that the continuation of σ be followed at node v. Under this definition of acceptability, relative egalitarianism needs to be applied only once at the initial node in order to determine the best social strategy for the entire multi-stage problem (T, π, R). Any multi-stage problem arising at any subsequent node is completely degenerate because the set of acceptable alternatives reduces to a singleton. To illustrate this point, consider the classical puzzle of a mother who must decide how to allocate a treat among her two children, A and B (Machina, 1989, Grant, 1995). A concern for fairness leads the mother to prefer to flip a coin rather than allocate the treat to either child. After the flip, she gives the treat to the lucky winner, say A, simply because flipping the coin again would violate A's newly acquired right to get the treat. This is a perfectly consistent sequence of ethically optimal decisions.

To sum up, relative egalitarianism is applicable if social alternatives are defined as complete plans of actions and respect of past commitments is taken as a social value. Defining feasible alternatives as complete plans of action seems unavoidable. The alternative approach which would take $\Delta(V_{+1}(v))$, the set of lotteries over the nodes immediately following v, as the set of feasible alternatives at v is unsatisfactory because individual preferences over $\Delta(V_{+1}(v))$ cannot be inferred without ambiguity from the preferences originally defined over $\Delta(Z(T))$.

The necessity to restrict the set of acceptable strategies at decision nodes other than the initial node is not proper to relative egalitarianism. Relative utilitarianism faces it too. Consider a tree T with two decision nodes v_0, v_1, no chance node, and four terminal nodes a, b, c, d. The nodes d and v_1 immediately follow v_0 and the nodes a, b, c immediately follow v_1. There are two individuals. Their preferences R_1, R_2 over $\Delta(\{a, b, c, d\})$ are represented

by the von Neumann-Morgenstern utility functions u_1^0, u_2^0 such that

$$u_1^0(x) = \begin{cases} 1 \text{ if } x = a, \\ \frac{2}{3} \text{ if } x = b, \\ \frac{1}{6} \text{ if } x = c, \\ 0 \text{ if } x = d, \end{cases} \quad u_2^0(x) = \begin{cases} \frac{5}{6} \text{ if } x = a, \\ 1 \text{ if } x = b, \\ \frac{2}{3} \text{ if } x = c, \\ 0 \text{ if } x = d. \end{cases}$$

At the initial node v_0, the feasible pure alternatives are the pure strategies for T, namely, in obvious notation, $(v_1, a), (v_1, b), (v_1, c)$, and d. The feasible alternatives are the strategies for T and the preferences of individuals 1 and 2 over these alternatives are derived from their preferences over $\Delta(\{a, b, c, d\})$ by identifying each strategy with the lottery it generates over the terminal nodes. If all feasible alternatives are acceptable, relative utilitarianism recommends the pure alternative (v_1, a) because it yields the highest sum of normalized utilities, namely, with some abuse of notation, $u_1^0(v_1, a) + u_2^0(v_1, a) = 1 + \frac{5}{6} = \frac{11}{6}$.

At node v_1, however, the set of feasible alternatives has changed: it is now the set of strategies for T_{v_1}, namely, $\Delta(\{a, b, c\})$. If all feasible alternatives at v_1 are acceptable, relative utilitarianism recommends to maximize the sum of the normalized von Neumann-Morgenstern utility functions representing the individuals' preferences *restricted to* $\Delta(\{a, b, c\})$. These utility functions are such that

$$u_1^1(x) = \begin{cases} 1 \text{ if } x = a, \\ \frac{3}{5} \text{ if } x = b, \\ 0 \text{ if } x = c, \end{cases} \quad u_2^1(x) = \begin{cases} \frac{1}{2} \text{ if } x = a, \\ 1 \text{ if } x = b, \\ 0 \text{ if } x = c. \end{cases}$$

(This follows from the definition of normalized utilities. For instance, $u_1^1(b) = \frac{3}{5}$ because $bI_1(\frac{3}{5}a + \frac{2}{5}c)$, which holds since $u_1^0(b) = \frac{3}{5}u_1^0(a) + \frac{2}{5}u_1^0(c)$.) Relative utilitarianism now prefers b to a since $u_1^1(b) + u_2^1(b) = \frac{3}{5} + 1 > u_1^1(a) + u_2^1(a) = 1 + \frac{1}{2}$. Thus it recommends at v_1 to deviate from the strategy it chose at v_0. This time inconsistency arises because relative utilitarianism uses different numerical representations of individual preferences at different decision nodes[9].

To conclude this discussion, we address the related criticism that relative egalitarianism leads to violations of the stochastic dominance principle. There exist problems where the relative leximin rule deems a lottery between two pure alternatives strictly better than a third pure alternative even though it finds each pure outcome of the lottery strictly worse than that third alternative. Consider the problem of allocating money between two equally deserving risk-neutral individuals. Suppose that the set X contains the three allocations $(1, 0), (0, 1), (0.4, 0.4)$, where the first coordinate of each vector denotes the payment to individual 1 and the second denotes the payment to individual 2. According to the relative leximin rule, the lottery yielding the allocations $(1, 0)$ and $(0, 1)$ with equal probability is

[9] *Classical* utilitarianism does not. Nor would ordinal variants of utilitarianism under which the numerical representation of preferences would be independent of the set of acceptable alternatives. Under such variants, however, the social preference over the acceptable set would depend on the profile of preferences outside that set.

strictly better than the allocation $(0.4, 0.4)$ even though both $(1, 0)$ and $(0, 1)$ are strictly worse than $(0.4, 0.4)$.

We stress that this social preference is not inconsistent in itself: the equal-chances lottery is simply regarded as a better compromise than the allocation $(0.4, 0.4)$. A difficulty seems to arise only because we think of a multi-stage scenario of the following type. A choice is offered between the equal-chances lottery and the allocation $(0.4, 0.4)$. If the lottery is chosen, and once its outcome is known, a new choice is offered between this outcome and the allocation $(0.4, 0.4)$. It seems that relative egalitarianism would reject $(0.4, 0.4)$ in the first stage only to accept it in the second. This is correct only if $(0.4, 0.4)$ is regarded as an acceptable alternative in the second stage. But $(0.4, 0.4)$ is not the continuation of the strategy originally chosen by relative egalitarianism. Therefore, as we argued earlier, it should not be acceptable in the second stage. Sticking with the outcome of the lottery is the only ethically correct alternative.

6.2. Relative egalitarianism and quadratic social welfare functions

Epstein and Segal (1992) are interested in the following problem. Given a fixed set of lotteries $\Delta(X)$, how should we aggregate a given profile of preferences $R_1, ..., R_n$ represented by the von Neumann-Morgenstern utility functions $u_1, ..., u_n$ over $\Delta(X)$ into a social ordering R_*? They propose a set of axioms guaranteeing that there exists a strictly increasing and strictly quasiconcave quadratic (Bergson-Samuelson) social welfare function W defined over $\{(u_1(x), ..., u_n(x)) \mid x \in \Delta(X)\}$ such that, for all $a, b \in \Delta(X)$, aR_*b if and only if $W((u_1(a), ..., u_n(a)) \geq W((u_1(b), ..., u_n(b))$. A quadratic social welfare function takes the form

$$W(z_1, ..., z_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} z_i z_j + \sum_{i=1}^{n} b_i z_i, \tag{6.1}$$

where a_{ij} and b_i are real numbers such that $a_{ij} = a_{ji}$ for all i, j. A key axiom, guaranteeing the strict quasiconcavity of W, is Randomization Preference: if $a, b \in \Delta(X)$, aI_*b, and aP_ib, bP_ja for some individuals i, j, then $\left(\frac{1}{2}a + \frac{1}{2}b\right) P_*b$.

Our paper shares with Epstein and Segal's work a common concern for ex ante fairness. In particular, our Preference for Lotteries axiom is essentially a restatement of Epstein and Segal's Randomization Preference.

There are two major differences between Epstein and Segal's contribution and ours. A first difference is that our main result, Theorem 2, uses a variant of Randomization Preference expressing a social preference for any type of compromise rather than just lotteries. Preference for Compromise leads to a form of welfare egalitarianism rather than simply welfare inequality aversion.

The second difference is more important. We shall argue that Epstein and Segal's result is of little practical help to aggregate ordinal preferences, and that this weakness is particularly serious in an Arrovian multi-profile framework. This, of course, is precisely the criticism that is addressed by Dhillon and Mertens (1999) to Harsanyi's (1955) theorem and motivates their definition of relative utilitarianism. Let us explain it in detail.

All axioms used by Epstein and Segal are restrictions on the preference profile $(R_1, ..., R_n)$, the social ordering R_*, and the relationship between the two; none bears on the numerical representations of the individual preferences. These axioms may therefore be reformulated in our framework as conditions on the aggregation rule $\mathbf{R}$ in a completely straightforward manner: see, for instance, the restatement of Randomization Preference in Section 5. It follows from Epstein and Segal's theorem that an aggregation rule $\mathbf{R}$ satisfies these reformulated conditions if and only if, for every problem $(X, R) \in \mathcal{P}$, there exist von Neumann-Morgenstern utility functions $u_1^{(X,R)}, ..., u_2^{(X,R)}$ representing $R_1, ..., R_n$ and a strictly increasing and strictly quasiconcave quadratic welfare function $W^{(X,R)}$ defined on $\left\{(u_1^{(X,R)}(x), ..., u_n^{(X,R)}(x)) \mid x \in \Delta(X)\right\}$ such that, for all $a, b \in \Delta(X)$, $a\mathbf{R}(X, R)b$ if and only if $W^{(X,R)}((u_1^{(X,R)}(a), ..., u_n^{(X,R)}(a)) \geq W^{(X,R)}((u_1^{(X,R)}(b), ..., u_n^{(X,R)}(b))$.

This restatement shows that two crucial questions remain unanswered: which numerical representations $u_i^{(X,R)}$ of individual preferences should we choose and which welfare function $W^{(X,R)}$ should we use? Even if the social choice problem (X, R) is fixed, as in Epstein and Segal's context, choosing the parameters a_{ij} and b_i for all $i, j \in N$ in formula (6.1) does not solve the ordinal aggregation problem as long as the numerical representations of preferences are left unspecified. Interpersonal comparisons are needed in order to eliminate the indeterminacy. Moreover, this indeterminacy becomes particularly serious in our variable-problem framework because the numerical representations of preferences and the welfare function are allowed to vary arbitrarily with the social choice problem under consideration.

Theorem 3 may be regarded as an answer to the first question above and a partial response to the second: the von Neumann-Morgenstern numerical representations should be (0,1) normalized and the social welfare function –more precisely, the social ordering over utility space that it represents– should be independent of the social choice problem. Theorem 4 is a complete answer in the sense that it pins down that social ordering.

7. Proofs

We begin by establishing three lemmas. The first lemma shows that Pareto Indifference and Independence of Inessential Expansions imply a strong form of neutrality. Let $\Pi(A)$ denote the set of permutations on A. If $(X, R) \in \mathcal{P}, \pi \in \Pi(A)$, and $a \in \Delta(X)$, then $a^\pi \in \Delta(\pi(X))$ is the lottery on $\pi(X)$ given by $a^\pi(\pi(x)) = a(x)$ for all $x \in X$ and the preference profile $R^\pi \in (\mathcal{R}_*(\pi(X)))^N$ is defined by $a^\pi R_i^\pi b^\pi \Leftrightarrow aR_ib$ for all $i \in N$ and $a, b \in \Delta(X)$.

Neutrality. For all $(X, R) \in \mathcal{P}$, $a, b \in \Delta(X)$ and $\pi \in \Pi(A)$, $a\mathbf{R}(X, R)b \Leftrightarrow a^\pi\mathbf{R}(\pi(X), R^\pi)b^\pi$.

Denoting by $\Pi(X)$ the set of permutations on $X \in \mathcal{A}$, Neutrality implies that for all $(X, R) \in \mathcal{P}$, $a, b \in \Delta(X)$ and $\pi \in \Pi(X)$, $a\mathbf{R}(X, R)b \Leftrightarrow a^\pi\mathbf{R}(X, R^\pi)b^\pi$.

Lemma 1. *If the aggregation rule* $\mathbf{R}$ *satisfies Pareto Indifference and Independence of Inessential Expansions, then* $\mathbf{R}$ *satisfies Neutrality.*

Proof. Let $\mathbf{R}$ satisfy Pareto Indifference and Independence of Inessential Expansions. Let $(X, R) \in \mathcal{P}$, $a, b \in \Delta(X)$ and $\pi \in \Pi(A)$. We prove that $a\mathbf{R}(X, R)b \Rightarrow a^\pi \mathbf{R}(\pi(X), R^\pi)b^\pi$. The converse implication follows immediately since $a = (a^\pi)^{\pi^{-1}}$, $b = (b^\pi)^{\pi^{-1}}$, $X = \pi^{-1}(\pi(X))$, and $R = (R^\pi)^{\pi^{-1}}$. Let us thus assume that

$$a\mathbf{R}(X, R)b. \tag{7.1}$$

Step 1. *We prove that* $a^\pi \mathbf{R}(\pi(X), R^\pi)b^\pi$ *if* $\pi(X) \cap X = \emptyset$.

Let $\overline{X} = X \cup \pi(X)$. For each $i \in N$, let $\overline{R}_i$ be the von Neumann-Morgenstern preference over $\Delta(\overline{X})$ which coincides with R_i on $\Delta(X)$ and is such that $x\overline{I}_i\pi(x)$ for all $x \in X$. This is well defined because $\pi(X) \cap X = \emptyset$. Observe that $\overline{R}_i$ coincides with R_i^π on $\Delta(\pi(X))$. Moreover, $\overline{a}\overline{R}_i x\overline{R}_i\underline{a}$ for all $\overline{a} \in \overline{A}(X, R_i) \cup \overline{A}(\pi(X), R_i^\pi)$, $x \in \overline{X}$, and $\underline{a} \in \underline{A}(X, R_i) \cup \underline{A}(\pi(X), R_i^\pi)$. Let $\overline{R} = (\overline{R}_1, ..., \overline{R}_n)$. Applying Independence of Inessential Expansions to (7.1),

$$a\mathbf{R}(\overline{X}, \overline{R})b. \tag{7.2}$$

Since $a^\pi\overline{I}_i a$ and $b^\pi\overline{I}_i b$ for all $i \in N$, Pareto Indifference implies $a^\pi\mathbf{I}(\overline{X}, \overline{R})a$ and $b^\pi\mathbf{I}(\overline{X}, \overline{R})b$. Hence from (7.2),

$$a^\pi\mathbf{R}(\overline{X}, \overline{R})b^\pi. \tag{7.3}$$

Applying Independence of Inessential Expansions to (7.3) and recalling that $\overline{R}$ coincides with R^π on $\Delta(\pi(X))$, we obtain $a^\pi\mathbf{R}(\pi(X), R^\pi)b^\pi$.

Step 2. *We prove that* $a^\pi\mathbf{R}(\pi(X), R^\pi)b^\pi$.

Choose $\rho \in \Pi(A)$ such that $\rho(X) \cap X = \rho(X) \cap \pi(X) = \emptyset$. By Step 1, (7.1) implies

$$a^\rho\mathbf{R}(\rho(X), R^\rho)b^\rho. \tag{7.4}$$

Next consider the permutation $\pi \circ \rho^{-1} \in \Pi(A)$. Since $(\pi \circ \rho^{-1})(\rho(X)) \cap \rho(X) = \emptyset$, Step 1 and (7.4) imply

$$(a^\rho)^{\pi\circ\rho^{-1}}\mathbf{R}((\pi \circ \rho^{-1})(\rho(X)), (R^\rho)^{\pi\circ\rho^{-1}})(b^\rho)^{\pi\circ\rho^{-1}}. \tag{7.5}$$

By definition, $(\pi \circ \rho^{-1})(\rho(X)) = \pi(X)$. Moreover, $(a^\rho)^{\pi\circ\rho^{-1}} = a^\pi$ since $(a^\rho)^{\pi\circ\rho^{-1}}(\pi(x)) = (a^\rho)^{\pi\circ\rho^{-1}}((\pi \circ \rho^{-1})(\rho(x))) = a^\rho(\rho(x)) = a(x)$ for all $x \in X$. Likewise, $(b^\rho)^{\pi\circ\rho^{-1}} = b^\pi$ and $(R^\rho)^{\pi\circ\rho^{-1}} = R^\pi$. Hence (7.5) reduces to $a^\pi\mathbf{R}(\pi(X), R^\pi)b^\pi$.■

The reader may have noticed that the above proof does not use the full force of Independence of Inessential Expansions. In fact, the axiom can be replaced with Independence of Redundant Alternatives in the statement of Lemma 1.

The second lemma proves a simple but useful property of the normalized von Neumann-Morgenstern representation of preferences.

Lemma 2. *For all* $(X, R) \in \mathcal{P}, i \in N, \pi \in \Pi(X)$ *and* $a \in \Delta(X)$, $u(a, X, R_i) = u(a^\pi, X, R_i^\pi)$.

Proof. Fix $(X, R) \in \mathcal{P}, i \in N, \pi \in \Pi(X)$ and $a \in \Delta(X)$. If $R_i = R_0$, then $R_i^\pi = R_0$ and therefore $u(a, X, R_i) = u(a^\pi, X, R_i^\pi) = 1$. If $R_i \neq R_0$, let $u(a, X, R_i) = \alpha$. By definition of

$u(., X, R_i)$, this means that $aI_i\,[\alpha\overline{a} + (1-\alpha)\underline{a})]$ for all $\overline{a} \in \overline{A}(X, R_i)$ and $\underline{a} \in \underline{A}(X, R_i)$. By definition of R_i^π, it follows that $a^\pi I_i^\pi[\alpha\overline{a}+ (1-\alpha)\underline{a}]^\pi = [\alpha\overline{a}^\pi + (1-\alpha)\underline{a}^\pi]$ for all $\overline{a} \in \overline{A}(X, R_i)$ and $\underline{a} \in \underline{A}(X, R_i)$. But $\overline{a} \in \overline{A}(X, R_i) \Leftrightarrow \overline{a}^\pi \in \overline{A}(X, R_i^\pi)$ and $\underline{a} \in \underline{A}(X, R_i) \Leftrightarrow \underline{a}^\pi \in \underline{A}(X, R_i^\pi)$. Therefore $a^\pi I_i^\pi \left[\alpha\overline{b} + (1-\alpha)\underline{b}\right]$ for all $\overline{b} \in \overline{A}(X, R_i^\pi)$ and $\underline{b} \in \underline{A}(X, R_i^\pi)$, that is, $u(a^\pi, X, R_i^\pi) = \alpha$.■

The third lemma establishes a normalized version of the so-called Suppes-Sen principle. Given $(X, R) \in \mathcal{P}$, we recall that $N_0(X, R) = \{i \in N \mid R_i = R_0\}$ and we let $\Pi_0(N, X, R) = \{\sigma \in \Pi(N) \mid \sigma(N_0(X, R)) = N_0(X, R)\}$.

Lemma 3. *Let the aggregation rule* $\mathbf{R}$ *satisfy the Pareto Principle, Anonymity, Independence of Inessential Expansions, and let* $(X, R) \in \mathcal{P}$, $a, b \in \Delta(X)$, *and* $\sigma \in \Pi_0(N, X, R)$. *If* $u(a, X, R_{\sigma(i)}) \geq u(b, X, R_i)$ *for all* $i \in N$, *then* $a\mathbf{R}(X, R)b$. *If in addition* $u(a, X, R_{\sigma(j)}) > u(b, X, R_j)$ *for some* $j \in N$, *then* $a\mathbf{P}(X, R)b$.

Proof. Let $\mathbf{R}$ satisfy the Pareto Principle, Anonymity, Independence of Inessential Expansions, and let $(X, R) \in \mathcal{P}$. By Lemma 1, $\mathbf{R}$ satisfies Neutrality.

Step 1. *We enlarge* (X, R) *to a symmetric problem* $(\overline{X}, \overline{R})$.

For each $x \in X$ and each $\theta \in \Pi_0(N, X, R)$, choose some alternative $x_\theta \in A$ such that $x_{id} = x$ and $x_\theta \neq x_{\theta'}$ if $\theta \neq \theta'$ (where id is the identity mapping). Let $\overline{X} = \{x_\theta \mid \theta \in \Pi_0(N, X, R)\}$.

For each $i \in N$, let $\overline{u}_i : \Delta(\overline{X}) \to [0, 1]$ be the (unique) von Neumann-Morgenstern utility function such that, for each $x_\theta \in \overline{X}$,

$$\overline{u}_i(x_\theta) = u(x, X, R_{\theta^{-1}(i)}). \tag{7.6}$$

Let $\overline{R}_i \in \mathcal{R}_*(\overline{X})$ be the preference represented by $\overline{u}_i$. Observe that $\overline{R}_i$ is the complete indifference relation on $\Delta(\overline{X})$ if R_i is the complete indifference relation on $\Delta(X)$. More generally, $\overline{R}_i$ coincides with R_i on $\Delta(X)$ and $\overline{a}\overline{R}_i x_\theta \overline{R}_i \underline{a}$ for all $\overline{a} \in \overline{A}(X, R_i), x_\theta \in \overline{X}$, $\underline{a} \in \underline{A}(X, R_i)$. Note that $\overline{u}_i$ is the normalized von Neumann-Morgenstern representation of $\overline{R}_i$, that is,

$$\overline{u}_i(.) = u(., \overline{X}, \overline{R}_i). \tag{7.7}$$

Let $\overline{R} = (\overline{R}_1, ..., \overline{R}_n)$. The preference profile $\overline{R}$ is highly symmetric. In particular, for all $i \in N, x_\theta \in \overline{X}$ and $\tau \in \Pi_0(N, X, R)$,

$$\overline{u}_{\tau(i)}(x_{\tau\circ\theta}) = \overline{u}_i(x_\theta). \tag{7.8}$$

Indeed, $\overline{u}_{\tau(i)}(x_{\tau\circ\theta}) = u(x, X, R_{(\tau\circ\theta)^{-1}(\tau(i))}) = u(x, X, R_{(\theta^{-1}\circ\tau^{-1})(\tau(i))}) = u(x, X, R_{\theta^{-1}(i)}) = \overline{u}_i(x_\theta)$.

Step 2. *We show how* $\overline{R}$ *can be mapped onto itself by combining a permutation on* N *with one on* $\overline{X}$.

With each $\sigma \in \Pi_0(N, X, R)$ we associate a permutation $\pi_\sigma \in \Pi(\overline{X})$ defined as follows: for all $x_\theta \in \overline{X}$,

$$\pi_\sigma(x_\theta) = x_{\sigma\circ\theta}. \tag{7.9}$$

This is a permutation because $x_{\sigma\circ\theta} \neq x_{\sigma\circ\theta'}$ whenever $\theta \neq \theta'$. Notice also that

$$\pi_\sigma^{-1}(x_\theta) = x_{\sigma^{-1}\circ\theta} \tag{7.10}$$

since $\pi_\sigma(x_{\sigma^{-1}\circ\theta}) = x_{\sigma\circ\sigma^{-1}\circ\theta} = x_\theta$.

We claim that

$$\left(\sigma\overline{R}\right)^{\pi_\sigma} = \overline{R} \text{ for all } \sigma \in \Pi_0(N, X, R). \tag{7.11}$$

To see why, fix $\sigma \in \Pi_0(N, X, R)$, $x_\theta \in \overline{X}$ and $i \in N$. We have

$$\begin{aligned}
u(x_\theta, \overline{X}, (\sigma\overline{R})_i^{\pi_\sigma}) &= u(\pi_\sigma^{-1}(x_\theta), \overline{X}, (\sigma\overline{R})_i) \\
&= u(x_{\sigma^{-1}\circ\theta}, \overline{X}, (\sigma\overline{R})_i) \\
&= u(x_{\sigma^{-1}\circ\theta}, \overline{X}, \overline{R}_{\sigma^{-1}(i)}) \\
&= \overline{u}_{\sigma^{-1}(i)}(x_{\sigma^{-1}\circ\theta}) \\
&= \overline{u}_i(x_\theta) \\
&= u(x_\theta, \overline{X}, \overline{R}_i).
\end{aligned}$$

These equalities hold, respectively, because of Lemma 2, (7.10), the definition of $(\sigma\overline{R})_i$, (7.7), (7.8), and (7.7) again. This proves that $\left(\sigma\overline{R}\right)_i^{\pi_\sigma} = \overline{R}_i$ for all $i \in N$, hence (7.11).

Step 3. *We show that $x\mathbf{I}(\overline{X}, \overline{R})x_\sigma$ for all $x \in X$ and $\sigma \in \Pi_0(N, X, R)$.*

Fix $\sigma \in \Pi_0(N, X, R)$. Let k be a a positive integer such that

$$\sigma^k = id, \tag{7.12}$$

where σ^k denotes the k-repeated composition of σ with itself. (Such an integer exists: for instance, we may take k equal to the product of the lengths of all the cycles of σ). Fix $x \in X$. Then

$$x\mathbf{R}(\overline{X}, \overline{R})x_\sigma \tag{7.13}$$

or

$$x_\sigma\mathbf{R}(\overline{X}, \overline{R})x. \tag{7.14}$$

Suppose (7.13) holds. By Anonymity,

$$x\mathbf{R}(\overline{X}, \sigma\overline{R})x_\sigma. \tag{7.15}$$

Letting $\pi_\sigma \in \Pi(\overline{X})$ be the permutation defined in (7.9), we get

$$\begin{aligned}
& \pi_\sigma(x)\mathbf{R}(\overline{X}, (\sigma\overline{R})^{\pi_\sigma})\pi_\sigma(x_\sigma) \\
\Rightarrow\ & x_\sigma\mathbf{R}(\overline{X}, (\sigma\overline{R})^{\pi_\sigma})x_{\sigma\circ\sigma} \\
\Rightarrow\ & x_\sigma\mathbf{R}(\overline{X}, \overline{R})x_{\sigma\circ\sigma}
\end{aligned}$$

by Neutrality, (7.9) and (7.11). Repeating this argument k times, we obtain

$$x_\sigma\mathbf{R}(\overline{X}, \overline{R})x_{\sigma\circ\sigma}\mathbf{R}(\overline{X}, \overline{R})...\mathbf{R}(\overline{X}, \overline{R})x_{\sigma^k}.$$

By (7.12), $x_\sigma \mathbf{R}(\overline{X}, \overline{R})x$ which together with (7.13) yields $x\mathbf{I}(\overline{X}, \overline{R})x_\sigma$.

The same argument holds, *mutatis mutandis*, if (7.14) holds instead of (7.13).

Step 4. *We establish the lemma for the case of pure alternatives.*

Let $x, y \in X, \sigma \in \Pi_0(N, X, R)$ and assume $u(x, X, R_{\sigma(i)}) \geq u(y, X, R_i)$ for all $i \in N$, which we rewrite

$$u(x, X, R_i) \geq u(y, X, R_{\sigma^{-1}(i)}) \text{ for all } i \in N. \tag{7.16}$$

Let $(\overline{X}, \overline{R})$ be the enlarged problem defined in Step 1. From (7.16) and (7.6),

$$\overline{u}_i(x) \geq \overline{u}_i(y_\sigma) \text{ for all } i \in N,$$

that is, $x\overline{R}_i y_\sigma$ for all $i \in N$. By the Pareto Principle, $x\mathbf{R}(\overline{X}, \overline{R})y_\sigma$. From Step 3, however, $y_\sigma \mathbf{I}(\overline{X}, \overline{R})y$. Hence $x\mathbf{R}(\overline{X}, \overline{R})y$. By Independence of Inessential Expansions, $x\mathbf{R}(X, R)y$.

If in addition $u(x, X, R_{\sigma(j)}) > u(y, X, R_j)$ for some $j \in N$, the argument is easily adapted to show that $x\mathbf{P}(X, R)y$.

Step 5. *We conclude the proof.*

Let $a, b \in \Delta(X), \sigma \in \Pi_0(N, X, R)$ and assume $u(a, X, R_{\sigma(i)}) \geq u(b, X, R_i)$ for all $i \in N$. Choose distinct pure alternatives $a', b' \in A \setminus X$ and let $X' = X \cup \{a', b'\}$. For each $i \in N$, let R'_i be the von Neumann-Morgenstern preference on $\Delta(X')$ which coincides with R_i on $\Delta(X')$ and is such that $a' I'_i a$ and $b' I'_i b$. Let $R' = (R'_1, ..., R'_n)$.

By construction,

$$u(a', X', R'_i) = u(a, X, R_i) \geq u(b, X, R_{\sigma^{-1}(i)}) = u(b', X', R'_{\sigma^{-1}(i)}) \text{ for all } i \in N.$$

Using this inequality instead of (7.16), we may repeat the argument in Step 4 with a', b', X', R' replacing x, y, X, R and obtain $a'\mathbf{R}(X', R')b'$. By the Pareto Principle, $a\mathbf{I}(X', R')a'$ and $b\mathbf{I}(X', R')b'$. Hence $a\mathbf{R}(X', R')b$. By Independence of Inessential Expansions, $a\mathbf{R}(X, R)b$.

If in addition $u(a, X, R_{\sigma(j)}) > u(b, X, R_j)$ for some $j \in N$, the argument is easily adapted to show that $a\mathbf{P}(X, R)b$.■

We are now equipped to prove Theorem 1.

Proof of Theorem 1. Let the aggregation rule $\mathbf{R}$ satisfy the Pareto Principle, Anonymity, Independence of Inessential Expansions, and Preference for Compromise. By Lemma 1, $\mathbf{R}$ also satisfies Neutrality. Let $(X, R) \in \mathcal{P}$, $a, b \in \Delta(X)$, and assume $\min_{i \in N} u(a, X, R_i) = \alpha > \beta = \min_{i \in N} u(b, X, R_i)$. For all $c \in \Delta(X)$ and $i \in N$, write $u_i(c) = u(c, X, R_i)$. Without loss of generality, assume $u_n(b) = \beta$. Suppose, contrary to the claim, that

$$b\mathbf{R}(X, R)a. \tag{7.17}$$

Step 1. *We derive a contradiction under the assumptions that*

$$u_1(b) = ... = u_{n-1}(b) = 1, \tag{7.18}$$

and

$$u_1(a) < ... < u_n(a). \tag{7.19}$$

Choose $a' \in A \setminus X$ and let $X' = X \cup \{a'\}$. For $i = 1, ..., n-1$, let $u'_i : \Delta(X') \to [0,1]$ be the von Neumann-Morgenstern utility function such that $u'_i(x) = u_i(x)$ for all $x \in X$ and $u'_i(a') = u_{i+1}(a)$. Let $u'_n : \Delta(X') \to [0,1]$ be the von Neumann-Morgenstern utility function such that $u'_n(x) = u_n(x)$ for all $x \in X$ and $u'_n(a') = \frac{\alpha+\beta}{2}$. For each $i \in N$, let R'_i be the preference on $\Delta(X')$ represented by u'_i. By Independence of Inessential Expansions, (7.17) implies $b\mathbf{R}(X', R')a$. Since $bR'_i a' P'_i a$ for $i = 1, ..., n-1$ and $aR'_n a' P'_n b$, Preference for Compromise implies $a'\mathbf{R}(X', R')a$. But Lemma 3 implies $a\mathbf{P}(X', R')a'$. (Indeed, (7.18) and (7.19) imply $R_i \neq R_0$ for all $i \in N$. Therefore $\Pi_0(N, X, R) = \Pi(N)$ and the permutation $\sigma(i) = i + 1 (\mathrm{mod}\, n)$ belongs to $\Pi_0(N, X, R)$. Then $u(a, X', R'_{\sigma(i)}) = u(a', X', R'_i)$ for $i = 1, ..., n-1$ while $u(a, X', R'_{\sigma(n)}) = \alpha > \frac{\alpha+\beta}{2} = u(a', X', R'_n)$, hence $a\mathbf{P}(X', R')a'$ by Lemma 3.)

Step 2. *We derive a contradiction under assumption* (7.18) *only.*

Choose $a'' \in A \setminus X$ and let $X'' = X \cup \{a''\}$. Let $u''_i : \Delta(X'') \to [0,1]$, $i = 1, ..., n$, be von Neumann-Morgenstern utility functions such that $u''_i(x) = u_i(x)$ for all $x \in X$ and

$$\beta \leq u''_1(a'') < ... < u''_n(a'') \leq \alpha. \tag{7.20}$$

Such functions exist because $\beta < \alpha$. For each $i \in N$, let R''_i be the preference on $\Delta(X'')$ represented by u''_i. By Independence of Inessential Expansions, (7.17) implies $b\mathbf{R}(X'', R'')a$. By the Pareto Principle, $a\mathbf{R}(X'', R'')a''$. Hence $b\mathbf{R}(X'', R'')a''$.

We may now repeat the argument in Step 1 (with X'', R'', a'' replacing X, R, a and (7.20) replacing (7.19)) to obtain a contradiction.

Step 3. *We drop both assumptions* (7.18) *and* (7.19) *and derive a contradiction.*

Choose $\overline{b} \in A \setminus X$ and let $\overline{X} = X \cup \{\overline{b}\}$. Let $\overline{u}_i : \Delta(\overline{X}) \to [0,1]$, $i = 1, ..., n$, be the von Neumann-Morgenstern utility functions such that $\overline{u}_i(x) = u_i(x)$ for all $x \in X$ and

$$\overline{u}_n(\overline{b}) = \beta < \overline{u}_1(\overline{b}) = ... = \overline{u}_{n-1}(\overline{b}) = 1. \tag{7.21}$$

For each $i \in N$, let $\overline{R}_i$ be the preference on $\Delta(\overline{X})$ represented by $\overline{u}_i$. By Independence of Inessential Expansions, (7.17) implies $b\mathbf{R}(\overline{X}, \overline{R})a$. By the Pareto Principle, $\overline{b}\mathbf{R}(\overline{X}, \overline{R})b$. Hence $\overline{b}\mathbf{R}(\overline{X}, \overline{R})a$.

We may now repeat the argument in Step 2 (with $\overline{X}, \overline{R}, \overline{b}$ replacing X, R, b) to obtain a contradiction.■

Proof of Corollary to Theorem 1. For a proof of the "if" statement, see step 1 of the proof of Theorem 2. To prove the "only if" statement, combine Theorem 1 and Lemma 3.■

The proof of Theorem 2 relies on one more lemma. This lemma is restricted to problems where preferences have multiple best and worst elements. Formally, let $\mathcal{P}^* = \{(X, R) \in$

$\mathcal{P} \mid \left|\overline{A}(X,R_i)\right| \geq 2$ and $|\underline{A}(X,R_i)| \geq 2$ for all $i \in N\}$. If $(X,R) \in \mathcal{P}^*$, $\sigma \in \Pi_0(N,X,R)$, $i \in N$, and $a \in X$, let $u_i^{a,\sigma} : \Delta(X) \to [0,1]$ be the von Neumann-Morgenstern utility function such that

$$u_i^{a,\sigma}(x) = \begin{cases} u(x,X,R_i) \text{ if } x \in X \setminus \{a\}, \\ u(a,X,R_{\sigma^{-1}(i)}) \text{ if } x = a. \end{cases}$$

Note that $u_i^{a,\sigma}$ is a normalized von Neumann-Morgenstern utility function because $\left|\overline{A}(X,R_i)\right|$, $|\underline{A}(X,R_i)| \geq 2$ and because $\sigma \in \Pi_0(N,X,R)$. The (a,σ)-*transform of* $R_i \in \mathcal{R}_*(X)$ is the preference $R_i^{a,\sigma} \in \mathcal{R}_*(X)$ whose normalized von Neumann-Morgenstern representation is $u_i^{a,\sigma}$. Observe that $R_i^{a,\sigma} = R_0$ if and only if $R_i = R_0$.

Lemma 4. *Let the aggregation rule* $\mathbf{R}$ *satisfy the Pareto Principle, Anonymity, and Independence of Inessential Expansions. If* $(X,R) \in \mathcal{P}^*$, $a \in X$ *and* $\sigma \in \Pi_0(N,X,R)$, *then for all* $x,y \in X$, $x\mathbf{R}(X,R)y \Leftrightarrow x\mathbf{R}(X,R^{a,\sigma})y$.

Proof. Let $\mathbf{R}$ satisfy the Pareto Principle, Anonymity, and Independence of Inessential Expansions. By Lemma 1, $\mathbf{R}$ also satisfies Neutrality. Let $(X,R) \in \mathcal{P}^*$, $a,x,y \in X$ and $\sigma \in \Pi_0(N,X,R)$. We prove that $x\mathbf{R}(X,R)y \Rightarrow x\mathbf{R}(X,R^{a,\sigma})y$. The converse implication follows immediately since $R = (R^{a,\sigma})^{a,\sigma^{-1}}$. Let us thus assume

$$x\mathbf{R}(X,R)y. \tag{7.22}$$

If $x = y$, it is trivial that $x\mathbf{R}(X,R^{a,\sigma})y$. From now on, suppose $x \neq y$.

Choose $\overline{a} \in A \setminus X$ and define $\overline{X} = X \cup \{\overline{a}\}$. For each $i \in N$, let $\overline{u}_i : \Delta(\overline{X}) \to [0,1]$ be the von Neumann-Morgenstern utility function such that

$$\overline{u}_i(x) = \begin{cases} u(x,X,R_i) \text{ if } x \in X, \\ u(a,X,R_{\sigma^{-1}(i)}) \text{ if } x = \overline{a}, \end{cases}$$

and let $\overline{R}_i$ be the preference on $\Delta(\overline{X})$ represented by $\overline{u}_i$. Thus, $u(.,\overline{X},\overline{R}_i) = \overline{u}_i$ for each $i \in N$ and the profile $\overline{R} = (\overline{R}_1,...,\overline{R}_n)$ coincides with R on $\Delta(X)$. Applying Independence of Inessential Expansions to (7.22), we have

$$x\mathbf{R}(\overline{X},\overline{R})y.$$

Define $\pi \in \Pi(\overline{X})$ by $\pi(a) = \overline{a}, \pi(\overline{a}) = a$ and $\pi(b) = b$ for all $b \in \overline{X} \setminus \{a,\overline{a}\}$ and consider the profile $\overline{R}^\pi$ on $\Delta(\overline{X})$. Observe that for all $i \in N$,

$$u(x,\overline{X},\overline{R}_i^\pi) = u(\pi^{-1}(x),\overline{X},\overline{R}_i) = \begin{cases} u(x,X,R_i) \text{ if } x \in \overline{X} \setminus \{a\}, \\ u(a,X,R_{\sigma^{-1}(i)}) \text{ if } x = a, \end{cases}$$

that is, $\overline{R}_i^\pi$ coincides with $R_i^{a,\sigma}$ on $\Delta(X)$. Now, distinguish two cases.

Case 1. $x \neq a$ and $y \neq a$. We get

$$\begin{aligned} x\mathbf{R}(\overline{X},\overline{R})y &\Rightarrow \pi(x)\mathbf{R}(\overline{X},\overline{R}^\pi)\pi(y) \\ &\Rightarrow x\mathbf{R}(\overline{X},\overline{R}^\pi)y \\ &\Rightarrow x\mathbf{R}(X,R^{a,\sigma})y, \end{aligned}$$

successively by Neutrality, the definition of π, and Independence of Inessential Expansions and the fact that $\overline{R}^{\pi}$ coincides with $R^{a,\sigma}$ on $\Delta(X)$.

Case 2. $x = a$ or $y = a$. We assume $x = a \neq y$; the case $x \neq a = y$ is treated similarly. By Lemma 3, $a\mathbf{I}(\overline{X}, \overline{R})\overline{a}$. Hence,

$$\begin{aligned} a\mathbf{R}(\overline{X}, \overline{R})y &\Rightarrow \overline{a}\mathbf{R}(\overline{X}, \overline{R})y \\ &\Rightarrow \pi(\overline{a})\mathbf{R}(\overline{X}, \overline{R}^{\pi})\pi(y) \\ &\Rightarrow a\mathbf{R}(\overline{X}, \overline{R}^{\pi})y \\ &\Rightarrow a\mathbf{R}(X, R^{a,\sigma})y, \end{aligned}$$

completing the proof.■

With Lemma 4 in hand we may now prove Theorem 2.

Proof of Theorem 2.
Step 1. *We check that* $\mathbf{R}^L$ *satisfies the axioms in Theorem 2.*

The Pareto Principle and Anonymity are obviously met. Independence of Inessential Expansions holds because $u(x, X, R_i) = u(x, X', R'_i)$ for all $x \in X$ and $i \in N$ whenever $(X, R), (X', R')$ satisfy the premises of the axiom.

To check that $\mathbf{R}^L$ satisfies Preference for Compromise, let $(X, R) \in \mathcal{P}$, $a, b, c \in \Delta(X)$, $\varnothing \subsetneqq S \subsetneqq N$ be such that $a\mathbf{R}^L(X, R)b$, aR_icP_ib for all $i \in S$ and bR_jcP_ja for all $j \in N \setminus S$. We claim that

$$\min_{i \in N} u(c, X, R_i) > \min_{i \in N} u(b, X, R_i), \tag{7.23}$$

which in turn implies $c\mathbf{P}^L(X, R)b$ (that is, $\mathbf{R}^L$ satisfies the strict version of Preference for Compromise mentioned in the last paragraph of Section 4). Suppose (7.23) does not hold. Let $j \in N$ be such that $u(c, X, R_j) = \min_{i \in N} u(c, X, R_i) \leq \min_{i \in N} u(b, X, R_i)$. Since $a\mathbf{R}^L(X, R)b$, we get $u(c, X, R_j) \leq \min_{i \in N} u(b, X, R_i) \leq \min_{i \in N} u(a, X, R_i)$. In particular, $u(c, X, R_j) \leq u(b, X, R_j)$ and $u(c, X, R_j) \leq u(a, X, R_j)$, that is, bR_jc and aR_jc. Hence $j \notin S$ and $j \notin N \setminus S$, a contradiction.

We omit the standard argument showing that $\mathbf{R}^L$ satisfies Separability.

Step 2. *We show that any aggregation rule satisfying the axioms coincides with* $\mathbf{R}^L$.

Let $\mathbf{R}$ satisfy the Pareto Principle, Anonymity, Independence of Inessential Expansions, Preference for Compromise, and Separability. By Lemma 1, $\mathbf{R}$ also satisfies Neutrality.

Step 2.1. *We prove that for all* $(X, R) \in \mathcal{P}$, $\mathbf{R}(X, R)$ *coincides with* $\mathbf{R}^L(X, R)$ *when comparing any two pure alternatives.*

Fix $(X, R) \in \mathcal{P}$ and $a, b \in X$. Suppose first that $a\mathbf{I}^L(X, R)b$. Then $(u(a, X, R_1), ..., u(a, X, R_n)) \sim^L (u(b, X, R_1), ..., u(b, X, R_n))$, hence there exists $\sigma \in \Pi_0(N, X, R)$ such that $u(a, X, R_{\sigma(i)}) = u(b, X, R_i)$ for all $i \in N$. Lemma 3 implies $a\mathbf{I}(X, R)b$.

Suppose now that $a\mathbf{P}^L(X,R)b$. This means that there exist $k \in N$ and $\theta, \tau \in \Pi(N)$ such that

$$\begin{aligned}
u(a,X,R_{\theta(1)}) &\leq \ldots \leq u(a,X,R_{\theta(n)}),\\
u(b,X,R_{\tau(1)}) &\leq \ldots \leq u(b,X,R_{\tau(n)}),\\
u(a,X,R_{\theta(i)}) &= u(b,X,R_{\tau(i)}) \text{ for all } i \in N \text{ such that } i < k,\\
u(a,X,R_{\theta(k)}) &> u(b,X,R_{\tau(k)}).
\end{aligned}$$

Let us assume that $\theta(i) = \tau(i)$ for all $i \in N_0(X,R)$. This is without loss of generality because $u(a,X,R_i) = u(b,X,R_i) = 1$ for $i \in N_0(X,R)$. If $k = 1$, Theorem 1 implies $a\mathbf{P}(X,R)b$. From now on, assume $k > 1$. Suppose, by way of contradiction, that

$$b\mathbf{R}(X,R)a. \tag{7.24}$$

Choose distinct $a_0, a_1 \in A \setminus X$ and let $\overline{X} = X \cup \{a_0, a_1\}$. For each $i \in N$, let $\overline{R}_i$ be the von Neumann-Morgenstern preference on $\Delta(\overline{X})$ which coincides with R_i on $\Delta(X)$ and is such that $a_0 \overline{I}_i \underline{a}$ for $\underline{a} \in \underline{A}(X,R_i)$ and $a_1 \overline{I}_i \overline{a}$ for $\overline{a} \in \overline{A}(X,R_i)$. We have

$$\begin{aligned}
u(a,\overline{X},\overline{R}_{\theta(1)}) &\leq \ldots \leq u(a,\overline{X},\overline{R}_{\theta(n)}),\\
u(b,\overline{X},\overline{R}_{\tau(1)}) &\leq \ldots \leq u(b,\overline{X},\overline{R}_{\tau(n)}),\\
u(a,\overline{X},\overline{R}_{\theta(i)}) &= u(b,\overline{X},\overline{R}_{\tau(i)}) \text{ for all } i \in N \text{ such that } i < k,\\
u(a,\overline{X},\overline{R}_{\theta(k)}) &> u(b,\overline{X},\overline{R}_{\tau(k)}).
\end{aligned}$$

From (7.24) and Independence of Inessential Expansions,

$$b\mathbf{R}(\overline{X},\overline{R})a. \tag{7.25}$$

Notice that $(\overline{X},\overline{R}) \in \mathcal{P}^*$. Define $\sigma \in \Pi(N)$ by $\sigma(i) = \tau(\theta^{-1}(i))$ for all $i \in N$. Because $\theta(i) = \tau(i)$ whenever $R_i = R_0$, $\sigma \in \Pi_0(N,X,R)$. For each $i \in N$, let $\overline{R}_i^{a,\sigma}$ be the (a,σ)-transform of $\overline{R}_i$ as defined just before Lemma 4. By definition, $u(a,\overline{X},\overline{R}_{\tau(i)}^{a,\sigma}) = u(a,\overline{X},\overline{R}_{\sigma^{-1}(\tau(i))}) = u(a,\overline{X},\overline{R}_{\theta(\tau^{-1}(\tau(i)))}) = u(a,\overline{X},\overline{R}_{\theta(i)})$ while $u(b,\overline{X},\overline{R}_{\tau(i)}^{a,\sigma}) = u(b,\overline{X},\overline{R}_{\tau(i)})$. Therefore

$$\begin{aligned}
u(a,\overline{X},\overline{R}_{\tau(1)}^{a,\sigma}) &\leq \ldots \leq u(a,\overline{X},\overline{R}_{\tau(n)}^{a,\sigma}),\\
u(b,\overline{X},\overline{R}_{\tau(1)}^{a,\sigma}) &\leq \ldots \leq u(b,\overline{X},\overline{R}_{\tau(n)}^{a,\sigma}),\\
u(a,\overline{X},\overline{R}_{\tau(i)}^{a,\sigma}) &= u(b,\overline{X},\overline{R}_{\tau(i)}^{a,\sigma}) \text{ for all } i \in N \text{ such that } i < k,\\
u(a,\overline{X},\overline{R}_{\tau(k)}^{a,\sigma}) &> u(b,\overline{X},\overline{R}_{\tau(k)}^{a,\sigma}).
\end{aligned}$$

By Lemma 4, (7.25) implies

$$b\mathbf{R}(\overline{X},\overline{R}^{a,\sigma})a. \tag{7.26}$$

Define the preference profile $\overline{R}'$ on $\Delta(X')$ by $\overline{R}'_{\tau(i)} = R_0$ for all $i \in N$ such that $i < k$ and $\overline{R}'_{\tau(i)} = \overline{R}_{\tau(i)}^{a,\sigma}$ for all $i \in N$ such that $i \geq k$. By Separability (or repeated application of Weak Separability), (7.26) implies

$$b\mathbf{R}(\overline{X},\overline{R}')a.$$

But $\min_{i\in N} u(a,\overline{X},\overline{R}'_i) = u(a,\overline{X},\overline{R}'_{\tau(k)}) = u(a,\overline{X},\overline{R}^{a,\sigma}_{\tau(k)}) > u(b,\overline{X},\overline{R}^{a,\sigma}_{\tau(k)}) = u(b,\overline{X},\ \overline{R}'_{\tau(k)})$ $= \min_{i\in N} u(b,\overline{X},\ \overline{R}'_i)$, contradicting Theorem 1.

Step 2.2. *We prove that for all $(X,R)\in\mathcal{P}$, $\mathbf{R}(X,R)$ coincides with $\mathbf{R}^L(X,R)$ when comparing any two alternatives.*

Fix $(X,R)\in\mathcal{P}$ and $a,b\in\Delta(X)$. Suppose $a\mathbf{I}^L(X,R)b$. Choose distinct pure alternatives $a',b'\in A\setminus X$ and let $X'=X\cup\{a',b'\}$. For each $i\in N$, let R'_i be the von Neumann-Morgenstern preference on $\Delta(X')$ which coincides with R_i on $\Delta(X)$ and is such that $a'I'_ia$ and $b'I'_ib$. Then $a'\mathbf{I}^L(X',R')b'$. By Step 2.1, $a'\mathbf{I}(X',R')b'$. By the Pareto Principle, $a\mathbf{I}(X',R')b$. By Independence of Inessential Expansions, $a\mathbf{I}(X,R)b$. The argument carries over to show that $a\mathbf{P}^L(X,R)b$ implies $a\mathbf{P}(X,R)b$.■

As explained in Section 5, we prove Theorem 3 as a corollary to Theorem 4.

Proof of Theorem 4.
Step 1. *We prove the "if" statement.*

Fix a collection of orderings $\succsim_S$ on $[0,1]^N$ and suppose that for all $S\subseteq N$, all $(X,R)\in\mathcal{P}_S$ and all $a,b\in X$, $a\mathbf{R}(X,R)b \Leftrightarrow (u(a,X,R_1),...,\ u(a,X,R_n))\succsim_S (u(b,X,R_1)\ ,...,\ u(b,X,R_n))$. It is obvious that $\mathbf{R}$ satisfies Pareto Indifference. To check Independence of Inessential Expansions, fix $(X,R),(X',R')$ satisfying the premises of the axiom. Then for each $i\in N$ and $c\in\Delta(X)$, $u(c,X,R_i)=u(c,X',R'_i)$. Moreover, since R'_i is the complete indifference relation on $\Delta(X')$ if and only if R_i is the complete indifference relation on $\Delta(X)$, we have $N_0(X,R)=N_0(X',R')$. Therefore, for all $a,b\in\Delta(X)$,

$$\begin{aligned} a\mathbf{R}(X,R)b &\Leftrightarrow (u(a,X,R_1),...,u(a,X,R_n))\succsim_{N_0(X,R)}(u(b,X,R_1),...,u(b,X,R_n)) \\ &\Leftrightarrow (u(a,X',R'_1),...,u(a,X',R'_n))\succsim_{N_0(X',R')}(u(b,X',R'_1),...,u(b,X',R'_n)) \\ &\Leftrightarrow a\mathbf{R}(X',R')b. \end{aligned}$$

Step 2. *We prove the "only if" statement.*

Let $\mathbf{R}$ satisfy Pareto Indifference and Independence of Inessential Expansions. By Lemma 1, $\mathbf{R}$ also satisfies Neutrality. For each $S\subseteq N$, define the binary relations $\succ_S,\sim_S$ $,\succsim_S$ on $[0,1]^N$ as follows:

(i) $v\succ_S w$ if and only if there exist $(X,R)\in\mathcal{P}_S$ and $a,b\in\Delta(X)$ such that $u(a,X,R_i)=v_i$ and $u(b,X,R_i)=w_i$ for all $i\in N$ and $a\mathbf{P}(X,R)b$,

(ii) $v\sim_S w$ if and only if there exist $(X,R)\in\mathcal{P}_S$ and $a,b\in\Delta(X)$ such that $u(a,X,R_i)=v_i$ and $u(b,X,R_i)=w_i$ for all $i\in N$ and $a\mathbf{I}(X,R)b$,

(iii) $v\succsim_S w$ if and only if $v\succ_S w$ or $v\sim_S w$.

The relations $\succ_S,\sim_S,\succsim_S$ are equivalently defined by replacing $\Delta(X)$ with X in statements (i) and (ii). To see why, fix $v,w\in[0,1]^N$ and suppose there exist $(X,R)\in\mathcal{P}_S$ and $a,b\in\Delta(X)$ such that $u(a,X,R_i)=v_i$ and $u(b,X,R_i)=w_i$ for all $i\in N$ and $a\mathbf{P}(X,R)b$ (respectively, $a\mathbf{I}(X,R)b$). Choose distinct pure alternatives $a',b'\in A\setminus X$, let $X'=X\cup\{a',b'\}$ and, for each $i\in N$, let R'_i be the von Neumann-Morgenstern preference on $\Delta(X')$ which coincides with R_i on $\Delta(X)$ and is such that $a'I'_ia$ and $b'I'_ib$. Then $(X',R')\in$

$\mathcal{P}_S$, $a', b' \in X'$, $u(a', X', R'_i) = v_i$ and $u(b', X', R'_i) = w_i$ for all $i \in N$ and, using Independence of Inessential Expansions and Pareto Indifference, $a'\mathbf{I}(X', R')a\mathbf{P}(X', R')b\mathbf{I}(X', R')b'$ (respectively, $a'\mathbf{I}(X', R')a\mathbf{I}(X', R')b\mathbf{I}(X', R')b'$).

Step 2.1. *We claim that each $\succsim_S$ is consistent in Suzumura's (1976) sense*: if there exist $v^1, ..., v^m \in [0,1]^N$ such that $v^1 \succsim_S ... \succsim_S v^m \succsim_S v^1$, then $v^1 \sim_S ... \sim_S v^m \sim_S v^1$. Suppose, on the contrary, that, say, $v^1 \succsim_S ... \succsim_S v^m \succ_S v^1$. Then there exist $(X^1, R^1), ..., (X^m, R^m) \in \mathcal{P}_S$ and $a^1, b^1 \in X^1, ..., a^m, b^m \in X^m$ such that

$$a^k\mathbf{R}(X^k, R^k)b^k \text{ for } k = 1, ..., m-1 \text{ and } a^m\mathbf{P}(X^m, R^m)b^m, \tag{7.27}$$

and

$$u(a^k, X^k, R^k_i) = v^k_i \text{ and } u(b^k, X^k, R^k_i) = v^{k+1(\operatorname{mod} m)}_i \text{ for all } i \in N \text{ and } k = 1, ..., m. \tag{7.28}$$

By Neutrality, we may assume that $X^1, ..., X^m$ are pairwise disjoint. Let $X = \cup_{k=1}^m X^k$. For each $i \in N$, let $u_i : \Delta(X) \to [0,1]$ be the von Neumann-Morgenstern utility function such that

$$u_i(x) = u(x, X^{k(x)}, R_i^{k(x)}) \text{ for all } x \in X, \tag{7.29}$$

where $k(x)$ is the unique k such that $x \in X^k$. Let R_i be the preference on $\Delta(X)$ represented by u_i and let $R = (R_1, ..., R_n)$.

Note that R coincides with R^k on $\Delta(X^k)$ for each k. Moreover, since $N_0(X^1, R^1) = ... = N_0(X^m, R^m)$ $(= S)$, (7.29) implies that $\overline{a}R_ixR_i\underline{a}$ for all $i \in N$, all $\overline{a} \in \cup_{k=1}^m \overline{A}(X^k, R^k_i)$, all $x \in X$, and all $\underline{a} \in \cup_{k=1}^m \underline{A}(X^k, R^k_i)$. Applying Independence of Inessential Expansions to (7.27),

$$a^k\mathbf{R}(X, R)b^k \text{ for } k = 1, ..., m-1 \text{ and } a^m\mathbf{P}(X, R)b^m. \tag{7.30}$$

On the other hand, (7.28) and (7.29) imply that $b^kI_ia^{k+1(\operatorname{mod} m)}$ for all $i \in N$ and $k = 1, .., m$. By Pareto Indifference,

$$b^k\mathbf{I}(X, R)a^{k+1(\operatorname{mod} m)} \text{ for } k = 1, .., m,$$

which together with (7.30) contradicts the transitivity of $\mathbf{R}(X, R)$.

Step 2.2. If $S \neq \varnothing$, the relation $\succsim_S$ need not be complete. Since $\succsim_S$ is consistent, however, it has an ordering extension (by Suzumura, 1976): denote it $\succsim'_S$. By the very definition of $\succsim_S$, we have that for all $(X, R) \in \mathcal{P}_S$ and $a, b \in \Delta(X)$, $a\mathbf{R}(X, R)b \Leftrightarrow (u(a, X, R_1), ..., u(a, X, R_n)) \succsim'_S (u(b, X, R_1), ..., u(b, X, R_n))$.■

Proof of Theorem 3. We leave the proof of the "if" statement to the reader. To prove the "only if" statement, fix an aggregation rule $\mathbf{R}$ satisfying the required axioms. By Theorem 4, there exists a collection of orderings $\succsim_S$ on $[0,1]^N$ such that for all $S \subseteq N$, all $(X, R) \in \mathcal{P}_S$ and all $a, b \in \Delta(X)$,

$$a\mathbf{R}(X, R)b \Leftrightarrow (u(a, X, R_1), ..., u(a, X, R_n)) \succsim_S (u(b, X, R_1), ..., u(b, X, R_n)). \tag{7.31}$$

Step 1. *We claim that for all* $(X,R) \in \mathcal{P}$ *and all* $a,b \in \Delta(X)$, $a\mathbf{R}(X,R)b \Leftrightarrow (u(a,X,R_1),...,u(a,X,R_n)) \succsim_\emptyset (u(b,X,R_1)\ ,...,\ u(b,X,R_n))$.

Suppose not. Then there exist $S \subseteq N$, $(X,R) \in \mathcal{P}_S$, and $a,b \in \Delta(X)$ such that one of the following statements holds:

$$a\mathbf{P}(X,R)b \text{ and } (u(b,X,R_1),...,u(b,X,R_n)) \succsim_\emptyset (u(a,X,R_1),...,u(a,X,R_n)), \tag{7.32}$$

$$a\mathbf{R}(X,R)b \text{ and } (u(b,X,R_1),...,u(b,X,R_n)) \succ_\emptyset (u(a,X,R_1),...,u(a,X,R_n)). \tag{7.33}$$

Assume (7.32). Without loss of generality, suppose also that $|X| \geq 3$. (If $|X| < 3$, simply choose $\overline{X} \in \mathcal{A}$ such that $X \subseteq \overline{X}$ and $\left|\overline{X}\right| \geq 3$. For each $i \in N$ let $\overline{R}_i$ be a von Neumann-Morgenstern preference on $\Delta(\overline{X})$ coinciding with R_i on $\Delta(X)$ and such that $\overline{a}\overline{R}_i x \overline{R}_i \underline{a}$ for all $\overline{a} \in \overline{A}(X,R_i)$, $x \in \overline{X}$, $\underline{a} \in \underline{A}(X,R_i)$. Then $a\mathbf{P}(\overline{X},\overline{R})b$ by Independence of Inessential Expansions and $(u(b,\overline{X},\overline{R}_1),...,u(b,\overline{X},\overline{R}_n)) = (u(b,X,R_1),...,u(b,X,R_n)) \succsim_\emptyset (u(a,X,R_1),...,u(a,X,R_n)) = (u(a,\overline{X},\overline{R}_1),...,u(a,\overline{X},\overline{R}_n))$, so that the argument below would apply with $(\overline{X},\overline{R})$ instead of (X,R).)

Let $R' = (R'_1,...,R'_n)$ be a profile of preferences over $\Delta(X)$ such that

$$\begin{aligned} R'_i &\neq R_0 \text{ and } aI'_i bR'_i x \text{ for all } x \in X \text{ and all } i \in S, \\ R'_i &= R_i \text{ for all } i \in N \setminus S. \end{aligned}$$

Such a profile exists because $\left|\overline{X}\right| \geq 3$. By construction, $(X,R') \in \mathcal{P}_\emptyset$ and

$$\begin{aligned} u(a,X,R'_i) &= u(a,X,R_i) = u(b,X,R'_i) = u(b,X,R_i) = 1 \text{ for all } i \in S, \\ u(a,X,R'_i) &= u(a,X,R_i) \text{ and } u(b,X,R'_i) = u(b,X,R_i) \text{ for all } i \in N \setminus S, \end{aligned}$$

so that (7.32) implies $(u(b,X,R'_1),...,u(b,X,R'_n)) \succsim_\emptyset (u(a,X,R'_1),...,\ u(a,X,R'_n))$.

But by Separability, $a\mathbf{P}(X,R)b$ implies $a\mathbf{P}(X,R')b$, contradicting (7.31) for $S = \emptyset$. Essentially the same argument applies if we assume (7.33) instead of (7.32).

Step 2. *We show that* $\succsim_\emptyset$ *is a strictly monotonic, symmetric, convex, and separable ordering.*

Let us first check that $\succsim_\emptyset$ is an ordering. To prove completeness, fix distinct vectors $v,w \in [0,1]^N$. Choose $a,b \in A$, $X \in \mathcal{A}$ containing a,b and such that $|X| \geq 3$, and R a profile of preferences over $\Delta(X)$ such that i) $u(a,X,R_i) = v_i$ and $u(b,X,R_i) = w_i$ for all $i \in N$ and ii) $(X,R) \in \mathcal{P}_\varnothing$. Restriction ii) can be satisfied because $|X| \geq 3$. Since $\mathbf{R}(X,R)$ is complete, $a\mathbf{R}(X,R)b$ or $b\mathbf{R}(X,R)a$. By definition of $\succsim_\emptyset$, $v \succsim_\emptyset w$ or $w \succsim_\emptyset v$. Reflexivity and transitivity of $\succsim_\emptyset$ are proved in a similar way.

To check that $\succsim_\emptyset$ is strictly monotonic, fix $v,w \in [0,1]^N$ such that $v > w$. As in the previous paragraph, we may choose $a,b \in A$ and $(X,R) \in \mathcal{P}_\varnothing$ such that $u(a,X,R_i) = v_i$ and $u(b,X,R_i) = w_i$ for all $i \in N$. Since $(u(a,X,R_1),...,\ u(a,X,R_n)) = v > w = (u(b,X,R_1),...,\ u(b,X,R_n))$, the Pareto Principle implies $a\mathbf{P}(X,R)b$. It follows from the definition of $\succsim_\emptyset$ that $v \succsim_\emptyset w$.

To check that $\succsim_\emptyset$ is symmetric, fix $v,w \in [0,1]^N$, $\sigma \in \Pi(N)$, and suppose $v \succsim_\emptyset w$. Again, let $a,b \in A$ and $(X,R) \in \mathcal{P}_\varnothing$ be such that $u(a,X,R_i) = v_i$ and $u(b,X,R_i) = w_i$ for

all $i \in N$. By definition of $\succsim_\emptyset$, $a\mathbf{R}(X,R)b$. By Anonymity, $a\mathbf{R}(X,\sigma R)b$. But for all $i \in N$, $u(a,X,(\sigma R)_i) = u(a,X,R_{\sigma^{-1}(i)}) = v_{\sigma^{-1}(i)} = (\sigma v)_i$ and, likewise, $u(b,X,(\sigma R)_i) = (\sigma w)_i$. Therefore, by definition of $\succsim_\emptyset$, $a\mathbf{R}(X,\sigma R)b$ implies $\sigma v \succsim_\emptyset \sigma w$.

To check that $\succsim_\emptyset$ is convex, fix $v, w \in [0,1]^N$ such that $v \succsim_\emptyset w$ and let $\lambda \in (0,1)$. Let $a, b \in A$ and $(X,R) \in \mathcal{P}_\varnothing$ be such that $u(a,X,R_i) = v_i$ and $u(b,X,R_i) = w_i$ for all $i \in N$. By definition of $\succsim_\emptyset$, $a\mathbf{R}(X,R)b$. By Preference for Lotteries, $(\lambda a + (1-\lambda)b)\mathbf{R}(X,R)b$. But for all $i \in N$,

$$\begin{aligned} u(\lambda a + (1-\lambda)b, X, R_i) &= \lambda u(a,X,R_i) + (1-\lambda)u(b,X,R_i) \\ &= \lambda v_i + (1-\lambda)w_i \\ &= (\lambda v + (1-\lambda)w)_i, \end{aligned}$$

so that by definition of $\succsim_\emptyset$, $(\lambda a + (1-\lambda)b)\mathbf{R}(X,R)b$ implies $\lambda v + (1-\lambda)w \succsim_\emptyset w$.

Finally, let us check that $\succsim_\emptyset$ is separable. Fix $S \subseteq N$ and $v, v', w, w' \in [0,1]^N$ and such that $v_i = v'_i$ and $w_i = w'_i$ for all $i \in S$ and $v_j = w_j$ and $v'_j = w'_j$ for all $j \in N \setminus S$. Suppose $v \succsim_\emptyset w$. Let $a, b \in A$ and $(X,R), (X,R') \in \mathcal{P}_\varnothing$ be such that $R_i = R'_i$ for all $i \in S$, $u(a,X,R_i) = v_i$ and $u(b,X,R_i) = w_i$ for all $i \in S$, and $u(a,X,R_j) = u(b,X,R_j) = v_j$ and $u(a,X,R'_j) = u(b,X,R'_j) = v'_j$ for all $j \in N \setminus S$. By definition of $\succsim_\emptyset$, $a\mathbf{R}(X,R)b$. By Separability, $a\mathbf{R}(X,R')b$. By definition of $\succsim_\emptyset$ again, $v' \succsim_\emptyset w'$.■

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