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Opérateurs de Heun, ansatz de Bethe et représentations de $su(3)$

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Résumé

Le présent mémoire contient deux articles reliés par le formalisme de l'ansatz de Bethe. Dans le premier article, l'opérateur de Heun de type Lie est identifié comme une spécialisation de la matrice de transfert d'un modèle de BC-Gaudin à un site dans un champ magnétique. Ceci permet de le diagonaliser à l'aide de l'ansatz de Bethe algébrique modifié. La complétude du spectre est démontrée en reliant les racines de Bethe aux zéros des solutions polynomiales d'une équation différentielle de Heun inhomogène. Le deuxième article aborde le sujet des représentations irréductibles de l'algèbre de Lie $su(3)$ dans la réduction $su(3) \supset so(3) \supset so(2)$. Cette manière de construire les représentations irréductibles de $su(3)$ porte une ambiguïté qui empêche de distinguer totalement les vecteurs de base, ce qui mène à un problème d'étiquette manquante. Dans cet esprit, l'algèbre des deux opérateurs fournissant cette étiquette est examinée. L'opérateur de degré 4 dans les générateurs de $su(3)$ est diagonalisé en se servant des techniques de l'ansatz de Bethe analytique.

Mots-clés : opérateur de Heun–Lie , algèbre de Lie $su(3)$, problème d'étiquette manquante, ansatz de Bethe, représentations, modèles intégrables.

Abstract

This Master's thesis contains two articles linked by the formalism of the Bethe ansatz. In the first article, the Lie-type Heun operator is identified as a specialization of the transfer matrix of a one-site BC-Gaudin model in a magnetic field. This allows its diagonalization by means of the modified algebraic Bethe ansatz. The completeness of the spectrum is proven by relating the Bethe roots to the zeros of the polynomial solutions of an inhomogeneous differential Heun equation. The second article deals with the subject of irreducible representations of the Lie algebra $su(3)$ in the reduction $su(3) \supset so(3) \supset so(2)$. This way of constructing the irreducible representations of $su(3)$ carries an ambiguity in distinguishing the basis vectors, also known as a missing label problem. In this spirit, the algebra of the two operators providing the missing label is examined. The operator of degree 4 in the generators of $su(3)$ is diagonalized using the techniques of the analytical Bethe ansatz.

Keywords : Heun operator of Lie type, Lie algebra of $su(3)$, missing label problem, Bethe ansatz, representations, integrable models.

Table des matières

Résumé 5

Abstract 7

Remerciements 11

Introduction 13

0.1. Opérateur de Heun algébrique 14

0.2. Problème d'étiquette manquante 15

0.3. Objectifs et résultats 17

0.4. Contributions de l'auteure 19

Chapitre 1. Heun operator of Lie type and the modified algebraic Bethe ansatz 21

1.1. Introduction 21

1.2. Heun operators and the Gaudin magnet 23

1.2.1. Algebraic Heun operator of Lie type 23

1.2.2. BC-Gaudin magnet in magnetic field 24

1.3. Modified algebraic Bethe ansatz 25

1.4. Bethe roots and Heun operator 29

1.4.1. Inhomogeneous Heun differential equation 29

1.4.2. Eigenvalues of the algebraic Heun operator 31

1.4.3. Bargmann realization of the algebraic Heun operator of Lie type 31

1.5. Homogeneous case 32

1.6. Representations of the rotation group $O(3)$ and Heun operator 35

1.7. Entanglement entropy for the Krawtchouk chain and Heun operator 36

1.8. Conclusion and outlook 39

1.A. Bethe ansatz from the lowest weight 40

References 41

Chapitre 2. The $SU(3) \supset SO(3)$ missing label problem and the analytical Bethe ansatz 45

2.1. Introduction 45

2.2. The $SU(3) \supset SO(3)$ missing label operators and their algebra 47

2.3. Differential realization of the missing label operators x and y 49

2.3.1. Irreducible representations of $su(3)$ 49

2.3.2. The Bargmann–Moshinsky basis 50

2.3.3. The operators x and y in the Bargmann–Moshinsky basis 51

2.4. Analytical Bethe ansatz 53

2.4.1. Yangian $Y(gl(3))$, Twisted Yangian $Y(gl(3); so(3))$ and a Bethe subalgebra 54

2.4.2. Evaluation representation of $Y(gl(3))$ 55

2.4.3. Bethe ansatz 56

2.5. Conclusion 59

References 59

Conclusion 63

Références bibliographiques 65

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Introduction

Le présent mémoire touche à deux sujets : le premier étant l'opérateur de Heun algébrique, et le deuxième portant sur les représentations irréductibles de l'algèbre de Lie $su(3)$. À première vue, les deux semblent distincts, mais l'identification d'objets d'intérêt des deux sujets dans un cadre de systèmes intégrables permet d'employer le formalisme de l'ansatz de Bethe. D'une part, c'est l'opérateur de Heun de type Lie qui est diagonalisé à l'aide de l'ansatz de Bethe algébrique modifié. D'autre part, le spectre d'un des deux opérateurs d'étiquette manquante dans la chaine $su(3) \supset so(3) \supset so(2)$ est déduit en se servant de l'ansatz de Bethe analytique.

Les systèmes dits intégrables occupent une place centrale en physique théorique, et leur étude remonte, au moins, au temps de Liouville (1809 -1882). Un système intégrable est caractérisé par l'existence de symétries, de structures algébriques associées et de solutions explicites. Ceux-ci offrent des modèles qui permettent de valider et d'explorer des idées et des explications en lien avec différents phénomènes.

L'intérêt pour les structures algébriques et leurs représentations possède une longue tradition dans un contexte physique ; des systèmes (classiques ou quantiques) qui semblent complexes peuvent être simplifiés et mieux compris à l'aide de leur groupe de symétries. De plus, les symétries permettent de concevoir des modèles physiques qui peuvent mieux décrire la réalité. L'avantage de ce formalisme est son universalité ; plusieurs problèmes physiques peuvent être traités avec les mêmes techniques.

Le formalisme de l'ansatz de Bethe est l'une des plus importantes approches servant à résoudre des systèmes intégrables. Le nom de l'ansatz remonte à H. Bethe, qui a proposé une manière originale de construire les vecteurs propres de l'Hamiltonien d'une chaine de spin 1/2 à une dimension [**6**]. De nombreux variants de l'ansatz de Bethe [**19**, **24**, **29**] ont été développés au fil des années, et cette méthode a pris une importance primordiale dans la résolution de systèmes quantiques à N corps. Même si le calcul, par diagonalisation numérique, des valeurs et des vecteurs propres du système en question peut être plus simple, la construction des vecteurs propres à l'aide de l'ansatz de Bethe permet d'étiqueter ces derniers avec un ensemble de nombres quantiques décrivant des propriétés physiques.

Deux versions de l'ansatz de Bethe sont employées dans le présent mémoire. La première est une version modifiée de l'ansatz de Bethe algébrique, et nous permet de diagonaliser l'opérateur de Heun de type Lie. La deuxième est l'ansatz de Bethe analytique, qui nous permet de diagonaliser un des deux opérateurs fournissant l'étiquette manquante à la réduction $su(3) \supset so(3)$. Dans les deux prochaines sections, on présentera un survol des principaux concepts qui font le sujet du présent mémoire, soit l'opérateur de Heun algébrique et le problème d'étiquette manquante.

0.1. Opérateur de Heun algébrique

L'opérateur de Heun algébrique a été récemment introduit dans [15] en faisant une généralisation de l'équation de Heun classique. L'équation de Heun [16, 25] est une équation différentielle linéaire ordinaire du deuxième ordre avec 4 singularités régulières en $0, 1, d, \infty$. Cette dernière est donnée par

$$M\psi(x) = 0, \tag{0.1.1}$$

avec

$$M = \partial_x^2 + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-d}\right)\partial_x + \frac{\alpha\beta x - q}{x(x-1)(x-d)}. \tag{0.1.2}$$

La condition supplémentaire $\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0$ est imposée pour assurer que les exposants du point singulier à l'infini soient (α, β).

Une autre manière de présenter l'opérateur M est la suivante. On considère l'opérateur hypergéométrique L donné par :

$$L = x(1-x)\partial_x^2 + (\alpha_1 x + \alpha_2)\partial_x \tag{0.1.3}$$

et l'opérateur multiplication par x : $X\psi(x) = x\psi(x)$. L'opérateur M peut être écrit sous la forme (voir ci dessous) [14]

$$M = \tau_1 LX + \tau_2 XL + \tau_3 L + \tau_4 X + \tau_0 \tag{0.1.4}$$

où les paramètres τ_i sont reliés à ceux de l'équation de Heun comme : $\gamma = \alpha_2$, $\delta = -\alpha_1 - \alpha_2$, $\epsilon = 2\tau_1/(\tau_1+\tau_2)$, $d = -\tau_3/(\tau_1+\tau_2)$, $\alpha\beta = -(\tau_1\alpha_1 + \tau_4)/(\tau_1+\tau_2)$, $q = (\tau_1\alpha_2 + \tau_0)/(\tau_1+\tau_2)$. Un aspect important des opérateurs L et X est le fait qu'il décrivent un problème bispectral. Plus précisément, il est connu que les fonctions propres de L sont données par les polynômes de Jacobi $P_n(x)$:

$$LP_n(x) = \lambda_n P_n(x), \qquad \lambda_n = n(1 + \alpha_1 - n). \tag{0.1.5}$$

De plus, ces polynômes sont orthogonaux, donc ils vérifient une relation de récurrence à trois termes :

$$P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x) = xP_n(x) \tag{0.1.6}$$

pour certains coefficients b_n, c_n.

On remarque qu'il y a deux représentations possibles pour les opérateurs L et X. Dans la première, L est un opérateur différentiel agissant sur l'argument x des polynômes de Jacobi (équation 0.1.3) et X est la multiplication par x. Dans la deuxième, L est la multiplication par λ_n et X est un opérateur tridiagonal aux différences (équation 0.1.6). Les opérateurs L, X forme donc une paire bispectrale. La généralisation de la construction 0.1.4 pour toute paire bispectrale permet de définir ce qu'on appelle opérateur de Heun algébrique, introduit pour la première fois dans [**15**], dans le cadre d'un problème d'analyse de signal.

Il existe une famille de polynômes qui offre de tels problèmes bispectraux. Il s'agit des polynômes hypergéométriques du schéma de Askey (ref. exemple [**20**]). Pour ces derniers, les opérateurs bispectraux correspondent à la relation de récurrence et à l'équation différentielle (ou aux différences) que ces polynômes vérifient. De plus, l'opérateur de récurrence X et l'opérateur différentiel ou aux différences Y génèrent une algèbre quadratique connue sous le nom d'algèbre d'Askey–Wilson [**31**].

Une autre propriété algébrique sous-jacente aux opérateurs de Heun est l'algèbre de Heun, construite à partir de la paire (W, X) ou (W, Y). L'opérateur de Heun algébrique construit à partir d'une paire X, Y décrivant une famille de polynômes orthogonaux du schéma d'Askey sera désigné par le même nom que cette dernière. Ainsi, pour la paire X,Y correspondant aux polynômes de Racah, l'opérateur W est appelé l'opérateur de Heun–Racah et l'algèbre générée par (W, X) ou (W, Y) est appelée l'algèbre de Heun–Racah [**5**]. L'étude systématique de ces opérateurs de Heun a été entreprise par Luc Vinet et ses collaborateurs [**3**, **4**, **9**, **30**]. Ainsi l'opérateur de Heun de type Askey-Wilson a été examiné en détail dans [**3**]. Il a récemment été observé que cet opérateur intervient dans l'algèbre de réflexion et cela a permis d'en effectuer la diagonalisation à l'aide de l'ansatz de Bethe algébrique [**2**]. L'opérateur de Heun le plus simple est celui dit de type Lie [**9**]. Dans le cas où l'algèbre sous-jacente et $su(2)$, il concorde avec l'opérateur de Heun associé aux polynômes de Krawtchouk. C'est ce dernier qui est diagonalisé au chapitre 1 à l'aide de l'Ansatz de Bethe algébrique modifié.

0.2. Problème d'étiquette manquante

Les méthodes de la théorie des représentations apparaissent de multiples façons dans la description de symétries en physique. C'est en particulier le cas lorsque l'on s'intéresse à réduire des représentations irréductibles d'un groupe à des représentations irréductibles d'un ou des sous-groupes. Il est commun d'examiner les représentations irréductibles de l'algèbre de Lie g plutôt que de son groupe de Lie G. Le passage aux représentations du groupe est fait grâce à l'application exponentielle. On adoptera cette approche pour la suite du mémoire. On considère donc les représentations irréductibles de l'algèbre g réduite selon la chaine

$$g \supset g' \supset g'' \dots \tag{0.2.1}$$

Afin d'étiqueter les vecteurs de base de la représentation, on se sert des opérateurs de Casimir de chaque sous-algèbre. Toutefois, ces opérateurs peuvent ne pas suffire pour entièrement distinguer les vecteurs de base. Cette dégénérescence est connue sous le nom de problème d'étiquette manquante.

En effet, si l'algèbre de départ g est une algèbre de Lie semi-simple, cette dernière possède $\mathcal{N}(g) = l$ opérateurs de Casimir, où l est la dimension de la sous-algèbre de Cartan (aussi appelé le rang de l'algèbre). En plus des l étiquettes fournies par les opérateurs de Casimir de g, un théorème de G. Racah [22] montre qu'il faut i étiquettes pour distinguer les vecteurs de base, avec

$$i = \frac{dim(g) - l}{2}. \tag{0.2.2}$$

Dans le cas où g est réduite selon la chaine (0.2.1), les opérateurs de Casimir des sous-algèbres peuvent fournir une partie ou la totalité de ces étiquettes, d'où la possibilité d'une étiquette manquante. Un prototype de tels chaines qui exergue un problème d'étiquette manquante est celle de $su(3)$:

$$su(3) \supset so(3) \supset so(2). \tag{0.2.3}$$

L'algèbre $su(3)$ est de dimension 8 et possède deux opérateurs de Casimir, notés g_2, g_3. Il faut donc encore $i = \frac{8-2}{2} = 3$ étiquettes pour définir une base pour les représentations irréductibles. L'élément de Casimir $\mathbf{L}^2$ de $so(3)$ et celui de $so(2)$, noté L_3, fournissent deux étiquettes supplémentaires. Il y a donc 1 étiquette manquante.

Une première façon de lever la dégénérescence pour des représentations irréductibles de $su(3)$ réduites selon la chaine (0.2.3) est de construire une base polynomiale qui contient un cinquième paramètre entier fournissant l'étiquette manquante. Ce dernier mène à distinguer les différentes copies d'une représentation irréductible de $so(3)$ dans une représentation irréductible donnée de $su(3)$. Une revue assez détaillée de telles bases a été faite dans [21]. Toutefois, aucune de ces bases n'est orthogonale et l'étiquette ne correspond pas à un opérateur hermitien. Il est donc difficile de lui assigner un sens physique.

La deuxième manière de lever la dégénérescence est de chercher un cinquième opérateur dans l'enveloppante universelle de $su(3)$ qui commute avec l'ensemble $(g_2, g_3, \mathbf{L}^2, L_3)$. Il existe deux tels opérateurs, le premier est de degré 3 et le deuxième est de degré 4 dans les générateurs, qu'on notera x et y respectivement. L'inconvénient de cette base est que le spectre des opérateurs x et y n'est pas simple. Pour trouver leurs valeurs propres, la manière standard est de les développer dans une base polynomiale puis de les diagonaliser numériquement. La nouveauté du présent travail est l'identification d'une deuxième méthode de diagonalisation pour y, soit à l'aide de l'ansatz de Bethe analytique.

0.3. Objectifs et résultats

Ce mémoire est constitué de deux chapitres contenant chacun un article. Le premier traite l'opérateur de Heun de type Lie, une spécialisation de l'opérateur de Heun algébrique introduit dans [15]. L'identification de cet opérateur dans un modèle de Gaudin à un site dans un champ magnétique, permet de le diagonaliser à l'aide d'une généralisation des techniques de l'ansatz de Bethe algébrique [29]. Le modèle de Gaudin a été d'abord introduit dans [11] pour décrire une chaine de spin intégrable associée à l'algèbre de Lie $sl(2)$ avec interaction à longue portée. Ce système a été ensuite examiné dans le cadre de la méthode de diffusion quantique inverse [26]. Plusieurs généralisations ont été également introduites : en remplaçant l'algèbre $sl(2)$ par une algèbre de Lie simple quelconque [12], ainsi que dans le cadre des matrices r solutions à l'équation de Yang-Baxter classique standard (voir p. ex. [27]) et non standard [7, 28].

Les grandes lignes de l'ansatz de Bethe utilisé sont les suivantes. On considère la matrice r du modèle de Gaudin en question, solution à l'équation de Yang–Baxter. Cette dernière assure l'intégrabilité du modèle, c'est-à-dire la commutativité de la matrice de transfert pour différents paramètres. Il est à noter que c'est l'équation de Yang–Baxter dans sa version classique et non standard qui est employée dans le présent travail. Une spécialisation de la matrice de transfert donne lieu à l'opérateur de Heun de type Lie. La matrice de transfert est écrite en termes des opérateurs d'annihilation et de création de l'algèbre sous-jacente $su(2)$, ce qui permet de l'écrire comme la trace du carré d'une autre matrice, appelé matrice K. Cette étape s'avère importante pour pouvoir poursuivre les calculs. La version modifiée de l'ansatz de Bethe consiste à décaler la matrice K par une matrice scalaire qui n'affecte pas l'intégrabilité du système. La prochaine étape est d'agir avec la matrice de transfert sur un vecteur simple, appelé le pseudo vide. Les états excités sont construits par la suite à partir du pseudo vide à l'aide des opérateurs de création et contiennent un ensemble de paramètres, appelés racines de Bethe, à déterminer. En exigeant que l'action de la matrice de transfert sur les vecteurs de base soit diagonale, on obtient des contraintes sur les racines de Bethe, connues sous le nom d'équations de Bethe. La résolution de ces équations permet de trouver les valeurs propres de l'opérateur de Heun en question.

Le spectre de l'opérateur de Heun étudié est prouvé complet de la manière suivante. On se rappelle que dans une représentation de spin s de $su(2)$, il faut $2s + 1$ vecteurs indépendants pour définir la base. La résolution des équations de Bethe s'avère en effet équivalente à trouver exactement $2s + 1$ solutions polynomiales à l'équation différentielle de Heun avec un terme inhomogène. Les zéros de chaque solution polynomiale sont reliés à un ensemble de racines de Bethe.

Enfin, deux spécialisations de cet opérateur sont discutées. La première est reliée aux représentations du groupe des rotations $O(3)$. En effet, il existe deux bases de représentations connues pour le groupe $O(3)$. Dans la première, les vecteurs de bases sont des fonctions propres d'une des trois générateurs de moments angulaires, ainsi que de l'opérateur de Casimir L^2. Cette base correspond à la réduction $O(3) \supset O(2)$. Dans la deuxième base, les vecteurs sont des fonctions propres de L^2 et d'un deuxième opérateur E de degré deux dans les générateurs (ainsi que des opérateurs discrets). C'est l'opérateur E qui figure comme un cas spécial de l'opérateur de Heun de type Lie. La deuxième application est relié au calcul d'entropie d'intrication de fermions sur une chaine de Krawtchouk. Cette dernière décrit une chaine de fermions libres qui sautent entre des sites adjacents du réseau et sont soumis à un champ magnétique non uniforme. L'opérateur de Heun commute avec la matrice de corrélation partielle correspondant aux $l+1$ premiers sites de la chaine. La diagonalisation de l'opérateur de Heun permet donc de déduire le spectre de la matrice de corrélation partielle.

Le deuxième chapitre comprend un article qui traite le sujet des représentations de l'algèbre de Lie $su(3)$ dans la réduction $su(3) \supset so(3) \supset so(2)$. Comme expliqué précédemment, cette chaine de sous algèbres présente un problème d'étiquette manquante. En plus des opérateurs de Casimir des sous-groupes qui étiquettent les vecteurs de base, une étiquette supplémentaire est nécessaire. Une façon de remédier au problème est de chercher un invariant de $so(3)$ dans l'algèbre enveloppante universelle de $su(3)$. Il s'avère qu'il existe deux tels opérateurs [**1**, **18**, **23**], un de degré 3 et l'autre de degré 4 dans les générateurs. Il y a près de 60 ans, Racah et Lehrer [**23**] ont examiné les liens algébriques entre ces deux opérateurs, et ont stipulé que ces derniers prennent une forme cubique, sans indiquer les coefficients explicites devant chaque terme. Ces coefficients sont présentés dans l'article. Ensuite, des expressions pour les opérateurs x, y dans la base polynomiale de Bargmann–Moshinsky [**1**] sont fournies, celles de y étant calculées pour la première fois. Ceci donne la possibilité diagonaliser ces deux éléments numériquement.

Une nouvelle façon de diagonaliser l'opérateur y à l'aide du formalisme de l'ansatz de Bethe analytique [**24**] est présentée. On y parvient à l'aide d'une identification de y dans un sous algèbre abélienne (aussi appelé sous algèbre de Bethe) du yangian twisté dans l'enveloppante universelle de $gl(3)$ [**17**]. Les générateurs de cette sous-algèbre jouent le rôle de matrices de transfert. L'équation de Yang–Baxter classique est remplacée par sa version quantique. La première étape est de calculer l'action de l'élément à diagonaliser sur un pseudo-vide. Ensuite, l'ansatz consiste à habiller la valeur propre du pseudo-vide avec des fonctions d'habillage qui dépendent d'un ensemble de racines de Bethe à déterminer. En imposant l'analyticité des valeurs propres, on obtient un ensemble d'équations de Bethe qui relient les racines. Leur résolution permet de trouver toutes les valeurs propres.

0.4. Contributions de l'auteure

Les idées du présent mémoire ont été initialement proposées par Luc Vinet, qui a encadré et dirigé l'auteure, et a contribué à l'élaboration de ses recherches. Ce travail est également le résultat d'une étroite collaboration avec Nicolas Crampé.

L'étude de la diagonalisation de l'opérateur de Heun Lie dans le premier article a débuté avec l'examen de deux cas particuliers, soit ceux présentés à la fin de l'article. L'auteure s'est penchée sur l'étude des représentations irréductibles de $O(3)$, tandis que Pierre-Antoine Bernard a examiné le calcul d'entropie des fermions sur une chaine de Krawtchouk. Grâce aux savoirs de Nicolas Crampé au sujet de l'ansatz de Bethe, il a pu orienter les deux coauteurs à faire la majorité des calculs propres à ces spécialisations. Il a ensuite réussi à généraliser et reproduire ces résultats pour l'opérateur de Heun Lie générique. Il a enfin rédigé la grande partie de l'article. Luc Vinet a été présent dans les discussions tout au long du travail, et a contribué à rédiger certaines parties de l'article.

Le deuxième article portant sur les représentations de l'algèbre $su(3)$ dans la réduction $su(3) \supset so(3)$ était principalement rédigé par l'auteure, qui a effectué la majorité des calculs. Luc Vinet a eu l'idée de se pencher sur le sujet après avoir remarqué la réapparition des relations algébriques entre les opérateurs d'étiquette manquante, dans une récente étude de l'algèbre de symétrie du centralisateur de l'action diagonale de $U(sl(3))$ dans $U(sl(3))^{\otimes 2}$ [8]. Nicolas Crampé a notamment contribué à la construction des fonctions d'habillage de la valeur propre et la diagonalisation de y avec l'ansatz de Bethe, et a fourni un programme informatique qui facilite les calculs qui y sont reliés.

Chapitre 1

Heun operator of Lie type and the modified algebraic Bethe ansatz

Par Pierre-Antoine Bernard, Nicolas Crampé, Dounia Shaaban Kabakibo et Luc Vinet.
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Abstract.The generic Heun operator of Lie type is identified as a certain BC-Gaudin magnet Hamiltonian in a magnetic field. By using the modified algebraic Bethe ansatz introduced to diagonalize such Gaudin models, we obtain the spectrum of the generic Heun operator of Lie type in terms of the Bethe roots of inhomogeneous Bethe equations. We show also that these Bethe roots are intimately associated to the roots of polynomial solutions of the differential Heun equation. We illustrate the use of this approach in two contexts: the representation theory of $O(3)$ and the computation of the entanglement entropy for free Fermions on the Krawtchouk chain.

1.1. Introduction

In the spirit of [5] where the diagonalization of the Heun–Askey–Wilson operator has been performed by using the modified algebraic Bethe ansatz, we indicate in the present paper how the Heun operator of Lie type can be diagonalized by a similar method. We show that the particular Heun operator introduced in [19] can be identified with the BC-Gaudin magnet Hamiltonian in an external magnetic field with one site in a spin s representation of $\mathfrak{su}(2)$. By a slight generalization of the modified algebraic Bethe ansatz used in [13], we succeed in diagonalizing the generic Heun operator of Lie type. We then look at two mathematical and physical problems where the Heun operator of Lie type appears and the results apply.

The algebraic Heun operators are generalizations, introduced in [21], of the differential Heun operator which is the standardized form of the Fuchsian second order differential equation with four regular singularities. These operators are the most general bilinear combination of the bispectral pair associated to the orthogonal polynomials of the Askey scheme and they share the name of these polynomials. The differential Heun operator is recovered in this framework when we consider the dual pair associated to the Jacobi polynomials [22]. The Heun–Askey–Wilson operator has been studied in [6] whereas the Heun–Racah, the Heun–Bannai–Ito and the Heun–Hahn operators have been examined in [9, 30]. These operators have found applications in different contexts. For example, they give a nice algebraic interpretation of the time-band limiting problem [21] and provide the commuting tridiagonal operator [20] that allows to compute the entanglement entropy for free Fermion models [16, 17]. This paper focuses on the simplest algebraic Heun operators which are those of Lie type studied in [19]. In [5], the Heun–Askey–Wilson operator has been identified in the transfer matrix of the XXZ spin chain which allows to use the methods of quantum integrable models to diagonalize this operator. We prove here a similar result: the algebraic Heun operator of Lie type is associated to the BC-Gaudin model in an external magnetic field. In the spin s representation of $\mathfrak{su}(2)$, this identification allows to use the algebraic Bethe ansatz to diagonalize this Heun operator.

The usual algebraic Bethe ansatz (or quantum inverse scattering method) has been developed in [27] to diagonalize integrable model with periodic boundary conditions. Its generalization to open boundary conditions has been introduced in [26]. The problem of how to apply this method for generic open boundary has remained unsolved for 30 years. In [7], the modified algebraic Bethe ansatz was finally introduced to compute the eigenvectors of the XXX spin chain with generic boundaries associated to the eigenvalues found in [10]. This modified method has also been used to diagonalize the totally asymmetric exclusion process [12] and the XXZ spin chain [2,8] and it provides the spectrum in terms of inhomogeneous Bethe equations. In this paper, we generalize the result of [13] where the BC-Gaudin model is diagonalized by the modified algebraic Bethe ansatz. An extension is required to take into account the external magnetic field.

The Bethe equations consist in N algebraic relations between N unknowns that need to be solved in order to obtain the eigenvalues and the eigenvectors. These equations usually have multiple solutions and one must demonstrate that all the eigenvalues are obtained in this manner to prove that the ansatz provides the complete spectrum. One way to tackle the problem is to show that obtaining the solutions of Bethe equations is equivalent to finding the polynomial solutions of a differential (or difference) equation. For a second order differential equation, this problem is referred to the Heine–Stieljes problem and is well-studied. This program has already been used with success in [23, 32] for various quantum integrable models. We show in this fashion that the spectrum obtained by the modified algebraic Bethe ansatz of the algebraic Heun operator is complete. The associated differential equation is an inhomogeneous differential Fuchsian equation with 4 regular singularities (*i.e.* an inhomogeneous differential Heun operator). We show moreover, by using the Bargmann realization of the Lie algebra $\mathfrak{su}(2)$, that the eigenvalues problem is also associated to a homogeneous differential Fuchsian equation with 5 regular singularities.

The plan of the paper is as follows. Section 1.2 offers a review of the Heun operators of Lie type [19] and of the algebraic framework needed to study the BC-Gaudin magnet. Section 1.3 provides the main steps of the modified algebraic Bethe ansatz for the Gaudin model. In Section 1.4, we study in more details the inhomogeneous Bethe equations and associate their solutions, the Bethe roots, to the roots of a polynomial solution of the differential Heun equation with an inhomogeneous term. Section 1.5 focuses on a particular case where the parameters of the algebraic Heun operator satisfy some constraints. It is seen in this case that there exists an alternative to compute the spectrum that leads to homogeneous Bethe equations. The last two sections discuss the use of the Heun operator of Lie type in two different contexts: first, in the representation theory of $O(3)$ (see Section 1.6) and second, in the computation of the entanglement entropy for free Fermions on a Krawtchouk chain (see Section 1.7). We conclude with an outlook in Section 1.8. Useful formulas are gathered in Appendix 1.A.

1.2. Heun operators and the Gaudin magnet

1.2.1. Algebraic Heun operator of Lie type

The Lie algebra $\mathfrak{su}(2)$ is generated by J_1, J_2 and J_3 subject to the following defining relations

$$[J_1,J_2] = iJ_3, \qquad [J_2,J_3] = iJ_1, \qquad [J_3,J_1] = iJ_2. \tag{1.2.1}$$

We set

$$X = \alpha J_1 + \beta J_2, \quad \text{and} \qquad Y = J_1. \tag{1.2.2}$$

The Heun operator of Lie type is the following element of the universal enveloping algebra $U(\mathfrak{su}(2))$

$$W = r_1[X,Y] + r_2\{X,Y\} + r_3 X + r_4 Y + r_5, \tag{1.2.3}$$

where r_i are free parameters and $\{X,Y\} = XY + YX$ is the anti-commutator. This Heun operator can be rewritten as follows in terms of the Lie generators (up to a renormalization)

$$W = \rho_1 J_1 + \rho_2 J_2 + \rho_3 J_3 + \{J_1,J_2\} + \rho_4 J_1^2 + \rho_5 \tag{1.2.4}$$

where ρ_i ($1 \leq i \leq 5$) are free real parameters. The normalization of W is chosen so that the coefficient in front of the anti-commutator $\{J_1,J_2\}$ be one. The parameters ρ_i have been taken real to ensure that W is Hermitian if J_i are represented by Hermitian matrices. The generators X and W satisfy the Heun algebra of Lie type studied in [19].

1.2.2. BC-Gaudin magnet in magnetic field

Let us introduce the r-matrix

$$r_{12}(u,v) = \frac{1}{(u-v)(uv-1)} \begin{pmatrix} u(1-v^2) & 0 & 0 & -2(u-v) \\ 0 & -u(1-v^2) & -2v(uv-1) & 0 \\ 0 & -2u(uv-1) & -u(1-v^2) & 0 \\ -2uv(u-v) & 0 & 0 & u(1-v^2) \end{pmatrix} \tag{1.2.5}$$

solution of the non-standard classical Yang-Baxter equation

$$[\ r_{13}(u_1,u_3)\ ,\ r_{23}(u_2,u_3)\ ] = [\ r_{21}(u_2,u_1)\ ,\ r_{13}(u_1,u_3)\ ] + [\ r_{23}(u_2,u_3)\ ,\ r_{12}(u_1,u_2)\ ]\ , \tag{1.2.6}$$

where $r_{12}(u) = r(u) \otimes \mathbb{I}$, $r_{23}(u) = \mathbb{I} \otimes r(u)$, $r_{21}(u)$ is obtained by conjugating $r_{12}(u)$ by the permutation operator, swapping spaces 1 and 2, and so on. This r-matrix has been used to give the FRT presentations of the Onsager algebra [3] and of the classical Askey–Wilson algebra [4]. Let us recall that the parameters u and v are usually referred to as spectral parameters. This r-matrix can be obtained from the one of the 6-vertex model under a twist by a matrix solution of the classical reflection equation (see [3]).

Let us now define the K-matrix containing as follows the generators of $\mathfrak{su}(2)$

$$K(u) = \frac{2}{(1-au)(u-a)} \begin{pmatrix} u(a^2-1)J_3 & (a^2u-2a+u)J_1 - iu(a^2-1)J_2 \\ -u(a^2-2au+1)J_1 + iu(a^2-1)J_2 & -u(a^2-1)J_3 \end{pmatrix}, \tag{1.2.7}$$

where a is a free parameter called the inhomogeneity parameter. The K-matrix satisfies the classical reflection equation

$$[\ K_1(u)\ ,\ K_2(v)\ ] = [\ r_{21}(v,u)\ ,\ K_1(u)\ ] + [\ K_2(v)\ ,\ r_{12}(u,v)\ ]. \tag{1.2.8}$$

In fact, this classical reflection equation is equivalent to the defining relations (1.2.1) of $\mathfrak{su}(2)$.

We introduce also a scalar matrix

$$c(u) \quad = \quad \left(\frac{2iu(a^2-1)\rho_1}{a(u^2-1)} + \frac{2(au-1)(a-u)\rho_2}{a(u^2-1)} \right) j_1 + 2i\rho_2 j_2 + 2i\rho_3 j_3 + \rho_3 \mathbb{1}, \tag{1.2.9}$$

where j_i are the 2 by 2 matrices representing J_i given explicitly by $j_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $j_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $j_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and ρ_i are the parameters entering in the definition (1.2.4) of the Heun operator. This matrix $c(x)$ is solution of

$$0 = [\ r_{21}(v,u)\ ,\ c_1(u)\ ] + [\ c_2(v)\ ,\ r_{12}(u,v)\ ]. \tag{1.2.10}$$

Such a scalar solution has been used to study the Gaudin model in a magnetic field (see [**28**]). It is easy to see that $\check{K}(x) = K(u) + c(u)$ satisfies also (1.2.8).

We introduce at this point the transfer matrix

$$t(u) = tr(\check{K}(u)^2). \tag{1.2.11}$$

The important feature of the transfer matrix is its commutativity property for different spectral parameters:

$$[t(u), t(v)] = 0. \tag{1.2.12}$$

We are now in a position to give the first result of this paper: amongst the conserved quantities of the Gaudin model (with one site) in a magnetic field, there is the Heun operator of Lie type (1.2.4). More explicitly

$$\frac{a}{8i(1-a^2)u} t(u)\Big|_{u=0} = W, \tag{1.2.13}$$

with $\rho_4 = \frac{2i(a^2+1)}{a^2-1}$ and $\rho_5 = \frac{\rho_1\rho_2}{2} + \frac{i\rho_2^2(1+a^2)}{2(1-a^2)}$. Since ρ_4 is a real parameter, a must be a pure phase. The parameter ρ_5 is given in terms of the other parameters but we can add the identity operator to recover the generic Heun operator.

The second result of this paper is the diagonalization of the Heun operator of Lie type W with the Bethe ansatz.

1.3. Modified algebraic Bethe ansatz

In this section, we recall the results of reference [**13**] and use them to diagonalize the operator W by the algebraic Bethe ansatz.

Gauge transformations. Let us introduce the following matrix

$$M(u) = \begin{pmatrix} 1/2 & 1/u \\ -u/2 & 1 \end{pmatrix}. \tag{1.3.1}$$

It allows to transform the r-matrix (1.2.5), the K-matrix (1.2.7) and the matrix $c(x)$ as follows

$$\widetilde{r}(u,v) = M_1(u)^{-1}M_2(v)^{-1}r(u,v)M_1(u)M_2(v) \tag{1.3.2}$$

$$= \frac{1}{(uv-1)((u-v)}\begin{pmatrix} -v(u^2-1) & 0 & \frac{2(uv-1)((u-v)}{u} & 0 \\ 0 & v(u^2-1) & -2v(v^2-1) & -\frac{2(uv-1)((u-v)}{u} \\ -\frac{u(uv-1)((u-v)}{2} & -\frac{2u^2(v^2-1)}{v} & v(u^2-1) & 0 \\ 0 & \frac{u(uv-1)((u-v)}{2} & 0 & -v(u^2-1) \end{pmatrix}, \tag{1.3.3}$$

and

$$\widetilde{K}(u) = M(u)^{-1}K(u)M(u), \quad \widetilde{c}(u) = M(u)^{-1}c(u)M(u). \tag{1.3.4}$$

It is easy to show that the matrices with tilde satisfy relations similar to those obeyed by $r(u,v)$, $K(u)$ and $c(u)$. Namely, one gets

$$[\ \widetilde{K}_1(u)\ ,\ \widetilde{K}_2(v)\ ] = [\ \widetilde{r}_{21}(v,u)\ ,\ \widetilde{K}_1(u)\ ] + [\ \widetilde{K}_2(v)\ ,\ \widetilde{r}_{12}(u,v)\ ]. \tag{1.3.5}$$

Moreover the gauge transformation does not modify the transfer matrix

$$t(u) = tr((\widetilde{K}(u) + \widetilde{c}(u))^2). \tag{1.3.6}$$

We introduce the new $\mathfrak{su}(2)$ generators: $\widetilde{J}_1 = -\frac{(a^2-1)(a^2-4)}{8a^2}J_1 + i\frac{(a^2+1(a^2-4)}{8a^2}J_2 + \frac{a^2+4}{4a}J_3$, $\widetilde{J}_2 = i\frac{(a^2-1)(a^2+4)}{8a^2}J_1 + \frac{(a^2+1)(a^2+4)}{8a^2}J_2 - i\frac{a^2-4}{4a}J_3$ and $\widetilde{J}_3 = -\frac{a^2-1}{2a}J_1 + i\frac{a^2-1}{2a}J_2$ so that

$$\widetilde{K}(u) = \begin{pmatrix} \frac{2a(u^2-1)}{(au-1)(a-u)}\widetilde{J}_3 & \frac{4}{u}\widetilde{J}_3 + \frac{2(a^2-1)a}{(au-1)(a-u)}\widetilde{J}_- \\ -u\widetilde{J}_3 + \frac{2u^2(a^2-1)}{a(au-1)(a-u)}\widetilde{J}_+ & -\frac{2a(u^2-1)}{(au-1)(a-u)}\widetilde{J}_3 \end{pmatrix}, \tag{1.3.7}$$

where $\widetilde{J}_+ = \widetilde{J}_1 + i\widetilde{J}_2$ and $\widetilde{J}_- = \widetilde{J}_1 - i\widetilde{J}_2$. We thus recover a particular case of the K-matrix used in [**13**] (see relations (4.1) and (4.2) in [**13**] for $L \to 1$, $x \to u$, $v_1 \to a$, $\alpha \to 1$, $\beta \to 0$ and $\gamma \to 0$).

Commutation relations. As is familiar in the context of algebraic Bethe ansatz, we define the operators $\widetilde{A}(u)$, $\widetilde{B}(u)$, $\widetilde{C}(u)$ and $\widetilde{D}(u)$ as follows

$$\widetilde{K}(u) + \widetilde{c}(u) = \begin{pmatrix} \widetilde{A}(u) & \widetilde{B}(u) \\ \widetilde{C}(u) & \widetilde{D}(u) \end{pmatrix}. \tag{1.3.8}$$

Only the special case $\widetilde{D}(u) = -\widetilde{A}(u)$ has been treated in [**13**]. It does not occur here because of the additional term $\widetilde{c}(u)$. We must therefore slightly generalizes the results of [**13**]. The commutation relations of these operators are computed from relation (1.3.5) and are given by

$$[\widetilde{A}(u),\widetilde{A}(v)]=[\widetilde{D}(u),\widetilde{D}(v)]=[\widetilde{A}(u),\widetilde{D}(v)]=0, \tag{1.3.9}$$

$$B(u,n)B(v,n+1)=B(v,n)B(u,n+1),\quad C(u,n)C(v,n-1)=C(v,n)C(u,n-1), \tag{1.3.10}$$

$$[\widetilde{A}(u),B(v,n)]=-[\widetilde{D}(u),B(v,n)]=\frac{2(u^2-1)}{(u-v)(vu-1)}(vB(v,n)-uB(u,n)), \tag{1.3.11}$$

$$[\widetilde{A}(u),C(v,n)]=-[\widetilde{D}(u),C(v,n)]=\frac{2(u^2-1)v^2}{(u-v)(vu-1)}\left(\frac{1}{u}C(u,n)-\frac{1}{v}C(v,n)\right), \tag{1.3.12}$$

$$\begin{aligned}C(u,n)B(v,n)&=B(v,n+1)C(u,n+1)+8n\frac{u}{v}\\&\quad+\frac{2u}{(u-v)(uv-1)v}\left(v(u^2-1)(\widetilde{A}(u)-\widetilde{D}(u))-u(v^2-1)(\widetilde{A}(v)-\widetilde{D}(v))\right),\end{aligned} \tag{1.3.13}$$

where $B(u,n)=\widetilde{B}(u)-\frac{2(2n-1)}{u}$ and $C(u,n)=\widetilde{C}(u)+\frac{(2n-1)u}{2}$.
Shifted transfer matrix. Following [13], we define the shifted transfer matrix

$$t(u,n)=\widetilde{A}(u)^2+B(u,n+1)C(u,n+1)+C(u,n)B(u,n)+\widetilde{D}(u)^2+2. \tag{1.3.14}$$

We can see that $t(u,0)=t(u)$ is the transfer matrix we are interested in. This shifted transfer matrix has the following commutation relation with $B(v,n)$:

$$\begin{aligned}t(u,n-1)B(v,n)-B(v,n)t(u,n)&=4B(v,n)\left(\frac{v(u^2-1)}{(u-v)(uv-1)}\left(\widetilde{A}(u)-\widetilde{D}(u)+4\frac{u^2+1}{u^2-1}\right)-2\right)\\&\quad-4\frac{u}{v}B(u,n)\left(\frac{u(v^2-1)}{(u-v)(uv-1)}\left(\widetilde{A}(v)-\widetilde{D}(v)+4\frac{v^2+1}{v^2-1}\right)-4(n-1)\right).\end{aligned} \tag{1.3.15}$$

Representation of $\mathfrak{su}(2)$. Denote by ω_s (with $s\in\mathbb{N}/2$) the highest weight vector of the spin s representation of $\mathfrak{su}(2)$ which satisfies

$$\widetilde{J}_3\,\omega_s=s\,\omega_s,\qquad \widetilde{J}_+\,\omega_s=0. \tag{1.3.16}$$

From the explicit form of the operators, one deduces that

$$\widetilde{A}(u)\omega_s=\alpha(u)\omega_s=\left(\rho_3+\frac{2as(u^2-1)}{(au-1)(a-u)}-\frac{i(a^2-1)(u^2+1)\rho_1}{2a(u^2-1)}-\frac{(a-u)^2+(au-1)^2}{2a(u^2-1)}\rho_2\right)\omega_s, \tag{1.3.17}$$

$$\widetilde{D}(u)\omega_s=\delta(u)\omega_s=\left(\rho_3-\frac{2as(u^2-1)}{(au-1)(a-u)}+\frac{i(a^2-1)(u^2+1)\rho_1}{2a(u^2-1)}+\frac{(a-u)^2+(au-1)^2}{2a(u^2-1)}\rho_2\right)\omega_s, \tag{1.3.18}$$

$$C(u,n)\omega_s=u\gamma_n\;\omega_s=u\left(-s+n-\frac{1}{2}-\frac{i(a^2-1)\rho_1}{4a}-\frac{(a^2+1)\rho_2}{4a}+\frac{i\rho_3}{2}\right)\omega_s. \tag{1.3.19}$$

Bethe vectors. We construct as follows the Bethe vectors that depend on the parameters $\boldsymbol{z} = \{z_1, z_2, \dots z_M\}$:

$$\mathbb{V}(\boldsymbol{z}) = B(z_1,1)B(z_2,2)\dots B(z_M,M)\omega_s. \tag{1.3.20}$$

Due to relation (1.3.10), the entries of the vector $\mathbb{V}(\boldsymbol{z})$ do not depend on the order of the parameters z_i. After some algebraic manipulations, we can express the action of the transfer matrix on the Bethe vector as follows

$$\begin{aligned} t(u)\mathbb{V}(\boldsymbol{z}) &= 2u\gamma_{M+1}B(z_1,1)B(z_2,2)\dots B(z_M,M)B(u,M+1)\omega_s \\ &+\mathcal{W}(u,\boldsymbol{z})\mathbb{V}(\boldsymbol{z}) \\ &-\sum_{k=1}^{M}\frac{16u^2(z_k^2-1)}{(u-z_k)(uz_k-1)z_k}\mathcal{U}_k(\boldsymbol{z})\mathbb{V}(\boldsymbol{z}_k,u) \end{aligned} \tag{1.3.21}$$

where $\mathbb{V}(\boldsymbol{z}_k,u) = B(z_1,1)\dots B(z_{k-1},k-1)B(u,k)B(z_{k+1},k+1)\dots B(z_M,M)\omega_s$ and

$$\mathcal{W}(u,\boldsymbol{z}) = \lambda(u) + \sum_{k=1}^{M}\frac{16z_k(u^2-1)}{(u-z_k)(uz_k-1)}\left(\frac{\alpha(u)-\delta(u)}{4} + \frac{u^2+1}{u^2-1} + \sum_{\substack{p=1\\p\neq k}}^{M}\frac{(u^2-1)z_kz_p}{u(z_k-z_p)(z_kz_p-1)}\right), \tag{1.3.22}$$

$$\lambda(u) = \alpha^2(u) + \delta^2(u) + 2 + 2u(\alpha'(u) - \delta'(u)) + 2\frac{u^2+1}{u^2-1}(\alpha(u)-\delta(u)), \tag{1.3.23}$$

$$\mathcal{U}_k(\boldsymbol{z}) = \frac{\alpha(z_k)-\delta(z_k)}{4} + \frac{z_k^2+1}{z_k^2-1} + \sum_{\substack{p=1\\p\neq k}}^{M}\frac{z_p(z_k^2-1)}{(z_k-z_p)(z_kz_p-1)}. \tag{1.3.24}$$

The prime in $\alpha'(u)$ and $\delta'(u)$ stands for the derivative with respect to u. In the usual algebraic Bethe ansatz, the first line in (1.3.21) is not present.

Modified algebraic Bethe ansatz and inhomogeneous Bethe equations. For generic values of the parameters, one gets $\gamma_{M+1} \neq 0$ for any M (the particular case where there exists a M such that $\gamma_M = 0$ is the object of Section 1.5). One must compute $B(z_1,1)B(z_2,2)\dots B(z_M,M)B(u,M+1)\omega_s$. For a generic M there is no simple formula but for $M = 2s$, one gets (see [13] for a proof of this result):

$$2uB(z_1,1)B(z_2,2)\dots B(z_{2s},2s)B(u,2s+1)\omega_s = \overline{\mathcal{W}}(u,\boldsymbol{z})\mathbb{V}(\boldsymbol{z}) - \sum_{k=1}^{2s}\frac{16u^2(z_k^2-1)}{(u-z_k)(uz_k-1)z_k}\overline{\mathcal{U}}_k(\boldsymbol{z})\mathbb{V}(\boldsymbol{z}_k,u) \tag{1.3.25}$$

where

$$\overline{\mathcal{W}}(u,\boldsymbol{z}) = -8\gamma_{2s+1}^*\prod_{p=1}^{2s}\frac{(u-a)(au-1)z_p}{a(u-z_p)(uz_p-1)}, \tag{1.3.26}$$

$$\overline{\mathcal{U}}_k(\boldsymbol{z}) = -\frac{\gamma_{2s+1}^*}{2(z_k^2-1)}\prod_{\substack{p=1\\p\neq k}}^{2s}\frac{(z_k-a)(az_k-1)z_p}{a(z_k-z_p)(z_kz_p-1)}, \tag{1.3.27}$$

and * is the complex conjugate. Then, the eigenvalues of the transfer matrix are

$$\mathcal{W}(u,\boldsymbol{z}) + \gamma_{2s+1}\overline{\mathcal{W}}(u,\boldsymbol{z}), \tag{1.3.28}$$

where $\mathcal{W}(u,\boldsymbol{z})$ is given by (1.3.22) with $M = 2s$ and $\overline{\mathcal{W}}(u,\boldsymbol{z})$ is given by (1.3.26) if $\boldsymbol{z}$ satisfies the inhomogeneous Bethe equations

$$\frac{\alpha(z_k) - \delta(z_k)}{4} + \frac{z_k^2+1}{z_k^2-1} + \sum_{\substack{p=1\\p\neq k}}^{2s} \frac{z_p(z_k^2-1)}{(z_k-z_p)(z_k z_p-1)} = \frac{|\gamma_{2s+1}|^2}{2(z_k^2-1)} \prod_{\substack{p=1\\p\neq k}}^{2s} \frac{z_p(z_k-a)(az_k-1)}{a(z_k-z_p)(z_k z_p-1)}. \tag{1.3.29}$$

1.4. Bethe roots and Heun operator

1.4.1. Inhomogeneous Heun differential equation

In this section, we study in more details the inhomogeneous Bethe equations obtained above.

Finding the Bethe roots of the inhomogeneous Bethe equations given by (1.3.29) is equivalent to computing the roots of the monic polynomial solution of degree $2s$ of the following differential Heun equation with an inhomogeneous term

$$y''(X) + \left(\frac{a_0}{X} + \frac{a_1}{X-1} + \frac{a_2}{X-A}\right) y'(X) + \frac{a_3(X-\mu)}{X(X-1)(X-A)} y(X) = |\gamma_{2s+1}|^2 \frac{(X-A)^{2s}}{X(X-1)} \tag{1.4.1}$$

where

$$A = -\frac{(a-1)^2}{4a}, \qquad a_0 = 1 - \frac{i(a^2-1)\rho_1}{4a} - \frac{(a-1)^2\rho_2}{4a}, \qquad a_1 = 1 - \frac{i(a^2-1)\rho_1}{4a} - \frac{(a+1)^2\rho_2}{4a},$$
$$a_2 = -2s, \qquad a_3 = |\gamma_{2s+1}|^2 - 2s(2s-1+a_0+a_1+a_2)\ . \tag{1.4.2}$$

The roots Z_i of the polynomials $y(X)$ and the Bethe roots z_i are linked by $Z_i = \frac{1}{4}(2-z_i-1/z_i)$. The parameter μ in the previous relation must be chosen so that there exists a polynomial solution. We come back to this point below.

The proof of this statement is quite standard since it is a generalization of the Heine–Stieljes problem. Let us recall the main steps here. Suppose that for a given μ there exists a polynomial solution $y(x)$ of degree $2s$ and denote by Z_i the $2s$ roots of this polynomial. The Heun differential equation (1.5.3) at $X = Z_i$ simplifies to

$$y''(Z_i) + \left(\frac{a_0}{Z_i} + \frac{a_1}{Z_i-1} + \frac{a_2}{Z_i-A}\right) y'(Z_i) = |\gamma_{2s+1}|^2 \frac{(Z_i-A)^{2s}}{Z_i(Z_i-1)}. \tag{1.4.3}$$

Defining z_i through $Z_i = \frac{1}{4}(2-z_i-1/z_i)$, one can show that (1.4.3) implies that z_i satisfy the Bethe equations.

The inverse is also true. For a given solution z_i of the Bethe equations (1.3.29), we define the following monic polynomial of degree $2s$:

$$y(X) = \prod_{i=1}^{2s}(X - \frac{1}{4}(2 - z_i - 1/z_i)). \tag{1.4.4}$$

One can prove that (1.4.3) holds for this polynomial and one deduces that the following polynomial

$$\begin{aligned} &X(X-1)(X-A)y''(X) + ((X-1)(X-A)a_0 + a_1X(X-A) + a_2X(X-1))\, y'(X) \\ &- |\gamma_{2s+1}|^2(X-A)^{2s+1} \end{aligned} \tag{1.4.5}$$

of degree $2s+1$ vanishes for $X = Z_i = \frac{1}{4}(2 - z_i - 1/z_i)$ which are the roots of $y(X)$. Therefore it is equal to $p(X)y(X)$ for some polynomial $p(x)$ of degree 1. By looking now at the term of degree $2s+1$ in equation (1.4.5), we conclude that $p(x)$ is of the form $p(x) = a_3(X - \mu)$ (with a_3 given by (1.4.2)) and that $y(X)$ satisfies the inhomogeneous differential Heun equation (1.4.1).

The goal is hence to find μ such that (1.4.1) has a monic polynomial solution of degree $2s$. To arrive at that, we start by putting the polynomial solution

$$y(X) = X^{2s} + \sum_{n=0}^{2s-1} c_n X^n \tag{1.4.6}$$

in the Heun equation (1.4.1). We obtain the following constraints for the coefficients c_n, for $0 \le n \le 2s+1$,

$$\begin{aligned} &A(n+1)(n+a_0)c_{n+1} - n((1+A)(a_0+n-1) + a_1A + a_2)c_n + |\gamma_n|^2 c_{n-1} \\ &= \mu a_3 c_n + |\gamma_{2s+1}|^2 \binom{2s+1}{n} (-A)^{2s+1-n} , \end{aligned} \tag{1.4.7}$$

with the conventions $c_{2s} = 1$ and $c_{-1} = c_{2s+1} = c_{2s+2} = 0$. The previous relation is directly satisfied for $n = 2s+1$ given the explicit value of a_3. Relation (1.4.7) for $n = 2s$ gives c_{2s-1} as a polynomial of degree 1 with respect to μ. Then, by recurrence, relation (1.4.7) for $n = 2s - p$ ($p = 1, \ldots 2s-1$) gives c_{2s-p-1} as a polynomial of degree $p+1$ with respect to μ. Finally, relation (1.4.7) for $n = 0$ implies that the resulting polynomial $P_{2s+1}(\mu)$ of degree $2s+1$ in μ must vanish.

The previous discussion allows to conclude that for each solution μ of the following relation

$$P_{2s+1}(\mu) = 0, \tag{1.4.8}$$

there exists a polynomial $y_\mu(X)$ solution of the Heun equation (1.4.1) of degree $2s$. The $2s$ roots of the polynomial $y_\mu(X)$ provide a solution of the Bethe equations (1.3.29). As

$P_{2s+1}(\mu)$ is of degree $2s+1$, there are $2s+1$ different μ and we obtain $2s+1$ different solutions of the Bethe equations. This proves that the spectrum obtained from the modified algebraic Bethe ansatz is complete.

Let us remark that in the discussion above, we have assumed that $P_{2s+1}(\mu)$ has simple roots. We can expect this to be true for generic parameters. As far as we know, there is no closed formula for $P_{2s+1}(\mu)$ but it is easy to compute this polynomial from the relations (1.4.7).

1.4.2. Eigenvalues of the algebraic Heun operator

Using the identification (1.2.13) between the Gaudin model Hamiltonian and the algebraic Heun operator, we can conclude that the eigenvalues of the algebraic Heun W are given by

$$w = \frac{2is|\gamma_{2s}|^2(a-1)}{a+1} - \frac{4ai|\gamma_{2s}|^2}{1-a^2}\sum_{j=1}^{2s} Z_j + \frac{\rho_1\rho_2}{2} + \frac{s(a^2+1)\rho_1 - is(a^2-1)\rho_2}{2a} + \frac{i(a^2+1)(4s^2-\rho_2^2)}{2(a^2-1)} \tag{1.4.9}$$

where $\{Z_i\}$ are the roots of a polynomial solution of the inhomogeneous differential Heun equation (1.4.1) or equivalently $Z_i = \frac{1}{4}(2 - z_i - 1/z_i)$ with $\{z_i\}$ the Bethe roots of (1.3.29).

In addition, remarking that $-\sum_{i=1}^{2s} Z_i$ is equal to the coefficient c_{2s-1} in (1.4.6), we can express this sum in terms of μ. Indeed, as explained before, relation (1.4.7) for $n = 2s$ gives c_{2s-1} as a polynomial in μ of degree 1. Therefore, we can conclude that

$$w = \frac{4aia_3\mu}{1-a^2} + \rho_1\left(\frac{\rho_2}{2} - s\right) - \frac{i(a^2+1)(\rho_2^2 - 2s(s+1))}{2(a^2-1)} + \frac{i(a-1)(\rho_2 - |\gamma_{2s+1}|^2)}{(a+1)} + \frac{2is(s-1)a}{a^2-1}. \tag{1.4.10}$$

This relation allows to view w, the eigenvalues of the algebraic Heun operator, as the parameters in the differential inhomogeneous Heun operator (1.4.1) such that this equation has a polynomial solution. Similar results can be obtained directly from the Bargmann representation of $\mathfrak{su}(2)$ as explained in the following subsection.

1.4.3. Bargmann realization of the algebraic Heun operator of Lie type

It is well-known that the spin s representation of $\mathfrak{su}(2)$ can be realized in terms of differential operators acting on the space of univariate polynomials of order less or equal to $2s$:

$$J_1 = \frac{1-z^2}{2}\frac{d}{dz} + sz, \qquad J_2 = \frac{1+z^2}{2i}\frac{d}{dz} - \frac{sz}{i}, \qquad J_3 = s - z\frac{d}{dz}. \tag{1.4.11}$$

Indeed, the actions on the monomials

$$|s,m\rangle = \frac{z^{s-m}}{\sqrt{(s-m)!(s+m)!}}, \quad \text{for } -s \leq m \leq s, \tag{1.4.12}$$

are given by

$$J_{\pm}|s,m\rangle = (J_1 \pm iJ_2)\,|s,m\rangle = \sqrt{(s\mp m)(s\pm m+1)}|s,m\pm 1\rangle, \qquad J_3|s,m\rangle = m|s,m\rangle, \tag{1.4.13}$$

which is the standard spin s representation of $\mathfrak{su}(2)$.

The algebraic Heun operator W in this realization becomes:

$$W = \frac{i(z^2-1)(z^2a^2-1)}{(a^2-1)}\frac{d^2}{dz^2} + \left(\frac{1-2s}{2}(z^3(\rho_4+2i)-\rho_4 z) - z^2\frac{\rho_1+i\rho_2}{2} - z\rho_3 + \frac{\rho_1-i\rho_2}{2}\right)\frac{d}{dz}$$
$$+ z^2\left(s^2\rho_4 + 2is^2 - \frac{s\rho_4}{2} - is\right) + sz(\rho_1+i\rho_2) + s\rho_3 + \frac{s\rho_4}{2} + \rho_5 \tag{1.4.14}$$

with $\rho_4 = \frac{2i(a^2+1)}{a^2-1}$ and $\rho_5 = \frac{\rho_1\rho_2}{2} + \frac{i\rho_2^2(1+a^2)}{2(1-a^2)}$. Then, the eigenvalues w of the algebraic Heun operator W can be seen as the eigenvalues of this differential operator

$$W\phi(z) = w\phi(z). \tag{1.4.15}$$

Thus, w is obtained by asking that there is a polynomial solution $\phi(z)$ of degree less or equal to $2s$ of this differential equation. This operator W leads to a Fuchsian second order differential equation with five regular singularities $\{1,-1,1/a,-1/a,\infty\}$, but in the special case where $\rho_1=\rho_2=0$, it reduces to the differential Heun operator. Indeed, let us perform the change of variable

$$y = a^2z^2 \quad \text{and} \qquad \psi(y) = \phi(a^2z^2). \tag{1.4.16}$$

In this new variable when $\rho_1=\rho_2=0$, (1.4.15) becomes the Heun differential equation

$$\frac{d^2\psi}{dy^2} + \left(\frac{1}{2y} + \frac{1-2s-i\rho_3}{2(y-1)} + \frac{1-2s+i\rho_3}{2(y-a^2)}\right)\frac{d\psi}{dy}$$
$$+ \frac{2(2s-1)sy + is\rho_3(1-a^2) + s(a^2+1)}{4y(y-1)(y-a^2)}\psi = \frac{iw(1-a^2)}{4y(y-1)(y-a^2)}\psi. \tag{1.4.17}$$

Comparing this differential Heun operator obtained from the Bargmann realisation and the one in (1.4.1) obtained from the Bethe ansatz, we see that one singularity of the former is located at a^2 whereas for the latter one singularity is at $A = -\frac{(a-1)^2}{4a}$.

1.5. Homogeneous case

In the previous sections, we have explained how the modified algebraic Bethe ansatz can be used to diagonalize the transfer matrix of the Gaudin model and the Heun operator, that leads to inhomogeneous Bethe equations. However, for particular values of the parameters, there exist homogeneous Bethe equations which diagonalize also this model. Indeed, let us suppose that s, a, ρ_1, ρ_2 and ρ_3 are such that there is an integer $0 \leq \mathcal{M} \leq 2s-1$ satisfying $\gamma_{\mathcal{M}+1} = 0$ *i.e.*

$$\begin{cases} \mathcal{M} = s - \dfrac{1}{2} + \dfrac{i(a^2-1)\rho_1}{4a} + \dfrac{(a^2+1)\rho_2}{4a} \\ \rho_3 = 0 \end{cases}, \tag{1.5.1}$$

where the real parameter ρ_3 is set to zero to ensure that the imaginary part of the relation $\gamma_{\mathcal{M}+1} = 0$ is verified. In this case, some eigenvalues of the transfer matrix are given by $\mathcal{W}(u,\boldsymbol{z})$ (relation (1.3.22) for $M = \mathcal{M}$), if $\boldsymbol{z}$ satisfies the Bethe equations $\mathcal{U}_k(\boldsymbol{z}) = 0$ *i.e.*

$$\frac{1}{4}(\alpha(z_k) - \delta(z_k)) + \frac{z_k^2+1}{z_k^2-1} + \sum_{\substack{p=1 \\ p\neq k}}^{\mathcal{M}} \frac{z_p(z_k^2-1)}{(z_k-z_p)(z_k z_p - 1)} = 0. \tag{1.5.2}$$

In this case, the Bethe equations are called homogeneous. As before, finding the Bethe roots z_k, solutions of the Bethe equations (1.5.2) is the same as finding polynomial solutions of a differential equation. More precisely, it amounts to finding polynomial solutions of degree $\mathcal{M}$ of the homogeneous Heun differential equation

$$y''(X) + \left(\frac{a_0}{X} + \frac{a_1}{X-1} + \frac{a_2}{X-A}\right) y'(X) + \mathcal{M}^2 \frac{(X-\mu)}{X(X-1)(X-A)} y(X) = 0, \tag{1.5.3}$$

for a suitable μ, where

$$A = -\frac{(a-1)^2}{4a}, \qquad a_0 = s + \frac{1}{2} - \mathcal{M} + \frac{\rho_2}{2}, \qquad a_1 = s + \frac{1}{2} - \mathcal{M} - \frac{\rho_2}{2}, \qquad a_2 = -2s. \tag{1.5.4}$$

If $\{Z_i\}$ are the $\mathcal{M}$ roots of a polynomial solution of the Heun differential equation, then, for each Z_i, take z_i as one solution of $Z_i = \frac{1}{4}(2 - z_i - 1/z_i)$, then these $\{z_i\}$ are solutions of the Bethe equations (1.5.2).

The parameter μ must be chosen such that (1.5.3) has a polynomial solution of degree $\mathcal{M}$. If we take the polynomial solution of the form

$$y(X) = X^{\mathcal{M}} + \sum_{n=0}^{\mathcal{M}-1} c_n X^n \tag{1.5.5}$$

upon inserting this expression in the Heun equation (1.5.3), we obtain the following constraints for the coefficients c_n, for $0 \leq n \leq \mathcal{M}+1$,

$$A(n+1)(n+a_0)c_{n+1} - n((1+A)(a_0+n-1) + a_1 A + a_2)c_n + (\mathcal{M}+1-n)^2 c_{n-1} = \mu \mathcal{M}^2 c_n, \tag{1.5.6}$$

with the conventions $c_{\mathcal{M}} = 1$ and $c_{-1} = c_{\mathcal{M}+1} = c_{\mathcal{M}+2} = 0$. Again, these equations are consistent only if μ is a root of the polynomial $P_{\mathcal{M}+1}(\mu)$ of degree $\mathcal{M}+1$ in μ.

When the constraints (1.5.1) are satisfied, the eigenvalues of the algebraic Heun operator W (1.2.4) can also be deduced from those of the Gaudin transfer matrix thanks to formula

(1.2.13), they are given by

$$w = \frac{i}{a^2-1}\Big(s(a^2+1) - 2\mathcal{M}a - a\rho_2\Big) + 4\frac{ia}{a^2-1}\sum_{i=1}^{\mathcal{M}} Z_i\ , \tag{1.5.7}$$

where $\{Z_i\}$ are the roots of a polynomial solution of the differential Heun equation (1.5.3) or equivalently $Z_i = \frac{1}{4}(2 - z_i - 1/z_i)$ with $\{z_i\}$ the Bethe roots of (1.5.2).

Since $-\sum_{i=1}^{\mathcal{M}} Z_i$ is equal to the coefficient $c_{\mathcal{M}-1}$ in (1.4.6), we can express this sum in terms of μ as follows

$$i\sum_{i=1}^{\mathcal{M}} Z_i = -ic_{\mathcal{M}-1} = -i(\mu\mathcal{M}^2 + \mathcal{M}((1+A)(a_0+\mathcal{M}-1) + a_1A + a_2))\ . \tag{1.5.8}$$

Then, the eigenvalues of the algebraic Heun operator W (1.2.4) become

$$w = \frac{4ia\mu\mathcal{M}^2}{1-a^2} + \frac{i}{1-a^2}\Big(((a^2+1)s + a\rho_2)(2\mathcal{M}+1) - \mathcal{M}^2(a-1)^2\Big)\,, \tag{1.5.9}$$

where μ are the roots of the polynomials $P_{\mathcal{M}+1}(\mu)$.

The previous construction provides $\mathcal{M}+1$ eigenvalues. They belong to the spin range $\{s-\mathcal{M}, s-\mathcal{M}+1, \dots, s\}$. We remark that this spin domain is stabilized by the algebraic Heun operator since (when relations (1.5.1) hold)

$$W_{\mathcal{M}+1,\mathcal{M}+2}W_{\mathcal{M}+2,\mathcal{M}+1} = 0. \tag{1.5.10}$$

To obtain the second part of the spectrum, we must start from the Bethe vectors $\overline{\mathbb{V}}(\overline{\boldsymbol{z}})$ constructed from the lowest weight of $\mathfrak{su}(2)$. We give the definition and the useful formulas in Appendix 1.A. We note that for $M = 2s-1-\mathcal{M}$, one gets $\beta_{-M} = 0$ and relation (1.A.7) simplifies since the first line vanishes. Therefore, the usual Bethe ansatz works and one finds that $\overline{\mathcal{W}}(u,\overline{\boldsymbol{z}})$ (see (1.A.8)) for $M = 2s-1-\mathcal{M}$ is an eigenvalue if $\overline{\boldsymbol{z}}$ satisfies the Bethe equations,

$$\frac{\overline{\delta}(z_k) - \overline{\alpha}(z_k)}{4} + \frac{\overline{z}_k^2+1}{\overline{z}_k^2-1} + \sum_{\substack{p=1\\p\neq k}}^{2s-1-\mathcal{M}} \frac{\overline{z}_p(\overline{z}_k^2-1)}{(\overline{z}_k-\overline{z}_p)(\overline{z}_k\overline{z}_p-1)} = 0, \tag{1.5.11}$$

for $1 \le k \le 2s-1-\mathcal{M}$. Once more, finding the Bethe roots $\overline{\boldsymbol{z}}$ is equivalent to finding a polynomial solution of a differential Heun equation. In this way, we prove that we obtain $2s-\mathcal{M}$ solutions for spins running from $-s$ to $s-\mathcal{M}-1$ in unit steps.

From the Bethe vectors $\mathbb{V}(\boldsymbol{z})$ and $\overline{\mathbb{V}}(\overline{\boldsymbol{z}})$, we obtain the complete spectrum of the Gaudin model or of the algebraic Heun operators when relations (1.5.1) hold.

1.6. Representations of the rotation group $O(3)$ and Heun operator

A special and important case of the Heun operator of Lie type arises when $\rho_1 = \rho_2 = \rho_3 = 0$:

$$W|_{\rho_1=\rho_2=\rho_3=0} = \{J_1, J_2\} + \frac{2i(a^2+1)}{a^2-1} J_1^2. \tag{1.6.1}$$

It corresponds to the situation where there is no external magnetic field for the Gaudin magnet (*i.e.* the scalar matrix $c(u)$ given in (1.2.9) vanishes). Consider the following rotation of the generators of $\mathfrak{su}(2)$

$$J_1 = \cos(\theta)\bar{J}_1 - \sin(\theta)\bar{J}_2, \quad \text{and} \qquad J_2 = \sin(\theta)\bar{J}_1 + \cos(\theta)\bar{J}_2, \tag{1.6.2}$$

with $a = e^{2i\theta}$. In terms of the generators $\bar{J}_1$ and $\bar{J}_2$, the algebraic Heun operator reads

$$E := \frac{4i(1-a)}{1+a} W|_{\rho_1=\rho_2=\rho_3=0} = 4(\bar{J}_1{}^2 + r\bar{J}_2{}^2), \qquad r = \left(\frac{1-a}{1+a}\right)^2. \tag{1.6.3}$$

The operator E occurs in many physical and mathematical contexts. It is seen to be equivalent (up to an affine transformation) to the Hamiltonian of the quantum Euler top [29]. It also appears in the representation theory of the group $O(3)$ and its universal covering group $SU(2)$ as follows. As is very familiar, the standard representation basis is defined by the joint eigenvectors of the Casimir element $C = J_1^2 + J_2^2 + J_3^2$ and of the generator J_3. It is said to be of subgroup type since it corresponds to the group reduction $O(3) \supset O(2)$ with one subgroup generator, J_3, diagonalized. Close to 50 years ago, Patera and Winternitz stressed [25] the existence of a second interesting basis stemming from the classification of second order polynomials in the generators. This second basis is provided by the eigenfunctions of C and $E = 4(J_1^2 + rJ_2^2)$ (and of the discrete operators X and PZ, where X and Z correspond to reflections in the yz and xy planes and P is the parity operator). It is not of subgroup type as E is not the generator of any subgroup of $O(3)$. Ways to obtain the eigenvalues of E are described in [25] but none are providing a closed formula. Our construction via the Bethe ansatz strikingly advances the characterization of the eigenvalues ϵ of E. Indeed, by specializing formulas (1.3.29) and (1.4.9), one gets that the eigenvalues of E are given by

$$\epsilon = \frac{2s(4s^2+1)(a^2+1)}{(a+1)^2} - \frac{4a(s-\frac{1}{2})^2}{(a+1)^2} \sum_{p=1}^{2s} (z_p + z_p^{-1}), \tag{1.6.4}$$

with the Bethe roots satisfying

$$\frac{as(z_k^2-1)}{(az_k-1)(a-z_k)} + \frac{z_k^2+1}{z_k^2-1} + \sum_{\substack{p=1\\p\neq k}}^{2s} \frac{z_p(z_k^2-1)}{(z_k-z_p)(z_kz_p-1)} = \frac{(2s+1)^2}{8(z_k^2-1)} \prod_{\substack{p=1\\p\neq k}}^{2s} \frac{z_p(z_k-a)(az_k-1)}{a(z_k-z_p)(z_kz_p-1)}. \tag{1.6.5}$$

Let us also remark that if s is a half-integer, the result of Section 1.5 can also be used to characterize the spectrum of E. In this case the eigenvalues read

$$\epsilon = \frac{4s(a^2+1)}{(a+1)^2} - \frac{4a}{(a+1)^2}\sum_{p=1}^{s-\frac{1}{2}}(z_p + z_p^{-1}), \tag{1.6.6}$$

with the Bethe roots satisfying

$$\frac{as(z_k^2-1)}{(az_k-1)(a-z_k)} + \frac{z_k^2+1}{z_k^2-1} + \sum_{\substack{p=1\\p\neq k}}^{s-1/2}\frac{z_p(z_k^2-1)}{(z_k-z_p)(z_kz_p-1)} = 0. \tag{1.6.7}$$

The specialization used here of relation (1.4.17) for the Bargmann realization of the Heun operator reproduces the result of [25]. We should point out that the supplementary condition $0 < r < 1$ is imposed in [25] which implies that $a = \frac{1-\sqrt{r}}{1+\sqrt{r}}$. In the construction above we have required that a be a pure phase but the results can easily be generalized and the relations (1.6.4)-(1.6.7) are still valid for any a.

1.7. Entanglement entropy for the Krawtchouk chain and Heun operator

In this section, we present another interesting problem to which the results of this paper can be applied, namely the determination of the entanglement entropy for free Fermions on Krawtchouk chains. These chains are interesting in many respects. They emerge from the projection of spin systems defined on hypercubes [11] and they are known to allow for perfect state transfer [1]. They are usually introduced in the following way. Let us consider the free Fermion inhomogeneous Hamiltonian with nearest neighbor interactions and local magnetic fields defined by:

$$\hat{\mathcal{H}} = \frac{\beta}{2}\sum_{n=0}^{2s-1}\sqrt{(n+1)(2s-n)}(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) - \alpha\sum_{n=0}^{2s}(n-s)c_n^\dagger c_n, \tag{1.7.1}$$

where $\{c_m^\dagger, c_n\} = \delta_{mn}$ and where α and β are free parameters. We can choose the normalization of $\mathcal{H}$ such that $\alpha^2+\beta^2 = 1$ and take $\alpha = \cos(2\theta)$ and $\beta = \sin(2\theta)$. In what follows, it will be more convenient to rewrite (1.7.1) in terms of the matrices J_1 and J_3 from an irreducible representation of $\mathfrak{su}(2)$ of dimension $2s+1$. Indeed, we have that

$$\hat{\mathcal{H}} = \begin{pmatrix} c_0^\dagger & \dots & c_{2s}^\dagger \end{pmatrix} \hat{H} \begin{pmatrix} c_0 \\ \vdots \\ c_{2s} \end{pmatrix}, \tag{1.7.2}$$

where

$$\hat{H} = \cos(2\theta)J_3 + \sin(2\theta)J_1. \tag{1.7.3}$$

We easily find an orthonormal basis $\{|\omega_k\rangle\}_{k\in\{0,\dots,2s\}}$ of $\mathbb{C}^{2s+1}$ such that $\hat{H}\,|\omega_k\rangle = \omega_k\,|\omega_k\rangle$. The eigenvalues are given by $\omega_k = k - s$ and the diagonalized Hamiltonian is

$$\hat{\mathcal{H}} = \sum_{k=0}^{2s} \omega_k \tilde{c}_k^\dagger \tilde{c}_k,$$

where the $\tilde{c}_k = \sum_{n=0}^{2s} \langle n|\omega_k\rangle\, c_n$ also respect $\{\tilde{c_j}, \tilde{c_k}^\dagger\} = \delta_{jk}$. The overlap coefficients are given in terms of Krawtchouk polynomials [24]:

$$\langle n|\omega_k\rangle = \sqrt{\binom{2s}{n}\binom{2s}{2s-k}}(\sin\theta)^{2s}\ |\cot\theta|^{k-n} K_n(2s-k;\sin^2\theta, 2s). \tag{1.7.4}$$

The ground state $|\Psi_0\rangle\rangle$ is defined by filling up the Fermi sea. If $|0\rangle\rangle$ represents the vacuum state which is annihilated by all the $\tilde{c}_k$, we have

$$|\Psi_0\rangle\rangle = \tilde{c}_K^\dagger \tilde{c}_{K-1}^\dagger \dots \tilde{c}_1^\dagger \tilde{c}_0^\dagger\ |0\rangle\rangle. \tag{1.7.5}$$

K is taken to be the largest integer k such that states associated to a negative energy ω_k are filled. A natural question to ask about such chain is the following: if we split it in two, what is the entanglement between the two parts? This information is contained in the entanglement entropy:

$$S = \mathrm{tr}(\rho \ln(\rho)), \tag{1.7.6}$$

where ρ is the reduced density matrix associated to one of the two parts. Here, we take the subsystem to be the first $l+1$ sites of the chain. In particular, the projection operator π_ℓ over the subsystem is taken to be

$$\pi_\ell = \sum_{n=0}^{\ell} |n\rangle\langle n|\,. \tag{1.7.7}$$

For a Krawtchouk chain in its ground state $|\Psi_0\rangle\rangle$, it is known that the value of S can also be extracted from the eigenvalues of the chopped correlation matrix [17]. This matrix is obtained by first considering the complete correlation matrix, which is the $(2s+1)\times(2s+1)$ matrix having for entries:

$$\hat{C}_{mn} = \langle\langle \Psi_0|c_m^\dagger c_n|\Psi_0\rangle\rangle. \tag{1.7.8}$$

It is useful to note that $\hat{C}$ can be expressed as:

$$\hat{C} = \sum_{k=0}^{K} |\omega_k\rangle\langle\omega_k|\,. \tag{1.7.9}$$

Then, the chopped correlation matrix associated to the first $\ell+1$ sites in the chain is obtained by considering the submatrix of $\hat{C}$ defined as:

$$C = \pi_\ell\ \hat{C}\ \pi_\ell = |\hat{C}_{mn}|_{0\le m,n\le \ell}. \tag{1.7.10}$$

Given (1.7.4), we see that

$$C_{mn} = \sum_{k=0}^{K} \sqrt{\binom{2s}{n}\binom{2s}{m}} \binom{2s}{k} (\sin^2\theta)^{2s} |\cot\theta|^{2k-n+m} K_m(2s-k)K_n(2s-k), \qquad (1.7.11)$$

where the last two parameters of the Krawtchouk polynomials are kept implicit.

In general, we see that the submatrices of (1.7.9) have a rather complicated expression and do not prove easy to diagonalize. However, considerations of bispectrality in the context of the time and band limiting problem have shown how to identify a tridiagonal operator T, in fact an algebraic Heun operator [17, 21], that has the property of commuting with the chopped correlation matrix. It is expressed as

$$T = \{\hat{H}, J_3\} + \mu J_3 + \nu \hat{H}, \qquad (1.7.12)$$

where $\mu = -2K - 1 + 2s$ and $\nu = 2\ell + 1 - 2s$ [16]. Thus, one could instead diagonalize T and use the results to extract the spectrum of C. This is where the modified algebraic Bethe ansatz comes into play. If we make the identification

$$-\sin(2\theta)J_1 + \cos(2\theta)J_3 \to \tilde{J}_1, \quad \cos(2\theta)J_1 + \sin(2\theta)J_3 \to \tilde{J}_2 \qquad (1.7.13)$$

for the generators and

$$\begin{aligned} &\rho_1 = \cot(2\theta)\mu + \frac{\nu}{\sin(2\theta)}, \qquad \rho_2 = \mu, \qquad \rho_3 = 0, \\ &\rho_4 = 2\cot 2\theta \qquad \text{and} \qquad \rho_5 = \frac{\mu\nu}{2\sin 2\theta} \end{aligned} \qquad (1.7.14)$$

for the parameters, we see that

$$T = \sin 2\theta (W - \rho_5), \qquad (1.7.15)$$

where W is defined in (1.2.3). We thus see that the spectrum of T is given, up to an affine transformation, by the one of W.

Moreover, it is observed that this W belongs to the homogeneous case (discussed in Section 5). Indeed, one can check that we here have $\mathcal{M} = \ell$ when considering the Bethe vector constructed with the lowest weight. Therefore, the $\ell + 1$ eigenvalues $t_{\bar{z}}$ of T will be given by:

$$t_{\bar{z}} = s\cos(2\theta) + \frac{\mu}{2} - \frac{1}{2}\sum_{i=1}^{\ell}(\bar{z}_i + \frac{1}{\bar{z}_i}), \qquad (1.7.16)$$

where the $\bar{\boldsymbol{z}}$ are solutions to the Bethe equations given by (1.5.11):

$$\frac{e^{2i\theta}s(\bar{z}_k^2 - 1)}{(e^{2i\theta}\bar{z}_k - 1)(e^{2i\theta} - \bar{z}_k)} + \frac{(\bar{z}_k^2 + 1)(1 - \frac{\nu}{2}) - \mu\bar{z}_k}{(\bar{z}_k^2 - 1)} + \sum_{\substack{p=1 \\ p\neq k}}^{l} \frac{\bar{z}_p(\bar{z}_k^2 - 1)}{(\bar{z}_k - \bar{z}_p)(\bar{z}_k\bar{z}_p - 1)} = 0. \qquad (1.7.17)$$

At this point, it is possible to recover the spectrum of C. Usually, this is done by acting with C on the eigenvectors of T, *i.e.* the Bethe vectors for W [14]. Since these are also eigenvectors of the chopped correlation matrix, we can read the eigenvalues of C from the results. We here want to present an alternative way of obtaining the spectrum. It amounts to constructing a polynomial P of order ℓ such that

$$C = P(T) = \sum_{j=0}^{\ell} a_j T^j. \tag{1.7.18}$$

If we have P, then it is easy to see that $P(t_{\boldsymbol{z}})$ gives the eigenvalue of C associated to $\boldsymbol{z}$. To prove the existence of P, it is sufficient to know that $[C,T] = 0$ is verified and to notice that, since T is tridiagonal, the first $\ell + 1$ powers of T are linearly independent. The fact that T is tridiagonal also allows to determine P. Indeed, since we have that

$$\langle 0| T^r |m\rangle = 0 \qquad \text{if } r < m, \tag{1.7.19}$$

we are led to

$$a_\ell = \langle 0| C |\ell\rangle \,/\, \langle 0| T^\ell |\ell\rangle \tag{1.7.20}$$

and to the recurrence relation:

$$a_{\ell-j} = \frac{\Big[\langle 0| C |\ell - j\rangle - \sum_{r=0}^{j-1} a_{\ell-r} \langle 0| T^{\ell-r} |\ell - j\rangle \Big]}{\langle 0| T^{\ell-j} |\ell - j\rangle}. \tag{1.7.21}$$

Given this relation, constructing P is straightforward and obtaining the eigenvalues of C and the entanglement entropy follows.

1.8. Conclusion and outlook

This paper has woven threads between the algebraic Heun operator of Lie type, the differential Heun operator, a Fuchsian second order differential equation with five regular singularities, the inhomogeneous Bethe equation and the Gaudin magnet. It showed in particular that there exist two equivalent ways to compute the eigenvalues of the algebraic Heun operator of Lie type by studying the polynomial solution of two differential equations, one inhomogeneous with four singularities and the other homogeneous with five singularities. This type of equivalence has already been noticed in a different context [15] and certainly deserves further investigation.

We have examined the Gaudin model with only one site. It is well-known however that the model with N sites remains integrable; the Bethe equations are also given in [13]. In the framework of the Bargmann realization, a differential equation with N variables is obtained in that case. A natural question is to compare this differential equation with the one obtained from the Bethe equations. This would provide a generalization to N variables of the results of this paper.

We showed that the tools of quantum integrable systems and the algebraic Heun operators of Lie type are relevant to the representation theory of $\mathfrak{su}(2)$. We expect that other quantum integrable systems and algebraic Heun operators of different types will find their way in the representation theory of higher rank Lie algebras. For example, algebraic Heun–Hahn or Racah–Hahn operators appear in [18] in the study of the diagonal centralizer of two $\mathfrak{su}(3)$.

Finally, we demonstrated the usefulness of the Bethe ansatz to compute the entanglement entropy of free Fermion chain with couplings given by the recurrence coefficients of the Krawtchouk polynomials. It would be interesting to use this approach for other chains. Algebraic Heun operators of different type which commute with the chopped correlation matrix associated to different inhomogeneous free Fermion chains have been introduced in [16,17]. The computation of their spectra with the Bethe ansatz could be used to determine the entanglement entropy at least in the thermodynamical limit. We plan on returning to some of these questions.

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1.A. Bethe ansatz from the lowest weight

The commutation relation of the shifted transfer matrix with $C(v,n)$ is given by

$$\begin{aligned} t(u,n)C(v,n) - C(v,n)t(u,n-1) = {}& 4C(v,n)\left(\frac{v(u^2-1)}{(u-v)(uv-1)}\left(\widetilde{D}(u) - \widetilde{A}(u) + 4\frac{u^2+1}{u^2-1}\right) - 2\right) \\ & -4\frac{v}{u}C(u,n)\left(\frac{u(v^2-1)}{(u-v)(uv-1)}\left(\widetilde{D}(v) - \widetilde{A}(v) + 4\frac{v^2+1}{v^2-1}\right) + 4n\right). \end{aligned} \tag{1.A.1}$$

Let $\overline{\omega}_s$ (with $s \in \mathbb{N}/2$) denote the lowest weight vector of the spin s representation of $\mathfrak{su}(2)$ which satisfies

$$\widetilde{J}_3\overline{\omega}_s = -s\overline{\omega}_s, \qquad \widetilde{J}_-\overline{\omega}_s = 0. \tag{1.A.2}$$

From the explicit form of the operators, one deduces that

$$\widetilde{A}(u)\overline{\omega}_s = \overline{\alpha}(u)\overline{\omega}_s = \Big(\rho_3 - \frac{2as(u^2-1)}{(au-1)(a-u)} - \frac{i(a^2-1)(u^2+1)\rho_1}{2a(u^2-1)} - \frac{(a-u)^2+(au-1)^2}{2a(u^2-1)}\rho_2\Big)\overline{\omega}_s, \tag{1.A.3}$$

$$\widetilde{D}(u)\overline{\omega}_s = \overline{\delta}(u)\overline{\omega}_s = \Big(\rho_3 + \frac{2as(u^2-1)}{(au-1)(a-u)} + \frac{i(a^2-1)(u^2+1)\rho_1}{2a(u^2-1)} + \frac{(a-u)^2+(au-1)^2}{2a(u^2-1)}\rho_2\Big)\overline{\omega}_s, \tag{1.A.4}$$

$$B(u,n)\overline{\omega}_s = \frac{1}{u}\beta_n\overline{\omega}_s = \frac{4}{u}\left(-s-n+\frac{1}{2}+\frac{i(a^2-1)\rho_1}{4a}+\frac{(a^2+1)\rho_2}{4a}+\frac{i\rho_3}{2}\right)\overline{\omega}_s. \tag{1.A.5}$$

We construct as follows the Bethe vectors, that depend on the parameters $\overline{\boldsymbol{z}} = \{\overline{z}_1, \overline{z}_2, \dots, \overline{z}_M\}$,

$$\overline{\mathbb{V}}(\overline{\boldsymbol{z}}) = C(\overline{z}_1,0)C(\overline{z}_2,-1)\dots C(\overline{z}_M,-M+1)\overline{\omega}_s. \tag{1.A.6}$$

Due to relation (1.3.10), the entries of the vector $\overline{\mathbb{V}}(\overline{\boldsymbol{z}})$ do not depend on the order of the parameters $\overline{z}_i$. The action of the transfer matrix on the Bethe vector $\overline{\mathbb{V}}(\overline{\boldsymbol{z}})$ is given by

$$\begin{aligned} t(u)\overline{\mathbb{V}}(\overline{\boldsymbol{z}}) \;=\; & \frac{2}{u}\beta_{-M}C(\overline{z}_1,0)C(\overline{z}_2,-1)\dots C(\overline{z}_M,-M+1)C(u,-M)\overline{\omega}_s \\ & +\overline{\mathcal{W}}(u,\overline{\boldsymbol{z}})\overline{\mathbb{V}}(\overline{\boldsymbol{z}}) \\ & -\sum_{k=1}^{M}\frac{16\overline{z}_k(\overline{z}_k^2-1)}{(u-\overline{z}_k)(u\overline{z}_k-1)}\overline{\mathcal{U}}_k(\boldsymbol{z})\overline{\mathbb{V}}(\overline{\boldsymbol{z}}_k,u) \end{aligned} \tag{1.A.7}$$

where $\overline{\mathbb{V}}(\boldsymbol{z}_k,u) = C(\overline{z}_1,0)\dots C(\overline{z}_{k-1},-k+2)C(u,-k+1)C(\overline{z}_{k+1},-k)\dots C(\overline{z}_M,-M+1)\overline{\omega}_s$ and

$$\overline{\mathcal{W}}(u,\overline{\boldsymbol{z}}) = \overline{\lambda}(u) + \sum_{k=1}^{M}\frac{16\overline{z}_k(u^2-1)}{(u-\overline{z}_k)(u\overline{z}_k-1)}\left(\frac{\overline{\delta}(u)-\overline{\alpha}(u)}{4} + \frac{u^2+1}{u^2-1} + \sum_{\substack{p=1\\p\neq k}}^{M}\frac{(u^2-1)\overline{z}_k\overline{z}_p}{u(\overline{z}_k-\overline{z}_p)(\overline{z}_k\overline{z}_p-1)}\right), \tag{1.A.8}$$

$$\overline{\lambda}(u) = \overline{\alpha}^2(u) + \overline{\delta}^2(u) + 2 - 2u(\overline{\alpha}'(u) - \overline{\delta}'(u)) - 2\frac{u^2+1}{u^2-1}(\overline{\alpha}(u) - \overline{\delta}(u)), \tag{1.A.9}$$

$$\overline{\mathcal{U}}_k(\overline{\boldsymbol{z}}) = \frac{\overline{\delta}(z_k)-\overline{\alpha}(z_k)}{4} + \frac{\overline{z}_k^2+1}{\overline{z}_k^2-1} + \sum_{\substack{p=1\\p\neq k}}^{M}\frac{\overline{z}_p(\overline{z}_k^2-1)}{(\overline{z}_k-\overline{z}_p)(\overline{z}_k\overline{z}_p-1)}. \tag{1.A.10}$$

References

[1] M.C. Albanese, M. Christandl, N. Datta, and A. Ekert, *Mirror inversion of quantum states in linear registers*, Phys. Rev. Lett. 93 (2004) 230502 and `arXiv:quant-ph/0405029`.

[2] J. Avan, S. Belliard, N. Grosjean, and R.A. Pimenta, *Modified algebraic Bethe ansatz for XXZ chain on the segment - III - Proof,* Nucl. Phys. B899 (2015) 229-246 and `arXiv:1506.02147`.

[3] P. Baseilhac, S. Belliard, and N. Crampe, *FRT presentation of the Onsager algebras,* Lett. Math. Phys. 108 (2018) 2189 and `arXiv:1709.08555`.

[4] P. Baseilhac, and N. Crampe, *FRT presentation of classical Askey-Wilson algebras,* Lett. Math. Phys. 109 (2019) 2187 and `arXiv:1806.07232`.

[5] P. Baseilhac, and R.A. Pimenta, *Diagonalization of the Heun–Askey–Wilson operator, Leonard pairs and the algebraic Bethe ansatz,* Nucl. Phys. B949 (2019) 114824 and `arXiv:1909.02464`.

[6] P. Baseilhac, S. Tsujimoto, L. Vinet, and A. Zhedanov, *The Heun–Askey–Wilson algebra and the Heun operator of Askey–Wilson type,* Annales Henri Poincaré 20 (2019) 3091-3112 and `arXiv:1811.11407`.

[7] S. Belliard, and N. Crampe, *Heisenberg XXX Model with General Boundaries: Eigenvectors from Algebraic Bethe Ansatz,* SIGMA 9 (2013) 072 and `arXiv:1309.6165`.

[8] S. Belliard, and R.A. Pimenta, *Modified algebraic Bethe ansatz for XXZ chain on the segment - II - general cases,* Nucl. Phys. B894 (2015) 527-552 and `arXiv:1412.7511`.

[9] G. Bergeron, N. Crampe, S. Tsujimoto, L. Vinet, and A. Zhedanov, *The Heun–Racah and Heun–Bannai–Ito algebras,* J. Math. Phys. 61 (2020) 081701 and `arXiv:2003.09558`.

[10] J. Cao, W.L. Yang, K. Shi, and Y. Wang, *Off-diagonal Bethe ansatz solution of the XXX spin-chain with arbitrary boundary conditions*, Nucl. Phys. B875 (2013) 152-165 and `arXiv:1306.1742`.

[11] M. Christandl, N. Datta, T.C. Dorlas, A. Ekert, A. Kay, and A.J. Landahl, *Perfect transfer of arbitrary states in quantum spin networks*, Phys. Rev. A032312 (2005) 71 and `arXiv:quant-ph/0411020`.

[12] N. Crampe, *Algebraic Bethe ansatz for the totally asymmetric simple exclusion process with boundaries,* J. Phys. A: Math. Theor. 48 (2015) 08FT01 and `arXiv:1411.7954`.

[13] N. Crampe, *Algebraic Bethe ansatz for the XXZ Gaudin models with generic boundary,* SIGMA 13 (2017) 094 and `arXiv:1710.08490`.

[14] N. Crampe, K. Guo, and L. Vinet, *Entanglement of free Fermions on Hadamard graphs,* Nucl. Phys. B960 (2020) 115176 and `arXiv:2008.04925`.

[15] N. Crampe, and R.I. Nepomechie, *Equivalent T-Q relations and exact results for the open TASEP,* J. Stat. Mech. (2018) 103105 and `arXiv:1806.07748`.

[16] N. Crampe, R.I. Nepomechie, and L.Vinet, *Free-Fermion entanglement and orthogonal polynomials*, J. Stat. Mech. (2019) 093101 and `arXiv:1907.00044`.

[17] N. Crampe, R.I. Nepomechie, and L.Vinet, *Entanglement in Fermionic Chains and Bispectrality*, Roman Jackiw 80th Birthday Festschrift (World Scientific, 2020) and `arXiv:2001.10576`.

[18] N. Crampe, L. Poulain d'Andecy, and L. Vinet, *A Calabi–Yau algebra with E_6 symmetry and the Clebsch–Gordan series of $sl(3)$*, `arXiv:2005.13444`.

[19] N. Crampe, L. Vinet, and A. Zhedanov, *Heun algebras of Lie type,* Proc. Amer. Math. Soc. 148 (2020) 1079-1094 and `arXiv:1904.10643`.

[20] V. Eisler, and I. Peschel, *Properties of the entanglement Hamiltonian for finite free-fermion chains*, J. Stat. Mech. 10 (2018) 104001 and `arXiv:1805.00078`.

[21] F.A. Grünbaum, L. Vinet, and A. Zhedanov, *Algebraic Heun Operator and Band-Time Limiting,* Comm. Math. Phys. 364 (2018) 1041-1068 and `arXiv:1711.07862`.

[22] F.A. Grünbaum, L. Vinet, and A. Zhedanov, *Tridiagonalization and the Heun equation,* J. Math. Phys. 58 (2017) 1-17 and `arXiv:1602.04840`.

[23] M.E.H. Ismail, S.S. Lin, and S.S. Roan, *Bethe Ansatz Equations of XXZ Model and q-Sturm-Liouville Problems,* `arXiv:math-ph/0407033`.

[24] R. Koekoek, and R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue,* `arXiv:math/9602214`.

[25] J. Patera, and P. Winternitz, *A new basis for the representations of the rotation group. Lamé and Heun polynomials*, J. Math. Phys. **14** (1973) 1130-1139.

[26] E.K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. A: Math. Gen.21 (1988) 2375-2389.

[27] E.K. Sklyanin, L.A. Takhtadzhyan, and L.D. Faddeev, *Quantum inverse problem method. I*, Theor. Math. Phys. 40 (1979) 688–706.

[28] T. Skrypnyk, *Generalized Gaudin systems in a magnetic field and non-skew-symmetric r-matrices,* J. Phys. A: Math. Theor. 40 (2007) 13337.

[29] A.V. Turbiner, *The Heun operator as a Hamiltonian,* J. Phys. A: Math. Theor. 49 (2016) 26LT01 and `arXiv:1603.02053`.

[30] L. Vinet, and A. Zhedanov, *The Heun operator of Hahn type,* Proc. Amer. Math. Soc. 147 (2019) 2987-2998 and `arXiv:1808.00153`.

[32] P.B. Wiegmann, and A.V. Zabrodin, *Algebraization of difference eigenvalue equations related to $Uq(sl_2)$*, Nucl. Phys. B 451 (1995) 699 and `cond-mat/9501129`.

Chapitre 2

The $SU(3) \supset SO(3)$ missing label problem and the analytical Bethe ansatz

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Abstract. The missing label for basis vectors of $SU(3)$ representations corresponding to the reduction $SU(3) \supset SO(3)$ can be provided by the eigenvalues of $SO(3)$ scalars in the enveloping algebra of $su(3)$. There are only two such independent elements of degree three and four. It is shown how the one of degree four can be diagonalized using the analytical Bethe ansatz.

2.1. Introduction

This paper bears on the missing label problem for the basis states of $SU(3)$ representations corresponding to the reduction $SU(3) \supset SO(3) \supset SO(2)$. This is a question that has been studied for a long time and a topic on which there is a vast literature. The reader may consult [**24**] for a review of where the subject stood roughly 50 years ago and for finding many references. The problem has not lost its interest through the years in view of its practical importance and has kept being explored. Among more recent reports we may cite [**10**, **27**, **34**, **36**] where additional references will be found and, in particular [**19**], which connects the missing label problem to quantum groups and integrable models and will prove of relevance in the following.

In a nutshell, the problem has to do with the fact that generally, the irreducible representations (irreps) of $SO(3)$ are not multiplicity free in the irreps of $SU(3)$. Hence the values of the two Casimir elements of $SU(3)$, and the quantum numbers that label the basis vectors of $SO(3)$ irreps do not suffice to characterize the basis vectors of $SU(3)$ irreps corresponding to the reduction $SU(3) \supset SO(3) \supset SO(2)$. In cases where the same $SO(3)$ irrep occurs more that once, clearly an additional multiplicity index is needed.

Of the various resolutions of the problem that have been proposed, the one we will concentrate on has the missing label provided by the eigenvalues of an operator belonging to the centralizer of $so(3)$ in the enveloping algebra $U(su(3))$ of the Lie algebra of $SU(3)$. It has been shown [21] that the integrity basis for this centralizer consists of two elements x and y of degree three and four respectively in the generators of $su(3)$. This approach has the merit of yielding an orthogonal set of basis vectors at the expense of having labels -the eigenvalues- that are not integer and somewhat difficult to compute. The operator x was first introduced in [4], while y can be traced back to lecture notes by Racah [30]. We shall also make use of the resolution provided by Bargmann and Moshinsky who constructed an $SU(3) \supset SO(3)$ basis bearing their names in terms of elementary permissible diagrams. In this BM basis, the additional label is integer but the vectors are not orthogonal.

Our attention will be focused on the spectrum of y. Numerical results were obtained in [21] by relying on the Gelf'and–Tseitlin states. Another method based on shift operators is described in [13,18]. We shall here add to this venerable topic by showing how the fourth order labelling operator y can be diagonalized by calling upon the analytical Bethe ansatz (BA).

Recently, the Bethe ansatz and its variants, which are cornerstones in the study of (quantum) integrable models, have found their way in representation theory. They have in particular appeared in two situations that bear a very close relation to the problem of finding the spectrum of an $SO(3)$ scalar in $U(su(3))$: i. the diagonalization in [9] using the algebraic Bethe ansatz of the Lie–Heun operator introduced in [28] as labeling operator of a non-subgroup basis of $O(3)$. ii. the determination in [12] with the nested Bethe ansatz of the eigenvalues of an operator providing the missing label in the $SU(3)$ Clebsch–Gordan problem.

With respect to the diagonalization of y using Bethe ansatz techniques, as already alluded to, a useful observation is the one made in [19] where the operator y was found to lie in a Bethe subalgebra of the twisted Yangian $Y(gl(3),so(3))$. Given this identification, we shall show that it is possible to diagonalize y using the analytical Bethe ansatz. This method has been developed previously to find the spectrum of different quantum integrable spin chains (see for example [3,22,31–33,35]). It lays on assuming a particular form of the eigenvalues. The Bethe equations are obtained by imposing their analyticity, which explains the name of the method. We are going in fact to generalise this method in order to diagonalize the whole

Bethe subalgebra. The hypothesis we make for the form of the eigenvalue is inspired from work done in [25] where the Bethe subalgebra of the Yangian has been diagonalized using the nested algebraic Bethe ansatz.

The paper will unfold as follows. Section 2 will introduce the missing label operators x and y of third and fourth order and recall the remarkable relations found by Lehrer and Racah that they obey. It is worth stressing that strikingly, the algebra thus defined has also appeared recently [11] as the centralizer of the diagonal embedding of $U(sl(3))$ in $U(sl(3))^{\otimes 2}$. Section 3 will look at the actions of x and y in the BM basis which will be found to be tridiagonal. Complementing the results of Bargmann and Moshinsky who had obtained the matrix elements of x, we provide those of y. Section 4 will present the extension of the analytical Bethe ansatz that is required to diagonalize y. The results of section 3 will be used to offer a comparison between some of the eigenvalues computed with the Bethe ansatz approach and the corresponding ones obtained from the direct diagonalization of the tridiagonal matrix expressing y in the BM basis. Throughout the paper, we shall often interchangeably refer to Lie groups and their algebras denoting the former by capital letters and the latter by lower case ones.

2.2. The $SU(3) \supset SO(3)$ missing label operators and their algebra

In this section, we establish notations and introduce the missing label operators x and y in $U(su(3))$. We then present the double commutator relations that these operators verify.

The complexified Lie algebra $su(3)$ is generated by E_{ij} $(1 \leq i, j \leq 3)$ satisfying the defining relations

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}, \tag{2.2.1}$$

with the extra condition

$$\sum_{i=1}^{3} E_{ii} = 0. \tag{2.2.2}$$

When considering the subgroup chain $SU(3) \supset SO(3)$, it is convenient to use a basis of $su(3)$ [30] generated by the five independent components T_{mn} $(1 \leq m, n \leq 3)$ of the symmetrical traceless tensor $\mathbf{T}$ and the three components L_k $(1 \leq k \leq 3)$ of the angular momentum vector $\mathbf{L}$ generating the $so(3)$ subalgebra. They satisfy the following commutation rules:

$$\begin{aligned}
[L_k, L_n] &= i\epsilon_{knp} L_p, \\
[L_k, T_{mn}] &= i(\epsilon_{kmp} T_{pn} + \epsilon_{knp} T_{mp}), \\
[T_{mn}, T_{pq}] &= i(\delta_{mp}\epsilon_{nqr} + \delta_{mq}\epsilon_{npr} + \delta_{np}\epsilon_{mqr} + \delta_{nq}\epsilon_{mpr}) L_r,
\end{aligned}$$

where ϵ_{kmn} is the Levi-Civita symbol and the sums over the repeated indices are understood. The previous generators can be written in terms of E_{ij}. Indeed, the $so(3)$ generators are written as follows

$$L_1 = \frac{E_{12} - E_{32} + E_{21} - E_{23}}{\sqrt{2}}, \quad L_2 = -\frac{i(E_{12} + E_{32} - E_{21} - E_{23})}{\sqrt{2}}, \quad L_3 = E_{11} - E_{33}, \quad (2.2.3)$$

and the five independent components of $\mathbf{T}$ are

$$T_{11} = E_{11} + E_{33} + E_{13} + E_{31}, \qquad T_{22} = E_{11} + E_{33} - E_{13} - E_{31}, \qquad T_{12} = i(E_{31} - E_{13}),$$
$$T_{13} = -\frac{E_{12} + E_{32} + E_{21} + E_{23}}{\sqrt{2}}, \quad T_{23} = \frac{i(E_{12} - E_{32} - E_{21} + E_{23})}{\sqrt{2}}.$$

The two Casimir elements of $su(3)$ are

$$g_2 = L_k L_k + \frac{1}{2} T_{mn} T_{nm}, \qquad g_3 = \frac{1}{3} T_{mn} T_{np} T_{pm} - L_m T_{mn} L_n. \qquad (2.2.4)$$

Their values characterize the finite irreps of $SU(3)$. For the chain $SU(3) \supset SO(3)$, one also considers the Casimir element of $so(3)$:

$$\mathbf{L}^2 = L_k L_k. \qquad (2.2.5)$$

A supplementary operator is needed in order to distinguish the different copies of the $SO(3)$ irrep within a given $SU(3)$ irrep. The degeneracy can be lifted using one of the two following operators belonging to the universal enveloping algebra of $su(3)$ [4, 21, 30]:

$$\bar{x} = L_m T_{mn} L_n, \qquad \bar{y} = L_m T_{mn} T_{np} L_p. \qquad (2.2.6)$$

One generator of $SO(3)$, the angular momentum projection L_3, completes the characterization of the basis vectors.

For reasons of simplicity, we consider the following shifted operators:

$$x = \frac{1}{16}(\bar{x} + \frac{g_3}{4}), \qquad y = -\frac{1}{64}\Big(\bar{y} + 3 - g_2 - \frac{\mathbf{L}^2}{3}(9 + g_2 - \mathbf{L}^2)\Big). \qquad (2.2.7)$$

Then by using the defining relations of $su(3)$, one can show that x and y satisfy the following relations:

$$[x,[x,y]] = -6y^2 + a_2 x^2 + a_5 x + a_8,$$
$$[y,[y,x]] = 2x^3 + a_2\{x,y\} + a_5 y - a_6 x - a_9, \qquad (2.2.8)$$

where the coefficients are given in terms of $g_2, g_3, \mathbf{L}^2$ as

$$
\begin{aligned}
a_2 &= \frac{g_2 + 2\mathbf{L}^2 - 18}{8}, \\
a_5 &= \frac{(4\mathbf{L}^2 - g_2)g_3}{256}, \\
a_6 &= \frac{1}{6144}\left(8g_2(3-\mathbf{L}^2)(2g_2-\mathbf{L}^2-21) + 9g_3^2 - 16\mathbf{L}^2((\mathbf{L}^2-18)\mathbf{L}^2+63)+1296\right), \\
a_8 &= \frac{1}{98304}\Big(16\left(\mathbf{L}^2-3\right)^2\left(g_2^2+\left(\mathbf{L}^2-9\right)\left(\mathbf{L}^2-3\right)\right) - 16\left(2\mathbf{L}^2-3\right)\left(\left(\mathbf{L}^2\right)^2-6\mathbf{L}^2-18\right)g_2 \\
&\quad + 3g_3^2(18-10\mathbf{L}^2+g_2) + 2592(\mathbf{L}^2-1)\Big), \\
a_9 &= \frac{1}{196608}\left(4g_2\left(\mathbf{L}^2-3\right)\left(2g_2-7\mathbf{L}^2-3\right) - 3g_3^2 - 16\mathbf{L}^2\left(\left(\mathbf{L}^2-18\right)\mathbf{L}^2+9\right)\right)g_3.
\end{aligned}
$$

These relations have been previously given in [30] without explicit expressions for the coefficients a_i. Let us note that a more general version of this algebra also appears in [11]. In the latter paper, a central element Ω has been discovered. By specializing this result to our case, one gets that

$$\Omega = b_1 x + b_2 y + b_3 x^2 + b_4\{x,y\} + b_5 y^2 + b_7 xyx - x^4 + 4y^3 + [x,y]^2 \tag{2.2.9}$$

with

$$b_1 = 6a_5 + 2a_9\,,\quad b_2 = -2a_6 - 2a_8\,,\quad b_3 = 6a_2 + a_6\,,\quad b_4 = -a_5\,,\quad b_5 = 8a_2 - 24\,,\quad b_7 = -2a_2 + 12\,, \tag{2.2.10}$$

commutes with x and y. It also commutes with $g_2, g_3, \mathbf{L}^2, L_3$ and is proportional to the identity in a given irrep. This is indeed compatible with equation (139c) of [30] stating that the square of the commutator $[x,y]$ is a function of $x, y, \mathbf{L}^2, g_2, g_3$.

2.3. Differential realization of the missing label operators x and y

In this section, we review a well-known polynomial basis for the irreducible representations of $SU(3) \supset SO(3)$, namely the Bargmann–Moshinsky basis. We present expressions for the matrix elements of x, y in this basis.

2.3.1. Irreducible representations of $su(3)$

A finite dimensional irreducible representation of $su(3)$ is labeled by a pair of non-negative integers (λ, μ). These give the Cartan labels of the highest weight vector $V_{\lambda,\mu}$ satisfying:

$$E_{ii} V_{\lambda,\mu} = \alpha_{ii} V_{\lambda,\mu} \qquad \text{for } 1 \le i \le 3, \tag{2.3.1}$$

$$E_{ij} V_{\lambda,\mu} = 0 \qquad \text{for } 1 \le i < j \le 3, \tag{2.3.2}$$

with

$$\alpha_{11} = \frac{\mu + 2\lambda}{3}, \quad \alpha_{22} = \frac{\mu - \lambda}{3}, \quad \alpha_{33} = -\alpha_{11} - \alpha_{22} = -\frac{2\mu + \lambda}{3}. \tag{2.3.3}$$

This representation correspond to a Young diagram with $\lambda + \mu$ boxes in the first row and μ boxes in the second row. The dimension of these irreps is

$$d_{(\lambda,\mu)} = \frac{1}{2}(\mu + 1)(\lambda + 1)(\lambda + \mu + 2). \tag{2.3.4}$$

For a detailed construction of $su(3)$ irreps using the highest weight vector, see for example [17].

The two Casimir operators of $su(3)$ become proportionnal to the identity in this irrep and are related to λ, μ by:

$$g_2 = \frac{4}{3}(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu), \qquad g_3 = \frac{8}{27}(\lambda - \mu)(3 + \lambda + 2\mu)(3 + 2\lambda + \mu). \tag{2.3.5}$$

A given $so(3)$ representation characterized by the integer L (such that $\mathbf{L}^2 = L(L+1)$), appears $d^L_{(\lambda,\mu)}$ times in the $su(3)$ irrep. This degeneracy has been computed in [29] and is given by:

$$d^L_{(\lambda,\mu)} = \left[\frac{\lambda + \mu - L + 2}{2}\right]_{\geq 0} - \left[\frac{\lambda - L + 1}{2}\right]_{\geq 0} - \left[\frac{\mu - L + 1}{2}\right]_{\geq 0}, \tag{2.3.6}$$

where $[\dots]_{\geq 0}$ is zero for any negative number and gives the positive integer part otherwise. One can then check that we indeed have:

$$d_{(\lambda,\mu)} = \sum_L (2L + 1) d^L_{(\lambda,\mu)}. \tag{2.3.7}$$

2.3.2. The Bargmann–Moshinsky basis

The Bargmann–Moshinsky (BM) basis [4] consists in a set of states for the general irrep (λ,μ) of $SU(3) \supset SO(3)$ reduction which depends on the supplementary label α that distinguishes the different copies of $SO(3)$ irreps. The highest states of all $SO(3)$ multiplets in this basis are given by

$$\left|\begin{matrix}\lambda & \mu & \\ \alpha & L & L\end{matrix}\right\rangle_B = \begin{cases} \eta_1^{\lambda-2\alpha}\ \eta_{12}^{L-\lambda+2\alpha}\ s_1^{\alpha}\ s_{12}^{\frac{1}{2}(\mu-L+\lambda-2\alpha)}, & \lambda + \mu - L \text{ even} \\ \eta_1^{\lambda-2\alpha-1}\ \eta_{12}^{L-\lambda+2\alpha}\ w\ s_1^{\alpha}\ s_{12}^{\frac{1}{2}(\mu-L+\lambda-2\alpha-1)}, & \lambda + \mu - L \text{ odd} \end{cases} \tag{2.3.8}$$

where:

$$\begin{aligned} &\eta_{12} = \eta_1\zeta_2 - \zeta_1\eta_2, && \zeta_{12} = \eta_1\xi_2 - \xi_1\eta_2, && \xi_{12} = \zeta_1\xi_2 - \xi_1\zeta_2, \\ &s_1 = \zeta_1^2 - 2\eta_1\xi_1, && s_{12} = \zeta_{12}^2 - 2\eta_{12}\xi_{12}, && w = \eta_1\zeta_{12} - \zeta_1\eta_{12}. \end{aligned} \tag{2.3.9}$$

In this realization, the generators E_{ij} of the $su(3)$ algebra are given by

$$E_{ij} = v_i\frac{\partial}{v_j} + w_i\frac{\partial}{w_j} - \frac{\delta_{ij}}{3}(\lambda + 2\mu) \tag{2.3.10}$$

with

$$v = (-\eta_1,\zeta_1,\xi_1), \qquad w = (-\eta_2,\zeta_2,\xi_2). \tag{2.3.11}$$

The rest of the basis elements

$$\left|\begin{matrix}\lambda & \mu & \\ \alpha & L & m\end{matrix}\right\rangle_B \qquad \text{where} \qquad L_3\left|\begin{matrix}\lambda & \mu & \\ \alpha & L & m\end{matrix}\right\rangle_B = m\left|\begin{matrix}\lambda & \mu & \\ \alpha & L & m\end{matrix}\right\rangle_B, \tag{2.3.12}$$

are obtained by repeatedly acting with $L_- = \frac{iL_2 - L_1}{\sqrt{2}}$. The highest weight state $V_{\lambda,\mu}$ in a given $su(3)$ irrep is obtained by setting $m = L = \lambda + \mu$ and $\alpha = 0$ in the BM vectors. The label α is a positive integer and is chosen such as all exponents are positive. It thus lays in the range

$$\max\left(0,\frac{\lambda-L}{2}+\frac{1-(-1)^{\lambda-L}}{4}\right) \le \alpha \le \min\left(\frac{\lambda-1}{2}+\frac{(-1)^{\mu+L}}{4}((-1)^{\lambda}+1),\frac{\lambda+\mu-L}{2}+\frac{(-1)^{\lambda+\mu+L}-1}{4}\right). \tag{2.3.13}$$

Remark 1. *The degeneracy $d^L_{(\lambda,\mu)}$ can also be found using equation (2.3.13):*

$$d^L_{(\lambda,\mu)} = \min\left[\frac{L}{2}+\frac{1}{4}+\frac{(-1)^L}{4}\left((-1)^{\lambda}+(-1)^{\mu}+(-1)^{\lambda+\mu}\right), \frac{\mu}{2}+\frac{1}{2}+\frac{(-1)^{\lambda+L}}{4}(1+(-1)^{\mu}),\right.$$
$$\left.\frac{\lambda+1}{2}+\frac{(-1)^{\mu+L}}{4}((-1)^{\lambda}+1), \frac{\lambda+\mu-L}{2}+\frac{(-1)^{\lambda+\mu+L}+3}{4}\right]. \tag{2.3.14}$$

The group of linear transformations acting on the three parameters λ, μ, L and preserving the size of the missing label matrices is the group of symmetry of the square. It is generated by the two permutations $r = (1,4)$ and $s = (13)(24)$ of the four possible values of $d^L_{(\lambda,\mu)}$ and are explicitly given by:

$$\begin{aligned} r:\ & L \to \lambda+\mu-L+1-\frac{(-1)^{\lambda+L}+(-1)^{\mu+L}}{2} \\ & \lambda \to \lambda \\ & \mu \to \mu \end{aligned} \quad , \quad \begin{aligned} s:\ & L \to \lambda+\frac{1-(-1)^{\lambda+L}}{2} \\ & \lambda \to L-\frac{1-(-1)^{\lambda+L}}{2} \\ & \mu \to \lambda+\mu-L+\frac{1-(-1)^{\lambda+L}}{2}. \end{aligned} \tag{2.3.15}$$

2.3.3. The operators x and y in the Bargmann–Moshinsky basis

As mentioned earlier, it is not possible to construct a basis for $SU(3) \supset SO(3)$ such that the operators x and y have analytical eigenfunctions [30]. Nevertheless, they both take the form of tridiagonal matrices in the BM basis. The resulting matrices can then be diagonalized numerically to find the spectrum.

The two non trivial expansions are:

$$x\left|\begin{matrix}\lambda & \mu & \\ \alpha & L & m\end{matrix}\right\rangle_B = \sum_{\alpha'}\left|\begin{matrix}\lambda & \mu & \\ \alpha' & L & m\end{matrix}\right\rangle_B \beta_{\alpha'\alpha}, \qquad y\left|\begin{matrix}\lambda & \mu & \\ \alpha & L & m\end{matrix}\right\rangle_B = \sum_{\alpha'}\left|\begin{matrix}\lambda & \mu & \\ \alpha' & L & m\end{matrix}\right\rangle_B \gamma_{\alpha'\alpha}. \tag{2.3.16}$$

The non vanishing coefficients $\beta_{\alpha'\alpha}$ have been computed in [4]. In our notation, they read as:

$\lambda+\mu+L$ even:

$$\beta^{(e)}_{\alpha+1,\alpha} = \frac{1}{8}(2\alpha-\lambda)(2\alpha-\lambda+1)(2\alpha-\mu+L-\lambda), \tag{2.3.17}$$

$$\beta^{(e)}_{\alpha-1,\alpha} = \frac{1}{4}\alpha(2\alpha+L-\lambda)(2\alpha+L-\lambda-1), \tag{2.3.18}$$

$$\begin{aligned}\beta^{(e)}_{\alpha,\alpha} = \frac{1}{216}\Big(&27(2\alpha-\lambda+L)\left(8\alpha^2-2\alpha(3\lambda-L+\mu)+\lambda^2+\mu(\lambda+L)+L+1\right)\\ &-9L(L+1)\left(\lambda+2\mu+\frac{3}{2}\right)+(\lambda-\mu)(2\lambda+\mu+3)(\lambda+2\mu+3)+27L\mu\Big),\end{aligned} \tag{2.3.19}$$

$\lambda+\mu+L$ odd:

$$\beta^{(o)}_{\alpha+1,\alpha} = \frac{1}{8}(2\alpha-\lambda+1)(2\alpha-\lambda+2)(2\alpha-\mu+L-\lambda+1), \tag{2.3.20}$$

$$\beta^{(o)}_{\alpha-1,\alpha} = \frac{1}{4}\alpha(2\alpha+L-\lambda)(2\alpha+L-\lambda-1), \tag{2.3.21}$$

$$\begin{aligned}\beta^{(o)}_{\alpha,\alpha} = \frac{1}{216}\Big(&27(2\alpha-\lambda+L)\left(8\alpha^2-2\alpha(3\lambda-L+\mu-3)+\lambda^2+\lambda(\mu-1)+L\mu+2L+5\right)\\ &-9L(L+1)\left(\lambda+2\mu+\frac{3}{2}\right)+(\lambda-\mu)(2\lambda+\mu+3)(\lambda+2\mu+3)+27\lambda+27L(\mu-1)+54\Big),\end{aligned} \tag{2.3.22}$$

We furthermore carried on the same computation for the operator y and obtained the following analytical expressions for its non vanishing elements:

$\lambda+\mu+L$ even:

$$\gamma^{(e)}_{\alpha+1,\alpha}=\frac{1}{6}(6\alpha-\lambda+\mu+3L+6)\beta^{(e)}_{\alpha+1,\alpha}, \tag{2.3.23}$$

$$\gamma^{(e)}_{\alpha-1,\alpha}=\frac{1}{3}(3\alpha-2\lambda-\mu-3)\beta^{(e)}_{\alpha-1,\alpha}, \tag{2.3.24}$$

$$\begin{aligned}\gamma^{(e)}_{\alpha,\alpha}=\frac{1}{192}\Big(&384\alpha^4-128\alpha^3(5\lambda-3L+\mu)\\&+16\alpha^2\left(23\lambda^2+\lambda(8\mu-30L-6)+6L^2-3L(2\mu+1)-\mu^2-6\mu+3\right)\\&-8\alpha\Big(10\lambda^3-2\lambda^2(11L-2\mu+6)+2\lambda\left(5L^2+L(2-4\mu)-\mu^2-6\mu+2\right)+L^2(2\mu+3)\\&+L(2\mu-3)+2\mu\Big)+4\lambda^2\left(3L^2+L(7-2\mu)-\mu^2-6\mu+2\right)-8\lambda^3(2L+3)+4\lambda^4\\&+4\lambda\left(L^2(\mu-1)+L(3\mu-1)+3(\mu+1)\right)-L^4-2L^3+5L^2+6L+4\mu^2+12\mu-9\Big),\end{aligned} \tag{2.3.25}$$

$\lambda+\mu+L$ odd:

$$\gamma^{(o)}_{\alpha+1,\alpha}=\frac{1}{6}(6\alpha-\lambda+\mu+3L+9)\beta^{(o)}_{\alpha+1,\alpha}, \tag{2.3.26}$$

$$\gamma^{(o)}_{\alpha-1,\alpha}=\frac{1}{6}(6\alpha-4\lambda-2\mu-6)\beta^{(o)}_{\alpha-1,\alpha}, \tag{2.3.27}$$

$$\begin{aligned}\gamma^{(o)}_{\alpha,\alpha}=\frac{1}{192}\Big(&384\alpha^4-128\alpha^3(5\lambda-3L+\mu-3)\\&+16\alpha^2\left(23\lambda^2+\lambda(-30L+8\mu-36)+6L^2+L(15-6\mu)-\mu^2-12\mu+12\right)\\&-8\alpha\Big(\lambda^2(-22L+4\mu-34)+10\lambda^3+2\lambda\left(5L^2-4L(\mu-4)-\mu^2-10\mu+13\right)+L^2(2\mu-3)\\&+L(10\mu-9)+10\mu-6\Big)+4\lambda^2\left(3L^2+L(15-2\mu)-\mu^2-8\mu+14\right)-8\lambda^3(2L+5)+4\lambda^4\\&+4\lambda\left(L^2(\mu-5)+L(7\mu-13)+9\mu-5\right)-8L^2\mu-L^4-2L^3+5L^2-24L\mu+6L+4\mu^2\\&-4\mu-9\Big).\end{aligned} \tag{2.3.28}$$

Remark 2. *With an appropriate identification of parameters, it can be shown that the matrix elements of x and y have the same form as those of the missing label operators in the two fold tensor product of $su(3)$ found in [**12**].*

2.4. Analytical Bethe ansatz

In this section, the definitions of the Yangian of $gl(3)$ and of the twisted Yangian $Y(gl(3);so(3))$ are recalled (see e.g. [**23**]). The connection, discovered in [**19**], between an abelian subalgebra of $Y(gl(3);so(3))$, called Bethe subalgebra, and the missing label operator y is reviewed. Exploiting this result, the spectrum of y is found using the analytical Bethe ansatz. This is achieved by generalising the method used in [**1**,**2**] to diagonalize the transfer matrix associated with open spin chains with soliton non preserving boundary conditions.

2.4.1. Yangian $Y(gl(3))$, Twisted Yangian $Y(gl(3); so(3))$ and a Bethe subalgebra

The Yangian $Y(gl(3))$ is defined [14, 15] as the complex associative unital algebra generated by the generators $\{t_{ij}^{(n)} | 1 \le i,j \le 3, n \in \mathbb{Z}_{\ge 0}\}$ subject to the defining relations:

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)}, \tag{2.4.1}$$

where $r,s \in \mathbb{Z}_{\ge 0}$ and $t_{ij}^{(0)} = \delta_{ij}$. In order to encompass all the defining relations of the Yangian, it is known that we can use the FRT presentation [16]. It is based on the following R-matrix [6–8, 37]

$$R(u) = u\, \mathbb{I}_3 \otimes \mathbb{I}_3 - \mathbb{P}, \tag{2.4.2}$$

where $\mathbb{I}_3$ is the 3×3 identity matrix, $\mathbb{P}$ is the permutation operator

$$\mathbb{P} = \sum_{i,j=1}^{3} e_{ij} \otimes e_{ji}, \tag{2.4.3}$$

and e_{ij} is a 3×3 matrix with zeros everywhere and 1 at the entry (i,j). It provides a solution to the Yang–Baxter equation:

$$R_{12}(u-v)\ R_{13}(u)\ R_{23}(v) = R_{23}(v)\ R_{13}(u)\ R_{12}(u-v), \tag{2.4.4}$$

with $R_{12} = R \otimes \mathbb{I}_3$, $R_{23} = \mathbb{I}_3 \otimes R$ and so on. We then define the following T-matrix:

$$T(u) = \sum_{i,j=1}^{3} e_{ij} \otimes t_{ij}(u), \tag{2.4.5}$$

with $t_{ij}(u)$ the following formal series in $1/u$:

$$t_{ij}(u) = \delta_{ij} + \sum_{k=1}^{\infty} \frac{t_{ij}^{(k)}}{u^k}. \tag{2.4.6}$$

The commutation relations (2.4.1) can then by compactly written in the form of an FRT equation [16]

$$R(u-v) T_1(u) T_2(v) = T_2(v) T_1(u) R(u-v). \tag{2.4.7}$$

We now introduce the following generalised transposition t, related to the usual transposition T by:

$$A^t = V^{-1} A^T V, \quad \text{where} \quad V = \text{antidiag}(1,1,\dots,1). \tag{2.4.8}$$

The $so(3)$ twisted Yangian is a subalgebra of $Y(gl(3))$, noted as $Y(gl(3); so(3))$, and is generated by elements $s_{ij}^{(k)}$ put in the following formal series

$$S(u) = \sum_{i,j}^{3} \sum_{k=0}^{\infty} e_{ij} \otimes \frac{s_{ij}^{(k)}}{u^k} = T(u) T^t(-u). \tag{2.4.9}$$

The commutation relations of the twisted Yangian can be compactly encoded as:

$$R(u-v)S_1(u)\widetilde{R}(-u-v)S_2(v) = S_2(v)\widetilde{R}(-u-v)S_1(u)R(u-v), \tag{2.4.10}$$

where

$$\widetilde{R}(u) = R^{t_1}(u), \tag{2.4.11}$$

and t_1 is the generalised transposition taken in the first space only.

One can then construct an Abelian subalgebra of $Y(gl(3); so(3))$, also called a Bethe subalgebra [26], generated by the coefficients in $u^{-1}, u^{-2}, \ldots$ of the following 3 elements $A_1(u), A_2(u), A_3(u)$:

$$A_1(u) = tr_1[S_1(u)], \tag{2.4.12}$$

$$A_2(u) = tr_{12}[(\mathbb{A}_{12}\ S_1(u)\widetilde{R}_{12}(-2u+1)S_2(u-1)], \tag{2.4.13}$$

$$\begin{aligned} A_3(u) = tr_{123}[\mathbb{A}_{123}\ S_1(u)\widetilde{R}_{12}(-2u+1)\widetilde{R}_{13}(-2u+2) \\ \times S_2(u-1)\widetilde{R}_{23}(-2u+3)S_3(u-2)], \end{aligned} \tag{2.4.14}$$

where the $\mathbb{A}$s are antisymmetrization operators:

$$\mathbb{A}_{12} = \mathbb{I} - \mathbb{P}_{12} \tag{2.4.15}$$

$$\mathbb{A}_{123} = \mathbb{I} - \mathbb{P}_{12} - \mathbb{P}_{13} - \mathbb{P}_{23} + \mathbb{P}_{12}\mathbb{P}_{23} + \mathbb{P}_{13}\mathbb{P}_{23} \tag{2.4.16}$$

and $tr_1[\ldots]$ is the trace taken over the first space, $tr_{12}[\ldots]$ is the trace taken over the first and second spaces, and so on. The three series A_i satisfy the following commutation relations [26]

$$[A_i(u), A_j(v)] = 0 \qquad \text{for} \qquad i,j = 1, \ldots 3, \tag{2.4.17}$$

which proves that the coefficients of the series are mutually commutative and form an Abelian subalgebra.

2.4.2. Evaluation representation of $Y(gl(3))$

The construction of representations of the Yangians is based on the following algebra homomorphism from $Y(gl(3))$ to the universal enveloping algebra of $gl(3)$

$$T_{ij} \longmapsto \delta_{ij} + \frac{E_{ij}}{u}. \tag{2.4.18}$$

It is called the evaluation representations of $Y(gl(3))$. In this realisation, the element $A_1(u)$ becomes

$$A_1(u) = 3 - \frac{2E_{32}E_{12} + 2E_{31}E_{13} + 2E_{21}E_{23} + 2E_{33}E_{11} + E_{22}^2 + E_{11} - E_{33}}{u^2} \tag{2.4.19}$$

The same computation can be done for $A_2(u)$ and $A_3(u)$. Then the elements (2.4.12)-(2.4.14) can be rewritten as follows [19]:

$$A_1(u) = 3 - \frac{g_2 - 2\mathbf{L}^2}{2u^2}, \tag{2.4.20}$$

$$A_2(u) = (1-2u)\left(\frac{32y}{(u-1)^2u^2} + 6 + \frac{(\mathbf{L}^2 - g_2)(12u^2 + \mathbf{L}^2 - 12u)}{6(u-1)^2u^2} + \frac{g_2^2 - 8\mathbf{L}^2 - 4g_2 + 12}{8(u-1)^2u^2}\right), \tag{2.4.21}$$

$$A_3(u) = 36 - \frac{18g_2}{u(u-2)} + \frac{9g_2^2}{4u^2(u-2)^2} - \frac{9g_3^2}{16u^2(u-1)^2(u-2)^2}. \tag{2.4.22}$$

Note that $A_2(u)$ contains the operator y. Therefore, the diagonalization of $A_2(u)$ will provide the eigenvalues of y.

2.4.3. Bethe ansatz

The idea behind the analytical Bethe ansatz is to first find a particular eigenvalue of the operator to be diagonalized (in our case, by acting on the $su(3)$ highest weight vector), and then to assume that the general formula for the eigenvalue function is a dressed version of the latter. The case for $A_1(u)$ appears as a special case of integrable quantum spin chains with soliton non preserving boundary conditions, which have been examined using the analytical Bethe ansatz in [1]. Knowing that $A_1(u)$ is already diagonal in a given $su(3)$ irreducible representation, it would seem of no interest to use the analytical Bethe ansatz at all. However, the exact same dressing functions found for $A_1(u)$ can be used to diagonalize $A_2(u)$, which contains the missing label operator y. We note however that this method does not allow the diagonalization of the operator x as it does not appear in the context of such integrable systems. We now present (without proof) the steps in the application of the analytical Bethe ansatz.

Using equations (2.3.1)-(2.3.2), as well as the expression for $A_1(u)$ in terms of E_{ij} given in (2.4.19), the action of $A_1(u)$ on the vector of highest weight in an $su(3)$ irrep (λ, μ) reads as:

$$A_1(u)V_{\lambda,\mu} = \Lambda_1^0(u)V_{\lambda,\mu}, \tag{2.4.23}$$

where

$$\Lambda_1^0(u) = 3 - \frac{2\alpha_{33}\alpha_{11} + \alpha_{22}^2 + \alpha_{11} - \alpha_{33}}{u^2} \tag{2.4.24}$$

and α_{ii} are defined as (2.3.3). As expected, this eigenvalue is equal to (2.4.20), where g_2 takes its value (2.3.5) and $\mathbf{L}^2 = (\lambda+\mu)(\lambda+\mu+1)$. This eigenvalue can be decomposed as follows:

$$\Lambda_1^0(u) = \frac{2u-1}{2u^3}\sigma_1(u) + \frac{\sigma_2(u)}{u^2} + \frac{2u+1}{2u^3}\sigma_3(u) \tag{2.4.25}$$

where

$$\sigma_1(u) = (u+\alpha_{11})(u-\alpha_{33}), \qquad \sigma_2(u) = (u+\alpha_{22})(u-\alpha_{22}), \qquad \sigma_3(u) = (u+\alpha_{33})(u-\alpha_{11}). \tag{2.4.26}$$

Then the ansatz used in [2] is that the general eigenvalue $\Lambda_1(u)$ has the following "dressed" form

$$\Lambda_1(u) = \frac{2u-1}{2u^3} D_1(u)\sigma_1(u) + \frac{D_2(u)\sigma_2(u)}{u^2} + \frac{2u+1}{2u^3} D_3(u)\sigma_3(u), \tag{2.4.27}$$

with

$$D_1(u) = D_3(-u) = \prod_{j=1}^{M} \frac{u-u_j-\frac{1}{2}}{u-u_j+\frac{1}{2}}, \tag{2.4.28}$$

$$D_2(u) = \prod_{j=1}^{M} \frac{(u-u_i+\frac{3}{2})(u+u_j-\frac{3}{2})}{(u-u_j+\frac{1}{2})(u+u_j-\frac{1}{2})}. \tag{2.4.29}$$

It seems that the dressing functions create additional poles. In order for the residues at these poles to vanish, constrains between the parameters u_j are necessary. These equations are known as Bethe equations and their solutions are called Bethe roots. In our case they read as follows:

$$\frac{(u_p-\frac{1}{2}-\alpha_{33})(u_p-\frac{1}{2}+\alpha_{11})}{(u_p-\frac{1}{2})^2-\alpha_{22}^2} = \prod_{\substack{j=1\\ j\neq p}}^{M} \frac{(u_p-u_j+1)(u_p+u_j-2)}{(u_p+u_j-1)(u_p-u_j-1)}, \qquad 1\le p\le M. \tag{2.4.30}$$

The physical Bethe roots correspond to the Bethe roots that are pairwise distinct. The different physical Bethe roots will provide the different eigenvalues. The ansatz also states that the whole spectrum is obtained in this way.

The parameter M is related to the eigenvalue of the total angular momentum in a given $so(3)$ representation. Indeed, one can repeatedly apply L'Hospital rule to compute the following limit and get:

$$\lim_{u\to\infty}(u^2\Lambda_1(u) - 3u^2) = g_2 - 2\mathbf{L}^2, \tag{2.4.31}$$

where $\mathbf{L}^2 = L(L+1) = (\lambda+\mu-M)(\lambda+\mu-M+1)$.

We now proceed in a similar fashion to compute the eigenvalue of $A_2(u)$. Acting on the highest weight, we first get:

$$A_2(u)V_{\lambda,\mu} = \Lambda_2^0 V_{\lambda,\mu}, \tag{2.4.32}$$

where the eigenvalue can be decomposed as follows

$$\Lambda_2^0 = \frac{2(1-2u)}{u^2(u-1)^2}\sigma_1(u)\sigma_3(u-1) - \frac{4}{u^2(u-1)}\sigma_1(u)\sigma_2(u-1) - \frac{4}{u(u-1)^2}\sigma_2(u)\sigma_3(u-1). \tag{2.4.33}$$

We remark that the same functions $\sigma_i(u)$ appear in the eigenvalues $\Lambda_1^0(u)$ and $\Lambda_2^0(u)$. The ansatz to get all the eigenvalues $\Lambda_1(u)$ consists in dressing each $\sigma_i(u)$ by $D_i(u)$. We then make use of the same dressing functions for Λ_2, namely:

$$\Lambda_2 = \frac{2(1-2u)}{u^2(u-1)^2} D_1(u)\sigma_1(u)D_3(u-1)\sigma_3(u-1) - \frac{4}{u^2(u-1)} D_1(u)\sigma_1(u)D_2(u-1)\sigma_2(u-1)$$
$$- \frac{4}{u(u-1)^2} D_2(u)\sigma_2(u)D_3(u-1)\sigma_3(u-1). \quad (2.4.34)$$

The Bethe roots u_i are solution the same Bethe equations (2.4.30) as those for $A_1(u)$, insuring that the residues at the poles of the dressing functions $D_i(u)$ vanish. To support this ansatz, let us mention that the same type of dressing appears in the algebraic Bethe ansatz for the Bethe subalgebra of the Yangian studied in [**25**].

As explained previously, using equation (2.4.21), we can extract the eigenvalue of the missing label operator y from $\Lambda_2(u)$. As a check, we solved the Bethe equations for a few low values of λ, μ. We wrote them in terms of the elementary symmetric polynomials and solved them using the Gröbner basis as explained in [**20**]. One verifies that the obtained eigenvalues of the operator y (see table 2.1) coincide with the ones obtained from direct diagonalization of the tridiagonal matrix y given in (2.3.23)-(2.3.28).

(λ,μ)	$(\alpha_{11},\alpha_{22},\alpha_{33})$	L	M	Elementary symmetric polynomials w.r.t. the Bethe roots $(e_1, e_2, \ldots, e_M)$	Eigenvalues of y
(1,0)	$(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$	1	0	-	$\frac{5}{64}$
(2,1)	$(\frac{5}{3}, -\frac{1}{3}, -\frac{4}{3})$	1	2	$\left(\frac{2}{3}, -\frac{13}{36}\right)$	$\frac{31}{192}$
		2	1	$(-\frac{5}{18})$	$-\frac{35}{64}$
		3	0	-	$\frac{37}{64}$
(2,2)	(2, 0, −2)	0	4	$(\frac{4}{3}, -\frac{97}{90}, -\frac{11}{5}, \frac{521}{3600})$	$\frac{29}{64}$
		2	2	$(\frac{2}{3}, -\frac{7}{12})$	$\frac{29}{64}$
				$(-\frac{6}{7}, -\frac{29}{196})$	$-\frac{99}{64}$
		3	1	$(-\frac{1}{2})$	$-\frac{75}{64}$
		4	0	-	$\frac{69}{64}$

Table 2.1. Examples of solutions to Bethe equations (2.4.30) and the corresponding eigenvalue of y.

2.5. Conclusion

Summing up, we have shown how the analytical Bethe ansatz could be generalized so as to provide a powerful method to diagonalize the $SO(3)$ scalar operator of degree four in the $su(3)$ generators whose eigenvalues give the basis vectors of irreducible modules corresponding to the $SU(3) \supset SO(3) \supset SO(2)$ non-canonical subgroup chain. This adds significant new understanding to an important and thoroughly studied problem by showing how the Bethe ansatz techniques can advance its resolution.

There are other well known and physically important occurrences of missing label problems such as the ones corresponding to the subgroup chains: $SU(4) \supset SU(2) \otimes SU(2)$, $O(5) \supset SU(2) \otimes U(1)$, $O(5) \supset O(3)$, etc. It would be of significant interest to explore further the applicability of the Bethe ansatz methods to the resolution of these and other instances where the degeneracies arise.

The algebra generated by x and y with relations given in (2.2.8) brings reminiscences of algebras of Heun–Askey–Wilson type [5] and must encode the properties of the overlaps between the two basis that are made out of eigenvectors of x and y respectively. It would be of interest to understand this more deeply.

In this vein, understanding the structure of the algebra associated to the overlaps between the basis corresponding to the canonical chain $SU(3) \supset SU(2) \otimes U(1)$ and the non-canonical one, $SU(3) \supset SO(3)$, is worth attention and would open up many new avenues. We shall keep looking into those questions.

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References

[1] D. Arnaudon, N. Crampe, A. Doikou, L. Frappat and E. Ragoucy, *Analytical Bethe Ansatz for closed and open gl(n)-spin chains in any representation*, J. Stat. Mech. (2005) P02007 and `math-ph/0411021`.

[2] D. Arnaudon, N. Crampe, A. Doikou, L. Frappat and E. Ragoucy, *Analytical Bethe Ansatz for open spin chains with soliton non preserving boundary conditions*, Int. J. Mod. Phys. A **21** (2006) 1537.

[3] S. Artz, L. Mezincescu and R. I. Nepomechie, *Spectrum of the Transfer Matrix for the* $U_q(B_n)$*-invariant* $A_{2n}^{(2)}$ *Open Spin Chain*, Int. J. Mod. Phys. A **10** (1995) 1937.

[4] V. Bargmann and M. Moshinsky, *Group theory of harmonic oscillators (II). The integrals of Motion for the quadrupole-quadrupole interaction*, Nuclear Phys. **23** (1961) 177.

[5] P. Baseillac, S. Tsujimoto, L. Vinet and A.Zhedanov, *The Heun–Askey–Wilson algebra and the Heun operator of Askey–Wilson type*, Annales Henri Poincaré **20** (2019) 3091.

[6] R. J. Baxter, *Partition function of the eight-vertex lattice model*, Ann. Phys. **70** (1972) 193.

[7] R. J. Baxter, *Asymptotically degenerate maximum eigenvalues of the eight-vertex model transfer matrix and interfacial tension*, J. Stat. Phys. **8** (1973) 25.

[8] R. J. Baxter, *Exactly solved models in statistical mechanics* (Academic Press, 1982).

[9] P.-A. Bernard, N. Crampe, D. Shaaban Kabakibo and L. Vinet, *Heun operator of Lie type and the modified algebraic Bethe ansatz*, J. Math. Phys. **083501** (2021) 62.

[10] R. Campoamor-Stursberg, *Internal labelling operators and contractions of Lie algebras*, Journal of Physics A: Math. and Theor. **40** (2007) 14773.

[11] N. Crampe, L. Poulain d'Andecy and L. Vinet, *A Calabi–Yau algebra with E_6 symmetry and the Clebsch–Gordan series of $sl(3)$*, Journal of Lie Theory **31** (2021) 1085.

[12] N. Crampe, L. Poulain d'Andecy and L. Vinet, *The missing label of $\mathfrak{su}(3)$ and its symmetry*, `arXiv:2110.03521`.

[13] H. De Meyer, G. Vanden Berghe and J. Van der Jeugt, On the spectra of $SO(3)$ scalars in the enveloping algebra of $SU(3)$, J. Math. Phys **26** (1985) 3109.

[14] V. G. Drinfel'd, *Hopf algebras and the quantum Yang–Baxter equation*, Soviet. Math. Dokl. **32** (1985) 254.

[15] V. G. Drinfel'd, *A new realization of Yangians and quantized affine algebras*, Soviet. Math. Dokl. **36** (1988) 212.

[16] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. **1** (1990) 193.

[17] W. Fulton and J. Harris, *Representation theory, a first course*, Springer-Verlag, Graduate Texts in Mathematics, **129** (1991).

[18] J. Hughes, *$SU(3)$ in an $O(3)$ basis, II. Solution of the state labelling problem*, J. Phys. A: Math. Nucl. Gen **6** (1973) 281.

[19] P. D. Jarvis and R. B. Zhang, *Resolution of the $GL(3) \supset O(3)$ state labelling problem via the $O(3)$-invariant Bethe subalgebra of the twisted Yangian*, J. Phys. A: Math. Gen. **38** (2005) L219.

[20] Y. Jiang and Y. Zhang, *Algebraic geometry and Bethe ansatz. Part I. The quotient ring for BAE*, J. High Energ. Phys. **2018** (2018), 3, 87.

[21] B. R. Judd, W. Miller Jr., J. Patera and P. Winternitz, *Complete sets of commuting operators and $O(3)$ scalars in the enveloping algebra of $SU(3)$*, J. Math. Phys. **15** (1974) 1787.

[22] L. Mezincescu and R. I. Nepomechie, *Analytical Bethe Ansatz for quantum-algebra-invariant spin chains*, Nucl. Phys. B **372** (1992), 3, 597.

[23] A. Molev, M. Nazarov and G. Olshanski, *Yangians and classical Lie algebras*, Russian Math. Survey **51** (1996) 205 and `hep-th/9409025`.

[24] M. Moshinsky, J. Patera, R. T. Sharp and P. Winternitz, *Everything you always wanted to know about SU(3) $\supset$ 0(3)*, Annals of Physics **95** (1975) 139.

[25] E. Mukhin, V. Tarasov and A. Varchenko, *Bethe eigenvectors of higher transfer matrices*, J. Stat. Mech. Theory Exp. **8** (2006) P08002 and `arXiv:math/0605015`.

[26] M. Nazarov and G. Olshanski, *Bethe subalgebras in twisted Yangians*, Commun. Math. Phys. **178** (1996) 483–506.

[27] F. Pan, S. Yuan, K. D. Launey, and J. P. Draayer, *A new procedure for constructing basis vectors of* $SU(3) \supset SO(3)$, Nucl. Phys. A **952** (2016) 78.

[28] J. Patera and P. Winternitz, *A new basis for the representations of the rotation group. Lamé and Heun polynomials*, J. Math. Phys. **14** (1973) 1130-1139.

[29] G. Racah, *On the decomposition of tensors by Contraction*, Rev. Mod. Phys. **21** (1949) 494.

[30] G. Racah, *Lectures on Lie groups*, Group Theoretical Concepts and Methods in Elementary Particle Physics, F. Giirsey, Ed., Gordon and Breach, New York (1964) 1.

[31] N. Yu. Reshetikhin, *A method of functional equations in the theory of exactly solvable quantum systems*, Lett. Math. Phys. **7** (1983) 205.

[32] N. Yu. Reshetikhin, *The functional equation method in the theory of exactly soluble quantum systems*, Sov. Phys. JETPS **57** (1983) 691.

[33] N. Yu. Reshetikhin, *The spectrum of the transfer matrices connected with Kac–Moody algebras*, Lett. Math. Phys. **14** (1987) 235.

[34] V. N. Tolstoy, $SU(3)$ *symmetry for Orbital Angular Momentum and Method of External Projection Operators*, Physics of Atomic Nuclei **69** (2006) 1058.

[35] V. I. Vichirko and N.Yu. Reshetikhin, *Excitation spectrum of the anisotropic generalization of an* SU_3 *magnet*, Theor. Math. Phys. **56** (1983) 805.

[36] S. I. Vinitsky, C. Burdik, A. A. Gusev, A. Deveikis, A. Gozdz, A. Pędrak and P. M. Krassovitskiy, *On generation of the Bargmann–Moshinsky basis of* $SU(3)$ *group*, J. Phys: Conf. Ser. **1194** (2019) 012109.

[37] C. N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Rev. Lett. **19** (1967) 1312.

Conclusion

Dans l'ensemble, ce mémoire traite l'opérateur de Heun de type Lie ainsi que le problème d'étiquette manquante de l'algèbre $su(3)$. L'ansatz de Bethe sert de trait d'union entre les deux sujets. En premier lieu, une généralisation des techniques de l'ansatz de Bethe algébrique permet de diagonaliser l'opérateur de Heun de type Lie. Des liens entre ce dernier et l'équation différentielle de Heun mène à démontrer la complétude du spectre. Des applications de ces résultats en théorie des représentation de $O(3)$ et le calcul d'entropie d'intrication de fermions libres sur une chaine de Krawtchouk sont également abordées. En deuxième lieu, la diagonalisation d'un sous algèbre abélien du yangian twisté $Y(gl(3); so(3))$ conduit à la diagonalisation de l'opérateur d'étiquette manquante y de degré quatre dans les générateurs de la chaine $su(3) \supset so(3)$.

Un aspect qui a été examiné durant l'étude des représentations de $su(3)$ et qui ne fait pas l'objet de l'article présenté est le lien algébrique entre la réduction $su(3) \supset so(3) \supset so(2)$ et la réduction $su(3) \supset s[u(2) \otimes u(1)] \supset s[u(1) \otimes u(1)]$. Cette dernière constitue la chaine habituelle utilisée pour construire les représentations irréductibles de $su(3)$. Les vecteurs de base sont ainsi étiquetées par les deux éléments de Casimir g_2, g_3 de $su(3)$, l'élément de Casimir de $su(2)$, noté I^2, ainsi que les deux éléments de Cartan I_3 et Y correspondant aux sous algèbres $u(1)$. Cette base intervient en physique des particules. Les éléments I^2 et I_3 décrivent l'isospin, tandis que Y donne l'hypercharge. En termes des générateurs de $gl(3)$, cette sous-algèbre $su(2)$ s'écrit comme

$$I_3 = \frac{E_{11} - E_{33}}{2}, \qquad I_+ = -E_{13}, \qquad I_- = -E_{31}. \tag{2.5.1}$$

L'élément de Casimir est donné par $\mathbf{I}^2 = \frac{1}{2}(I_+I_- + I_-I_+) + I_3^2$. L'hypercharge Y s'exprime de la façon suivante :

$$Y = \frac{1}{3}(E_{11} + E_{33} - 2E_{22}). \tag{2.5.2}$$

Il existe en fait un “chevauchement” algébrique entre les deux bases. Plus précisément, les deux opérateurs d'étiquette manquante x et y de la chaine $su(3) \supset so(3)$ peuvent être exprimés en termes de $\mathbf{L}^2$, Y, L_3, g_2, g_3 et leurs commutateurs. Pour l'opérateur x, cette relation s'écrit comme:

$$x = \frac{1}{32}\Big(\frac{3}{16}[[[\mathbf{L}^2,Y],Y],[\mathbf{L}^2,Y]] - \frac{1}{2}[\mathbf{L}^2,[\mathbf{L}^2,Y]] + 3(2+g_2+2L_3^2)Y + \frac{9}{4}\{Y,[[\mathbf{L}^2,Y],Y]\} - 5\{Y,\mathbf{L}^2\} - 12Y^3 - g_3\Big). \tag{2.5.3}$$

On note aussi que $L_3 = 2I_3$. Il existe une relation similaire pour y mais elle est assez élaborée et elle n'est donc pas présentée ici. D'autres relations algébriques d'intérêt incluent

$$[[[\mathbf{L}^2,Y],Y],Y] = 4[\mathbf{L}^2,Y], \tag{2.5.4}$$

$$[[[x,Y],Y],Y] = 4[x,Y], \tag{2.5.5}$$

$$[[[\mathbf{L}^2,\mathbf{I}^2],\mathbf{I}^2],\mathbf{I}^2] = 2\{\mathbf{I}^2,[\mathbf{L}^2,\mathbf{I}^2]\} = 2[\{\mathbf{L}^2,\mathbf{I}^2\},\mathbf{I}^2], \tag{2.5.6}$$

$$[[[x,\mathbf{I}^2],\mathbf{I}^2],\mathbf{I}^2] = 2\{[x,\mathbf{I}^2],\mathbf{I}^2\} = 2[\{x,\mathbf{I}^2\},\mathbf{I}^2]. \tag{2.5.7}$$

On observe que les équations (2.5.4) et (2.5.5) prennent la même forme que les relations de Donald–Grady [**10**]. Il serait d'intérêt de donner toutes les relations de commutation et de caractériser entièrement l'algèbre de chevauchement tel qu'il a été fait pour le cas de l'algèbre de Hahn et les coefficients de Clebsch–Grodan [**13**].

Pour conclure, il serait particulièrement intéressant de pouvoir faire une identification similaire à celle de l'opérateur d'étiquette manquante y pour l'opérateur x dans un contexte de systèmes intégrables. Ceci donnerait la possibilité de le diagonaliser à l'aide de l'ansatz de Bethe. Les techniques d'ansatz de Bethe employées à travers ces deux articles pourraient certainement s'appliquer à d'autres sujets connexes. En effet, l'ansatz de Bethe algébrique a été auparavant employé pour diagonaliser l'opérateur de Heun–Askey–Wilson [**2**]. Ces idées pourront servir de pistes de réflexion pour de futurs projets faisant suite au présent mémoire.

Références bibliographiques

[1] V. Bargmann and M. Moshinsky, *Group theory of harmonic oscillators (II). The integrals of Motion for the quadrupole-quadrupole interaction*, Nuclear Phys. **23** (1961) 177.

[2] P. Baseilhac and R. A. Pimenta, *Diagonalization of the Heun–Askey–Wilson operator, Leonard pairs and the algebraic Bethe ansatz*, Nucl. Phys. B **949** (2019) 114824.

[3] P. Baseilhac, S. Tsujimoto, L. Vinet and A. Zhedanov, *The Heun–Askey–Wilson algebra and the Heun operator of Askey–Wilson type*, Ann. Henri Poincaré **20** (2019), 3091.

[4] P. Baseilhac, L. Vinet and A. Zhedanov, *The q-Heun operator of big q-Jacobi type and the q-Heun algebra*, Ramanujan J. **52** (2020) 367.

[5] G. Bergeron, N. Crampe, S. Tsujimoto, L. Vinet, and A. Zhedanov, *The Heun–Racah and Heun–Bannai–Ito algebras*, J. Math. Phys. **61** (2020) 081701.

[6] H. Bethe, *Zur Theorie der Metalle. Eigenwerte und Eigenfunktionen der linearen Atomkett*, Z. Phys. **71** (1931) 205.

[7] N. Crampe, *Algebraic Bethe ansatz for the XXZ Gaudin models with generic boundary*, SIGMA **13** (2017) 094.

[8] N. Crampe, L. Poulain d'Andecy and L. Vinet, *A Calabi–Yau algebra with E_6 symmetry and the Clebsch–Gordan series of $sl(3)$*, `arXiv:2005.13444`.

[9] N. Crampe, L. Vinet, and A. Zhedanov, *Heun algebras of Lie type*, Proc. Amer. Math. Soc. (2019).

[10] L. Dolan and M. Grady, *Conserved charges from self-duality*, Phys. Rev. D **25** (1982) 1587.

[11] M. Gaudin, *Diagonalisation d'une classe d'hamiltoniens de spin*, J. Phys. **37** (1976) 1087.

[12] M. Gaudin, *La fonction d'onde de Bethe*, chapter 13 Masson, Paris, (1983).

[13] Ya. A. Granovskii, and A. S. Zhedanov, *Nature of the symmetry group of the $6j$-symbol*, Zh. Eksp. Teor. Fiz. **94** (1988) 49.

[14] F. A. Grünbaum, L. Vinet, and A. Zhedanov, *Tridiagonalization and the Heun equation*, Journal of Mathematical Physics, **58** (2017), 3, 031703.

[15] F. A. Grünbaum, L. Vinet, and A. Zhedanov, *Algebraic Heun Operator and Band-Time Limiting*, Communications in Mathematical Physics, **364** (2018), 3, 1041.

[16] K. Heun, *Zur Theorie der Riemann'schen Functionen zweiter Ordnung mit vier Verzweigungspunkten*, Mathematische Annalen, **33** (1888), 2, 161.

[17] P. D. Jarvis and R. B. Zhang, *Resolution of the $GL(3) \supset O(3)$ state labelling problem via the $O(3)$-invariant Bethe subalgebra of the twisted Yangian*, J. Phys. A: Math. Gen. **38** (2005) L219.

[18] B. R. Judd, W. Miller Jr., J. Patera and P. Winternitz, *Complete sets of commuting operators and $O(3)$ scalars in the enveloping algebra of $SU(3)$*, J. Math. Phys. **15** (1974) 1787.

[19] P.P. Kulish and N. Yu. Reshetikhin, *Diagonalisation of gl(n) invariant transfer matrices and quatum N-wave system (Lee model)*, J. Phys. A **16** (1983) L591.

[20] R. Koekoek, P.A. Lesky, and R.F. Swarttouw, *Hypergeometric orthogonal polynomials and their qanalogues*, Springer, 1-st edition, 2010.

[21] M.Moshinsky, J.Patera, R. T. Sharp, and P. Winternitz, *Everything you always wanted to know about SU(3) $\supset$ 0(3),* Annals of Physics **95** (1975) 139.

[22] G. Racah, *Lectures on Lie groups*, Group Theoretical Concepts and Methods in Elementary Particle Physics, F. Giirsey, Ed., Gordon and Breach, New York (1964) 1.

[23] G. Racah, lecture notes, *Group theory and spectroscopy*, Springer Tracts Mod. Phys. **37** (1965) 28.

[24] N. Yu. Reshetikhin, *A method of functional equations in the theory of exactly solvable quantum systems*, Lett. Math. Phys. **7** (1983) 205.

[25] A. Ronveaux (Ed.), *Heun's Differential Equations*, Oxford University Press, Oxford, 1995.

[26] E. K. Sklyanin, *Separation of variables in the Gaudin model*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **164** (1987) 151; translation in J. Soviet Math. **47** (1989) 2473.

[27] E. K. Sklyanin and T. Takebe, *Algebraic Bethe ansatz for the XYZ Gaudin model*, Phys. Lett. A **219** (1996) 217.

[28] T. Skrypnyk, *Generalized Gaudin spin chains, nonskew symmetric r-matrices, and reflection equation algebras*, J. Math. Phys. **48**(2007) 113521.

[29] E.K. Sklyanin, L.A. Takhtadzhyan, and L.D. Faddeev, *Quantum inverse problem method. I*, Theor. Math. Phys. **40** (1979) 688.

[30] L. Vinet and A. Zhedanov, *The Heun operator of Hahn type*, Proc. Amer. Math. Soc. **147** (2019), 2987.

[31] A. S. Zhedanov, *"Hidden symmetry" of the Askey-Wilson polynomials,* Theoret. Math. Phys. **89** (1991) 1146.