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Hamiltonian Floer theory on surfaces

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Résumé

Dans cette thèse, nous développons de nouveaux outils pour relier les dynamiques qualitatives des systèmes hamiltoniens sur des surfaces aux propriétés algébriques de leurs *complexes de Floer* - un objet algébrique qui encode l'information sur la façon dont les orbites 1-périodiques d'un système sont reliées par des cylindres satisfaisant une équation différentielle partielle elliptique appelée *l'équation de Floer*.

L'idée principale est de considérer — pour un hamiltonian $H \in C^\infty(S^1 \times \Sigma)$ sur une surface symplectique (Σ, ω) — les graphes des orbites contractiles 1-périodiques de l'isotopie $(\phi_t^H)_{t\in[0,1]}$ comme définissant une *tresse* P^H dans $S^1 \times \Sigma$. En choisissant des capuchons pour chacune de ces orbites 1-périodiques, nous obtenons un objet que nous appelons *une tresse encapuchonnée* $\hat{P}^H$, qui est muni d'une fonction d'indexation $\mu_{CZ} : \hat{P}^H \to \mathbb{Z}$ obtenue en assignant à chaque brin (encapuchonné) l'indice de Conley-Zehnder de l'orbite encapuchonnée associée. L'idée est alors de s'interroger sur la relation entre l'information topologique encodée dans la tresse encapuchonnée indexée $(\hat{P}^H, \mu_{CZ})$ et la structure du complexe de Floer $CF_*(H,J)$ pour une structure presque complexe générique J. À cette fin, nous aurons recours à : un nouvel invariant relatif pour les paires de tresses encapuchonnées que nous appelons *le nombre d'enlacement homologique*, un cercle d'idées concernant le comportement asymptotique des courbes pseudo-holomorphes développé par Hofer-Wysocki-Zehnder dans leur série d'articles [**8**], [**10**], [**12**] et aussi [**11**] (ainsi qu'un raffinement supplémentaire dans le cas relatif dû à Siefring dans [**32**]), et une nouvelle technique en basses dimensions pour la construction de morphismes de continuation de Floer qui ont un comportement prescrit.

En conséquence de ces techniques, nous établissons l'existence — pour des systèmes hamiltoniens génériques sur une surface fermée arbitraire — de certaines feuilletages singulières spéciaux sur $S^1 \times \Sigma$ dont le comportement est étroitement lié à la fois à la dynamique sous-jacente et à la structure du complexe de Floer du système. La construction de tels feuilletages dans le cas particulier

des pseudo-rotations d'un disque, par des méthodes très différentes des nôtres, a été au cœur des progrès significatifs récents de Bramham dans [**3**] sur une célèbre question de Katok concernant les systèmes conservatifs de basse dimension et d'entropie nulle. Ces feuilletages fournissent également, pour les systèmes hamiltoniens lisses génériques, une construction Floer-théorique des feuilletages positivement transversaux sur Σ qui ont été construits originellement (pour les homéomorphismes de surface généraux) par Le Calvez à travers d'une extension substantielle de la théorie de Brouwer classique pour les homéomorphismes de surface dans [**16**].

En plus de fournir un pont géométrique entre la dynamique d'une isotopie hamiltonienne et l'information algébrique contenue dans son complexe de Floer, les techniques développées dans cette thèse permettent également de donner une caractérisation — purement en termes de la dynamique de l'isotopie hamiltonienne sous-jacente — des cycles de Floer dans $CF_*(H,J)$ qui représentent la classe fondamentale de la surface et qui de plus se trouvent dans l'image d'un morphisme de PSS au niveau des chaines.

Finalement, ces techniques permettent de définir une nouvelle famille d'invariants d'un système hamiltonien (sur une variété symplectique arbitraire) qui se comporte formellement de manière similaire à une famille bien étudiée de tels invariants connue comme les *invariants spectraux de Oh-Schwarz*. L'avantage de nos nouveaux invariants est que nous sommes capable de calculer explicitement les plus importants d'entre eux pour des systèmes hamiltoniens génériques sur des surfaces arbitraires, ce uniquement en termes de topologie relative des orbites périodiques du système (avec leurs indices de Conley-Zehnder). Ceci généralise un résultat de Humilière-Le Roux-Seyfaddini dans [**13**] dans lequel ils ont donné une caractérisation dynamique du principal invariant spectral de Oh-Schwarz dans le cas de systèmes hamiltoniens autonomes sur des surfaces de genre positif.

Mots-clefs: Topologie symplectique, systèmes hamiltoniens, théorie de Floer hamiltonienne, courbes pseudo-holomorphes, tresses, systèmes dynamiques de basse dimension, enlacement, feuilletages positivement transverses, invariants spectraux.

Abstract

In this thesis, we develop novel tools for relating the qualitative dynamics of Hamiltonian systems on surfaces to the algebraic properties of their *Floer complexes* — an algebraic object which encodes information about the ways in which a system's 1-periodic orbits are connected by cylinders satisfying an elliptic partial differential equation known as *Floer's equation*.

The main idea is to consider — for a generic Hamiltonian $H \in C^\infty(S^1 \times \Sigma)$ on a symplectic surface (Σ,ω) — the graphs of the contractible time-1 periodic orbits of the isotopy $(\phi_t^H)_{t\in[0,1]}$ as defining a *braid* P^H in $S^1 \times \Sigma$. Upon choosing cappings for each such 1-periodic orbit, we obtain an object which we term a *capped braid* $\hat{P}^H$, which comes equipped with an indexing function $\mu_{CZ} : \hat{P}^H \to \mathbb{Z}$ given by assigning to each (capped) strand of the braid the Conley-Zehnder index of the associated capped orbit. The idea is then to enquire into the relation of the topological information encoded in the indexed capped braid $(\hat{P}^H,\mu_{CZ})$ and the structure of the Floer complex $CF_*(H,J)$ for a generic J. The main tools employed to this end are: a novel relative invariant for pairs of capped braids which we term the *homological linking number*, a circle of ideas about the asymptotic behaviour of pseudo-holomorphic curves pioneered by Hofer-Wysocki-Zehnder in their series of papers [**8**], [**10**], [**12**] as well as in [**11**] (along with a further refinement to the relative case by Siefring in [**32**]), and a novel technique for the construction of regular Floer continuation maps in low-dimensions having prescribed behaviour.

As a consequence of these techniques, we establish the existence — for generic Hamiltonian systems on an arbitrary closed surface Σ — of certain special singular foliations on $S^1 \times \Sigma$ whose behaviour is tightly related to both the underlying dynamics, as well as the structure of the system's Floer complex. The construction of such foliations (by very different methods) in the particular case of pseudo-rotations on a disk was the crux of Bramham's recent significant progress in [**3**] on a famous question due to Katok about low-dimensional conservative systems with vanishing entropy. These foliations also provide, for generic smooth Hamiltonian systems,

a Floer-theoretic construction of the positively transverse foliations on Σ which were originally constructed (for general surface homeomorphisms) by Le Calvez through a significant extension of classical Brouwer theory for surface homeomorphisms in [**16**].

In addition to providing a geometric bridge between the dynamics of a Hamiltonian isotopy and the algebraic information contained in its associated Floer complex, the techniques developed in this dissertation also permit a characterization — purely in terms of the dynamics of the underlying Hamiltonian isotopy — of those Floer cycles in $CF_*(H,J)$ which represent the fundamental class of the surface, and which moreover lie in the image of some chain-level PSS map.

Finally, these techniques permit the definition of a new family of invariants of a Hamiltonian system (on an arbitrary symplectic manifold) which behave formally similarly to a well-studied family of such invariants known as 'Oh-Schwarz spectral invariants' (and which agree with them in all known cases). The advantage of these novel spectral invariants is that we are able to explicitly compute the most important of these spectral invariants for generic Hamiltonian systems on arbitrary surfaces purely in terms of the relative topology of the system's periodic orbits (together with their Conley-Zehnder indices). This considerably generalizes a result by Humilière-Le Roux-Seyfaddini in [**13**] in which they gave a dynamical characterization of the main Oh-Schwarz spectral invariant in the case of time-independent Hamiltonian systems on surfaces with positive genus.

Keywords: Symplectic topology, Hamiltonian systems, Hamiltonian Floer theory, Pseudo-holomorphic curves, Braids, Low-dimensional dynamical systems, Linking, Positively transverse foliations, Spectral invariants

Contents

Résumé 5

Abstract 7

Acknowledgments 11

Introduction 15

0.1. Structure of this thesis 20

Chapter 1. Capped braids and homological linking 25

1.1. Capped Braids 26

1.2. Linking of capped loops with close strands 29

1.3. The homological linking number for capped braids 30

Chapter 2. Elements of Floer theory and linking 37

2.1. Floer theory 37

2.1.1. Continuation maps 42

2.1.2. The PSS isomorphism 43

2.1.3. The Gromov trick 45

2.2. Asymptotic analysis for pseudoholomorphic cylinders 45

2.2.1. Winding of eigenvectors of $A_{x,J}$ 49

Chapter 3. Constructing chain-level continuation maps with prescribed behaviour 55

3.1. Perturbing positively intersecting strands to solve Floer's equation near the intersection 57

3.2. Proof of Theorem 3.0.4 60

3.3. From models of cobordisms to regular models 62

3.4. Appendix 63

3.4.1. A short fibration argument 63

3.4.2. A proof of Lemma 3.3.2 65

Chapter 4. Application: A new spectral invariant and its computation on surfaces 69

4.1. A new spectral invariant: definitions and properties 70

4.1.1. Basic Properties of c_{im} 70

4.1.2. A new spectral norm 78

4.2. Computing $c_{im}([\Sigma]; H)$ on surfaces 79

4.3. Dynamical Consequences 84

4.4. On the equivalence $c_{OS} = c_{im}$ 88

Chapter 5. Application: Positively transverse foliations from Floer theory 93

5.1. Foliated sectors and Floer moduli spaces as leaf spaces 94

5.2. The restricted complex associated to a capped braid 97

5.2.1. Maximal unlinkedness relative the Morse range 97

5.2.2. Dominating Morse functions and dominating homotopies 99

5.3. Construction and properties of $\mathcal{F}^{\hat{X}}$ 105

5.3.1. $\mathcal{F}^{\hat{X}}$ as negative gradient flow-lines of the restricted action functional 110

5.3.2. $\mathcal{F}^{\hat{X}}$ as a positively transverse foliation 110

5.4. Consequences for the structure of Hamiltonian isotopies 113

5.4.1. Comparison to Le Calvez's theory of transverse foliations 114

5.5. Appendix: Stefan-Sussmann Foliations 116

References 119

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Introduction

The goal of this thesis is to develop a certain picture of Hamiltonian dynamics on surfaces via a set of tools which are well-suited to the study of the relationship between their dynamics and the structure of their Floer complexes. More precisely, recall that an isotopy

$$\phi^H : [0,1] \times \Sigma \to \Sigma$$

of a symplectic surface (Σ,ω) is called *Hamiltonian* if it is realized as the flow of the periodic family of vector fields $(X_t^H)_{t\in S^1}$, where each vector field satisfies

$$\omega(X_t^H,\cdot) = -dH_t(\cdot)$$

for some smooth family of functions $(H_t : \Sigma \to \mathbb{R})_{t\in S^1}$. Modulo some genericity conditions, to each such isotopy and any family $(J_t)_{t\in S^1}$ of ω-compatible almost complex structures, we may associate a complex $CF_*(H,J)$ which is generated by the set $\widetilde{Per}_0(H)$ of capped 1-periodic orbits of ϕ^H. This complex has a $\mathbb{Z}$-grading μ_{CZ} which counts the amount of 'symplectic winding' that occurs locally about the orbit throughout the isotopy, and whose differential counts the number of 'Floer cylinders' $u : S^1 \times \mathbb{R} \to \Sigma$ which connect these orbits. Together with μ_{CZ}, any collection of capped orbits $\hat{X} \subset \widetilde{Per}_0(H)$ such that the underlying orbits x,y of any two elements $\hat{x}, \hat{y} \in \hat{X}$ are distinct forms a geometric object which we call an *indexed capped braid* $(\hat{X}, \mu_{CZ})$. We can think of the topology of the indexed capped braids which make up $\widetilde{Per}_0(H)$ as encoding the topological structure of 1-periodic orbits of $(\phi_t^H)_{t\in[0,1]}$ along with the coarse local structure of the isotopy near these orbits. The broad question broached by this work may be stated as follows

What relations does the topological structure of $\widetilde{Per}_0(H)$ impose on the (filtered) algebraic structure of $CF_*(H,J)$ and vice versa?

Clearly, this question is only particularly meaningful in low-dimensions where the orbits may twist about one another in homotopically interesting ways, but in this situation it proves surprisingly fruitful and sheds considerable light on the possible dynamics of generic Hamiltonian isotopies on surfaces. To wit, we will say that a capped braid $\hat{X}$ is *unlinked* if the capping disks of the strands may be chosen such that their graphs in $D^2 \times \Sigma$ are disjoint. An indexed capped braid $\hat{X} \subset \widetilde{Per}_0(H)$ is *maximally unlinked relative the Morse range* if $\hat{X}$ is unlinked, every (capped) strand of $\hat{X}$ has index lying in the set $\{-1, 0, 1\}$ and $\hat{X}$ is maximal among all collections of capped orbits in $\widetilde{Per}_0(H)$ having these two properties. We write $murm(H)$ to denote the set of such capped braids. As a consequence of the theory developed herein, we obtain the following structural result for generic Hamiltonians H on an arbitrary symplectic surface.

Theorem A. *Let $H \in C^\infty(S^1 \times \Sigma)$ be a non-degenerate Hamiltonian, and let $J \in C^\infty(S^1; \mathcal{J}_\omega(\Sigma))$ be such that (H,J) is Floer regular. For any capped braid $\hat{X} \in murm(H)$, we may construct an oriented singular foliation $\mathcal{F}^{\hat{X}}$ of $S^1 \times \Sigma$ with the following properties*

(1) The singular leaves of $\mathcal{F}^{\hat{X}}$ are precisely the graphs of the orbits in $\hat{X}$.

(2) The regular leaves are annuli parametrized by maps

$$\begin{aligned} \check{u} : \mathbb{R} \times S^1 &\to S^1 \times \Sigma \\ (s,t) &\mapsto (t,u(s,t)). \end{aligned}$$

for $u \in \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J)$, for $\hat{x}, \hat{y} \in \hat{X}$.

(3) The vector field $\check{X}^H(t,z) = \partial_t \oplus X^H_t(z)$ is positively transverse to every regular leaf of $\mathcal{F}^{\hat{X}}$.

Similar foliations play a crucial role in Bramham's recent celebrated construction of periodic approximations for irrational pseudo-rotations of the disk in [**3**]. Our approach gives both the existence of such foliations for generic Hamiltonians, and moreover ties their behaviour directly to the structure of the Floer complex and the dynamics of the Hamiltonian isotopy.

The structure of the foliations $\mathcal{F}^{\hat{X}}$ could be *a priori* rather complicated, however, with Theorem A in hand, we can define the $\hat{X}$*-restricted action functional* $A^{\hat{X}} \in C^\infty(S^1 \times \Sigma)$ by $A^{\hat{X}}(t,u(s,t)) = \mathcal{A}_H(\hat{u}_s)$, for $\mathcal{A}_H : \widetilde{\mathcal{L}_0}(\Sigma) \to \mathbb{R}$ the *Hamiltonian action functional* on the space of capped loops (the capping of u_s is naturally induced by the cappings of the limiting orbits). $A^{\hat{X}}$ turns out to be a Morse-Bott function, and if we define $A^{\hat{X}}_t \in C^\infty(\Sigma)$, $t \in S^1$, to be its restriction to the fiber $\{t\} \times \Sigma$, we obtain an S^1-family of Morse functions, such that the negative gradient flow of $A^{\hat{X}}_t$ provides a singular foliation which coincides with the foliation $\mathcal{F}^{\hat{X}}_t$ given by intersecting $\mathcal{F}^{\hat{X}}$ with the fiber over $t \in S^1$. Sliding the fiber $\{0\} \times \Sigma$ along the circles $t \mapsto (t,u_s(t))$ provides a loop $(\psi^{\hat{X}}_t)_{t \in S^1}$, and we prove

Theorem B. *For every $t \in S^1$, $\mathcal{F}_t^{\hat{X}}$ is a singular foliation of Morse type. Moreover, the loop $(\psi_t^{\hat{X}})_{t\in S^1}$ is a contractible loop such that the orbits of $(\psi^{\hat{X}})^{-1} \circ \phi^H$ are positively transverse to the foliation $\mathcal{F}_0^{\hat{X}}$.*

We thereby reduce the study of the qualitative dynamics of the isotopy ϕ^H to the much better understood situation of dynamics which are positively transverse to a Morse-type foliation. Note that similar foliations (with a weaker notion of positive transversality) have been constructed by Le Calvez in [**16**] for Hamiltonian homeomorphisms, and play a central role in the forcing theory developed in [**18**]. The above result can be viewed as giving a Floer-theoretic construction of certain of Le Calvez's foliations, along with additional insight into their structure in the smooth case. We also obtain as a corollary the following somewhat surprising structural result about the topology of the braid generated by the 1-periodic orbits of H (a capped braid is said to be *linked* if it is not unlinked):

Theorem C. *Let H be non-degenerate, $\hat{X} = \{[x_i(t), w_i(se^{2\pi i t})]\}_{i=1}^k \in murm(H)$ and for any $m \in \mathbb{Z}_{>0}$, denote by $\hat{X}^{\sharp m} = \{[x_i(mt), w_i(se^{2m\pi i t}]\}_{i=1}^k \subset \widetilde{Per}_0(H^{\sharp m})$ its m-fold iterate. For any $\hat{y} \in \widetilde{Per}_0(H^{\sharp m})$ with $y(t) \neq x_i(mt)$ for all $i = 1, \ldots, k$ and $t \in S^1$, $\hat{X}^{\sharp m} \cup \hat{y}$ is linked. In particular every $\hat{X} \in murm(H)$ is maximally unlinked as a subset of $\widetilde{Per}_0(H)$.*

Remark that this is essentially the same relationship as exists between the periodic orbits of an *autonomous* Morse Hamiltonian and its critical points. Taken together with the observations that every capped orbit $\hat{x} \in \widetilde{Per}_0(H)$ with index in the Morse range lies in *some* $\hat{X} \in murm(H)$ and that any two orbits in the Morse range which are connected by a Floer cylinder are unlinked (see Corollary 5.1.4 in Section 5.1), the above results imply that the problem of understanding the structure of $CF_*(H,J)$ in the Morse range may be decomposed into the problem of understanding the finitely many Morse-type foliations $\{\mathcal{F}_0^{\hat{X}}\}_{\hat{X}\in murm(H)}$ on Σ which are associated to the Morse functions $\{A_0^{\hat{X}}\}_{\hat{X}\in murm(H)}$. Moreover, for each $\hat{X} \in murm(H)$, the qualitative dynamics of the isotopy $(\psi^{\hat{X}})^{-1} \circ \phi^H$ relative to the foliation $\mathcal{F}_0^{\hat{X}}$ are similar in fundamental ways to the dynamics of an autonomous Morse Hamiltonian relative to the foliations provided by its negative gradient trajectories.

The techniques developed in this thesis also enable a second, logically distinct (though thematically related), line of enquiry, examining the relationship between the qualitative dynamics of a Hamiltonian isotopy and those aspects of the filtered Floer complex which are 'probeable' by chain-level PSS maps. To be more precise, recall that there is a natural way to identify the quantum homology of a symplectic manifold with the Floer homology of a generic Floer pair (H,J) given by the

PSS isomorphism originally introduced in [**25**] (see also [**28**]). This isomorphism is induced on homology by choosing some ancillary data $\mathcal{D}$ — which includes a Morse-Smale pair (f,g) and so provides a Morse-theoretic model for the quantum chain complex $QC_*(f,g) = C^{Morse}(f,g) \otimes \Lambda_\omega$ whose homology computes the quantum homology of (M,ω) — and this data is then used to construct a chain morphism

$$\Phi_{\mathcal{D}}^{PSS} : QC_*(f,g) \mapsto CF_{*-n}(H,J).$$

The induced map on homology is an isomorphism and is independent of the ancillary data $\mathcal{D}$. Our next result provides, in the case of surfaces, a purely topological characterization of those non-trivial Floer cycles in $CF_*(H,J)$ which lie in the image of some chain-level PSS map and which represent the fundamental class. Note that for $\sigma = \sum a_{\hat{x}}\hat{x} \in CF_*(H,J)$, supp $\hat{\sigma} := \{\hat{x} : \ a_{\hat{x}} \neq 0\}$ may be thought of as a capped braid.

Theorem D. *Let $\sigma \in CF_1(H,J)$. σ is a non-trivial cycle such that $\sigma \in$ im $\Phi_{\mathcal{D}}^{PSS}$ for some regular PSS data $\mathcal{D}$ if and only if supp σ is a maximal positive capped braid relative index* 1.

See Section 4.2 for the definition of a 'maximal positive capped braid relative index 1'. The salient point is that this condition is purely topological and depends only on the qualitative dynamics of the underlying Hamiltonian system. This result is something of a novelty, since in general, it is very difficult (unless we are in the case of a C^2 small autonomous Hamiltonian) to identify candidates for cycles which represent the fundamental class in Floer homology. The above theorem provides us with a wealth of different such candidates having nice dynamical properties. Our methods also permit a similar type of characterization to be given for cycles which lie in the image of some chain-level PSS map and which represent the point class, although this characterization is somewhat more involved and we do not present it here. Interestingly though, there seem to be fundamental obstructions to using the same approach to characterize Floer cycles of middle degree lying in the image of some chain-level PSS map.

Theorem D motivates us to examine, for general symplectic manifolds, the quantity $c_{im}(\alpha; H)$ obtained by examining the infimal action level required to represent some non-zero quantum homology class α in the filtered Floer complex of H via some chain-level PSS map. It turns out that these quantities define spectral invariants which are intimately related to the geometry of Hamiltonian fibrations over disks and which we term the *PSS-image spectral invariants*. Theorem D thereby provides us with a purely topological formula for $c_{im}([\Sigma]; H)$ on arbitrary surfaces (here $\hat{X} \in mp_{(1)}(H)$ if $\hat{X} = \{\hat{x}_1, \ldots, \hat{x}_k\}$ is a maximally positive capped braid relative index 1)

Corollary. *Let H be non-degenerate.*

$$c_{im}([\Sigma];H) = \inf_{\hat{X}\in mp_{(1)}(H)} \sup_{\hat{x}\in\hat{X}} \mathcal{A}_H(\hat{x}).$$

The above result is quite similar in spirit to the characterization of the usual Oh-Schwarz spectral invariant associated to the fundamental class (see Section 2.1.2 for the definition) established by Humilière-Le Roux-Seyfaddini in [**13**] for autonomous Hamiltonians on aspherical surfaces, and, in fact we will see in Section 4.4 that their work implies that the two quantities agree in this setting (indeed, in many ways this work grows out of an attempt to understand what is going on in [**13**] in more directly geometric terms). Intriguingly, the class of orbits over which the minimax procedure is performed in the work of Humilière-Le Roux-Seyfaddini is quite different from the class of orbits which we consider; it would be interesting to understand the geometry behind why this is so.

In virtue of their definition, the PSS-image spectral invariants share many of the desirable formal properties of the usual Oh-Schwarz spectral invariants (see Section 2.1.2 for the definition), and so one can perform many of the same arguments with them, with the added advantage that the behaviour of the PSS-image spectral invariants are more tightly linked to the dynamical behaviour of the system under study — at least on surfaces. For instance, we may use the PSS-image spectral invariants to define a symplectically bi-invariant norm on $Ham(M,\omega)$ by

$$\gamma_{im}(\phi) := \inf\ c_{im}([M];H) + c_{im}([M];\bar{H}),$$

where the infimum is taken over Hamiltonians H such that $\phi_1^H = \phi$ and which are normalized so that $\int_M H_t\ \omega^n = 0$ for all $t \in S^1$. In the case of surfaces, the fact that the PSS-image spectral invariants share enough of the formal properties satisfied by the Oh-Schwarz spectral invariants allows Seyfaddini's argument establishing the C^0-continuity of the Oh-Schwarz spectral norm on surfaces (see Theorem 3 in [**31**]) to carry through form γ_{im}. As a consequence, we obtain

Theorem E. *On surfaces, the symplectically bi-invariant norm γ_{im} is both C^0-continuous and Hofer-continuous. Moreover, if ϕ is non-degenerate and $\Sigma \neq S^2$, then*

$$\gamma_{im}(\phi) = \inf_{\hat{X}\in mp_{(1)}(H)} \sup_{\hat{x}\in\hat{X}} \mathcal{A}_H(\hat{x}) - \sup_{\hat{X}\in mn_{(-1)}(H)} \inf_{\hat{x}\in\hat{X}} \mathcal{A}_H(\hat{x}),$$

for H any normalized Hamiltonian such that $\phi_1^H = \phi$.[1] *(Here $mn_{(-1)}(H)$ denotes the set of all capped braids $\hat{X} \subset \widetilde{Per}_0(H)$ which are 'maximally negative relative index -1'. See Section 4.2 for the definition).*

[1]There is of course a similar dynamical formula for γ_{im} on the sphere which is slightly more involved, taking into account the non-triviality of $\pi_1(Ham(S^2))$. See Corollary 4.3.3

The main interest in the above theorem, apart from the intrinsic interest of having a dynamically defined symplectically bi-invariant norm on Hamiltonian diffeomorphisms, is that the structure of the sets $mp_{(1)}(H)$ and $mn_{(-1)}(H)$ can change dramatically as one interpolates between Hamiltonians (in either the C^0 or the Hofer topology), however, the above theorem guarantees that the quantity obtained by the specified mini-max procedure over these collections remains continuous regardless. Finally, as a last example of the sort of information that can be drawn from Theorem D, we may exploit the computability of our newfound spectral invariants together with their close relationship to the Oh-Schwarz spectral invariants, in order to extract the following dynamical controls over the commutator lengths of homotopy classes $\tilde{\phi} \in \widetilde{Ham}(S^2,\omega)$ from Entov's work in [**5**].

Theorem F. *Assume that $H \in C^\infty(S^1 \times S^2)$ is non-degenerate and normalized so that $\int H_t\omega = 0$ for all $t \in S^1$, then*

$$\min\Big\{\min_{\hat{X} \in mp_{(1)}(H)} \max_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x}), -\max_{\hat{X} \in mn_{(-1)}(H)} \min_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x})\Big\} < -kArea(S^2,\omega)$$

only if the commutator length of $\tilde{\phi}^H$ in $\widetilde{Ham}(S^2)$ is strictly greater than $2k+1$.

0.1. Structure of this thesis

Chapters 1, 2 and 3 develop the main conceptual and technical tools in this work, which we then apply in Chapters 4 and 5 to deduce the results laid out in the previous section. These latter two chapters are independent and may be read in any order, but both depend heavily on the first three chapters.

Chapter 1 introduces the basic notion of *capped braids* and related basic language. The most essential development in this chapter is the definition of a relative invariant for capped braids which associates to any pair of capped k-braids $(\hat{X},\hat{Y})$ an integer $L_0(\hat{X},\hat{Y}) \in \mathbb{Z}$. We call this quantity the *homological linking number* of $\hat{X}$ and $\hat{Y}$. If we think of the (uncapped) braids X and Y as being graphs (suggestively called 'strands') in $S^1 \times \Sigma$ of loops in Σ), then roughly speaking, $L_0(\hat{X},\hat{Y})$ counts the number of signed intersections that occur between the strands if we start with the braid X and attempt to continuously deform it to the braid Y, subject to a certain homotopy condition coming from the cappings of $\hat{X}$ and $\hat{Y}$.

Chapter 2 introduces the elements of Floer theory of which we will have need, and explains the relevance of the homological linking number to the study of the Floer theory of low-dimensional systems. More precisely, one of the central geometric ideas in this work is that we may interpret collections of Floer-type cylinders as providing such deformations between the capped braid $\hat{X}$ formed by the capped orbits at their negative ends and the capped braid $\hat{Y}$ formed by the capped orbits at their positive ends. When we do this, the Gromov trick combined with positivity of intersections in dimension 4 implies that the homological linking number $L_0(\hat{X},\hat{Y})$ must be non-negative, which gives strong topological restrictions on the putative existence of collections of Floer cylinders in terms of the relative topologies of the capped braids formed by their negative and positive ends. There is a slight deficiency in this story however, in the sense that in general one may have multiple Floer cylinders emerging from or converging to the same orbit.

In order to deal with cylinders which emerge from or converge to the same orbit, we make use of the analysis of the relative asymptotic behaviour of pseudo-holomorphic curves developed by Siefring in [**32**], which pairs with work of Hofer-Wysocki-Zehnder in [**10**] to connect the Conley-Zehnder index of an orbit to bounds on the winding behaviour of pairs of cylinders which emerge from or converge to that orbit. In the contact setting, these sorts of bounds (in the non-relative case), along with the insight that under appropriate index conditions on the asymptotic orbits families of pseudoholomorphic curves automatically form local foliations in the symplectization of a contact manifold, go back to the pioneering work of Hofer-Wysocki-Zehnder in [**8**], [**10**], [**12**] and [**11**]. Siefring has also more recently put this circle of ideas to use in [**33**] to define an intersection number for arbitrary pseudoholomorphic curves in 4-dimensional symplectic cobordisms which is invariant under homotopy. In any case, we may thereby split the analysis of the behaviour of general collections of Floer cylinders into two portions: an asymptotic portion on the ends where the linking behaviour of cylinders asymptotic to the same orbit is controlled by the Conley-Zehnder index, and a compact portion which interpolates between two capped braids such that their intersections are controlled by the homological linking number of the capped braids at each end of this compact portion.

Chapter 3 is the last of the technical chapters, and introduces our technique for designing Floer continuation maps such that certain moduli spaces are non-empty. The basic idea is to show that, given any collection of smooth cylinders which topologically *could* be a collection of (s-dependent) Floer-type cylinders (in the sense that their graphs intersect only positively and they exhibit the appropriate asymptotic winding behaviour), we may — after a series of suitable small perturbations — always find regular Floer homotopy data $(\mathcal{H},\mathbb{J})$ such that the perturbed cylinders are all $(\mathcal{H},\mathbb{J})$-Floer.

In Chapter 4, we introduce the PSS-image spectral invariants and prove Theorem D and its various consequences. The main line of the proof of Theorem D is as follows. The fact that the cycle in question lies in the image of some chain-level PSS map implies the existence of various PSS disks, which may be thought of as providing a deformation from a trivial capped braid to the support of the cycle in question. The theory from Chapters 1 and 2 may then be used to force the support of this cycle to be positive, and when combined with the cycle's homological non-triviality this theory forces the support to be *maximally* positive (relative index 1). Conversely, in order to deduce that *every* such capped braid lies in the image of some chain level PSS map, it is sufficient to construct an appropriate continuation map from a small Morse function which sends the fundamental class of the Morse function to a cycle supported on the capped braid in question. The technique for designing continuation maps with prescribed behaviour developed in Chapter 3 may be seen to provide precisely such a continuation map.

Chapter 5 proves Theorems A, B and C. Theorem A is the main theorem from which the others follow as ready consequences. The rough structure of its proof is as follows. The theory developed in Chapter 2 may be used to show that if two capped orbits $\hat{x}, \hat{y}$ have Conley-Zehnder indices lying in the Morse range with $\widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J)$, then maps of the form $(s,t) \mapsto (t,u(s,t))$ for $u \in \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J)$ foliate some subset $W(\hat{x},\hat{y}) \subset S^1 \times \Sigma$. This fact was essentially discovered in the contact setting in [**12**] (see also [**11**]). We then study conditions under which collections $\hat{X}$ of capped orbits having indices lying in the Morse range may be chosen so that these foliated subsets piece together to form the desired foliation $\mathcal{F}^{\hat{X}}$. In order to do this, it turns out to be convenient to associate a 'restricted complex' $C_*(\hat{X})$ to a capped braid $\hat{X} \subset \widetilde{Per}_0(H)$ whose 'differential' counts the Floer cylinders running between capped orbits in $\hat{X}$ of index difference 1. In general, this is not actually a differential and $C_*(\hat{X})$ is not actually a chain complex. However, we show that when $\hat{X} \in murm(H)$, $CF_*(\hat{X}; H,J) := C_*(\hat{X})$ *is* a chain complex (although it is not, in general a *sub*complex of $CF_*(H,J)$). In order to study $CF_*(\hat{X}; H,J)$, we introduce a useful class of continuation maps which we call $\hat{X}$*-dominating* continuation maps from (resp. to) $\hat{X}$*-dominating* Morse functions. Roughly speaking, these are continuation maps such that for each $\hat{x} \in \hat{X}$, we can find some continuation moduli space $\mathcal{M}(\hat{p},\hat{x}; \mathcal{H},\mathbb{J})$ (resp. $\mathcal{M}(\hat{x},\hat{p}; \mathcal{H},\mathbb{J})$) which is non-empty. The techniques of Chapter 3 are used to establish their existence. The crucial point is that if $h_{\mathcal{H},\mathbb{J}}$ is a $\hat{X}$-dominating continuation map from a $\hat{X}$-dominating Morse function f, then the map

$$\pi^{\hat{X}} \circ h_{\mathcal{H},\mathbb{J}} : CF_*(f,J^-) \to CF_*(H,J) \to CF(\hat{X}; H,J),$$

where $\pi^{\hat{X}} : CF_*(H,J) \to CF(\hat{X};H,J)$ is the map which is the identity of $\Lambda_\omega\langle\hat{x}\rangle_{\hat{X}}$ and sends everything else to 0, is a chain map which induces an injection on homology. Moreover, such maps interact nicely with the cap action on the Floer chain complex in the sense that for a generic smooth cycle $\alpha^\#$,

$$(\pi^{\hat{X}} \circ h_{\mathcal{H},\mathbb{J}})(\alpha^\# \cap \hat{z}) = \alpha^\# \cap (\pi^{\hat{X}} \circ h_{\mathcal{H},\mathbb{J}})(\hat{z})$$

at the chain level. Consequently, if $\alpha^\#$ is a generic point $p \in \Sigma$ this implies that, when $\sum \hat{z} \in CF_*(f,J^-)$ represents the fundamental class

$$p \cap (\pi^{\hat{X}} \circ h_{\mathcal{H},\mathbb{J}})(\sum \hat{z}) = (\pi^{\hat{X}} \circ h_{\mathcal{H},\mathbb{J}})(p \cap \sum \hat{z}).$$

Since $[p \cap \sum \hat{z}] = [pt] \in QH_0(\Sigma) \simeq HF_{-1}(f)$ and $\pi^{\hat{X}} \circ h_{\mathcal{H},\mathbb{J}}$ is injective on homology, the left-hand side of the above equation is non-zero. This implies that through any generic $(t,p) \in S^1 \times \Sigma$, we can find $\hat{x},\hat{y} \in \hat{X}$ and a Floer cylinder $u \in \widetilde{\mathcal{M}}(\hat{x},\hat{y};H,J)$ such that $(t,u(0,t)) = (t,p)$. Consequently, when $\hat{X} \in murm(H)$, the union of the subspaces $W(\hat{x},\hat{y})$ over all $\hat{x},\hat{y} \in \hat{X}$ forms a foliated open, dense set of $S^1 \times \Sigma$. We may then use an argument pioneered in [**11**] to establish that, in fact, all of $S^1 \times \Sigma$ is foliated. The positive transversality property for $\mathcal{F}^{\hat{X}}$ is a direct consequence of the ω-compatibility of the almost complex structure and the fact that the regular leaves are parametrized by Floer cylinders.

Chapter 1

Capped braids and homological linking

In this chapter we introduce the basic notion of *capped braids*, their appropriate notion of equivalence (0-*homotopy*), and an important relative invariant of a pair of capped braids with the same number of strands which we call the *homological linking number*. These notions will later serve to encode the topology of the capped 1-periodic orbits of a non-degenerate Hamiltonian system, and give controls on the behaviour of Floer-type cylinders which may run between various collections of such capped orbits.

The basic definitions required for our work with capped braids are presented in Section 1.1. Section 1.2 explains a mild generalization of the classical winding number of loops in $\mathbb{R}^2$ to capped loops whose underlying loops are close in the loop space. This generalization becomes important later on in the work when we consider the asymptotics of Floer-type cylinders which emerge or converge to the same orbit. Section 1.3 introduces the homological linking number and establishes its basic properties including how it recovers the generalized winding number in the setting of Section 1.3.

For the duration of this thesis, Σ will always denote a smooth symplectic surface (Σ,ω), $\mathcal{L}(M)$ the (smooth) loop space of the manifold M, $\mathcal{L}_0(M)$ its space of contractible loops and $\widetilde{\mathcal{L}_0}(M)$ its Novikov covering space (see [**19**], Section 12.1).

Throughout this work, there is a certain amount of juggling of different perspectives on the same objects that will be necessary. In particular, though our main objects of study are isotopies on some surface Σ — and so the initial arena in which the action takes place is 2 dimensional — it will frequently be useful to work on the 3-dimensional mapping torus

$$\check{\Sigma} := S^1 \times \Sigma$$

as well as the 4-dimensional space

$$\widetilde{\Sigma} := \mathbb{R} \times S^1 \times \Sigma.$$

In order to have our notation be suggestive of these perspectival shifts, maps taking values in $\check{\Sigma}$ will frequently be adorned with a $\check{\cdot}$, while maps taking values in $\widetilde{\Sigma}$ will be adorned with a $\tilde{\cdot}$. The most frequent use-cases for this notation will be the following. For $x \in \mathcal{L}(M)$, we write $\check{x}(t) := (t,x(t))$ for its graph in $\check{M} = S^1 \times M$. For $u : I \times S^1 \to M$, where $I \subseteq \mathbb{R}$, we write $\tilde{u}(s,t) := (s,t,u(s,t))$ for its graph in $\widetilde{M} = I \times S^1 \times M$ and $\check{u}(s,t) := (t,u(s,t))$ for the projection of this graph onto $\check{M}$.

1.1. Capped Braids

Definition 1.1.1. For any $k \in \mathbb{N}$, we define the **k-configuration space**

$$C_k(\Sigma) := \{(z_1, \ldots, z_k) \in \Sigma^k : (i \neq j) \Rightarrow z_i \neq z_j\}$$

Definition 1.1.2. An **(ordered) k-braid** is an element $X = (x_1, \ldots, x_k) \in \mathcal{L}(C_k(\Sigma))$. Denote by $B^k(\Sigma)$ the space of ordered k-braids. The loop x_i is called the **i-th strand of** X, for $i = 1, \ldots, k$.

Definition 1.1.3. An **unordered k-braid** is an element $[X] \in \mathcal{L}(C_k(\Sigma))/S_k$, where S_k acts by permutation of coordinates. Such unordered braids may be identified with certain finite subsets of $\mathcal{L}(\Sigma)$.

Remark. We raise the distinction between ordered and unordered braids here mainly to flag for the reader that we will make no real effort outside of this section to separate these two concepts. In particular, we will routinely treat ordered braids as finite subsets of $\mathcal{L}(\Sigma)$ and perform set-wise operations on them, when properly speaking we should be speaking of the unordered braids which they represent. We will moreover speak simply of 'braids' relying on the context to make clear whether these braids are ordered or unordered. For the remainder of this section, we will make a clear distinction between ordered and unordered braids, mainly to convince the suspicious reader that nothing essential is lost in making this elision.

Definition 1.1.4. The **graph** $\check{X}$ of an (ordered) k-braid is the set-valued map $\check{X}(t) = \sqcup_{i=1}^k \check{x}(t) \subseteq S^1 \times \Sigma$, $t \in S^1$. The graph of an unordered braid $[X]$ is the graph of some (hence every) representative X of $[X]$.

Definition 1.1.5. An ordered k'-braid $Y \in B^{k'}(\Sigma)$ is an ordered **sub-braid** of $X \in B^k(\Sigma)$ if $Y \subseteq X$, as an ordered set. An unordered braid $[Y]$ is a sub-braid of $[X]$ if $[Y] \subseteq [X]$ as sets. There is an obvious partial ordering on the collection of sub-braids of X (resp. of $[X]$).

Definition 1.1.6. $X \in B^k(\Sigma)$ is **contractible** if each strand of X is a contractible loop. We write $B^k_0(\Sigma)$ for the space of contractible ordered k-braids. $[X] \in B^k(\Sigma)/S_k$ is contractible if some (hence every) representative X of $[X]$ is contractible.

Definition 1.1.7. A continuous map $h : [0,1] \to B^k(\Sigma)$ with $h(0) = X$, $h(1) = Y$ is a **braid homotopy from** X **to** Y. When such a map exists, we shall say that X and Y are **braid homotopic**, denoted $X \simeq Y$. The map $(s,t) \mapsto h_i(s,t)$ is called the **i-th strand of** h. To any braid homotopy, we associate its **graph** $\tilde{h}(s,t) = \sqcup_{i=1}^k \tilde{h}_i(s,t) \subseteq [0,1] \times S^1 \times \Sigma$, $(s,t) \in [0,1] \times S^1$.

Definition 1.1.8. An ordered braid X will be said to be **trivial** if all of its strands are constant maps. We will sometimes write $0 \in B^k(\Sigma)$ to stand for some fixed but arbitrary trivial braid, when the particular choices of the constant maps are unimportant. An unordered braid $[X]$ is trivial if some (hence every) ordered representative is trivial.

Definition 1.1.9. $X \in B^k(\Sigma)$ is **unlinked** if $X \simeq 0$. An ordered braid is **linked** if it is not unlinked. An unordered braid is unlinked (resp. linked) if some, hence every, ordered representative is unlinked (resp. linked).

Definition 1.1.10. A continuous map $h : [0,1] \to \mathcal{L}(\Sigma)^k$ with $h(0) = X \in B^k(\Sigma)$, $h(1) = Y \in B^k(\Sigma)$ will be called a **braid cobordism** if there exists some $\delta > 0$ such that $h(s) \in B^k(\Sigma)$, $\forall s \in (0,\delta) \cup (1-\delta,1)$.

Remark. We will frequently find ourselves concerned with maps $h : I \to \mathcal{L}(\Sigma)^k$, where $I = \mathbb{R}$ or $I = [a,b]$ for some $a,b \in \mathbb{R}$, and in the case that $I = \mathbb{R}$, it will always be the case that h extends continuously to a map $\bar{\mathbb{R}} \to \mathcal{L}(\Sigma)^k$ such that on some neighbourhood of $\pm\infty$, the graphs of the strands of h do not intersect. In such a case, we will speak freely of 'the' braid cobordism **induced** by h, which is simply any braid cobordism $h \circ \varphi$, where $\varphi : \bar{I} \to [0,1]$ is any orientation-preserving diffeomorphism.

Definition 1.1.11. An (ordered) **capped k-braid** $\hat{X}$ is an equivalence class $[X,\vec{w}]$ where $X \in B^k_0(\Sigma)$ and $\vec{w} = (w_1, \cdots, w_k)$ with $w_i : D^2 \to \Sigma$ a capping disk for the i-the strand of X, subject to the equivalence relation $[X,\vec{w}] \sim [X', \vec{w}']$ if and only if $X = X'$ and $[w_i]\#(-[w_i']) = 0 \in \pi_2(\Sigma)$ for each $i = 1, \cdots, k$. The space of ordered capped k-braids is denoted by $\widetilde{B}^k_0(\Sigma)$. The capped loop $\hat{x}_i = [x_i,w_i] \in \widetilde{\mathcal{L}_0}(\Sigma)$ is called the **i-th strand** of $\hat{X}$. The notion of capped sub-braids $\hat{Y} \subseteq \hat{X}$ is defined in the obvious way.

Remark. The distinction between ordered capped braids and unordered capped braids obtains here as well, and we adopt parallel conventions as those discussed in the case of braids in Remark 1.1.

$\pi_2(\Sigma)^k$ acts on $\widetilde{B}_0^k(\Sigma)$ by the obvious 'gluing of spheres':

$$(A_1,\ldots,A_k)\cdot([x_1,w_1],\ldots,[x_k,w_k]) = ([x_1,A_1\#w_1],\ldots,[x_k,A_k\#w_k]),$$

where here we abuse notation slightly by thinking of $A_i \in \pi_2(\Sigma, x_i(0))$ as being both a homotopy class of maps, as well as a particular choice of a representative from that class. This action does not descend to an action on $\widetilde{B}_0^k(\Sigma)/S_k$. However, if we denote by $Fix_{S_k}(\pi_2(\Sigma)^k) \simeq \pi_2(\Sigma)$ the set of fixed points of the action of the symmetric group on $\pi_2(\Sigma)^k$ by permutation of coordinates, we obtain a well-defined induced action by $Fix_{S_k}(\pi_2(\Sigma)^k)$ on unordered braids given by $(A,\ldots,A)\cdot[\hat{X}] = [(A,\ldots,A)\cdot\hat{X}]$, for $A \in \pi_2(\Sigma)$.

Definition 1.1.12. A trivial braid $0 \in B^k(\Sigma)$ has a naturally associated capping $\hat{0} \in \widetilde{B}_0^k(\Sigma)$ given by capping each strand of 0 with the constant capping. We call any such braid a **trivial capped braid**. When the particular components of a trivial capped braid are unimportant, we denote some fixed but arbitrary capped braid by the symbol $\hat{0}$. An unordered capped braid is said to be **trivial** if some (hence every) ordered representative is trivial.

Definition 1.1.13. For $A = (A_1,\cdots,A_k) \in \pi_2(\Sigma)^k$, an ordered braid cobordism h from X to Y will be called an A**-cobordism from** $[X,\vec{w}]$ **to** $[Y,\vec{v}]$ if $[w_i]\#[h_i]\#(-[v_i]) = A_i$, for all $i = 1,\cdots,k$. Whenever such a map exists, $[X,\vec{w}]$ and $[Y,\vec{v}]$ will be said to be A**-cobordant**. This notion descends to unordered capped braids provided that $A \in Fix_{S_k}(\pi_2(\Sigma)^k)$.

Definition 1.1.14. If $u : [0,1] \to \mathcal{L}_0(\Sigma)$ is a homotopy from x to y, then for any choice of cappings $\hat{x} = [x,w_x]$ and $\hat{y} = [y,w_y]$, u is an A-cobordism from $\hat{x}$ to $\hat{y}$, for $A = [w_x]\#[u]\#(-[w_y]) \in \pi_2(\Sigma)$. Moreover, for any $s \in [0,1]$, there are two natural choices of cappings for the loop $u_s \in \mathcal{L}_0(\Sigma)$. Namely, if we write $\alpha^s(\tau) := u(s\cdot\tau,t)$ and $\beta(\tau) := u(1-(1-s)\cdot\tau,t)$ for $s \in [0,1]$, then we may associate to u_s either of the cappings $[u_s,w_x\#\alpha^s]$ or $[u_s,w_y\#\beta^s]$, and these two cappings are obviously related by $A\cdot[u_s,w_y\#\beta^s] = [u_s,w_x\#\alpha^s]$. Consequently, if u is a 0-homotopy between $\hat{x}$ and $\hat{y}$, these two cappings agree and we may associate a unique capping

$$\hat{u}_s := [u_s, w_x\#\alpha^s] = [u_s, w_y\#\beta^s]$$

to each u_s in this case. We will call such a capping **the natural capping** of u_s whenever u is such a 0-homotopy.

Definition 1.1.15. An A-cobordism h from $\hat{X} = [X,\vec{w}]$ to $\hat{Y} = [Y,\vec{v}]$ is called an A-homotopy if h is in addition a braid homotopy from X to Y. In such a situation, we will say that $\hat{X}$ and $\hat{Y}$ are A**-homotopic**, and we will denote the relation by $\hat{X} \simeq_A \hat{Y}$. This notion descends to unordered capped braids, provided that $A \in Fix_{S_k}(\pi_2(\Sigma))$.

Definition 1.1.16. $\hat{X} \in \widetilde{B}_0^k(\Sigma)$ is **unlinked** if $\hat{X} \simeq_0 \hat{0}$. An unordered capped braid $[\hat{X}]$ is unlinked if some (hence every) ordered representative is unlinked. The notion of linkedness for a capped braid or for a pair of capped braids is defined as in the case of braids.

1.2. Linking of capped loops with close strands

For use in the sequel, we explain here a minor adaptation of the classical linking number of two loops in the plane $x,y : S^1 \to \mathbb{R}^2$ to two capped loops $\hat{x}$ and $\hat{y}$ such that the underlying loops x and y lie sufficiently close to each other in an arbitrary symplectic surface (Σ,ω).

To any $x \in \mathcal{L}_0(\Sigma)$, we may associate the set

$$S_x := \{y \in \mathcal{L}_0(\Sigma) : \exists t \in S^1 \text{ such that } x(t) = y(t)\},$$

with the property that $\mathcal{L}_0(\Sigma) \setminus S_x$ consists of precisely those loops y such that $(x,y) \in \mathcal{L}_0(\Sigma)^2$ is a braid.

We fix some family $J = (J_t)_{t\in S^1}$ of ω-compatible almost complex structures, and let $g_J = (g_{J_t})_{t\in S^1}$ denote the associated family of compatible metrics. This data provides us with an exponential neighbourhood $\mathcal{O} \subset \mathcal{L}_0(M)$ of x, along with a diffeomorphism

$$\begin{aligned} Exp : U \subset \Gamma^\infty(x^*T\Sigma) &\to \mathcal{O} \\ \xi &\mapsto \{t \mapsto exp^{J_t}_{x(t)}(\xi(t))\} \end{aligned}$$

from a neighbourhood of the zero section onto $\mathcal{O}$.

Remark that any choice of a lift $\hat{x} = [x,\alpha] \in \widetilde{\mathcal{L}_0}(\Sigma)$ of x gives rise to a lift $\tilde{\mathcal{O}}_\alpha$ of $\mathcal{O}$, and $T_{\hat{x}}\widetilde{\mathcal{L}_0}(\Sigma) \simeq (x^*T\Sigma,J,\omega)$ comes equipped with a homotopically unique unitary trivialization

$$T_{\hat{x}} : S^1 \times (\mathbb{R}^2, J_0, \omega_0) \to (x^*T\Sigma,J,\omega),$$

provided by any unitary trivialization which extends over the capping. For any $y \in \mathcal{O} \setminus S_x$ and any capping $\hat{x}_\alpha := [x,\alpha]$ of x, let $\hat{y}_\alpha$ denote the unique lift of y lying in $\widetilde{\mathcal{O}}_\alpha$. We define the linking number of $\hat{x}_\alpha$ and $\hat{y}_\alpha$ as

$$\ell(\hat{x}_\alpha,\hat{y}_\alpha) := wind((T_{\hat{x}}^{-1} \circ \widetilde{Exp}^{-1})(\hat{y}_\alpha)),$$

where $wind(\xi)$ denotes the classical winding number of a non-vanishing family of vectors $t \mapsto \xi(t)$ for $t \in S^1$ in $\mathbb{R}^2$. Note that for $A \in \pi_2(\Sigma)$, we clearly have

$$\ell(A \cdot \hat{x}_\alpha, A \cdot \hat{y}_\alpha) = \ell(\hat{x}_\alpha,\hat{y}_\alpha) + c_1(A),$$

and it is moreover not hard to show that ℓ is symmetric in its arguments. In order to extend this definition to arbitrary cappings of the loops x and y, let $A, B \in \pi_2(\Sigma)$ and define

$$\ell(A \cdot \hat{x}_\alpha, B \cdot \hat{y}_\alpha) := \ell(\hat{x}_\alpha,\hat{y}_\alpha) + \frac{1}{2}(c_1(A) + c_1(B)).$$

It is not hard to check that this definition does not depend on the choice of α, nor the choice of compatible almost complex structure $J = (J_t)_{t \in S^1}$, and agrees with the previous definition in the case that $\hat{x}$ and $\hat{y}$ are close in $\widetilde{\mathcal{L}_0}(\Sigma)$. A more geometric view of this formula will be provided in the following two sections.

1.3. The homological linking number for capped braids

Definition 1.3.1. Let $\hat{X}, \hat{Y} \in \widetilde{B}_0^k(\Sigma)$ and $A \in \pi_2(\Sigma)^k$. We define the **homological (A)-linking number of $\hat{Y}$ relative to $\hat{X}$**

$$L_A(\hat{X};\hat{Y}) := \sum_{1 \le i < j \le k} \#(\tilde{h}_i \pitchfork \tilde{h}_j),$$

where $h = (h_1, \cdots, h_k)$ is any A-cobordism from $\hat{X}$ to $\hat{Y}$ such that the graphs of the strands of h in $[0,1] \times S^1 \times \Sigma$ are all pairwise transverse, and $\#(\tilde{h}_i \pitchfork \tilde{h}_j)$ denotes the signed count of the intersections of the graphs $\tilde{h}_i$ and $\tilde{h}_j$ (recall that in our setting Σ carries the orientation induced by ω).

Remark. The above definition may be generalized straightforwardly by replacing the cylinder $[0,1] \times S^1$ with a surface S_{g,k^-,k^+} of genus g, having k^- negatively oriented boundary components and k^+ positively oriented boundary components. This provides a family of homotopy invariants for collections of k^- 'input' and k^+ 'output' capped braids in the obvious way. Much of the theory developed in this thesis can be adapted in a straightforward way to use such invariants to extract information about the field-theoretic operations in Floer theory described in [**25**] at the chain level for Hamiltonian isotopies on surfaces, but we will not pursue this extension in this thesis.

The following proposition summarizes the main properties of the homological linking number which we will need in our investigations.

Proposition 1.3.2. *For any $\hat{X}, \hat{Y}, \hat{Z} \in \widetilde{B}_0^k(\Sigma)$ and $A, B \in \pi_2(\Sigma)^k$ we have that:*

(1) $L_A(\hat{X},\hat{Y})$ *is well-defined.*

(2) For any $\sigma \in S_k$, $L_{\sigma \cdot A}(\sigma \cdot \hat{X}; \sigma \cdot \hat{Y}) = L_A(\hat{X}; \hat{Y})$.

(3) $L_A(\hat{X},\hat{Y}) + L_B(\hat{Y},\hat{Z}) = L_{A+B}(\hat{X},\hat{Z})$.

(4) If $\hat{X}$ *and* $\hat{Y}$ *are* A*-homotopic, then* $L_A(\hat{X},\hat{Y}) = 0$.

(5) $L_A(\hat{X},\hat{Y}) = -L_{-A}(\hat{Y},\hat{X})$.

(6) $L_A(\hat{X},B \cdot \hat{Y}) = L_{A+B}(\hat{X},\hat{Y})$.

(7) $L_0(\hat{X},A \cdot \hat{X}) = (k-1) \sum_{i=1}^{k} \frac{c_1(A_i)}{2}$.

PROOF. (1) That $L_A(\hat{X},\hat{Y})$ is well-defined follows from the standard transversality arguments that are typical in differential topology. Alternately, one may simply note that for any braid cobordism $h = (h_1, \cdots, h_k)$ and any $i = 1, \cdots, k$, the graph of h_i in $[0,1] \times S^1 \times \Sigma$ defines a compact surface with boundary $S_i \subset [0,1] \times S^1 \times \Sigma$ which induces a well-defined element of

$$[S_i] \in H_2([0,1] \times S^1 \times \Sigma; \check{X} \sqcup \check{Y}),$$

where $\check{X}$ and $\check{Y}$ denote the graphs of the braids X and Y respectively, thought of as submanifolds lying in $\{0\} \times S^1 \times \Sigma$ and $\{1\} \times S^1 \times \Sigma$, respectively. The intersection product of such classes is well-defined and

$$L_A(\hat{X},\hat{Y}) = \sum_{1 \leq i < j \leq k} [S_i] \cup [S_j],$$

which, since $[0,1] \times S^1 \times (\Sigma,\omega)$ is canonically oriented by $(ds \wedge dt) \wedge \omega$, is obviously precisely what is computed by the sum of pairwise intersection numbers of the graphs when these are transverse.

(2) This statement follows upon remarking that for any $\sigma \in S_k$ and any $A \in \pi_2(\Sigma)^k$, $h = (h_1, \ldots, h_k)$ is an A-cobordism from $\hat{X}$ to $\hat{Y}$ if and only if $\sigma \cdot h := (h_{\sigma^{-1}(1)}, \ldots, h_{\sigma^{-1}(k)})$ is a $(\sigma \cdot A)$-cobordism from $\sigma \cdot \hat{X}$ to $\sigma \cdot \hat{Y}$.

(3) This is straightforward and follows directly from concatenating an A-cobordism from $\hat{X}$ to $\hat{Y}$ with a B-cobordism from $\hat{Y}$ to $\hat{Z}$.

(4) An A-cobordism $h = (h_1, \cdots, h_k)$ is an A-homotopy precisely when the graphs of the h_i, $i = 1, \cdots, k$ are all disjoint. Clearly this implies $L_A(\hat{X},\hat{Y}) = 0$.

(5) This follows immediately from noting that h is an A-cobordism from $\hat{X}$ to $\hat{Y}$ if and only if $\bar{h}(s,t) := h(1-s,t)$ is a $(-A)$-cobordism from $\hat{Y}$ to $\hat{X}$.

(6) This follows immediately from the equivalence of the homotopy conditions

$$\begin{aligned} \alpha_i \# [h_i] \# - \beta_i = A_i \# B_i = A_i + B_i, \text{ and} \\ \alpha_i \# [h_i] \# - (A_i \cdot \beta_i) = B_i. \end{aligned}$$

That is, h is an $A+B$-cobordism from $\hat{X}$ to $\hat{Y}$ if and only if h is also a B-cobordism from $\hat{X}$ to $A \cdot \hat{Y}$.

(7) We note first that item (3) implies that

$$L_0(\hat{0},\hat{X}) + L_0(\hat{X},A \cdot \hat{X}) + L_0(A \cdot \hat{X}, A \cdot \hat{0}) = L_0(\hat{0},A \cdot \hat{0}).$$

Next, items (5) and (6) imply that

$$\begin{aligned} L_0(A \cdot \hat{X}, A \cdot \hat{0}) &= L_A(A \cdot \hat{X},\hat{0}) \\ &= -L_{-A}(\hat{0}, A \cdot \hat{X}) \\ &= -L_0(\hat{0},\hat{X}), \end{aligned}$$

whence we need only show that the desired formula holds when $\hat{X} = \hat{0}$. To reduce to an even simpler case, let us write A as

$$\begin{aligned} (A_1, \cdots, A_k) &= (A_1,0, \cdots, 0) + (0,A_2,0, \cdots, 0) + \cdots + (0,0, \cdots, A_k) \\ &= A'_1 + \cdots + A'_k. \end{aligned}$$

By items (3) and (6), demonstrating the desired equality is therefore equivalent to showing that

$$L_0(\hat{0},A'_i \cdot \hat{0}) = (k-1) \cdot \frac{c_1(A_i)}{2}$$

for any $i = 1, \cdots, k$. In what follows, let $(\hat{p}_1, \cdots, \hat{p}_k) = \hat{0}$ represent the trivial capped braid. Since the statement is trivial when $\Sigma \neq S^2$, as then $\pi_2(\Sigma) = 0$ and every capped braid is 0-homotopic to itself, we now suppose $\Sigma = S^2$. For $m \in \mathbb{Z}$, if $u_i : (S^2,*) \to (\Sigma,p_i)$ represents $A_i = m[S^2] \in \pi_2(\Sigma,p_i)$, we may pull u_i back along the quotient $[0,1] \times S^1 \to S^2$ (given by collapsing the boundary circles to points) to a map which we will denote

$$h_i : [0,1] \times S^1 \to \Sigma.$$

If we take h to be the 0-cobordism from $\hat{0}$ to $A'_i \cdot \hat{0}$ given by h_i as the i-th strand and the constant strand $h_j(s,t) \equiv p_j$ for all other strands $j \neq i$, then the important point is that

$PD(c_1) = 2[S^2]$ and hence the intersection of the graph of h_i with the constant cylinder $h_j(s,t) \equiv p_j$ for $j \neq i$ contributes precisely

$$\begin{aligned}(u_{i*}[S^2]) \cap [p_j] &= m[S^2] \\ &= \frac{c_1(A_i)}{2}\end{aligned}$$

to the sum defining $L_0(\hat{0},A_i' \cdot \hat{0})$, and such intersections are the only ones that occur, since all other strands are constant and disjoint. The desired equality follows.

$\square$

Proposition 1.3.3. *For $A \in \pi_2(\Sigma)^k$, and $[\hat{X}], [\hat{Y}] \in \widetilde{B}_0^k(\Sigma)/S_k$, the function $L_A([\hat{X}]; [\hat{Y}]) := L_A(\hat{X}; \hat{Y})$, is well-defined.*

PROOF. The previous proposition implies that for any $\sigma, \tau \in S_k$, we have

$$L_A(\sigma \cdot \hat{X}; \tau \cdot \hat{Y}) = L_A(\hat{0}; \hat{0}) - L_0(\hat{0}; \sigma \cdot \hat{X}) + L_0(\hat{0}; \tau \cdot \hat{Y}),$$

so it suffices to show that the expression $L_0(\hat{0}; \sigma \cdot \hat{X})$ is independent of $\sigma \in S_k$. To see this, note that item 2 of the previous proposition, together with the fact that $0 \in Fix_{S_k}(\pi_2(\Sigma)^k)$ implies that $L_0(\hat{0}; \sigma \cdot \hat{X}) = L_0(\sigma^{-1} \cdot \hat{0}; \hat{X})$. Moreover, it is easy to see that $\hat{0} = (\hat{p}_1, \ldots, \hat{p}_k)$ is 0-homotopic to $\sigma \cdot \hat{0} = (\hat{p}_{\sigma^{-1}(1)}, \ldots, p_{\sigma^{-1}(k)})$ for any $\sigma \in S_k$ (simply choose k paths $s \mapsto \gamma_i(s) \in \Sigma$, $s \in [0,1]$ from p_i to $p_{\sigma^{-1}(i)}$ such that $\gamma_i(s) = \gamma_j(s)$ implies $i = j$ for all $s \in [0,1]$ and define the i-th strand of the 0-homotopy to be $h_i(s,t) = \gamma_i(s)$) and consequently, $L_0(\sigma^{-1} \cdot \hat{0}; \hat{X}) = L_0(\hat{0}; \hat{X})$, which is independent of $\sigma \in S_k$. $\square$

Proposition 1.3.4. *Let $\hat{X} = (\hat{x}_1,\hat{x}_2) \in \widetilde{B}_0^2(\Sigma)$ with x_2 lying in some exponential of neighbourhood of x_1 in $\mathcal{L}_0(\Sigma)$ (as in Section 1.2), then $L_0(\hat{0}; \hat{X}) = \ell(\hat{x}_1,\hat{x}_2)$.*

PROOF. As $L_0(\hat{0}; \hat{X})$ depends on $\hat{X}$ only up to a 0-homotopy, we may assume without loss of generality that x_1 is a constant loop. Moreover, noting that if $A,B \in \pi_2(\Sigma)$, then

$$\begin{aligned}L_0(\hat{0}; (A,B) \cdot \hat{X}) - L_0(\hat{0}; \hat{X}) &= \frac{1}{2}(c_1(A) + c_1(B)) \\ &= \ell(A \cdot \hat{x}_1, B \cdot \hat{x}_2) - \ell(\hat{x}_1,\hat{x}_2),\end{aligned}$$

and so it suffices to prove the statement in the case in which $\hat{x}_1$ is a trivially capped constant loop and $\hat{x}_2$ lies inside an exponential neighbourhood of $\tilde{x}_1$ in $\widetilde{\mathcal{L}_0}(\Sigma)$.

As discussed in Section 1.2, $T_{\hat{x}}\widetilde{\mathcal{L}_0}(\Sigma)$ is naturally identified (up to a homotopy of trivializations) with $\Gamma^\infty(S^1 \times \mathbb{R}^2)$ and so (in the notation of that section) we may write $\tilde{x}_2$ in local coordinates as

$$v(t) := (T_{\hat{x}_1}^{-1} \circ \widetilde{Exp}^{-1})(\tilde{x}_2),$$

and we have that $\ell(\hat{x}_1,\hat{x}_2) = wind(v)$ by definition. By the capped braid homotopy-invariance of the homological linking number, and the homotopy invariance of the winding number in $\mathbb{R}^2 \setminus \{0\}$, we may assume that

$$v(t) = r_0 e^{2\pi i l t} \in \mathbb{C},\ \forall t \in [0,1],$$

for some small $r_0 > 0$ and $l = \ell(\hat{x}_1;\hat{x}_2)$. Taking the homotopies $h_1(s,t) \equiv 0$, $h_2(s,t) = (1 - s)\frac{-r_0}{2} + s\frac{r_0}{2}e^{2\pi i l t}$, we see that their graphs $\tilde{h}_1$ and $\tilde{h}_2$ intersect only if $l \neq 0$, and in that case intersections occur when $s = \frac{1}{2}$ and $t = 0, \frac{1}{l}, \cdots, \frac{l-1}{l}$. Moreover, using polar coordinates (r,θ) on D^2, since

$$\partial_s h_2 = \partial_r$$
$$\partial_t h_2 = l\partial_\theta,$$

each intersection is transverse and has orientation $sign(l)$. Consequently

$$L_0(\hat{0};\hat{X}) = l = \ell(\hat{x}_1,\hat{x}_2),$$

proving the claim. □

The following proposition is not used anywhere in this thesis, but we include it because it provides some intuitive justification for the relationship between the homological linking number of capped braids and symplectic measurements on surfaces: the area functional on the space of capped loops may be computed at a given capped loop $\hat{x}$ as simply being the average linking number of $\hat{x}$ with a generic set of points in the surface (viewed as trivially capped loops).

Proposition 1.3.5. *Suppose that $\hat{\gamma} = [\gamma,w]$ is a capped loop such that γ is smooth, then*

$$\int_{D^2} w^*\omega = \int_{\Sigma\setminus im\ \gamma} \ell(\hat{\gamma},\hat{x})\omega,$$

where for $x \in \Sigma$, $\hat{x} = [x,x]$ denotes the trivially capped constant loop based at x.

PROOF. Suppose without loss of generality that $w : D^2 \to \Sigma$ is a smooth map with $w(0) = \gamma(0)$. The main point is to notice that for any $x \in \Sigma \setminus \mathrm{im}\ \gamma$ such that w is transversal to x, we have that $\ell(\hat{\gamma},\hat{x}) = deg(w)_x$, where $deg(w)_x$ denotes the local degree of w at x. Indeed, transversality of w to x implies the transversality of the maps into $[0,1] \times S^1 \times \Sigma$ defined by $\tilde{w}(s,t) = (s,t,w(se^{2\pi i t}))$

and $\tilde{x}(s,t) = (s,t,x)$, for $(s,t) \in [0,1] \times S^1$, and obviously the algebraic count of the intersection number between these two graphs is identical with $deg(w)_x$. One immediate consequence is that $\ell(\hat{\gamma},\hat{x}) = 0$ for all $x \notin$ im w, so it suffices to show that

$$\int_{D^2} w^*\omega = \int_{\text{im } w \setminus \text{im } \gamma} \ell(\hat{\gamma},\hat{x})\omega.$$

To this end, denote by $S(w) \subseteq D^2$ the set of points such that Dw is not of full rank and define $G(w) \subseteq D^2$ to be $G(w) = w^{-1}(\text{im } \gamma) \setminus S(w)$. Note that $G(w)$ is a set of measure 0, since we may realize $G(w)$ as the projection onto $D^2 \setminus S(w)$ of $(w|_{D^2 \setminus S(w)} \times \gamma)^{-1}(\triangle)$ which is a submanifold of $D^2 \setminus S(w)$ of codimension 2. Next note that we must have $\int_{S(w)} w^*\omega = 0$, since $w^*\omega$ vanishes on $S(w)$. Consequently, writing $N := S(w) \cup G(w)$, we note that im $\gamma \subseteq w(N)$ and so it suffices to establish that

$$\int_{D^2 \setminus N} w^*\omega = \int_{w(D^2 \setminus N)} \ell(\hat{\gamma},\hat{x})\omega = \int_{w(D^2 \setminus N)} deg(w)_x\omega.$$

The local degree is a locally constant function of x, and so we obtain

$$\int_{w(D^2 \setminus N)} deg(w)_x\omega = \sum_{C \in \pi_0(w(D^2 \setminus N))} deg(w)|_C \int_C \omega = \sum_{C \in \pi_0(w(D^2 \setminus N))} \int_{w^{-1}(C)} w^*\omega = \int_{D^2 \setminus N} w^*\omega$$

as claimed. $\square$

Chapter 2

Elements of Floer theory and linking

This chapter serves to collect and review the necessary facts that we will need from Hamiltonian Floer theory, along with the analysis of the asymptotic behaviour of Floer-type cylinders that proves crucial to our study. Section 2.1 collects the Floer-theoretic preliminaries. Section 2.2 explains how we may combine a result formulated by Siefring in [**32**], which relates the asymptotic behaviour or Floer-type cylinders tending to an orbit x to the eigenvectors of the so-called *asymptotic operator* associated to x, with Hofer-Wysocki-Zehnder's study in [**10**] of the winding behaviour of such eigenvectors, in order to obtain information about the asymptotics of Floer-type cylinders. We then explain how we may combine this information with the homological linking number of Section 1.3 to obtain control over the relative topology of collections of capped braids having Floer-type cylinders running between them. The results in this chapter form the technical core of the rest of the work.

2.1. Floer theory

In this section, we give a rapid overview of the elements of Floer theory of which we will have need, mainly to fix notation and conventions. For a more detailed treatment, see [**1**] or [**27**] for standard accounts of Hamiltonian Floer theory (see also [**9**] for its adaptation to the semi-positive case) and [**28**], [**25**], [**30**] or [**14**] for a more detailed treatment of how Floer theory fits into a field theory over surfaces. Throughout, we assume that (M^{2n},ω) is a strongly semi-positive compact symplectic manifold (ie. $2-n \leq c_1(A) < 0$ implies $\omega(A) \leq 0$ for all $A \in \pi_2(M)$). $\mathcal{J}(M,\omega)$ denotes the space of all smooth ω-compatible almost complex structures. For convenience, we work only with $\mathbb{Z}_2$-coefficients, but this restriction is inessential.

A smooth (not necessarily autonomous) Hamiltonian function

$$\begin{aligned} H : S^1 \times M &\to \mathbb{R} \\ (t,x) &\mapsto H_t(x) \end{aligned}$$

induces a time-dependent vector field $(X_H^t)_{t\in[0,1]}$ on M defined by the relation

$$\omega(X_H^t, -) = -dH_t.$$

The Hamiltonian isotopy obtained as the flow by this vector field is denoted $\phi^H := (\phi_t^H)_{t\in[0,1]}$. A Hamiltonian H is said to be **normalized** if $\int_M H_t\omega^n = 0$ for all $t \in S^1$. There is a group structure on the set of Hamiltonian functions $C^\infty(S^1 \times M)$ given by the operation

$$(H\#K)(t,x) := H(t,x) + K(t,(\phi_t^H)^{-1}(x)),$$

which is such that $\phi_t^{H\#K} = \phi_t^H \circ \phi_t^K$. The inverse of H with respect to this relation is given by

$$\bar{H}(t,x) = -H(t,\phi_t^H(x)),$$

which generates the isotopy $t \mapsto (\phi_t^H)^{-1}$.

Recall that $\widetilde{\mathcal{L}_0}(M)$ denotes the Novikov covering of the loop space. That is to say, elements $[\gamma, v] \in \widetilde{\mathcal{L}_0}(M)$ are capped loops (γ,v) in M subject to the equivalence relation $(\gamma_1,v_1) \sim (\gamma_2,v_2)$ precisely when $\gamma_1 = \gamma_2$ and $[v_1\#\bar{v}_2] \in \ker c_1(M) \cap \ker[\omega] \subset \pi_2(M)$.

The Hamiltonian H defines a corresponding **action functional** on the Novikov covering of the loop space

$$\begin{aligned} \mathcal{A}_H : \widetilde{\mathcal{L}_0}(M) &\to \mathbb{R} \\ [\gamma,v] &\mapsto \int_0^1 H_t(\gamma(t))\, dt - \int_{D^2} v^*\omega. \end{aligned}$$

We write

$$\begin{aligned} &\widetilde{Per_0}(H) := Crit\ \mathcal{A}_H, \text{ and} \\ &Per_0(H) := \pi(\widetilde{Per_0}(H)) \subseteq \mathcal{L}_0(M), \end{aligned}$$

noting that the latter consists precisely of the contractible 1-periodic orbits of ϕ^H, while the former consists of capped such periodic orbits.

H is said to be **non-degenerate** if for all $x \in Per_0(H)$, $(D\phi_1^H)_{x(0)}$ has no eigenvalues equal to 1. When H is non-degenerate, there exists a well-defined **Conley-Zehnder index**

$$\mu = \mu_{CZ} : \widetilde{Per}_0(H) \to \mathbb{Z}.$$

See [**26**] for details on the definition of μ. We shall normalize the Conley-Zehnder index by insisting that if H is a C^2-small Morse function and x a critical point of H, then

$$\mu(\hat{x}) = \mu_{Morse}(x) - n,$$

where μ_{Morse} is the Morse index of x, and $\hat{x}$ denotes the trivial capping of the constant orbit x. For $k \in \mathbb{Z}$, and any $\hat{P} \subseteq \widetilde{Per}_0(H)$ we define $\hat{P}_{(k)}$ to be the collection of capped orbits in $\hat{P}$ with Conley-Zehnder index k.

Given $\hat{x}^{\pm} = [x^{\pm}, w^{\pm}] \in \widetilde{Per}_0(H)$, we write $C^\infty(\mathbb{R} \times S^1; M)_{\hat{x}^-,\hat{x}^+}$ for the subspace of $C^\infty(\mathbb{R} \times S^1; M)$ consisting of cylinders which induce a 0-homotopy from $\hat{x}^-$ to $\hat{x}^+$. Letting

$$\mathcal{E} \to C^\infty(\mathbb{R} \times S^1; M)_{\hat{x}^-,\hat{x}^+}$$

be the infinite dimensional vector bundle with fiber $\mathcal{E}_u = \Gamma^\infty(u^*TM)$ at u, any smooth S^1-family $J = (J_t)_{t \in S^1} \subseteq \mathcal{J}(M,\omega)$, permits the definition of the **Floer operator**, which is the section

$$\begin{aligned} \mathcal{F}_{H,J} : C^\infty(\mathbb{R} \times S^1; M)_{\hat{x}^-,\hat{x}^+} &\to \mathcal{E} \\ u &\mapsto \partial_s u + J(\partial_t u - X_H). \end{aligned}$$

After passing to appropriate Banach space completions (see Section 8.2 of [**1**] for instance), $\mathcal{F}_{H,J}$ defines a Fredholm operator with index $\mu(\hat{x}^-) - \mu(\hat{x}^+)$. The intersection of $\mathcal{F}_{H,J}$ with the 0-section gives rise to **Floer's equation**

$$\partial_s u + J_t(\partial_t u - X_H^t) = 0 \tag{2.1.1}$$

for smooth maps $u : \mathbb{R} \times S^1 \to M$. If we define the **energy** of $u \in C^\infty(\mathbb{R} \times S^1; M)$ by

$$E(u) := \int_{\mathbb{R} \times S^1} \|\partial_s u\|_{J_t}^2 \, dt \, ds,$$

then the finite energy solutions of Floer's equation may be thought of as the projections to M of negative gradient flow lines of $\mathcal{A}_H$ with respect to the L^2-metric on $\widetilde{\mathcal{L}_0}(M)$ induced by J. It follows easily from this that if $u \in C^\infty(\mathbb{R} \times S^1; M)_{\hat{x}^-,\hat{x}^+}$ is such a finite energy solution, then $E(u) = \mathcal{A}_H(\hat{x}^-) - \mathcal{A}_H(\hat{x}^+)$.

For any $\hat{x}^{\pm} \in \widetilde{Per}_0(H)$, we define $\widetilde{\mathcal{M}}(\hat{x}^-,\hat{x}^+; H,J)$ to be the zero set of $\mathcal{F}_{H,J}$ on $C^\infty(\mathbb{R} \times S^1; M)_{\hat{x}^-,\hat{x}^+}$. It carries an obvious $\mathbb{R}$-action given by translation in the s-coordinate. The **reduced moduli space** is defined by

$$\mathcal{M}(\hat{x},\hat{y}; H,J) := \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J)/\mathbb{R}.$$

In order to define the appropriate genericity condition on pairs (H,J) such that we may define the Floer complex unproblematically, we follow [**30**] in introducing the following sets. For $k \in \mathbb{Z}_{\geq 0}$, let V_k be the set of pairs $(t,p) \in S^1 \times M$ such that $p \in \text{im } v$ for v some non-constant J_t-holomorphic sphere with $c_1(v) \leq k$.

Definition 2.1.1. A pair (H,J) with H and J as above, H non-degenerate, are said to be **Floer regular** if

(1) for every $x \in Per_0(H)$, $(t,x(t)) \notin V_1(J)$ for all $t \in S^1$;
(2) for all $\hat{x}^{\pm} \in \widetilde{Per}_0(H)$ with $\mu(\hat{x}^-) - \mu(\hat{x}^+) \leq 2$, the linearization $(D\mathcal{F}_{H,J})_u$ of the Floer operator at $u \in C^\infty(\mathbb{R} \times S^1; M)_{\hat{x}^-,\hat{x}^+}$ is surjective for all $u \in \widetilde{\mathcal{M}}(\hat{x}^-,\hat{x^+}; H,J)$;
(3) for all $\hat{x}^{\pm} \in \widetilde{Per}_0(H)$ with $\mu(\hat{x}^-) - \mu(\hat{x}^+) \leq 2$ and all $u \in \widetilde{\mathcal{M}}(\hat{x}^-,\hat{x^+}; H,J)$, $(t,u(s,t)) \notin V_0(J)$ for all $(s,t) \in \mathbb{R} \times S^1$.

For H non-degenerate, let $\mathcal{J}^{nd}(H) \subseteq C^\infty(S^1; \mathcal{J}(M,\omega))$ denote the space of S^1-families of complex structures such that (H,J) is Floer regular. $\mathcal{J}^{nd}(H)$ is residual in $C^\infty(S^1; \mathcal{J}(M,\omega))$.

If (H,J) is Floer regular, then $\mathcal{M}(\hat{x},\hat{y}; H,J)$ is a compact manifold of dimension 0 whenever $\mu(\hat{x}) - \mu(\hat{y}) = 1$, and in this case we may define the **Floer chain complex** $CF(H,J)$ to be the set of formal sums of the form

$$\sum_{\hat{x} \in \widetilde{Per}_0(H)} a_{\hat{x}} \hat{x},$$

where $a_{\hat{x}} \in \mathbb{Z}_2$ for all $\hat{x} \in \widetilde{Per}_0(H)$ and which moreover verifies the *Novikov condition*: for all $c \in \mathbb{R}$,

$$\#\{a_{\hat{x}} \neq 0 : \mathcal{A}_H(\hat{x}) \geq c\} < \infty.$$

$CF(H,J)$ is then graded by μ and has a differential defined on generators by

$$\partial_{H,J}\hat{x} := \sum_{\mu(\hat{x})-\mu(\hat{y})=1} n(\hat{x},\hat{y})\hat{y},$$

with $n(\hat{x},\hat{y})$ being the mod 2 count of elements in $\mathcal{M}(\hat{x},\hat{y};H,J)$. The homology of this complex $HF_*(H)$ is the **Floer homology of** H and is independent of the choice of J.

The Floer complex has the structure of a *filtered complex*, with the filtration coming from the action functional. Explicitly, for $\sigma = \sum_{\hat{x}\in\widetilde{Per}_0(H)} a_{\hat{x}}\hat{x} \in CF_*(H,J)$, we define

$$\text{supp } \sigma := \{\hat{x} \in \widetilde{Per}_0(H) : a_{\hat{x}} \neq 0\},$$

and we define the **level of** σ to be

$$\lambda_H(\sigma) := \sup_{\hat{x}\in\text{supp } \sigma} \mathcal{A}_H(\hat{x}).$$

Let $H_2^S(M)$ denote the image in $H_2(M;\mathbb{Z})$ of the Hurewicz morphism, and let $\Gamma_\omega := H_2^S(M)/\ker c_1 \cap \ker[\omega]$. We define the **Novikov ring**

$$\Lambda_\omega := \{\sum_{A\in\Gamma_\omega} \lambda_A e^A : \lambda_A \in \mathbb{Z}_2, \#\{\lambda_A \neq 0, \omega(A) \leq c\} < \infty, \text{ for all } c \in \mathbb{R}\}.$$

This is a graded commutative ring with grading given by declaring $deg(e^A) := 2c_1(A)$. $CF_*(H,J)$ is a Λ_ω-module where the action of Λ_ω is defined on generators $\hat{x} = [x,v]$ of $CF_*(H,J)$ and e^A of Λ_ω by $e^A \cdot \hat{x} := [x,A\#v]$, and extended linearly. Note that we have the relations

$$\mu(e^A \cdot \hat{x}) = \mu(\hat{x}) - 2c_1(A),$$
$$\mathcal{A}_H(e^A \cdot \hat{x}) = \mathcal{A}_H(\hat{x}) - \omega(A).$$

Remark that the Λ_ω action needn't preserve the filtration.

It is a standard fact in Floer theory that if $f \in C^\infty(M)$ is a sufficiently C^2-small Morse function and $J \in \mathcal{J}(M,\omega)$ is such that (f,g_J) is Morse-Smale, then the Floer chain complex of (f,J) may be identified (after a grading shift) with the **quantum chain complex** of (f,g_J), which is by definition the Morse complex of (f,g_J) with coefficients in the Novikov ring, ie.

$$CF_*(f,J) = QC_{*+n}(f,g_J) := (C^{Morse}(f,g_J) \otimes \Lambda_\omega)_{*+n}.$$

Taking homology then gives a natural identification with the quantum homology of (M,ω):

$$HF_*(f) = H_{*+n}(M;\Lambda_\omega) = QH_{*+n}(M,\omega).$$

2.1.1. Continuation maps

Definition 2.1.2. For X a smooth manifold, a function $F \in C^\infty(\mathbb{R} \times X)$ is said to be T**-adapted** for $T \in (0,\infty)$ if $(\partial_s F)_{s_0} \equiv 0$ for all $|s_0| \geq T$. F is said to be **adapted** if it is T-adapted for some T. For $X = C^\infty(S^1 \times M)$, and $H^\pm \in C^\infty(S^1 \times M)$, we denote by $\mathscr{H}(H^-,H^+)$ the space of adapted homotopies $\mathcal{H}$ having $\lim_{s\to\pm\infty} \mathcal{H}(s) \equiv H^\pm$. We make a similar definition for $\mathscr{J}(J^-,J^+)$ in the case where $X = C^\infty(S^1; \mathcal{J}(M,\omega))$.

Definition 2.1.3. A pair $(\mathcal{H},\mathbb{J})$ is an **adapted homotopy of Floer data** from (H^-,J^-) to (H^+,J^+) if $\mathcal{H} = (H^s_t) \in \mathscr{H}(H^-,H^+)$ and $\mathbb{J} = (J^s_t) \in \mathscr{J}(J^-,J^+)$. We will write $\mathscr{H}\mathscr{J}(H^-,J^-;H^+,J^+)$ for the collection of all such adapted homotopies, often omitting the dependence on $(H^\pm,J^\pm)$ if it is clear from context.

Just as in the s-independent case, for any adapted homotopy of Floer data $(\mathcal{H},\mathbb{J})$, we obtain a corresponding Floer operator $\mathcal{F}_{\mathcal{H},J}$. For any pair $\hat{x}^\pm \in \widetilde{Per}_0(H^\pm)$, consideration of the zeros of $\mathcal{F}_{\mathcal{H},\mathbb{J}}$ along $C^\infty(\mathbb{R} \times S^1; M)_{\hat{x}^-,\hat{x}^+}$ gives rise to the s-dependent Floer equation

$$\partial_s u + J^s_t(\partial_t u - X_{H^s_t}) = 0, \tag{2.1.2}$$

and everything proceeds as before, with the proviso that now, if $u \in C^\infty(\mathbb{R} \times S^1; M)_{\hat{x}^-,\hat{x}^+}$ solves Equation 2.1.2, then its energy is given by

$$E(u) = \mathcal{A}_{H^-}(\hat{x}^-) - \mathcal{A}^+_H(\hat{x}^+) + \int_{-\infty}^{\infty}\int_0^1 (\partial_s\mathcal{H})(s,t,u(s,t))\, dtds. \tag{2.1.3}$$

The moduli space $\mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H},J)$ is defined to be the zero set of $\mathcal{F}_{\mathcal{H},J}$ on $C^\infty(\mathbb{R} \times S^1; M)_{\hat{x}^-,\hat{x}^+}$.

Remark. When $(H^-,J^-) = (H^+,J^+)$, then the s-independent homotopy $(\mathcal{H},\mathbb{J}) = (H^-,J^-) = (H^+,J^+)$ is a special case of an adapted homotopy. In this case, $\mathcal{M}(\hat{x},\hat{y};\mathcal{H},J) = \widetilde{\mathcal{M}}(\hat{x},\hat{y};H^\pm,J^\pm)$. In the sequel, when we speak of adapted homotopies of Floer data, this case is included.

As in the s-independent case, in order to formulate the appropriate generic regularity criterion, we introduce the following set: we define $V_0(\mathbb{J})$ to be the set of tuples $(s,t,p) \in \mathbb{R} \times S^1 \times M$ such that $p \in \text{im } v$ for v a non-constant $\mathbb{J}^s_t$-holomorphic sphere with $c_1(v) \leq 0$.

Definition 2.1.4. Given $(H^\pm,J^\pm)$ Floer regular, $\hat{x}^\pm \in \widetilde{Per}_0(H^\pm)$, and $(\mathcal{H},\mathbb{J}) \in \mathscr{H}\mathscr{J}$, we will say that $(\mathcal{H},\mathbb{J})$ is $(\hat{x}^-,\hat{x}^+)$**-regular** if

(1) the linearization $(D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u$ of the Floer operator at $u \in C^\infty(\mathbb{R} \times S^1; M)_{\hat{x}^-,\hat{x}^+}$ is surjective for all $u \in \widetilde{\mathcal{M}}(\hat{x}^-,\hat{x^+}; H,J)$;

(2) if $\mu(\hat{x}^-) - \mu(\hat{x}^+) \leq 1$, then $(s,t,u(s,t)) \notin V_0(\mathbb{J})$ for all $(s,t) \in \mathbb{R} \times S^1$ and all $u \in \mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H},\mathbb{J})$.

We denote the collection of all such adapted homotopies by $\mathscr{HJ}^{reg}_{\hat{x}^-,\hat{x}^+}$. $(\mathcal{H},\mathbb{J})$ will be said to be **Floer-regular** if it is $(\hat{x}^-,\hat{x}^+)$-regular whenever $\mu(\hat{x}^-)-\mu(\hat{x}^+)\leq 1$. We denote the space of Floer-regular adapted homotopies from (H^-,J^-) to (H^+,J^+) by $\mathscr{HJ}^{reg}(H^-,J^-;H^+,J^+)$, suppressing the dependence on $(H^{\pm},J^{\pm})$ when no confusion will arise.

For any fixed $\mathbb{J}\in\mathscr{J}(J^-,J^+)$, the set $\mathscr{H}^{reg}(\mathbb{J};H^-,H^+)\subseteq\mathscr{H}(H^-,H^+)$ of adapted homotopies $\mathcal{H}$ such that $(\mathcal{H},\mathbb{J})$ is Floer regular is residual.

For $(\mathcal{H},\mathbb{J})\in\mathscr{HJ}^{reg}$, the spaces $\mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H},J)$ are all compact manifolds of dimension 0 whenever $\mu^{H^-}(\hat{x}^-)=\mu^{H^+}(\hat{x}^+)$, and so we may define the **continuation morphism**

$$h_{\mathcal{H},J}:CF_*(H^-,J^-)\to CF_*(H^+,J^+)$$

on generators by setting

$$h_{\mathcal{H},\mathbb{J}}(x^-):=\sum_{\mu(\hat{x}^-)-\mu(\hat{x}^+)=0}n(\hat{x}^-,\hat{x}^+)\hat{x}^+,$$

where $n(\hat{x}^-,\hat{x}^+)$ is the mod 2 count of elements in the moduli space $\mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H},\mathbb{J})$. The continuation morphism is a morphism of complexes, which descends to an isomorphism at the level of homology. Moreover any two continuation maps between (H^-,J^-) and (H^+,J^+) define the same map at the level of homology, and further these isomorphisms satisfy the obvious composition law

$$h_{21}\circ h_{10}=h_{20},$$

where $h_{ji}:HF(H_i)\to HF(H_j)$.

2.1.2. The PSS isomorphism

There is another type of morphism of chain complexes, in some way related to the continuation morphisms, which will concern us in this work. Introduced in [**25**] (see also [**28**]), PSS maps may be viewed as a variant on Floer continuation maps, with the exception that they consider adapted homotopies of Floer data from $(0,J^-)$ — for $J^-\in\mathcal{J}(M,\omega)$ an autonomous almost complex structure —to the Floer pair (H,J) whose Floer complex we are studying. The fact that the 0 function is a heavily degenerate Hamiltonian forces some modifications to the definition of the chain morphism. Explicitly, we write

$$\mathscr{HJ}^{PSS}(H,J):=\{(J^-;\mathcal{H},J):J^-\in\mathcal{J}(M,\omega),\ (\mathcal{H},\mathbb{J})\in\mathscr{HJ}(0,J^-;H,J)\}$$

and the set of all **PSS data** for the pair (H,J) is

$$PSS(H,J) := \{(f,g;J^-;\mathcal{H},\mathbb{J}) \in C^\infty(M) \times Met(M) \times \mathscr{HJ}^{PSS}(H,J) :\ (f,g) \text{ is Morse-Smale}\},$$

where $Met(M)$ denotes the space of smooth Riemannian metrics on M. There is a residual set $PSS_{reg}(H,J) \subset PSS_{reg}(H,J)$ of **regular PSS data** such that for any $\mathcal{D} = (f,g;J^-;\mathcal{H},\mathbb{J}) \in PSS_{reg}(H,J)$, we may define a morphism of chain complexes

$$\Phi^{PSS}_{\mathcal{D}} : (C^{Morse}(f,g) \otimes \Lambda_\omega)_{*+n} \to CF_*(H,J)$$
$$p \otimes e^A \mapsto \sum_{A \in \Gamma_\omega} \sum_{\hat{x}} n(p,(-A)\cdot\hat{x})\hat{x},$$

where for $q \in Crit(f)_{k-n}$, $\hat{y} = [y,v] \in \widetilde{Per}_0(H)_k$, $n(q,\hat{y})$ denotes the $\mathbb{Z}/2\mathbb{Z}$-count of elements in the 0-dimensional moduli space

$$\mathcal{M}(q,\hat{y};\mathcal{D})$$

of finite energy maps $u \in C^\infty(\mathbb{R} \times S^1; M)$ which are $(\mathcal{H},\mathbb{J})$-Floer and which satisfy

$$\lim_{s\to-\infty} u(s,t) \in W^u(q;f,g),$$
$$\lim_{s\to\infty} u(s,t) = y(t), \text{ and}$$
$$[\bar{u}]\#[v] = 0 \in \Gamma_\omega,$$

where $W^u(q;f,g)$ is the unstable manifold of q with respect to (f,g) and $\bar{u} : D^2 \to M$ is the disc map obtained by completing u to a continuous map $\bar{u} : \bar{\mathbb{R}} \times S^1 \to M$ (note that since u has finite energy and is by hypothesis J^--holomorphic in some neighbourhood of $\{-\infty\} \times S^1$ for some fixed $J^- \in \mathcal{J}(M,\omega)$, removal of singularities for pseudo-holomorphic maps implies that there is some $m \in W^u(q,f,g)$ such that $\lim_{s\to-\infty} u(s,t) = m$ for all $t \in S^1$). The map $\Phi^{PSS}_{\mathcal{D}}$ descends to a map in homology

$$\Phi^{PSS}_* : QH_{*+n}(M,\omega) \to HF_*(H)$$

which is independent of the regular PSS data. It turns out (see [**25**] for more discussion on this point as well as the definitions of the relevant operations) that Φ^{PSS}_* is an isomorphism which intertwines the quantum product with the pair-of-pants product in Floer homology. This isomorphism permits the definition of the **Oh-Schwarz spectral invariants**

$$c_{OS} : QH_*(M,\omega) \setminus \{0\} \times C^\infty(S^1 \times M) \to \mathbb{R}$$
$$(\alpha,H) \mapsto c_{OS}(\alpha;H).$$

Introduced by Schwarz in the symplectically aspherical case [**29**] and extended by Oh to more general symplectic manifolds in [**21**], these invariants are defined by

$$c_{OS}(\alpha; H) := \inf\{\lambda_H(\sigma) : \sigma \in CF_*(H,J),\ [\sigma] = \Phi_*^{PSS}(\alpha)\}.$$

A cycle $\sigma \in CF_*(H,J)$ such that $[\sigma] = \Phi_*^{PSS}(\alpha)$ and $\lambda_H(\sigma) = c_{OS}(H;\alpha)$ is called **tight** (for $c_{OS}(\alpha; H)$). It is a non-trivial fact that such cycles always exist (see [**36**] or [**22**]).

The Oh-Schwarz spectral invariants are C^0-continuous in their Hamiltonian argument, take values in the spectrum of their Hamiltonian argument, and satisfy a bevy of formal properties (see Theorem III in [**21**] for a representative list, for example) which make them useful in studying the behaviour of Hamiltonian diffeomorphisms.

2.1.3. The Gromov trick

We conclude this by recalling the so-called 'Gromov trick', which forms the basis of much of this thesis by establishing that we may use pseudo-holomorphic techniques to analyze the graphs of Floer-type cylinders.

Theorem 2.1.5 (1.4.C'. in [**7**])**.** *Let $(\mathcal{H},\mathbb{J})$ be an adapted homotopy of Floer data, then there exists a unique almost complex structure $\tilde{\mathbb{J}}$ on $\mathbb{R} \times S^1 \times M$ with the property that a graph*

$$\begin{aligned} \tilde{u} : \mathbb{R} \times S^1 &\to \mathbb{R} \times S^1 \times M \\ (s,t) &\mapsto (s,t,u(s,t)) \end{aligned}$$

is $(j_0, \tilde{\mathbb{J}})$-holomorphic if and only if u satisfies Equation 2.1.2, where j_0 denotes the standard complex structure on the cylinder.

2.2. Asymptotic analysis for pseudoholomorphic cylinders

The main analytic fact that gives us control over the asymptotic winding behaviour of Floer cylinders, as well as that of vector fields lying in the kernel of the Floer differential, is the following theorem which describes the asymptotic behaviour of solutions to an appropriately perturbed Cauchy-Riemann equation. This result is originally due to [**20**], although the version we reproduce here for the convenience of the reader is from the appendix of [**32**]

Theorem 2.2.1. *Let $w : [0,\infty) \times S^1 \to \mathbb{R}^{2n}$ satisfy the equation*

$$\partial_s w + J_0 \partial_t w + (S(t) - \Delta(s,t))w = 0, \tag{2.2.1}$$

where $S : S^1 \to End(\mathbb{R}^{2n})$ is a smooth family of symmetric matrices and

$$\Delta : [0,\infty) \times S^1 \to End(\mathbb{R}^{2n})$$

is smooth. Suppose that for $\beta \in \mathbb{N}^2$, there exist constants $M_\beta, d > 0$ such that

$$|(\partial^\beta \Delta)(s,t)| \leq M_\beta e^{-ds}, \text{ and}$$
$$|(\partial^\beta w)(s,t)| \leq M_\beta e^{-ds}.$$

Then either $w \equiv 0$ or $w(s,t) = e^{\lambda s}(\xi(t) + r(s,t))$, where λ is a negative eigenvalue of the self-adjoint operator

$$\mathbf{A} : H^1(S^1;\mathbb{R}^{2n}) \subseteq L^2(S^1;\mathbb{R}^{2n}) \to L^2(S^1;\mathbb{R}^{2n})$$
$$h \mapsto -J_0(\partial_t - J_0 S)h,$$

$\xi : S^1 \to \mathbb{R}^{2n}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ, and r satisfies the decay estimates

$$|(\partial^\beta r)(s,t)| \leq e^{-d's} M'_\beta$$

for $d', M'_\beta > 0$, for all $\beta \in \mathbb{N}^2$.

This theorem is useful in the following setting. Let (H,J) be Floer regular. To any $x \in Per_0(H)$, we may assign the **asymptotic operator**

$$A_{x,J} : \Gamma(x^*TM) \to \Gamma(x^*TM)$$

as follows. Viewing $\xi \in \Gamma(x^*TM)$ as a section of the vertical tangent bundle $\mathcal{V}|_{\check{x}} \leq T(S^1 \times M)|_{\check{x}}$ along the graph $\check{x}$ of x, we let $\check{X}_H := \partial_t \oplus X_H \in \mathcal{X}(S^1 \times M)$, and we view $J = (J_t)_{t \in S^1}$ as an endomorphism of the vertical tangent bundle by setting $\check{J}_{t,x} := J_t(x)$. $A_{x,J}$ is then defined by setting $A_{x,J}(\xi) := -\check{J}\mathcal{L}_{\check{X}_H}\xi$, where $\mathcal{L}_X Y$ denotes the Lie derivative of Y along X. $A_{x,J}$ extends to an unbounded self-adjoint operator with discrete spectrum (still denoted $A_{x,J}$) from $W^{1,2}(x^*TM)$ to $L^2(x^*TM)$.

By taking an exponential chart as in Section 1.2 on a neighbourhood $\tilde{\mathcal{O}}$ of $\hat{x} \in \widetilde{Per}_0(H)$, Floer's equation may be written in the local coordinates provided by this chart in the form of Equation 2.2.1, with $A_{x,J}$ being sent via these coordinates to $\mathbf{A}$. Following [**32**], we define

Definition 2.2.2. Let $x \in \mathcal{L}_0(M)$ and suppose that $\lim_{s\to\infty} u_s \equiv x$ for a map $u : \mathbb{R} \times S^1 \to M$. For any $R > 0$, a map

$$U^+ : [R, \infty) \to \Gamma(x^*TM)$$

will be said to be a **positive asymptotic representative** of u if $u(s,t) = Exp(U^+(s))(t)$ for all $(s,t) \in [R,\infty) \times S^1$, where Exp is as in Section 1.2. The notion of a **negative asymptotic representative** of u,

$$U^- : (-\infty,-R] \to \Gamma(x^*TM)$$

is defined in the obvious analogous manner.

Every Floer-type cylinder considered in this thesis admits, due to exponential convergence at the ends, essentially unique positive and negative asymptotic representatives, determined up to a restriction of the domains of $U^\pm$ to larger values of $|R|$.

The main result that we will need from [**32**] (paraphrased for our setting) is the following

Theorem 2.2.3. *Let (H,J) be Floer regular, $x \in Per_0(H)$ and let u,v solve Equation 2.1.2 for $s >> 0$ (resp. for $s << 0$), where the adapted homotopy used in defining Equation 2.1.2 satisfies $(H^+,J^+) = (H,J)$ (resp. $(H^-,J^-) = (H,J)$). Suppose moreover that u_s and v_s both converge to x as $s \to \infty$ (resp. $s \to -\infty$). Let U and V be positive (resp. negative) asymptotic representatives of u and v respectively. Then either $U \equiv V$ or there exists a strictly negative (resp. strictly positive) eigenvalue $\lambda \in \sigma(A_{x,J})$ and an eigenvector ξ with eigenvalue λ such that*

$$(U-V)(s,t) = e^{\lambda s}(\xi(t) + r(s,t)),$$

where the remainder term satisfies the decay estimates $|\nabla_s^i \nabla_t^i r(s,t)| \leq M_{ij} e^{-ds}$ for all $(i,j) \in \mathbb{N}^2$ and some $M_{i,j},d > 0$ (resp. $d < 0$).

Whenever u,v and x are as above, we will write $\xi^+_{u,v}$ (resp. $\xi^-_{u,v}$) for the eigenvectors of $A_{x,J}$ whose existence is guaranteed by the above theorem. We will call $\xi^\pm_{u,v}$ the **positive (resp. negative) asymptotic eigenvector of** v **relative** u. Note that the above result only requires that u and v solve Equation 2.1.2 on some neighbourhood of $s = \infty$ (resp. $s = -\infty$), and that, for $(\mathcal{H},\mathbb{J}) \in \mathscr{HJ}$, the trivial cylinder $v(s,t) = x(t)$ is always a solution to Equation 2.1.2 outside some compact set. We will write $\xi^\pm_u := \xi^\pm_{u,x}$ and call these the (positive and negative) **asymptotic eigenvectors of** u.

This asymptotic information becomes especially useful when combined with the following fact (see [**10**] p. 285 or [**32**] p.1637).

Proposition 2.2.4. *If $\xi \in \Gamma(x^*TM)$ is an eigenvector of $A_{x,J}$, then $\xi(t) \neq 0$ for all $t \in S^1$.*

Corollary 2.2.5. *Let $u,v : \mathbb{R} \times S^1 \to M$ be distinct finite energy solutions of Equation 2.1.2, then there is a compact subset $K \subseteq \mathbb{R} \times S^1$ such that $u(s,t) = v(s,t)$ only if $(s,t) \in K$.*

These results become even stronger in the case when $\dim M = 2$, as in this case Proposition 2.2.4 implies that eigenvectors of the asymptotic operator have a well-defined winding number, once we fix a trivialization of x^*TM via a choice of capping disk. More precisely, when $M = \Sigma$, if $\hat{x} \in \widetilde{\mathcal{L}_0}(\Sigma)$, and $T_{\hat{x}} : S^1 \times (\mathbb{R}^2,\omega_0) \to (x^*T\Sigma,\omega)$ is a symplectic trivialization as in Section 1, then for any eigenvector ξ of $A_{x,J}$, the map $t \mapsto T_{\hat{x}}(t)^{-1}\xi(t)$ has a well-defined winding number $wind(\xi;\hat{x})$, by Proposition 2.2.4. Proposition 1.3.4 then implies

Corollary 2.2.6. *Let u,v be distinct finite energy solutions of Equation 2.1.2 with*

$$\lim_{s\to-\infty} u_s = \lim_{s\to-\infty} v_s = x.$$

Then there exists $R > 0$ such that for all $s < -R$ and any capping $\hat{x} = [x,\alpha]$, we have

$$\ell(\hat{v}_s^\alpha,\hat{u}_s^\alpha) = wind(\xi_{u,v}^-;\hat{x}),$$

where $\hat{u}_s^\alpha$ (resp. $\hat{v}_s^\alpha$) denotes the capping of u_s (resp. v_s) such that $[x,\alpha]$ and $\hat{u}_s$ (resp. $\hat{v}_s$) are 0-homotopic. The analogous statement when $\lim_{s\to\infty} u_s = \lim_{s\to\infty} v_s = x$ also holds.

If we combine the positivity of intersection of holomorphic curves in dimension 4 with the foregoing discussion, we arrive at the principal point of this section

Lemma 2.2.7. *Let $(H^\pm,J^\pm)$ be Floer regular, $(\mathcal{H},\mathbb{J}) \in \mathscr{HJ}$ and let $u,v \in C^\infty(\mathbb{R} \times S^1;\Sigma)$ be distinct finite energy solutions to Equation 2.1.2 for $(\mathcal{H},\mathbb{J})$. Then for any lifts $\hat{u},\hat{v}$ of $u,v : \mathbb{R} \to \mathcal{L}_0(\Sigma)$, the function $\ell_{\hat{u},\hat{v}}(s) := \ell(\hat{u}_s,\hat{v}_s)$ is non-decreasing, locally constant, and well-defined for all but finitely many values $s \in \mathbb{R}$. Moreover, for $s,s' \in dom(\ell_{\hat{u},\hat{v}})$, with $s < s'$, $\ell_{\hat{u},\hat{v}}(s) \neq \ell_{\hat{u},\hat{v}}(s')$ if and only if there exists $s_0 \in (s,s')$ and some $t_0 \in S^1$ such that $u(s_0,t_0) = v(s_0,t_0)$.*

PROOF. That $\ell_{\hat{u},\hat{v}}$ has only finitely many points at which it is ill-defined follows the fact that, by definition, $\ell(\hat{u}_{s_0},\hat{v}_{s_0})$ is undefined only when there exists $t_0 \in S^1$ such that $u(s_0,t_0) = v(s_0,t_0)$. By Corollary 2.2.5, the set of all such $(s_0,t_0) \in \mathbb{R} \times S^1$ must lie inside some compact set, and we may then apply Theorem 2.1.5 to choose an almost complex structure on $\mathbb{R} \times S^1 \times \Sigma$ such that the graphs $\tilde{u}$ and $\tilde{v}$ are pseudoholomorphic, whence all such intersections must be isolated, and so finite in number. That $\ell_{\hat{u},\hat{v}}$ is non-decreasing follows by applying the positivity of intersections for holomorphic curves in dimension 4 at these intersections, which implies moreover that any such intersection contributes strictly positively to the change in $\ell_{\hat{u},\hat{v}}(s)$ as s increases and passes from one connected component of $dom(\ell_{\hat{u},\hat{v}})$ to another. $\square$

Definition 2.2.8. For $(H^{\pm},J^{\pm})$ Floer regular, $(\mathcal{H},\mathbb{J}) \in \mathscr{H}\mathscr{J}$ and $u,v \in \mathcal{M}(\hat{x},\hat{y};\mathcal{H},\mathbb{J})$, $u \neq v$, we define

$$\ell_{\pm\infty}(u,v) := \lim_{s\to\pm\infty} \ell(\hat{u}_s,\hat{v}_s),$$

where $\hat{u}_s$ and $\hat{v}_s$ are the natural cappings of u_s and v_s (cf. Definition 1.1.14).
Note that the previous lemma implies that these quantities exist and are finite. Indeed, if $\hat{u}_s$ and $\hat{v}_s$ tend to $\hat{x}$ as $s \to \pm\infty$, then $\ell_{\pm\infty}(u,v) = wind(\xi^{\pm}_{u,v};\hat{x})$ by Corollary 2.2.6, while if $\lim_{s\to\pm\infty} u_s = \hat{x}$ and $\lim_{s\to\pm\infty} v_s = \hat{y}$ with $x \neq y$, then $\ell_{\pm\infty}(u,v) = \ell(\hat{x},\hat{y})$.

2.2.1. Winding of eigenvectors of $A_{x,J}$

We summarize here some necessary facts from [**10**] on the winding numbers of eigenvectors of $A_{x,J}$ which appeared in the previous subsection (while [**10**] works in the aspherical case, our previous discussion makes clear how this winding number depends on the choice of cappings of x and y and this is all that is needed to extend the results to capped orbits). For a loop $x \in \mathcal{L}_0(\Sigma)$, let $\pi_2(\Sigma;x)$ denote the set of homotopy classes of capping disks for x.

Proposition 2.2.9. *Let $x \in Per_0(H)$ with H non-degenerate and $J : S^1 \to \mathcal{J}(\Sigma,\omega)$ arbitrary. There is a well-defined function*

$$W = W_{x,J} : \pi_2(\Sigma;x) \times \sigma(A_{x,J}) \to \mathbb{Z}$$
$$(\alpha,\lambda) \mapsto wind(T^{-1}_{[x,\alpha]} \circ \xi),$$

where $\xi \in \Gamma(x^*T\Sigma)$ is any eigenvector with eigenvalue λ. Moreover, W satisfies the following properties

(1) For any $\alpha \in \pi_2(\Sigma;x)$, $\lambda < \lambda' \;\Rightarrow\; W(\alpha,\lambda) \leq W(\alpha,\lambda')$.
(2) For any $\alpha \in \pi_2(\Sigma;x)$, and any $k \in \mathbb{Z}$, $\sum_{\lambda\in W_\alpha^{-1}(k)} \dim E_\lambda = 2$,
where $W_\alpha(\lambda) = W(\alpha,\lambda)$, and E_λ is the eigenspace associated to the eigenvalue λ.
(3) For any $A \in \pi_2(\Sigma)$, $W(A\cdot\alpha,\lambda) = W(\alpha,\lambda) + c_1(A)$.

In view of the control over the sign of the eigenvalue provided by Theorem 2.2.3, combined with the monotonicity of the winding number provided by item (1) of the above proposition, we make the following

Definition 2.2.10. For (H,J) and $\hat{x} = [x,\alpha] \in Per_0(H)$ as above, define

$$a(\hat{x}) = a(\hat{x}; H) := \sup_{\lambda \in \sigma(A_{x,J}) \cap (-\infty,0)} W(\alpha,\lambda)$$
$$b(\hat{x}) = b(\hat{x}; H) := \inf_{\lambda \in \sigma(A_{x,J}) \cap (0,\infty)} W(\alpha,\lambda).$$

Remark that we have, by the monotonicity of W and by Theorem 2.2.3, that

Corollary 2.2.11. *Let $(H^\pm,J^\pm)$ be Floer regular, $(\mathcal{H},\mathbb{J}) \in \mathscr{HJ}$, $x_i^\pm \in \widetilde{Per}_0(H^\pm)$ for $i = 0,1$.*

(1) If $u_i \in \mathcal{M}(\hat{x}_0^-,\hat{x}_i^+;\mathcal{H},\mathbb{J})$, $i = 0,1$, then $b(\hat{x}_0^-) \leq \ell_{-\infty}(u_0,u_1)$.

(2) If $u_i \in \mathcal{M}(\hat{x}_i^-,\hat{x}_0^+;\mathcal{H},\mathbb{J})$, $i = 0,1$, then $\ell_\infty(u_0,u_1) \leq a(\hat{x}_0^+)$.

The result which relates this discussion to the behaviour of the Floer complex is the following

Theorem 2.2.12 ([**10**] Theorem 3.10)**.**

$$-\mu(\hat{x}) = 2a(\hat{x}) + p(\hat{x}), \tag{2.2.2}$$

where $p([x,\alpha]) = 0$ if there exists $\lambda \in \sigma(A_{x,J}) \cap (0,\infty)$ such that $W(\alpha,\lambda) = a(\hat{x})$ and $p([x,\alpha]) = 1$ otherwise.

Remark. Note that our sign convention for the Conley-Zehnder index is the negative of the convention adopted in [**10**].

Corollary 2.2.13.

$$-\mu(\hat{x}) = a(\hat{x}) + b(\hat{x})$$

PROOF. The point is that $b(\hat{x}) = a(\hat{x}) + p(\hat{x})$. To see this write $\hat{x} = [x,\alpha]$, and let $\lambda = \max\{\nu \in \sigma(A_{x,J}) : \nu < 0\}$ be such that $W(\alpha,\lambda) = a(\hat{x})$.

Suppose first that $p(\hat{x}) = 0$, then by the definition of $p(\hat{x})$, there exists $\lambda' \in \sigma(A_{x,J}) \cap (0,\infty)$ such that $W(\alpha,\lambda') = a(\hat{x})$. By item (2) of Proposition 2.2.9, it follows that E_λ and $E_{\lambda'}$ are 1-dimensional and that $W(\alpha,\nu) = a(\hat{x})$ for $\nu \in \sigma(A_{x,J})$ if and only if $\nu \in \{\lambda,\lambda'\}$. The monotonicity of the winding number expressed in item (1) of Proposition 2.2.9 then implies that if $\nu \in \sigma(A_{x,J}) \cap (0,\infty)$ is distinct from λ', then $W(\alpha,\lambda') < W(\alpha,\nu)$, and consequently we see that

$$W(\alpha,\lambda') = b(\hat{x}).$$

Suppose next that $p(\hat{x}) = 1$ so that, by the monotonicity of item (1) in Proposition 2.2.9, we have that $W(\alpha,\lambda') > a(\hat{x})$ for all $\lambda' \in \sigma(A_{x,J}) \cap (0,\infty)$. Item (2) of Proposition 2.2.9 implies that there exists $\lambda' \in \sigma(A_{x,J})$ such that $W(\alpha,\lambda') = a(\hat{x}) + 1$ and the monotonicity of the winding number implies that such λ' must satisfy $\lambda' > \lambda$. Since λ is by definition the largest negative eigenvalue of $A_{x,J}$, it follows from the non-degeneracy of $A_{x,J}$ (which is equivalent to the non-degeneracy of $x \in Per_0(H)$ as a 1-periodic orbit of H) that λ' is positive, and so clearly we must have $b(\hat{x}) = W(\alpha,\lambda') = a(\hat{x}) + 1$. □

Lemma 2.2.14. *Let $k \in \mathbb{Z}$.*

(1) If $\mu(\hat{x};H) \in \{2k-1, 2k\}$, then $a(\hat{x};H) = -k$.

(2) If $\mu(\hat{x};H) \in \{2k, 2k+1\}$, then $b(\hat{x};H) = -k$.

PROOF. We prove (1) with the argument for (2) being entirely analogous (but additionally making use of the fact that $b(\hat{x}) = a(\hat{x}) + p(\hat{x})$ established in the proof of Corollary 2.2.13). Theorem 2.2.12 states that $-\mu(\hat{x}) = 2a(\hat{x}) + p(\hat{x})$, where $p(\hat{x}) \in \{0,1\}$ is the parity of $\mu(\hat{x})$. Consequently, if $\mu(\hat{x}) = 2k-1$, we see that

$$1 - 2k = 2a(\hat{x}) + 1$$

and therefore $a(\hat{x}) = -k$. Similarly, if $\mu(\hat{x}) = 2k$, then

$$-2k = 2a(\hat{x})$$

and so $a(\hat{x}) = -k$, as claimed. □

Corollary 2.2.15. *Let $(H^{\pm},J^{\pm})$ be Floer regular, $(\mathcal{H},\mathbb{J}) \in \mathscr{HJ}(H^-,J^-;H^+,J^+)$ and suppose that $\hat{x}^{\pm} \in \widetilde{Per}_0(H^{\pm})$ satisfy $\mu(\hat{x}^{\pm}) = 2k+1$ for some $k \in \mathbb{Z}$. Then $|\mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H},\mathbb{J})| \in \{0,1\}$.*

PROOF. Suppose for the sake of contradiction that there exist $u,v \in \mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H},\mathbb{J})$, $u \neq v$. Then we have by Lemma 2.2.7 and Lemma 2.2.14

$$-k = b(\hat{x}^-) \leq \ell_{-\infty}(u,v) \leq \ell_{\infty}(u,v) \leq a(\hat{x}^+) = -k-1,$$

which is a contradiction. The lemma follows. □

Recall from Section 2.1 that to any $(\mathcal{H},\mathbb{J}) \in \mathscr{HJ}(H^-,J^-;H^+,J^+)$, we associate the operator $\mathcal{F}_{\mathcal{H},\mathbb{J}} : C^{\infty}(\mathbb{R} \times S^1;\Sigma) \to \mathcal{E}$. Whenever $\mathcal{F}_{\mathcal{H},\mathbb{J}}(u) = 0$, for $u \in C^{\infty}(\mathbb{R} \times S^1;\Sigma)_{\hat{x}^-,\hat{x}^+}$, $T_u\mathcal{E}$ splits canonically as $T_uC^{\infty}(\mathbb{R} \times S^1;\Sigma)_{\hat{x}^-,\hat{x}^+} \oplus \mathcal{E}_u$. In such a case, we denote by $D\mathcal{F}_{\mathcal{H},\mathbb{J}}$ the projection of the differential of $\mathcal{F}_{\mathcal{H},\mathbb{J}}$ onto $\mathcal{E}_u$, and we call $D\mathcal{F}_{\mathcal{H},\mathbb{J}}$ the **linearized Floer operator**. The transversality of $\mathcal{F}_{\mathcal{H},\mathbb{J}}$ to the 0-section of $\mathcal{E}$ at u is equivalent to the surjectivity of $(D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u$, which is in turn related to the behaviour of its kernel by the Fredholm property. The following result is essentially proved in [**10**] as Proposition 5.6 and serves to give significant control over elements in the kernel of the linearized Floer operator. We give a simple proof here in the Floer-theoretic setting for the convenience of the reader.

Proposition 2.2.16. *Let $(H^{\pm},J^{\pm})$ be Floer regular, let $(\mathcal{H},\mathbb{J}) \in \mathscr{HJ}$, $u \in \mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H},\mathbb{J})$, and let $\xi \in \ker(D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u$. Suppose that $\xi \not\equiv 0$ and denote by $Z(\xi)$ the algebraic count of the number of zeros of ξ, then $Z(\xi)$ is finite and satisfies the inequality $0 \leq Z(\xi) \leq a(\hat{x}^+) - b(\hat{x}^-)$.*

PROOF. It is a standard result in Floer theory (see for instance [**27**], Section 2.2) that for any $u \in \mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H},\mathbb{J})$, any element $\xi \in \ker(D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u$ may be expressed (with respect to the unitary trivialization $\Phi : \mathbb{R} \times S^1 \times (\mathbb{R}^2,J_0) \to u^*(T\Sigma,J)$ along u induced by the cappings of $\hat{x}^-$ and $\hat{x}^+$) as solving an equation of the form

$$\partial_s \xi + J_0 \partial_t \xi + S\xi = 0,$$

where we may write S on the positive and negative ends as $S^{\pm}(s,t) = \Phi^{-1}A_{x^{\pm},J} - \Delta^{\pm}(s,t)$, with $\Delta^{\pm}$ satisfying the decay estimates of Theorem 2.2.1. Consequently, any $\xi \in \ker(D\mathcal{F})_u$ must be non-vanishing outside of some compact neighbourhood of $\mathbb{R} \times S^1$, and the Carlemann similarity principle, combined with positivity of intersections of holomorphic curves in dimension 4, implies that $Z(\xi)$ is finite and non-negative.

To see that $Z(\xi) \leq a(\hat{x}^+) - b(\hat{x}^-)$, we take $R > 0$ sufficiently large so that ξ is non-vanishing outside of $(-R,R) \times S^1$ and consider the homotopy of 2-braids in $\mathbb{R}^2$ induced by $h(s) = (0, \Phi^{-1}\xi_s) \in \mathcal{L}_0(\mathbb{R}^2)^2$, $s \in [-R,R]$. Theorem 2.2.1 implies that for $R > 0$ sufficiently large,

$$\ell(0,\xi_{-R}) = wind(\Phi^{-1}\xi_{-R}) \geq b(\hat{x}^-),\ and$$
$$\ell(0,\xi_{R}) = wind(\Phi^{-1}\xi_{R}) \leq a(\hat{x}^+),$$

(since $\mathbb{R}^2$ is aspherical, we omit any mention of cappings), and the algebraic count zeros of ξ correspond to the algebraic count of the intersections of the graphs of the strands of h from which the proposition follows. $\square$

The following corollary is essentially the linear analogue of Corollary 2.2.15.

Corollary 2.2.17. *Let $(H^{\pm},J^{\pm})$ be Floer regular, let $(\mathcal{H},\mathbb{J}) \in \mathscr{HJ}$, and let $\hat{x}^{\pm} \in \widetilde{Per}_0(H^{\pm})$ have $\mu(\hat{x}^{\pm}) = 2k+1$ for some $k \in \mathbb{Z}$. Then $(\mathcal{H},\mathbb{J})$ is $(\hat{x}^-,\hat{x}^+)$-regular.*

PROOF. If $(H^-,J^-) = (H^+,J^+)$ and $(\mathcal{H},\mathbb{J})$ is the constant homotopy of Floer data, then this is automatic by Floer regularity of $(H^{\pm},J^{\pm})$. The statement is also vacuously true if $\mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H},\mathbb{J})$ is empty. So we may suppose that $(\mathcal{H},\mathbb{J})$ is not $\mathbb{R}$-invariant and that there exists some $u \in \mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H},\mathbb{J})$. To see that u is regular, note that $D\mathcal{F}_{\mathcal{H},\mathbb{J}}$ has Fredholm index 0, and so to prove that $(D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u$ is surjective, it suffices to show that its kernel vanishes. Suppose to the contrary that

$\xi \in \ker(D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u$ is a non-trivial vector field along u. By the previous proposition, we must have that the algebraic count of its zeros $Z(\xi)$ satisfies

$$0 \leq Z(\xi) \leq a(\hat{x}^+) - b(\hat{x}^-) = -k - (k-1) = -1,$$

where the second to last inequality follows from Lemma 2.2.14. Clearly this is a contradiction, so we conclude that $\ker(D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u = 0$ and so u is regular. $\square$

In the case of even index orbits, we have a somewhat weaker conclusion that will still be of use to us

Corollary 2.2.18. *Let $(H^\pm,J^\pm)$ be Floer regular, let $(\mathcal{H},\mathbb{J}) \in \mathscr{HJ}$, and let $\hat{x}^\pm \in \widetilde{Per}_0(H^\pm)$ have $\mu(\hat{x}^\pm) = 2k$ for some $k \in \mathbb{Z}$. Then for every $u \in \mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H},\mathbb{J})$,*

$$\dim\ker(D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u \leq 1$$

PROOF. As in the proof of the previous corollary, it suffices to consider the case where $(\mathcal{H},\mathbb{J})$ is not $\mathbb{R}$-invariant and $\mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H},\mathbb{J})$ is non-empty. Letting $u \in \mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H},\mathbb{J})$, consider the behaviour of some $\eta_1,\eta_2 \in \ker(D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u$. Let

$$\Phi : \mathbb{R} \times S^1 \times \mathbb{C} \to u^*T\Sigma$$

be a unitary trivialization of the tangent bundle along u which extends over the cappings of $\hat{x}^\pm$. Since η_1 and η_2 satisfy an appropriate perturbed Cauchy-Riemann equation, Theorem 2.2.1 implies that $v_i(s,t) := (\Phi^{-1} \circ \eta_i)(s,t)$, $i = 1,2$ satisfies

$$\lim_{s\to-\infty} v_i(s,t) = \Phi^{-1}(-\infty,t) \circ \xi_i(t)$$

for $\xi_i \in \Gamma^\infty((x^-)^*T\Sigma)$ an eigenvector of the asymptotic operator A_{x^-,J^-} associated to x^-, whose associated eigenvalue is positive. Let λ_0 be the smallest positive eigenvalue of A_{x^-,J^-} and note that every eigenvector $\xi' \in E_{\lambda_0}$ is such that $\Phi^{-1}(-\infty,t) \circ \xi'(t)$ has winding number $-k$ by Corollary 2.2.13, and by the same reasoning as in the proof of that corollary, $\dim E_{\lambda_0} = 1$. We claim first that $\xi_i \in E_{\lambda_0}$ for $i = 1,2$. Indeed, suppose not, then without loss of generality we may suppose that $\xi_1 \in E_\lambda$ for $\lambda > \lambda_0$. This implies that $wind(\Phi^{-1}(-\infty,t) \circ \xi_1(t)) \geq -k+1$, by points (1) and (2) of Proposition 2.2.9 combined with the fact that $a(\hat{x}^-) = b(\hat{x}^-)$ (so the only other eigenspace whose eigenvector have winding $-k$ is the eigenspace associated to the largest negative eigenvalue of A_{x^-,J^-}). Whence,

$$wind(\Phi^{-1}(-R,t) \circ \eta_1(-R,t)) \geq -k+1$$

for $R > 0$ sufficiently large. As in the proof of Proposition 2.2.16, we have that

$$Z(\eta_1) \leq a(\hat{x}^+) - wind(\Phi^{-1}(-R,t) \circ \eta_1(-R,t))$$

for $R > 0$ sufficiently large, but this implies

$$Z(\eta_1) \leq -k - (-k + 1) = -1$$

which violates the positivity of $Z(\eta_1)$. So $\xi_i \in E_{\lambda_0}$ for $i = 1,2$.

Now consider $\delta := \eta_1 - \eta_2$. δ once again satisfies a perturbed Cauchy-Riemann equation satisfying the required asymptotic decay conditions, because η_1 and η_2 do. Let $\xi_\delta(t) = \lim_{s \to -\infty} \delta(s,t)$ be the associated negative asymptotic eigenvector. We claim that $\xi_\delta \in E_{\lambda_0}$. Indeed, if not then we may repeat the above argument with δ in the place of η_1 and ξ_δ in the place of ξ_1 to derive a contradiction. It follows that the map which sends $\eta \in \ker(D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u$ to its negative asymptotic eigenvector in E_{λ_0} is linear, and it is injective by Theorem 2.2.1 (since the negative asymptotic eigenvector of η is zero if and only if η vanishes on the negative end, and thus by the Carlemann similarity principle, η vanishes everywhere). Since $\dim E_{\lambda_0} = 1$, this proves the claim. $\square$

Chapter 3

Constructing chain-level continuation maps with prescribed behaviour

In this chapter, we introduce a technique for designing chain-level continuation maps such that certain 0-dimensional moduli spaces may be guaranteed to be non-empty. For the remainder of the section, we fix Floer regular $(H^{\pm},J^{\pm}) \in C^{\infty}(S^1 \times M) \times C^{\infty}(S^1; \mathcal{J}(M,\omega))$

Definition 3.0.1. We will say that a finite collection of smooth maps $\{u_i : \mathbb{R} \times S^1 \to \Sigma\}$, $i = 1,\ldots,k$ is a **model for a continuation cobordism** (from (H^-,J^-) to (H^+,J^+)) if there exists $(\mathcal{H},\mathbb{J}) \in \mathscr{H}\mathscr{J}(H^-,J^-;H^+,J^+)$ such that u_i is an $(\mathcal{H},\mathbb{J})$-Floer cylinder with finite energy for each $i = 1,\ldots,k$. Such a model for a continuation cobordism will be called an $(\mathcal{H},\mathbb{J})$**-model**.

If we are doing Floer theory on a surface, then the results of Section 2.2 tell us that the graphs $\tilde{u}_i$, when restricted to a sufficiently large compact set $[-K,K] \times S^1$, define a braid cobordism having only positive intersections (outside this compact set, the maps u_i solve the s-independent Floer equations on the ends, and their behaviour is controlled by the winding behaviour of the eigenvectors of the asymptotic operator as discussed in the previous chapter). It will be convenient to make the following

Definition 3.0.2. Let $h : [0,1] \to \mathcal{L}(C_k(\Sigma))$ be a braid cobordism. We will say that h is a **positive** (resp. **negative**) cobordism if for $1 \leq i < j \leq k$, the graphs $\tilde{h}_i, \tilde{h}_j : [0,1] \times S^1 \to [0,1] \times S^1 \times \Sigma$ are transverse, and every intersection is positive (resp. negative).

The main result of this chapter is to show that this essentially topological condition, combined with the obvious necessary condition on the asymptotic behaviour of the strands, is sufficient to guarantee the existence of some regular homotopy of Floer data $(\mathcal{H},J)$ such that each cylinder is $(\mathcal{H},J)$-Floer.

Definition 3.0.3. We will say that a finite collection of smooth maps $u_i : \mathbb{R} \times S^1 \to \Sigma$, $i = 1, \ldots, k$, is a **pre-model for a continuation cobordism** (from (H^-, J^-) to (H^+, J^+)) if they satisfy the following

(1) There exists $K > 0$ such that the maps $u_i|_{[-K,K]}$ are the strands of a positive braid cobordism such that the graphs of these strands intersect transversally.
(2) For the same K as above, $u_i|_{(-\infty,-K]}$ has finite (H^-, J^-)-energy and satisfies the (H^-, J^-)-Floer equation, while $u_i|_{[K,\infty)}$ has finite (H^+, J^+) energy and satisfies the (H^+, J^+)-Floer equation, for each $i = 1, \ldots, k$.

And we aim to show

Theorem 3.0.4. *If $u_i : \mathbb{R} \times S^1 \to \Sigma$, $i = 1, \ldots, k$ defines a pre-model for a continuation cobordism from (H^-, J^-) to (H^+, J^+), then there exists $u_i' : \mathbb{R} \times S^1 \to \Sigma$, $i = 1, \ldots, k$ which is a model for a continuation cobordism from (H^-, J^-) to (H^+, J^+) and such that each u_i' differs from u_i only by a small homotopy inside a compact set of $(-K, K) \times S^1$ (where here $K > 0$ is as in the definition of a pre-model for a continuation cobordism).*

The principal virtue of this result is that the existence or non-existence of pre-models may largely be reduced to the question of the relative topologies of the collections of capped orbits (*qua* capped braids) being connected by these cylinders, along with asymptotic information provided by the asymptotic operator.

The argument itself is straightforward and rather hands-on. Our goal is to show that given a pre-model for a cobordism from (H^-, J^-) to (H^+, J^+), we may build a homotopy of Floer data $(\mathcal{H}, J) \in \mathscr{H}\mathscr{J}(H^-, J^-; H^+, J^+)$ such that each u_i is $(\mathcal{H}, J)$-Floer. When $(u_1(s_0, t_0), \ldots, u_k(s_0, t_0)) \in C^k(\Sigma)$, it is not difficult to simply choose $(\mathcal{H}^{s_0}_{t_0}, J^{s_0}_{t_0})$ such that each u_i satisfies the relevant Floer-equation at (s_0, t_0), and this assignment may be made smoothly in (s, t) as long as $u_i(s, t) \neq u_j(s, t)$ for $i \neq j$. The difficulty, therefore, arises when the graphs of the cylinders intersect. To resolve this, in Section 3.1 we explain how to perturb the cylinders in a neighbourhood V of an intersection point to another pre-model which satisfies the Floer equations for a judiciously chosen pair $(\mathcal{H}^V, J^V)$ on V. In Section 3.2, we prove Theorem 3.0.4 by performing this perturbation at every intersection point, and then arguing that we may extend the locally judiciously chosen s-dependent Hamiltonian-almost complex structure pairs to an adapted homotopy of Floer data $(\mathcal{H}, J)$ such that the resulting perturbed cylinders are all $(\mathcal{H}, J)$-Floer. Finally, in Section 3.3, we address the question of obtaining model cobordisms for *regular* homotopies of Floer data from the existence of the model cobordisms constructed in Theorem

3.0.4. This gives method for designing Floer continuation maps which have prescribed chain-level behaviour.

3.1. Perturbing positively intersecting strands to solve Floer's equation near the intersection

For $\kappa > 0$, let $D(\kappa) \subset \mathbb{C}$ denote the closed disk of radius κ, centered at the origin. We write $D = D(1)$ for the closed unit disk.

Lemma 3.1.1. *Let (M^4,J) be a smooth almost complex 4-manifold, and let $\mathcal{F}$ be a smooth oriented codimension 2 foliation of M such that $T\mathcal{F}$ is J-invariant. Suppose that $u,v : D \to M$ are smooth embeddings which are positively transverse to the leaves of $\mathcal{F}$ and which intersect positively at $u(0) = v(0) = p \in M$. Suppose moreover that v is J-holomorphic on some neighbourhood of p. Then for any C^0-neighbourhood $\mathcal{U}$ of u in $C^\infty(D; M)$, there exists $\kappa > 0$, $\delta > 0$ and $u' \in \mathcal{U}$ such that*

(1) $u'(0) = p$,
(2) u' is J-holomorphic on $D(\kappa)$,
(3) $u' = u$ on $D \setminus D(\kappa + \delta)$,
(4) u' is positively transverse to the leaves of $\mathcal{F}$.

PROOF. By the local existence theorem for holomorphic curves (see theorem $3.1.1$ in [**34**], for instance), there exists a $\rho > 0$ and a J-holomorphic map $f : D(\rho) \to M$ such that $f(0) = p$ and $(\partial_s f)_p = (\partial_s u)_p$. Up to taking ρ to be smaller and restricting the domain of f, we may assume that f is an embedding. Note that, $(\partial_s u)_p = (\partial_s f)_p \neq 0$ and hence $(\partial_t f)_p \neq 0$, so f and v intersect transversally and positively, since both are J-holomorphic. Note that similar reasoning shows that f is positively transverse to the leaf of $\mathcal{F}$ passing through p, and so f is positively transverse to $\mathcal{F}$ for all $z \in D(\rho)$ sufficiently close to 0. We will construct u' by interpolating between u and f. In order to construct this interpolation, it will be useful to introduce a convenient local coordinate system.

To this end, let us note that there is a neighbourhood $U \subset M$ of p, an $\epsilon > 0$, and an orientation-preserving chart

$$\phi : U \to \mathbb{C} \times \mathbb{C},$$

such that for all $z \in D(\epsilon)$, $(\phi \circ v)(z) = (z,0)$. Moreover, if the foliation $\mathcal{F}$ is as in the statement of the lemma, then the fact that v intersects positively and transversally with the leaves of $\mathcal{F}$ implies that ϕ may be chosen such that for any leaf $F \in \mathcal{F}$, $\phi(F \cap U) \subset \{z_0\} \times \mathbb{C}$ for some $z_0 \in \mathbb{C}$. That is, ϕ locally diffeomorphically sends leaves of $\mathcal{F}$ into the fibers of the projection map onto the first coordinate. Since u and f are both positively transverse to v, up to shrinking $\epsilon > 0$, for $z \in D(\epsilon)$ we may write

$$f_{loc}(z) := (\phi \circ f)(z) = (z, h_0(z)),$$
$$u_{loc}(z) := (\phi \circ u)(z) = (z, h_1(z))$$

for $h_0, h_1 : D(\epsilon) \to \mathbb{C}$ smooth functions such that $h_0(0) = h_1(0) = 0$, $(\partial_s h_0)_0 = (\partial_s h_1)_0 \neq 0$, and $\{(\partial_s h_i)_0, (\partial_t h_i)_0\}$ is a positively-oriented basis for $\mathbb{C}$ for $i = 0,1$. Next, for $\tau \in [0,1]$, let us define $h_\tau : D(\epsilon) \to \mathbb{C}$ by

$$h_\tau(z) := (1-\tau)h_0(z) + \tau h_1(z).$$

Note that $(\partial_s h_\tau)_0$ is constant in τ, and so it is easy to see that

$$\{(\partial_s h_\tau)_0, (\partial_t h_\tau)_0\} = \{(\partial_s h_0)_0, (1-\tau)(\partial_t h_0)_0 + \tau(\partial_t h_1)_0\}$$

is positively oriented, as the set of all vectors $w \in \mathbb{C}$ such that $\{(\partial_s h_0)_0, w\}$ is positively oriented is clearly a convex set and contains both $(\partial_t h_0)_0$ and $(\partial_t h_1)_0$ by hypothesis. Consequently, for each $\tau \in [0,1]$ there exists $\epsilon_\tau \in (0,\epsilon)$, varying continuously with τ, such that $h_\tau(z) = 0$ if and only if $z = 0$ for all $z \in D(\epsilon_\tau)$. Posing

$$\epsilon' := \inf_{\tau \in [0,1]} \epsilon_\tau,$$

we see that $\epsilon' > 0$ and the maps $u_\tau(z) := (z, h_\tau(z))$ intersect $D(\epsilon') \times \{0\}$ only in $(0,0)$ for all $\tau \in [0,1]$. Fix $\kappa \in (0,\epsilon')$ and some $\delta \in (0, \epsilon' - \kappa)$, and let $\beta : D(\epsilon') \to [0,1]$ be a smooth radially non-decreasing function which is identically 0 on $D(\kappa)$ and identically 1 outside of $D(\kappa + \delta)$. For $z \in D(\epsilon')$, we define

$$u'_{loc}(z) := u_{\beta(z)}(z) = (1 - \beta(z))f_{loc} + \beta(z)u_{loc}.$$

Note that $\phi^{-1} \circ u'_{loc}$ then obviously satisfies the first three items listed in the lemma, by construction, and is moreover transparently positively transverse to leaves of $\mathcal{F}$ by construction, since ϕ^{-1} sends graphs of maps $\mathbb{C} \to \mathbb{C}$ to submanifolds transverse to leaves of $\mathcal{F}$ by our choice of ϕ. We then obtain the claimed $u' : D \to M$ by setting

$$u'(z) = \begin{cases} (\phi^{-1} \circ u'_{loc})(z) & z \in D(\epsilon') \\ u(z) & z \in D \setminus D(\epsilon'). \end{cases} \tag{3.1.1}$$

It is then clear from construction that u' satisfies the four properties listed in the lemma. Moreover, because $f(0) = u(0) = p$ and f and u are both locally Lipchitz (being smooth functions), u' may obviously be taken to be as C^0-close to u as we wish, simply by taking κ and δ smaller in the preceding argument if necessary. $\square$

Corollary 3.1.2. *Let $u,v : \mathbb{R} \times S^1 \to \Sigma$ be smooth maps such that their graphs $\tilde{u}, \tilde{v} : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1 \times \Sigma$ intersect positively and transversally at $(s_0,t_0,p) \in \mathbb{R} \times S^1 \times \Sigma$. Then for any neighbourhood $V \subset \mathbb{R} \times S^1$ of (s_0,t_0), there exists an open set $U \subset \bar{U} \subset V$ containing (s_0,t_0), smooth maps*

$$\mathcal{H}_{\bar{U}} : \bar{U} \to C^\infty(\Sigma),$$
$$J_{\bar{U}} : \bar{U} \to \mathcal{J}(\Sigma,\omega),$$

and a smooth map $u' : \mathbb{R} \times S^1 \to \Sigma$ agreeing with u outside of V, such that $\tilde{u}'$ intersects $\tilde{v}$ positively and transversally at (s_0,t_0,p), and such that both u' and v solve the $(\mathcal{H}_{\bar{U}},J_{\bar{U}})$-Floer equation for $(s,t) \in \bar{U}$.

PROOF. All that needs to be shown here is that for V a small enough neighbourhood of (s_0,t_0), we may always choose $\mathcal{H}_V : V \to C^\infty(\Sigma)$ and $J_V : V \to \mathcal{J}(\Sigma,\omega)$ such that v is $(\mathcal{H}_V,J_V)$-Floer on V. Once this is established, we simply apply the above lemma to $\tilde{u}$ and $\tilde{v}$ using Gromov's trick to construct an almost complex structure $\tilde{J}^{\mathcal{H}}$ on $V \times \Sigma$ such that maps solve the $(\mathcal{H}_V,J_V)$-Floer equations if and only if their graphs are $\tilde{J}^{\mathcal{H}}$ holomorphic over V. The above lemma gives us an open set $U \subset \bar{U} \subset V$ containing (s_0,t_0) and a map $\tilde{u}' : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1 \times \Sigma$ which may be taken to be transverse to the fibers of the projection map $\mathbb{R} \times S^1 \times \Sigma \to \mathbb{R} \times S^1$ — and thus may be taken to be the graph of some function $u' : \mathbb{R} \times S^1 \to \Sigma$ — which is $\tilde{J}^{\mathcal{H}}$ holomorphic on $\bar{U}$, and which agrees with $\tilde{u}$ outside of V. We then simply take $\mathcal{H}_{\bar{U}} := \mathcal{H}_V|_{\bar{U}}$, $J_{\bar{U}} := J_V|_{\bar{U}}$.

To see that $\mathcal{H}_V : V \to C^\infty(\Sigma)$ and $J_V : V \to \mathcal{J}(\Sigma,\omega)$ may be chosen as claimed, let J_V be arbitrary, and define

$$X : V \to v^*T\Sigma|_V$$
$$(s,t) \mapsto (\partial_t v - J\partial_s v)_{(s,t)}.$$

It's straightforward to see that we may choose $\mathcal{H}_V$ to satisfy $X^{\mathcal{H}_V}(v(s,t)) = X(s,t)$ for all $(s,t) \in V$. Indeed, define

$$\mathcal{H}(X) := \{(s,t,H) \in V \times C^\infty(\Sigma) : X^H(v(s,t)) = X(s,t)\}.$$

It is not difficult to verify that $\pi : \mathcal{H}(X) \to V$ is a locally trivial fibration (see Corollary 3.4.2 in the appendix to this section), with π being the obvious projection map. The fiber over $z \in V$ is diffeomorphic to the subspace of functions on Σ having a critical point at $v(z)$. This shows that the fibers are contractible, and so there exists a section $\mathcal{H}^V$. By construction, v is $(\mathcal{H}_V,J_V)$-Floer on V. $\square$

3.2. Proof of Theorem 3.0.4

We note first that for any pre-model for a continuation cobordism, there are only ever finitely many intersection points where the strands must be perturbed to make them locally satisfy Floer's equation for some choice of $(\mathcal{H},J) \in \mathscr{HJ}(H^-,J^-;H^+,J^+)$ in a neighbourhood of the intersection point.

Proposition 3.2.1. *If $u_i : \mathbb{R} \times S^1 \to \Sigma$, $i = 1,\dots,k$ is a pre-model for a continuation cobordism, then*

$$I := \{(s,t) \in [-K,K] \times S^1 : u_i(s,t) = u_j(s,t), 1 \leq i < j \leq k\},$$

where K is as in the definition of a pre-model for a continuation cobordism, contains only finitely many points and lies in the interior of $[-K,K] \times S^1$.

PROOF. That I is disjoint from the boundary of $[-K,K] \times S^1$ follows directly from the fact that, by hypothesis $h(s) := (u_1^s,\dots,u_k^s)$, $s \in [-K,K]$ defines a braid cobordism (here u_i^s denotes the loop $t \mapsto u_i(s,t)$). As a consequence, $(u_1^{\pm K},\dots,u_k^{\pm K})$ are braids, and therefore their graphs do not intersect. Consequently, I has no accumulation points on $\{-K,K\} \times S^1$, and therefore, there exists some $\delta > 0$ such that $I \subset [-K+\delta, K-\delta] \times S^1$. Since the intersections of the graphs of the u_i are assumed to be transverse, all points in I must be isolated, and consequently, I is a finite collection of points. $\square$

Let us now prove Theorem 3.0.4.

PROOF. By Lemma 3.2.1, $I \subset [-K,K] \times S^1$ consists of only finitely many points, and the graphs of the u_i do not intersect over $\{-K,K\} \times S^1$. By Corollary 3.1.2 for each $z \in I$, there exists an arbitrarily small open set U_z containing z, local sections

$$\mathcal{H}^z_{loc} : \bar{U}_z \to C^\infty(\Sigma)$$
$$J^z_{loc} : \bar{U}_z \to \mathcal{J}(\Sigma,\omega),$$

and small perturbations u_i' of each u_i, $i = 1, \dots, k$ such that each u_i' satisfies the $(\mathcal{H}^z_{loc},J^z_{loc})$-Floer equation on $\bar{U}_z$. Let us write

$$V := ([-K,K] \times S^1) \setminus \bigcup_{z \in I_K} U_z.$$

Our goal is to find smooth maps

$$\mathcal{H}^V : V \to C^\infty(\Sigma)$$
$$J^V : V \to \mathcal{J}(\Sigma,\omega),$$

which agree with $\mathcal{H}^z_{loc}$ and J^z_{loc} respectively on $\partial\bar{U}_z$, which in addition agree with $H^\pm$ and with $J^\pm$ respectively on $\{\pm K\} \times S^1$, and such that, moreover, each u_i' is $(\mathcal{H}^V,J^V)$-Floer on V. Clearly, if this can be done, these data will patch together to give a homotopy $(\mathcal{H},J)$ of Floer data between (H^-,J^-) and (H^+,J^+) such that each u_i' is $(\mathcal{H},J)$-Floer on all of $\mathbb{R} \times S^1$, and we will be done.

To see that $(\mathcal{H}^V,J^V)$ may be chosen as desired we argue essentially identically to the proof of Corollary 3.1.2. First take J^V to be any smooth map $J^V : V \to \mathcal{J}(\Sigma,\omega)$ which restricts to the desired almost complex structures on ∂V. Clearly, this can be done, since the space of ω-compatible almost complex structures is contractible. Next, for each $i = 1,\dots,k$, we define a section of $(u_i'|_V)^*T\Sigma$ by

$$X_i : V \mapsto (u_i'|_V)^*T\Sigma$$
$$(s,t) \mapsto (\partial_t u_i' - J^V \partial_s u_i')_{(s,t)},$$

and we define $\mathcal{H}(\vec{X}) \to V$ to be the fiber bundle with fiber

$$\mathcal{H}(\vec{X})_{s,t} := \{H \in C^\infty(\Sigma) : X^H(u_i'(s,t)) = X_i(s,t),\ \forall (s,t) \in V,\ \forall i = 1, \dots, k\}.$$

It is not difficult to see that $\mathcal{H}(\vec{X})$ is a locally trivial fibration (apply Corollary 3.4.2 from the appendix in the setting where the underlying symplectic manifold is the configuration space $C_k(\Sigma)$) and that each $\mathcal{H}^z_{loc}|_{\partial\bar{U}_z}$ defines a section of $\mathcal{H}(\vec{X})$ over $\partial\bar{U}_z$, in addition to $(\pm K, t) \mapsto H^\pm_t$ defining a section over $\{\pm K\} \times S^1$. Moreover, each fiber of $\mathcal{H}(\vec{X})$ is diffeomorphic to the space of functions on Σ having critical points at k distinct points. In particular, $\mathcal{H}(\vec{X})$ has contractible fibers and therefore admits a smooth section extending the given aforementioned data on ∂V, completing the proof. □

3.3. From models of cobordisms to regular models

Our ultimate goal is to use the topological construction of pre-models for cobordisms to guarantee the more rigid existence of Floer continuation maps with prescribed behaviour. As such, we will need to consider issues of regularity.

Definition 3.3.1. Let $(H^{\pm},J^{\pm})$ be Floer regular. A model for a continuation cobordism from (H^-,J^-) to (H^+,J^+) will be called **regular** if it is an $(\mathcal{H},\mathbb{J})$-model for some $(\mathcal{H},\mathbb{J}) \in \mathscr{H}\mathscr{J}_{reg}(H^-,J^-;H^+,J^+)$.

The following lemma follows essentially by combining Corollaries 2.2.17 and 2.2.18 with results which are likely folkloric: that local transversality of the s-dependent Floer operator at some cylinder u only requires perturbations with support contained in an appropriately dense open neighbourhood of the graph of u, and the fact that near a minimally degenerate Floer cylinder of index 0 (ie. when $\dim\ker(D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u) = 1$), the universal moduli space locally has the structure of a 'fold' singularity over the space of perturbation data. I am unaware of a place in the published literature where such results are stated, so for the convenience of the reader, a detailed proof sketch is provided in the appendix to this chapter (see Section 3.4.2).

Lemma 3.3.2. *Let $(H^{\pm},J^{\pm})$ be Floer-regular, $(\mathcal{H},J) \in \mathscr{H}\mathscr{J}(H^-,J^-;H^+,J^+)$, $\hat{x}^{\pm} \in \widetilde{Per}_0(H^{\pm})$, and $u \in C^\infty(\mathbb{R}\times S^1;\Sigma)_{\hat{x}^-,\hat{x}^+}$. Suppose that $\mathcal{F}_{\mathcal{H},J}(u) = 0$ and that $ind\ (D\mathcal{F}_{\mathcal{H},J})_u = 0$. Let $N \subset \mathbb{R}\times S^1 \times \Sigma$ denote a neighbourhood of im $\tilde{u}$ and let $U \subset N$ be an open dense subset of N such that $U \cap$ im $\tilde{u}$ is dense in im $\tilde{u}$. For any neighbourhood $\mathcal{U} \subset C^\infty(\mathbb{R}\times S^1;\Sigma)_{\hat{x}-,\hat{x}^+}$, there exists $\mathcal{H}' \in \mathscr{H}(H^-;H^+)$ and $u' \in \mathcal{U}$ such that supp $\mathcal{H}-\mathcal{H}' \subset U$, $\mathcal{F}_{\mathcal{H}',J}(u') = 0$, and $(D\mathcal{F}_{\mathcal{H}',J})_{u'}$ is surjective.*

With this lemma in hand, we can readily prove

Theorem 3.3.3. *Let $(H^{\pm},J^{\pm})$ be Floer regular and let $u_i : \mathbb{R}\times S^1 \to \Sigma$, $i = 1,\ldots,k$ define an $(\mathcal{H},\mathbb{J})$-model for a continuation cobordism for some $(\mathcal{H},\mathbb{J}) \in \mathscr{H}\mathscr{J}(H^-,J^-;H^+,J^+)$. Suppose that for each $i = 1,\ldots,k$, the index of the linearized Floer operator satisfies*

$$ind\ (D\mathcal{F}_{\mathcal{H},\mathbb{J}})_{u_i} = 0,$$

then for any choice of neighbourhoods $\mathcal{U}_i$ of $u_i \in C^\infty(\mathbb{R}\times S^1;\Sigma)_{\hat{x}_i,\hat{y}_i}$, $i = 1,\ldots,k$, there exists a regular model for a continuation cobordism $\{u_i''\}_{i=1}^k$ such that $u_i'' \in \mathcal{U}_i$ for $i = 1,\ldots,k$.

PROOF. For each $i = 1, \ldots, k$, let $\mathcal{N}(\text{im } \tilde{u}_i)$ be an open neighbourhood of im $\tilde{u}_i$ and let

$$U_i := \mathcal{N}(\text{im } \tilde{u}_i) \setminus \bigcup_{j \neq i} \text{im } \tilde{u}_i.$$

Since graphs $\tilde{u}_i, \tilde{u}_j$ have only finitely many intersections for $i \neq j$, U_i clearly satisfies the hypotheses of Lemma 3.3.2 with $u = u_i$ and $\mathcal{U} = \mathcal{U}_i$ for each $i = 1, \ldots, k$. It follows that there exists $h_i \in C^\infty(\mathbb{R} \times S^1 \times \Sigma)$ with compact support contained in U_i and $u_i' \in \mathcal{U}_i$ such that $\mathcal{F}_{\mathcal{H}+h_i,J}(u_i') = 0$, and u_i' is a regular point of $\mathcal{F}_{\mathcal{H}+h_i,J}$. Note that, because the support of h_i is compactly contained in U_i which is disjoint from im $\tilde{u}_j$ for each $j \neq i$, there exists an open neighbourhood V_j of im $\tilde{u}_j$ such supp $h_i \cap V_j = \emptyset$. Consequently, since the local behaviour of the Floer operator near a map depends only on the behaviour of the continuation data near the graph of that map, if we set

$$\mathcal{H}' = \mathcal{H} + \sum_{i=1}^{k} h_i$$

we have that $\mathcal{F}_{\mathcal{H}',J}(u_i') = 0$ for each $i = 1, \ldots, k$, and each u_i' is a regular point of $\mathcal{F}_{\mathcal{H}',J}$. Finally, in light of the regularity of each u_i' for $\mathcal{F}_{\mathcal{H}',J}$ and the index condition on the Floer operator, we may perturb $\mathcal{H}'$ to some $\mathcal{H}'' \in \mathscr{H}(H^-,H^+)$ such that $(\mathcal{H}'',J) \in \mathscr{H}\mathscr{J}_{reg}(H^-,J^-;H^+,J^+)$, such that there exist maps $u_i'' \in \mathcal{U}_i$ with $\mathcal{F}_{\mathcal{H}'',J}(u_i'') = 0$ for $i = 1, \ldots, k$. This completes the proof. $\square$

As an easy consequence of the previous theorem, we obtain

Corollary 3.3.4. *Let $(H^\pm,J^\pm)$ be Floer regular, and let $\hat{x}_i^\pm \in \widetilde{Per}_0(H^\pm)$, $i = 1, \ldots, k$, be such that $\mu(\hat{x}_i^-) = \mu(\hat{x}_i^+)$. If there exists a model for a continuation cobordism $\{u_i\}_{i=1}^k$ with $u_i \in C^\infty_{\hat{x}_i^-,\hat{x}_i^+}(\mathbb{R} \times S^1; \Sigma)$, then there exists $(\mathcal{H},\mathbb{J}) \in \mathscr{H}\mathscr{J}_{reg}(H^-,J^-;H^+,J^+)$ such that*

$$\mathcal{M}(\hat{x}_i^-,\hat{x}_i^+;\mathcal{H},\mathbb{J}) \neq \emptyset$$

for all $i = 1, \ldots, k$.

3.4. Appendix

3.4.1. A short fibration argument

Proposition 3.4.1. *Let (M,ω) be a symplectic manifold. The space*

$$\mathcal{E} := \{(m,v;H) \in TM \times C^\infty(M) : X^H(m) = v\}$$

equipped with the canonical projection $\pi : \mathcal{E} \to TM$ is a locally trivial fibration.

PROOF. Let $(m,v) \in TM$ be arbitrary, and let $\phi : (U,\omega) \to (\mathbb{R}^{2n},\omega_0)$ be a Darboux neighbourhood of $m \in M$ with $\phi(m) = 0$. Letting $\kappa > 0$ be such that $\phi(U) = B^{2n}(\kappa)$ and fixing some $\epsilon > 0$, we will construct a local section for $\mathcal{E}$ over the set $T\phi^{-1}(B^{2n}(\kappa - \epsilon))$ which implies the claim.

Consider the map

$$\begin{aligned} SG_0 : C^\infty(\mathbb{R}^{2n}) &\to T_0\mathbb{R}^{2n} = \mathbb{R}^{2n} \\ H &\mapsto X^H(0). \end{aligned}$$

SG_0 is a linear surjection, and so we may select a right inverse

$$C : T_0\mathbb{R}^{2n} \to C^\infty(\mathbb{R}^{2n}).$$

We may suppose without loss of generality that in fact C takes values in the set $C_0^\infty(B^{2n}(\epsilon))$ of functions with support compactly contained in $B^{2n}(\epsilon)$. We may define

$$\begin{aligned} \mathcal{C}_{loc} : B^{2n}(\kappa - \epsilon) \times \mathbb{R}^{2n} &\to C_0^\infty(B^{2n}(\kappa)) \\ (x,v) &\mapsto A_x(C(v)), \end{aligned}$$

where

$$\begin{aligned} A_x : C^\infty(\mathbb{R}^{2n}) &\to C^\infty(\mathbb{R}^{2n}) \\ H(p) &\mapsto H(p - x) \end{aligned}$$

is the pullback of affine translation by $x \in \mathbb{R}^{2n}$. It's clear that $\mathcal{C}_{loc}$ is smooth by construction. Writing $V = \phi^{-1}(B^{2n}(\kappa - \epsilon))$,

$$\mathcal{C} := \phi^*(\mathcal{C}_{loc} \circ D\phi|_{TV}) : TV \to C^\infty(M)$$

defines a smooth map such that if $G = \mathcal{C}(m',v')$ for $(m',v') \in TV$, then $X^G(m') = v'$. That is, $\mathcal{C}$ is a smooth right inverse for $\pi : \mathcal{E} \to TM$ on TV, as desired. $\square$

Let N be a smooth (possibly open) manifold, $f : N \to M$ a smooth map and $Y : N \to f^*TM$ a vector field along f. We define

$$\mathcal{H}(Y) := \{(n,H) \in N \times C^\infty(M) : X^H(f(n)) = Y(n)\}.$$

Corollary 3.4.2. *$\pi : \mathcal{H}(Y) \to N$ is a locally trivial fibration.*

PROOF. It is straightforward to check that $\mathcal{H}(Y)$ is nothing but the pullback bundle $Y^*\mathcal{E}$. Since $(\mathcal{E},\pi_{TM})$ is a locally trivial fibration, so is $\mathcal{H}(Y)$. $\square$

3.4.2. A proof of Lemma 3.3.2

Let $(H^{\pm},J^{\pm})$ be Floer regular and $\hat{x}^{\pm} \in \widetilde{Per}_0(H^{\pm})$. We fix $(\mathcal{H},\mathbb{J}) \in \mathscr{HJ}(H^-,J^-;H^+,J^+)$. If $\mathcal{F}_{\mathcal{H},\mathbb{J}}(u) = 0$, ind $(D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u) = 0$ then Lemma 3.3.2 is immediately true if $\ker(D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u = 0$, by taking $u' = u$ and $\mathcal{H}' = \mathcal{H}$. By Corollaries 2.2.17 and 2.2.18, it suffices to consider the case where u is such that

$$\dim\ker(D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u = \text{corank }\ (D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u = 1.$$

Let $\lambda \mapsto \mathcal{H}^{\lambda}$, $\lambda \in [-1,1]$ be a smooth path in $\mathcal{H}(H^-,H^+)$ such that $(\mathcal{H}^{\lambda},\mathbb{J})$ is a regular homotopy of Floer data for $\lambda = \pm 1$ and such that $\mathcal{H}^0 = \mathcal{H}$. Our goal is to show that if the path $\mathcal{H}^{\lambda}$ is chosen appropriately generically, then, using only perturbations in a neighbourhood U of im $\tilde{u}$ which may be taken to avoid the graphs of finitely many other Floer-type cylinders, the space $\bigcup_{\lambda\in[-1,1]} \mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H}^{\lambda},\mathbb{J})$ may be taken to be a non-empty 1-manifold which contains u. This clearly implies that for any neighbourhood $\mathcal{O} \subset C^{\infty}(\mathbb{R}\times S^1;M)_{\hat{x}^-,\hat{x}^+}$, there exists $u' \in \mathcal{O}$ and $\mathcal{H}' \in \mathcal{H}(H^-,H^+)$ such that $(\mathcal{H}',\mathbb{J}) \in \mathscr{HJ}_{reg}$, supp $\mathcal{H}' - \mathcal{H} \subset U$ and $\mathcal{F}_{\mathcal{H}',\mathbb{J}}(u') = 0$.

Fix a neighbourhood $N \subset \mathbb{R}\times S^1 \times M$ of im $\tilde{u}$, and let $U \subset N$ be an open set which is dense in N and such that $U \cap$ im $\tilde{u}$ is dense in $\tilde{u}$.

Let us define $C^{\infty}_{\epsilon}(U;0)$ to be the set of compactly supported smooth functions $h \in C^{\infty}_0((-1,1)\times U)$ such that $h^0 \equiv 0$, $h^{\lambda}(s,t,x) = h(\lambda,s,t,x)$ for $\lambda \in (-1,1)$ and which satisfy

$$\|h\|_{\epsilon} := \sum_{k=0}^{\infty} \epsilon_k \|h\|_{C^k} < \infty,$$

where $\epsilon = (\epsilon_k)_{k=0}^{\infty}$ is some sequence of positive numbers which decreases sufficiently quickly at infinity so that $(C^{\infty}_{\epsilon}(U;0), \|\cdot\|_{\epsilon})$ is dense in the space of smooth functions on U which satisfy $h^0 \equiv 0$. For $\hat{x}^{\pm} \in \widetilde{Per}_0(H^{\pm})$ and $p > 2$, let $W^{1,p}(\hat{x}^-,\hat{x}^+)$ be the usual Banach space completion of $C^{\infty}(\mathbb{R}\times S^1;M)_{\hat{x}^-,\hat{x}^+}$ and consider the infinite-dimensional vector bundle
$\mathcal{E} \to W^{1,p}(\hat{x}^-,\hat{x}^+) \times C^{\infty}_{\epsilon}(U;0)$ where

$$\mathcal{E} := \{(u,h,Y) : (u,h) \in W^{1,p}(\hat{x}^-,\hat{x}^+) \times C^{\infty}_{\epsilon}(U;0),\ Y \in L^p(u^*TM)\},$$

and consider the universal moduli space

$$Z(\hat{x}^-,\hat{x}^+) := \{(\lambda,u,h) : \lambda \in (-1,1), h \in C^{\infty}_{\epsilon}(U;0), u \in \mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H}+h^{\lambda},\mathbb{J})\}$$

which is cut out by the intersection of the map

$$\sigma : (-1,1) \times W^{1,p}(\hat{x}^-,\hat{x}^+) \times C^\infty_\epsilon(U;0) \to \mathcal{E}$$
$$(\lambda,u,h) \mapsto \sigma_\lambda(u,h) = \partial_s u + \mathbb{J}(\partial_t u - X^{\mathcal{H}+h^\lambda})$$

with the zero section. By our hypothesis on the corank of $D\mathcal{F}_{\mathcal{H},\mathbb{J}}$ at u, and the density of compactly supported smooth functions in L^p-space, there exists $\xi \in L^p(u^*TM) \cap C^\infty_0(u^*TM)$ such that

$$L^p(u^*TM) = \text{im } (D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u \oplus \langle \xi \rangle. \tag{3.4.1}$$

It is then easy to see that we may suppose without loss of generality (up to perturbing the path $\lambda \mapsto \mathcal{H}^\lambda$ near $\lambda = 0$) that

$$(\frac{\partial \mathcal{H}^\lambda}{\partial \lambda})_{\lambda=0}(u) = \mathbb{J}\xi.$$

Definition 3.4.3. If $\lambda \mapsto \mathcal{H}^\lambda \in \mathcal{H}(H^-,H^+)$ is such that $(\mathcal{H}^\lambda,\mathbb{J})$ is a regular homotopy of Floer data for $\lambda = \pm 1$, $\mathcal{H}^0 = \mathcal{H}$, and

$$L^p(u^*TM) = \text{im } (D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u \oplus \langle -\mathbb{J}(\frac{\partial \mathcal{H}^\lambda}{\partial \lambda})_{\lambda=0}(u) \rangle,$$

then we will say that the path $(\mathcal{H}^\lambda)_{\lambda \in [-1,1]}$ is **transverse to** u **at** 0.

The main point is the following

Proposition 3.4.4. *If $(\mathcal{H}^\lambda)_{\lambda \in [-1,1]}$ is transverse to u at 0, then there are neighbourhoods $\Lambda \subset [-1,1]$ of 0, $\mathcal{O} \subset W^{1,p}(\hat{x}^-,\hat{x}^+)$ of u and $\mathcal{V} \subset C^\infty_\epsilon(U;0)$ of 0 such that the restriction*

$$\sigma|_{\Lambda \times \mathcal{O} \times \mathcal{V}} : \Lambda \times \mathcal{O} \times \mathcal{V} \to \mathcal{E}$$

is transverse to the zero section of $\mathcal{E}$.

PROOF. Since the set of surjective Fredholm maps is open and the Floer operator is continuously differentiable, it will suffice to show that the vertical differential of σ is surjective at $(0,u,0) \in [-1,1] \times W^{1,p}(\hat{x}^-,\hat{x}^+) \times C^\infty_0(U;0)$. To see this, compute the vertical differential at a point $(\lambda_0,v,h_0) \in Z(\hat{x}^-,\hat{x}^+)$

$$(D\sigma)_{(\lambda_0,v,h_0)}(a,Y,h) = D(\mathcal{F}_{\mathcal{H}+h_0,\mathbb{J}})_v(Y) - \mathbb{J}(aX^{\partial_\lambda(\mathcal{H}+h_0)^{\lambda_0}} + X^{h^{\lambda_0}}).$$

Note then that if $\lambda_0 = 0$ we have that $X^{h^{\lambda_0}} = 0$, since $h \in C^\infty_\epsilon(U;0)$ and so $h^0 = 0$ by hypothesis. If we further impose that $v = u$ and $h = 0$, we see that

$$(D\sigma)_{(0,u,0)}(a,Y,h) = D(\mathcal{F}_{\mathcal{H},\mathbb{J}})_u(Y) + a\xi$$

where

$$L^p(u^*TM) = \text{im } (D\mathcal{F}_{\mathcal{H},\mathbb{J}})_u \oplus \langle\xi\rangle,$$

since $\lambda \mapsto \mathcal{H}^\lambda$ is transversal to u at 0 by hypothesis. This proves the claim. $\square$

Finally, we claim

Proposition 3.4.5. *If $(\mathcal{H}^\lambda)_{\lambda\in[-1,1]}$ is transverse to u at 0, then there are neighbourhoods $\mathcal{O} \subset W^{1,p}(\hat{x}^-,\hat{x}^+)$ of u and $\mathcal{V} \subset C^\infty_\epsilon(U;0)$ of 0 such that the restriction*

$$\sigma|_{(-1,1)\times\mathcal{O}\times\mathcal{V}} : (-1,1) \times \mathcal{O} \times \mathcal{V} \to \mathcal{E}$$

is transverse to the zero section of $\mathcal{E}$.

PROOF. This basically follows from the previous proposition in combination with the usual arguments used to establish transversality in this setting, with the only caveat being that we must show that away from $\lambda = 0$, transversality can be obtained via perturbations with support in U. This is essentially a consequence of the usual proof in any case, but we will sketch out the argument for the convenience of the reader.

The previous proposition gives us open neighbourhoods $\Lambda \subset [-1,1]$ of 0, $\mathcal{O} \subset W^{1,p}(\hat{x}^-,\hat{x}^+)$ of u and $\mathcal{V} \subset C^\infty_\epsilon(U;0)$ of 0 such that the restriction $\sigma|_{\Lambda\times\mathcal{O}\times\mathcal{V}}$ is transverse to the zero section of $\mathcal{E}$. Up to shrinking $\mathcal{O}$, we may suppose that if $v \in \mathcal{O}$, then im $\tilde{v} \subset$ int N. We will suppose that this is the case from now on. In particular, any $v \in \mathcal{O}$ has the property that $U \cap$ im $\tilde{v}$ is dense in im $\tilde{v}$. Clearly, it will suffice to show that σ is transverse to the zero section of $\mathcal{E}$ at any point (λ_0,v,h_0) with $\lambda_0 \in (-1,1) \setminus \{0\}$. Indeed as the vertical differential of σ is

$$(D\sigma)_{(\lambda_0,v,h_0)}(a,Y,h) = D(\mathcal{F}_{\mathcal{H}+h_0,\mathbb{J}})_v(Y) - \mathbb{J}(aX^{\partial_\lambda(\mathcal{H}+h_0)^{\lambda_0}} + X^{h^{\lambda_0}}),$$

we may consider its restriction to the subspace $W^{1,p}(v^*TM) \times C^\infty_\epsilon(U;0)$, which is the tangent space of the fiber of $(-1,1) \times W^{1,p}(\hat{x}^-,\hat{x}^+) \times C^\infty_\epsilon(U;0)$ over λ_0. We claim that the map

$$F : (D\sigma)_{(\lambda_0,v,h_0)}|_{W^{1,p}(v^*TM)\times C^\infty_\epsilon(U;0)} : W^{1,p}(v^*TM) \times C^\infty_\epsilon(U;0) \to L^p(v^*TM)$$
$$(Y,h) \mapsto D(\mathcal{F}_{\mathcal{H}+h_0,\mathbb{J}})_v(Y) - \mathbb{J}X^{h^{\lambda_0}}$$

is surjective. That this is so follows along standard lines. If we suppose that the above map is not surjective, then for q Hölder conjugate to p, there will exist some non-zero $Z \in L^q(v^*TM)$ which annihilates the image of F under the usual duality pairing of $L^q(v^*TM)$ with the dual space to $L^p(v^*TM)$. A standard argument implies that Z lies in the kernel of the adjoint of $(D\mathcal{F}_{\mathcal{H}+h_0,\mathbb{J}})_v$, and thus Z is smooth by elliptic regularity. Since Z is non-zero, we may view Z as a vector field along $\tilde{v}$ by setting

$$\tilde{Z}(s,t) = (0,0,Z(s,t)) \in T_{\tilde{v}(s,t)}\mathbb{R} \times S^1 \times M.$$

If $Z \neq 0$, then there is some open set $W \subset \mathbb{R} \times S^1$ on which Z is non-vanishing, and so $\tilde{Z}$ is non-vanishing along $\tilde{v}(W) \subset \mathbb{R} \times S^1 \times M$. But $U \cap \text{im } \tilde{v}$ is open and dense in im $\tilde{v}$ and so we may clearly take some $h \in C^\infty_\epsilon(U;0)$ such that supp $h \cap \text{im } \tilde{v} \subset \tilde{v}(W)$, which is not annihilated by Z. This is a contradiction, so it follows that F is surjective. $\square$

As a consequence of the above, the Sard-Smale theorem implies that there is set $\mathcal{R} \subset C^\infty_\epsilon(U;0)$ which is open and dense in a neighbourhood of 0 such that

$$\bigcup_{\lambda \in [-1,1]} \mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H}^\lambda + h^\lambda;\mathbb{J})$$

is a manifold in a neighbourhood of $(0,u) \in \mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H}^0;\mathbb{J})$, and the standard arguments in Floer theory then show that the dimension of this manifold is 1.

Chapter 4

Application: A new spectral invariant and its computation on surfaces

In this chapter, we bring to bear the theory developed in the preceding chapters in order to give an explicit dynamical characterization — for any non-degenerate Hamiltonian H on a symplectic surface (Σ,ω) — of those Floer cycles which both represent a non-trivial homology class in $HF_1(H)$ *and* which lie in the image of some PSS map at the chain level.

This dynamical characterization motivates us to study the quantity obtained by modifying the definition of the Oh-Schwarz spectral invariants by looking at the minimal action required to represent a given homology class by a cycle which lies in the image of some PSS map at the chain level. It turns out that this defines a family of action selectors, which we call the *PSS-image spectral invariants*. These novel spectral invariants share many of the same properties as the Oh-Schwarz spectral invariants, including a triangle inequality for the fundamental class, and bound the Oh-Schwarz spectral invariants from above.

The chapter begins with a general study of the PSS-image spectral invariants in Section 4.1, establishing their basic formal properties, whose proofs are largely analogous to the corresponding proofs for the the Oh-Schwarz spectral invariants. In Section 4.2, we compute the PSS-image spectral invariant of a non-degenerate Hamiltonian on a surface by deriving the aforementioned dynamical characterization of Floer cycles which represent the fundamental class and which lie in the image of some PSS map at the chain level. Section 4.3 presents two fairly immediate dynamical consequences of this characterization, and the chapter closes with a brief discussion of the relationship between the Oh-Schwarz spectral invariants and the PSS-image spectral invariants in Section 4.4.

4.1. A new spectral invariant: definitions and properties

Let (H,J) be Floer-regular. Recall that the set $PSS(H,J)$ of *PSS data* for (H,J) is the collection of tuples

$$\mathcal{D} = (f,g;\mathcal{H},\mathbb{J}) \in C^\infty(M) \times \mathfrak{Met}(M) \times \mathscr{H}(0;H) \times \mathcal{J}^{PSS}(J),$$

where (f,g) is a Morse-Smale pair. There is a residual set $PSS_{reg}(H,J)$ of *regular PSS data* such that $\mathcal{D} \in \mathcal{D}^{PSS}_{reg}(H,J)$, we may define a chain-level PSS map

$$\Phi^{PSS}_{\mathcal{D}} : QC_{*+n}(f,g) \to CF_*(H,J)$$

which is Λ_ω-linear and induces a canonical isomorphism at the level of homology.

Definition 4.1.1. Let (H,J) be Floer-regular. For $\alpha \in QH_*(M,\omega)$, $\alpha \neq 0$, we define the **PSS-image spectral invariant**

$$c_{im}(\alpha; H,J) := \inf\{\lambda_H(\sigma) : \exists \mathcal{D} \in PSS_{reg}(H,J),\ \exists \sigma \in \text{im } \Phi^{PSS}_{\mathcal{D}},\ \text{such that } [\sigma] = (\Phi^{PSS}_{\mathcal{D}})_*\alpha\}$$

In general, it is not immediately obvious that $c_{im}(\alpha; H) \neq -\infty$ for all $\alpha \in QH_*(M,\omega) \setminus \{0\}$. To see that this is so, recall the definition of the Oh-Schwarz spectral invariant $c_{OS}(\alpha; H)$ for $0 \neq \alpha \in QH_*(M)$ from Section 2.1.2. These spectral invariants are always finite (see theorem 5.3 in [**21**], for instance). Moreover, it is an obvious consequence of the definitions that we have

Proposition 4.1.2. *For any $\alpha \in QH_*(M) \setminus \{0\}$ and any $J \in \mathcal{J}_\omega(M)$ such that (H,J) is Floer regular*

$$c_{OS}(\alpha; H) \leq c_{im}(\alpha; H,J).$$

Consequently, we obtain the finiteness of $c_{im}(\alpha; H,J)$ for all non-zero quantum homology classes α.

4.1.1. Basic Properties of c_{im}

Proposition 4.1.3. *Let (H,J), (K,J') be Floer regular pairs. For any $\alpha \in QH_*(M,\omega)$, $\alpha \neq 0$, we have*

$$\int_0^1 \min_{x\in M}(K-H)(t,x)\, dt \leq c_{im}(\alpha; K,J') - c_{im}(\alpha; H,J) \leq \int_0^1 \max_{x\in M}(K-H)(t,x)\, dt$$

PROOF. Let $\delta > 0$ be arbitrary. By definition, there exists some $\sigma \in CF_*(H,J)$, some $\mathcal{D} = (f,g;\mathcal{H},\mathbb{J}) \in PSS_{reg}(H,J)$, and some $\sigma_f \in QC_*(f,g)$ which represents $\alpha \in QH_*(M,\omega)$ such that

(1) $\Phi_D^{PSS}(\sigma_f) = \sigma$, and

(2) $\lambda_H(\sigma) \leq c_{im}(\alpha; H,J) + \delta$.

Let $(\mathcal{H}',\mathbb{J}') \in \mathscr{H}\mathscr{J}(H,J;K,J')$ be a regular homotopy such that $\mathcal{H}'$ is arbitrarily close to the linear homotopy

$$\mathcal{H}^{lin}_{s,t} = (1-\beta(s))H_t + \beta(s)K_t,$$

for $\beta : \mathbb{R} \to [0,1]$ some smooth non-decreasing function which is 0 for s sufficiently small and 1 for s sufficiently large. Equation 2.1.3 for solutions of the s-dependent Floer equation combined with the positivity of the energy functional implies that

$$\lambda_K(h_{\mathcal{H}'}(\sigma)) \leq \lambda_H(\sigma) + \int_0^1 \max_{x\in M}(K-H)(t,x)\; dt + \epsilon,$$

where $\epsilon > 0$ may be made arbitrarily small by choosing $\mathcal{H}'$ sufficiently close to $\mathcal{H}^{lin}$. The usual gluing arguments of Floer theory (see Section 4.4 of [**28**]) imply that there exists $\mathcal{D}' = \mathcal{H}'\#\mathcal{D} \in PSS_{reg}(K,J')$ such that

$$\Phi^{PSS}_{\mathcal{D}'} = h_{\mathcal{H}'} \circ \Phi^{PSS}_{\mathcal{D}},$$

whence we see that

$$c_{im}(\alpha;K,J') \leq c_{im}(\alpha;H,J) + \int_0^1 \max_{x\in M}(K-H)(t,x)\; dt + \epsilon + \delta$$

for arbitrary small $\epsilon,\delta > 0$ which implies the upper bound. The reverse inequality follows by swapping the roles of H and K in the above. $\square$

Corollary 4.1.4. *$c_{im}(\alpha;H) = c_{im}(\alpha;H,J)$ is well-defined and independent of the choice of regular J.*

Proposition 4.1.5. *For any $H,K \in C^\infty(S^1 \times M)$ and any $\alpha \in QH_*(M,\omega) \setminus \{0\}$*

(1) if $r : [0,1] \to \mathbb{R}$ is smooth, then

$$c_{im}(\alpha;H+r) = c_{im}(\alpha;H) + \int_0^1 r(t)\; dt.$$

(2) $c_{im}(\psi_*\alpha;\psi_*H) = c_{im}(\alpha;H)$ for any symplectic diffeomorphism ψ.

(3) $|c_{im}(\alpha;H) - c_{im}(\alpha;K)| \leq \|H-K\|_{L^{1,\infty}}$.

(4) (Weak triangle inequality) $c_{im}(\alpha;H\#K) \leq c_{im}(\alpha;H) + c_{im}([M];K)$.

PROOF. As one might expect, the proofs of these properties follow largely the same lines as the proofs of the corresponding properties for the Oh-Schwarz spectral invariants with only very minor modifications to the details in those places where we must establish the existence of a witnessing PSS chain map to prove that the corresponding property holds for $c_{im}(\alpha; -)$.

(1) H and $H + r$ induce the same Hamiltonian vector fields, and (H,J) and $(H + r,J)$ induce the same Floer equations. Consequently, we have the canonical identification of Floer complexes

$$CF_*(H,J) = CF_*(H + r,J). \tag{4.1.1}$$

Moreover, if $\mathcal{D} = (f,g;\mathcal{H},\mathbb{J}) \in PSS_{reg}(H,J)$, then we may view $\mathcal{H}$ as a T-adapted homotopy of Hamiltonians from 0 to H. Hence, if

$$\beta : \mathbb{R} \to [0,1]$$

is a smooth non-decreasing function such that $\beta(s) = 0$ for $s \in (-\infty, -T - 1]$ and $\beta(s) = 1$ for $s \in [T,\infty)$, then

$$\mathcal{H}_\beta(s,t,x) := \mathcal{H}(s,t,x) + \beta(s)r(t)$$

is a $(T + 1)$-adapted homotopy from 0 to $H + r$ such that the pair $(\mathcal{H}_\beta,\mathbb{J})$ defines the same Floer equation as $(\mathcal{H},\mathbb{J})$, and therefore the map

$$\Phi^{PSS}_{\mathcal{D}_\beta} : QC_{*+n}(f,g) \to CF_*(H + r,J)$$

induced by $\mathcal{D}_\beta := (f,g;\mathcal{H},\mathbb{J}) \in PSS_{reg}(H + r,J)$ agrees with $\Phi^{PSS}_{\mathcal{D}}$ under the canonical identification of 4.1.1. The desired statement follows.

(2) If $\mathcal{D} = (f,g;\mathcal{H},\mathbb{J})$ and $\psi \in Symp(M,\omega)$, then by the naturality of the negative gradient flow equation with respect to push-forwards by diffeomorphisms, and the naturality of Floer's equation with respect to symplectomorphisms, we have that $\mathcal{D} \in PSS_{reg}(H,J)$ if and only if $\psi_*\mathcal{D} = (\psi_*f,\psi_*g;\psi_*H,\psi_*\mathbb{J}) \in PSS_{reg}(\psi_*H, \psi_*J)$, and the relevant PSS moduli spaces are identified via the map

$$\begin{gathered}\mathcal{M}(p,[x,w];\mathcal{D}) \to \mathcal{M}(\psi(p),[\psi \circ x, \psi \circ w];\psi_*\mathcal{D}) \\ u \mapsto \psi \circ u.\end{gathered}$$

Consequently, we see that for $\sigma_f \in QC_{*+n}(f,g)$

$$\Phi^{PSS}_{\mathcal{D}}(\sigma_f) = \sigma \in CF_*(H,J)$$

if and only if

$$\Phi^{PSS}_{\psi_*\mathcal{D}}(\psi_*\sigma_f) = \psi_*\sigma \in CF_*(\psi_*H,\psi_*J),$$

where $\psi_*(p \otimes e^A) = \psi(p) \otimes e^{\psi_* A}$ defines the Λ_ω-linear map

$$\psi_* : C^{Morse}(f,g) \otimes \Lambda_\omega \to C^{Morse}(\psi_* f,\psi_* g) \otimes \Lambda_\omega$$

on generators, and $\psi_*([x,w]) = [\psi \circ x, \psi \circ w]$ defines the linear map

$$\psi_* : CF_*(H,J) \to CF_*(\psi_*H,\psi_*J)$$

on generators. Finally, it's clear that $\sigma_f \in C^{Morse}(f,g) \otimes \Lambda_\omega$ represents $\alpha \in QH_*(M,\omega)$ if and only if $\psi_*\sigma_f$ represents $\psi_*\alpha \in QH_*(M,\omega)$.

(3) This is a direct consequence of Proposition 4.1.3.

(4) We fix Floer regular pairs (H,J_1) and (K,J_2). The fundamental point, established in Section 4 of [**29**] (see also Section 6.2 of [**21**]) is that if $(S_{2,1},j)$ denotes an model Riemannian surface with two negative cylindrical ends

$$\phi_i^- : ((-\infty,0] \times S^1,j_0) \to (Z_i^-,j) \subset S_{2,1},\ i = 1,2,$$

and one positive cylindrical end

$$\phi^+ : ((-\infty,0] \times S^1,j_0) \to (Z^+,j) \subset S_{2,1},$$

then for any $\delta > 0$, there exist smooth maps

$$\mathcal{H}^S : S_{2,1} \to C^\infty(M), \text{ and}$$
$$\mathbb{J}^S : S_{2,1} \to \mathcal{J}(M,\omega)$$

such that

(a) $(\mathcal{H}^S \circ \phi^-, \mathbb{J}^S \circ \phi_i^-)(s,t)$ equals (H,J_1) for all s sufficiently small if $i = 1$ and equals (K,J_2) for all such s if $i = 2$

(b) $(\mathcal{H}^S \circ \phi^+, \mathbb{J}^S \circ \phi^+)(s,t)$ equals $(H\#K, J_3)$ for all s sufficiently large, where $J_3 \in \mathcal{J}_\omega(M)$ is such that $(H\#K, J_3)$ is Floer regular.

(c) For $[x_1,w_1] \in \widetilde{Per}_0(H)$, $[x_2,w_2] \in \widetilde{Per}_0(K)$ and $[x_3,w_3] \in \widetilde{Per}_0(H\#K)$, if $u : S_{2,1} \to M$ is any smooth map with $\lim_{s\to-\infty}(u \circ \phi_i^-)(s,t) = x_i$, $i = 1,2$, $\lim_{s\to\infty}(u \circ \phi^+)(s,t) = x_3(t)$ and the map $S^2 \to M$ formed by gluing w_1 to u along Z_1^-, w_2 to u along Z_2^- and $\bar{w}_3$ along Z^+ is a torsion element of $H_2(M;\mathbb{Z})$, then

$$\mathcal{A}_{H\#K}([x_3,w_3]) \le \mathcal{A}_H([x_1,w_1]) + \mathcal{A}_K([x_2,w_2]) + \delta.$$

(d) If we denote by $n(\hat{x}_1,\hat{x}_2;\hat{x}_3)$ the count of elements in the zero-dimensional moduli spaces $\mathcal{M}(\hat{x}_1,\hat{x}_2;\hat{x}_3;\mathcal{H}^S,\mathbb{J}^S)$ which consist of smooth maps $u : S_{2,1} \to M$ satisfying the conditions of the previous point and which, moreover satisfy, in any conformal coordinates (s,t) on $(S_{2,1},j)$, the $(\mathcal{H}^S,\mathbb{J}^S)$-Floer equations

$$\partial_s u + \mathbb{J}^S(\partial_t u - X^{\mathcal{H}^S}) = 0, \tag{4.1.2}$$

then the map

$$\begin{aligned}\Phi^S : CF_*(H,J_1) \otimes CF_*(K,J_2) &\to CF_*(H\#K,J_3)\\ \hat{x} \otimes \hat{y} &\mapsto \sum_{\mu(\hat{z})=\mu(\hat{x})+\mu(\hat{y})} n(\hat{x},\hat{y};\hat{z})\hat{z}\end{aligned}$$

defines a map which descends to the Pair-of-Pants product on Floer homology.

(e) If $\alpha^\# = \Phi^{PSS}(\alpha) \in HF_*(H)$ and $\beta^\# = \Phi^{PSS}(\beta) \in HF_*(K)$, then $(\Phi^S)_*(\alpha \otimes \beta) = \Phi^{PSS}(\alpha * \beta) \in HF_*(H\#K)$.

The statement is then proven by letting $\sigma_H \in CF_*(H,J_1)$ be a witness for $c_{im}(\alpha;H)$ and $\sigma_K \in CF_*(H,J_2)$ a witness for $c_{im}([M];K)$. The previous points imply that $\sigma_{H\#K} := \Psi^S(\sigma_H \otimes \sigma_K)$ represents the class α under the natural isomorphism $QM_{*+n}(M,\omega) \simeq HF_*(H\#K)$ and supp $\Psi^S(\sigma_H \otimes \sigma_K)$ is composed of capped orbits with action bounded above by

$$\lambda_H(\sigma_H) + \lambda_K(\sigma_K) + \delta = c_{im}(\alpha;H) + c_{im}([M];K) + \delta.$$

If we were dealing with the Oh-Schwarz spectral invariants, then we would be done, but to establish the statement for c_{im}, we must exhibit some regular PSS data $\mathcal{D} \in PSS_{reg}(H\#K,J_3)$ such that $\sigma_{H\#K} \in$ im $\Psi^{PSS}_{\mathcal{D}}$. If we let $\mathcal{D}_H = (f_1,g_1;\mathcal{H},\mathbb{J}_1) \in PSS_{reg}(H,J_1)$ and $\mathcal{D}_K = (f_2,g_2;\mathcal{K},\mathbb{J}_2) \in PSS_{reg}(K,J_2)$ be PSS data such that $\sigma_H \in$ im $\Phi^{PSS}_{\mathcal{D}_H}$ and $\sigma_K \in$ im $\Phi^{PSS}_{\mathcal{D}_K}$, then the standard gluing theorems from Floer theory imply that we may glue the PSS data $\mathcal{D}_H$ and $\mathcal{D}_K$ along the negative cylindrical ends Z_1^- and Z_2^- respectively to obtain regular PSS data

$$\mathcal{D}_{H\#K} := \mathcal{D}_H\#_{Z_1^-}\mathcal{D}_K\#_{Z_2^-}(\mathcal{H}^S,\mathbb{J}^S) \in PSS_{reg}(H\#K;J_3)$$

having the property that we have the bijection of moduli spaces

$$\begin{aligned}\mathcal{M}(p,\hat{x};\mathcal{D}_H) \times \mathcal{M}(q,\hat{y};\mathcal{D}_K) \times \mathcal{M}(\hat{x},\hat{y};\hat{z};\mathcal{H}^S,\mathbb{J}^S) &\to \mathcal{M}(p,\hat{z};\mathcal{D}_{H\#K})\\ (u,v,w) &\mapsto \bar{u}\#_{Z_1^-}\bar{v}\#_{Z_2^-}w.\end{aligned}$$

Remark (4). It is precisely here that we use the fact that $[\sigma_K] = \Phi_*^{PSS}[M]$. In general (that is, when $[\sigma_K] = \Phi_*^{PSS}\beta$ for $\beta \in QH_{*+n}(M,\omega) \setminus \{0\}$), the three moduli spaces above glue together to give

$$\mathcal{M}^{\beta}(p,\hat{z};\mathcal{D}_{H\#K}) \subset \mathcal{M}(p,\hat{z};\mathcal{D}_{H\#K}),$$

which is the subspace of maps $u \in \mathcal{M}(p,\hat{z};\mathcal{D}_{H\#K})$ which additionally satisfy $u(s_0,t_0) \in$ im $\beta^{\#}$ for an appropriately chosen marked point $(s_0,t_0) \in \mathbb{R} \times S^1$ and for $\beta^{\#} : \cup\Delta^k \to M$ a smooth cycle composed of unstable manifolds of the Morse-Smale pair (f_2,g_2) which represent β in the quantum homology of M. When $\beta = [M]$, this condition is vacuous and we generically obtain

$$\mathcal{M}^{[M]}(p,\hat{z};\mathcal{D}_{H\#K}) = \mathcal{M}(p,\hat{z};\mathcal{D}_{H\#K}).$$

Returning to the proof, the above bijection of moduli spaces implies that

$$\Phi_{H\#K}^{PSS}(\sigma_H) = \Phi^S(\sigma_H \otimes \sigma_K) = \sigma_{H\#K},$$

which proves the claim.

□

Remark. We note that it is precisely the point discussed in Remark 4 that keeps us from being able to establish the full triangle inequality

$$c_{im}(\alpha * \beta; H\#K) \leq c_{im}(\alpha; H) + c_{im}(\beta; K)$$

for the PSS-image spectral invariant. Indeed, for any $\beta \neq [M]$, the maps $(u,v,w) \in \mathcal{M}(p,\hat{x};\mathcal{D}_H) \times \mathcal{M}(q,\hat{y};\mathcal{D}_K) \times \mathcal{M}(\hat{x},\hat{y};\hat{z};\mathcal{H}^S,\mathbb{J}^S)$ will glue together to give a disk with origin lying in a cycle $\alpha^{\#}$ representing α, an additional marked point lying in a cycle $\beta^{\#}$ as before and with boundary on the orbit $z \in Per_0(H\#K)$. In general, when $\beta \neq [M]$, we will have $\deg(\alpha) + n \neq \mu(\hat{z})$ and so the maps in the moduli spaces 'glued moduli spaces'

$$\mathcal{M}^{glued}(p,\hat{x};q,\hat{y};\hat{x},\hat{y};\hat{z}) \simeq \mathcal{M}(p,\hat{x};\mathcal{D}_H) \times \mathcal{M}(q,\hat{y};\mathcal{D}_K) \times \mathcal{M}(\hat{x},\hat{y};\hat{z};\mathcal{H}^S,\mathbb{J}^S)$$

which contribute to $\sigma_{H\#K}$ cannot be identified with a subset of the maps in the moduli space

$$\mathcal{M}(p,\hat{z};\mathcal{D}_{\mathcal{H}\#K})$$

which contribute to $\Phi_{\mathcal{D}_{H\#K}}^{PSS}(\alpha^{\#} * \beta^{\#})$, since the latter consist of maps $u : \mathbb{R} \times S^1 \to M$ which, roughly speaking, satisfy $\lim_{s\to-\infty} u(s,t) \in$ im $(\alpha\#\beta)^{\#}$. It is, however, plausible that a judicious choice of a different Morse-Smale pair for the PSS data $\mathcal{D}_{H\#K}$ (instead of the pair (f_1,g_1)) could be made so that $\mathcal{M}(p,\hat{z};\mathcal{D}_{\mathcal{H}\#K})$ agrees with $\mathcal{M}^{glued}(p,\hat{x};q,\hat{y};\hat{x},\hat{y};\hat{z})$. Since the weaker form of the triangle inequality suffices for our purposes, however, we do not pursue this matter further here. The above proposition implies that $c_{im}(\alpha;H)$ is C^0-continuous in its Hamiltonian argument. By the density on non-degenerate Hamiltonians in the space of all Hamiltonians, we may extend the definition of $c(\alpha;H)$ to any $H \in C^\infty(S^1 \times M)$. The above properties then obviously still hold by approximation even if the Hamiltonians in question are not presumed non-degenerate.

Moreover, if we define, for $\alpha = \sum \alpha_A e^A \in QH_*(M)$ its **valuation**

$$\nu(\alpha) := \max\{-\omega(A) : \alpha_A \neq 0\},$$

then we see that the PSS-image spectral invariants satisfy the same normalization condition as the Oh-Schwarz spectral invariants. Namely

Proposition 4.1.6. *For $\alpha = \sum \alpha_A e^A \in QH_*(M) \setminus \{0\}$, we have*

$$c_{im}(\alpha;0) = \nu(\alpha).$$

PROOF. Let $(\epsilon_k)_{k\in\mathbb{N}}$ be a sequence of small positive real numbers tending to 0 as $k \to \infty$. Let $f \in C^\infty(M)$ be a sufficiently small Morse function such that we may choose $J \in \mathcal{J}_\omega(M)$ so that for all $k \in \mathbb{N}$, $(\epsilon_k f, J)$ is Floer regular and we have the canonical identification $CF_*(\epsilon_k f,J) = QC_{*+n}(\epsilon_k f,g_J)$. It's clear that if $\sigma_A \in C^{Morse}_{*+n+2c_1(A)}(\epsilon_k f,g_J)$ represents α_A in homology, then

$$\begin{aligned}
\lambda_{\epsilon_k f}(\sigma_A \otimes e^A) &:= \lambda_{\epsilon_k f}(e^A \cdot \sigma_A) \\
&= \max_{[x,w]\in \text{supp } \sigma_A} \mathcal{A}_{\epsilon_k f}([x,A\#w]) \\
&= \max_{[x,w]\in \text{supp } \sigma_A} \mathcal{A}_{\epsilon_k f}([x,A]) - \omega(A) \\
&= \lambda_{\epsilon_k f}(\sigma_A) - \omega(A) \\
&= \epsilon_k f(x) - \omega(A), \text{ for some } x \in Crit(f).
\end{aligned}$$

Consequently, if we consider $\sigma = \sum \sigma_A \otimes e^A \in (C^{Morse}(\epsilon_k f,g_J) \otimes \Lambda_\omega)_{*+n}$ representing $\alpha = \sum \alpha_A e^A$ in homology, we see that

$$\begin{aligned}
\lambda_{\epsilon_k f}(\sigma) &= \max_{\sigma_A \neq 0} \lambda_{\epsilon_k f}(\sigma_A \otimes e^A) \\
&= \max_{\sigma_A \neq 0}\{\epsilon_k \max_{[x,x]\in \text{supp } \sigma_A} f(x) - \omega(A)\}.
\end{aligned}$$

Of course, for each A such that $\sigma_A \neq 0$, we have

$$\min_{x\in M} f(x) \leq \epsilon_k \max_{[x,x]\in \text{supp}\, \sigma_A} f(x) \leq \max_{x\in M} f(x).$$

Whence, we see that

$$\max_{\sigma_A\neq 0}\{\epsilon_k \min_{x\in M} f(x) - \omega(A)\} \leq \lambda_{\epsilon_k f}(\sigma) \leq \max_{\sigma_A\neq 0}\{\epsilon_k \max_{x\in M} f(x) - \omega(A)\},$$

and so for all $k \in \mathbb{N}$

$$\epsilon_k \min_{x\in M} f(x) + \max_{\sigma_A\neq 0}\{-\omega(A)\} \leq c_{im}(\alpha; \epsilon_k f) \leq \epsilon_k \max_{x\in M} f(x) + \max_{\sigma_A\neq 0}\{-\omega(A)\}.$$

Taking $k \to \infty$ proves the claim. □

Corollary 4.1.7. *For any $\alpha \in QH_*(M,\omega)$, $\alpha \neq 0$, we have*

$$\int_0^1 \min_{x\in M} H(t,x)\, dt + \nu(\alpha) \leq c_{im}(\alpha; H) \leq \int_0^1 \max_{x\in M} H(t,x)\, dt + \nu(\alpha).$$

In particular, if $\alpha \in H_*(M)$, then

$$\int_0^1 \min_{x\in M} H_t\, dt \leq c_{im}(\alpha; H) \leq \int_0^1 \max_{x\in M} H_t\, dt$$

PROOF. By continuity of the function $H \mapsto c_{im}(\alpha; H)$ and the density of non-degenerate Hamiltonians, the inequality stated in Proposition 4.1.3 holds for arbitrary Hamiltonians, regardless of whether or not they are non-degenerate. Thus, we obtain the inequality

$$\int_0^1 \min_{x\in M} H(t,x)\, dt \leq c_{im}(\alpha; H) - c_{im}(\alpha; 0) \leq \int_0^1 \max_{x\in M} H(t,x)\, dt.$$

This combines with the previous proposition to give

$$\int_0^1 \min_{x\in M} H(t,x)\, dt \leq c_{im}(\alpha; H) - \nu(\alpha) \leq \int_0^1 \max_{x\in M} H(t,x)\, dt,$$

which readily rearranges to give the desired inequality. □

The PSS-image spectral invariant also shares the following properties of the Oh-Schwarz spectral invariant

Proposition 4.1.8. *If (M,ω) is rational in the sense that the subgroup $\omega(\pi_2(M)) \subset \mathbb{R}$ is discrete, then*

(1) $c_{im}(\alpha; H) \in Spec(H)$ for all $\alpha \in QH_*(M,\omega) \setminus \{0\}$.
(2) If $H \in C^\infty(S^1 \times M)$ is assumed to be normalized so that $\int H_t\omega^n = 0$ for all $t \in S^1$, then $c_{im}(\alpha; H)$ depends only on the homotopy class of the isotopy $(\phi_t^H)_{t\in[0,1]}$ relative endpoints.

PROOF. The proofs of (1) and (2) are identical to the proofs of the corresponding statements for the Oh-Schwarz spectral invariants. We refer the interested reader to Theorem II and its proof in [**21**].

□

4.1.2. A new spectral norm

The fact that the PSS-image spectral invariants satisfy the triangle inequality for the fundamental class allows us to define an associated bi-invariant pseudo-norm on $\widetilde{Ham}(M,\omega)$ which descends to a non-degenerate bi-invariant norm on $Ham(M,\omega)$.

Definition 4.1.9. Let (M,ω) be a compact rational semi-positive symplectic manifold. We define the function

$$\begin{aligned}\tilde{\nu}_{im} : C^\infty(S^1 \times M) \to \mathbb{R} \\ H \mapsto c_{im}([M];H) + c_{im}([M];\bar{H}),\end{aligned}$$

Which descends to the **PSS-image pseudo-norm** on $\widetilde{Ham}(M,\omega)$ by defining

$$\begin{aligned}\tilde{\nu}_{im} : \widetilde{Ham}(M,\omega) \to \mathbb{R} \\ \tilde{\phi}^H \mapsto c_{im}([M];H) + c_{im}([M];\bar{H}),\end{aligned}$$

where $\int H_t\omega^n = 0$ for all $t \in S^1$.

Proposition 4.1.10. *Let $H \in C^\infty(S^1 \times M)$, then*

(1) $\tilde{\nu}_{im}(H) = \tilde{\nu}_{im}(\bar{H})$*,*

(2) $\tilde{\nu}_{im}(H\#F) \leq \tilde{\nu}_{im}(H) + \tilde{\nu}_{im}(F)$, $\forall H,F \in C^\infty(S^1 \times M)$*. In particular*

$$0 \leq \tilde{\nu}_{im}(H).$$

(3) $\tilde{\nu}_{im}(H) \leq \int_0^1 \max_{x\in M} H_t - \min_{x\in M} H_t \, dt$

PROOF. (1) is obvious. (2) follows from the computation

$$\begin{aligned}\tilde{\nu}_{im}(H\#F) &= c_{im}([M];H\#F) + c_{im}([M];\bar{F}\#\bar{H}) \\ &\leq c_{im}([M];H) + c_{im}([M];F) + c_{im}([M];\bar{F}) + c_{im}([M];\bar{H}) \\ &= \tilde{\nu}_{im}(H) + \tilde{\nu}_{im}(F),\end{aligned}$$

where the inequality in the penultimate line follows from the weak triangle-inequality for the PSS-image spectral invariants. To establish property (3), note that Corollary 4.1.7 implies

$$\begin{aligned} c([M];H)+c([M];\bar{H}) &\leq \int_0^1 \max_{x\in M} H_t\,dt + \int_0^1 \max_{x\in M} \bar{H}_t\,dt \\ &= \int_0^1 \max_{x\in M} H_t\,dt - \int_0^1 \min_{x\in M} H_t\,dt, \end{aligned}$$

which proves the claim. □

Definition 4.1.11. To obtain a norm on the group of Hamiltonian diffeomorphisms, we define the **PSS-image spectral norm**

$$\begin{aligned} \nu_{im} : Ham(M,\omega) &\to [0,\infty) \\ \phi &\mapsto \inf_{\tilde{\phi}} \tilde{\nu}_{im}(\tilde{\phi}), \end{aligned}$$

where the infimum is taken over all $\tilde{\phi} \in \widetilde{Ham}(M,\omega)$ which project to $\phi \in Ham(M,\omega)$ under the natural projection $\widetilde{Ham}(M,\omega) \to Ham(M,\omega)$.

Proposition 4.1.12. *ν_{im} is a norm. Moreover, for all $\phi \in Ham(M,\omega)$,*

$$\nu_{OS}(\phi) \leq \nu_{im}(\phi) \leq \|\phi\|_{Hof},$$

where ν_{OS} denotes the Oh-Schwarz spectral norm and $\|\cdot\|_{Hof}$ denotes the Hofer norm.

PROOF. That ν_{im} is a pseudonorm is an immediate consequence of Proposition 4.1.10. To establish the stated inequality note that Proposition 4.1.10 along with the fact that the Oh-Schwarz spectral invariants bound the PSS-image spectral invariants from below imply

$$c_{OS}([M];H)+c_{OS}([M];\bar{H}) \leq \tilde{\nu}_{im}(H) \leq \int_0^1 \max_{x\in M} H_t - \min_{x\in M} H_t\,dt$$

for all Hamiltonians $H \in C^\infty(S^1 \times M)$. Taking the infimum over all normalized Hamiltonians which generate an isotopy with time-1 map equal to ϕ gives the claimed inequality. That ν_{im} is non-degenerate and hence a norm then follows from the fact that ν_{OS} is a norm (and hence non-degenerate). □

4.2. Computing $c_{im}([\Sigma];H)$ on surfaces

In this section, we restrict our attention to the case of $M = \Sigma$ an arbitrary orientable surface, and we use the theory developed in Chapters 1-3 to give a topological characterization of all homologically non-trivial $\mathbb{Z}/2\mathbb{Z}$-cycles in $CF_1(H,J)$ which lie in the image of some PSS-map. Specifically, we prove

Theorem 4.2.1. *Let $\sigma \in CF_1(H,J)$. σ is a non-trivial cycle with $\sigma \in$ im $\Phi_{\mathcal{D}}^{PSS}$ for some $\mathcal{D} \in PSS_{reg}(H,J)$ if and only if supp σ is a maximal positive capped braid relative index* 1.

Let us spell out the topological condition that is used in the above characterization. Recall from Definition 3.0.2 that a braid cobordism $h : [0,1] \to \mathcal{L}(C_n(\Sigma))$ is said to be *positive* (resp. *negative*) if whenever the graphs of the strands of h intersect, they do so positively (resp. negatively).

Definition 4.2.2. A capped k-braid $\hat{X}$ is said to be **positive** (resp. **negative**) if there exists a positive 0-cobordism from some, hence any, trivial capped braid $\hat{0}$ to $\hat{X}$.

Definition 4.2.3. For $H \in C^\infty(S^1 \times \Sigma)$ non-degenerate, a collection of capped 1-periodic orbits $\hat{X} \subset \widetilde{Per}_0(H)$ will be said to be **maximally positive** (resp. **negative**) **relative index** 1 (resp. **relative index** -1) if

(1) $\hat{X} \subset \widetilde{Per}_0(H)_{(1)}$ (resp. $\hat{X} \subset \widetilde{Per}_0(H)_{(-1)}$) ,

(2) $\hat{X}$ is a positive (resp. negative) capped braid, and

(3) $\hat{X}$ is maximal among all subsets of $\widetilde{Per}_0(H)$ satisfying the two previous items.

We denote by $mp_{(1)}(H)$ (resp. $mn_{(-1)}(H)$) the set of all such capped braids.

This leads immediately to the following characterization of $c_{im}([\Sigma]; -)$ in the non-degenerate case

Theorem 4.2.4. *Let H be non-degenerate.*

$$c_{im}([\Sigma]; H) = \inf_{\hat{X} \in mp_{(1)}(H)} \sup_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x})$$

We begin with the following lemma

Lemma 4.2.5. *Let $\mathcal{D} \in PSS(H,J)$, and let $u_i : D^2 \to \Sigma$, $i = 1, \ldots, k$, solve the PSS-equation induced by $\mathcal{D}$. Suppose that for some $(s_0,t_0) \in D^2$ and some $p \in \Sigma$,*

$$u_i(s_0,t_0) = p, \text{ for all } i = 1, \ldots, k.$$

Then there exists an open neighbourhood $U \subset D^2$ of (s_0,t_0) and smooth maps $v_i : D^2 \to \Sigma$, $i = 1, \ldots, k$ such that

(1) $v_i(s,t) = u_i(s,t)$ for all $(s,t) \notin U$,

(2) $v_i(s_0,t_0) \neq p$ for all $i = 1, \ldots, k$, and

(3) any pairwise intersections of the graphs $\tilde{v}_i$ and $\tilde{v}_j$, $1 \leq i < j \leq k$, are both transverse and positive.

PROOF. This is a straightforward consequence of the intersection theory for pseudoholomorphic curves in 4-dimensional almost complex manifolds as worked out in Appendix E of [**19**]. In slightly more detail, the proof of Proposition E.2.2 of [**19**] shows that if $f : (\Sigma,j) \to (M,J)$ is any simple pseudoholomorphic map with (Σ,j) any Riemannian surface and (M,J) any 4-dimensional almost complex manifold, then for any self -intersection point $(z_0,z_1) \in \Sigma \times \Sigma \setminus \Delta$ with $f(z_0) = f(z_1)$, there exist (disjoint) neighbourhoods $U_i \subset \Sigma$ of z_i, $i = 0,1$, and a perturbation f' of f differing from f only on $U_0 \bigcup U_1$ such that f' has only positive and transverse self-intersections for $(z_0',z_1') \in U_0 \times U_1$. Noting that for each $i = 1, \ldots, k$, the graph $\tilde{u}_i$ is $\tilde{J}^{\mathcal{D}}$-holomorphic, where $\tilde{J}^{\mathcal{D}}$ is the almost complex structure on $D^2 \times \Sigma$ associated to the PSS data $\mathcal{D}$ by the Gromov trick, we may apply this proposition with $f = \sqcup_{i=1}^k \tilde{u}_i$. The perturbation may be chosen small enough such that each component of $f' = \sqcup \tilde{v}_i$ is still graphical, and without loss of generality (up to another perturbation of f'), we may suppose that self-intersections of f' do not occur at $(s_0,t_0,p) \in D^2 \times \Sigma$. Setting $v_i := \pi_\Sigma \circ \tilde{v}_i$ provides the desired maps. □

Corollary 4.2.6. *Let $\mathcal{D} = (f,g;\mathcal{H},\mathbb{J}) \in PSS_{reg}(H,J)$ and suppose that $\sigma \in$ im $\Phi_{\mathcal{D}}^{PSS} \subset CF_*(H,J)$. Then supp σ is a positive capped braid.*

PROOF. Write $\hat{X} = \{\hat{x}_1, \ldots, \hat{x}_k\} =$ supp σ. Because $\sigma \in$ im $\Phi_{\mathcal{D}}^{PSS}$ for each $\hat{x} \in \hat{X}$, there exists $p \in Crit(f)$ such that $\mathcal{M}(p,\hat{x};\mathcal{D}) \neq \emptyset$. For $i = 1, \ldots, k$, we choose $u_i \in \mathcal{M}(p_i,\hat{x}_i;\mathcal{D})$. Owing to the Gromov trick, and positivity of intersections in dimension 4, the algebraic intersection number of the graphs $\tilde{u}_i$ and $\tilde{u}_j$ is positive for any $1 \leq i < j \leq k$. Moreover, using Lemma 4.2.5, we may suppose without loss of generality that none of the graphs intersect over $0 \in D^2$. As a consequence, if we set $v^i(s,t) = u(se^{2\pi i t})$, for $i = 1, \ldots, k$, the map $s \mapsto (v_s^1, \ldots, v_s^k) \in \mathcal{L}_0(\Sigma)^k$ defines a positive 0-cobordism from the trivial capped braid $(\widehat{u_1(0)}, \ldots, \widehat{u_k(0)})$ to the capped braid $\hat{X}$. □

Proposition 4.2.7. *Let $\mathcal{D} \in PSS_{reg}(H,J)$ and suppose that $\sigma \in CF_1(H,J)$ is such that supp σ is a positive capped braid. If σ is a cycle satisfying $(\Phi_{\mathcal{D}}^{PSS})_*([\Sigma]) = [\sigma]$, then supp σ is maximally positive relative index* 1.

PROOF. Suppose that $(\Phi_{\mathcal{D}}^{PSS})_*([\Sigma]) = [\sigma]$, but that $\hat{X} =$ supp σ is not maximally positive relative index 1. Then there must exist $\hat{y} \in \widetilde{Per}_0(H)_{(1)}$ with $\ell(\hat{x},\hat{y}) \geq 0$ for all $\hat{x} \in \hat{X}$, but $\hat{y} \notin \hat{X}$. Choose $f \in C^\infty(\Sigma)$ to be a small Morse function with a unique maximum $M \in \Sigma$, and let $J^+ \in \mathcal{J}(\Sigma,\omega)$ be such that (f,J^+) is Floer-regular. Let $u : \mathbb{R} \times S^1 \to \Sigma$ be a smooth cylinder such that $u(s,t) = y(t)$ for all $s \in \mathbb{R}$ sufficiently small, and $u(s,t) = M$ for all $s \in \mathbb{R}$ sufficiently large. u is obviously a pre-model for a continuation cobordism, and so by (the proof of) Theorem 3.0.4, there exists a homotopy of Floer data $(\mathcal{H},J) \in \mathscr{HJ}(H,J^-;f,J^+)$ such that u solves the $(\mathcal{H},J)$ Floer equation. By Corollary 2.2.17, u is regular, and so perturbing $(\mathcal{H},J)$ to a regular pair $(\mathcal{H}',J')$, and perturbing u to u', solving the $(\mathcal{H}',J')$ Floer equation if necessary, we conclude that

$$\mathcal{M}(\hat{y},\hat{M};\mathcal{H}',J') \neq \emptyset.$$

Consequently, for all $\hat{x} \in \hat{X}$, we must have that

$$\mathcal{M}(\hat{x},\hat{M};\mathcal{H}',J') = \emptyset,$$

for if $v \in \mathcal{M}(\hat{x},\hat{M};\mathcal{H}',J')$, then we must have

$$0 \leq \ell(\hat{x},\hat{y}) = \ell_{-\infty}(u',v) \leq \ell_{\infty}(u',v) \leq a(\hat{M}) = -1,$$

which is a contradiction. Consequently, $h_{\mathcal{H}'}(\sigma) = 0$, but this is absurd, since σ represents a non-trivial homology class in $CF_*(H,J^-)$, and so $h_{\mathcal{H}'}(\sigma) \neq 0$, since $h_{\mathcal{H}'}$ is an isomorphism on homology. It follows that no $\hat{y}$ as above may exist and hence $\hat{X} \in mp_{(1)}(H)$. □

PROOF OF THEOREM 4.2.1. Corollary 4.2.6 together with Proposition 4.2.7 immediately implies that every cycle representing the fundamental class and lying in the image of some PSS map must be a maximal positive braid relative index 1. Let us show the converse. Let $\hat{X} = \{\hat{x}_1, \ldots, \hat{x}_k\} \subset \widetilde{Per}_0(H)_{(1)}$ be maximally positive relative index 1, and fix some $J \in C^\infty(S^1;\mathcal{J}_\omega(\Sigma))$ such that (H,J) is Floer regular. We will show that there exists $\mathcal{D} = (f,g;\mathcal{H},\mathbb{J}) \in PSS_{reg}(H,J)$ such that f is a Morse function and

$$\Phi^{PSS}_{\mathcal{D}}\Big(\sum_{\substack{M\in Crit(f)\\ \mu^{Morse}(M)=2}} M\Big) = \sum_{\hat{x}\in\hat{X}} \hat{x}.$$

To do this, note first that it will suffice to show that there exists some Floer regular pair (f',J^-) with $f' \in C^\infty(\Sigma)$ a C^2-small Morse function and a regular homotopy of Floer data $(\mathcal{H}',\mathbb{J}') \in \mathscr{HJ}(f',J^-;H,J)$ such that

$$h_{\mathcal{H}'}\Big(\sum_{\hat{p}\in\widetilde{Per}_0(f')_{(1)}} \hat{p}\Big) = \sum_{\hat{x}\in\hat{X}} \hat{x}.$$

Indeed, if $(\mathcal{H}',\mathbb{J}')$ as above exists, then because $\sum_{\hat{p}\in\widetilde{Per}_0(f')_{(1)}} \hat{p}$ represents the unique non-trivial cycle in $CF_1(f',J^-)$, *any* choice of $\mathcal{D}' \in PSS^{reg}(f',J^-)$ will satisfy

$$\Phi^{PSS}_{\mathcal{D}'}(\sum_{\substack{M\in Crit(f)\\ \mu^{Morse}(M)=2}} M) = \sum_{\hat{p}\in\widetilde{Per}_0(f')_{(1)}} \hat{p}$$

at the chain level, and using the gluing results of [**28**], we may define $\mathcal{D} \in PSS_{reg}(H,J)$ as an appropriate gluing $\mathcal{D} := \mathcal{D}'\#(\mathcal{H}',\mathbb{J}')$ such that at the chain level, we have

$$PSS_{\mathcal{D}} = h_{\mathcal{H}'} \circ PSS_{\mathcal{D}'}.$$

With the above understood, note next that since $\hat{X}$ is a positive braid, there exists a positive 0-cobordism $s \mapsto (v^1_s,\ldots,v^k_s) \in \mathcal{L}_0(\Sigma)^k$ from some trivial braid $(\hat{p}_1,\ldots,\hat{p}_k)$ to $\hat{X}$, where $\hat{p}_i$ denotes the trivially capped constant loop based at p_i. We choose $f' \in C^\infty(\Sigma)$ to be any C^2-small Morse function having local maxima precisely at the points p_i, $i = 1,\ldots,k$. Let $J^- \in \mathcal{J}_\omega(\Sigma)$ be any compatible almost complex structure such that (f',J^-) is Floer regular. Next, we define a pre-model for a continuation cobordism from (f',J^-) to (H,J). Fix some $K > 0$ and for each $i = 1,\ldots,k$, define

$$u_i(s,t) := \begin{cases} p_i & s \in (-\infty,-K] \\ v^i_{s+K}(t) & s \in (-K,-K+1) \\ x_i(t) & s \in [K+1,\infty). \end{cases}$$

It is not hard to verify that $\{u_i\}_{i=1}^k$ defines a pre-model for a continuation cobordism from (f',J^-) to (H,J). Theorem 3.0.4 implies that we may perturb this to a continuation cobordism $\{u'_i\}_{i=1}^k$ for some $(\mathcal{H}',\mathbb{J}') \in \mathscr{HJ}(f',J^-;H,J)$. Since each u'_i is asymptotic at both ends to a non-degenerate orbit of Conley-Zehnder index 1, Corollary 2.2.17 implies that without loss of generality, we may take $(\mathcal{H}',\mathbb{J}')$ to be regular. We claim that for each $1 \le i,j \le k$,

$$|\mathcal{M}(\hat{p}_i,\hat{x}_j;\mathcal{H}',\mathbb{J}')| = \begin{cases} 0 & \text{if } i \ne j \\ 1 & \text{if } i = j \end{cases}.$$

The latter case is guaranteed by Corollary 2.2.15 together with the existence of $u'_i \in \mathcal{M}(\hat{p}_i,\hat{x}_i;\mathcal{H}',\mathbb{J}')$. To establish the former case, note that since $\{\hat{p}_1,\ldots,\hat{p}_k\}$ is a trivial capped braid, we certainly have $\ell(\hat{p}_i,\hat{p}_j) = 0$ for all $i \ne j$. Suppose for the sake of contradiction that there exists $v \in \mathcal{M}(\hat{p}_i,\hat{x}_j;\mathcal{H}',\mathbb{J}')$. Then we must have

$$0 = \ell(\hat{p}_i,\hat{p}_j) = \ell_{-\infty}(v,u'_j) \le \ell_\infty(v,u'_j) \le a(\hat{x}_j) = -1,$$

which is a contradiction. It follows that

$$\hat{X} \subset \text{supp } h_{\mathcal{H}'}(\sum_{i=1}^{k} \hat{p}_i).$$

To conclude that the above containment is in fact an equality, note that in combination with the gluing trick outlined at the beginning of this proof, Proposition 4.2.7 implies that supp $h_{\mathcal{H}'}(\sum_{i=1}^{k} \hat{p}_i) \in mp_{(1)}(H)$, but $\hat{X} \in mp_{(1)}(H)$ by hypothesis, and so maximality of $\hat{X}$ implies

$$\hat{X} = \text{supp } h_{\mathcal{H}'}(\sum_{i=1}^{k} \hat{p}_i)$$

as desired. $\square$

4.3. Dynamical Consequences

Theorem 4.3.1. *For any closed symplectic surface* (Σ,ω)*, and any non-degenerate* $\phi \in Ham(\Sigma,\omega)$*,*

$$\nu_{im}(\phi) = \inf_{\hat{X} \in mp_{(1)}(H)} \sup_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x}) - \sup_{\hat{X} \in mn_{(-1)}(H)} \inf_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x})$$

for any Hamiltonian H *with* $\phi_1^H = \phi$*.*

PROOF. As the PSS-image spectral invariants are well-defined on $\widetilde{Ham}(M,\omega)$

$$\nu_{im}(\phi) = \inf_{\tilde{g} \in \pi_1(Ham(M,\omega), id_M)} \tilde{\nu}_{im}(\tilde{g} \cdot \tilde{\phi})$$

for $\tilde{\phi}$ any lift of ϕ to $\widetilde{Ham}(M,\omega)$. The theorem follows directly from the fact that

$$\begin{aligned}\tilde{\nu}_{im}(\tilde{\phi}^H) &= c_{im}([\Sigma];H) + c_{im}([\Sigma];\bar{H}) \\ &= \inf_{\hat{X} \in mp_{(1)}(H)} \sup_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x}) - \sup_{\hat{X} \in mn_{(-1)}(H)} \inf_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x}),\end{aligned}$$

where the identification of the second term in the last line with $c_{im}([\Sigma];\bar{H})$ follows from the fact that $\bar{H}$ is homotopic relative endpoints to the Hamiltonian $\tilde{H}(t,x) = -H(1-t,x)$ which generates the isotopy $\phi_{1-t}^H \circ (\phi_1^H)^{-1}$, whose time-1 flowlines are precisely the time reversal of the time-1 flowlines of the isotopy induced by H. Consequently, there is a natural bijection

$$\begin{aligned}\widetilde{Per}_0(H) &\mapsto \widetilde{Per}_0(\tilde{H}) \\ [x(t),w(se^{2\pi it})] &\mapsto [x(1-t),w(se^{-2\pi it})]\end{aligned}$$

which sends capped orbits with Conley-Zehnder index k to capped orbits with Conley-Zehnder index $-k$. It is easy to see that positive braids are sent to negative braids under the above bijection, which establishes the desired formula. $\square$

Corollary 4.3.2. *If $\pi_2(\Sigma) = 0$, then for any non-degenerate $\phi \in Ham(\Sigma,\omega)$,*

$$\nu_{im}(\phi) = \inf_{\hat{X} \in mp_{(1)}(H)} \sup_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x}) - \sup_{\hat{X} \in mn_{(-1)}(H)} \inf_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x})$$

for any normalized Hamiltonian H with $\phi_1^H = \phi$.

Corollary 4.3.3. *If $\Sigma = S^2$, then for any non-degenerate $\phi \in Ham(\Sigma,\omega)$,*

$$\nu_{im}(\phi) = \min\{ \inf_{\hat{X} \in mp_{(1)}(K)} \sup_{\hat{x} \in \hat{X}} \mathcal{A}_K(\hat{x}) - \sup_{\hat{X} \in mn_{(-1)}(K)} \inf_{\hat{x} \in \hat{X}} \mathcal{A}_K(\hat{x}) : \ K \in \{H, G\# H\}\},$$

where H is any normalized Hamiltonian H with $\phi_1^H = \phi$ and G is any normalized Hamiltonian such that $(\phi_t^G)_{t\in[0,1]}$ is a non-contractible loop in $Ham(S^2,\omega)$.

Another interesting consequence of our dynamical characterization of the PSS-image spectral invariants is that it permits us to use the work of Entov in [**5**] to obtain computable bounds on the commutator length of the isotopy $(\phi_t^H)_{t\in[0,1]}$ in $\widetilde{Ham}(S^2,\omega)$. Recall that for any group G, the **commutator length** of $g \in G$ is defined to be

$$cl(g) := \inf\{k : g = \Pi_{i=1}^k [f_i,h_i], \ f_i, h_i \in G\}.$$

It is a classical result due to Banyaga that for compact symplectic manifolds the group of Hamiltonian diffeomorphisms is perfect, and so every element $\tilde{\phi}^H \in \widetilde{Ham}(S^2,\omega)$ has a finite commutator length. We will show

Proposition 4.3.4. *Assume that $H \in C^\infty(S^1 \times S^2)$ is non-degenerate and normalized, then*

$$\min\Big\{ \min_{\hat{X} \in mp_{(1)}(H)} \max_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x}), - \max_{\hat{X} \in mn_{(-1)}(H)} \min_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x})\Big\} < -kArea(S^2,\omega)$$

only if $cl(\tilde{\phi}^H) > 2k+1$.

This result essentially says that if a Hamiltonian isotopy on a surface is to be 'simple' in the sense that it has small commutator length in $\widetilde{Ham}$, then the actions of orbits forming maximally positive (relative index 1) capped braids cannot be uniformly small, and nor can the actions of orbits forming maximally negative (relative index -1) capped braids be uniformly large.

PROOF. Morally speaking, this is essentially a direct application of Theorem 2.5.1 in [**5**], combined with our dynamical characterization of $c_{im}([S^2]; H)$, but since Entov uses the Oh-Schwarz spectral invariants with rather different conventions than we do, and since we moreover do not know if $c_{im} = c_{OS}$, we will explain how to deduce this result from Entov's arguments for the reader unwilling to take such assurances on faith. Note that here we work with Floer and quantum homologies with coefficients in the Novikov ring with *rational* coefficients, but all one needs to note in order to work in this setting (beyond the usual tracking of orientations) is that Theorem 4.2.1 remains valid for Floer homology with $\mathbb{Z}$-coefficients, and this is an immediate consequence of Corollary 2.2.15.

We will summarize the three main points of relevance for us from [**5**]. Let (M,ω) be any semi-positive symplectic manifold of dimension $2n$, (H,J) be a Floer regular pair with H normalized as in the statement of the proposition, and for any $g \in \mathbb{Z}_{\geq 0}$, let $(S_{g,1},j)$ denote a Riemannian surface of genus g with 1 positive cylindrical end. Entov (following Schwarz in [**28**]) establishes that there exist almost complex structures $\breve{J}$ on $S_{g,1} \times M$ such that

(1) for $\hat{x} = [x,w_x] \in \widetilde{Per}_0(H)$ with $\mu(\hat{x}) = n(1-2g)$ the moduli space

$$\mathcal{M}_g(\hat{x};\breve{J})$$

consisting of $\breve{J}$ pseudo-holomorphic sections $u : S_{g,1} \to S_{g,1} \times M$ such that $\pi_M \circ u$ is asymptotic to x along the positive end and $[\pi_M \circ u]\#(-[w_x]) \in H_2(M;\mathbb{Z})$ is torsion, is compact and 0-dimensional. (This simply follows from [**28**], see also [**25**]).

(2) If $\mathcal{M}_g(\hat{x};\breve{J}) \neq \emptyset$ for some $\hat{x} \in \widetilde{Per}_0(H)_{(n(1-2g))}$ with $\mathcal{A}_H(\hat{x}) > 0$, then

$$cl(\tilde{\phi}^H) > g.$$

(This is essentially a special case of the content of Proposition $6.3.1$ in [**5**]. See also the proof of Theorem $2.5.1$ in [**5**]).

(3) The sum

$$\theta_{\breve{J}} = \sum_{\mu(\hat{x})=n(1-2g)} \#\mathcal{M}_g(\hat{x};\breve{J})\hat{x} \in CF_{n(1-2g)}(H,J)$$

represents a non-trivial homology class. (See Propositions $7.0.3$ and $7.0.4$ is [**5**]. In fact, Entov characterizes this homology class precisely as the homology class $(\Phi_*^{PSS} \circ PD)(E^g)$, where E^g is the g-th (quantum) power of the Euler class in the quantum cohomology with rational Novikov coefficients $QH^*(M,\omega)$ and $PD : QH^*(M,\omega) \to QH_{2n-*}(M,\omega)$ is isomorphism given by Poincaré duality. But we will not need the full force of this).

Specializing now to the case where $M = S^2$, we see that for $g = 2k+1$ and $k \in \mathbb{Z}_{\geq 0}$, then the only nontrivial homology class (up to multiplication by a non-zero rational constant, which doesn't affect the value of the spectral invariants) in $QH_{-4k}(S^2,\omega)$ is $[pt] \otimes e^{k[S^2]}$ (recall that the

PSS isomorphism $\Phi_*^{PSS} : QH_{*+n}(M,\omega) \to HF_*(H)$ shifts the grading by n). Thus

$$[\theta_{\breve{J}}] = \lambda \Phi_*^{PSS}([pt] \otimes e^{k[S^2]})$$

for some $\lambda \in \mathbb{Q}$, $\lambda \neq 0$. Now, Poincaré duality for Hamiltonian Floer homology (for rational symplectic manifolds) arises from the isomorphism

$$CF_m(\tilde{H},\tilde{J}) \to Hom(CF_{-m}(H,J);\mathbb{Q})$$
$$\tilde{x} \mapsto \hat{x}^*,$$

where $\tilde{J}(t) = J(1-t)$ and $\tilde{H}(t,x) = -H(1-t,x)$ is the Hamiltonian discussed in the preceding proof, with $\hat{x} = [x(t),w(se^{2\pi it})] \mapsto \tilde{x} = (x(1-t),w(se^{-2\pi it})$ the correspondence discussed also in the previous proof. This isomorphism induces an isomorphism at the level of homology, giving

$$HF_{-m}(\tilde{H}) \simeq HF^m(H).$$

In particular, this implies that if

$$\Theta \in CF_{4k+1}(\tilde{H},\tilde{J})$$

is any chain such $[\Theta] \in HF_{4k+1}(\tilde{H},\tilde{J})$ is non-trivial, then

$$\langle \Theta, \theta_{\breve{J}} \rangle \neq 0.$$

In particular, there exists some $\tilde{x} \in \text{supp}\ \Theta$ such that $\hat{x} \in \theta_{\breve{J}}$. This implies that

$$\begin{aligned} \lambda_H(\theta_{\breve{J}}) &\geq \min_{\tilde{x} \in \text{supp}\ \Theta} \mathcal{A}_H(\hat{x}) \\ &= -\max_{\tilde{x} \in \text{supp}\ \Theta} \mathcal{A}_{\tilde{H}}(\tilde{x}) \\ &= -\lambda_{\tilde{H}}(\Theta). \end{aligned}$$

If we take Θ to be a witness for $c_{im}((\Phi_*^{PSS})^{-1}([\Theta]);\tilde{H})$, then we obtain

$$\lambda_H(\theta_{\breve{J}}) \geq -c_{im}((\Phi_*^{PSS})^{-1}([\Theta]);\tilde{H}).$$

Next, note that there is really only one thing that $(\Phi_*^{PSS})^{-1}([\Theta]) \in QH_{4k+2}(S^2,\omega)$ can be (up to a non-zero rational multiple), namely

$$(\Phi_*^{PSS})^{-1}([\Theta]) = \lambda[S^2] \otimes e^{-k[S^2]},\ \lambda \in \mathbb{Q} \setminus \{0\}.$$

Thus, we infer that if

$$c([S^2] \otimes e^{-k[S^2]};\tilde{H}) < 0,$$

then $\lambda_H(\theta_{\check{J}}) > 0$ and so $cl(\tilde{\phi}^H) > 2k+1$. We may then use Proposition 4.1.5 to compute

$$c([S^2] \otimes e^{-k[S^2]}; \tilde{H}) = c([S^2]; \tilde{H}) + kArea(S^2,\omega).$$

Whence we see that

$$\begin{aligned} -kArea(S^2,\omega) &> c([S^2]; \tilde{H}) \\ &= -\max_{\hat{X} \in mn_{(-1)}(H)} \min_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x}). \end{aligned}$$

implies that $cl(\tilde{\phi}^H) > 2k+1$. To see that

$$\min_{\hat{X} \in mp_{(1)}(H)} \max_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x}) < -kArea(S^2,\omega)$$

also implies that $cl(\tilde{\phi}^H) > 2k+1$, simply note that $cl(\tilde{\phi}^{\tilde{H}}) = cl(\tilde{\phi}^H)$ and apply the above argument reversing the roles of H and $\tilde{H}$. $\square$

4.4. On the equivalence $c_{OS} = c_{im}$

Proposition 4.1.2 assures us that

$$c_{OS}(\alpha; H) \leq c_{im}(\alpha; H),$$

and the PSS-image spectral invariants satisfy many of the same formal properties as the Oh-Schwarz spectral invariants. One is naturally led to ask when these two invariants coincide, and what to make of their difference. By the definition of the PSS-image spectral invariants, $c_{im}(\alpha; H) - c_{OS}(\alpha; H)$ represents the obstruction to our ability to find a tight cycle $\sigma \in CF_*(H,J)$ for $c_{OS}(\alpha; H)$ which additionally lies in the image of some PSS map at the chain level. Note that, on a surface, our knowledge that the image of a chain-level PSS map must be form a positive capped braid, while in general, the support of Floer cycles need not be positive implies that, at least in low dimensions, there are genuine obstructions to the equivalence of these two quantities. Thus, in some sense, while the Oh-Schwarz spectral invariants arise from purely (filtered) homological considerations, the PSS-image spectral invariants in principle encode some additional geometric information about the degree to which this filtration information may be probed by PSS-type maps. In fact, it turns out that one can also interpret the difference of the functions

$$c_{OS}(\alpha; -), c_{im}(\alpha; -) : C^\infty(S^1 \times M) \to \mathbb{R}$$

as measuring the failure of the PSS-image spectral invariants to satisfy the Poincaré duality relation discovered by Entov-Poterovich for the Oh-Schwarz spectral invariants in [**6**].

Proposition 4.4.1. *Let* $\alpha,\beta \in QH_*(M,\omega)\setminus\{0\}$ *be such that*

$$c_{OS}(\alpha;H) = -\inf_{\Pi(\alpha,\beta)\neq 0} c_{OS}(\beta;\bar{H}),$$

where Π *is the bilinear pairing described in Section 2.3 of* [**6**] *(see also Section 20.4 of* [**23**]*), then for all* $H \in C^\infty(S^1\times M)$*,*

$$c_{im}(\alpha;H) - c_{OS}(\alpha;H) = c_{im}(\beta;\bar{H}) - c_{OS}(\beta;\bar{H}) = 0$$

if and only if

$$c_{im}(\alpha;H) = -c_{im}(\beta;\bar{H}).$$

PROOF. The proof is a straight-forward consequence of Entov-Polterovich's Poincaré duality relation for the Oh-Schwarz spectral invariants. Indeed, since the PSS-image spectral invariants always bound the Oh-Schwarz spectral invariants from above, we have

$$-c_{im}(\beta;\bar{H}) \leq -c_{OS}(\beta;\bar{H}) = c_{OS}(\alpha;H) \leq c_{im}(\alpha;H),$$

thus it is clear that if $c_{im}(\alpha;H) = -c_{im}(\beta;\bar{H})$, we must have

$$c_{im}(\alpha;H) - c_{OS}(\alpha;H) = c_{im}(\beta;\bar{H}) - c_{OS}(\beta;\bar{H}) = 0.$$

The converse statement is immediate. $\square$

One may also show that $c_{OS}([\Sigma];H)$ and $c_{im}([\Sigma];H)$ coincide for all autonomous $H \in C^\infty(\Sigma)$ when $\Sigma \neq S^2$. Indeed, in [**13**], Humilière-Le Roux-Seyfaddini showed that when (Σ,ω) is a closed aspherical symplectic surface, then any function $c : C^\infty(S^1\times\Sigma)\to\mathbb{R}$ satisfying

(1) **(Spectrality)** $c(H) \in Spec(H)$ for all $H \in C^\infty(S^1\times\Sigma)$
(2) **(Nontriviality)** There exists a topological disk D and a Hamiltonian H supported in D such that $c(H)\neq 0$
(3) **(Continuity)** c is continuous with respect to the C^∞ topology on $C^\infty(S^1\times\Sigma)$
(4) **(Max formula)** If $H_i \in C^\infty(S^1\times\Sigma)$, $i=1,\ldots,k$ are supported in pairwise disjoint disks then

$$c(H_1+\ldots+H_k) = \max\{c(H_1),\ldots,c(H_k)\}$$

agrees with $c_{OS}([\Sigma];-)$ on the space of autonomous Hamiltonians. By Proposition 4.1.5, $c_{im}([\Sigma];-)$ satisfies the first and third properties, while the fact that $c_{OS}([\Sigma],-)$ satisfies the non-triviality condition and bounds $c_{im}([\Sigma];-)$ from below implies that the non-triviality condition holds for the PSS-image spectral invariant. The following proposition may be deduced from the proof given in Section 5.2.2 of [**13**] of the max formula for $c_{OS}([\Sigma];-)$ simply by noting that the arguments provided therein apply equally well to $c_{im}([\Sigma];-)$, however, we also supply an independent proof for the convenience of the reader (although the reader should note that this proof is in some sense morally the same as what is done in [**13**], and we simply exploit our dynamical characterization of $c_{im}([\Sigma];-)$ in order to clarify what is being measured by the spectral invariant).

Proposition 4.4.2. *Let* (Σ,ω) *be a closed aspherical symplectic surface and let* $H_i \in C^\infty(S^1\times\Sigma)$, $i=1,\dots,k$ *be supported in pairwise disjoint disks, then*

$$c_{im}([\Sigma];H_1+\ldots+H_k)=\max\{c_{im}([\Sigma];H_1),\ldots,c_{im}([\Sigma];H_k)\}$$

PROOF. Let $D_1,\ldots,D_k$ be pairwise disjoint embedded disks in Σ with supp $H_i\subset D_i$. Up to enlarging each disk slightly while preserving pairwise disjointness, we may assume that H_i is compactly supported on the interior of D_i. Let U_i be a tubular neighbourhood of ∂D_i which is disjoint from supp H_j for every $j=1,\dots,k$ and fix a diffeomorphism

$$\phi_i : U_i \to (-1,1)\times S^1,$$

with $\phi_i(0,t)\in\partial D_i$ for all $t\in S^1\simeq\mathbb{R}/\mathbb{Z}$. Let $h\in C^\infty(S^1)$ be a perfect Morse function having unique maximum $h(0)=1$ at 0 and unique minimum at $\frac{1}{2}$, and let $\rho:(-1,1)\to[0,1]$ be a smooth, compactly supported function with a unique critical point at 0, and $\rho(0)=1$. Set $k(r,t)=\rho(r)h(t)$ for $(r,t)\in(-1,1)$. For any $\epsilon>0$ sufficiently small, we may choose a perturbation H_i^ϵ of H_i with $\|H_i^\epsilon-H_i\|_{C^2}<\epsilon$, such that there exists $\delta_i>0$ such that $(H_i^\epsilon\circ\phi_i^{-1})(r,t)=\frac{\epsilon}{2\|k\|_{C^2}}k(r,t)$ for all $(r,t)\in(-\delta_i,1)\times S^1$ and such that H_i^ϵ is non-degenerate on the interior of its support. We will let $G_i^\epsilon=H_i^\epsilon+f_i$ be the non-degenerate Hamiltonian obtained by adding a small function $f\in C^\infty(\Sigma)$ which vanishes in a neighbourhood of D_i, has only non-degenerate critical points on the interior of its support and satisfies the smallness condition $\|f_i\|_{C^2}<\frac{\epsilon}{3\|k\|_{C^2}}$. We define also $G_{tot}^\epsilon=H_1^\epsilon+\ldots+H_k^\epsilon+f_{tot}$ where $f_{tot}\in C^\infty(\Sigma)$ satisfies $\|f_{tot}\|_{C^2}<\frac{\epsilon}{3\|k\|_{C^2}}$, has only non-degenerate critical points on the interior of its support and vanishes on a neighbourhood of $\bigcup_{i=1}^k D_i$. Clearly, it will suffice to show that

$$c_{im}([\Sigma];G_{tot}^\epsilon)=\max\{c_{im}([\Sigma];G_1^\epsilon),\ldots,c_{im}([\Sigma];G_k^\epsilon)\}$$

for all sufficiently small $\epsilon>0$. Since all the Hamiltonians involved in the above expression are non-degenerate, we may compute relevant spectral invariants through dynamical considerations by Theorem 4.2.4. Let us consider first the structure of the set $mp_{(1)}(G_i^\epsilon)$. Write $p_i=\phi_i^{-1}(0,0)$ and remark that if $x\in Per_0(G_i^\epsilon)_{(1)}$, then one of the following holds

(1) $x \in Per_0(H_i^\epsilon)_{(1)}$, and x lies inside of $V_i = D_i \bigcup \phi_i^{-1}((-1, -\delta_i] \times S^1)$,

(2) $x = p_i$,

(3) $x \in Crit(f_i)$ and $x \in$ intsupp f_i.

We note that any orbit in one of the above classes is unlinked with any orbit in either of the other two classes. For orbits in the last two classes, this is obvious, while if one of the orbits comes from the first class, then this follows from the fact that such an orbit may be contracted to a point entirely inside of V_i, which shows that it is unlinked with any orbit in one of the latter two classes. Let us denote by $mp_{(1)}(H_i^\epsilon; V_i)$ the set of all braids $X \subset Per_0(H_i^\epsilon)_{(1)}$ which are maximally positive relative having all strands of Conley-Zehnder index 1 and also such that all strands are contained in V_i, and write $Crit(f_i; \text{intsupp } f_i)$ for all critical points of f_i contained in the interior of the support of f_i. Note also that if $X \in mp_{(1)}(H_i^\epsilon; V_i)$, then it is not hard to see that there exists a positive cobordism from a trivial braid lying inside V_i to X such that each strand of the cobordism is contained entirely in V_i. These considerations imply that the map

$$\begin{aligned} mp_{(1)}(H_i^\epsilon; V_i) &\to mp_{(1)}(G_i^\epsilon) \\ X_i &\mapsto X_i \cup \{p_i\} \cup Crit(f_i; \text{int supp } f_i) \end{aligned}$$

is a bijection. Let $X = X_i \cup \{p_i\} \cup Crit(f_i; \text{int supp } f_i)$ with $X_i \in mp_{(1)}(H_i^\epsilon; V_i)$, and note that by construction all $x \in Crit(f_i; \text{int supp } f_i)$ have action less than $\mathcal{A}_{G_i^\epsilon}(p_i) = \frac{\epsilon}{2\|k\|_{C^2}}$. Consequently, we see that

$$c_{im}([\Sigma]; G_i^\epsilon) = \min_{X_i \in mp_{(1)}(H_i^\epsilon; V_i)} \max_{x \in X_i} \lambda_{H_i^\epsilon}(x + p_i).$$

Next, let us consider the structure of the set $mp_{(1)}(G_{tot}^\epsilon)$. In a manner entirely analogous to that which preceded, we see that there is a bijection

$$\begin{aligned} \Pi_{i=1}^k mp_{(1)}(H_i^\epsilon; V_i) &\to mp_{(1)}(G_{tot}^\epsilon) \\ (X_1, \ldots, X_k) &\mapsto \bigcup_{i=1}^k X_i \cup \{p_1, \ldots, p_k\} \cup Crit(f_{tot}; \text{int supp } f_{tot}), \end{aligned}$$

and if $X = \bigcup_{i=1}^k X_i \cup \{p_1, \ldots, p_k\} \cup Crit(f_{tot}; \text{int supp } f_{tot})$ for $X_i \in mp_{(1)}(H_i^\epsilon; V_i)$, then the fact that the critical points of f_{tot} are taken to have small action implies similarly to before that we have

$$\begin{aligned} \max_{x \in X} \mathcal{A}_{G_{tot}^\epsilon}(x) &= \max_{i=1,\ldots,k} \max_{x \in X_i} \lambda_{G_{tot}^\epsilon}(x + p_i) \\ &= \max_{i=1,\ldots,k} \max_{x \in X_i} \lambda_{H_i^\epsilon}(x + p_i). \end{aligned}$$

Taking the minimum of this last expression over all tuples $(X_1, \ldots, X_k) \in \Pi_{i=1}^k mp_{(1)}(H_i^\epsilon; V_i)$ gives

$$\begin{aligned}\min_{X \in mp_{(1)}(G^\epsilon_{tot})} \max_{x \in X} \mathcal{A}_{G^\epsilon_{tot}}(x) &= \max_{i=1,\ldots,k} \min_{X_i \in mp_{(1)}(H_i^\epsilon; V_i)} \max_{x \in X_i} \lambda_{H_i^\epsilon}(x + p_i) \\ &= \max\{c_{im}([\Sigma]; G_1^\epsilon), \ldots, c_{im}([\Sigma]; G_k^\epsilon)\},\end{aligned}$$

which proves the claim. □

Corollary 4.4.3. *For $\Sigma \neq S^2$ and any (time-independent) $H \in C^\infty(\Sigma)$,*

$$c_{im}([\Sigma]; H) = c_{OS}([\Sigma]; H).$$

Chapter 5

Application: Positively transverse foliations from Floer theory

In this chapter, we turn our attention to the structure of the moduli spaces of Floer cylinders for a Floer regular pair (H,J), and their relation to the capped braid-theoretic topology of $\widetilde{Per}_0(H)$. The main result in this chapter is that there is a certain topological condition that one may place on capped braids formed by collections of elements of $\widetilde{Per}_0(H)$ — which we call being *maximally unlinked relative the Morse range* — such that whenever $\hat{X} \subset \widetilde{Per}_0(H)$ satisfies this property, we may construct a singular foliation $\mathcal{F}^{\hat{X}}$of $S^1 \times \Sigma$ having singular leaves precisely the graphs of loops x such that $\hat{x} \in \hat{X}$ and with regular leaves parametrized by annuli of the form $(s,t) \mapsto (t,u(s,t))$ for $u \in \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J)$ some Floer cylinder, with $\hat{x}, \hat{y} \in \hat{X}$. Owing to the fact that the regular leaves are parametrized by (projections of the graphs of) Floer cylinders, $\mathcal{F}^{\hat{X}}$ is *positively transverse* to the vector field $\partial_t \oplus X^H$, and can be used to induce a singular foliation $\mathcal{F}_0^{\hat{X}}$ on Σ itself, which is *positively transverse in the sense of Le Calvez* (see Le Calvez's work in [**16**] and [**17**]). Thus, the results in this chapter may be viewed as a way to construct foliations of Le Calvez-type by purely Floer-theoretic considerations. Moreover, we show that $\mathcal{F}_0^{\hat{X}}$ is a singular foliation *of Morse type*, and that $\mathcal{F}^{\hat{X}}$ may be viewed as a Morse-Bott foliation associated to a finite-dimensional reduction of the action functional $A^{\hat{X}} \in C^\infty(S^1 \times \Sigma)$. This provides us with Morse-theoretic models for those parts of the Hamiltonian Floer complex lying in the homologically non-trivial range.

Section 5.1 studies the relationship of $\widetilde{M}(\hat{x},\hat{y}; H,J)$ to the topology of the capped braid $\{\hat{x}, \hat{y}\}$ and deduces conditions under which the maps $(s,t) \mapsto (t,u(s,t)) \in S^1 \times \Sigma$, $u \in \widetilde{M}(\hat{x},\hat{y}; H,J)$, provide a smooth foliation of some subset $W(\hat{x},\hat{y}) \subset S^1 \times \Sigma$. We call $W(\hat{x},\hat{y})$ a *foliated sector*. That such moduli spaces can be used to construct such foliated sectors is essentially due to Hofer-Wysocki-Zehnder in [**12**] (see also [**11**]). In an effort to understand how these foliated sectors may be glued together to form a singular foliation of $S^1 \times \Sigma$, Section 5.2 introduces the notion of capped braids $\hat{X} \subset \widetilde{Per}_0(H)$ which are *maximally unlinked relative the Morse range*, and associates to each such capped braid a chain complex $CF_*(\hat{X}; H,J)$ which is not quite a subcomplex of $CF_*(H,J)$, but whose differential counts Floer cylinders which run between orbits in $\hat{X}$ — equivalently, once the existence of the singular foliation $\mathcal{F}^{\hat{X}}$ is established, the differential counts the 'rigid' leaves of $\mathcal{F}^{\hat{X}}$ which run between the graphs of orbits of index difference 1. Section 5.3 establishes the existence of the foliation $\mathcal{F}^{\hat{X}}$ and some of its basic properties, which imply in particular that $CF_*(\hat{X}; H,J)$ has a Morse-theoretic model, given by a finite-dimensional reduction of the action functional. Section 5.4 concludes with some novel consequences for the structure of Hamiltonian isotopies on surfaces, and a short discussion of the relationship between the singular foliations we produce and those appearing in Le Calvez's theory of transverse foliations.

5.1. Foliated sectors and Floer moduli spaces as leaf spaces

We begin with some observations on the relationship between the topology of the capped braid $(\hat{x},\hat{y})$ for $\hat{x},\hat{y} \in \widetilde{Per}_0(H)$, and the existence of Floer cylinders running between them.

Lemma 5.1.1. *Let $(\mathcal{H},J)$ be an adapted homotopy of Floer data with $(H^\pm,J^\pm)$ Floer regular. Suppose that $\hat{x}^\pm \in \widetilde{Per}_0(H^-) \cap \widetilde{Per}_0(H^+)$, with $x^- \neq x^+$, and that $\mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H},J), \mathcal{M}(\hat{x}^-,\hat{x}^-;\mathcal{H},J)$, and $\mathcal{M}(\hat{x}^+,\hat{x}^+;\mathcal{H},J)$ are all non-empty, then $b(\hat{x}^-;H^-) \leq \ell(\hat{x}^-,\hat{x}^+) \leq a(\hat{x}^+;H^+)$.*

PROOF. That $b(\hat{x}^-;H^-) \leq \ell(\hat{x}^-,\hat{x}^+)$ follows from applying Corollary 2.2.11 with $u_0 \in \mathcal{M}(\hat{x}^-,\hat{x}^+;\mathcal{H},J)$ and $u_1 \in \mathcal{M}(\hat{x}^-,\hat{x}^-;\mathcal{H},J)$. The second inequality uses $u_1 \in \mathcal{M}(\hat{x}^+,\hat{x}^+;\mathcal{H},J)$ instead. $\square$

Applying the preceding lemma in the case where $(\mathcal{H},J)$ is s-independent yields

Corollary 5.1.2. *Let (H,J) be Floer regular and suppose that $\widetilde{\mathcal{M}}(\hat{x},\hat{y};H,J) \neq \emptyset$, $x \neq y$, then*

$$b(\hat{x}) \leq \ell(\hat{x},\hat{y}) \leq a(\hat{y}).$$

Applying Lemma 2.2.7 to the constant cylinder $u \in \widetilde{\mathcal{M}}(\hat{x}',\hat{x}';H,J)$ and $v \in \widetilde{\mathcal{M}}(\hat{x},\hat{y};H,J)$ shows

Proposition 5.1.3. *Let (H,J) be Floer regular and let $\hat{x},\hat{y} \in \widetilde{Per}_0(H)$ with $\hat{y} \in$ supp $\partial_{H,J}\hat{x}$, then for all $\hat{x}' \in \widetilde{Per}_0(H)$, $x' \notin \{x,y\}$, we have $\ell(\hat{x},\hat{x}') \leq \ell(\hat{y},\hat{x}')$.*

Lemma 2.2.14 combines with Corollary 5.1.2 to give

Corollary 5.1.4. *If $\mu(\hat{x}), \mu(\hat{y}) \in \{2k-1, 2k, 2k+1\}$ for some $k \in \mathbb{Z}$ and $\widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J) \neq \emptyset$, then $\ell(\hat{x},\hat{y}) = -k$.*

The main geometric input for this section is the following (cf. Theorem 5.6 in [**12**])

Proposition 5.1.5. *Let (H,J) be Floer regular, and $\hat{x},\hat{y} \in \widetilde{Per}_0(H)$. If $2k-1 \leq \mu(\hat{x}),\mu(\hat{y}) \leq 2k+1$, for some $k \in \mathbb{Z}$, then the map $\widetilde{Ev} : \mathbb{R} \times S^1 \times \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J) \to \mathbb{R} \times S^1 \times \Sigma$ defined by $\widetilde{Ev}(s,t,u) := \tilde{u}(s,t) = (s,t,u(s,t))$ is a diffeomorphism onto its image.*

PROOF. We may suppose that $\widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J) \neq \emptyset$, or else the proposition is vacuously true. Moreover, if $\hat{x} = \hat{y}$, then the statement is obvious. Thus, we may suppose that $\mu(\hat{x}) - \mu(\hat{y}) \in \{1,2\}$. We will show that $\widetilde{Ev}$ is a proper injective immersion. To see that $\widetilde{Ev}$ is one-to-one, note that $\widetilde{Ev}(s,t,u) = \widetilde{Ev}(s',t',v)$ if and only if $u \neq v$ and the graphs of $\tilde{u}$ and $\tilde{v}$ intersect over $(s,t) = (s',t')$. Thus, by Lemma 2.2.7 $\widetilde{Ev}$ fails to be injective only if there exist $u,v \in \widetilde{M}(\hat{x},\hat{y}; H,J)$, $u \neq v$ such that

$$b(\hat{x}) \leq \ell_{-\infty}(u,v) < \ell_{\infty}(u,v) \leq a(\hat{y}),$$

The hypothesized constraints on the Conley-Zehnder indices of $\hat{x}$ and $\hat{y}$ imply that $\mu(\hat{x}) \in \{2k+1, 2k\}$ and $\mu(\hat{y}) \in \{2k,2k-1\}$, and so by Lemma 2.2.14 we deduce that $b(\hat{x}) = -k = a(\hat{y})$, which contradicts the above inequality. Thus, $\widetilde{Ev}$ is injective.

That $\widetilde{Ev}$ is proper essentially follows from compactness results in Floer theory; if $\{(s_n,t_n,u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R} \times S^1 \times \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J)$ is some sequence which eventually leaves any compact set, then either $s_n \to \pm\infty$ and (s_n,t_n,u_n) converges to a point on either the graph $\tilde{x}(s,t) = (s,t,x(t))$ or on the graph $\tilde{y}(s,t) = (s,t,y(t))$, or $(s_n)_{n\in\mathbb{N}}$ remains bounded, in which case (s_n,t_n,u_n) must converge to a point on the graph of some broken Floer cylinder between x and y. In either case, the sequence (s_n,t_n,u_n) eventually leaves every compact subset of im $\widetilde{Ev}$.

It remains to show that $\widetilde{Ev}$ is an immersion when $\mu(\hat{x}) - \mu(\hat{y}) \in \{1,2\}$. We note that

$$T(\mathbb{R} \times S^1 \times \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J)) = T(\mathbb{R} \times S^1) \oplus (\ker(D\mathcal{F}))|_{\widetilde{\mathcal{M}}(\hat{x},\hat{y};H,J)},$$

and since $d\widetilde{Ev}$ may be computed with respect to this splitting as

$$(d\widetilde{Ev})_{(s,t,u)}(\partial_s,\partial_t,\xi) = (\partial_s,\partial_t,\xi(s,t)) \in T_{\tilde{u}(s,t)}\mathbb{R} \times S^1 \times \Sigma,$$

the problem reduces to showing that any $\xi \in \ker(D\mathcal{F}_{H,J})_u$ with $\xi \neq 0$ is a nowhere-vanishing vector field along u, where $D\mathcal{F}_{H,J}$ denotes the vertical part of the linearized Floer operator. This follows by combining Proposition 2.2.16 with Corollary 5.1.4 to deduce that $Z(\xi) = 0$ whenever ξ is not identically zero. $\square$

Note that as a consequence of the previous lemma, whenever $\mu(\hat{x}), \mu(\hat{y}) \in \{2k-1, 2k, 2k+1\}$ for $k \in \mathbb{Z}$, then $\widetilde{Ev}(\mathbb{R} \times S^1 \times \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J))$ carries a smooth 2-dimensional foliation $\widetilde{\mathcal{F}}^{\hat{x},\hat{y}}$, the leaves of which are nothing but the graphs $\tilde{u}$ of $u \in \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J)$.

Definition 5.1.6. For $\hat{x},\hat{y} \in \widetilde{Per}_0(H)$, **the connecting subspace** of $\hat{x}$ and $\hat{y}$ will denote the subspace $W(\hat{x},\hat{y}) := \{(t,u(s,t)) \in S^1 \times \Sigma : s \in \mathbb{R}, u \in \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J)\}$.

Remark that if we write $\widetilde{W}(\hat{x},\hat{y}) := \widetilde{Ev}(\mathbb{R} \times S^1 \times \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J))$, then the map $\breve{\pi} : \mathbb{R} \times S^1 \times \Sigma \to S^1 \times \Sigma$ restricts to a projection $\breve{\pi} : \widetilde{W}(\hat{x},\hat{y}) \to W(\hat{x},\hat{y})$, with fiber $\breve{\pi}^{-1}(t,p) = \{\widetilde{Ev}(s,t,u) : u(s,t) = p\}$, which under the hypotheses of Proposition 5.1.5 may be identified via $\widetilde{Ev}$ with the orbit of any $(s_0,t,u_0) \in \mathbb{R} \times S^1 \times \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J)$ such that $u_0(s_0,t) = p$ under the $\mathbb{R}$-action $\tau \cdot (s,t,u) = (s-\tau,t,u^\tau)$, for $\tau \in \mathbb{R}$, where $u^\tau(s,t) = u(s+\tau,t)$. Consequently, $\widetilde{Ev}$ descends to a well-defined map $\check{Ev}([s,t,u]) = (t,u(s,t)) \in W(\hat{x},\hat{y})$, $[s,t,u] \in (\mathbb{R} \times S^1 \times \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J))/\mathbb{R}$.

Hence, under the hypotheses of Proposition 5.1.5, $\breve{\pi}$ restricts to a submersion on $\widetilde{W}(\hat{x},\hat{y})$ with fiber diffeomorphic to $\mathbb{R}$. Moreover, if we choose a section $\sigma : \mathcal{M}(\hat{x},\hat{y}; H,J) \to \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J)$, then we may thereby (non-canonically) identify

$$\phi_\sigma : \mathbb{R} \times S^1 \times \mathcal{M}(\hat{x},\hat{y}; H,J) \xrightarrow{\simeq} (\mathbb{R} \times S^1 \times \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J))/\mathbb{R}$$
$$(s,t,[u]) \mapsto [s,t,\sigma([u])].$$

Finally, to understand the behaviour of the foliation $\widetilde{\mathcal{F}}^{\hat{x},\hat{y}}$ under this projection, note that $\ker d\breve{\pi} = \langle \partial_s \rangle$, and that since the tangent space of any leaf of $\widetilde{\mathcal{F}}$ is given by $\langle \partial_s + (\partial_s u)_{u(s,t)}, \partial_t + (\partial_t u)_{u(s,t)} \rangle$, where $\partial_s u \in \ker(d\mathcal{F})_u$, and is therefore nowhere-vanishing whenever u is not an orbit cylinder by our index constraint. So the leaves of the foliation are nowhere tangent to the fibers of the projection map whenever $\hat{x} \neq \hat{y}$. As a consequence, we deduce

Corollary 5.1.7. *Let $\hat{x},\hat{y} \in \widetilde{Per}_0(H)$ satisfy $\mu(\hat{x}),\mu(\hat{y}) \in \{2k-1,2k,2k+1\}$, for some $k \in \mathbb{Z}$ then for any section $\sigma : \mathcal{M}(\hat{x},\hat{y}; H,J) \to \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J)$, as above,*

$$\check{Ev} \circ \phi_\sigma : \mathbb{R} \times S^1 \times \mathcal{M}(\hat{x},\hat{y}; H,J) \mapsto W(\hat{x},\hat{y})$$
$$(s,t,[u]) \mapsto (t,\sigma([u])(s,t))$$

is a smooth embedding. Moreover, writing $\sigma([u]) = u_\sigma$, the partition $\mathcal{F}^{\hat{x},\hat{y}} := \{im\ \check{u}_\sigma\}_{[u]\in\mathcal{M}(\hat{x},\hat{y};H,J)}$ is a smooth 2-dimensional foliation of $W(\hat{x},\hat{y})$ whenever $\hat{x} \neq \hat{y}$.

5.2. The restricted complex associated to a capped braid

Let (H,J) be a non-degenerate Floer pair. To any capped braid $\hat{X} \subseteq \widetilde{Per}_0(H)$, we may associate the submodule $C_*(\hat{X}) := \Lambda_\omega\langle\hat{x}\rangle_{\hat{x}\in\hat{X}}$ of $CF_*(H,J)$, which comes with the projection $\pi^{\hat{X}} : CF_*(H,J) \to C_*(\hat{X})$ associated to any splitting $CF_*(H,J) = C_*(\hat{X}) \oplus C_*(\hat{Y})$, for $\hat{Y} \subseteq \widetilde{Per}_0(H)$ any capped braid such that $Per_0(H) = X \sqcup Y$. $C_*(\hat{X})$ is not generally a subcomplex of $CF_*(H,J)$, since there is no reason that Floer cylinders should only run between strands of $\hat{X}$. However, we will see that if we define the **restricted differential** $\partial^{\hat{X}} := \pi^{\hat{X}} \circ \partial_{H,J}$, then under suitable conditions on $\hat{X}$,

$$CF_*(\hat{X}; H,J) := (C_*(\hat{X}), \partial^{\hat{X}})$$

is a chain complex.

5.2.1. Maximal unlinkedness relative the Morse range

Definition 5.2.1. For any capped braid $\hat{X} \subseteq \widetilde{Per}_0(H)$, we define

$$Pos(\hat{X}) := \{\sigma \in CF_*(H,J) : \forall\hat{\gamma} \in \text{supp}\,\sigma, \forall\hat{x} \in \hat{X},\ \ell(\hat{x},\hat{\gamma}) \geq 0\},$$
$$Pos^*(\hat{X}) := \{\sigma \in Pos(\hat{X}) : \forall\hat{\gamma} \in \text{supp}\,\sigma, \exists\hat{x} \in \hat{X}, \text{such that } \ell(\hat{x},\hat{\gamma}) > 0\}.$$

We define $Neg(\hat{X})$ and $Neg^*(\hat{X})$ in the obvious manner simply by reversing the inequalities in the above.

Definition 5.2.2. Let $\hat{X} \subseteq \widetilde{Per}_0(H)$ be a capped braid for some Hamiltonian H. $\hat{X}$ will be said to be **maximally unlinked** if it is unlinked and if for any $\hat{y} \in \widetilde{Per}_0(H)$ either $\hat{y} \in \hat{X}$ or $\hat{y}$ and $\hat{X}$ are linked. We write $mu(H)$ for the collection of all such capped braids.

Definition 5.2.3. Let $\hat{X} \subseteq \widetilde{Per}_0(H)$ be a capped braid for some Hamiltonian H. $\hat{X}$ of X will be said to be **maximally unlinked relative the Morse range** $\hat{X}$ is unlinked, $\mu(\hat{x}) \in \{-1,0,1\}$ for all $\hat{x} \in \hat{X}$, and moreover if for any $\hat{y} \in \widetilde{Per}_0(H)$ such that $\mu(\hat{y}) \in \{-1,0,1\}$, either $\hat{y} \in \hat{X}$ or $\hat{y}$ and $\hat{X}$ are linked. We write $murm(H)$ for the collection of all capped braids $\hat{X} \subseteq \widetilde{Per}_0(H)$ which are maximally unlinked relative the Morse range.

The next lemma is a direct consequence of the definitions.

Lemma 5.2.4. *Let $\hat{X} \subseteq \widetilde{Per}_0(H)$ be an unlinked braid, then $Pos^*(\hat{X}), Neg^*(\hat{X}) \subseteq \ker\pi^{\hat{X}}$.*

The following situation will occur frequently enough that it will be useful to isolate it as a

Lemma 5.2.5. *Let $(H^{\pm},J^{\pm})$ be Floer regular, $(\mathcal{H},J) \in \mathscr{HJ}$, and $\hat{x}_i \in \widetilde{Per}_0(H^-) \cap \widetilde{Per}_0(H^+)$, $i = 1,\dots k$. Suppose that $\mu(\hat{x}_1) \in \{0,1\}$, $\hat{X} = (\hat{x}_1,\dots,\hat{x}_k)$ is unlinked, and $u^i \in \mathcal{M}(\hat{x}_i,\hat{x}_i;\mathcal{H},J)$ for $i = 1,\dots,k$. If there exists $v \in \mathcal{M}(\hat{x}_1,\hat{x}^+;\mathcal{H},J)$, with $\hat{x}^+ \in \widetilde{Per}_0(H^+)$ such that $\hat{X} \cup \{\hat{x}^+\}$ is linked, then $\hat{x}^+ \in Pos^*(\hat{X})$.*

PROOF. Note that we have $0 = \ell(\hat{x}_1,\hat{x}_i) \leq \ell(\hat{x}^+,\hat{x}_i)$ for $i = 2,\dots,k$ by Lemma 2.2.7 and $0 = b(\hat{x}_1) \leq \ell(\hat{x}_1,\hat{x}^+)$ by Lemma 5.1.1, so we need only show that $\ell(\hat{x}^+,\hat{x}_i) > 0$ for some $i = 1,\dots,k$. We write $h(s) = (u^1_s,\dots,u^k_s,v_s)$, $s \in \mathbb{R}$. h does not induce a braid cobordism, because u_1 and v degenerate to the same orbit as $s \to -\infty$, however Lemma 2.2.7 implies that for $R > 0$ sufficiently large, $h|_{(-R,\infty)}$ induces a braid cobordism from $\hat{X} \cup \{\hat{v}_{-R}\}$ to $\hat{X} \cup \{\hat{y}\}$. Since $b(\hat{x}_1) \geq 0$, there are two possibilities: either $0 < \ell_{-\infty}(u^1,v)$, or $\ell_{-\infty}(u^1,v) = 0$. In the former case, Lemma 2.2.7 immediately implies that $0 < \ell_{\infty}(u^1,v) = \ell(\hat{x}_1,\hat{x}^+)$, and we are done.

We may therefore assume that $\ell_{-\infty}(u^1,v) = 0$. In this case, $\hat{X} \cup \{\hat{v}_{-R}\}$ is unlinked. Indeed by Corollary 2.2.5, $R > 0$ may be chosen such that $\tilde{v}$ has no intersections with $\tilde{u}_i$, $i = 1,\dots,k$ for $s < -R$, and the property of being unlinked is invariant under 0-homotopies, so $\hat{X} \cup \{\hat{v}_{-R}\}$ is unlinked only if $\{\hat{x}_1,\hat{v}_{-R}\}$ is unlinked. But we may take $R > 0$ sufficiently large such that $\hat{v}_{-R} \in \widetilde{\mathcal{L}_0}(\Sigma)$ lies in an exponential neighbourhood of $\hat{x}_1$, and in this neighbourhood the homological linking number reduces to the classical winding number by Proposition 1.3.4, and so that $\{\hat{x}_1,\hat{v}_{-R}\}$ is unlinked follows directly from the fact that the winding number classifies homotopy classes of loops into $\mathbb{R}^2 \setminus 0$.

Thus, $\hat{X} \cup \{\hat{v}_{-R}\}$ is unlinked, while $\hat{X} \cup \{\hat{x}^+\}$ is linked, whence the graphs of some of the strands of h must intersect. Since $\ell_{-\infty}(u^i,u^j) = \ell_{\infty}(u^i,u^j) = \ell(\hat{x}_i,\hat{x}_j) = 0$ for $i \neq j$, it follows from Lemma 2.2.7 that the graphs of u^i and u^j are disjoint for $i \neq j$. Thus, there exists some $i = 1,\dots,k$ such that the graphs of u^i and v intersect, so $0 < \ell_{\infty}(u^i,v) = \ell(\hat{x}_i,\hat{x}^+)$, as claimed. $\square$

Lemma 5.2.6. *Let $\hat{X} \in murm(H)$, then for all $\hat{x} \in \hat{X}$, $\partial_{H,J}\hat{x} \in \mathbb{Z}_2\langle\hat{x}\rangle_{\hat{x}\in\hat{X}} \oplus Pos^*(\hat{X})$.*

PROOF. Let $\hat{x} \in \hat{X}$, and $\hat{y} \in \operatorname{supp} \partial_{H,J}\hat{x}$. Either $\mu(\hat{x}) = -1$ or $\mu(\hat{x}) \in \{0,1\}$. If $\mu(\hat{x}) = -1$, then $b(\hat{x}) = 1$ and so Corollary 5.1.2 implies that $1 \leq \ell(\hat{x},\hat{y})$ while Proposition 5.1.3 implies that $0 \leq \ell(\hat{x}',\hat{y})$ for all $\hat{x}' \in \hat{X}$, $\hat{x}' \neq \hat{x}$, and so $\hat{y} \in Pos^*(\hat{X})$. If $\mu(\hat{x}) \in \{0,1\}$, then Corollary 5.1.2 and Proposition 5.1.3 imply that $0 \leq \ell(\hat{x}',\hat{y})$ for all $\hat{x}' \in \hat{X}$. To see that $\hat{y} \in \hat{X} \cup Pos^*(\hat{X})$, note that either $\hat{X} \cup \{\hat{y}\}$ is unlinked, in which case $\hat{y} \in \hat{X}$ by the maximality of $\hat{X}$, or $\hat{X} \cup \{\hat{y}\}$ is linked, in which case Lemma 5.2.5 directly implies that $\hat{y} \in Pos^*(\hat{X})$. □

Proposition 5.1.3 immediately implies

Lemma 5.2.7. *Let $\hat{X} \subseteq \widetilde{Per}_0(H)$ be any capped braid, then $\partial_{H,J} Pos^*(\hat{X}) \subseteq Pos^*(\hat{X})$.*

Theorem 5.2.8. *Let $\hat{X} \in murm(H)$, then $CF_*(\hat{X}; H,J)$ is a chain complex. That is, $\partial^{\hat{X}} \circ \partial^{\hat{X}} = 0$.*

PROOF. First, consider that Σ is either an aspherical surface, in which case $\Lambda_\omega = \mathbb{Z}_2$, or else a sphere, in which case Σ has minimal Chern number 2, and so in this case $CF_*(\hat{X}; H,J)$ vanishes in any degree congruent to 2 mod 4. Thus, by the Λ_ω-equivariance of the Floer boundary map, it suffices in all cases to prove that $(\partial^{\hat{X}})^2$ vanishes in the Morse range. To wit, by the previous two lemmas, we see that for any $\hat{x} \in \hat{X}$,

$$\partial_{H,J}\hat{x} = \partial^{\hat{X}}\hat{x} + (\pi^{Pos^*(\hat{X})} \circ \partial_{H,J})(\hat{x}),$$

where $\pi^{Pos^*(\hat{X})}$ denotes projection onto $Pos^*(\hat{X})$. Thus, since $\partial_{H,J}^2 = 0$, $(\partial^{\hat{X}})^2\hat{x} + \sigma = 0$, where $\sigma \in Pos^*(\hat{X})$. It follows that $(\partial^{\hat{X}})^2\hat{x} = 0$. □

We will write $HF_*(\hat{X}; H)$ for the homology of the complex $CF_*(\hat{X}; H,J)$ when $\hat{X} \in murm(H)$.

5.2.2. Dominating Morse functions and dominating homotopies

Throughout this section, let (H,J) be a fixed Floer regular pair.

Definition 5.2.9. Let $\hat{X} = \{\hat{x}_1, \ldots, \hat{x}_k\} \subseteq \widetilde{Per}_0(H)$. We will say that a C^2-small Morse function $f \in C^\infty(\Sigma)$ is $\hat{X}$**-dominating** if there exist cappings $w_i : D^2 \to \Sigma$ for x_i, $i = 1, \ldots, k$ such that

(1) $w_i(0) \neq w_j(0)$ for $i \neq j$,
(2) $w_i(0) \in Crit(f)$, for each $i = 1, \ldots, k$ and
(3) $\mu_{CZ}(\widehat{w_i(0)}; f) = \mu_{CZ}(\hat{x}_i; H)$ for $i = 1, \ldots, k$, where $\widehat{w_i(0)}$ denotes the trivially capped constant orbit of f based at $w_i(0)$.

The point $p_i := w_i(0) \in Crit(f)$, will be called the **corresponding critical point** of $\hat{x}_i \in \hat{X}$ for $i = 1, \ldots, k$.

Remark. (1) Remark that the C^2-smallness condition on f in the definition of of $\hat{X}$-dominating functions forces

$$\mu_{CZ}(\widehat{w_i(0)}; f) = \mu_{CZ}(\hat{x}_i; H) \in \{-1, 0, 1\}$$

for $i = 1, \ldots, k$.

(2) It is obvious that $\hat{X}$-dominating Morse functions exist for any $\hat{X} \subset \widetilde{Per_0}(H)$ such that $\mu(\hat{x}) \in \{-1,0,1\}$ for all $\hat{x} \in \hat{X}$. Indeed, take any 0-cobordism from a trivial capped k-braid to $\hat{X}$ and use local models for Morse critical points to introduce critical points with the desired index at the points which are the images of the constant orbits making up the trivial k-braid in question. Of course, there is no reason that a $\hat{X}$-dominating Morse function should have *all* its critical points arising in such a way (although we shall see these latter functions exist when $\hat{X} \in murm(H)$).

Definition 5.2.10. Let $\hat{X} = \{\hat{x}_1, \ldots, \hat{x}_k\} \subset \widetilde{Per_0}(H)$. We will say that an adapted homotopy of Floer data $(\mathcal{H},\mathbb{J})$ is $\hat{X}$**-dominating (as a homotopy to** (H,J)**)** if there exists a Floer regular pair (f,J^-) for f a $\hat{X}$-dominating Morse function such that

(1) $(\mathcal{H},\mathbb{J}) \in \mathscr{HJ}_{reg}(f,J^-;H,J)$, and
(2) $\mathcal{M}(\hat{p}_i,\hat{x}_i;\mathcal{H},\mathbb{J}) \neq \emptyset$ for each $i = 1, \ldots, k$, where here $\hat{p}_i$ is the trivially capped critical point of f corresponding to $\hat{x}_i$.

Whenever f is a $\hat{X}$-dominating Morse function, and (f,J^-) is Floer regular, we will write $\mathscr{HJ}^{\hat{X}}(f,J^-;H,J)$ for the set of all $\hat{X}$-dominating homotopies from (f,J^-) to (H,J).

Similarly, we will say that $(\mathcal{H},\mathbb{J})$ is $\hat{X}$**-dominating (as a homotopy from** (H,J)**)** if there exists a Floer regular pair (f,J^+) for f a $\hat{X}$-dominating Morse function such that

(1) $(\mathcal{H},\mathbb{J}) \in \mathscr{HJ}_{reg}(H,J;f,J^+)$, and
(2) $\mathcal{M}(\hat{x}_i,\hat{p}_i;\mathcal{H},\mathbb{J}) \neq \emptyset$ for each $i = 1, \ldots, k$.

Whenever f is a $\hat{X}$-dominating Morse function, and (f,J^+) is Floer regular, we will write $\mathscr{HJ}^{\hat{X}}(H,J;f,J^+)$ for the set of all $\hat{X}$-dominating homotopies from (H,J) to (f,J^+).

Proposition 5.2.11. *Let* $\hat{X} \subset \widetilde{Per_0}(H)$.

(1) There exist $\hat{X}$*-dominating homotopies of Floer data to* (H,J) *if and only if* $\hat{X}$ *is a positive capped braid and* $\mu(\hat{x}) \in \{-1,0,1\}$ *for all* $\hat{x} \in \hat{X}$.
(2) There exist $\hat{X}$*-dominating homotopies of Floer data from* (H,J) *if and only if* $\hat{X}$ *is a negative capped braid and* $\mu(\hat{x}) \in \{-1,0,1\}$ *for all* $\hat{x} \in \hat{X}$.

PROOF. It is obviously a necessary condition for the existence of a $\hat{X}$-dominating homotopy of Floer data that $\hat{X}$ be a positive capped braid and have all strands lying in the Morse range. Let us show that it is sufficient.

Let $\hat{X} \subset \widetilde{Per}_0(H)$ be a positive capped braid and suppose that all strands lie in the Morse range. Since all strands of $\hat{X}$ lie in the Morse range, there exists a $\hat{X}$-dominating Morse function. Moreover, since $\hat{X}$ is a positive capped braid, we may assume that the capping disks w_i appearing in the definition of a $\hat{X}$-dominating Morse function are such that for $i = 1, \dots, k$, the maps

$$v_i(s,t) := w_i(se^{2\pi it}), (s,t) \in [0,1] \times S^1$$

define a positive 0-cobordism from $\{\widehat{w_1(0)}, \dots, \widehat{w_k(0)}\}$ to $\hat{X}$ and that for $i = 1, \dots, k$ the maps

$$u_i(s,t) := \begin{cases} v_i(0,t), & (s,t) \in (-\infty,0) \times S^1, \\ v_i(s,t), & (s,t) \in [0,1] \times S^1, \\ v_i(1,t), & (s,t) \in (1,\infty) \end{cases}$$

define a pre-model for a continuation cobordism from (f,J^-) to (H,J) for $J^- \in C^\infty(S^1; \mathcal{J}(\Sigma,\omega))$ such that (f,J^-) is Floer regular. By Corollary 3.3.4, it follows that a $\hat{X}$-dominating homotopy to (H,J) exists. The proof of the second point is entirely dual to the preceding proof. $\square$

Remark. Note that if $\hat{X} \subset \widetilde{Per}_0(H)$ is unlinked and has all strands in the Morse range, then $\hat{X}$-dominating homotopies exist both *to* and *from* (H,J).

Proposition 5.2.12. *Let (H,J) be Floer regular and suppose that $\hat{X} \in murm(H)$. If $(\mathcal{H},\mathbb{J}) \in \mathscr{H}\mathscr{J}^{\hat{X}}(f,J^-;H,J)$, then the continuation map*

$$h_{\mathcal{H}} : CF^*(f,J^-) \to CF^*(H,J)$$

satisfies the following.

(1) For all $\hat{x}_i \in \hat{X}_{(1)} \cup \hat{X}_{(-1)}$, $h_{\mathcal{H}}(\hat{p}_i) = \hat{x}_i + \sigma$, where supp $\sigma \subseteq Pos^*(\hat{X})$.
(2) For all $\hat{q} \in \widetilde{Per}_0(f) \setminus \{\hat{p}_1, \dots, \hat{p}_k\}$ with $\mu(\hat{q}) = 1$, $h_{\mathcal{H}}(\hat{q}) \in Pos^*(\hat{X})$.
(3) For all $\hat{q} \in \widetilde{Per}_0(f) \setminus \{\hat{p}_1, \dots, \hat{p}_k\}$ with $\mu(\hat{q}) = -1$, $h_{\mathcal{H}}(\hat{q}) \in \mathbb{Z}_2\langle \hat{x} \rangle_{\hat{X}_{(-1)}} \oplus Pos^*(\hat{X})$.
(4) For all $\hat{q} \in \widetilde{Per}_0(f)$ with $\mu(\hat{q}) \in \{-1,0,1\}$, $h_{\mathcal{H}}(\hat{q}) \in \mathbb{Z}_2\langle \hat{x} \rangle_{\hat{x} \in \hat{X}} \oplus Pos^*(\hat{X})$.

PROOF. We write $\hat{X} = \{\hat{x}_1, \dots, \hat{x}_k\}$. To prove (1) note that since $(\mathcal{H},\mathbb{J})$ is $\hat{X}$-dominating, for each $\hat{x}_i \in \hat{X}_{(1)} \cup \hat{X}_{(-1)}$ has

$$\mathcal{M}(\hat{p}_i,\hat{x}_i; \mathcal{H},\mathbb{J}) \neq \emptyset.$$

By Corollary 2.2.15, this implies that $|\mathcal{M}(\hat{p}_i,\hat{x}_i; \mathcal{H},\mathbb{J})| = 1$.

Next, consider $\hat{x}_j \in \hat{X}$, $i \neq j$ having $\mu(\hat{x}_j) = \mu(\hat{x}_i)$ and suppose that

$$\mathcal{M}(\hat{p}_i,\hat{x}_j;\mathcal{H},\mathbb{J}) \neq \emptyset.$$

If $\mu(\hat{x}_i) = \mu(\hat{x}_j) = 1$, then we consider $u \in \mathcal{M}(\hat{p}_j,\hat{x}_j;\mathcal{H},\mathbb{J})$ and $v \in \mathcal{M}(\hat{p}_i,\hat{x}_j;\mathcal{H},\mathbb{J})$. We note that

$$0 = \ell_{-\infty}(u,v) \leq \ell_{\infty}(u,v) \leq b(\hat{x}_j) = -1,$$

which gives a contradiction. If $\mu(\hat{x}_i) = \mu(\hat{x}_j) = -1$, then we consider $u \in \mathcal{M}(\hat{p}_i,\hat{x}_i;\mathcal{H},\mathbb{J})$ and $v \in \mathcal{M}(\hat{p}_i,\hat{x}_j;\mathcal{H},\mathbb{J})$. We then obtain

$$1 = a(\hat{p}_i) \leq \ell_{-\infty}(u,v) \leq \ell_{\infty}(u,v) = \ell(\hat{x}_i,\hat{x}_j) = 0,$$

another contradiction. So all such moduli spaces are empty.

Finally, let $\hat{y} \in \widetilde{Per}_0(H) \setminus \hat{X}$ have $\mu(\hat{y}) = \mu(\hat{x}_i)$ and suppose that

$$\mathcal{M}(\hat{p}_i,\hat{x}_j;\mathcal{H},\mathbb{J}) \neq \emptyset.$$

In this case, Lemma 5.2.5 implies that $\hat{y} \in Pos^*(\hat{X})$. This proves (1).

To prove item (2),let $\hat{x}_i \in \hat{X}$ have $\mu(\hat{x}_i) = \mu(\hat{q}) = 1$ and suppose that

$$\mathcal{M}(\hat{q},\hat{x}_i;\mathcal{H},\mathbb{J}) \neq \emptyset.$$

Let $u \in \mathcal{M}(\hat{p}_i,\hat{x}_i;\mathcal{H},\mathbb{J})$ and $v \in \mathcal{M}(\hat{q},\hat{x}_i;\mathcal{H},\mathbb{J})$ arguing as we have above together with the facts that $\ell(\hat{q},\hat{p}_i) = 0$ while $b(\hat{x}_i) = -1$ gives a contradiction, so all such moduli spaces are empty. If $\mathcal{M}(\hat{q},\hat{y};\mathcal{H},\mathbb{J}) \neq \emptyset$, then taking $v \in \mathcal{M}(\hat{q},\hat{y};\mathcal{H},\mathbb{J})$ and for each $\hat{x}_i \in \hat{X}$, letting $u_i \in \mathcal{M}(\hat{p}_i,\hat{x}_i;\mathcal{H},\mathbb{J})$, we see that $\{u_i\}_{i=1}^k \cup \{v\}$ is a regular model for a cobordism from (f,J^-) to (H,J), and so in particular induces a positive cobordism. That $\ell(\hat{x}_i,\hat{y}) > 0$ for some $i = 1,\ldots,k$ follows from the fact that since $\hat{X} \in murm(H)$ and $\mu(\hat{y}) = 1$, $\hat{X} \cup \{\hat{y}\}$ is linked, and so the above cobordism must have some intersections.

The proof of (3) is word-for-word the same as the proof of the proof of (2), only in this case, it is possible that moduli spaces of the form $\mathcal{M}(\hat{q},\hat{x}_i;\mathcal{H},\mathbb{J})$ are non-empty for $\hat{x}_i \in \hat{X}_{(-1)}$.

To prove item (4), we note first that items (1) - (3) suffice to establish (4) in the event that $\mu(\hat{q}) = \pm 1$, so we can, and do, assume that $\mu(\hat{q}) = 0$. Let $\hat{\gamma} \in \widetilde{Per}_0(H) \setminus \hat{X}$ and suppose that $\mathcal{M}(\hat{q},\hat{\gamma};\mathcal{H},\mathbb{J}) \neq \emptyset$. We will show that $\hat{\gamma} \in Pos^*(\hat{X})$. There are two cases: either $\hat{q} = \hat{p}_i$ for some $\hat{x}_i \in \hat{X}_{(0)}$, or else $\hat{q} \neq \hat{p}_i$ for any such $\hat{x}_i$. Let us first suppose that $\hat{q} = \hat{p}_i$ for some

$\hat{x}_i \in \hat{X}_{(0)}$. In this case, Lemma 5.2.5 immediately gives that $\hat{\gamma} \in Pos^*(\hat{X})$. If $\hat{q} \neq \hat{p}_i$ for any $\hat{x}_i \in \hat{X}_{(0)}$ then consideration of the braid cobordism between the unlinked capped braid $\hat{X} \cup \{\hat{q}\}$ and the linked capped braid $\hat{X} \cup \{\hat{\gamma}\}$ given $\{u_i\}_{i=1}^k \cup \{v\}$ for $u_i \in \mathcal{M}(\hat{p}_i,\hat{x}_i;\mathcal{H},\mathbb{J})$, $i = 1,\ldots,k$ and $v \in \mathcal{M}(\hat{q},\hat{\gamma};\mathcal{H},J)$, yields (since $\hat{X}$ is maximally unlinked relative the Morse range and so $\hat{X} \cup \{\hat{\gamma}\}$ is linked) that the graph of v in $\mathbb{R} \times S^1 \times \Sigma$ must intersect the graph of some u_i at least once, and any intersections may only contribute positively to the change in linking number as all the maps solve Equation 2.1.2 for $(\mathcal{H},\mathbb{J})$. Consequently, $\hat{\gamma} \in Pos^*(\hat{X})$. □

Proposition 5.2.13. *If $(\mathcal{H}',\mathbb{J}') \in \mathscr{H}\mathscr{J}^{\hat{X}}(H,J;f,J^-)$, then $\sigma \in \ker h_{\mathcal{H}'}$ for all $\sigma \in Pos^*(\hat{X})$ such that*

$$-N+1 \leq \mu(\hat{\gamma}) \leq N-1,\ \forall \hat{\gamma} \in \textit{supp}\ \sigma.$$

(Here N denotes the minimal Chern number of the surface).

PROOF. Remark that if $\mu(\hat{\gamma})$ satisfies the stated bounds, then the only moduli spaces that may contribute to $h_{\mathcal{H}'}$ are of the form $\mathcal{M}(\hat{\gamma},\hat{q};\mathcal{H}',\mathbb{J}')$ for $\hat{q}$ a trivially capped critical point of f. We claim that when $\hat{\gamma} \in Pos^*(\hat{X})$, then $\mathcal{M}(\hat{\gamma},\hat{q};\mathcal{H}',\mathbb{J}') = \emptyset$ for all such $q \in Crit(f)$. Indeed, suppose for a contradiction that there exists $u \in \mathcal{M}(\hat{\gamma},\hat{q};\mathcal{H}',\mathbb{J}')$ for some $q \in Crit(f)$. Since $\hat{\gamma} \in Pos^*(\hat{X})$, there exists some $\hat{x}_i \in \hat{X}$ such that $\ell(\hat{x}_i,\hat{\gamma}) > 0$. Moreover, by hypothesis, $(\mathcal{H}',\mathbb{J}')$ is $\hat{X}$-dominating, and so there exists $v \in \mathcal{M}(\hat{x}_i,\hat{p}_i;\mathcal{H}',\mathbb{J}')$. Consequently

$$0 < \ell(\hat{x}_i,\hat{\gamma}) = \ell_{-\infty}(v,u) \leq \ell_{\infty}(v,u),$$

but either $\hat{p}_i \neq \hat{q}$, in which case $\ell_\infty(v,u) = \ell(\hat{p}_i,\hat{q}) = 0$, or else $\hat{p}_i = \hat{q}$, in which case $\ell_\infty(v,u) \leq a(\hat{p}_i)$ and $a(\hat{p}_i) \leq 0$, since $\mu(\hat{p}_i) \in \{-1,0,1\}$. In either case, $\ell_\infty(v,u) \leq 0$, which gives a contradiction. □

Proposition 5.2.14. *(1) Let $(\mathcal{H},\mathbb{J}) \in \mathscr{H}\mathscr{J}^{\hat{X}}(f,J^-;H,J)$, then the map $\pi^{\hat{X}} \circ h_{\mathcal{H}}$ is a morphism of chain complexes.*
(2) Let $(\mathcal{H}',\mathbb{J}') \in \mathscr{H}\mathscr{J}^{\hat{X}}(H,J;f,J^+)$, then the map $h_{\mathcal{H}'}|_{CF_*(\hat{X};H,J)}$ is a morphism of chain complexes.

PROOF. As in the proof of Theorem 5.2.8, it suffices to prove that the maps are chain maps in the Morse range. Attending first the map $\pi^{\hat{X}} \circ h_{\mathcal{H}}$, we therefore consider $\hat{p} \in \widetilde{Per}_0(f)_{(k)}$ for $k \in \{-1,0,1\}$. We note that $h_{\mathcal{H}}(\hat{p}) \in \mathbb{Z}_2\langle\hat{x}\rangle_{\hat{x}\in\hat{X}} \oplus Pos^*(\hat{X})$ by Proposition 5.2.12 and $\partial_{H,J} Pos^*(\hat{X}) \subseteq$

$Pos^*(\hat{X}) \subseteq \ker \pi^{\hat{X}}$ by Lemmas 5.2.4 and 5.2.7. Consequently, we see that

$$\begin{aligned}(\partial^{\hat{X}} \circ h_{\mathcal{H}})(\hat{p}) &= (\partial^{\hat{X}} \circ \pi^{\hat{X}} \circ h_{\mathcal{H}})(\hat{p}) + (\partial^{\hat{X}} \circ \pi^{Pos^*(\hat{X})} \circ h_{\mathcal{H}})(\hat{p}) \\ &= (\partial^{\hat{X}} \circ \pi^{\hat{X}} \circ h_{\mathcal{H}})(\hat{p}) + (\pi^{\hat{X}} \circ \partial_{H,J^+} \circ \pi^{Pos^*(\hat{X})} \circ h_{\mathcal{H}})(\hat{p}) \\ &= (\partial^{\hat{X}} \circ \pi^{\hat{X}} \circ h_{\mathcal{H}})(\hat{p})\end{aligned}$$

Thus, since $h_{\mathcal{H}}$ is a chain map with respect to the full Floer differential, we compute

$$(\pi^{\hat{X}} \circ h_{\mathcal{H}})(\partial_{f,J^-}\hat{p}) = (\pi^{\hat{X}} \circ \partial_{H,J^+})(h_{\mathcal{H}}(\hat{p})) = (\partial^{\hat{X}} \circ h_{\mathcal{H}})(\hat{p}) = \partial^{\hat{X}}((\pi^{\hat{X}} \circ h_{\mathcal{H}})(\hat{p})),$$

which shows that $\pi^{\hat{X}} \circ h_{\mathcal{H}}$ is a chain map.

To see that $h_{\mathcal{H}'}|_{CF_*(\hat{X};H,J)}$ is a chain map, consider $\hat{x} \in \hat{X}$. It will suffice by the Λ_ω-equivariance of Floer continuation maps to show that

$$h_{\mathcal{H}'}(\partial^{\hat{X}}\hat{x}) = (\partial_{f,J^-} \circ h_{\mathcal{H}'})(\hat{x}).$$

Using the fact that $h_{\mathcal{H}'}$ is a chain map with respect to the usual Floer differential and that

$$\partial_{H,J}\hat{x} = \partial^{\hat{X}}\hat{x} + \sigma,$$

for $\sigma \in Pos^*(\hat{X})$, we compute

$$\begin{aligned}(\partial_{f,J^-} \circ h_{\mathcal{H}'})(\hat{x}) &= h_{\mathcal{H}'}(\partial_{H,J}\hat{x}) \\ &= h_{\mathcal{H}'}(\partial^{\hat{X}}\hat{x}) + h_{\mathcal{H}'}(\sigma).\end{aligned}$$

Note that $\sigma \in CF_k(H,J)$ for $k \in \{0,-1,-2\}$. If $k = -2$, then $h_{\mathcal{H}'}(\sigma) = 0$, as $CF_{-2}(f,J^-) = 0$, while if $k \in \{0,-1\}$, then since $\sigma \in Pos^*(\hat{X})$, Proposition 5.2.13 implies that $h_{\mathcal{H}'}(\sigma) = 0$. In either case, we see that

$$(\partial_{f,J^-} \circ h_{\mathcal{H}'})(\hat{x}) = h_{\mathcal{H}'}(\partial^{\hat{X}}\hat{x}),$$

as desired. □

Lemma 5.2.15. *For any* $(\mathcal{H},\mathbb{J}) \in \mathscr{H}\mathscr{J}^{\hat{X}}(f,J^-;H,J)$ *and any* $(\mathcal{H}',\mathbb{J}') \in \mathscr{H}\mathscr{J}^{\hat{X}}(H,J;f,J^+)$*, the map* $h_{\mathcal{H}'} \circ (\pi^{\hat{X}} \circ h_{\mathcal{H}})$ *induces the identity map on homology.*

PROOF. Once more, it suffices to show that $h_{\mathcal{H}'} \circ (\pi^{\hat{X}} \circ h_{\mathcal{H}})$ induces an isomorphism on homology in degrees lying in the Morse range. Since, for any $\hat{p} \in CF_k(f,J^-)$, with $k \in \{-1,0,1\}$, we have that $h_{\mathcal{H}}(\hat{p}) \in \mathbb{Z}_2\langle\hat{x}\rangle_{\hat{x}\in\hat{X}} \oplus Pos^*(\hat{X})$, and $Pos^*(\hat{X}) \subseteq \ker\pi^{\hat{X}} \cap \ker h_{\mathcal{H}'}$, so we compute

$$\begin{aligned}(h_{\mathcal{H}'} \circ h_{\mathcal{H}})(\hat{p}) &= h_{\mathcal{H}'}((\pi^{\hat{X}} \circ h_{\mathcal{H}})(\hat{p}) + (\pi^{Pos^*(\hat{X})} \circ h_{\mathcal{H}})(\hat{p})) \\ &= (h_{\mathcal{H}'} \circ \pi^{\hat{X}} \circ h_{\mathcal{H}})(\hat{p}) \\ &= h_{\mathcal{H}'}|_{CF_*(\hat{X};H,J^+)} \circ (\pi^{\hat{X}} \circ h_{\mathcal{H}})(\hat{p}).\end{aligned}$$

But it is a standard fact in Floer theory that $h_{\mathcal{H}'} \circ h_{\mathcal{H}}$ induces the identity map on homology, and so it must be that case that the composition

$$HF_k(f) \xrightarrow{(\pi^{\hat{X}} \circ h_{\mathcal{H}})_*} HF_k(\hat{X};H) \xrightarrow{(h_{\mathcal{H}'})_*} HF_k(f)$$

is the identity map for $k \in \{-1,0,1\}$. ☐

We will use the above fact to bootstrap ourselves into a much finer-grained understanding of the structure of $CF_*(\hat{X};H,J)$ in the coming section.

5.3. Construction and properties of $\mathcal{F}^{\hat{X}}$

The purpose on this section is to prove the existence of the foliation in the following theorem.

Theorem 5.3.1. *Let $H \in C^\infty(S^1 \times \Sigma)$ be a non-degenerate Hamiltonian, and let $J \in C^\infty(S^1;\mathcal{J}_\omega(\Sigma))$ be such that (H,J) is Floer regular. For any capped braid $\hat{X} \in murm(H)$, we may construct an oriented singular foliation $\mathcal{F}^{\hat{X}}$ of $S^1 \times \Sigma$ with the following properties*

1. *The singular leaves of $\mathcal{F}^{\hat{X}}$ are precisely the graphs of the orbits in $\hat{X}$.*
2. *The regular leaves are annuli parametrized by maps*

$$\begin{aligned}\check{u} : \mathbb{R} \times S^1 &\to S^1 \times \Sigma \\ (s,t) &\mapsto (t,u(s,t)).\end{aligned}$$

for $u \in \widetilde{\mathcal{M}}(\hat{x},\hat{y};H,J)$, for $\hat{x},\hat{y} \in \hat{X}$.

3. *The vector field $\check{X}^H(t,z) = \partial_t \oplus X^H_t(z)$ is positively transverse to every regular leaf of $\mathcal{F}^{\hat{X}}$.*

The positive transversality property will be established in Section 5.3.2. Our construction of the singular foliation in Theorem 5.3.1 proceeds by establishing that a generic point in $S^1 \times \Sigma$ lies inside the foliated sector $W(\hat{x},\hat{y})$ for some $\hat{x} \in \hat{X}_{(1)},\hat{y} \in \hat{X}_{(-1)}$. The remaining points lie in the closure of these sectors, and so lie either on leaves parametrized by broken cylinders, or the graphs of orbits in X. To establish existence of the requisite leaves, we make use of the cap action of a point on the Floer complex, first introduced in detail in [**15**] (see also [**25**]).

Given a singular homology class $\alpha \in H_k(M;\mathbb{Z}_2)$, we represent α by a smooth cycle $\alpha^\# : \cup\Delta^k \to M$, and for any (H,J) and any $t \in S^1$, we may consider, for any $\hat{x},\hat{y} \in \widetilde{Per}_0(H)$, the moduli space

$$\mathcal{M}^{\alpha^\#,t}(\hat{x},\hat{y};H,J) := \{u \in \widetilde{\mathcal{M}}(\hat{x},\hat{y};H,J) : u(0,t) \in \text{im } \alpha^\#\}.$$

For $t \in S^1$, we will say that the smooth chain $\alpha^\#$ is $(H,J;t)$**-generic** when the evaluation map $ev_t(u,q) := (u(0,t),\alpha^\#(q)) \in M \times M$, $(u,q) \in \widetilde{\mathcal{M}}(\hat{x},\hat{y};H,J) \times \cup\Delta^k$, is transversal to the diagonal whenever $\mu(\hat{x}) - \mu(\hat{y}) \leq (2n-k)+1$. Such chains form a residual set for fixed H if we permit generic perturbations of J, and in such a case we define the **cap product** of α on $HF_*(H)$ (at time t) at the chain level by defining, for $\hat{x} \in \widetilde{Per}_0(H)$,

$$\alpha^\# \cap_t \hat{x} := \sum_{\substack{\hat{y}\in\widetilde{Per}_0(H):\\ \mu(\hat{x})-\mu(\hat{y})=2n-k}} n^{\alpha^\#,t}(\hat{x},\hat{y})\hat{y},$$

where $n^{\alpha^\#,t}(\hat{x},\hat{y})$ is the mod 2 count of the number of elements in $\mathcal{M}^{\alpha^\#,t}(\hat{x},\hat{y};H,J)$. The cap action descends to homology, and is independent at the homology level of all choices. Moreover, for generic adapted homotopies of Floer data, the cap action commutes with continuation maps *at the chain level*. That is, for generic $(\mathcal{H},J)$ we have

$$h_{\mathcal{H}}(\alpha^\# \cap_t \hat{x}) = \alpha^\# \cap_t h_{\mathcal{H}}(\hat{x}), \tag{5.3.1}$$

whenever the Floer pairs $(H^\pm,J^\pm)$ at the ends of the homotopy are such that the relevant moduli spaces are transversal. It follows from the above that, under the identification of $HF_*(H)$ with $QH_{*+n}(M)$, the cap action on $HF_*(H)$ is identified with the standard cap action of the homology of M on its quantum homology.
A rather important point for us will be that the cap action interacts nicely with respect to the chain maps $\pi^{\hat{X}} \circ h_{\mathcal{H}}$ and $h_{\mathcal{H}'}|_{CF_*(\hat{X};H,J)}$ introduced in the previous section.

Proposition 5.3.2. *Let* $\hat{X} \in murm(H)$, $(\mathcal{H},\mathbb{J}) \in \mathscr{HJ}^{\hat{X}}(f,J^-;H,J)$, $(\mathcal{H}',\mathbb{J} \in \mathscr{HJ}^{\hat{X}}(H,J;f,J^+)$ *and suppose that* $\alpha^\#$ *represents* α *as above and* $\alpha^\#$ *is both* $(\mathcal{H},\mathbb{J};t)$*-generic and* $(\mathcal{H}',\mathbb{J}';t)$*-generic, then*

$$(\pi^{\hat{X}} \circ h_{\mathcal{H}})(\alpha^\# \cap_t \hat{p}) = \pi^{\hat{X}}(\alpha^\# \cap_t (\pi^{\hat{X}} \circ h_{\mathcal{H}})(\hat{p})),\ \forall \hat{p} \in \widetilde{Per}_0(f) \text{ and} \tag{5.3.2}$$

$$(h_{\mathcal{H}'} \circ \pi^{\hat{X}})(\alpha^\# \cap_t \pi^{\hat{X}}(\hat{y})) = \alpha^\# \cap_t (h_{\mathcal{H}'} \circ \pi^{\hat{X}})(\hat{y}),\ \forall \hat{y} \in \widetilde{Per}_0(H) \tag{5.3.3}$$

PROOF. A straightforward computation shows that the map induced by capping with a smooth cycle $\alpha^{\#}$ at time t is Λ_ω-equivariant at the chain level, in the sense that for any $A \in \Gamma_\omega$

$$\alpha^{\#} \cap_t (e^A \cdot \hat{x}) = e^A \cdot (\alpha^{\#} \cap_t \hat{x}).$$

Since the maps $\pi^{\hat{X}}$ and $h_{\mathcal{H}}$ are also Λ_ω-equivariant and f is by hypothesis C^2-small, so that

$$CF_*(f,J^-) = Crit(f) \otimes \Lambda_\omega,$$

as Λ_ω-modules, we may reason similarly as in the proof of Theorem 5.2.8, and reduce to the case where $\mu(\hat{p}) \in \{-1,0,1\}$ (since any $\hat{q} \in \widetilde{Per}_0(f)$ may be written as $e^A \cdot \hat{p}$ for some $\hat{p}$ with Conley-Zehnder index in the Morse range, and so if the desired relations hold in the Morse range, then by Λ_ω-equivariance of the involved maps, the desired relations hold for all $\hat{q} \in \widetilde{Per}_0(f)$). Thus, we may, and do, suppose going forward that $\mu(\hat{p}) \in \{-1, 0,1\}$.

Note that Proposition 5.2.12 implies that we may write $h_{\mathcal{H}}(\hat{p}) = \sigma + \beta$ for $\sigma \in CF_*(\hat{X}; H,J)$ and $\beta \in Pos^*(\hat{X})$, so that the right-hand side of $(5.3.2)$ may be computed as

$$\begin{aligned}\pi^{\hat{X}}(\alpha^{\#} \cap_t (\pi^{\hat{X}} \circ h_{\mathcal{H}})(\hat{p})) &= \pi^{\hat{X}}(\alpha^{\#} \cap_t \pi^{\hat{X}}(\sigma + \beta)) \\ &= \pi^{\hat{X}}(\alpha^{\#} \cap_t \sigma).\end{aligned}$$

In order to establish that this agrees with the quantity on the left-hand side of $(5.3.2)$, the central point is to remark that capping with $\alpha^{\#}$ preserves $Pos^*(\hat{X})$, because linking in the Floer complex is non-decreasing along Floer cylinders. Indeed, if $\hat{y} \in Pos^*(\hat{X})$ and there exists $u \in \mathcal{M}^{\alpha^{\#},t}(\hat{y},\hat{y}'; H,J)$ for some $\hat{y}' \in \widetilde{Per}_0(H)$, then for each $\hat{x} \in \hat{X}$, Lemma 2.2.7 with $v(s,t) = x(t)$ implies that $\ell(\hat{y},\hat{x}) \leq \ell(\hat{y}',\hat{x})$. Thus $\alpha^{\#} \cap_t \beta \in Pos^*(\hat{X})$ and the left-hand side of $(5.3.2)$ may be computed as

$$(\pi^{\hat{X}} \circ h_{\mathcal{H}})(\alpha^{\#} \cap_t \hat{p}) = \pi^{\hat{X}}(\alpha^{\#} \cap_t \sigma + \alpha^{\#} \cap_t \beta) = \pi^{\hat{X}}(\alpha^{\#} \cap_t \sigma),$$

where we use Lemma 5.2.4 in the last equality. This establishes Equation $(5.3.2)$. Equation $(5.3.3)$ is proved similarly, needing only the additional remark that for $\hat{x} \in \hat{X}$, $\alpha^{\#} \cap_t \hat{x} \in \mathbb{Z}_2\langle \hat{x} \rangle_{\hat{x} \in \hat{X}} \oplus Pos^*(\hat{X})$, which follows by the same reasoning as above, using that $b(\hat{x}) \geq 0$ when $\mu(\hat{x})$ lies in the Morse range. $\square$

The following proposition is essentially tautological.

Proposition 5.3.3. *Let (H,J) be Floer regular, $t \in S^1$, and suppose that $p \in \Sigma$ is $(H,J;t)$-generic (for the point class in homology). Then $p \in W(\hat{x},\hat{y})$ implies that $\mu(\hat{x}) - \mu(\hat{y}) \geq 2$, and $\hat{y} \in$ supp $(p \cap_t \hat{x})$ if and only if $(t,p) \in W(\hat{x},\hat{y})$.*

Combining the above with Corollary 5.1.7, allows us to conclude

Corollary 5.3.4. *Suppose that $\mu(\hat{x}) = 2k+1$ for $k \in \mathbb{Z}$ and p is $(H,J;t)$-generic, then $\hat{y} \in$ supp $(p \cap_t \hat{x})$ if and only if there exists an open neighbourhood of $(t,p) \in S^1 \times \Sigma$ which is foliated by leaves of $\mathcal{F}^{\hat{x},\hat{y}}$.*

Recall from Section 5.1 that $W(\hat{x},\hat{y}) = \{(t,u(s,t)) \in S^1 \times \Sigma : u \in \widetilde{\mathcal{M}}(\hat{x},\hat{y};H,J)\}$, and write $\mathcal{W}(\hat{X})$ for the union of all $W(\hat{x},\hat{y})$ where $\hat{x} \in \hat{X}_{(1)}$, $\hat{y} \in \hat{X}_{(-1)}$.

Lemma 5.3.5. *Let (H,J) be Floer regular and $\hat{X} \in murm(H)$, then $\mathcal{W}(\hat{X})$ is open and dense in $S^1 \times \Sigma$.*

PROOF. Let $(\mathcal{H},\mathbb{J}) \in \mathscr{HJ}^{\hat{X}}(f,J^-;H,J)$. We fix $t \in S^1$ arbitrarily and let $p \in \Sigma$ be $(\mathcal{H},\mathbb{J};t)$-generic. We let $\sigma \in CF_1(f,J^-)$ represent the fundamental class $[\Sigma] \in QH_2(\Sigma) \simeq HF_1(f)$, and we note that we must have $[p \cap_t \sigma] = [pt] \in QH_0(\Sigma)$, which is in particular not 0. But Lemma 5.2.15 implies that $(\pi^{\hat{X}} \circ h_{\mathcal{H}})_*$ is injective on homology, and so

$$0 \neq (\pi^{\hat{X}} \circ h_{\mathcal{H}})(p \cap_t \sigma) = \pi^{\hat{X}}(p \cap_t (\pi^{\hat{X}} \circ h_{\mathcal{H}})(\sigma)),$$

and this implies that for every such generic p, there must exist some $\hat{x} \in$ supp $(\pi^{\hat{X}} \circ h_{\mathcal{H}})(\sigma) \subseteq \hat{X}_{(1)}$ and some $\hat{y} \in \hat{X}_{(-1)}$ such that $\hat{y} \in$ supp $p \cap_t \hat{x}$. Hence for every $t \in S^1$, there is a generic set of $p \in \Sigma$ such that p is $(\mathcal{H},\mathbb{J};t)$-generic and so (t,p) lies inside the open 3-dimensional connecting submanifold $W(\hat{x},\hat{y})$ for some such $\hat{x}, \hat{y} \in \hat{X}$, which proves the lemma. $\square$

Lemma 5.3.6. *Let (H,J) be Floer regular, and $\hat{X} \subseteq \widetilde{Per}_0(H)$ any unlinked capped braid. Then for any $\hat{\gamma} \in \widetilde{Per}_0(H)$ such that $\mathcal{M}(\hat{x}^+,\hat{\gamma};H,J) \times \mathcal{M}(\hat{\gamma},\hat{x}^-;H,J) \neq \emptyset$ for some $\hat{x}^+ \in \hat{X}_{(1)}$, $\hat{x}^- \in \hat{X}_{(-1)}$, $\hat{X}$ and $\hat{\gamma}$ are unlinked.*

PROOF. Suppose that $\hat{X} \cup \{\hat{\gamma}\}$ is linked, then Lemma 5.2.5 implies that $\hat{\gamma} \in Pos^*(\hat{X})$, and so $\ell(\hat{\gamma},\hat{x}) > 0$ for some $\hat{x} \in \hat{X}$, but then Proposition 5.1.3 implies that $\ell(\hat{x}^-,\hat{x}) > 0$, which contradicts the assumption that $\hat{X}$ is unlinked. $\square$

Inductively applying Lemma 5.3.6 yields

Corollary 5.3.7. *Suppose that (H,J) is Floer regular, and let $\hat{X} \subseteq \widetilde{Per}_0(H)$ be such that $\hat{X}$ is unlinked and $\hat{X} = \hat{X}_{(1)} \cup \hat{X}_{(-1)}$, then $\hat{X}$ and $\hat{\Upsilon}$ are unlinked, where $\hat{\Upsilon}$ is the capped braid consisting of all $\hat{\gamma} \in \widetilde{Per}_0(H)$ with $\mu(\hat{\gamma}) = 0$ such that $\mathcal{M}(\hat{x}^+,\hat{\gamma};H,J) \times \mathcal{M}(\hat{\gamma},\hat{x}^-;H,J) \neq \emptyset$, $\hat{x}^\pm \in \hat{X}$.*

Finally, we are ready to prove the existence of the advertised foliation. We write

$$\mathcal{M}^{H,J}(\hat{X}) := \bigcup_{\hat{x},\hat{y}\in\hat{X}} \mathcal{M}(\hat{x},\hat{y};H,J).$$

Note that we include the case where $\hat{x} = \hat{y}$ in the above union. In such a case $[u] \in \mathcal{M}(\hat{x},\hat{x};H,J)$ may simply be identified with the loop x.

Theorem 5.3.8 (Existence part of Theorem 5.3.1)**.** *Let (H,J) be a non-degenerate Floer pair, and $\hat{X} \in murm(H)$, then the collection of submanifolds $\mathcal{F}^{\hat{X}} := \cup_{[u]\in\mathcal{M}^{H,J}(\hat{X})}\{im\ \breve{u}\}$ forms a Stefan-Sussmann foliation of $S^1 \times \Sigma$.*

PROOF. We adapt a strategy used in [**11**] that shows that the foliation $\widetilde{\mathcal{F}}^{\hat{X}}$ with leaves given by the graphs $\tilde{u}$ of all the $u \in \widetilde{\mathcal{M}}(\hat{x},\hat{y};H,J)$, $\hat{x},\hat{y} \in \hat{X}$, is a smooth 2-dimensional foliation if $\mathbb{R}\times S^1\times\Sigma$, from which it follows immediately that $\mathcal{F}^{\hat{X}}$ is a Steffan-Sussmann foliation. Indeed, in this event, $\mathcal{F}^{\hat{X}}$ integrates the distribution $\mathcal{D}^{\hat{X}} = \breve{\pi}_*\widetilde{\mathcal{D}}^{\hat{X}}$, where $\widetilde{\mathcal{D}}^{\hat{X}}$ is the distribution integrated by $\widetilde{\mathcal{F}}^{\hat{X}}$, and this realizes $\mathcal{D}^{\hat{X}}$ in a way that is manifestly smooth in the sense of generalized distributions (see Definition 5.5.2).

To see that $\widetilde{\mathcal{F}}^{\hat{X}}$ is a smooth foliation, note that by Lemma 5.3.5, the set $\mathcal{W}(\hat{X})$ is open and dense in $S^1\times\Sigma$. This implies, by the $\mathbb{R}$-invariance of solutions to Equation 2.1.1, that the set of points $\widetilde{W}(\hat{X})$ lying on the graph $\tilde{u}$ of some $u \in \widetilde{\mathcal{M}}(\hat{x},\hat{y};H,J)$, $\hat{x} \in \hat{X}_{(1)}, \hat{y} \in \hat{X}_{(-1)}$ is open and dense in $\mathbb{R}\times S^1\times\Sigma$. Consequently, the partition $\bigcup \widetilde{\mathcal{F}}^{\hat{x},\hat{y}}$, where the union runs over all $\hat{x} \in \hat{X}_{(1)}, \hat{y} \in \hat{X}_{(-1)}$, gives a smooth foliation of an open, dense set of $\mathbb{R}\times S^1\times\Sigma$. Consequently we may argue just as in [**11**] in the paragraphs following the proof of lemma 6.10 (p. 231-232); all of the remaining leaves in $\widetilde{\mathcal{F}}^{\hat{X}}$ are graphs of constant orbits or of cylinders u which connect orbits of index difference equal to 1. In either case, by standard compactness theorems of Floer theory those graphs which form the leaves of the foliation of $\widetilde{\mathcal{W}}(\hat{X})$ converge modulo reparametrization in the C^∞_{loc}-topology either to the graphs $(s,t) \mapsto (s,t,x(t))$ of the orbits x for $\hat{x} \in \hat{X}$, or to graphs of cylinders connecting orbits of index difference 1, which come in pairs $(u,v) \in \widetilde{\mathcal{M}}(\hat{x},\hat{\gamma};H,J) \times \widetilde{\mathcal{M}}(\hat{\gamma},\hat{y};H,J)$, $\hat{x} \in \hat{X}_{(1)}, \hat{\gamma} \in \hat{X}_{(0)}, \hat{y} \in \hat{X}_{(-1)}$. By Corollary 5.3.7 and Lemma 5.3.6, the capped braid formed by the collection of all the $\hat{\gamma} \in \widetilde{Per}_0(H)$ on which such pairs break are unlinked with $\hat{X}$, and so lie in $\hat{X}$ by maximality. Consequently, the graphs of such broken trajectories cannot intersect, nor can they intersect any leaf of $\widetilde{\mathcal{F}}^{\hat{X}}$ in the dense set $\widetilde{\mathcal{W}}(\hat{X})$. Since every point in $\mathbb{R}\times S^1\times\Sigma$ lies in the closure of $\widetilde{\mathcal{W}}(\hat{X})$, every such point much lie on the graph of an orbit in X or the graph of such a broken cylinder. It follows that the union of all the leaves in $\widetilde{\mathcal{F}}^{\hat{X}}$ thus fits together into a smooth foliation on all of $\mathbb{R} \times S^1 \times \Sigma$, and so the theorem follows. $\square$

5.3.1. $\mathcal{F}^{\hat{X}}$ as negative gradient flow-lines of the restricted action functional

For $\hat{X} \in murm(H)$, denote

$\mathfrak{M}_{\hat{X}} = \mathfrak{M}_{\hat{X};H,J} := \{\hat{\alpha} \in \widetilde{\mathcal{L}_0}(\Sigma) : \exists \hat{x}, \hat{y} \in \hat{X}, \exists u \in \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J),$ such that $\hat{u}_s = \hat{\alpha},$ for some $s \in \mathbb{R}\}$

Proposition 5.3.9. *The map $Ev : S^1 \times \mathfrak{M}_{\hat{X}} \to S^1 \times \Sigma$, given by $Ev(t,\hat{\alpha}) = (t,\alpha(t))$ is a diffeomorphism.*

Proof. The generalized distribution $\mathcal{D}^{\hat{X}}_{(t,u(s,t))} = \langle \partial_s u, \partial_t \oplus \partial_u \rangle$ which is integrated by $\mathcal{F}^{\hat{X}}$ contains the one-dimensional distribution $\mathcal{D}^{\mathfrak{M}}_{(t,u(s,t))} = \langle \partial_t \oplus \partial_t u \rangle$, which is easily seen to be smooth near the singular fibers by employing the local model for leaves of a Stefan-Sussmann foliation of Section 5.5. Consequently, $\mathcal{D}^{\mathfrak{M}}$ is a smooth foliation which integrates precisely to the graphs of the maps $\alpha : S^1 \to \Sigma$ for $\hat{\alpha} \in \mathfrak{M}_{\hat{X}}$. This is obviously equivalent to the proposition. $\square$

Definition 5.3.10. For $\hat{X} \in murm(H)$, define **the ($\hat{X}$-)restricted action functional** $A^{\hat{X}} \in C^\infty(S^1 \times \Sigma)$ by $A^{\hat{X}} := \mathcal{A}_H \circ Ev^{-1}$. Additionally, for each $t \in S^1$, we define $A^{\hat{X}}_t := \iota_t^* A^{\hat{X}}$, where $\iota_t : \Sigma \hookrightarrow S^1 \times \Sigma$ is the inclusion of the fiber over $t \in S^1$.
Note that each $A^{\hat{X}}_t$ is automatically Morse, since the Hessian of $A^{\hat{X}}_t$ at $x(t)$ for $\hat{x} \in \hat{X}$ obviously inherits the non-degeneracy of the Hessian of $\mathcal{A}_H$ at $\hat{x}$. In fact, our construction clearly identifies Floer trajectories connecting orbits in $\hat{X}$ with negative gradient flow lines of the $A^{\hat{X}}_t$, giving us Morse models for the foliation $\mathcal{F}^{\hat{X}}$.

Proposition 5.3.11. *If (H,J) is Floer regular, $\hat{X} \in murm(H)$ and $\epsilon > 0$ is sufficiently small, then for every $t \in S^1$, and every $\hat{x},\hat{y} \in \hat{X}$, there is a natural identification $\widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J) \cong \widetilde{\mathcal{M}}(\hat{x},\hat{y}; \epsilon A^{\hat{X}}_t,J_t)$ given by $u(s,t) \mapsto u(\epsilon s, t)$.*

Corollary 5.3.12. *Let (H,J) be Floer regular and $\hat{X} \in murm(H)$. Then for every $t \in S^1$, and any $\epsilon > 0$ sufficiently small, $CF_*(\hat{X}; H,J) \cong (C^{Morse}(A^{\hat{X}}_t, g_{J_t}) \otimes \Lambda_\omega)_{*+1} \cong CF_*(\epsilon A^{\hat{X}},J_t)$.*

5.3.2. $\mathcal{F}^{\hat{X}}$ as a positively transverse foliation

Each regular leaf of the foliation $\mathcal{F}^{\hat{X}}$ arises naturally as the image of an embedding $\hat{u} : \mathbb{R} \times S^1 \hookrightarrow S^1 \times \Sigma$ for u, the standard orientation on the cylinder induces the orientation $\partial_s u \wedge \partial_t u$ on each regular leaf, so we may view $\mathcal{F}^{\hat{X}}$ in a natural way as an oriented singular foliation.

Definition 5.3.13. Let $\mathcal{F}$ be an oriented codimension 1 Steffan-Sussmann foliation of an oriented d-dimensional manifold (M^d, o_M). We will say that a smooth path $\alpha : [0,1] \to M$ is **positively transverse** to $\mathcal{F}$ if the following dichotomy holds, either

(1) α is contained in a singular leaf of $\mathcal{F}$, or

(2) for every $t \in [0,1]$, $\{(\partial_t \alpha)_t, v_1, \ldots, v_{d-1}\}$ is an oriented basis for $(T_{\alpha(t)}M, o_M)$, where $\{v_1, \ldots, v_{d-1}\}$ is an oriented basis for the tangent space of the regular leaf of $\mathcal{F}$ passing through $\alpha(t)$.

Definition 5.3.14. Let $\mathcal{F}$ be an oriented codimension 1 Steffan-Sussmann foliation on an oriented d-dimensional manifold (M^d, o_M) and let $X \in \mathcal{X}(M)$ be a vector field generating an isotopy $(\phi_t^X)_{t\in\mathbb{R}}$. We say that X (or $(\phi_t^X)_{t\in\mathbb{R}}$) is **positively transverse to** $\mathcal{F}$ if every integral curve of X is positively transverse to $\mathcal{F}$.

Proposition 5.3.15. *Let (H,J) be Floer regular, $\hat{X} \in murm(H)$, and $\check{X}_H := \partial_t \oplus X_H \in \mathcal{X}(S^1 \times \Sigma)$, then $\mathcal{F}^{\hat{X}}$ is positively transverse to $\check{X}_H$.*

PROOF. As the singular leaves of $\mathcal{F}^{\hat{X}}$ are orbits of $\check{X}_H$, it suffices to consider points $(t,p) \in S^1 \times \Sigma$ lying on regular leaves. In such a case, since u solves Equation 2.1.1, the basis formed by $\{\check{X}_H, \partial_s \check{u}, \partial_t \check{u}\}$ is easily seen to be orientation-equivalent to the basis $\{\partial_t, \partial_s u, J_t \partial_s u\}$, which is a positively oriented basis, as $J_t \in \mathcal{J}(\Sigma,\omega)$ for all $t \in S^1$. ☐

The previous proposition tells us that to any non-degenerate Hamiltonian H and each $\hat{X} \in murm(H)$, we may associate a foliation on $S^1 \times \Sigma$ with respect to which the graph of the isotopy is well-behaved in a certain sense. However, if we're willing to modify the isotopy by a contractible loop, then we can in fact do better and obtain a positively transverse singular foliation on Σ itself. To see this, consider the distribution $\mathcal{D}^{\mathfrak{M}}_{(t,u(s,t))} = \langle \partial_t \oplus \partial_t u \rangle$ introduced in the proof of Proposition 5.3.9. As noted therein, $\mathcal{D}^{\mathfrak{M}}$ integrates to a smooth 1-dimensional foliation by the graphs of the loops $t \mapsto u_s(t)$ for $\hat{u}_s \in \mathfrak{M}_{\hat{X}}$. This induces a natural loop of diffeomorphisms $(\psi_t^{\hat{X}})_{t\in S^1}$ given by sliding the fiber $\{0\} \times \Sigma$ along the foliation which integrates $\mathcal{D}^{\mathfrak{M}}$. In other words, we have the isotopy $\psi_t^{\hat{X}}(p) = u_p(s,t)$, $t \in S^1$, where $u_p \in \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J)$, $\hat{x},\hat{y} \in \hat{X}$, is any Floer cylinder such that $u_p(s,0) = p$. It follows from Corollary 5.1.7 and the fact that if $\hat{x} = \hat{y}$ then $u(s,t) = x(t)$ that $\psi^{\hat{X}}$ is well-defined.

Proposition 5.3.16. $\psi := (\psi_t^{\hat{X}})_{t\in S^1}$ *is a contractible loop.*

PROOF. ψ defines a loop of diffeomorphisms based at the identity by construction. If the genus of Σ is strictly greater than 1, then the proposition follows from the fact that $Diff_0(\Sigma)$ is contractible.

If $\Sigma = \mathbb{T}^2$, $Diff_0(\Sigma)$ has as a strong deformation retract the group of diffeomorphisms given by the action of the torus on itself. In particular, contractible loops in $Diff_0(\mathbb{T}^2)$ are precisely the loops having contractible orbits. Note that for any $\hat{x} = [x,w] \in \hat{X}$, $\psi_t^{\hat{X}}(x(0)) = x(t)$, and so $\psi^{\hat{X}}$ has contractible periodic orbits, and hence is contractible if $\Sigma = \mathbb{T}^2$.

In the case that $\Sigma = S^2$, it is an easy exercise to see that if $(f_t)_{t\in S^1}$ is a loop based at the identity in $Diff(S^2)$, then the homotopy class of $(f_t)_{t\in S^1}$ in $\pi_1(Diff(S^2))$ is classified by the parity of the winding number of the loop

$$v(t) := ((\Phi_{\hat{z}}^t)^{-1} \circ Df_t \circ \Phi_{\hat{z}}^t)(v_0),\ t \in S^1$$

for any $v_0 \in \mathbb{R}^2 \setminus \{0\}$, where $\hat{z} \in \widetilde{\mathcal{L}_0}(S^2)$ is any capped 1-periodic orbit of $(f_t)_{t\in S^1}$ and

$$\Phi_{\hat{z}} : S^1 \times \mathbb{R}^2 \to z^*TS^2$$

is any trivialization of the tangent bundle along z which extends over the chosen capping disk. Since the capped loops $\hat{x} \in \hat{X}$ are all capped 1-periodic orbits of $\psi^{\hat{X}}$ by construction, it suffices to fix some $\hat{x} = [x,w] \in \hat{X}$ and to compute the winding number of some vector in $T_{x(0)}S^2$ under the linearized flow of $\psi^{\hat{X}}$ along x. Suppose without loss of generality that $\mu(\hat{x}; H) = 1$ and let $\xi_1, \xi_2 \in x_0^*T\Sigma$ be a basis of eigenvectors of $A_{x,J}$ with winding number 0 relative the capping w, then by the construction of $\psi^{\hat{X}}$, $(D\psi_t)(\xi_i(0)) = \xi_i(t)$ for $t \in S^1$ by the asymptotic estimates of Theorem 2.2.3. Consequently the linearized winding of $\psi^{\hat{X}}$ along x relative the capping w is even, and so $\psi^{\hat{X}}$ is a contractible loop. $\square$

Note that $\mathcal{F}^{\hat{X}}$ is everywhere transverse to the fibers $\{t\} \times \Sigma$ of $S^1 \times \Sigma$, and so may be viewed as an S^1-family of (singular) foliations on Σ. Let us write $\mathcal{F}_t^{\hat{X}}$ for the foliation obtained on Σ by intersecting $\mathcal{F}^{\hat{X}}$ with $\{t\} \times \Sigma$.

Theorem 5.3.17. *Let (H,J) be a Floer regular pair, $\hat{X} \in murm(H)$, then the orbits of the isotopy $(\psi^{\hat{X}})^{-1} \circ \phi^H$ are positively transverse to the foliation $\mathcal{F}_0^{\hat{X}}$*

PROOF. Writing $\psi = \psi^{\hat{X}}$, observe that the vector field $(Z_t)_{t\in[0,1]}$ which generates the isotopy $\psi^{-1} \circ \phi^H$ is easily computed via the chain rule as $(Z_t)_{u(s,0)} = (\psi_t^{-1})_*(X_t^H - \partial_t u)_{u(s,t)}$. Note moreover that the definition of ψ_t implies that

$$(\psi_t)_*(\partial_s u)_{u(s,0)}) = (\partial_s u)_{u(s,t)}$$

for all $u \in \mathfrak{M}^{\hat{X}}$ and all $(s,t) \in \mathbb{R} \times S^1$. Consequently, because for all $t \in S^1$ ψ_t is an orientation-preserving diffeomorphism, we see that

$$sgn\ \omega_{u(s,0)}(Z_t,\partial_s u) = sgn\ (\psi_t^*\omega)(Z_t,\partial_s u) = sgn\ \omega_{u(s,t)}(X_t^H - \partial_t u,\partial_s u) = sgn\ \omega_{u(s,t)}(-J_t\partial_s u,\partial_s u)$$

from which the claim follows. □

Theorem B is an immediate consequence of Proposition 5.3.11 and the preceding Theorem.

5.4. Consequences for the structure of Hamiltonian isotopies

Definition 5.4.1. For (H,J) non-degenerate and $\hat{X} \in murm(H)$, we define the **Piexoto graph** of $\mathcal{F}^{\hat{X}}$ to be the directed graph $\Gamma(\mathcal{F}^{\hat{X}})$ whose vertex set is $\hat{X}$ and such that there is a directed edge from $\hat{x}$ to $\hat{y}$ only if $\mu(\hat{x}) - \mu(\hat{y}) = 1$, and in this case there is an edge from $\hat{x}$ to $\hat{y}$ for each element in $\mathcal{M}(\hat{x},\hat{y};H,J)$.

Remark. Note that since $\mathcal{F}_t^{\hat{X}}$ may be realized as the singular foliation obtained by the negative gradient flow of $(A^{\hat{X}},g_{J_t})$, $\Gamma(\mathcal{F}^{\hat{X}})$ may be naturally identified with the Piexoto graph (see [**24**]) of $(A_t^{\hat{X}},g_{J_t})$

Definition 5.4.2. Let (H,J) be non-degenerate and $\hat{X} \in murm(H)$. To any capped loop $\hat{\gamma} \in \widetilde{\mathcal{L}_0}(\Sigma)$ such that (γ, x) is a braid for all $\hat{x} \in \hat{X}$, we may define the **linking cochain** $\ell_{\hat{\gamma}}(\hat{x}) := \ell(\hat{\gamma},\hat{x}) \in \mathbb{Z}$ for any $\hat{x} \in V(\Gamma(\mathcal{F}^{\hat{X}})) = \hat{X}$, as well as the **intersection cochain** $I_\gamma : E(\Gamma(\mathcal{F}^{\hat{X}})) \to \mathbb{Z}$, where $I_\gamma(u)$ counts the signed intersection number of (some transverse perturbation of) the maps $\breve{u}(s,t) = (t,u(s,t))$ and $\breve{\gamma}(t) = (t,\gamma(t))$.

The following relation between these two quantities is immediate from the definition of the homological linking number.

Proposition 5.4.3. *Let (H,J) be non-degenerate and $\hat{X} \in murm(H)$. For any capped loop $\hat{\gamma} \in \widetilde{\mathcal{L}_0}(\Sigma)$ such that (γ, x) is a braid for all $\hat{x} \in \hat{X}$, we have $I_\gamma = \delta\ell_\gamma$ (ie. $I_\gamma(u) = \ell_\gamma(\hat{y}) - \ell_\gamma(\hat{x})$, where $u \in \widetilde{\mathcal{M}}(\hat{x},\hat{y};H,J)$).*

For any Hamiltonian H, we write $H^{\natural m} := H\#\ldots\#H$ for the m-fold concatenated Hamiltonian which generates $(\phi_t^H)^{\sharp m}$. For $\hat{x} = [x(t),w(se^{2\pi it})] \in \widetilde{\mathcal{L}_0}(\Sigma)$, we write $\hat{x}^{\sharp m} := [x(mt),w(se^{2\pi imt})]$, and for $\hat{X} = \{\hat{x}_i\}_{i=1}^k$ we write $\hat{X}^{\sharp m} := \{\hat{x}_i^{\sharp m}\}_{i=1}^k$.

Note that the fact that the isotopy $(\varphi_t)_{t\in[0,1]} := ((\psi^{\hat{X}})_t^{-1} \circ \phi_t^H)_{t\in[0,1]}$ is positively transverse to the singular foliation $\mathcal{F}_0^{\hat{X}}$ on Σ implies that for all $m \in \mathbb{Z}_{>0}$, its m-fold concatenation $(\varphi_t^{\sharp m})_{t\in[0,1]}$ is positively transverse to $\mathcal{F}_0^{\hat{X}}$. As a consequence, if $\hat{\gamma} \in \widetilde{\mathcal{L}_0}(\Sigma)$ is such that $(\gamma,x^{\sharp m})$ is a braid for all

$\hat{x}^{\sharp m} \in \hat{X}^{\sharp m}$, then we have that

$$I_\gamma(u) = \ell_{\hat{\gamma}}(\hat{y}^{\sharp m}) - \ell_{\hat{\gamma}}(\hat{x}^{\sharp m}),$$

where $u \in \widetilde{\mathcal{M}}(\hat{x},\hat{y}; H,J)$. We obtain as an immediate corollary

Corollary 5.4.4. *Let (H,J) be non-degenerate, $\hat{X} \in murm(H)$ and let $\hat{\gamma} \in \widetilde{Per}_0(H^{\natural k})$ for $k \in \mathbb{Z}_{>0}$, then $I_{\hat{\gamma}}$ is non-negative on every edge of $\Gamma(\mathcal{F}^{\hat{X}})$. Moreover, for every $\hat{x},\hat{z} \in \hat{X}$, $\ell(\hat{x},\hat{\gamma}) < \ell(\hat{z},\hat{\gamma})$ if and only if $\check{\gamma}(t) \in W(\hat{x},\hat{z})$ for some $t \in S^1$.*

Definition 5.4.5. For a Hamiltonian H, we will say that a capped braid $\hat{X} \subseteq \widetilde{Per}_0(H)$ is **strongly linking** if for any $\hat{\gamma} \in \widetilde{Per}_0(H)$, $\ell(\hat{\gamma},\hat{x}) = 0$ for all $\hat{x} \in \hat{X}$ implies that $\hat{\gamma} \in \hat{X}$. Denote by $usl(H)$ the collection of all $\hat{X} \subseteq \widetilde{Per}_0(H)$ such that $\hat{X}$ is both unlinked and strongly linking.

Clearly, $usl(H) \subseteq mu(H)$.

Theorem G. *Let H be non-degenerate. If $\hat{X} \in murm(H)$, then $\hat{X}^{\sharp m} \in usl(H^{\sharp m})$ for all $m \in \mathbb{Z}_{m>0}$. In particular, every $\hat{X} \in murm(H)$ is maximally unlinked as a subset of $\widetilde{Per}_0(H)$.*

PROOF. Let $\hat{X} \in murm(H)$, fix some J such that (H,J) is Floer regular, and suppose for a contradiction that $\hat{\gamma} \in \widetilde{Per}_0(H^{\natural k}) \setminus \hat{X}^{\sharp m}$ but $\ell(\hat{x}^{\sharp m},\hat{\gamma}) = 0$ for all $\hat{x}^{\sharp m} \in \hat{X}^{\sharp m}$. We may in fact suppose that $\hat{\gamma} \notin \pi_2(\Sigma) \cdot \hat{X}^{\sharp m}$, since if $\hat{\gamma} = A \cdot \hat{x}$ for some $A \in \pi_2(\Sigma)$, then for any $\hat{y}^{\sharp m} \in \hat{X}^{\sharp m}$, $\hat{x}^{\sharp m} \neq \hat{y}^{\sharp m}$, Proposition 1.3.2 implies $\ell(\hat{\gamma},\hat{y}^{\sharp m}) = \ell(\hat{\gamma},\hat{y}^{\sharp m}) + \frac{c_1(A)}{2} = \frac{c_1(A)}{2}$, since $\hat{X}$ is unlinked, which easily implies that $\hat{X}^{\sharp m}$ is unlinked as well.

So we may as well assume that $\gamma \neq x$ for any $\hat{x}^{\sharp m} \in \hat{X}^{\sharp m}$. Since $\mathcal{F}^{\hat{X}}$ foliates $S^1 \times \Sigma$, it is necessary that $\check{\gamma}(0) \in W(\hat{x},\hat{y})$ for some $\hat{x},\hat{y} \in \hat{X}$ and so by Corollary 5.4.4 implies that $\ell(\hat{x}^{\sharp m},\hat{\gamma}) < \ell(\hat{y}^{\sharp m},\hat{\gamma})$, a contradiction. $\square$

5.4.1. Comparison to Le Calvez's theory of transverse foliations

Definition 5.4.6. Let $I = (\phi_t)_{t\in[0,1]}$ be an isotopy of homeomorphisms based at the identity. We define

$$Fix(I) := \bigcap_{t\in[0,1]} Fix(\phi_t),$$

and we say that I is a **maximal isotopy** if for every $x \in Fix(\phi_1)$, the loop $t \mapsto \phi_t(x)$ is not contractible in $\Sigma \setminus Fix(I)$.

Definition 5.4.7 (cf. [**18**]). Given a continuous oriented singular 1-dimensional foliation $\mathcal{F}$ of Σ, a continuous path $\gamma : [0,1] \to \Sigma$ is said to be **positively transverse** to $\mathcal{F}$ if its image is disjoint from $Sing(\mathcal{F})$ and for each $t_0 \in [0,1]$, there exists an orientation preserving homeomorphism of a neighbourhood of $\gamma(t_0)$ to a neighbourhood of $0 \in \mathbb{R}^2$ which sends $\mathcal{F}$ to the standard vertical foliation of $\mathbb{R}^2$, oriented downward, and sends γ to a map whose x-coordinate is increasing in a neighbourhood of t_0.

In [**16**], Le Calvez developed a theory which associates to any maximal isotopy $I = (\phi_t)_{t\in[0,1]}$, an oriented singular continuous 1-dimensional foliation $\mathcal{G}^I$ on Σ having singular points on precisely the points $x \in Fix(I)$, and moreover having the property that the dynamics of I are *homotopically positively transverse* to the leaves of $\mathcal{G}^I$, in the following sense

Definition 5.4.8. We say that an isotopy I **homotopically positively transverse** to an oriented singular continuous foliation $\mathcal{G}$ if, for all $x \in \Sigma \setminus Fix(I)$, the path $x \mapsto \phi_t(x)$ is homotopic relative endpoints inside of $\Sigma \setminus Fix(I)$ to a path which is positively transverse to $\mathcal{G}$.

Remark. Le Calvez and those working with his theory use the term 'positively transverse' rather than 'homotopically positively transverse' as we have used here. We introduce this term here merely to disambiguate between the other notion of positive transversality (for smooth isotopies) which we have already employed in this thesis.

The reader will surely notice the similarity of Le Calvez's result to the results expounded in the previous section. Let us compare them more closely on the domain of overlap of these two theories, namely, the case where ϕ_1 is a smooth, non-degenerate Hamiltonian diffeomorphism.

As we have shown, to any non-degenerate pair (H,J) and any $\hat{X} \in murm(H)$, we may associate a new isotopy $(\psi_t^{\hat{X}})^{-1} \circ \phi_t^H$ having

$$Fix((\psi_t^{\hat{X}})^{-1} \circ \phi_t^H) = \bigcup_{\hat{x}\in\hat{X}} \{x(0)\}.$$

The fact that $murm(H) \subset mu(H)$ (by Theorem G) implies that this isotopy is maximal, we know that it is positively transverse (in our initial sense) to the foliation $\mathcal{F}_0^{\hat{X}}$ of Σ. This implies positive transversality in the sense of Le Calvez, but is at least superficially a rather stronger condition, in that positive transversality in Le Calvez's sense allows us to homotope the orbits of the maximal isotopy, individually, in order to achieve positive transversality in the usual sense, whereas here, every orbit is already positively transverse in the usual sense. Said another way, if $\hat{X}$ is a trivial capped braid, so that the isotopy $(\phi_t^H)_{t\in[0,1]}$ is already a maximal isotopy, then it is already positively transverse to $\mathcal{F}_0^{\hat{X}}$ in the sense of Le Calvez, and $(\psi^{\hat{X}})^{-1}$

provides us with a choice of homotopy for each orbit $t \mapsto \phi_t^H(x)$ in such a way that these homotopies fit together in a coherent way so as to be induced by composition with a contractible loop.

In [**16**], Le Calvez also observed that in the case of Hamiltonian homeomorphisms satisfying some minor niceness requirements of their fixed point sets which always applies in our setting (see also the work of Béguin-Crovisier-Le Roux in [**2**] in which the authors remove the need for this niceness condition), the foliations $\mathcal{G}^I$ constructed by his methods are always *gradient-like*, in the sense that they admit discrete Lyapunov functions which are decreasing along the leaves of the foliation. These Lyapunov functions are, essentially, winding numbers with the maximally unlinked set of orbits used to construct the foliation. Thus, we can see the results of our previous section as recovering Le Calvez's theory in the smooth case — with the restricted action functional $A_0^{\hat{X}}$ playing the role of the smooth analogue to the winding numbers with the unlinked set $\hat{X}$ as a discrete Lyapunov function — on the proviso that $\hat{X} \in murm(H)$, and offering certain refinements on the structure of these fixed points along with a certain strengthening of the transversality condition.

We should note, however, that in general $murm(H) \neq mu(H)$, and so even in the smooth case, Le Calvez's method provides foliations which do not arise via our methods. It would be interesting to understand the exact nature of the discrepancy between $murm(H)$ and $mu(H)$ in terms of the differences in the topologies between the corresponding Le Calvez foliations $\mathcal{G}^{\phi^H}$ and those foliations $\mathcal{F}^{\hat{X}}$ that we have constructed here. It seems likely that the foliations $\mathcal{F}^{\hat{X}}$ which arise by our methods correspond to the Le Calvez foliations $\mathcal{G}^{\phi^H}$ which arise from the *torsion-low* maximal isotopies introduced by Yan in [**37**] (I would like to thank Vincent Humilière for directing my attention to this).

5.5. Appendix: Stefan-Sussmann Foliations

It will be convenient in what follows to have some explicit language with which to speak about singular foliations. To that end, we will make use of some elementary notions from the theory of *Stefan-Sussmann foliations* (cf. [**4**] and the references therein) which will be suitable to our purposes.

As in the non-singular theory, one may think of (singular) foliations as being partitions of the ambient space into integral submanifolds of some (generalized) distributions. To that end, let

$$\mathcal{G}^k(M) \to M$$

denote the k-*Grassmannian* of M, having fiber $Gr(k,T_xM)$ over $x \in M$, and let

$$\mathcal{G}^*(M) := \sqcup_{k=0}^{n} \mathcal{G}^k(M),$$

where $n = \dim M$, denote the *total Grassmannian* of M.

Definition 5.5.1. A **(generalized) distribution** on a manifold M is a section

$$D : M \to \mathcal{G}^*(M).$$

A local section $X : M \to TM$ is said to **belong to** D if $X(x) \in D(x)$ for all $x \in dom(X)$. The set of all smooth local sections $X \in \mathcal{X}_{loc}(M)$ belonging to D is denoted by Δ_D.

Definition 5.5.2. A generalized distribution D is said to be **smooth** if for every $x \in M$,

$$D(x) = \text{span} \langle\ X(x)\ \rangle_{X \in \Delta_D}.$$

Naturally, we will want to define objects which integrate these generalized smooth distributions. To that end, following [**35**], we shall define

Definition 5.5.3. A **(smooth)** k**-leaf** of M is a subset $L \subset M$ equipped with a differentiable structure σ such that

(1) (L,σ) is a connected k-dimensional immersed submanifold of M and
(2) for any continuous map $f : N \to M$ such that $f(N) \subset L$ and N a locally connected topological space, we have that

$$f : N \to (L,\sigma)$$

is continuous.

Definition 5.5.4. A (C^∞)**-singular (Stefan-Sussmann) foliation** of M is a partition $\mathcal{F}$ of M into smooth leaves of M such that for every $x \in M$, there exists a local smooth chart

$$\varphi : U \xrightarrow{\cong} \mathcal{O}(x) \subset M$$

from $U \subset \mathbb{R}^n$ an open neighbourhood of $0 \in \mathbb{R}^n$, such that

(1) $U = V \times W$ for V an open neighbourhood of 0 in $\mathbb{R}^k$ and W an open neighbourhood of 0 in $\mathbb{R}^{n-k}$, where k is the dimension of the smooth leaf $L_x \in \mathcal{F}$ containing x.
(2) $\varphi(0,0) = x$.
(3) For any leaf $L \in \mathcal{F}$,

$$L \cap \varphi(U \times W) = \varphi(U \times l),$$

where $l := \{w \in W : \varphi(0,w) \in L\}$.

Definition 5.5.5. A smooth generalized distribution D is said to be **integrable** if for every $x \in M$ there exists an immersed submanifold $L \subset M$, such that

(1) $x \in L$, and

(2) $T_y L \subset D(y)$ for all $y \in L$.

Such an immersed submanifold is called an **integral submanifold of** D.

The main point is the following (due to [**35**])

Theorem 5.5.6. *If D is a smooth integrable generalized distribution and $\mathcal{F}_D$ is the partition of M formed by taking the collection maximal connected integral submanifolds of D, then $\mathcal{F}_D$ is a smooth singular Stefan-Sussmann foliation.*

For a singular foliation $\mathcal{F}$, we let

$$\begin{aligned} d(-,\mathcal{F}) : M &\to \mathbb{Z}_{\geq 0} \\ x &\mapsto \dim L_x \end{aligned}$$

denote the function which keeps track of the dimension of the leaf of $\mathcal{F}$ passing through $x \in M$. It's not hard to see that $d(-,\mathcal{F})$ is lower semi-continuous.

Definition 5.5.7. A smooth singular foliation is said to have **codimension** k if

$$n - k = \max_{x \in M} d(x,\mathcal{F}).$$

For a codimension k smooth singular foliation, we define the **domain** of $\mathcal{F}$ to be

$$dom(\mathcal{F}) := \{x \in M : d(x, \mathcal{F}) = n - k\},$$

while we define the **singular set** of $\mathcal{F}$ to be

$$sing(\mathcal{F}) := M \setminus dom(\mathcal{F}).$$

A leaf of $\mathcal{F}$ is said to be **regular** if it is of maximal dimension, otherwise it is said to be **singular**. $\mathcal{F}$ is said to be **oriented** if every regular leaf of $\mathcal{F}$ is in addition equipped with an orientation, and the local charts about points on the regular leaves may be taken to be orientation-preserving.

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