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Production externalities: cooperative and non-cooperative approaches

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Sommaire

Cette thèse a pour sujet les externalités de production. Trois cas particuliers sont étudiés, où les coûts et les décisions de production sont intimement reliés. La thèse est divisée en trois chapitres. Les deux premiers chapitres traitent ces externalités de production avec l'approche coopérative. Il est supposé que les producteurs coopèrent et que la production est réalisée conjointement. Nous sommes intéressés par les conditions sous lesquelles les agents acceptent volontairement de coopérer, et par les méthodes de partage du coût commun si la coopération a lieu. Dans le troisième chapitre, l'approche est non-coopérative, et il est supposé que les producteurs sont en compétition.

Le premier chapitre examine les situations où les externalités de production proviennent d'une structure de réseau. Nous étudions les problèmes où des agents, situés à différents points dans l'espace, doivent être reliés à une source pour obtenir un bien. Le coût d'un arc entre deux agents est une fonction du flux passant dans cet arc. Nous sommes intéressés par les conditions sous lesquelles il existe toujours au moins une allocation stable des coûts, c'est-à-dire telle qu'aucune coalition n'a d'incitatifs à quitter unilatéralement la grande coalition. Le noyau est l'ensemble de toutes ces allocations. Même si nous trouvons qu'en général, si les fonctions de coût sont concaves, le noyau peut être vide, pour certains problèmes simples spécifiques, il est toujours non-vide. En particulier, nous introduisons une nouvelle famille de problèmes, appelés problèmes de congestion quasi-symétriques et ordonnés, qui a toujours un noyau non-vide. Quand toutes les fonctions de coûts sont convexes, le jeu qui en résulte a toujours au moins une allocation stable.

Le deuxième chapitre étudie les problèmes de partage de coûts lorsque les externalités de production sont définies de manière générale : les gains de coopération peuvent venir de la présence d'autres agents, comme lorsque les agents partagent leurs technologies. Un modèle simple est construit, où les économies d'échelle sont éliminées dans le but de se concentrer sur les effets du partage technologique. Nous cherchons à trouver des règles de partage de coûts équitables. Nous utilisons comme axiome central la propriété que si un agent n'améliore jamais la technologie des coalitions auxquelles il se joint, il ne devrait recevoir aucune part du gain de coopération. Avec des propriétés de linéarité et de symétrie, cet axiome caractérise une famille bien définie de règles. De cette famille, nous proposons une règle dérivée de la célèbre valeur de Shapley. Nous trouvons qu'il s'agit de la seule règle de la famille satisfaisant une propriété de limite supérieure sur les allocations individuelles ou une propriété de monotonie lorsque la technologie s'améliore. Nous obtenons aussi une règle distincte en utilisant une propriété faisant en sorte qu'aucune coalition n'a d'incitatifs à manipuler les demandes individuelles de ses membres.

Dans le troisième chapitre, le sujet d'analyse est un duopole où les firmes produisent des biens différenciés verticalement. La qualité de ces produits peut être améliorée par les contributions automatiques faites par les usagers, comme dans le cas d'une compagnie de logiciels enregistrant les bogues et défaillances subis par ses utilisateurs. Cette technologie d'amélioration de la qualité est en contraste avec le concept de code source ouvert, où les utilisateurs doivent décider de contribuer. Nous n'avons pas besoin de modéliser cette décision et mettons l'emphase sur les effets sur la compétition, que nous modélisons par un jeu à deux étapes. Nous trouvons que les équilibres ne génèrent pas tous une différenciation maximale. Il s'avère aussi qu'une avance initiale n'est jamais renversée, mais que la menace de renversement peut être suffisante pour réduire les prix. Nous examinons aussi les effets sur le bien-être, et trouvons qu'augmenter les contributions des usagers augmente toujours le bien-être social. En particulier, dans une région des paramètres, augmenter les contributions est une amélioration au sens de Pareto.

Mots clés : externalités de production; théorie des jeux; partage de coûts; noyau; concavité permutationnelle; valeur de Shapley; axiome de l'agent négligeable; différenciation verticale; contributions des usagers.

Summary

The subject of this thesis is production externalities. Three particular cases where production costs and decisions are closely interrelated are studied. The thesis is divided in three chapters. The first two chapters deal with these production externalities with a cooperative approach. It is assumed that producers cooperate and production is done jointly. We are interested in conditions under which this cooperation happens without any outside intervention, and in the methods to share the common cost if cooperation does happen. In the third chapter, the approach is non-cooperative and producers are assumed to compete.

The first chapter examines situations where the production externalities come from a network structure. We study problems where agents, located at different points in space, have to be connected to a source to obtain a good. The cost of an arc between two agents is a function of the flow that goes through this arc. We are interested in conditions under which there always exists at least one stable cost allocation, which is such that no coalition has incentives to secede from the grand coalition. The core is the set of all such allocations. We find that in general, if the cost functions are concave, the core can be empty. However, in specific simple problems, it is always non-empty. In particular, we introduce a new family of problems, called the ordered quasi-symmetric congestion problems, that always have a non-empty core. When all cost functions are convex, the resulting game always has at least one stable allocation.

The second chapter studies cost sharing problems where production externalities are defined in a general manner: gains from cooperation can come from the presence of other agents, such as when agents share their technologies. A simple model is built, where economies of scale are eliminated in order to focus on technology sharing. We are interested in finding fair cost allocation rules. We use as the key axiom the property that, if an agent does not improve the technology of any coalition he joins, he should not get any part of the gain from cooperation. With properties of linearity and symmetry, this axiom characterizes a well-defined set of rules. From this set, we propose a rule derived from the familiar Shapley value. We show that it is the only rule in that set satisfying an upper-limit property on individual cost allocations or a monotonicity property when technology improves. We also derive a distinct rule using a property that ensures that no coalition has an incentive to manipulate the individual demands of its members.

In the third chapter, the subject of analysis is a duopoly where firms produce vertically-differentiated goods. The quality of these products can be improved through automatic contributions from their users, such as software companies tracking bugs and crashes incurred by its users. This quality-improvement technology is in contrast with open-sourcing, where agents must decide to contribute. We do not need to model why users contribute, and can therefore focus on the effects on competition, that we model as a two-stage game. We find that equilibria are not always such that we have maximal differentiation. We also find that an initial lead is never reversed, but the threat of leapfrogging can be enough to reduce prices. We also look at welfare effects, and find that increasing user contributions always increase total welfare. In particular, in one parameter region, increasing user contributions is Pareto-improving.

Keywords: production externalities; game theory; cost sharing; core; permutational concavity; Shapley value; Dummy axiom; vertical differentiation; user contributions.

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Introduction générale

La production de biens et services génère dans bien des cas des externalités, qui peuvent être positives ou négatives. Les exemples sont nombreux : pollution, congestion, effets de réseaux, apprentissage par la production, dispersion de technologies, etc. Ces externalités peuvent être supportées par les consommateurs et/ou les autres producteurs. Le premier cas a été traité abondamment dans la littérature depuis Pigou (1920). On distingue dans ce cas les coûts privés et sociaux. La solution habituelle consiste à utiliser des taxes ou des subsides pour réaligner coûts privés et sociaux. La présente thèse s'intéresse au deuxième cas, où les coûts et/ou les décisions de production des différents compétiteurs sont étroitement reliés. Nous offrons deux approches de résolution. Dans les deux premiers essais, l'approche est coopérative ; les agents s'unissent pour la production conjointe de biens. Dans le troisième essai, l'approche est non-coopérative ; dans la plus pure tradition de l'organisation industrielle, les firmes sont en compétition.

L'analyse de problèmes d'externalités dans la production via l'approche coopérative permet de répondre à deux importantes questions : sous quelles circonstances les agents vont-ils volontairement accepter de coopérer, et s'il y a coopération, comment peut-on diviser équitablement les coûts communs de production? Dans des modèles légèrement différents, le premier essai traite la question de stabilité, alors que le deuxième tente de trouver des solutions équitables.

Plusieurs idées de la théorie des jeux coopératifs ont d'abord été mentionnées dans la littérature sur l'analyse coûts-bénéfices de projets communs. Notamment, les notions de stabilité et de partage de coûts se retrouvent dans les analyses des projets de réservoirs d'eau à usages multiples de la *Tennessee Valley Authority* (Ransmeier (1942)). L'analyse formelle débute avec la monographie de von Neumann et Morgenstern (1944). Si l'ensemble de joueurs est N, un jeu coopératif avec utilité transférable possède une fonction caractéristique qui assigne à chaque coalition $S \subseteq N$ une valeur (ou un coût). La notion de noyau, cet ensemble d'allocations telles qu'aucune coalition n'a intérêt à quitter le groupe pour réaliser le projet de façon indépendante, est développée par Gillies (1953) et Shapley (1953). Bondareva (1963) et Shapley (1967) montrent de façon indépendante que les jeux coopératifs avec utilité transférable ont un noyau non-vide si et seulement si le jeu est dit balancé. Un jeu est balancé si la somme des coûts des différentes coalitions, pondérés par un ensemble balancé de poids, est au moins aussi grande que le coût total de la grande coalition N. Des poids sont balancés si pour tout joueur i, la somme des poids associés aux coalitions auxquelles i appartient est égale à un.

Shapley (1971) trace ensuite le lien entre les jeux concaves et le noyau. Un jeu de coûts est dit concave si le coût d'ajouter un agent à une coalition S est au moins aussi élevé que lorsqu'il joint une coalition T, si $S \subseteq T$. Alors, non seulement le jeu est stable, mais le noyau a une forme particulière. Pour une permutation donnée des agents, l'allocation via les coûts incrémentaux est telle qu'on assigne à chaque agent le coût supplémentaire lorsqu'il se joint à la coalition formée par les joueurs avant lui dans la permutation choisie. Dans un jeu concave, l'allocation via les coûts incrémentaux fait partie du noyau, et ce pour chaque permutation des agents. En plus, ces allocations constituent les points extrêmes du noyau. En générant les combinaisons convexes de ces allocations, on obtient l'ensemble du noyau.

Le noyau pouvant être vide, certaines extensions toujours non-vides ont été proposées, notamment par Shapley et Shubik (1966) et Maschler, Peleg et Shapley (1979). Alors que le concept de jeu coopératif a été appliqué à des jeux avec plus de structure, des raffinements propres à ces structures ont aussi été proposés. Voir Bilbao (2000) pour une revue de ces raffinements.

Dans le premier essai, le jeu coopératif a une structure particulière : les externalités de production proviennent d'une structure de réseaux. Les agents sont situés à des endroits différents et doivent être connectés à une source pour obtenir les biens demandés. La construction d'aqueducs, de gazoducs ainsi que de réseaux de distribution d'électricité ou d'information constituent quelques exemples de problèmes économiques avec structures de réseaux. Les structures de réseaux ont été étudiées d'abord par les chercheurs en recherche opérationnelle, qui s'intéressaient aux méthodes de résolution et d'approximation des problèmes d'optimisation sous-jacents. Les spécialistes en théorie des jeux se sont ensuite servis de ces résultats pour discuter des problèmes de partage de coûts.

Le premier et plus célèbre cas étudié a été celui des arbres de coûts couvrants (cost spanning trees) où la source est unique et le coût de construction d'un arc entre deux agents est fixe. Claus et Kleitman (1973) ont été les premiers à s'intéresser au partage de coûts dans ce cas. Bird (1976) et Granot et Huberman (1981) ont proposé une allocation qui est toujours dans le noyau. L'allocation proposée pouvant être qualifiée d'inéquitable, Kar

(2002), Dutta et Kar (2004) et Bergantinos et Vidal-Puga (2007) proposent et caractérisent de nouvelles règles.

Diverses extensions ont été proposées, notamment suite à l'ajout de contraintes de capacité (Skorin-Kapov (1993)) ou d'applications à des problèmes de transport ou de communications (par exemple, Kalai et Zemel (1982) et Tamir (1991)). Les modèles d'appariement constituent un autre exemple de problèmes économiques avec structures de réseaux. Sharkey (1995) fait une revue de tous ces modèles économiques avec structures de réseaux.

Le premier essai étudie la stabilité de problèmes de réseaux, dans un cas plus général que les arbres de coûts couvrants. De façon générale, nous examinons des problèmes où des agents, situés à différents endroits dans l'espace, doivent être connectés à une source pour obtenir un bien. Le coût sur un arc entre deux agents est une fonction du flux passant dans cet arc. Alors que les problèmes avec arbres couvrants imposent que le coût de ces arcs ne soient que des coûts fixes, nous permettons n'importe quelle fonction de coûts non-décroissante et continue sur $\mathbb{R}_{++}$. Les problèmes où toutes les fonctions de coût sont concaves (convexes) sont appelés des problèmes concaves (convexes). La notion de demande n'est pas présente dans les modèles avec arbres couvrants, tandis que notre modèle permet aux agents d'avoir des demandes différentes, ce qui affecte les coûts. Ce problème, appelé *problème de réseau à source unique et sans contraintes de capacité*, a été étudié en recherche opérationnelle. En général, il n'existe pas d'algorithme pour trouver une solution optimale exacte. Guisewite et Pardalos (1991) offrent une revue des méthodes d'approximations.

Quant, Borm et Reijnierse (2006) sont les premiers à examiner le problème avec les outils de la théorie des jeux. Ils montrent dans un problème légèrement moins général que les problèmes concaves peuvent ne pas avoir de solutions stables et que les problèmes convexes en ont toujours au moins une. Ils annoncent aussi que le résultat s'étend à l'ensemble des problèmes de réseau à source unique et sans contrainte de capacité, un résultat que nous prouvons formellement. Dans le cas convexe, bien que chaque arc présente des déséconomies d'échelle, les agents ont toujours intérêt à se regrouper. Cela est dû au fait que pour éviter les déséconomies d'échelle, les agents tentent de diviser les flux sur le plus grand nombre d'arcs possibles. Comme l'ajout d'un agent à un groupe de k individus donne accès à $k + 1$ nouveaux arcs, les gains de coopération sont très importants.

Dans le cas concave, après avoir offert quelques exemples nouveaux et simples de cas où le noyau est vide, nous examinons la sous-famille de problèmes concaves dont le noyau est toujours non-vide. Il est déjà connu que les problèmes à arbres couvrants font partie de cette sous-famille. Après avoir discuté de quelques exemples simples de problèmes concaves et stables, et à défaut d'avoir pu offrir une caractérisation complète de cette sous-famille, une toute nouvelle classe est proposée : les problèmes de congestion quasi-symétriques et ordonnés. Ils sont tels que tous les agents ont la même demande et que les fonctions de coûts sur les arcs entre eux sont identiques. Les agents sont ordonnés selon les coûts sur l'arc les reliant à la source. Si l'agent i vient avant l'agent j, alors le coût incrémental de la k^e unité sur l'arc entre la source et i ne doit pas être plus grand que sur l'arc entre la source et j, et ce, pour tout entier positif k.

La stabilité de ces problèmes provient du fait que le jeu de coût associé est permutationnellement concave, un concept introduit par Granot et Huberman (1982). Bien que plus faible que la propriété de concavité, il est suffisant pour garantir un noyau non-vide. En effet, le concept de concavité permutationnelle est tel qu'un jeu le respectant possède une permutation des agents pour laquelle les coûts incrémentaux, par rapport à cette permutation, sont décroissants. Assigner à chaque agent son coût incrémental, selon cette permutation, donne une allocation stable. Par opposition, ces résultats tiennent pour n'importe quelle permutation dans un jeu concave. Les problèmes à arbres de coûts couvrants constituent un exemple de tels problèmes. Le concept de concavité permutationnelle a été utilisé pour prouver la stabilité de divers problèmes provenant de la recherche opérationnelle (Meca, Timmer, Garcia-Jurado et Borm (2004), van Velzen (2006), van Velzen et Hamers (2003)). Tout comme dans le cas des problèmes à arbres couvrants, l'allocation stable trouvée pourrait être qualifiée d'inéquitable, laissant la porte ouverte à la recherche d'autres méthodes de répartition des coûts.

Cette question d'équité dans le partage des coûts a été étudiée de façon parallèle à celle sur la stabilité. Commençant avec Shapley (1953) qui propose la valeur qui porte son nom, une vaste littérature visant à proposer et à axiomatiser des concepts de solution aux jeux coopératifs s'est développée. Appliquées aux jeux de coûts, ces solutions sont appelées règles de partage de coûts.

Il convient de distinguer trois modèles standards de jeux de coûts. Le modèle classique est celui sous forme coalitionnelle, où tout ce qui est connu est le coût total pour chaque coalition. Billera, Heath et Raanan (1978) ont introduit le jeu continu, où les agents sont caractérisés par une demande et où une fonction transforme en coûts n'importe quel point de demande dans $\mathbb{R}_+^N$. Moulin (1995) propose le jeu discret, où les demandes sont dans $\mathbb{N}^N$.

Parmi les règles de partage proposées, mentionnons la valeur de Shapley. Elle consiste en une moyenne des coûts incrémentaux et possède de nombreuses propriétés. Dans le jeu sous forme coalitionnelle, il s'agit de la seule règle satisfaisant les axiomes d'Additivité, de Symétrie et de l'Agent Négligeable. L'axiome d'Additivité indique que s'il est possible de diviser les coûts en sous-fonctions (production, livraison, administration, etc.), partager les coûts totaux ou le faire indépendamment sur chacune de ces sous-fonctions devrait donner le même résultat. Bien que de nature mathématique, cette propriété est désirable et fait en sorte qu'il n'y a pas de discussions sur le niveau d'agrégation auquel doit s'effectuer le partage de coûts. La propriété de symétrie indique que si deux agents sont identiques au point de vue de leurs coûts incrémentaux, alors ils devraient se voir allouer la même part des coûts. Un agent est dit négligeable si ses coûts incrémentaux sont toujours nuls. L'axiome de l'Agent Négligeable fait en sorte qu'un tel agent se verra attribué une allocation égale à zéro. Cette propriété rend les agents responsables pour les coûts causés par leurs demandes. Moulin et Sprumont (2006) discutent des implications de cette propriété et comparent avec des propriétés ne générant qu'une responsabilité partielle.

Le deuxième essai considère un jeu de coûts sous une forme nouvelle, inspiré, sans en être une généralisation, de la structure de réseaux du premier essai. Les problèmes de partage de coûts considérés sont tels qu'il existe une externalité dans la production ; les gains de coopération peuvent venir de la simple présence des agents, par exemple, lorsqu'il y a partage de technologies ou coopération pour la formation d'un réseau. Ceci est en contraste avec les modèles traditionnels, où l'hypothèse sous-jacente est que la technologie de production est de nature publique, et donc identique pour chacune des coalitions. Nous considérons donc ce nouveau type de partage, où la technologie est définie au sens très large ; il peut s'agir de mise en commun de techniques de production, de ressources, de pouvoirs d'achat ou de négociation. Cela implique que le coût ne dépend plus seulement du niveau de demandes à produire, mais aussi de l'identité des agents contribuant à sa production.

Un modèle simple est construit, où les économies d'échelle sont éliminées pour isoler les effets du partage technologique. Les agents sont caractérisés par une demande, et à chaque coalition est associée une technologie de production. Dans le cadre du modèle, cette technologie consiste en un coût moyen et est telle que lorsqu'une coalition devient plus grande, son coût moyen ne peut augmenter. Le coût coalitionnel est alors le coût moyen multiplié par la somme des demandes des agents dans la coalition.

Nous sommes intéressés par les différents concepts de solution pour diviser le coût entre les différents agents et proposons de nouvelles règles. Ces règles sont axiomatisées via la propriété d'Agent Technologiquement Négligeable. Cet axiome est tel que si un agent n'améliore jamais la technologie d'une coalition à laquelle il se joint, il n'obtient aucune part du gain de coopération. La propriété d'Agent Technologiquement Négligeable rend les agents responsables pour leur technologie. Cela est désirable quand les agents ont investi temps, argent ou ressources pour la développer. Cet axiome est bien sûr fortement lié à la notion d'Agent Négligeable, mais ce sont deux axiomes indépendants. Conceptuellement, l'axiome est encore plus près de la propriété de Séparabilité. Cette propriété indique que si le coût incrémental d'un agent est toujours égal à une constante, il devrait se voir attribuer cette constante. Comme il ne contribue alors aucunement aux gains d'échelle, il n'en reçoit aucune part.

La section 3 de ce deuxième essai caractérise les règles satisfaisant cet axiome d'Agent Technologiquement Négligeable, combiné à une application de l'axiome de Symétrie au modèle considéré, et deux applications de la propriété d'Additivité. La première, la Linéarité Technologique, est une extension directe et fait en sorte que si les coûts moyens peuvent être divisés en sous-fonctions, le niveau d'agrégation choisi n'est pas important. La deuxième, la Linéarité par rapport aux Demandes, implique que chaque unité de demande peut faire l'objet d'un partage de coût indépendant. C'est une propriété naturelle étant donnée l'absence d'économies d'échelle. Elle fait aussi en sorte que les agents n'ont pas intérêt à divulguer un niveau de demande différent du niveau véritable, pour ensuite demander un réajustement sur lequel serait effectué un nouveau partage de coûts.

Les axiomes étant semblables à ceux utilisés pour caractériser la valeur de Shapley, il est peu surprenant que cette caractérisation reprenne des étapes de la caractérisation obtenue par Weber (1988). Nous obtenons une famille de règles différenciées par la façon dont elles répondent à la question suivante : si un agent j est le seul à avoir une demande non nulle, lorsque nous calculons la part des coûts de l'agent i, quel poids devrions-nous accorder aux améliorations technologiques générées par l'agent i pour des coalitions ne contenant pas l'agent j? La réponse à cette question consiste en un vecteur de $|N|-2$ composantes. Les sections 4 et 5 offrent ensuite des caractérisations de deux règles appartenant à cette famille.

La *valeur de la contribution technologique* partage les coûts de la façon suivante : pour chaque agent i, nous définissons un jeu qui assigne à chaque coalition ne contenant pas i la variation du coût moyen quand cette coalition se joint à i. La valeur de Shapley est ensuite appliquée sur chacun de ces jeux et multipliée par la demande de l'agent i, pour exprimer la valeur créée. Un agent se voit ensuite allouer ces valeurs créées dans chacun des jeux, auxquelles on additionne son coût de faire cavalier seul. Pour caractériser cette solution, il suffit d'ajouter la propriété de Rationalité Individuelle, qui implique qu'aucun des agents n'a intérêt à quitter unilatéralement la grande coalition, ou la propriété de Monotonie Technologique, telle que si la technologie d'une coalition s'améliore, aucune des membres de cette coalition n'en souffre.

La *règle de la quasi-moyenne des contributions* alloue les coûts ainsi : un jeu est construit où chaque coalition se voit attribuer le coût de produire l'ensemble du vecteur de demande en utilisant sa technologie. La valeur de Shapley est ensuite appliquée pour partager les gains technologiques. Finalement, les allocations individuelles sont ajustées selon les différences entre le niveau de demande individuelle et le niveau moyen. La caractérisation de cette règle est obtenue en ajoutant la propriété de Non Réassignation. Cette propriété est telle qu'aucune coalition n'a intérêt à manipuler la distribution de la demande de ses membres afin de réduire le coût total qui lui est attribué. C'est donc une importante propriété incitative.

Le troisième essai utilise l'approche non-coopérative pour étudier les externalités de production. En organisation industrielle, tout modèle d'oligopole, à commencer par ceux de Bertrand et Cournot, comportent une part d'externalités et d'interrelations entre les firmes. Nous sommes intéressés à des interrelations un peu plus importantes, où non seulement les décisions d'une firme influencent directement celles de ses adversaires, mais où on retrouve aussi un effet indirect sur les paramètres de base du modèle, comme la fonction de coûts, la demande du marché ou la qualité des produits. De tels exemples sont nombreux. Nous y retrouvons notamment les marchés avec économies de réseau (par exemple Katz et Shapiro (1985), Farrell et Klemperer (2007)), des marchés avec apprentissage par la production (Cabral et Riordan (1994)) et des marchés avec différenciation verticale et détermination endogène de la qualité (Shaked et Sutton (1982)).

Le cas considéré s'inspire d'un phénomène récent. Avec l'expansion de l'Internet, les usagers sont de plus en plus appelés à contribuer directement à la qualité d'un produit. Beaucoup d'attention à été portée sur les logiciels à code source ouvert, où chaque utilisateur peut directement participer, de façon volontaire, à sa programmation. Des sites à contenus générés par les utilisateurs, tels YouTube et Wikipedia, sont conceptuellement semblables. L'attention académique s'est essentiellement portée sur les déterminants de cette participation (Lerner et Tirole (2005)) et sur l'organisation des participants et l'effet sur la productivité. Voir à ce sujet la revue de von Krogh et von Hippell (2006).

Nous nous intéressons à un cas légèrement différent, où la contribution des consommateurs est automatique. Par exemple, une firme productrice de logiciels peut compiler automatiquement les instances de bogues ou défaillances observées par ses utilisateurs, ce qui lui permet d'améliorer la qualité future de son produit. Pareillement, une firme produisant un logiciel anti-virus se sert des informations transmises par ses utilisateurs pour améliorer la protection offerte. C'est ce que nous appelons les contributions automatiques des usagers. Dans de tels cas, la question des déterminants de la participation n'est plus pertinente. Nous pouvons alors nous concentrer sur les effets de ces contributions sur la concurrence.

Ce cas diffère des effets de réseaux par le fait que les externalités sont sur les consommateurs futurs uniquement, et parce que les consommateurs sont tout à fait libres de passer d'un fournisseur à l'autre entre les périodes. Il diffère aussi du cas de l'apprentissage par la production, où l'externalité est sur le coût et se transmet généralement à toute l'industrie. Il s'agit aussi d'un cas particulier de modèle dynamique de différenciation verticale endogène, mais avec la particularité que l'amélioration technologique se fait via les ventes, donc via une technologie rivale. Les firmes ont intérêt à être différenciées, mais aussi à être celle avec la qualité la plus élevée. Comme elles doivent lutter entre elles, via les prix, pour améliorer la qualité de leur produit, les effets nets sur les prix, les profits et le bien-être des consommateurs sont incertains.

Le modèle est tel que les firmes se font compétition sur deux périodes. Les qualités initiales sont données, et dans la première période, la phase de développement, les firmes se font compétition en prix. Les quantités vendues affectent la qualité du produit à la période suivante, la phase de maturité. Le concept d'équilibre étant l'équilibre de Nash parfait en sous-jeu, nous trouvons d'abord les prix optimaux à la deuxième période avant de remonter à la phase de développement. Une caractérisation complète des équilibres en stratégies pures est offerte.

Comme dans la plupart des modèles d'oligopoles dynamiques avec asymétries (Maskin et Tirole (1988), Athey et Schmutzler (2001)), nous trouvons que ces asymétries tendent à augmenter ; la firme ayant initialement la qualité la plus élevée accentuera son avance. Toutefois, contrairement à Tirole (1988) dans son modèle où le choix des qualités est sans coût, nous n'obtenons pas toujours une différenciation complète. Dans certains cas, les deux firmes vendent une quantité strictement positive dans la phase de développement.

Dans un cadre général, Athey et Schmutzler (2001) trouvent que même si l'asymétrie grandissante a un effet négatif sur le bien-être social, si les investissements ou efforts faits pour accentuer cette asymétrie améliorent le bien-être des consommateurs, l'effet net peut être positif. Dans notre cas, l'effet net sur le bien-être social est toujours positif. En plus, l'effet est positif pas seulement sur l'ensemble des périodes, mais à chaque période. Il est même possible qu'améliorer marginalement le niveau des contributions soit une amélioration parétienne.

Comme dans la plupart des modèles d'oligopoles dynamiques, nous trouvons des possibilités de tarification sous le coût marginal. Par contre, dans certains cas, la firme dominée choisit de ne pas lutter dans la phase de développement dans le but de profiter d'une différenciation maximale ensuite. Il est alors possible que le prix payé à la première période par certains consommateurs augmente.

Diverses extensions, reliées aux asymétries entre firmes, à une variation dans le temps des préférences des consommateurs et à la couverture du marché sont ensuite discutées.

Chapter 1

On the core of single-source uncapacitated network flow problems

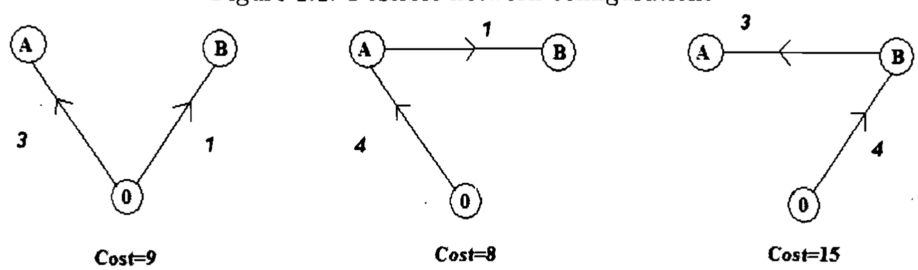
1 Introduction

We are interested in problems where agents located at different points in space demand a good that they can only get from a common source. To obtain this good, they must be connected to the source so that the good can be delivered to them. An efficient network is built, and the cost of this network must be shared among the agents. Applications of this model include gas and oil pipelines, electricity distribution networks, delivery routes and communication networks.

Consider this example.

Example 1 *2 countries, A and B, consider cooperating on the construction of a gas pipeline, that would supply them from a foreign country, that we call the source (0). Country A needs a supply of 3 millions cubic feet per day. The needs of country B are of 1 million cubic feet per day. There are three possible segments to the pipeline: one between the source and country A, one between the source and country B, and one between countries A and B. In this last segment, the flow can be from country A to country B, or in the opposite direction. If* x *is the flow in million cubic feet per day, the cost, in millions of dollars, on the segment between the source and A is* $3 + x$ *for a strictly positive flow and* 0 *else. On the segment between the source and B, it is* $3x$*, and between A and B, in any direction, it is* x*. We then have the following possible network configurations (flows are in italics):*

Figure 1.1: Possible network configurations



The second configuration is optimal.

This model, called the *single-source uncapacitated network flow (SSUNF) problem*, has been studied extensively in the operations research literature, where the focus has been on the search for efficient approximation methods. A natural way to define a game from this problem is to assign to each coalition its minimal cost to fulfill the demand of its agents. We call this game the *SSUNF game*. We are interested in the conditions under which the problem generates a game that has a non-empty core; that is, when there are cost allocations such that no subset of agents is better off by leaving the coalition. Such allocations are said to be *stable allocations.* Stability is a crucial concept, as it is important in understanding which coalition will form on its own and which will require outside intervention.

SSUNF games are part of the large family of transferable utility games already studied in the cooperative game-theoretic literature. It shares characteristics with some of the early models from this literature, where agents have a demand for a single good, and share a production technology to commonly produce the amount of good needed to satisfy the demand of every agent. Cost is then shared among these agents. Bondareva (1963) and Shapley (1967) independently showed that a cooperative game with transferable utility has a non-empty core if and only if the game is balanced. A game is balanced if the sum of costs over all coalitions, weighted by a balanced set of weights, is at least as large as the cost for the coalition composed of all agents. A balanced set of weights is such that for any player i, the sum of weights over the coalitions that include i equals one.

Generalizations of this model have been made, notably to cases where the goods are specific to agents, by Billera and Heath (1982), Mirman and Tauman (1982) and Samet and Tauman (1982). In their framework, a characteristic function assigns to any demand vector in $\mathbb{R}_+^N$ a cost value in $\mathbb{R}_+$. In our proposed setting, this cost for each coalition is directly dependent on the underlying network structure.

Game theoretic tools have been used on many problems that have network characteristics. See Sharkey (1995) for a review of many such models.

SSUNF games have been studied in Quant et al. (2006). They show that if all cost functions are convex, the resulting game always has at least one stable allocation. While they prove the result on restricted versions of SSUNF games and mention that it can be extended to all SSUNF games, we provide an explicit proof of the result.[1]

Quant et al. (2006) also show that if all cost functions are concave, the game can have an empty core. As a complement to this result, we offer simpler examples where concave

[1] When this paper was written, the author was aware of a working paper by Quant, Borm and Reijnierse in which they studied restricted versions of SSUNF games. It is only after the author had independently proven the result for SSUNF games that Quand, Borm and Reijnierse mentioned the extension.

SSUNF problems generate unstable games.

While we have this negative result, some families of concave SSUNF games are known to always be stable. *Cost spanning tree games* are the most studied example. Cost structures are such that there is only a fixed cost associated with each arc. Bird (1976) was the first to study cost spanning tree problems with game-theoretic tools, and he proposed a cost allocation that is always in the core. An algorithm to generate all core allocations was obtained by Granot and Huberman (1984).

For more general concave SSUNF problems, difficulties arise because of the non-availability of exact algorithms to build optimal networks.[2] We try to shed light on the family of concave SSUNF problems generating games with non-empty cores by introducing a new family of problems, called the *ordered quasi-symmetric congestion problems.* In this set of problems, agents all demand the same quantity. Agents are ordered according to costs on the arc between the source and them. If agent i comes before agent j in that order, the incremental cost of a kth unit on the arc between the source and agent i is no larger than on the arc between the source and agent j, for all positive integers k. For all other arcs, cost functions are identical.

An example of an ordered quasi-symmetric problem would be a situation where neighbors want an internet connection. While they could all get individual connections through an Internet Service Provider (ISP), they could set up a network in which one of them is subscribing to an ISP, while the others gain access via wireless connections to the subscriber. We can suppose that all agents demand roughly the same amount of bandwidth. The cost to connect to the wireless network is identical for all agents. The ISP offers quantity discounts, as the average price of bandwidth goes down with usage. Most ISP also bundle internet connections with other telecommunication services (phone, television) which can result in agents facing roughly proportional price schedules.

We show that this family of problems generates permutationally concave games, a concept introduced in Granot and Huberman (1982). It is a weakening of the concept of concavity of a game, but it nonetheless guarantees the existence of at least one stable allocation. The well-studied class of concave games are such that the cost of adding an agent to a coalition never increases when that coalition grows. See also Driessen (1990) for a discussion of the

[2] See Guisewite and Pardalos (1991) for a review of approximation methods.

link between stability and permutational concavity.

The concept of permutational concavity has been used to show non-emptiness of the core of minimum cost spanning forest games (Alidaee (1994)), which are a generalization of cost spanning trees where there are multiple sources. It is also used in games generated by different operations research problems, such as inventory games (Meca et al. (2004)) and various sequencing games (van Velzen (2006), van Velzen and Hamers (2003)). Cost spanning tree problems also generate permutationally concave games.

The paper is organized as follows. Section 2 defines SSUNF problems. In Section 3, we study the core of concave SSUNF problems and find that in general, it can be empty. For some simple SSUNF problems, however, the core is always non-empty. Section 4 introduces a family of concave SSUNF problems, the ordered quasi-symmetric congestion problems. In Section 5, we show that ordered quasi-symmetric congestion problems generate games for which the core is always non-empty. In Section 6, we look at the case of convex SSUNF problems, and we provide an explicit proof of the stability result announced by Quant et al. (2006). Section 7 concludes.

2 The model

Let $N = \{1, ..., n\}$ be the set of agents. All agents are located at different points in space, called nodes. Each node is identified by the agent occupying it. There is a special node called the source, denoted by 0, which does not contain any agent. Each agent has a demand for a common good. The demand profile is $Q = (q_1, ..., q_n) \in \mathbb{R}^n_+$. A connection to the source is needed in order to obtain the good. The set of nodes is $N_0 = N \cup \{0\}$ and the set of arcs between nodes is $A = \{(i,j) \mid i,j \in N_0,\ i \neq j\}$.

The cost of a flow of z_{ij} units of good on the arc (i,j), that is from node i to node j, is $c_{ij}(z_{ij})$. The cost function $c_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing and $c_{ij}(0) = 0$. c_{ij} is also continuous on $\mathbb{R}_{++}$. We do not impose continuity at 0. In particular, we allow for functions with fixed costs, so that this model includes as particular cases cost spanning trees and linear functions with intercept. We call $\gamma = (c_{ij})_{i,j \in N_0}$ the cost structure.

We assume that there is a feasible arc between each pair of nodes in N_0.[3] This allows us

[3] This is not a restrictive assumption as we can approximate a problem with non-feasible arcs by assigning cost functions to these arcs that generate an arbitrarily large cost to any flow.

to define a problem by the set of agents, the demand profile and the cost structure; that is, (S, Q, γ) with $S \subseteq N$. The problem implies delivering quantities q_i for all $i \in S$ and nothing to all other agents, using only arcs between nodes in S_0, while facing the cost structure γ. A problem where the cost structure only contains concave cost functions is said to be a *concave problem.* Concave problems are natural in many settings. The resulting economies of scale are characteristic of set-up costs of large physical networks, like gas and oil pipelines and water and electricity distribution networks, among many others. Costs for a small flow implies large investments to set the network up, but building a slightly larger network adds little to the cost. On the other end of the spectrum, we will also consider cost structures where all cost functions are convex, called *convex problems.* This structure is natural in many data transmission networks and generally in models where congestion comes into play. If we think of the cost in term of time of transmission, larger flows imply congestion and can induce exponential growth in the time of transmission.

If $c_{ij}(x) = c_{ji}(x)$ for all $i, j \in N_0$ and all $x \in \mathbb{R}_+$, then we say that the cost structure is *symmetric.* While simpler, this assumption can be naturally dropped in many settings, for instance if gravity affects costs. Then, if there is a slope between i and j, costs on the arc (i, j) and (j, i) need not be symmetric.

A provision plan is a matrix $Z \in \mathbb{R}_+^A$ which assigns a flow $z_{ij} \geq 0$ to every arc (i, j) in A. For a plan Z, define $supp(Z) = \{(i, j) \in A \mid z_{ij} > 0\}$, the set of arcs used in the plan Z. A path $r_{ij}(Z)$ from i to j in Z is a sequence $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ of arcs in $supp(Z)$, all distinct and such that $i = i_0$ and $j = i_l$. Define $R_{ij}(Z)$ as the set of all paths $r_{ij}(Z)$. We say that a plan Z is a tree if for all agents i, $R_{0i}(Z)$ is a singleton, that is for any agent, there is a unique path between the source and i.

Formally, for the problem (S, Q, γ), with $S \subseteq N$, we define the set of *feasible provision plans* as the set of plans Z satisfying these properties:[4]

$$\left.\begin{array}{c} \sum_{i \in S} z_{0i} \geq \sum_{i \in S} q_i \\ \sum_{j \in N_0 \setminus \{i\}} (z_{ji} - z_{ij}) \geq q_i \text{ for all } i \in S \\ z_{ij} = z_{ji} = 0 \text{ for all } i \in N \setminus S, \text{ all } j \in N_0 \end{array}\right\} \quad (1.1)$$

We will denote by $F(S, Q)$ the set of plans which respect the conditions (1.1) with equal-

[4] If we impose that $z_{i0} = 0$ for all $i \in N$, then the first condition is implied by the second. While the two set of conditions are equivalent, the proposed one is more illustrative.

ities. We can restrict our attention to $F(S,Q)$ since the functions c_{ij} are non-decreasing. In addition, it is clear that plans in which we have flows $z_{ij} > 0$ and $z_{ji} > 0$ are always at least as costly as a plan Z' that has $z'_{ij} = \max(z_{ij} - z_{ji}, 0)$ for all $(i,j) \in A$.

The first condition in (1.1) implies that the total flow going out of the source must be equal to what is consumed within the network. The second condition ensures that subtracting what goes out of a node from what goes in leaves enough quantity to respond to the demand of the agent at this node. The third condition says that agents in S are constrained to use only the nodes that are in S_0.

The optimization problem faced by a coalition S is to find a feasible plan $Z \in F(S,Q)$ that minimizes the cost $c(Z) = \sum_{(i,j)\in A} c_{ij}(z_{ij})$. An efficient plan Z^S is in $\arg\min_{Z\in F(S,Q)} c(Z)$.

Let $C(S,Q,\gamma) = \min\{c(Z) \mid Z \in F(S,Q)\}$ denote the resulting minimal cost.[5] The mapping $C(\cdot,Q,\gamma) : \mathfrak{S} \cup \{N\} \to \mathbb{R}_+$ is the TU-cooperative game generated by (N,Q,γ), with $\mathfrak{S} = \{S \mid \varnothing \neq S \subset N\}$,[6] that we call the single-source uncapacitated network flow (SSUNF) game.

The game $C(\cdot,Q,\gamma)$ is sub-additive, that is $C(S,Q,\gamma) + C(T,Q,\gamma) \geq C(S \cup T,Q,\gamma)$ for all $S,T \subset N$ such that $S \cap T = \varnothing$. For sets S and T, the sum of their respective efficient plans is a feasible plan for $S \cup T$. Since S and T are disjoint, this plan will have cost $C(S,Q,\gamma) + C(T,Q,\gamma)$. Therefore, by definition of $C(\cdot,Q,\gamma)$, we have $C(S,Q,\gamma) + C(T,Q,\gamma) \geq C(S \cup T,Q,\gamma)$.

However, $C(\cdot,Q,\gamma)$ is not always monotonic. It is possible that adding an agent allows the coalition to use lower cost arcs that will reduce the overall cost, as illustrated by this simple example.

Example 2 *Suppose a network with 2 agents and a source. We have* $Q = (1,1)$, $c_{01}(x) = x$, $c_{02}(x) = 10x$, $c_{12}(x) = c_{21}(x) = 2$ *for all* $x \in \mathbb{R}_{++}$ *and* $c_{ij}(0) = 0$ *for all* $(i,j) \in A$.

We have $C(\{2\},Q,\gamma) = 10$ *and* $C(\{1,2\},Q,\gamma) = \min\{10+1, 2+2, 20+2\} = 4$. *Thus* $C(\{2\},Q,\gamma) > C(\{1,2\},Q,\gamma)$.

A cost allocation for the problem (N,Q,γ) is a vector y assigning a cost share to each agent $i \in N$. The efficiency condition imposes that $\sum_{i\in N} y_i = C(N,Q,\gamma)$. We will not impose any other condition on y. In particular, we will not impose $y_i \geq 0$. As we saw in Example

[5] Since cost functions are non-decreasing and $c_{ij}(0) = 0$ for all $i,j \in N_0$, the existence of a minimum is guaranteed.

[6] Throughout this thesis, $\subset$ is meant as $\subsetneq$.

2, in some problems an agent can generate important economies to other agents by letting them use his node to connect to the source. In such problems, it would be conceivable to subsidize this agent.

The core of the problem (N, Q, γ) is the set of cost allocations such that no subset S of N is better off by leaving the grand coalition N.

Formally, we define the core as

$$core(Q,\gamma) = \left\{ y \left| \begin{array}{c} \sum_{i \in S} y_i \leq C(S,Q,\gamma) \text{ for all } S \subset N \\ \sum_{i \in N} y_i = C(N,Q,\gamma) \end{array} \right. \right\}.$$

We say that a problem (N, Q, γ) is stable (unstable) if $core(Q, \gamma)$ is non-empty (empty).

3 Concave SSUNF problems

Building the network is the first step in determining the stability of a problem. It is a crucial step, as knowledge of the structure of the efficient plans is needed to talk about stability. Unfortunately for general concave SSUNF problems, we do not have an algorithm that can exactly solve all cases.

We know, from a classic result from Zangwill (1968) that we will always have in a concave problem a tree among the efficient plans. The only way to find the exact solution is thus to enumerate all feasible trees and choose one that minimizes costs. For large problems, this will likely be impossible. Finding the core necessitates solving 2^N efficiency problems, one for each coalition, so this becomes a problem even for games with few agents.

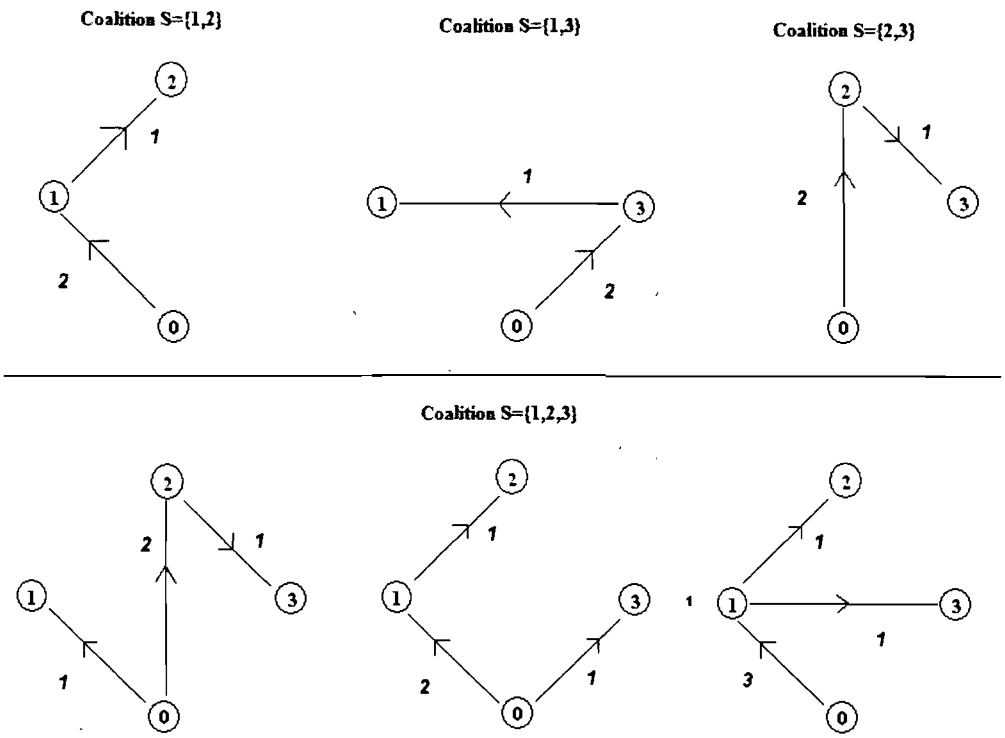
3.1 Examples showing possible instability

Quant et al. (2006) show, with a 6-player example, that concave SSUNF problems can be unstable. We offer two simpler examples, that illustrate that even with only 3 players stability is difficult to achieve.

Example 3 (N, Q, γ) *is a concave problem with a non-symmetric cost structure such that* $N = \{1, 2, 3\}$ *and* $Q = (1, 1, 1)$. *For* $x > 0$, *we have the following cost functions:* $c_{01}(x) = 3$, $c_{02}(x) = 6$, $c_{03}(x) = 5$, $c_{12}(x) = 1 + 3x$, $c_{21}(x) = 4x$, $c_{13}(x) = 1 + 4x$, $c_{31}(x) = 2x$, $c_{23}(x) = 1 + 2x$ *and* $c_{32}(x) = 5 + x$.

Figure 1.2 illustrates efficient plan(s) for each coalition of a least two agents. Flows are indicated in italics.

Figure 1.2: Optimal plans for non-singleton coalitions in Example 3



We can verify that the function $C(\cdot, Q, \gamma)$ *takes the following values:*

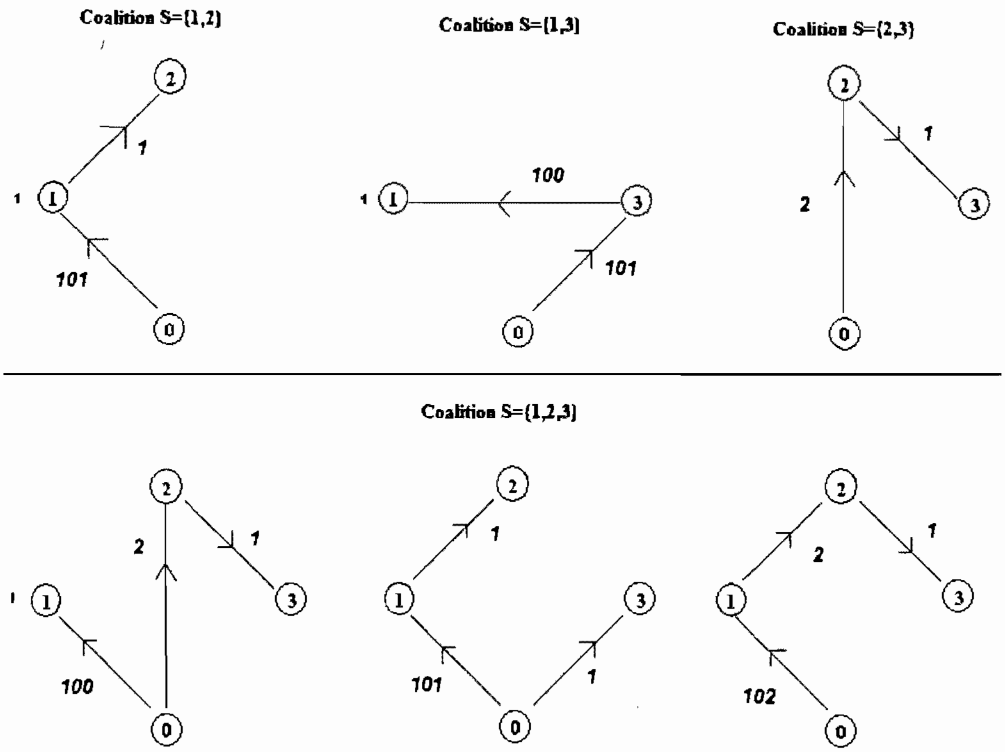
S	$C(S,Q,\gamma)$
$\{1\}$	*3*
$\{2\}$	*6*
$\{3\}$	*5*
$\{1,2\}$	*7*
$\{1,3\}$	*7*
$\{2,3\}$	*9*
$\{1,2,3\}$	*12*

Using these values, we know that, for the core to be non-empty, we must have $y_1 + y_2 \leq 7$, $y_1 + y_3 \leq 7$ *and* $y_2 + y_3 \leq 9$. *Adding these 3 constraints, we get* $y_1 + y_2 + y_3 \leq 11.5 < 12 = C(N, Q, \gamma)$. *Thus, the core is empty.*

Example 4 *(N, Q, γ) is a concave problem with a symmetric cost structure such that $N = \{1, 2, 3\}$ and $Q = (100, 1, 1)$. For $x > 0$, we have the following cost functions: $c_{01}(x) = 6$, $c_{02}(x) = 2$, $c_{03}(x) = 4$, $c_{12}(x) = c_{21}(x) = 1 + \frac{x}{2}$, $c_{23}(x) = c_{32}(x) = 3 + \frac{x}{2}$, $c_{13}(x) = c_{31}(x) = 5 + \frac{1-\epsilon}{100}x$ with $\epsilon \in \,]0, 0.009]$.*

Figure 1.3 illustrates efficient plan(s) for each coalition of a least two agents. Flows are indicated in italics.

Figure 1.3: Optimal plans for non-singleton coalitions in Example 4



We can verify that the function $C(\cdot, Q, \gamma)$ takes the following values:

S	$C(S,Q,\gamma)$
$\{1\}$	6
$\{2\}$	2
$\{3\}$	4
$\{1,2\}$	7.5
$\{1,3\}$	$10-\epsilon$
$\{2,3\}$	5.5
$\{1,2,3\}$	11.5

Since we must have $y_1 + y_2 \leq 7.5$, $y_1 + y_3 \leq 10 - \epsilon$ and $y_2 + y_3 \leq 5.5$, adding these 3 constraints gives us $y_1 + y_2 + y_3 \leq 11.5 - \frac{\epsilon}{2} < 11.5 = C(N,Q,\gamma)$. Thus, the core is empty.

3.2 Simple stable concave SSUNF problems

As seen in the previous subsection, in the general concave case, the core of the associated cost game can be empty. In addition, algorithms to find exact optimal solutions do not exist. However, some simple concave problems are easily solved and their natural solutions lead us to conclude that these problems are always stable.

First, if the cost functions do not depend on the magnitude of the flows, that is they are of the form $c_{ij}(x) = k_{ij}$ if $x > 0$, we are back to the cost spanning tree format and we can apply Prim's algorithm, which builds an efficient plan arc by arc, selecting at each step an arc that connects at the smallest cost an agent not previously connected. We know in this format that the core is always non-empty (Bird (1976)).

If the costs depend on the magnitude of the flows, the simplest problem is the *linear problem*, where γ is such that we have $c_{ij}(x) = a_{ij}x$ for all $(i,j) \in A$ and all $x \in \mathbb{R}_+$. To find the optimal plan in such problems, one can look at each agent i separately, select an optimal path between 0 and i, and sum these paths. Formally, take a plan Z such that $z_{jk} > 0$ for all $(j,k) \in A$. For agent i, select a path $P_i \in \arg\min_{r_{0i}(Z) \in R_{0i}(Z)} \sum_{(j,k)\in r_{0i}(Z)} a_{jk}$ and construct a plan Z^i such that $z^i_{jk} = q_i$ if $(j,k) \in P_i$ and 0 otherwise. Then, the optimal plan for the coalition N is $Z^* = \sum_{i\in N} Z^i$. This is called the shortest path algorithm.

We can go one step further, in the case of the *linear problem with equal fixed cost*, where γ is such that we have $c_{ij}(x) = b + a_{ij}x$ for all $(i,j) \in A$ and $x \in \mathbb{R}_{++}$, with $b \geq 0$ being a fixed cost that is the same on all arcs. We can keep using the shortest path algorithm, but this time adding the constraint that we choose the optimal paths P_i such that for any $S \subseteq N$, $\sum_{i \in S} Z^i$ form a tree. Define γ^l as a cost structure generating a linear problem, and γ^f as a cost structure where we add an equal fixed cost b to all cost functions in γ^l. We then have, for all $S \subseteq N$, $C(S, Q, \gamma^f) = C(S, Q, \gamma^l) + mb$ where $m \leq |S|$ is the number of agents in S having strictly positive demands.

Since in this problem all arcs have the same fixed cost, it does not become a factor in the network design, other than limiting the total number of arcs that we build.

In the linear problem, a very natural way to allocate the cost is to assign to each agent the cost of his path, that is $y_i^l = c(Z_i)$ for all $i \in N$. One can easily see that this allocation is always in $core(Q, \gamma^l)$.

Using the fact that $C(S, Q, \gamma^f) = C(S, Q, \gamma^l) + mb$, where $m \leq |S|$ is the number of agents in S having strictly positive demands, and the fact that $core(Q, \gamma^l)$ is non-empty, it is clear that assigning, for all $i \in N$, $y_i^f = y_i^l + b$ if $q_i > 0$ and $y_i^f = y_i^l = 0$ otherwise, will generate an allocation in $core(Q, \gamma^f)$. In the case where all agents have strictly positive demands, we have $core(Q, \gamma^f) = core(Q, \gamma^l) + b$.

Furthermore, we can easily prove that if we have the triangular inequality, that is $a_{ij} \leq a_{ik} + a_{kj}$ for all $i, j, k \in N_0$, then an efficient plan connects all agents directly to the source. Then, the function $C(\cdot, Q, \gamma)$ is additive and $core(Q, \gamma)$ is the singleton $y_i = c_{0i}(q_i)$.

For the next sets of simple stable concave SSUNF games, we need some definitions. We will say that an arc (i, j) is *primary* if $i = 0$ and $j \in N$, that is if it is between the source and an agent $i \in N$. Similarly, we will say that an arc (i, j) is *secondary* if $i, j \in N$. We say that an agent i is *connected directly* in plan Z if $(0, i) \in supp(Z)$. Otherwise, he is connected indirectly.

Another simple case is when secondary arcs are extremely costly compared to primary arcs. Suppose that c_{0i} is a concave function for all $i \in N$ and that $c_{ij}(x) = k$ if $x > 0$ for all $i, j \in N$, with $k > \max_{i \in N} c_{0i}\left(\sum_{j \in N} q_j\right)$. Costs on secondary arcs, k, are so large that it is clear that the efficient way to connect all agents is by using only the primary arcs. We then have, for any γ having this structure and any $S \subseteq N$, that $C(S, Q, \gamma) = \sum_{i \in S} c_{0i}(q_i)$. Clearly the game is additive, and $core(Q, \gamma)$ is the singleton $y_i = c_{0i}(q_i)$ for all $i \in N$.

We can extend this result to conclude that whenever secondary arcs have arbitrarily large costs compared to primary arcs, we will have this efficient plan where all agents are directly connected to the source, in which case the game is additive and $core(Q, \gamma)$ a singleton.

The opposite case is when secondary arcs have no costs for any flow. We say that a problem is a *concave problem with free secondary arcs* if γ is such that we have $c_{0i}(x) = f_i(x)$ for all $i \in N$, with $f_i(x)$ concave, and $c_{ij}(x) = 0$ for all $i, j \in N$ and all $x \in \mathbb{R}_+$.

In such a problem, it is optimal to have only one agent directly connected to the source, say agent i, and all other agents connected to this agent i. Since in this model we allow different concave cost functions on arcs $(0, j)$, we will have that if agent i is the one that is directly connected to the source in the problem $C(S, Q, \gamma)$, then $i \in \arg\min_{j \in S} f_j(\sum_{k \in S} q_k)$. Thus for any concave problem with a cost structure with free secondary arcs, $C(S, Q, \gamma) = \min_{j \in S} f_j(\sum_{k \in S} q_k)$.

To prove that the core of this game is non-empty, define the more restrictive game $C'(S, Q, \gamma) = F(\sum_{j \in S} q_j)$ with $F(x) \equiv \min_{i \in N} f_i(x)$ for all $x \in \mathbb{R}_+$ and all $S \subseteq N$. Since in a concave problem with free secondary arcs all f_i are concave functions and the minimum of concave functions is concave, F is concave. Thus, incremental cost cannot be increasing in the size of S, and $core(C'(S, Q, \gamma))$ is non-empty. It is also clear that for $S \subset N$, we have $C'(S, Q, \gamma) \leq C(S, Q, \gamma)$ and $C'(N, Q, \gamma) = C(N, Q, \gamma)$. Since the conditions to be in $core(C(\cdot, Q, \gamma))$ are less strict than for $core(C'(\cdot, Q, \gamma))$, allocations in $core(C'(\cdot, Q, \gamma))$ are in $core(C(\cdot, Q, \gamma))$. Therefore, $core(Q, \gamma)$ is non-empty for any concave problem with free secondary arcs.

Thus, at the opposite end of the spectrum, we know that if costs on secondary arcs are arbitrarily small compared to costs on primary arcs, $core(Q, \gamma)$ will be non-empty.

4 Ordered quasi-symmetric congestion problems

We introduce a new and non-trivial set of problems that generate stable games. We look at a subset of concave problems that have the following characteristics. Agents all demand the same quantity, that we normalize to one unit. For any $i, j, k, l \in N$ and $m \in \mathbb{R}_+$, we have $c_{ij}(m) = c_{kl}(m)$; cost functions on secondary arcs are identical. In particular, define $\theta = c_{ij}(1)$. For all $i, j \in N$, let $\alpha^i_j = c_{0i}(j) - c_{0i}(j-1)$ be the marginal cost of unit j along arc $(0, i)$. Agents are ordered according to these marginal costs on their primary arc. Formally, for all $k \in N$, if $i < j$ then $\alpha^i_k \leq \alpha^j_k$. The case where cost functions on primary arcs are all proportional is an obvious example of this structure. The concavity of the cost functions imposes that for all $i, j, k \in N$, $\alpha_j^i \geq \alpha_k^i$ if $j < k$. We call such a problem an *ordered quasi-symmetric congestion problem.* This terminology comes from the fact that costs on secondary arcs are identical, while agents are ordered according to costs on primary arcs. Since demand plays no role, an ordered quasi-symmetric congestion problem is defined by (N, γ).

An optimal plan for an ordered quasi-symmetric congestion problem can be found using the following Algorithm:

Algorithm 1 *i) Let* $r_n = \alpha_2^1 + \theta - \alpha_1^n$*. For all* $i \in N \backslash \{1, n\}$*, compute* $r_i = \alpha_{2+n-i}^1 + \theta - \alpha_1^i + \sum_{j=i+1}^n r_j$.

ii) If $min_{i \in N \backslash \{1\}} r_i \geq 0$*, let, for all* $i \in N$*,* $z_{0i} = 1$ *and* $z_{ij} = 0$ *else, define* $I = \varnothing$ *and terminate. Otherwise, move to step iii).*

iii) Pick the smallest $j \in N \backslash \{1\}$ *such that* $r_j = min_{i \in N \backslash \{1\}} r_i$*. Then,* $z_{01} = n - j + 1$*, for all* $i \geq j$*,* $z_{1i} = 1$*, for all* $1 < i < j$*,* $z_{0i} = 1$ *and* $z_{ik} = 0$ *otherwise. Define* $I = \{j, j+1, ..., n\}$.

I is the set of agents connected indirectly in the optimal plan selected by the algorithm. Let $D = N \backslash I$ be the set of agents connected directly. More generally, let I_S denote the set of agents connected indirectly in the problem (S, γ).

For every $i \in \{2, ..., n\}$, the algorithm computes r_i, the difference in cost between the network where agents $\{i, i+1, ..., n\}$ are connected indirectly and the network where all agents are connected directly. Steps ii) and iii) then select the optimal network.

This gives us an expression for the cost of coalition N, $C(N, \gamma) = \sum_{k \in N \backslash I} \alpha_1^k + \theta |I| + \sum_{k=2}^{|I|+1} \alpha_k^1$.

In order to prove that this algorithm generates efficient plans, we need the following notation.

Notation 1 *For any* $i \in N$*, let* $[i]$ *denote the set* $\{1, ..., i\}$.

Theorem 1 *For any ordered quasi-symmetric congestion problem, a plan Z found using Algorithm 1 is an optimal plan.*

Proof. Following Zangwill (1968), we can restrict our attention to trees.

Suppose that $m \geq 2$ agents are connected on a path to agent i, who is connected to the source. Cost is $c_{0i}(m+1) + \sum_{l=1}^m c_{ij}(m - l + 1)$. This plan is always (weakly) dominated by a plan where the m agents are directly connected to i, as cost is $c_{0i}(m+1) + mc_{ij}(1)$.

Moreover, if the set of agents S have to be connected indirectly, it is always optimal to connect them to the same agent, as cost functions are concave. Moreover, as $c_{01}(x) \leq c_{0i}(x)$ for all $i \in N$ and all $x \in \mathbb{R}_+$, agent 1 should always be connected directly to the source ($z^*_{01} \geq 1$).

We can limit our attention to the set of plans where we have a set of agents $D \subseteq N$ such that $1 \in D$ that are directly connected to the source and a set $I = N \backslash D$ that are connected indirectly.

Also, notice that in order for Z^* to be optimal, if $i \in I$, than for all $j > i$, $j \in I$. Suppose that there exists a $j > i$ such that $j \in D$. Then, $c(Z^*) = c_{01}(|I|+1)+|I|\,\theta+\sum_{k\in D\backslash\{j\}} \alpha_1^k+\alpha_1^j$. It is weakly dominated by the plan Z' where $j \in I'$, $i \in D'$, $D\backslash\{j\} = D'\backslash\{i\}$ and $I\backslash\{i\} = I'\backslash\{j\}$. The cost is then $c(Z') = c_{01}(|I'|+1)+|I'|\,\theta+\sum_{k\in D'\backslash\{i\}} \alpha_1^k+\alpha_1^i$. Thus, in an optimal plan, there exists an $i \in N$ such that $D = [i]$.

For all $i \in N$, let Z^i be the plan where $D = [i]$.

Step ii) of the algorithm verifies if Z^n is optimal. If not, step iii) finds an optimal Z^i by looking at all of them. ■

Theorem 1 tells us that the agents connected indirectly are always those at the end of the ranking. Therefore, if $I = \{i, i+1, ..., n\}$, then $C(N,\gamma) = \sum_{k=1}^{i-1} \alpha_1^k + \theta(n-i+1) + \sum_{k=1}^{n-i+2} \alpha_k^1$. An ordered quasi-symmetric congestion game might have more than one optimal plan, but the algorithm picks the one with the largest set of (consecutive) players connected indirectly. Thus, $C(N,\gamma) < \sum_{k=1}^{j-1} \alpha_1^k + \theta(n-j+1) + \sum_{k=1}^{n-i+2} \alpha_k^1$ for all $j < i$ and $C(N,\gamma) \leq \sum_{k=1}^{j-1} \alpha_1^k + \theta(n-j+1) + \sum_{k=1}^{n-i+2} \alpha_k^1$ for all $j > i$.

For any coalition $S = \{s_1, ..., s_{|S|}\} \subseteq N$ where agents are ordered in the same way as in N, we have, by Theorem 1,

$$C(S,\gamma) = \sum_{k\in S\backslash I_S} \alpha_1^k + \theta\,|I_S| + \sum_{k=2}^{|I_S|+2} \alpha_k^{s_1}.$$

5 Permutational concavity and stability of ordered quasi-symmetric congestion problems

The concept of concavity of a game is closely related to the core. A game is said to be concave if adding an agent to a coalition S is at least as costly as adding him to a coalition

T, for $S \subset T$. These games have a core that is easily characterized: if we take an ordering of the agents and assign to each of them his incremental cost, we obtain an extreme point of the core. In fact, every extreme point can be obtained in that manner (Shapley (1971)). In addition, every game having a core with this characteristic must be a concave game (Ichiishi (1981)).

We use a related concept, permutational concavity, introduced by Granot and Huberman (1982). The formal definition follows.

Definition 1 *A game C is permutationally concave if there exists a labeling of the players $1, 2, ..., n$ such that*

$$C([j] \cup S) - C([j]) \leq C([i] \cup S) - C([i])$$

for all $S \subseteq N \backslash [j]$ and $j \geq i$.

This is in contrast with a concave game, for which the above definition holds for any labeling of the agents. Alternatively, in a concave game, we have

$$C(S \cup T) - C(T) \leq C(S \cup T') - C(T')$$

for all $T' \subset T$ and all $S \subseteq N \backslash T$.

While permutational concavity is a weakening of the concept of concavity, it nonetheless guarantees the existence of at least one stable allocation.

Theorem 2 *(Granot and Huberman (1982)) A permutationally concave game has a non-empty core.*

The proof of this result consists in showing that the allocation $y_i = C([i]) - C([i-1])$ is always in the core. Again, by contrast, while this allocation is in the core of a permutationally concave game only when agents are correctly ordered, in a concave game, this allocation will be in the core for any ordering of the agents.

We show that ordered quasi-symmetric congestion games are stable, by demonstrating that they are permutationally concave.

Theorem 3 *Any ordered quasi-symmetric congestion game (N, γ) has a non-empty core.*

We require the following lemma.

Lemma 1 *For ordered quasi-symmetric congestions problems (S,γ) and (T,γ), with $S \subseteq T \subseteq N$ such that $S = (s_1, ..., s_{|S|})$ and $T = (t_1, ..., t_{|T|})$, if their elements are ordered in the same manner as in N and if $s_1 = t_1$, then $I_S \subseteq I_T$.*

Proof. Suppose that $I_S = \{s_{l+1}, ..., s_{|S|}\}$, with $s_{l+1} = t_{p+1} = j \in N$. Since $S \subseteq T$, we have $l \leq p$. By Theorem 1, agents connected indirectly in an optimal plan are the agents ranked highest. We first show that $|I_S| \leq |I_T|$.

By our assumption on I_S, $C(S,\gamma) = \sum_{k=1}^{l} \alpha_1^{s_k} + \theta(|S| - l) + \sum_{k=2}^{|S|-l+1} \alpha_k^1$.

Taking into account the non optimal plan to connect agents $\{s_{m+1}, ..., s_{|S|}\}$ indirectly to the source, with $m \in \{l+1, ..., |S|\}$, we have

$$C(S,\gamma) \leq \sum_{k=1}^{m} \alpha_1^{s_k} + \theta(|S| - m) + \sum_{k=2}^{|S|-m+1} \alpha_k^1 \text{ and hence, with the definition of } C(S,\gamma)$$

$$\theta(m-l) + \sum_{k=|S|-m+2}^{|S|-l+1} \alpha_k^1 \leq \sum_{k=l+1}^{m} \alpha_1^{s_k}. \tag{1.2}$$

With respect to coalition T, consider the plan to connect the $|S| - l$ last agents, that is $\{t_{|T|-|S|+l+1}, ..., t_{|T|}\}$, indirectly to the source. The cost of such a plan is $\sum_{k=1}^{|T|-|S|+l} \alpha_1^{t_k} + \theta(|S| - l) + \sum_{k=2}^{|S|-l+1} \alpha_k^1$. As an alternative, consider a plan where the set of agents connected indirectly is $\{t_{|T|-|S|+m+1}, ..., t_{|T|}\}$, with $m \in \{l+1, ..., |S|\}$. We will have that the first plan has a lesser or equal cost if

$$\sum_{k=1}^{|T|-|S|+l} \alpha_1^{t_k} + \theta(|S| - l) + \sum_{k=2}^{|S|-l+1} \alpha_k^1 \leq \sum_{k=1}^{|T|-|S|+m} \alpha_1^{s_{tk}} + \theta(|S| - m) + \sum_{k=2}^{|S|-m+1} \alpha_k^1$$

which simplifies to $\theta(m-l) + \sum_{k=|S|-m+2}^{|S|-l+1} \alpha_k^1 \leq \sum_{k=|T|-|S|+l+1}^{|T|-|S|+m} \alpha_1^{t_k}$. (1.3)

By the assumption that $s_{l+1} = t_{p+1} = j$ and since $p \leq l + |T| - |S|$, we have that $t_{|T|-|S|+k} \geq s_k$, and by the ordering assumption, that $\alpha_1^{t_{k+|T|-|S|}} \geq \alpha_1^{s_k}$ for all $k \in \{l+1, ..., m\}$. Therefore, $\sum_{k=|T|-|S|+l+1}^{|T|-|S|+m} \alpha_1^{t_k} \geq \sum_{k=l+1}^{m} \alpha_1^{s_k}$, and by (1.2), (1.3) holds. Hence, for coalition T, a plan where the last $|S| - l$ agents are connected indirectly is (weakly) better than any plans containing less agents connected indirectly. Thus, $|I_S| \leq |I_T|$.

By our assumption on I_S, we obtain, by letting $m = l + 1$ in (1.2), that

$$\theta + \alpha^1_{|S|-l+1} \leq \alpha_1^{s_{l+1}} = \alpha_1^j. \tag{1.4}$$

With respect to coalition T, we know that $|I_T| \geq |S| - l$. We have that a plan where the last $|S| - l + 1$ agents are connected indirectly to the source is (weakly) better than a plan where the last $|S| - l$ agents are connected indirectly if

$$\begin{aligned} \sum_{k=1}^{|T|-|S|+l-1} \alpha_1^{t_k} + \theta(|S| - l + 1) + \sum_{k=2}^{|S|-l+2} \alpha_k^1 \quad &\leq \quad \sum_{k=1}^{|T|-|S|+l} \alpha_1^{t_k} + \theta(|S| - l) + \sum_{k=2}^{|S|-l+1} \alpha_k^1 \\ \text{which simplifies to } \theta + \alpha^1_{|S|-l+2} \quad &\leq \quad \alpha_1^{t_{|T|-|S|+l}}. \end{aligned} \tag{1.5}$$

By the concavity assumption, $\alpha^1_{|S|-l+2} \leq \alpha^1_{|S|-l+1}$. By the same argument as in the first step, $\alpha_1^{t_{|T|-|S|+l}} \geq \alpha_1^{s_{l+1}}$. Therefore, by (1.4), (1.5) holds.

More generally, with respect to coalition T, we will have that a plan where the last $r + 1$ agents are connected indirectly to the source is (weakly) better than a plan where the last r agents are connected indirectly if

$$\begin{aligned} \sum_{k=1}^{|T|-r-1} \alpha_1^{t_k} + \theta(r + 1) + \sum_{k=2}^{r+2} \alpha_k^1 \quad &\leq \quad \sum_{k=1}^{|T|-r} \alpha_1^{t_k} + \theta r + \sum_{k=2}^{r+1} \alpha_k^1 \\ \text{which simplifies to } \theta + \alpha^1_{r+2} \quad &\leq \quad \alpha_1^{t_{|T|-r}}. \end{aligned}$$

For all $r \geq |S| - l$, $\alpha^1_{r+2} \leq \alpha^1_{|S|-l+1}$ by the concavity assumption (that is, $\alpha_k^1 \leq \alpha_l^1$ for all $k \geq l$). We have $\alpha_1^{t_{|T|-r}} \geq \alpha_1^{s_{l+1}} = \alpha_1^j$ if $|T| - r \geq p + 1$ (using the fact that $s_{l+1} = t_{p+1} = j$ and the ordering assumption). Therefore, for all $r \in \{|S| - l, ..., |T| - p - 1\}$, we have, using (1.4), that

$$\theta + \alpha^1_{r+2} \leq \theta + \alpha^1_{|S|-l+2} \leq \alpha_1^{t_{|T|-|S|+l}} \leq \alpha_1^{t_{|T|-r}}.$$

Thus, for coalition T, a plan where the last $|T| - p - 1$ agents connect indirectly to the source is (weakly) better than a plan where $k < |T| - p - 1$ connect indirectly. Formally, $\{t_{p+1}, ..., t_{|T|}\} \subseteq I_T$. By our definition, $S \subseteq T$ and $I_S = \{s_{l+1}, ..., s_{|S|}\}$, with $s_{l+1} = t_{p+1} = j$. Therefore, $I_S \subseteq I_T$. ∎

We are now ready for the proof of Theorem 3.

Proof of Theorem 3. Fix a pair $i, j \in N$ of players with $j \geq i$ and let $S \subseteq N \setminus [j]$. For notational simplicity, let $I_k = I_{[k]}$ and $I_{k,S} = I_{\{[k]\cup S\}}$ for $k \in \{i,j\}$. We also assume that the cost structure γ is fixed, and thus omit it from the notation. By Lemma 1, either $I_{i,S} \subseteq S$ or $S \subsetneq I_{i,S}$. That is, with respect to the unified coalition $[i] \cup S$, either the indirectly connected players are part of S or cover S. We distinguish these two cases.

Case 1: $I_{i,S} \subseteq S$. Then $I_i = \varnothing$ because $I_i \subseteq [i] \subseteq N \backslash S$, whereas $I_i \subseteq I_{i,S} \subseteq S$ due to Lemma 1 and the relevant assumption. Hence, by their optimal plans, we have

$$\begin{aligned} C([i]) &= \sum_{k\in[i]} \alpha_k^1 \text{ as well as } C([i]\cup S) = \sum_{k\in(S\backslash I_{i,S})\cup[i]} \alpha_1^k + \theta\,|I_{i,S}| + \sum_{k=2}^{|I_{i,S}|+1} \alpha_k^1 \text{ while} \\ C([j]) &= \sum_{k\in[j]\backslash I_j} \alpha_1^k + \theta\,|I_j| + \sum_{k=2}^{|I_j|+1} \alpha_k^1. \end{aligned}$$

Notice that $I_j \cap I_{i,S} = \varnothing$ since $I_{i,S} \subseteq S$, while $I_j \subseteq [j] \subseteq N\backslash S$. With respect to coalition $[j] \cup S$, take into account the plan to connect agents of $\{[j]\backslash I_j\} \cup \{S \backslash I_{i,S}\}$ directly to the source, and the remaining agents $I_j \cup I_{i,S}$ indirectly to the source. In view of this non-optimal plan for coalition $[j] \cup S$, it follows that

$$C([j]\cup S) \le \sum_{k\in[j]\backslash I_j} \alpha_1^k + \sum_{k\in S\backslash I_{i,S}} \alpha_1^k + \theta\left(|I_j| + |I_{i,S}|\right) + \sum_{k=2}^{|I_j|+|I_{i,S}|+1} \alpha_k^1$$

and in particular, $C([j]\cup S) - C([j]) \le C([i]\cup S) - C([i])$ due to concavity of the cost function c_{01} (that is, $\alpha_k^1 \le \alpha_l^1$ for all $k \ge l$).

Case 2: $S \subseteq I_{i,S}$. Then $S \subseteq I_{i,S} \subseteq I_{j,S}$ by Lemma 1. Denote $I_{k,S}^k = I_{k,S} \cap [k]$ for $k \in \{i,j\}$. Thus, by their optimal plans, we have for $l \in \{i,j\}$

$$C([l]\cup S) = \sum_{k=1}^{l-|I_{l,S}^l|} \alpha_1^k + \theta\left(|S| + \left|I_{l,S}^l\right|\right) + \sum_{k=2}^{|S|+\left|I_{l,S}^l\right|+1} \alpha_k^1 \text{ as well as} \tag{1.6}$$

$$C([l]) = \sum_{k=1}^{l-|I_l|} \alpha_1^k + \theta\,|I_l| + \sum_{k=2}^{|I_l|+1} \alpha_k^1. \tag{1.7}$$

We distinguish two subcases, depending on the relative sizes of I_j and $I_{i,S}^i$.

Subcase 2(i): $|I_j| \ge \left|I_{i,S}^i\right|$. Then $\left|I_{j,S}^j\right| \ge |I_j| \ge \left|I_{i,S}^i\right| \ge |I_i|$ by Lemma 1.

With respect to coalition $[j] \cup S$, taking into account the non-optimal plan to connect agents of $I_j \cup S$ indirectly to the source, we have

$$\begin{aligned} C([j] \cup S) &\leq \sum_{k=1}^{j-|I_j|} \alpha_1^k + \theta(|S| + |I_j|) + \sum_{k=2}^{|S|+|I_j|+1} \alpha_k^1 \text{ and hence, by } (1.7), \\ C([j] \cup S) - C([j]) &\leq \theta|S| + \sum_{k=|I_j|+2}^{|S|+|I_j|+1} \alpha_k^1. \end{aligned}$$

With respect to coalition $[i]$, taking into account the non-optimal plan to connect agents of $I_{i,S}^i$ indirectly to the source, we have

$$\begin{aligned} C([i]) &\leq \sum_{k=1}^{i-|I_{i,S}^i|} \alpha_1^k + \theta\left|I_{i,S}^i\right| + \sum_{k=2}^{|I_{i,S}^i|+1} \alpha_k^1 \text{ and hence, by } (1.6), \\ C([i] \cup S) - C([i]) &\geq \theta|S| + \sum_{k=|I_{i,S}^i|+2}^{|S|+|I_{i,S}^i|+1} \alpha_k^1. \end{aligned}$$

We conclude that $C([j] \cup S) - C([j]) \leq C([i] \cup S) - C([i])$ due to concavity of the cost function c_{01}(that is, $\alpha_k^1 \leq \alpha_l^1$ for all $k \geq l$), together with $|I_j| \geq \left|I_{i,S}^i\right|$.

Subcase 2(ii): $|I_j| \leq \left|I_{i,S}^i\right|$. Then, $\left|I_{j,S}^j\right| \geq \left|I_{i,S}^i\right| \geq |I_j| \geq |I_i|$ by Lemma 1.

With respect to coalition $[j] \cup S$, taking into account the non-optimal plan to connect the last $\left(\left|I_{i,S}^i\right| + |S|\right)$ agents indirectly to the source, we have, by noting $\left|I_{i,S}^i\right| \leq \left|I_{j,S}^j\right| \leq j$,

$$\begin{aligned} C([j] \cup S) &\leq \sum_{k=1}^{j-|I_{i,S}^i|} \alpha_1^k + \theta\left(|S| + \left|I_{i,S}^i\right|\right) + \sum_{k=2}^{|S|+|I_{i,S}^i|+1} \alpha_k^1 \text{ and hence by } (1.6), \\ C([j] \cup S) - C([j]) &\leq \theta\left(|S| + \left|I_{i,S}^i\right| - |I_j|\right) + \sum_{k=|I_j|+2}^{|S|+|I_{i,S}^i|+1} \alpha_k^1 - \sum_{k=j-|I_{i,S}^i|+1}^{j-|I_j|} \alpha_1^k. \end{aligned}$$

With respect to coalition $[i]$, taking into account the non-optimal plan to connect the last $|I_j|$ agents indirectly to the source, we have, by noting that $|I_j| \leq \left|I_{i,S}^i\right| \leq i$,

$$\begin{aligned} C([i]) &\leq \sum_{k=1}^{i-|I_j|} \alpha_1^k + \theta|I_j| + \sum_{k=2}^{|I_j|+1} \alpha_k^1 \text{ and hence, by } (1.7), \\ C([i] \cup S) - C([i]) &\geq \theta\left(|S| + \left|I_{i,S}^i\right| - |I_j|\right) + \sum_{k=|I_j|+2}^{|S|+|I_{i,S}^i|+1} \alpha_k^1 - \sum_{k=i-|I_{i,S}^i|+1}^{i-|I_j|} \alpha_1^k. \end{aligned}$$

We conclude that $C([j] \cup S) - C([j]) \leq C([i] \cup S) - C([i])$ due to the ordering assumption $\alpha_1^k \geq \alpha_1^l$ for all $k \geq l$, that is the cost functions satisfy $c_{0k}(1) \geq c_{0l}(1)$ for all $k \geq l$. ∎

As a special case, we look at coalition-symmetric congestion problems, which are ordered quasi-symmetric congestion problems where for all $i, j, k \in N$, $c_{0i}(k) = c_{0j}(k)$. Therefore, we have $\alpha_k^i = \alpha_k^j = \alpha_k$. This game is permutationally concave and the allocation $y_i = C([i]) - C([i-1])$ is in the core. Since all players are symmetric, it is true for any ordering of the agents. Therefore, the game is not only permutationally concave, but also concave. The structure of the network is also simplified.

Proposition 1 *For a coalition-symmetric problem (N, γ), there exists $\bar{k} \in N$ such that for all $S \subseteq N$ containing $m \leq \bar{k}$ agents, the set of efficient plans contains a plan where all agents are connected directly to the source. For all $S \subseteq N$ containing $m > \bar{k}$ agents, the set of efficient plans contains a plan where one agent is connected directly to the source and all other agents are connected to this agent.*

Proof. From Theorem 1, we know that there is an efficient plan where a set of agents is connected directly to the source, with other players all connected to the same agent. Since all players are symmetric, only the number of agents in each set matters. Let $Z_S^l \in F(S)$ be such a plan for coalition S where the number of agents connected directly to the source is $l \leq |S|$.

Suppose that Z_S^m is an efficient plan, with $1 < m < |S|$. We have that $c(Z_S^m) \leq c(Z_S^{m-1})$, or

$$m\alpha_1 + (|S| - m)\,\theta + \sum_{i=2}^{|S|-m+1} \alpha_k \leq (m-1)\alpha_1 + (|S| - m + 1)\,\theta + \sum_{i=2}^{|S|-m+2} \alpha_k$$

which simplifies to $\alpha_1 \leq \theta + \alpha_{|S|-m+2}$.

We also have that $c(Z_S^m) \leq c(Z_S^{m+1})$, or $\alpha_1 \geq \theta + \alpha_{|S|-m+1}$. Therefore, we need $\alpha_1 \in \left[\theta + \alpha_{|S|-m+1}, \theta + \alpha_{|S|-m+2}\right]$. By the concavity assumption, $\alpha_{|S|-m+1} \geq \alpha_{|S|-m+2}$. Thus the interval is empty, and Z_S^m is efficient, only if $\alpha_{|S|-m+1} = \alpha_{|S|-m+2}$. In that case, $c(Z_S^m) = c(Z_S^{m-1}) = c(Z_S^{m+1})$; if Z_S^m is efficient, then so are Z_S^{m+1} and Z_S^{m-1}. We repeat the process and conclude that if Z_S^m is optimal, then Z_S^1 and $Z_S^{|S|}$ are also efficient plans. Therefore, among the set of efficient plans, we always have Z_S^1 or $Z_S^{|S|}$.

By Lemma 1, we have that if $Z_S^{|S|}$ is an efficient plan for S, than $Z_T^{|T|}$ is an efficient plan for T, with $T \subseteq S$. By symmetry of the agents, we can extend this result to all T such that

$|T| \leq |S|$. Therefore, there exists a $\bar{k} \in N$ such that for all $S \subseteq N$ containing $m \leq \bar{k}$ agents, the set of efficient plans contains $Z_S^{|S|}$. For all $S \subseteq N$ containing $m > \bar{k}$ agents, the set of efficient plans contains Z_S^1. ■

If $(|S|-1)(\alpha_1 - \theta) \leq \sum_{l=2}^{|S|} \alpha_l$, then the set of efficient plans for coalition S contains a plan where all agents are connected directly to the source. If $(|S|-1)(\alpha_1 - \theta) \geq \sum_{l=2}^{|S|} \alpha_l$, then the set of efficient plans for coalition S contains a plan where one agent is connected directly to the source and all other agents are connected directly to this agent.

In ordered quasi-symmetric congestion problems, agents are ordered according to the costs of their direct connection. However, we can construct a related family of problems yielding permutationally concave games by ordering agents according to the costs of indirect connections. Formally, we keep the assumption that demands are identical, but we specify the following cost structure: for all $i, j, k \in N, c_{0i}(k) = c_{0j}(k)$ and $\alpha_k^{ij} = c_{ij}(k) - c_{ij}(k-1)$. We retain that $\alpha_k^{ij} = \alpha_k^{ji}$, but costs are now ordered in a lexicographic manner:

$$\alpha_k^{12} \leq \alpha_k^{13} \leq ... \leq \alpha_k^{1n} \leq \alpha_k^{23} \leq ... \leq \alpha_k^{2n} \leq ... \leq \alpha_k^{n-1,n}.$$

This problem may have limited economic relevance but, in the same manner as for ordered quasi-symmetric congestion problems, it can be shown that it generates permutationally concave games.

We can also order agents according to their demands, leaving the rest of the structure symmetric. Suppose that (N, Q, γ) is such that for all $i, j \in N$, all $x \in \mathbb{R}_+$, $c_{0i}(x) = f_p(x)$ and $c_{ij}(x) = f_s(x)$, with $f_k(0) = 0$ and f_k concave for $k = p, s$. Suppose also that $q_1 \geq ... \geq q_n$. The resulting game is also permutationally concave.

More generally, while we can add the set of ordered quasi-symmetric congestion problems and the simple problems of Section 3.2 to the family of stable concave SSUNF problems, the family has yet to be fully characterized.

6 Convex SSUNF games

In opposition to concave problems, nothing guarantees on convex problems that there will be a tree among the efficient plans. In fact, to avoid the decreasing returns to scale associated with the convex cost functions, there is an incentive to split flows on many arcs.

For convex problems, we have an algorithm that approximates an efficient plan. In fact, we can approximate the minimal cost as close as wanted, although at the cost of many operations. This algorithm is named the cycle-cancelling method and has been extensively studied in the operations research literature. See for example Goldberg and Tarjan (1989). It has also been mentioned, in a slightly different form, in the economic literature by Quant and Reijnierse (2004).

Here is an overview of the method: starting with a feasible plan $Z \in F(N, Q)$ such that there are no $i, j \in N$ such that $z_{ij} > 0$ and $z_{ji} > 0$, we define a marginal cost vector $l(Z)$ that assigns the marginal cost l_{ij} of one unit of flow on each arc. Formally,

$$l_{ij}(z_{ij}) = \begin{cases} c_{ij}(z_{ij}+1) - c_{ij}(z_{ij}) \text{ if } z_{ji} = 0 \\ c_{ji}(z_{ji}-1) - c_{ji}(z_{ji}) \text{ if } z_{ji} > 0 \end{cases} \quad \text{for all } (i,j) \in A$$

We define a cycle $\boldsymbol{C}$ on plan Z as a path $r_{ii}(Z)$ with $i \in N_0$. We say that $\boldsymbol{C}$ is a negative cycle on the network Z if $\sum_{(i,j) \in supp(\boldsymbol{C})} l_{ij}(z_{ij}) < 0$.

Thus starting with the feasible plan Z, if there are no negative cycles, Z is efficient. If there are negative cycles, we pick one at random and define Z' such that $z'_{ij} = z_{ij} + 1$ if $(i, j) \in supp(\boldsymbol{C})$ and $z'_{ij} = z_{ij}$ otherwise. We calculate $l(Z')$ and continue eliminating negative cycles and updating the marginal cost vector until there are no more negative cycles. The resulting plan is the approximation of an optimal plan.

To get a better approximation, we have to define $l(Z)$ as the marginal cost of x units, with $0 < x < 1$. In problems with a small number of agents, once the approximation is obtained, one can use this information to infer the direction of the flows ($z_{ij} > 0$ or $z_{ji} > 0$) and then solve a standard optimization problem to get an exact efficient plan.

6.1 Stability result

Before stating the main result of this section, let's first define some notation and state a Lemma on feasible plans that will be used later on.

Let $\lambda \in [0,1]^{\mathfrak{S}}$ be balanced:

$$\sum_{S \in \mathfrak{S} : i \in S} \lambda^S = 1 \text{ for all } i \in N$$

We have the property that a sum of feasible plans over a balanced collection, weighted by the vector λ, is a feasible plan for the grand coalition N. Note that this is in no way related to the cost structure.

Lemma 2 *For a fixed Q, a balanced λ, and a set of plans Z^S such that $Z^S \in F(S,Q)$ for all $S \in \mathfrak{S}$, we have, for $Z = \sum_{S\in\mathfrak{S}} \lambda^S Z^S$, that $Z \in F(N,Q)$.*

Proof. We must verify that the resulting plan respects with equalities the conditions (1.1) to be a feasible plan. We can see that the first condition is verified:

$$\begin{aligned}\sum_{i\in N} z_{0i} &= \sum_{S\in\mathfrak{S}} \lambda^S \sum_{i\in S} z_{0i}^S \\ &= \sum_{S\in\mathfrak{S}} \lambda^S \sum_{i\in S} q_i \\ &= \sum_{i\in N} q_i\end{aligned}$$

where the second equality comes from the fact that $Z^S \in F(S,Q)$. The last equality comes from the fact that $\sum_{S\in\mathfrak{S}:i\in S} \lambda^S = 1$ for every $i \in N$.

The second condition is also verified. For any $i \in N$, we have:

$$\begin{aligned}\sum_{j\in N_0\setminus\{i\}} (z_{ji} - z_{ij}) &= \sum_{S\in\mathfrak{S}} \lambda^S \left(\sum_{j\in N_0\setminus\{i\}} (z_{ji}^S - z_{ij}^S) \right) \\ &= \sum_{S\in\mathfrak{S}:i\in S} \lambda^S \left(\sum_{j\in N_0\setminus\{i\}} (z_{ji}^S - z_{ij}^S) \right) \\ &= \sum_{S\in\mathfrak{S}:i\in S} \lambda^S q_i \\ &= q_i\end{aligned}$$

where the second equality comes from the fact that $Z^S \in F(S,Q)$. The last equality comes from the fact that λ is balanced.

The third condition is trivially satisfied. Therefore, $Z \in F(N,Q)$. ■

The following is an explicit proof of the result announced by Quant et al. (2006).

Theorem 4 *If the cost structure γ is such that all c_{ij} are convex functions, then $core(Q,\gamma)$ is non-empty.*

Proof. Step 1. Fix the problem (N, Q, γ) and take a balanced λ, and a set of efficient plans Z^S such that $Z^S \in \arg\min_{X \in F(S,Q)} c(X)$ for all $S \in \mathfrak{S}$. Define $Z = \sum_{S \in \mathfrak{S}} \lambda^S Z^S$.

We have that $c_{ij}(z_{ij}) = c_{ij}\left(\sum_{S \in \mathfrak{S}} \lambda^S z_{ij}^S\right) = c_{ij}\left(\sum_{S \in \mathfrak{S}: i,j \in S_0} \lambda^S z_{ij}^S\right)$ since by definition of the game, for $(i,j) \in A$, the arc (i,j) can only be used by the coalition S if $i, j \in S_0$.

By definition of a balanced collection, $\sum_{S \in \mathfrak{S}: i \in S} \lambda^S = 1$ for every $i \in N$. We have that $\sum_{S \in \mathfrak{S}: i,j \in S_0} \lambda^S \leq \sum_{S \in \mathfrak{S}: i \in S} \lambda^S = 1$ for all $i \in N$ and all $j \in S_0$.

Define $\beta_{ij} = 1 - \sum_{S \in \mathfrak{S}: i,j \in S_0} \lambda^S$

Therefore:

$$c_{ij}(z_{ij}) = c_{ij}\left(\sum_{S \in \mathfrak{S}: i,j \in S_0} \lambda^S z_{ij}^S + \beta_{ij} * 0\right).$$

We also have that $\sum_{S \in \mathfrak{S}} \lambda^S c_{ij}\left(z_{ij}^S\right) = \sum_{S \in \mathfrak{S}: i,j \in S_0} \lambda^S c_{ij}\left(z_{ij}^S\right)$ since if i or j is not in S_0, we must have $z_{ij}^S = 0$ and $c_{ij}(0) = 0$. We then have that

$$\sum_{S \in \mathfrak{S}} \lambda^S c_{ij}\left(z_{ij}^S\right) = \sum_{S \in \mathfrak{S}: i,j \in S_0} \lambda^S c_{ij}\left(z_{ij}^S\right) + \beta_{ij} c_{ij}(0)$$

since $c_{ij}(0) = 0$.

By the convexity of the cost functions

$$c_{ij}\left(\sum_{S \in \mathfrak{S}: i,j \in S_0} \lambda^S z_{ij}^S + \beta_{ij} * 0\right) \leq \sum_{S \in \mathfrak{S}: i,j \in S_0} \lambda^S c_{ij}\left(z_{ij}^S\right) + \beta_{ij} c_{ij}(0).$$

Thus,

$$c_{ij}\left(\sum_{S \in \mathfrak{S}} \lambda^S z_{ij}^S\right) \leq \sum_{S \in \mathfrak{S}} \lambda^S c_{ij}\left(z_{ij}^S\right) \text{ for all } (i,j) \in A. \tag{1.8}$$

Step 2. We have

$$\begin{aligned} c(Z) &= \sum_{(i,j)\in A} c_{ij}(z_{ij}) \\ &= \sum_{(i,j)\in A} c_{ij}\left(\sum_{S\in\mathfrak{S}} \lambda^S z_{ij}^S\right) \\ &\leq \sum_{(i,j)\in A}\sum_{S\in\mathfrak{S}} \lambda^S c_{ij}(z_{ij}^S) \\ &= \sum_{S\in\mathfrak{S}}\sum_{(i,j)\in A} \lambda^S c_{ij}(z_{ij}^S) \\ &= \sum_{S\in\mathfrak{S}} \lambda^S c(Z^S) \end{aligned}$$

where the inequality follows from equation (1.8).

Step 3. Let Z^* be an efficient plan for the problem (N,Q,γ): $Z^* \in \arg\min_{Z\in F(N,Q)} c(Z)$. By lemma 2, $Z \in F(N,Q)$. Then $c(Z^*) \leq c(Z)$.

Since $c(Z) \leq \sum_{S\in\mathfrak{S}} \lambda^S c(Z^S)$, we have that:

$$C(N,Q,\gamma) = c(Z^*) \leq \sum_{S\in\mathfrak{S}} \lambda^S c(Z^S) = \sum_{S\in\mathfrak{S}} \lambda^S C(S,Q,\gamma)$$

which is the condition to have a balanced game.

Since a game is balanced if and only if the core is non-empty (Bondareva (1963), Shapley (1967)), the proof is complete. ∎

The non-emptiness of the core in this case might seem counter-intuitive at first, since convex cost functions generate decreasing returns to scale. However, the gains in cooperation stem from the fact that a larger coalition will have more arcs on which to split flows and in this way limit the damage caused by the decreasing returns to scale. Since it is never optimal for a coalition to use both arcs (i,j) and (j,i), the number of usable arcs for a coalition S is $|S| + \frac{|S|!}{2}$ for $|S| > 1$. Thus, for $|S| = 1,2,3,4,...$ the number of usable arcs is $1,3,6,16,...$ In all convex problems, this is sufficient to generate enough cooperation gains to guarantee a non-empty core.

However, the game resulting from a convex problem need not be a concave game, as illustrated by this example:

Example 5 (N,Q,γ) *is a convex problem with* $N = \{1,2,3\}$, $Q = (1,2,5)$, $c_{01}(x) = x^2$,

$c_{02}(x) = 4x^2$, $c_{03}(x) = 10x^2$, $c_{12}(x) = c_{21}(x) = 2x^2$, $c_{13}(x) = c_{31}(x) = 5x^2$, $c_{23}(x) = c_{32}(x) = 50x^2$.

We can easily find that $C(\{1\}, Q, \gamma) = 1$, $C(\{2\}, Q, \gamma) = 16$, $C(\{3\}, Q, \gamma) = 250$.

Using the cycle-cancelling method and then solving a standard optimization problem, we find that $C(\{1,2\}, Q, \gamma) = 10$ *(with* $z^*_{01} = 2$ *and* $z^*_{02} = z^*_{12} = 1$*) and* $C(\{1,3\}, Q, \gamma) = 100,9375$ *with* ($z^*_{01} = 65/16$, $z^*_{03} = 31/16$, $z^*_{13} = 49/16$).

For the grand coalition (1, 2, 3), *we get* $C(N, Q, \gamma) = 111,1173$. *We have*

$$C(\{1,2\}, Q, \gamma) - C(\{1\}, Q, \gamma) = 9 < 10,1798 = C(N, Q, \gamma) - C(\{1,3\}, Q, \gamma)$$

and thus the game is not concave.

Here, agent 2 gains by joining agent 1 because he then has access to arcs with cheaper cost functions. However, when he joins coalition $\{1,3\}$, agent 3 has created congestion on arc $(0,1)$, while making available arcs $(0,3)$, $(1,3)$ and $(3,2)$ that are costly. Thus, cooperations gains for agent 2 decrease.

7 Conclusion

Starting from a single-source uncapacitated network flow problem, we defined a game that associates to each coalition the minimal cost to supply all of its agents with their demand, while using only nodes contained within the coalition.

In the case where the cost functions are concave, we offered new examples showing that the problem can have a non-empty core. We looked at simpler problems that always have non-empty cores. In particular, we introduced the ordered quasi-symmetric congestion problems, that always generate stable games. A complete characterization of the family of concave problems with non-empty cores has yet to be found.

We also offered an explicit proof of the stability of convex SSUNF problems, a result mentioned by Quant et al. (2006). However the resulting game might not be a concave game, leaving open the question of the characterization of the core elements.

Chapter 2

Cost sharing with multiple technologies

1 Introduction

In cost sharing problems, agents cooperate and combine their demands, at a cost. The central planner's main task is then to divide that cost among the participating agents.

In their simplest form, cost sharing problems have in common the following characteristics: agents demand different levels of a homogeneous private good and have access to a commonly owned technology to produce this good. With this technology, we can associate a cost to any level of demand. The gains (or losses) associated with cooperation arise from the varying returns to scale exhibited by the technology. However, we argue that in many cases, at least part of the gains from cooperation derive from the presence of other agents, regardless of their levels of demand.

The motivating example is the formation of a network, where a coalition can only use the locations of its members to build its network. As a coalition grows, more efficient networks can be built as the number of possible network configurations increases, yielding gains from cooperation.

Another clear example is when agents have privately-owned technologies of production. When agents cooperate, in addition to combining their demands, they also combine their technologies. A possibly complex process aggregates these technologies to form a new efficient technology for the coalition. In particular, technologies can be complementary, in which case all agents enjoy gains from this technology sharing.

We find many examples of such cooperation in environmental problems. Consider the case of the Kyoto Protocol that came into force on February 16th, 2005. Under its terms, participating countries have until 2012 to reach reduction targets on greenhouse gases emissions. Countries that have not reached their targets can buy emission credits from other countries that have exceeded their target reductions.

Countries that signed the protocol were assigned individual targets but are free to regroup with other countries. If they choose to do so, emissions must be reduced within the zone without regard to where in the zone the reductions are made. This cooperation allows countries to cooperate technologically, to coordinate their programs and to create larger markets for their firms to trade emission credits. More importantly, it allows a more efficient allocation of reductions, to where it is less costly to do so. Countries of the European Union have regrouped in such a manner. For example, both Germany and Spain had to reduce their emissions by 8%, compared to 1990 levels, under the Kyoto Protocol. However, the EU assigned a target of -21% to Germany and +15% to Spain.

While the United States are not part of the Kyoto Protocol, some of its states have adopted legislations to reduce greenhouses gases. In 1997, Oregon introduced a legislation that requires newly-constructed or expanded power plants to meet standards in CO2 emissions. If they fail to meet standards, they must pay an amount of \$0.85 per pound in excess. Alternatively they can finance, alone or with partners, a project that will at least offset the excess emissions. Similar legislations have been adopted in Massachusetts and Washington and are under study in other states.

Consider the following example:

Example 1 *Two firms from Oregon have plans for new power plants that do not meet the standards. Meeting the standards would prove too costly for both firms. Under current plans, firm 1 would have excess emissions of* $x_1 = 15$ *million pounds. For firm 2, excess emissions would be of* $x_2 = 10$ *million pounds.*

They have privately-owned technologies for offset projects and we suppose that there are no outside firms with whom they can form a partnership. The technology of firm 1 is such that in its offset project, its average (and marginal) cost per pound of emission is $c_1 = \$0.60$. *Firm 2 has some technology for an offset project, but finds it more profitable to pay the fine. Thus* $c_2 = 0.85\$$. *However, if they cooperate, their technologies are complements and average cost falls to* $c_{12} = \$0.40$.

Firms are responsible for their technologies, that result from past investments in R&D. Demands are inelastic as firms must comply with the law.

Cost sharing problems are usually studied with cost functions that assign to each level of demand a cost, that is $C : x \mapsto C(x)$ where x is the level of demand. Clearly, this structure is inappropriate for this problem. In this example, the structure is more complex, as the same level of demand can generate different costs depending on who produces it. The correct model assumes that $C : (x, S) \mapsto C(x, S)$ is defined for any demand level x and any coalition S.

In our example we have $C(25, \{1\}) = \$15M$, $C(25, \{2\}) = \$21.25M$ and $C(25, \{1, 2\}) = \$10M$, so for a given level of demand, the cost depends on who cooperates in its production.

In practice, technology plays an important role in many situations of cooperation. Research joint ventures are the obvious examples, where cooperation can improve efficiency of the research even without increasing investment (La Manna (2006)). One can also think of countries cooperating in a space agency, or different departments of a company not only putting together their requests for a project, but also some of the workers and tools they have at their disposal to complete the task.

Technology is defined here in the broadest sense. Possible definitions of technology sharing in our context include the sharing of experience, lists of suppliers or negotiation powers.

The assumption that agents share private technologies has been used in the output-sharing literature, where agents combine their inputs and where the task is to allocate the output among the participants. Leroux (2007) studies a problem where the technology to be shared results from the aggregation of private production processes. He focuses on incentives issues, whereas we study fairness.

By integrating technologies, we are able to characterize cost sharing rules using new properties related to the effects of technology on gains from cooperation. While models where costs depend on demands could be studied by looking only at total coalition costs, using the extra information contained in the cost function allows us to focus on different properties. In the same manner, we could ignore technological cooperation. However, this extra information can be used, as gains from cooperation now arise from different sources. In fact, with technology sharing, there are two ways to create cooperation gains; one related to demand and the other to technology. Revisiting Example 1 will highlight this feature.

Example 1 revisited. *Everything is as in Example 1, except that when firms* 1 *and* 2 *cooperate, we have that the average cost is* $c_{12} = c_1 = \$0.60$.

Cost for firm 1: $c_1 x_1 = \$9M$

Cost for firm 2: $c_2 x_2 = \$8.5M$

Total cost (with cooperation): $c_1(x_1 + x_2) = \$15M$

Therefore, there is a cooperation gain of $c_1 x_1 + c_2 x_2 - c_1(x_1 + x_2) = (c_2 - c_1)x_2 = \$2.5M$.

The cooperation gain comes from the fact that cooperation allows production of the 10 units of firm 2 at a lower marginal cost. Therefore, we can clearly identify two sources for this gain: 1) the technology improvement that we can attribute to firm 1, and 2) the level of agent 2's demand, which allows this new technology to be applied on a larger scale.

The Shapley value splits the gain in half and gives allocations of $\$7.75M$ to firm 1 and $\$7.25M$ to firm 2. While the Shapley value is a good predictor of what could happen if the two were allowed to bargain for this gain, in this case, it is difficult from a normative point of view to accept the proposed allocation. In fact, here, we have the unattractive feature that firm 2 gets a strictly positive share of the cooperation gain because it is polluting.

In this example, it is natural to dismiss this second type of contribution and to put all weight on the technology improvement as the source of the cooperation gain. We would then allocate no part of the gain to firm 2.

More generally, if an agent can be added to any (non-empty) coalition without ever improving the technology of that coalition, then he should be allocated no part of the cooperation gain and should pay his stand-alone cost. This will be the central axiom used in the paper, that we will call the *Technological Dummy property.*

In more general problems, this property is not always desirable, notably when agents are not completely responsible for their technology (for example if it is a physical skill). However, when we are close to the conditions of the previous example, or more generally when we put a lot of weight on rewarding technology, the property is natural.

This property is a close relative of the familiar Dummy property in the classic cost sharing literature, that says that if an agent can be added to any coalition at no extra cost, then this agent should not pay anything. This property was introduced in the original paper by Shapley (1953) where it was one of the main equity postulates. It conveys the simple idea that agents should not pay for costs for which they are not responsible. They are, therefore, fully responsible for the costs of their demands. This property has been adapted to models with continuous demands. In that context, it implies that if there is an agent such that adding one unit of his good to any demand profile can be done at no extra cost, then this agent should not pay anything. See Moulin and Sprumont (2006) for a discussion of the implications of this property and a comparison to properties yielding partial responsibility.

Conceptually, the property is even closer to the stronger Separability property for a model with continuous demands for differentiated goods. This property says that if there is an agent such that the extra cost of adding one unit of his good to any demand profile is always equal to a, then this agent should pay a times his demand. In that case, there is never any gain related to scale when the agent joins a coalition and the property imposes that he is not credited with any cooperation gain. This property does not add anything new in the model with publicly-owned technology and continuous demands as it is implied by the usual properties of Additivity, Non-negativity and the classic Dummy property. The cost sharing rules satisfying these properties have been characterized by Friedman (2004) and Haimanko (2000). See Wang (1999) for the case with indivisible demands and Weber (1988) for the case of the stand-alone game.

Combined with adaptations of classic properties of linearity (on both demands and technologies) and symmetry, the Technological Dummy property generates a well-defined class of cost sharing rules. We then define a property of monotonicity in the technology and the classic stand-alone property. It turns out that any one of these properties, in combination with the four mentioned before, is sufficient to characterize a unique rule, that we call the *technological contribution value.*

This rule is as follows. For each agent i, we define a game on the set of agents other than i. Each coalition is assigned the change in average cost when it joins agent i. The Shapley value is then applied to this game, and multiplied by the demand of agent i, to express a created value. An agent is allocated this created value in the different games, to which we add his stand-alone cost.

We also characterize a rule, named the quasi-average contribution rule, using the No Reshuffling property. This property says that no coalition should have an incentive to reallocate demand among its members in order to reduce their aggregate cost share. We first build a game where the cost of a coalition is its cost to produce the whole vector of demand, using its technology. The Shapley value is applied to this game to share the technological gains. Then, individual shares are adjusted according to the difference between individual and average demands.

The paper is organized as follows: Section 2 formally defines the framework. Section 3 presents the characterization of the set of rules satisfying the Technological Dummy property and a set of basic properties. In Section 4, we offer two characterizations for one of these rules, the technological contribution value. In Section 5, we define and characterize the quasi-average contribution rule using the No Reshuffling property. Section 6 discusses possible extensions. Existence and independence of properties are proven in the appendix.

2 The model

Let $N = \{1, ..., n\}$ be the set of agents. Let $\mathcal{N} = \{S \mid \varnothing \neq S \subseteq N\}$ be the set of non-empty subsets of N, i.e., the set of possible, non-empty coalitions. Agents have demands for a single, common good. Let $x \in \mathbb{R}_+^N$ be the demand vector. Demands are assumed to be inelastic. If $S \in \mathcal{N}$ and $x \in \mathbb{R}_+^N$, write $x(S) := \sum_{i \in S} x_i$.

The technology for producing the good varies with the set of participating agents, but is assumed to be always linear. By making this assumption, we can focus on the implications of technology sharing, as no part of the cooperation gains arise from returns to scale. Let $\Gamma = \left\{c \in \mathbb{R}_+^{\mathcal{N}} \mid S \subseteq T \Rightarrow c(S) \geq c(T)\right\}$ be the set of cost vectors. $c(S)$ denotes the (constant) marginal cost of coalition S. Therefore, marginal cost does not increase when a coalition grows. The total cost for coalition S is thus $c(S)x(S)$.

A problem is an ordered pair $(c, x) \in \Gamma \times \mathbb{R}_+^N$; that is, the cost and demand vectors.

A rule is a map $y : \Gamma \times \mathbb{R}_+^N \to \mathbb{R}^N$ satisfying the budget balance condition $\sum_{i \in N} y_i(c, x) = c(N)x(N)$. Note that since $c(S)x(S)$ can decrease when we add agents to coalition S, we do not impose that $y_i \geq 0$. There is ample justification in this setting to subsidize an agent if he allows other agents to significantly improve on their technologies.

3 The implications of the Technological Dummy Property

The Technological Dummy property, discussed in the introduction, is a powerful axiom. To define it formally, we need the following definition:

For any $c \in \Gamma$, if $c(S \cup \{i\}) = c(S)$ for all $\varnothing \neq S \subseteq N \backslash \{i\}$, we say that i is a *technological dummy in c*.

Technological Dummy property: For any $c \in \Gamma$ and any $x \in \mathbb{R}_+^N$, if $i \in N$ is a technological dummy in c, then $y_i(c, x) = c(\{i\})x_i$.

We combine the Technological Dummy property with a symmetry property, as well as two linearity properties.

We use a weak form of symmetry, such that if two agents are completely identical, both in their demands and in how they influence the technology, they are allocated the same amount.

Equal Shares for Equals: For any $c \in \Gamma$, any $x \in \mathbb{R}^N_+$ and any $i, j \in N$, if $c(S \cup \{i\}) = c(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$ and $x_i = x_j$, then $y_i(c, x) = y_j(c, x)$.

The next two properties have no apparent ethical content, but prevent problems coming from manipulation and allow easy implementation.

The first property is along the lines of the classic Additivity property that is standard in the literature (see Moulin (2002)). With the following property, if we compute cost shares on the problem (c, x), but technology changes to $c' = c + \delta$, then adding the cost shares on the problem (δ, x) or computing them again on (c', x) will yield the same results. In our context, without the assumption of non-negativity of the cost shares, we use the stronger property that the rule be linear in the cost vector.[1]

Technological Linearity: For any $x \in \mathbb{R}^N_+$ and any c^1, $c^2 \in \Gamma$, $\beta_1, \beta_2 \in \mathbb{R}$, such that $\beta_1 c^1 + \beta_2 c^2 \in \Gamma$, $y(\beta_1 c^1 + \beta_2 c^2, x) = \beta_1 y(c^1, x) + \beta_2 y(c^2, x)$.

We will also use a similar property for demand. Suppose that agents have agreed on the cost sharing method and shares are computed using the demand vector x. If demands increase to $x' = x + \delta$, we can either add the cost shares obtained when using δ as the demand vector or compute directly the cost shares using x'. Because of the assumption of constant returns to scale, the property is not particularly strong. In fact, without it, agents could manipulate their cost shares through the demand vector.

Demand Linearity: For any $c \in \Gamma$, $x^1, x^2 \in \mathbb{R}^N_+$ and λ_1, $\lambda_2 \in \mathbb{R}_+$, $y(c, \lambda_1 x^1 + \lambda_2 x^2) = \lambda_1 y(c, x^1) + \lambda_2 y(c, x^2)$.

Together, these four properties characterize a well-defined set of rules.

3.1 Characterization

First, note that for any problem with only two agents, the Technological Dummy property and Technological Linearity are sufficient to characterize a unique rule, with no symmetry axiom needed. To see this, observe that any $c \in \Gamma$ can be written as the sum of two technological vectors c^1 and c^2 such that one agent is a technological dummy in c^1 and the other agent is a technological dummy in c^2. There are many ways to do this, one simple way being to define c^1 such that $c^1(\{1\}) = c(\{1\}) - c(\{1, 2\})$ and $c^1(\{2\}) = c^1(\{1, 2\}) = 0$. We then define $c^2 = c - c^1$. The Technological Dummy property, combined with the budget-balance

[1] In standard models, Linearity is used in most proofs. However, Linearity is implied by Additivity under the assumption that cost shares are non-negative.

condition, defines a unique rule on c^1 and c^2. By Technological Linearity (in fact a weaker property of additivity is sufficient), we have also defined a unique rule on c.

However, on problems with larger sets of agents, the budget-balance condition together with the Technological Dummy property will not yield a unique rule.

Before moving on, we need to define marginal contributions rules, for which cost allocations depend linearly on those marginal contributions.

For a set of agents N, let v be a function that assigns a value to all $S \in \mathcal{N}$. Let $z \in \mathbb{R}^N$ be a rule such that $\sum_{i \in N} z_i(v) = v(N)$. We say that z is a *marginal contributions rule* with weight p, if for all $i \in N$, $z_i(v) = \sum_{\varnothing \neq S \subseteq N \setminus \{i\}} p_i(S) \left[v(S \cup \{i\}) - v(S) \right] + p_i(\varnothing) v(\{i\})$, with $p_i \in \mathbb{R}^{2^{N-1}}$ and $p = (p_i)_{i \in N}$.

We are now ready to introduce the family of rules satisfying the four properties defined above. The characterization is obtained in the following manner: by Demand Linearity, we can allocate the cost of each unit of demand independently. By Technological Linearity and the Technological Dummy property, the rule to share the cost of one unit of demand of agent i is a marginal contributions rule, for all $i \in N$. Equal Shares for Equals and the Technological Dummy property impose restrictions on the weights of the marginal contributions rules. The first steps follow closely Weber (1988), who then characterizes the Shapley value.

Theorem 1 *A rule y satisfies the Technological Dummy property, Technological Linearity, Demand Linearity and Equal Shares for Equals if and only if it is of the following form: for all $k \in N$, $y_k(c,x) = \sum_{i \in N} x_i \eta_k^i(c)$ with $\eta^i(\cdot)$ a marginal contributions rule with weight p^i, such that, for all $j \in N \setminus \{i\}$*

$$\begin{aligned} p_j^i(\varnothing) &= 0 \\ p_j^i(S) &= \theta_{|S|} \text{ for all } \varnothing \neq S \subseteq N \setminus \{i,j\} \\ p_j^i(S \cup \{i\}) &= \frac{|S|!(n-|S|-2)!}{(n-1)!} - \theta_{|S|} \text{ for all } S \subseteq N \setminus \{i,j\} \end{aligned}$$

and

$$\begin{aligned} p_i^i(\varnothing) &= 1 \\ p_i^i(S) &= |S|\,\theta_{|S|-1} - (n-|S|-1)\,\theta_{|S|} \text{ for all } \varnothing \neq S \subset N \setminus \{i\}. \\ p_i^i(N \setminus \{i\}) &= (n-1)\theta_{n-2} \end{aligned}$$

with the convention that $\theta_0 = 0$, *and* $\theta = [\theta_1, \theta_2, ..., \theta_{n-2}] \in \mathbb{R}^{n-2}$.

While Theorem 1 will serve as a lemma in the following sections, it is by itself instructive, as it clearly identifies the restrictions imposed by the four properties. The proof is in the appendix.

Rules defined in Theorem 1 differ only in how they respond to the following question: If agent j is the only one with a positive demand, when computing the cost allocation of agent i, what weight should be put on his marginal technological contributions to coalitions that do not include agent j (i.e. what should θ be)?

4 The technological contribution value

We define a new rule called the *technological contribution value* and denote it by y^*.

Fix a problem (c, x). For every $i \in N$, define the function $v^i(\cdot, c)$ as follows:

$$v^i(S, c) = c(\{S \cup i\}) - c(\{i\}) \text{ for each } S \subseteq N \setminus \{i\}.$$

The function $v^i(\cdot, c)$ is a TU-game on the player set $N \setminus \{i\}$. Given the problem (c, x), it gives, for each coalition not containing i, the change in the marginal cost when a coalition joins agent i. Notice that since $c(S) \geq c(T)$ for all $S \subseteq T \subseteq N$, we have that $v^i(S, c) \geq v^i(T, c)$ for any $S \subseteq T \subseteq N \setminus \{i\}$ and $v^i(S, c) \leq 0$ for all $S \subseteq N \setminus \{i\}$, all $i \in N$ and all $c \in \Gamma$.

The *technological contribution value* of the problem (c, x) is given by

$$y_i^*(c, x) = c(\{i\})x_i + \sum_{j \in N \setminus \{i\}} x_j Sh_i(v^j(\cdot, c)),$$

for all $i \in N$, where $Sh(v^j(\cdot, c))$ is the Shapley value of the game $v^j(\cdot, c)$.

Thus, for agent i, $c(\{i\})x_i$ is the maximum amount he can pay. This amount will be reduced by his share of the value created by the technological improvements generated for other agents. For each agent $j \neq i$, the share allocated to agent i is his Shapley value on the game v^j multiplied by the demand of agent j.

In the representation of Theorem 1, the technological contribution value is the rule such that $\theta = (0, ..., 0)$. To see this, suppose that x is such that $x_k = 1$ and $x_i = 0$ for all $i \in N \setminus \{k\}$. Then, $y_j^*(c, x) = Sh_j(v^k(\cdot, c)) = \sum_{S \subseteq N \setminus \{j,k\}} \frac{|S|!(n-|S|-2)!}{(n-1)!} [c(S \cup \{j, k\})$ - $c(S \cup \{k\})]$.

This implies that $p_j^k(S)=0$ and $p_j^k(S\cup\{j\})=\frac{|S|!(n-|S|-2)!}{(n-1)!}$ for all $\varnothing\neq S\subseteq N\backslash\{j,k\}$, and therefore that $\theta=(0,...,0)$.

We offer two characterizations of the technological contribution value, each obtained by adding one additional axiom to the four presented in the previous section.

From the definition of the technological contribution value, $c(\{i\})x_i$ is the upper-bound of what agent i can pay. The classic stand-alone property requires that no agent should be allocated more than $c(\{i\})x_i$, his stand-alone cost. Since c is anti-monotonic, this is a meaningful property in our context. It is a minimal stability property that makes sure that no one prefers to leave unilaterally the coalition N.

Stand-alone property: For any $c\in\Gamma$, $x\in\mathbb{R}_+^N$ and $i\in N$, $y_i(c,x)\leq c(\{i\})x_i$

The technological contribution value, because of the properties of the Shapley value, also satisfies a monotonicity property. That is, starting from a cost vector $c\in\Gamma$, if the average cost decreases for a coalition S and stays the same for all other coalitions, then if an agent is part of S, he should not see his cost share increase following this change. The property is similar in nature to the various definitions of coalitional monotonicity first defined by Young (1985).

Technological Monotonicity: For any $i\in N$, if $c,c'\in\Gamma$ are such that there is a $S\subseteq N\backslash\{i\}$ with $c(S\cup\{i\})>c'(S\cup\{i\})$ and $c(T)=c'(T)$ for all $T\neq S\cup\{i\}$, then $y_i(c,x)\geq y_i(c',x)$ for any $x\in\mathbb{R}_+^N$.

With these properties, we are ready for the main characterization results.

Theorem 2 *A rule y satisfies the Technological Dummy property, Technological Linearity, Demand Linearity, Equal Shares for Equals and the Stand alone property if and only if $y=y^*$.*

Proof. Lemma 1 (in appendix) shows that y^* satisfies the five properties. We show that they imply a unique rule.

Fix $i\in N$. For each $S\subseteq N$ and $k\leq|S|$, we define a cost vector c_k^S such that coalitions that are part of S and have no more than k members have an average cost of 1, with all other coalitions having an average cost of 0. Formally, let c_k^S be such that $c_k^S(T)=1$ if $T\subseteq S$ and $|T|\leq k$, and 0 otherwise. By Theorem 1, $y_i(c_{n-1}^N,e^i)=1-p_i^i(N\backslash\{i\})=1-(n-1)\theta_{n-2}$. Since $c_{n-1}^N(\{i\})=1$, by the Stand-alone property, $y_i(c_{n-1}^N,e^i)=1-(n-1)\theta_{n-2}\leq 1$ and therefore $\theta_{n-2}\geq 0$.

Suppose that for $k \in \{2, ..., n-2\}$ we have shown that $\theta_k \geq 0$. We show that it implies $\theta_{k-1} \geq 0$.

By Theorem 1,

$$\begin{aligned} y_i(c_k^N, e^i) &= 1 - \sum_{\substack{S \subseteq N \setminus \{i\} \\ |S| = k}} p_i^i(S) \\ &= 1 - \frac{(n-1)!}{k!(n-k-1)!} \left[k\theta_{k-1} - (n-k-1)\theta_k \right]. \end{aligned}$$

Since $c_k^N(\{i\}) = 1$, by the Stand-alone property,

$$y_i(c_k^N, e^i) = 1 - \frac{(n-1)!}{k!(n-k-1)!} \left[k\theta_{k-1} - (n-k-1)\theta_k \right] \leq 1$$

and therefore $(n-k-1)\theta_k \leq k\theta_{k-1}$. By the induction argument, $\theta_k \geq 0$, which allows us to conclude that $\theta_{k-1} \geq 0$. Therefore, $\theta_k \geq 0$ for all $k \in \{1, ..., n-2\}$.

By Theorem 1,

$$\begin{aligned} y_i(c_k^{N \setminus \{i\}}, e^i) &= -\sum_{\substack{S \subseteq N \setminus \{i\} \\ |S| \leq k}} p_i^i(S) \\ &= -\sum_{l=1}^{k} \frac{(n-1)!}{l!(n-l-1)!} \left[l\theta_{l-1} - (n-l-1)\theta_l \right] \end{aligned}$$

for any $k \leq n-1$. Developing the terms, we obtain that $y_i(c_k^{N \setminus \{i\}}, e^i) = \frac{(n-1)!}{k!(n-k-2)!}\theta_k$. Since $c_k^{N \setminus \{i\}}(\{i\}) = 0$, by the Stand-alone property, $y_i(c_k^{N \setminus \{i\}}, e^i) = \frac{(n-1)!}{k!(n-k-2)!}\theta_k \leq 0$ and therefore $\theta_k \leq 0$ for all $k \in \{1, ..., n-2\}$.

Combining the previous results, we obtain that $\theta = [0, 0, ..., 0]$. ■

Theorem 3 *A rule y satisfies the Technological Dummy property, Technological Linearity, Demand Linearity, Equal Shares for Equals and Technological Monotonicity if and only if* $y = y^*$.

Proof. Lemma 1 (in appendix) shows that y^* satisfies the five properties. We show that they imply a unique rule.

Fix $i \in N$ and take a $S \subseteq N \setminus \{i\}$. Suppose that $c, c' \in \Gamma$ are such that $c(T) = c'(T)$ for all $T \neq S \cup \{i\}$ and $c(S \cup \{i\}) < c'(S \cup \{i\})$. By Technological Monotonicity, we must have $y_i(c, e^i) \leq y_i(c', e^i)$, which can be written as $y_i(c', e^i) - y_i(c, e^i) \geq 0$. By Theorem 1, $y_i(c', e^i) - y_i(c, e^i) = p_i^i(S)\left[c'(S \cup \{i\}) - c(S \cup \{i\})\right]$. We therefore need $p_i^i(S) \geq 0$ for all $S \subseteq N \backslash \{i\}$.

Looking at the form of $p_i^i(S)$ in Theorem 1, this is only possible if $\theta = [0, 0, ..., 0]$. To see this, note that for $i \neq k$, $p_i^i(\{k\}) = -(n - |S| - 1)\theta_1$. We therefore need $\theta_1 \leq 0$. However, for i, j, k distinct, $p_i^i(\{j, k\}) = 2\theta_1 - (n - 3)\theta_2$, which implies that $\theta_2 \leq 0$. Note also that $p_i^i(S) = |S|\,\theta_{|S|-1} - (n - |S| - 1)\,\theta_{|S|}$ for all $S \subset N \backslash \{i\}$, which implies $\theta_k \leq 0$ for all $k = 1, ..., n-2$. However, $p_i^i(N \backslash \{i\}) = (n-1)\theta_{n-2}$. Since we must have $p_i^i(N \backslash \{i\}) \geq 0$, we need $\theta_{n-2} \geq 0$. This is only possible if $\theta = [0, 0, ..., 0]$. ■

Following the terminology of Weber (1988), we say that a marginal contributions rule z with weight p is a probabilistic value if, for all $i \in N$

$$\begin{aligned} \sum_{S \subseteq N \backslash \{i\}} p_i(S) &= 1 \\ p_i(S) &\geq 0 \text{ for all } S \subseteq N \backslash \{i\}. \end{aligned}$$

In all rules y defined in Theorem 1, y was a weighted sum of marginal contributions rules that satisfied the first condition above but not necessarily the second one. Following the proof of Theorem 3, it is clear that the technological contribution value is the only rule among the set defined in Theorem 1 that is a weighted sum of probabilistic values.

5 No reshuffling and the quasi-average contribution rule

While the technological contribution value has nice properties, we can look at rules defined in Theorem 1 that satisfy different properties. We now focus on a non-manipulability property violated by the technological contribution value, the No Reshuffling property. This property says that the aggregate cost share of agents in S should depend only on their aggregate demand $x(S)$. A rule that does not satisfy the No Reshuffling property could be the subject of manipulation. Agents in S could reshuffle their individual demands within S, perform side transfers and possibly benefit from the operation.

This property was introduced by Moulin (1987) in a bankruptcy model. Also, it was used by Sprumont (2005) to characterize the discrete version of the Aumann-Shapley cost-sharing rule.

No Reshuffling: For any $x, x' \in \mathbb{R}^N$ and $S \in \mathcal{N}$, if $x(S) = x'(S)$ and $x_i = x'_i$ for all $i \in N \backslash S$, then $\sum_{j \in S} y_j(c,x) = \sum_{j \in S} y_j(c,x')$ for all $c \in \Gamma$.

Among the rules defined in Theorem 1, the rule y', called the *quasi-average contribution rule*, is the only one satisfying No Reshuffling. It is defined as follows.

For a problem (c, x), define $\bar{c} = \sum_{S \in \mathcal{N}} (-1)^{|S|+1} c(S)$. Then, define $c' \in \mathbb{R}^{2^N}$ such that $c'(\varnothing) = \bar{c}$ and $c'(S) = c(S)$ for all $S \in \mathcal{N}$. Then, for all $i \in N$, $y'_i(c,x) = \bar{c} x_i + Sh_i(c')x(N)$. Alternatively, for each $i \in N$, $\bar{c} = c(\{i\}) + \sum_{\varnothing \neq S \subseteq N \backslash \{i\}} (-1)^{|S|} \left[c(S \cup \{i\}) - c(S) \right]$. Then, for all $i \in N$, $y'_i(c,x) = \bar{c} \left[x_i - \frac{x(N)}{n} \right] + Sh_i(c)x(N)$.

We thus start by computing the Shapley value of the game generated by the technological function c applied to the whole demand. We then adjust cost allocations by comparing demands to the average demand, and applying the difference to $\bar{c}$. As seen in the first representation, $\bar{c}$ can be viewed as the freely available technology.

We can see that $\sum_{i \in S} y'_i(c,x) = \bar{c}x(S) + x(N) \sum_{i \in S} Sh_i(c) - \frac{|S|}{n} \bar{c} x(N)$. The aggregate cost share of S depends only on $x(S)$ and $x(N)$.

The quasi-average contribution rule is the rule in Theorem 1 such that $\theta_k = \frac{k!(n-k-1)!}{n!} + \frac{(-1)^{k+1}}{n}$ for all $k \in \{1, ..., n-2\}$. To see this, recall that $\theta_{|S|} = p^i_j(S)$ for $i \neq j$ and $S \subseteq N \backslash \{i, j\}$. $p^i_j(S)$ is the weight put on the term $x_i c(S \cup \{j\})$ in the allocation of agent j. Since

$$Sh_j(c')x(N) = x(N) \left[\sum_{\varnothing \neq S \subseteq N \backslash \{j\}} \frac{|S|!\,(n-|S|-1)!}{n!} \left(c(S \cup \{j\} - c(S)) + \frac{1}{n} \left(c(\{j\}) - \bar{c} \right) \right],$$

the term $x_i c(S \cup \{j\})$ appears twice: first multiplied by $\frac{|S|!(n-|S|-1)!}{n!}$ and a second time in the term $\frac{1}{n} \left(c(\{j\}) - \bar{c} \right)$. Putting this together, we obtain that $p^i_j(S) = \frac{|S|!(n-|S|-1)!}{n!} + \frac{(-1)^{|S|+1}}{n} = \theta_{|S|}$.

Theorem 4 *A rule y satisfies the Technological Dummy property, Technological Linearity, Demand Linearity, Equal Shares for Equals and No Reshuffling if and only if $y = y'$.*

Proof. Lemma 2 (in appendix) shows that y' satisfies the five properties. We show that they imply a unique rule.

Fix a technological vector c and $i, j \in N$. By Theorem 1,

$$y_k(c, e^i) = \eta^i_k(c) = \sum_{\varnothing \neq S \subseteq N \backslash \{k\}} p^i_k(S) \left[c(S \cup \{k\}) - c(S) \right] + p^i_k(\varnothing) c(\{k\}).$$

We consider the possible reshuffling of the demands of agents i and j. We have that $y_i(c,e^i) + y_j(c,e^i) = \eta_i^i(c) + \eta_j^i(c) = \sum_{\varnothing \neq S \subseteq N} w^i_{\{i,j\}}(S)c(S)$, where $w^i_{\{i,j\}}(S) \in \mathbb{R}$ is the weight assigned to $c(S)$ in the allocations of $\{i,j\}$ when i has all the demand. Formally, since $\eta_k^i = \sum_{\varnothing \neq S \subseteq N\setminus\{k\}} p_k^i(S)\left[c(S\cup\{k\}) - c(S)\right] + p_k^i(\varnothing)c(\{k\})$, we have, for all $S \subseteq N\setminus\{i,j\}$, that

$$
\begin{aligned}
w^i_{\{i,j\}}(S) &= -p_i^i(S) - p_j^i(S)\\
w^i_{\{i,j\}}(S\cup\{i\}) &= p_i^i(S) - p_j^i(S\cup\{i\})\\
w^i_{\{i,j\}}(S\cup\{j\}) &= -p_i^i(S\cup\{j\}) + p_j^i(S)\\
w^i_{\{i,j\}}(S\cup\{i,j\}) &= p_i^i(S\cup\{j\}) + p_j^i(S\cup\{i\}).
\end{aligned}
$$

One possible reshuffling is to shift demand to agent j. No Reshuffling imposes that $y_i(c,e^i) + y_j(c,e^i) = y_i(c,e^j) + y_j(c,e^j)$. It clearly imposes that $w^i_{\{i,j\}}(S) = w^j_{\{i,j\}}(S)$ for all $\varnothing \neq S \subseteq N$.

Let $S = \{i\}$. Then, by Theorem 1, $w^i_{\{i,j\}}(\{i\}) = p_i^i(\varnothing) - p_j^i(\{i\}) = 1 - \frac{1}{n-1}$ while $w^j_{\{i,j\}}(\{i\}) = p_i^j(\varnothing) - p_j^j(\{i\}) = (n-2)\theta_1$. We obtain that $\theta_1 = \frac{1}{n-1}$.

Let $S \subseteq N\setminus\{j\}$ be such that $i \in S$ and $|S| \geq 2$. Then, by Theorem 1, $w^i_{\{i,j\}}(S) = p_i^i(S\setminus\{i\}) - p_j^i(S) = (|S|-1)\,\theta_{|S|-2} - (n-|S|-1)\theta_{|S|-1} - \frac{(|S|-1)!(n-|S|-1)!}{(n-1)!}$ while $w^j_{\{i,j\}}(S) = p_i^j(S\setminus\{i\}) - p_j^j(S) = \theta_{|S|-1} - |S|\,\theta_{|S|-1} + (n-|S|-1)\theta_{|S|}$. This can be rewritten as

$$
(|S|-1)\,\theta_{|S|-2} - (n-2\,|S|)\theta_{|S|-1} - (n-|S|-1)\theta_{|S|} = \frac{(|S|-1)!(n-|S|-1)!}{(n-1)!} \qquad (2.1)
$$

Since $\theta_0 = 0$ and $\theta_1 = \frac{1}{n-1}$, we can proceed recursively to find a unique value for all θ_k, $k = 2, \ldots n-2$. ■

While the facts that (2.1) yields a unique value for θ and that y' satisfies the properties of Theorem 4 are sufficient for the proof, it is illustrative to show that the θ associated with y' and given at the beginning of this section is compatible with (2.1). Let $A(S) = (|S|-1)\,\theta_{|S|-2} - (n-2\,|S|)\theta_{|S|-1} - (n-|S|-1)\theta_{|S|}$, take an S that has an even number of members and replace in (2.1) to obtain

$$\begin{aligned}
A(S) &= (|S|-1)\left[\frac{(|S|-2)!\,(n-|S|+1)!}{n!}-\frac{1}{n}\right] \\
&\quad -(n-2|S|)\left[\frac{(|S|-1)!\,(n-|S|)!}{n!}+\frac{1}{n}\right]-(n-|S|-1)\left[\frac{|S|!\,(n-|S|-1)!}{n!}-\frac{1}{n}\right] \\
&= (|S|-1)\frac{(|S|-2)!\,(n-|S|+1)!}{n!}-(n-2|S|)\frac{(|S|-1)!\,(n-|S|)!}{n!} \\
&\quad -(n-|S|-1)\frac{|S|!\,(n-|S|-1)!}{n!}-\frac{1}{n}\left[(|S|-1)+n-2|S|-(n-|S|-1)\right] \\
&= \frac{1}{n!}\left[\begin{array}{c}(|S|-1)!\,(n-|S|+1)!-(n-2|S|)\,(|S|-1)!\,(n-|S|)! \\ -(n-|S|-1)\,|S|!\,(n-|S|-1)!\end{array}\right] \\
&= \frac{(|S|-1)!(n-|S|-1)!}{n!}\left[\begin{array}{c}(n-|S|+1)\,(n-|S|)-(n-2|S|)\,(n-|S|) \\ -(n-|S|-1)\,|S|\end{array}\right] \\
&= \frac{(|S|-1)!(n-|S|-1)!}{n!}n \\
&= \frac{(|S|-1)!(n-|S|-1)!}{(n-1)!}.
\end{aligned}$$

6 Discussions and extensions

6.1 Linearity

The Technological Linearity property imposes structure on the admissible set of rules, allowing the complete characterization of Theorem 1. While the benefit is important, it comes at a cost, as we lose fair non-linear rules. Consider the following fairness property:

Proportional Shares for Symmetric Technology: If $|S|=|T|$ implies $c(S)=c(T)$ then $y_i(c,x)=c(N)x_i$ for all $i\in N$ and $x\in\mathbb{R}^n_+$.

The property states that if all agents are symmetric with respect to the technology vector, then the cost should be split in proportion to demand.

No rule satisfies Proportional Shares for Symmetric Technology, Technological Dummy and Technological Linearity. To see this, consider the vector $c^{N\backslash\{i\}}$ such that $c^{N\backslash\{i\}}(S)=1$ if $S\subseteq N\backslash\{i\}$ and 0 otherwise. Agents in $N\backslash\{i\}$ are technological dummies in $c^{N\backslash\{i\}}$. By Technological Dummy and budget balance, we have, for any $x\in\mathbb{R}^n_+, y_j(c^{N\backslash\{i\}};x)=x_j$ for all $j\neq i$ and $y_i(c^{N\backslash\{i\}},x)=-x(N\backslash\{i\})$. Next, consider the cost vector $c'=\sum_{i\in N}c^{N\backslash\{i\}}$. Then, $c'(S)=n-|S|$ for all $\varnothing\neq S\subseteq N$. By Proportional Shares for Symmetric Technology, we have

$y_i(c', x) = 0$. However, by Technological Linearity, we have $y_i(c', x) = \sum_{j \in N} y_i(c^{N\setminus\{j\}}, x) = (n-1)x_i - x(N\setminus\{i\})$ which in general is not equal to 0.

With Technological Linearity, a rule satisfying Technological Dummy will give credits to an agent for the savings his technology generates for the other agents. These savings depend on the demands of those agents. This makes it incompatible with Proportional Shares for Symmetric Technology, where an agent's share depends only on his own demand in the case where technologies are symmetric.

However, in itself, Technological Dummy and Proportional Shares for Symmetric Technology are not incompatible. For example, one can build the following rule:

$$\tilde{y}(c, x) = \begin{cases} y^{pr}(c, x) \text{ if } c(S) = c(T) \text{ when } |S| = |T| \\ y^*(c, x) \text{ otherwise} \end{cases}$$

where y^{pr} is the proportional rule. Alternatively, one could replace y^* by any rule from Theorem 1, and $\tilde{y}$ would satisfy Technological Dummy and Proportional Shares for Symmetric Technology, as well as Demand Linearity and Equal Shares for Equals. The family of rules satisfying these four properties would however contain many other rules. A complete characterization has yet to be obtained without Technological Linearity.

6.2 Returns to scale

The model presented here is obviously simple, as it assumes constant returns to scale and deals with non-differentiated goods. We used this model to show the fundamental differences when dealing with technology sharing and how the Technological Dummy property restricts the set of potential rules. The most general model would have a cost function $C(S, x)$ that assigns a cost to each coalition S and each possible profile of demand $x \in \mathbb{R}_+^N$, where $C(S, x)$ represents the cost of producing x using the technology of coalition S.

Without the constant returns to scale assumption, Demand Linearity loses its appeal. We then have two different effects on cost: the technology sharing effect, but also the returns to scale effect. Since it seems a challenge to separate those effects, it is not clear how to apply the idea of the Technological Dummy property in that context. A weak version could apply the Technological Dummy principle only when there are constant returns to scale and homogeneous goods. Formally, if there exists a function $c : \mathcal{N} \to \mathbb{R}_+$ such that

$C(S,x) = c(S)\sum_{i\in S} x_i$ for all $S \in \mathcal{N}$ and all $x \in \mathbb{R}_+^N$, and if $c(S \cup \{i\}) = c(S)$ for all $\varnothing \neq S \subseteq N \setminus \{i\}$, then $y_i = c(\{i\})\, x_i$. Rules that only look at the final cost function C, such as the Shapley value on C, do not satisfy this property.

Appendix

A.1 Proof of Theorem 1

Proof. Step 1: Define a linear operator for each agent

Let $e^1, ..., e^n$ denote the standard basis vectors for $\mathbb{R}^N$. Applying Demand Linearity, it follows that:

$$y_i(c, x) = \sum_{k=1}^{n} x_k y_i(c, e^k).$$

Fix k and define an operator $\eta^k : \Gamma \to \mathbb{R}^N$ where

$$\eta_i^k(c) = y_i(c, e^k).$$

Applying Technological Linearity, we have that η_i^k is a linear function in Γ.

Following the Theorem 1 of Weber (1988), there is a collection of constants

$$\left\{a_i^k(S) : \varnothing \neq S \subseteq N\right\} \text{ such that}$$

$$\eta_i^k(c) = \sum_{\varnothing \neq S \subseteq N} a_i^k(S) c(S).$$

Step 2: Show that η^k is a marginal contributions rule

For all $\varnothing \neq S \subseteq N$, define a cost vector d_S such that we have an average cost of 0 if S is part of the coalition that produces the good, and 1 otherwise. Formally, $d_S(T) = 0$ if $S \subseteq T$ and 1 otherwise. Note that for all $\varnothing \neq S \subseteq N$, $d_S \in \Gamma$ and that agents in $N \backslash S$ are technological dummies in d_S. Define d such that $d(S) = 1$ for all $S \in \mathcal{N}$. Note that all agents in N are technological dummies in d.

We have that $\eta_i^k(d) = \sum_{\varnothing \neq S \subseteq N} a_i^k(S) = e_i^k$ (by Technological Dummy (TD) Property) $= \eta_i^k(d_{N \backslash \{i\}}) = \sum_{\varnothing \neq S \subseteq N} a_i^k(S) - a_i^k(N) - a_i^k(N \backslash \{i\})$.

Therefore $a_i^k(N) + a_i^k(N \backslash \{i\}) = 0$.

Suppose that we have shown that $a_i^k(T) + a_i^k(T \cup \{i\}) = 0$ for every $T \subset N \backslash \{i\}$ such that $|T| \geq m \geq 2$. Take any fixed $\varnothing \neq S \subseteq N\backslash\{i\}$ with $|S| = m-1$. Then,

$$\begin{aligned}
\eta_i^k(d_S) &= \sum_{\varnothing \neq T \subseteq N} a_i^k(T) - \sum_{\substack{\varnothing \neq T \subseteq N \\ S \subseteq T}} a_i^k(T) \\
&= \sum_{\varnothing \neq T \subseteq N} a_i^k(T) - \left\{ \sum_{\substack{T \subseteq N\backslash\{i\} \\ S \subset T}} a_i^k(T \cup \{i\}) + a_i^k(T) \right\} - \left(a_i^k(S \cup \{i\}) + a_i^k(S)\right) \\
&= \sum_{\varnothing \neq T \subseteq N} a_i^k(T) - \left(a_i^k(S \cup \{i\}) + a_i^k(S)\right) \text{ (induction argument)}
\end{aligned}$$

since by definition, for $i \in N\backslash S$, we have $\eta_i^k(d_S) = e_i^k$ by Technological Dummy. Since $\sum_{\varnothing \neq T \subseteq N} a_i^k(T) = e_i^k$, we have $a_i^k(S \cup \{i\}) + a_i^k(S) = 0$ for all S such that $|S| = m-1$.

Therefore, there is a collection of constants $\left\{p_i^k(S) : S \subseteq N\backslash\{i\}\right\}$ such that:

$$\eta_i^k(c) = \sum_{\varnothing \neq S \subseteq N\backslash\{i\}} p_i^k(S)\left[c(S \cup \{i\}) - c(S)\right] + p_i^k(\varnothing)c(\{i\}).$$

Step 3: Find values for the weights p^i

Fix $i \in N$.

For each $S \subseteq N$ and $k \leq |S|$, we define a cost vector c_k^S such that coalitions that are part of S and have no more than k members have an average cost of 1, with all other coalitions having an average cost of 0. Formally, let c_k^S be such that $c_k^S(T) = 1$ if $T \subseteq S$ and $|T| \leq k$, and 0 otherwise. Note that agents in S are technological dummies in $c_{|S|}^S$.

For all $l \in N$, we have that $y_l(c_n^N, e^i) = \eta_l^i(c_n^N)$. By TD, we must have $y_i(c_n^N, e^i) = 1$ and $y_j(c_n^N, e^i) = 0$ for $j \in N\backslash\{i\}$. However, for $l = i, j$, we have $\eta_l^i(c_n^N) = p_l^i(\varnothing)$. Therefore, we obtain:

$$p_i^i(\varnothing) = 1,\ p_j^i(\varnothing) = 0,\ \text{for } j \in N\backslash\{i\}. \tag{A.1}$$

Take a $S \ni i$. By TD, we have that $y_i(c_{|S|}^S, e^i) = 1$, $y_j(c_{|S|}^S, e^i) = 0$ for all $j \in S\backslash\{i\}$. By Budget Balance and Equal Shares for Equals, $y_k(c_{|S|}^S, e^i) = -\frac{1}{n-|S|}$ for all $k \in N\backslash S$.

We have that $y_k(c_{|S|}^S, e^i) = \eta_k^i(c_{|S|}^S) = -\sum_{\varnothing \neq T \subseteq S} p_k^i(T) = -\frac{1}{n-|S|}$.

With $S = \{i\}$, we obtain that $p_k^i(\{i\}) = \frac{1}{n-1}$. With $S = \{i,j\}$ and $j \in N\backslash\{i,k\}$, we derive $p_k^i(\{i,j\}) + p_k^i(\{j\}) = \frac{1}{(n-1)(n-2)}$. Proceeding recursively in this manner we find that, for all $j \in N\setminus\{i\}$, all $S \subseteq N\setminus\{i,j\}$,

$$p_j^i(S) + p_j^i(S \cup \{i\}) = \frac{|S|!(n-|S|-2)!}{(n-1)!}. \tag{A.2}$$

The combinatorial format comes from the fact that when we take S such that $|S| = k$ in $-\sum_{\varnothing \neq T \subseteq S} p_k^i(T)$, the relations $p_j^i(T)+p_j^i(T\cup\{i\})$, with $|T| = l \leq k$, appear $\frac{(k-1)!}{l!(k-l-1)!}$ times.

For $j,k \in N\setminus\{i\}$ and $\varnothing \neq S \subseteq N\setminus\{j,k\}$, we have, by Equal Shares for Equals, that $y_j(c_{|S|}^S, e^i) = y_k(c_{|S|}^S, e^i)$. However, for $l = j,k$, $y_l(c_{|S|}^S, e^i) = \eta_l^i(c_{|S|}^S) = -\sum_{\varnothing \neq T \subseteq S} p_l^i(T)$. Taking $S = \{m\}$, with $m \in N\setminus\{j,k\}$, we find $p_j^i(\{m\}) = p_k^i(\{m\})$. Suppose that we have shown that $p_j^i(T) = p_k^i(T)$ for all $T \subseteq N\setminus\{j,k\}$ such that $|T| < r \leq n-2$. We show that it implies that $p_j^i(T) = p_k^i(T)$ for all $T \subseteq N\setminus\{j,k\}$ such that $|T| \leq r$. Take $S \subseteq N\setminus\{j,k\}$ such that $|S| = r$. For $l = j,k$, $y_l(c_{|S|}^S, e^i) = \eta_l^i(c_{|S|}^S) = -\sum_{\substack{\varnothing \neq T \subseteq S \\ |T|<r}} p_l^i(T) - p_l^i(S)$. By the recursive argument, it follows directly that $p_j^i(S) = p_k^i(S)$. Therefore, for all $j,k \in N\setminus\{i\}$ and all $S \subseteq N\setminus\{j,k\}$,

$$p_j^i(S) = p_k^i(S). \tag{A.3}$$

For $j,k \in N\setminus\{i\}$ and $\varnothing \neq S \subseteq N\setminus\{j,k\}$, we have, by Equal Shares for Equals, that $y_j(c_{|S|+1}^{S\cup\{j,k\}}, e^i) = y_k(c_{|S|+1}^{S\cup\{j,k\}}, e^i)$. However, $y_j(c_{|S|+1}^{S\cup\{j,k\}}, e^i) = \eta_j^i(c_{|S|+1}^{S\cup\{j,k\}}) = -p_j^i(S\cup\{k\}) = -p_k^i(S\cup\{j\}) = \eta_k^i(c_{|S|+1}^{S\cup\{j,k\}}) = y_k(c_{|S|+1}^{S\cup\{j,k\}}, e^i)$. Therefore, for $j,k \in N\setminus\{i\}$ and all $S \subseteq N\setminus\{j,k\}$, we have

$$p_j^i(S\cup\{k\}) = p_k^i(S\cup\{j\}). \tag{A.4}$$

By Equal Shares for Equals, we have that $y_j(c_m^{N\setminus\{i\}}, e^i) = y_k(c_m^{N\setminus\{i\}}, e^i)$ for all $j,k \in N\setminus\{i\}$ and $m < n-1$. However, for $l = j,k$, $y_l(c_m^{N\setminus\{i\}}, e^i) = \eta_l^i(c_m^{N\setminus\{i\}}) = -\sum_{\substack{S \subseteq N\setminus\{i,j\} \\ |S|=m}} p_l^i(S)$. Combining this result with (A.3) and (A.4), we obtain, for all $j,k \in N\setminus\{i\}$, all $S \subseteq N\setminus\{i,j\}$, all $T \subseteq N\setminus\{i,k\}$ such that $|S| = |T|$, that

$$p_j^i(S) = p_k^i(T). \tag{A.5}$$

Define $\theta_{|S|}^i = p_j^i(S)$ for $S \subseteq N\setminus\{i,j\}$ and $j \in N\setminus\{i\}$.

We obtain, using (A.1), (A.2) and (A.5), for $j \in N\setminus\{i\}$

$$\begin{aligned} p_j^i(\varnothing) &= 0 \\ p_j^i(S) &= \theta^i_{|S|} \text{ for all } S \subseteq N\setminus\{i,j\} \\ p_j^i(S\cup\{i\}) &= \frac{|S|!(n-|S|-2)!}{(n-1)!} - \theta^i_{|S|} \text{ for all } S \subseteq N\setminus\{i,j\}. \end{aligned} \tag{A.6}$$

It remains to define values for $p_i^i(S)$, $S \subseteq N\setminus\{i\}$.

For each $\varnothing \neq S \subset N\setminus\{i\}$ and $j \in S$, we have, by TD, that $y_j(c^S_{|S|}, e^i) = 0$. We also have that, for all $k \in N\setminus S$, $y_k(c^S_{|S|}, e^i) = \eta^i_k(c^S_{|S|}) = -\sum_{\varnothing\neq T\subseteq S} p^i_k(T)$.

By (A.6), $y_k(c^S_{|S|}, e^i) = -\sum_{l=1}^{|S|} \theta^i_l \frac{|S|!}{l!(|S|-l)!}$ for all $k \in N\setminus(S\cup\{i\})$ (since $p^i_k(T)$ depends only on the cardinality of T). By budget balance, it follows that $\sum_{j\in N} y_j(c^S_{|S|}, e^i) = 0$. Therefore, $y_i(c^S_{|S|}, e^i) = -\sum_{\varnothing\neq T\subseteq S} p^i_i(S) = (n-|S|-1)\sum_{l=1}^{|S|} \theta^i_l \frac{S|!}{l!(|S|-l)!}$. With $S = \{j\}$, we have that $p^i_i(\{j\}) = -(n-2)\theta^i_1$. For $S = \{j,k\}$, we obtain $p^i_i(\{j\}) + p^i_i(\{k\}) + p^i_i(\{j,k\}) = -(n-3)(2\theta^i_1 + \theta^i_2)$. Since $p^i_i(\{j\}) = p^i_i(\{k\}) = -(n-2)\theta^i_1$, we obtain $p^i_i(\{j,k\}) = 2\theta^i_1 - (n-3)\theta^i_2$. Proceeding recursively in this manner, we obtain that

$$p^i_i(S) = |S|\,\theta^i_{|S|-1} - (n-|S|-1)\,\theta^i_{|S|}$$

for any $\varnothing \neq S \subset N\setminus\{i\}$, with the convention that $\theta^i_0 = 0$.

Therefore,

$$\begin{aligned} p_i^i(\varnothing) &= 1 \\ p_i^i(S) &= |S|\,\theta^i_{|S|-1} - (n-|S|-1)\,\theta^i_{|S|} \text{ for } \varnothing \neq S \subseteq N\setminus\{i\} \\ p_i^i(N\setminus\{i\}) &= (n-1)\theta^i_{n-2}. \end{aligned}$$

Step 4: Show that $\theta^i_k = \theta^j_k$

Fix $i, j \in N$. By Equal Shares for Equals, we have that $y_i(c_1^{\{i,j\}}, e^i + e^j) = y_j(c_1^{\{i,j\}}, e^i + e^j)$.

For $l, m \in \{i, j\}$ and $l \neq m$, we have, by Step 3, that

$$\begin{aligned} y_l(c_1^{\{i,j\}}, e^i + e^j) &= y_l(c_1^{\{i,j\}}, e^i) + y_l(c_1^{\{i,j\}}, e^j) \text{ (by Demand Linearity)} \\ &= p_l^l(\varnothing) + p_l^m(\varnothing) - p_l^l(\{m\}) - p_l^m(\{m\}) \\ &= 1 + 0 - \theta_1^l - \frac{1}{n-1}. \end{aligned}$$

Directly, we have that $\theta_1^i = \theta_1^j = \theta_1$.

Now take k such that $1 < k \leq n-2$ and suppose that we have shown that $\theta_{k-1}^i = \theta_{k-1}^j$. We show that this implies $\theta_k^i = \theta_k^j$.

Take S such that $|S| = k+1$ and $i, j \in S$. By Equal Shares for Equals, we have $y_i(c_k^S, e^i + e^j) = y_j(c_k^S, e^i + e^j)$.

For $l, m \in \{i, j\}$ and $l \neq m$, from Step 3

$$\begin{aligned} y_l(c_k^S, e^i + e^j) &= y_l(c_k^S, e^i) + y_l(c_k^S, e^j) \text{ (by Demand Linearity)} \\ &= p_l^l(\varnothing) + p_l^m(\varnothing) - p_l^l(S \setminus \{l\}) - p_l^m(S \setminus \{l\}) \\ &= 1 + 0 - k\theta_{k-1}^l + (n-k-1)\theta_k^l - \frac{(k-1)!(n-k-1)!}{(n-1)!} + \theta_{k-1}^m. \end{aligned}$$

By induction, the result is that $\theta_k^i = \theta_k^j = \theta_k$.

Therefore, y must be as claimed.

We leave it to the reader to verify that all such rules satisfy the four properties. ■

A.2 Properties of y^* and y'

Lemma 1 *y^* is a budget balanced rule that satisfies the Technological Dummy property, Technological Linearity, Demand Linearity, Equal Shares for Equals, the Stand-alone property and Technological Monotonicity.*

Proof. Fix $i, j \in N$.

Budget balance:

$$\begin{aligned}
\sum_{k\in N} y_k^*(c,x) &= \sum_{k\in N} c(\{k\})\,x_k + \sum_{k\in N}\sum_{l\in N\setminus\{k\}} x_l Sh_k(v^l(\cdot,c)) \\
&= \sum_{l\in N} c(\{l\})\,x_l + \sum_{l\in N}\sum_{k\in N\setminus\{l\}} x_l Sh_k(v^l(\cdot,c)) \\
&= \sum_{l\in N} c(\{l\})\,x_l + \sum_{l\in N} x_l(c(N) - c(\{l\})) \\
&= c(N)\sum_{l\in N} x_l = c(N)x(N)
\end{aligned}$$

Technological Dummy: Suppose that c is such that $c(S\cup\{i\}) = c(S)$ for all $\varnothing \neq S \subseteq N\setminus\{i\}$. For any $k\in N\setminus\{i\}$ and any $S\in N\setminus\{i,k\}$, $v^k(S\cup\{i\},c) = c(S\cup\{i,k\}) - c(\{k\}) = c(S\cup\{k\}) - c(\{k\}) = v^k(S,c)$. We also have that $v^k(\{i\},c) = c(\{i,k\}) - c(\{k\}) = 0$. Thus, all marginal contributions of agent i are equal to zero and $Sh_i(v^k(\cdot,c)) = 0$ for all $k\in N\setminus\{i\}$. Therefore, $y_i^*(c,x) = c(\{i\})x_i$.

Technological Linearity: For $c^1, c^2 \in \Gamma$ and $\beta_1, \beta_2 \in \mathbb{R}$ such that $\beta_1 c^1 + \beta_2 c^2 \in \Gamma$, we have that, for any $j\in N$, and any $\varnothing \neq S \subseteq N\setminus\{j\}$

$$\begin{aligned}
v^j(S,\beta_1 c^1 + \beta_2 c^2) &= \beta_1 c^1(S\cup\{j\}) + \beta_2 c^2(S\cup\{j\}) - \beta_1 c^1(\{j\}) - \beta_2 c^2(\{j\}) \\
&= \beta_1 v^j(S,c^1) + \beta_2 v^j(S,c^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
y_i^*(\beta_1 c^1 + \beta_2 c^2, x) &= \beta_1 c^1(\{i\}) + \beta_2 c^2(\{i\}) + \sum_{j\in N\setminus\{i\}} x_j Sh_i(v^j(\cdot,\beta_1 c^1 + \beta_2 c^2)) \\
&= \beta_1 c^1(\{i\}) + \beta_2 c^2(\{i\}) + \sum_{j\in N\setminus\{i\}} x_j \left[\beta_1 Sh_i(v^j(\cdot,c^1)) + \beta_2 Sh_i(v^j(\cdot,c^2))\right] \\
&= \beta_1 y_i^*(c^1,x) + \beta_2 y_i^*(c^2,x).
\end{aligned}$$

Demand Linearity: Follows directly from the fact that demands enter y^* in a linear fashion.

Equal Shares for Equals: Suppose that we have c such that $c(S\cup\{i\}) = c(S\cup\{j\})$ for all $S \subseteq N\setminus\{i,j\}$ and x such that $x_i = x_j$. Clearly, the first terms in $y_i^*(c,x)$ and $y_j^*(c,x)$ are equal. For any $S \subseteq N \backslash \{i,j\}$ and any $k \in N \backslash \{i,j\}$

$$v^k(S \cup \{i\}, c) = c(S \cup \{i,k\}) - c(\{k\}) = c(S \cup \{j,k\}) - c(\{k\}) = v^k(S \cup \{j\}, c).$$

Therefore, $Sh_i(v^k(\cdot,c)) = Sh_j(v^k(\cdot,c)$ for any $k \in N \backslash \{i,j\}$. For any $S \subseteq N \backslash \{i,j\}$, we have

$$\begin{aligned} v^i(S \cup \{j\}, c) - v^i(S,c) &= c(S \cup \{i,j\}) - c(\{i\}) - c(S \cup \{i\}) + c(\{i\}) \\ &= c(S \cup \{i,j\}) - c(S \cup \{i\}) \\ v^j(S \cup \{i\}, c) - v^j(S,c) &= c(S \cup \{i,j\}) - c(\{j\}) - c(S \cup \{j\}) + c(\{j\}) \\ &= c(S \cup \{i,j\}) - c(S \cup \{j\}). \end{aligned}$$

We thus have that $v^i(S \cup \{j\}, c) - v^i(S,c) = v^j(S \cup \{i\}, c) - v^j(S,c)$ for any $S \subseteq N \backslash \{i,j\}$, which implies that $Sh_j(v^i(\cdot,c)) = Sh_i(v^j(\cdot,c))$. Therefore, $y_i^*(c,x) = y_j^*(c,x)$.

Stand-alone property: Follows directly from the fact that $Sh_i(v^j,c) \leq 0$ for all $i,j \in N$ and all $c \in \Gamma$, and from the fact that $x \in \mathbb{R}_+^N$.

Technological Monotonicity: Fix $x \in \mathbb{R}_+^N$ and $S \in \mathcal{N}$ such that $i \in S$. Take $c, c' \in \Gamma$ such that $c(S) \geq c'(S)$ and $c(T) = c'(T)$ for all $T \in \mathcal{N} \backslash S$. For any $k \in N \backslash \{i\}$, $v^k(S \backslash \{k\}, c) = c(S) - c(\{k\}) \geq c'(S) - c(\{k\}) = v^k(S \backslash \{k\}, c')$, while $v^k(T,c) = v^k(T,c')$ for all $T \neq S \backslash \{k\}$. By the properties of the Shapley value, $Sh_i(v^k(\cdot,c)) \geq Sh_i(v^k(\cdot,c'))$. Then, $y_i^*(c,x) \geq y_i^*(c',x)$. ■

Lemma 2 *y' is a budget balanced rule that satisfies the Technological Dummy property, Technological Linearity, Demand Linearity, Equal Shares for Equals and No Reshuffling.*

Proof. Fix $c \in \Gamma$, $x \in \mathbb{R}_+^n$, $i \in N$.

Budget-balance: $\sum_{i \in N} y_i'(c,x) = \bar{c}x(N) + \sum_{i \in N} Sh_i(c)x(N) - \bar{c}x(N) = c(N)x(N)$.

Technological dummy property: If i is a technological dummy in c, $c(S \cup \{i\}) - c(S) = 0$ for all $\varnothing \neq S \subseteq N \backslash \{i\}$. Immediately, we have that $\bar{c} = c(\{i\})$. All marginal contributions are equal to zero, except $c(\{i\})$. Therefore, $Sh_i(c) = \frac{\bar{c}}{n}$ and $y_i'(c,x) = c(\{i\})x_i$.

Technological Linearity: Let $c = \beta_1 c^1 + \beta_2 c^2$ such that $c, c^1, c^2 \in \Gamma$, $\beta_1, \beta_2 \in \mathbb{R}_+$. Clearly, $\bar{c} = \beta_1 \bar{c}^1 + \beta_2 \bar{c}^2$. It follows, by the properties of the Shapley value, that $y_i'(c,x) = \beta_1 y_i'(c^1,x) + \beta_2 y_i'(c^2,x)$.

Demand Linearity: Follows directly since x_i and $x(N)$ enter y' linearly.

Equal Shares for Equals: If $c(S \cup \{i\}) = c(S \cup \{j\})$ for all $S \subseteq N \setminus \{i,j\}$, then $Sh_i(c) = Sh_j(c)$. Therefore, if $x_i = x_j$, then $y'_i(c,x) = y'_j(c,x)$.

No Reshuffling: Fix $S \in \mathcal{N}$.

$\sum_{i \in S} y'_i(c,x) = \bar{c}x(S) + x(N) \sum_{i \in S} Sh_i(c) - \frac{|S|}{n}\bar{c}x(N)$. It depends only on $x(S)$ and $x(N)$.

■

A.3 Independence of properties

We show that if $n \geq 3$, Theorems 2, 3 and 4 are tight. We find a budget-balanced rule satisfying all but one of the properties used.

Define $C(S,c,x) = c(S)x(S)$, the function that assigns the total cost to each coalition. Define $y^1(c,x) = Sh(C(\cdot,c,x))$ as the Shapley value on the stand-alone game C. One can verify that y^1 satisfies Budget-balance and all properties in Theorems 2 and 3 but the Technological Dummy property (see example 2).

For all $i \in N$, define

$$y_i^2(c,x) = c(\{i\})x_i + \frac{\sum_{\varnothing \neq S \subseteq N \setminus \{i\}} c(S \cup \{i\}) - c(S)}{\sum_{j \in N} \sum_{\varnothing \neq S \subseteq N \setminus \{j\}} c(S \cup \{j\}) - c(S)} \left[c(N)x(N) - \sum_{j \in N} c(\{j\})x_j \right],$$

if $\sum_{j \in N} \sum_{\varnothing \neq S \subseteq N \setminus \{j\}} c(S \cup \{j\}) - c(S) \neq 0$ and $y_i^2(c,x) = c(\{i\})x_i$ otherwise. One can verify that y^2 satisfies Budget-balance and all properties in Theorems 2 and 3, but Technological Linearity.

For all $i,j \in N$ and all $x \in \mathbb{R}_+^N$, let $S_i^j(x) = \{k \in N \setminus \{j\} \mid x_k < x_i\}$ and $\bar{S}_i^j(x) = \{k \in N \setminus \{j\} \mid x_k = x_i\}$. Also, for all $\varnothing \neq T \subseteq \bar{S}_i^j(x)$, define $w_i^j(T,c,x) = c(T \cup S_i^j(x)) - c(S_i^j(x))$. For all $k \in N$, define $\theta_k^j(c,x) = Sh_k(w_k^j(\cdot,c,x))$ where $Sh_k(w_k^j(\cdot,c,x))$ is the Shapley value of the TU-game $w_k^j(\cdot,c,x)$, defined for the set of agents $\bar{S}_k^j(x)$.

For all $c \in \Gamma$ and all $x \in \mathbb{R}_+^N$, define

$$y_i^3(c,x) = c(\{i\})x_i + \sum_{j \neq i} \theta_i^j(c,x)x_j.$$

One can verify that y^3 satisfies Budget-balance and all properties in Theorems 2 and 3 but Demand Linearity.

Let π be an ordering of agents in N, and define $S_i^\pi \subset N$ as the set of agents coming before i in the ordering π. Then, define y^4 as follows:

$$y_i^4 = c(\{i\})\,x_i + \sum_{j\in N\setminus\{i\}} x_j\left(c(S_i^\pi \cup \{i,j\}) - c(S_i^\pi \cup \{j\})\right).$$

One can verify that y^4 satisfies Budget-balance and all properties in Theorems 2 and 3 but Equal Shares for Equals.

The following table, where "+" signifies that the property is satisfied, and "-" that it is not, summarizes the results:

	y^*	y'	y^1	y^2	y^3	y^4
Technological Dummy property	+	+	-	+	+	+
Technological Linearity	+	+	+	-	+	+
Demand Linearity	+	+	+	+	-	+
Equal Shares for Equals	+	+	+	+	+	-
Stand-alone property	+	-	+	+	+	+
Technological Monotonicity	+	-	+	+	+	+

For all $c \in \Gamma$ all $x \in \mathbb{R}_+^N$ and all $i \in N$, define

$$y_i^5(c,x) = \frac{c(N)x(N)}{n}.$$

One can verify that y^5 satisfies Budget-balance and all properties in Theorem 4 but the Technological Dummy property.

For all $c \in \Gamma$ all $x \in \mathbb{R}_+^N$ and all $i \in N$, define

$$y_i^6(c,x) = \bar{c}\left(x_i - \frac{x(N)}{n}\right) + c(\{i\})\frac{x(N)}{n} + \frac{x(N)\left(c(N) - c(N\setminus\{i\})\right)}{\sum_{j\in N}\left(c(N) - c(N\setminus\{j\})\right)}\left[c(N) - \sum_{j\in N}\frac{c(\{j\})}{n}\right]$$

if $\sum_{j\in N}\left(c(N) - c(N\setminus\{j\})\right) \neq 0$ and $y_i^6(c,x) = y'(c,x)$ else.

One can verify that y^6 satisfies Budget-balance and all properties in Theorem 4 but Technological Linearity.

For all $i,j \in N$ and $x \in \mathbb{R}_+^N$, let $S_i(x) = \{j \in N \mid x_j < x_i\}$ and $\bar{S}_i(x) = \{k \in N \mid x_j = x_i\}$. Also, for all $\varnothing \neq T \subseteq \bar{S}_i(x)$, define $w_i(T,c,x) = c(T \cup S_i(x)) - c(S_i(x))$. Define $\theta_i(c,x) =$

$Sh_i(w_i(\cdot,c,x))$ where $Sh_i(w_i(\cdot,c,x))$ is the Shapley value on the TU-game $w_i(\cdot,c,x)$, defined for the set of agents $\bar{S}_i(x)$.

For all $c \in \Gamma$ and all $x \in \mathbb{R}_+^N$, define

$$y_i^7(c,x) = \bar{c}(\{i\})x_i + \theta_i(c',x)x(N).$$

One can verify that y^7 satisfies Budget-balance and all properties in Theorem 4 but Demand Linearity.

For all $c \in \Gamma$, take an ordering of the agents such that $c(\{1\}) \leq c(\{2\}) \leq ... \leq c(\{n\})$. For all $i \in N$, let $S_i = \{j \in N \mid j < i\}$. Then, for all $x \in \mathbb{R}_+^N$, define

$$y_i^8(c,x) = \bar{c}\left(x_i - \frac{x(N)}{n}\right) + c(\{i\})\frac{x(N)}{n} + x(N)\left(c'(S_i \cup \{i\}) - c'(S_i)\right).$$

One can verify that y^8 satisfies Budget-balance and all properties in Theorem 4 but Equal Shares for Equals.

The following table summarizes the results:

	y^*	y'	y^5	y^6	y^7	y^8
Technological Dummy property	+	+	-	+	+	+
Technological Linearity	+	+	+	-	+	+
Demand Linearity	+	+	+	+	-	+
Equal Shares for Equals	+	+	+	+	+	-
No Reshuffling	-	+	+	+	+	+

Chapter 3

Automatic user contributions and price competition

1 Introduction

Studies of vertical differentiation models with endogenous quality have focused on cases where quality can be improved through R&D (see for instance Lehmann-Grube (1997)). We consider a different form of innovation, made through user contributions. The recent phenomenon of the open-source movement, where a large group of users devote time and energy to improve software, with its code being freely available, has attracted the most attention. However, users can provide various types of contributions to a product: the presence of an extra user can affect instantaneously the welfare of other users (network externality), a user can freely choose to work on the improvement of the quality of the product (voluntary contribution), or a user improves the future quality of the product just by consuming it (automatic contribution).

The first type includes classical examples such as cell phones and competition between technological standards (for instance Blu-Ray vs HD-DVD), but also community-based websites like Facebook. Examples of the second type include the previously mentioned case of open-source softwares, but also websites presenting user-generated content like YouTube and Wikipedia. The third type of contribution includes many recent examples such as a software company automatically tracking bugs and crashes, search engines tracking the links clicked by their users to improve the quality of the search results and anti-virus software companies tracking malicious activities on their users' systems to improve the protection offered. A more traditional way to interpret automatic user contributions would be cases where each sales made help the firm improve the product, for instance through an improvement in after-sales services.

The paper focuses on the third kind of user contribution, which has received little attention but is gaining in prevalence. This type of contribution differs from voluntary contributions by the fact that users do not decide whether to contribute or not. While in the open-sourcing literature much attention has been devoted to the determinants of contributions (Lerner and Tirole (2005), Roberts, Hann, and Slaughter (2006), Shah (2006), Bagozzi and Dholakia (2006)) or on the way the contributors are organized (Grewal, Lilien, and Mallapragada (2006), Baldwin and Clark (2006)), this is a non-factor with automatic user contributions. We can focus on how agents choose among different products, with contributions being a function of sales. Note also that open-sourcing firms are usually modelled as non-profit maximizing (although the open-sourcing and free software movements are not the same), while the examples we have of automatic contributions point to profit-maximizing behavior.

Automatic contributions differ from network externalities by the timing of the benefit to other users. Network externalities are usually such that agents create instant benefits for other users and agents have large switching costs to move from one product to another. The behavior of a consumer therefore depends on his expectations of future sales while, with automatic contributions, a consumer only cares about current qualities and prices.

Note that automatic contributions are related to but different from learning-by-doing, which usually has an impact on cost and not on quality. Learning-by-doing usually has a positive spillover on the whole industry, whereas here firms compete for contributions.

We model automatic contributions in the following way. Firms play a 2-period game and start with given qualities of their product. In the first period, the development phase, they choose a price, and make some sales. The sales made in the development phase affect the quality of the good in the second period, the maturity phase. There are no switching costs, and the good is not durable, so agents can switch products without being penalized. We model firms as profit-maximizing. Moreover, prices will be loosely defined here, and can include indirect costs incurred by the users such as ad banners, use of E-mail for commercial purposes, etc. This also allows us to have negative prices (users can win prizes, etc.). A recent example of these negative prices is Microsoft's reward point system for its search engine Live Search.[1]

The proposed model is part of the literature on vertical differentiation with endogenous quality initiated by Shaked and Sutton (1982). Our main distinction is the fact that the quality improvement technology here is rival, as firms fight for customers and their contributions. Common conclusions of vertical differentiation models include different qualities observed in equilibrium and a high-quality advantage. In particular, Tirole (1988) obtains maximal differentiation in a model with covered markets and costless quality investment. With automatic user contributions, we show that maximal differentiation is a possible out-

[1] "Microsoft sweetens pay-to-search deal", Yi-Wyn Yen, CNNmoney.com, September 29, 2008, http://techland.blogs.fortune.cnn.com/2008/09/29/microsoft-sweetens-its-pay-to-search-deal-2/, viewed on October 1, 2008.

come but, depending on the parameters, is not always an equilibrium.

More generally, the model is part of the large literature on price competition in dynamic oligopolies. Maskin and Tirole (1988) provide a general framework. Among the many applications of this model, we find dynamic oligopolies with learning-by-doing (Cabral and Riordan (1994)) or with network and switching costs effects (Farrell and Klemperer (2007)). As in these models, we find evidence of pricing under marginal cost.

We are interested in how quality evolves and if there can be lead reversals. Related contributions include Fudenberg, Gilbert, Stiglitz, and Tirole (1983) and Harris and Vickers (1987), who study possible leapfrogging in patent races, and Budd, Harris, and Vickers (1993), who build a model to study if asymmetry tends to grow in a dynamic oligopoly where competition depends on efforts (R&D, advertising, etc.) from the firms. Athey and Schmutzler (2001) build a general model of dynamic competition with investment, which encompasses many of the previous models. The general conclusions are that it requires large advantages, in terms of investment effectiveness, for the follower or significant uncertainty to obtain leapfrogging. In general, the leading firm tends to increase its dominance. These results will still hold with automatic contributions.

Athey and Schmutzler (2001) also look at welfare effects, and conclude that while increasing dominance of the leading firm has a negative effect on welfare, if investments have positive impacts on consumers (cost reductions, quality improvements), it is possible for the total welfare effect to be positive. In our model with user contributions, compared to a vertical differentiation model without quality improvements, this total welfare effect is always positive. In fact, automatic user contributions improve welfare in each period. Also, the (implicit) private cost for the firm to improve quality might be negative, as the incentives to obtain maximal differentiation might reduce price competition. We also find that in some parameter regions, marginally increasing user contributions is Pareto-improving. These welfare effects are of interest because they determine if an industry or consumer groups, or even the government, might have incentives to intervene in a market to either push or restrict automatic contributions.

The paper is as follows. In Section 2, we formally define the model. In Section 3, we fully characterize the subgame perfect Nash equilibria of this dynamic game. We look at equilibrium prices and discuss how they evolve with the initial quality lead and time preferences.

We also examine when these prices are lower than in a model without user contributions and when they are less than marginal cost. Welfare and individual effects are studied in Section 4. In that section, we also examine the implicit cost of quality improvements. Extensions to the model are proposed in Section 5. These include dealing with asymmetries, increasing preferences for quality and market coverage. Section 6 concludes.

2 The model

Two firms a and b compete in prices over two periods. Their products are differentiated in quality. However, we suppose that the quality of these products evolves between the periods, because of the contributions of users. Therefore, period 1 is labelled the development phase and period 2 is the maturity phase. We suppose that these contributions are automatic, or proportional to sales, in the first period. We focus on the effect contributions have on price competition.

Let p_i, x_i, q_i, π_i denote, respectively, the price, quantity, quality and profits of firm i in period 1, and p_i', x_i', q_i', π_i' represent the same variables in period 2. We suppose that all quantities and qualities are in $\mathbb{R}_+$, with prices and profits in $\mathbb{R}$. We suppose that the marginal cost of production is 0. Time is discounted by $\beta \in [0,1]$. Let $\Pi_i = \pi_i + \beta\pi_i'$ be the total profits of firm i.[2]

Let q_a and q_b be given. We suppose that $q_a \geq q_b$, i.e. firm a has an initial quality lead. Let $q_i' = q_i + \lambda x_i$, for $i = a, b$, with $\lambda \in [0,1]$ being the quality improvement parameter. λ can be interpreted as the amount of automatic contributions extracted from users or as the efficiency it has in improving quality. Let $\Delta q = q_a - q_b$ and $\Delta q' = |q_a' - q_b'|$.

Each consumer k is characterized by a marginal willingness to pay for quality γ_k, uniformly distributed over the support $[0,1]$, with the population normalized to 1. Let u_i^k be the utility of consumer k when he buys the product of firm i: $u_i^k = v + \gamma_k q_i - p_i$. A consumer buys one unit or buys nothing. If he buys nothing, his utility is 0. The valuation v is high enough that each agent buys one unit at each period. Note that each consumer has a negligible effect on aggregate variables and thus can be treated as myopic.

[2] Since marginal cost is equal to zero, negative prices allow for pricing under marginal cost. The assumption on marginal cost eliminates variables and allows for a clearer exposition.

3 Equilibria

3.1 The maturity phase

The equilibrium concept is subgame perfect Nash-equilibrium, so we proceed by backward induction and start by looking at the second period. Suppose that $\Delta q'$ is known. We denote by L the leading firm (in terms of quality) in the maturity phase, with F being the follower. Note that even though we suppose that firm a has the lead initially, that is $q_a \geq q_b$, this order need not hold in the second period. This subgame in the maturity phase is exactly the classic game with vertical differentiation when qualities do not evolve. We look at two cases: $\Delta q' > 0$ and $\Delta q' = 0$.

Case i) $\Delta q' > 0$

Let $\bar{\gamma}$ be the willingness to pay for quality such that the consumer with $\bar{\gamma}$ is indifferent between the products of firm L and F. We find

$$\bar{\gamma} = \frac{p'_L - p'_F}{\Delta q'}.$$

From this, we obtain x'_L and x'_F,

$$(x'_L, x'_F) = \begin{cases} (0,1) \text{ if } p'_L \geq \Delta q' + p'_F \\ \left(1 - \frac{p'_L - p'_F}{\Delta q'}, \frac{p'_L - p'_F}{\Delta q'}\right) \text{ if } \Delta q' + p'_F \geq p'_L \geq p'_F \\ (1,0) \text{ if } p'_L \leq p'_F \end{cases}.$$

Maximizing second period profits $\pi'_i = p'_i x'_i$, we obtain the best responses $p_i'^{br}$. For firm L, we have[3]

$$p_L'^{br} = \begin{cases} 0 \text{ if } p'_F \leq -\Delta q' \\ \frac{\Delta q' + p'_F}{2} \text{ if } -\Delta q' \leq p'_F \leq \Delta q' \\ p'_F \text{ if } p'_F \geq \Delta q' \end{cases}$$

[3]Formally, in the first segment, firm L has no sales, so it could choose any price $p \geq 0$. Since any price $p > 0$ cannot be part of an equilibrium, we only keep $p = 0$ as the best response. Similiarly, in all following best response functions, the same simplification has been made for cases where a firm has no sales.

while, for firm F,

$$p_F'^{br} = \begin{cases} 0 \text{ if } p_L' \leq 0 \\ \frac{p_L'}{2} \text{ if } 0 \leq p_L' \leq 2\Delta q' \\ \Delta q' \text{ if } p_L' \geq 2\Delta q' \end{cases}.$$

We find that there exists a unique pure strategy Nash equilibrium $(p_L^*, p_F^*) = \left(\frac{2\Delta q'}{3}, \frac{\Delta q'}{3}\right)$ such that

$$\begin{aligned} x_L'^* &= \frac{2}{3} \\ x_F'^* &= \frac{1}{3} \\ \pi_L'^* &= \frac{4\Delta q'}{9} \\ \pi_F'^* &= \frac{\Delta q'}{9}. \end{aligned}$$

Case ii) $\Delta q' = 0$

We directly obtain that, for $i = L, F$

$$x_i' = \begin{cases} 1 \text{ if } p_i' < p_j' \\ 1/2 \text{ if } p_i' = p_j' \\ 0 \text{ if } p_i' > p_j' \end{cases}.$$

Products are identical, and we obtain the Bertrand Paradox. That is, $p_i'^* = 0$, $\pi_i'^* = 0$ for $i = a, b$.

3.2 The development phase

We now consider choices in the first period. The starting value of Δq and the choice of prices at period 1 determine the outcome of the whole game. There are two possible cases: when initial qualities are identical and when firm a has an initial lead.

Case i) Identical qualities

We obtain a unique ϵ-Nash equilibrium in pure strategies, which is such that one of the firm has the strategy p^* while the other has the strategy $p^* + \epsilon$, with ϵ close to zero.

Theorem 1 *If $\Delta q = 0$, there is a unique ϵ-Nash equilibrium in pure strategies, such that $p_i^* = -\frac{\beta\lambda}{3}$ for $i = a, b$.*

Proof. Suppose that $p_a = p_b$ is an equilibrium. Then, both firms sell 1/2 units and have the same quality in period 2. Therefore, their second-period profits are equal to 0, and their global profits are $\frac{p_a}{2}$. If firm a deviates and charges $p_a - \epsilon$, it gets the whole market and achieves maximum differentiation for a global profit of $p_a - \epsilon + \frac{4\beta\lambda}{9}$, which is higher than $\frac{p_a}{2}$ when ϵ is small and $p_a \geq -\frac{8\beta\lambda}{9}$. In the case where $p_a = p_b < -\frac{8\beta\lambda}{9}$, profits are negative. Firms have incentives to increase prices. Therefore, there is no equilibrium where $p_a = p_b$.

Now suppose that $p_i < p_j$. We then have that $\Pi_i = p_i + \frac{4\beta\lambda}{9}$ and $\Pi_j = \frac{\beta\lambda}{9}$. Firm j has no incentive to increase its price as it already sells nothing in period 1. However, it could decrease its price to $p_i - \epsilon$. That would yield a deviating profit of $\Pi_j^d = p_i - \epsilon + \frac{4\beta\lambda}{9}$. For ϵ close to zero, firm j will have no incentive to deviate if $p_i \leq -\frac{\beta\lambda}{3}$.

Firm i has no incentive to lower its price, as it already sells to all consumers in period 1. However, it could increase its price to $p_j + \epsilon$. This would yield a deviating profit of $\Pi_i^d = \frac{\beta\lambda}{9}$. For ϵ close to zero, firm i will have no incentive to deviate if $p_i \geq -\frac{\beta\lambda}{3}$. Therefore, the only possibility is to have $p_i^* = -\frac{\beta\lambda}{3} < p_j^*$.

Putting all of this together, we obtain a unique ϵ-Nash equilibria in pure strategies such that $p_i^* = -\frac{\beta\lambda}{3}$ for $i = a, b$. ■

We therefore have the result that one firm lets its competitor have the whole market in the first period (although at a negative price). Both firms then benefit from maximal differentiation in the second period. The leading firm enjoys higher second-period profits, and these more than offset losses in the first period. In fact, both firms end up with exactly the same (positive) profits.

Case ii) Non-identical qualities

Suppose that $0 < \Delta q$, that is, firm a has the initial lead in quality. Note that if $\Delta q \leq \lambda$, firm b can potentially take the lead in period 2.

We first derive separate best response functions, conditional on a firm being the leader or the follower in the second period. Of course, if $\Delta q > \lambda$, we only consider best responses where firm a is the leader and firm b the follower.

If firm a maintains leadership in the second period, its profit function is:

$$\pi_a^L = p_a\left(1 - \frac{p_a - p_b}{\Delta q}\right) + \frac{4\beta}{9}\left(\Delta q + \lambda\left(1 - 2\frac{p_a - p_b}{\Delta q}\right)\right),$$

with the following conditions:

$$0 \leq 1 - \frac{p_a - p_b}{\Delta q} \leq 1$$
$$\Delta q + \lambda\left(1 - 2\frac{p_a - p_b}{\Delta q}\right) \geq 0.$$

We obtain the following best response :

$$p_a^L = \begin{cases} p_b + \frac{\Delta q}{2} + \frac{\Delta q^2}{2\lambda} \text{ if } p_b \leq -\frac{8\beta\lambda}{9} - \frac{\Delta q^2}{\lambda} \\ \frac{p_b}{2} + \frac{\Delta q}{2} - \frac{4\beta\lambda}{9} \text{ if } -\frac{8\beta\lambda}{9} - \frac{\Delta q^2}{\lambda} \leq p_b \leq -\frac{8\beta\lambda}{9} + \Delta q \\ p_b \text{ if } p_b \geq -\frac{8\beta\lambda}{9} + \Delta q \end{cases}$$

yielding

$$\pi_a^L = \begin{cases} \frac{\Delta q}{4} + \frac{\Delta q^2}{2\lambda} + \frac{\Delta q^3}{4\lambda^2} + \frac{p_b}{2} + \frac{\Delta q p_b}{2\lambda} \text{if } p_b \leq -\frac{8\beta\lambda}{9} - \frac{\Delta q^2}{\lambda} \\ \frac{64\beta^2\lambda^2 + 81\Delta q^2 + 144\beta\Delta q^2 + 144\beta\lambda p_b + 162\Delta q p_b + 81 p_b^2}{324\Delta q} \text{ if } -\frac{8\beta\lambda}{9} - \frac{\Delta q^2}{\lambda} \leq p_b \leq -\frac{8\beta\lambda}{9} + \Delta q \\ p_b + \frac{4\beta}{9}(\Delta q + \lambda) \text{ if } p_b \geq -\frac{8\beta\lambda}{9} + \Delta q \end{cases}.$$

Note that the first segment of these functions is such that $\Delta q' = 0$.

If firm a surrenders leadership in the second period, its profit function is:

$$\pi_a^F = p_a\left(1 - \frac{p_a - p_b}{\Delta q}\right) + \frac{\beta}{9}\left(-\Delta q + \lambda\left(2\frac{p_a - p_b}{\Delta q} - 1\right)\right),$$

with the following conditions:

$$0 \leq 1 - \frac{p_a - p_b}{\Delta q} \leq 1$$
$$-\Delta q + \lambda\left(2\frac{p_a - p_b}{\Delta q} - 1\right) \geq 0.$$

We obtain the following best response:

$$p_a^F = \begin{cases} p_b + \Delta q \text{ if } p_b \le \frac{2\beta\lambda}{9} - \Delta q \\ \frac{p_b}{2} + \frac{\Delta q}{2} + \frac{\beta\lambda}{9} \text{ if } \frac{2\beta\lambda}{9} - \Delta q \le p_b \le \frac{2\beta\lambda}{9} - \frac{\Delta q^2}{\lambda} \\ p_b + \frac{\Delta q}{2} + \frac{\Delta q^2}{2\lambda} \text{ if } p_b \ge \frac{2\beta\lambda}{9} - \frac{\Delta q^2}{\lambda} \end{cases}$$

yielding

$$\pi_a^F = \begin{cases} \frac{\beta}{9}(-\Delta q + \lambda) \text{ if } p_b \le \frac{2\beta\lambda}{9} - \Delta q \\ \frac{4\beta^2\lambda^2 + 81\Delta q^2 - 36\beta\Delta q^2 - 36\beta\lambda p_b + 162\Delta q p_b + 81 p_b^2}{324\Delta q} \text{ if } \frac{2\beta\lambda}{9} - \Delta q \le p_b \le \frac{2\beta\lambda}{9} - \frac{\Delta q^2}{\lambda} \\ \frac{\Delta q}{4} + \frac{\Delta q^2}{2\lambda} + \frac{\Delta q^3}{4\lambda^2} + \frac{p_b}{2} + \frac{\Delta q p_b}{2\lambda} \text{ if } p_b \ge \frac{2\beta\lambda}{9} - \frac{\Delta q^2}{\lambda} \end{cases}.$$

Note that the third segment of these functions is such that $\Delta q' = 0$.

If firm b remains a follower in the second period, its profit function is:

$$\pi_b^F = p_b\left(\frac{p_a - p_b}{\Delta q}\right) + \frac{\beta}{9}\left(\Delta q + \lambda\left(1 - 2\frac{p_a - p_b}{\Delta q}\right)\right)$$

with the following conditions:

$$0 \le \frac{p_a - p_b}{\Delta q} \le 1$$
$$\Delta q + \lambda\left(1 - 2\frac{p_a - p_b}{\Delta q}\right) \ge 0.$$

We obtain the following best response:

$$p_b^F = \begin{cases} p_a \text{ if } p_a \le \frac{2\beta\lambda}{9} \\ \frac{p_a}{2} + \frac{\beta\lambda}{9} \text{ if } \frac{2\beta\lambda}{9} \le p_a \le \frac{2\beta\lambda}{9} + \Delta q + \frac{\Delta q^2}{\lambda} \\ p_a - \frac{\Delta q}{2} - \frac{\Delta q^2}{2\lambda} \text{ if } p_a \ge \frac{2\beta\lambda}{9} + \Delta q + \frac{\Delta q^2}{\lambda} \end{cases}$$

yielding

$$\pi_b^F = \begin{cases} \frac{\beta}{9}(\Delta q + \lambda) \text{ if } p_a \le \frac{2\beta\lambda}{9} \\ \frac{4\beta^2\lambda^2 + 36\beta\lambda\Delta q + 36\beta\Delta q^2 - 36\beta\lambda p_a + 81 p_a^2}{324\Delta q} \text{ if } \frac{2\beta\lambda}{9} \le p_a \le \frac{2\beta\lambda}{9} + \Delta q + \frac{\Delta q^2}{\lambda} \\ -\frac{\Delta q}{4} - \frac{\Delta q^2}{2\lambda} - \frac{\Delta q^3}{4\lambda^2} + \frac{p_a}{2} + \frac{\Delta q p_a}{2\lambda} \text{ if } p_a \ge \frac{2\beta\lambda}{9} + \Delta q + \frac{\Delta q^2}{\lambda} \end{cases}.$$

Note that the third segment of these functions is such that $\Delta q' = 0$.

If firm b takes the leadership, its profit function is:

$$\pi_b^L = p_b \left(\frac{p_a - p_b}{\Delta q} \right) + \frac{4\beta}{9} \left(-\Delta q + \lambda \left(2\frac{p_a - p_b}{\Delta q} - 1 \right) \right),$$

with the following conditions:

$$0 \leq \frac{p_a - p_b}{\Delta q} \leq 1$$
$$-\Delta q + \lambda \left(2\frac{p_a - p_b}{\Delta q} - 1 \right) \geq 0.$$

We obtain the following best response:

$$p_b^L = \begin{cases} p_a - \frac{\Delta q}{2} - \frac{\Delta q^2}{2\lambda} \text{ if } p_a \leq -\frac{8\beta\lambda}{9} + \Delta q + \frac{\Delta q^2}{\lambda} \\ \frac{p_a}{2} - \frac{4\beta\lambda}{9} \text{ if } -\frac{8\beta\lambda}{9} + \Delta q + \frac{\Delta q^2}{\lambda} \leq p_a \leq -\frac{8\beta\lambda}{9} + 2\Delta q \\ p_a - \Delta q \text{ if } p_a \geq -\frac{8\beta\lambda}{9} + 2\Delta q \end{cases}$$

yielding

$$\pi_b^L = \begin{cases} -\frac{\Delta q}{4} - \frac{\Delta q^2}{2\lambda} - \frac{\Delta q^3}{4\lambda^2} + \frac{p_a}{2} + \frac{\Delta q p_a}{2\lambda} \text{ if } p_a \leq -\frac{8\beta\lambda}{9} + \Delta q + \frac{\Delta q^2}{\lambda} \\ \frac{64\beta^2\lambda^2 - 144\beta\lambda\Delta q - 144\beta\Delta q^2 + 144\beta\lambda p_a + 81 p_a^2}{324\Delta q} \text{ if } -\frac{8\beta\lambda}{9} + \Delta q + \frac{\Delta q^2}{\lambda} \leq p_a \leq -\frac{8\beta\lambda}{9} + 2\Delta q \\ p_a - \Delta q + \frac{4\beta}{9}(\lambda - \Delta q) \text{ if } p_a \geq -\frac{8\beta\lambda}{9} + 2\Delta q \end{cases}.$$

Note that the first segment of these function is such that $\Delta q' = 0$.

Lemma 1 *p_i^L, p_i^F, π_i^L, π_i^F are continuous functions for $i = a, b$.*

Although we have different segments, it is easy to check for continuity.

From these best responses, we need to build an unconditional best response function that encompasses the decision to take the lead or not. To do so, we must compare profits when firm i takes the leadership and when it is a follower. Let p_i^{br} be the unconditional best response function of firm i. We find that firms use a trigger strategy; for low prices, they want to be a follower while for high prices they prefer to be in the lead.

Lemma 2 *For $i = a, b$ and $i \neq j$, there exists $\bar{p}_i$ such that*

$$\begin{aligned} p_i &\leq \bar{p}_i \rightarrow p_j^{br}(p_i) = p_j^F(p_i) \\ p_i &\geq \bar{p}_i \rightarrow p_j^{br}(p_i) = p_j^L(p_i) \end{aligned}$$

with $\bar{p}_b \in \left[-\frac{8\beta\lambda}{9} - \frac{\Delta q^2}{\lambda}, \frac{2\beta\lambda}{9} - \frac{\Delta q^2}{\lambda}\right]$ *and* $\bar{p}_a \in \left[-\frac{8\beta\lambda}{9} + \Delta q + \frac{\Delta q^2}{\lambda}, \frac{2\beta\lambda}{9} + \Delta q + \frac{\Delta q^2}{\lambda}\right]$.

Proof. First, notice that the first segment of π_a^L and the third segment of π_a^F are identical. Therefore, for $p_b \leq -\frac{8\beta\lambda}{9} - \frac{\Delta q^2}{\lambda}$, $\pi_a^L \leq \pi_a^F$ and for $p_b \geq \frac{2\beta\lambda}{9} - \frac{\Delta q^2}{\lambda}$, $\pi_a^F \leq \pi_a^L$.

We thus have, for $p_b \leq -\frac{8\beta\lambda}{9} - \frac{\Delta q^2}{\lambda}$, $p_a^{br} = p_a^F$ and for $p_b \geq \frac{2\beta\lambda}{9} - \frac{\Delta q^2}{\lambda}$, $p_a^{br} = p_a^L$. Notice that in the first segment of π_a^F, $\frac{\partial \pi_a^F}{\partial p_b} = 0$, while in the second segment, $\frac{\partial \pi_a^F}{\partial p_b} = \frac{1}{2} - \frac{\beta\lambda}{9} + \frac{p_b}{2}$. For π_a^L, we have that, in the second segment, $\frac{\partial \pi_a^L}{\partial p_b} = \frac{1}{2} + \frac{4\beta\lambda}{9} + \frac{p_b}{2}$ and in the third segment $\frac{\partial \pi_a^L}{\partial p_b} = 1$. Profits increase monotonically with p_b in both functions, but at a faster rate in π_a^L. Therefore, there exists a point $\bar{p}_b$ such that $p_b \leq \bar{p}_b \rightarrow p_a^{br}(p_b) = p_a^F(p_b)$ and $p_b \geq \bar{p}_b \rightarrow p_a^{br}(p_b) = p_a^L(p_b)$. Moreover, this point must be in the interval $\bar{p}_b \in \left[-\frac{8\beta\lambda}{9} - \frac{\Delta q^2}{\lambda}, \frac{2\beta\lambda}{9} - \frac{\Delta q^2}{\lambda}\right]$. The proof follows for $\bar{p}_a$. ■

Lemma 3 $\bar{p}_b \leq \bar{p}_a$ *and any subgame perfect pure strategy Nash equilibrium* (p_a^*, p_b^*) *must be such that* $p_a^* \in \left[p_a^{br}(\bar{p}_b), \bar{p}_a\right]$, *and* $p_b^* \in \left[\bar{p}_b, p_b^{br}(\bar{p}_a)\right]$.

Proof. $\bar{p}_b \leq \bar{p}_a$ follows from the fact that for any p, $\pi_a^L(p) \geq \pi_b^L(p)$ and $\pi_a^F(p) \leq \pi_b^F(p)$. Suppose that $p_a^* > \bar{p}_a$. Then, both firms want to be the leader. There cannot be an equilibrium. Suppose that $p_b^* < \bar{p}_b$. Then, both firms want to be the follower. There cannot be an equilibrium. ■

Corollary 1 *In all pure strategy subgame perfect Nash Equilibria, firm a maintains leadership. That is,* $q_a' > q_b'$.

Lemma 4 *Equilibrium prices are homogeneous of degree 1 in* $(\Delta q, \lambda)$.

We can therefore express equilibrium regions as functions of $\theta = \frac{\Delta q}{\lambda}$.

We say that an equilibrium is a *maximal differentiation equilibrium* if it is such that $x_a = 1$. This is in contrast with a *submaximal differentiation equilibrium*, which is such that $x_a < 1$. We are now ready to describe equilibria.

Theorem 2 *If $\beta \leq \frac{9\theta}{10}$, then the unique pure-strategy Subgame Perfect Nash equilibrium is a submaximal differentiation equilibrium such that $p_a^* = -\frac{14\beta\lambda}{27} + \frac{2\Delta q}{3}$ and $p_b^* = -\frac{4\beta\lambda}{27} + \frac{\Delta q}{3}$.*

Proof. We first check that (p_a^*, p_b^*) is an equilibria. To do so, we need to verify that firm a keeps the lead ($\Delta q' > 0$), that quantities sold are in $[0,1]$, that firm a has no incentive to let firm b take the lead, and that firm b has no incentive to take the lead (if $\Delta q < \lambda$).

First, we have that $x_a^* = 1 - \frac{p_a^* - p_b^*}{\Delta q} = \frac{2}{3} + \frac{10\beta\lambda}{27\Delta q}$ and $x_b^* = \frac{p_a^* - p_b^*}{\Delta q} = \frac{1}{3} - \frac{10\beta\lambda}{27\Delta q}$. Since, $x_a^* > x_b^*$, we have $\Delta q' > \Delta q > 0$.

Second, in order to have $0 \leq x_i^* \leq 1$, we need $\frac{10\beta\lambda}{27\Delta q} \leq \frac{1}{3}$, which simplifies to $\beta \leq \frac{9\theta}{10}$.

Third, $\Pi_a(p_a^*, p_b^*) = \left(\frac{2\Delta q}{3} - \frac{14\beta\lambda}{27}\right)\left(\frac{2}{3} + \frac{10\beta\lambda}{27\Delta q}\right) + \frac{4\beta}{9}\left(\Delta q + \lambda\left(\frac{1}{3} + \frac{20\beta\lambda}{27\Delta q}\right)\right)$. We need to verify that firm a has lower profits if it changes its price to be a follower. Firm a can achieve this by changing its price to the first or second segment of p_a^F. In the first segment, $\Pi_a(p_b^* + \Delta q, p_b^*) = \frac{\beta}{9}(\lambda - \Delta q)$, while in the second it is

$$\Pi_a\left(\frac{p_b^*}{2} + \frac{\Delta q}{2} + \frac{\beta\lambda}{9}, p_b^*\right) = \left(\frac{2\Delta q}{3} + \frac{\beta\lambda}{27}\right)\left(\frac{2}{3} - \frac{5\beta\lambda}{27\Delta q}\right) + \frac{\beta}{9}\left(-\Delta q + \lambda\left(\frac{10\beta\lambda}{27\Delta q} - \frac{1}{3}\right)\right).$$

We therefore need $\Pi_a(p_a^*, p_b^*) \geq \max\left[\Pi_a(p_b^* + \Delta q, p_b^*), \Pi_a(\frac{p_b^*}{2} + \frac{\Delta q}{2} + \frac{\beta\lambda}{9}, p_b^*)\right]$. This holds under the condition that $\beta \leq \frac{9\theta}{10}$.

Fourth, $\Pi_b(p_a^*, p_b^*) = \left(\frac{\Delta q}{3} - \frac{4\beta\lambda}{27}\right)\left(\frac{1}{3} - \frac{10\beta\lambda}{27\Delta q}\right) + \frac{\beta}{9}\left(\Delta q + \lambda\left(\frac{1}{3} + \frac{20\beta\lambda}{27\Delta q}\right)\right)$. We need to verify that firm b has lower profits if it changes its price to be a leader. Firm b can achieve this by changing its price to the second or third segment of p_b^L. In the second segment, $\Pi_b\left(p_a^*, \frac{p_a^*}{2} - \frac{4\beta\lambda}{9}\right) = \left(\frac{\Delta q}{3} - \frac{19\beta\lambda}{27}\right)\left(\frac{1}{3} + \frac{5\beta\lambda}{27\Delta q}\right) + \frac{4\beta}{9}\left(-\Delta q + \lambda\left(\frac{10\beta\lambda}{27\Delta q} - \frac{1}{3}\right)\right)$. In the third segment, $\Pi_b(p_a^*, p_a^* - \Delta q) = \frac{\Delta q}{9}(-4\beta - 3) - \frac{2\beta\lambda}{27} < 0$. We therefore need $\Pi_b(p_a^*, p_b^*) \geq \max\left[\Pi_b\left(p_a^*, \frac{p_a^*}{2} - \frac{4\beta\lambda}{9}\right), \Pi_b(p_a^*, p_a^* - \Delta q)\right]$. This holds under the condition that $\beta \leq \frac{9\theta}{10}$.

Therefore, (p_a^*, p_b^*) is an equilibrium. We need to show uniqueness. The only other possible equilibrium is a maximal differentiation equilibrium. Then, from p_a^L, we need $p_a = p_b = p \geq -\frac{8\beta\lambda}{9} + \Delta q$, and from p_b^F, that $p_b \geq p_a = p \leq \frac{2\beta\lambda}{9}$. Therefore, $p \in \left[-\frac{8\beta\lambda}{9} + \Delta q, \frac{2\beta\lambda}{9}\right]$. Under the condition $\beta \leq \frac{9\theta}{10}$, the interval is empty. There exists no maximal differentiation equilibrium. ∎

Theorem 3 *If $\beta \geq \frac{9\theta}{10}$, then the unique pure-strategy Subgame Perfect Nash equilibria are maximal differentiation equilibria such that $p_a^* = p_b^* = p^*$. More precisely:*

i) If $\beta \geq \frac{9\theta}{5(1-\theta)}$, $p^* \in \left[-\frac{5\beta\Delta q}{9} - \frac{\beta\lambda}{3}, \Delta q + \frac{5\beta\Delta q}{9} - \frac{\beta\lambda}{3}\right]$.

ii) If $\beta \leq \frac{9\theta}{5(1-\theta)}$, $p^* \in \left[-\frac{8\beta\lambda}{9} + \Delta q, \frac{2\beta\lambda}{9}\right]$.

Proof. In order for (p^*, p^*) to be an equilibrium, we need to verify that firm a has no incentive to raise its price to sell less than 1 ($p^* \geq -\frac{8\beta\lambda}{9} + \Delta q$), that firm b has no incentive to lower its price to sell more than 0 ($p^* \leq \frac{2\beta\lambda}{9}$), that firm a has no incentive to become the follower, and that firm b has no incentive to become the leader (if $\Delta q \leq \lambda$).

To first two conditions yield $p^* \in \left[-\frac{8\beta\lambda}{9} + \Delta q, \frac{2\beta\lambda}{9}\right]$. This interval is non-empty if $\beta \geq \frac{9\theta}{10}$. This is a necessary condition to have a maximal differentiation equilibrium.

Then, note that $\Pi_a(p^*, p^*) = p^* + \frac{4\beta}{9}(\Delta q + \lambda)$. Firm a can become the follower in two ways: by increasing its price to $p^* + \Delta q$ so that it sells 0, or by raising it to $\frac{p^*}{2} + \frac{\Delta q}{2} + \frac{\beta\lambda}{9}$, so that it still sells a positive amount. From our best response function p_a^F, we know that the best way for firm a to be a follower is by charging $p^* + \Delta q$ if $p^* \leq \frac{2\beta\lambda}{9} - \Delta q$, and $\frac{p^*}{2} + \frac{\Delta q}{2} + \frac{\beta\lambda}{9}$ if $p^* \in \left[\frac{2\beta\lambda}{9} - \Delta q, \frac{2\beta\lambda}{9} - \frac{\Delta q^2}{\lambda}\right]$. We also have that $\Pi_a(p^* + \Delta q, p^*) = \frac{\beta}{9}(-\Delta q + \lambda)$ and $\Pi_a(\frac{p^*}{2} + \frac{\Delta q}{2} + \frac{\beta\lambda}{9}, p^*) = \frac{4\beta^2\lambda^2 + 81\Delta q^2 - 36\beta\Delta q^2 - 36\beta\lambda p^* + 162\Delta q p^* + 81(p^*)^2}{324\Delta q}$. Under the assumptions of our model and in the relevant interval, $\Pi_a(p^*, p^*) \geq \Pi_a(\frac{p^*}{2} + \frac{\Delta q}{2} + \frac{\beta\lambda}{9}, p^*)$. We have that $\Pi_a(p^*, p^*) \geq \Pi_a(p^* + \Delta q, p^*)$ if $p^* \geq -\frac{5\beta\Delta q}{9} - \frac{\beta\lambda}{3}$, a condition that is more restrictive than $p^* \geq -\frac{8\beta\lambda}{9} + \Delta q$ if $\beta \geq \frac{9\theta}{5(1-\theta)}$.

Then, note that $\Pi_b(p^*, p^*) = \frac{\beta}{9}(\Delta q + \lambda)$. Firm b can become the leader in two ways: by lowering its price to $p^* - \Delta q$ so that it sells 1, or by lowering it to $\frac{p^*}{2} - \frac{4\beta\lambda}{9}$, so that it still sells less than 1. From our best response function p_b^L, we know that the best way for firm b to be a leader is by charging $p^* - \Delta q$ if $p^* \geq -\frac{8\beta\lambda}{9} + 2\Delta q$, and $\frac{p^*}{2} - \frac{4\beta\lambda}{9}$ if $p^* \in \left[-\frac{8\beta\lambda}{9} + \Delta q + \frac{\Delta q^2}{\lambda}, -\frac{8\beta\lambda}{9} + 2\Delta q\right]$. We also have that $\Pi_b(p^*, p^* - \Delta q) = p^* - \Delta q + \frac{4\beta}{9}(\lambda - \Delta q)$ and $\Pi_b(p^*, \frac{p^*}{2} - \frac{4\beta\lambda}{9}) = \frac{64\beta^2\lambda^2 - 144\beta\lambda\Delta q - 144\beta\Delta q^2 + 144\beta\lambda p^* + 81(p^*)^2}{324\Delta q}$. Under the assumptions of our model and in the relevant interval, $\Pi_b(p^*, p^*) \geq \Pi_b(p^*, \frac{p^*}{2} - \frac{4\beta\lambda}{9})$. We have that $\Pi_b(p^*, p^*) \geq \Pi_b(p^*, p^* - \Delta q)$ if $p^* \leq \Delta q + \frac{5\beta\Delta q}{9} - \frac{\beta\lambda}{3}$, a condition that is more restrictive than $p^* \leq \frac{2\beta\lambda}{9}$ if $\beta \geq \frac{9\theta}{5(1-\theta)}$. Note that when the lead cannot be reversed ($\Delta q > \lambda$), we always have $\beta \leq \frac{9\theta}{5(1-\theta)}$.

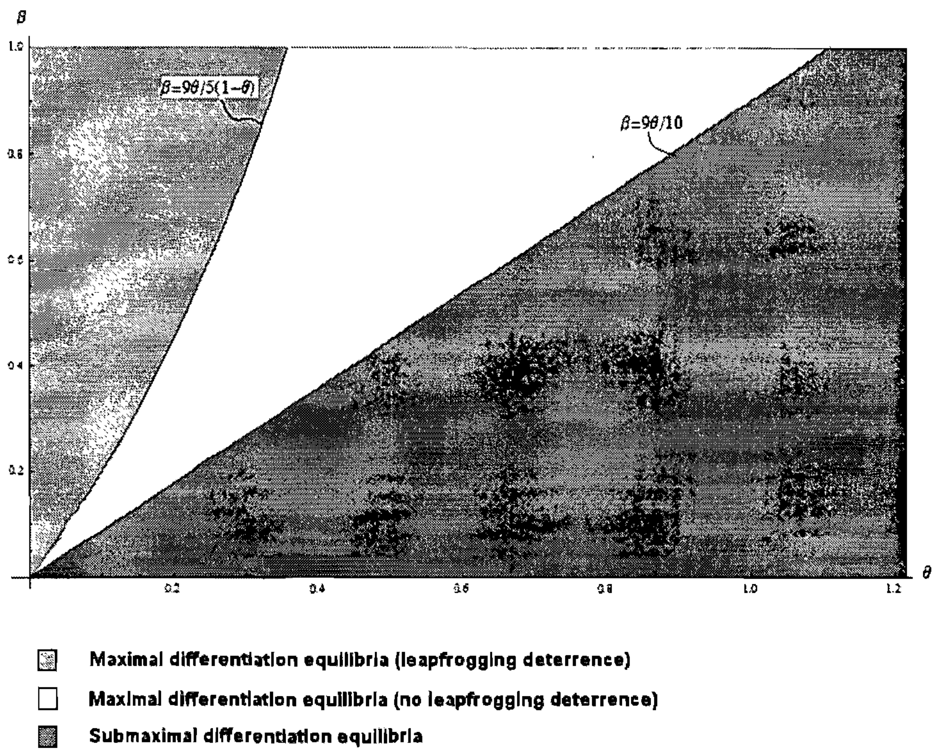
Putting all of this together, if $\beta \geq \frac{9\theta}{5(1-\theta)}$, $p^* \in \left[-\frac{5\beta\Delta q}{9} - \frac{\beta\lambda}{3}, \Delta q + \frac{5\beta\Delta q}{9} - \frac{\beta\lambda}{3}\right]$, while if $\beta \leq \frac{9\theta}{5(1-\theta)}$, $p^* \in \left[-\frac{8\beta\lambda}{9} + \Delta q, \frac{2\beta\lambda}{9}\right]$. Note that both intervals are non-empty when $\beta \geq \frac{9\theta}{10}$.

To show uniqueness, note that the only other possible equilibrium is a submaximal differentiation equilibrium as in Theorem 3. Such an equilibrium is only possible if $\beta \leq \frac{9\theta}{10}$. If $\beta = \frac{9\theta}{10}$, both types of equilibria are identical. ∎

Note that for all values of the parameters, not only does firm a keep its lead, but it also increases it.

Figure 3.1 summarizes the results of the previous theorems, as it describes equilibria as functions of the lead reversibility $\left(\theta = \frac{\Delta q}{\lambda}\right)$ and time preferences (β).

Figure 3.1: Equilibria type depending on the parameters

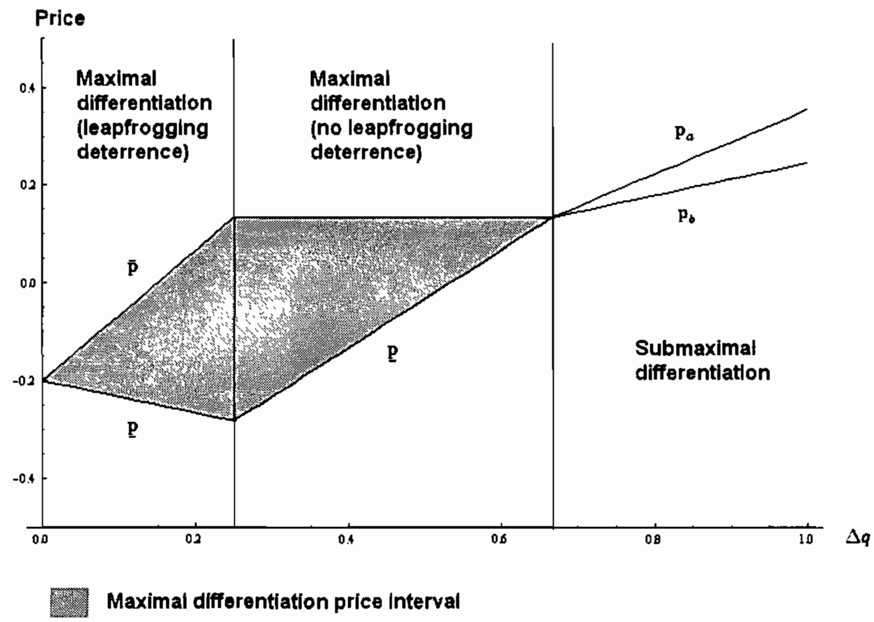


We see that when we have an easily reversible lead and patient firms, we have maximal differentiation equilibria, with multiple possible equilibrium prices. For the most easily reversible leads, those equilibrium prices are constrained because firms must deter the possibility of leapfrogging. When the lead is relatively safe and/or when firms are impatient, we have a submaximal differentiation equilibrium where both firms have strictly positive demands in the development phase.

To see this price evolution, Figure 3.2 plots equilibrium prices as functions of the initial quality lead, Δq.

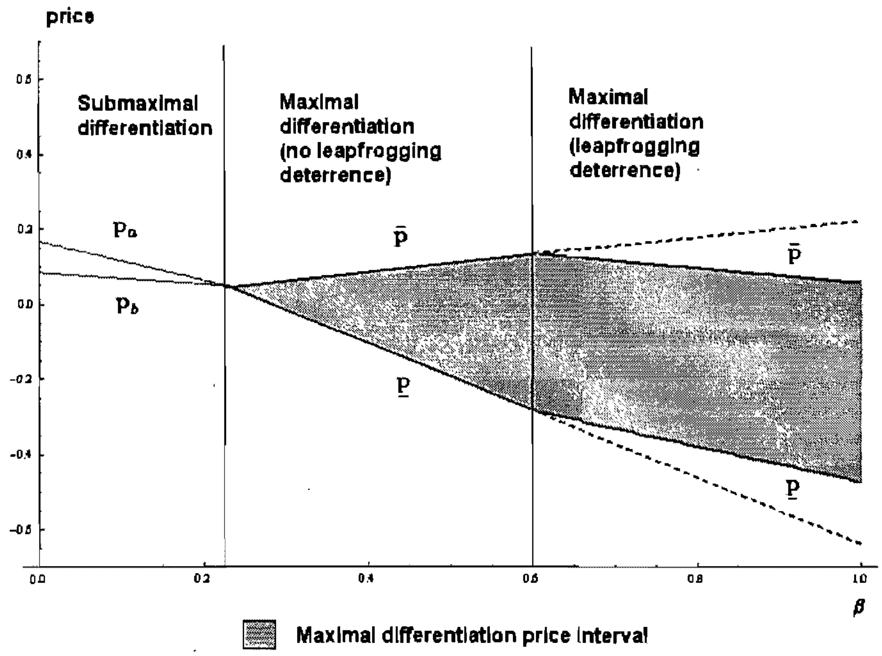
Suppose that two firms are producing softwares, that can be improved through automatic bug tracking, with firm a having an initial quality lead. If that lead is small, firm a will

Figure 3.2: Equilibrium prices as functions of the initial quality difference when $\beta = \frac{3}{5}$ and $\lambda = 1$



capture the market in the development phase and achieve maximal differentiation. For a very small initial lead, the price interval is also very small, as the maximum price has to be low so that firm b does not try to take the lead, and the minimal price has to be high enough to not give incentives to firm a to try to become the follower. Let $\bar{p}$ and $\underline{p}$ be, respectively, the highest and lowest equilibrium prices. Note that when $\Delta q \to 0$, $\bar{p} \to -\frac{\beta\lambda}{3}$ and $\underline{p} \to -\frac{\beta\lambda}{3}$, the equilibrium price found in Theorem 1 for identical initial qualities. As the initial lead increases, the price interval also increases. As the threat of leapfrogging becomes less credible, $\bar{p}$ increases and $\underline{p}$ decreases. As Δq continues to increase, we reach a point where the leapfrogging threat is no longer credible. The only constraint on prices is to make sure that firm a still has an incentive to sell to all the customers in the development phase; that is, $\underline{p}$ must be high enough. As the initial lead increases, maximal differentiation becomes less attractive, leading to an increase in $\underline{p}$. Note that in this interval, firm b is never tempted to steal a few customers, as it prefers maximal differentiation. Therefore, $\bar{p}$ remains unchanged. If the initial quality lead is large enough, maximal differentiation is less attractive to both firms and cannot be achieved in equilibrium. Firms have different prices with, of course, the leading firm having the higher price. Both firms increase their prices as the initial lead increases, with firm a increasing its price at a faster rate. However, the difference in quantities sold, $\frac{1}{3} + \frac{20\beta\lambda}{27\Delta q}$, is decreasing in Δq. As Δq tends to infinity, the difference in quantities sold tends to $\frac{1}{3}$, just as if there were no automatic user contributions. Observe that it implies that, everywhere, firm a sells more than it would do so without user contributions.

Figure 3.3: Equilibrium prices as functions of time preferences when $\Delta q = \frac{1}{4}$ and $\lambda = 1$



In Figure 3.3, we see equilibrium prices as functions of β. For a small β, gains in the maturity phase are not valued highly, and firms behave almost as if there were no automatic contributions. As β increases, firm a becomes more aggressive in prices. We eventually reach a point where firm a takes the whole market, and where multiple prices can generate this maximal differentiation equilibria. As β keeps increasing, both firms value highly the gains obtained by maximal differentiation and the price interval widens. Eventually, second-period profits are so appealing that leapfrogging becomes a credible threat. To deter it, $\bar{p}$ starts decreasing, while $\underline{p}$ keeps on decreasing, but at a slower rate. The price interval keeps growing, but very slowly. The dashed lines represent what would be the price interval if it were not for this lead reversal possibility.

The fact that the profits of firm b are increasing in the second-period quality difference makes the effect of user contributions on prices unclear. While firm a has strong incentives to be aggressive and increase its lead, firm b might decide to concede the market in the first period in order to reap higher profits in the second period. Let $p_a^{NC} \equiv \frac{2\Delta q}{3}$ and $p_b^{NC} \equiv \frac{\Delta q}{3}$ be the equilibrium prices when there are no user contributions. We have the following result.

Theorem 4 *For every possible triplet of parameters $(\beta, \lambda, \Delta q)$, with $\beta > 0$ and $\lambda > 0$, equilibrium prices (p_a^*, p_b^*) are such that*

i) All possible p_a^* are lower than p_a^{NC}.

ii) There always exists a p_b^* lower than p_b^{NC}.

iii) For $\beta \in \left(\frac{3\theta}{2}, \frac{6\theta}{3-5\theta}\right)$, there exists a p_b^* higher than p_b^{NC}.

Proof. i) For $\beta \leq \frac{9\theta}{10}$, we are in the submaximal differentiation region, where $p_a^* = -\frac{14\beta\lambda}{27} + \frac{2\Delta q}{3} < p_a^{NC}$. For $\beta \geq \frac{9\theta}{10}$, we are in the maximal differentiation region. We need to check that $\bar{p} \leq p_a^{NC}$.

For $\beta \in \left[\frac{9\theta}{10}, \frac{9\theta}{5(1-\theta)}\right]$, $\bar{p} = \frac{2\beta\lambda}{9}$. We have $\bar{p} \leq p_a^{NC} \Leftrightarrow \beta \leq 3\theta$. In our relevant interval, and with $\beta \leq 1$, this always holds.

For $\beta \geq \frac{9\theta}{5(1-\theta)}$, $\bar{p} = \Delta q + \frac{5\beta\Delta q}{9} - \frac{\beta\lambda}{3}$. We have $\bar{p} \leq p_a^{NC} \Leftrightarrow \beta \geq \frac{3\theta}{3-5\theta}$. In our relevant interval, and with $\beta \leq 1$, this always holds.

ii) For $\beta \leq \frac{9\theta}{10}$, we are in the submaximal differentiation region, where $p_b^* = -\frac{4\beta\lambda}{27} + \frac{\Delta q}{3} < p_b^{NC}$. For $\beta \geq \frac{9\theta}{10}$, we are in the maximal differentiation region. We need to check that $\underline{p} \leq p_b^{NC}$.

For $\beta \in \left[\frac{9\theta}{10}, \frac{9\theta}{5(1-\theta)}\right]$, $\underline{p} = -\frac{8\beta\lambda}{9} + \Delta q$. We have $\underline{p} \leq p_b^{NC} \Leftrightarrow \beta \geq \frac{3\theta}{4}$, which is always true in the relevant interval,

For $\beta \geq \frac{9\theta}{5(1-\theta)}$, $\underline{p} = -\frac{5\beta\Delta q}{9} - \frac{\beta\lambda}{3} < 0 \leq p_b^{NC}$.

iii) In maximal differentiation regions, we need to find conditions under which $\bar{p} > p_b^{NC}$.

For $\beta \in \left[\frac{9\theta}{10}, \frac{9\theta}{5(1-\theta)}\right]$, $\bar{p} = \frac{2\beta\lambda}{9}$. We have $\bar{p} > p_b^{NC} \Leftrightarrow \beta \geq \frac{3\theta}{2}$. Note that $\frac{9\theta}{10} \leq \frac{3\theta}{2} \leq \frac{9\theta}{5(1-\theta)}$.

For $\beta \geq \frac{9\theta}{5(1-\theta)}$, $\bar{p} = \Delta q + \frac{5\beta\Delta q}{9} - \frac{\beta\lambda}{3}$. We have $\bar{p} > p_b^{NC} \Leftrightarrow \beta < \frac{6\theta}{3-5\theta}$. Note that $\frac{6\theta}{3-5\theta} \geq \frac{9\theta}{5(1-\theta)}$. ■

Going back to our example of software producing firms, we get that automatic bug tracking leads to a lower price for firm a, which has the higher quality product. There always exists some equilibrium prices where firm b also charges a lower price. There also exists a region where firm b can have a higher price than in the situation where there are no user contributions. However, this region is such that firm b sells nothing. Still, without user contributions, firm b sells to consumers such that $\gamma \in \left[0, \frac{1}{3}\right]$, who might end up buying at a higher price. The interval where this might happen is such that firm b values sufficiently second-period profits, so that it is less aggressive in order to obtain maximal differentiation, but not too strongly, so that it doesn't have strong incentives to try to reverse the lead.

Finally, note that there are strong possibilities of obtaining first-period prices lower than marginal cost (i.e., negative). This is possible as firms are ready to sell at a loss in the first period in order to gain consumers and increase the quality difference and the profits in the second-period. In fact, if $\beta > \frac{9\theta}{3-5\theta}$, the whole price interval is negative. At the extreme, if $\beta < \frac{9\theta}{8}$, which is the whole sub-maximal differentiation region and part of the maximal differentiation region, prices are always positive. In between, $\bar{p} \geq 0$ and $\underline{p} \leq 0$. This pricing under marginal cost is consistent with some observed behavior. Many softwares or services are initially free, while in the development phase, before moving to a subscriber-only pricing scheme, with beta versions of softwares being the prime example.

4 Welfare effects

We are interested in the effects user contributions have on welfare, an important issue for public policies.

4.1 Global effects

We are first interested in the global effects on consumers and on the total welfare. We start by looking at cases where $\beta \leq \frac{9\theta}{10}$, where we have submaximal differentiation. We have three groups of consumers, depending on their valuation of quality γ_l. The first group buys from firm b in both periods. It is comprised of agents in $\left[0, \frac{1}{3} - \frac{10\beta\lambda}{27\Delta q}\right]$. They have utility $U_{b,b}(\gamma_k) = \gamma_k q_b + \frac{4\beta\lambda}{27} - \frac{\Delta q}{3} + \beta\left(\gamma_k\left(q_b + \lambda\left(\frac{1}{3} - \frac{10\beta\lambda}{27\Delta q}\right)\right) - \frac{1}{3}\left(\Delta q + \frac{\lambda}{3} + \frac{20\beta\lambda^2}{27\Delta q}\right)\right)$. The second group buys from firm a in period 1 and from firm b in period 2. It is comprised of agents in $\left[\frac{1}{3} - \frac{10\beta\lambda}{27\Delta q}, \frac{1}{3}\right]$. They have utility

$$U_{a,b}(\gamma_k) = \gamma_k q_a + \frac{14\beta\lambda}{27} - \frac{2\Delta q}{3} + \beta\left(\gamma_k\left(q_b + \lambda\left(\frac{1}{3} - \frac{10\beta\lambda}{27\Delta q}\right)\right) - \frac{1}{3}\left(\Delta q + \frac{\lambda}{3} + \frac{20\beta\lambda^2}{27\Delta q}\right)\right).$$

The third group buys from firm a in both periods. Its members have utility $U_{a,a}(\gamma_k) = \gamma_k q_a + \frac{14\beta\lambda}{27} - \frac{2\Delta q}{3} + \beta\left(\gamma_k\left(q_a + \lambda\left(\frac{2}{3} + \frac{10\beta\lambda}{27\Delta q}\right)\right) - \frac{2}{3}\left(\Delta q + \frac{\lambda}{3} + \frac{20\beta\lambda^2}{27\Delta q}\right)\right)$.

Putting it all together, total utility (TU) is

$$TU = \int_0^{\frac{1}{3}-\frac{10\beta\lambda}{27\Delta q}} U_{b,b}(\gamma)d\gamma + \int_{\frac{1}{3}-\frac{10\beta\lambda}{27\Delta q}}^{\frac{1}{3}} U_{a,b}(\gamma)d\gamma + \int_{\frac{1}{3}}^{1} U_{a,a}(\gamma)d\gamma.$$

Taking the derivative with respect to λ, the parameter of efficiency of user contributions, we obtain

$$\frac{\partial TU}{\partial \lambda} = \frac{5\beta}{1458\Delta q}\left(153\Delta q - 116\beta\lambda\right).$$

It is non-negative if $\beta \leq \frac{153\theta}{116}$. Since $\beta \leq \frac{9\theta}{10}$, this is always verified. Increasing λ (while remaining in the zone $\beta \leq \frac{9\theta}{10}$) always increases total consumer utility.

In this zone, firms have the following profit functions

$$\begin{aligned}
\Pi_a &= \left(\frac{2}{3} + \frac{10\beta\lambda}{27\Delta q}\right)\left(\frac{2\Delta q}{3} - \frac{14\beta\lambda}{27}\right) + \frac{4\beta}{9}\left(\Delta q + \frac{\lambda}{3} + \frac{20\beta\lambda^2}{27\Delta q}\right)\\
\Pi_b &= \left(\frac{1}{3} - \frac{10\beta\lambda}{27\Delta q}\right)\left(\frac{\Delta q}{3} - \frac{4\beta\lambda}{27}\right) + \frac{\beta}{9}\left(\Delta q + \frac{\lambda}{3} + \frac{20\beta\lambda^2}{27\Delta q}\right)
\end{aligned}$$

yielding $\frac{\partial \Pi_a}{\partial \lambda} = \frac{4\beta}{729\Delta q}\left(9\Delta q + 50\beta\lambda\right) \geq 0$ and $\frac{\partial \Pi_b}{\partial \lambda} = \frac{\beta}{729\Delta q}\left(-99\Delta q + 200\beta\lambda\right)$. This gives $\frac{\partial \Pi_b}{\partial \lambda} \geq 0$ if $\beta \geq \frac{99\theta}{200}$.

Total welfare is $W = TU + \Pi_a + \Pi_b$. We obtain $\frac{\partial W}{\partial \lambda} = \frac{\beta}{1458\Delta q}\left(639\Delta q + 220\beta\lambda\right) > 0$.

Second, we look at the case where $\beta \geq \frac{9\theta}{10}$, where we have maximal differentiation equilibria. All agents buy from firm a in the 1st period and agents in $\left[0, \frac{1}{3}\right]$ buy from firm b in the second period, with the rest buying from firm a. Therefore,

$$TU = \int_0^1 \left(\gamma q_a - p^*\right)d\gamma + \int_0^{\frac{1}{3}} \left(\gamma q_b - \frac{\Delta q + \lambda}{3}\right)d\gamma + \int_{\frac{1}{3}}^1 \left(\gamma(q_a + \lambda) - \frac{2(\Delta q + \lambda)}{3}\right)d\gamma.$$

Taking the derivative with respect to λ, we obtain

$$\frac{\partial TU}{\partial \lambda} = -\frac{\partial p^*}{\partial \lambda} - \frac{\beta}{9}.$$

In this zone, firms have the following profit functions:

$$\Pi_a = p^* + \frac{4\beta}{9}(\Delta q + \lambda)$$
$$\Pi_b = \frac{\beta}{9}(\Delta q + \lambda)$$

yielding $\frac{\partial \Pi_a}{\partial \lambda} = \frac{\partial p^*}{\partial \lambda} + \frac{4\beta}{9}$ and $\frac{\partial \Pi_b}{\partial \lambda} = \frac{\beta}{9} \geq 0$.

Note that total welfare does not depend on p^*, and that we have $\frac{\partial W}{\partial \lambda} = \frac{4\beta}{9} \geq 0$.

We consider two different paths for $p^* : p^* = \bar{p}$ and $p^* = \underline{p}$.

We start with $p^* = \bar{p}$. If $\frac{9\theta}{10} \leq \beta \leq \frac{9\theta}{5(1-\theta)}$, $\bar{p} = \frac{2\beta\lambda}{9}$. Therefore, $\frac{\partial TU}{\partial \lambda} = -\frac{\beta}{3} \leq 0$ and $\frac{\partial \Pi_a}{\partial \lambda} = \frac{2\beta}{3} \geq 0$. If $\beta \geq \frac{9\theta}{5(1-\theta)}$, $\bar{p} = \Delta q + \frac{5\beta\Delta q}{9} - \frac{\beta\lambda}{3}$. Therefore, $\frac{\partial TU}{\partial \lambda} = \frac{2\beta}{9} \geq 0$ and $\frac{\partial \Pi_a}{\partial \lambda} = \frac{\beta}{9} \geq 0$.

Consider again the example of the software-producing firms. Suppose that we start with a low level of automatic bug tracking, λ, where we have a submaximal differentiation equilibrium (See Figure 1). By increasing λ, we eventually reach a maximal differentiation equilibrium with $\beta \leq \frac{9\theta}{5(1-\theta)}$, before moving on to the case where $\beta \geq \frac{9\theta}{5(1-\theta)}$. Therefore, starting from a small level of tracking λ, when we increase it, consumer utility is initially increasing, decreasing when we reach maximal differentiation, but increasing again when prices start to fall. Profits of the leading firm a are always increasing, while profits for firm b are initially decreasing, but become increasing before we even reach the maximal differentiation equilibria region. Their profits then remain increasing throughout. Total welfare is always increasing. Therefore, we can also conclude that, compared to a situation where there are no quality improvements, moving to a situation with quality-improving user contributions always improves total welfare. Also, a government interested in increasing total welfare could always try to increase λ.

Next, we look at the case $p^* = \underline{p}$. If $\frac{9\theta}{10} \leq \beta \leq \frac{9\theta}{5(1-\theta)}$, $\underline{p} = -\frac{8\beta\lambda}{9} + \Delta q$. Therefore, $\frac{\partial TU}{\partial \lambda} = \frac{7\beta}{9} \geq 0$ and $\frac{\partial \Pi_a}{\partial \lambda} = -\frac{4\beta}{9} \leq 0$. If $\beta \geq \frac{9\theta}{5(1-\theta)}$, $\underline{p} = -\frac{5\beta\Delta q}{9} - \frac{\beta\lambda}{3}$. Therefore, $\frac{\partial TU}{\partial \lambda} = \frac{2\beta}{9} \geq 0$ and $\frac{\partial \Pi_a}{\partial \lambda} = \frac{\beta}{9} \geq 0$.

In this case, when we increase λ, consumer utility is always increasing. Profits for firm a are initially increasing but start to decrease when we reach the maximal differentiation region. Profits start increasing again when we reach the leapfrogging deterrence region. Profits for firm b and total welfare behave as in the previous case.

4.2 Individual effects

Leaving aside aggregate effects, we now focus on individual effects of the level of contributions λ on consumer utility. For $\beta \geq \frac{9\theta}{10}$ we once again have to choose the price path and consider $p^* = \bar{p}$ and $p^* = \underline{p}$. However, for $\beta \geq \frac{9\theta}{5(1-\theta)}$, since $\frac{\partial \bar{p}}{\partial \lambda} = \frac{\partial \underline{p}}{\partial \lambda}$, it does not matter which of these paths is chosen. The following table gives the marginal utilities and the conditions for them to be non-negative in the different cases.

Table 1 : Individual marginal utilities

depending on equilibria regions, price paths and user type

γ	$\frac{\partial U}{\partial \lambda}$	Conditions for $\frac{\partial U}{\partial \lambda} \geq 0$
	$\beta \leq \frac{9\theta}{10}$	
$\left[0, \frac{1}{3} - \frac{10\beta\lambda}{27\Delta q}\right]$	$\frac{\beta\left(3+27\gamma-\frac{60\gamma\beta}{\theta}-\frac{40\beta}{\theta}\right)}{81}$	if $\beta \leq \frac{3\theta}{40}$: always non-negative if $\beta \in \left[\frac{3\theta}{40}, \frac{9\theta}{20}\right]$: $\frac{9\theta-10\beta}{27\theta} \geq \gamma \geq \frac{40\beta-3\theta}{27\theta-60\beta}$ if $\beta \in \left[\frac{9\theta}{20}, \frac{9\theta}{10}\right]$: always non-positive
$\left[\frac{1}{3} - \frac{10\beta\lambda}{27\Delta q}, \frac{1}{3}\right]$	$\frac{\beta\left(33+27\gamma-\frac{60\gamma\beta}{\theta}-\frac{40\beta}{\theta}\right)}{81}$	if $\beta \leq \frac{9\theta}{20}$: always non-negative if $\beta \in \left[\frac{9\theta}{20}, \frac{33\theta}{40}\right]$: $\frac{9\theta-10\beta}{27\theta} \leq \gamma \leq \frac{40\beta-33\theta}{27\theta-60\beta}$ if $\beta \in \left[\frac{33\theta}{40}, \frac{9\theta}{10}\right]$: always non-positive
$\left[\frac{1}{3}, 1\right]$	$\frac{2\beta\left(12+27\gamma+\frac{30\gamma\beta}{\theta}-\frac{40\beta}{\theta}\right)}{81}$	if $\beta \leq \frac{7\theta}{10}$: always non-negative if $\beta \in \left[\frac{7\theta}{10}, \frac{9\theta}{10}\right]$: $\gamma \geq \frac{40\beta-12\theta}{27\theta+30\beta}$
	$\frac{9\theta}{10} \leq \beta \leq \frac{9\theta}{5(1-\theta)}$, $p^* = \bar{p}$	
$\left[0, \frac{1}{3}\right]$	$-\frac{5\beta}{9}$	Always non-positive
$\left[\frac{1}{3}, 1\right]$	$\beta\left(\gamma - \frac{8}{9}\right)$	$\gamma \geq \frac{8}{9}$
	$\frac{9\theta}{10} \leq \beta \leq \frac{9\theta}{5(1-\theta)}$, $p^* = \underline{p}$	
$\left[0, \frac{1}{3}\right]$	$\frac{5\beta}{9}$	Always non-negative
$\left[\frac{1}{3}, 1\right]$	$\beta\left(\gamma + \frac{2}{9}\right)$	Always non-negative
	$\beta \geq \frac{9\theta}{5(1-\theta)}$	
$\left[0, \frac{1}{3}\right]$	0	Always non-negative
$\left[\frac{1}{3}, 1\right]$	$\beta\left(\gamma - \frac{1}{3}\right)$	Always non-negative

Combining these results, we can formulate the following theorem.

Theorem 5 *For $p^* = \bar{p}$ or $p^* = \underline{p}$, increasing λ is a Pareto-improving measure if and only if $\beta \geq \frac{9\theta}{5(1-\theta)}$.*

While increasing the contribution level λ adds only marginally to the quality q'_a, it has a larger (negative) effect on first-period price because of the threat of lead reversal. This price decrease in the first period, plus increased quality for those buying from firm a, are sufficient to compensate for second-period price increases. For firms, price increases in the second period more than offset this first-period price decrease.

Note also that for $\beta \leq \frac{3\theta}{40}$, all consumers would agree to increase λ. For a larger β, the set of consumers willing to lobby for more user contributions is composed of agents that highly value quality. There are always some consumers that would prefer λ to increase.

4.3 Costs of improving quality

We now examine the cost at which we obtain the quality improvement in the maturity phase. In particular, we compare the automatic user contributions case to the one where improvements come from R&D. In an equivalent two-period model, a firm would invest in the first period, reaping the benefits in the second period. Since there are no strategic components to prices, the leading firm charges $\frac{2\Delta q}{3}$ in the first period and $\frac{2\Delta q'}{3}$ in the second period, while the trailing firm charges $\frac{\Delta q}{3}$ and $\frac{\Delta q'}{3}$. Therefore, the leading firm a, if it increases its quality by z_a, will have profits of $\frac{4\Delta q}{9} - C(z_a) + \beta\frac{4\Delta q'}{9}$, where $C(\cdot)$ is the investment needed. Similarly, the trailing firm would have profits of $\frac{\Delta q}{9} - C(z_b) + \beta\frac{\Delta q'}{9}$ for an improvement in quality of z_b.

With user contributions, this cost is indirect. To improve quality, firms modify strategically their prices, thus changing their first-period profits. We can, therefore, define the (implicit) private cost of improving quality as the forgone profits in period 1, compared to the situation with no user contributions. In the maximal differentiation case, only firm a improves its quality (by λ). We obtain:

$$C_a(\lambda) = \frac{4\Delta q}{9} - p^*.$$

Interestingly, this cost is not always positive. In the leapfrogging deterrence region, if $\beta < \frac{5\theta}{3-5\theta}$, there exist equilibrium prices such that this cost is negative. In the no leapfrogging deterrence region, we have the same result if $\beta > 2\theta$. This result is closely related to Theorem 4, as it is the combined effort of both firms to restrict competition in order to obtain maximal differentiation that allows it. It is interesting that, potentially, firm a can increase its profits in both periods while increasing its quality by λ.

We do, however, have to consider the fact that firm b also contributes to the quality improvement, as it also sees its profits affected. In the maximal differentiation region, it loses its first period profits of $\frac{\Delta q}{9}$. Let $C_F(z_a, z_b)$ be the total private cost of improving quality. We obtain

$$C_F(\lambda, 0) = \frac{5\Delta q}{9} - p^*.$$

There is still the possibility of negative total private cost, this time if $\frac{5\theta}{2} < \beta < \frac{4\theta}{3-5\theta}$.

The consumers are also affected by the strategic choice of prices, as under automatic user contributions, some of them that would have bought from firm b end up buying from firm a, which has higher quality. Let u^{NUC} be the first-period utility of consumers when there are no user contributions, and let $u^{UC}(z_a, z_b)$ when there are. In the maximal differentiation region, we have

$$\begin{aligned} u^{NUC} &= \int_0^{\frac{1}{3}} \left(\gamma q_b - \frac{\Delta q}{3}\right) d\gamma + \int_{\frac{1}{3}}^1 \left(\gamma q_a - \frac{2\Delta q}{3}\right) d\gamma = \frac{11 q_b}{18} - \frac{q_a}{9} \\ u^{UC}(\lambda, 0) &= \int_0^1 (\gamma q_a - p^*)\, d\gamma = \frac{q_a}{2} - p^*. \end{aligned}$$

Let $B(z_a, z_b) = u^{UC}(z_a, z_b) - u^{NUC}$ be the benefits obtained by consumers when there are user contributions. We have

$$B(\lambda, 0) = \frac{11\Delta q}{18} - p^* > 0.$$

We can now define the social cost of quality improvements, $C_S(z_a, z_b)$, as $C_F(z_a, z_b) - B(z_a, z_b)$. We find

$$C_S(\lambda, 0) = -\frac{\Delta q}{18} \leq 0.$$

Therefore, socially, the immediate cost of improving quality is always negative when we are in the maximal differentiation region. We now look at the submaximal differentiation region, in which $z_a = \frac{2\lambda}{3} + \frac{10\beta\lambda^2}{27\Delta q}$ and $z_b = \frac{\lambda}{3} - \frac{10\beta\lambda^2}{27\Delta q}$. We obtain

$$\begin{aligned} C_a\left(\frac{2\lambda}{3} + \frac{10\beta\lambda^2}{27\Delta q}\right) &= \frac{8\beta\lambda}{81} + \frac{140\beta^2\lambda^2}{729\Delta q} \geq 0 \\ C_b\left(\frac{\lambda}{3} - \frac{10\beta\lambda^2}{27\Delta q}\right) &= \frac{14\beta\lambda}{81} - \frac{140\beta^2\lambda^2}{729\Delta q} \geq 0. \end{aligned}$$

Therefore, $C_F\left(\frac{2\lambda}{3} + \frac{10\beta\lambda^2}{27\Delta q}, \frac{\lambda}{3} - \frac{10\beta\lambda^2}{27\Delta q}\right) = \frac{22\beta\lambda}{81}$. Private costs are positive as the extra competition brought by the user contributions decreases prices.

Consumers still gain, even though not all consumers buy from firm a. We have

$$\begin{aligned} u^{UC}\left(\frac{2\lambda}{3} + \frac{10\beta\lambda^2}{27\Delta q}, \frac{\lambda}{3} - \frac{10\beta\lambda^2}{27\Delta q}\right) &= \int_0^{\frac{1}{3} - \frac{10\beta\lambda}{27\Delta q}} \left(\gamma q_b - \left(\frac{\Delta q}{3} - \frac{4\beta\lambda}{27}\right)\right) d\gamma \\ &\quad + \int_{\frac{1}{3} - \frac{10\beta\lambda}{27\Delta q}}^{1} \left(\gamma q_a - \left(\frac{2\Delta q}{3} - \frac{14\beta\lambda}{27}\right)\right) d\gamma \\ &= \frac{11 q_b}{18} - \frac{q_a}{9} + \frac{32\beta\lambda}{81} + \frac{50\beta^2\lambda^2}{729\Delta q}. \end{aligned}$$

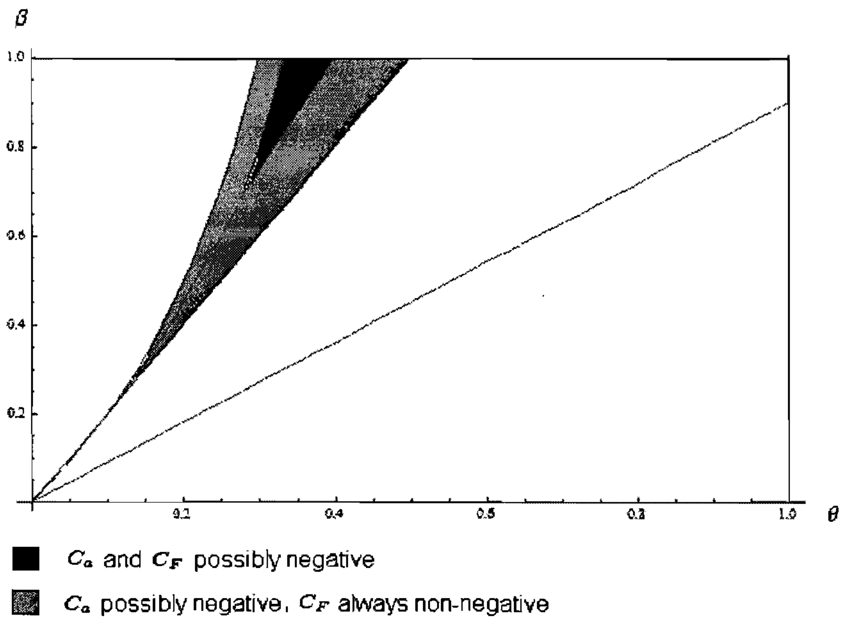
Therefore, $B\left(\frac{2\lambda}{3} + \frac{10\beta\lambda^2}{27\Delta q}, \frac{\lambda}{3} - \frac{10\beta\lambda^2}{27\Delta q}\right) = \frac{32\beta\lambda}{81} + \frac{50\beta^2\lambda^2}{729\Delta q} \geq 0$. Again, consumers gain from the increases in qualities. Putting it all together, we obtain

$$C_S\left(\frac{2\lambda}{3} + \frac{10\beta\lambda^2}{27\Delta q}, \frac{\lambda}{3} - \frac{10\beta\lambda^2}{27\Delta q}\right) = -\frac{\beta\lambda}{9} - \frac{50\beta^2\lambda^2}{729\Delta q} \leq 0.$$

Once again, socially, the first-period cost of improving quality is negative. We can verify that second-period welfare is increasing in λ. Therefore, we can conclude that not only do automatic user contributions increase total welfare, they also increase it in every period. We can also conclude that the social cost of quality improvements with automatic user contributions is always negative. Private costs are positive in the submaximal differentiation region, but might be negative when there is maximal differentiation. Figure 3.4 shows where we might have negative private costs.

Note that even though automatic user contributions are efficient the extent of quality improvement that can be done using them is limited. There is a strong possibility that quality improvements would be larger with R&D. Therefore, it would be wise to push for as much automatic user contributions as possible and then complement these improvements with R&D to improve welfare even further.

Figure 3.4: Regions where private costs of improving quality might be negative



5 Extensions

5.1 Asymmetry and lead reversals

While previous results showed that there are no equilibria where the leader changes between periods, this was in part due to the fact that firms are symmetric in every aspect except initial quality. Obviously, by making firms asymmetric, we could find situations where there can be lead reversals. Suppose that $0 \leq \lambda_a < \lambda_b \leq 1$. Since firm b is more efficient than firm a in using contributions from its users, we might have some equilibria where b takes the lead.

While the conditions for this type of equilibria to happen are complicated in the general case, we can focus on a particular example. Suppose that $\Delta q = \frac{1}{3}$, $\beta = 1$ and $\lambda_b = 1$. If $\lambda_a \leq \frac{1}{5}$, there is an equilibrium where b takes the lead but firm a still has a positive demand in period 1. If $\lambda_a \leq 0.3289$, there are equilibria where firm b takes the lead and sells to all consumers in period 1. For any larger λ_a, firm a always keeps the lead. Therefore, we need considerable asymmetries to obtain lead reversals.

Similar lead reversals can be obtained if we add asymmetry to the time preference parameters or to the marginal costs of production.

5.2 Increasing preferences for quality

Until now we have assumed that consumer tastes stay constant over time. However, it is possible that as a product evolves and once the novelty effect is gone, agents come to value more its quality. One way to model this is to retain the assumption that θ_l, the parameter giving the preference of agent l for quality, is still in $[0,1]$ in period 1 but, in period 2, it is in $[\mu, 1+\mu]$, with $\mu > 0$. Solving backward, in the second period we have that $p_L^* = \frac{p_F}{2} + (1-\mu)\frac{\Delta q'}{2}$ while $p_F^* = \frac{p_L}{2} - \mu\frac{\Delta q'}{2}$, yielding an equilibrium of $p^* = \left(\frac{2\Delta q}{3} - \mu\Delta q', \frac{\Delta q}{3} - \mu\Delta q'\right)$. Sales are, respectively, $\frac{2}{3}+\mu$ and $\frac{1}{3}-\mu$. Directly, we have that if $\mu \geq \frac{1}{3}$, firm F is eliminated from the market in the second period. When $\mu < \frac{1}{3}$, we have that $\pi_L' = \Delta q'(\frac{4}{9}-\mu^2)$ and $\pi_F' = \Delta q'(\frac{1}{3}-\mu)^2$.

We obtain a submaximal differentiation equilibrium if $\beta \leq \frac{9\theta+12\mu}{10}$, where $p_a^* = \frac{2\Delta q}{3} - \frac{14\beta\lambda}{27} - \frac{4\beta\lambda\mu}{9} + 2\beta\lambda\mu^2$ and $p_b^* = \frac{\Delta q}{3} - \frac{4\beta\lambda}{27} - \frac{8\beta\lambda\mu}{9} + 2\beta\lambda\mu^2$. Sales are respectively $\frac{2}{3} + \frac{4\beta\lambda}{27\Delta q} - \frac{4\beta\lambda\mu}{9\Delta q}$ and $\frac{1}{3} - \frac{4\beta\lambda}{27\Delta q} + \frac{4\beta\lambda\mu}{9\Delta q}$. We therefore get less sales for firm a (and thus less differentiation) than in the corresponding case where preferences are not increasing. The price for firm b is always lower, while firm a's price is lower if $\mu < \frac{2}{9}$.

As in our model, for $\beta \geq \frac{9\theta+12\mu}{10}$, we have maximal differentiation equilibria with $p_a^* = p_b^* = p^*$. The interval for p^* will vary depending on the parameter region. If $\beta \leq \frac{9\theta}{(1-\theta)(5-6\mu)}$, this interval is $\left[\Delta q - 2\beta\lambda\left(\frac{4}{9}-\mu^2\right), 2\beta\lambda\left(\frac{1}{3}-\mu\right)^2\right]$. If $\beta \geq \frac{9\theta}{(1-\theta)(5-6\mu)}$, this interval is

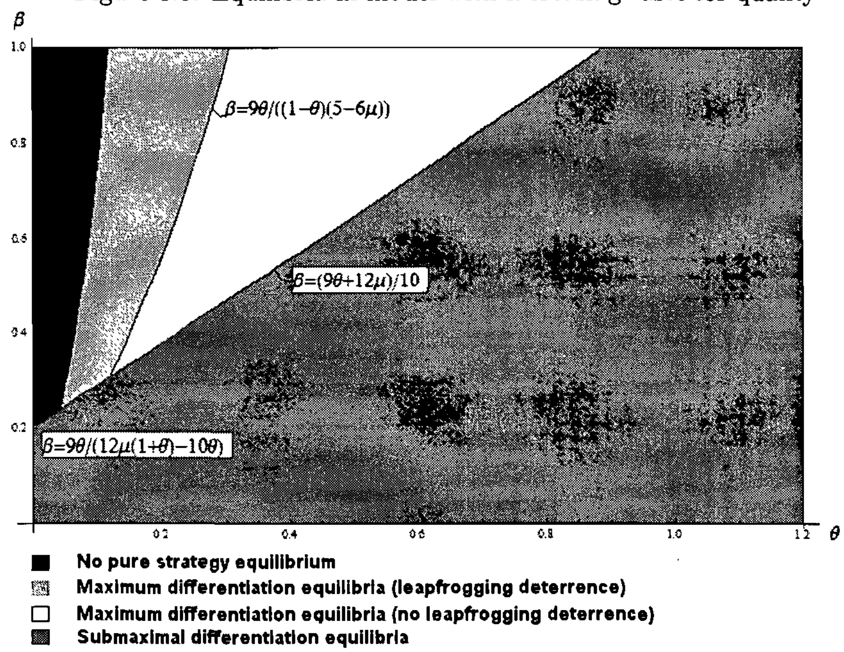
$$\left[-\frac{5\beta\Delta q}{9} - \frac{\beta\lambda}{3} + 2\beta\mu\left(\lambda\mu + \frac{\Delta q}{3} + \frac{\lambda}{3}\right), \Delta q - \frac{5\beta\Delta q}{9} - \frac{\beta\lambda}{3} + 2\beta\mu\left(\lambda\mu - \frac{\Delta q}{3} - \frac{\lambda}{3}\right)\right].$$

The first interval is always non-empty, but if $12\mu(1+\theta) \geq 10\theta$ and $\beta > \frac{9\theta}{12\mu(1+\theta)-10\theta}$, the second interval is empty. When the intervals are empty, no pure strategy subgame perfect Nash equilibrium exists. This is because the follower now sees its profit being considerably smaller, giving more incentives to try to become the leader. Therefore, if the initial difference in quality is small and β is high, both firms will try to become the leader, resulting in a "no equilibrium" zone. Figure 3.5 shows these zones when $\mu = \frac{1}{6}$.

5.3 Market coverage

We have assumed that the market is always covered, i.e. that all agents buys from firm a or firm b. Another possibility is for some agents to not buy at all. By letting $v = 0$, we have

Figure 3.5: Equilibria in model with increasing taste for quality



that an agent i buying a product of quality q_k at price p_k has a utility of $\theta_i q_k - p_k$. If he buys nothing, he has a utility of 0. Firm b, therefore, sells to consumers in the interval $\left[\frac{p_b}{q_b}, \frac{p_a - p_b}{\Delta q}\right]$. This problem becomes much less tractable, as second-period profits are $\pi'_L = \frac{4(q'_L)^2 \Delta q'}{(4q'_L - q'_F)^2}$ and $\pi'_F = \frac{q'_L q'_F \Delta q'}{(4q'_L - q'_F)^2}$. The profit functions are not only of a much more complicated form, but they also depend on q'_L and q'_F, and not only on $\Delta q'$.

We can still see intuitively the effects of this change. The follower now loses clients from both sides: consumers with high quality valuation buy from the leader, and consumers with low quality valuation do not buy at all. While in our model being the leader gave 4 times more profits in the maturity phase, here it gives $4\frac{q'_L}{q'_F} > 4$ times the profits, making the strategy to stay or become the leader more profitable. However, firm b now has more trouble attracting consumers in the first period, making the lead reversal strategy much more costly. Since firm a has a greater incentive to stay the leader, it is to be expected that not only firm a will always keep the lead, but the leapfrogging deterrence maximal differentiation region would be strongly reduced. Note also that while, in our model, we have maximum differentiation equilibria, here it seems probable that in the corresponding case, firm a would eliminate firm b from the market in period 1, but would not necessarily sell to all consumers as this becomes increasingly costly. Like in the case with increasing preferences for quality, it is possible that some "no equilibrium" zone will appear.

6 Conclusion

Markets with automatic user contributions have distinct characteristics from well-studied models of network externalities, open-sourcing and learning-by-doing. We proposed a simple model that can be applied to study markets for softwares, search engines and anti-viruses, among others. We built a model to study price competition in a two-stage game and showed that maximal differentiation does not always happen in equilibrium. While there are never lead reversals, for certain values of the parameters, the threat of such lead reversals has a negative effect on prices. Prices can be initially under marginal cost, a behavior observed in the development phase of many softwares. Automatic user contributions improve total welfare, and increasing them might even be Pareto-improving.

While we offered a few extensions to the model, more work needs to be done. In particular, firms usually do not use only one method to improve quality. The study of competition with firms improving quality through a mix of R&D, open sourcing and automatic contributions is still to be done. One interesting case is the competition between firms specializing in different quality-improvement methods.

Among other interesting questions still open are the effect of automatic user contributions on entry deterrence or market exits (in which the welfare-improving result might become ambiguous), and the dynamics in an infinite-horizon model.

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