

Université de Montréal

**Generalised ladder operators, degeneracy and coherent  
states in two-dimensional quantum mechanics**

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Cette thèse intitulée

## Generalised ladder operators, degeneracy and coherent states in two-dimensional quantum mechanics

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# Résumé

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Dans cette thèse, nous discutons de la dégénérescence et de la construction d'états cohérents généralisés dans les systèmes quantiques en deux dimensions d'espace. Nous développons un schéma pour obtenir des spectres non dégénérés et des combinaisons linéaires appropriées des états propres d'énergie correspondants. Lorsque la dégénérescence dans le spectre d'énergie est linéaire dans les nombres quantiques, nous définissons des opérateurs d'échelle généralisés qui conduisent à une chaîne d'états avec un ensemble naturel de coefficients. De plus, nous récupérons des relations de complétude pour les états généralisés. Lorsque le spectre d'énergie est quadratique dans les nombres quantiques, nous utilisons certains résultats de la théorie des nombres pour catégoriser la dégénérescence et, par conséquent, les combinaisons linéaires appropriées des états propres d'énergie associés. En particulier, nous étudions des oscillateurs harmoniques bidimensionnels isotropes et anisotropes ainsi que le potentiel Morse bidimensionnel et son partenaire supersymétrique non séparable. Dans tous les cas, nous construisons des états cohérents et discutons certains aspects de leur caractère non classique. On retrouve une certaine compression dans les quadratures conjuguées, une dépendance non triviale des variances des quadratures vis-à-vis des paramètres introduits lors de la définition des spectres non dégénérés, et un problème de localisation pour les fonctions d'onde. Comme application, nous étudions le problème de la quantification et de l'analyse semi-classique de l'espace des phases en deux dimensions en exploitant la complétude des familles généralisées d'états cohérents comprimés en deux dimensions.

**Mots clés:** Mécanique quantique bidimensionnelle; États cohérents; États comprimés; Dégénérescence; Oscillateurs isotropes; Oscillateurs anisotropes; Opérateurs d'échelles; Potentiel de Morse; Quantification.



# Abstract

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In this thesis we discuss degeneracy and the construction of generalised coherent states in two-dimensional quantum systems. We develop a scheme for defining non-degenerate spectra and the corresponding averaged energy eigenstates. When the degeneracy in the spectrum is linear in the quantum numbers, we are able to define generalised ladder operators which lead to a chain of states with a natural set of coefficients. Additionally, we are able to recover completeness relations for the generalised states. On the other hand, when the spectrum is quadratic in the quantum numbers, we utilise some results from number theory to categorise the degeneracy and correspondingly the averaged energy eigenstates. In particular we study the two-dimensional isotropic and anisotropic oscillators as well the two-dimensional Morse potential and its non-separable supersymmetric partner. In all cases, we compute the coherent states and discuss certain aspects of their non-classicality. We find squeezing between conjugate quadratures, non-trivial dependence of the quadrature variances on the parameters introduced when defining the non-degenerate spectra, and non-localisation of wavefunctions. As an application, we study the problem of quantisation and semiclassical phase space analysis in two dimensions by exploiting the completeness of generalised families of two-dimensional squeezed coherent states.

**Keywords:** Two-dimensional quantum mechanics; Coherent states; Squeezed states; Degeneracy; Isotropic oscillator; Anisotropic oscillator; Ladder operators; Morse potential; Quantisation.





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# List of publications

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## Papers included in the thesis

- **J. Moran** and V. Hussin; *Coherent States for the Isotropic and Anisotropic 2D Harmonic Oscillators*, Quantum Rep. **1**, 2 (2019)
- **J. Moran**, V. Hussin, and I. Marquette; *Non-linear ladder operators and coherent states for the 2:1 oscillator*, J. Phys. A: Math. Theor. **54**, 275301 (2021)
- **J. Moran**; *Degeneracy and coherent states of the two-dimensional Morse potential*, Eur. Phys. J. Plus **136**, 716 (2021)
- **J. Moran** and V. Hussin; *A New Method for Constructing Squeezed States for the Isotropic 2D Harmonic Oscillator*, Quantum Theory and Symmetries: Proceedings of the 11th International Symposium (2021)
- **J. Moran** and V. Hussin; *Coherent states of the two-dimensional non-separable supersymmetric Morse potential*, [submitted to: *The European Physical Journal Plus*]
- J.-P. Gazeau, V. Hussin, **J. Moran**, and K. Zelaya; *Two-mode squeezed state quantisation and semiclassical portraits*, [in preparation]

## Papers not included in the thesis

- J.-P. Gazeau, V. Hussin, **J. Moran**, and K. Zelaya; *Quantum and semi-classical aspects of confined systems with variable mass*, J. Phys. A: Math. Theor. **53**, 505306 (2020)
- J.-P. Gazeau, V. Hussin, **J. Moran**, and K. Zelaya; *Generalized Susskind–Glogower coherent states*, J. Math. Phys. **62**, 072104 (2021)



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*Dedicated to the memory of my mother*





# Introduction

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Since the beginnings of quantum theory there has been great interest in the boundary between a classical and quantum system. In their first incarnation, Schrödinger's minimal uncertainty wavepackets [1] proved to be as classical as a quantum state could be. These wavepackets minimise the Heisenberg uncertainty relation and maintain a localised Gaussian wavefunction under unitary time evolution. Moreover, the expectation values of the wavepackets mimic the classical equations of motion for the harmonic oscillator. The minimal uncertainty wavepackets later became known as coherent states and found a renewed meaning in the context of quantum optics. In the sixties, Glauber [2] and Sudarshan [3] formalised what we describe today as coherent states. This formalisation generalised their definition as minimal uncertainty wavepackets to include equivalent definitions as eigenstates of the annihilation operator, and the unitary displacement by an exponential of a linear antihermitian combination of the creation and annihilation operators, of the ground state. A further generalisation of the coherent states is generated by the action of a unitary squeezing operator which is quadratic in the ladder operators. The squeezing refers to the reduction in one quadrature variance at the expense of the increase in the conjugate quadrature variance.

Coherent states have been applied in many contexts throughout physics. Coherent states underpin the mathematical description of laser light [4], their completeness as a basis for the Hilbert space give rise to applications in path integral techniques for condensed matter and many-body systems [5]. Furthermore, as will be explored in this thesis, their completeness facilitates their use in general quantisation problems [6, 7, 8].

In a more recent newsworthy application, squeezed states were used in the LIGO detection of gravitational waves [9]. The experiment is highly sensitive to quantum fluctuations in the measurement, and employing squeezed states allowed reliable measurement by mitigating the uncertainty introduced by the quantum-ness of nature. Elsewhere, squeezed states are used in continuous variable quantum information. The use of an optical device like a beam splitter requires non-classical input in order to generate a non-classical output [10], this non-classical output is entangled and is an essential resource in continuous variable quantum information processing, just as entanglement is integral to discrete variable quantum information.

The three definitions of coherent state have spawned swathes of coherent state analysis in mathematical physics for many different systems. Nowadays coherent states defined as generalised eigenstates of generalised annihilation operators are called Barut-Girardello coherent states [11], while the notion of unitary displacement operators has been generalised to include arbitrary Lie group elements as unitary exponential operators. Such coherent states are called group-theoretic coherent states [12]. When neither definition is applicable, coherent states may be constructed as a particular superposition of energy eigenstates with a view to defining a well localised state, in this case the coherent states are aptly named generalised coherent states [13, 14]. These definitions are the modern day departing for defining new coherent states for new systems.

For multidimensional systems the problem of coherent state analysis opens up further. While it is always possible to take products of many one-dimensional coherent states to define a multidimensional coherent state, we do not learn anything new about the generalities that come with multidimensional systems. Perhaps the best known example is the two-mode squeezed state [15], the resulting state is entangled and therefore not separable as a product of two states. Definitions of generalised coherent states as superpositions of eigenstates become obscure in the presence of degeneracy [16]. Degeneracy does not typically occur in one-dimensional systems by the non-degeneracy theorem [17].

Two-dimensional quantum systems are an interesting subset of multidimensional systems. Examples of physically important two-dimensional systems include: quasiparticles known as anyons with neither Fermi-Dirac or Bose-Einstein statistics that can only exist in two-dimensional systems [18]; the nanomaterial graphene which is a one atom thick layer of graphite with applications throughout electronics [19, 20]; polarisation states of gravitational waves in an inflationary universe [21]. When dealing with two-dimensional quantum systems we typically begin with its Hamiltonian and the first system we study is the harmonic oscillator. The harmonic oscillator is ubiquitous in all domains of physics, and the Morse potential, describing anharmonic interactions, appears in a variety of problems in quantum physics: in molecular dynamics, the Morse potential more accurately describes the phenomenology of bond breaking with a finite bound state spectrum [22]; the two-dimensional Morse potential is used as a perturbation expansion basis for studying triatomic molecular dynamics [23]. Morse oscillators also appear in problems in quantum gravity [24].

In this thesis we approach the problem of defining generalised coherent states for two-dimensional systems from the viewpoint of studying the degeneracy of the system. We define a non-degenerate spectrum as an average of its degenerate contributions and where possible, generalised ladder operators to act on the non-degenerate spectrum. In this way we are able to remove the ambiguity that the degeneracy introduces, leading to well-defined classes of coherent states rich in the non-trivial features that two-dimensional systems present. The procedure we describe is generalisable to any system with any degeneracy structure.

This thesis is organised as follows. In chapter 1 we review some of the essential features of the canonical coherent states of the harmonic oscillator including their mathematical formulation, uncertainty relations and aspects of non-classicality. Following this, in chapter 2 we turn our attention to the two-dimensional harmonic oscillator. We construct a non-degenerate spectrum and the  $\mathfrak{su}(2)$  coherent states by defining generalised ladder operators connecting these states. Equipped with the non-degenerate basis we define Schrödinger-type coherent states by direct analogy with the one-dimensional definitions and we find that they factorise as the product of two one-dimensional coherent states. We extend some of the ideas to the anisotropic harmonic oscillator following Chen's definitions [25], and compute their Schrödinger-type coherent states. We then formally define generalised squeezed states by defining a unitary operator quadratic in the generalised ladder operators. We find that under this construction a two-mode mixing occurs and the resulting states are non-separable. Chapter 3 is devoted to the specific example of the 2 : 1 anisotropic oscillator to account for some of the missing coherent state features in Chen's definition, namely, completeness and ladder operators. In order to address the modified degeneracy structure, we must include non-linearities in the definition of the generalised ladder operators, and we find that the resulting  $\mathfrak{su}(2)$ -like coherent states satisfy a resolution of the identity, thereby leading to a set of generalised coherent states with all their mathematical features recovered.

To generalise the discussion further, in chapter 4 we turn our attention to the two-dimensional Morse potential whose spectrum is quadratic. The two-dimensional quadratic spectrum presents an interesting challenge because we find accidental degeneracies. Accidental degeneracies are best understood in the context of number theory. We determine that for irrational values of the system's principle parameter,  $p$ , the degeneracies are at most twofold and thus accidental degeneracies will not appear. Proceeding with this we define a non-degenerate spectrum and set of basis states, then we discuss the construction of generalised coherent states and their properties. Continuing in the same fashion, in chapter 5, we discuss the non-separable, singular, supersymmetric partner of the two-dimensional Morse potential [26]. We are able to utilise the non-degenerate states constructed in the preceding chapter to develop a non-degenerate set of states for the partner potential. Once again we construct the generalised coherent states and we find that the wavefunction is unable to localise and there is significant squeezing between the position and momentum quadratures.

Lastly, in chapter 6, as an application, we study quantisation and semiclassical phase space analysis in two dimensions using two families of squeezed states: separable and non-separable. The completeness of the generalised families of two-dimensional squeezed states allows us to quantise any classical phase space function in the basis of squeezed states, and moreover, allows us to study their semiclassical portraits by averaging against the same

states. When non-separable squeezed states are used to quantise classical phase space functions, a mixing between the resulting quantum operators in the two modes occurs. This is a non-trivial feature that does not exist in one-dimensional quantisation problems.

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# Chapter 1

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## Mathematical preliminaries

### 1.1. The quantum harmonic oscillator

The one-dimensional quantum harmonic oscillator is defined by the following Hamiltonian [1]

$$H = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2. \quad (1.1.1)$$

The Hamiltonian (1.1.1) is related to its corresponding classical Hamiltonian through canonical quantisation, where we have promoted the initial phase space variables  $(x,p) \rightarrow (\hat{X}, \hat{P})$  to quantum operators. In the position basis  $\hat{P} = -i\hbar\frac{\partial}{\partial x}$  is the quantum momentum operator and  $\hat{X} = x$  is the quantum position operator. These are non-commuting operators which satisfy the Weyl-Heisenberg algebra  $[\hat{X}, \hat{P}] = i\hbar\mathbb{1}$ . The solutions to the time independent Schrödinger equation  $H\psi = E\psi$  in the Fock, or, number basis are given by  $H|n\rangle = E_n|n\rangle$ . The position representation of these solutions is obtained by computing  $\langle x|n\rangle \equiv \psi_n(x)$ , where  $|x\rangle$  is the eigenstate of the position operator with eigenvalue  $x$ ,

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \quad (1.1.2)$$

and the associated energy spectra  $E_n = \omega\hbar(n + \frac{1}{2})$ .

A motivation for the study of coherent states is to find solutions which are ‘more’ classical, that is, they more closely resemble the solutions to the classical harmonic oscillator. When we compute the dispersions  $\Delta\hat{X} = \sqrt{\langle\hat{X}^2\rangle - \langle\hat{X}\rangle^2}$  and  $\Delta\hat{P} = \sqrt{\langle\hat{P}^2\rangle - \langle\hat{P}\rangle^2}$  for a number state  $|n\rangle$  we find the following Heisenberg uncertainty relation

$$\Delta\hat{X}\Delta\hat{P} = \frac{\hbar}{2}(2n+1) \geq \frac{\hbar}{2}, \quad (1.1.3)$$

which grows with  $n$ . Note that there is no such restriction on the dispersions in classical physics, it is a remnant of the fact that  $\hat{X}$  and  $\hat{P}$  are non-commuting observables and as such it is impossible to specify the associated eigenvalues simultaneously.

It is necessary to understand the algebraic solutions to the harmonic oscillator in order to define the coherent states. For brevity we will use units with  $m, \omega = 1$  and  $\hbar = 1$ , the units can always be reintroduced through dimensional analysis. Defining the following creation and annihilation operators respectively [2]

$$\hat{X} = \frac{1}{\sqrt{2}}(a^- + a^+), \quad (1.1.4)$$

$$\hat{P} = \frac{1}{\sqrt{2}i}(a^- - a^+) \quad (1.1.5)$$

and consequently their inverse relations,

$$a^+ = \frac{1}{\sqrt{2}}(\hat{X} - i\hat{P}), \quad (1.1.6)$$

$$a^- = \frac{1}{\sqrt{2}}(\hat{X} + i\hat{P}), \quad (1.1.7)$$

which obey the following algebra

$$[a^-, a^+] = \mathbb{1}. \quad (1.1.8)$$

We can use these new definitions to rewrite (1.1.1) as

$$H = a^+ a^- + \frac{1}{2}, \quad (1.1.9)$$

and we find the following action of the operators on the Fock states as

$$a^- |n\rangle = \sqrt{n} |n-1\rangle, \quad a^+ |n\rangle = \sqrt{n+1} |n+1\rangle, \quad (1.1.10)$$

therefore the Fock space is generated by

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}} |0\rangle. \quad (1.1.11)$$

We also obtain the orthogonality and completeness relations

$$\langle m | n \rangle = \delta_{nm}, \quad (1.1.12)$$

and

$$\sum_{n=0}^{\infty} |n\rangle \langle n| = \mathbb{1}, \quad (1.1.13)$$

respectively.

This justifies the nomenclature,  $a^-$  annihilates an excitation (or particle, up to a normalisation) in  $|n\rangle$  and  $a^+$  creates one. It is important to note that the annihilation operator destroys the vacuum such that  $a^- |0\rangle = 0$ , this is necessary for consistency. These were originally introduced by Dirac [2] as a way to calculate eigenvalues of (1.1.1) without solving the complete differential equation.



## 1.2. Coherent states of the harmonic oscillator

Coherent states of the harmonic oscillator have been defined in several equivalent ways. The first definition of the coherent states we will introduce are as eigenstates of the annihilation operator with complex eigenvalue  $z$  [3],

$$a^- |z\rangle = z |z\rangle. \quad (1.2.1)$$

If we expand  $|z\rangle$  in a Fock basis as  $|z\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n|z\rangle$  and using (1.1.11) we find the normalised Fock expansion,

$$|z\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle. \quad (1.2.2)$$

Such series converge  $\forall z \in \mathbb{C}$ , so the coherent states represent a continuously parametrised set of states.

The second equivalent definition of the coherent states is by the action of the unitary displacement operator,  $D(z)$ , on the oscillator ground state,

$$D(z) |0\rangle = |z\rangle. \quad (1.2.3)$$

We may reverse-engineer (1.2.2) to find the form of the displacement operator

$$\begin{aligned} |z\rangle &= e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{(za^+)^n}{n!} |0\rangle \\ &= e^{-\frac{|z|^2}{2} + za^+} |0\rangle, \end{aligned} \quad (1.2.4)$$

We make use of the annihilation of the vacuum,  $a^- |0\rangle = 0$ , and the Baker-Campbell-Hausdorff identity [1],

$$e^A e^B = e^{A+B + \frac{[A,B]}{2} + \dots}, \quad (1.2.5)$$

to write (1.2.4) as

$$\begin{aligned} |z\rangle &= e^{-\frac{|z|^2}{2} + za^+} e^{-\bar{z}a^-} |0\rangle \\ &= e^{-\frac{|z|^2}{2} + za^+ - \bar{z}a^- + \frac{|z|^2}{2}} |0\rangle, \end{aligned} \quad (1.2.6)$$

establishing the form of  $D(z)$  [3]

$$|z\rangle = e^{(za^+ - \bar{z}a^-)} |0\rangle \equiv D(z) |0\rangle. \quad (1.2.7)$$

This allows us to interpret  $|z\rangle$  as the vacuum state displaced by some complex parameter  $z$  from the origin. The displacement operator is unitary,  $D(z)D^\dagger(z) = \mathbb{1}$ , and it follows that

$$D(z) = D^\dagger(-z). \quad (1.2.8)$$

The third definition of the coherent states is as the minimal uncertainty states  $\forall z$  with equal variance in  $\hat{X}$  and  $\hat{P}$  [3],

$$\left\{ \text{states } |z\rangle \quad \text{s.t.} \quad \Delta\hat{X}\Delta\hat{P} = \frac{1}{2}, \Delta\hat{X} = \Delta\hat{P} \right\}. \quad (1.2.9)$$

This can be interpreted as a consequence of the second definition of the coherent states; the ground state is a minimal uncertainty state and the coherent states are a displacement of the ground state by the parameter  $z$ . The equations (1.2.1), (1.2.2) and (1.2.9) are the defining equations for the coherent states of the quantum harmonic oscillator and they are all equivalent.

Furthermore we have the property of overcompleteness of the family of coherent states, that is, they form a complete basis for the Hilbert space but the coherent states are not mutually orthogonal,  $\langle z'|z\rangle \neq 0$ . We introduce the notation on the complex plane  $\int_{\mathbb{C}} d^2z = \int_{\mathbb{R}} d\text{Im}(z) \int_{\mathbb{R}} d\text{Re}(z) = \int_0^{2\pi} d\theta \int_0^\infty |z|d|z|$  for  $z = |z|e^{i\theta}$ , then the completeness relation is given by [3]

$$\begin{aligned} \int d^2z |z\rangle \langle z| &= \int d^2z e^{-|z|^2} \sum_{n,m} \frac{z^n}{\sqrt{n!}} \frac{\bar{z}^m}{\sqrt{m!}} |n\rangle \langle m| \\ &= \frac{1}{2} \int d(r^2) \int d\theta e^{-r^2} \sum_{n,m} \frac{r^{n+m}}{\sqrt{n!}\sqrt{m!}} e^{i(n-m)\theta} |n\rangle \langle m| \\ &= \pi \int d(r^2) e^{-r^2} \sum_n \frac{r^{2n}}{n!} |n\rangle \langle n| \\ &= \pi \sum_n \underbrace{\int d(r^2) (r^2)^n e^{-(r^2)}}_{n!} \frac{1}{n!} |n\rangle \langle n| \\ &= \pi \mathbb{1}. \end{aligned} \quad (1.2.10)$$

The completeness of the Fock space is used in the last line, so the coherent states form a complete basis for the Hilbert space. This property is also called the resolution of the identity, it means that the sum (in our case the integral) of the projectors form the identity operator in the Hilbert space.

Resolution of the identity has far reaching consequences, for example we can construct a quantisation map using the coherent states to associate a quantum operator  $\hat{A}$  with a classical phase space observable  $A(z, \bar{z})$  (where  $z$  is to be associated with phase space variables  $(x, p)$  in the next section) in the following way

$$\hat{A} = \frac{1}{\pi} \int d^2z A(z, \bar{z}) |z\rangle \langle z|, \quad (1.2.11)$$

this is known as Klauder-Berezin quantisation [3].

The coherent states are overcomplete as they have a nonzero inner product between two different coherent states

$$\begin{aligned}\langle z_1 | z_2 \rangle &= e^{-\frac{|z_1|^2}{2} - \frac{|z_2|^2}{2}} \sum_{n,m} \frac{\bar{z}_1^n}{\sqrt{n!}} \frac{z_2^m}{\sqrt{m!}} \langle n | m \rangle \\ &= e^{-\frac{|z_1|^2}{2} - \frac{|z_2|^2}{2} + \bar{z}_1 z_2} \neq 0,\end{aligned}\tag{1.2.12}$$

using the orthogonality of Fock states  $\langle n | m \rangle = \delta_{nm}$ . It means that no two coherent states are orthogonal.

Moreover, we may compute the position space wavefunction of the coherent states,

$$\begin{aligned}\langle x | z \rangle &= e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \langle x | n \rangle \\ &= e^{-\frac{|z|^2}{2}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{4\pi}} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{z}{\sqrt{2}} \right)^n H_n(x),\end{aligned}\tag{1.2.13}$$

where we recognise the summation as the generating function for the Hermite polynomials [4],

$$\sum_{n=0}^{\infty} \frac{y^n}{n!} H_n(x) = e^{2xy - \frac{y^2}{2}}.\tag{1.2.14}$$

After discarding non- $x$ -dependent phases we find

$$\psi_z(x) = \langle x | z \rangle = \left( \frac{1}{\pi} \right)^{\frac{1}{4}} e^{-\frac{1}{2}(x - \sqrt{2}\text{Re}(z))^2} e^{i\sqrt{2}\text{Im}(z)x}.\tag{1.2.15}$$

The preceding definitions allow us to verify the Heisenberg uncertainty relations. Considering the canonical position operator (1.1.4) and making use of  $[a^-, a^+] = \mathbb{1}$ , the position dispersion in the coherent state basis is given by

$$\begin{aligned}(\Delta \hat{X})^2 &= \langle z | \left( \hat{X}^2 - \langle \hat{X} \rangle^2 \right) | z \rangle \\ &= \frac{1}{2}(z + \bar{z})^2 - \frac{1}{2}(z + \bar{z})^2 + \frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}\tag{1.2.16}$$

Similarly for the canonical momentum operator (1.1.5) we find  $(\Delta P)^2 = \frac{1}{2}$ , thus  $\Delta X \Delta P = \frac{1}{2}$ , minimising the Heisenberg uncertainty relation.

The coherent states remain coherent states under time evolution, that is,

$$|z\rangle \rightarrow e^{-i\hat{H}t} |z\rangle = e^{-\frac{i\omega t}{2}} |ze^{-i\omega t}\rangle,\tag{1.2.17}$$

where the external phase is unphysical. This leads us to the time-dependent expectation value of the position and momentum operators with  $z = |z|e^{i\theta}$ , as [3]

$$\langle \hat{X}(t) \rangle = \sqrt{\frac{2\hbar}{m\omega}} \text{Re}(z(t)) = \sqrt{\frac{2\hbar}{m\omega}} |z| \cos(\omega t - \theta)\tag{1.2.18}$$

and

$$\langle \hat{P}(t) \rangle = \sqrt{2\hbar m\omega} \operatorname{Im}(z(t)) = -\sqrt{2\hbar m\omega} |z| \sin(\omega t - \theta) = m \frac{d\langle \hat{X}(t) \rangle}{dt} \quad (1.2.19)$$

respectively, where we have reintroduced dimensionful units for clarity. Suppressing the dimensional units once again, we can then write  $z(t) = \frac{1}{\sqrt{2}}(x(t) + ip(t))$  with  $\langle \hat{X}(t) \rangle = x(t)$  and  $\langle \hat{P}(t) \rangle = p(t)$  satisfying the classical equations of motion for the harmonic oscillator. Of course these states are still technically quantum because there is an implicit  $\hbar$  dependence throughout, but this analysis gives us a useful, classical interpretation of the coherent parameter  $z$ . In the literature of quantum optics the coherent states are referred to as ‘classical’ for these reasons.

### 1.2.1. Application to the electromagnetic field

The electromagnetic field may be thought of as an infinite assembly of harmonic oscillators, with an infinite number of degrees of freedom. In order to describe the electromagnetic field in terms of the coherent states we introduce the following multimode notation,

$$|\{z_k\}\rangle = |z_1\rangle \otimes |z_2\rangle \otimes \dots, \quad k = 1, 2, \dots \quad (1.2.20)$$

In this notation, the multimode generalisations of the preceding results is clear, each coherent state  $|z_k\rangle$  in the ensemble independently satisfies the basic coherent state properties with its ladder operators  $a_k^\pm$ . The quantum Hamiltonian of the system is given by

$$H = \frac{1}{2} \sum_k (2a_k^+ a_k^- + 1). \quad (1.2.21)$$

Considering a free field confined to a periodic cubic box of side length  $L$  and in the Coulomb gauge ( $\nabla \cdot \vec{A}(t, \vec{x}) = 0$ ), the quantised vector potential is found to be [5]

$$\vec{A}(t, \vec{x}) = \frac{1}{L^{\frac{3}{2}}} \sum_\lambda \sum_k \sqrt{\frac{\hbar}{2\omega_k}} \left( a_k^\lambda(t) e^{i\vec{k} \cdot \vec{x}} + (a_k^\lambda)^\dagger(t) e^{-i\vec{k} \cdot \vec{x}} \right) \hat{\varepsilon}^\lambda(\vec{k}), \quad (1.2.22)$$

where  $a_k^\lambda(t) = a_k^\lambda e^{-i\omega_k t}$  is the time-dependent annihilation operator that annihilates a photon of momentum  $k$  at time  $t$ , and  $\hat{\varepsilon}^\lambda(\vec{k})$  is the polarization vector with  $\lambda$  labelling the polarization modes. The coherent states are eigenstates of the following expression

$$\sum_\lambda a_k^\lambda(t) \hat{\varepsilon}^\lambda(\vec{k}) |\{z_k\}\rangle = \sum_\lambda z_k(t) \hat{\varepsilon}^\lambda(\vec{k}) |\{z_k\}\rangle, \quad (1.2.23)$$

which was generalized to include arbitrary polarization, it does not affect the analysis. The vector potential is related to the physical electric field through Maxwell’s equations [6], specifically,  $\vec{E}(t, x) = -\frac{\partial \vec{A}(t, x)}{\partial t}$ , yielding

$$\vec{E}(t, x) = \frac{i}{L^{\frac{3}{2}}} \sum_\lambda \sum_k \sqrt{\frac{\hbar\omega_k}{2}} \left( a_k^\lambda(t) e^{i\vec{k} \cdot \vec{x}} - (a_k^\lambda)^\dagger(t) e^{-i\vec{k} \cdot \vec{x}} \right) \hat{\varepsilon}^\lambda(\vec{k}). \quad (1.2.24)$$

By writing the electric field as a sum of positive and negative frequency parts,  $\vec{E}(t,x) = \vec{E}^+(t,x) + \vec{E}^-(t,x)$ , we see by (1.2.23) that the following holds

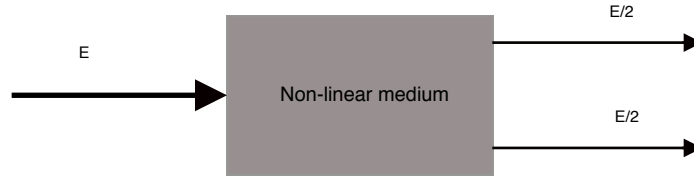
$$\vec{E}^+(t,x) |\{z_k\}\rangle = \vec{E}_{\text{cl}}^+(t,x) |\{z_k\}\rangle, \quad (1.2.25)$$

where  $\vec{E}_{\text{cl}}^+(t,x)$  solves the classical Maxwell equations.

The description of the electromagnetic field in terms of coherent states  $|\{z_k\}\rangle$  is considered to be an ideal laser in optics, it describes a perfect monochromatic emission. In practice a real-world laser would be described as a statistical mixture of coherent states with random phases [7].

### 1.3. Squeezed coherent states of the harmonic oscillator

Beginning with a physical motivation, the next class of states we wish to study, the squeezed coherent states (or squeezed states for short), emerge from second-order interactions. Second-order non-linear optical media were first observed by P.A. Franken et. al. by firing a maser at a quartz crystal [8] this is a parametric process as shown in the diagram below.



**Fig. 1.1.** Photon of energy  $E$  enters the crystal and two photons of energy  $\frac{E}{2}$  leave the crystal, this is an example of degenerate parametric down conversion. It is degenerate because two identical photons are produced.

This process yields an interaction Hamiltonian with input field  $b$  and output fields  $a$

$$H_I = i\chi^{(2)} \left( b^-(a^+)^2 - b^+(a^-)^2 \right), \quad (1.3.1)$$

where  $\chi^{(2)}$  is a coupling associated with this second-order process, and the factor of  $i$  ensures that the operator is hermitian. If we approximate the input field as a coherent state (laser, maser),  $|z\rangle$ , and assume it is classical, so that we may replace the quantum operator by the associated complex number, (1.3.1) simplifies to

$$H_I = i\chi^{(2)} \left( z(a^+)^2 - \bar{z}(a^-)^2 \right). \quad (1.3.2)$$

The time evolution operator associated with this interaction is given by

$$e^{-iH_I t} = e^{t\chi^{(2)}(z(a^+)^2 - \bar{z}(a^-)^2)} = e^{\frac{1}{2}(\xi a^{+2} - \bar{\xi} a^{-2})} \equiv S(\xi), \quad (1.3.3)$$

where  $\xi = 2\chi^{(2)}zt$ . This naturally defines the squeezing operator  $S(\xi)$  which will be the mathematical object of interest in this section, and counterpart of the displacement operator  $D(z)$ .

Squeezed states have similar properties to coherent states only they are more general. The defining feature is that, while the Heisenberg uncertainty relation is still satisfied (under the condition that  $\xi \in \mathbb{R}$ , which will be shown in due course), there is no longer equal dispersion between position and momentum. This is encoded by a squeezing parameter  $\xi$  and the origin of the ‘squeezing’. The squeezed states can be defined as a displacement of the squeezed vacuum in the following way [7]

$$|\alpha, \xi\rangle = D(\alpha)S(\xi)|0\rangle, \quad (1.3.4)$$

where  $D(\alpha)$  is the same displacement operator (1.2.7), and the squeezing operator  $S(\xi)$  (1.3.3).

### 1.3.1. Transformation properties of $D(\alpha)$ and $S(\xi)$

We need to show how the creation and annihilation operators transform under the action of  $D(\alpha)$  and  $S(\xi)$  before computing uncertainty relations. Using the Baker-Campbell-Hausdorff formula  $e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \dots$  [1], considering the displacement operator,

$$D^\dagger(\alpha)a^-D(\alpha) = a^- + \alpha\mathbf{1}, \quad (1.3.5)$$

similarly we find

$$D^\dagger(\alpha)a^+D(\alpha) = a^+ + \bar{\alpha}\mathbf{1}. \quad (1.3.6)$$

Looking now at the squeezing operator, by writing the complex squeezing parameter  $\xi$  in polar form  $\xi = re^{i\theta}$  we use the same Baker-Campbell-Hausdorff formula to find

$$S^\dagger(\xi)a^-S(\xi) = a^- \cosh r + a^+ e^{i\theta} \sinh r, \quad (1.3.7)$$

this is the Bogoliubov transformation [9]. We get the transformation for  $a^+$  by taking the hermitian conjugate,

$$S^\dagger(\xi)a^+S(\xi) = a^+ \cosh r + a^- e^{-i\theta} \sinh r. \quad (1.3.8)$$

For the derivations that follow it is useful to write down the transformations of bilinears of  $a^-, a^+$ ,

$$\begin{aligned} S^\dagger(\xi)a^{-2}S(\xi) &= a^{-2} \cosh^2 r + (2a^+a^- + 1)e^{i\theta} \cosh r \sinh r + a^{+2}e^{2i\theta} \sinh^2 r \\ S^\dagger(\xi)a^{+2}S(\xi) &= a^{+2} \cosh^2 r + (2a^+a^- + 1)e^{-i\theta} \cosh r \sinh r + a^{-2}e^{-2i\theta} \sinh^2 r \\ S^\dagger(\xi)a^+a^-S(\xi) &= a^+a^- \cosh^2 r + a^{+2}e^{i\theta} \sinh r \cosh r + a^{-2}e^{-i\theta} \sinh r \cosh r + \sinh^2 r (a^+a^- + 1). \end{aligned} \quad (1.3.9)$$

The definition of squeezed states by acting first with the squeezing operator and then with the displacement operator is somewhat arbitrary. While the two operators do not commute, they are equivalent up to a relabelling of parameters with the following braiding relations

$$D(\alpha)S(\xi) |0\rangle = S(\xi)S^\dagger(\xi)D(\alpha)S(\xi) |0\rangle, \quad (1.3.10)$$

now we compute the term  $S^\dagger(\xi)D(\alpha)S(\xi)$  extending the results from (1.3.7, 1.3.8),

$$S^\dagger(\xi)D(\alpha)S(\xi) = \exp(\beta a^+ - \bar{\beta} a^-) = D(\beta), \quad (1.3.11)$$

for  $\beta = \alpha \cosh r - \bar{\alpha} e^{i\theta} \sinh r$ , thus

$$D(\alpha)S(\xi) = S(\xi)D(\beta). \quad (1.3.12)$$

We see that we can use the definitions almost interchangeably, if we take note of the simple change of parameters, this however does not affect the physics because it just amounts to a rescaling of the coordinate system.

### 1.3.2. Uncertainty relations

We can demonstrate that the squeezed states almost saturate the Heisenberg inequality but no longer with equal dispersion in position and momentum, but they always saturate the more general Schrödinger-Robertson uncertainty relation [10]. By considering the states defined as  $|\alpha, \xi\rangle = D(\alpha)S(\xi) |0\rangle$  with  $\xi = r e^{i\theta}$ , we make use of the transformations found in (1.3.1) and compute the expectation value of the position operator between two squeezed states

$$\langle \alpha, \xi | \hat{X} | \alpha, \xi \rangle = \frac{1}{\sqrt{2}} \langle \alpha, \xi | (a^- + a^+) | \alpha, \xi \rangle = \frac{1}{\sqrt{2}} (\alpha + \bar{\alpha}), \quad (1.3.13)$$

with normally ordered terms vanishing. Now we need to compute the average of  $\hat{X}^2$  which is given by

$$\langle \alpha, \xi | \hat{X}^2 | \alpha, \xi \rangle = \frac{1}{2} \left( 2 \cos \theta \cosh r \sinh r + 2 \sinh^2 r + (\alpha + \bar{\alpha})^2 + 1 \right), \quad (1.3.14)$$

where we used the relations in (1.3.5) and (1.3.9). From this we calculate the following dispersion

$$(\Delta \hat{X})^2 = \langle \beta, \xi | \left( \hat{X}^2 - \langle \hat{X} \rangle^2 \right) | \beta, \xi \rangle = \frac{1}{2} + \cosh r \sinh r \cos \theta + \sinh^2 r. \quad (1.3.15)$$

Similarly for the dispersion in momentum

$$(\Delta \hat{P})^2 = \langle \beta, \xi | \left( \hat{P}^2 - \langle \hat{P} \rangle^2 \right) | \beta, \xi \rangle = \frac{1}{2} - \cosh r \sinh r \cos \theta + \sinh^2 r, \quad (1.3.16)$$

then we find the uncertainty relation

$$(\Delta \hat{X})^2 (\Delta \hat{P})^2 = \frac{1}{4} + \sinh^2 r \cosh^2 r \sin^2 \theta. \quad (1.3.17)$$

From this we deduce that the Heisenberg uncertainty relation is saturated in the case that we have a real squeezing parameter  $\xi \in \mathbb{R}$ . In the  $\theta = 0$  case we see that we get  $(\Delta X)^2 = \frac{1}{2}e^{2r}$  and  $(\Delta P)^2 = \frac{1}{2}e^{-2r}$ . This is as we expect, we have squeezed the variance in one quadrature at the expense of increasing the uncertainty in the conjugate quadrature in such a way that the Heisenberg inequality is still saturated. Similarly, taking  $\theta = \pi$  reverses the squeezing in the quadratures. It is well known that squeezed states in general do not saturate the Heisenberg uncertainty relation for  $\xi \in \mathbb{C}$  [7], but it is often of interest to study real squeezing.

For the more general case  $\xi \in \mathbb{C}$ , the extra terms that arise in the Heisenberg uncertainty relation are a result of the covariance between  $\hat{X}, \hat{P}$ . In such a case the Schrödinger-Robertson uncertainty relation is always minimised. To see this we begin by defining the covariance matrix [10]

$$\Gamma[\hat{X}, \hat{P}] = \begin{pmatrix} (\Delta\hat{X})^2 & \Delta(\hat{X}\hat{P}) \\ \Delta(\hat{P}\hat{X}) & (\Delta\hat{P})^2 \end{pmatrix}, \quad (1.3.18)$$

where the covariance for non-commuting observables is defined as

$$\Delta(\hat{X}\hat{P}) = \frac{\langle \hat{X}\hat{P} + \hat{P}\hat{X} \rangle}{2} - \langle \hat{X} \rangle \langle \hat{P} \rangle \underset{[\hat{X}, \hat{P}] = i}{=} \frac{\langle 2\hat{X}\hat{P} - i \rangle}{2} - \langle \hat{X} \rangle \langle \hat{P} \rangle = \Delta(\hat{P}\hat{X}), \quad (1.3.19)$$

and the uncertainty constraint is encoded in the determinant of (1.3.18)

$$\det \Gamma[\hat{X}, \hat{P}] \geq \frac{1}{4} |\langle [\hat{X}, \hat{P}] \rangle|^2 = \frac{1}{4}, \quad (1.3.20)$$

and computing in the same fashion the covariance  $\Delta(\hat{X}\hat{P}) = -i \sin \theta \cosh r \sinh r$ , we see that (1.3.20) is minimised for all squeezed states

$$\det \Gamma[\hat{X}, \hat{P}] = \frac{1}{4}, \quad (1.3.21)$$

thus the squeezed states are always minimal uncertainty states of the Schrödinger-Robertson uncertainty relation.

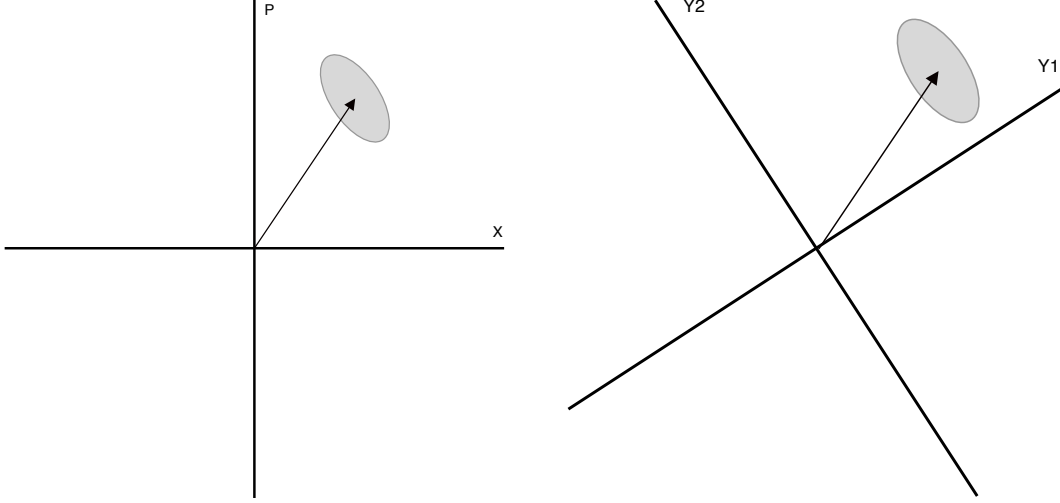
In quantum optics it is useful to work in generalised quadratures. These quadratures generalise the phase space we have been considering so far, the two quadrature components (in our case  $(\hat{X}, \hat{P})$ ) satisfy the canonical commutation relation. Considering the case where we are dealing with an arbitrary squeezing angle, it is useful to work with rotated quadratures using the following transformation [7]

$$\hat{X} + i\hat{P} \rightarrow \hat{Y}_1 + i\hat{Y}_2 = (\hat{X} + i\hat{P})e^{-i\frac{\theta}{2}}, \quad (1.3.22)$$

for squeezing angle  $\theta$ . In the phase space picture, the squeezed states have the interpretation of being displaced, rotated error ellipses of the squeezed vacuum (see figure 1.2). The uncertainties in the new quadratures now saturate their respective Heisenberg uncertainty relation,  $(\Delta\hat{Y}_1)^2(\Delta\hat{Y}_2)^2 = \frac{1}{4}$ . In almost all examples other than the harmonic oscillator, it



is convenient to work with generalised quadratures which need not necessarily refer to the physical position and momentum of the system, see, for example [11].



**Fig. 1.2.** Original quadrature variables on the left and rotated quadrature variables on the right. Notice how the rotated quadratures are aligned with the semi-major and semi-minor axes of the error ellipse.

### 1.3.3. Squeezed coherent state eigenequation

In order to establish an eigenequation for the squeezed coherent states, consider the following,

$$a^- |0\rangle = D(\alpha)S(\xi)a^- S^\dagger(\xi)D^\dagger(\alpha) |\alpha, \xi\rangle. \quad (1.3.23)$$

The transformations here appear in the form  $S(\xi)a^- S^\dagger(\xi)$  (as opposed to  $S^\dagger(\xi)a^- S(\xi)$ ) but this just amounts to the transformation  $r \rightarrow -r$  in the Bogoliubov transformations (1.3.7) and (1.3.8). Using the transformation properties we obtain [12]

$$D(\alpha)S(\xi)a^- S^\dagger(\xi)D^\dagger(\alpha) |\alpha, \xi\rangle = ((a^- - \alpha) \cosh r - (a^+ - \bar{\alpha})e^{i\theta} \sinh r) |\alpha, \xi\rangle = 0, \quad (1.3.24)$$

so

$$(a^- - a^+ e^{i\theta} \tanh r) |\alpha, \xi\rangle = (\alpha - \bar{\alpha} e^{i\theta} \tanh r) |\alpha, \xi\rangle. \quad (1.3.25)$$

We see that it is possible to write this in a simplified form [13]

$$(a^- + \gamma a^+) |z, \gamma\rangle = z |z, \gamma\rangle, \quad (1.3.26)$$

if we make the following identifications

$$\begin{aligned} z &= \alpha - \bar{\alpha} e^{i\theta} \tanh r, \\ \gamma &= -e^{i\theta} \tanh r. \end{aligned} \quad (1.3.27)$$

Note that it is possible to invert these formulae

$$\begin{aligned}\xi &= e^{i\theta} \tanh^{-1}(-e^{-i\theta}\gamma), \\ \alpha &= \frac{1}{1 - \tanh^2 r} (z + \bar{z}e^{i\theta} \tanh r),\end{aligned}\tag{1.3.28}$$

where taking  $r \rightarrow 0$  reproduces the results derived for coherent states. We also see that when taking the eigenequation definition of squeezed states, the squeezing parameter  $|\gamma| < 1$ , because  $|\tanh r| < 1$ .

As we did with the coherent states, we can derive the Fock space expansion for the squeezed states by using the eigenvalue equation (1.3.26). We look for an expansion of the form [13]

$$|z, \gamma\rangle = \frac{1}{\sqrt{\mathcal{N}(z, \gamma)}} \sum_{n=0}^{\infty} \frac{Z(z, \gamma, n)}{\sqrt{n!}} |n\rangle,\tag{1.3.29}$$

substituting into (1.3.26) we obtain the following recursions,

$$\begin{cases} Z(z, \gamma, 0) = 1 & \text{for } n = 0, \\ Z(z, \gamma, n+1) + \gamma n Z(z, \gamma, n-1) = z Z(z, \gamma, n) & \text{for } n > 0. \end{cases}\tag{1.3.30}$$

We see that in the limit of no squeezing,  $\gamma = 0$ , we recover the coherent states  $Z(z, 0, n) = z^n$ . At this point we notice that the recursion is of a similar form to that of the Hermite polynomials,  $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$  [14]. With the argument of the polynomials being complex the recursion is still valid. Indeed if we make the following transformation [13]

$$Z(z, \gamma, n) \rightarrow \left(\frac{\gamma}{2}\right)^{\frac{n}{2}} H\left(\frac{z}{\sqrt{2\gamma}}, n\right),\tag{1.3.31}$$

we find the Fock space expansion

$$|z, \gamma\rangle = \frac{1}{\sqrt{\mathcal{N}(z, \gamma)}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(\frac{\gamma}{2}\right)^{\frac{n}{2}} H_n\left(\frac{z}{\sqrt{2\gamma}}\right) |n\rangle.\tag{1.3.32}$$

To determine the normalisation function  $\mathcal{N}(z, \gamma)$  we require  $\langle z, \gamma | z, \gamma \rangle = 1$ . Computing the inner product,

$$\langle z, \gamma | z, \gamma \rangle = \frac{1}{\mathcal{N}(z, \gamma)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{|\gamma|}{2}\right)^n H_n\left(\frac{z}{\sqrt{2\gamma}}\right) H_n\left(\frac{\bar{z}}{\sqrt{2\bar{\gamma}}}\right),\tag{1.3.33}$$

we make use of Mehler's identity [4],

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\rho}{2}\right)^n H_n(x) H_n(y) = \frac{1}{\sqrt{1 - \rho^2}} \exp\left(-\frac{\rho^2(x^2 + y^2) - 2\rho xy}{1 - \rho^2}\right),\tag{1.3.34}$$

to find

$$\mathcal{N}(z, \gamma) = \frac{1}{\sqrt{1 - |\gamma|^2}} \exp\left(\frac{1}{1 - |\gamma|^2} \left(|z|^2 - |\gamma|^2 \operatorname{Re}\left(\frac{z^2}{\gamma}\right)\right)\right).\tag{1.3.35}$$

It is worth noting that in many places in the literature the normalisation function is written in terms of the variables  $\alpha, r, \theta$  in which case (1.3.35) may be expressed as

$$\mathcal{N}(\alpha, r, \theta) = \cosh r \exp\left(|\alpha|^2 - \tanh r \operatorname{Re}\left(\alpha^2 e^{-i\theta}\right)\right). \quad (1.3.36)$$

Using Mehler's identity once again we can find the explicit position representation of the wavefunction

$$\langle x|z, \gamma\rangle = \left(\frac{1}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{(1-\gamma)\mathcal{N}(z, \gamma)}} \exp\left(-\frac{\left(\gamma + \frac{1}{2}\right)x^2}{1-\gamma}\right) \exp\left(-\frac{z^2}{2(1-\gamma)}\right) \exp\left(\sqrt{2}xz\right), \quad (1.3.37)$$

which satisfies the coherent state limit (1.2.15), up to a constant phase, when  $\gamma \rightarrow 0$ .

### 1.3.4. Completeness relation and time evolution for the squeezed states

For the coherent states we found the resolution of the identity with respect to a uniform measure of  $\frac{1}{\pi}$ . If we were to naively substitute the Fock space expanded squeezed states into

$$\int_{\mathbb{C}} \frac{d^2\alpha}{\pi} \mu(\alpha) |\alpha, \xi\rangle \langle \alpha, \xi|, \quad (1.3.38)$$

and attempt to determine the measure  $\mu(\alpha)$ , the problem can become tricky and requires results on the holomorphic Hermite polynomials and their orthogonality relations [15]. Fortunately there is a simpler way to construct the measure, because we need only integrate over one of the parameters  $\alpha, \xi$ , we can exploit the completeness of the coherent states and the unitarity of the squeezing operator to observe that

$$\int_{\mathbb{C}} \frac{d^2\alpha}{\pi} |\alpha, \xi\rangle \langle \alpha, \xi| = S(\xi) \left( \int_{\mathbb{C}} \frac{d^2\alpha}{\pi} |\alpha\rangle \langle \alpha| \right) S^\dagger(\xi) = S(\xi) \mathbf{1} S^\dagger(\xi) = \mathbf{1}. \quad (1.3.39)$$

Thus the squeezed coherent states are also an overcomplete family of states. In other forms we may integrate over the squeezing parameter  $\gamma$  in (1.3.27) defined on the complex disk  $|\gamma| < 1$  in which case we can use measures for  $SU(1,1)$  coherent states, see for example [16].

Just as the coherent states remain coherent states under unitary time evolution (1.2.17), the squeezed states also remain squeezed states under time evolution. Explicitly the time dependence affects the parameters in the following way:

$$|z, \gamma; t\rangle = e^{-iHt} |z, \gamma\rangle = e^{\frac{-i\omega t}{2}} \left| e^{-i\omega t} z, e^{-2i\omega t} \gamma \right\rangle. \quad (1.3.40)$$

## 1.4. Quasiprobability distributions

It is valuable to study quantum mechanics in phase space. Due to the uncertainty of quantum mechanics and the non-commutativity of its phase space variables, distributions

constructed on the phase space will not always have a proper probabilistic interpretation, for this reason they are known as quasiprobability distributions.

The Weyl Transform of an operator  $\hat{A}$  was originally defined as [17]

$$\tilde{A}(x,p) = \int_{\mathbb{R}} dy e^{-ipy} \left\langle x + \frac{y}{2} \left| \hat{A} \right| x - \frac{y}{2} \right\rangle, \quad (1.4.1)$$

with the operator expressed in the  $x$  basis as  $\langle x' | \hat{A} | x \rangle$ . The critical feature being that this transform converts an operator into a function on phase space.

In the density operator formalism, the principle quantum object is defined by

$$\hat{\rho} = \begin{cases} |\psi\rangle \langle\psi|, & \text{pure,} \\ \sum_i p_i |\psi_i\rangle \langle\psi_i|, \quad \sum_i p_i = 1, & \text{mixed.} \end{cases} \quad (1.4.2)$$

A state may be pure, in which case the density operator is given by the outerproduct of the known wavefunction with itself, otherwise the state is mixed and is given by a statistical ensemble of pure states with probabilities  $p_i$ . We calculate the expectation values of observables by taking the trace with the density matrix of the system we are interested in as

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle = \text{tr}[\hat{\rho} \hat{A}] = \frac{1}{2\pi} \int dx dp \tilde{\rho} \tilde{A}, \quad (1.4.3)$$

leading to the standard definition of the Wigner function

$$W(x,p) = \frac{\tilde{\rho}}{2\pi} = \frac{1}{2\pi} \int dy e^{-ipy} \left\langle x + \frac{y}{2} \left| \hat{\rho} \right| x - \frac{y}{2} \right\rangle, \quad (1.4.4)$$

as being the Weyl transformation of the density operator. Because the state of the quantum system encodes its probabilistic content, the square absolute value of the wavefunction or the density operator, we interpret the Wigner function as a quasiprobability function on the quantum phase space. Indeed for a pure state  $\hat{\rho} = |\psi\rangle \langle\psi|$  the marginals of (1.4.4) read

$$\int_{\mathbb{R}} dp W(x,p) = |\psi(x)|^2, \quad (1.4.5)$$

$$\int_{\mathbb{R}} dx W(x,p) = |\psi(p)|^2. \quad (1.4.6)$$

It is possible to obtain a more general construction of quantum phase space, in work pioneered by Cahill and Glauber [18], the Cahill-Glauber phase space was introduced in the coherent state basis as a symplectic Fourier transform of the characteristic function  $\chi_s = \exp\left(s \frac{|\beta|^2}{2}\right) \text{tr}(\hat{\rho} D(\beta))$  [19] [20],

$$W_s(\alpha) = \frac{1}{\pi^2} \int d^2\beta \exp\left(s \frac{|\beta|^2}{2}\right) \text{tr}(\hat{\rho} D(\beta)) \exp(\bar{\beta}\alpha - \beta\bar{\alpha}). \quad (1.4.7)$$

Here  $s$  refers to the operator ordering,  $s = 1$  corresponds to normal ordering and yields the  $P$  distribution [22, 22]  $s = 0$  corresponds to symmetric (Weyl) ordering and gives the Wigner function [23] and finally  $s = -1$  corresponds to anti-normal ordering and gives the

$Q$  function [24]. The average of products of the ladder operators in ordering  $s$  is then given by

$$\langle [(a^+)^n (a^-)^m]_s \rangle = \frac{1}{\pi} \int d^2\alpha \bar{\alpha}^n \alpha^m W_s(\alpha). \quad (1.4.8)$$

### 1.4.1. Wigner and $P$ distributions in the coherent state basis

Focusing on the Wigner function and  $P$  distribution, we wish to obtain forms which are more suitable for practical calculation. The  $P$  distribution is defined to be diagonal in the coherent states

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi} P(\alpha) |\alpha\rangle \langle\alpha|, \quad (1.4.9)$$

this can be inverted to yield [25]

$$P(\alpha) = e^{|\alpha|^2} \int \frac{d^2\beta}{\pi} \langle -\beta | \hat{\rho} | \beta \rangle e^{|\beta|^2} e^{-\bar{\alpha}\beta + \alpha\bar{\beta}}. \quad (1.4.10)$$

The  $P$  distribution for coherent states is a well defined Dirac delta function, for non-classical states it may be more singular than a delta function [18]. Similarly we can obtain the Wigner function from (1.4.7) [26],

$$W(\alpha) = e^{2|\alpha|^2} \int \frac{d^2\beta}{\pi} \langle -\beta | \hat{\rho} | \beta \rangle e^{2(\alpha\bar{\beta} - \bar{\alpha}\beta)}, \quad (1.4.11)$$

or equivalently

$$W(\alpha) = 2 \int \frac{d^2\beta}{\pi} P(\beta) \exp\{-2|\beta - \alpha|^2\}. \quad (1.4.12)$$

Expressed as a convolution of the  $P$  distribution, this particular form is only useful when we have access to the  $P$  distribution (which may not exist). An advantage that the Wigner function has over the  $P$  distribution is that it is always well defined even if it can become negative for certain states.

### 1.4.2. Wigner functions for coherent, squeezed and cat states

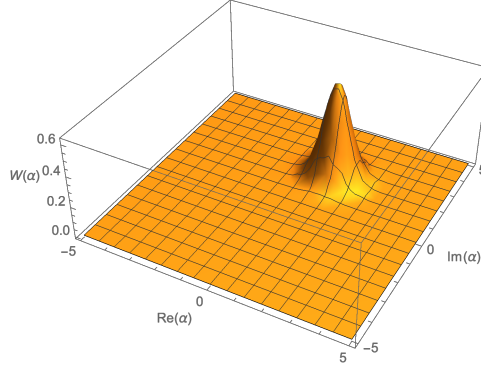
We can read off the  $P$  distribution immediately for a pure coherent state  $z$ , its density operator is described by  $\hat{\rho} = \frac{1}{\pi} |z\rangle \langle z|$ . Note that we include a factor of  $\pi$  in the definition of this density operator to ensure  $\text{tr} \hat{\rho} = 1$ , it arises due to the overcompleteness of the states established in (1.2.10). The  $P$  function is defined to be diagonal in the coherent state basis [22], it is therefore, by construction,

$$P_{|z\rangle}(\alpha) = \delta^2(\alpha - z). \quad (1.4.13)$$

Now we can establish the Wigner function for these states using (1.4.12)

$$W_{|z\rangle}(\alpha) = \frac{2}{\pi} \exp\{-2|\alpha - z|^2\}. \quad (1.4.14)$$

Heuristically this result makes sense. Coherent states were introduced in such a way that their wavefunction kept a gaussian localisation. The Wigner function is related to the square of the wavefunction and so we see that the Wigner function behaves as a Gaussian centred at  $z$ . The  $P$  distribution is naturally constructed in terms of the coherent states of the harmonic oscillator and thus simplifies the computation.



**Fig. 1.3.** The Wigner function for a coherent state of the harmonic oscillator centred at fixed  $z = 1 + 2i$ . We see that it is a two-dimensional Gaussian and is always positive.

The Wigner function is well behaved for all coherent states. When looking for non-classicality we look for negative values of the Wigner function. It is well known that the coherent states of the harmonic oscillator exhibit no non-classicality. The positivity of the Wigner function is evidence of this [27].

Consider now a pure squeezed state  $\hat{\rho} = \frac{1}{\pi} |z, \xi\rangle \langle z, \xi|$ . In calculating its Wigner function it is useful to write down a couple of definitions regarding the displacement and squeezing operators first, namely

$$D(\alpha)D(\beta) = e^{\frac{\alpha\bar{\beta} - \bar{\alpha}\beta}{2}} D(\alpha + \beta) \quad (1.4.15)$$

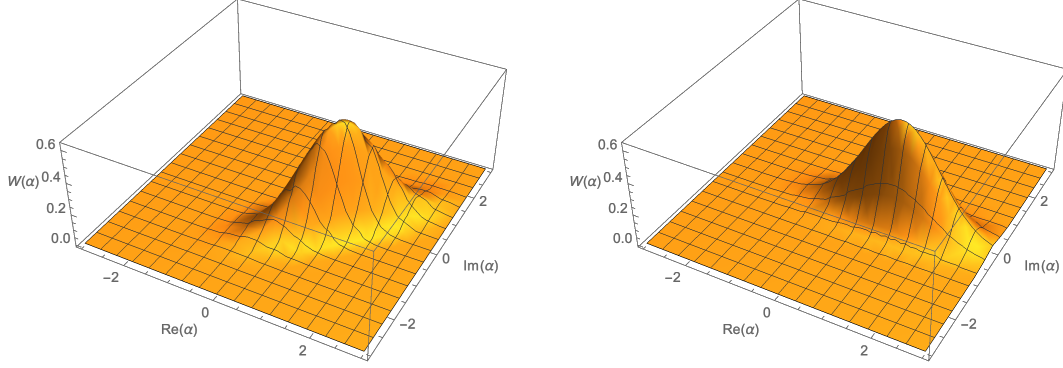
and we may write  $S(\xi)$  as

$$\exp\left(\frac{e^{i\theta}}{2} \tanh r a^{+2}\right) \exp\left(-\log \cosh r \left(a^+ a^- + \frac{1}{2}\right)\right) \exp\left(-\frac{e^{-i\theta}}{2} \tanh r a^{-2}\right), \quad (1.4.16)$$

which comes from a  $\mathfrak{su}(1,1)$  decomposition of the squeezing operator [28]. We also make use of (1.2.8). Using this decomposition and computing (1.4.11) we obtain

$$W(\alpha)_{|z, \xi} = \frac{2}{\pi} \frac{e^{2|\alpha|^2}}{\cosh r} e^{-|z|^2} \exp\left(\tanh r \operatorname{Re}\left[e^{i\theta} \bar{z}^2\right]\right) \exp\left(\frac{-|\sigma|^2 + \tanh r \operatorname{Re}\left[e^{i\theta} \bar{\sigma}^2\right]}{1 - \tanh^2 r}\right), \quad (1.4.17)$$

where  $\sigma = z - 2\alpha - e^{i\theta} \bar{z} \tanh r$ .



**Fig. 1.4.** Wigner functions for squeezed states with  $z = 1 + 0i$ ,  $r = 0.5$  and  $\theta = \frac{\pi}{2}$  (left), and  $z = 1 + 0i$ ,  $r = 0.5$  and  $\theta = 0$  (right). For the case of real squeezing notice that the squeezing is entirely in the  $\text{Im}(\alpha)$  (or  $p$ ) direction, resulting in larger spread in the  $\text{Re}(\alpha)$  (or  $x$ ) direction.

We see in figure 1.4 that the Wigner function for the squeezed states does resemble that of the coherent states only squeezed according to  $\xi$ . We observe further that taking the limit  $r \rightarrow 0$  in (1.4.17), we obtain the result previously derived for the coherent states (1.4.14).

Lastly, consider a cat state given by  $|\text{cat}\rangle \propto |z\rangle + |-z\rangle$  [7]. Up to a normalisation, these states describes a quantum superposition of two macroscopically distinct states. The coherent states are a useful way to describe cat states as the coherent states themselves are the most classical states we can construct in quantum mechanics. The etymology of “cat state” is related to Schrödinger’s cat, and the macroscopic superpositions can be interpreted as alive ( $|z\rangle$ ) and dead ( $|-z\rangle$ ). The normalised state is given by

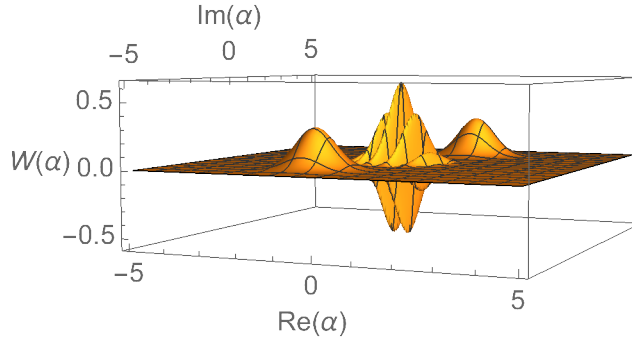
$$|\text{cat}\rangle = \sqrt{\frac{1}{2(1 + e^{-2|z|^2})}}(|z\rangle + |-z\rangle), \quad (1.4.18)$$

and the density operator of the system is given by

$$\hat{\rho} = \frac{1}{\pi} |\text{cat}\rangle \langle \text{cat}| = \frac{1}{\pi} \frac{1}{2(1 + e^{-2|z|^2})} (|z\rangle \langle z| + |-z\rangle \langle -z| + |-z\rangle \langle z| + |z\rangle \langle -z|). \quad (1.4.19)$$

We use this to compute the following Wigner function,

$$W_{|\text{cat}\rangle}(\alpha) = \frac{1}{2(1 + e^{-2|z|^2})} \left( \frac{2}{\pi} e^{-2|\alpha-z|^2} + \frac{2}{\pi} e^{-2|\alpha+z|^2} + \frac{4}{\pi} e^{-2|\alpha|^2} \cosh(2\alpha\bar{z} - 2\bar{\alpha}z) \right). \quad (1.4.20)$$



**Fig. 1.5.** The Wigner function for a cat state with  $z = 1 + 3i$ .

We see in figure 1.5 that the Wigner function has two emergent peaks corresponding to  $|z\rangle$  and  $| -z\rangle$  and it becomes negative around the origin. This is due to the interference term  $W_c(\alpha)$ . Even though the coherent states themselves are classical, the violation of classical probabilistic interpretation arises due to the superposition of quantum states, while the states  $|z\rangle$  and  $| -z\rangle$  might be classical (in the greatest possible sense in quantum theory) their superposition most certainly is not - superposition states do not exist in classical physics and so we should expect odd behaviour.

## 1.5. Non-classicality

Non-classicality is a term used to distinguish features of a quantum system that do not occur in its classical analogue. Non-classicality can manifest itself through several phenomena such as photon anti-bunching [29] and entanglement [30]. It is however not always straightforward to determine whether a system is non-classical or not. We have seen the negativity of the Wigner function of the cat state which clearly has no classical analogue (negative probability is forbidden classically), but the Wigner function of the squeezed states remained always positive. An alternative route to studying non-classicality is by analysing number statistics of the state of interest, in some cases we are able to detect emergent non-classicality not detected by the Wigner function. Another indication of non-classicality can be inferred from how far the uncertainty relations deviate from their minimum, but again, the squeezed states minimise this yet are still non-classical.

We will introduce the Mandel  $Q$  parameter as well as the notion of entanglement in the next sections. Tests of non-classicality are not limited to these however, there exist very recent techniques such as operator ordering sensitivity [31], where the fluctuations in operator ordering parameter  $s$ , introduced with the quasiprobability distributions, serve to classify non-classicality. In order to obtain a complete description of non-classicality in a system, one must analyse several measures of non-classicality.



### 1.5.1. Entanglement

Of central importance to quantum physics is the emergence of entanglement, a phenomenon that does not exist in the classical realm. In heuristic terms it is an instantaneous connection between space-like separated quanta. In terms of the wavefunction, the total wavefunction of the multimode system does not factorise into the product of wavefunctions on each mode. In quantum information theory it is often useful to create entanglement for the purposes of cryptography and teleportation of information [32]. In its most primitive mathematical sense, a system is entangled if it can't be written as a tensor product of its composite states acting on their respective Hilbert spaces. Using two qubit Hilbert spaces as an example,  $\mathcal{H}_a = \{|0\rangle_a, |1\rangle_a\}$  and  $\mathcal{H}_b = \{|0\rangle_b, |1\rangle_b\}$  and their composition  $\mathcal{H}_a \otimes \mathcal{H}_b$ , the following Bell state is entangled

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}} (|0\rangle_a \otimes |1\rangle_b + |1\rangle_a \otimes |0\rangle_b) \neq |\psi\rangle_a \otimes |\phi\rangle_b, \quad \forall |\psi\rangle_a \in \mathcal{H}_a, |\phi\rangle_b \in \mathcal{H}_b, \quad (1.5.1)$$

while the following product state is not entangled

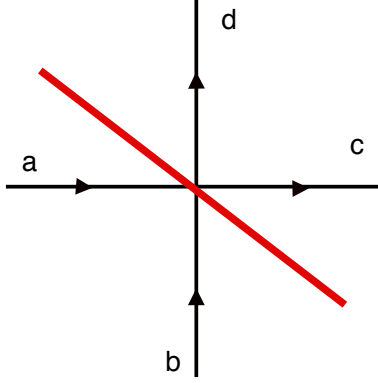
$$\begin{aligned} |\Phi\rangle &= \frac{1}{2} (|0\rangle_a \otimes |1\rangle_b + |1\rangle_a \otimes |0\rangle_b + |0\rangle_a \otimes |0\rangle_b + |1\rangle_a \otimes |1\rangle_b) \\ &= \frac{1}{\sqrt{2}} (|0\rangle_a + |1\rangle_a) \otimes \frac{1}{\sqrt{2}} (|0\rangle_b + |1\rangle_b) \\ &= |\psi\rangle_a \otimes |\phi\rangle_b. \end{aligned} \quad (1.5.2)$$

A useful tool in quantum optics and quantum information processing is the beam splitter which is described mathematically by the following operator [33],

$$B(\theta, \varphi) = \exp\left(\frac{\theta}{2} [a^+ b^- e^{i\varphi} - a^- b^+ e^{-i\varphi}]\right) \quad (1.5.3)$$

with  $a^+$  and  $a^-$  referring to the creation and annihilation operators of one input field, and  $b^+$ ,  $b^-$  referring to the operators on the other input field. The beam splitter acts in the following way

$$\begin{aligned} a^- &\rightarrow Ba^-B^\dagger = a^- \cos \frac{\theta}{2} - b^- e^{i\varphi} \sin \frac{\theta}{2} = c^- \\ b^- &\rightarrow Bb^-B^\dagger = b^- \cos \frac{\theta}{2} + a^- e^{-i\varphi} \sin \frac{\theta}{2} = d^-. \end{aligned} \quad (1.5.4)$$



**Fig. 1.6.** A schematic of a 50/50 beam splitter with input fields  $a, b$  and output fields  $c, d$ .

When two pure input coherent states are used we obtain the following [34]

$$B(\theta, \varphi) (|\alpha\rangle_a \otimes |\beta\rangle_b) = \left| \alpha \cos \frac{\theta}{2} + \beta e^{i\varphi} \sin \frac{\theta}{2} \right\rangle_a \otimes \left| -\alpha e^{-i\varphi} \sin \frac{\theta}{2} + \beta \cos \frac{\theta}{2} \right\rangle_b, \quad (1.5.5)$$

using  $B(|0\rangle_a \otimes |0\rangle_b) = |0\rangle_a \otimes |0\rangle_b$  we see that the action of the beam splitter on two input coherent states does not generate any entanglement. In fact, this can be seen as a manifestation of a larger result: non-classicality is a prerequisite for the generation of entanglement by a beam splitter [33].

Because the displacement operator does not generate any entanglement, we consider the action of the beam splitter on the product of two squeezed vacuum states,

$$B(\theta, \varphi) (|\xi\rangle_a \otimes |\gamma\rangle_b) = BS_a(\xi)S_b(\gamma) (|0\rangle_a \otimes |0\rangle_b). \quad (1.5.6)$$

We need to calculate the action of the beam splitter on the squeezing operator  $BS_a(\xi)B^\dagger$ . The calculation may be simplified by using the Euler angle representation of the  $SU(2)$  decomposition of the beam splitter operator [35]

$$B(\theta, \varphi) = e^{-i\varphi L_3} e^{-i\theta L_2} e^{-i\varphi L_3} = \exp\left(\frac{\theta}{2} [a^+ b^- e^{i\varphi} - a^- b^+ e^{-i\varphi}]\right), \quad (1.5.7)$$

where  $L_i$  satisfy the  $\mathfrak{su}(2)$  algebra  $[L_i, L_j] = i\epsilon_{ijk} L_k$  and are given explicitly by the following representation

$$L_1 = \frac{1}{2}(a^+ b^- + a^- b^+), \quad L_2 = \frac{1}{2i}(a^+ b^- - a^- b^+), \quad L_3 = \frac{1}{2}(a^+ a^- - b^+ b^-). \quad (1.5.8)$$

We simplify further by considering a 50/50 ( $\theta = \frac{\pi}{2}$ ) beam splitter with a relative phase  $\varphi = \frac{l\pi}{2}$  for  $l \in \mathbb{N}$  and real squeezing so that we restrict ourselves to the case where the squeezing is in either quadrature but not an arbitrary combination of both. In this case we compute the following expression [33]

$$B\left(\frac{\pi}{2}, \varphi\right) (|\xi\rangle_a \otimes |\gamma\rangle_b) = BS_a(|\xi|)S_b(|\gamma|) (|0\rangle_a \otimes |0\rangle_b), \quad (1.5.9)$$

which yields

$$S_a \left( -\frac{1}{2}(|\xi| + e^{2i\varphi}|\gamma|) \right) S_b \left( -\frac{1}{2}(e^{-2i\varphi}|\xi| + |\gamma|) \right) S_{ab} \left( -\frac{1}{2}(e^{i\varphi}|\xi| + e^{-i\varphi}|\gamma|) \right) (|0\rangle_a \otimes |0\rangle_b). \quad (1.5.10)$$

This introduces the two-mode squeezing operator  $S_{ab}(\eta) = \exp(\eta a^+ b^+ - \bar{\eta} a^- b^-)$  [36]. The two-mode squeezing operator represents the entanglement generated by the action of the beam splitter on squeezed state inputs.

### 1.5.2. Mandel $Q$ parameter

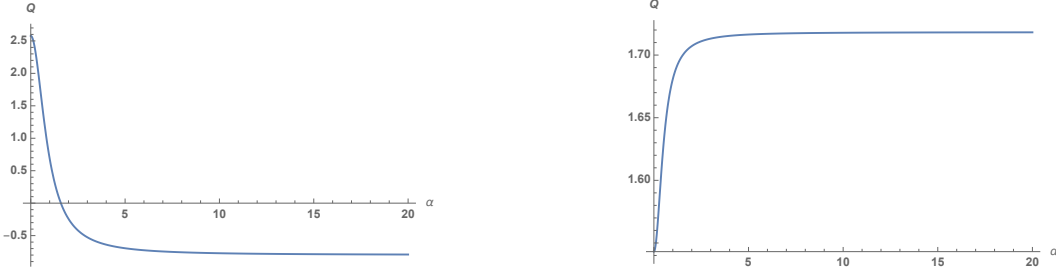
Entanglement can only occur in systems of dimension higher than two, so for systems of dimension one it is not a meaningful measure of non-classicality. When we are dealing with just one Hilbert space we require different notions of non-classical behaviour. This leads us to assess the number statistics of the system. The coherent states are the reference point for classicality in a quantum system. A useful measure of non-classicality is then the Mandel  $Q$  parameter [37], it is defined in the following way

$$Q = \frac{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2}{\langle \hat{N} \rangle} - 1, \quad (1.5.11)$$

it is bounded from below by  $-1$  (corresponding to a number state  $|n\rangle$ ) and it measures the deviation of the photon number statistics from a Poisson distribution, i.e. those of the coherent states. The canonical coherent states satisfy  $Q = 0$ , thus defining our criteria: for  $Q < 0$  we have sub-poissonian statistics and for  $Q > 0$  we have super-poissonian statistics. The case when  $Q > 0$  (photon bunching) we cannot determine whether the state is non-classical, but for  $-1 < Q < 0$  the statistics are sub-poissonian and this corresponds to the classically forbidden phenomenon of photon anti-bunching [38].

For squeezed states we obtain a slightly more general expression, we reuse the result  $\langle \hat{N} \rangle = \langle \alpha, \xi | a^\dagger a | \alpha, \xi \rangle = |\alpha|^2 + \sinh^2 r$  and the transformations in (1.3.9) to compute the second moment of  $\hat{N}$ ,

$$Q_{|\alpha, \xi\rangle}(\alpha, \xi) = -\frac{\sinh r \left[ e^{i\theta} \bar{\alpha}^2 \cosh r + \sinh r \left( 2|\alpha|^2 + \cosh(2r) \right) + \alpha^2 e^{-i\theta} \cosh r \right]}{|\alpha|^2 + \sinh^2 r}. \quad (1.5.12)$$



**Fig. 1.7.** Two plots of  $Q_{|\alpha,\xi\rangle}$ , on the left  $r = -0.8, \theta = 0$ , and on the right  $r = 0.5, \theta = 0$ . We take  $\alpha$  to be real without loss of generality.

We see that for certain values of the squeezing it is possible to obtain a negative Mandel parameter (or sub-poissonian statistics), this is indicative of the purely quantum phenomenon of photon antibunching. Photon antibunching refers to the way in which photons group when received at a detector, they are more equally spaced than coherent states of light (which are totally randomly distributed). In contrast to bunching of thermal light, antibunching cannot be described by a classical wave electromagnetism and requires a quantum particle theory of light [39]. We also verify that in the limiting case of the squeezing  $r \rightarrow 0$  we recover  $Q = 0$  as with the coherent states.

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## Chapter 2

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# Coherent and squeezed states for the two-dimensional harmonic oscillator

*This chapter is based on the following two publications:*

- **J. Moran** and V. Hussin; *Coherent States for the Isotropic and Anisotropic 2D Harmonic Oscillators*, Quantum Rep. 1 2 (2019)
- **J. Moran** and V. Hussin; *A New Method for Constructing Squeezed States for the Isotropic 2D Harmonic Oscillator*, Quantum Theory and Symmetries: Proceedings of the 11th International Symposium (2021)

The main contributions of James Moran for these articles are:

- Performed all calculations in both papers
- Wrote and edited both papers

Véronique Hussin made some initial calculations on finding degenerate contributions; assisted with developing the scope of the papers; assisted with proofreading and editing.

## 2.1. Introduction

In this chapter we extend the formalism of the canonical coherent and squeezed states to the two-dimensional harmonic oscillator. Degeneracy in the spectrum of the Hamiltonian is one of the first problems we encounter when trying to define a new type of coherent states for the two-dimensional oscillator. Klauder described coherent states of the hydrogen atom [1] which preserved many of the usual properties required by coherent state analysis [2]. Fox and Choi proposed the Gaussian Klauder states [3], an alternative method for producing coherent states for more general systems with degenerate spectra. An analysis of the connection between the two definitions was studied in [4].

When labelling energy eigenstates of a two-dimensional system,  $|n,m\rangle$ , there exist several representations of the state space, in this chapter we present a motivation for an  $SU(2)$  representation of the state space. Discussions of alternate state space representations as well as the application to two-dimensional magnetism may be found in [5], [6]. When generalising beyond two dimensions, there exist many more state space representations leading many definitions of coherent states in higher dimensions.

In this chapter we develop a natural approach for constructing coherent states for two-dimensional oscillators in both the isotropic and commensurate anisotropic settings. We aim to minimally extend the standard definitions of the canonical coherent and squeezed states in one dimension, in order to construct a set of coherent and squeezed states that maintain the simplicity of the one-dimensional definitions, but capture the richness afforded to us in two dimensions.

We begin by recounting the energy eigenstates of the two-dimensional isotropic oscillator in section 2.2. Following this, in section 2.3 we address the degeneracy in the energy spectrum by constructing a single-indexed non-degenerate spectrum of the two-dimensional system labelled by  $\nu$ . We define a pair of generalised ladder operator formed from a linear combination of the one-dimensional ladder operators with complex coefficients which preserve the canonical commutation relations, from which the  $\mathfrak{su}(2)$  coherent states follow naturally. After establishing the  $\mathfrak{su}(2)$  coherent states as a Fock basis for the two-dimensional system, in section 2.4 we extend the definitions of the canonical coherent states by replacing the one-dimensional Fock basis with the two-dimensional  $\mathfrak{su}(2)$  coherent states to define what we refer to as Schrödinger-type coherent states. All definitions of the canonical coherent states are systematically recovered for the two-dimensional states in this scheme, including the resolution of the identity in section 2.5.

We turn our attention to the anisotropic oscillator, in section 2.6 we modify the  $\mathfrak{su}(2)$  coherent states according to Chen's hypothesis [7] to produce coherent states for the commensurate anisotropic oscillator, and we discuss the emergent properties and the correspondence



of their wavefunctions to Lissajous figures in configuration space. We complete this analysis by discussing the associated Schrödinger-type coherent states in section 2.7.

Lastly, in section 2.8 we develop squeezed states for the two-dimensional system by replacing the definitions from the one-dimensional squeezed states with definitions involving the generalised ladder operators. We find that a non-trivial coupling is induced between the two modes and the resulting wavefunction represents the most general permissible Gaussian wavefunction. All the while the definitions from the one-dimensional case are preserved.

### 2.1.1. Definitions from one dimension

In order to keep this chapter self contained, we recapitulate the essential forms of the canonical coherent and squeezed states in one dimension discussed in the previous chapter. For complex parameters  $z, \xi$ , the displacement and squeezing operators are

$$D(z) = e^{za^+ - \bar{z}a^-} \quad (2.1.1)$$

and

$$S(\xi) = e^{\frac{1}{2}(\xi a^{+2} - \bar{\xi} a^{-2})} \quad (2.1.2)$$

respectively. The canonical coherent states satisfy the following essential relations:

$$a^- |z\rangle = z |z\rangle; \quad (2.1.3)$$

$$|z\rangle = D(z) |0\rangle; \quad (2.1.4)$$

$$|z\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle; \quad (2.1.5)$$

$$\Delta \hat{Q} \Delta \hat{P} = \frac{1}{2}, \quad \forall |z\rangle, \quad \text{with} \quad \Delta \hat{Q} = \Delta \hat{P}; \quad (2.1.6)$$

$$\int_{\mathbb{C}} \frac{d^2 z}{\pi} |z\rangle \langle z| = \sum_{n=0}^{\infty} |n\rangle \langle n| = \mathbb{I}_{\mathcal{H}}, \quad d^2 z = d \operatorname{Re}(z) d \operatorname{Im}(z). \quad (2.1.7)$$

The squeezed coherent states satisfy the following essential relations:

$$|\alpha, \xi\rangle = D(\alpha) S(\xi) |0\rangle = S(\xi) D(\beta) |0\rangle = |\beta, \xi\rangle; \quad (2.1.8)$$

$$(a^- + \gamma a^+) |z, \gamma\rangle = z |z, \gamma\rangle, \quad (2.1.9)$$

$$|z, \gamma\rangle = \frac{1}{\sqrt{\mathcal{N}(z, \gamma)}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(\frac{\gamma}{2}\right)^{\frac{n}{2}} H_n\left(\frac{z}{\sqrt{2\gamma}}\right) |n\rangle; \quad (2.1.10)$$

$$\int_{\mathbb{C}} \frac{d^2 \beta}{\pi} |\beta, \xi\rangle \langle \beta, \xi| = \sum_{n=0}^{\infty} |n\rangle \langle n| = \mathbb{I}_{\mathcal{H}}, \quad d^2 \beta = d \operatorname{Re}(\beta) d \operatorname{Im}(\beta); \quad (2.1.11)$$

the relationships between the parameters are given by:

$$\begin{aligned}
\xi &= r e^{i\theta}, \\
\beta &= \alpha \cosh r - \bar{\alpha} e^{i\theta} \sinh r, \\
z &= \alpha - \bar{\alpha} e^{i\theta} \tanh r, \\
\gamma &= -e^{i\theta} \tanh r.
\end{aligned} \tag{2.1.12}$$

## 2.2. The two-dimensional harmonic oscillator

For a two-dimensional isotropic oscillator we have the quantum Hamiltonian

$$\hat{H} = \frac{\hat{P}_a^2}{2} + \frac{\hat{P}_b^2}{2} + \frac{\hat{Q}_a^2}{2} + \frac{\hat{Q}_b^2}{2} = \hat{H}_a + \hat{H}_b \tag{2.2.1}$$

where we have once again set  $\hbar = 1$ , the mass  $m = 1$ , and the frequency  $\omega = 1$ . The Hamiltonian (2.2.1) is the sum of two one-dimensional oscillator Hamiltonians labelled by  $a$  and  $b$  respectively.

We solve the time independent Schrödinger equation  $\hat{H}|\Psi\rangle = E|\Psi\rangle$  and obtain the usual energy eigenstates (or Fock states), this time labelled by two numbers,  $n, m$ , such that  $|\Psi\rangle = |n\rangle \otimes |m\rangle \equiv |n, m\rangle$  with eigenvalue

$$E_{n,m} = n + m + 1, \quad n, m \in \mathbb{Z}^{\geq 0}. \tag{2.2.2}$$

These states may be generated by the action of two sets of raising and lowering operators in the following way [8]

$$\begin{aligned}
a^- |n, m\rangle &= \sqrt{n} |n-1, m\rangle, & a^+ |n, m\rangle &= \sqrt{n+1} |n+1, m\rangle; \\
b^- |n, m\rangle &= \sqrt{m} |n, m-1\rangle, & b^+ |n, m\rangle &= \sqrt{m+1} |n, m+1\rangle.
\end{aligned} \tag{2.2.3}$$

The states  $|n, m\rangle$  in configuration space have the following wavefunction

$$\langle x, y | n, m \rangle = \psi_n(x) \psi_m(y) = \frac{1}{\sqrt{2^{n+m} n! m!}} \sqrt{\frac{1}{\pi}} e^{-\frac{x^2}{2} - \frac{y^2}{2}} H_n(x) H_m(y), \tag{2.2.4}$$

where  $\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{1}{\pi}\right)^{\frac{1}{4}} e^{-\frac{x^2}{2}} H_n(x)$  is the wavefunction of the one-dimensional oscillator and  $H_n(x)$  are the Hermite polynomials.

Because the problem is two-dimensional, we have two sets of position and momentum quadratures. For the physical position and momentum operators,  $\hat{Q}_s = \frac{1}{\sqrt{2}}(s^+ + s^-)$ ,  $\hat{P}_s = \frac{1}{\sqrt{2}i}(s^- - s^+)$ , respectively in the  $s$  mode, the states  $|n, m\rangle$  satisfy the following

$$(\Delta \hat{Q}_a)_{|n, m\rangle}^2 = (\Delta \hat{P}_a)_{|n, m\rangle}^2 = \frac{1}{2} + n, \tag{2.2.5}$$

$$(\Delta \hat{Q}_b)_{|n, m\rangle}^2 = (\Delta \hat{P}_b)_{|n, m\rangle}^2 = \frac{1}{2} + m. \tag{2.2.6}$$

Because the Hamiltonian is coordinate separable, the energy eigenstates are therefore product separable and as a result we see no influence of the  $a$  mode on the  $b$  mode quadratures and vice versa. The  $a(b)$  mode satisfies the Heisenberg uncertainty relation which grows linearly in  $n(m)$ .

In what follows we will construct two new ladder operators as linear combinations of the operators in (2.2.3) and proceed to define a single indexed Fock state for the two-dimensional system which yields the  $\mathfrak{su}(2)$  coherent states, as well as extending the definitions in section 2.1.1 to obtain Schrödinger-type coherent states for the two-dimensional system.

### 2.3. $\mathfrak{su}(2)$ coherent states

We extend the definitions of the ladder operators presented in section 2.2 to apply to the 2D oscillator. Introducing a set of states  $\{|\nu\rangle\}$ , and defining a new set of ladder operators through their action on the set,

$$\mathcal{A}^- |\nu\rangle = \sqrt{\nu} |\nu - 1\rangle, \quad \mathcal{A}^+ |\nu\rangle = \sqrt{\nu + 1} |\nu + 1\rangle, \quad \langle \nu | \nu \rangle = 1, \quad \nu = 0, 1, 2, \dots \quad (2.3.1)$$

The states  $|\nu\rangle$  are constructed in such a way as to preserve all the features of the one dimensional Fock states. These states have a linear increasing spectrum  $E_\nu = \nu + 1$  to be compared with (2.2.2).

We may build the states by hand starting with the only non-degenerate state, the ground state,  $|0\rangle \equiv |0,0\rangle$  and we take simple linear combinations of the 1D ladder operators

$$\begin{aligned} \mathcal{A}_{\alpha,\beta}^+ &= \alpha a^+ \otimes \mathbb{I}_b + \mathbb{I}_a \otimes \beta b^+, \\ \mathcal{A}_{\alpha,\beta}^- &= \bar{\alpha} a^- \otimes \mathbb{I}_b + \mathbb{I}_a \otimes \bar{\beta} b^-, \\ [\mathcal{A}_{\alpha,\beta}^-, \mathcal{A}_{\alpha,\beta}^+] &= (|\alpha|^2 + |\beta|^2) \mathbb{I}_a \otimes \mathbb{I}_b \equiv \mathbb{I}, \end{aligned} \quad (2.3.2)$$

for  $\alpha, \beta \in \mathbb{C}$  and  $\mathbb{I}_a \otimes \mathbb{I}_b = \mathbb{I}_b \otimes \mathbb{I}_a \equiv \mathbb{I}$ . We find that the normalisation condition  $|\alpha|^2 + |\beta|^2 = 1$  in (2.3.2) allows us to preserve the canonical commutation relation of the one dimensional system. Constructing the states  $\{|\nu\rangle\}$  explicitly, starting with the ground state, gives us the following table:

$ \nu\rangle$	$ n,m\rangle$
$ 0\rangle$	$ 0,0\rangle$
$ 1\rangle$	$\alpha  1,0\rangle + \beta  0,1\rangle$
$ 2\rangle$	$\alpha^2  2,0\rangle + \sqrt{2}\alpha\beta  1,1\rangle + \beta^2  0,2\rangle$
$\vdots$	$\vdots$
$ \nu\rangle$	$\sum_{n,m}^{n+m=\nu} \alpha^n \beta^m \sqrt{\binom{\nu}{n}}  n,m\rangle$

**Table 2.1.** Construction of the states  $|\nu\rangle$  using the relation  $\mathcal{A}^+ |\nu\rangle = \sqrt{\nu + 1} |\nu + 1\rangle$ .

The states  $|\nu\rangle$  depend on  $\alpha, \beta$  and may be expressed as

$$|\nu\rangle_{\alpha,\beta} = \sum_{n=0}^{\nu} \alpha^n \beta^{\nu-n} \sqrt{\binom{\nu}{n}} |n, \nu-n\rangle. \quad (2.3.3)$$

For clarity we introduce a subscript label to the states,  $|\nu\rangle_{\alpha,\beta}$ , and find that they are precisely the  $\mathfrak{su}(2)$  coherent states in the Schwinger boson representation [2]. This makes sense from our construction, the degeneracy present in the spectrum  $E_{n,m}$  is an  $SU(2)$  degeneracy, and so we created states which averaged out the degenerate contributions to a given energy  $E_\nu$ .

The states  $|\nu\rangle_{\alpha,\beta}$  have the following orthogonality relations

$$\langle \mu |_{\gamma,\delta} |\nu\rangle_{\alpha,\beta} = (\bar{\gamma}\alpha + \bar{\delta}\beta)^\nu \delta_{\mu,\nu}, \quad (2.3.4)$$

which reduces to a more familiar relation, by construction, when  $\gamma = \alpha$  and  $\delta = \beta$ ,

$$\langle \mu |_{\alpha,\beta} |\nu\rangle_{\alpha,\beta} = \delta_{\mu,\nu}, \quad (2.3.5)$$

using the normalisation condition  $|\alpha|^2 + |\beta|^2 = 1$ .

In order to develop the exact position representation of the wavefunction,  $\langle x, y | \nu \rangle_{\alpha,\beta}$ , we make use of the following identity for a finite sum of Hermite polynomials,

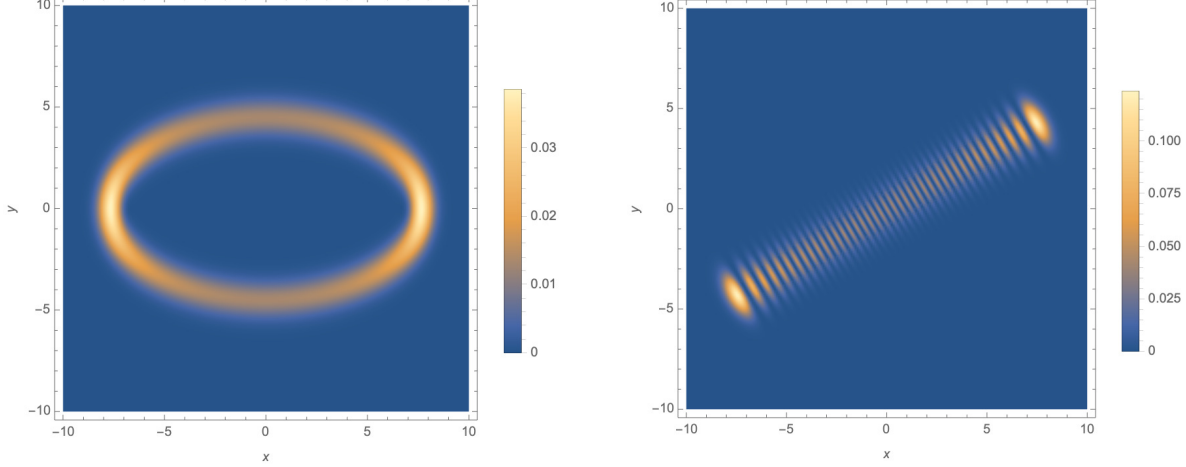
$$(1 + \gamma^2)^{\frac{\nu}{2}} H_\nu \left( \frac{\gamma x + y}{\sqrt{1 + \gamma^2}} \right) = \sum_{n=0}^{\nu} \binom{\nu}{n} \gamma^n H_n(x) H_{\nu-n}(y), \quad (2.3.6)$$

after which we find,

$$\langle x, y | \nu \rangle_{\alpha,\beta} = \frac{1}{\sqrt{\pi 2^\nu \nu!}} e^{-\frac{x^2}{2} - \frac{y^2}{2}} (\alpha^2 + \beta^2)^{\frac{\nu}{2}} H_\nu \left( \frac{\alpha x + \beta y}{\sqrt{\alpha^2 + \beta^2}} \right). \quad (2.3.7)$$

The wavefunction of the  $\mathfrak{su}(2)$  coherent states is expressible in terms of a single Hermite polynomial multiplied by a two dimensional Gaussian. Additionally, when  $\alpha, \beta$  are completely out of phase and their magnitudes are equal such that  $\alpha = i\frac{1}{\sqrt{2}}$  and  $\beta = \frac{1}{\sqrt{2}}$ , all but the highest power term vanish in the Hermite polynomials,

$$\langle x, y | \nu \rangle_{i\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}} = \frac{\sqrt{2^\nu}}{\sqrt{\pi \nu!}} e^{-\frac{x^2}{2} - \frac{y^2}{2}} i^\nu \left( \frac{1}{\sqrt{2}} x - i \frac{1}{\sqrt{2}} y \right)^\nu. \quad (2.3.8)$$



**Fig. 2.1.** Density plots of  $|\langle x,y|40\rangle_{\alpha,\beta}|^2$  for  $\alpha = \frac{\sqrt{3}}{2}e^{i\frac{\pi}{2}}, \beta = \frac{1}{2}$  (left) and  $\alpha = \frac{\sqrt{3}}{2}, \beta = \frac{1}{2}$  (right).

In figure 2.1 there are two plots of the probability density functions  $|\langle x,y|\nu\rangle_{\alpha,\beta}|^2$ . In the picture on the left there is an imaginary component to the relative phase between  $\alpha$  and  $\beta$ , this causes the emergence of an elliptical shape to the density. Conversely, on the right, when  $\alpha$  and  $\beta$  are exactly in phase (or out of phase) the probability density is concentrated on a line, the angle of the line to the  $x$  axis is determined by  $\tan\theta = \frac{|\beta|}{|\alpha|}$ . The probability densities of the quantum  $\mathfrak{su}(2)$  coherent states mimic the spatial distribution of a classical 2D isotropic oscillator, that is, ellipses in the  $(x,y)$  plane. Coherent states may be defined by their correspondence to classical physics, so in this sense the states  $|\nu\rangle_{\alpha,\beta}$  exist in the spirit of coherent states and their generalisations.

The states  $|\nu\rangle_{\alpha,\beta}$  have the following variances for the physical position and momentum operators:

$$(\Delta\hat{Q}_a)_{|\nu\rangle_{\alpha,\beta}}^2 = (\Delta\hat{P}_a)_{|\nu\rangle_{\alpha,\beta}}^2 = \frac{1}{2} + |\alpha|^2\nu, \quad (2.3.9)$$

$$(\Delta\hat{Q}_b)_{|\nu\rangle_{\alpha,\beta}}^2 = (\Delta\hat{P}_b)_{|\nu\rangle_{\alpha,\beta}}^2 = \frac{1}{2} + |\beta|^2\nu. \quad (2.3.10)$$

The results are essentially the same as those in (2.2.5) and (2.2.6), but they are tuned by the continuous parameters  $\alpha, \beta$  introduced in (2.3.2) allowing additional freedom in increasing uncertainty in one set of quadrature operators at the expense of reducing it in the other set.

## 2.4. Schrödinger-type two-dimensional coherent states

Though the basis states,  $|\nu\rangle_{\alpha,\beta}$ , developed thus far are themselves coherent states, we can go further and define what we will refer to as Schrödinger-type coherent states. Just as the states  $|\nu\rangle_{\alpha,\beta}$  were defined to preserve the properties of the one dimensional energy

eigenstates, the Schrödinger-type coherent states are defined so as to preserve the form of the one dimensional canonical coherent states.

Beginning with the Fock expansion, the Schrödinger-type coherent states,  $|\Psi\rangle_{\alpha,\beta}$  are:

$$|\Psi\rangle_{\alpha,\beta} = e^{-\frac{|\Psi|^2}{2}} \sum_{\nu=0}^{\infty} \frac{\Psi^\nu}{\sqrt{\nu!}} |\nu\rangle_{\alpha,\beta}. \quad (2.4.1)$$

These states have the following inner product relation

$$\langle \Psi' |_{\gamma,\delta} |\Psi\rangle_{\alpha,\beta} = e^{-\frac{|\Psi'|^2 + |\Psi|^2}{2}} e^{\bar{\Psi}'\Psi(\bar{\gamma}\alpha + \bar{\delta}\beta)}. \quad (2.4.2)$$

Because these states are constructed in direct analogy with the one dimensional definitions, we also find that they are eigenstates of the generalised lowering operator  $\mathcal{A}_{\alpha,\beta}^-$ ,

$$\mathcal{A}_{\alpha,\beta}^- |\Psi\rangle_{\alpha,\beta} = \Psi |\Psi\rangle_{\alpha,\beta}. \quad (2.4.3)$$

The expansion in (2.4.1) also implies the existence of a displacement operator as in the one dimensional case

$$\begin{aligned} |\Psi\rangle_{\alpha,\beta} &= e^{-\frac{|\Psi|^2}{2}} \sum_{\nu=0}^{\infty} \frac{\Psi^\nu}{\sqrt{\nu!}} |\nu\rangle_{\alpha,\beta} \\ &= e^{-\frac{|\Psi|^2}{2}} \sum_{\nu=0}^{\infty} \frac{\Psi^\nu}{\sqrt{\nu!}} \frac{\mathcal{A}_{\alpha,\beta}^{+\nu}}{\sqrt{\nu!}} |0\rangle_{\alpha,\beta} \\ &= e^{-\frac{|\Psi|^2}{2} + \Psi \mathcal{A}_{\alpha,\beta}^+} |0\rangle_{\alpha,\beta}, \end{aligned} \quad (2.4.4)$$

from which we may reconstruct a unitary operator

$$\mathcal{D}(\Psi) = \exp\left(\Psi \mathcal{A}_{\alpha,\beta}^+ - \bar{\Psi} \mathcal{A}_{\alpha,\beta}^-\right) \quad (2.4.5)$$

and thus recover our last definition of the canonical coherent states

$$|\Psi\rangle_{\alpha,\beta} = \mathcal{D}(\Psi) |0\rangle_{\alpha,\beta}. \quad (2.4.6)$$

A Baker-Campbell-Hausdorff identity along with the annihilation of the two dimensional vacuum,  $\mathcal{A}_{\alpha,\beta}^- |0\rangle_{\alpha,\beta} = 0$  allow us to rewrite  $\mathcal{D}(\Psi)$  in the following way

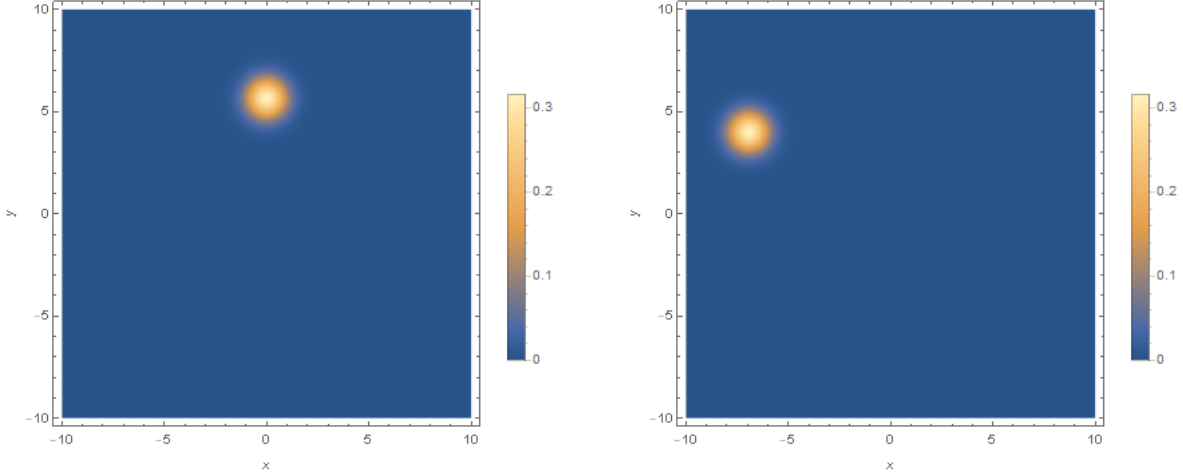
$$\begin{aligned} \mathcal{D}(\Psi) &= e^{\Psi \mathcal{A}_{\alpha,\beta}^+ - \bar{\Psi} \mathcal{A}_{\alpha,\beta}^-} \\ &= e^{(\alpha \Psi a^+ - \bar{\alpha} \bar{\Psi} a^-) + (\beta \Psi b^+ - \bar{\beta} \bar{\Psi} b^-)} \\ &= D_a(\alpha \Psi) D_b(\beta \Psi), \end{aligned} \quad (2.4.7)$$

thus splitting  $\mathcal{D}(\Psi)$  into the product of two one dimensional displacement operators acting on  $a$  and  $b$  independently. Consequently, the Schrödinger-type coherent states then factorise into two uncoupled one dimensional canonical coherent states

$$|\Psi\rangle_{\alpha,\beta} = |\alpha \Psi\rangle_a \otimes |\beta \Psi\rangle_b. \quad (2.4.8)$$

In the position representation, we find that the wavefunction factorises into the product of two equal width one dimensional Gaussians. Explicitly, using the position representation in one dimension (1.2.15), we get the position representation of the two-dimensional Schrödinger-type coherent states

$$\langle x,y|\Psi\rangle_{\alpha,\beta} = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{1}{2}[(x - \sqrt{2} \operatorname{Re}(\alpha\Psi))^2 + (y - \sqrt{2} \operatorname{Re}(\beta\Psi))^2]\right) e^{(i\sqrt{2}[x \operatorname{Im}(\alpha\Psi) + y \operatorname{Im}(\beta\Psi)])}. \quad (2.4.9)$$



**Fig. 2.2.** Density plots of  $|\langle x,y|\Psi\rangle_{\alpha,\beta}|^2$  for  $\Psi = 8, \alpha = \frac{\sqrt{3}}{2}e^{i\frac{\pi}{2}}, \beta = \frac{1}{2}$  (left) and  $\Psi = 8e^{i\frac{\pi}{4}}, \alpha = \frac{\sqrt{3}}{2}e^{i\frac{\pi}{2}}, \beta = \frac{1}{2}$  (right).

In figure 2.2 we see the probability densities  $|\langle x,y|\Psi\rangle_{\alpha,\beta}|^2$  are Gaussian in the  $(x,y)$  plane. The peak of the probability density is located at the coordinates  $(x,y) = (\sqrt{2} \operatorname{Re}(\alpha\Psi), \sqrt{2} \operatorname{Re}(\beta\Psi))$ .

The Schrödinger-type coherent states represent an infinite sum of the elliptical, or  $\mathfrak{su}(2)$  coherent states established previously, with a Poissonian probability of being in a state  $|\mu\rangle_{\alpha,\beta}$  given by

$$|\langle \mu |_{\alpha,\beta} | \Psi \rangle_{\alpha,\beta}|^2 = e^{-|\Psi|^2} \frac{|\Psi|^{2\mu}}{\mu!}, \quad (2.4.10)$$

analogously to the one dimensional canonical coherent states,  $|\langle n | z \rangle|^2 = e^{-|z|^2} \frac{|z|^{2n}}{n!}$ . Moreover, we recover the saturation of the Heisenberg uncertainty relation in the  $a$  and  $b$  modes independently, this follows from the factorisation of the states,

$$(\Delta \hat{Q}_a)_{|\Psi\rangle_{\alpha,\beta}} (\Delta \hat{P}_a)_{|\Psi\rangle_{\alpha,\beta}} = \frac{1}{2}, \quad (\Delta \hat{Q}_a)_{|\Psi\rangle_{\alpha,\beta}} = (\Delta \hat{P}_a)_{|\Psi\rangle_{\alpha,\beta}}, \quad (2.4.11)$$

$$(\Delta \hat{Q}_b)_{|\Psi\rangle_{\alpha,\beta}} (\Delta \hat{P}_b)_{|\Psi\rangle_{\alpha,\beta}} = \frac{1}{2}, \quad (\Delta \hat{Q}_b)_{|\Psi\rangle_{\alpha,\beta}} = (\Delta \hat{P}_b)_{|\Psi\rangle_{\alpha,\beta}}. \quad (2.4.12)$$

## 2.5. Resolution of the identity

The  $\mathfrak{su}(2)$  coherent states resolve the identity in the following way

$$\frac{\nu+1}{\pi^2} \int_{S^3} d^2\alpha d^2\beta \delta(|\alpha|^2 + |\beta|^2 - 1) |\nu\rangle_{\alpha,\beta} \langle\nu|_{\alpha,\beta} = \mathbb{I}_\nu, \quad (2.5.1)$$

where  $S^3$  refers to integration over the 3-sphere parametrised by  $|\alpha|^2 + |\beta|^2 = 1$ . The operator  $\mathbb{I}_\nu$  is the identity operator for the states  $\{|\nu\rangle_{\alpha,\beta}\}$ , in other words, the sum of projectors onto states with total occupation number  $n+m = \nu$ , for example  $\mathbb{I}_2 = |2,0\rangle\langle 2,0| + |1,1\rangle\langle 1,1| + |0,2\rangle\langle 0,2|$ .

We retrieve the identity operator for the entire Hilbert space by summing over  $\nu$

$$\sum_{\nu=0}^{\infty} \left( \frac{\nu+1}{\pi^2} \int_{S^3} d^2\alpha d^2\beta \delta(|\alpha|^2 + |\beta|^2 - 1) |\nu\rangle_{\alpha,\beta} \langle\nu|_{\alpha,\beta} \right) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |n,m\rangle \langle n,m| = \mathbb{I}_{\mathcal{H}}. \quad (2.5.2)$$

The resolution of the identity allows us to express any other state in the Hilbert space in terms of the states  $\{|\nu\rangle_{\alpha,\beta}\}$ . The energy eigenstates are then given by

$$|n,m\rangle = \sum_{\nu=0}^{\infty} \left\{ \frac{\nu+1}{\pi^2} \int_{S^3} d^2\alpha d^2\beta \delta(|\alpha|^2 + |\beta|^2 - 1) \sqrt{\binom{\nu}{n}} \bar{\alpha}^n \bar{\beta}^m |\nu\rangle_{\alpha,\beta} \right\}. \quad (2.5.3)$$

The Schrödinger-type two dimensional coherent states resolve the identity with a slightly modified measure. It is insufficient to combine the measures used for the one dimensional coherent states and  $\mathfrak{su}(2)$  coherent states in equations (2.1.11) and (2.5.1), doing so we would obtain

$$\frac{1}{\pi^2} \int_{S^3} d^2\alpha d^2\beta \delta(|\alpha|^2 + |\beta|^2 - 1) \int_{\mathbb{C}} \frac{d^2\Psi}{\pi} |\Psi\rangle_{\alpha,\beta} \langle\Psi|_{\alpha,\beta} = \sum_{\nu=0}^{\infty} \frac{\mathbb{I}_\nu}{\nu+1} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|n,m\rangle \langle n,m|}{n+m+1} \not\propto \mathbb{I}_{\mathcal{H}}. \quad (2.5.4)$$

However, the identity operator for the full Hilbert space can be retrieved by the inclusion of the positive term,  $|\Psi|^2$ , into the measure as follows

$$\frac{1}{\pi^2} \int_{S^3} d^2\alpha d^2\beta \delta(|\alpha|^2 + |\beta|^2 - 1) \int_{\mathbb{C}} \frac{d^2\Psi}{\pi} |\Psi|^2 |\Psi\rangle_{\alpha,\beta} \langle\Psi|_{\alpha,\beta} = \mathbb{I}_{\mathcal{H}}, \quad (2.5.5)$$

thus the Schrödinger-type coherent states for the two dimensional oscillator represent an overcomplete basis for the full Hilbert space of the two dimensional oscillator. The resolution of the identity means the states could have some application in coherent state quantisation in two dimensions [2].

## 2.6. Commensurate anisotropic $\mathfrak{su}(2)$ coherent states

In order to generalise coherent states to the commensurable frequency anisotropic oscillator, we introduce two integers  $p, q$  in the Hamiltonian as



$$\begin{aligned}\hat{H} &= \frac{\hat{P}_a^2}{2} + \frac{\hat{P}_b^2}{2} + \omega_a^2 \frac{\hat{Q}_a^2}{2} + \omega_b^2 \frac{\hat{Q}_b^2}{2} \\ &= \frac{\hat{P}_a^2}{2} + \frac{\hat{P}_b^2}{2} + p^2 \omega^2 \frac{\hat{Q}_a^2}{2} + q^2 \omega^2 \frac{\hat{Q}_b^2}{2},\end{aligned}\tag{2.6.1}$$

where the frequencies are related by  $\omega_a = p\omega$  and  $\omega_b = q\omega$ , and the ratio,  $\frac{p}{q}$ , represents the ratio of the two frequencies  $\frac{\omega_a}{\omega_b}$ . Without loss of generality, we will set the common frequency  $\omega = 1$  in what follows and choose  $p, q$  such that they are relative prime integers. In terms of ladder operators we may represent the Hamiltonian (2.6.1) as

$$\hat{H} = pa^+a^- + qb^+b^- + \frac{p+q}{2}\mathbb{I},\tag{2.6.2}$$

where

$$\begin{aligned}a^- &= \sqrt{\frac{p}{2}} \left( \hat{Q}_a + \frac{i}{p} \hat{P}_a \right), & a^+ &= \sqrt{\frac{p}{2}} \left( \hat{Q}_a - \frac{i}{p} \hat{P}_a \right), \\ b^- &= \sqrt{\frac{q}{2}} \left( \hat{Q}_b + \frac{i}{q} \hat{P}_b \right), & b^+ &= \sqrt{\frac{q}{2}} \left( \hat{Q}_b - \frac{i}{q} \hat{P}_b \right),\end{aligned}\tag{2.6.3}$$

and

$$\begin{aligned}[a^-, a^+] &= \mathbb{1}, & [b^-, b^+] &= \mathbb{1}, \\ [\hat{H}, a^\pm] &= \pm pa^\pm, & [\hat{H}, b^\pm] &= \pm qb^\pm.\end{aligned}\tag{2.6.4}$$

A hypothesis made by Chen [9] says that the integers  $p, q$  enter the quantum  $\mathfrak{su}(2)$  coherent states in the following way,

$$|\nu\rangle_{\alpha, \beta}^{p, q} = \sum_{n=0}^{\nu} \alpha^n \beta^{\nu-n} \sqrt{\binom{\nu}{n}} |pn, q(\nu-n)\rangle,\tag{2.6.5}$$

where the states are normalised in the usual way  $\langle \nu |_{\alpha, \beta}^{p, q} | \nu \rangle_{\alpha, \beta}^{p, q} = 1$  with the condition  $|\alpha|^2 + |\beta|^2 = 1$ .

Chen's states (2.6.5) suitably address the extension of our construction to the commensurate anisotropic oscillator, the states themselves resemble the  $\mathfrak{su}(2)$  coherent states but the anisotropy is accounted for in the Fock expansion. Energy eigenstates of (2.6.1) have eigenvalues  $E_{n, m} = p\left(n + \frac{1}{2}\right) + q\left(m + \frac{1}{2}\right)$  which do not have the same degenerate structure as in the isotropic case where  $p = q = 1$ , instead we are considering a superposition of states  $|pn, qm\rangle$  such that  $n + m = \nu$  for given  $p, q$ .

There are however some immediate limitations. Firstly, because the states (2.6.5) do not include all states in the full Hilbert space of the two dimensional oscillator, they will not permit a meaningful resolution of the identity. Secondly, the states are not presented with a generalised ladder operator to act on the two dimensional basis. Ladder operators may be retroactively fit to generate the states (2.6.5), but it requires the use of inverse square

roots of number operators and as such the algebra is poorly defined outside of the specific representation.

Proceeding with some analysis, the energy eigenvalues of the states  $|\nu\rangle_{\alpha,\beta}^{p,q}$  may be calculated from

$$\begin{aligned}
\langle \nu |_{\alpha,\beta}^{p,q} p a^+ a^- + q b^+ b^- + \frac{p+q}{2} \mathbb{I} | \nu \rangle_{\alpha,\beta}^{p,q} &= (p^2 - q^2) \left( \sum_{n=0}^{\nu} |\alpha|^{2n} |\beta|^{2(\nu-n)} \binom{\nu}{n} \right) + q^2 \nu + \frac{p+q}{2} \\
&= (p^2 - q^2) |\alpha|^2 \nu + q^2 \nu + \frac{p+q}{2} \\
&= p^2 |\alpha|^2 \nu + q^2 |\beta|^2 \nu + \frac{p+q}{2} \\
&\equiv E_{\nu}^{p,q},
\end{aligned} \tag{2.6.6}$$

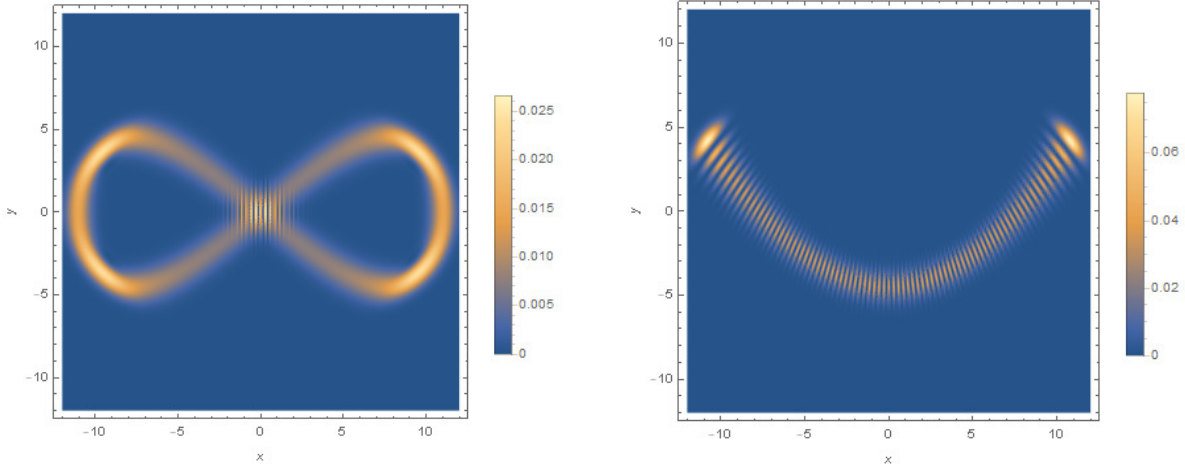
which was computed by observing that

$$\frac{\partial}{\partial |\alpha|^2} \sum_{n=0}^{\nu} |\alpha|^{2n} |\beta|^{2(\nu-n)} \binom{\nu}{n} = \frac{\partial}{\partial |\alpha|^2} (|\alpha|^2 + |\beta|^2)^{\nu}, \tag{2.6.7}$$

yielding

$$\sum_{n=0}^{\nu} |\alpha|^{2(n-1)} |\beta|^{2(\nu-n)} \binom{\nu}{n} n = \nu (|\alpha|^2 + |\beta|^2)^{\nu-1} = \nu. \tag{2.6.8}$$

The states  $|\nu\rangle_{\alpha,\beta}^{p,q}$  correspond to Lissajous-type probability densities in configuration space, a feature present in the classical spatial distribution of an anisotropic oscillator with commensurate frequencies [7] [10].



**Fig. 2.3.** Density plots of  $|\langle x, y | 40 \rangle_{\alpha,\beta}^{2,1}|^2$  for  $\alpha = \frac{\sqrt{3}}{2} e^{i\frac{\pi}{2}}, \beta = \frac{1}{2}$  (left) and  $\alpha = \frac{\sqrt{3}}{2}, \beta = \frac{1}{2}$  (right).

In figure 2.3 we have two types of Lissajous figure associated with the 2:1 oscillator. On the left is a closed figure-of-eight, and on the right an open figure. The frequency ratio  $\frac{p}{q}$  determines the type of Lissajous figure, and the relative phase between  $\alpha$  and  $\beta$  deforms

the figures such that when they are completely in (or out) of phase, the figure is open, and when there is an imaginary component to the relative phase, the figure is closed. Tables of Lissajous figures corresponding to different choices of  $p$  and  $q$  can be found in [11]. The correspondence of the quantum probability densities to the classical spatial distribution of a two-dimensional commensurate anisotropic oscillator confirms Chen's definition as a suitable description of coherent states.

The commensurate anisotropic  $\mathfrak{su}(2)$  coherent states have slightly modified variances compared with the isotropic case. Inverting the relations in (2.6.3) we obtain

$$(\Delta \hat{Q}_a)_{|\nu\rangle_{\alpha,\beta}}^{p,q} = \frac{1}{p} \left( \frac{1}{2} + |\alpha|^2 p \nu \right), \quad (\Delta \hat{P}_a)_{|\nu\rangle_{\alpha,\beta}}^{p,q} = p \left( \frac{1}{2} + |\alpha|^2 p \nu \right), \quad (2.6.9)$$

$$(\Delta \hat{Q}_b)_{|\nu\rangle_{\alpha,\beta}}^{p,q} = \frac{1}{q} \left( \frac{1}{2} + |\beta|^2 q \nu \right), \quad (\Delta \hat{P}_b)_{|\nu\rangle_{\alpha,\beta}}^{p,q} = q \left( \frac{1}{2} + |\beta|^2 q \nu \right), \quad (2.6.10)$$

thus we find the the parameter  $p$  serves to squeeze the uncertainty in the position quadrature,  $\hat{Q}_a$ , while increasing the uncertainty in the momentum quadrature,  $\hat{P}_a$ . The same is true in the  $b$  mode except the scaling is governed by  $q$ .

## 2.7. Commensurate anisotropic two-dimensional Schrödinger-type coherent states

As with the isotropic case, we can build two dimensional Schrödinger-type coherent states using the commensurate anisotropic  $\mathfrak{su}(2)$  coherent states as an expansion basis. Defining the states  $|\Psi\rangle_{\alpha,\beta}^{p,q}$

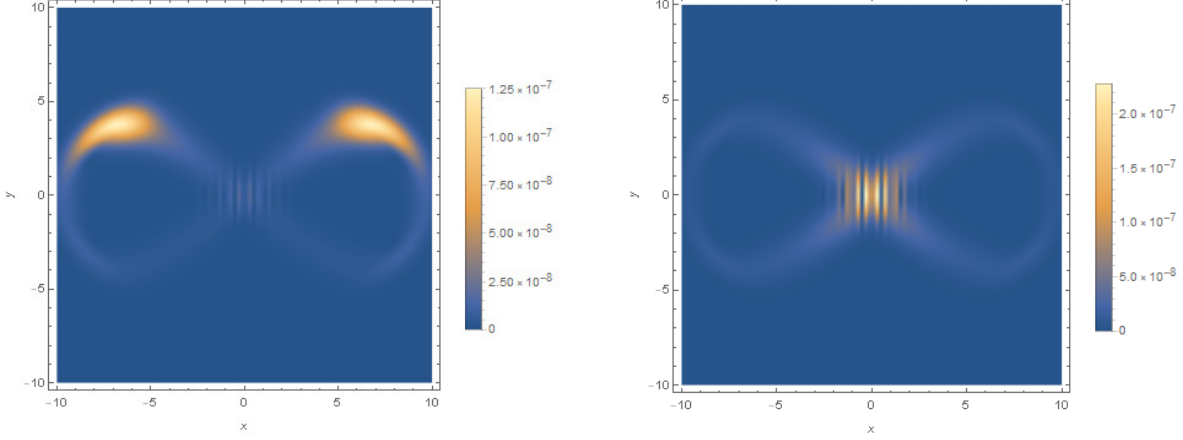
$$|\Psi\rangle_{\alpha,\beta}^{p,q} = e^{-\frac{|\Psi|^2}{2}} \sum_{\nu=0}^{\infty} \frac{\Psi^\nu}{\sqrt{\nu!}} |\nu\rangle_{\alpha,\beta}^{p,q}. \quad (2.7.1)$$

These Schrödinger-type coherent states are normalised  $\langle \Psi |_{\alpha,\beta}^{p,q} | \Psi \rangle_{\alpha,\beta}^{p,q} = 1$  with inner product

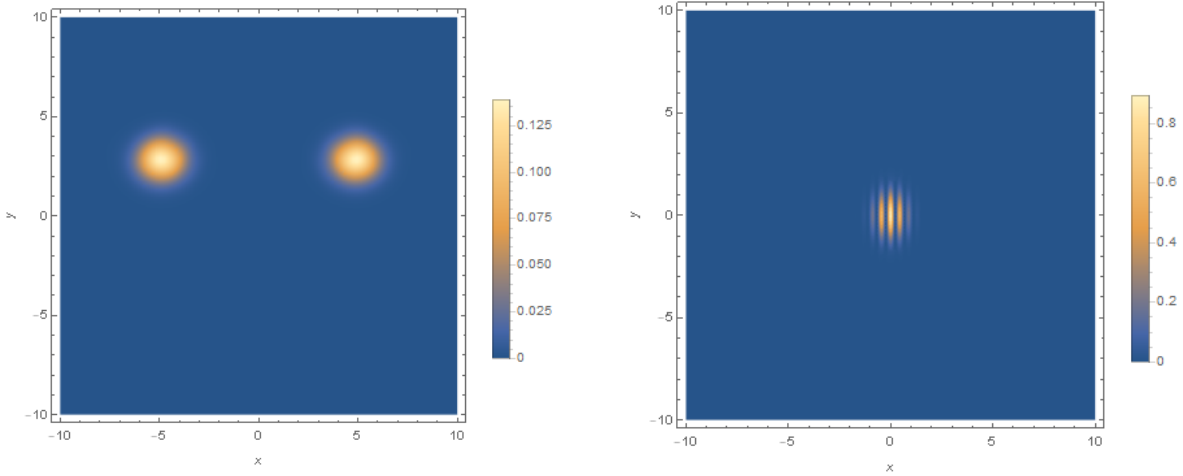
$$\langle \Psi' |_{\alpha,\beta}^{p,q} | \Psi \rangle_{\alpha,\beta}^{p,q} = e^{-\frac{|\Psi'|^2 + |\Psi|^2}{2}} e^{\bar{\Psi}' \Psi}. \quad (2.7.2)$$

Similarly to the isotropic case, (2.7.1) may be interpreted as the infinite sum of commensurate anisotropic  $\mathfrak{su}(2)$  coherent states, determined by  $p, q$ , with probability of being in a given coherent state (or Lissajous figure),  $|\mu\rangle_{\alpha,\beta}^{p,q}$ , given by

$$\left| \langle \mu |_{\alpha,\beta}^{p,q} | \Psi \rangle_{\alpha,\beta}^{p,q} \right|^2 = e^{-|\Psi|^2} \frac{|\Psi|^{2\mu}}{\mu!}. \quad (2.7.3)$$



**Fig. 2.4.** Density plots of  $|\langle x,y|\Psi\rangle_{\alpha,\beta}^{p,q}|^2$  for  $\Psi = 8, \alpha = \frac{\sqrt{3}}{2}e^{i\frac{\pi}{2}}, \beta = \frac{1}{2}$  (left) and  $\Psi = 8e^{i\frac{\pi}{2}}, \alpha = \frac{\sqrt{3}}{2}e^{i\frac{\pi}{2}}, \beta = \frac{1}{2}$  (right), with  $p = 2, q = 1$  in both instances. 30 terms are kept in the expansion of  $|\Psi\rangle_{\alpha,\beta}^{p,q}$ . We see the emergence of localisation onto parts of the  $\mathfrak{su}(2)$  coherent state used in the expansion.



**Fig. 2.5.** Density plots of  $|\langle x,y|\Psi\rangle_{\alpha,\beta}^{p,q}|^2$  for  $\Psi = 4, \alpha = \frac{\sqrt{3}}{2}e^{i\frac{\pi}{2}}, \beta = \frac{1}{2}$  (left) and  $\Psi = 4e^{i\frac{\pi}{2}}, \alpha = \frac{\sqrt{3}}{2}e^{i\frac{\pi}{2}}, \beta = \frac{1}{2}$  (right), with  $p = 2, q = 1$  in both instances. 30 terms are kept in the expansion of  $|\Psi\rangle_{\alpha,\beta}^{p,q}$ .

Figures 2.4 and 2.5 show four density plots for the probability density of the commensurate anisotropic two dimensional Schrödinger-type coherent states. We have used finitely many terms in the expansion of  $|\Psi\rangle_{\alpha,\beta}^{p,q}$  and so we can see the emergence of localisation, but the pictured graphs are not properly normalised as a result. An interesting difference between the isotropic and commensurate anisotropic Schrödinger-type coherent states is that for certain values of  $(\alpha, \beta, \Psi, p, q)$  the probability density can localise onto two or more separate points in space. This can be seen clearly in the left-most image in figure 2.5, unlike the

isotropic Schrödinger-type states which were seen to have Gaussian probability distributions in configuration space with a single maximum.

In the right-most density plot in figure 2.5 there is good localisation, but the probability distribution is fringed around the origin, this behaviour differs from the isotropic counterparts. The graphs in figure 2.4 are clearly far from normalisation (because larger  $\Psi$  was used) but they demonstrate how the first few terms in the expansion of  $|\Psi\rangle_{\alpha,\beta}^{p,q}$  begin to localise onto the Lissajous figure. The parameters  $(\alpha, \beta, p, q)$  determine the topology of the Lissajous figure, as described in section 2.6, while  $\arg \Psi$  controls the points on the Lissajous figure where the probability density will concentrate.

## 2.8. Two-dimensional squeezed states

Returning to the isotropic oscillator, we found the  $\mathfrak{su}(2)$  coherent states play the role of a single indexed Fock number basis for the two dimensional system. We were able to construct Schrödinger-type coherent states and found that they separated into the product of two one dimensional canonical coherent states.

An interesting proposition now is to address the construction of squeezed states by the same analogy to the one dimensional problem. Defining the generalised squeezing operator,

$$\mathcal{S}(\Xi) = \exp\left(\frac{1}{2}[\Xi \mathcal{A}_{\alpha,\beta}^{+2} - \bar{\Xi} \mathcal{A}_{\alpha,\beta}^{-2}]\right), \quad (2.8.1)$$

the generalized squeezed state is obtained through the action of generalised displacement and squeezing operators on the two dimensional vacuum as

$$|\Psi, \Xi\rangle_{\alpha,\beta} = \mathcal{D}(\Psi) \mathcal{S}(\Xi) |0\rangle_{\alpha,\beta}, \quad (2.8.2)$$

where the displacement operator,  $\mathcal{D}(\Psi)$ , is defined in (2.4.5).

Because we preserve the canonical commutation relations with the operators  $\mathcal{A}_{\alpha,\beta}^+, \mathcal{A}_{\alpha,\beta}^-$  we also get the expansion of the squeezed states by replacing the basis  $|n\rangle \rightarrow |\nu\rangle_{\alpha,\beta}$  in the one dimensional problem. Doing so leads to

$$|Z, \Gamma\rangle_{\alpha,\beta} = \frac{1}{\sqrt{\mathcal{N}(Z, \Gamma)}} \sum_{\nu=0}^{\infty} \frac{1}{\sqrt{\nu!}} \left(\frac{\Gamma}{2}\right)^{\frac{\nu}{2}} H_{\nu}\left(\frac{Z}{\sqrt{2\Gamma}}\right) |\nu\rangle_{\alpha,\beta}, \quad (2.8.3)$$

recalling the following relationships between the parameters

$$Z = \Psi - \bar{\Psi} e^{i\Theta} \tanh R, \quad (2.8.4)$$

$$\Gamma = -e^{i\Theta} \tanh R, \quad (2.8.5)$$

for

$$\Xi = R e^{i\Theta}. \quad (2.8.6)$$

The use of capital letter parameters is to indicate that we are referring to the two dimensional states. The normalisation function is also determined by direct analogy

$$\mathcal{N}(Z, \Gamma) = \frac{1}{\sqrt{1 - |\Gamma|^2}} \exp \left( \frac{1}{1 - |\Gamma|^2} \left( |Z|^2 - |\Gamma|^2 \operatorname{Re} \left( \frac{Z^2}{\Gamma} \right) \right) \right). \quad (2.8.7)$$

Using our previously obtained results for the wavefunction of the states  $\langle x, y | \nu \rangle_{\alpha, \beta}$  in (2.3.7) and Mehler's identity [12] we can derive the exact wavefunction of the two dimensional squeezed states,

$$\begin{aligned} \langle x, y | Z, \Gamma \rangle_{\alpha, \beta} &= \frac{1}{\sqrt{\pi \mathcal{N}(Z, \Gamma)}} \frac{e^{-\frac{x^2}{2} - \frac{y^2}{2}}}{\sqrt{1 - (\alpha^2 + \beta^2) \Gamma}} \\ &\times \exp \left( -\frac{\Gamma(\alpha^2 x^2 + \beta^2 y^2 + 2\alpha\beta xy)}{1 - (\alpha^2 + \beta^2) \Gamma} + \frac{\sqrt{2} Z(\alpha x + \beta y)}{1 - (\alpha^2 + \beta^2) \Gamma} - \frac{(\alpha^2 + \beta^2) Z^2}{2(1 - (\alpha^2 + \beta^2) \Gamma)} \right). \end{aligned} \quad (2.8.8)$$

Studying the exact form of the wavefunction we see that an immediate non-triviality arises: the states are not separable, that is,  $|Z, \Gamma\rangle_{\alpha, \beta} \neq |\xi_a\rangle_a \otimes |\xi_b\rangle_b$ . While the generalised displacement operator did not generate any entanglement the squeezing operator has. By taking the square of the generalised ladder operators we generate terms proportional to  $a^+ b^+$  and  $a^- b^-$  in the exponentials which are mixing the modes. These coupling terms are related to the two-mode squeezing operator [13]. In our method of generating these states, we preserve the ladder operator algebra and form of the one dimensional problem but produce the most general type of two dimensional Gaussian wavefunction including the mixing terms.

Restricting to the case of the squeezed vacuum,  $Z = \Psi = 0$ , the squeezing operator admits an  $\mathfrak{su}(1,1)$  decomposition [14] yielding

$$|\Xi\rangle_{\alpha, \beta} = \frac{1}{\sqrt{\cosh R}} \exp \left\{ \frac{e^{i\Theta}}{2} \tanh R (\alpha^2 a^{+2} + \beta^2 b^{+2} + \alpha\beta a^+ b^+) \right\} |0\rangle_{\alpha, \beta}, \quad (2.8.9)$$

in terms of the one-dimensional ladder operators. Equation (2.8.9) show that the generalised squeezed states do not factorise; the bilinear one-dimensional terms in the expansion of  $\mathcal{A}_{\alpha, \beta}^{+2}$  have induced a coupling between the  $a$  and  $b$  modes of the oscillator. This represents a non-trivial generalisation of the squeezed states to two dimensions, a two-mode-like squeezing was generated as a result of the construction in addition to the two single mode squeezings, but the two-dimensional squeezed states themselves retain most of the definitions from their one-dimensional counterparts.

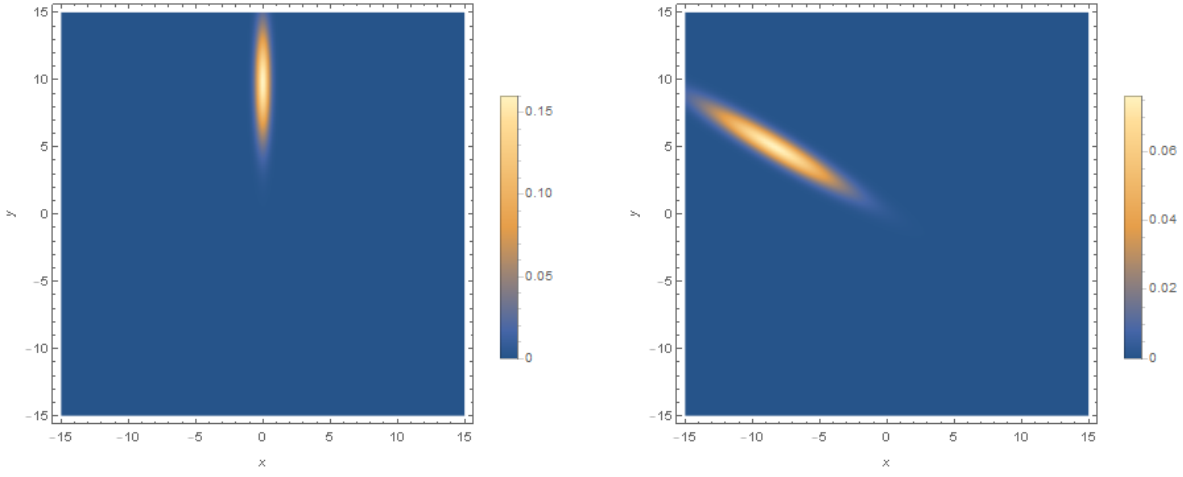
Another consequence of our construction is that our two-dimensional squeezed states must resolve the identity by unitarity of the squeezing operator  $\mathcal{S}(\Xi)$ . We know from (2.5.5) that the two-dimensional coherent states  $|\Psi\rangle_{\alpha, \beta}$  resolve the identity. To see this explicitly, notice that we proved in chapter 1 in equation (1.3.12) that there exists a braiding relation between the one-dimensional operators  $D(\alpha)$  and  $S(\xi)$  such that  $D(\alpha)S(\xi) = S(\xi)D(\beta)$ .

Because our two-dimensional operators satisfy the same algebra there exists an equivalent braiding relation, thus we find that for a two-dimensional squeezed state

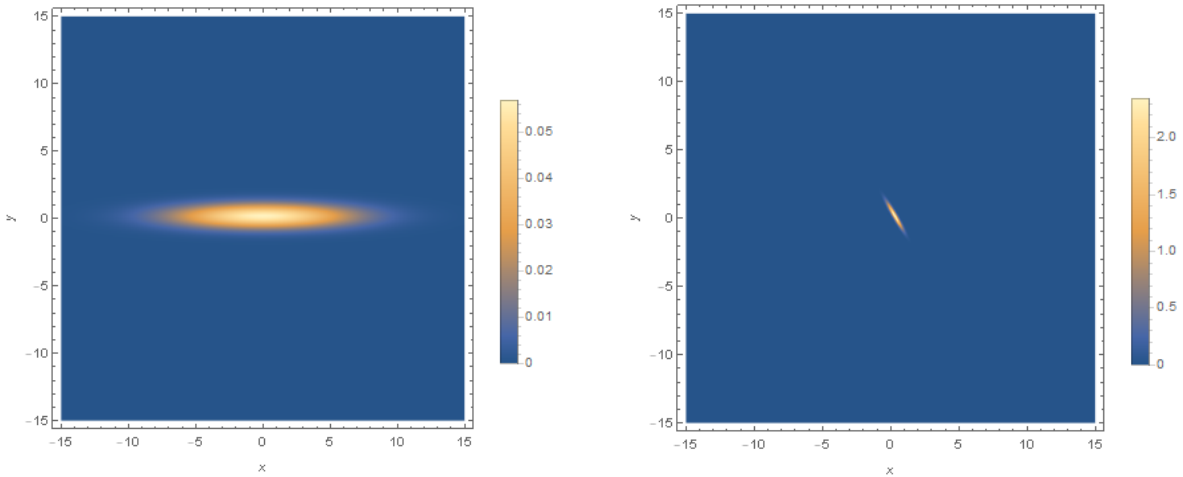
$$|\Phi, \Xi\rangle_{\alpha, \beta} = \mathcal{S}(\Xi) \mathcal{D}(\Phi) |0\rangle_{\alpha, \beta}, \quad (2.8.10)$$

we resolve the identity with respect to the following measure

$$\begin{aligned} \mathcal{S}(\Xi) \left\{ \frac{1}{\pi^2} \int_{S^3} d^2\alpha d^2\beta \delta(|\alpha|^2 + |\beta|^2 - 1) \int_{\mathbb{C}} \frac{d^2\Phi}{\pi} |\Phi|^2 |\Phi\rangle_{\alpha, \beta} \langle \Phi|_{\alpha, \beta} \right\} \mathcal{S}^\dagger(\Xi) \\ = \mathcal{S}(\Xi) \mathbb{I}_{\mathcal{H}} \mathcal{S}^\dagger(\Xi) \\ = \mathbb{I}_{\mathcal{H}}. \end{aligned} \quad (2.8.11)$$



**Fig. 2.6.** Density plots of  $|\langle x, y|Z, \Gamma\rangle_{\alpha, \beta}|^2$  for  $\alpha = \frac{\sqrt{3}}{2}e^{i\frac{\pi}{2}}, \beta = \frac{1}{2}, Z = 0.5, \Gamma = -\tanh(2)$  (left) and  $\alpha = \frac{\sqrt{3}}{2}e^{i\frac{\pi}{2}}, \beta = \frac{1}{2}, Z = 0.5, \Gamma = -e^{i\frac{\pi}{2}}\tanh(2)$  (right).



**Fig. 2.7.** Density plots of  $|\langle x, y|Z, \Gamma\rangle_{\alpha, \beta}|^2$  for  $\alpha = \frac{\sqrt{3}}{2}e^{i\frac{\pi}{2}}, \beta = \frac{1}{2}, Z = 0.5, \Gamma = \tanh(2)$  (left) and  $\alpha = \frac{\sqrt{3}}{2}, \beta = \frac{1}{2}, Z = 0.5, \Gamma = \tanh(2)$  (right).

In figure 2.6 we see two plots of the probability density functions of the squeezed states for fixed  $\alpha, \beta, Z$  and we vary the phase of  $\Gamma$ . On the left there is no phase associated with  $\Gamma$  and we find that there is greater squeezing in the  $\hat{Q}_a$  quadrature than in the  $\hat{Q}_b$  quadrature. Once we add a phase to  $\Gamma$  the density function rotates in the  $(x, y)$  plane and the squeezing is now more equally split between the  $\hat{Q}_a$  and  $\hat{Q}_b$  quadratures. We also find that the probability density function's peak does soften as some of the squeezing is pushed into the momentum quadratures.

Another interesting case occurs when we vary the phase of  $\alpha$  or  $\beta$  and fix all other parameters. In figure 2.7 we vary the phase of  $\alpha$  between the left and right images. For a purely imaginary value of  $\alpha$  the probability density function is not particularly well localised but is squeezed more significantly in the  $\hat{Q}_b$  direction than in the  $\hat{Q}_a$  direction. If however we consider a purely real value of  $\alpha$  we see significant squeezing in the  $\hat{Q}_a$  quadrature, while the  $\hat{Q}_b$  quadrature maintains the same width. In this situation, the momentum quadratures have acquired a larger variance while the position wavefunction has become very well localised. With the three complex parameters  $\Gamma, \alpha, \beta$  (subject to the normalisation constraint  $|\alpha|^2 + |\beta|^2 = 1$ ) we have a great deal of control over the squeezings in the four canonical quadratures.

In order to understand exactly how the squeezings occur in each quadrature we have to calculate the dispersions in each mode. We use the Baker-Campbell-Hausdorff identity  $e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \dots$  [15] to compute Bogoliubov transformations on the individual mode creation and annihilation operators. In the  $a$  mode the ladder operators are transformed as

$$\begin{aligned} \mathcal{S}^\dagger(\Xi) a^- \mathcal{S}(\Xi) &= (|\beta|^2 + |\alpha|^2 \cosh R) a^- + \alpha \bar{\beta} (\cosh R - 1) b^- \\ &\quad + e^{i\Theta} \sinh R (\alpha^2 a^+ + \alpha \beta b^+), \end{aligned} \quad (2.8.12)$$

$$\begin{aligned} \mathcal{S}^\dagger(\Xi) a^+ \mathcal{S}(\Xi) &= (|\beta|^2 + |\alpha|^2 \cosh R) a^+ + \bar{\alpha} \beta (\cosh R - 1) b^+ \\ &\quad + e^{-i\Theta} \sinh R (\bar{\alpha}^2 a^- + \bar{\alpha} \bar{\beta} b^-). \end{aligned} \quad (2.8.13)$$

Similarly for the  $b$  mode ladder operators we find

$$\begin{aligned} \mathcal{S}^\dagger(\Xi) b^- \mathcal{S}(\Xi) &= (|\alpha|^2 + |\beta|^2 \cosh R) b^- + \bar{\alpha} \beta (\cosh R - 1) a^- \\ &\quad + e^{i\Theta} \sinh R (\beta^2 b^+ + \alpha \beta a^+), \end{aligned} \quad (2.8.14)$$

$$\begin{aligned} \mathcal{S}^\dagger(\Xi) b^+ \mathcal{S}(\Xi) &= (|\alpha|^2 + |\beta|^2 \cosh R) b^+ + \alpha \bar{\beta} (\cosh R - 1) a^+ \\ &\quad + e^{-i\Theta} \sinh R (\bar{\beta}^2 b^- + \bar{\alpha} \bar{\beta} a^-). \end{aligned} \quad (2.8.15)$$

The transformations (2.8.12) and (2.8.13) are symmetric with (2.8.14) and (2.8.15) when one makes the replacements:  $\alpha \rightarrow \beta$  and  $a^\pm \rightarrow b^\pm$ . Using these transformations we can compute



the dispersions in the position and momentum quadratures:

$$\begin{aligned}(\Delta \hat{Q}_a)_{|\Xi\rangle_{\alpha,\beta}}^2 &= \frac{1}{2} + |\alpha|^2 \sinh^2 R + \operatorname{Re}(e^{i\Theta} \alpha^2) \sinh R \cosh R; \\(\Delta \hat{P}_a)_{|\Xi\rangle_{\alpha,\beta}}^2 &= \frac{1}{2} + |\alpha|^2 \sinh^2 R - \operatorname{Re}(e^{i\Theta} \alpha^2) \sinh R \cosh R,\end{aligned}\tag{2.8.16}$$

and similarly in the  $b$  mode

$$\begin{aligned}(\Delta \hat{Q}_b)_{|\Xi\rangle_{\alpha,\beta}}^2 &= \frac{1}{2} + |\beta|^2 \sinh^2 R + \operatorname{Re}(e^{i\Theta} \beta^2) \sinh R \cosh R; \\(\Delta \hat{P}_b)_{|\Xi\rangle_{\alpha,\beta}}^2 &= \frac{1}{2} + |\beta|^2 \sinh^2 R - \operatorname{Re}(e^{i\Theta} \beta^2) \sinh R \cosh R.\end{aligned}\tag{2.8.17}$$

These results also hold for the generalized squeezed states  $|\Psi, \Xi\rangle_{\alpha,\beta}$  because the action of the displacement operator has no effect on the variances. The results resemble those in equations (1.3.15) and (1.3.16) but are modified by  $\alpha, \beta$ . We see in the limit  $R \rightarrow 0$  we saturate the Heisenberg uncertainty relation in both modes. We also see clearly how the phase of  $\alpha, \beta$  dictate the overall squeezing between the position and momentum quadratures, it is due to the terms  $\pm \operatorname{Re}(e^{i\Theta} \alpha^2) \sinh R \cosh R$ . For the  $a$  mode, the reality (or lack thereof) of  $\alpha$  for fixed  $\Gamma$  is critical in determining whether the squeezing occurs in its position or momentum quadrature.

## 2.9. Conclusion

In this chapter we have described a method for constructing coherent states for the two-dimensional oscillator which relies on using the minimal set of definitions used to describe the coherent states of the one dimensional oscillator. We found that most of the properties of the one dimensional coherent states were also present in their two-dimensional isotropic Schrödinger-type counterparts: minimisation of the uncertainty principle, existence of a displacement operator, eigenstates of an annihilation operator, and correspondence to the classical system. As well, a suitable measure was found for the resolution of the identity.

Using the hypothesis of Chen, we generalised these results to the commensurate anisotropic two dimensional harmonic oscillator and found that their probability densities corresponded to Lissajous orbits. At present it is not clear how to extend these results to the non-commensurate case. The relative prime integers  $p, q$  enter the  $\mathfrak{su}(2)$  coherent states in a very natural way, but it seems that a different formalism altogether would be required when dealing with non-commensurable  $\omega_a, \omega_b$ , classically this would correspond to quasi-periodicity [16].

Lastly we presented a construction of generalised squeezed states for the two-dimensional oscillator in the same vein. Unlike the Schrödinger-type coherent states, the generalised squeezed states were found not to be separable into a product of two wavefunctions, one for each mode independently. Instead we obtained the most general type of squeezed state

(the most general Gaussian wavefunction in two dimensions), which includes a coupling between the  $(x,y)$  coordinates. The states have all the features of two-dimensional squeezed states, but we maintained the simplicity and analysis from the one-dimensional system. The preservation of the canonical commutation relation for our generalised ladder operators was key to simplifying our analysis.

To conclude this chapter, we remark that it is clear that some analysis in the anisotropic setting is lacking. We do not have the formal set of ladder operators nor do we have a resolution of the identity. The goal of the following chapter is to reintroduce these features into a description of  $\mathfrak{su}(2)$ -like coherent states for the 2:1 anisotropic oscillator, based on the same principle of addressing the degeneracy and constructing operators acting on a non-degenerate spectrum for the two-dimensional system.

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# Chapter 3

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## Non-linear ladder operators and coherent states for the 2:1 oscillator

by

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The main contributions of James Moran for this article are:

- Performed all calculations in the paper
- Wrote and edited the paper

Véronique Hussin made some initial calculations on finding degenerate contributions; suggested the need for a non-commuting binomial theorem; assisted with proofreading and editing.

Ian Marquette made some initial calculations on computing zero modes; developed some of the underlying algebraic formalism; assisted with proofreading and editing.

ABSTRACT. The 2:1 two-dimensional anisotropic quantum harmonic oscillator is considered and new sets of states are defined by means of normal-ordering non-linear operators through the use of non-commutative binomial theorems as well as solving recurrence relations. The states generated are good candidates for the natural generalisation of the  $\mathfrak{su}(2)$  coherent states of the two-dimensional isotropic oscillator. The two-dimensional non-linear generalised ladder operators lead to several chains of states which are connected in a non trivial way. The uncertainty relations of the defining chain of states are calculated and it is found that they admit a resolution of the identity and the spatial distribution of the wavefunction produces Lissajous figures in correspondence with the classical 2:1 oscillator.

**Keywords:** Ladder operators, Coherent states, Anisotropic oscillator, Two-dimensional quantum systems.

### 3.1. Introduction

The problem of quantum mechanical degeneracy in the 2:1 anisotropic oscillator with commensurable frequencies has been studied in great mathematical detail [1, 2, 3, 4, 5], and more recently a complete algebraic description of the symmetry generators of the quantum anisotropic oscillator with commensurable frequencies has been completed [6]. The 2:1 oscillator in particular has been studied in other branches of mathematical physics. It is, for example, the only case of the two-dimensional quantum simple harmonic oscillator (other than the isotropic case) where the system is second order super integrable [7]. It is separable in both Cartesian and parabolic coordinates and its eigenfunctions in parabolic coordinates are related to the confluent Heun equation [8]. Dunkl operator generalisations of the problem have also been considered [9].

In experimental settings, the understanding of multidimensional quantum anisotropic oscillators has been important in describing the states of deformed nuclei [10, 11], and the two-dimensional model has been suggested to be of use in semiconductor physics [12] by modelling electrons moving on an anisotropic lattice [13]. In mesoscopic physics it has been experimentally verified that mode locking of three-dimensional coherent waves of the anisotropic oscillator on parametric Lissajous surfaces form a nearly complete devil's staircase [14].

Multidimensional coherent states have also attracted some interest in the development of their general formalism [15] and in studying their classical limits [16, 17]. Work on coherent states for the anisotropic oscillator is limited, though some have been written down by ansatz [18], they are not presented with ladder operators or a resolution of the identity. Coherent states have many desirable properties in both the mathematical and physical sense. A particular curiosity is their closeness to their classical counterparts be it through minimised uncertainty relations (a canonical coherent state limit) or as we shall find in the example of the anisotropic oscillator, this closeness may be quantified by the reproduction of Lissajous figures in the probability distribution of purely quantum states.

In this paper we develop a scheme for constructing  $\mathfrak{su}(2)$ -like coherent states for the 2:1 two-dimensional anisotropic harmonic oscillator. The principle idea is to extend the construction from the isotropic setting presented in [19], where ladder operators were defined to organise the degenerate spectrum into a non-degenerate increasing spectrum, to the more general case where the frequencies between the two modes are different, but still commensurable. In doing so we need to introduce non-linear modifications to the ladder operators intended to remove the degeneracy, these operators also mix the two modes. We define a particular natural choice for these ladder operators which properly select all contributing states to the organised spectrum of the 2:1 oscillator.

Degeneracy in quantum systems must be addressed in order to properly define their corresponding coherent states [20]. This is because generalised definitions of coherent states rely on having a properly ordered spectrum. Suppose we have a non-degenerate discrete spectrum ordered in the following way

$$E_0 < E_1 < E_2 < \dots < E_k, \quad (3.1.1)$$

where  $k$  may be finite or infinite, and their associated eigenstates  $\{|n\rangle\}$  form an orthonormal basis

$$\sum_{n=0}^k |n\rangle \langle n| = \mathbb{1}. \quad (3.1.2)$$

Klauder showed that generalized coherent states may be expressed as [21]

$$|\xi, \gamma\rangle = M(\xi^2) \sum_{n=0}^k \frac{\xi^n}{\sqrt{\rho_n}} e^{-i\gamma E_n} |n\rangle, \quad (3.1.3)$$

where  $0 \leq \xi < \infty$  and  $-\infty < \gamma < \infty$ . The normalisation  $M(\xi^2)$  is chosen such that  $\langle \xi, \gamma | \xi, \gamma \rangle = 1$ , and the parameters  $\rho_n$  are solutions to a moment equation with a positive weight function  $k(u)$ ,

$$\rho_n = \int_0^\infty du u^n M^2(u) k(u). \quad (3.1.4)$$

Furthermore we have the completeness relation

$$\int d\mu(\xi, \gamma) |\xi, \gamma\rangle \langle \xi, \gamma| = \mathbb{1}, \quad (3.1.5)$$

where integration on the measure  $d\mu(\xi, \gamma)$  is defined as

$$\int d\mu(\xi, \gamma) = \lim_{\Gamma \rightarrow \infty} \frac{1}{2\Gamma} \int_0^\infty d\xi^2 k(\xi^2) \int_{-\Gamma}^\Gamma d\gamma. \quad (3.1.6)$$

The key property here is the resolution of the identity in equation (3.1.5). A resolution of the identity means we have a complete family of continuously parametrised states and therefore may represent any other state in the system in terms of the family of coherent states. This is often considered a defining property of generalised coherent states and it fundamentally holds under this construction due to the organisation of the spectrum.

If on the other hand the spectrum is degenerate we lose the property of the resolution of the identity because the phase factor  $e^{-i\gamma E_n}$  is degenerate. Performing the phase integrations in (3.1.6) will lead to terms of the form

$$\lim_{\Gamma \rightarrow \infty} \frac{1}{2\Gamma} \int_{-\Gamma}^{\Gamma} d\gamma e^{i\gamma(E_n - E_m)} \quad (3.1.7)$$

which should yield a term proportional to  $\delta_{nm}$  for non-degenerate energies, but in the event of degeneracy there exists  $n \neq m$  such that  $E_n = E_m$  and thus we lose proper interpretation of the integral. Work on addressing coherent states for degenerate spectra under this formalism has been completed by Crawford [22] where a factor of the degree of degeneracy is included in (3.1.3), as well as Fox and Choi [20] in which they add complex phases to the degenerate contributions in order to recover a well defined identity operator. More recently sets of ladder operators were defined in the example of the two-dimensional particle in a box [23] which describes a framework whereby the ladder operators act on the basis states and properly account for the degeneracy.

An ansatz for the form of the coherent states of the anisotropic quantum oscillator has been made [18], for the example of the 2:1 oscillator they would read

$$|\varphi_\nu\rangle = \sum_{k=0}^{\nu} \alpha^k \beta^{\nu-k} \sqrt{\binom{\nu}{k}} |k, 2(\nu - k)\rangle, \quad (3.1.8)$$

which generalises the form of the  $\mathfrak{su}(2)$  coherent states of the isotropic oscillator. The states (3.1.8) produce Lissajous figures in their spatial distribution. It is clear however that these states cannot resolve the identity on the full Hilbert space of states because they miss out states with an odd number in the second mode,  $|k, 2(\nu - k) + 1\rangle$ , in their construction. And while it is possible to retroactively fit ladder operators to generate these states, they often include terms such as inverse square roots of the number operator and as such are only well defined on particular representations.

This paper will be organised as follows. In Section 3.2 we will introduce the 2:1 two-dimensional anisotropic harmonic oscillator and describe the set up of the problem including the degeneracy in the spectrum and how we wish to organise it to remove the degeneracy. We define sets of states on the non-degenerate spectrum as linear superpositions of states with equal total energy. In section 4.3 we introduce some general definitions that we require the generalised ladder operators to satisfy. We explicitly define a set of non-linear ladder operators  $\mathcal{A}^+, \mathcal{A}^-$ , compute all zero modes associated with the operator  $\mathcal{A}^-$  and define chains of states associated to each zero mode by the action of  $\mathcal{A}^+$ . Section 3.4 is devoted to the principle chain of states generated from the lowest energy zero mode, which corresponds to the ground state  $|0,0\rangle$ . We show that they modify the form of the binomial coefficient of the  $\mathfrak{su}(2)$  coherent states, and we find that they admit a resolution of the identity. Following this in section 3.5 we discuss the uncertainty relations for the principle chain of states and find



that the uncertainties in each mode are intrinsically connected due to the parameters introduced in their construction. Finally, we conclude in section 3.6 by discussing the possibility of generalising the techniques developed in this paper to other systems.

## 3.2. Bosonic states and degenerate spectra

The quantum hamiltonian for the 2:1 oscillator whose frequency in its  $a$  mode is twice that of its frequency in its  $b$  mode is given in terms of the canonical position and momentum operators  $\hat{Q}_i, \hat{P}_i$  respectively by

$$\hat{H} = \frac{1}{2}\hat{P}_a^2 + \frac{1}{2}\hat{P}_b^2 + 2\hat{Q}_a^2 + \frac{1}{2}\hat{Q}_b^2, \quad (3.2.1)$$

where we have set the dimensionful quantities  $\hbar, \omega, m = 1$ , Planck's constant, the common frequency and the geometric mean mass respectively. The position and momentum operators satisfy the canonical commutation relations

$$[Q_a, P_b] = i\delta_{ab}\mathbb{1}, \quad [Q_a, Q_b] = 0, \quad [P_a, P_b] = 0, \quad (3.2.2)$$

where  $\mathbb{1}$  is the identity operator. For our purposes we are interested in the corresponding ladder operator formalism, achieved by defining the operators

$$\begin{aligned} a^- &= \left( \hat{Q}_a + \frac{i}{2}\hat{P}_a \right), & a^+ &= \left( \hat{Q}_a - \frac{i}{2}\hat{P}_a \right), \\ b^- &= \frac{1}{\sqrt{2}} \left( \hat{Q}_b + i\hat{P}_b \right), & b^+ &= \frac{1}{\sqrt{2}} \left( \hat{Q}_b - i\hat{P}_b \right), \end{aligned} \quad (3.2.3)$$

which can be used to rewrite the hamiltonian (3.2.1) as

$$\hat{H} = 2a^+a^- + b^+b^- + \frac{3}{2}\mathbb{1}. \quad (3.2.4)$$

The operators (3.2.3) and (3.2.4) satisfy the canonical commutation relations in the  $a$  and  $b$  modes separately,

$$\begin{aligned} [a^-, a^+] &= \mathbb{1}, & [b^-, b^+] &= \mathbb{1}, \\ [\hat{H}, a^\pm] &= \pm 2a^\pm, & [\hat{H}, b^\pm] &= \pm b^\pm, \end{aligned} \quad (3.2.5)$$

with other commutators vanishing. Notice that in our definitions we have absorbed the difference in frequencies into the definition of the ladder operators and not into their commutation relations, so we preserve  $[a^-, a^+] = [b^-, b^+] = \mathbb{1}$ .

The energy eigenstates are solutions to the time-independent Schrödinger equation

$$\hat{H} |n, m\rangle = E_{n, m} |n, m\rangle, \quad (3.2.6)$$

where  $n, m \in \mathbb{Z}^{\geq 0}$ . The ladder operators (3.2.3) have the following actions on the energy eigenstates

$$\begin{aligned} a^- |n, m\rangle &= \sqrt{n} |n-1, m\rangle, & a^+ |n, m\rangle &= \sqrt{n+1} |n+1, m\rangle, \\ b^- |n, m\rangle &= \sqrt{m} |n, m-1\rangle, & b^+ |n, m\rangle &= \sqrt{m+1} |n, m+1\rangle, \end{aligned} \quad (3.2.7)$$

and the eigenvalues  $E_{n,m}$  are given by

$$E_{n,m} = 2n + m + \frac{3}{2}. \quad (3.2.8)$$

Equation (3.2.8) is our first descriptor of the degeneracy present in the 2:1 oscillator, it is equivalent to a simple problem in number theory, namely, for  $\nu, n, m \in \mathbb{Z}^{\geq 0}$ , what values can  $n$  and  $m$  take satisfying the following equation

$$\nu = 2n + m. \quad (3.2.9)$$

In terms of  $\nu$  the spectrum  $E_{n,m} = E_\nu = \nu + \frac{3}{2}$  is well organised

$$E_0 < E_1 < E_2 < \dots, \quad (3.2.10)$$

and the states with energy  $E_\nu$  will be expressed as linear superpositions of all states with energy  $E_{n,m} = E_\nu$  as

$$|\varphi_\nu\rangle = \sum_{k=0}^{\lfloor \frac{\nu}{2} \rfloor} \Lambda_k(\nu, \{\alpha_i\}) |k, \nu - 2k\rangle. \quad (3.2.11)$$

$ \varphi_\nu\rangle$	contributing states
$ \varphi_0\rangle$	$ 0,0\rangle$
$ \varphi_1\rangle$	$ 0,1\rangle$
$ \varphi_2\rangle$	$ 1,0\rangle,  0,2\rangle$
$ \varphi_3\rangle$	$ 1,1\rangle,  0,3\rangle$
$\vdots$	$\vdots$
$ \varphi_\nu\rangle$	$\{ k, \nu - 2k\rangle \mid k = 0, 1, 2, \dots, \lfloor \frac{\nu}{2} \rfloor\}$

**Table 3.1.** States  $|n, m\rangle$  contributing to a state  $|\varphi_\nu\rangle$  with energy  $E_\nu$ .

Here  $\Lambda_k(\nu, \{\alpha_i\})$  are the complex expansion coefficients, with complex conjugate  $\bar{\Lambda}_k(\nu, \{\alpha_i\})$ , and they may depend on additional variables  $\{\alpha_i\}, i \in \{1, 2, \dots, m\}$ . The coefficients represent the weights we attribute to each Fock basis state, after which we sum over the number  $k$  to produce an averaged contribution to a state  $|\varphi_\nu\rangle$ .

We require that the states (3.2.11) satisfy the following properties: normalisation

$$\sum_{k=0}^{\lfloor \frac{\nu}{2} \rfloor} \bar{\Lambda}_k(\nu, \{\alpha_i\}) \Lambda_k(\nu, \{\alpha_i\}) = 1, \quad (3.2.12)$$

and the completeness relation

$$\int_{\mathbb{C}^m} d\mu(\{\alpha_i\}) \bar{\Lambda}_{k'}(\nu, \{\alpha_i\}) \Lambda_k(\nu, \{\alpha_i\}) = \delta_{k'k}, \quad (3.2.13)$$

for some measure  $d\mu(\{\alpha_i\})$  such that they resolve the identity on the subspace indexed by  $\nu$

$$\int_{\mathbb{C}^m} d\mu(\{\alpha_i\}) |\varphi_\nu\rangle \langle \varphi_\nu| = \mathbb{1}_\nu, \quad (3.2.14)$$

and over the total Hilbert space via

$$\sum_{\nu=0}^{\infty} \mathbb{1}_\nu = \mathbb{1}_{\mathcal{H}}. \quad (3.2.15)$$

Here  $\mathbb{1}_{\mathcal{H}} = \sum_{n,m}^{\infty} |n,m\rangle \langle n,m|$  refers to the identity operator on the full Hilbert space of states. This choice of partitioning the space of states  $\{|n,m\rangle\}$  is the most natural when considering the 2:1 oscillator.

There is some freedom in defining the set of states  $\{|\varphi_\nu\rangle\}$  as the contributing states can be weighted in any way such that (3.2.13) and (3.2.14) are satisfied, these represent a proper probabilistic interpretation and completeness, respectively. By defining sets of ladder operators which correctly pick out the states in table 3.1, the set of coefficients defining the states  $\{|\varphi_\nu\rangle\}$  are predetermined. These can always be normalised, the pertinent calculation is to check that a measure can be found for (3.2.14).

### 3.3. Ladder operators and chains of states

Now we turn our attention to the ladder operators which will generate the states (3.2.11). A defining feature of the generalised ladder operators of our system will be

$$[\hat{H}, \mathcal{A}^+] = \mathcal{A}^+, \quad [\hat{H}, \mathcal{A}^-] = -\mathcal{A}^-, \quad (3.3.1)$$

this ensures that if  $|\psi_\nu\rangle$  is an eigenstate of  $\hat{H}$  then so are  $\mathcal{A}^- |\psi_\nu\rangle$  and  $\mathcal{A}^+ |\psi_\nu\rangle$ . Operators in more than one variable allow for infinitely many zero modes, to this end, the generalised annihilation operator,  $\mathcal{A}^-$ , admits the following zero modes

$$\mathcal{A}^- |\varphi_0^{(n)}\rangle = 0, \quad (3.3.2)$$

where the index  $(n)$  enumerates zero modes and we define the principle zero mode  $|\varphi_0^{(0)}\rangle = |0,0\rangle$  to be the ground state of (3.2.4). The Fock spaces (chains of states) associated to each zero mode  $|\varphi_0^{(n)}\rangle$  are generated by

$$|\varphi_\nu^{(n)}\rangle = \frac{1}{\sqrt{[f(\nu)]!}} (\mathcal{A}^+)^{\nu} |\varphi_0^{(n)}\rangle, \quad (3.3.3)$$

where  $\nu$  describes the  $\nu$ -th state in the chain and the states generated from the principle zero mode,  $|\varphi_\nu^{(0)}\rangle$ , define the principle chain of states. The function  $f(\nu)$  is chosen such that

the states are normalised  $\langle \varphi_\nu^{(n)} | \varphi_\nu^{(n)} \rangle = 1$ , and the action on an intermediate state is given by

$$\mathcal{A}^+ |\varphi_\nu^{(n)}\rangle = \sqrt{f(\nu+1)} |\varphi_{\nu+1}^{(n)}\rangle. \quad (3.3.4)$$

Because we are interested in the constructive generation of states we do not define the action of  $\mathcal{A}^-$  on an intermediate state because we will choose  $\mathcal{A}^- = (\mathcal{A}^+)^\dagger$  and in general its action will lead to a superposition of states from different chains. Much of the framework described follows from the defining features of the ladder operators for the one-dimensional harmonic oscillator, but we relax the canonical commutation relation such that

$$[\mathcal{A}^-, \mathcal{A}^+] \neq \mathbf{1}, \quad (3.3.5)$$

the commutation relation will not be canonical, this allows us to define generalised ladder operators as non-linear combinations in  $a^-, a^+, b^-,$  and  $b^+$ .

With the framework described we find that a suitable pair of ladder operators are

$$\mathcal{A}^+ = \alpha b^+ + \beta a^+ b^-, \quad \mathcal{A}^- = \bar{\alpha} b^- + \bar{\beta} a^- b^+. \quad (3.3.6)$$

Their commutator yields

$$[\mathcal{A}^-, \mathcal{A}^+] = |\alpha|^2 \mathbf{1} + |\beta|^2 (b^+ b^- - a^+ a^-), \quad (3.3.7)$$

and it can be verified that these operators obey (3.3.1) with hamiltonian (3.2.4). Firstly we categorise all of the zero modes of the operator  $\mathcal{A}^-$  by solving (3.3.2) in the basis (3.2.11)

$$|\psi_0^{(n)}\rangle = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_j^{(n)}(\alpha, \beta) |j, n-2j\rangle, \quad n \in \mathbb{Z}^{\geq 0}, \quad (3.3.8)$$

assuming  $\gamma_j^{(n)}(\alpha, \beta) \neq 0$ . Considering  $n \rightarrow 2n'$  even, equation (3.3.2) is explicitly

$$\begin{aligned} & \sum_{j=0}^{n'} \left( \gamma_j^{(2n')}(\alpha, \beta) \bar{\alpha} \sqrt{2(n'-j)} |j, 2n'-2j-1\rangle \right. \\ & \left. + \gamma_j^{(2n')}(\alpha, \beta) \bar{\beta} \sqrt{j-1} \sqrt{2(n'-j)+1} |j-1, 2n'-2j+1\rangle \right) = 0. \end{aligned} \quad (3.3.9)$$

After relabelling the second summation index  $j \rightarrow j+1$  in (3.3.9) we find that the coefficients  $\gamma_j^{(2n')}(\alpha, \beta)$  satisfy the following recursion relation on the index  $j$ ,

$$\gamma_{j+1}^{(2n')}(\alpha, \beta) = -\frac{\bar{\alpha}}{\bar{\beta}} \frac{\sqrt{2(n'-j)}}{\sqrt{j+1} \sqrt{2(n'-j)-1}} \gamma_j^{(2n')}(\alpha, \beta). \quad (3.3.10)$$

This can be solved straightforwardly with the definition  $\gamma_0^{(2n')}(\alpha, \beta) = 1$  (this is just an overall multiplicative constant) to give

$$\begin{aligned}\gamma_j^{(2n')}(\alpha, \beta) &= \left(-\frac{\bar{\alpha}}{\bar{\beta}}\right)^j \prod_{k=0}^{j-1} \frac{\sqrt{2(n' - j + k + 1)}}{\sqrt{j - k} \sqrt{2(n' - j + k) + 1}}, \\ &= \left(-2\frac{\bar{\alpha}}{\bar{\beta}}\right)^j \left(\frac{n!}{(n' - j)!}\right) \sqrt{\frac{(2(n' - j))!}{j!(2n')!}}, \quad j \in \{0, 1, \dots, n'\}.\end{aligned}\tag{3.3.11}$$

Repeating the procedure for  $n \rightarrow 2n' + 1$  odd,

$$|\psi_0^{(2n'+1)}\rangle = \sum_{j=0}^{n'} \gamma_j^{(2n'+1)}(\alpha, \beta) |j, 2(n' - j) + 1\rangle,\tag{3.3.12}$$

solving equation (3.3.2) we find a similar recursion relation to the case where  $n$  is even, but with an added constraint,

$$\begin{aligned}\gamma_j^{(2n'+1)}(\alpha, \beta) \bar{\alpha} \sqrt{2(n' - j) + 1} + \gamma_{j+1}^{(2n'+1)}(\alpha, \beta) \bar{\beta} \sqrt{j + 1} \sqrt{2(n' - j)} &= 0, \\ \gamma_n^{(2n'+1)}(\alpha, \beta) &= 0.\end{aligned}\tag{3.3.13}$$

The last constraint  $\gamma_n^{(2n'+1)}(\alpha, \beta) = 0$  implies that all terms in the recursion vanish and as a result we have no zero modes associated with odd values of  $n$ , that is there are no zero modes of  $\mathcal{A}^-$  associated to odd values of  $\nu$ . Thus, all normalised zero modes are given by

$$|\varphi_0^{(2n)}\rangle = \frac{1}{\sqrt{\mathcal{N}_0^{(2n)}(\alpha, \beta)}} \sum_{j=0}^n \gamma_j^{(2n)}(\alpha, \beta) |j, 2(n - j)\rangle, \quad n \in \mathbb{Z}^{\geq 0},\tag{3.3.14}$$

with coefficients

$$\gamma_j^{(2n)}(\alpha, \beta) = \left(-2\frac{\bar{\alpha}}{\bar{\beta}}\right)^j \left(\frac{n!}{(n - j)!}\right) \sqrt{\frac{(2(n - j))!}{j!(2n)!}}, \quad j \in \{0, 1, \dots, n\}.\tag{3.3.15}$$

We have introduced the normalisation function  $\mathcal{N}_0^{(2n)}(\alpha, \beta)$  which ensures that  $\langle \varphi_0^{(2n)} | \varphi_0^{(2n)} \rangle = 1$ ,  $\forall n \in \mathbb{Z}^{\geq 0}$ . It is given in terms of the coefficients  $\gamma_j^{(2n)}(\alpha, \beta)$  by

$$\mathcal{N}_0^{(2n)}(\alpha, \beta) = \sum_{j=0}^n \left| \gamma_j^{(2n)}(\alpha, \beta) \right|^2.\tag{3.3.16}$$



To compute all states in the diagram we make use of a non-commutative binomial theorem [24] to obtain the normal ordered expansion of  $(\mathcal{A}^+)^{\nu}$

$$(\alpha b^+ + \beta a^+ b^-)^{\nu} = \sum_{k=0}^{\lfloor \frac{\nu}{2} \rfloor} \sum_{j=0}^{\nu-2k} \alpha^{j+k} \beta^{\nu-k-j} \frac{\nu!}{(\nu-2k-j)!k!j!2^k} \times (b^+)^j (a^+)^{\nu-k-j} (b^-)^{\nu-2k-j}, \quad (3.3.19)$$

and we define the  $\nu$ -th unnormalised state in the  $n$ -th chain by

$$|\psi_{\nu}^{(2n)}\rangle = (\mathcal{A}^+)^{\nu} |\varphi_0^{(2n)}\rangle. \quad (3.3.20)$$

Using (3.3.19) this yields

$$|\psi_{\nu}^{(2n)}\rangle = \frac{1}{\sqrt{\mathcal{N}_0^{(2n)}(\alpha, \beta)}} \sum_{m=0}^n \sum_{k=0}^{\lfloor \frac{\nu}{2} \rfloor} \sum_{j=\nu-2(k+n-m)}^{\nu-2k} \Gamma_{m,k,j}^{(2n),(\nu)}(\alpha, \beta) |m + \nu - k - j, 2(n-m) - \nu + 2k + 2j\rangle, \quad (3.3.21)$$

where the coefficients are given by

$$\begin{aligned} \Gamma_{m,k,j}^{(2n),(\nu)}(\alpha, \beta) &= \alpha^{j+k} \beta^{\nu-k-j} \gamma_m^{(2n)}(\alpha, \beta) \frac{\nu!}{(\nu-2k-j)!k!j!2^k} \\ &\times \sqrt{(m+1)_{\nu-k-j}} \sqrt{(2(n-m) - \nu + 2k + j + 1)_j} \\ &\times \sqrt{(2(n-m) - \nu + 2k + j + 1)_{\nu-2k-j}}. \end{aligned} \quad (3.3.22)$$

Here  $(x)_n = x(x+1)\dots(x+n-1)$  is the Pochhammer symbol and the functions  $\gamma_m^{(2n)}(\alpha, \beta)$  are defined in (3.3.15). We mention that the lower limit on the  $j$  summation is achieved by excluding states annihilated by  $(b^-)^{\nu-2k-j}$ . Again it is convenient to normalise the states so we scale (3.3.21) by

$$|\varphi_{\nu}^{(2n)}\rangle = \frac{1}{\sqrt{\mathcal{N}_{\nu}^{(2n)}(\alpha, \beta)}} |\psi_{\nu}^{(2n)}\rangle, \quad (3.3.23)$$

where  $\mathcal{N}_{\nu}^{(2n)}(\alpha, \beta) = \langle \psi_{\nu}^{(2n)} | \psi_{\nu}^{(2n)} \rangle$ . The function (3.3.4) is also defined

$$f(\nu) = \frac{\mathcal{N}_{\nu}^{(2n)}(\alpha, \beta)}{\mathcal{N}_{\nu-1}^{(2n)}(\alpha, \beta)}. \quad (3.3.24)$$

With every state now defined we are in a position to compute the action of  $\mathcal{A}^-$  on an arbitrary state, the states in a given row  $\nu$  provide a non-orthogonal basis for the  $\nu$ -th subspace. Using this basis we can compute

$$\mathcal{A}^- |\varphi_{\nu}^{(2n)}\rangle = \sum_{k=0}^{\lfloor \frac{\nu-1}{2} \rfloor} \Gamma_k^{(2n)} |\varphi_{\nu-1-2k}^{(2k)}\rangle \quad (3.3.25)$$

by means of inverting the Gram matrix [25]

$$\mathbf{G} = \begin{bmatrix} 1 & \langle \varphi_{\nu-1}^{(0)} | \varphi_{\nu-3}^{(2)} \rangle & \dots \\ \vdots & \ddots & \\ \langle \varphi_{\nu-1-2\lfloor \frac{\nu-1}{2} \rfloor}^{(2\lfloor \frac{\nu-1}{2} \rfloor)} | \varphi_{\nu-1}^{(0)} \rangle & \dots & 1 \end{bmatrix}, \quad (3.3.26)$$

whose matrix elements are  $G_{kj} = \langle \varphi_{\nu+1-2k}^{(2(k-1))} | \varphi_{\nu+1-2j}^{(2(j-1))} \rangle$  and the diagonal entries  $G_{kk} = 1$  because the states are normalised. The coefficients (3.3.25) are then determined by the inversion formula

$$\Gamma_k^{(2n)} = \sum_{j=0}^{\lfloor \frac{\nu-1}{2} \rfloor} G_{kj}^{-1} \langle \varphi_{\nu-1-2j}^{(2j)} | \mathcal{A}^- | \varphi_{\nu}^{(2n)} \rangle. \quad (3.3.27)$$

In general, for arbitrary  $\alpha, \beta$  the off diagonal elements of (3.3.26) are non-zero but curiously it may be shown that all the zero modes are orthogonal to their corresponding state in the first chain,

$$\langle \varphi_0^{(2\nu)} | \varphi_{2\nu}^{(0)} \rangle = \frac{\alpha^{2\nu}}{\sqrt{\mathcal{N}_0^{(2\nu)}(\alpha, \beta) \mathcal{N}_{2\nu}^{(0)}(\alpha, \beta)}} \sum_{k=0}^{\nu} (-1)^k \binom{\nu}{k} = 0. \quad (3.3.28)$$

### 3.4. Principle states

Focusing on the principle set of states generated from the principle zero mode (the ground state)  $|\varphi_0^{(0)}\rangle = |0,0\rangle$ , we find the  $n = 0$  limit of (3.3.21) leads to the unnormalised states

$$|\psi_{\nu}^{(0)}\rangle = \sum_{k=0}^{\lfloor \frac{\nu}{2} \rfloor} \alpha^{\nu-k} \beta^k \frac{\nu!}{\sqrt{(\nu-2k)! \sqrt{k!} 2^k}} |k, \nu-2k\rangle. \quad (3.4.1)$$

Normalising the states and absorbing a factor of  $\sqrt{\nu!}$  into the definition of the action of  $\mathcal{A}^+$  we write the normalised chain as

$$|\varphi_{\nu}^{(0)}\rangle = \frac{1}{\sqrt{\mathcal{N}_{\nu}^{(0)}(\alpha, \beta)}} \sum_{k=0}^{\lfloor \frac{\nu}{2} \rfloor} \alpha^{\nu-k} \beta^k \sqrt{\binom{\nu}{k}_2} |k, \nu-2k\rangle. \quad (3.4.2)$$

Here we have introduced a modified binomial coefficient

$$\binom{n}{k}_t = \frac{n!}{k!(n-tk)!t^{2k}}, \quad t \in \mathbb{Z}^{\geq 0}, \quad (3.4.3)$$

to elucidate the similarity of the states to the two-mode  $\mathfrak{su}(2)$  coherent states. Just as the  $\mathfrak{su}(2)$  coherent states may be built from generalised ladder operators picking out degenerate states of the isotropic oscillator [19], it is in this sense that we say the states (3.4.2) generalise the  $\mathfrak{su}(2)$  coherent states to the 2:1 oscillator.

The function  $f(\nu)$  (with the additional factor of  $\nu$ ) in (3.3.4) is defined as

$$f(\nu) = \nu \frac{\mathcal{N}_{\nu}^{(0)}(\alpha, \beta)}{\mathcal{N}_{\nu-1}^{(0)}(\alpha, \beta)}, \quad (3.4.4)$$



and we observe that the normalisation function may be expressed as

$$\mathcal{N}_\nu^{(0)}(\alpha, \beta) = \left( \frac{|\alpha||\beta|}{2} \right)^\nu \mathcal{H}_\nu \left( \frac{|\alpha|}{|\beta|} \right), \quad (3.4.5)$$

where  $\mathcal{H}_\nu(x) = \sum_{k=0}^{\lfloor \frac{\nu}{2} \rfloor} \frac{\nu!}{(\nu-2k)!k!} (2x)^{\nu-2k}$  are the pseudohermite polynomials [26]. This yields the desired normalisation condition

$$\langle \varphi_\mu^{(0)} | \varphi_\nu^{(0)} \rangle = \delta_{\mu\nu}. \quad (3.4.6)$$

Although we have insisted on taking  $\mathcal{A}^- = (\mathcal{A}^+)^\dagger$  and as such  $\mathcal{A}^- |\varphi_\nu^{(0)}\rangle \not\propto |\varphi_{\nu-1}^{(0)}\rangle$ , we can still connect the states exclusively in the  $n = 0$  chain by using the operator  $b^-$ , such that

$$b^- |\varphi_\nu^{(0)}\rangle = \alpha \sqrt{\nu} \sqrt{\frac{\mathcal{N}_{\nu-1}^{(0)}(\alpha, \beta)}{\mathcal{N}_\nu^{(0)}(\alpha, \beta)}} |\varphi_{\nu-1}^{(0)}\rangle. \quad (3.4.7)$$

A key advantage of these states, because they are built out of the whole space of states of the two-dimensional oscillator, is that we can resolve the identity both in the  $\nu$ -th subspace and over the full Hilbert space by

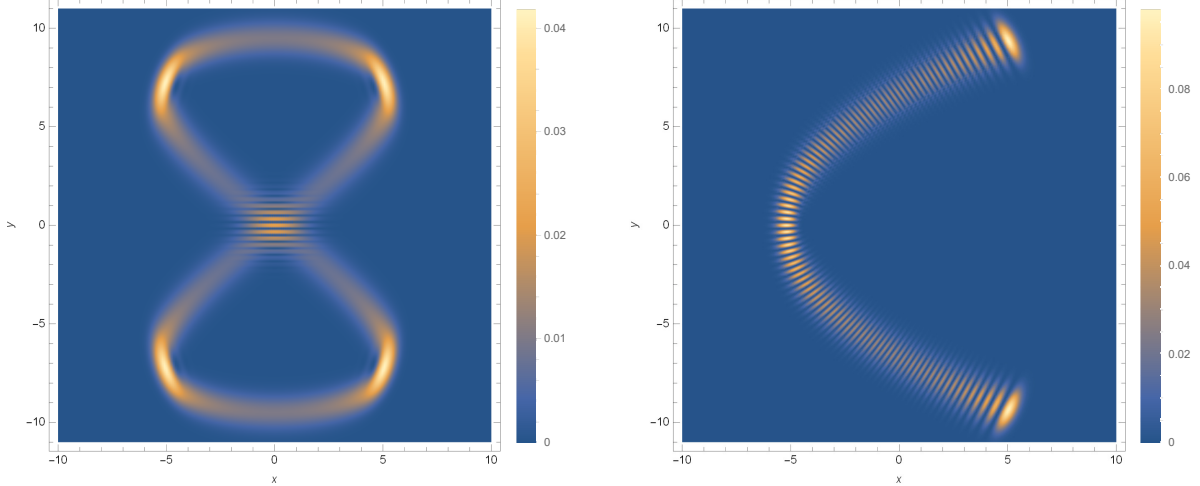
$$\frac{1}{8\pi^2\nu!} \int_{\mathbb{C}^2} d^2\alpha d^2\beta \mathcal{N}_\nu(\alpha, \beta) \frac{e^{-|\alpha| - \frac{|\beta|^2}{4}}}{|\alpha|^{\nu+1}} |\varphi_\nu^{(0)}\rangle \langle \varphi_\nu^{(0)}| = \mathbb{I}_\nu, \quad (3.4.8)$$

and

$$\sum_{\nu=0}^{\infty} \left( \frac{1}{8\pi^2\nu!} \int_{\mathbb{C}^2} d^2\alpha d^2\beta \mathcal{N}_\nu(\alpha, \beta) \frac{e^{-|\alpha| - \frac{|\beta|^2}{4}}}{|\alpha|^{\nu+1}} |\varphi_\nu^{(0)}\rangle \langle \varphi_\nu^{(0)}| \right) = \mathbb{I}_{\mathcal{H}}, \quad (3.4.9)$$

respectively. For details on the derivation of these results see appendix 3.A.

Thus the states  $\{|\varphi_\nu^{(0)}\rangle\}$  form a complete family of states for the Hilbert space of the two-dimensional oscillator, they are equipped with ladder operators and they reproduce the Lissajous figures in their spatial distribution, mimicking the behaviour of the classical 2:1 oscillator [18]. In this sense they are a good candidate for the generalisation of the  $\mathfrak{su}(2)$  coherent states of the two-dimensional isotropic oscillator.



**Fig. 3.2.**  $|\langle x, y | \varphi_{100}^{(0)} \rangle|^2$  with  $\alpha = 3, \beta = \frac{e^{i\frac{\pi}{2}}}{\sqrt{2}}$  (left) and  $\alpha = 3, \beta = \frac{1}{\sqrt{2}}$  (right).

Coherent states for harmonic oscillators in their typical presentation follow from three equivalent definitions: eigenstates of the annihilation operator, the action of a unitary displacement operator on the vacuum state, and by a particular infinite superposition of eigenstates [27]. In principle one could construct such displacement operators and eigenstates for the operators presented in (3.3.6), however, because their commutation relation is no longer canonical and  $\mathcal{A}^-$  does not act correctly as a lowering operator on the same chain of states generated by  $(\mathcal{A}^+)^{\nu} |\varphi_0^{(0)}\rangle$ , their generalisation is not straightforward. Additionally because the term  $\beta a^+ b^-$  in  $\mathcal{A}^+$  intrinsically couples the two modes, states generated by the exponential of these operators will not factorise into the product of two one-dimensional harmonic oscillator coherent states.

We stress that the comparison to be made with the states (3.4.2) is with the two-mode or Schwinger boson realisation of the  $\mathfrak{su}(2)$  coherent states.

### 3.5. Uncertainty relations

We can calculate the position and momentum uncertainties in the state  $|\varphi_{\nu}^{(0)}\rangle$ . In the  $a$  mode we get the following product uncertainty relation

$$((\Delta \hat{Q}_a)^2 (\Delta \hat{P}_a)^2)_{|\varphi_{\nu}^{(0)}\rangle} = \frac{1}{4} \left( 1 + \frac{1}{2} |\alpha|^2 |\beta|^2 \nu (\nu - 1) \frac{\mathcal{N}_{\nu-2}^{(0)}(\alpha, \beta)}{\mathcal{N}_{\nu}^{(0)}(\alpha, \beta)} \right)^2 \geq \frac{1}{4}. \quad (3.5.1)$$

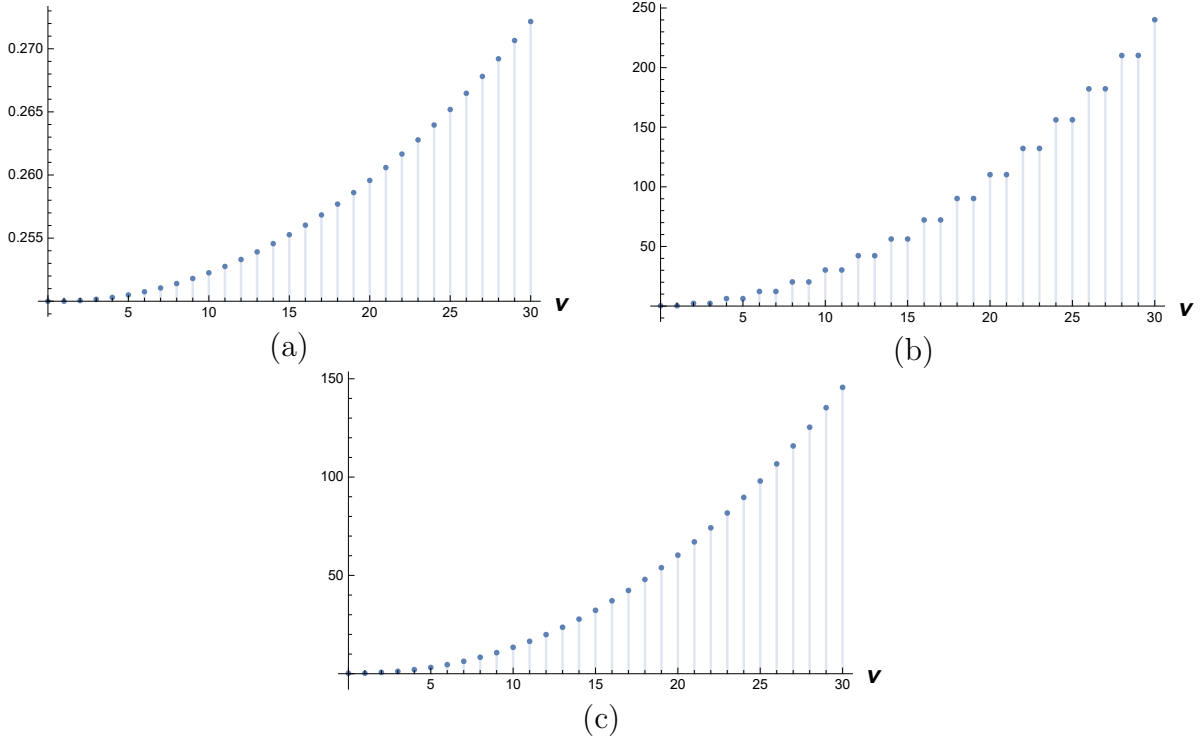
Similarly for the  $b$  mode we find

$$((\Delta \hat{Q}_b)^2 (\Delta \hat{P}_b)^2)_{|\varphi_{\nu}^{(0)}\rangle} = \left( \frac{1}{2} + |\alpha|^2 \nu \frac{\mathcal{N}_{\nu-1}^{(0)}(\alpha, \beta)}{\mathcal{N}_{\nu}^{(0)}(\alpha, \beta)} \right)^2. \quad (3.5.2)$$

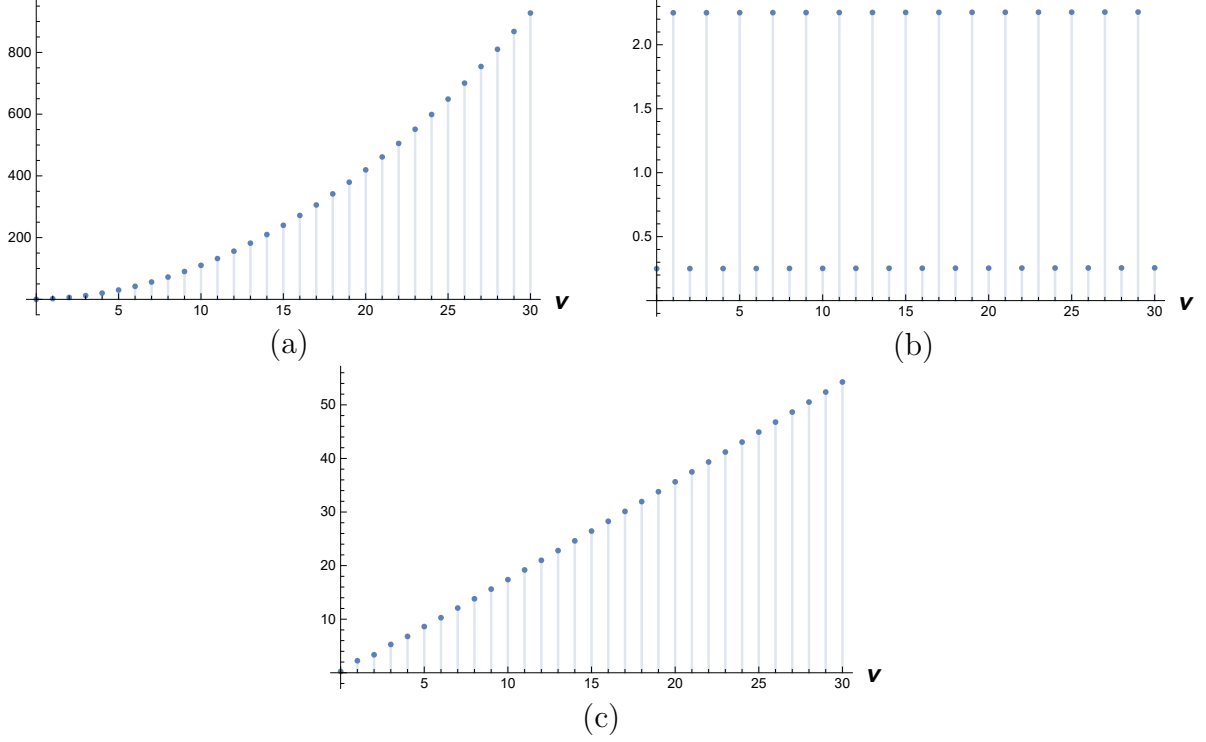
In order to prove these we write  $(\Delta\hat{Q}_s)^2(\Delta\hat{P}_s)^2$  for  $s = a, b$  in terms of their ladder operator representations (3.2.3) and find that the only non-zero contributions are of the form  $\langle\varphi_\nu^{(0)}|s^+s^-|\varphi_\nu^{(0)}\rangle$ . For  $\langle\varphi_\nu^{(0)}|a^+a^-|\varphi_\nu^{(0)}\rangle$  we find

$$\begin{aligned}\langle\varphi_\nu^{(0)}|a^+a^-|\varphi_\nu^{(0)}\rangle &= \frac{1}{\mathcal{N}_\nu^{(0)}(\alpha, \beta)} \sum_{k=0}^{\lfloor\frac{\nu}{2}\rfloor} |\alpha|^{2(\nu-k)} |\beta|^{2k} \frac{k}{(\nu-2k)!k!2^{2k}} \\ &= \frac{1}{\mathcal{N}_\nu^{(0)}(\alpha, \beta)} \sum_{k=0}^{\lfloor\frac{\nu}{2}\rfloor-1} |\alpha|^{2(\nu-1-k)} |\beta|^{2(k+1)} \frac{1}{(\nu-2-2k)!k!2^{2(k+1)}} \\ &= \frac{1}{4} |\alpha|^2 |\beta|^2 \nu(\nu-1) \frac{\mathcal{N}_{\nu-2}^{(0)}(\alpha, \beta)}{\mathcal{N}_\nu^{(0)}(\alpha, \beta)},\end{aligned}\tag{3.5.3}$$

where in the second line of (3.5.3) we changed the summation index  $k \rightarrow k+1$  and in the third line we used the fact that  $\lfloor\frac{\nu}{2}\rfloor-1 = \lfloor\frac{\nu-2}{2}\rfloor$  to rewrite the summation in terms of the normalisation function with index  $\nu-2$ . The same principle is used to compute terms of the form  $\langle\varphi_\nu^{(0)}|b^+b^-|\varphi_\nu^{(0)}\rangle$  allowing us to recover the uncertainty relations in (3.5.1) and (3.5.2).



**Fig. 3.3.**  $\left((\Delta\hat{Q}_a)^2(\Delta\hat{P}_a)^2\right)|\varphi_\nu^{(0)}\rangle$  as a function of  $\nu$ . (a)  $\alpha = 100, \beta = 1$ , (b)  $\alpha = 1, \beta = 100$ , (c)  $\alpha = 1, \beta = 1$ .



**Fig. 3.4.**  $\left((\Delta\hat{Q}_b)^2(\Delta\hat{P}_b)^2\right)_{\left|\varphi_{\nu}^{(0)}\right\rangle}$  as a function of  $\nu$ . (a)  $\alpha = 100, \beta = 1$ , (b)  $\alpha = 1, \beta = 100$ , (c)  $\alpha = 1, \beta = 1$ .

The states satisfy the physical condition on the position-momentum uncertainty relation in both the  $a$  and  $b$  modes,  $(\Delta\hat{Q}_i)^2(\Delta\hat{P}_i)^2 \geq \frac{1}{4}$ . In (a) of figure 3.3 we have approximated the limit  $|\alpha| \gg |\beta|$ , so the  $b$  mode is the dominant mode. In this limit the mixing of the modes is such that the effects of the  $a$  mode are diluted by that of the  $b$  mode and as a result we see that the product of uncertainties in the  $a$  mode increase relatively slowly when compared with (a) of figure 3.4, where the uncertainty relation increases much more rapidly as a function of  $\nu$ .

In (b) of figures 3.3 and 3.4 we approximate the limit  $|\beta| \gg |\alpha|$ , in this case the mixing term in  $\mathcal{A}^+$  is dominant and as such we see a staggering pattern in the uncertainties of the  $a$  mode, for an even value of  $\nu = 2n$ , we observe that the uncertainty in the state  $2n + 1$  is approximately equal. Meanwhile in the  $b$  mode the effect is such that for odd values of  $\nu$  the product of uncertainties  $\left((\Delta\hat{Q}_b)^2(\Delta\hat{P}_b)^2\right)_{\left|\varphi_{2\nu+1}^{(0)}\right\rangle} \sim 2.25$  while for even values  $\left((\Delta\hat{Q}_b)^2(\Delta\hat{P}_b)^2\right)_{\left|\varphi_{2\nu}^{(0)}\right\rangle} \sim 0.25$ , this is due to the fact that the state  $\left|\varphi_{2\nu+1}^{(0)}\right\rangle$  contains the same number of basis states  $|k, \nu - 2k\rangle$  as the state  $\left|\varphi_{2\nu}^{(0)}\right\rangle$  but with a larger number in the  $b$  mode and therefore there is a larger uncertainty associated to odd values of  $\nu$ . The staggering in the  $a$  mode is a result of the anisotropy of the system, because  $a^+$  is worth two

quanta (relative to  $b^+$  being worth one quantum), we only see a measurable increase in its uncertainty when  $\nu$  increases by two.

In (c) of figures 3.3 and 3.4 we choose  $\alpha = \beta$ , so that the mixing term has the same weight as the term adding only  $b$  modes, we find that the uncertainty in the  $a$  mode is parabolic in  $\nu$  while the uncertainty in  $b$  is linear.

## 3.6. Conclusion

In this article we have presented a new set of states for the 2:1 quantum anisotropic oscillator. They admit a ladder operator construction and a resolution of the identity, moreover, they reproduce the Lissajous figures of [18] and generalise the  $\mathfrak{su}(2)$  coherent states of the two dimensional isotropic oscillator. Furthermore we find that the mixing of the modes means that the uncertainty relations for each mode are codependent, and, for certain choices of parameters there are interesting staggering patterns on the respective uncertainties.

We also found in the general construction that we can build chains of states from higher energy zero modes defined by the operator  $\mathcal{A}^-$ , and as such building states by defining ladder operators rather than defining expansion coefficients leads to a rich structure in the space of states. We focused on the principle chain of states in this paper, but similar analyses may be completed for the other chains of states.

It would be interesting to consider other types of ladder operators, it is clear that  $\mathcal{A}^+$  is not a unique choice which picks out the appropriate energy eigenstates, though it appears to be the simplest one. Other choices of ladder operator will be, in general, harder to analytically normal order, though the solution to this may lie in extending some results in [28] to the ordering of multidimensional operators. As well, the inclusion of accidentally degenerate states into this formalism will require additional caution because they cannot be predicted by symmetry arguments and therefore the process of defining ladder operators to capture this may need some modification. Finally, applying these techniques to multidimensional systems other than the harmonic oscillator would be interesting, a system such as the 2D Morse potential which is a more realistic modelling of the vibrations of molecules [29], or the Pais-Uhlenbeck oscillator [30] which has been studied as one possible path to a theory of quantum gravity. The techniques presented in this paper may lead to interesting classes of states for these systems.

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### 3.A Resolution of the identity

To compute the resolution of the identity we write the parameters  $\alpha = |\alpha|e^{i\theta}$ ,  $\beta = |\beta|e^{i\phi}$ , in polar form and considering for some measure  $\mu_\nu(|\alpha|, |\beta|)$ ,

$$\int_0^\infty d|\alpha| |\alpha| \int_0^\infty d|\beta| |\beta| \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \mu_\nu(|\alpha|, |\beta|) \mathcal{N}_\nu(\alpha, \beta) \left| \varphi_\nu^{(0)} \right\rangle \left\langle \varphi_\nu^{(0)} \right|. \quad (3.A.1)$$

The angular integrations over  $\theta, \phi$ , yield a factor of  $4\pi^2$  multiplied by a Kronecker delta matching the summation indices of  $\left| \varphi_\nu^{(0)} \right\rangle \left\langle \varphi_\nu^{(0)} \right|$ . To address the radial integrations

$$\begin{aligned} & 4\pi^2 \nu! \int_0^\infty d|\alpha| \int_0^\infty d|\beta| \mu_\nu(|\alpha|, |\beta|) \\ & \times \sum_{k=0}^{\lfloor \frac{\nu}{2} \rfloor} |\alpha|^{2(\nu-k)+1} |\beta|^{2k+1} \frac{1}{(\nu-2k)! k! 2^{2k}} |k, \nu-2k\rangle \langle k, \nu-2k|, \end{aligned} \quad (3.A.2)$$

we make use of the following identities

$$\int_0^\infty dx x^{2k+1} e^{-cx^2} = \frac{k!}{2(c^{k+1})}, \quad k \in \mathbb{Z}, c > 0, \quad (3.A.3)$$

and

$$\int_0^\infty dx x^n e^{-dx} = \frac{n!}{d^{n+1}}, \quad n \in \mathbb{Z}^{\geq 0}, \text{Re}(d) > 0, \quad (3.A.4)$$

after which we observe that choosing the measure

$$\mu_\nu(|\alpha|, |\beta|) = \frac{1}{8\pi^2 \nu!} \frac{e^{-|\alpha| - \frac{|\beta|^2}{4}}}{|\alpha|^{\nu+1}}, \quad (3.A.5)$$

produces the correct factorial terms to cancel the denominator of (3.A.2) and we obtain

$$\sum_{k=0}^{\lfloor \frac{\nu}{2} \rfloor} |k, \nu-2k\rangle \langle k, \nu-2k| \equiv \mathbb{I}_\nu, \quad (3.A.6)$$

in agreement with (3.4.8).

To obtain the identity operator on the full Hilbert space we sum over each partition  $\nu$  using the reverse of the Cauchy product formula

$$\left(\sum_{n=0}^{\infty} x_n\right) \left(\sum_{m=0}^{\infty} y_m\right) = \sum_{k=0}^{\infty} \sum_{l=0}^k x_l y_{k-l}. \quad (3.A.7)$$

We consider  $\nu$  even and odd separately in (3.A.6). For  $\nu \rightarrow 2\nu'$  even

$$\sum_{\nu'=0}^{\infty} \sum_{k=0}^{\nu'} |k, 2\nu' - 2k\rangle \langle k, 2\nu' - 2k| = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |n, 2m\rangle \langle n, 2m|, \quad (3.A.8)$$

and for  $\nu \rightarrow 2\nu' + 1$  odd

$$\sum_{\nu'=0}^{\infty} \sum_{k=0}^{\nu'} |k, 2\nu' - 2k + 1\rangle \langle k, 2\nu' - 2k + 1| = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |n, 2m + 1\rangle \langle n, 2m + 1|. \quad (3.A.9)$$

The combination of (3.A.8) and (3.A.9) gives the desired result (3.4.9)

$$\sum_{\nu=0}^{\infty} \mathbb{I}_{\nu} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |n, m\rangle \langle n, m| = \mathbb{I}_{\mathcal{H}}. \quad (3.A.10)$$



# Chapter 4

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## Degeneracy and coherent states of the two-dimensional Morse potential

by

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- Performed all calculations in the paper
- Wrote and edited the paper

**ABSTRACT.** In this paper we construct coherent states for the two-dimensional Morse potential. We find the dependence of the spectrum on the physical parameters and use this to understand the emergence of accidental degeneracies. It is observed that, under certain conditions pertaining to the irrationality of the parameters, accidental degeneracies do not appear and as such energy levels are at most two-fold degenerate. After defining a non-degenerate spectrum and set of states for the 2D Morse potential, we construct generalised coherent states and discuss the spatial distribution of their probability densities and their uncertainty relations.

**Keywords:** Degeneracy, Coherent states, Two-dimensional quantum systems.

## 4.1. Introduction

The Morse potential was originally introduced as a means to model interactions in diatomic molecules [1], it is an exactly solvable model with eigenfunctions expressible in terms of Laguerre polynomials. In physical contexts the two-dimensional Morse product eigenfunctions have been used as a basis for perturbative solutions to a triatomic molecular Hamiltonian [2, 3, 4, 5, 6] and interacting Morse oscillators [7, 8]. Throughout physics the Morse potential is used in a variety of applications including the study of graphene [9, 10], spectroscopy [11, 12], Bose-Einstein condensation [13], theories of interacting electrons [14], nuclear physics [15], supersymmetric quantum mechanics [16], and molecular dynamics [17]. Coherent states for the 1D Morse potential have been studied [18], and while there is literature on defining coherent states for systems with degenerate spectra [19], and coherent states for the 2D square well have been analysed [20], so far coherent states for the 2D Morse potential have not been explicitly defined.

The 2D Morse potential, and more specifically, its supersymmetric generalisations have been studied in detail [21, 22, 23, 24, 25]. Because the supersymmetric partners of the 2D Morse share the same spectrum, they also give rise to accidentally degenerate states, the existence of which may be explained in terms of an operator constructed from supercharges [25]. For our purposes we do not need to invoke the framework of supersymmetry, instead we can study the degeneracy in terms of the rationality of the physical constants appearing in the definition of the potential.

Degeneracy arises in practically all multidimensional quantum systems and in the Morse potential the degeneracy is found to be quadratic in the principle quantum number. Categorising degeneracies in 2D systems with quadratic spectra has solutions found in number theory. This is distinct from the case where, for example, the spectrum is linear in the quantum number, we can associate a known symmetry group to the spectrum such as  $U(n)$  for the  $n$ -dimensional isotropic oscillator [26].

There are several quantum systems with quadratic spectra, including the Morse potential, Pöschl-Teller [27] and the particle in an infinite square box [28]. The prototypical 2D quadratic spectrum is that of the particle in a square box. Finding the degenerate energies is equivalent to finding numbers which are the sum of the squares of two integers [29]. Because the square box only admits bound states, in principle we need to find solutions to the sum of the squares of two integers to infinity.

The 2D Morse spectrum is different in a few key ways. Firstly, if we are just interested in the bound states of the system, we need only to find energies up to a certain finite value. Secondly, the behaviour of the degeneracies changes when the defining parameter in the Morse potential changes. As we will show, accidentally degenerate solutions do not exist when this parameter is irrational.

The understanding of the degeneracy of multidimensional quantum systems is important in the construction of generalised coherent states [19]. Indeed, the usual definitions require a spectrum to be ordered as

$$E_0 < E_1 < E_2 < \dots < E_M, \quad (4.1.1)$$

in order to fulfil the resolution of the identity [30]. For most systems it is important to work with complete sets of states, in the Morse potential however, we will only be studying the finite dimensional bound state spectrum and as a result do not require a resolution of the identity on the entire Hilbert space, though one in principle may be constructed on the finite dimensional bound state sector. This being said, the ordering of the spectrum is still required so that we can extend the formalism of generalised coherent states to the 2D Morse potential without modifying the existing definitions.

The paper is organised as follows. Firstly, in section 4.2 we define the energy eigenstates and eigenvalues of the 2D Morse potential as well its principle parameter,  $p$ . Following this in section 4.3 we discuss the nature of the degeneracies that may arise depending on the rationality of the principle parameter. Focussing on the case of irrational  $p$ , in section 4.4 we introduce two parameters,  $\gamma_1, \gamma_2$  which control the mixing of the degenerate contributions and define the cumulative wavefunctions. Lastly, in sections 4.5 and 4.6 we assess the behaviour of the generalised coherent states and their probability distributions as well as computing their uncertainty relations, we conclude by discussing future work that could be made on the subject.

## 4.2. The 2D Morse potential and the parameter $p$

The 2D isotropic Morse Hamiltonian is defined by

$$\hat{H} = \frac{1}{2m} (\hat{P}_x^2 + \hat{P}_y^2) + V_0 \left( e^{-2\beta\hat{Q}_x} + e^{-2\beta\hat{Q}_y} - 2 \left( e^{-\beta\hat{Q}_x} + e^{-\beta\hat{Q}_y} \right) \right), \quad (4.2.1)$$

where  $\hat{P}_x, \hat{P}_y$  are the momentum operators and  $\hat{Q}_x, \hat{Q}_y$  are the corresponding position operators. The Hamiltonian (4.2.1) is isotropic in the sense that the parameters in the  $x$  mode are equal to those in the  $y$  mode. The entire Hilbert space decomposes into the sum of a finite dimensional bound state part with discrete spectrum and an infinite dimensional unbound state part with continuous spectrum. In the present work we are concerned with only the bound states of the Morse oscillator.

The bound quantum states are found by solving the stationary Schrödinger equation

$$\hat{H} |n, m\rangle = E_{n, m} |n, m\rangle, \quad (4.2.2)$$

to obtain energy eigenvalues

$$E_{n, m} = -\frac{\hbar^2 \beta^2}{2m} \left( (p - n)^2 + (p - m)^2 \right) \quad (4.2.3)$$

where the parameters

$$\nu = \sqrt{\frac{8mV_0}{\beta^2\hbar^2}}, \quad p = \frac{\nu - 1}{2}, \quad (4.2.4)$$

have been defined. The parameter  $p$  (or equivalently  $\nu$ ) we refer to as the principle parameter of the Morse potential, as it is this combination of the physical constants that determine the behaviour of the degeneracy in the spectrum. The eigenvectors of (4.2.2) in their position representation are given by

$$\begin{aligned} \psi_{n,m}(x,y) &= \langle x,y|n,m\rangle \\ &= \mathcal{N}_{n,m} e^{-\frac{\tilde{x}}{2} - \frac{\tilde{y}}{2}} \tilde{x}^{p-n} \tilde{y}^{p-m} L_n^{2(p-n)}(\tilde{x}) L_m^{2(p-m)}(\tilde{y}), \end{aligned} \quad (4.2.5)$$

where the tilde variables are related to the canonical position variables by

$$\tilde{x} = \nu e^{-\beta x}, \quad \tilde{y} = \nu e^{-\beta y}, \quad (4.2.6)$$

$L_n^\alpha(z)$  are the Laguerre polynomials and the normalisation factor  $\mathcal{N}_{n,m}$  is given explicitly by

$$\mathcal{N}_{n,m} = \beta \sqrt{\frac{(\nu - 2n - 1)(\nu - 2m - 1)\Gamma(n + 1)\Gamma(m + 1)}{\Gamma(\nu - n)\Gamma(\nu - m)}}. \quad (4.2.7)$$

Additionally we have the completeness relation on the bound states

$$\sum_{n=0}^{\lfloor p \rfloor} \sum_{m=0}^{\lfloor p \rfloor} |n,m\rangle \langle n,m| = \mathbb{1}_{\text{BS}}. \quad (4.2.8)$$

For  $p$  not integer, the 1D Morse potential admits only a finite number,  $\lfloor p \rfloor + 1$ , of bound states where  $\lfloor r \rfloor$  is the integer part of  $r$ . As such the two-dimensional Morse system admits  $(\lfloor p \rfloor + 1) \times (\lfloor p \rfloor + 1)$  bound states. The quantum numbers  $n, m$  take on the finite number of values

$$n, m \in \{0, 1, \dots, \lfloor p \rfloor\}. \quad (4.2.9)$$

If  $p$  is integer, the zero energy state is not normalisable, this can be seen in the normalisation factor (4.2.7) which vanishes at  $p = n$ .

### 4.2.1. Ladder operators for the 1D Morse oscillator

Ladder operators for the 1D Morse potential have been explicitly realised as differential operators. Due to the bounded spectrum, a ladder operator construction built out of the compact  $SU(2)$  generators appears naturally [31]. Additionally, constructions based on the two-dimensional parameter space the of confluent hypergeometric functions are presented in [32, 33, 34, 35] where the eigenstates of the 1D Morse oscillator are connected by ladder operators not only in the quantum number  $n$  but also in terms of the parameter  $\nu$ .

In this paper we are considering the uncoupled two-dimensional Morse oscillator and in principle we may define sets of ladder operators for the  $x$  and  $y$  modes separately. However,

we are interested in the problem of the degeneracy in the quantum numbers,  $n, m$ , and this degeneracy is categorised by number theoretical means. A proper treatment of the explicit realisation of ladder operators for the spectrum defined in the following sections is relevant and deserves its own paper.

### 4.3. Analysis of the degeneracies of the energy spectrum

In this section we will focus on equation (4.2.3) and the role of the principle parameter  $p$  in determining the nature of the degeneracy in the spectrum. Assuming the principle parameter  $p$  to be a real number, the structure of the degeneracy depends on the rationality of  $p$ . It is not possible to say whether  $p$  is rational or not, it is built out of experimentally determined numbers  $m, V_0, \beta, \hbar$  which themselves may be rational (or the number determined by experiment is a rational approximation), but the square root of the ratio (4.2.4) may not be rational. We will demonstrate that accidental degeneracies are an inevitability when  $p$  is taken to be rational, but they can be eliminated by choosing  $p$  to be irrational.

Consider then the scaled bound state energy spectrum with parameter  $p$  defined in (4.2.3)

$$\varepsilon_{n,m} = - \left[ (p-n)^2 + (p-m)^2 \right], \quad n, m \in \{0, 1, \dots, [p]\}. \quad (4.3.1)$$

Writing the parameter  $p$  as the sum of its closest integer and a remainder

$$p = k + \epsilon, \quad k = [p], \quad \epsilon \in [0, 1), \quad (4.3.2)$$

we may rewrite (4.3.1) as

$$\varepsilon_{n,m}(k, \epsilon) = - \left[ (k-n)^2 + (k-m)^2 + 2\epsilon(2k-n-m) + 2\epsilon^2 \right]. \quad (4.3.3)$$

The three distinct cases in (4.3.3) we can discuss here are  $p$  integer,  $p$  rational and  $p$  irrational. In the notation we have introduced these are:

$$\text{Case I} \quad \varepsilon_{n,m}(k, 0) \quad p \text{ integer} \quad (4.3.4)$$

$$\text{Case II} \quad \varepsilon_{n,m} \left( k, \frac{r}{q} \right) \quad p \text{ rational} \quad (4.3.5)$$

$$\text{Case III} \quad \varepsilon_{n,m}(k, \epsilon) \quad p \text{ irrational,} \quad (4.3.6)$$

for  $\frac{r}{q} \in [0, 1) \subset \mathbb{Q}$  and  $\epsilon \in (0, 1) \subset \mathbb{R} \setminus \mathbb{Q}$ .

#### 4.3.1. Case I

When the remainder term  $\epsilon = 0$  we find

$$\varepsilon_{n,m}(k, 0) = - \left[ (k-n)^2 + (k-m)^2 \right], \quad (4.3.7)$$

where  $n, m \in \{0, 1, \dots, k-1\}$ . Solutions to this problem are well understood through Gaussian prime decomposition [29]. As an example, if we take  $k = 9$

$$\varepsilon_{n,m}(9,0) = - \left[ (9-n)^2 + (9-m)^2 \right], \quad (4.3.8)$$

we find that

$$\varepsilon_{2,8}(9,0) = \varepsilon_{8,2}(9,0) = \varepsilon_{4,4}(9,0). \quad (4.3.9)$$

In (4.3.9) the first two solutions are related by permuting the indices  $n, m$ , but neither are related to the third solution by any known symmetry. This problem is similar to that of the particle in a square box, though, because the quantum numbers  $n, m$  take on only finitely many values, some solutions may be discarded. For instance  $1^2 + 8^2 = 8^2 + 1^2 = 4^2 + 7^2 = 7^2 + 4^2$ , but if  $k = 7$  then the solutions  $(1,8), (8,1)$  lie outside of the bound state parameter range and as such should not be included as degenerate contributions.

To give a sense of scale to the problem of accidentally degenerate states, if we take the large example of  $k = 29$  then we have 841 different bound states. After removing the doubly degenerate states (states symmetric under interchange of indices) we have 435 states. Upon analysing the remaining states we find 361 distinct values for the energy. This implies that there are 74 accidentally degenerate (not counting the interchanging of their indices) states. The degrees of degeneracy also vary. Clearly this problem proliferates for larger  $p$ .

### 4.3.2. Case II

The next distinct case is for rational  $p$ , we may take the remainder term to be a rational number on the interval  $[0,1)$

$$p \in \mathbb{Q}, \quad \epsilon = \frac{r}{q} \in [0,1). \quad (4.3.10)$$

The spectral problem then takes the form

$$\varepsilon_{n,m} \left( k, \frac{r}{q} \right) = - \left[ (k-n)^2 + (k-m)^2 + 2\frac{r}{q}(2k-n-m) + 2\frac{r^2}{q^2} \right]. \quad (4.3.11)$$

The rational  $p$  case includes as a limiting case ( $\epsilon = 0$ ) the integer  $p$  degeneracy problem but also a more general class of accidental degeneracies. Consider the following example of a rational  $p$  degeneracy,

$$\varepsilon_{n,m} \left( 7, \frac{1}{2} \right) = - \left[ (7-n)^2 + (7-m)^2 + (14-n-m) + \frac{1}{2} \right]. \quad (4.3.12)$$

we find

$$\varepsilon_{2,6} \left( 7, \frac{1}{2} \right) = \varepsilon_{6,2} \left( 7, \frac{1}{2} \right) = \varepsilon_{3,4} \left( 7, \frac{1}{2} \right) = \varepsilon_{4,3} \left( 7, \frac{1}{2} \right). \quad (4.3.13)$$

Clearly the first two solutions are related by a permutation of indices, as are the last two, but these two sets of solutions do not have a known symmetry connecting them. This generalises

the previous problem where we found solutions represented as sum of two squares, this problem is the sum of two squares plus a fraction of the sum of the respective linear terms.

There is a subtle point to be made here, while accidental degeneracies can occur for rational  $p$ , it does not mean they will with certainty. If we want no accidental degeneracies to occur to simplify calculations, in order to implement the following results in a computer algebra system, it is necessary to find a rational value of  $p$  close enough to our initial  $p$  which does not produce accidental degeneracies. This usually means keeping more terms in the decimal expansion. For small enough values of  $p$  it is straight forward to determine how many accidental degeneracies occur with a computer and thus it is easy to verify whether a certain choice of rational  $p$  works well, though it is hard to make more general comments for arbitrarily large values of rational  $p$ .

### 4.3.3. Case III

Lastly, the case in which we will focus our attention from here on out is when  $p$  is irrational. For  $\epsilon$  some irrational number on  $[0,1)$  the spectrum (4.3.3) is

$$\varepsilon_{n,m}(k, \epsilon) = - \left[ (k-n)^2 + (k-m)^2 + 2\epsilon(2k-n-m) + 2\epsilon^2 \right]. \quad (4.3.14)$$

This equation has no accidentally degenerate solutions, only degenerate solutions obtained from the permutation of the indices  $n,m$ .

Indeed, for accidentally degenerate solutions to (4.3.14),  $(n,m)$  and  $(n',m')$ , we must satisfy the rational and irrational parts of the equation separately because an irrational multiple of a rational number cannot coincide with a rational number, that is to say

$$(k-n)^2 + (k-m)^2 = (k-n')^2 + (k-m')^2, \quad (4.3.15)$$

and

$$(k-n) + (k-m) = (k-n') + (k-m'), \quad (4.3.16)$$

must be satisfied. These equations may be thought of in terms of two triangles with the same length hypotenuse and perimeters. In order for (4.3.15) and (4.3.16) to be simultaneously satisfied we square (4.3.16) and substitute in (4.3.15), in doing so imply the area equation for the triangles

$$(k-n)(k-m) = (k-n')(k-m'). \quad (4.3.17)$$

Using the fact that  $(k-n) = (k-n') + (k-m') - (k-m)$ , substitution into (4.3.17) gives a quadratic equation in  $(k-m)$

$$\begin{aligned} (k-m)^2 - [(k-n') + (k-m')] (k-m) \\ + (k-n')(k-m') = 0 \end{aligned} \quad (4.3.18)$$

which admits the solutions  $m = n'$  or  $m = m'$ , after which  $n$  is uniquely determined by (4.3.16). Thus we have at most doubly degenerate eigenvalues in the spectrum and these are precisely the degeneracies found by permuting the indices  $n, m$ . This makes the degeneracy problem more tractable. Irrationality of the principle parameter  $p$  is the key to breaking the degeneracy symmetry.

## 4.4. Ordering of the 2D non-degenerate spectrum

After discussing the degeneracies we are in a position to be able to organise the spectrum of the full 2D system in terms of a single index. If we consider the principle parameter  $p$  to be irrational, we found that states are at most doubly degenerate. To this end whenever we encounter a doubly-degenerate eigenvalue we define the cumulative wavefunction with a pair of complex coefficients,  $\gamma_1, \gamma_2$ , such that

$$|\mu_i\rangle_p = \gamma_1 |n, m\rangle + \gamma_2 |m, n\rangle, \quad n > m, \quad \gamma_1, \gamma_2 \in \mathbb{C}. \quad (4.4.1)$$

Note that  $n > m$  ensures that we uniformly introduce the coefficients throughout the spectrum. The complex coefficients (4.4.1) are subject to

$$|\gamma_1|^2 + |\gamma_2|^2 = 1, \quad (4.4.2)$$

to preserve normalisation. Otherwise, for non degenerate states (states of the form  $|n, n\rangle$ ) we simply take the definition

$$|\mu_j\rangle_p = |n, n\rangle. \quad (4.4.3)$$

Implicitly, each state comes with a degeneracy index  $d_i$  which is equal to either 1 or 2 corresponding to non-degenerate and doubly degenerate contributions respectively. The states form a complete set with the following resolution of the identity

$$\frac{d_i}{\pi^2} \int_{S^3} d^2\gamma_1 d^2\gamma_2 \delta(|\gamma_1|^2 + |\gamma_2|^2 - 1) |\mu_i\rangle_p \langle \mu_i|_p = \mathbb{1}_i, \quad (4.4.4)$$

where the bound state identity operator is then recovered by

$$\sum_{i=0}^{\xi} \mathbb{1}_i = \mathbb{1}_{\text{BS}}. \quad (4.4.5)$$

Considering the spectrum (4.3.14), where for convenience we remove the constant term,  $\epsilon^2$ , to define a shifted energy

$$\tilde{\epsilon}_{n,m}(k, \epsilon) = - \left[ (k - n)^2 + (k - m)^2 + 2\epsilon(2k - n - m) \right], \quad (4.4.6)$$

which has maximum,  $\max \tilde{\epsilon}_{n,m}(k, \epsilon) = 0$ . We table values and order the energies as follows

$$\tilde{\epsilon}_{0,0}(k, \epsilon) < \tilde{\epsilon}_{1,0}(k, \epsilon) = \tilde{\epsilon}_{0,1}(k, \epsilon) < \dots < \tilde{\epsilon}_{k,k}(k, \epsilon) = 0 \quad (4.4.7)$$



once we remove duplicated energies, this inequality is in one to one correspondence with the single indexed spectrum

$$\varepsilon_{\mu_0}(k, \epsilon) < \varepsilon_{\mu_1}(k, \epsilon) < \dots < \varepsilon_{\mu_\xi}(k, \epsilon). \quad (4.4.8)$$

The number  $\xi$  can be computed for any value of  $k$ , it is just the total number of unique elements in the  $(k+1) \times (k+1)$  symmetric matrix with matrix elements  $\tilde{\varepsilon}_{n,m}(k, \epsilon)$  minus one (because we count the first state with index zero), it is given as

$$\xi = \frac{(k+1)(k+2)}{2} - 1. \quad (4.4.9)$$

The final consideration we have is the dependence of the ordering on the parameter  $\epsilon$ . Take the example  $k = 3$  and the energies

$$\tilde{\varepsilon}_{3,0}(3, \epsilon) = 9 + 6\epsilon, \quad \tilde{\varepsilon}_{1,1}(3, \epsilon) = 8 + 8\epsilon, \quad (4.4.10)$$

for  $\epsilon < 0.5$  we find  $\tilde{\varepsilon}_{3,0}(3, \epsilon) > \tilde{\varepsilon}_{1,1}(3, \epsilon)$ , while for  $\epsilon > 0.5$  we find  $\tilde{\varepsilon}_{3,0}(3, \epsilon) < \tilde{\varepsilon}_{1,1}(3, \epsilon)$ . This does prevent us for obtaining more general solutions for any irrational  $\epsilon$ . However, once  $\epsilon$  is fixed, the ordering is uniquely determined.

#### 4.4.1. Non-degenerate states for $p = 3\pi$

To illustrate the points made so far, we take the example of  $p = 3\pi \approx 9.42478$ , the bound state space is spanned by 100 different eigenfunctions and there are 55 distinct energy eigenvalues. This is our departing point, we will construct the set of states (4.4.1) which are associated to the distinct eigenvalues. In cases where doubly degenerate states appear we will introduce the parameters  $\gamma_1, \gamma_2$  that will control the mixing between the  $x$  and  $y$  modes to give one averaged contribution to the degenerate energy level.

The set

$$\mathcal{S} = \{|\mu_0\rangle_{3\pi}, |\mu_1\rangle_{3\pi}, \dots, |\mu_{54}\rangle_{3\pi}\}, \quad |\mathcal{S}| = 55, \quad (4.4.11)$$

provides all bound states of the problem under consideration. The states  $|\mu_i\rangle$  themselves are given in terms of the original eigenfunctions by

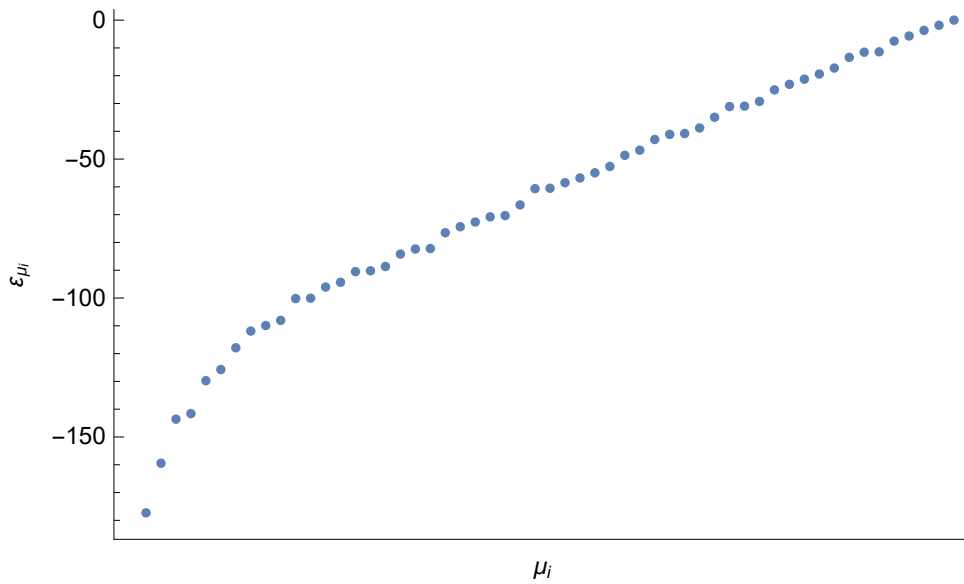
$$|\mu_i\rangle_{3\pi} = \begin{cases} \gamma_1 |n, m\rangle + \gamma_2 |m, n\rangle, & \text{for } n > m \\ |n, n\rangle, & \text{otherwise.} \end{cases} \quad (4.4.12)$$

Explicitly, the first few and last states are

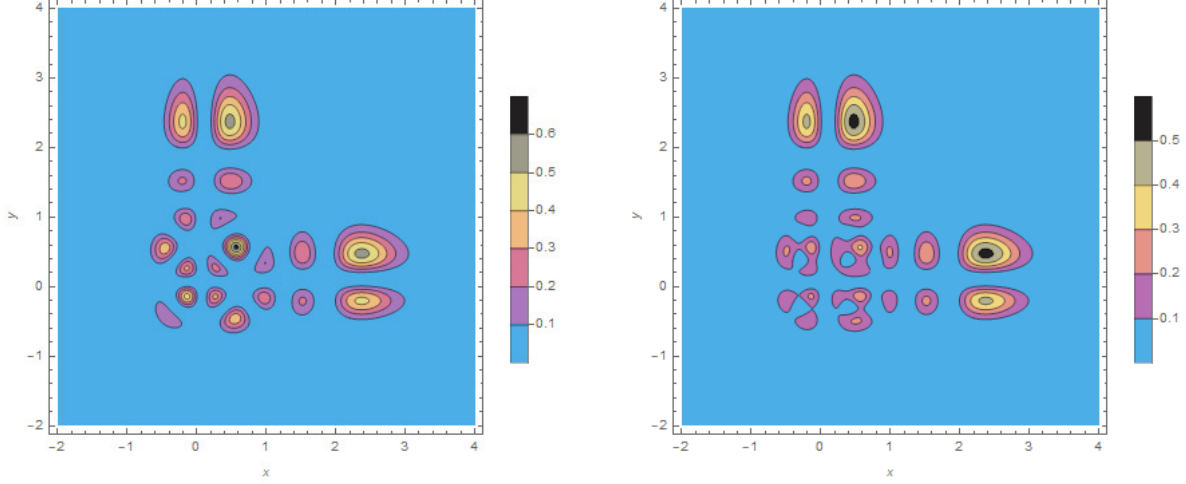
$$|\mu_i\rangle_{3\pi} = \begin{cases} |0,0\rangle, & i = 0 \\ \gamma_1 |1,0\rangle + \gamma_2 |0,1\rangle, & i = 1 \\ \gamma_1 |2,0\rangle + \gamma_2 |0,2\rangle, & i = 2 \\ |1,1\rangle, & i = 3 \\ \vdots & \\ |9,9\rangle, & i = 54. \end{cases} \quad (4.4.13)$$

The states  $|\mu_i\rangle$  have energies which correspond to arranging the following energy function in increasing order

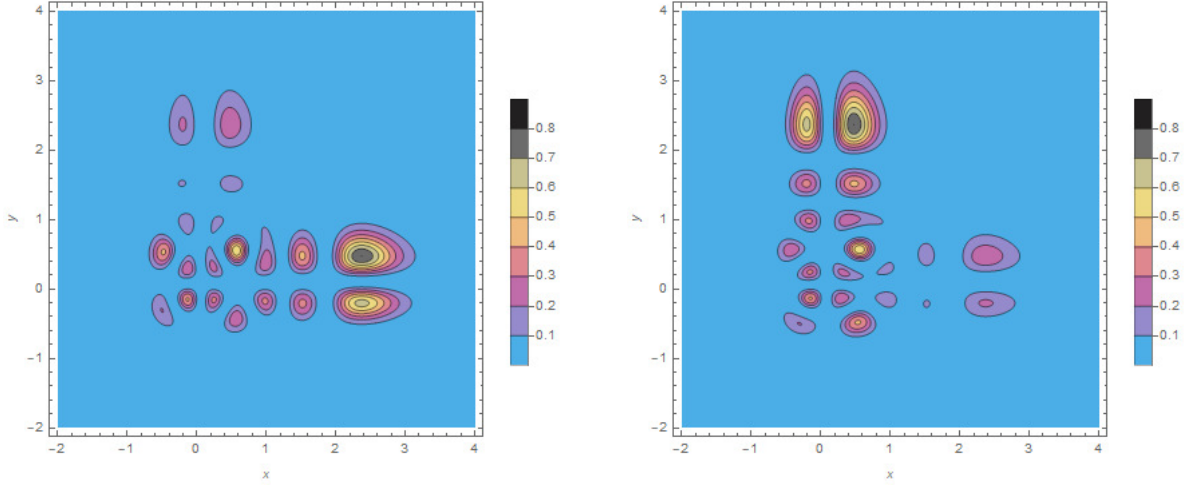
$$\varepsilon_{\mu_i}(9, 0.42478) = - \left[ (9 - n)^2 + (9 - m)^2 + 2(0.42478)(18 - n - m) \right], \quad (4.4.14)$$



**Fig. 4.1.** Energies  $\varepsilon_{\mu_0}, \dots, \varepsilon_{\mu_{54}}$  arranged in increasing order.



**Fig. 4.2.**  $|\langle x,y|\mu_{18}\rangle_{3\pi}|^2$ ,  $p = 3\pi$  with  $\gamma_1 = \frac{1}{\sqrt{2}}, \gamma_2 = \frac{1}{\sqrt{2}}$  (left) and  $\gamma_1 = \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}}, \gamma_2 = \frac{1}{\sqrt{2}}$  (right).



**Fig. 4.3.**  $|\langle x,y|\mu_{18}\rangle_{3\pi}|^2$ ,  $p = 3\pi$  with  $\gamma_1 = \frac{\sqrt{3}}{2}, \gamma_2 = \frac{1}{2}$  (left) and  $\gamma_1 = \frac{1}{2}, \gamma_2 = \frac{\sqrt{3}}{2}$  (right).

The graph in figure 4.1 shows the possible values that the energy function (4.4.14) can take. These values are arranged in increasing order of magnitude. The states (4.4.12) or (4.4.13) are in one to one correspondence with the spectrum (4.4.14) and together they define a non-degenerate basis for the bound state sector of the 2D Morse potential at  $p = 3\pi$ . In figure 4.2 we see the effect of adding complex phase to a doubly degenerate state,  $|\mu_{18}\rangle_{3\pi}$ , the phase alters the positioning of some of the ‘islands’ of non-zero probability near to the origin due to the modes being in or out of phase, but it preserves the overall structure of the probability density. If on the other hand we change the magnitudes of  $\gamma_1, \gamma_2$  so that they are not equal we find that this corresponds to a larger change in the probability density function as seen in figure 4.3. For  $\gamma_1 > \gamma_2$  the higher energy  $x$  mode is mixed with greater probability, and as a result most of the probability density occupies the islands which extend along the  $x$

axis. Similarly, for  $\gamma_2 > \gamma_1$ , the most of the probability densities occupies the islands which extend along the  $y$  axis.

## 4.5. Ladder operators and coherent states

Coherent states for the harmonic oscillator were first studied by Schrödinger as minimal uncertainty wavepackets [36]. They may be defined through several equivalent means: as eigenstates of the annihilation operator, the action of the unitary displacement operator on the vacuum, or by their Fock space expansion. The three equivalent harmonic oscillator coherent state definitions read

$$a^- |\alpha\rangle = \alpha |\alpha\rangle, \quad \text{Barut-Girardello} \quad (4.5.1)$$

$$\exp(\alpha a^+ - \bar{\alpha} a^-) |0\rangle = |\alpha\rangle, \quad \text{Displacement operator} \quad (4.5.2)$$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad \text{Fock expansion.} \quad (4.5.3)$$

Typically, for systems other than the harmonic oscillator, the three definitions do not coincide. Another consequences of these definitions is that the coherent states are minimal uncertainty with respect to the Heisenberg uncertainty relation with equal uncertainty in the position and momentum quadratures. In units of  $\hbar = 1$  this is

$$(\Delta \hat{Q})_{|\alpha\rangle}^2 (\Delta \hat{P})_{|\alpha\rangle}^2 = \frac{1}{4}, \quad \Delta \hat{Q} = \Delta \hat{P}. \quad (4.5.4)$$

For some more general classes of coherent states, such as the squeezed states, the condition that the uncertainty in the position and momentum quadratures be equal is relaxed but their product still minimises the uncertainty relation.

When defining generalised coherent states for systems other than the harmonic oscillator, if we are not using any definitions regarding ladder operators or displacement operators, we use extensions of the form of (4.5.3). The extensions are formed by replacing the Fourier coefficients with some set of coefficients which satisfy completeness relations and produce good localisation of the coherent state wavefunction. As well, we replace the Fock basis vectors with the Fock states of the system under consideration. These definitions exist typically for 1D systems with non-degenerate spectra. In our case we used the preceding section to define a non-degenerate spectrum for the 2D Morse oscillator and as such allow ourselves to use these definitions.

Using the set of non-degenerate states

$$\mathcal{S} = \left\{ |\mu_0\rangle_p, |\mu_1\rangle_p, \dots, |\mu_\xi\rangle_p \right\}, \quad |\mathcal{S}| = \xi + 1, \quad (4.5.5)$$

and defining the set of ladder operators,  $\mathcal{B}^+, \mathcal{B}^-$ , such that

$$\begin{aligned}\mathcal{B}^+ |\mu_i\rangle_p &= \sqrt{f(i+1)} |\mu_{i+1}\rangle_p, \\ \mathcal{B}^- |\mu_i\rangle_p &= \sqrt{f(i)} |\mu_{i-1}\rangle_p,\end{aligned}\tag{4.5.6}$$

subject to boundary conditions on the finite set of states, i.e.  $f(0) = f(\xi + 1) = 0$ . We can define a generalised coherent state from a Barut-Girardello type coherent state as an approximate eigenstate of  $\mathcal{B}^-$  with complex eigenvalue  $\Psi$ ,

$$\mathcal{B}^- |\Psi\rangle_p \approx \Psi |\Psi\rangle_p.\tag{4.5.7}$$

Expanding  $|\Psi\rangle_p$  in the basis  $|\mu_i\rangle_p$  we retrieve the well known generalised coherent states [37]

$$\begin{aligned}|\Psi\rangle_p &= \frac{1}{\sqrt{\mathcal{N}(\Psi)}} \sum_{n=0}^{\xi} \frac{\Psi^n}{\sqrt{[f(n)]!}} |\mu_n\rangle_p, \\ [f(n)]! &= \prod_{m=0}^n f(m),\end{aligned}\tag{4.5.8}$$

where the generalised factorial takes the usual definition  $[f(0)]! = 1$  and the normalisation function

$$\mathcal{N}(\Psi) = \sum_{n=0}^{\xi} \frac{|\Psi|^{2n}}{[f(n)]!},\tag{4.5.9}$$

ensures  ${}_p\langle\Psi|\Psi\rangle_p = 1$ . The reason this is an approximate eigenstate is because of the finite spectrum, the last term in the expansion (4.5.8),  $\frac{\Psi^\xi}{\sqrt{[f(\xi)]!}} |\mu_\xi\rangle$ , does not appear when computing (4.5.7). Nevertheless this term is typically very small and therefore does not contribute much, but by including it we can recover the finite spectrum version of the generalised coherent states defined in [30].

There is some freedom in the choice of function  $f(i)$ , but a natural choice in analogy to the harmonic oscillator is to use the difference in energies with respect to the ground states, that is

$$f(i) = \begin{cases} \varepsilon_{\mu_i}(k, \epsilon) - \varepsilon_{\mu_0}(k, \epsilon), & \text{for } i \in \{0, 1, \dots, \xi\} \\ 0, & \text{otherwise.} \end{cases}\tag{4.5.10}$$

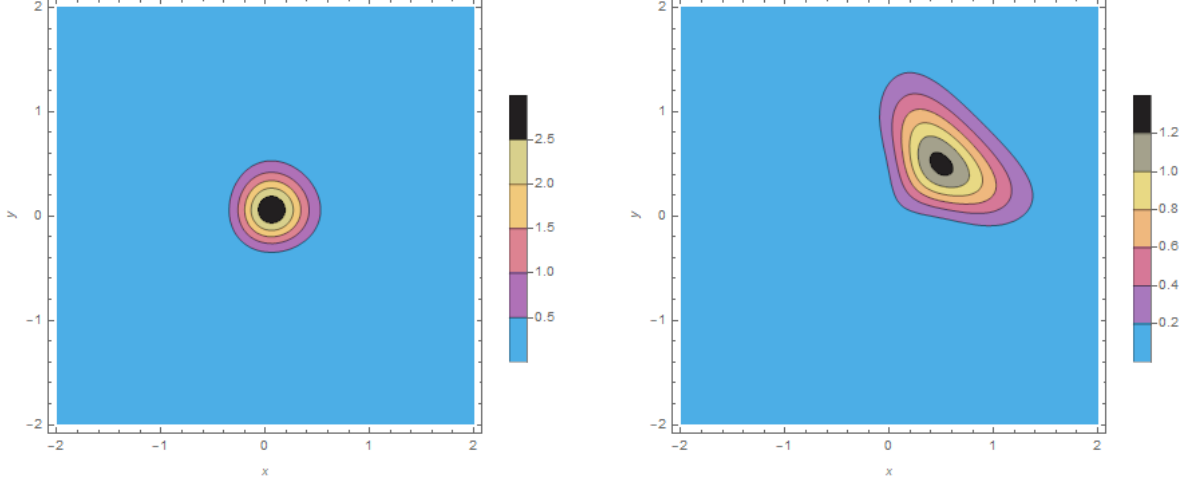
This function by definition satisfies the boundary conditions on the set, it appropriately annihilates the highest and lowest weight states. We will use this definition of coherent states from here on out.

### 4.5.1. Application to $p = 3\pi$

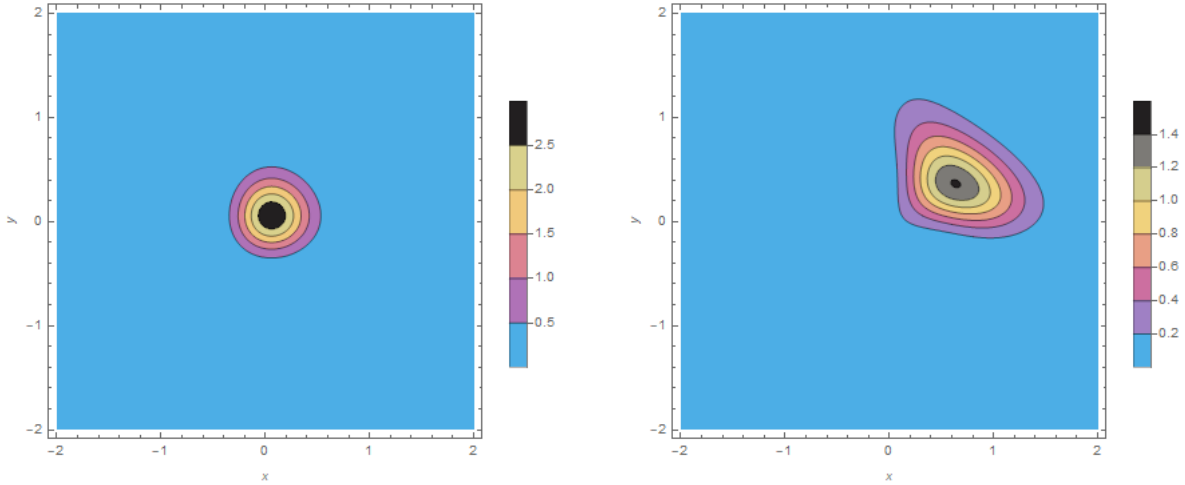
We now apply the formalism to our working example of  $p = 3\pi$ . Using the set of 55 states defined in (4.4.11) along with spectrum (4.4.14) we write the generalised coherent

state (4.5.8) as

$$|\Psi\rangle_{3\pi} = \frac{1}{\sqrt{\mathcal{N}(\Psi)}} \sum_{n=0}^{54} \frac{\Psi^n}{\sqrt{[\varepsilon_{\mu_n} - \varepsilon_{\mu_0}]!}} |\mu_n\rangle \quad (4.5.11)$$



**Fig. 4.4.** Spatial probability densities for the generalised coherent states,  $|\langle x,y|\Psi\rangle|^2$ , at  $\Psi = 0.1$  (left) and  $\Psi = 5$  (right). Both with  $\gamma_1 = \frac{1}{\sqrt{2}}, \gamma_2 = \frac{1}{\sqrt{2}}$ .



**Fig. 4.5.** Spatial probability densities for the generalised coherent states,  $|\langle x,y|\Psi\rangle|^2$ , at  $\Psi = 0.1$  (left) and  $\Psi = 5$  (right). Both with  $\gamma_1 = \frac{\sqrt{3}}{2}, \gamma_2 = \frac{1}{2}$ .

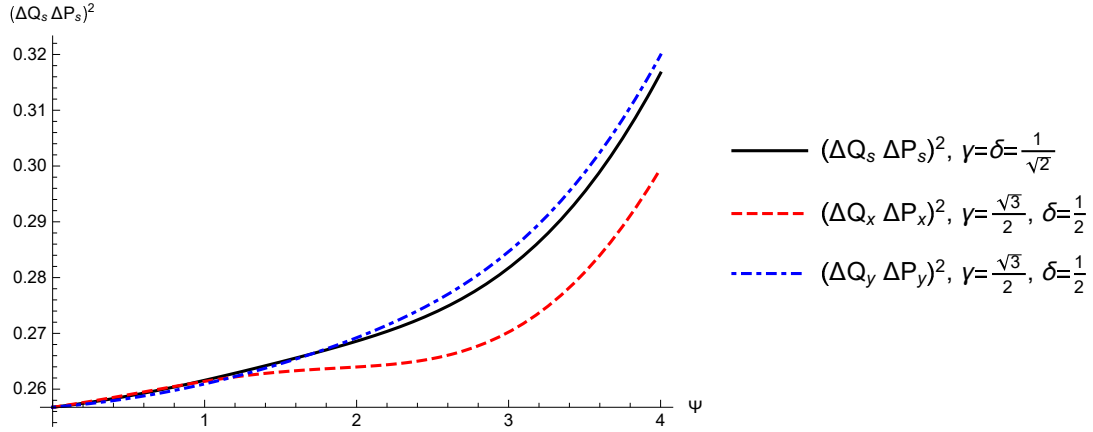
When we consider  $\Psi \in \mathbb{R}^+$  we find that the wavefunction has better localisation for smaller  $\Psi$ . This is expected, because the generalised coherent states are formed by a power series in  $\Psi$ , when  $\Psi$  is small, higher powers in the series contribute little to the wavefunction and the dominant contribution is the ground state.

The effect of the parameters  $\gamma_1, \gamma_2$  is minimal when the coherent state parameter,  $\Psi$ , is suitably chosen. If we set  $\gamma_1 \neq \gamma_2$  we do induce some asymmetry about the line  $y = x$  in the

probability distribution as seen in figure 4.5. This effect is more apparent for larger values of  $\Psi$  and almost indistinguishable from the  $\gamma_1 = \gamma_2$  case for small values of  $\Psi$ . The coherent states are most sensitive to changes in  $\Psi$ , but we have additional control over their behaviour by adjusting  $\gamma_1, \gamma_2$ .

For the generalised coherent states (4.5.11) we find that the Heisenberg uncertainty relation is satisfied, moreover, it is closer to its minimum for smaller values of  $\Psi$  as expected.

$$\left(\Delta\hat{Q}_s\right)_{|\Psi\rangle}^2 \left(\Delta\hat{P}_s\right)_{|\Psi\rangle}^2 \geq \frac{1}{4}, \quad s = x, y. \quad (4.5.12)$$



**Fig. 4.6.** Uncertainty relations in the  $x$  and  $y$  modes for the generalised coherent states in the symmetric,  $\gamma_1 = \gamma_2 = \frac{1}{\sqrt{2}}$ , and asymmetric,  $\gamma_1 = \frac{\sqrt{3}}{2}, \gamma_2 = \frac{1}{2}$ , regimes. Here  $s = x, y$ .

In figure 4.6 we plot the product uncertainty relations in the  $x$  and  $y$  modes with equal and unequal values of  $\gamma_1, \gamma_2$  for the generalised coherent states as a function of  $\Psi$ . We see that for the generalised coherent states the product of the uncertainties remains close to the minimum for values  $\Psi < 1$ , and the effect of asymmetry between  $\gamma_1, \gamma_2$  is minimal. For larger  $\Psi$  however, the wavefunctions begin to delocalise and we also observe a growth in the product of uncertainties. This is also reflected in the spatial distribution in figures 4.4 and 4.5. In the case where the parameters  $\gamma_1, \gamma_2$  are equal we do not introduce any asymmetry between the two modes and thus the uncertainty relations look identical for both the  $x$  and  $y$  modes.

When we include some asymmetry by setting the parameters  $\gamma_1 > \gamma_2$ , we find that the product of the uncertainties for the generalised coherent states in the  $x$  mode are smaller than those of the  $y$  mode when  $\Psi$  is large enough. This effect is most noticeable for  $\Psi > 1.4$ . Again the parameters  $\gamma_1, \gamma_2$  offer additional control over the behaviour of the generalised coherent states. We can, in effect, reduce the product uncertainties in one mode at the expense of increasing the product uncertainties in the other mode. The global behaviour is still determined by the coherence parameter  $\Psi$ .

## 4.6. Conclusion and outlook

We found a scheme for constructing a singled indexed set of non-degenerate states for the 2D Morse potential. We assessed three distinct forms the spectrum can take corresponding to the rationality of the principle parameter  $p$  and discussed their degeneracies. The critical observation is that the irrationality of the principle parameter  $p$  implies that the degeneracy in the 2D Morse potential is at most two-fold. This follows from a straightforward analysis of the spectrum. In restricting to irrational choices of  $p$  we make the problem of handling degeneracy in the system much more tractable, and correspondingly we only need to introduce two meaningful complex parameters,  $\gamma_1, \gamma_2$ , subject to a normalisation constraint in order to define a degeneracy free spectrum for the 2D system. The solution we have used is algorithmic in approach and the techniques discussed here should be applicable to any two-dimensional system with quadratically degenerate spectra.

We saw that the introduction of the parameters  $\gamma_1, \gamma_2$  serve to tune the concentration of the probability densities of the non-degenerate states in configuration space by controlling the weight of the contributing  $x$  and  $y$  modes. Furthermore, we introduced Barut-Girardello type generalised coherent states from a set of ladder operators acting on the non-degenerate set of states and found that they are well localised in their spatial distribution and approximately minimise the Heisenberg uncertainty relation for small values of the coherence parameter  $\Psi$ . Additionally, we found the effect of  $\gamma_1, \gamma_2$  to be more significant in the coherent states for larger values of  $\Psi$ .

We relied on an algorithmic method to deal with the degeneracy problem, but algebraic and symmetry approaches may offer further insight into the structure of the degeneracies and allow for a more detailed discussion of ladder operators on the states presented in this paper. Algebraic approaches may also be of interest in studying classical models [38, 39], and in the study of the scattering (unbound) states [40].

Finally, it would also be interesting to study the sets of basis and coherent states we could generate using the supersymmetry formalism [41, 42]. In particular, supersymmetric partners of the 2D Morse system have potential functions which are non-separable in configuration space [24, 21, 22, 23, 25], the study of such states will be saved for a separate work.

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# Chapter 5

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## Coherent states of the two-dimensional non-separable supersymmetric Morse potential

by

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The main contributions of James Moran for this article are:

- Performed all calculations in the paper
- Wrote and edited the paper

Véronique Hussin suggested the project based on their previous work on the subject; assisted with proofreading and editing.

ABSTRACT. Supersymmetry is a fundamental feature of a quantum theory described by a supersymmetric Hamiltonian comprised of several quantum mechanical Hamiltonians with different potentials that allows us to relate information about the states and spectra of the constituent Hamiltonians. In this paper we reconstruct Ioffe’s set of states for the singular non-separable two-dimensional Morse potential using supersymmetry from a non-degenerate set of states constructed for the initial separable Morse Hamiltonian. We define generalised coherent states, compute their uncertainty relations, and we find that the singularity in the partner Hamiltonian significantly affects the localisation of the coherent state wavefunction.

**Keywords:** Degeneracy, Coherent states, Supersymmetry, Two-dimensional quantum systems

## 5.1. Introduction

Supersymmetric quantum mechanics is a powerful tool in solving new quantum problems by their mathematical relationship to known solved problems [1, 2]. When two systems are related through supersymmetry they share nearly identical spectra and states from one system may be transformed into states of the other system by the action of differential operators known as supercharges. In one-dimensional quantum mechanics, supersymmetry has been used to study a plethora of new potentials including partners of: the Rosen-Morse potentials [3, 4], the truncated oscillator [5, 6, 7], and the singular oscillator [8]. In higher dimensional quantum systems, supersymmetry is less explored, though coherent states for the two-dimensional infinite well and its coordinate separable supersymmetric partners have been discussed [9]. There is considerably more freedom when defining the supercharges for higher dimensional systems [10], and features which are exclusive to systems of dimension larger than one, such as non-coordinate separability of the Hamiltonian, can appear. In the present work we deal with precisely such a case where the initial Hamiltonian is coordinate separable, while its partner Hamiltonian is not [11].

The Morse potential was originally introduced to model anharmonic interactions in diatomic molecules which allow the possibility of the bond between the molecules breaking at sufficiently high energy [12]. The uncoupled two-dimensional Morse potential has been used in the expansion of triatomic molecular interactions [13, 14, 15, 16]. More recently the Morse potential has been studied in the context of graphene [17, 18], spectroscopy [19], and supersymmetry [11, 20].

Describing coherent states for generalised quantum systems has been of interest since the canonical coherent states of the harmonic oscillator were formalised by Glauber and Sudarshan in their seminal works [21, 22]. For the canonical coherent states there exist three equivalent definitions: as eigenstates of the annihilation operator; the orbit of the displacement operator acting on the vacuum; a particular superposition of Fock basis states [23]. For systems beyond the harmonic oscillator the three definitions typically do not coincide and one must choose the most applicable definition. Coherent states for two-dimensional

harmonic oscillator systems with linear spectra have been discussed in [24, 25, 26] but in the present case we are dealing with a quadratically degenerate spectrum and must forego some of the nicer algebraic structures.

Generalised coherent states for the one-dimensional and two-dimensional Morse potentials have been studied in [27] and [28] respectively. For multidimensional quantum systems there exists a greater class of states when one departs from taking the tensor product of one-dimensional coherent states: the Hamiltonian may not be coordinate separable and thus its eigenstates may be entangled between its respective modes; the Hamiltonian may be coordinate separable but when defining a non-degenerate spectrum in order to define generalised coherent states [29] the new basis states become entangled.

The paper is structured as follows: In section 5.2 we review the supersymmetry formalism in two dimensions using second-order supercharges, after which we follow the work of Ioffe [11] and compare the two partner Hamiltonians in section 5.3. In section 5.4 we develop set of non-degenerate states for the initial Hamiltonian which depend on two mixing parameters  $\gamma_1, \gamma_2$ , and transform allowed antisymmetric combinations of these states into a reduced set of non-degenerate states for the partner Hamiltonian. Following this in section 5.5 we construct an explicit example of these energy eigenstates for a value of the principle parameter  $p = 3\pi$ , and then we develop coherent states for the partner Hamiltonian and discuss their properties in section 5.5.2. Lastly we conclude with some open questions and remarks about the work.

## 5.2. Second-order two-dimensional supersymmetry

Work on developing the supersymmetry formalism in two dimensions has been done by Ioffe and others [30, 31, 32]. In this context, the second-order supercharges are given by

$$Q^+ = (Q^-)^\dagger = g_{kl}(x,y)\hbar^2\partial_k\partial_l + c_k(x,y)\hbar\partial_k + b(x,y), \quad k,l \in \{x,y\}, \quad (5.2.1)$$

where repeated indices are summed over and  $g_{kl}(x,y), c_k(x,y)$  and  $b(x,y)$  are real-valued functions. Two partner Hamiltonians,  $H, \tilde{H}$ , are intertwined via

$$\tilde{H}Q^+ = Q^+H, \quad HQ^- = Q^-\tilde{H}. \quad (5.2.2)$$

Supposing we have the set of states of the initial Hamiltonian

$$\psi_{nm}(x,y) \equiv \langle x,y|n,m\rangle, \quad (5.2.3)$$

the implication of the intertwining relations (5.2.2) is that for a given eigenstate  $\psi_{nm}(x,y)$  of  $H$ ,  $Q^+\psi_{nm}(x,y)$  is an eigenstate of  $\tilde{H}$  with the same eigenvalue through

$$H|n,m\rangle = E_{n,m}|n,m\rangle \implies Q^+H|n,m\rangle = E_{n,m}[Q^+|n,m\rangle] = \tilde{H}[Q^+|n,m\rangle], \quad (5.2.4)$$

where we have suppressed the dependence on  $x,y$  for brevity. In general (and in our example) there is not a one-to-one correspondence between states belonging to both Hamiltonians,

certain eigenstates are not permissible because they become non-normalisable in the partner Hamiltonian [30, 31, 32]. That is to say while the intertwining relations (5.2.2) always imply mathematically the eigenvalue equations (5.2.4) we may encounter a situation where the transformed eigenfunction,  $Q^+ |n, m\rangle$ , is not physical.

In some cases additional normalisable eigenfunctions may be obtained for the partner Hamiltonian by transforming non-normalisable solutions of the initial Hamiltonian by the supercharge  $Q^+$  [33, 34]. In the present case, all normalisable solutions of the partner Hamiltonian were obtained by Ioffe [11].

General schemes for finding solutions for second-order supercharges defined in (5.2.1) do not exist. For the particular case of a metric,  $g_{kl}(x, y)$ , with Lorentz signature  $(1, -1)$  the intertwining relations can be simplified and the coefficient appearing in (5.2.1) can be found through a system of differential equations [30].

Introducing the generators

$$\mathcal{Q}^+ = \begin{pmatrix} 0 & Q^+ \\ 0 & 0 \end{pmatrix}, \quad \mathcal{Q}^- = \begin{pmatrix} 0 & 0 \\ Q^- & 0 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} \tilde{H} & 0 \\ 0 & H \end{pmatrix}, \quad (5.2.5)$$

we obtain the superalgebra defined by

$$[\mathbf{H}, \mathcal{Q}^\pm] = 0, \quad \{\mathcal{Q}^+, \mathcal{Q}^-\} = \mathcal{R}, \quad (5.2.6)$$

where  $[\cdot, \cdot]$  and  $\{\cdot, \cdot\}$  are the commutator and anticommutator, respectively. The existence of the fourth order operator  $\mathcal{R}$  is a consequence of defining second order supercharges [35]. Comparing this with the implementation of supersymmetry using first order supercharges where the anticommutator of the supercharges is just the superHamiltonian.

### 5.3. Initial and partner Morse Hamiltonians

The separable two-dimensional Morse Hamiltonian in coordinate representation is given by

$$H = -\frac{\hbar^2}{2m} (\partial_x^2 + \partial_y^2) + V_0 (e^{-2\beta x} + e^{-2\beta y} - 2(e^{-\beta x} + e^{-\beta y})) = H_x + H_y. \quad (5.3.1)$$

Hamiltonian (5.3.1) permits a finite number of bound states which are solutions to the time independent Schrödinger equation  $H |n, m\rangle = E_{n, m} |n, m\rangle$ ,

$$\psi_{n, m}(x, y) = \langle x, y | n, m \rangle = \mathcal{N}_{n, m} e^{-\frac{x}{2} - \frac{y}{2}} \tilde{x}^{p-n} \tilde{y}^{p-m} L_n^{2(p-n)}(\tilde{x}) L_m^{2(p-m)}(\tilde{y}). \quad (5.3.2)$$

Here  $L_n^\alpha(z)$  are the generalised Laguerre polynomials, the normalisation factor  $\mathcal{N}_{n, m}$  is given by

$$\mathcal{N}_{n, m} = \beta \sqrt{\frac{(\nu - 2n - 1)(\nu - 2m - 1)\Gamma(n + 1)\Gamma(m + 1)}{\Gamma(\nu - n)\Gamma(\nu - m)}}, \quad (5.3.3)$$

and the tilde variables are defined as

$$\tilde{x} = \nu e^{-\beta x}, \quad \tilde{y} = \nu e^{-\beta y}. \quad (5.3.4)$$

The parameter  $\nu$  and an additional parameter  $p$  are written in terms of the initial parameters by

$$\nu = \sqrt{\frac{8mV_0}{\beta^2 \hbar^2}}, \quad p = \frac{\nu - 1}{2}. \quad (5.3.5)$$

The bound state spectrum is given by

$$E_{n,m} = -\frac{\hbar^2 \beta^2}{2m} \left( (p-n)^2 + (p-m)^2 \right). \quad (5.3.6)$$

The spectrum is defined in such a way that states with negative energy are bound and states with zero or positive energy are unbound. For a discussion on the unbound states and spectrum, see, for example [36, 37, 38]. The bound state conditions imply that the quantum numbers  $n, m$  may take the following values [39]

$$n, m \in \{0, 1, \dots, \lfloor p \rfloor\}. \quad (5.3.7)$$

To facilitate counting arguments later, it is convenient to rewrite the parameter  $p$  in terms of its integer part plus a remainder,

$$p = k + \epsilon, \quad k = \lfloor p \rfloor, \quad \epsilon \in [0, 1), \quad (5.3.8)$$

where we further restrict to the case where  $\epsilon$  is irrational to ensure that states are at most doubly degenerate [28].

The partner Hamiltonian we are interested in studying is given by [11]

$$\tilde{H} = H + \frac{\hbar^2 \beta^2}{2m \sinh^2 \left( \frac{\beta}{2}(x-y) \right)}. \quad (5.3.9)$$

The two Hamiltonians (5.3.1) and (5.3.9) are related by (5.2.2) through the supercharges

$$Q^\pm = -H_x + H_y + D^\pm \quad (5.3.10)$$

where  $D^\pm$  is given by

$$D^\pm = \frac{\hbar^2 \beta^2}{2m} \coth \left( \frac{x-y}{2} \right) \mp \frac{\hbar^2 \beta}{2m} \left( (\partial_x - \partial_y) + \coth \left( \frac{x-y}{2} \right) (\partial_x + \partial_y) \right). \quad (5.3.11)$$

The supercharges  $Q^\pm$  are related to a particular choice of (5.2.1) where the metric  $g_{kl}(x, y)$  has a Lorentz signature on the indices,  $(1, -1)$ , thus the second order derivative terms are not mixed between the two modes and there is a relative minus sign between  $\partial_x^2$  and  $\partial_y^2$ .

So far we have kept factors of  $\hbar$  to show exactly how the terms appear in the Hamiltonians and supercharges, but from here on we set the dimensionful quantities  $\hbar = \beta = m = 1$ . The most striking features of the partner Hamiltonian (5.3.9) are its non-separability, and that the non-separable term is singular for  $y = x$ .

## 5.4. States of the initial and partner Morse Hamiltonians

Because the initial Hamiltonian is amenable to separation of variables its eigenstates are products of wavefunctions in each direction. In general, multidimensional quantum systems are degenerate in their energy spectrum, and in the view of addressing the construction of generalised coherent states it is important to have a non-degenerate increasing spectrum [40, 29]. For the finite quadratic spectrum (5.3.6) the degeneracies are obtained via number theoretic means and, moreover, they are found to be at most doubly degenerate for irrational values of the parameter  $p$ . In [28] general such degenerate combinations are constructed

$$|\mu_i\rangle_p = \begin{cases} \gamma_1 |n,m\rangle + \gamma_2 |m,n\rangle, & \text{for } |n-m| > 0, \\ |n,n\rangle, & \text{otherwise,} \end{cases} \quad (5.4.1)$$

for complex coefficients  $\gamma_1, \gamma_2$ , satisfying the normalisability condition  $|\gamma_1|^2 + |\gamma_2|^2 = 1$ . For the states  $|\mu_i\rangle_p$ , the index  $i$  organises them in increasing energy order, and the subscript  $p$  indicates the value of the parameter  $p$  introduced in (5.3.5).

Due to the singular nature of the partner Hamiltonian along the line  $y = x$  we cannot construct states in the partner Hamiltonian with arbitrary coefficients. A detailed analysis provided in [11] proves that the only way to compensate for this singularity to obtain normalisable states in the partner Hamiltonian is to transform perfectly antisymmetric combinations of the initial eigenfunctions in which case we obtain symmetric (about the line  $y = x$ ) eigenfunctions in the partner Hamiltonian. In the notation we are using this corresponds to the choice of parameters  $\gamma_1 = -\gamma_2 = \frac{1}{\sqrt{2}}$ .

Additionally, Ioffe's analysis shows that the states admissible by the partner Hamiltonian require that  $|n-m| > 1$  [11]. This follows from the superalgebra (5.2) and specifically the existence of the fourth order operator  $R = Q^- Q^+$ . The operator  $R$  commutes with the initial Hamiltonian,  $[R, H] = 0$  and we have that  $R|\mu_i\rangle_p = r_{n,m}|\mu_i\rangle_p$  where

$$r_{n,m} = \frac{1}{2} \left( (m-n)^2 - 1 \right) \left( (2p-m-n)^2 - 1 \right), \quad (5.4.2)$$

meanwhile the states in the partner system obtained by transforming the states of the initial system have norm given by (5.2.4)

$$\langle \mu_i |_p Q^- Q^+ |\mu_i\rangle_p. \quad (5.4.3)$$

We see immediately that the case where  $n = m + 1$  and vice versa lead to states in the partner Hamiltonian which are not normalisable because (5.4.2) vanishes. The case where  $n = m$  is excluded by definition when taking perfectly antisymmetric combinations of the eigenstates.



We take the convention that the positive coefficient appears in front of the term with  $n > m$ , thus we can exactly solve the partner Hamiltonian (5.3.9) to obtain the set of normalised states

$$|\nu_j\rangle_p = \frac{1}{\sqrt{\mathcal{N}_j}} Q^+ |\mu_i\rangle_p = \begin{cases} \frac{Q^+}{\sqrt{2\mathcal{N}_j}} (|n,m\rangle - |m,n\rangle), & \text{for } |n-m| > 1 \\ \emptyset, & \text{otherwise,} \end{cases} \quad (5.4.4)$$

where  $\mathcal{N}_j = |Q^+ |\mu_i\rangle_p|^2$  are normalisation constants,  $Q^+$  is defined in (5.3.10), and  $\emptyset$  means that nothing is to be done, i.e. do not generate an element for the set  $\{|\nu_j\rangle_p\}$ . The matching of the indices  $i, j$  in (5.4.4) is unique, but because the set  $\{|\mu_i\rangle_p\}$  is larger than  $\{|\nu_j\rangle_p\}$ , the correspondence is not  $i = j$ . Due to the isospectrality given to us by the supersymmetry formalism [41], the states  $|\nu_j\rangle_p$  are automatically arranged in increasing energy order because the initial states  $|\mu_j\rangle_p$  were prepared in this way.

The Morse potential admits only a finite number of bound states, as such it is instructive to count the number of bound states in the initial and partner Hamiltonians. We know that the number of states in the set  $|\{|\mu_i\rangle_p\}|$  is [28]

$$|\{|\mu_i\rangle_p\}| = \frac{(k+1)(k+2)}{2}, \quad (5.4.5)$$

recalling that  $k = \lfloor p \rfloor$ . From (5.4.4) it is clear that the size of the set  $|\{|\nu_i\rangle_p\}|$  must be smaller than (5.4.5). Imposing the condition  $|n-m| > 1$ , we count all pairs  $(n, m)$ . Adding

$m$	$n$
0	$2, \dots, k$
1	$3, \dots, k$
$\vdots$	$\vdots$
$k-2$	$k$

**Table 5.1.** Allowed combinations of quantum numbers  $(n, m)$  leading to normalisable states in the partner Hamiltonian.

up the possibilities across each row of table 5.1, we arrive at the following formula

$$|\{|\nu_i\rangle_p\}| = \sum_{j=1}^{k-1} (k-j) = \frac{k}{2}(k-1), \quad (5.4.6)$$

thus the number of ‘missing’ states between the two sets scales linearly in  $k$ :

$$|\{|\mu_i\rangle_p\}| - |\{|\nu_i\rangle_p\}| = 1 + 2k. \quad (5.4.7)$$

A key observation to be made is that bound states only exist in the partner Hamiltonian for  $k \geq 2$  or equivalently,  $p > 2$ .

Considering a scaled version of the spectrum (5.3.6)

$$\varepsilon_{n,m}(p) = \varepsilon_{n,m}(k,\epsilon) = - \left[ (p-n)^2 + (p-m)^2 \right], \quad n,m \in \{0,1,\dots,\lfloor p \rfloor\}. \quad (5.4.8)$$

We recall that in (5.3.8) we write  $p$  as the sum of its integer part plus an irrational remainder term,  $\epsilon$ , in which case the states of the initial Hamiltonian have a spectrum found by arranging the solutions of (5.4.8) in increasing order

$$\varepsilon_{0,0}(k,\epsilon) < \varepsilon_{1,0}(k,\epsilon) = \varepsilon_{0,1}(k,\epsilon) < \dots < \varepsilon_{k,k}(k,\epsilon), \quad (5.4.9)$$

such that the spectrum for the states (5.4.1) can be written as

$$\varepsilon_{\mu_0}(k,\epsilon) < \varepsilon_{\mu_1}(k,\epsilon) < \dots < \varepsilon_{\mu_{\max}}(k,\epsilon). \quad (5.4.10)$$

Then by the isospectrality afforded to us by supersymmetry, we have the spectrum for the states (5.4.4)

$$\varepsilon_{\nu_0}(k,\epsilon) < \varepsilon_{\nu_1}(k,\epsilon) < \dots < \varepsilon_{\nu_{\max}}(k,\epsilon). \quad (5.4.11)$$

We remark that  $\mu_{\max}$  and  $\nu_{\max}$  are determined by (5.4.5) and (5.4.6) respectively.

## 5.5. Application to $p = 3\pi$

We will now apply the formalism set up in the preceding sections to the example of  $p = 3\pi$ . It is worth mentioning that while the analysis of the degeneracies relies on the irrationality of the principle parameter,  $p$ , in order to implement calculations on a computer algebra system we do need to approximate  $p$  by a rational number. That being said, as long as states are at most double degenerate (which is easily verified on a computer), the analysis still holds.

### 5.5.1. Energy eigenstates

We begin by recounting the non-degenerate energy eigenstates for the initial system for  $p = 3\pi$ . The example used in [28] gives us the following 55 state set,  $\mathcal{S}$ , for the initial Hamiltonian

$$\mathcal{S} = \{ |\mu_0\rangle_{3\pi}, |\mu_1\rangle_{3\pi}, \dots, |\mu_{54}\rangle_{3\pi} \}, \quad |\mathcal{S}| = 55, \quad (5.5.1)$$

where

$$|\mu_i\rangle_{3\pi} = \begin{cases} |0,0\rangle, & i = 0 \\ \gamma_1 |1,0\rangle + \gamma_2 |0,1\rangle, & i = 1 \\ \gamma_1 |2,0\rangle + \gamma_2 |0,2\rangle, & i = 2 \\ |1,1\rangle, & i = 3 \\ \vdots & \\ |9,9\rangle, & i = 54. \end{cases} \quad (5.5.2)$$

Following this, we set  $\gamma_2 = -\gamma_1 = -\frac{1}{\sqrt{2}}$  and determine the 36 state set for the partner Hamiltonian,  $\tilde{\mathcal{S}}$ , using (5.4.4),

$$\tilde{\mathcal{S}} = \{|\nu_0\rangle_{3\pi}, |\nu_1\rangle_{3\pi}, \dots, |\nu_{35}\rangle_{3\pi}\}, \quad |\tilde{\mathcal{S}}| = 36. \quad (5.5.3)$$

States in the set  $\tilde{\mathcal{S}}$  preserve orthonormality by construction. Explicitly the first few and last states are

$$|\nu_i\rangle_{3\pi} = \begin{cases} \frac{1}{\sqrt{\mathcal{N}_0}} Q^+ \left[ \frac{1}{\sqrt{2}} (|2,0\rangle - |0,2\rangle) \right], & i = 0 \\ \frac{1}{\sqrt{\mathcal{N}_1}} Q^+ \left[ \frac{1}{\sqrt{2}} (|3,0\rangle - |0,3\rangle) \right], & i = 1 \\ \frac{1}{\sqrt{\mathcal{N}_2}} Q^+ \left[ \frac{1}{\sqrt{2}} (|4,0\rangle - |0,4\rangle) \right], & i = 2 \\ \frac{1}{\sqrt{\mathcal{N}_3}} Q^+ \left[ \frac{1}{\sqrt{2}} (|3,1\rangle - |1,3\rangle) \right], & i = 3 \\ \vdots & \\ \frac{1}{\sqrt{\mathcal{N}_{35}}} Q^+ \left[ \frac{1}{\sqrt{2}} (|9,7\rangle - |7,9\rangle) \right], & i = 35, \end{cases} \quad (5.5.4)$$

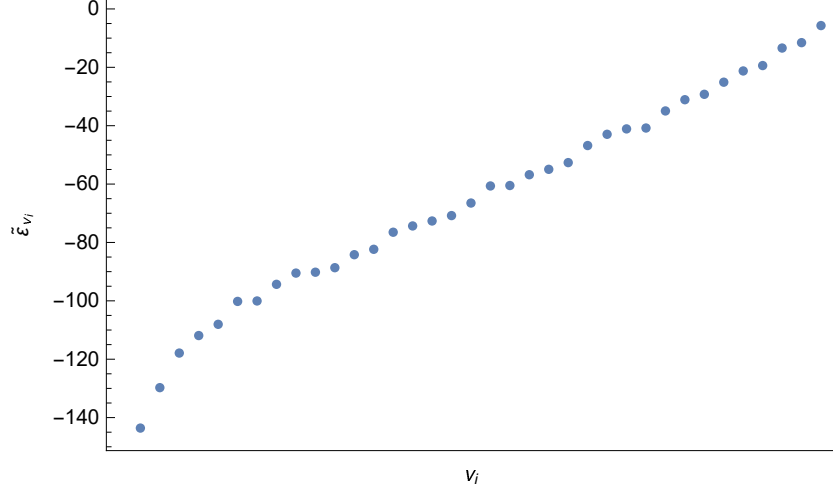
where the normalisation constants  $\mathcal{N}_i = |Q^+ |\mu_k\rangle|^2$  where the index  $i$  in the set  $\tilde{\mathcal{S}}$  is to be identified with the state with index  $k$  in the set  $\mathcal{S}$  from which it is generated. In addition we verify that the difference in size of the two sets of states satisfies (5.4.7) for  $k = 9$ .

As with the initial separable two-dimensional Hamiltonian, the organisation of the spectrum and states of the partner Hamiltonian is algorithmic, so there is no obvious pattern to be seen. This follows from the ordering of numbers as the sum of two squares. And since the partner Hamiltonian is isospectral to the initial Hamiltonian, the partner Hamiltonian also inherits this problem with the additional complication related to the restriction  $|n - m| > 1$  discussed in the previous section. Nevertheless, organising the states in this way can be done quite easily with most computer algebra systems.

Consider a scaled version of the spectrum (5.4.11) where the constant term  $-2\epsilon^2$  has been removed,

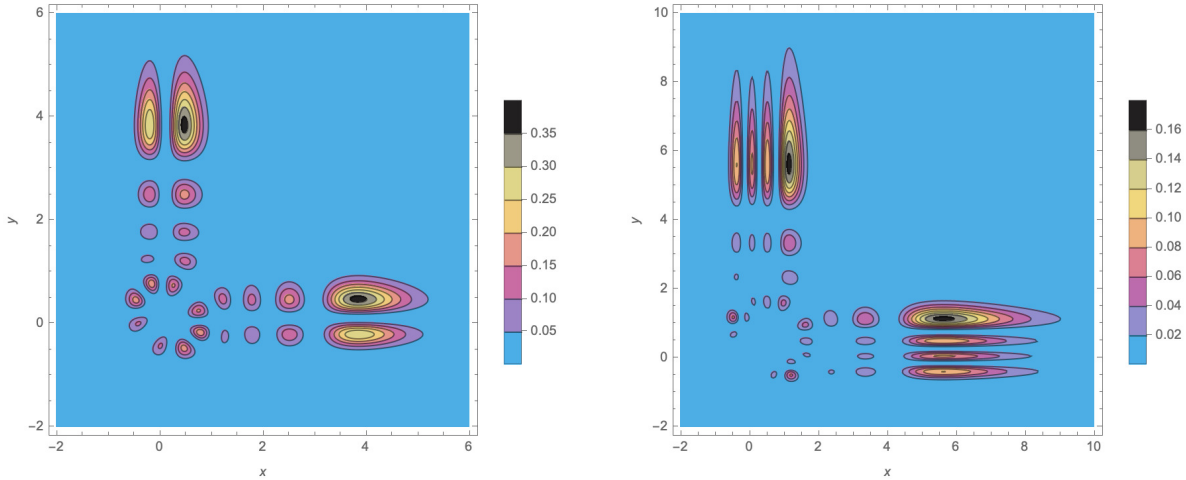
$$\tilde{\epsilon}_{\nu_i}(9, 0.42478) = - \left[ (9 - n)^2 + (9 - m)^2 + 2(0.42478) (18 - n - m) \right], \quad (5.5.5)$$

arranged in increasing order such that  $|n - m| > 1$ .



**Fig. 5.1.** Energies  $\tilde{\epsilon}_{\nu_0}, \dots, \tilde{\epsilon}_{\nu_{35}}$  for  $p = 3\pi$  arranged in increasing order.

In figure 5.1 we see the spectrum of the partner Hamiltonian arranged in increasing order. Notice that there does not appear to be a nice functional form to the spectrum, this is because the techniques required to understand the degeneracy of an arbitrary energy level are number theoretical and in general must be computed on a case by case basis.



**Fig. 5.2.** Probability densities of  $|\langle x, y | \nu_{15} \rangle_{3\pi}|^2$  (left) and  $|\langle x, y | \nu_{25} \rangle_{3\pi}|^2$  (right).

We see in figure 5.2 that the configuration space wavefunctions  $\langle x, y | \nu_i \rangle_{3\pi}$  (and their probability densities) are symmetric and non-singular about the line  $y = x$ . The eigenstates  $|\nu_i\rangle$  have similar patterns in their wavefunctions to those of the original Hamiltonian  $|\mu_j\rangle$  [28], the key differences being that the wavefunction  $\langle x, y | \nu_i \rangle_{3\pi}$  is always identically zero on the line  $y = x$ , and it is always symmetric about the line too. In contrast, the parameters  $\gamma_1, \gamma_2$  in  $\langle x, y | \mu_i \rangle_{3\pi}$  are free up to a normalisation and allow us to adjust the symmetry or antisymmetry of the wavefunctions. Additionally we find that for larger values of  $i$ ,  $|\langle x, y | \nu_i \rangle_{3\pi}|^2$  contain

more detailed structure in their probability densities and have more ‘islands’ of non-zero probability density, while for smaller  $i$  there is less detail to the structure and fewer ‘islands’ of probability.

### 5.5.2. Coherent states

Let us suppose the existence of some ladder operator on the set  $\tilde{\mathcal{S}}$ ,

$$\begin{aligned}\mathcal{B}^+ |\nu_i\rangle_p &= \sqrt{f(i+1)} |\nu_{i+1}\rangle_p, \\ \mathcal{B}^- |\nu_i\rangle_p &= \sqrt{f(i)} |\nu_{i-1}\rangle_p,\end{aligned}\tag{5.5.6}$$

where the functions  $f(i)$  must be positive and satisfy suitable boundary conditions. In practice the  $f(i)$  are often taken to be the relative difference of the energy to the ground state energy.

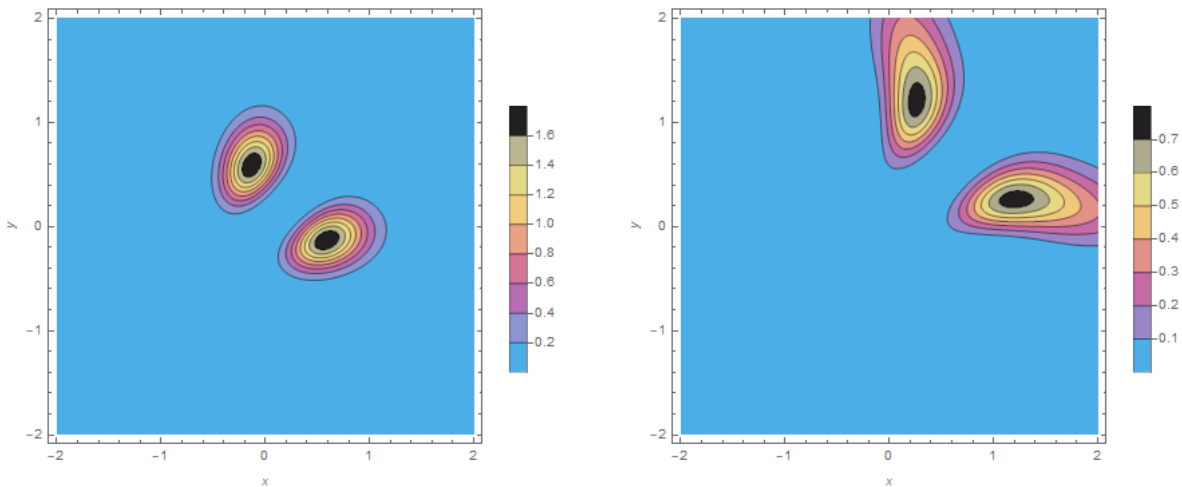
We can construct generalised coherent states as approximate eigenstates of  $\mathcal{B}^-$ ,

$$\mathcal{B}^- |\Phi\rangle_p \approx \Phi |\Phi\rangle_p.\tag{5.5.7}$$

Expanding the generalised coherent states  $|\Phi\rangle_p$  for  $p = 3\pi$  in the basis (5.5.3) we find the following

$$|\Phi\rangle_{3\pi} = \frac{1}{\sqrt{\mathcal{N}(\Phi)}} \sum_{n=0}^{35} \frac{\Phi^n}{\sqrt{[\tilde{\varepsilon}_{\nu_n} - \tilde{\varepsilon}_{\nu_0}]!}} |\nu_n\rangle.\tag{5.5.8}$$

We remark that the state  $|\Phi\rangle_p$  is only an approximate eigenstate of  $\mathcal{B}^-$  because the term proportional to  $|\nu_{\max}\rangle$  does not appear in the expansion (5.5.8), we add it in by hand. This is a general feature of ladder operator eigenstate definitions of coherent states with finite spectra.



**Fig. 5.3.**  $|\langle x,y|\Phi\rangle_{3\pi}|^2$ , with  $\Phi = 0.001$  (left) and  $\Phi = 5$  (right).

The behaviour of the generalised coherent states is qualitatively very different to the original 2D Morse problem [28]. A result of the non separability of the potential means that the probability density concentrates on two symmetric peaks around the line  $y = x$ . Even if we let  $\Phi$  become smaller, this separation remains (figure 5.3).

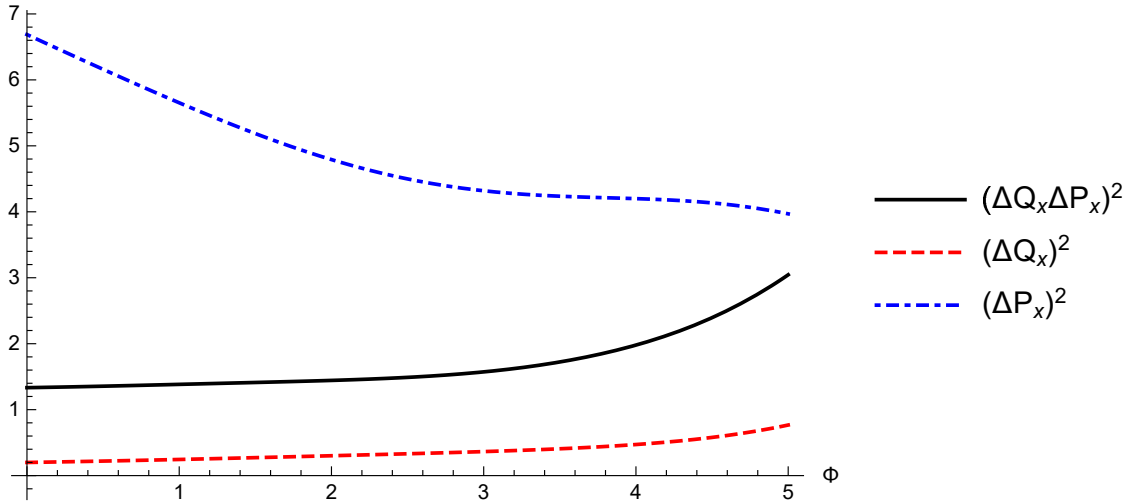
Moreover we can also analyse the the uncertainty relations for the coherent states. The variance of an operator,  $\hat{O}$ , is defined by

$$(\Delta\hat{O})^2 = \langle \hat{O}^2 - \langle \hat{O} \rangle^2 \rangle, \quad (5.5.9)$$

and since we are working in the position representation, the momentum and position operators (in the  $x$  direction), as well as the Heisenberg uncertainty relation take the form

$$\hat{P}_x = -i\frac{d}{dx}, \quad \hat{Q}_x = x, \quad (\Delta\hat{Q})^2 (\Delta\hat{P})^2 \geq \frac{1}{4}. \quad (5.5.10)$$

For canonical coherent states of the harmonic oscillator the variance in the position and momentum quadratures is equal,  $(\Delta\hat{Q})^2 = (\Delta\hat{P})^2 = \frac{1}{2}$ . Whenever  $(\Delta\hat{Q})^2 \neq (\Delta\hat{P})^2$  we have squeezing between the quadrature operators, furthermore when either quadrature has variance smaller than  $\frac{1}{2}$  we have sub-shot-noise squeezing.



**Fig. 5.4.** Uncertainty in the position and momentum quadratures and their product.

No matter the value of  $\Phi$ , the generalised coherent states  $|\Phi\rangle_p$  never minimise the Heisenberg uncertainty relation. This is due to the existence of the singularity on the line  $y = x$ . In quantum physics the notion of non-classicality is used to explain purely quantum phenomena (see, for example, [42, 43, 44]). In the present case we have a few markers of non-classicality: we have eigenstates of a non-separable Hamiltonian which are themselves non-separable, they cannot be written as the tensor product of two one-dimensional states  $|\Phi\rangle_p \neq |\phi_x\rangle \otimes |\phi_y\rangle$ ; the states are never minimal uncertainty; there is significant squeezing between the position and momentum quadratures.

In figure 5.4 we see that the squeezing in the position quadrature is below that of the canonical coherent states, that is  $(\Delta\hat{Q}_x)^2 < \frac{1}{2}$ . In the language of quantum optics this refers to sub-shot-noise squeezing, these techniques allow one to obtain higher resolution imaging by reducing the uncertainty in quadrature at the expense of increasing it in the conjugate quadrature [45, 46, 47, 48]. This indicates that despite the fact that the coherent states  $|\Phi\rangle_p$  are unable to localise onto a single point in space, there is little statistical variance in the accuracy of its position. Other examples of states which appear to localise at two spatially separated points include the well studied non-classical cat states [49].

## 5.6. Conclusion

In this paper, following the work of Ioffe [11], we constructed an explicit set of eigenstates for the non-separable singular two-dimensional Morse potential. We were able to connect these states with a set of non-degenerate states for the initial separable Morse potential [28]. We constructed coherent states for the non-separable Morse Hamiltonian and found that the configuration space wavefunction is unable to localise at the origin due to the singularity present in the potential. This extends work previously done in the domain of multidimensional coherent states. Not only are the states entangled, but they arise from a non-coordinate separable Hamiltonian which is singular along the line  $y = x$  resulting in strongly non-classical behaviour. The procedure we developed in this paper is algorithmic and in principle can be used to describe coherent states for any two-dimensional system with quadratically degenerate spectrum and its supersymmetric partners (if they exist).

Continuing the analysis we computed the uncertainty relation for the coherent states and while the states themselves were not minimal uncertainty, significant squeezing was found between the position and momentum quadratures with sub-shot-noise squeezing in the position quadrature. This indicates strong quantum behaviour similar to that of the highly non-classical cat states. The statistical variance in the position quadrature is smaller than that of the canonical coherent states, yet the wavefunction appears to localise onto two space-like separated regions. For systems of dimension larger than two, one should expect richer structure still, and the emergence of multipartite entangled systems.

Lastly, we remark that a more detailed study of the coherent states for interacting multidimensional quantum systems, such as the triatomic molecular Hamiltonian [13] and the Pais-Uhlenbeck oscillator [50], would be of interest going forward. It is clear that we can expect novel behaviour that might not be found from non-interacting multidimensional generalisations, though in practice their solution is much more difficult to obtain.

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# Chapter 6

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## Two-mode squeezed state quantisation and semiclassical portraits

by

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This article is in preparation.

The main contributions of James Moran for this article are:

- Performed some calculations
- Produced figures
- Co-wrote and edited the paper

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Véronique Hussin helped develop the scope of the paper; assisted with proofreading and editing.

Kevin Zelaya performed calculations; produced figures; co-wrote and edited the paper.

ABSTRACT. Quantisation with Gaussian type states offers certain advantages over other quantisation schemes, in particular, they can serve to regularise formally discontinuous classical functions leading to well defined quantum operators. In this work we define a squeezed state quantisation in two dimensions using several families of squeezed states. The completeness relations of the squeezed states are exploited in order to tackle the quantisation and semiclassical analysis of a constrained position dependent mass model with harmonic potential. The effects of the squeezing parameters on the resulting operators and phase space functions are studied, and configuration space trajectories are compared between the classical and semiclassical models.

**Keywords:** Quantisation, Semiclassical phase space analysis, Position dependent mass systems, Two-dimensional quantum systems.

## 6.1. Introduction

Coherent states and their generalisation, squeezed states, are ubiquitous in the study of quantum optics. They describe a set of minimal uncertainty states with respect to their generalised quadratures (in typical quantum systems these may refer to position and momentum), and the ‘squeezing’ refers to the reduction in one quadrature variance at the expense of an increase in the conjugate quadrature variance [1]. In multimode systems there exists an even greater variety of squeezed states because the squeezing can occur between four or more quadratures and their combinations. Multimode squeezed states are the most general type of Gaussian state permissible and have found use outside of optics as a resource in continuous variable quantum information processing for generating multipartite entanglement [2, 3]. Beyond Gaussian states, non-Gaussian states represent a further generalisation [4]. Schumaker investigated the most general two-mode Gaussian pure states [5], schemes designed to generalise this construction to the  $N$ -mode case were studied in [6], and a general presentation of a coupled three-mode squeezed vacuum was presented in [7].

Outside of optics and information theory, squeezed states have attractive mathematical properties, in particular they form an overcomplete basis in the Hilbert space of quantum states [8, 9]. Equipped with this property one may expand any state of a given system in the basis of squeezed states. Moreover, one may quantise a classical function in the squeezed state basis yielding an associated quantum operator as well as define an averaged value of the initial function with respect to the squeezed states yielding a semiclassical portrait. This is the precisely the purpose of this work, to extend the ideas of Klauder-Berezin coherent state quantisation [10] by defining a two-mode squeezed state quantisation in which we use several families of two-mode squeezed states and study the effect of their squeezing parameters on the resulting quantum operators and semiclassical phase space functions.

Let  $A$  and  $B$  be two observables,<sup>1</sup> so that  $[A,B] \equiv AB - BA = iC$  and  $C^\dagger = C$ , together with the corresponding Schrödinger-Robertson inequality

$$(\Delta A)^2(\Delta B)^2 \geq \left|\frac{1}{4}\langle C \rangle\right|^2 + \sigma(A,B), \quad \sigma(A,B) = \frac{\langle AB + BA \rangle}{2} - \langle A \rangle \langle B \rangle, \quad (6.1.1)$$

with mean  $\langle F \rangle = \langle \Psi | F | \Psi \rangle$ , variance  $(\Delta F)^2 = \langle F^2 \rangle - \langle F \rangle^2$ , and  $\sigma(A,B)$ , the *correlation function* [11]. Then, we say that  $|\Psi\rangle$ , with  $\|\Psi\| < \infty$ , is a squeezed state if one of the variances associated to the observables  $A$  and  $B$  takes values below the uncertainty minimum  $\sqrt{\frac{1}{4}|\langle C \rangle|^2 + \sigma(A,B)}$  while the second variance compensates by increasing such that the inequality (6.1.1) is always saturated. Note that the definition of squeezing is in reference to the observable whose variance is being ‘squeezed’.

Interestingly, if an underlying algebra can be identified with the observables  $A$  and  $B$ , the squeezed states can be constructed by the sequential action of unitary operators on a fiducial state. Such unitary operators are usually constructed as the exponential representation of the algebra elements. To this end, there exist a great deal of examples of squeezed states in the literature such as the coherent squeezed states, coherent and squeezed number states [12], second-order squeezed states [13], and Susskind-Glogower coherent states [14, 15] (also known as London coherent states [16]) to mention some. On the other hand, if a closed algebra is not available, one may proceed by solving an eigenvalue equation of the form  $(A+i\lambda B)|\psi\rangle = z|\psi\rangle$ , which minimises (6.1.1). See for instance [17, 18, 19]. The latter constructions have been extended to quantised electromagnetic fields composed of several modes. Some examples include two-mode [20, 21, 22, 23] and higher-mode in [7] constructions.

In this work, we exploit the overcompleteness of certain families of squeezed states as a means of quantisation in two dimensions. The paper is structured as follows. In section 6.2 we review the basics of quantisation with one-mode squeezed states and define their semiclassical portraits. Following this, in section 6.3 we generalise the notions of the preceding section to the two-mode case. We first define the most natural extension, the separable squeezed states, as the tensor product of two one-mode squeezed states acting on each mode independently, and then we define the non-separable squeezed states which cannot be factorised by a tensor product. We compare the quantisation of some classical functions in both cases and find that non-separability leads to mixing between quadrature operators between both modes. In section 6.4 we study the semiclassical portraits of a position dependent mass system in constrained geometry as an application, before concluding in section 6.5 with some remarks about extensions of the ideas presented in this paper to different problems.

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<sup>1</sup>In this work, we focus on the common definition of observables as defined by self-adjoint operators,  $A^\dagger = A$ .

## 6.2. One-mode squeezed state quantisation

Before proceeding to the two-mode quantisation, let us recapitulate some results in one-dimensional squeezed state quantisation. Firstly, the unitary displacement and squeezing operators are defined as follows

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}, \quad S(\xi) = e^{-\frac{1}{2}\xi a^{\dagger 2} + \frac{1}{2}\xi^* a^2}, \quad \alpha, \xi \in \mathbb{C}, \quad (6.2.1)$$

in terms of the boson operators  $a$  and  $a^\dagger$ , whose action on the elements of the elements of the Fock basis  $\{|n\rangle\}_{n=0}^\infty$  is given by

$$a|n+1\rangle = \sqrt{n+1}|n\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad n = 0, 1, \dots, \quad (6.2.2)$$

along with the annihilation of the vacuum state,  $a|0\rangle = 0$ . The squeezed coherent states,  $|\alpha; \xi\rangle$ , are then constructed through the action of the unitary operators (6.2.1) on the corresponding fiducial state  $|0\rangle$ ,

$$|\alpha; \xi\rangle = S(\xi)D(\alpha)|0\rangle. \quad (6.2.3)$$

Note that the alternative definition of squeezed states,  $|\xi; \beta\rangle = D(\beta)S(\xi)|0\rangle$ , is equivalent to (6.2.3) through a braiding relation and this amounts to a relabelling of the parameters. Following the customary procedure, one can disentangle the unitary operators in the product of exponential functions in terms of  $a$  and  $a^\dagger$  separately. Alternatively, we can determine the eigenvalue equation related to  $|\alpha; \xi\rangle$ . This is achieved by computing the unitary transformations on the boson ladder operators

$$S^\dagger(\xi)aS(\xi) = a \cosh |\xi| - a^\dagger \frac{\xi}{|\xi|} \sinh |\xi|, \quad D^\dagger(\alpha)aD(\alpha) = a + \alpha, \quad (6.2.4)$$

where a Baker–Campbell–Hausdorff identity [24] has been used. Thus, from the unitary transformation  $D^\dagger S^\dagger a S D$ , and after several calculations, we get the eigenvalue equation

$$(a + \tau a^\dagger)|\alpha; \xi\rangle = \alpha \sqrt{1 - |\tau|^2} |\alpha; \xi\rangle, \quad \tau = \frac{\xi}{|\xi|} \tanh |\xi|. \quad (6.2.5)$$

The latter is solved by expanding  $|\alpha; \xi\rangle$  in the Fock basis and solving the resulting second-order finite-difference equation [25]. This yields the normalised states [26]

$$|\alpha; \xi\rangle = (1 - |\tau|^2)^{1/4} e^{-\frac{|\alpha|^2}{2} + \frac{\alpha^2 \tau^* + \alpha^* 2\tau}{4}} \sum_{n=0}^{\infty} \frac{\tau^{n/2}}{(2^n n!)^{1/2}} H_n \left( \alpha \sqrt{\frac{1 - |\tau|^2}{2\tau}} \right) |n\rangle, \quad (6.2.6)$$

for  $\alpha \in \mathbb{C}$  and  $|\tau| < 1$  ( $\xi \in \mathbb{C}$ ).

In general, the squeezed states do not form an orthogonal set of states as they have a non-zero overlap,  $\langle \alpha'; \xi | \alpha; \xi \rangle \neq 0$ . Nevertheless they form an overcomplete set of states on the Hilbert space as they fulfil the resolution of the identity

$$\int_{\alpha \in \mathbb{C}} \frac{d^2 \alpha}{\pi} |\alpha; \xi\rangle \langle \alpha; \xi| = \mathbb{I}, \quad (6.2.7)$$

with  $\mathbb{I}$  the identity operator in the Fock space  $\mathcal{H} = \text{span}\{|n\rangle\}_{n=0}^{\infty}$ , and  $\mu(\alpha)$  the corresponding *measure* function. The factor  $\pi^{-1}$  has been introduced for convenience, and it does not modify the final result as it can be absorbed in the measure.

From the very definition of the squeezed states,  $|\alpha, \xi\rangle = S(\xi)D(\alpha)|0\rangle$ , we obtain a more simple form for the resolution of the identity, which reads

$$S(\xi) \left( \int_{\alpha \in \mathbb{C}} \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| \right) S^\dagger(\xi) = \mathbb{I}, \quad |\alpha\rangle = D(\alpha)|0\rangle, \quad (6.2.8)$$

with  $|\alpha\rangle$  the conventional Glauber-Sudarshan coherent states. By defining the squeezed coherent states using the convention in (6.2.3), the measure function is constant. In this form, we have shown that squeezed states form an overcomplete set  $\{|\alpha, \xi\rangle\}_{\alpha \in \mathbb{C}}$  with a uniform measure. The identity operator can be alternatively achieved through the orthogonality property related to the holomorphic Hermite polynomial [27, 28]. See appendix 6.A for a detailed proof.

The resolution of the identity ensures that every element  $|\phi\rangle \in \mathcal{H}$  can be expanded in the non-orthogonal basis  $\{|\psi(\alpha)\rangle\}_{\alpha \in \mathbb{C}}$  through

$$|\phi\rangle = \int_{\alpha \in \mathbb{C}} \frac{d^2\alpha}{\pi} \mathcal{F}_\phi(\alpha) |\alpha, \xi\rangle, \quad \mathcal{F}_\phi(\alpha, \xi) = \langle\alpha, \xi|\phi\rangle, \quad (6.2.9)$$

where  $\mathcal{F}_\phi(\alpha)$  is uniquely defined for each vector  $|\phi\rangle$ .

Throughout this manuscript, we will use an alternative representation for the resolution of the identity (6.2.7) that encodes information about the position and momentum observables. To this end, let us recall the following relationships:

$$\hat{x} = \lambda \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \quad \hat{p} = \frac{\hbar}{\lambda} \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}, \quad (6.2.10)$$

where  $\lambda > 0$  is a free parameter with units of length. From the latter, a relationship between the coherence parameter  $\alpha = \text{Re}(\alpha) + i\text{Im}(\alpha)$  and the expectation values  $q \equiv \langle\hat{x}\rangle$  and  $p \equiv \langle\hat{p}\rangle$  associated to the canonical position and momentum operators, respectively, with  $\langle\cdot\rangle \equiv \langle\alpha, \xi|\cdot|\alpha, \xi\rangle$ . By combining (6.2.4) with (6.2.10), and averaging in the squeezed state basis we obtain the symplectic transform

$$\begin{pmatrix} \text{Re}(\alpha) \\ \text{Im}(\alpha) \end{pmatrix} = \begin{pmatrix} \frac{1 + \text{Re}(\tau)}{\sqrt{1 - |\tau|^2}} & \frac{\text{Im}(\tau)}{\sqrt{1 - |\tau|^2}} \\ \frac{\text{Im}(\tau)}{\sqrt{1 - |\tau|^2}} & \frac{1 - \text{Re}(\tau)}{\sqrt{1 - |\tau|^2}} \end{pmatrix} \begin{pmatrix} q \\ \frac{\lambda p}{\hbar\sqrt{2}} \end{pmatrix}. \quad (6.2.11)$$

Note that, for  $\tau = 0$ , we recover the well-known relationships  $\text{Re}(\alpha) = \frac{q}{\lambda\sqrt{2}}$  and  $\text{Im}(\alpha) = \frac{\lambda p}{\hbar\sqrt{2}}$  for coherent states.

From (6.2.11), one may notice that  $\alpha$  is linear in the expectation values  $q = \langle\hat{x}\rangle$  and  $p = \langle\hat{p}\rangle$ . Thus, the complex-plane  $\alpha$  can be understood as an analogue of the classical phase

space manifold as every point  $(q,p) \in \mathbb{R}^2$  is in unique correspondence with  $\alpha \in \mathbb{C}$ . Moreover, the transformation from the point  $(q,p)$  to  $\alpha$  given in (6.2.11) is determined by a unimodular matrix, and thus the existence of the respective inverse transformation is guaranteed. The differential element in both frames is preserved, that is,  $d^2\alpha \rightarrow (2\hbar)^{-1}dqdp$ . With this identification, we can alternatively rewrite the resolution of the identity in terms of  $q$  and  $p$  as

$$\mathbb{I} = \int_{\mathbb{R}^2} \frac{dqdp}{2\pi\hbar} |q,p;\xi\rangle\langle q,p;\xi|, \quad |q,p;\xi\rangle \equiv |\alpha(q,p);\xi\rangle, \quad (6.2.12)$$

with  $\alpha(q,p)$  given in (6.2.11).

It is useful to determine the position representation for the squeezed states,  $\psi_{\{q,p;\xi\}}(x) = \langle x|q,p;\xi\rangle$ , as it facilitates the determination of some observables. From (6.2.6), together with  $\langle x|n\rangle = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{x^2}{2\lambda^2}} H_n(\frac{x}{\lambda})$ , and using the summation identities for Hermite polynomials [29] we obtain the normalised wavefunction

$$\psi(\alpha; \xi; x)(x) = \langle x|\alpha; \xi\rangle = \frac{(1 - |\tau|^2)^{1/4}}{\pi^{1/4} \sqrt{1 - \tau}} e^{-\frac{|\alpha|^2}{2}} e^{\frac{\alpha^2 \tau^* + \alpha^{*2} \tau}{4}} e^{-\frac{\alpha^2 (1 - |\tau|^2)}{2(1 - \tau)}} e^{-\frac{1}{2} \left(\frac{1 + \tau}{1 - \tau}\right) \frac{x^2}{\lambda^2}} e^{\frac{\sqrt{2(1 - |\tau|^2)} \alpha x}{1 - \tau}}. \quad (6.2.13)$$

Alternatively, we can rewrite (6.2.13) in terms of the expectation values  $q$  and  $p$  by using the relationships (6.2.11) to get, up to a complex-phase,

$$\psi(q,p;\xi,x) = \frac{1}{\pi^{1/4} \sqrt{\lambda}} \frac{(1 - |\tau|^2)^{1/4}}{|1 - \tau|^{1/2}} \exp\left(-\frac{\sigma_q^2}{2\lambda^2} (q - x)^2 + i \frac{\text{Im}(\tau)}{4\lambda^2} \left(q - \frac{\lambda^2}{\hbar} p\right)^2 + i \frac{p}{\hbar} \left(x - \frac{q}{2}\right)\right), \quad (6.2.14)$$

with  $\sigma_q^2$  a complex parameter given by

$$\sigma_q^2 = \frac{(1 - |\tau|^2) + 2i \text{Im}(\tau)}{|1 - \tau|^2}. \quad (6.2.15)$$

### 6.2.1. Quantisation

We now proceed to discuss one of the main results of this manuscript, the quantisation map using squeezed states. Although we summarise the results from the one-dimensional case, the results developed here extend to higher dimensions with relative ease. To this end, let us introduce an operation that maps a classical observable  $f(\alpha) \equiv f(q,p)$ , defined in the classical phase space manifold, into a linear operator  $\hat{A}_f$ , defined to act on elements of the vector space  $\mathcal{H}$ . This a procedure is known as a *quantisation map*, which requires a complete family of states, such as the squeezed states, such that to every classical observable we can associate a unique quantum observable. The map is defined by

$$f(q,p) \mapsto \hat{A}_f = \int_{\mathbb{R}^2} \frac{dqdp}{2\pi\hbar} f(q,p) |q,p;\xi\rangle\langle q,p;\xi|. \quad (6.2.16)$$

Although this definition is quite general, in some cases it can be computationally infeasible. To overcome this issue, we take advantage of the coordinate representation in order to



compute the action of the observable  $A_f$  on a test function,  $\Psi(x)$ , which is an arbitrary element  $|\Psi\rangle \in \mathcal{H}$ . This corresponds to the operation

$$(A_f^{(op)}) \Psi(x) \equiv \langle x | \hat{A}_f | \Psi \rangle = \int_{\mathbb{R}} dx' \mathcal{K}_f(\xi; x, x') \Psi(x'), \quad (6.2.17)$$

where  $A_f^{(op)}$  is the coordinate representation of  $\hat{A}_f$ , together with  $\mathcal{K}_f(\xi; x, x')$ , a kernel operator containing information about the action of  $\hat{A}_f$  on the test function  $\Psi(x)$ , determined through

$$\mathcal{K}_f(\xi; x, x') = \int_{\mathbb{R}^2} \frac{dqdp}{2\pi\hbar} f(q,p) \psi^*(q,p; \xi; x') \psi(q,p; \xi; x), \quad (6.2.18)$$

with  $\psi(q,p; \xi; x)$  the wavefunction given in (6.2.14).

To illustrate the use of the kernel representation (6.2.17), we consider two examples.

- First, let  $f(q,p) = q$  such that the kernel becomes  $\mathcal{K}_q(\xi; x, x') = \delta(x' - x)$ , where we have used some elementary properties of the Fourier transform while integrating with respect to  $p$ . In this form, we get  $A_q^{(op)} \Psi(x) \equiv x \Psi(x)$ , which means that  $q \mapsto \hat{A}_q = \hat{x} \equiv x$ , as expected.

- Similarly, for  $f(q,p) = p$ , and using some properties involving derivatives of the Fourier transform, we obtain the kernel  $\mathcal{K}_p(\xi; x, x') = -i\hbar \delta_{x'}(x' - x)$ , with the subscript index denoting the partial derivative with respect to  $x'$ . Such a kernel leads to  $A_p^{(op)} \Psi(x) \equiv \frac{\hbar}{i} \frac{\partial}{\partial x} \Psi(x)$ . That is, the quantisation of  $p$  becomes, in the  $x$ -representation, proportional to the derivative with respect to  $x$ ,  $p \mapsto \hat{A}_p = \hat{p} \equiv \frac{\hbar}{i} \frac{\partial}{\partial x}$ .

- The quantisation of  $f(q,p) = qp$ , the dilation operator, follows by analogy with the previous two cases, leading to  $\hat{A}_{q,p} = \frac{\hat{x}\hat{p} + \hat{p}\hat{q}}{2} - \frac{\text{Im}(\sigma_q^2)}{2\text{Re}(\sigma_q^2)}$ . The latter corresponds to the symmetrisation of the resulting quantum operator plus a constant term that depends explicitly on the squeezing parameter. We observe that for  $\tau \in \mathbb{R}$  or  $\tau = 0$ , we recover the conventional symmetrisation rule.

Before proceeding, it is worth mentioning that the definition (6.2.16) fulfills two fundamental properties required by any quantisation mechanism [30, 31]. Firstly, the quantisation map (6.2.16) must promote the classical function  $f(q,p) = 1$  into the identity operator  $\mathbb{I}$ . This is already guaranteed from the completeness relationship (6.2.12). Secondly, Dirac's correspondence rule should be recovered,  $\{q,p\}_{PB} = 1 \rightarrow [\hat{x}, \hat{p}] = i\hbar$ , with  $\{f(q,p), h(q,p)\}_{PB}$  the Poisson brackets [32]. From the previous two examples, it follows directly that  $[A_q^{(op)}, A_p^{(op)}] \Psi(x) = i\hbar \Psi(x)$ , which fulfils the correspondence rule.

## 6.2.2. Semiclassical portraits

Interestingly, as with the conventional coherent states, we can define a set of quantities that behave analogously to their classical counterparts. These quantities are known as *semiclassical portraits* [33, 34], which are defined as the expectation values of the corresponding quantum observables  $\hat{A}_f$  in the squeezed states basis. We thus introduce the *lower symbol*,

or semiclassical portrait, as

$$f(q,p) \mapsto \check{A}_f = \langle q,p; \xi | \hat{A}_f | q,p; \xi \rangle = \int_{\mathbb{R}^2} \frac{dq dp}{2\pi\hbar} f(q',p') |\langle q',p'; \xi | q,p; \xi \rangle|^2. \quad (6.2.19)$$

where the absolute value square overlap between squeezed states is given by

$$|\langle q',p'; \xi | q,p; \xi \rangle|^2 = e^{-\frac{\Delta_q^2}{2\lambda^2}(q-q')^2 - \frac{\lambda^2}{2\hbar^2}\Delta_p^2(p-p')^2 - 2\frac{\gamma}{\hbar}(q-q')(p-p')}, \quad (6.2.20)$$

with the Gaussian widths  $\Delta_q$  and  $\Delta_p$ , together with the coupling parameter  $\gamma$ , given in terms of the original parameters by

$$\Delta_q^2 = |\sigma_q|^2 = \frac{1 - |\tau|^2}{|1 - \tau|^2} + \frac{4 \operatorname{Im}(\tau)^2}{|1 - \tau|^2(1 - |\tau|^2)}, \quad \Delta_p^2 = \frac{|1 - \tau|^2}{1 - |\tau|^2}, \quad \gamma = \frac{\operatorname{Im}(\tau)}{1 - |\tau|^2}. \quad (6.2.21)$$

A handy formula can be derived for classical functions that depend only on position,  $f(q,p) = h(q)$ , in which case the integral (6.2.19) becomes

$$\check{A}_{h(q)} = \frac{1}{\sqrt{2\pi\lambda\Delta_p}} \int_{\mathbb{R}} dq' h(q') e^{-\frac{(q-q')^2}{2\lambda^2\Delta_p^2}}. \quad (6.2.22)$$

This may be thought of as a Gaussian regularisation of the classical function  $h(q)$ . This is particularly useful when dealing with discontinuous functions  $h(q)$ . Further examples will be discussed once we introduce the two-mode extension in the upcoming sections.

### 6.3. Families of two-mode squeezed states

We now turn our attention to the main purpose of the paper: quantisation for two-dimensional systems. As we discussed in section 6.2, the quantisation map depends on the choice of the family of overcomplete states used, and thus the quantisation for two-dimensional systems can be constructed in a similar vein to the one-dimensional case by implementing families of multimode states, such as multimode coherent states, such that they fulfil the resolution of the identity. Here, we define the larger Hilbert space  $\mathcal{H} = \operatorname{span}\{|n_1\rangle \otimes |n_2\rangle\}_{n_1, n_2=0}^{\infty}$ , with  $|n_j\rangle$  elements of the Fock basis on the  $j$  mode, and the corresponding identity operator in such a vector space reads as  $\mathbb{I} = \sum_{n_1, n_2=0}^{\infty} |n_1, n_2\rangle \langle n_1, n_2|$ , with  $|n_1, n_2\rangle \equiv |n_1\rangle \otimes |n_2\rangle$ . In this form, we introduce the set of multimode bosonic operators  $\{a_1, a_1^\dagger, a_2, a_2^\dagger\}$ , which fulfil the commutation relationships  $[a_j, a_k^\dagger] = \delta_{j,k}$ , for  $j, k = 1, 2$ . Moreover, the action of such multimode operators on the extended vector space  $\mathcal{H}$  is defined as

$$\begin{aligned} a_1 |n_1, n_2\rangle &= \sqrt{n_1} |n_1 - 1, n_2\rangle, & a_2 |n_1, n_2\rangle &= \sqrt{n_2} |n_1, n_2 - 1\rangle, \\ a_1^\dagger |n_1, n_2\rangle &= \sqrt{n_1 + 1} |n_1 + 1, n_2\rangle, & a_2^\dagger |n_1, n_2\rangle &= \sqrt{n_2 + 1} |n_1, n_2 + 1\rangle. \end{aligned} \quad (6.3.1)$$

The canonical position and momentum quadratures  $\hat{x}_j$  and  $\hat{p}_j$ , respectively, for the  $j$  mode are related to the multimode boson operators as

$$\hat{x}_j = \lambda_j \frac{a_j + a_j^\dagger}{\sqrt{2}}, \quad \hat{p}_j = \frac{\hbar}{\lambda_j} \frac{a_j - a_j^\dagger}{i\sqrt{2}}, \quad j = 1, 2, \quad (6.3.2)$$

where  $[\hat{x}_j, \hat{p}_k] = i\hbar\delta_{j,k}$  and  $[\hat{x}_j, \hat{x}_k] = [\hat{p}_j, \hat{p}_k] = 0$ , for  $j, k = 1, 2$ .

Throughout this section we focus on two particularly interesting cases. The first one being the most immediate extension by taking the tensor product of two independent one-mode squeezed states. In the second case, we consider a family of two-mode states that do not factorise as the tensor product of two one-mode squeezed states.

### 6.3.1. Separable two-dimensional squeezed states

Let us consider the conventional one-mode squeezed states introduced in section 6.2, and extend them into the extended vector space  $\mathcal{H}$  through the direct product of two squeezed, one in each mode, with the coherence and squeezing parameters in general being different in each mode. Henceforth, we refer to this specific construction as *separable squeezed states*, which are explicitly defined as

$$|\vec{\alpha}; \vec{\xi}\rangle = |\alpha_1, \xi_1\rangle \otimes |\alpha_2, \xi_2\rangle = S_2(\xi_2)S_2(\xi_1)D_2(\alpha_2)D_1(\alpha_1)|0, 0\rangle, \quad \alpha_j, \xi_j \in \mathbb{C}, \quad j = 1, 2, \quad (6.3.3)$$

with  $D_j(\alpha_j)$  and  $S_j(\xi_j)$  denoting the displacement and squeezing operators, respectively, defined on the  $j$  mode, and  $|0, 0\rangle$  the two-mode vacuum.

From the separable squeezed states (6.3.3), we can find a relationship between the coherent parameter and the expectation value of the canonical coordinates. This is done analogously to the one-mode case, and we find

$$\begin{pmatrix} \vec{r}_{\alpha_1} \\ \vec{r}_{\alpha_2} \end{pmatrix} = \begin{pmatrix} \mathbb{M}_1 & \mathbb{O}_{2 \times 2} \\ \mathbb{O}_{2 \times 2} & \mathbb{M}_2 \end{pmatrix} \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix}, \quad \mathbb{M}_j = \begin{pmatrix} \frac{1+\text{Re}(\tau_j)}{\sqrt{1-|\tau_j|^2}} & \frac{\text{Im}(\tau_j)}{\sqrt{1-|\tau_j|^2}} \\ \frac{\text{Im}(\tau_j)}{\sqrt{1-|\tau_j|^2}} & \frac{1-\text{Re}(\tau_j)}{\sqrt{1-|\tau_j|^2}} \end{pmatrix}, \quad \tau_j = \frac{\xi_j}{|\xi_j|} \tanh |\xi_j|, \quad (6.3.4)$$

with  $j = 1, 2$ ,  $\mathbb{O}_{2 \times 2}$  the null  $2 \times 2$  matrix, and

$$\vec{r}_{\alpha_j} = \begin{pmatrix} \text{Re}(\alpha_j) \\ \text{Im}(\alpha_j) \end{pmatrix}, \quad \vec{r}_j = \begin{pmatrix} q_j \\ \frac{\lambda_j p_j}{\hbar\sqrt{2}} \end{pmatrix}, \quad q_j = \langle \hat{x}_j \rangle, \quad p_j = \langle \hat{p}_j \rangle. \quad (6.3.5)$$

We remark that the limit  $\tau_j \rightarrow \infty$  refers to infinite squeezing in the  $j$  mode. The separable squeezed states minimise the Schrödinger-Robertson uncertainty relation for the physical position and momentum quadratures in each mode independently. That is,

$$(\Delta \hat{x}_j)^2 (\Delta \hat{p}_j)^2 = \frac{\hbar^2}{4} + \tilde{\sigma}(\hat{x}_j, \hat{p}_j), \quad j = 1, 2. \quad (6.3.6)$$

Additionally, the separable squeezed states admit a coordinate representation defined in terms of the eigenstates of the quadratures  $\hat{x}_1$  and  $\hat{x}_2$  in a similar manner to their one-dimensional counterparts. By considering the linear transformation (6.3.4), we rewrite the squeezed states in terms of  $q_j$  and  $p_j$  so that the normalised wavefunction  $\psi(\vec{q}, \vec{p}; \vec{\xi}, \vec{x}) = \langle \vec{x} | \vec{q}, \vec{p}; \vec{\xi} \rangle$ , with  $|\vec{x}\rangle = |x_1\rangle \otimes |x_2\rangle$ , takes the form

$$\psi(\vec{q}, \vec{p}; \vec{\xi}; \vec{x}) = \frac{\exp\left(-\frac{1}{2} \sum_{j=1}^2 \left[ \frac{\sigma_{q_j}^2}{\lambda_j^2} (q_j - x_j)^2 - i \frac{\text{Im}(\tau_j)}{2\lambda_j^2} \left( q_j - \frac{\lambda_j^2}{\hbar} p_j \right)^2 \right] + i \frac{\vec{p}}{\hbar} \cdot \left( \vec{x} - \frac{\vec{q}}{2} \right)\right)}{\sqrt{\lambda_1 \lambda_2 \Delta_{p_1} \Delta_{p_2} \pi^{1/4}}}, \quad (6.3.7)$$

where

$$\sigma_{q_j}^2 = \frac{(1 - |\tau_j|^2) + 2i \text{Im}(\tau_j)}{|1 - \tau_j|^2}, \quad \Delta_{p_j}^2 = \frac{|1 - \tau_j|^2}{1 - |\tau_j|^2}, \quad j = 1, 2. \quad (6.3.8)$$

### 6.3.2. Quantisation map and semiclassical portraits

In section 6.2, we showed that the one-mode squeezed states form an overcomplete family of states. This property is inherited by the two-dimensional case in the extended vector space  $\mathcal{H}$  through

$$\mathbb{I} = \int_{\mathbb{R}^4} \frac{d^2 \vec{q} d^2 \vec{p}}{(2\pi\hbar)^2} |\vec{q}, \vec{p}; \vec{\xi}\rangle \langle \vec{q}, \vec{p}; \vec{\xi}|, \quad d^2 \vec{q} = dq_1 dq_2, \quad d^2 \vec{p} = dp_1 dp_2. \quad (6.3.9)$$

The latter can be easily shown by factorising the separable squeezed states into its independent modes and then using the corresponding one-mode results.

In this form, the quantisation map is implemented straightforwardly through the integral transform

$$f(\vec{q}, \vec{p}) \mapsto \hat{A}_f = \int_{\mathbb{R}^4} \frac{d\vec{q} d\vec{p}}{(2\pi\hbar)^2} f(\vec{q}, \vec{p}) |\vec{q}, \vec{p}; \vec{\xi}\rangle \langle \vec{q}, \vec{p}; \vec{\xi}|, \quad (6.3.10)$$

which can be conveniently rewritten in terms of the coordinate representation as

$$A_f^{(op)} \Psi(\vec{x}) = \int_{\mathbb{R}^2} d\vec{x}' \mathcal{K}_f(\vec{\xi}; \vec{x}, \vec{x}') \Psi(\vec{x}'), \quad (6.3.11)$$

where the integral kernel is given by

$$\mathcal{K}_f(\vec{\xi}; \vec{x}, \vec{x}') = \int_{\mathbb{R}^2} \frac{d\vec{q} d\vec{p}}{2\pi\hbar} f(\vec{q}, \vec{p}) \psi^*(\vec{q}, \vec{p}; \vec{\xi}; \vec{x}') \psi(\vec{q}, \vec{p}; \vec{\xi}; \vec{x}). \quad (6.3.12)$$

Since the squeezed states are the tensor product of two one-dimensional states, the quantisation of a classical function of the form  $f(\vec{q}, \vec{p}) = f_1(q_1, p_1) f_2(q_2, p_2)$  produces an operator factorisable as  $f(q, p) \mapsto \hat{A}_{f_1} \otimes \hat{A}_{f_2}$ . In particular, for a classical function  $f(\vec{q}, \vec{p}) = h_1(q_1) h_2(q_2)$  we obtain a simplified kernel of the form

$$\mathcal{K}_{h_1 h_2}(\vec{\xi}; \vec{x}, \vec{x}') = \mathcal{K}_{h_1}(\xi_1; x_1, x'_1) \mathcal{K}_{h_2}(\xi_2; x_2, x'_2), \quad (6.3.13)$$

with

$$\mathcal{K}_{h_j}(\xi_j; x_j, x'_j) = \delta(x_j - x'_j) \frac{(1 - |\tau_j|^2)^{\frac{1}{2}}}{\pi^{\frac{1}{2}} \lambda_j |1 - \tau_j|} \int_{\mathbb{R}} dq_j h_j(q_j) e^{-\frac{\text{Re}[\sigma_{q_j}^2]}{\lambda_j} (q_j - x_j)^2}, \quad j = 1, 2. \quad (6.3.14)$$

In a similar vein, the construction of the corresponding semiclassical portraits follows straightforwardly from the one-dimensional case. That is, by averaging the quantised operators  $\hat{A}_f$  over the two-mode separable squeezed state basis we get

$$\check{A}_{f(\vec{q}, \vec{p})} = \int_{\mathbb{R}^4} \frac{d\vec{q}' d\vec{p}'}{(2\pi\hbar)^2} f(\vec{q}', \vec{p}') |\langle \vec{q}', \vec{p}'; \vec{\xi} | \vec{q}, \vec{p}; \vec{\xi} \rangle|^2, \quad (6.3.15)$$

where the squeezed state overlap is defined as the product of two one-dimensional squeezed state overlaps given in (6.2.20). In this form, we may distinguish the following cases:

- The semiclassical portrait of  $f(\vec{q}, \vec{p}) = h(\vec{q})$  leads to

$$\check{A}_{h(\vec{q})} = \frac{1}{2\pi\lambda_1\lambda_2\Delta_{p_1}\Delta_{p_2}} \int_{\mathbb{R}^2} dq'_1 dq'_2 h(\vec{q}') \exp(\vec{v}' \cdot \mathfrak{S} \vec{v}'), \quad (6.3.16)$$

where the vector  $\vec{v}' = \begin{pmatrix} q_1 - q'_1 \\ q_2 - q'_2 \end{pmatrix}$  and scaling matrix  $\mathfrak{S} = \begin{pmatrix} \frac{-1}{2\lambda_1^2[\Delta_{p_1}]^2} & 0 \\ 0 & \frac{-1}{2\lambda_2^2[\Delta_{p_2}]^2} \end{pmatrix}$ . This is a Gaussian regularisation of the classical observables, analogously to the kernel regularisation obtained in (6.3.14). • A general expression can be found if we consider a classical function that mixes one of the momenta with an arbitrary function of both positions,  $f(\vec{q}, \vec{p}) = p_j h(\vec{q})$ . In this setup we get

$$\check{A}_{p_j h(\vec{q})} = p_j \check{A}_{h(\vec{q})} + \frac{2\hbar\gamma_j}{\Delta_{p_j}^2 \lambda_j^2} (q_j \check{A}_{h(\vec{q})} - \check{A}_{q_j h(\vec{q})}), \quad j = 1, 2. \quad (6.3.17)$$

Clearly, for  $h(\vec{q}) = 1$ , we recover the expected result  $\check{A}_{p_j} = p_j$ .

• From the previous two examples, we may compute the semiclassical portrait related to a kinetic energy of the form  $f(\vec{q}, \vec{p}) = p_j^2 h(\vec{q})$ , with  $h(\vec{q}) = (m(\vec{q}))^{-1}$  playing the role of a position-dependent mass term. We obtain

$$\check{A}_{p_j^2 h(\vec{q})} = \left( p_j^2 + \frac{\hbar^2}{\Delta_{p_j}^2 \lambda_j^2} \right) \check{A}_{h(\vec{q})} + \frac{4\hbar^2 \gamma_j^2}{\Delta_{p_j}^4 \lambda_j^4} (q_j^2 \check{A}_{h(\vec{q})} - 2q_j \check{A}_{q_j h(\vec{q})} + \check{A}_{q_j^2 h(\vec{q})}) + \frac{4\hbar\gamma_j}{\Delta_{p_j}^2 \lambda_j^2} p_j (q_j \check{A}_{h(\vec{q})} - \check{A}_{q_j h(\vec{q})}), \quad (6.3.18)$$

for  $j = 1, 2$ . Contrary to the previous case, for  $h(\vec{q}) = 1$ , we obtain  $p_j^2 + \frac{\hbar^2}{\Delta_{p_j}^2 \lambda_j^2}$ , which contains an additive purely quantum term proportional to  $\hbar^2$ .

### 6.3.3. Non-separable two-mode squeezed states

The two-dimensional construction of squeezed states discussed in section (6.3.1) is the most immediate generalisation of the one-dimensional squeezed states. However, those states are a particular extension, and in multidimensional systems more general states can be constructed which cannot be decomposed into the tensor product of one-dimensional states. Such classes of states have been discussed in the literature for the two-dimensional case by using two-mode ladder operators so that the information of both modes is mixed [20, 23].

In this section we follow the construction introduced in [20], where a family of two-mode squeezed states are constructed with the aid of the mixing operator

$$U_{BS}(\phi) = e^{\phi(a_1^\dagger a_2 - a_1 a_2^\dagger)}, \quad \phi \in [0, 2\pi), \quad (6.3.19)$$

which is equivalent to the quantum representation of the *beam-splitter*. In this form, we may combine the beam-splitter with the one-mode displacement and squeezing operators  $D(\alpha_j)$  and  $S(\xi_j)$ , respectively, in order to construct the *non-separable squeezed states* [20]

$$|\vec{\alpha}; \vec{\xi}, \phi\rangle = G|0, 0\rangle, \quad \vec{\alpha} = (\alpha_1, \alpha_2), \quad \vec{\xi} = (\xi_1, \xi_2). \quad (6.3.20)$$

with  $G$  the unitary operator

$$G = D_1(\alpha_1)D_2(\alpha_2)U_{BS}(\phi)S_1(\xi_1)S_2(\xi_2), \quad \alpha_j, \xi_j \in \mathbb{C}, \quad j = 1, 2. \quad (6.3.21)$$

The order of the displacement and squeezing operators has been deliberately chosen so that the squeezing operators act first on the two-mode vacuum state  $|0, 0\rangle$ . This is due the fact that the beam-splitter operator acting on a nonclassical state, such as the two-mode squeezed vacuum, produces a non-separable state at the output. Therefore, if we were to act with the displacement operator first, we would get a separable state at the output, as the coherent states are classical in this respect. See [35] for details. In this form, the non-separability of the two-mode intertwined squeezed states is determined by the parameter  $\phi$ . For  $\phi = 0$ , we recover the separable states of section 6.2.

Now, from (6.3.21), we can find the unitary transformation of the boson ladder operators for both the modes,  $a_1$  and  $a_2$ , respectively. We make use of the well-known Bogoliubov transformations [36] to obtain

$$\begin{aligned} G^\dagger a_1 G &= \alpha_1 + \cos \phi \left( a_1 \cosh |\xi_1| - a_1^\dagger \frac{\xi_1}{|\xi_1|} \sinh |\xi_1| \right) + \sin \phi \left( a_2 \cosh |\xi_2| - a_2^\dagger \frac{\xi_2}{|\xi_2|} \sinh |\xi_2| \right), \\ G^\dagger a_2 G &= \alpha_2 + \cos \phi \left( a_2 \cosh |\xi_2| - a_2^\dagger \frac{\xi_2}{|\xi_2|} \sinh |\xi_2| \right) - \sin \phi \left( a_1 \cosh |\xi_1| - a_1^\dagger \frac{\xi_1}{|\xi_1|} \sinh |\xi_1| \right). \end{aligned} \quad (6.3.22)$$

From the latter it is evident that  $G$  indeed mixes the modes  $a_1$  and  $a_2$ , where, for  $\phi = 0$ , the transformation decouples  $a_1$  from  $a_2$ . The unitary transformation (6.3.22), combined with

the definition of the physical canonical quadratures (6.3.2), allows us to recover the same relationships between the canonical coordinates and complex parameters  $\alpha_i$  as in (6.3.5).

The unitary transformations (6.3.22) lead to a set of two eigenvalue equations whose eigenfunctions are the two-dimensional squeezed states  $|\vec{\alpha}; \vec{\xi}, \phi\rangle$ , see appendix 6.B for details. In this form, we obtain the corresponding wavefunction as

$$\psi(\vec{q}, \vec{p}; \vec{\xi}, \phi; \vec{x}) = \mathcal{N}(\vec{q}, \vec{p}; \vec{\xi}, \phi) e^{-\frac{\Delta_1}{\lambda_1^2} x_1^2 - \frac{\Delta_2}{\lambda_2^2} x_2^2 - \frac{\ell}{\lambda_1 \lambda_2} x_1 x_2 + \frac{\ell_1}{\lambda_1} x_1 + \frac{\ell_2}{\lambda_2} x_2}, \quad (6.3.23)$$

where  $\mathcal{N}(\vec{q}, \vec{p}; \vec{\xi}, \phi)$  is a normalisation factor, and the coefficients proportional to the bilinear terms in  $x_1, x_2$  are

$$\begin{aligned} \Delta_1 &= \frac{1 - \tau_1 \tau_2 - \cos(2\phi)(\tau_2 - \tau_1)}{2(1 - \tau_1)(1 - \tau_2)}, & \Delta_2 &= \frac{1 - \tau_1 \tau_2 + \cos(2\phi)(\tau_2 - \tau_1)}{2(1 - \tau_1)(1 - \tau_2)}, \\ \ell &= \frac{\sin(2\phi)(\tau_2 - \tau_1)}{(1 - \tau_1)(1 - \tau_2)}, & \tau_j &= \frac{\xi_j}{|\xi_j|} \tanh |\xi_j|, \quad j \in \{1, 2\}. \end{aligned} \quad (6.3.24)$$

These depend only on the squeezing and mixing parameters  $\xi_j$  and  $\phi$ , respectively. On the other hand, the coefficients proportional to the linear terms in  $x_1, x_2$  are given by

$$\ell_1 = -2\Delta_1 \frac{q_1}{\lambda_1} - \ell \frac{q_2}{\lambda_2} - i \frac{\lambda_1}{\hbar} p_1, \quad \ell_2 = \ell \frac{q_1}{\lambda_1} + 2\Delta_2 \frac{q_2}{\lambda_2} + i \frac{\lambda_2}{\hbar} p_2, \quad (6.3.25)$$

which have an explicit dependence on the phase-space variables  $q_1, q_2, p_1$ , and  $p_2$ .

After some calculations involving elementary integrals with Gaussian functions, we explicitly determine the normalisation factor as

$$\mathcal{N}(\vec{q}, \vec{p}; \vec{\xi}, \phi) = \left( \frac{\Delta}{4\pi^2 \lambda_1^2 \lambda_2^2} \right)^{\frac{1}{4}} e^{\frac{4}{\Delta} (\text{Re}(\ell) \text{Re}(\ell_1) \text{Re}(\ell_2) - \text{Re}(\ell_2)^2 \text{Re}(\Delta_1) - \text{Re}(\ell_1)^2 \text{Re}(\Delta_2))}, \quad (6.3.26)$$

with

$$\begin{aligned} \Delta &= 16 \text{Re}(\Delta_1) \text{Re}(\Delta_2) - 4 \text{Re}(\ell)^2 = \\ &= \frac{(1 - |\tau_1|^2)(1 - |\tau_2|^2)(1 + |\tau_1|^2 - 2 \text{Re}(\tau_1))(1 + |\tau_2|^2 - 2 \text{Re}(\tau_2))}{|1 - \tau_1|^2 |1 - \tau_2|^2}. \end{aligned} \quad (6.3.27)$$

For brevity we omit the Fock expansion as it cannot be conveniently simplified. Instead, we can use the wavefunction representation to prove that the resolution of the identity is satisfied with a uniform measure  $\mu(\vec{q}, \vec{p}; \vec{\xi}, \phi) = 1$ . See appendix 6.C for details. We thus have

$$\begin{aligned} \langle \tilde{\Psi} | \mathbb{I} | \Psi \rangle &= \int_{\mathbb{R}^4} d\vec{x}' d\vec{x} [\tilde{\Psi}(\vec{x}')]^* \Psi(\vec{x}) \int_{\mathbb{R}^4} \frac{d^2 \vec{\alpha}}{(2\pi \hbar)^2} \psi(\vec{\alpha}; \vec{\xi}, \phi; \vec{x}') \psi(\vec{\alpha}; \vec{\xi}, \phi; \vec{x}) = \\ &= \int_{\mathbb{R}^4} d\vec{x}' d\vec{x} [\tilde{\Psi}(\vec{x}')]^* \Psi(\vec{x}) \delta(\vec{x} - \vec{x}') = \langle \tilde{\Psi}' | \Psi \rangle. \end{aligned} \quad (6.3.28)$$

In this form, the quantisation and semiclassical picture of any function  $f(\vec{\alpha}) \equiv f(\vec{q}, \vec{p})$  can be determined in the same way as the separable case because the measure in both instances

Classical $f(\vec{q}, \vec{p})$	Separable SS $\hat{A}_f$	Non-separable SS $\hat{A}_f$
1	$\mathbb{I}$	$\mathbb{I}$
$q_1$	$\hat{x}_1$	$\left(1 + 8 \frac{\text{Re}(\ell)^2}{\Delta}\right) \hat{x}_1 + 16 \frac{\lambda_1}{\lambda_2} \frac{\text{Re}(\ell) \text{Re}(\Delta_2)}{\Delta} \hat{x}_2$
$q_2$	$\hat{x}_2$	$\left(1 + 8 \frac{\text{Re}(\ell)^2}{\Delta}\right) \hat{x}_2 + 16 \frac{\lambda_2}{\lambda_1} \frac{\text{Re}(\ell) \text{Re}(\Delta_1)}{\Delta} \hat{x}_1$
$q_1 q_2$	$\hat{x}_1 \hat{x}_2$	$\left(1 + \frac{2^9 \text{Re}(\Delta_1) \text{Re}(\Delta_2) \text{Re}(\ell)^2}{\Delta^3}\right) \hat{x}_1 \hat{x}_2 + \frac{4^2 \text{Re}(\Delta_1) \text{Re}(\ell)}{\Delta} \left(1 + 8 \frac{\text{Re}(\ell)^2}{\Delta}\right) \frac{\lambda_2}{\lambda_1} \hat{x}_1^2 + \frac{4^2 \text{Re}(\Delta_2) \text{Re}(\ell)}{\Delta} \left(1 + 8 \frac{\text{Re}(\ell)^2}{\Delta}\right) \frac{\lambda_1}{\lambda_2} \hat{x}_2^2 + \frac{8\lambda_1 \lambda_2 \text{Re}(\ell)}{\Delta^2} \left(3 + 16 \frac{\text{Re}(\ell)^2}{\Delta}\right)$

**Table 6.1.** Two-mode quantisation associated with separable and non-separable squeezed states for different classical functions  $f(\vec{q}, \vec{p})$ .

is the same. That is, we have the quantisation map

$$f(\vec{q}, \vec{p}) \mapsto \hat{A}_f = \int_{\mathbb{R}^4} \frac{d\vec{q}d\vec{p}}{(2\pi\hbar)^2} f(\vec{q}, \vec{p}) |\vec{q}, \vec{p}; \vec{\xi}, \phi\rangle \langle \vec{q}, \vec{p}; \vec{\xi}, \phi|, \quad (6.3.29)$$

and its alternative form through the kernel representation

$$A_f^{(op)} \Psi(\vec{x}) = \int_{\mathbb{R}^2} d\vec{x}' \mathcal{K}_f(\vec{\xi}, \phi; \vec{x}, \vec{x}') \Psi(\vec{x}'), \quad (6.3.30)$$

where

$$\mathcal{K}_f(\xi; \vec{x}, \vec{x}') = \int_{\mathbb{R}^2} \frac{d\vec{q}d\vec{p}}{2\pi\hbar} f(\vec{q}, \vec{p}) \psi^*(\vec{q}, \vec{p}; \vec{\xi}, \phi; \vec{x}') \psi(\vec{q}, \vec{p}; \vec{\xi}, \phi; \vec{x}). \quad (6.3.31)$$

In order to expose the differences between quantisations using the separable and non-separable squeezed states we consider a few examples. In Table. 6.1 we consider linear functions on the classical position  $q_1$  and  $q_2$ , where we observe that the separable case produces a factorisable quantisation, that is, for a classical function  $f(\vec{q}, \vec{p}) = q_1 q_2$  the resulting operator is the product of the independent quadratures  $\hat{x}_1$  and  $\hat{x}_2$ . However, the non-separable case shows that the resulting operator is not factorisable and becomes quadratic combinations of both quadratures  $\hat{x}_1$  and  $\hat{x}_2$ . Similarly, the function  $f(\vec{q}, \vec{p}) = q_j$ , for  $j = 1, 2$ , the resulting operator leads to a linear combination of both quadratures as well. In the limiting case  $\ell = 0$  (see cases above), the resulting quantisation reduces to that of separable squeezed states.

On the other hand, the semiclassical portrait is given by

$$f(\vec{q}, \vec{p}) \mapsto \check{A}_{f(\vec{q}, \vec{p})} = \langle \hat{A}_f \rangle = \int_{\mathbb{R}^4} \frac{d^2\vec{q}d^2\vec{p}}{(2\pi\hbar)^2} f(\vec{q}', \vec{p}') |\langle \vec{q}', \vec{p}'; \vec{\xi}, \phi | \vec{q}, \vec{p}; \vec{\xi}, \phi \rangle|^2. \quad (6.3.32)$$

where the overlap between two non-separable squeezed states is explicitly given by

$$|\langle \vec{q}', \vec{p}'; \vec{\xi}, \phi | \vec{q}, \vec{p}; \vec{\xi}, \phi \rangle|^2 = \exp\left(\frac{\vec{R}^T \cdot \tilde{\mathbb{M}} \cdot \vec{R}}{\Delta}\right), \quad (6.3.33)$$

$$\vec{R} = \left(\frac{q_1 - q'_1}{\lambda_1}, \frac{\lambda_1(p_1 - p'_1)}{\hbar}, \frac{q_2 - q'_2}{\lambda_2}, \frac{\lambda_2(p_2 - p'_2)}{\hbar}\right)^T,$$



with the matrix

$$\widetilde{\mathbb{M}} = \begin{pmatrix} \theta_1 & \frac{L_{11}}{2} & \frac{\theta_{12}}{2} & \frac{L_{12}}{2} \\ \frac{L_{11}}{2} & \Xi_1 & \frac{L_{21}}{2} & \frac{\Xi_{12}}{2} \\ \frac{\theta_{12}}{2} & \frac{L_{21}}{2} & \theta_2 & \frac{L_{22}}{2} \\ \frac{L_{12}}{2} & \frac{\Xi_{12}}{2} & \frac{L_{22}}{2} & \Xi_2 \end{pmatrix} \quad (6.3.34)$$

together with the coefficients

$$\begin{aligned} \theta_1 &= 4 \operatorname{Re}(\Delta_1) |\ell|^2 + 16 \operatorname{Re}(\Delta_2) |\Delta_1|^2 + 8 \operatorname{Re}(\ell) \operatorname{Re}(\Delta_1 \ell^*), \\ \theta_2 &= 16 \operatorname{Re}(\Delta_1) |\Delta_2|^2 + 4 \operatorname{Re}(\Delta_2) |\ell|^2 + 8 \operatorname{Re}(\ell) \operatorname{Re}(\Delta_2 \ell^*), \\ \theta_{12} &= 16 \operatorname{Re}(\Delta_1) \operatorname{Re}(\ell \Delta_2) + 4 \operatorname{Re}(\Delta_2) |\ell|^2 - 8 \operatorname{Re}(\ell) \operatorname{Re}(\Delta_2 \ell^*), \\ \Xi_1 &= 4 \operatorname{Re}(\Delta_2), \quad \Xi_2 = 4 \operatorname{Re}(\Delta_1), \quad \Xi_{12} = 4 \operatorname{Re}(\ell), \\ L_{11} &= -16 \operatorname{Im}(\Delta_1) \operatorname{Re}(\Delta_2) - 4 \operatorname{Im}(\ell) \operatorname{Re}(\ell), \\ L_{12} &= -8 \operatorname{Im}(\ell) \operatorname{Re}(\Delta_1) - 8 \operatorname{Im}(\Delta_1) \operatorname{Re}(\ell), \\ L_{21} &= 8 \operatorname{Im}(\ell) \operatorname{Re}(\Delta_2) + 8 \operatorname{Im}(\Delta_2) \operatorname{Re}(\ell), \\ L_{22} &= -16 \operatorname{Im}(\Delta_2) \operatorname{Re}(\Delta_1) - 4 \operatorname{Im}(\ell) \operatorname{Re}(\ell). \end{aligned} \quad (6.3.35)$$

Notice that for the non-separable states (6.3.33), besides mixing the canonical positions  $q_1$  and  $q_2$  among themselves, they also mix the canonical position  $q_1$  with both of the canonical momenta  $p_1$  and  $p_2$ . The same is true vice-versa for  $q_2$ .

From the coefficients in (6.3.35) we can identify two interesting limiting cases:

- If the squeezing parameters are both equal,  $\tau_1 = \tau_2$ , we get  $\ell = 0$ . We therefore have  $\theta_{12} = \Xi_{12} = L_{12} = L_{21} = 0$ , and thus the overlap (6.3.33) just couples  $q_1$  with  $p_1$ , and  $q_2$  with  $p_2$ . That is, the semiclassical canonical position observables  $q_1$  and  $q_2$  couple only with their respective canonical momenta.

- If  $\tau_1, \tau_2 \in \mathbb{R}$ , we get  $\operatorname{Im}(\Delta_1) = \operatorname{Im}(\Delta_2) = \operatorname{Im}(\ell) = 0$ . Therefore, the canonical position  $q_1$  couples with  $q_2$ , and the canonical momentum  $p_1$  couples with  $p_2$ .

- For  $\phi = 0$  and  $\tau_1, \tau_2 \in \mathbb{R}$ , the matrix  $\widetilde{\mathbb{M}}$  becomes diagonal and no coupling among the semiclassical observables is generated. This corresponds to the separable squeezed state limiting case.

To illustrate these results, let us consider the classical function  $f(\vec{q}, \vec{p}) \equiv h(\vec{q})$  which leads to

$$\check{A}_{h(\vec{q})} = \frac{\pi \hbar^2}{2\sqrt{\Delta}} \int_{\mathbb{R}} \frac{d\vec{q}'}{(2\pi \hbar)^2} h(\vec{q}') e^{-\frac{\mathfrak{c}_1}{\lambda_1^2 \Delta} (q_1 - q_1')^2 - \frac{\mathfrak{c}_2}{\lambda_2^2 \Delta} (q_2 - q_2')^2 + \frac{\mathfrak{c}_{12}}{\lambda_1 \lambda_2} (q_1 - q_1')(q_2 - q_2')}, \quad (6.3.36)$$

where

$$\begin{aligned} \mathfrak{c}_1 &= \theta_1 - \frac{\Xi_1 L_{12}^2 + \Xi_2 L_{11}^2 + \Xi_{12} L_{11} L_{12}}{4\Delta}, \quad \mathfrak{c}_2 = \theta_2 - \frac{\Xi_1 L_{22}^2 + \Xi_2 L_{21}^2 + \Xi_{12} L_{21} L_{22}}{4\Delta} \\ \mathfrak{c}_{12} &= \theta_{12} + \frac{2\Xi_1 L_{12} L_{22} + 2\Xi_2 L_{11} L_{21} + \Xi_{12} (L_{11} L_{22} + L_{12} L_{21})}{4\Delta}. \end{aligned} \quad (6.3.37)$$

Notice that the Gaussian function in (6.3.36), besides regularising the classical function  $h(q)$ , it couples the canonical position  $q_1$  with  $q_2$ . This will lead to an anisotropic semiclassical portrait  $\check{A}_{h(\vec{q})}$  even if the original classical function is isotropic.

## 6.4. Position-dependent mass models

In this section, we apply the discussion from the previous sections to a specific problem. In particular, we focus on a position-dependent mass (PDM) model defined in a classical constrained geometry. Before proceeding with our specific model, we require some generalities in both the classical and semiclassical cases. To begin with, let us consider a two-dimensional classical Hamiltonian of the form

$$H = H_1 + H_2, \quad H_j = \frac{p_j^2}{2m_j(q_j)} + V_j(q_j), \quad j = 1, 2, \quad (6.4.1)$$

which is separable as the sum of two one-dimensional Hamiltonians. From the Hamilton equations of motion [32], we obtain the canonical momentum  $p_j = m_j(q_j)\dot{q}_j$ , from which, the corresponding equation of motion for the position coordinate becomes

$$\frac{d^2 q_j}{dt^2} + \frac{1}{2m_j(q_j)} \left( \frac{\partial m_j(q_j)}{\partial q_j} \right) \left( \frac{dq_j(t)}{dt} \right)^2 + \frac{1}{m_j(q_j)} \left( \frac{\partial V_j(q_j)}{\partial q_j} \right) = 0, \quad j = 1, 2. \quad (6.4.2)$$

Note that, in the constant mass case,  $m'_j = 0$ , the equations of motion (6.4.2) reduce to the Newton equation of motion,  $m_j \ddot{q}_j = -\frac{\partial V_j(q_j)}{\partial q_j}$ .

From the setup described in section 6.3.1, the corresponding semiclassical portrait can be determined. In particular, the semiclassical Hamiltonian becomes

$$\check{H}(\vec{q}, \vec{p}) = \frac{1}{2} \check{A}_{p_1^2 \mathfrak{m}_1(q_1)} + \frac{1}{2} \check{A}_{p_2^2 \mathfrak{m}_2(q_2)} + \check{A}_{V_1(q_1)} + \check{A}_{V_2(q_2)}, \quad \mathfrak{m}_j(q_j) = (m_j(q_j))^{-1}, \quad (6.4.3)$$

where a general formula for  $\check{A}_{p_j^2 h(\vec{q})}$  is given in (6.3.18). Interestingly, the semiclassical portrait admits a symplectic structure similar to that of the classical model. That is, from the semiclassical Hamiltonian (6.4.3), we can determine the evolution of  $q_j(t)$  and  $p_j(t)$  through the Hamilton equations of motion (see [34] for a detailed proof)

$$\dot{q}_j(\vec{q}, \vec{p}) = \frac{\partial \check{H}(\vec{q}, \vec{p})}{\partial p_j}, \quad -\dot{p}_j(\vec{q}, \vec{p}) = \frac{\partial \check{H}(\vec{q}, \vec{p})}{\partial q_j}, \quad j = 1, 2, \quad (6.4.4)$$

where  $\dot{q}_j \equiv \frac{dq_j}{dt}$  and  $\dot{p}_j \equiv \frac{dp_j}{dt}$ . In the latter, the time derivatives are functions of  $q_j$  and  $p_j$ . Nevertheless, by analogy to the classical case above discussed, we may cast the equations of motion so that  $q_j(t)$  is a function of time. To this end, we use (6.3.18) to obtain

$$\dot{q}_j = p_j \check{A}_{\mathfrak{m}_j} + \frac{4\hbar\gamma_j}{\Delta_{p_j}^2 \lambda_j^2} \left( q_j \check{A}_{\mathfrak{m}_j} - \check{A}_{q_j \mathfrak{m}_j} \right), \quad (6.4.5)$$

from which one may determine a relation between the semiclassical momentum  $p_j$  and the velocity  $\dot{q}_j$ . To determine the equation of motion for  $q_j$  we use the time evolution relation for any semiclassical observable  $\frac{df}{dt} = \{f, \check{H}\}_{PB} + \frac{\partial f}{\partial t}$ , with  $\{f, g\}_{PB}$  the Poisson brackets. Using the latter with  $\dot{q}$ , and after some calculations, we get a nonlinear coupled second-order differential equation for  $q_1$  and  $q_2$ . An explicit form will be shown in the following section.

We have the general equations to determine the dynamics at both the classical and semiclassical levels. Their solutions are specified by the mass, potential energy, and initial conditions. One may foresee that the resulting equations of motion are in general nonlinear, and we thus have to rely on numerical calculations in most cases.

### 6.4.1. Variable mass oscillator in constrained geometry

In order to implement the results obtained so far, let us consider the PDM Hamiltonian introduced in [34], which is in turn contained in the family of non-linear oscillators in [37]. We thus introduce the corresponding two-dimensional classical Hamiltonian

$$H = H_1 + H_2, \quad H_j = \frac{p_j^2}{2m(q_j)} + \bar{V}_j q_j^2, \quad m(q_j) = \frac{m_0}{1 - \Lambda_j^2 q_j^2}, \quad \bar{V}_j, \Lambda_j \in \mathbb{R}, \quad j = 1, 2, \quad (6.4.6)$$

where  $m_0 > 0$  is the mass and an external oscillator interaction has been added, which can be turned off by fixing  $\bar{V}_j = 0$ . Notice that the model is only well-defined inside the interval  $q_j \in (-\Lambda_j^{-1}, \Lambda_j^{-1})$ , as outside of such interval the mass takes negative values. Considering the latter, we constrain the model to be defined only in the physically allowed regions. This is done by implementing a characteristic function  $\chi_E(\vec{q})$  of the form

$$\chi_E(\vec{q}) = \begin{cases} 1 & q_1 \in \left(-\frac{1}{\Lambda_1}, \frac{1}{\Lambda_1}\right), q_2 \in \left(-\frac{1}{\Lambda_2}, \frac{1}{\Lambda_2}\right), \\ 0 & \text{otherwise} \end{cases}, \quad (6.4.7)$$

so that the redefined nonlinear oscillator Hamiltonian becomes

$$H_\chi(\vec{q}, \vec{p}) = \chi_E \left( \frac{p_1^2}{2m_{\Lambda_1}(q_1)} + \frac{p_2^2}{2m_{\Lambda_2}(q_2)} + \bar{V}_1 q_1^2 + \bar{V}_2 q_2^2 \right). \quad (6.4.8)$$

That is, we have introduced the characteristic function so that the dynamics are constrained to the rectangle defined by  $\chi_E$ .

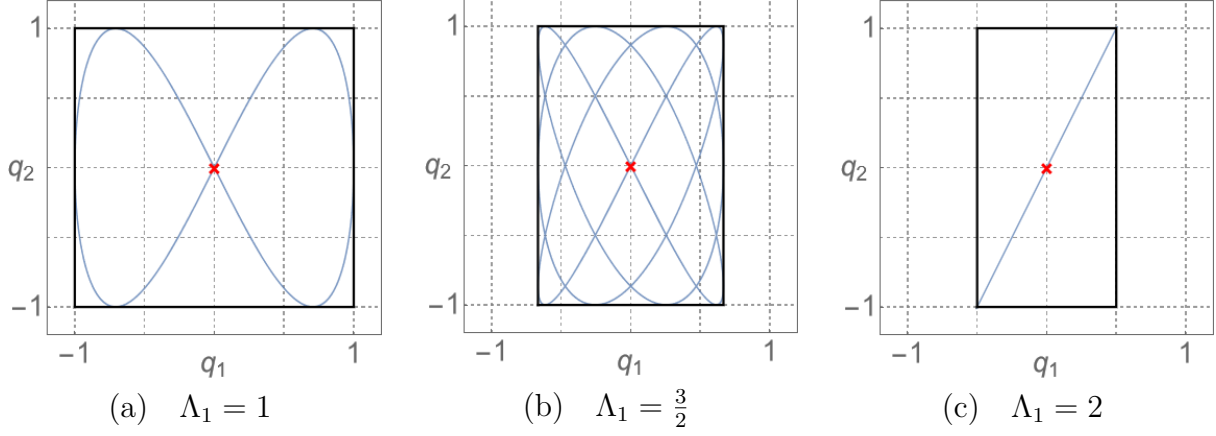
In particular, for a null oscillator interaction,  $V_1 = V_2 = 0$ , the equations of motion (6.4.2) for the Hamiltonian (6.4.6) can be determined in a closed form, leading to the solutions

$$q_j(t) = \frac{1}{\Lambda_j} \sin \left( \frac{\Lambda_j v_{0;j} t}{\sqrt{1 - \Lambda_j^2 q_{0;j}^2}} + \arcsin(\Lambda_j q_{0;j}) \right), \quad (6.4.9)$$

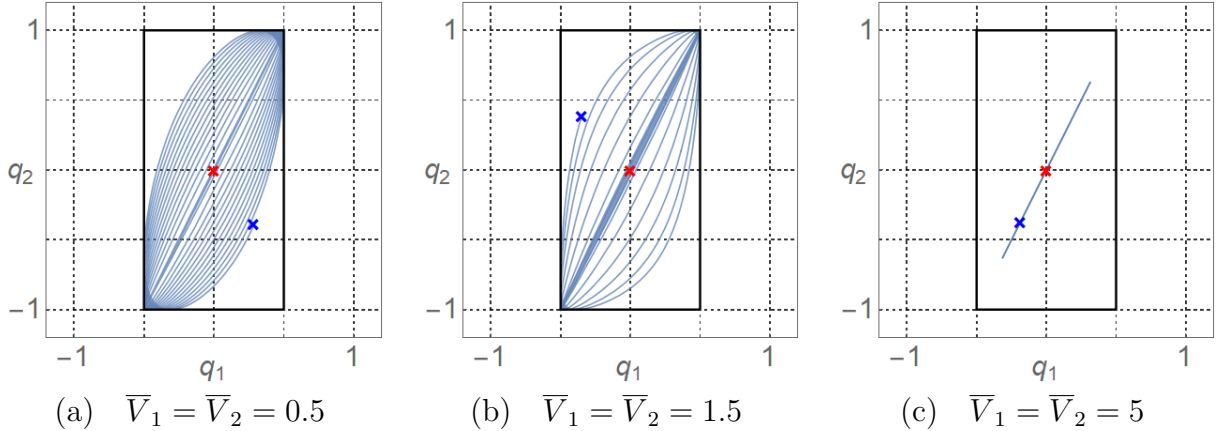
with  $q_{0;j} \equiv q_j(t=0)$  and  $v_{0;j} \equiv \dot{q}_j(t=0)$  the initial position and velocity, respectively.

Interestingly, despite the lack of a trapping interaction, the solutions for  $q_j(t)$  describe bounded and oscillatory trajectories, which is reminiscent of the dynamics of the harmonic

oscillator. The corresponding dynamics in the  $q_1 - q_2$  plane is depicted in figure 6.1 for several geometries, governed by the parameters  $\Lambda_j$ , and fixed initial conditions  $q_{0;j}$  and  $v_{0;j}$ . In all the cases, the initial positions have been placed at the origin,  $q_{0;1} = q_{0;2} = 0$ , whereas the ratio between the initial velocities,  $\frac{v_{0;1}}{v_{0;2}}$ , and the rectangle lengths,  $\Lambda_1/\Lambda_2$ , have been chosen so that they are both rational numbers. From the exact solution (6.4.9), it can be seen that the oscillation frequency in each direction reduces to  $\Lambda_j v_{0;j}$ , and so the ratio of the frequencies is a rational number. This explains the closed trajectories observed figures 6.1a-6.1c.



**Fig. 6.1.** Trajectories on the  $(q_1, q_2)$  plane for a classical particle described by the Hamiltonian (6.4.8). In every case, we have fixed  $m_0 = 1$  and  $\bar{V}_1 = \bar{V}_2 = 0$  (null external oscillator interaction), together with the initial conditions  $q_{0;1} = q_{0;2} = 0$ ,  $v_{0;1} = 1$ , and  $v_{0;2} = 2$ . The particle is confined to the rectangle characterized by  $\Lambda_2 = 1$  and the indicated values of  $\Lambda_1$ . The red cross indicates the particle's starting position of  $(0,0)$ .



**Fig. 6.2.** Trajectories on the  $(q_1, q_2)$  plane for a classical particle described by the Hamiltonian (6.4.8). In every case, we have fixed  $m_0 = 1$ ,  $\Lambda_1 = 2$ ,  $\Lambda_2 = 1$ , with the initial conditions  $q_{0;1} = q_{0;2} = 0$ ,  $v_{0;1} = 1$ ,  $v_{0;2} = 2$ , which corresponds to the setup in figure 6.1c. In addition, we have considered the external oscillator strength  $\bar{V}_1 = \bar{V}_2 = 0.5$  (a),  $\bar{V}_1 = \bar{V}_2 = 1.5$  (b), and  $\bar{V}_1 = \bar{V}_2 = 5$  (c). The red and blue cross represent the initial and final position, respectively, for a time interval  $t \in (0, 65)$ .

On the other hand, the presence of the external oscillator interaction prevents us from obtaining a closed expression, and the dynamics cannot not be foreseen a priori. Still, it is expected that trajectories in this case would no longer be closed, as the additional nonlinearities in the equation of motion would break the strict balance required to obtain closed trajectories. Such behaviour is depicted in figure 6.2, where solutions for  $q_j(t)$  have been determined by numerical means. In these figures, we can see that the trajectories keep spreading from the null-interaction case (figure 6.1c). For a large enough oscillator strength (figure 6.2c), the trajectory seems to match the one related to null-interaction case; however, the length of the trajectory shortens. This behaviour is due the additional confinement produced by the external interaction, which constrains the particle in a smaller region inside the rectangle.

### 6.4.2. Semiclassical dynamics

Now, let us determine the semiclassical dynamics related to the classical truncated Hamiltonian (6.4.8). To determine the semiclassical counterpart, we consider the example where  $\gamma_1 = \gamma_2 = 0$  in (6.3.18), with mass functions in (6.4.3) gaining some dependence on the other coordinate by,

$$\mathfrak{M}_j(q_j) \rightarrow \mathfrak{M}_j(q_j)\chi_{E_i}(q_i) = \frac{1}{m_0}(1 - \Lambda_j^2 q_j^2)\chi_{E_j}(q_j)\chi_{E_i}(q_i), \quad i, j = 1, 2, \quad i \neq j. \quad (6.4.10)$$

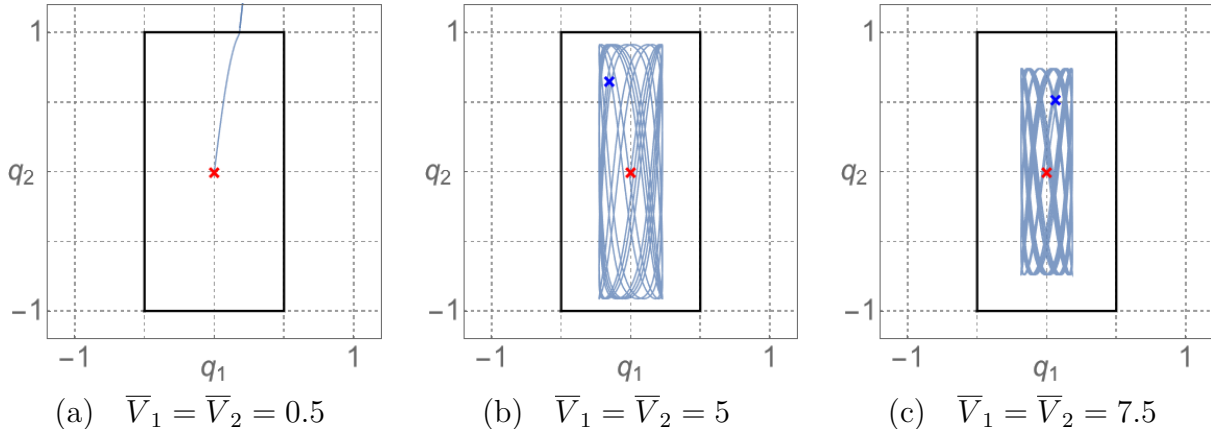
From these we obtain the semiclassical Hamiltonian

$$\begin{aligned} \check{H} = & \left( \frac{p_1^2}{2} + \frac{\hbar^2}{2\Delta_{p_1}^2 \lambda_1^2} \right) \check{A}_{\mathfrak{M}_1(q_1)} \check{A}_{\chi_{E_2}(q_2)} + \left( \frac{p_2^2}{2} + \frac{\hbar^2}{2\Delta_{p_2}^2 \lambda_2^2} \right) \check{A}_{\mathfrak{M}_2(q_2)} \check{A}_{\chi_{E_1}(q_1)} + \\ & \bar{V}_1 \check{A}_{q_1^2 \chi_{E_1}(q_1)} \check{A}_{\chi_{E_2}(q_2)} + \bar{V}_2 \check{A}_{q_2^2 \chi_{E_2}(q_2)} \check{A}_{\chi_{E_1}(q_1)}, \quad (6.4.11) \end{aligned}$$

where we have used the fact that the two-dimensional window function  $\chi_E$  factorises as  $\chi_E = \chi_{E_1}\chi_{E_2}$  in the rectangular geometry. The explicit form of the semiclassical portrait of the window functions  $\chi_{E_j}$ , as well as some other results necessary to this computation, are presented in appendix 6.D. As we preserve the Hamiltonian structure, one may also derive the equations of motion for the semiclassical system. For  $q_1(t)$  we find,

$$\begin{aligned} \ddot{q}_1 = & \frac{\dot{q}_1^2}{2\check{A}_{\mathfrak{M}_1(q_1)}} \left( \frac{\partial \check{A}_{\mathfrak{M}_1(q_1)}}{\partial q_j} \right) - \frac{\dot{q}_2^2}{2} \frac{\check{A}_{\mathfrak{M}_1(q_1)} \check{A}_{\chi_{E_2}(q_2)}}{\check{A}_{\mathfrak{M}_2(q_2)} \check{A}_{\chi_{E_1}(q_1)}} \left( \frac{\partial \check{A}_{\chi_{E_1}(q_1)}}{\partial q_1} \right) + \frac{\dot{q}_1 \dot{q}_2}{\check{A}_{\chi_{E_2}(q_2)}} \left( \frac{\partial \check{A}_{\chi_{E_2}(q_2)}}{\partial q_2} \right) - \\ & \frac{\hbar^2}{\Delta_{p_1}^2 \lambda_1^2} \check{A}_{\mathfrak{M}_1(q_1)} \check{A}_{\chi_{E_2}(q_2)}^2 \left( \frac{\partial \check{A}_{\mathfrak{M}_1(q_1)}}{\partial q_1} \right) - \frac{\hbar^2}{\Delta_{p_2}^2 \lambda_2^2} \check{A}_{\mathfrak{M}_1(q_1)} \check{A}_{\mathfrak{M}_2(q_2)} \check{A}_{\chi_{E_2}(q_2)} \left( \frac{\partial \check{A}_{\chi_{E_1}(q_1)}}{\partial q_1} \right) - \\ & \bar{V}_1 \check{A}_{\mathfrak{M}_1(q_1)} \check{A}_{\chi_{E_2}(q_2)}^2 \left( \frac{\partial \check{A}_{q_1^2 \chi_{E_1}(q_1)}}{\partial q_1} \right) - \bar{V}_2 \check{A}_{\mathfrak{M}_1(q_1)} \check{A}_{q_2^2 \chi_{E_2}(q_2)} \check{A}_{\chi_{E_2}(q_2)} \left( \frac{\partial \check{A}_{\chi_{E_1}(q_1)}}{\partial q_1} \right). \quad (6.4.12) \end{aligned}$$

A similar differential equation holds for  $q_2(t)$  by interchanging the subscript indices 1 and 2. This is a system of two coupled non-linear second order differential equations and it is not possible to solve it by analytical means. For the generation of the dynamics we resort to numerical methods.



**Fig. 6.3.** Trajectories on the  $(q_1, q_2)$  plane obtained from the semiclassical Hamiltonian (6.4.3). In every case we use units of  $\hbar = 0.1$ , and we have fixed  $m_0 = 1$ ,  $\Lambda_1 = 2$ ,  $\Lambda_2 = 1$ , with the initial conditions  $q_{0;1} = q_{0;2} = 0$ ,  $v_{0;1} = 1$ ,  $v_{0;2} = 2$ , which corresponds to the setup in figure 6.2c. The regularisation parameters are  $\lambda_1 = \lambda_2 = 0.2$  and  $\tau_1 = \tau_2 = 0.2$ . In addition, we have considered the external oscillator strength  $\bar{V}_1 = \bar{V}_2 = 0.5$  (a),  $\bar{V}_1 = \bar{V}_2 = 5$  (b), and  $\bar{V}_1 = \bar{V}_2 = 7.5$  (c). The red and blue cross represent the initial and final position, respectively, for a time interval  $t \in (0, 30)$ .

A curious result of the regularisation is an induced force due to the wall. The regularisation of the wall has rendered it finite and the particle can escape. This is a clear departure from the classical case where the particle is always confined within the restricted region. In figure 6.3a we see that for a particle fired towards the top right corner of the confined region, its initial energy is enough to overcome the confining potential strength and thus it escapes. Conversely, in figures 6.3b and 6.3c, the external potential strength is great enough to capture the particle. In the bounded case, the particle never quite reaches the boundary due to the smoothing of the wall. In all cases we have to keep in mind that the terms in (6.4.3) which are proportional to  $\hbar$  and its powers are the purely quantum corrections, and as such we must consider the relative strength between the classical parts of the Hamiltonian and the quantum parts. The classical effects in principle should be the dominant contributions and the quantum effects should contribute less. For this reason we choose the scale  $\hbar = 0.1$ .

## 6.5. Conclusion

In this work we have looked at quantisation in two dimensions using general families of two-dimensional squeezed states. We found the dependence of the quantised operators and their semiclassical portraits on the squeezing parameters introduced, and found that

in the case where the squeezed states are coupled (entangled), an anisotropy is induced in the resulting operators and portraits. This offers additional control over the strength and direction of the regularisation of discontinuous functions when compared with standard coherent state quantisation. In principle the additional control available may allow one to quantise a theory with more precision than would be available under different quantisation schemes.

In section 6.2 we reviewed a quantisation scheme in one dimension using conventional squeezed states. In section 6.3 we defined two distinct families of two-dimensional squeezed states: coordinate separable squeezed states taken as the product of two one-dimensional squeezed states, and the non-separable squeezed states which are not separable as a product of two one-dimensional squeezed states. It was found that the quantisation of the classical position functions  $q_1$  and  $q_2$  led to the expected quantum operators  $\hat{x}_1$  and  $\hat{x}_2$  respectively for the separable squeezed states. On the other hand, for the non-separable squeezed states, the quantisations of the individual classical position functions became linear combinations of the quantum position operators for both modes. In section 6.4 we applied the preceding formalism to a position-dependent mass model in two dimensions using the separable squeezed states, and studied a comparison between the classical and semiclassical portraits using squeezed state quantisation.

For future work, it would be interesting to look at quantisation in non-rectangular confined regions. It would seem that there are a number of physical systems one could approach when one can conveniently regularise confined regions in quantisation problems, such as the hadron bag model [38], and quantum motion on non-rectangular surfaces. The anisotropic effect of the squeezing parameters would allow one to distort the restricted region to obtain semiclassical billiard-like dynamics in non-trivial geometries. Additionally, exploring quantisation in higher-dimensional systems should lead to interesting results, the classes of generalised coherent and squeezed states will proliferate and as such so will the ways in which one can quantise a problem.

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## 6.A Resolution of the identity through holomorphic Hermite polynomials

In this appendix we prove the resolution of the identity associated to the one-mode squeezed states written in their Fock expansion given in (6.2.6). In such a case, the coefficients are written in terms of complex Hermite polynomials. Thus, to determine their completeness, we consider the holomorphic Hermite polynomials in two variables  $H_n(x + iy)$  [27, 28], which satisfy the orthogonality relationship

$$\int_{\mathbb{R}^2} dx dy H_n(x + iy) H_m(x - iy) e^{-ax^2 - by^2} = \frac{\pi}{\sqrt{ab}} 2^n n! \left( \frac{a+b}{ab} \right)^n \delta_{n,m}, \quad (6.A.1)$$

where the constants  $a$  and  $b$  are constrained by

$$0 < a < b, \quad \frac{1}{a} - \frac{1}{b} = 1. \quad (6.A.2)$$

To simplify the notation, we use  $u_1 = \text{Re}(\alpha)$  and  $u_2 = \text{Im}(\alpha)$  throughout this section. Substituting (6.2.6) into (6.2.7) leads to

$$\int_{\mathbb{R}^2} \frac{du_1 du_2}{\pi} \mu(u_1, u_2) |\alpha(u_1, u_2); \xi\rangle \langle \alpha(u_1, u_2); \xi| = (1 - |\tau|^2)^{\frac{1}{2}} \sum_{n,m=0}^{\infty} \frac{\tau^{\frac{n}{2}} (\tau^*)^{\frac{m}{2}}}{(2^{n+m} n! m!)^{\frac{1}{2}}} \mathcal{F}_{n,m} |n\rangle \langle m|, \quad (6.A.3)$$

where

$$\mathcal{F}_{n,m} = \int_{\mathbb{R}^2} \frac{du_1 du_2}{\pi} \mu(u_1, u_2) e^{-(1-\text{Re}(\tau))u_1^2 - (1+\text{Re}(\tau))u_2^2 + 2\text{Im}(\tau)u_1 u_2} H_n(z_1 + iz_2) H_m(z_1 - iz_2), \quad (6.A.4)$$

and the functions  $z_1 \equiv z_1(u_1, u_2)$  and  $z_2 \equiv z_2(u_1, u_2)$  are defined through the following linear transformation:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \mathbb{M} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \mathbb{M} = \frac{\sqrt{1 - |\tau|^2}}{2|\tau|} \begin{pmatrix} (|\tau| + \text{Re}(\tau))^{\frac{1}{2}} & (|\tau| - \text{Re}(\tau))^{\frac{1}{2}} \\ -(|\tau| - \text{Re}(\tau))^{\frac{1}{2}} & (|\tau| + \text{Re}(\tau))^{\frac{1}{2}} \end{pmatrix}. \quad (6.A.5)$$

In order to use the orthogonality (6.A.1), we have make a change of variable into  $z_1$  and  $z_2$ . In this case the differential element in the new variables is given by  $du_1 du_2 \rightarrow \det(\mathbb{M}^{-1}) dz_1 dz_2$ . Making these substitutions, Eq. (6.A.4) becomes

$$\mathcal{F}_{n,m} = \frac{2|\tau|}{1 - |\tau|^2} \int_{\mathbb{R}^2} \frac{dz_1 dz_2}{\pi} \mu(z_1, z_2) e^{-\left(\frac{2|\tau|}{1+|\tau|}\right)z_1^2 - \left(\frac{2|\tau|}{1-|\tau|}\right)z_2^2} H_n(z_1 + iz_2) H_m(z_1 - iz_2), \quad (6.A.6)$$

from which we realise that, in order to use the orthogonality of the holomorphic Hermite polynomials, the measure must be uniform and take the form  $\mu(z_1, z_2) = \mu_0$ . Moreover, from (6.A.6) we identify

$$a \equiv \frac{2|\tau|}{1 + |\tau|}, \quad b \equiv \frac{2|\tau|}{1 - |\tau|}, \quad (6.A.7)$$

making it clear that the constraints in (6.A.2) are fulfilled for all  $|\tau| < 1$ , or equivalently  $\xi \in \mathcal{C}$ . In this form, Eq. (6.A.1) leads to

$$\mathcal{F}_{n,m} = \frac{\mu_0}{(1 - |\tau|^2)^{\frac{1}{2}}} \left(\frac{2}{|\tau|}\right)^n n! \delta_{n,m}. \quad (6.A.8)$$

Finally, by substituting (6.A.8) into (6.A.3), with  $\mu_0 = 1$ , we get

$$\int_{\mathbb{R}^2} \frac{du_1 du_2}{\pi} |\alpha(u_1, u_2); \xi\rangle \langle \alpha(u_1, u_2); \xi| = \sum_{n=0}^{\infty} |n\rangle \langle n| = \mathbb{I}, \quad (6.A.9)$$

recovering the resolution of the identity for the one-mode squeeze states with respect to the uniform measure  $\mu(\alpha) \equiv \mu(u_1, u_2) = 1$ .

## 6.B Determining $\psi(\vec{q}, \vec{p}; \vec{\xi}, \phi; \vec{x})$

In this appendix, we detail the steps followed to get the wavefunction representation associated to the two-mode states (6.3.20). The wavefunction can be determined without explicitly expanding  $|\vec{\alpha}; \vec{\xi}, \phi\rangle$  in the two-mode Fock basis. We exploit the unitary transformations of the boson operators  $a_1$  and  $a_2$  generated by  $G$  to determine an eigenvalue equation, which in turns lead to a partial differential equation for the wavefunction.

To begin with, we use  $G$  in (6.3.21) together with the Baker–Campbell–Hausdorff formula  $e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \dots$  to get the unitary transformations

$$G^\dagger a_1 G = \cos \phi \left( a_1 \cosh |\xi_1| - a_1^\dagger \frac{\xi_1}{|\xi_1|} \sinh |\xi_1| \right) + \sin \phi \left( a_2 \cosh |\xi_2| - a_2^\dagger \frac{\xi_2}{|\xi_2|} \sinh |\xi_2| \right) + \alpha_1, \quad (6.B.1)$$

$$G^\dagger a_2 G = \cos \phi \left( a_2 \cosh |\xi_2| - a_2^\dagger \frac{\xi_2}{|\xi_2|} \sinh |\xi_2| \right) - \sin \phi \left( a_1 \cosh |\xi_1| - a_1^\dagger \frac{\xi_1}{|\xi_1|} \sinh |\xi_1| \right) + \alpha_2, \quad (6.B.2)$$

where the transformations for the the creation operators follow straightforwardly from the latter by applying the adjoint operation and exploiting the unitarity of  $G$ . On the other hand, we may compute the following alternative unitary transformations:

$$G a_1^\dagger G^\dagger = \cos \phi \left( (a_1^\dagger - \alpha_1^*) \cosh |\xi_1| + (a_1 - \alpha_1) \frac{\xi_1^*}{|\xi_1|} \sinh |\xi_1| \right) - \sin \phi \left( (a_2^\dagger - \alpha_2^*) \cosh |\xi_1| + (a_2 - \alpha_2) \frac{\xi_1^*}{|\xi_1|} \sinh |\xi_1| \right), \quad (6.B.3)$$

$$G a_2^\dagger G^\dagger = \cos \phi \left( (a_2^\dagger - \alpha_2^*) \cosh |\xi_2| + (a_2 - \alpha_2) \frac{\xi_2^*}{|\xi_2|} \sinh |\xi_2| \right) + \sin \phi \left( (a_1^\dagger - \alpha_1^*) \cosh |\xi_2| + (a_1 - \alpha_1) \frac{\xi_2^*}{|\xi_2|} \sinh |\xi_2| \right). \quad (6.B.4)$$

Now, by recalling that  $|\vec{\alpha}; \vec{\xi}, \phi\rangle = G|0,0\rangle$ , we apply (6.B.1) on  $|0,0\rangle$ , then multiply on the left by  $G$  in order to get

$$a_1 |\vec{\alpha}; \vec{\xi}, \phi\rangle = \left( \alpha_1 - \frac{\xi_1}{|\xi_1|} \sinh |\xi_1| \cos \phi \left( G a_1^\dagger G^\dagger \right) - \frac{\xi_2}{|\xi_2|} \sinh |\xi_2| \sin \phi \left( G a_2^\dagger G^\dagger \right) \right) |\vec{\alpha}; \vec{\xi}, \phi\rangle, \quad (6.B.5)$$

which, with the aid of (6.B.3)-(6.B.4), leads us to the eigenvalue equation

$$\left( \mathcal{A}_1 a_1 + \mathcal{A}_2 a_1^\dagger + \mathcal{A}_3 a_2 + \mathcal{A}_4 a_2^\dagger \right) |\vec{\alpha}; \vec{\xi}, \phi\rangle = z_1 |\vec{\alpha}; \vec{\xi}, \phi\rangle, \quad (6.B.6)$$

where the coefficients are given by

$$\begin{aligned}
\mathcal{A}_1 &= 1 + \sinh^2 |\xi_1| \cos^2 \phi + \sinh^2 \xi_2 \sin^2 \phi, \\
\mathcal{A}_2 &= \frac{\xi_2}{|\xi_2|} \sinh |\xi_2| \cosh |\xi_2| \sin^2 \phi + \frac{\xi_1}{|\xi_1|} \sinh |\xi_1| \cosh |\xi_1| \cos^2 \phi, \\
\mathcal{A}_3 &= \left( -\sinh^2 |\xi_1| + \sinh^2 |\xi_2| \right) \sin \phi \cos \phi, \\
\mathcal{A}_4 &= \left( \frac{\xi_2}{|\xi_2|} \sinh |\xi_2| \cosh |\xi_2| - \frac{\xi_1}{|\xi_1|} \sinh |\xi_1| \cosh |\xi_1| \right) \cos \phi \sin \phi,
\end{aligned} \tag{6.B.7}$$

with  $z_1 = \mathcal{A}_1 \alpha_1 + \mathcal{A}_2 \alpha_1^* + \mathcal{A}_3 \alpha_2 + \mathcal{A}_4 \alpha_2^*$  a complex eigenvalue. Following the same steps, from the unitary transformation of  $a_2$ , we have a second eigenvalue equation of the form

$$\left( \mathcal{B}_1 a_1 + \mathcal{B}_2 a_1^\dagger + \mathcal{B}_3 a_2 + \mathcal{B}_4 a_2^\dagger \right) |\vec{\alpha}; \vec{\xi}, \phi\rangle = z_2 |\vec{\alpha}; \vec{\xi}, \phi\rangle, \tag{6.B.8}$$

where  $\mathcal{B}_1 = \mathcal{A}_3^*$ ,  $\mathcal{B}_2 = \mathcal{A}_4$ , together with

$$\begin{aligned}
\mathcal{B}_3 &= 1 + \sinh^2 |\xi_2| \cos^2 \phi + \sinh^2 \xi_1 \sin^2 \phi, \\
\mathcal{B}_4 &= \frac{\xi_2}{|\xi_2|} \sinh |\xi_2| \cosh |\xi_2| \cos^2 \phi + \frac{\xi_1}{|\xi_1|} \sinh |\xi_1| \cosh |\xi_1| \sin^2 \phi.
\end{aligned} \tag{6.B.9}$$

and  $z_2 = \mathcal{B}_1 \alpha_1 + \mathcal{B}_2 \alpha_1^* + \mathcal{B}_3 \alpha_2 + \mathcal{B}_4 \alpha_2^*$ .

From (6.3.2), we may revert the relationships between the boson operators and the canonical position and momentum observables such that we get

$$a_j = \frac{\hat{x}_j}{\sqrt{2}\lambda_j} + i \frac{\lambda_j \hat{p}_j}{\sqrt{2}\hbar}, \quad a_j^\dagger = \frac{\hat{x}_j}{\sqrt{2}\lambda_j} - i \frac{\lambda_j \hat{p}_j}{\sqrt{2}\hbar}, \quad j = 1, 2, \tag{6.B.10}$$

which, once substituted in both eigenvalue equation (6.B.6) and (6.B.8), leads to two eigenvalue equations linear in both position  $\hat{x}_j$  and momentum  $\hat{p}_j$ . That is,

$$\begin{aligned}
\left[ \left( \frac{\mathcal{A}_1 + \mathcal{A}_2}{\sqrt{2}\lambda_1} \right) \hat{x}_1 + i \left( \frac{\mathcal{A}_1 - \mathcal{A}_2}{\sqrt{2}\hbar/\lambda_1} \right) \hat{p}_1 + \left( \frac{\mathcal{A}_3 + \mathcal{A}_4}{\sqrt{2}\lambda_2} \right) \hat{x}_2 + i \left( \frac{\mathcal{A}_3 - \mathcal{A}_4}{\sqrt{2}\hbar/\lambda_2} \right) \hat{p}_2 - z_1 \right] |\vec{\alpha}; \vec{\xi}, \phi\rangle &= 0, \\
\left[ \left( \frac{\mathcal{B}_1 + \mathcal{B}_2}{\sqrt{2}\lambda_1} \right) \hat{x}_1 + i \left( \frac{\mathcal{B}_1 - \mathcal{B}_2}{\sqrt{2}\hbar/\lambda_1} \right) \hat{p}_1 + \left( \frac{\mathcal{B}_3 + \mathcal{B}_4}{\sqrt{2}\lambda_2} \right) \hat{x}_2 + i \left( \frac{\mathcal{B}_3 - \mathcal{B}_4}{\sqrt{2}\hbar/\lambda_2} \right) \hat{p}_2 - z_2 \right] |\vec{\alpha}; \vec{\xi}, \phi\rangle &= 0,
\end{aligned} \tag{6.B.11}$$

From the latter, and using the coordinate representation  $\hat{x}_j \equiv x_j$  and  $\hat{p}_j \equiv -i\hbar\partial_{x_j}$ , we get a set of two first-order partial differential equations for  $\psi(\vec{\alpha}; \vec{\xi}, \phi; \vec{x})$ , which are solved by introducing a Gaussian ansatz of the form

$$\psi(\vec{\alpha}; \vec{\xi}, \phi; \vec{x}) = \mathcal{N} e^{-\frac{\Delta_1}{\lambda_1^2} x_1^2 - \frac{\Delta_2}{\lambda_2^2} x_2^2 - \frac{\ell}{\lambda_1 \lambda_2} x_1 x_2 + \frac{\ell_1}{\lambda_1} x_1 + \frac{\ell_2}{\lambda_2} x_2}, \tag{6.B.12}$$

with  $\mathcal{N}$  the normalization factor, and the unknown coefficients  $\Delta_j$ ,  $\ell_j$ , and  $\ell$ , for  $j = 1, 2$ , are to be determined once we substitute (6.B.12) into both eigenvalue equations.

This leads to a system of six equations involving the above-mentioned five unknown coefficients, the system is therefore overdetermined. Nevertheless, the extra equation provides a

compatibility condition that tells us whether the ansatz is correct. After several calculations, it can be shown that compatibility condition is fulfilled, and the ansatz (6.B.12) provides a valid solution. Thus, after solving the remaining five equations, we get the coefficients

$$\begin{aligned}\frac{\ell_1}{\sqrt{2}} &= \frac{-((\tau_1 - \tau_2) \cos 2\phi - \tau_1 \tau_2 + 1) \operatorname{Re}(\alpha_1) + ((\tau_1 - \tau_2) \sin 2\phi) \operatorname{Re}(\alpha_2)}{(1 - \tau_1)(1 - \tau_2)} - i \operatorname{Im}(\alpha_1), \\ \frac{\ell_2}{\sqrt{2}} &= \frac{-((\tau_1 - \tau_2) \sin 2\phi) \operatorname{Re}(\alpha_1) - ((\tau_1 - \tau_2) \cos 2\phi + \tau_1 \tau_2 - 1) \operatorname{Re}(\alpha_2)}{(1 - \tau_1)(1 - \tau_2)} + i \operatorname{Im}(\alpha_2),\end{aligned}\tag{6.B.13}$$

together with  $\Delta_1$ ,  $\Delta_2$ , and  $\ell$  given in (6.3.24). To recover the coefficients given in (6.3.25), we rewrite  $\alpha_j$  in (6.B.13) in terms of  $q_j$  and  $p_j$ , with  $j = 1, 2$ , through the relationships obtained in (??). The normalisation constant in (6.3.26) follows straightforwardly by using elementary integrals involving Gaussian functions.

## 6.C Resolution of the identity for the non-separable states

In this appendix, we explain the intermediate steps needed to recover the resolution of the identity associated with the non-separable two-mode squeezed states. The squeezed states should verify the property  $\langle \Psi' | \mathbb{I} | \Psi' \rangle = \langle \Psi' | \Psi \rangle$ , where

$$\mathbb{I} \equiv \int_{\mathbb{R}^4} \frac{d\vec{q}d\vec{p}}{(2\pi\hbar)^4} \mu(\vec{q}, \vec{p}; \vec{\xi}, \phi) |\vec{q}, \vec{p}; \vec{\xi}, \phi\rangle \langle \vec{q}, \vec{p}; \vec{\xi}, \phi|, \quad d\vec{q} = dq_1 dq_2, \quad d\vec{p} = dp_1 dp_2, \tag{6.C.1}$$

to be considered an overcomplete family of states. In (6.C.1),  $\mu(\vec{q}, \vec{p}; \vec{\xi}, \phi)$  stands for the measure required to satisfy the resolution of the identity. Since the wavefunction associated to  $|\vec{q}, \vec{p}; \vec{\xi}, \phi\rangle$  takes the form of a two-variable non-separable Gaussian (6.B.12), we consider a uniform measure  $\mu(\vec{q}, \vec{p}; \vec{\xi}, \phi) = \tilde{\mu}(\vec{\xi}, \phi)$ , which accounts for any remaining constants that might appear after solving the resolution of the identity.

Thus, using the coordinate representation in the condition  $\langle \Psi' | \mathbb{I} | \Psi \rangle$  and combining with (6.C.1), we are led to

$$\begin{aligned}\langle \tilde{\Psi} | \mathbb{I} | \Psi \rangle &= \int_{\mathbb{R}^4} d\vec{x}' d\vec{x} [\tilde{\Psi}(\vec{x}')]^* \Psi(\vec{x}) \tilde{\mu}(\vec{\xi}, \phi) \left[ \left( \frac{4 \operatorname{Re}(\Delta_1) \operatorname{Re}(\Delta_2) - \operatorname{Re}(\ell)^2}{\pi^2} \right)^{\frac{1}{2}} \times \right. \\ &\quad e^{-\Delta_1 x_1^2 - \Delta_1^* x_1'^2 - \Delta_2 x_2^2 - \Delta_2^* x_2'^2 + \ell x_1 x_2 + \ell^* x_1' x_2'} \int_{\mathbb{R}^4} \frac{d\vec{q}d\vec{p}}{(2\pi\hbar)^2} e^{-\frac{i}{\hbar} p_1 (x_1 - x_1')} e^{-\frac{i}{\hbar} p_2 (x_2 - x_2')} \\ &\quad \left. e^{-\frac{1}{\Delta} \left( \frac{\eta_1}{\lambda_1^2} q_1^2 + \frac{\eta_2}{\lambda_2^2} q_2^2 + \frac{\eta_{12}}{\lambda_1 \lambda_2} q_1 q_2 \right)} e^{-\left( \frac{\ell x_2 + \ell^* x_2'}{\lambda_1 \lambda_2} + \frac{2\Delta_1 x_1 + 2\Delta_1^* x_1'}{\lambda_1^2} \right) q_1} e^{-\left( \frac{\ell x_1 + \ell^* x_1'}{\lambda_1 \lambda_2} + \frac{2\Delta_2 x_2 + 2\Delta_2^* x_2'}{\lambda_2^2} \right) q_2} \right],\end{aligned}\tag{6.C.2}$$

with  $d\vec{x} = dx_1 dx_2$ , together with

$$\eta_1 = 8 \operatorname{Re}(\Delta_1) \left( 3 \operatorname{Re}(\ell)^2 + 4 \operatorname{Re}(\Delta_1) \operatorname{Re}(\Delta_2) \right), \quad \eta_2 = 8 \operatorname{Re}(\Delta_2) \left( 3 \operatorname{Re}(\ell)^2 + 4 \operatorname{Re}(\Delta_1) \operatorname{Re}(\Delta_2) \right), \quad (6.C.3)$$

and  $\Delta$  given in (6.3.27).

From the term inside square brackets in (6.C.2), it is clear that integrating over  $p_1$  and  $p_2$  leads to  $2\pi\hbar\delta(x_1 - x'_1)$  and  $2\pi\hbar\delta(x_2 - x'_2)$ , respectively. Moreover, by integrating over  $q_1$  and  $q_2$  one can conclude that the term in square brackets reduces to  $\delta(x_1 - x'_1)\delta(x_2 - x'_2)$ . This is straightforward, as it involves elementary integrals on Gaussian functions, and will be left to the reader to verify. We thus get

$$\langle \tilde{\Psi} | \mathbb{I} | \Psi \rangle = \tilde{\mu}(\vec{\xi}, \phi) \int_{\mathbb{R}^4} d\vec{x}' d\vec{x} [\tilde{\Psi}(\vec{x}')]^* \Psi(\vec{x}) \delta(\vec{x} - \vec{x}') = \tilde{\mu}(\vec{\xi}, \phi) \langle \Psi' | \Psi \rangle, \quad (6.C.4)$$

from which it is clear that  $\tilde{\mu}(\vec{\xi}, \phi) = 1$  in order to fulfil the resolution of the identity. We therefore verify that the non-separable two-mode squeezed states form an overcomplete family.

## 6.D Some useful formulae

Here we summarise the expressions for the semiclassical portraits used in section 6.4.1. For a simplified notation, we use the reparametrised variables

$$z_{a_j} = \frac{q_j - a_j}{\sqrt{2}\lambda_j\Delta_{p_j}}, \quad z_{b_j} = \frac{q_j - b_j}{\sqrt{2}\lambda_j\Delta_{p_j}}, \quad j = 1, 2. \quad (6.D.1)$$

In this form, the semiclassical portrait of the characteristic function becomes

$$\check{A}_{\chi_E}(\vec{q}) = \check{A}_{\chi_{E_1}}(q_1)\check{A}_{\chi_{E_2}}(q_2), \quad \check{A}_{\chi_{E_j}}(q_j) = \frac{1}{2} \left( \operatorname{Erfc}(z_{b_j}) - \operatorname{Erfc}(z_{a_j}) \right). \quad (6.D.2)$$

The semiclassical portrait related to  $f(\vec{q}, \vec{p}) = q_j \chi_E(\vec{q})$  is

$$\check{A}_{q_1^k \chi_E}(\vec{q}) = \check{A}_{q_1^k \chi_{E_1}}(q_1)\check{A}_{\chi_{E_2}}(q_2), \quad \check{A}_{q_2^k \chi_E}(\vec{q}) = \check{A}_{q_2^k \chi_{E_2}}(q_2)\check{A}_{\chi_{E_1}}(q_1), \quad (6.D.3)$$

for  $k = 1, 2, 3, 4$ , with the functions  $\check{A}_{q_j^k \chi_{E_j}}(q_j)$  given by

$$\check{A}_{q_j \chi_{E_j}}(q_j) = q_j \check{A}_{\chi_{E_j}}(q_j) + \frac{\Delta_{p_j} \lambda_j}{\sqrt{2\pi}} \left( e^{-z_{a_j}^2} - e^{-z_{b_j}^2} \right), \quad (6.D.4)$$

$$\check{A}_{q_j^2 \chi_{E_j}}(q_j) = (q_j^2 + \Delta_{p_j}^2 \lambda_j^2) A_{\chi_{E_j}} + \frac{\Delta_{p_j} \lambda_j}{\sqrt{2\pi}} \left( (a_j + q_j) e^{-z_{a_j}^2} - (b_j + q_j) e^{-z_{b_j}^2} \right), \quad (6.D.5)$$

$$\begin{aligned} \check{A}_{q_j^3 \chi_{E_j}}(q_j) &= (q_j^3 + 3\Delta_{p_j}^2 \lambda_j^2) A_{\chi_{E_j}} + \\ &\frac{\Delta_{p_j} \lambda_j}{\sqrt{2\pi}} \left( (a_j^2 + a q_j + q_j^2 + 2\Delta_{p_j}^2 \lambda_j^2) e^{-z_{a_j}^2} - (b_j^2 + b q_j + q_j^2 + 2\Delta_{p_j}^2 \lambda_j^2) e^{-z_{b_j}^2} \right), \end{aligned} \quad (6.D.6)$$

$$\begin{aligned}
\check{A}_{q_j \chi_{E_j}}(q_j) = & \left( q_j^4 + 6q_j^2 \Delta_{p_j}^2 \lambda_j^2 + 3\Delta_{p_j}^4 \lambda_j^4 \right) A_{\chi_{E_j}} + \\
& \frac{\Delta_{p_j} \lambda_j}{\sqrt{2\pi}} \left( (a_j^3 + a^2 q_j + a^2 q_j + q_j^3 + \Delta_{p_j}^2 \lambda_j^2 (5q_j + 3a_j)) e^{-z_{a_j}^2} - \right. \\
& \left. (b_j^3 + b^2 q_j + b^2 q_j + q_j^3 + \Delta_{p_j}^2 \lambda_j^2 (5q_j + 3b_j)) e^{-z_{b_j}^2} \right). \quad (6.D.7)
\end{aligned}$$





# Conclusion

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In this thesis we have presented a generalised construction of coherent states for a variety of two-dimensional quantum systems. The fundamental idea throughout was to address the degeneracy in the spectrum, construct a non-degenerate spectrum and, where possible, define ladder operators for the new spectrum. A primary goal of this work was to ensure that we captured the richness of two-dimensional quantum systems such as entanglement and broader notions of non-classicality. Some of these features would not be discoverable by simply taking tensor products of one-dimensional states to construct two-dimensional states. The states we presented were constructed starting from very basic principles, and as such, are applicable to a variety of quantum bound state problems.

We studied the examples of the harmonic oscillator and Morse potential because of their importance and widespread application in physics. Applying the technique of averaging degenerate contributions we were able to generate generalised two-mode squeezed states of the harmonic oscillator, a new set of  $\mathfrak{su}(2)$ -type coherent states for the 2:1 anisotropic oscillator, and coherent states for the two-dimensional separable Morse potential and its non-separable supersymmetric partner. In each case study we explored aspects of non-classicality including squeezing, non-separability of the wavefunction and non-localisation of the wavefunction. Some non-classical features of the two-dimensional states for these systems could be missed if one were to neglect the problem of degeneracy. The applicability of these techniques to any system with degeneracy in its bound state spectrum is evident.

In chapter 1 we outlined some of the basic features of coherent and squeezed states of the harmonic oscillator in one dimension. The spirit of chapter 2 was to generalise the definitions from the one-dimensional oscillator into two dimensions. The complications introduced were the lack of singled-indexed spectrum and lack of ladder operators to act on said spectrum. In this regard we succeeded in overcoming these complications, we defined generalised ladder operators as linear combinations of the one-dimensional ladder operators which act on a single-indexed spectrum containing averaged contributions from degenerate states. From these generalised ladder operators the definitions from the one-dimensional case

followed naturally, and we recovered the most general type squeezed state, including two-mode squeezing. We also considered Chen's definition of coherent states for the anisotropic oscillator [1], as well as their Schrödinger-type variant under our formalism.

The purpose of chapter 3 was to fill in the gaps in the analysis of coherent states for two-dimensional anisotropic oscillators. While the states of Chen [1] provide a very reasonable ansatz of  $\mathfrak{su}(2)$ -like coherent states for anisotropic oscillators, they clearly miss some of more attractive mathematical properties required by many definitions of generalised coherent states. In our work we were able to construct ladder operators which necessitated the use of non-linear combinations of the one-dimensional ladder operators in order to account for the degeneracy in the spectrum. Moreover, we recovered the identity operator for the states demonstrating that they form a complete set.

So far in the thesis we had only dealt with degeneracies which are linear in the quantum numbers, such degeneracies have known symmetry groups. Chapter 4 was devoted to extending our construction to a two-dimensional system whose spectrum is quadratic in the quantum numbers. In particular we studied the Morse potential bound states. It was found that in general accidental degeneracies appear which make the extension of the analysis from the preceding chapters somewhat difficult. We were able to determine that if the principle parameter of the model,  $p$ , is irrational, then states in the system are at most twofold degenerate. Equipped with this we constructed a set of generalised coherent states as approximate eigenstates of the annihilation operator on the non-degenerate spectrum and found the dependence of the uncertainties on the mixing parameters  $\gamma_1, \gamma_2$ .

Following this, in chapter 5, we invoked the supersymmetry formalism. We studied the non-separable Morse potential presented by Ioffe [2] and were able to transform the non-degenerate states developed in the preceding chapter by the action of the supercharge  $\mathcal{Q}^+$  into states of the partner Hamiltonian. We then constructed the generalised coherent states and found that due to the singularity along the line  $y = x$  in the potential, the coherent state wavefunction was unable to localise. This along with the squeezing in between position and momentum quadratures led us to conclude that the states generated were significantly non-classical.

Lastly, in chapter 6, we exploited one of the properties of coherent states we have stressed throughout this thesis: the resolution of the identity. We studied the effects of quantisation with coupled and uncoupled two-dimensional squeezed states and discussed the particular example of a position dependent mass system. The use of non-separable squeezed states generated a correlation between the quantised observables in each mode. We saw, for instance, that the quantisation of the classical position  $q_1$  led to a quantum operator which is a linear combination of  $\hat{x}_1$  and  $\hat{x}_2$ . Such features would not be exposed without the aid of more general classes of two-dimensional coherent states.

There are several avenues which deserve thorough exploration. Of course there exist many two-dimensional systems which may still benefit from the analysis presented in this thesis. But perhaps the most tempting generalisation is to  $N$ -dimensional systems. The ideas presented in chapter 2 seem ready to be generalised in this case, and one might expect to generate an  $N$ -dimensional tensor product of one-dimensional coherent states. More interestingly, one might expect to define the most general possible coupled squeezed state in  $N$  dimensions while maintaining a direct mathematical analogy with the well-understood one-dimensional construction. We would naturally generate all possible bilinear couplings between each mode in the system. For more general multilinear couplings one would have to explore defining new operators altogether.

With respect to the ideas presented in chapter 3 there is clearly a lot left to be done in two dimensions. Generalisation to the case where the commensurable frequencies are relative primes  $p : q$ , would seem to be the starting point. It would seem that higher order nonlinearities in the generalised ladder operators would be required to capture the degenerate states leading to more difficult analysis. Nevertheless, generating complete sets of  $\mathfrak{su}(2)$ -like coherent states for an arbitrary commensurable frequency anisotropic oscillators seems attainable. Additionally, a construction of Schrödinger-type coherent states by defining a unitary exponential operator out of  $\mathcal{A}^-$ ,  $\mathcal{A}^+$  will lead to an entirely different class of coherent states for the problem.

The clearest omission in the analysis of chapters 4 and 5 was the explicit construction of ladder operators to act on the non-degenerate spectra. For spectra quadratic in the quantum numbers we can always apply the same principles of averaging degenerate contributions as we have stressed throughout this thesis, but constructing explicit generalised ladder operators for the non-degenerate spectrum out of the one-dimensional ladder operators or otherwise has proven difficult. In this case the organisation of the spectrum has solutions in number theory and is not related to a well understood Lie symmetry group. If one were to be able to construct such explicit representations, it would open the door to new classes of coherent states for the system. Additionally, in the spirit of the transition from chapter 4 to chapter 5, the states we generated were compatible with supersymmetry, and it would be interesting to explore the generation of non-degenerate eigenstates for more general non-separable partner Hamiltonians.

The ideas presented in chapter 6 deserve to be explored further in the context of quantisation in non-rectangular restricted geometries. Mathematically speaking, some computations become more challenging, but the geometry may represent a more physical example. A particularly interesting example would be to study semiclassical billiard dynamics in the generalised squeezed state formalism. We know that the squeezed states generate anisotropy in the regularised restricted region and understanding the impact on the dynamics of a point particle confined to the region represents a curious mathematical problem. We must

mention that the generalisation to higher dimensions can give very precise control of the regularisation strength and coupling between coordinates which may have applications in multidimensional materials. The most general multidimensional squeezed state quantisation scheme requires the most general class of squeezed states, so it is integral to further understand the generalisations of the work presented in chapter 2.

Multidimensional quantum systems afford us much richer physics. The mathematical underpinnings of coherent state analysis have proved critical in our understanding throughout physics and it seems that some formalism in the multidimensional case is missing. The goal of this thesis was to reliably produce a coherent state formalism in two-dimensional systems which captured the nuances that arise in dimensions greater than one. To this extent we succeeded, and we advocate for the understanding of coherent states from the viewpoint of degeneracy. We live in an interesting time in the development of quantum information science, and we believe that a detailed understanding of generalised multimode coherent states will be of great use moving forward.

## References

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