Université de Montréal

Nouvelles perspectives sur les algèbres de type Askey–Wilson

Dualité de Howe, opérateurs de Sklyanin–Heun et centralisateurs

par

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Résumé

Cette thèse se divise en trois parties qui peuvent être toutes regroupées autour d'une même bannière : l'étude de structures algébriques reliées aux algèbres de type Askey–Wilson. Alors que dans la première partie on s'efforce d'obtenir des interprétations duales (au sens de Howe) de ces algèbres, dans les autres parties on étudie des généralisations de ces algèbres. Des dégénérations de l'algèbre de Sklyanin, générées par des blocs plus fondamentaux que ceux générant les algèbres de type Askey–Wilson, sont étudiées dans la deuxième partie et des généralisations de plus haut rang des algèbres de type Askey–Wilson sont étudiées dans la troisième partie.

Dans la première partie, en invoquant la dualité de Howe, deux interprétations duales sont obtenues pour les algèbres de Racah, Bannai–Ito, Askey–Wilson, Higgs, Hahn, q-Hahn et dual -1 Hahn. La façon dont la dualité de Howe opère est rendue explicite par l'examen de processus de réduction dimensionnelle. Un modèle superintégrable 2D de mécanique quantique superconforme dont l'algèbre de symétrie est celle de type dual -1 Hahn est également introduit et solutionné.

Dans la deuxième partie, des algèbres générées par des opérateurs de contiguïté et d'échelle encodant des propriétés de familles de polynômes sont étudiées. Ces opérateurs appartiennent à la classe des opérateurs de Sklyanin–Heun, qui peuvent être définis sur plusieurs grilles diverses. On découvre qu'ils génèrent des dégénérations de l'algèbre de Sklyanin. On démontre que les représentations irréductibles de dimension finie de ces algèbres ont pour base des familles de para-polynômes. Les grilles linéaires, quadratiques, exponentielles et d'Askey–Wilson sont étudiées et mènent respectivement aux polynômes orthogonaux des familles de para-Krawtchouk, para-Racah, q-para-Krawtchouk et q-para-Racah. Enfin, la façon dont les polynômes de para-Krawtchouk et d'autres familles de polynômes orthogonaux sont reliées aux représentations tridiagonales du plan de Jordan déformé est présentée.

Dans la dernière partie, on explore des généralisations à plus haut rang pour les algèbres de Racah et Askey–Wilson. Pour ce faire, on étudie les réalisations de ces algèbres en termes de Casimirs intermédiaires. Le rôle de la matrice R tressée est élucidé : celle-ci permet de relier divers Casimirs intermédiaires entre eux par conjugaison. Un isomorphisme entre l'algèbre de skein du crochet de Kauffman de la sphère à 4 trous et l'algèbre engendrée

par les Casimir intermédiaires dans $U_q(\mathfrak{sl}_2)^{\otimes 3}$ est présenté et permet d'interpréter de façon diagrammatique la conjugaison par la matrice R tressée mentionnée ci-haut. Finalement, une présentation du centralisateur $Z_n(\mathfrak{sl}_2)$ de $U(\mathfrak{sl}_2)$ dans $U(\mathfrak{sl}_2)^{\otimes n}$ par générateurs et relations est obtenue et on montre que ce centralisateur est isomorphe à un quotient (obtenu explicitement) de l'algèbre de Racah de plus haut rang $R(n)$.

Mots-clés

- Algèbre d'Askey–Wilson
- Dualité de Howe
- Opérateurs de Sklyanin–Heun
- Centralisateurs
- Réduction dimensionnelle
- Algèbre de Sklyanin
- Para-polynômes
- Algèbre de skein du crochet de Kauffman
- Matrice R
- Théorie des invariants classique

Abstract

This thesis is divided in three parts which all orbit around the same theme: the study of algebraic structures related to the algebras of Askey–Wilson type. In the first part we obtain two interpretations that are dual in the sense of Howe for the algebras of Askey–Wilson type. Meanwhile, the other two parts are concerned with generalizations of these algebras. In the second part, we study degenerations of the Sklyanin algebra, which are built out of generators that are more fundamental than those of the Askey–Wilson algebra. In the last part, generalizations of the Askey–Wilson type algebras to higher rank are studied.

In the first part, dual interpretations are obtained for the Racah, Bannai–Ito, Askey–Wilson, Higgs, Hahn, q-Higgs and dual -1 Hahn algebras by invoking Howe duality. The way that this Howe duality operates is made explicit through the examination of a dimensional reduction procedure. A 2D superintegrable superconformal quantum mechanics model, whose symmetry algebra is the one of dual -1 Hahn type, is also introduced and solved.

In the second part, we study algebras that are generated by contiguity and ladder operators that encode properties of families of orthogonal polynomials. We show that these operators belong to the Sklyanin–Heun class of operators, which can be defined for various grids. We also show how their algebraic relations correspond to those of degenerations of the Sklyanin algebra. Then, we show how various families of para-polynomials support finite-dimensional irreducible representations of these degenerate algebras. From the linear, quadratic, exponential and Askey–Wilson grids, we are respectively led to the para-Krawtchouk, para-Racah, q-para-Krawtchouk and q-para-Racah polynomials. Later, we connect the para-Krawtchouk polynomials (and other families of orthogonal polynomials) to tridiagonal representations of the deformed Jordan plane.

In the final part, we explore higher rank generalizations of the Racah and Askey–Wilson algebras. To that end, their realizations in terms of intermediate Casimir elements are studied. The role of the braided R-matrix is understood as follows: it connects various intermediate Casimir elements through conjugation. We obtain an isomorphism between the Kauffman bracket skein algebra of the four-punctured sphere and the algebra generated by the intermediate Casimir elements in $U_q(\mathfrak{sl}_2)^{\otimes 3}$. This leads to a diagrammatic interpretation of the conjugation by the braided R-matrix mentioned in the above. Lastly, a presentation of the centralizer $Z_n(\mathfrak{sl}_2)$ of $U(\mathfrak{sl}_2)$ in $U(\mathfrak{sl}_2)^{\otimes n}$ by generators and relations is obtained and we show that this centralizer is isomorphic to a quotient (which we provide explicitly) of the higher rank Racah algebra $R(n)$.

Keywords

- Askey–Wilson algebra
- Howe duality
- Sklyanin–Heun operators
- Centralizers
- Dimensional reduction
- Sklyanin algebra
- Para-polynomials
- Kauffman bracket skein algebra
- R-matrix
- Classical invariant theory

Table des matières

Liste des figures

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Introduction

Prologue

Nous sommes en 1981. Le jeune homme a 23 ans et est en train de compléter son service militaire dans l'armée soviétique. Mais il n'est pas parti les mains vides. En sa possession, un livre sur les fonctions spéciales qu'il étudie dans son temps libre [1]. Un chapitre en particulier pique sa curiosité : celui sur les familles de polynômes orthogonaux de Meixner, Krawtchouk et Charlier.

Le protagoniste s'intéresse aux opérateurs encodant les relations de récurrence et les équations aux différences caractérisant ces trois familles de polynômes. En calculant les relations algébriques entre ces opérateurs, il reconnaît les relations définissantes des algèbres de Lie $\mathfrak{su}(1,1)$ et $\mathfrak{su}(2)$.

C'est là le point de départ de ce qui suit.

1982. Notre protagoniste est rentré de son service militaire et débute ses travaux de doctorat à Donetsk sous la supervision de Ya Granovskii. Il tente d'étendre les idées qu'il a eu à la suite de ses observations sur les polynômes orthogonaux et les algèbres de Lie. Une première publication en 1984 [2] explique comment obtenir les trois familles mentionnées ci-haut à partir des algèbres de Lie $\mathfrak{su}(1,1)$ et $\mathfrak{su}(2)$. Cependant, il note que son approche ne fonctionne plus pour les familles de polynômes situées plus haut dans le tableau d'Askey [Figure 1].

En quelques mots, la raison pour laquelle ça bloque, c'est qu'il n'est pas possible de retrouver le spectre quadratique des familles de polynômes plus hautes dans le tableau d'Askey uniquement à l'aide d'algèbres de Lie. Il faut des outils plus puissants.

La solution à laquelle parvient éventuellement le protagoniste : étudier des algèbres quadratiques. Ici, il faut apprécier que ce type d'algèbre est hors du commun à cette époque. Les algèbres de Lie sont des objets bien connus apparus à la fin du 19^{e} siècle et la classification des algèbres de Lie semi-simples a été complétée par Cartan un peu avant le 20^{e} siècle. En revanche, presque rien n'est connu sur la classe des algèbre quadratiques du type étudié par le protagoniste, à l'exception peut-être de l'algèbre de Sklyanin [3] et certains travaux indépendants sur les algèbres de symétrie de certains systèmes superintégrables [4].

Ces algèbres quadratiques sont la clé pour retrouver le spectre quadratique mentionné précédemment. Les représentations tridiagonales de l'algèbre dite de Racah font alors intervenir les polynômes de Racah, qui trônent au sommet du tableau d'Askey [5]. Nous sommes alors en 1989. Autre curiosité : ces algèbres quadratiques apparaissent dans l'étude du problème de Racah des algèbres $\mathfrak{su}(1,1)$ et $\mathfrak{su}(2)$...

Ce n'est pas la fin de l'histoire.

Il existe également un q-tableau d'Askey [Figure 2] qui rassemble des généralisations des familles du tableau d'Askey par un paramètre q. Au sommet de ce q-tableau se trouvent les polynômes d'Askey–Wilson. Dans la même veine que ce qui a été fait pour les polynômes et l'algèbre de Racah, notre protagoniste se demande : existe-t-il des structures algébriques similaires associées aux polynômes d'Askey–Wilson?

Deux publications en 1991 et 1993 [6, 7] répondent à l'affirmative. Une algèbre cubique, baptisée *algèbre d'Askey–Wilson* par le protagoniste, possède des représentations tridiagonales sur les polynômes d'Askey–Wilson et est de plus reliée au problème de Racah de $\mathfrak{sl}_q(2)$.

Dans les années qui suivent, les algèbres de Racah et d'Askey–Wilson prennent de l'ampleur et se mettent à apparaître dans un nombre grandissant de domaines des mathématiques et de la physique.

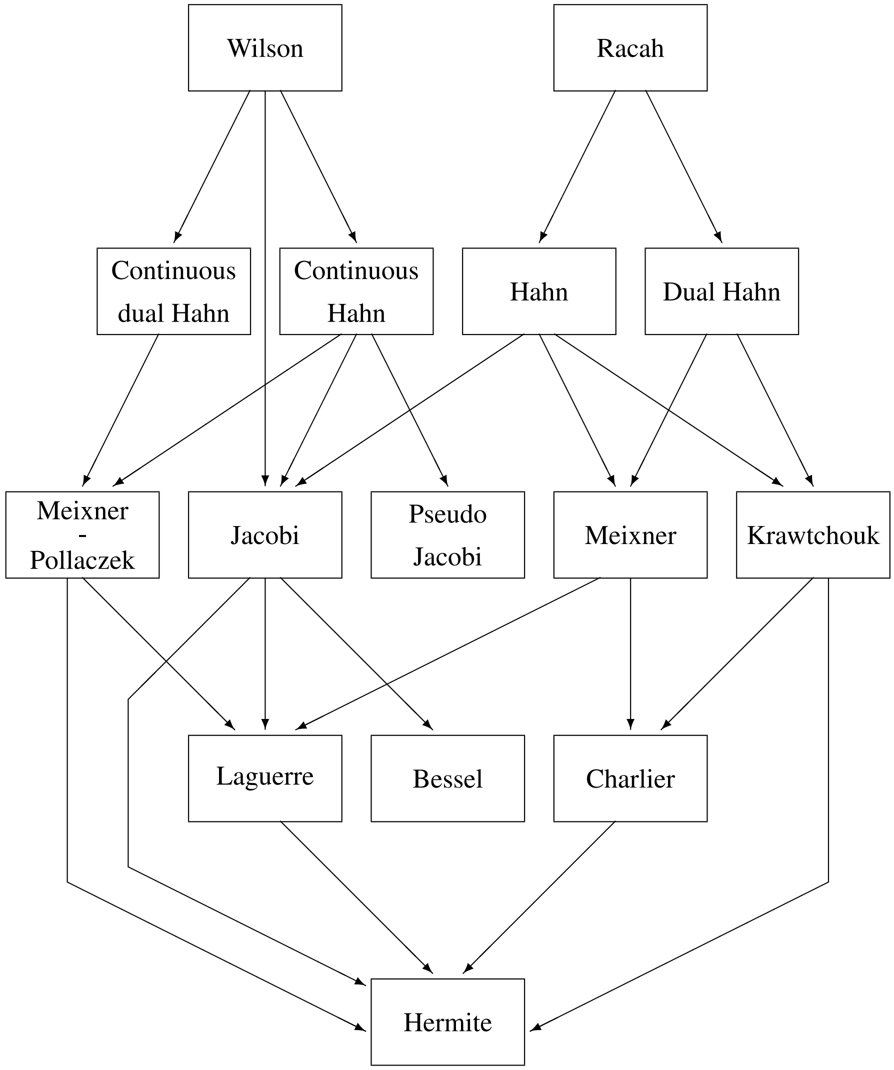

Fig. 1. Tableau d'Askey des familles de polynômes orthogonaux classiques hypergéométriques. Les flèches indiquent l'existence de limites permettant de passer d'une famille à une autre. Les familles situées en haut du tableau peuvent donc être considérées comme les plus générales.

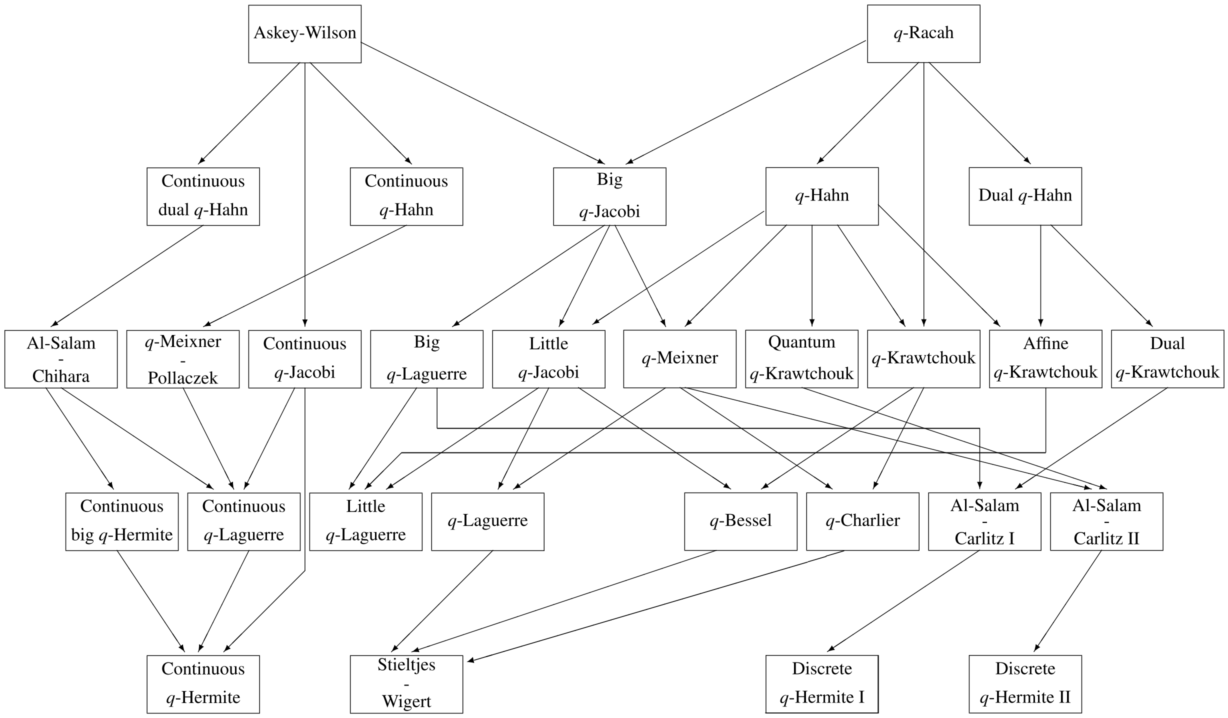

FIG. 2. q-tableau d'Askey des familles de polynômes orthogonaux hypergéométriques basiques. Les flèches indiquent l'existence de limites permettant de passer d'une famille à une autre. Les familles situées en haut du tableau peuvent donc être considérées comme les plus générales.

Sur l'importance de l'algèbre et de la théorie des groupes en physique

Pourquoi parler d'algèbre et de théorie des groupes dans une thèse de physique?

Il existe une riche tradition d'interactions entre la physique théorique et le domaine de l'algèbre en mathématiques : à tour de rôle, un domaine inspire des avancées dans l'autre, et les multiples connexions entre ces domaines sont riches et bénéfiques.

Dès les premiers balbutiements de la mécanique quantique déjà, des éléments d'algèbre, de théorie des groupes et de théorie des représentations apparaissaient un peu partout dans la théorie. Assez vite, certains physiciens se rendent compte que des arguments de symétrie peuvent servir à expliquer les espacements entre les niveaux d'énergie et les dégénérescences observées dans des expériences de spectroscopie. L'algèbre et la théorie des groupes sont le langage mathématique naturel pour parler de symétries.

Petit à petit, les articles de pionniers tels que Wigner et Weyl commencent à cimenter l'importance et à formaliser le rôle de la théorie des groupes (et de leurs représentations) en mécanique quantique. Toutefois, le formalisme mathématique utilisé est peu familier pour la plupart des physiciens. Un bon nombre voit cette formalisation mathématique de la théorie quantique d'un mauvais œil; l'appellation *Gruppenpest*, ou « Peste de la théorie des groupes » due à Ehrenfest est emblématique de cette d'opposition envers une abstraction mathématique grandissante de la théorie quantique [8].

Malgré tout, cette résistance disparaît éventuellement en raison des multiples succès rencontrés par la théorie des groupes en physique théorique. Pour n'en nommer que quelques uns, mentionnons l'élucidation de l'effet Zeeman anomal, les progrès en chimie physique et en spectroscopie, les développements dans l'étude des représentations irréductibles du groupe de Poincaré, la classification des particules élémentaires, ainsi que le développement de la théorie électrofaible et des théories de jauge.

Dans son livre *The Theory of Groups and Quantum Mechanics* [9], Weyl écrit :

> It has recently been recognized that group theory is of fundamental importance for quantum physics; it here reveals the essential features which are not contigent on a special form of the dynamical laws nor on special assumptions concerning the forces involved.[1]

Les constatations de symétrie et de covariance sont si puissantes qu'elles fixent la forme des lois physiques, sans même qu'on n'aie besoin de supposer quoi que ce soit sur la forme des

[1]« On a récemment reconnu l'importance fondamentale de la théorie des groupes en physique quantique; celle-ci révèle des éléments essentiels de la théorie qui ne sont pas conditionnels à la forme des lois dynamiques ou des forces impliquées » (traduction libre).

forces physiques en action. C'est d'ailleurs ainsi qu'on arrive aux équations de Schrödinger et de Dirac en partant des postulats de covariance de Galilée et de Poincaré.

L'étude de la théorie des groupes en mécanique quantique est fructueuse. Weyl en particulier travaille à l'interface des deux domaines et se met à étudier de plus près le groupe symmétrique et le groupe des transformations linéaires. Dans les problèmes de couplage de 2 spins, il est bien connu que les propriétés de symétrie/antisymétrie de la fonction d'onde totale sont reliées au spin total résultant de la combinaison. Ce lien entre spin et symétrie sous permutation est en fait un cas particulier d'un phénomène plus général, étudié par Weyl et baptisé « réciprocité entre les représentations du groupe symétrique et du groupe général linéaire ». De nos jours, on le connaît sous le nom de « dualité de Schur–Weyl » et c'est un résultat central en théorie des représentations.

La dualité de Schur–Weyl a vu le jour à la suite de progrès en mathématiques inspirés en partie par des questions en physique. L'inverse se produit aussi parfois. Un des facteur limitant certains progrès en physique est la capacité à effectuer des calculs pour solutionner des problèmes, une difficulté fondamentalement mathématique. À cet égard, des avancées mathématiques menant à des prouesses calculatoires permettent l'étude de modèles de plus en plus sophistiqués. Mentionnons ici la solution du modèle d'Ising en 2D par Onsager [10] en 1944 : un tour de force algébrique qui a par la suite grandement stimulé les domaines des transitions de phase et l'étude des phénomènes critiques en physique.

De nos jours, la théorie physique moderne ayant rencontré le plus grand succès dans les dernières décennies est le Modèle Standard de physique des particules, et ses symétries, qui s'expriment dans le langage de l'algèbre et la théorie des groupes, jouent un rôle essentiel dans cette théorie : ce sont elles qui encodent les forces fondamentales de la nature.

Intégrabilité et algèbre

Un autre domaine relativement récent en physique qui utilise l'algèbre à profusion est le domaine des systèmes intégrables. Une tâche standard en physique théorique est de concocter des modèles afin de rendre compte de phénomènes observés. Ces modèles sont, par exemple, un système d'équations mathématiques décrivant la dynamique ou certaines propriétés d'un système physique qu'on veut encoder. Si les solutions du modèle concocté parviennent à bien reproduire ce qui est observé expérimentalement, alors on pourra utiliser ce modèle comme point de départ pour développer une théorie physique du système.

Les modèles en physique jouent un rôle important dans l'élucidation de phénomènes, mais en général ils ne sont pas exactement résolubles. La plupart du temps, on ne peut que les solutionner numériquement ou bien en faisant des approximations. Toutefois, pour un faible nombre de modèles, leurs solutions exactes peuvent être obtenues explicitement :

ceux-ci sont dits *exactement résolubles*. Les systèmes associés sont dits intégrables et ont beaucoup d'utilités en physique théorique.

Précisons un peu ce qui est entendu par système intégrable (pour une revue récente, voir [11]). En mécanique classique, si un système possédant n degrés de liberté possède n constantes du mouvement indépendantes et qui Poisson-commutent[2] simultanément, on dit que ce système est *intégrable*. Si ce système possède plus de n constantes du mouvement, on dit qu'il est *superintégrable*. Enfin, un système est *maximalement superintégrable* s'il possède le nombre maximal de constantes du mouvement, soit $2n-1$, et qu'une d'entre elles, l'Hamiltonien, commute avec toutes les autres.

Les systèmes maximalement superintégrables peuvent typiquement être solutionnés algébriquement en tirant parti de leurs nombreuses symétries. C'est la présence de ces nombreuses symétries qui explique la solvabilité du système. Ces symétries sont très riches et peuvent être plus générales que les symétries de groupe (associées à des algèbres de Lie) mentionnées à la section précédente : elles peuvent former des algèbres quadratiques, cubiques, ou même polynomiales [4].

Qu'en est-il des systèmes intégrables en mécanique quantique? Malgré un volume élevé de travaux et un nombre de définitions proposées pour définir la notion de système quantique intégrable, il ne semble toujours pas y avoir de consensus à ce jour sur « la bonne » définition [12]. Cela contraste avec la situation en mécanique classique où la théorie est bien établie. Toutefois, tous ces travaux ne sont pas en vain et ont mené à une pluie de résultats en physique et en mathématiques. En particulier, dans la foulée de l'étude des modèles sur réseau utilisant la machinerie de la matrice de transfert ainsi que dans l'étude des systèmes quantiques intégrables par l'entremise des méthodes de « scattering inverse » [3], Drinfeld [13] et Jimbo [14] ont identifié des nouvelles structures algébriques, les groupes quantiques. Ces structures sont très riches : il s'agit d'algèbres de Hopf quasi-triangulaires non-commutatives qui sont des déformations d'algèbres enveloppantes universelles d'algèbres de Lie. Dans cet exemple-ci, c'est la physique qui inspire des progrès en algèbre.

Au-delà de leurs riches propriétés algébriques, les systèmes superintégrables ont d'autres utilités en physique. Il est souvent utile d'approximer des systèmes plus complexes par des perturbations de systèmes superintégrables (on peut penser ici à l'omniprésence d'oscillateurs harmoniques en physique), lesquels sont exactement résolubles.

Les systèmes maximalement superintégrables semblent être tous exactement résolubles. Il s'agit d'une conjecture pour laquelle aucun contre-exemple n'a été trouvé à ce jour. Les solutions de systèmes intégrables font intervenir des polynômes orthogonaux et des fonctions spéciales : d'autres connexions entre les mathématiques, l'algèbre et la physique théorique. On peut résumer le principe des dernières pages par l'heuristique suivant :

[2] c'est-à-dire que leur crochet de Poisson s'annule

[3] ou bien « problème inverse de la diffusion quantique »?

Si on comprend davantage d'algèbres et les fonctions spéciales qui leur sont associées, et que ces algèbres sont liées à des modèles en physique, alors on peut solutionner davantage de problèmes en physique.

Les deux prochaines sections illustreront ceci par deux exemples.

Un premier exemple : l'oscillateur harmonique quantique 2D

L'oscillateur harmonique quantique 2D a pour Hamiltonien

$$H = \tfrac{1}{2}\{a_1, a_1^\dagger\} + \tfrac{1}{2}\{a_2, a_2^\dagger\} = N_1 + N_2 + 1, \tag{0.0.1}$$

où les N_i sont donnés par

$$N_i = a_i^\dagger a_i, \qquad i, j = 1, 2. \tag{0.0.2}$$

Ici, $\{A, B\} = AB + BA$ dénote l'anticommutateur et les $a_i^\dagger$, a_i sont respectivement les opérateurs d'échelle de création et d'annihilation :

$$a_i = \frac{1}{\sqrt{2}}\left(x_i + \frac{\partial}{\partial x_i}\right), \qquad a_i^\dagger = \frac{1}{\sqrt{2}}\left(x_i - \frac{\partial}{\partial x_i}\right), \qquad i, j = 1, 2. \tag{0.0.3}$$

Ceux-ci respectent les relations de commutation canoniques de l'algèbre de Weyl :

$$[a_i, a_j^\dagger] = \delta_{ij}, \qquad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \qquad i, j = 1, 2, \tag{0.0.4}$$

où $[A, B] = AB - BA$ dénote le commutateur. Quelles sont les symétries de ce système? On trouve assez facilement que les K_{ij}

$$K_{ij} = a_i^\dagger a_j, \qquad i, j = 1, 2, \tag{0.0.5}$$

commutent avec l'Hamiltonien (0.0.1) :

$$[H, K_{ij}] = 0 \tag{0.0.6}$$

et génèrent l'algèbre de symétrie du système. C'est la construction de Schwinger [15]. Puisqu'on exclut l'Hamiltonien des générateurs, prenons plutôt comme base

$$A_+ = a_1^\dagger a_2, \qquad A_- = a_1 a_2^\dagger, \qquad D = N_1 - N_2. \tag{0.0.7}$$

Un calcul direct permet d'obtenir

$$\begin{aligned} [D, A_\pm] &= \pm 2A_\pm, \\ [A_+, A_-] &= D, \end{aligned} \tag{0.0.8}$$

et d'identifier l'algèbre de Lie $\mathfrak{su}(2)$ comme algèbre de symétrie du système. Cette algèbre de Lie est étroitement reliée aux rotations. Le système d'oscillateur harmonique 2D possède donc des propriétés d'invariance sous rotations. Dans le prochain exemple, on obtiendra une algèbre qui n'a plus d'interprétation géométrique directe.

Un autre exemple : l'oscillateur singulier 2D

Ajoutons maintenant des termes centrifuges à l'Hamiltonien (0.0.1) pour obtenir l'Hamiltonien d'un oscillateur singulier 2D

$$H_k = \tfrac{1}{2}[J_-^{(1)}, J_+^{(1)}] + \tfrac{1}{2}[J_-^{(2)}, J_+^{(2)}] = J_0^{(1)} + J_0^{(2)}, \tag{0.0.9}$$

avec

$$J_\pm^{(i)} = \frac{1}{4}\left[\left(x_i \mp \frac{\partial}{\partial x_i}\right)^2 + \frac{k_i}{x_i{}^2}\right], \qquad J_0^{(i)} \ = \frac{1}{4}\left[-\frac{\partial^2}{\partial x_i{}^2} - \frac{k_i}{x_i{}^2} + x_i{}^2\right]. \tag{0.0.10}$$

Dans la limite k_1, $k_2 \to 0$, on retrouve l'oscillateur harmonique 2D de la section précédente. Ce modèle peut être vu comme une déformation du modèle précédent par des paramètres k_1, k_2, et est donc un modèle légèrement plus sophistiqué. Mais cette légère sophistication nous amène à utiliser des outils algébriques plus complexes.

Cette fois-ci, les relations de commutation sont

$$\begin{gathered}[J_-^{(i)}, J_+^{(j)}] = 2J_0^{(i)}\delta_{ij}, \qquad [J_0^{(i)}, J_\pm^{(j)}] = \pm J_\pm^{(i)}\delta_{ij}, \\ [J_0^{(i)}, J_0^{(j)}] = [J_-^{(i)}, J_-^{(j)}] = [J_+^{(i)}, J_+^{(j)}] = 0,\end{gathered} \qquad i, j = 1, 2. \tag{0.0.11}$$

L'algèbre de symétrie de ce système est générée par des combinaisons à la Schwinger analogues à celles dans l'exemple précédent[4] :

$$A_+ = 4J_+^{(1)}J_-^{(2)}, \qquad A_- \ = 4J_-^{(1)}J_+^{(2)}, \qquad D = 2(J_0^{(1)} - J_0^{(2)}). \tag{0.0.12}$$

Cette fois-ci les relations algébriques entre les générateurs ne sont plus linéaires. Un calcul direct donne

$$\begin{aligned}[][D, A_\pm] &= \pm 4A_\pm, \\ [A_+, A_-] &= -D^3 + D\alpha_1 + \alpha_2,\end{aligned} \tag{0.0.13}$$

où les α_i sont des termes centraux donnés par

$$\alpha_1 = 4H^2 - (3 + 2(k_1 + k_2)), \qquad \alpha_2 = 4(k_1 - k_2)H. \tag{0.0.14}$$

L'algèbre (0.0.13) est une extension centrale de l'algèbre de Higgs [16] et est un exemple d'algèbre quadratique qui apparaît lors de l'étude des symétries de modèles en physique.

Algèbre quadratique ai-je dit? À première vue, les relations (0.0.13) semblent cubiques et non quadratiques. Toutefois, l'algèbre de Higgs est isomorphe à l'algèbre de Hahn, qui est

[4]Pour simplifier les expressions ci-bas, on a ajusté la normalisation.

clairement quadratique. Sous le changement de variables (inversible) suivant :

$$\begin{aligned}
&K_1 = \tfrac{1}{2}D,\\
&K_2 = -\tfrac{1}{4}(A_+ + A_- + \tfrac{1}{2}D^2) + \tfrac{1}{8}\alpha_1,\\
&K_3 = -\tfrac{1}{2}(A_+ - A_-),
\end{aligned} \tag{0.0.15}$$

les relations (0.0.13) prennent la forme (d'une extension centrale) de l'algèbre de Hahn :

$$\begin{aligned}
&[K_1, K_2] = K_3,\\
&[K_2, K_3] = -2\{K_1, K_2\} - \tfrac{1}{4}\alpha_2,\\
&[K_3, K_1] = -2{K_1}^2 - 4K_2 + \tfrac{1}{2}\alpha_1.
\end{aligned} \tag{0.0.16}$$

L'exemple de l'oscillateur singulier 2D démontre qu'en étudiant un modèle légèrement plus sophistiqué, on voit déjà le besoin d'introduire des structures algébriques plus générales que les algèbres de Lie.

L'algèbre et les polynômes de Hahn

D'où l'algèbre de Hahn tire-t-elle son nom? Il existe une famille de polynômes orthogonaux classiques appelés polynômes de Hahn. Ces polynômes orthogonaux sont données par l'expression suivante [17] :

$$Q_n(x) := Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix}-n, n+\alpha+\beta+1, -x\\ -N, \alpha+1\end{matrix}; 1\right), \qquad n = 0, 1, \ldots, N, \tag{0.0.17}$$

où la fonction hypergéométrique ${}_rF_s$ est donnée par

$${}_rF_s\left(\begin{matrix}a_1, \ldots, a_r\\ b_1, \ldots, b_s\end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!} \tag{0.0.18}$$

et $(a)_k = a(a+1)\cdots(a+k-1)$ est le symbole de Pochhammer.

Ces polynômes possèdent de nombreuses propriétés remarquables. En particulier, ils obéissent à une relation de récurrence à trois termes

$$\begin{aligned}
-xQ_n(x) &= A_nQ_{n+1}(x) - [A_n + C_n]Q_n(x) + C_nQ_n(x),\\
A_n &= \frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},\\
C_n &= \frac{n(n+\alpha+\beta+N+1)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)},
\end{aligned} \tag{0.0.19}$$

ainsi qu'à une équation aux différences

$$\begin{aligned}
n(n+\alpha+\beta+1)Q_n(x) &= B(x)Q_n(x+1) - [B(x) + D(x)]Q_n(x) + D(x)Q_n(x-1),\\
B(x) &= (x+\alpha+1)(x-N),\\
D(x) &= x(x-\beta-N-1).
\end{aligned} \tag{0.0.20}$$

Ces deux équations peuvent être vues comme des équations aux valeurs propres pour deux opérateurs, un premier opérateur associé à la relation de récurrence

$$X = A_n T_+ - [A_n + C_n]I + C_n T_-, \qquad T_\pm f(n) = f(n \pm 1), \tag{0.0.21}$$

ainsi qu'un second opérateur associé à l'équation aux différences

$$D = B(x)\Delta_+ - [B(x) + D(x)]I + D(x)\Delta_-, \qquad \Delta_\pm f(x) = f(x \pm 1), \tag{0.0.22}$$

ayant respectivement $-x$ et $n(n + \alpha + \beta + 1)$ comme valeurs propres. Ici I est l'opérateur identité.

Les $Q_n(x)$ sont simultanément les fonctions propres de deux problèmes aux valeurs propres en deux variables différentes : n et x. Dans une telle situation, on dit que X et D sont une *paire bispectrale d'opérateurs*, ou encore des *opérateurs bispectraux*, et que les $Q_n(x)$ satisfont la propriété de *bispectralité*.

Les deux opérateurs X et D ne commutent pas. Les relations algébriques auxquelles ils obéissent sont celles de l'algèbre de Hahn, nommée ainsi en référence aux polynômes du même nom qui solutionnent leur problème bispectral. Sous le choix

$$\begin{aligned} K_1 &= -2X - \tfrac{1}{2}(2N + \beta - \alpha), \\ K_2 &= D + \tfrac{1}{4}(\alpha + \beta)(\alpha + \beta + 2), \end{aligned} \tag{0.0.23}$$

on obtient les relations de l'algèbre de Hahn :

$$\begin{aligned} [K_1, K_2] &= K_3, \\ [K_2, K_3] &= -2\{K_1, K_2\} + \delta_1, \\ [K_3, K_1] &= -2{K_1}^2 - 4K_2 + \delta_2, \end{aligned} \tag{0.0.24}$$

avec paramètres

$$\begin{aligned} \delta_1 &= \tfrac{1}{2}(2N + \alpha + \beta + 2)(\alpha^2 - \beta^2), \\ \delta_2 &= \tfrac{1}{2}(\alpha - \beta)^2 + 2N(2N + \alpha + \beta + 2). \end{aligned} \tag{0.0.25}$$

L'algèbre de Hahn est une algèbre quadratique qui apparaît dans une multitude de contextes et a été également vue dans l'exemple de l'oscillateur singulier 2D à la section précédente.

Tous ces exemples ont pour but de faire valoir que les algèbres quadratiques (ou cubiques, même polynomiales) sont des structures d'intérêt en physique qui apparaissent également dans l'étude des fonctions spéciales en mathématiques.

Les algèbres de type Askey–Wilson

Les polynômes orthogonaux hypergéométriques et hypergéométriques basiques sont respectivement classifiés dans le tableau d'Askey [Figure 1] et le q-tableau d'Askey [Figure 2]. Un certain nombre de familles -1 de polynômes orthogonaux obtenues des familles du

q-tableau d'Askey par des limites $q \to -1$ particulières ont été définies et constituent le -1-tableau d'Askey.

Toutes ces familles de polynômes sont bispectrales, c'est-à-dire qu'elles satisfont à la propriété de bispectralité introduite précédemment. Pour chaque famille, il est donc possible de considérer la paire bispectrale d'opérateurs de récurrence et aux différences et de déterminer les relations de commutation qu'ils respectent. Les relations obtenues définissent une algèbre abstraite qu'on nomme en fonction de la famille de polynômes à partir de laquelle elle a été obtenue. Dépendemment de la famille étudiée, les relations de commutation s'expriment linéairement (par exemple, pour les familles de Meixner, Krawtchouk et Charlier), quadratiquement (familles de Hahn, Racah) ou cubiquement (famille d'Askey–Wilson) en termes des générateurs.

Le cas le plus général est obtenu en considérant les polynômes d'Askey–Wilson et mène à l'algèbre d'Askey–Wilson, baptisée et obtenue pour la première fois par Alexei Zhedanov en [6] et dont l'origine historique a été racontée dans le Prologue.

Les objets autour desquels gravitent tous les travaux de cette thèse sont ce que j'appelle les *algèbres de type Askey–Wilson*.

Définition. *Dans cette thèse, on entendra par* ***algèbres de type Askey–Wilson*** *chacune des algèbres abstraites (ainsi que leurs extensions centrales) obtenues en calculant les relations de commutation de la paire d'opérateurs de bispectralité associée à une famille de polynômes orthogonaux du q-tableau d'Askey ou de ses limites. Ceci comprend les familles du tableau d'Askey et du -1-tableau d'Askey.*

D'autres algèbres de type Askey–Wilson qui apparaîtront dans cette thèse sont les algèbres de q-Hahn, Racah, Bannai–Ito, duale -1-Hahn, ainsi que certaines de leurs extensions centrales.

Comme les exemples de l'algèbre de Hahn et de l'oscillateur singulier 2D l'illustrent bien, ces algèbres de type Askey–Wilson sont omniprésentes et connectent des domaines variés en physique et en mathématiques. En général, quand ces algèbres apparaissent, les polynômes associés sont également présents. En plus de leur rôle comme algèbre de symétrie de systèmes physiques et de leur présence en théorie des polynômes orthogonaux, ces algèbres apparaissent en théorie des représentations dans l'étude du recouplement de représentations irréductibles des algèbres $U_q(\mathfrak{sl}_2)$, $\mathfrak{osp}(1|2)$ et $\mathfrak{sl}_2$. Sans être exhaustif, on peut rajouter que : ces algèbres sont des cas particuliers des algèbres de Painlevé, appartiennent à la classe des algèbres de Calabi–Yau, elles peuvent être vues comme des troncations de l'algèbre de q-Onsager, elles peuvent être obtenues par l'équation de réflexion dans le contexte de modèles intégrables et du formalisme de la matrice R, elles apparaissent dans la classification des paires de Leonard en combinatoire algébrique, elles sont reliées aux algèbres de Hecke doublement affines, elles sont reliées à la dualité de Schur–Weyl quantique, elles apparaissent dans l'étude des modèles ASEP et elles sont présentes en théorie des noeuds. Plus de détails et des références sont donnés dans l'introduction de l'article de revue *The Askey–Wilson algebra and its avatars* [18] présenté au Chapitre 15. Leurs nombreuses apparitions dans des domaines aussi variés permettent de connecter ces domaines et sont un signe indéniable de la richesse de ces algèbres.

Structure de cette thèse

Tous les travaux dans cette thèse tournent autour de ces algèbres de type Askey–Wilson et peuvent être séparés assez naturellement en trois parties. Chaque partie débutera par une mise en contexte qui précèdera les articles.

La première partie porte sur la dualité de Howe et les algèbres de type Askey–Wilson. Les articles dans la première partie utilisent le concept de paires duales en théorie des représentations (également connu sous le nom de dualité de Howe) afin de fournir deux interprétations duales aux algèbres de type Askey–Wilson. Un modèle superintégrable de mécanique quantique superconforme qui a été étudié dans la foulée de ces travaux est également présenté et solutionné.

La deuxième partie porte sur des structures algébriques qui sont en un sens plus fondamentales que les algèbres de type Askey–Wilson. Tout comme l'algèbre d'Askey–Wilson a été obtenue à l'origine à partir des opérateurs de bispectralité, ces nouvelles structures algébriques sont obtenues à partir d'opérateurs associés à des familles de polynômes orthogonaux mais qui sont cette fois-ci plus élémentaires. Les algèbres obtenues sont des dégénérations de l'algèbre de Sklyanin. De nouvelles connexions entre les fonctions spéciales, les opérateurs de Heun algébriques et les algèbres de Sklyanin dans le domaine des systèmes intégrables sont mises de l'avant par l'introduction du concept d'opérateurs de Sklyanin–Heun.

La troisième partie porte sur la relation entre les algèbres de type Askey–Wilson et les centralisateurs. Des outils comme la matrice R universelle, les théorèmes fondamentaux de théorie des invariants et les algèbres de skein sont employés afin de préciser de quelle façon les algèbres de type Askey–Wilson peuvent être vues comme des centralisateurs. Également, on y obtient des ensembles de relations définissantes pour certains centralisateurs.

Contributions de l'auteur

Cette section a pour but de détailler les contributions de Julien Gaboriaud pour chaque article inclus dans la thèse, conformément aux exigences de la FESP. Chaque article est le fruit d'un effort collectif qui est parfois difficile à départager. Voici toutefois un sommaire tentatif des contributions des coauteurs pour chaque article. La rédaction réfère à l'écriture de la première version de l'article.

- Chapitre 1. Idée originale par LV, SV et AZ. Calculs par JG. Rédaction par JG et LV.
- Chapitre 2. Idée originale par LV, SV et AZ. Calculs par JG. Rédaction par JG et LV.
- Chapitre 3. Idée originale par LV, SV et AZ. Calculs par JG. Rédaction par JG et LV.
- Chapitre 4. Idée originale par LV, SV et AZ. Calculs par LF, JG et LV. Rédaction par JG et LV.
- Chapitre 5. Idée originale par LF et LV. Calculs par LF, JG et ER. Rédaction par LF, JG et LV.
- Chapitre 6. Idée originale par LF, ER et LV. Calculs par LF et JG. Rédaction par JG et LV.
- Chapitre 7. Idée originale par LV et SV. Calculs par JG. Rédaction par LV.
- Chapitre 8. Idée originale par JG et LV. Calculs par JG. Rédaction par JG et LV.
- Chapitre 9. Idée originale par LV. Calculs par PAB et JG. Rédaction par JG et LV.
- Chapitre 10. Idée originale par ST, LV et AZ. Calculs par JG et LV. Rédaction par JG et LV.
- Chapitre 11. Idée originale par LV et AZ. Calculs par JG. Collaboration de GB. Rédaction par JG.
- Chapitre 12. Idée originale par LV et AZ. Calculs par GB. Collaboration de JG. Rédaction par GB.
- Chapitre 13. Idée originale par LV et AZ. Calculs par AB, GB et AB. Rédaction par GB et JG.

- Chapitre 14. Idée originale par NC. Calculs par NC et MZ. Contribution de JG à certaines preuves. Rédaction par NC, LV et MZ.
- Chapitre 15. Idée originale par NC, JG, ER et LV. Calculs par NC, LF, JG et LPD. Rédaction par NC, JG, LPD.
- Chapitre 16. Idée originale par NC, JG et LV. Calculs par NC, JG et LPD. Rédaction par NC, JG.

Partie 1

Algèbres de type Askey–Wilson et dualité de Howe

Introduction

La théorie des paires duales réductives, également connue sous le nom de *dualité de Howe*, a été développée par Roger Howe dans les années 1970. Tout a commencé avec une prépublication de Howe en 1976, *Remarks on classical invariant theory* [19], qui a beaucoup circulé mais n'a été publié au final qu'en 1989. Dans cet article, Howe remarque qu'un nombre de résultats épars en théorie des invariants classiques peuvent être formulés de façon assez uniforme par l'introduction du concept de paires duales réductives.

Cette unification laisse envisager que ce nouveau concept est d'une grande importance; entres autres, il permet également de jeter un éclairage nouveau sur les identités de Capelli, la théorie des harmoniques sphériques, la cohomologie du groupe unitaire ainsi que diverses structures mathématiques typiques qui apparaissent en physique théoriques telles que l'équation d'onde, l'équation de Laplace, les équations de Dirac et les équations de Maxwell [20].

Howe développe initialement sa théorie pour les groupes classiques (GL_n, O_n, Sp_n, U_n). Par la suite, des travaux subséquents identifient des analogues des paires duales faisant intervenir des groupes quantiques [21–23] et des superalgèbres de Lie [24] et le concept de paire duale prend de l'importance en théorie des représentations.

Qu'est-ce qu'une paire duale? Voici tout d'abord une définition pour les groupes et algèbres de Lie.

Définition 0.0.1. *[20] Soit S un groupe de Lie et soient G, G' deux sous-groupes de S. Alors (G, G') forment une* ***paire duale*** *de sous-groupes de S is G est le centralisateur de G' dans S et vice-versa. Dans ce cas, on dit également que la paire $(\mathfrak{g}, \mathfrak{g}')$ d'algèbres de Lie des groupes (G, G') est une paire duale dans l'algèbre de Lie $\mathfrak{s}$ de S.*

Définition 0.0.2. *Lorsque les deux membres de la paire duale (G, G') sont réductibles, on dit que (G, G') forment une* ***paire duale réductive****.*

Si l'un des deux membres de la paire duale est un groupe compact, on obtient le résultat suivant.

Théorème 0.0.3. *[25, 26] Soit $\mathcal{H}$ un espace de Hilbert qui supporte des représentations de S. Alors, les actions de G et G' sur $\mathcal{H}$ commutent et sont complètement réductibles. L'espace de Hilbert admet la décomposition (sans multiplicité) suivante :*

$$\mathcal{H} = \bigoplus_{\lambda} \Gamma^{(\lambda)} \otimes \Gamma'^{(\lambda)}, \tag{0.0.26}$$

où les $\Gamma^{(\lambda)}$ et les $\Gamma'^{(\lambda)}$ sont des modules irréductibles de G et G' respectivement.

Ces définitions sont données ici pour les groupes et algèbres de Lie mais peuvent être étendues de façon appropriée pour des superalgèbres de Lie et des algèbres quantiques.

Il est intéressant de noter la similarité avec l'énoncé de la dualité de Schur–Weyl [27].

Théorème 0.0.4. *Les actions des groupes symmétrique S_k et du groupe général linéaire GL_n sur l'espace $(\mathbb{C}^n)^{\otimes k}$ commutent et cet espace se décompose en une somme directe sans* *multiplicité de produits tensoriels de modules irréductibles des deux groupes*

$$(\mathbb{C}^n)^{\otimes k} = \bigoplus_{\lambda} \pi_k^{(\lambda)} \otimes \rho_n^{(\lambda)}, \tag{0.0.27}$$

où $\pi_k^{(\lambda)}$ sont des représentations irréductibles de S_k étiquettées par des diagrammes de Young λ appropriés et les $\rho_n^{(\lambda)}$, des représentations irréductibles de GL_n également étiquettées par λ.

Cette similarité n'est pas une coïncidence : on peut utiliser l'énoncé de la dualité de Howe pour prouver l'énoncé de la dualité de Schur–Weyl [27]. Également, les Premier et Second Théorèmes Fondamentaux de la théorie des invariants sont des conséquences de cet énoncé [27]. Les avancées amenées par la dualité de Howe s'inscrivent dans l'héritage de Weyl et de ses travaux à l'interface des domaines de la physique théorique et de la théorie des représentations.

Tel que mentionné dans le Prologue, un nombre d'algèbres de type Askey–Wilson peuvent être réalisées comme commutants et dans ce cadre, leurs générateurs sont réalisés par des Casimirs intermédiaires. La dualité de Howe permet d'associer des représentations irréductibles de deux algèbres membres d'une paire duale réductive. De par le lemme de Schur, il n'est donc guère surprenant qu'au niveau algébrique, cette association des représentations irréductibles se traduise par une correspondance entre les Casimirs des deux algèbres qui étiquettent ces représentations irréductibles. C'est ce qui est utilisé ici. Grâce à cette correspondance, un énoncé portant sur le Casimir d'un membre de la paire duale peut être interprété de façon duale en terme du Casimir de l'autre membre de la paire (si on se place dans un contexte où la dualité de Howe prend place).

La question principale à laquelle répond à l'affirmative la première partie de cette thèse est:

Peut-on obtenir des interprétations duales (dans le sens de Howe) des algèbres de type Askey–Wilson?

L'astuce ici est de se placer dans une réalisation où la dualité de Howe prend place à plusieurs niveaux simultanément. Les Casimirs intermédiaires qui réalisent les algèbres de type Askey–Wilson comme centralisateurs sont tous mis simultanément en correspondance avec les Casimirs associés aux autres algèbres des paires duales. Par la suite, une réinterprétation de ces autres Casimirs donne lieu à une nouvelle interprétation (duale à la première) des algèbres de type Askey–Wilson.

Dans cette première partie de la thèse, des interprétations duales sont fournies pour les algèbres de Racah aux Chapitres 1 et 2, Bannai–Ito au Chapitre 3, Askey–Wilson au Chapitre 6, Hahn au Chapitre 4, dual -1 Hahn au Chapitre 8 et q-Hahn au Chapitre 5. De façon remarquable, l'astuce derrière la construction dans tous ces articles est en l'essence la même; cette idée s'applique autant pour les paires duales réductives faisant intervenir des algèbres de Lie que celles faisant intervenir des superalgèbres de Lie et des algèbres quantiques. L'article au Chapitre 9 présente cette observation tout en survolant l'ensemble des résultats énumérés ci-haut. Un article sur la mécanique quantique superconforme faisant intervenir l'algèbre duale -1 Hahn est également inclus au Chapitre 7 dans cette première partie et a été rédigé dans la foulée des travaux sur les interprétations duales de l'algèbre duale -1 Hahn.

Chapitre 1

The Racah algebra as a commutant and Howe duality

Par Julien Gaboriaud, Luc Vinet, Stéphane Vinet et Alexei Zhedanov.
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Abstract: The Racah algebra encodes the bispectrality of the eponym polynomials. It is known to be the symmetry algebra of the generic superintegrable model on the 2-sphere. It is further identified in the commutant of the $\mathfrak{o}(2) \oplus \mathfrak{o}(2) \oplus \mathfrak{o}(2)$ subalgebra of $\mathfrak{o}(6)$ in oscillator representations of the universal algebra of the latter. How this observation relates to the $\mathfrak{su}(1,1)$ Racah problem and the superintegrable model on the 2-sphere is discussed on the basis of the Howe duality associated to the pair $\big(\mathfrak{o}(6), \mathfrak{su}(1,1)\big)$.

Keywords: Racah algebra, commutant, Howe duality, superintegrability

1.1. Introduction

The Racah algebra $\mathcal{R}$ has three generators K_1, K_2, K_3 that are subjected to the relations:

$$\begin{aligned} [K_1, K_2] = K_3, \qquad\qquad & [K_2, K_3] = {K_2}^2 + \{K_1, K_2\} + dK_2 + e_1, \\ & [K_3, K_1] = {K_1}^2 + \{K_1, K_2\} + dK_1 + e_2, \end{aligned} \tag{1.1.1}$$

where $[A, B] = AB - BA$, $\{A, B\} = AB + BA$ and d, e_1, e_2 are central. This algebra has appeared in many guises and we here add to its understanding with the identification of a new realization. We shall indeed show that $\mathcal{R}$ arises in the commutant of the $\mathfrak{o}(2) \oplus \mathfrak{o}(2) \oplus \mathfrak{o}(2)$ subalgebra of $\mathfrak{o}(6)$ in the representations of $U(\mathfrak{o}(6))$ on the Hilbert space of states of six oscillators.

The Racah algebra was introduced as an encoding of the bispectrality properties of the Racah polynomials [1], see for example [2]. It is also intimately connected to the recouplings of $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ representations [3, 4] since the Racah coefficients for these Lie algebras are expressed in terms of the Racah polynomials. We shall review this last aspect in the next section because it is relevant for the results we want to present in this paper.

The Racah algebra has also been found [5] to be the symmetry algebra of the generic superintegrable model on the two-sphere with Hamiltonian H given by

$$H = \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2 + \frac{a_1}{{x_1}^2} + \frac{a_2}{{x_2}^2} + \frac{a_3}{{x_3}^2} \tag{1.1.2}$$

where

$$\mathcal{J}_k = \epsilon_{kij}\ x_i \frac{\partial}{\partial x_j}, \qquad i,\, j,\, k = 1,\, 2,\, 3, \qquad {x_1}^2 + {x_2}^2 + {x_3}^2 = 1 \tag{1.1.3}$$

and a_1, a_2, a_3 are parameters. We shall also bring this fact to bear on our discussion.

The Racah algebra has further been shown to have a natural embedding in $\mathfrak{su}(2)$ [6, 7] and to be related to distance-regular graphs [8]. It also extends to arbitrary ranks [9] with a connection to multivariate Racah polynomials of the Tratnik type [10, 11] and higher dimensional superintegrable models [12]. We shall here add to this the commutant realization mentioned above and explain how Howe duality [13–15] relates this observation to the fact that $\mathcal{R}$ is also in the commutant in $U(\mathfrak{su}(1,1)^{\otimes 3})$ of the addition of three $\mathfrak{su}(1,1)$.

The paper will unfold as follows. As already indicated, we shall review in Section 1.2 the occurrence of the Racah algebra in the recoupling of three irreducible representations of $\mathfrak{su}(1,1)$. This is where $\mathcal{R}$ will be in the commutant of $\mathfrak{su}(1,1)$ in $U(\mathfrak{su}(1,1)^{\otimes 3})$ with the intermediate Casimir operators as generators. Our main result will be the object of Section 1.3 where the connection with the Racah algebra and the Lie algebra $\mathfrak{o}(6)$ will be made. The link between this last incarnation of $\mathcal{R}$ as the commutant of the maximal Abelian subalgebra $\mathfrak{o}(2) \oplus \mathfrak{o}(2) \oplus \mathfrak{o}(2)$ of $\mathfrak{o}(6)$ in the universal algebra of the latter and the realization stemming from the Racah problem of $\mathfrak{su}(1,1)$ will be discussed in Section 2.4. This will be done by considering six harmonic oscillators and the dual reductive pair $\big(\mathfrak{o}(6), \mathfrak{sp}(2)\big)$ in $\mathfrak{sp}(12)$ that acts on the Hilbert space of their collective states. This Howe duality will be invoked to put in correspondance the $\mathfrak{o}(6)$ and $\mathfrak{sp}(2) \simeq \mathfrak{su}(1,1)$ pictures for the Racah algebra. In the last section, we shall complete the analysis by rederiving the results pertaining to the symmetry of the generic superintegrable model on the 2-sphere. To that end, we shall carry out the dimensional reduction of the six-dimensional (6D) oscillator under the action of $O(2) \otimes O(2) \otimes O(2)$ to obtain the Hamiltonian from the total Casimir operator for the addition of six metaplectic representations of $\mathfrak{sp}(2)$ and the constants of motion from the proper intermediate Casimirs with the knowledge that they realize the Racah algebra. Summary and outlook will form the Conclusion.

1.2. The Racah problem for $\mathfrak{su}(1,1)$ and the Racah algebra

The Lie algebra $\mathfrak{su}(1,1)$ has generators $J_0, J_\pm$ that obey the following commutation relations:

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -2J_0 \tag{1.2.1}$$

and its Casimir operator is given by:

$$C = {J_0}^2 - J_+J_- - J_0. \tag{1.2.2}$$

Consider now the addition of 3 irreducible representations of $\mathfrak{su}(1,1)$ for which the initial Casimir operators take values $C^{(i)} = \lambda_i$, $i = 1,2,3$, and let us write

$$J_0^{(123)} = J_0^{(1)} + J_0^{(2)} + J_0^{(3)}, \qquad J_\pm^{(123)} = J_\pm^{(1)} + J_\pm^{(2)} + J_\pm^{(3)} \tag{1.2.3}$$

with the superindex denoting on which of the three factors in $\mathfrak{su}(1,1)^{\otimes 3}$ the operator is acting. In addition to the initial Casimir operator $C^{(i)}$ we also have the intermediate Casimir operators associated to the addition of two representations

$$C^{(ij)} = \left(J_0^{(i)} + J_0^{(j)}\right)^2 - \left(J_+^{(i)} + J_+^{(j)}\right)\left(J_-^{(i)} + J_-^{(j)}\right) - \left(J_0^{(i)} + J_0^{(j)}\right) \tag{1.2.4}$$

with $(ij) = (12), (23), (31)$ and also the total Casimir operator, given by:

$$C^{(123)} = \left(J_0^{(123)}\right)^2 - J_+^{(123)}J_-^{(123)} - J_0^{(123)}. \tag{1.2.5}$$

Take $C^{(123)} = \lambda_4$. Let $V^{(\lambda_i)}$ denote irreducible representation spaces of $\mathfrak{su}(1,1)$, and look at the decomposition of $V^{(\lambda_1)} \otimes V^{(\lambda_2)} \otimes V^{(\lambda_3)}$ in irreducibles $V^{(\lambda_4)}$. The Racah problem for $\mathfrak{su}(1,1)$ is about determining the unitary transformations between the bases corresponding to the steps $(1\oplus 2)\oplus 3$ and $1\oplus(2\oplus 3)$ that respectively diagonalize the intermediate Casimirs $C^{(12)}$ and $C^{(23)}$

$$\begin{aligned} C^{(12)} &= 2J_0^{(1)}J_0^{(2)} - \left(J_+^{(1)}J_-^{(2)} + J_-^{(1)}J_+^{(2)}\right) + \lambda_1 + \lambda_2, \\ C^{(23)} &= 2J_0^{(2)}J_0^{(3)} - \left(J_+^{(2)}J_-^{(3)} + J_-^{(2)}J_+^{(3)}\right) + \lambda_2 + \lambda_3. \end{aligned} \tag{1.2.6}$$

These intermediate Casimir operators realize the Racah algebra $\mathcal{R}$, since the relations (1.1.1) are satisfied by $K_1 = -\frac{1}{2}C^{(12)}$ and $K_2 = -\frac{1}{2}C^{(23)}$ with $d = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$, $e_1 = \frac{1}{4}(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)$, $e_2 = \frac{1}{4}(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_3)$.

1.3. The Racah algebra and $\mathfrak{o}(6)$

We will now observe that $\mathcal{R}$ is in the commutant in $U(\mathfrak{o}(6))$ of a subalgebra of $\mathfrak{o}(6)$ in the oscillator representation. The algebra $\mathfrak{o}(6)$ has 15 generators $L_{\mu\nu} = -L_{\nu\mu}$, $\mu, \nu = 1, ..., 6$

obeying

$$[L_{\mu\nu}, L_{\rho\sigma}] = \delta_{\nu\rho}L_{\mu\sigma} - \delta_{\nu\sigma}L_{\mu\rho} - \delta_{\mu\rho}L_{\nu\sigma} + \delta_{\mu\sigma}L_{\nu\rho} \tag{1.3.1}$$

and it possesses the following quadratic Casimir:

$$\mathcal{C} = \sum_{\mu<\nu} {L_{\mu\nu}}^2. \tag{1.3.2}$$

We will use the realization

$$L_{\mu\nu} = \xi_\mu \frac{\partial}{\partial \xi_\nu} - \xi_\nu \frac{\partial}{\partial \xi_\mu}, \qquad \mu \neq \nu, \qquad \mu, \nu = 1, ..., 6. \tag{1.3.3}$$

Pick the $\mathfrak{o}(2) \oplus \mathfrak{o}(2) \oplus \mathfrak{o}(2)$ subalgebra of $\mathfrak{o}(6)$ generated by the commutative set $\{L_{12},\, L_{34},\, L_{56}\}$. We want to focus on the commutant of this Abelian subalgebra in $U(\mathfrak{o}(6))$. It is fairly easy to see that it will be generated by the following two invariants:

$$K_1 = \frac{1}{8}\Big({L_{12}}^2 + {L_{13}}^2 + {L_{14}}^2 + {L_{23}}^2 + {L_{24}}^2 + {L_{34}}^2\Big) \tag{1.3.4}$$

$$K_2 = \frac{1}{8}\Big({L_{34}}^2 + {L_{35}}^2 + {L_{36}}^2 + {L_{45}}^2 + {L_{46}}^2 + {L_{56}}^2\Big). \tag{1.3.5}$$

Define K_3 by $[K_1, K_2] = K_3$. One then finds that

$$\begin{aligned} K_3 = \frac{1}{16}\Big(&{L_{35}}^2 + {L_{36}}^2 + {L_{45}}^2 + {L_{46}}^2 - {L_{13}}^2 - {L_{14}}^2 - {L_{23}}^2 - {L_{24}}^2 - {L_{15}}^2 - {L_{16}}^2 - {L_{25}}^2 - {L_{26}}^2 \\ &+ L_{13}L_{35}L_{15} + L_{13}L_{36}L_{16} + L_{23}L_{35}L_{25} + L_{23}L_{36}L_{26} + L_{14}L_{45}L_{15} \\ &+ L_{14}L_{46}L_{16} + L_{24}L_{45}L_{25} + L_{24}L_{46}L_{26}\Big). \end{aligned} \tag{1.3.6}$$

Working out the commutation relations of K_3 with K_1 and K_2, we find that they correspond to those of a central extension of the Racah algebra with L_{12}, L_{34}, L_{45} and $\mathcal{C}$ playing role of structure constants. Indeed one obtains

$$\begin{aligned} &[K_1, K_2] = K_3 \\ &[K_2, K_3] = {K_2}^2 + \{K_1, K_2\} - \tfrac{1}{8}K_2\Big(\mathcal{C}+{L_{12}}^2+{L_{34}}^2+{L_{56}}^2\Big) - \tfrac{1}{64}\Big(\mathcal{C} - {L_{12}}^2 - 4\Big)\Big({L_{34}}^2-{L_{56}}^2\Big) \\ &[K_3, K_1] = {K_1}^2 + \{K_1, K_2\} - \tfrac{1}{8}K_2\Big(\mathcal{C}+{L_{12}}^2+{L_{34}}^2+{L_{56}}^2\Big) - \tfrac{1}{64}\Big(\mathcal{C} - {L_{56}}^2 - 4\Big)\Big({L_{34}}^2-{L_{12}}^2\Big) \end{aligned} \tag{1.3.7}$$

where the parameters $d = -\frac{1}{8}(\mathcal{C} + {L_{12}}^2 + {L_{34}}^2 + {L_{56}}^2)$, $e_1 = -\frac{1}{64}(\mathcal{C} - {L_{12}}^2 - 4)({L_{34}}^2 - {L_{56}}^2)$, $e_2 = -\frac{1}{64}(\mathcal{C} - {L_{56}}^2 - 4)({L_{34}}^2 - {L_{12}}^2)$ are obviously central.

Let us indicate how this result is obtained. Take for example the commutator $[K_2, K_3]$ ($[K_3, K_1]$ is treated similarly). On the one hand, it is readily observed that the r.h.s in (1.3.7) only contains terms of the form ${L_{\mu\nu}}^2$. On the other hand, using the $\mathfrak{o}(6)$ commutation relations (1.3.1) and the explicit expressions (9.3.3), (1.3.6) for K_2, K_3, one obtains:

$$\begin{aligned}
128[K_2,K_3] = &\{{L_{35}}^2,{L_{13}}^2\}+\{{L_{35}}^2,{L_{23}}^2\}+\{{L_{36}}^2,{L_{13}}^2\}+\{{L_{36}}^2,{L_{23}}^2\}\\
&+\{{L_{45}}^2,{L_{14}}^2\}+\{{L_{45}}^2,{L_{24}}^2\}+\{{L_{46}}^2,{L_{14}}^2\}+\{{L_{46}}^2,{L_{24}}^2\}\\
&-\{{L_{35}}^2,{L_{15}}^2\}-\{{L_{35}}^2,{L_{25}}^2\}-\{{L_{36}}^2,{L_{16}}^2\}-\{{L_{36}}^2,{L_{26}}^2\}\\
&-\{{L_{45}}^2,{L_{15}}^2\}-\{{L_{45}}^2,{L_{25}}^2\}-\{{L_{46}}^2,{L_{16}}^2\}-\{{L_{46}}^2,{L_{26}}^2\}\\
&+4({L_{13}}^2+{L_{23}}^2+{L_{14}}^2+{L_{24}}^2-{L_{15}}^2-{L_{25}}^2-{L_{16}}^2-{L_{26}}^2)\\
&-2L_{15}L_{35}L_{36}L_{16}-2L_{13}L_{35}L_{56}L_{16}-2L_{25}L_{35}L_{36}L_{26}-2L_{23}L_{35}L_{56}L_{26}\\
&-2L_{16}L_{36}L_{35}L_{15}+2L_{13}L_{36}L_{56}L_{15}-2L_{26}L_{36}L_{35}L_{25}+2L_{23}L_{36}L_{56}L_{25}\\
&-2L_{15}L_{45}L_{46}L_{16}-2L_{14}L_{45}L_{56}L_{16}-2L_{25}L_{45}L_{46}L_{26}-2L_{24}L_{45}L_{56}L_{26}\\
&-2L_{16}L_{46}L_{45}L_{15}+2L_{14}L_{46}L_{56}L_{15}-2L_{26}L_{46}L_{45}L_{25}+2L_{24}L_{46}L_{56}L_{25}\\
&+2L_{14}L_{45}L_{35}L_{13}-2L_{14}L_{34}L_{35}L_{15}+2L_{24}L_{45}L_{35}L_{23}-2L_{24}L_{34}L_{35}L_{25}\\
&+2L_{14}L_{46}L_{36}L_{13}-2L_{14}L_{34}L_{36}L_{16}+2L_{24}L_{46}L_{36}L_{23}-2L_{24}L_{34}L_{36}L_{26}\\
&+2L_{13}L_{35}L_{45}L_{14}+2L_{13}L_{34}L_{45}L_{15}+2L_{23}L_{35}L_{45}L_{24}+2L_{23}L_{34}L_{45}L_{25}\\
&+2L_{13}L_{36}L_{46}L_{14}+2L_{13}L_{34}L_{46}L_{16}+2L_{23}L_{36}L_{46}L_{24}+2L_{23}L_{34}L_{46}L_{26}.
\end{aligned} \tag{1.3.8}$$

The terms of the type $L_{\mu\nu}L_{\rho\nu}L_{\rho\sigma}L_{\mu\sigma}$ thus need to be re-expressed. The key to rewriting them with factors involving only the ${L_{\mu\nu}}^2$'s is to make use of the identity

$$L_{\mu\nu}L_{\rho\sigma}+L_{\mu\rho}L_{\sigma\nu}+L_{\mu\sigma}L_{\nu\rho}=0 \tag{1.3.9}$$

which is directly proved in the realization (1.3.3) (and which in fact remains true for Dunkl angular momenta ([16])), and to also take its square, which yields

$$\{L_{\mu\nu}L_{\rho\sigma},\ L_{\mu\rho}L_{\nu\sigma}\}={L_{\mu\nu}}^2{L_{\rho\sigma}}^2+{L_{\mu\rho}}^2{L_{\nu\sigma}}^2-{L_{\mu\sigma}}^2{L_{\nu\rho}}^2. \tag{1.3.10}$$

Combining these two identities and calling upon other elementary formulas such as

$$[{L_{\sigma\mu}}^2+{L_{\sigma\nu}}^2,\ L_{\mu\nu}]=0 \tag{1.3.11}$$

allows one to equate (1.3.8) with the r.h.s in (1.3.7), which completes the proof.

1.4. The Racah algebra and Howe duality

We shall now show how the result in the previous section can be explained by identifying the Howe pair in play in the system we have considered. In the last two sections we showed that the Racah algebra is in the commutant of $\mathfrak{su}(1,1)$ in $U(\mathfrak{su}(1,1)^{\otimes 3})$ and of $\mathfrak{o}(2)\oplus\mathfrak{o}(2)\oplus\mathfrak{o}(2)$ in $U(\mathfrak{o}(6))$. The connection between these two observations can be traced to Howe duality.

It is known [15] that $\mathfrak{o}(n)$ and $\mathfrak{sp}(2d)$ form a dual pair in $\mathfrak{sp}(2dn)$, i.e. these two subgroups are mutual commutants. This implies that $\mathfrak{o}(6)$ and $\mathfrak{sp}(2) \simeq \mathfrak{su}(1,1)$ have dual actions on the Hilbert space of six oscillators. That means that their irreducible representations can be paired and this can be done through the Casimirs.

Consider the following 6 oscillator realizations of $\mathfrak{sp}(2)$:

$$J_+^{(\mu)} = \frac{1}{2}\xi_\mu^2, \qquad J_-^{(\mu)} = \frac{1}{2}\frac{d^2}{d\xi_\mu^2}, \qquad J_0^{(\mu)} = \frac{1}{2}\xi_\mu \frac{d}{d\xi_\mu} + \frac{1}{4}, \qquad \mu = 1,2,3,4,5,6. \tag{1.4.1}$$

We shall add these six representations by first coupling the three pairs $(\mu\nu) = (12), (34), (56)$ and shall write:

$$J^{(\mu\nu)} = J^{(\mu)} + J^{(\nu)}, \qquad J^{(123456)} = J^{(12)} + J^{(34)} + J^{(56)}. \tag{1.4.2}$$

Recall that the $\mathfrak{sp}(2)$ Casimir is $C = {J_0}^2 - J_+J_- - J_0$. The connection between the Casimirs of combined metaplectic representations with elements in $U(\mathfrak{o}(6))$ is readily obtained:

$$C^{(\mu\nu)} = -\tfrac{1}{4}({L_{\mu\nu}}^2 + 1), \tag{1.4.3}$$

$$C^{(123456)} = -\tfrac{1}{4}\Big(\sum_{\mu<\nu} {L_{\mu\nu}}^2 - 3\Big) = -\tfrac{1}{4}\mathcal{C} + \tfrac{3}{4}, \tag{1.4.4}$$

$$C^{(1234)} = -\tfrac{1}{4}({L_{12}}^2 + {L_{13}}^2 + {L_{14}}^2 + {L_{23}}^2 + {L_{24}}^2 + {L_{34}}^2) = -2K_1, \tag{1.4.5}$$

$$C^{(3456)} = -\tfrac{1}{4}({L_{34}}^2 + {L_{35}}^2 + {L_{36}}^2 + {L_{45}}^2 + {L_{46}}^2 + {L_{56}}^2) = -2K_2. \tag{1.4.6}$$

In addition to observing that $C^{(123456)}$ and the Casimir $\mathcal{C}$ of $\mathfrak{o}(6)$ are affinely related, we see that the intermediate $\mathfrak{sp}(2)$ Casimirs correspond to the generators of the commutant of $\{L_{12}, L_{34}, L_{56}\}$ in $U(\mathfrak{o}(6))$. We know from Section 1.3 that the intermediate $\mathfrak{sp}(2)$ Casimirs realize the commutation relations of the Racah algebra. This will hence be the case also for the commutant generators and we have here our duality connection.

1.5. The Racah algebra and the generic superintegrable model on S^2

We can now complete the picture by performing the dimensional reduction from $\mathbb{R}^6$ to $\mathbb{R}^+ \times S^2$ [17–19] to obtain the generic superintegrable model (introduced in Section 2.1) with Hamiltonian H and to recover as well its symmetries. Make the following change of variables:

$$\begin{aligned} \xi_{2i-1} &= x_i \cos\theta_i, \\ \xi_{2i} &= x_i \sin\theta_i, \end{aligned} \qquad L_{2i-1,2i} = \xi_{2i-1}\frac{\partial}{\partial \xi_{2i}} - \xi_{2i}\frac{\partial}{\partial \xi_{2i-1}} = \frac{\partial}{\partial \theta_i}, \qquad i = 1,2,3. \tag{1.5.1}$$

Eliminate the ignorable $\theta_i s$ by separating these variables and setting ${L_{2i-1,2i}}^2 \sim {k_i}^2$. After performing the gauge transformation $\mathcal{O} \mapsto \widetilde{\mathcal{O}} = x_i^{\frac{1}{2}}\ \mathcal{O}\ x_i^{-\frac{1}{2}}$ one obtains

$$\widetilde{J_+}^{(2i-1,2i)} = \frac{1}{2}{x_i}^2, \quad \widetilde{J_-}^{(2i-1,2i)} = \frac{1}{2}\left(\frac{d^2}{d{x_i}^2} + \frac{a_i}{{x_i}^2}\right), \quad \widetilde{J_0}^{(2i-1,2i)} = \frac{1}{2}\left(x_i\frac{d}{dx_i} + \frac{1}{2}\right), \tag{1.5.2}$$

with $a_i = {k_i}^2 + \frac{1}{4}$. The reduced Casimirs

$$\widetilde{C}^{(\mu,\nu,\rho,\sigma)} = (\widetilde{J_0})^2 - \widetilde{J_+}\widetilde{J_-} - \widetilde{J_0} \qquad \text{with} \qquad \widetilde{J} = \widetilde{J}^{(\mu,\nu)} + \widetilde{J}^{(\rho,\sigma)}, \tag{1.5.3}$$

are easily computed and have the following expressions:

$$\begin{aligned}
\widetilde{C}^{(1234)} &= -\frac{1}{4}\left[{\mathcal{J}_3}^2 + a_1\frac{{x_2}^2}{{x_1}^2} + a_2\frac{{x_1}^2}{{x_2}^2} + a_1 + a_2 + 1\right], \\
\widetilde{C}^{(3456)} &= -\frac{1}{4}\left[{\mathcal{J}_1}^2 + a_2\frac{{x_3}^2}{{x_2}^2} + a_3\frac{{x_2}^2}{{x_3}^2} + a_2 + a_3 + 1\right], \\
\widetilde{C}^{(1256)} &= -\frac{1}{4}\left[{\mathcal{J}_2}^2 + a_3\frac{{x_1}^2}{{x_3}^2} + a_1\frac{{x_3}^2}{{x_1}^2} + a_1 + a_3 + 1\right], \qquad \mathcal{J}_k = \epsilon_{kij}\, x_i\frac{d}{dx_j}.
\end{aligned} \tag{1.5.4}$$

Using $\widetilde{J}^{(123456)} = \widetilde{J}^{(12)} + \widetilde{J}^{(34)} + \widetilde{J}^{(56)}$ it is seen that

$$\widetilde{C}^{(123456)} = \widetilde{C}^{(1234)} + \widetilde{C}^{(3456)} + \widetilde{C}^{(1256)} - \widetilde{C}^{(12)} - \widetilde{C}^{(34)} - \widetilde{C}^{(56)}. \tag{1.5.5}$$

Since $\widetilde{C}^{(2i-1,2i)} = -\frac{1}{4}\left(a_i + \frac{3}{4}\right)$ are constants, the invariant can be taken to be given by the sum of the first three terms in $\widetilde{C}^{(123456)}$. Assuming ${x_1}^2 + {x_2}^2 + {x_3}^2 = 1$, this is recognized to be the Hamiltonian (2.1.1) of the generic model (up to an affine transformation). The Casimirs are essentially the conserved quantities:

$$Q_i = {\mathcal{J}_i}^2 + a_j\frac{{x_k}^2}{{x_j}^2} + a_k\frac{{x_j}^2}{{x_k}^2}, \qquad i,j,k \in \{1,2,3\} \quad \text{cyclic} \tag{1.5.6}$$

and they generate $\mathcal{R}$ which is hence the symmetry algebra.

1.6. Conclusion

The Racah algebra $\mathcal{R}$ embodies the theory of the Racah polynomials. The ubiquity of these orthogonal polynomials explains the diverse roles that $\mathcal{R}$ plays and motivates, with an eye to generalizations, the examination of all facets of this algebra. It is hence quite nice that we could find a new characterization of $\mathcal{R}$. Put in simple terms, our findings can be summarized as follows. We have shown that the Racah algebra is realized by polynomials in the generalized angular momenta in six dimensions that are invariant under rotations in three non-intersecting planes. We have further indicated that this picture is dual, in the sense of Howe, to the one where the Racah algebra is realized by the Casimir operators in the addition of three irreducible representations of $\mathfrak{su}(1,1)$. This was done by exploiting the correspondence between the representations of $\mathfrak{o}(6)$ and those of $\mathfrak{sp}(2)$ acting on the state space of a 6D harmonic oscillator. The analysis provided an illuminating context within which the Racah symmetry of the generic superintegrable model on the 2-sphere is naturally obtained by dimensional reduction. This suggests numerous potential extensions.

It should be possible to extend all the results of this paper to higher dimensions, namely, to Racah algebras with rank superior to one. These algebras have already been introduced using the recoupling model, that is, as the ones realized by the various Casimir operators arising the the addition of four and more $\mathfrak{su}(1,1)$ representations [9]. In view of our observations, it is natural to expect that these could be shown to be in duality with commutants of the n-torus in $\mathfrak{o}(2n)$ with $n \geq 4$.

There are also two other important rank one algebras that share properties with the Racah algebra: the Askey–Wilson (AW) and the Bannai–Ito (BI) algebras. The AW algebra [20] accounts for the bispectral properties of the Askey–Wilson polynomials. It is the object of much attention and like the Racah algebra it arises in particular in a Racah problem, this time for the quantum algebra $U_q(\mathfrak{sl}(2))$ [21]. The Bannai–Ito polynomials are most simply defined as a $q \to -1$ limit of the Askey–Wilson polynomials [22]. They are also bispectral and the BI algebra [23] encodes these defining features. The Racah problem of relevance in this case is the one associated to the Lie superalgebra $\mathfrak{osp}(1|2)$ leading to a realization of the BI algebra again, in terms of the intermediate Casimir operators [24]. The BI algebra has also been shown to be the symmetry algebra of a superintegrable model on the two-sphere involving reflection operators [25] as well as of a Dirac–Dunkl equation in $\mathbb{R}^3$ [26]. This is observed by realizing the three $\mathfrak{osp}(1,2)$ that are added in terms of Dunkl operators [27] and using a Clifford algebra in the latter problem. For a review of the BI algebra and its applications see [28]. For both the AW and BI cases, it would be quite interesting to determine if there is a Howe duality setting that would allow to develop for these algebras a commutant interpretation similar to the one that we have found for the Racah algebra. How dimensional reduction would then operate would be revealing with respect to superintegrable models. The AW and BI algebras of higher ranks could then as well lend themselves to similar descriptions. We plan to examine all these questions in the near future.

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Chapitre 2

The generalized Racah algebra as a commutant

Par Julien Gaboriaud, Luc Vinet, Stéphane Vinet et Alexei Zhedanov.
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Abstract: The Racah algebra $R(n)$ of rank $(n-2)$ is realized in the commutant of the $\mathfrak{o}(2)^{\oplus n}$ subalgebra of $\mathfrak{o}(2n)$ in oscillator representations of the universal algebra of $\mathfrak{o}(2n)$. This result is shown to be related in a Howe duality context to the realization of $R(n)$ in the algebra of Casimir operators arising in recouplings of n copies of $\mathfrak{su}(1,1)$. These observations provide a natural framework to carry out the derivation by dimensional reduction of the generic superintegrable model on the $(n-1)$ sphere which is invariant under $R(n)$.

2.1. Introduction

The Racah algebra $R(3)$ of rank 1 [1, 2] encodes the bispectrality properties of the Racah polynomials [3] and is the symmetry algebra of the generic superintegrable model on the 2-sphere with Hamiltonian H given by [4]

$$H = \sum_{1\leq i<j\leq 3} {\mathcal{J}_{ij}}^2 + \sum_{i=1}^{3} \frac{a_i}{{x_i}^2} \tag{2.1.1}$$

where

$$\mathcal{J}_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \qquad {x_1}^2 + {x_2}^2 + {x_3}^2 = 1 \tag{2.1.2}$$

and a_1, a_2, a_3 are parameters. For a review the reader is referred to [5]. Of particular relevance is the fact that $R(3)$ was seen to be in the commutant in $U(\mathfrak{su}(1,1)^{\otimes 3})$ of the embedding of $\mathfrak{su}(1,1)$ in the three-fold tensor product of this algebra with itself, or in other words, that it is generated by the invariant operators arising in this Racah problem. This observation provided a way to generalize $R(3)$ to Racah algebras of arbitrary rank $(n-2)$ [6] by extending the picture to n factors and identifying structure relations between the various Casimir operators arising in the possible recouplings. It follows that $R(n)$ thus defined is the symmetry algebra of the superintegrable model on the $(n-1)$-sphere obtained by straighforwardly extending to n variables the model on S^2 defined above.

We have found recently [7] that $R(3)$ can be realized in the commutant of the subalgebra $\mathfrak{o}(2) \oplus \mathfrak{o}(2) \oplus \mathfrak{o}(2) \subset \mathfrak{o}(6)$ in oscillator representations of the enveloping algebra of $\mathfrak{o}(6)$. We further observed that this description of $R(3)$ could be related to the one associated to the Racah problem for $\mathfrak{su}(1,1)$ through the Howe duality corresponding to the pair $(\mathfrak{o}(6), \mathfrak{su}(1,1))$. This provided a natural background for obtaining the superintegrable Hamiltonian (2.1.1) with $R(3)$ as symmetry algebra, under the dimensional reduction of a six-dimensional harmonic oscillator problem. We here wish to indicate how these results extend for $R(n)$, that is, for arbitrary ranks and dimensions.

The paper is structured as follows. In Section 2.2, we review how the Racah algebra $R(n)$ is obtained from the Casimir operators in the n-fold tensor product of $\mathfrak{su}(1,1)$ Lie algebras. Structure relations satisfied by these Casimirs are provided. In Section 2.3, we show that the generators of the commutant of the $\mathfrak{o}(2)^{\oplus n}$ subalgebra of $\mathfrak{o}(2n)$ satisfy the relations of $R(n)$. In Section 2.4, we invoke Howe duality to explain how the pairings between representations of $\mathfrak{o}(2n)$ and those of $\mathfrak{su}(1,1)$ underpin the connection between the tensorial and the commutant pictures of $R(n)$. How the $R(n)$-invariant superintegrable model on $S^{(n-1)}$ is obtained from an n-dimensional harmonic oscillator by modding out the action of the n-torus group is described in Section 2.5 and conclusions form Section 4.7.

2.2. The generalized Racah algebra and tensor products of $\mathfrak{su}(1,1)$

Let us recall how the generalized Racah algebra $R(n)$ is defined from the n-fold tensor product of $\mathfrak{su}(1,1)$. The $\mathfrak{su}(1,1)$ algebra has 3 generators, J_0, $J_\pm$ obeying the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -2J_0. \tag{2.2.1}$$

The Casimir element is given by

$$C = {J_0}^2 - J_+J_- - J_0. \tag{2.2.2}$$

Let $[n] = \{1, 2, \ldots, n\}$ denote the set of the n first integers and consider the tensor product $\mathfrak{su}(1,1)^{\otimes n}$. Coproduct embeddings of $\mathfrak{su}(1,1)$ in $\mathfrak{su}(1,1)^{\otimes n}$ are labelled by subsets $A \subset [n]$

with the generators mapped to

$$J^A = \sum_{i \in A} J^i \tag{2.2.3}$$

and where the superindex denotes on which factor of $\mathfrak{su}(1,1)^{\otimes n}$ the operator J^i is acting. Correspondingly, the Casimirs are sent to

$$C^A = \left(J_0^A\right)^2 - J_+^A J_-^A - J_0^A. \tag{2.2.4}$$

The generalized Racah algebra $R(n)$ is taken to be an algebra realized by all these intermediate Casimirs C^A since this is the case for $R(3)$.

It is important to note that not all intermediate Casimirs C^A are independent; indeed one has

$$C^A = \sum_{\{i,j\} \subset A} C^{ij} - (|A| - 2) \sum_{i \in A} C^i. \tag{2.2.5}$$

where $|A|$ stands for the cardinality of A. In order to characterize $R(n)$, given that the elements C^i are central, it therefore suffices to provide all the iterated commutators between the C^{ij}'s with $i \neq j$ until closure is achieved. This has been carried out in [6]. It is convenient to introduce P^{ij} and F^{ijk}:

$$P^{ij} = C^{ij} - C^i - C^j, \qquad F^{ijk} = \frac{1}{2}[P^{ij}, P^{jk}]. \tag{2.2.6}$$

The defining relations of the Racah algebra $R(n)$ then read

$$[P^{ij}, P^{jk}] = 2F^{ijk}, \tag{2.2.7a}$$

$$[P^{jk}, F^{ijk}] = P^{ik}P^{jk} - P^{jk}P^{ij} + 2P^{ik}C^j - 2P^{ij}C^k, \tag{2.2.7b}$$

$$[P^{kl}, F^{ijk}] = P^{ik}P^{jl} - P^{il}P^{jk}, \tag{2.2.7c}$$

$$[F^{ijk}, F^{jkl}] = F^{jkl}P^{ij} - F^{ikl}\left(P^{jk} + 2C^j\right) - F^{ijk}P^{jl}, \tag{2.2.7d}$$

$$[F^{ijk}, F^{klm}] = F^{ilm}P^{jk} - P^{ik}F^{jlm}, \tag{2.2.7e}$$

where $i, j, k, l, m \in [n]$ are all different.

In the rank 1 case, (3.7.1c), (3.7.1d) and (3.7.1e) are redundant and the standard Racah algera $R(3)$ is fully described by (3.7.1a) and (3.7.1b). Note that the presentation that results from the specialization of these equations to $n = 3$ is the equitable one. The relation between this presentation and the standard one used in [7] is given explicitly in [8]. The rank 2 Racah algebra (which has been studied in detail in [9]) only requires (3.7.1a)-(3.7.1c) to be characterized, while the relations (3.7.1d) and (3.7.1e) have to be added in order to define Racah algebras of rank 3 or higher.

2.3. The generalized Racah algebra and $\mathfrak{o}(2n)$

Let us now indicate how the relations (3.7.1) given above are satisfied by the generators in $U(\mathfrak{o}(2n))$ of the commutant of n copies of $\mathfrak{o}(2)$ sitting in $\mathfrak{o}(2n)$. The algebra $\mathfrak{o}(2n)$ has $n(2n-1)$ generators $L_{\mu\nu} = -L_{\nu\mu}$, $\mu, \nu = 1, ..., 2n$ obeying

$$[L_{\mu\nu}, L_{\rho\sigma}] = \delta_{\nu\rho}L_{\mu\sigma} - \delta_{\nu\sigma}L_{\mu\rho} - \delta_{\mu\rho}L_{\nu\sigma} + \delta_{\mu\sigma}L_{\nu\rho} \tag{2.3.1}$$

and possesses the following quadratic Casimir:

$$\mathcal{C} = \sum_{1\leq\mu<\nu\leq n} {L_{\mu\nu}}^2. \tag{2.3.2}$$

We will use the realization

$$L_{\mu\nu} = \xi_\mu \frac{\partial}{\partial \xi_\nu} - \xi_\nu \frac{\partial}{\partial \xi_\mu}, \qquad \mu \neq \nu, \qquad \mu, \nu = 1, \dots, 2n. \tag{2.3.3}$$

Pick the $\mathfrak{o}(2)^{\oplus n}$ subalgebra of $\mathfrak{o}(2n)$ generated by the commutative set $\{L_{12},\ L_{34},\ \dots,\ L_{2n-1,2n}\}$. We want to focus on the commutant in $U(\mathfrak{o}(2n))$ of this Abelian algebra.

It is easy to see that the set of invariants $\{G^i\}_{1\leq i\leq n}$, $\{K^{ij}\}_{1\leq i<j\leq n}$,

$$G^i = {L_{2i-1,2i}}^2, \tag{2.3.4}$$

$$K^{ij} = {L_{2i-1,2i}}^2 + {L_{2i-1,2j-1}}^2 + {L_{2i-1,2j}}^2 + {L_{2i,2j-1}}^2 + {L_{2i,2j}}^2 + {L_{2j-1,2j}}^2, \tag{2.3.5}$$

is sufficient to generate this commutant and it happens to realize the generalized Racah algebra. Indeed, with the following redefinitions

$$C^i = -\frac{1}{4}G^i + \frac{1}{4}, \tag{2.3.6}$$

$$C^{ij} = -\frac{1}{4}K^{ij}, \tag{2.3.7}$$

$$P^{ij} = -\frac{1}{4}K^{ij} + \frac{1}{4}\left(G^i + G^j\right) + \frac{1}{2}, \tag{2.3.8}$$

$$F^{ijk} = \frac{1}{32}[K^{ij}, K^{jk}], \tag{2.3.9}$$

a long but straightforward calculation in the realization (2.3.3) shows that the defining relations (3.7.1) of the algebra $R(n)$ are obeyed.

2.4. The $\mathfrak{su}(1,1)$ and $\mathfrak{o}(2n)$ descriptions of $R(n)$ and Howe duality

In the last two sections we indicated that the generalized Racah algebra $R(n)$ is in the commutant of $\mathfrak{su}(1,1)$ in $U(\mathfrak{su}(1,1)^{\otimes n})$ and of $\mathfrak{o}(2)^{\oplus n}$ in oscillator representations of $U(\mathfrak{o}(2n))$. The connection between these two descriptions is rooted in Howe duality.

It is known [10–13] that $\mathfrak{o}(2n)$ and $\mathfrak{sp}(2)$ form a dual pair in $\mathfrak{sp}(4n)$, with these two subalgebras being their mutual commutants. This implies that $\mathfrak{o}(2n)$ and $\mathfrak{sp}(2) \simeq \mathfrak{su}(1,1)$ have dual actions on the Hilbert space of $2n$ oscillator states. That means that their irreducible representations can be paired and this can be done through the Casimirs in the following way.

Consider the $2n$ copies of the metaplectic realization of $\mathfrak{sp}(2)$:

$$J_+^{(\mu)} = \frac{1}{2}{\xi_\mu}^2, \qquad J_-^{(\mu)} = \frac{1}{2}\frac{\partial^2}{\partial {\xi_\mu}^2}, \qquad J_0^{(\mu)} = \frac{1}{2}\left(\frac{1}{2} + \xi_\mu \frac{\partial}{\partial \xi_\mu}\right), \qquad \mu = 1, 2, \dots 2n. \quad (2.4.1)$$

We first add these $2n$ representations by coupling them pairwise

$$J^{(\mu;\nu)} = J^{(\mu)} + J^{(\nu)}. \quad (2.4.2)$$

In what follows, we will always assume that the pairs denoted $(\mu;\nu)$ are such that $(\mu;\nu) = (2i-1; 2i)$, $i = 1, \dots, n$. Now take $A \subset [n]$ to be any subset that is the union of N such pairs:

$$A = \bigcup_{i=1}^{N} \{\mu_i; \nu_i\}, \quad (2.4.3)$$

with $|A| = 2N$ and $1 \leq N \leq n$. The $\mathfrak{su}(1,1)$ realization associated to such a subset A reads

$$J_+^A = \frac{1}{2}\sum_{\mu\in A} {\xi_\mu}^2, \qquad J_-^A = \frac{1}{2}\sum_{\mu\in A}\frac{\partial^2}{\partial {\xi_\mu}^2}, \qquad J_0^A = \frac{1}{2}\left(\frac{|A|}{2} + \sum_{\mu\in A}\xi_\mu\frac{\partial}{\partial\xi_\mu}\right). \quad (2.4.4)$$

It is then straightforward to show that the Casimir for an embedding labelled by the subset A is given by

$$C^A = \left(J_0^A\right)^2 - J_+^A J_-^A - J_0^A = \frac{|A|(|A|-4)}{16} - \sum_{\substack{\mu<\nu \\ \mu,\nu\in A}} \frac{\left(L_{\mu\nu}\right)^2}{4}. \quad (2.4.5)$$

As already noted, not all C^A's are independent. The translation of (2.2.5) shows that all C^A's can be rewritten as

$$C^A = \sum_{\substack{(\mu;\nu),(\rho;\sigma)\in A \\ \mu<\nu<\rho<\sigma}} C^{(\mu;\nu)(\rho;\sigma)} - \frac{|A|-4}{2}\sum_{(\mu;\nu)\in A} C^{(\mu;\nu)} \quad (2.4.6)$$

with

$$C^{(\mu;\nu)} = -\frac{1}{4}\left({L_{\mu\nu}}^2 + 1\right), \qquad C^{(\mu;\nu)(\rho;\sigma)} = -\frac{1}{4}\left({L_{\mu\nu}}^2 + {L_{\mu\rho}}^2 + {L_{\mu\sigma}}^2 + {L_{\nu\rho}}^2 + {L_{\nu\sigma}}^2 + {L_{\rho\sigma}}^2\right). \quad (2.4.7)$$

This shows that all higher order Casimirs can be reexpressed in terms of those of lowest orders.

We thus observe that the intermediate $\mathfrak{sp}(2)$ Casimirs correspond (up to an affine transformation) to the generators of the commutant of $\{L_{1,2}, \dots, L_{2n-1,2n}\}$ in $U(\mathfrak{o}(2n))$. We know from Section 2.3, that the intermediate $\mathfrak{sp}(2)$ Casimirs realize the commutation relations of the generalized Racah algebra. This will hence be the case also for the commutant generators and we have here our duality connection.

2.5. The generalized Racah algebra and the generic superintegrable model on S^{n-1}

We can now complete the picture by performing the dimensional reduction from $\mathbb{R}^{2n}$ to $\mathbb{R}^{+} \times S^{n-1}$ to obtain the generic superintegrable model with Hamiltonian H (introduced in Section 2.1) and to recover its symmetries. Starting from the oscillator representation (2.4.1), make the following change of variables:

$$\begin{aligned} \xi_{2i-1} &= x_i \cos\theta_i, \\ \xi_{2i} &= x_i \sin\theta_i, \end{aligned} \qquad L_{2i-1,2i} = \xi_{2i-1}\frac{\partial}{\partial \xi_{2i}} - \xi_{2i}\frac{\partial}{\partial \xi_{2i-1}} = \frac{\partial}{\partial \theta_i}, \qquad i = 1,\ldots,n. \tag{2.5.1}$$

Eliminate the ignorable θ_i's by separating these variables and setting ${L_{2i-1,2i}}^2 \sim {k_i}^2$. After performing the gauge transformation $\mathcal{O} \mapsto \widetilde{\mathcal{O}} = x_i^{1/2}\ \mathcal{O}\ x_i^{-1/2}$ one obtains the reduced system

$$\widetilde{J}_+^{(2i-1,2i)} = \frac{1}{2}{x_i}^2, \qquad \widetilde{J}_-^{(2i-1,2i)} = \frac{1}{2}\left(\frac{\partial^2}{\partial {x_i}^2} + \frac{a_i}{{x_i}^2}\right), \qquad \widetilde{J}_0^{(2i-1,2i)} = \frac{1}{2}\left(x_i\frac{\partial}{\partial x_i} + \frac{1}{2}\right), \tag{2.5.2}$$

with $a_i = {k_i}^2 + \frac{1}{4}$. Defining $\widetilde{J}^i \equiv \widetilde{J}^{(2i-1,2i)}$, the reduced Casimirs

$$\widetilde{C}^i = \left(\widetilde{J}_0^{\,i}\right)^2 - \widetilde{J}_+^{\,i}\widetilde{J}_-^{\,i} - \widetilde{J}_0^{\,i}, \tag{2.5.3}$$

$$\widetilde{C}^{ij} = \left(\widetilde{J}_0^{\,i} + \widetilde{J}_0^{\,j}\right)^2 - \left(\widetilde{J}_+^{\,i} + \widetilde{J}_+^{\,j}\right)\left(\widetilde{J}_-^{\,i} + \widetilde{J}_-^{\,j}\right) - \left(\widetilde{J}_0^{\,i} + \widetilde{J}_0^{\,j}\right), \tag{2.5.4}$$

are easily computed and have the following expressions:

$$\begin{aligned} \widetilde{C}^i &= -\frac{1}{4}\left(a_i + \frac{3}{4}\right), \\ \widetilde{C}^{ij} &= -\frac{1}{4}\left[{\mathcal{J}_{ij}}^2 + a_i\frac{{x_j}^2}{{x_i}^2} + a_j\frac{{x_i}^2}{{x_j}^2} + a_i + a_j + 1\right], \end{aligned} \qquad \mathcal{J}_{ij} = x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}, \qquad i < j. \tag{2.5.5}$$

Using the fact that $\widetilde{J}^{[n]} = \sum_{i=1}^{n}\widetilde{J}^i$, the total Casimir $\widetilde{C}^{[n]}$ is obtained:

$$\widetilde{C}^{[n]} = -\frac{1}{4}\sum_{1\le i<j\le n}{\mathcal{J}_{ij}}^2 - \frac{1}{4}\left(\sum_{i=1}^{n}{x_i}^2\right)\sum_{j=1}^{n}\frac{a_j}{{x_j}^2} + \frac{n(n-4)}{16}, \tag{2.5.6}$$

and assuming $\sum_{i=1}^{n}{x_i}^2 = 1$, one thereby obtains the Hamiltonian of the generic model on S^{n-1} (up to an affine transformation). The basic intermediate Casimirs are essentially the conserved quantities:

$$Q_{ij} = \mathcal{J}_{ij}{}^2 + a_i \frac{x_j{}^2}{x_i{}^2} + a_j \frac{x_i{}^2}{x_j{}^2}, \qquad 1 \leq i < j \leq n \tag{2.5.7}$$

and they generate $R(n)$ which is hence the symmetry algebra of the superintegrable model on the $(n-1)$ sphere. (Note that the Q_{ij}'s are affinely related to the P^{ij}'s in the relations (2.2.6).)

2.6. Conclusion

Summing up, we have shown that the generalized Racah algebra $R(n)$ can be defined in the commutant of the $\mathfrak{o}(2)^{\oplus n}$ subalgebra of $\mathfrak{o}(2n)$ in oscillator representations of $U(\mathfrak{o}(2n))$. This offers an alternative to the definition of $R(n)$ in the algebra of the intermediate Casimirs associated to the $\mathfrak{su}(1,1)$ embeddings in $\mathfrak{su}(1,1)^{\otimes n}$. We have related these two pictures in the context of Howe duality and obtained the generic $R(n)$-invariant superintegrable model on S^{n-1} through the dimensional reduction scheme stemming from the analysis. This has provided a generalization to arbitrary ranks and dimensions of the study carried in [7] for the standard Racah algebra.

We wish to remark that since $\mathfrak{o}(nd)$ and $\mathfrak{sp}(2)$ form a dual pair in $\mathfrak{sp}(2nd)$, it is also possible to realize the generalized Racah algebra as the commutant of the $\mathfrak{o}(d)^{\oplus n}$ subalgebra of $\mathfrak{o}(nd)$. We have concentrated on the case $d = 2$ because it offers the simplest situation that allows to obtain the superintegrable system on S^{n-1} by dimensional reduction.

In the near future, we plan on exploring similarly the Askey–Wilson (AW) and the Bannai–Ito (BI) algebras which share features with the Racah algebra since both encode the bispectrality properties of the eponym polynomials and appear through tensor products of $U_q(\mathfrak{sl}(2))$ [14] and $\mathfrak{osp}(1|2) \simeq \mathfrak{sl}_{-1}(2)$ [15] respectively. Moreover, the BI algebra is the symmetry algebra of a superintegrable model on the sphere involving reflection operators [16] as well as a Dirac–Dunkl equation in $\mathbb{R}^3$ [6]. It would be of interest to build on the work of the present paper to obtain a Howe duality setting for the interpretation of the AW and BI algebras as commutants; moreover extensions along the lines of this paper would shed interesting light on the higher rank versions of these algebras.

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Chapitre 3

The dual pair $Pin(2n) \times \mathfrak{osp}(1|2)$, the Dirac equation and the Bannai–Ito algebra

Par Julien Gaboriaud, Luc Vinet, Stéphane Vinet et Alexei Zhedanov.
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Abstract: The Bannai–Ito algebra can be defined in the centralizer of the coproduct embedding of $\mathfrak{osp}(1|2)$ in $\mathfrak{osp}(1|2)^{\otimes n}$. It will be shown that it is also in the commutant of a maximal Abelian subalgebra of $\mathfrak{o}(2n)$ in a spinorial representation and an embedding of the Racah algebra in this commutant will emerge. The connection between the two pictures for the Bannai–Ito algebra will be traced to the Howe duality which is embodied in the $Pin(2n) \times \mathfrak{osp}(1|2)$ symmetry of the massless Dirac equation in $\mathbb{R}^{2n}$. Dimensional reduction to $\mathbb{R}^n$ will provide an alternative to the Dirac–Dunkl equation as a model with Bannai–Ito symmetry.

3.1. Introduction

The Bannai–Ito algebra $B(n)$ can be presented in terms of generators and relations [1, 2]. Let $[n] = \{1, 2, \dots, n\}$ denote the set of the n first integers and $S = \{s_1, \dots, s_k\}$ be an ordered k-subset of $[n]$. The generators Γ_S of $B(n)$ are labelled by all subsets for $k = 0, 1, \dots, n$. For any two subsets A and B of $[n]$, the relations between the generators Γ_A and Γ_B that define $B(n)$ are:

$$\{\Gamma_A, \Gamma_B\} = \Gamma_{(A\cup B)\setminus(A\cap B)} + 2\Gamma_{A\cap B}\Gamma_{A\cup B} + 2\Gamma_{A\setminus(A\cap B)}\Gamma_{B\setminus(A\cap B)}, \tag{3.1.1}$$

where $\{X, Y\} = XY + YX$. By convention $\Gamma_\emptyset = -1/2$, and the generators associated to a set are simply labelled by the indices $\Gamma_{\{i_1,\ldots,i_k\}} \equiv \Gamma_{i_1\ldots i_k}$. Moreover, Γ_i, $i \in [n]$ and $\Gamma_{[n]}$ are central.

For the rank one case which occurs when $n = 3$, the relations for $B(3)$ are seen to be [1, 3]

$$\{\Gamma_{ij}, \Gamma_{jk}\} = \Gamma_{ik} + 2\Gamma_j\Gamma_{ijk} + 2\Gamma_i\Gamma_k, \tag{3.1.2}$$

where $i, j, k \in [3]$ are all distinct.

We shall present in this paper a defining context for $B(n)$ (and $B(3)$) as a commutant and relate this result to a Howe duality framework.

The Bannai–Ito algebra $B(3)$ was initially introduced in [4] to encode the bispectral properties of the Bannai–Ito polynomials which were discovered by the researchers whose name they bear in a classification problem in algebraic combinatorics [5]. $B(3)$ was later seen [6] to be isomorphic to a degenerate double affine Hecke algebra (DAHA) of type $(C_1^\vee, C_1)$. A key connection between the Lie superalgebra $\mathfrak{osp}(1|2)$ and the Bannai–Ito algebra was also made [3]. Indeed, the Bannai–Ito polynomials were identified as forming the Racah coefficients for $\mathfrak{osp}(1|2)$ [7]. It was then shown quite naturally that $B(3)$ is realized by the intermediate Casimir operators arising in the threefold tensor product of $\mathfrak{osp}(1|2)$ with itself. Extending this construction to the n-fold tensor product $\mathfrak{osp}(1|2)^{\otimes n}$ led to the definition of $B(n)$ [1]. (More details will be given on this later.) A number of applications for $B(n)$ were subsequently found. For instance, the conserved quantities of a superintegrable model on the $(n-1)$-sphere [8–11] were seen to satisfy the commutation relations (3.1.1). Of particular relevance to the present study is the fact that $B(n)$ is also the symmetry algebra of the (massless) Dirac–Dunkl [1, 12] equation in $\mathbb{R}^n$.

The Bannai–Ito algebra $B(n)$ has much in common with the Racah algebra $R(n)$ [13]. Like $B(n)$, $R(n)$ can be defined in a tensorial fashion in the algebra formed by the intermediate Casimir operators associated to the n-fold tensor product of the Lie algebra $\mathfrak{sl}(2)$ with itself. $R(n)$ is also the symmetry algebra of a certain generic superintegrable model on S^{n-1} (without reflections). The relation between the rank one version $R(3)$ and the Racah polynomials [14] is quite similar to the one between the Bannai–Ito algebra $B(3)$ and the Bannai–Ito polynomials. $R(3)$ has three generators K_1, K_2, K_3 which are subjected to the relations

$$\begin{aligned} [K_1, K_2] = K_3, \qquad\qquad & [K_2, K_3] = {K_2}^2 + \{K_1, K_2\} + dK_2 + e_1, \\ & [K_3, K_1] = {K_1}^2 + \{K_1, K_2\} + dK_1 + e_2, \end{aligned} \tag{3.1.3}$$

with d, e_1, e_2 central. The defining relations for $R(n)$ [13] are given in Section 4.4 where an embedding of $R(n)$ into $B(n)$ will be identified.

We have recently observed that $R(3)$ is in the commutant in oscillator representations of the universal enveloping algebra $U(\mathfrak{o}(6))$ of the subalgebra $\mathfrak{o}(2) \oplus \mathfrak{o}(2) \oplus \mathfrak{o}(2)$ of the Lie algebra $\mathfrak{o}(6)$ of rotations in six dimensions [15]. This was extended to $R(n)$ in [16]. We shall provide here an analogous description of $B(n)$ and shall focus first on $B(3)$.

After recalling in Section 3.2 how $B(3)$ can be viewed in the centralizer of the coproduct embedding of $\mathfrak{osp}(1|2)$ into $\mathfrak{osp}(1|2)^{\otimes 3}$, we shall show in Section 3.3 that $B(3)$ is in the commutant of $\mathfrak{o}(2) \oplus \mathfrak{o}(2) \oplus \mathfrak{o}(2)$ in the enveloping algebra of the spinorial representation of $\mathfrak{o}(6)$ associated to the Clifford algebra in $\mathbb{R}^6$. By considering the (massless) Dirac equation, a Howe duality setting will be brought up to explain the connection between the results of the two previous sections. Dimensional reduction will be performed in Section 3.5 to complete the picture and will result in a new class of model with Bannai–Ito symmetry. In Section 3.6 the generalization of these results to the Bannai–Ito algebra $B(n)$ will be obtained for $n > 3$. As mentioned above, an embedding of the higher rank $R(n)$ Racah algebra in the $B(n)$ Bannai–Ito algebra will be explicitly given in Section 4.4, thus linking the construction presented here to the one in [16]. Brief concluding remarks will follow. Finally, in Appendix 3.A we indicate that our results offer as a byproduct a derivation through dimensional reduction of the superconformal quantum Hamiltonian of Fubini and Rabinovici which is known to be invariant under $\mathfrak{sl}(2|1)$ [17].

3.2. The superalgebra $\mathfrak{osp}(1|2)$ and the Bannai–Ito algebra as a centralizer

The superalgebra $\mathfrak{osp}(1|2)$ can be presented as follows. Let J_0, $J_\pm$ be respectively the even and odd generators of the algebra, obeying the relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad \{J_+, J_-\} = 2J_0. \tag{3.2.1a}$$

The $\mathbb{Z}_2$-grading of the superalgebra can be encoded through a grading involution $\mathscr{S}$, which commutes with even elements and anticommutes with odd elements:

$$[\mathscr{S}, J_0] = 0, \qquad \{\mathscr{S}, J_\pm\} = 0. \tag{3.2.1b}$$

The sCasimir operator of $\mathfrak{osp}(1|2)$ given by

$$S = \frac{1}{2}\left([J_-, J_+] - 1\right) \tag{3.2.2}$$

commutes with all the odd elements and anticommutes with all the even ones, that is $[S, J_0] = 0$, $\{S, J_\pm\} = 0$. It is then straightforward to define a Casimir of $\mathfrak{osp}(1|2)$ by combining the sCasimir with the involution:

$$\Gamma = S\,\mathscr{S} = \frac{1}{2}\left([J_-, J_+] - 1\right)\mathscr{S}. \tag{3.2.3}$$

This Casimir Γ commutes with all the elements of $\mathfrak{osp}(1|2)$.

There is a coassociative algebra morphism, the coproduct Δ, which acts as follows:

$$\begin{aligned} &\Delta : \mathfrak{osp}(1|2) \to \mathfrak{osp}(1|2) \otimes \mathfrak{osp}(1|2) \\ &\Delta(J_0) = J_0 \otimes 1 + 1 \otimes J_0 \qquad = J_0^{(1)} + J_0^{(2)}, \\ &\Delta(J_\pm) = J_\pm \otimes \mathscr{S} + 1 \otimes J_\pm \quad = J_\pm^{(1)} \mathscr{S}^{(2)} + J_\pm^{(2)}, \\ &\Delta(\mathscr{S}) = \mathscr{S} \otimes \mathscr{S} \qquad\qquad = \mathscr{S}^{(1)} \mathscr{S}^{(2)}, \end{aligned} \tag{3.2.4}$$

where the superindex denotes on which factor of the tensor product the generator is acting.

Now consider the product of three copies of $\mathfrak{osp}(1|2)$. The generators corresponding to embeddings in two factors are

$$\begin{gathered} J_0^{(ij)} = J_0^{(i)} + J_0^{(j)}, \qquad \mathscr{S}^{(ij)} = \mathscr{S}^{(i)} \mathscr{S}^{(j)}, \qquad i, j = 1, 2, 3, \\ J_\pm^{(12)} = J_\pm^{(1)} \mathscr{S}^{(2)} + J_\pm^{(2)}, \quad J_\pm^{(23)} = J_\pm^{(2)} \mathscr{S}^{(3)} + J_\pm^{(3)}, \quad J_\pm^{(13)} = J_\pm^{(1)} \mathscr{S}^{(2)} \mathscr{S}^{(3)} + J_\pm^{(3)}. \end{gathered} \tag{3.2.5}$$

Note the presence of $\mathscr{S}^{(2)}$ in $J_\pm^{(13)}$. Applying the coproduct twice yields

$$\begin{aligned} &\Delta^{(2)}(J_0) = J_0^{(1)} + J_0^{(2)} + J_0^{(3)} = J_0^{(123)}, \\ &\Delta^{(2)}(J_\pm) = J_\pm^{(1)} \mathscr{S}^{(2)} \mathscr{S}^{(3)} + J_\pm^{(2)} \mathscr{S}^{(3)} + J_\pm^{(3)} = J_\pm^{(123)}, \\ &\Delta^{(2)}(\mathscr{S}) = \mathscr{S}^{(1)} \mathscr{S}^{(2)} \mathscr{S}^{(3)} = \mathscr{S}^{(123)}, \\ &\quad \text{with } \ \Delta^{(2)} = (\Delta \otimes 1) \circ \Delta. \end{aligned} \tag{3.2.6}$$

The intermediate Casimirs Γ_A associated to embeddings of $\mathfrak{osp}(1|2)$ in $\mathfrak{osp}(1|2)^{\otimes 3}$ labelled by $A \subset [3]$ can then be obtained from the sets above:

$$\begin{aligned} \Gamma_i &= \frac{1}{2}\left([J_-^{(i)}, J_+^{(i)}] - 1\right) \mathscr{S}^{(i)}, \\ \Gamma_{ij} &= \frac{1}{2}\left([J_-^{(ij)}, J_+^{(ij)}] - 1\right) \mathscr{S}^{(ij)}, \end{aligned} \qquad \Gamma_{123} = \frac{1}{2}\left([J_-^{(123)}, J_+^{(123)}] - 1\right) \mathscr{S}^{(123)}. \tag{3.2.7}$$

Even though the intermediate Casimirs Γ_A commute with the action of $\mathfrak{osp}(1|2)$, they do not all commute with each other. A direct computation shows that they precisely obey the commutation relations (3.1.2) of $B(3)$ thereby proving that this algebra is in the centralizer of $\Delta^{(2)}(\mathfrak{osp}(1|2))$ in $U(\mathfrak{osp}(1|2)^{\otimes 3})$.

3.3. The rank 1 Bannai–Ito algebra as a commutant

We will now show how the Bannai–Ito algebra arises in the commutant of the subalgebra $\mathfrak{o}(2) \oplus \mathfrak{o}(2) \oplus \mathfrak{o}(2)$ of $\mathfrak{o}(6)$ in spinorial representations of $U(\mathfrak{o}(6))$.

Let $\mathcal{C}\ell_6$ be the Clifford algebra generated by the elements $\gamma_1, \dots, \gamma_6$ verifying the relations

$$\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}. \tag{3.3.1}$$

Denote by $\ell_{\mu\nu}$, $\mu, \nu = 1, \ldots, 6$ the generators of $\mathfrak{o}(6)$ which obey

$$[\ell_{\mu\nu}, \ell_{\rho\sigma}] = \delta_{\nu\rho}\ell_{\mu\sigma} - \delta_{\nu\sigma}\ell_{\mu\rho} - \delta_{\mu\rho}\ell_{\nu\sigma} + \delta_{\mu\sigma}\ell_{\nu\rho}. \tag{3.3.2}$$

We shall consider the following representation of the algebra of $Pin(6)$ where

$$J_{\mu\nu} = -iL_{\mu\nu} + \frac{1}{2}\Sigma_{\mu\nu}, \tag{3.3.3}$$

with

$$L_{\mu\nu} = x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu}, \tag{3.3.4}$$

and $\Sigma_{\mu\nu} = i\gamma_\mu\gamma_\nu$ the spin operators.

We are interested in the commutant of the $\mathfrak{o}(2) \oplus \mathfrak{o}(2) \oplus \mathfrak{o}(2)$ subalgebra of $\mathfrak{o}(6)$ represented by the set of elements $\{J_{12}, J_{34}, J_{56}\}$. First note that:

$$[J_{\mu\nu}, L_{\mu\rho}\Sigma_{\mu\rho} + L_{\nu\rho}\Sigma_{\nu\rho}] = 0. \tag{3.3.5}$$

Remark also that $[J_{\mu\nu}, \Sigma_{\mu\nu}] = 0$ and finally that $[J_{\mu\nu}, \Sigma_{\rho\sigma}] = 0$ for μ, ν, ρ, σ all different.

It is observed that the commutant of the span of $\{J_{12}, J_{34}, J_{56}\}$ is generated by the operators (see the remark below):

$$\begin{aligned}
M_1 &= (L_{12}\gamma_1\gamma_2 + L_{13}\gamma_1\gamma_3 + L_{14}\gamma_1\gamma_4 + L_{23}\gamma_2\gamma_3 + L_{24}\gamma_2\gamma_4 + L_{34}\gamma_3\gamma_4)\,\Sigma_{12}\Sigma_{34}\\
M_2 &= (L_{34}\gamma_3\gamma_4 + L_{35}\gamma_3\gamma_5 + L_{36}\gamma_3\gamma_6 + L_{45}\gamma_4\gamma_5 + L_{46}\gamma_4\gamma_6 + L_{56}\gamma_5\gamma_6)\,\Sigma_{34}\Sigma_{56}\\
M_3 &= (L_{12}\gamma_1\gamma_2 + L_{15}\gamma_1\gamma_5 + L_{16}\gamma_1\gamma_6 + L_{25}\gamma_2\gamma_5 + L_{26}\gamma_2\gamma_6 + L_{56}\gamma_5\gamma_6)\,\Sigma_{12}\Sigma_{56}.
\end{aligned} \tag{3.3.6}$$

We can now see that these realize the (rank 1) Bannai–Ito algebra. It is convenient to first introduce the shortened notation $\overline{j} \equiv \{2j-1, 2j\}$, $j \in \mathbb{N}$. Take now the elements

$$\begin{aligned}
\Gamma_{\overline{1}\,\overline{2}} &= M_1 + \tfrac{3}{2}\Sigma_{\overline{1}}\Sigma_{\overline{2}},\\
\Gamma_{\overline{2}\,\overline{3}} &= M_2 + \tfrac{3}{2}\Sigma_{\overline{2}}\Sigma_{\overline{3}}, \qquad \Gamma_{\overline{j}} = \left(L_{2j-1,2j}\,\gamma_{2j-1}\gamma_{2j} + \tfrac{1}{2}\right)\Sigma_{2j-1,2j} = J_{2j-1,2j}.\\
\Gamma_{\overline{1}\,\overline{3}} &= M_3 + \tfrac{3}{2}\Sigma_{\overline{1}}\Sigma_{\overline{3}},
\end{aligned} \tag{3.3.7}$$

A straightforward calculation in the realization (3.3.4) shows that one has the defining relations of the rank 1 Bannai–Ito algebra $B(3)$

$$\begin{aligned}
\{\Gamma_{\overline{1}\,\overline{2}}, \Gamma_{\overline{2}\,\overline{3}}\} &= \Gamma_{\overline{1}\,\overline{3}} + 2\Gamma_{\overline{2}}\Gamma_{\overline{1}\,\overline{2}\,\overline{3}} + 2\Gamma_{\overline{3}}\Gamma_{\overline{1}},\\
\{\Gamma_{\overline{2}\,\overline{3}}, \Gamma_{\overline{1}\,\overline{3}}\} &= \Gamma_{\overline{1}\,\overline{2}} + 2\Gamma_{\overline{3}}\Gamma_{\overline{1}\,\overline{2}\,\overline{3}} + 2\Gamma_{\overline{1}}\Gamma_{\overline{2}},\\
\{\Gamma_{\overline{1}\,\overline{3}}, \Gamma_{\overline{1}\,\overline{2}}\} &= \Gamma_{\overline{2}\,\overline{3}} + 2\Gamma_{\overline{1}}\Gamma_{\overline{1}\,\overline{2}\,\overline{3}} + 2\Gamma_{\overline{2}}\Gamma_{\overline{3}},
\end{aligned} \tag{3.3.8}$$

where $\Gamma_{\overline{1}\,\overline{2}\,\overline{3}}$ denotes the (Casimir) element

$$\Gamma_{\overline{1}\,\overline{2}\,\overline{3}} = \left(\sum_{1\le\mu<\nu\le 6} -iL_{\mu\nu}\Sigma_{\mu\nu} + \frac{5}{2}\right)\Sigma_{\overline{1}}\Sigma_{\overline{2}}\Sigma_{\overline{3}} \tag{3.3.9}$$

and $\Gamma_{\overline{1}\,\overline{2}\,\overline{3}}$, $\Gamma_{\overline{1}}$, $\Gamma_{\overline{2}}$, $\Gamma_{\overline{3}}$ are all central.

Proof: Let us explain how the expression for $\{\Gamma_{\overline{1}\,\overline{2}}, \Gamma_{\overline{2}\,\overline{3}}\}$ is derived. (The other anticommutators are obtained in a similar way.) Making use of the Clifford algebra properties (3.3.1) as well as the $\mathfrak{o}(6)$ commutation relations (8.3.6), one obtains

$$\begin{aligned}
\{\Gamma_{\overline{1}\,\overline{2}}, \Gamma_{\overline{2}\,\overline{3}}\} = {} & 3(L_{12}\gamma_1\gamma_2 + 2L_{34}\gamma_3\gamma_4 + L_{56}\gamma_5\gamma_6) + 2L_{12}\gamma_1\gamma_2 L_{56}\gamma_5\gamma_6 \\
& + 2(L_{15}\gamma_1\gamma_5 + L_{16}\gamma_1\gamma_6 + L_{25}\gamma_2\gamma_5 + L_{26}\gamma_2\gamma_6) + \tfrac{9}{2} \\
& + L_{13}\gamma_1\gamma_3 + L_{14}\gamma_1\gamma_4 + L_{23}\gamma_2\gamma_3 + L_{24}\gamma_2\gamma_4 \\
& + L_{35}\gamma_3\gamma_5 + L_{36}\gamma_3\gamma_6 + L_{45}\gamma_4\gamma_5 + L_{46}\gamma_4\gamma_6 \\
& + 2L_{34}\gamma_3\gamma_4\Big(L_{12}\gamma_1\gamma_2 + L_{13}\gamma_1\gamma_3 + L_{14}\gamma_1\gamma_4 + L_{23}\gamma_2\gamma_3 + L_{24}\gamma_2\gamma_4 \\
& \quad + L_{34}\gamma_3\gamma_4 + L_{35}\gamma_3\gamma_5 + L_{36}\gamma_3\gamma_6 + L_{45}\gamma_4\gamma_5 + L_{46}\gamma_4\gamma_6 + L_{56}\gamma_5\gamma_6\Big) \\
& - 2(L_{13}L_{45} + L_{14}L_{53})\gamma_3\gamma_4\gamma_1\gamma_5 - 2(L_{13}L_{46} + L_{14}L_{36})\gamma_3\gamma_4\gamma_1\gamma_6 \\
& - 2(L_{23}L_{45} + L_{24}L_{53})\gamma_3\gamma_4\gamma_2\gamma_5 - 2(L_{23}L_{46} + L_{24}L_{36})\gamma_3\gamma_4\gamma_2\gamma_6.
\end{aligned} \tag{3.3.10}$$

The additional identity [18]

$$L_{ab}L_{cd} + L_{ac}L_{db} + L_{ad}L_{bc} = 0 \tag{3.3.11}$$

satisfied in our realization is then the required tool in order to rewrite (3.3.10) as the r.h.s of (3.3.8) using the definitions (3.3.7) and (3.3.9).

Remark: It is fairly obvious that the elements $\{G^i\}_{1\leq i\leq n}$, $\{K^{ij}\}_{1\leq i<j\leq n}$,

$$\begin{aligned}
G^i &= {L_{2i-1,2i}}^2, \\
K^{ij} &= {L_{2i-1,2i}}^2 + {L_{2i-1,2j-1}}^2 + {L_{2i-1,2j}}^2 + {L_{2i,2j-1}}^2 + {L_{2i,2j}}^2 + {L_{2j-1,2j}}^2,
\end{aligned} \tag{3.3.12}$$

also belong to the commutant of $\{J_{12}, J_{34}, J_{56}\}$. It can be seen that they are algebraically dependent on the Bannai–Ito generators given above. It is interesting to observe however that these G^i and K^{ij} realize the Racah algebra $R(3)$ as shown in [15]. This highlights the fact that the Racah algebra can be embedded in the Bannai–Ito algebra. The explicit embedding and a generalization to the higher rank algebras will be given in Section 4.4. Operators $\widetilde{G}^i$ and $\widetilde{K}^{ij}$ obtained by replacing L_{ij} by Σ_{ij} also belong obviously to the commutant of $\{J_{12}, J_{34}, J_{56}\}$ but lead to trivial operators.

3.4. The Dirac model and Howe duality

We now wish to shed light on the result of the previous two sections by casting in a Howe duality context the observation that the Bannai–Ito algebra arises in the commutant of both $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2)^{\otimes 3})$ and $\mathfrak{o}(2)^{\oplus 3}$ in the considered spinorial representations of $U(\mathfrak{o}(6))$.

To that end, we shall introduce a Dirac model where $Pin(6)$ and $\mathfrak{osp}(1|2)$ act as a dual reductive pair [19] on the eigenfunctions so that their respective irreducible representations can be paired through connections between the Casimir operators.

The Dirac operator $\underline{D}$ as well as $\underline{x}$ and $\mathbb{E}$ are defined in six dimensions as follows:

$$\underline{D} = \sum_{\mu=1}^{6} \gamma_\mu \partial_\mu \ , \qquad \underline{x} = \sum_{\mu=1}^{6} \gamma_\mu x_\mu \ , \qquad \mathbb{E} = \sum_{\mu=1}^{6} x_\mu \partial_\mu, \tag{3.4.1}$$

with $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$.

These operators have $\mathfrak{osp}(1|2)$ as their dynamical algebra. Indeed, with the $\mathbb{Z}_2$-grading involution $\mathscr{S}$ given by

$$\mathscr{S} = i^{6/2} \prod_{\mu=1}^{6} \gamma_\mu, \tag{3.4.2}$$

and

$$J_- = -i\underline{D}, \qquad J_+ = -i\underline{x}, \qquad J_0 = \mathbb{E} + 3, \tag{3.4.3}$$

the presentation (3.2.1) of the $\mathfrak{osp}(1|2)$ algebra is realized, with the total Casimir given by

$$\Gamma_{[6]} = \frac{1}{2} \left([J_-, J_+] - 1 \right) \mathscr{S}. \tag{3.4.4}$$

It should be noted that to any subset $A \subset [6]$ there corresponds a realization of $\mathfrak{osp}(1|2)$. More precisely, if one defines

$$J_-^A = -i \sum_{\mu \in A} \gamma_\mu \partial_\mu \ , \qquad J_+^A = -i \sum_{\mu \in A} \gamma_\mu x_\mu \ , \qquad J_0^A = \tfrac{|A|}{2} + \sum_{\mu \in A} x_\mu \partial_\mu, \tag{3.4.5}$$

these generators obey the $\mathfrak{osp}(1|2)$ relations (3.2.1) with the involution given by

$$\mathscr{S}^A = i^{|A|/2} \prod_{\mu \in A} \gamma_\mu \tag{3.4.6}$$

when $|A|$ is even. The Casimir has then the expression

$$\Gamma_A = \frac{1}{2} \Big([J_-^A, J_+^A] - 1 \Big) \mathscr{S}^A. \tag{3.4.7}$$

We will first couple the six representations pairwise (each pair will correspond to an $\mathfrak{osp}(1|2)$ and we can hence use the previous observations for $|A|$ even). We will refer to those pairs using the shortened index notation, $\overline{j} = \overline{1}, \overline{2}, \overline{3}$. The $\overline{j}$'s will now label the representations we want to pair next, in this two-step process.

The Casimirs associated to one or two such indices are easily calculated using the expression (3.4.7) and they are found to be

$$
\begin{aligned}
\Gamma_{\bar{j}} &= -iL_{\bar{j}} + \tfrac{1}{2}\Sigma_{\bar{j}}\,, \\
\Gamma_{\bar{i}\,\bar{j}} &= \Big(L_{2i-1,2i}\,\gamma_{2i-1}\,\gamma_{2i} + L_{2i-1,2j-1}\,\gamma_{2i-1}\,\gamma_{2j-1} + L_{2i-1,2j}\,\gamma_{2i-1}\,\gamma_{2j} \\
&\qquad + L_{2i,2j-1}\,\gamma_{2i}\,\gamma_{2j-1} + L_{2i,2j}\,\gamma_{2i}\,\gamma_{2j} + L_{2j-1,2j}\,\gamma_{2j-1}\,\gamma_{2j} + \tfrac{3}{2}\Big)\Sigma_{2i-1,2i}\,\Sigma_{2j-1,2j}.
\end{aligned}
\tag{3.4.8}
$$

They are immediately recognized as the generators (3.3.7) of the commutant of the set $\{J_{12}, J_{34}, J_{56}\}$ in the spinorial representations of $U(\mathfrak{o}(6))$ that were identified in the previous section.

It is here interesting to point out how the coproduct structure of $\mathfrak{osp}(1|2)$ occurs in the Dirac operator. In two dimensions, the gamma matrices are given in terms of the Pauli matrices:

$$
\gamma_1 = i\sigma_1, \qquad \gamma_2 = i\sigma_2, \qquad \{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}. \tag{3.4.9}
$$

The involution is simply $\mathscr{S} = i\gamma_1\gamma_2 = \sigma_3$ and coincides with the spin operator which we denoted Σ_{12}.

We observed that the Dirac equation in 4D provides an $\mathfrak{osp}(1|2)$ made out ot two subsystems each realizing also an $\mathfrak{osp}(1|2)$. It must hence result from the coproduct mapping:

$$
\underline{D}_{[2]} \mapsto \Delta(\underline{D}_{[2]}) = \underline{D}_{[2]} \otimes \mathscr{S} + 1 \otimes \underline{D}_{[2]} = \underline{D}_{[2]}^{(1)}\mathscr{S}^{(2)} + \underline{D}_{[2]}^{(2)}. \tag{3.4.10}
$$

This connects with the construction of higher dimensional gamma matrices. Indeed, starting with a realization of the Clifford algebra $\mathcal{C}\ell_2$ (generated for example by the 2 γ_i's in (3.4.9)), a systematic way to construct a realization of a Clifford algebra in two additional dimensions (involving 4 $\hat{\gamma}_i$'s) is [20] to take

$$
\begin{aligned}
\hat{\gamma}_1 &= \gamma_1 \otimes (i\gamma_1\gamma_2) &&= (i\sigma_1) \otimes \sigma_3 \\
\hat{\gamma}_2 &= \gamma_2 \otimes (i\gamma_1\gamma_2) &&= (i\sigma_2) \otimes \sigma_3 \\
\hat{\gamma}_3 &= 1 \otimes \gamma_1 &&= \ 1 \ \otimes (i\sigma_1) \\
\hat{\gamma}_4 &= 1 \otimes \gamma_2 &&= \ 1 \ \otimes (i\sigma_2).
\end{aligned}
\tag{3.4.11}
$$

This construction can be iterated as many times as needed for higher dimensional Clifford algebra realizations in even dimensions.

With this choice of gamma matrices, the Dirac operator in 4D reads:

$$
\begin{aligned}
\underline{D}_{[4]} &= \gamma_1\partial_1 + \gamma_2\partial_2 + \gamma_3\partial_3 + \gamma_4\partial_4 \\
&= \Big(\partial_1(i\sigma_1) + \partial_2(i\sigma_2)\Big) \otimes \sigma_3 + 1 \otimes \Big(\partial_1(i\sigma_1) + \partial_2(i\sigma_2)\Big) \\
&= \underline{D}_{[2]} \otimes \mathscr{S} + 1 \otimes \underline{D}_{[2]},
\end{aligned}
\tag{3.4.12}
$$

which checks with the expected coproduct result. The algebra involution $\mathscr{S}$ is realized by the σ_3 matrix and its occurence is made manifest in this fashion.

3.5. Dimensional reduction

It is instructive to perform the dimensional reduction of the six-dimensional Dirac operator to $\mathbb{R}^3$. Introduce the cylindrical coordinates

$$\begin{aligned} x_{2j-1} &= \rho_j \cos\theta_j, \\ x_{2j} &= \rho_j \sin\theta_j, \end{aligned} \qquad j = 1, 2, 3. \tag{3.5.1}$$

We then have transformed expressions for $\underline{D}$, $\underline{x}$ and $\mathbb{E}$. In particular,

$$\underline{D} = \sum_{j=1}^{3} \left(\check{\gamma}_{2j-1} \frac{\partial}{\partial \rho_j} + \check{\gamma}_{2j} \frac{1}{\rho_j} \frac{\partial}{\partial \theta_j} \right), \tag{3.5.2}$$

where

$$\begin{aligned} \check{\gamma}_{2j-1} &= \quad \cos\theta_j \, \gamma_{2j-1} + \sin\theta_j \, \gamma_{2j}, \\ \check{\gamma}_{2j} &= -\sin\theta_j \, \gamma_{2j-1} + \cos\theta_j \, \gamma_{2j}. \end{aligned} \tag{3.5.3}$$

We can now bring the $\check{\gamma}_\mu$'s back to their original form (the γ_μ's) by means of a rotation in spin space. Let

$$S = \prod_{j=1}^{3} \exp\left(-\frac{i\theta_j}{2} \Sigma_{\bar{j}} \right), \qquad \Sigma_{\bar{j}} = i\gamma_{2j-1}\gamma_{2j}, \tag{3.5.4}$$

a straightforward calculation shows that

$$S^{-1} \check{\gamma}_\mu S = \gamma_\mu, \qquad 1 \leq \mu \leq 6. \tag{3.5.5}$$

This rotation however leads to additional terms in the expression of $\underline{D}$, which we can also eliminate with a gauge transformation depending on the radii and of the form $e^B = \prod_{i=1}^{n} f_i(\rho_i)$. Requiring that after this additional transformation

$$\widetilde{\underline{D}} = \sum_{j=1}^{3} \left(\gamma_{2j-1} \frac{\partial}{\partial \rho_j} + \gamma_{2j} \frac{1}{\rho_j} \frac{\partial}{\partial \theta_j} \right) \tag{3.5.6}$$

imposes that

$$e^B = \prod_{j=1}^{3} \frac{1}{\sqrt{\rho_j}}. \tag{3.5.7}$$

The following transformation

$$\mathcal{O} \mapsto \widetilde{\mathcal{O}} = e^{-B} S^{-1} \mathcal{O} S e^B, \tag{3.5.8}$$

with e^B and S given by (3.5.7) and (3.5.4) respectively, is thus to be carried.

The angular momentum J_{12} (3.3.3) is one of the elements in the set whose commutant we looked for. It is transformed into

$$J_{12} \mapsto \widetilde{J}_{12} = e^{-B} S^{-1} \left(-i \frac{\partial}{\partial \theta_1} + \frac{1}{2} \Sigma_{12} \right) S e^{B} \tag{3.5.9}$$

$$= -i \frac{\partial}{\partial \theta_1} + \frac{1}{2} \Sigma_{12} + (-i) \left(\frac{-i}{2} \Sigma_{12} \right) = -i \frac{\partial}{\partial \theta_1} \tag{3.5.10}$$

and similar results hold for J_{34} and J_{56}.

We also have

$$\underline{x} \mapsto \underline{\widetilde{x}} = \sum_{j=1}^{3} \rho_j \, \gamma_{2j-1}, \tag{3.5.11}$$

$$\mathbb{E} \mapsto \widetilde{\mathbb{E}} = \sum_{j=1}^{3} \rho_j \frac{\partial}{\partial \rho_j}, \tag{3.5.12}$$

$$\Sigma_{\overline{j}} \mapsto \widetilde{\Sigma}_{\overline{j}} = \Sigma_{\overline{j}}. \tag{3.5.13}$$

Fixing $\widetilde{J}_{2j-1,2j} \sim k_j$ as a result of separation of variables, we can rewrite

$$\widetilde{\underline{D}} = \sum_{j=1}^{3} \left(\gamma_{2j-1} \frac{\partial}{\partial \rho_j} + \gamma_{2j} \frac{i k_j}{\rho_j} \right). \tag{3.5.14}$$

Note that these reduced operators still generate the same dynamical algebra since this is not altered by conjugation or separation of variables.

It is now interesting to investigate what is the effect of the reduction on the Casimirs operators. Recall that the reduced Casimir associated to the subset $\{\overline{i_1}, \ldots, \overline{i_n}\} = A \subset [6]$, with $\overline{j} \equiv \{2j-1, 2j\}$, is given by

$$\widetilde{\Gamma}_A = \frac{1}{2} \left([\underline{\widetilde{x}}_A, \, \widetilde{\underline{D}}_A] - 1 \right) \widetilde{\Sigma}_A, \qquad \widetilde{\Sigma}_A = \prod_{k=1}^{n} \Sigma_{\overline{i_k}}. \tag{3.5.15}$$

The reduced Casimirs:

$$\widetilde{\Gamma}_{\overline{i}}, \qquad \widetilde{\Gamma}_{\overline{i}\,\overline{j}}, \qquad \widetilde{\Gamma}_{\overline{i}\,\overline{j}\,\overline{k}}, \tag{3.5.16}$$

will satisfy the Bannai–Ito relations

$$\{\widetilde{\Gamma}_{\overline{i}\,\overline{j}}, \, \widetilde{\Gamma}_{\overline{j}\,\overline{k}}\} = \widetilde{\Gamma}_{\overline{i}\,\overline{k}} + 2\widetilde{\Gamma}_{\overline{j}}\widetilde{\Gamma}_{\overline{i}\,\overline{j}\,\overline{k}} + 2\widetilde{\Gamma}_{\overline{i}}\widetilde{\Gamma}_{\overline{k}} \tag{3.5.17}$$

with

$$\widetilde{\Gamma}_{\overline{j}} = \frac{1}{2} \left([\underline{\widetilde{x}}_{\overline{j}}, \, \widetilde{\underline{D}}_{\overline{j}}] - 1 \right) \Sigma_{\overline{j}} = \widetilde{J}_{\overline{j}} = k_j. \tag{3.5.18}$$

This should be compared with the system studied in [1], where the Dirac operator was given in terms of Dunkl derivatives D_i of the $\mathbb{Z}_2$ type

$$D_i = \partial_i + \frac{k_i}{x_i}(1 - R_i), \qquad R_i f(x_i) = f(-x_i). \tag{3.5.19}$$

The Bannai–Ito algebra was also seen to be the symmetry algebra in that case, with the one-index Casimirs Γ_j equal to the deformation parameters k_j. Here, we started with ordinary partial derivatives in twice as many dimensions and found the one-index Casimirs taking the values of the angular momenta that are diagonalized in the dimensional reduction. Hence, the reduced system obtained here offers a new model with Bannai–Ito symmetry in addition to the Dirac-Dunkl one.

3.6. The higher rank Bannai–Ito algebra as a commutant

Let us now show how one can extend the result of the previous sections to the higher rank Bannai–Ito algebra $B(n)$.

Take any triple of pairwise disjoint subsets of $[2n]$ called K, L, and M. There is an obvious isomorphism

$$\mathfrak{osp}^K(1|2) \otimes \mathfrak{osp}^L(1|2) \otimes \mathfrak{osp}^M(1|2) \cong \mathfrak{osp}(1|2) \otimes \mathfrak{osp}(1|2) \otimes \mathfrak{osp}(1|2), \tag{3.6.1}$$

so that the Casimir elements Γ_K, Γ_L, Γ_M, $\Gamma_{K\cup L}$, $\Gamma_{K\cup M}$, $\Gamma_{L\cup M}$, and $\Gamma_{K\cup L\cup M}$ will generate $B(3)$ and hence obey

$$\{\Gamma_{K\cup L}, \Gamma_{L\cup M}\} = \Gamma_{K\cup M} + 2\Gamma_L \Gamma_{K\cup L\cup M} + 2\Gamma_K \Gamma_M. \tag{3.6.2}$$

Now we wish to know $\{\Gamma_A, \Gamma_B\}$ for any two subsets A and B. To that end, take $K = A \setminus B$, $L = A \cap B$, $M = B \setminus A$ to see that in view of (3.6.2) the corresponding Casimirs satisfy

$$\{\Gamma_A, \Gamma_B\} = \Gamma_{(A\cup B)\setminus(A\cap B)} + 2\Gamma_{A\cap B}\Gamma_{A\cup B} + 2\Gamma_{A\setminus(A\cap B)}\Gamma_{B\setminus(A\cap B)}, \tag{3.6.3}$$

which are the structure relations of $B(n)$ given in (3.1.1) [2].

Underneath this quick derivation of (3.6.3) is the fact that the entire algebra $B(n)$ is realized by the Casimirs associated to 2-subsets since the general relations are inferred from those of $B(3)$. Let us map as we have done the 2-subsets $\{2i-1, 2i\}$, $i = 1, \ldots, n$ of $[2n]$ to the elements $\bar{i}$ of $[\overline{n}]$ so that obviously $\{\bar{i}, \bar{j}\} \in [\overline{n}]$ corresponds to $\{2i-1, 2i, 2j-1, 2j\} \in [2n]$. It follows that the structure relations for the 4-subsets in $[2n]$ are those of 2-subsets in $[\overline{n}]$. Working with the set of n integers viewed in this way and in light of the preceeding remark, it will suffice to examine the generators $\Gamma_{\bar{i}\,\bar{j}}$, namely the $\Gamma_{2i-1,2i,2j-1,2j}$ of the Dirac model in $2n$ dimensions.

We can now use Howe's duality to conclude that $B(n)$ is in the commutant of the subalgebra $\mathfrak{o}(2)^{\oplus n}$. Let $\mathcal{C}\ell_{2n}$ be the Clifford algebra generated by γ_μ, $\mu = 1, \ldots, 2n$ with relations as in (3.3.1). The spatial rotation generators are the $L_{\mu\nu}$ verifying the $\mathfrak{o}(2n)$ commutation relations (8.3.6).

Replacing 6 by $2n$ in equations (3.4.1), (3.4.2) provides operators $\underline{D}$, $\underline{x}$, $\mathbb{E}$ and $\mathscr{S}$ realizing $\mathfrak{osp}(1|2)$ in $2n$ dimensions.

Setting $J_- = -i\underline{D}$, $J_+ = -i\underline{x}$, $J_0 = \mathbb{E} + n$, the presentation of $\mathfrak{osp}(1|2)$ coincides with (3.2.1) and the total Casimir is $\Gamma_{[2n]} = \frac{1}{2}\left([J_-, J_+] - 1\right)\mathscr{S}$. The Casimirs associated to one or two labels $\bar{j}$ are easily calculated with the help of formula (3.4.7) which remains valid for $A \subset [2n]$. One finds exactly the same operators $\Gamma_{\bar{j}}$ and $\Gamma_{\bar{i}\bar{j}}$ with expressions as in (3.4.8) for $i, j = 1, \ldots, n$.

These Casimir operators are immediately recognized as the generators of the commutant of the set $\{J_{12}, \ldots, J_{2n-1,2n}\}$ in the spinorial representation of $U(\mathfrak{o}(2n))$.

Since we know that the intermediate Casimirs realize the commutation relations of the Bannai–Ito algebra, it is necessarily also the case for the generators of the commutant. This therefore confirms our claim to the effect that $B(n)$ is in the commutant of $\mathfrak{o}(2)^{\oplus n}$.

We can again perform the dimensional reduction of the $2n$-dimensional model to obtain a n-dimensional system whose symmetry algebra is $B(n)$. Write $\underline{D}$ in the cylindrical coordinates (3.5.1) with $j = 1, \ldots, n$ and the $\check{\gamma}_\mu$'s defined as in (3.5.3). Exactly as was done in six dimensions, rotate the $\check{\gamma}_\mu$'s to their original form with the help of an operator S extending (3.5.4) and accompany this with the gauge transformation defined by $e^B = \prod_{j=1}^n \rho_j^{-1/2}$, thus performing $\mathcal{O} \mapsto \widetilde{\mathcal{O}} = e^{-B} S^{-1} \mathcal{O} S e^B$. One then obtains for $\widetilde{\underline{D}}$, $\widetilde{\underline{x}}$ and $\widetilde{\mathbb{E}}$ the same expressions as in (3.5.6), (3.5.11) and (3.5.12) with the sum extending to n instead of stopping at 3.

The angular momenta $J_{2j-1,2j}$ (3.3.3) are then mapped to $\widetilde{J}_{2j-1,2j} = -i\frac{\partial}{\partial\theta_j}$. Fixing $\widetilde{J}_{2j-1,2j} \sim k_j$ once the ignorable variable is eliminated, we can rewrite $\widetilde{\underline{D}}$ as in (3.5.14) again extending the sum to n.

Note that the reduced operators still generate the same dynamical algebra.

The reduction of the Casimirs is as described from (3.5.15) to (3.5.18), except that A is now a subset of $[2n]$.

The reduced model thus obtained offers a new n-dimensional system in addition to the Dirac–Dunkl one, with the $B(n)$ Bannai–Ito algebra as its symmetry algebra. The one-index Casimirs in the different models respectively take the values of the angular momenta and the deformation parameters in the Dunkl-derivatives.

3.7. An embedding of the $R(n)$ Racah algebra in the $B(n)$ Bannai–Ito algebra

The higher rank Racah algebra $R(n)$ is an associative algebra with generators $\{C^i\}_{1\leq i\leq n}$, $\{P^{ij}\}_{1\leq i<j\leq n}$, and defining relations [13]:

$$[P^{ij}, P^{jk}] = 2F^{ijk}, \tag{3.7.1a}$$

$$[P^{jk}, F^{ijk}] = P^{ik}P^{jk} - P^{jk}P^{ij} + 2P^{ik}C^j - 2P^{ij}C^k, \tag{3.7.1b}$$

$$[P^{kl}, F^{ijk}] = P^{ik}P^{jl} - P^{il}P^{jk}, \tag{3.7.1c}$$

$$[F^{ijk}, F^{jkl}] = F^{jkl}P^{ij} - F^{ikl}\Big(P^{jk} + 2C^j\Big) - F^{ijk}P^{jl}, \tag{3.7.1d}$$

$$[F^{ijk}, F^{klm}] = F^{ilm}P^{jk} - P^{ik}F^{jlm}, \tag{3.7.1e}$$

where $i, j, k, l, m \in [n]$ are all different.

In [16] we identified the generators of the commutant of $\mathfrak{o}(2)^{\oplus n}$ (in oscillator representations of $U(\mathfrak{o}(2n))$) which are the invariants $\{G^i\}_{1\leq i\leq n}$, $\{K^{ij}\}_{1\leq i<j\leq n}$ given in (3.3.12). With the following redefinitions

$$\begin{aligned}
C^i &= -\frac{1}{4}G^i - \frac{1}{4}, \\
C^{ij} &= -\frac{1}{4}K^{ij}, \\
P^{ij} &= C^{ij} - C^i - C^j = -\frac{1}{4}K^{ij} + \frac{1}{4}\Big(G^i + G^j\Big) + \frac{1}{2},
\end{aligned} \tag{3.7.2}$$

a long but straightforward calculation in the oscillator realization showed that the defining relations (3.7.1) of the algebra $R(n)$ were obeyed.

It has been seen in [21] that $R(3)$ admits an embedding in $B(3)$. We already noted in Section 3.3 that the commutant picture brought this inclusion to the fore and we shall exploit it here together with the results in [16] to explicitly provide the embedding of $R(n)$ in $B(n)$.

The key point is that the intermediate Casimirs C^i and C^{ij} realizing the $R(n)$ algebra can be obtained from the intermediate sCasimirs of the $B(n)$ algebra.

We will only need to use the sCasimirs associated to 2 or 4 indices. They are given as follows:

$$S_{\mu\nu} = \Big(L_{\mu\nu}\gamma_\mu\gamma_\nu + \tfrac{1}{2}\Big) \tag{3.7.3}$$

$$S_{\mu\nu\rho\sigma} = \Big(L_{\mu\nu}\gamma_\mu\gamma_\nu + L_{\mu\rho}\gamma_\mu\gamma_\rho + L_{\mu\sigma}\gamma_\mu\gamma_\sigma + L_{\nu\rho}\gamma_\nu\gamma_\rho + L_{\nu\sigma}\gamma_\nu\gamma_\sigma + L_{\rho\sigma}\gamma_\rho\gamma_\sigma + \tfrac{3}{2}\Big) \tag{3.7.4}$$

and a straightforward calculation in the realization (3.3.4) allows to recover the C^i's and C^{ij}'s as defined in (3.7.2) through the following formulae:

$$C^i = \frac{1}{4}\left(S_{2i-1,2i}{}^2 - S_{2i-1,2i} - \tfrac{3}{4}\right), \tag{3.7.5}$$

$$C^{ij} = \frac{1}{4}\left(S_{2i-1,2i,2j-1,2j}{}^2 - S_{2i-1,2i,2j-1,2j} - \tfrac{3}{4}\right). \tag{3.7.6}$$

This readily gives the embedding of $R(n)$ inside $B(n)$.

3.8. Conclusion

This paper has offered a novel presentation of the Bannai–Ito algebra $B(n)$ in the commutant of $\mathfrak{o}(2) \oplus \cdots \oplus \mathfrak{o}(2)$ in the spinorial representation of $\mathfrak{o}(2n)$ associated to the Clifford algebra $C\ell_{2n}$. It has also indicated how this picture can be elegantly related to the definition of $B(n)$ in the centralizer in $U(\mathfrak{osp}(1|2)^{\otimes n})$ of the coproduct embedding of the Lie superalgebra $\mathfrak{osp}(1|2)$ in $\mathfrak{osp}(1|2)^{\otimes n}$ in the framework of the Howe duality associated to $\Big(Pin(2n)\,,\mathfrak{osp}(1|2)\Big)$. This called for the introduction of a model involving the (massless) Dirac equation in $2n$ dimensions and identifying the connection between the Casimir operators that describe the paired representations of the two mutually commuting algebras on the solution space.

Invariance under the subalgebra of $Pin(2n)$ allowed for dimensional reduction of the Dirac equation under separation of variables. This resulted in a model in $\mathbb{R}^n$ with Bannai–Ito symmetry without reflection operators that hence differ from the Dirac–Dunkl equation already known to possess the same symmetry.

The commutant picture for $B(n)$ made manifest the fact that the Racah algebra $R(n)$ can be embedded in $B(n)$. This observation had been made for $n = 3$ [21] and could here be explicitly extended.

Looking ahead it would be interesting to understand how various contractions of $B(n)$ (and $R(n)$) play out within the commutant presentation. The relation with superintegrable systems would certainly be worth exploring [22, 23].

The Racah algebra is associated to $\mathfrak{sl}(2)$ and the Bannai–Ito algebra to $\mathfrak{osp}(1|2)$. The Askey–Wilson algebra [24] is similarly related to the quantum algebra $U_q(\mathfrak{sl}(2))$. The rank 1 algebra encodes the bispectrality of the Askey–Wilson polynomials. Efforts are now deployed to construct the extensions to arbitrary ranks [25, 26]. It is natural to think that the Askey–Wilson algebra also admits a dual commutant presentation. We plan on examining this matter which could shed useful light on the higher rank construction. We hope to report on these questions we have raised in the near future.

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3.A. Connection with superconformal quantum mechanics

As an aside to our discussion, we wish to observe that the superconformal quantum Hamiltonian with $\mathfrak{sl}(2|1)$ symmetry presented in [17] can be obtained by dimensional reduction from the two-dimensional harmonic oscillator. The Dirac operator in 2D and the corresponding position and Euler operators are given by

$$\underline{D} = \gamma_1\partial_1 + \gamma_2\partial_2, \qquad \underline{x} = \gamma_1 x_1 + \gamma_2 x_2, \qquad \mathbb{E} = x_1\partial_1 + x_2\partial_2. \tag{3.A.1}$$

They generate the $\mathfrak{osp}(1|2)$ dynamical algebra (3.2.1), precisely realized upon defining $J_- = -i\underline{D}$, $J_+ = -i\underline{x}$, $J_0 = \mathbb{E} + 1$, and taking the algebra involution to be $\mathscr{S} = \Sigma_{12} = i\gamma_1\gamma_2$. The 2D harmonic oscillator Hamiltonian is the algebra element

$$H_{\text{h.osc.}} = \underline{D}^2 - \underline{x}^2 = -\left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2}\right) + \left({x_1}^2 + {x_2}^2\right) \tag{3.A.2}$$

and hence possesses this $\mathfrak{osp}(1|2)$ symmetry.

Performing the dimensional reduction $(x_1, x_2) \to \rho$, carrying the transformation $e^B S$ and fixing the angular momentum to be $J_{12} \sim k$, as explained in section 3.5, one obtains

$$\widetilde{\underline{D}} = \gamma_1\frac{\partial}{\partial\rho} + \gamma_2\frac{i\,k}{\rho}, \qquad \widetilde{\underline{x}} = \gamma_1\rho, \qquad \widetilde{\mathbb{E}} = \rho\frac{\partial}{\partial\rho}, \qquad \widetilde{\Sigma}_{12} = \Sigma_{12}. \tag{3.A.3}$$

With the gamma matrices realized in terms of the Pauli matrices as

$$\gamma_1 = i\sigma_1, \qquad \gamma_2 = i\sigma_2, \qquad \Sigma_{12} = \sigma_3, \tag{3.A.4}$$

the 2D harmonic oscillator Hamiltonian is "reduced" to

$$\widetilde{H}_{\text{h.osc.}} = -\frac{\partial^2}{\partial\rho^2} + \frac{k(k-\sigma_3)}{\rho^2} + \rho^2, \tag{3.A.5}$$

which is identified with the superconformal quantum mechanical model introduced and analyzed by Fubini and Rabinovici [17]. The supercharges are given by $\widetilde{\underline{D}}$ and $\widetilde{\underline{x}}$. In [17] the Hamiltonian (3.A.5) is actually observed to have the larger $\mathfrak{sl}(2|1)$ or $\mathfrak{osp}(2|2)$ symmetry.

This follows from the fact that Σ_{12} is an additional even symmetry which generates two supplementary supercharges when commuted with $\underline{D}$ and $\underline{x}$.

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Chapitre 4

The Higgs and Hahn algebras from a Howe duality perspective

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Abstract: The Hahn algebra encodes the bispectral properties of the eponymous orthogonal polynomials. In the discrete case, it is isomorphic to the polynomial algebra identified by Higgs as the symmetry algebra of the harmonic oscillator on the 2-sphere. These two algebras are recognized in the commutant of a $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ subalgebra of $\mathfrak{o}(4)$ in the oscillator representation of the universal algebra $U(\mathfrak{u}(4))$. This connection is further related to the embedding of the (discrete) Hahn algebra in $U(\mathfrak{su}(1,1)) \otimes U(\mathfrak{su}(1,1))$ in light of the dual action of the pair $\Big(\mathfrak{o}(4), \mathfrak{su}(1,1)\Big)$ on the state vectors of four harmonic oscillators. The two-dimensional singular oscillator is naturally seen by dimensional reduction to have the Higgs algebra as its symmetry algebra.

4.1. Introduction

This paper is concerned with the Higgs algebra [1–3] and the discrete version of the Hahn algebra [4, 5] which actually designate two different but isomorphic presentations of the same algebra [3, 6]. We aim to establish that this algebra arises in the commutant of a $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ subalgebra of $\mathfrak{o}(4)$ in the oscillator representation of the universal algebra $\mathcal{U}(\mathfrak{u}(4))$. We will moreover point out that this relation that the Higgs and Hahn algebras have with $\mathfrak{o}(4)$ is in duality, in the sense of Howe [7–9], with the one they are known to have with $\mathfrak{su}(1,1) \otimes \mathfrak{su}(1,1)$ [10, 11]. Let us start with some background.

The Hahn algebra has three generators $\widehat{K}_1$, $\widehat{K}_2$ and $\widehat{K}_3$ subjected to the relations

$$\begin{aligned}
&[\widehat{K}_1, \widehat{K}_2] = \widehat{K}_3, \\
&[\widehat{K}_2, \widehat{K}_3] = a\{\widehat{K}_1, \widehat{K}_2\} + b\widehat{K}_2 + c_1\widehat{K}_1 + d_1, \\
&[\widehat{K}_3, \widehat{K}_1] = a\widehat{K}_1{}^2 + b\widehat{K}_1 + c_2\widehat{K}_2 + d_2,
\end{aligned} \tag{4.1.1}$$

where $\{A, B\} = AB + BA$ and a, b, c_1, c_2, d_1, d_2 are structure constants. We assume $a \neq 0$ (otherwise (4.1.1) would be equivalent to the Lie algebra $\mathfrak{sl}(2)$). This algebra describes the eigenvalue problems of both the discrete and continuous Hahn polynomials [12]. We shall henceforth consider the discrete case where $c_2 < 0$ and which is realized by the bispectral operators of the Hahn polynomials (see [13] for instance). In this case, upon performing the affine transformation

$$\begin{aligned}
\widehat{K}_1 &= \frac{1}{2}\sqrt{-c_2}\, K_1 - \frac{b}{2a}, \\
\widehat{K}_2 &= -\frac{1}{2}aK_2 - \frac{c_1}{2a},
\end{aligned} \tag{4.1.2}$$

one can cast the commutation relations in the form

$$\begin{aligned}
&[K_1, K_2] = K_3, \\
&[K_2, K_3] = -2\{K_1, K_2\} + \delta_1, \\
&[K_3, K_1] = -2K_1{}^2 - 4K_2 + \delta_2,
\end{aligned} \tag{4.1.3}$$

with δ_1, δ_2 constants (or central elements).

The Hahn algebra admits an embedding in $U(\mathfrak{su}(1,1)) \otimes U(\mathfrak{su}(1,1))$ that we shall describe in details later as it is germane to our analysis. This observation underscores its connection to the Clebsch–Gordan problem for $\mathfrak{su}(1,1)$ (and $\mathfrak{su}(2)$).

The Higgs algebra can be viewed as a polynomial deformation of $\mathfrak{su}(2)$. It has three generators D, A_+, A_- satisfying the following commutation relations:

$$\begin{aligned}
&[D, A_\pm] = \pm 4A_\pm, \\
&[A_+, A_-] = -D^3 + \alpha_1 D + \alpha_2,
\end{aligned} \tag{4.1.4}$$

with α_1, α_2 central elements. That the Higgs algebra is isomorphic to the discrete Hahn algebra is readily seen by taking

$$\begin{aligned}
K_1 &= \frac{1}{2}D, \\
K_2 &= -\frac{1}{4}\left(A_+ + A_- + \frac{1}{2}D^2\right) + \frac{\alpha_1}{8}, \\
K_3 &= [K_1, K_2] = -\frac{1}{2}(A_+ - A_-),
\end{aligned} \tag{4.1.5}$$

and observing that the commutation relations (9.4.1) then follow from (4.1.4) with

$$\delta_1 = -\frac{\alpha_2}{4}, \qquad \delta_2 = \frac{\alpha_1}{2}. \tag{4.1.6}$$

Historically, the algebra defined in (4.1.4) was found by Higgs, hence the name, as the one realized by the conserved quantities of the Coulomb problem and harmonic oscillator on the two-sphere. It can be viewed as a deformed $\mathfrak{su}(2)$ algebra [14] or a truncation of the quantum algebra $U_q(\mathfrak{sl}(2))$ [6]. This algebra has been identified as the symmetry algebra of the Hartmann [4] and of certain ring-shaped potentials [5] as well as the singular oscillator in two dimensions [2, 3]. The Higgs algebra has moreover emerged in the Heisenberg quantization of identical particles [15]. Furthermore, it has been seen to coincide with the finite quantum W-algebra $W(\mathfrak{sp}(4), 2\mathfrak{sl}(2))$ [16, 17]. (For a review of finite W-algebras and their applications, see [18].)

Similarly to the Hahn case, the Racah algebra [3, 10, 19] is realized by the bispectral operators of the corresponding polynomials. It admits an embedding in $U(\mathfrak{su}(1,1))^{\otimes 3}$ with the intermediate Casimir elements representing the generators. The Hahn algebra can be obtained through a contraction of the standard presentation of the Racah algebra in a way that parallels the limit that takes the Racah polynomials into those of Hahn [12]. A generalization of the Racah algebra to higher ranks is found in [20].

Recently the Racah algebra has been interpreted in a Howe duality framework and shown to be in a commutant [21] in the enveloping algebra of $\mathfrak{o}(6)$, the Lie algebra of the rotation group in six dimensions. An extension of this result to the generalized Racah algebra is given in [22]. An analogous treatment of the Bannai-Ito algebra [23–25], which is in a sense a supersymmetric version of the Racah algebra, was also achieved in [26]. These advances raised the question of how to describe the Higgs algebra from a Howe duality perspective. The answer to this question will be provided here with the significant merit of expanding and interconnecting the various descriptions of the Higgs and Hahn algebras.

The remainder of the paper is organized as follows. As preparation background, familiar results on the metaplectic representation of $\mathfrak{su}(1,1)$ and the embedding of $\mathfrak{u}(4)$ in the Heisenberg-Weyl algebra will be reviewed in Section 4.2. The Higgs algebra will be obtained in the commutant of $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ in $U(\mathfrak{u}(4))$ in Section 4.3. The embedding of the Hahn algebra into $U(\mathfrak{su}(1,1)) \otimes U(\mathfrak{su}(1,1))$ will be described in Section 4.4. The two pictures of the Higgs/Hahn algebra presented in Sections 4.3 and 4.4 will be connected via the Howe dual pair $\big(\mathfrak{o}(4), \mathfrak{su}(1,1)\big)$ that acts on the state vectors of the four-dimensional oscillator. Dimensional reduction will be used in Section 4.6 to recover the fact that the symmetries of the singular oscillator in two dimensions generate the Higgs algebra. The paper will end with a summary of the findings and an outlook.

4.2. $\mathfrak{su}(1,1)$, $\mathfrak{u}(4)$ and oscillators

We shall be dealing with the Heisenberg–Weyl algebra $W(n)$ generated by n pairs of oscillator operators a_i, $a_i^\dagger$, $i=1,\ldots,n$, that satisfy

$$[a_i, a_j^\dagger] = \delta_{ij}, \qquad i,j = 1,\ldots,n. \tag{4.2.1}$$

The number operators $N_i = a_i^\dagger a_i$ are such that

$$[N_i, a_j] = -a_i\delta_{ij}, \qquad [N_i, a_j^\dagger] = a_i^\dagger \delta_{ij}. \tag{4.2.2}$$

In the position coordinates x_i, $i=1,\ldots,n$ these operators read

$$a_i = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_i} + x_i\right), \quad a_i^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x_i} + x_i\right), \quad N_i = -\frac{1}{2}\frac{\partial^2}{\partial {x_i}^2} + \frac{1}{2}{x_i}^2 - \frac{1}{2}. \tag{4.2.3}$$

The Lie algebra $\mathfrak{su}(1,1)$ has generators J_0, J_+, J_- obeying the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -2J_0. \tag{4.2.4}$$

Its Casimir element is given by

$$C = {J_0}^2 - J_+J_- - J_0. \tag{4.2.5}$$

Owing to the fact that $\mathfrak{su}(1,1)$ has a trivial coproduct, $J_0^{(12)} = J_0^{(1)} + J_0^{(2)}$, $J_\pm^{(12)} = J_\pm^{(1)} + J_\pm^{(2)}$ with $J_\bullet^{(1)} = J_\bullet \otimes 1$ and $J_\bullet^{(2)} = 1 \otimes J_\bullet$, defines an embedding of $\mathfrak{su}(1,1)$ into $\mathfrak{su}(1,1)\otimes\mathfrak{su}(1,1)$. This fact and the notation extend to $\mathfrak{su}(1,1)^{\otimes n}$.

The metaplectic representation of $\mathfrak{su}(1,1)$ is defined by the following map in $W(1)$:

$$\mathcal{J}_0^{(i)} = \frac{1}{2}\left(a_i^\dagger a_i + \frac{1}{2}\right), \qquad \mathcal{J}_+^{(i)} = \frac{1}{2}a_i^{\dagger 2}, \qquad \mathcal{J}_-^{(i)} = \frac{1}{2}{a_i}^2. \tag{4.2.6}$$

It consists in the direct sum of two irreducible $\mathfrak{su}(1,1)$ representations on the spaces spanned respectively by the eigenstates of $N_i = a_i^\dagger a_i$ with either even or odd eigenvalues. The Casimir element C has value $-3/16$ in that representation. In the following we shall consider $\mathcal{J}_\bullet^{(1234)} = \mathcal{J}_\bullet^{(12)} + \mathcal{J}_\bullet^{(34)}$ which provides an embedding of $\mathfrak{su}(1,1)$ into $W(4)$ as per the remarks above.

The Lie algebra $\mathfrak{u}(4)$ with generators E_{ij}, $i,j=1,\ldots,4$ admits the following realization à la Schwinger in $W(4)$:

$$E_{ij} = a_i^\dagger a_j, \qquad i,j = 1,\ldots,4. \tag{4.2.7}$$

The Hamiltonian of the isotropic harmonic oscillator in four dimensions:

$$H = N_1 + N_2 + N_3 + N_4 + 2 \tag{4.2.8}$$

is central, $[H, a_i^\dagger a_j] = 0$, and should be excluded from the 16 independant $a_i^\dagger a_j$ to deal with $\mathfrak{u}(4)$ per se. In this oscillator representation, the $\mathfrak{o}(4)$ subalgebra of $\mathfrak{u}(4)$ is spanned by the infinitesimal rotation generators

$$L_{jk} = \frac{i}{2}(a_j a_k^\dagger - a_j^\dagger a_k) = -\frac{i}{2}\left(x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}\right) \tag{4.2.9}$$

with commutation relations

$$[L_{jk}, L_{\ell m}] = \frac{i}{2}(L_{j\ell}\delta_{km} - L_{k\ell}\delta_{jm} + L_{km}\delta_{j\ell} - L_{jm}\delta_{k\ell}), \qquad j,k,\ell,m = 1,\dots,4. \tag{4.2.10}$$

4.3. The Higgs algebra as a commutant in $U(\mathfrak{u}(4))$

We are now ready to obtain our first main result, namely that the Higgs algebra can be defined as a commutant. Pick the $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ subalgebra of $\mathfrak{o}(4)$ generated by L_{12} and L_{34}; clearly $[L_{12}, L_{34}] = 0$. We want to concentrate on the commutant of this subalgebra in $U(\mathfrak{u}(4))$. We are thus looking for polynomials in the generators $a_i^\dagger a_j$, $i,j = 1,\dots,4$ that are invariant under rotations in both the $(1-2)$- and $(3-4)$-planes. It is not difficult to convince oneself that an integrity basis for that set is provided by the three operators

$$\begin{aligned} A_+ &= (a_1^{\dagger 2} + a_2^{\dagger 2})({a_3}^2 + {a_4}^2), \\ A_- &= ({a_1}^2 + {a_2}^2)(a_3^{\dagger 2} + a_4^{\dagger 2}), \\ D &= (N_1 + N_2) - (N_3 + N_4). \end{aligned} \tag{4.3.1}$$

$A_\pm$ and D are manifestly invariant under the rotations generated by L_{12} and L_{34} and they clearly commute with H (thus belonging to $U(\mathfrak{u}(4))$). All other elements of the commutant are built from those.

Let us now determine the commutation relations of these generators. It is immediate to see that

$$[D, A_\pm] = \pm 4 A_\pm. \tag{4.3.2}$$

There remains to evaluate $[A_+, A_-]$. Observe first that one has the following identities:

$$a_i^{\dagger 2} {a_i}^2 = {N_i}^2 - N_i \tag{4.3.3}$$

as well as

$${a_i}^2 a_j^{\dagger 2} + a_i^{\dagger 2} {a_j}^2 = 2N_i N_j + N_i + N_j - 4{L_{ij}}^2, \qquad i,j = 1,\dots,4. \tag{4.3.4}$$

A straightforward computation yields

$$\begin{aligned} [A_+, A_-] = {} & 4\left(a_1^{\dagger 2}{a_1}^2 + a_1^{\dagger 2}{a_2}^2 + a_2^{\dagger 2}{a_1}^2 + a_2^{\dagger 2}{a_2}^2\right)(N_3 + N_4 + 1) \\ & - 4(N_1 + N_2 + 1)\left(a_3^{\dagger 2}{a_3}^2 + a_3^{\dagger 2}{a_4}^2 + a_4^{\dagger 2}{a_3}^2 + a_4^{\dagger 2}{a_4}^2\right) \end{aligned} \tag{4.3.5}$$

which with the help of (4.3.3) and (4.3.4) is readily converted to

$$\begin{aligned}[A_+, A_-] = 4\left[(N_1+N_2)^2 - 4{L_{12}}^2\right](N_3+N_4+1) \\ - 4(N_1+N_2+1)\left[(N_3+N_4)^2 - 4{L_{34}}^2\right].\end{aligned} \tag{4.3.6}$$

Since

$$N_1 + N_2 = \frac{1}{2}(H + D - 2), \qquad N_3 + N_4 = \frac{1}{2}(H - D - 2), \tag{4.3.7}$$

upon substituting and after some algebra, one obtains

$$[A_+, A_-] = -D^3 + \left[H^2 + 8\left({L_{12}}^2 + {L_{34}}^2\right) - 4\right] D - 8\left({L_{12}}^2 - {L_{34}}^2\right) H. \tag{4.3.8}$$

Since H, L_{12}, L_{34} commute with all the generators, we thus conclude comparing with (4.1.4) that indeed the Higgs algebra is in the commutant in $U(\mathfrak{u}(4))$ of $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ with the structure "constants" given by

$$\begin{aligned}\alpha_1 &= H^2 + 8\left({L_{12}}^2 + {L_{34}}^2\right) - 4, \\ \alpha_2 &= -8\left({L_{12}}^2 - {L_{34}}^2\right) H.\end{aligned} \tag{4.3.9}$$

This provides a most simple characterization of the Higgs algebra.

We can translate these results in terms of the Hahn presentation. Substituting (4.3.1) in (4.1.5), using formula (4.3.4) and keeping in mind the expression for α_1 given in (4.3.9), one arrives at the following nice expressions

$$\begin{aligned}K_1 &= \frac{1}{2}\left[(N_1+N_2) - (N_3+N_4)\right], \\ K_2 &= {L_{12}}^2 + {L_{13}}^2 + {L_{14}}^2 + {L_{23}}^2 + {L_{24}}^2 + {L_{34}}^2, \\ K_3 &= [K_1, K_2],\end{aligned} \tag{4.3.10}$$

knowing that these operators will satisfy the commutation relations of the Hahn algebra given in (9.4.1) with

$$\begin{aligned}\delta_1 &= -\frac{\alpha_2}{4} = 2\left({L_{12}}^2 - {L_{34}}^2\right) H, \\ \delta_2 &= \ \ \frac{\alpha_1}{2} = \frac{1}{2}H^2 + 4\left({L_{12}}^2 + {L_{34}}^2\right) - 2.\end{aligned} \tag{4.3.11}$$

4.4. The embedding of the Hahn algebra into $U(\mathfrak{su}(1,1)) \otimes U(\mathfrak{su}(1,1))$

Let us here indicate how the Hahn algebra is embedded in the tensor product of $U(\mathfrak{su}(1,1))$ with itself. Let $\Delta : \mathfrak{su}(1,1) \to \mathfrak{su}(1,1) \otimes \mathfrak{su}(1,1)$ be the coproduct homomorphism with $\Delta(J_\bullet) = J_\bullet^{(12)} = J_\bullet^{(1)} + J_\bullet^{(2)}$ in the superscript notation introduced in Section 4.2.

Consider the following identification [10, 11]:

$$\begin{aligned} K_1 &= J_0^{(1)} - J_0^{(2)}, \\ K_2 &= \Delta(C) = \left[J_0^{(12)}\right]^2 - J_+^{(12)} J_-^{(12)} - J_0^{(12)}, \end{aligned} \tag{4.4.1}$$

that is K_2 is the image of the Casimir element under the coproduct. It is clear that the computation of the overlaps coefficients between the eigenbases of those two operators corresponds to the Clebsch–Gordan problem for $\mathfrak{su}(1,1)$.

A simple calculation gives

$$K_2 = C^{(1)} + C^{(2)} + 2J_0^{(1)} J_0^{(2)} - J_+^{(1)} J_-^{(2)} - J_-^{(1)} J_+^{(2)} \tag{4.4.2}$$

with $C^{(1)} = C \otimes 1$, $C^{(2)} = 1 \otimes C$ in keeping with the adopted notation. Let $K_3 = [K_1, K_2]$, one finds

$$K_3 = -2\left(J_+^{(1)} J_-^{(2)} - J_-^{(1)} J_+^{(2)}\right). \tag{4.4.3}$$

One can now proceed to determine the commutators of K_3 with K_1 and K_2 and one gets:

$$\begin{aligned} [K_3, K_1] &= -2{K_1}^2 - 4K_2 + 2\left(J_0^{(1)} + J_0^{(2)}\right)^2 + 4\left(C^{(1)} + C^{(2)}\right), \\ [K_2, K_3] &= -2\{K_1, K_2\} + 4\left(J_0^{(1)} + J_0^{(2)}\right)\left(C^{(1)} - C^{(2)}\right). \end{aligned} \tag{4.4.4}$$

While the first is immediately obtained, a little bit of algebra involving the $\mathfrak{su}(1,1)$ commutation relations and its Casimir operator gives the second.

Note that $J_0^{(1)} + J_0^{(2)}$ is central since it commutes with K_1 and K_2 by construction.

We recognize in (4.4.4) the commutation relations (9.4.1) of the (centrally extended) Hahn algebra with

$$\begin{aligned} \delta_1 &= 4\left(J_0^{(1)} + J_0^{(2)}\right)\left(C^{(1)} - C^{(2)}\right), \\ \delta_2 &= 2\left(J_0^{(1)} + J_0^{(2)}\right)^2 + 4\left(C^{(1)} + C^{(2)}\right). \end{aligned} \tag{4.4.5}$$

We thus have with the formulas (4.4.1), the embedding of the Hahn algebra in $U(\mathfrak{su}(1,1)) \otimes U(\mathfrak{su}(1,1))$. What relation this has to do with the commutant picture will be adressed next.

4.5. The Howe duality connection

We shall now indicate that the two descriptions of the Hahn algebra presented in Section 4.3 and 4.4 can be connected through Howe duality. It is known (see in particular [9]) that there is a pairing between the representations of $\mathfrak{o}(4)$ and $\mathfrak{su}(1,1)$ that act in a mutually commuting way (see (4.5.3)) on the state space of the four-dimensional harmonic oscillator. We shall exploit this to show that the embedding of the Hahn algebra in the double tensor product of the universal enveloping algebra of one algebra of the pair, $\mathfrak{su}(1,1)$, is in duality with embedding in the commutant (in the universal algebra of $\mathfrak{u}(4)$) of the $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ subalgebra of the other algebra of the pair $\mathfrak{o}(4)$.

Let us consider the addition of four metaplectic representations (4.2.6) grouped in two pairs, that is take

$$\mathcal{J}_{\bullet}^{(1234)} = \mathcal{J}_{\bullet}^{(12)} + \mathcal{J}_{\bullet}^{(34)} \tag{4.5.1}$$

with

$$\begin{aligned} \mathcal{J}_0^{(ij)} &= \frac{1}{2}\left[N_i + N_j + 1\right], \\ \mathcal{J}_+^{(ij)} &= \frac{1}{2}\left(a_i^{\dagger 2} + a_j^{\dagger 2}\right), \\ \mathcal{J}_-^{(ij)} &= \frac{1}{2}\left({a_i}^2 + {a_j}^2\right). \end{aligned} \tag{4.5.2}$$

Note that

$$[L_{ij}, \mathcal{J}_{\bullet}^{(1234)}] = 0 \qquad \forall i,j = 1,\ldots,4. \tag{4.5.3}$$

We shall put $\mathcal{J}_{\bullet}^{(12)}$ and $\mathcal{J}_{\bullet}^{(34)}$ in correspondance with the $J_{\bullet}^{(1)}$ and $J_{\bullet}^{(2)}$ of Section 4.4. In this model,

$$\mathcal{K}_1 = \mathcal{J}_0^{(12)} - \mathcal{J}_0^{(34)} = \frac{1}{2}\left[(N_1 + N_2) - (N_3 + N_4)\right] \tag{4.5.4}$$

which is identical with the expression in (4.3.10) for K_1 arising from the commutant approach. For $\mathcal{K}_2$ we have

$$\mathcal{K}_2 = \mathcal{C}^{(1234)} = \left[\mathcal{J}_0^{(12)} + \mathcal{J}_0^{(34)}\right]^2 - \left(\mathcal{J}_+^{(12)} + \mathcal{J}_+^{(34)}\right)\left(\mathcal{J}_-^{(12)} + \mathcal{J}_-^{(34)}\right) - \left(\mathcal{J}_0^{(12)} + \mathcal{J}_0^{(34)}\right). \tag{4.5.5}$$

Using (4.5.2), this becomes

$$\mathcal{K}_2 = \frac{1}{4}H^2 - \frac{1}{2}H - \frac{1}{4}\left(a_1^{\dagger 2} + a_2^{\dagger 2} + a_3^{\dagger 2} + a_4^{\dagger 2}\right)\left({a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2\right) \tag{4.5.6}$$

and with the help of formulas (4.3.3) and (4.3.4), we have

$$\mathcal{K}_2 = \frac{1}{4}H^2 - \frac{1}{2}H - \frac{1}{4}(A_+ + A_-) - \frac{1}{4}\left((N_1+N_2)^2 + (N_3+N_4)^2 - 4{L_{12}}^2 - 4{L_{34}}^2\right). \tag{4.5.7}$$

This can be rewritten as

$$\mathcal{K}_2 = -\frac{1}{4}\left(A_+ + A_- + \frac{1}{2}D^2\right) + \frac{1}{8}H^2 + {L_{12}}^2 + {L_{34}}^2 - \frac{1}{2} \tag{4.5.8}$$

which coincides with the expression that was found when looking for generators commuting with L_{12} and L_{34}. Recall that we also found that the expression (9.3.3) can identically be reexpressed as $\mathcal{K}_2 = {L_{12}}^2 + {L_{13}}^2 + {L_{14}}^2 + {L_{23}}^2 + {L_{24}}^2 + {L_{34}}^2$ which makes it also manifest that $\mathcal{K}_2$, calculated as a $\mathfrak{su}(1,1)$ Casimir, belongs to the commutant of $\{L_{12}, L_{34}\}$ in $U(\mathfrak{o}(4)) \subset U(\mathfrak{u}(4))$.

A similar computation shows that the $\mathfrak{su}(1,1)$ Casimir for the representation $\mathcal{J}_\bullet^{(ij)}$ is given by the square of the corresponding rotation generator in $\mathfrak{o}(4)$, namely

$$\mathcal{C}^{(ij)} = {L_{ij}}^2 - \frac{1}{4}. \tag{4.5.9}$$

It follows that the structure constants become on the basis of (4.4.5):

$$\begin{aligned}
\delta_1 &= 4\left(\mathcal{J}_0^{(12)} + \mathcal{J}_0^{(34)}\right)\left(\mathcal{C}^{(12)} - \mathcal{C}^{(34)}\right) \\
&= 2H\left({L_{12}}^2 - {L_{34}}^2\right), \\
\delta_2 &= 2\left(\mathcal{J}_0^{(12)} + \mathcal{J}_0^{(34)}\right)^2 + 4\left(\mathcal{C}^{(12)} + \mathcal{C}^{(34)}\right) \\
&= \frac{1}{2}H^2 + 4\left({L_{12}}^2 + {L_{34}}^2\right) - 2,
\end{aligned} \tag{4.5.10}$$

in perfect correspondance with (4.3.11). Of course $\mathcal{K}_3 = [\mathcal{K}_1, \mathcal{K}_2]$.

Owing to the pairing of the $\mathfrak{su}(1,1)$ and $\mathfrak{o}(4)$ representations under Howe duality, it is found that the embedding of the Hahn algebra into $U(\mathfrak{su}(1,1)) \otimes U(\mathfrak{su}(1,1))$ leads to its description as a commutant in $U(\mathfrak{u}(4))$.

4.6. Dimensional reduction and the singular oscillator in two dimensions

We shall now carry the dimensional reduction of the four-dimensional isotropic harmonic oscillator under the $O(2) \times O(2)$ action to identify in this way the Higgs/Hahn symmetry of the singular oscillator in the plane.

Make the change of variables

$$x_{2j-1} = \rho_j \cos\theta_j, \qquad x_{2j} = \rho_j \sin\theta_j, \qquad j = 1,2. \tag{4.6.1}$$

Eliminate the θ_i's by separating the variables with

$$L_{2j-1,2j} = -\frac{i}{2}\frac{\partial}{\partial\theta_j}. \tag{4.6.2}$$

Take the eigenvalues of this operator equal to $-\frac{i}{2}k_j$. After performing the gauge transformation $\mathcal{O} \to \widetilde{\mathcal{O}} = (\rho_1\rho_2)^{1/2}\mathcal{O}(\rho_1\rho_2)^{-1/2}$ one sees that the $\mathfrak{su}(1,1)$ operators become:

$$\begin{aligned}
\widetilde{\mathcal{J}}_0^{(2i-1,2i)} &= \frac{1}{4}\left[-\frac{\partial^2}{\partial{\rho_i}^2} - \frac{a_i}{{\rho_i}^2} + {\rho_i}^2\right], \\
\widetilde{\mathcal{J}}_\pm^{(2i-1,2i)} &= \frac{1}{4}\left[\left(\rho_i \mp \frac{\partial}{\partial\rho_i}\right)^2 + \frac{a_i}{{\rho_i}^2}\right],
\end{aligned} \qquad a_i = {k_i}^2 + \frac{1}{4}, \qquad i = 1,2. \tag{4.6.3}$$

The Hamiltonian of the singular oscillator in two dimensions is thus given by

$$\widetilde{H} = 2\left[\widetilde{\mathcal{J}}_0^{(12)} + \widetilde{\mathcal{J}}_0^{(34)}\right] = -\frac{1}{2}\left(\frac{\partial^2}{\partial {\rho_1}^2} + \frac{\partial^2}{\partial {\rho_2}^2}\right) + \frac{1}{2}\left({\rho_1}^2 + {\rho_2}^2 - \frac{a_1}{{\rho_1}^2} - \frac{a_2}{{\rho_2}^2}\right). \tag{4.6.4}$$

The constants of motion are clearly

$$\begin{aligned} K_1 &= \widetilde{\mathcal{J}}_0^{(12)} - \widetilde{\mathcal{J}}_0^{(34)}, \\ K_2 &= \widetilde{C}^{(1234)}, \\ K_3 &= [K_1, K_2]. \end{aligned} \tag{4.6.5}$$

We know from our construction that these will close to form the Hahn algebra. The (reduced) Casimir $\widetilde{C}^{(1234)} = \left(\widetilde{J}_0\right)^2 - \widetilde{J}_+\widetilde{J}_- - \widetilde{J}_0$, with $\widetilde{J}_\bullet = \widetilde{\mathcal{J}}_\bullet^{(12)} + \widetilde{\mathcal{J}}_\bullet^{(34)}$ is easily computed and one finds

$$\begin{aligned} K_2 &= -\frac{1}{4}\left[\left(\rho_1\frac{\partial}{\partial\rho_2} - \rho_2\frac{\partial}{\partial\rho_1}\right)^2 + a_1\left(\frac{{\rho_2}^2}{{\rho_1}^2} + 1\right) + a_2\left(\frac{{\rho_1}^2}{{\rho_2}^2} + 1\right) + 1\right], \\ K_3 &= \frac{1}{4}\left[\left(2\rho_1\frac{\partial}{\partial\rho_1} + 1\right)\left(\frac{\partial^2}{\partial{\rho_2}^2} + {\rho_2}^2 + \frac{a_2}{{\rho_2}^2}\right) - \left(2\rho_2\frac{\partial}{\partial\rho_2} + 1\right)\left(\frac{\partial^2}{\partial{\rho_1}^2} + {\rho_1}^2 + \frac{a_1}{{\rho_1}^2}\right)\right]. \end{aligned} \tag{4.6.6}$$

This approach, which combines the commutant viewpoint via the dimensional reduction under the torus group action and the $\mathfrak{su}(1,1)$ embedding through the metaplectic representation, provides an alternative and straightforward way of showing that the Hahn algebra is the symmetry algebra of the singular oscillator.

4.7. Conclusion

This paper has provided a synthetic description of the Higgs and Hahn algebras in light of Howe duality. With the understanding that the Higgs and the (discrete) Hahn algebras are isomorphic, we have shown that this algebra can be viewed as a commutant in $U(\mathfrak{u}(4))$. It has also been recalled that it can be embedded in the tensor product of $U(\mathfrak{su}(1,1))$ with itself. The two approaches have been linked in view of the fact that $\mathfrak{o}(4)$ and $\mathfrak{su}(1,1)$ form a dual pair on the state space of the harmonic oscillator in four dimensions. This has also provided context to identify the Hahn symmetry of the singular oscillator in two dimensions through dimensional reduction.

In this respect, one might think of obtaining the higher rank Hahn algebras and by that token the symmetries of the singular oscillator in higher dimensions, by considering the commutant of the sum of n $\mathfrak{o}(2)$'s in $U(\mathfrak{u}(2n))$. Take for instance $n = 3$. the resulting commutant in $U(\mathfrak{u}(6))$ would have as subalgebras two Hahn algebras associated to the $(\overline{12})$ and $(\overline{23})$ coordinate sectors as well as the Racah algebra also, since we know [21] it is the commutant of $\mathfrak{o}(2) \oplus \mathfrak{o}(2) \oplus \mathfrak{o}(2)$ in $U(\mathfrak{o}(6)) \subset U(\mathfrak{u}(6))$. The entire mixed Hahn–Racah algebra will be an interesting deformation of $\mathfrak{su}(3)$. Its analysis would certainly warrant particular attention as this algebra will encompass in particular the properties of the connection coefficients for the various separated solutions of singular oscillators in higher dimensions [27, 28]. We plan to return to this question from this angle.

We would also wish to determine if some Howe duality operates in the case of the algebras, like the Askey–Wilson one, associated to q-polynomials. Examining the q-Hahn algebra to that end in the wake of the present study might prove illuminating and is in our plans.

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Chapitre 5

The q-Higgs and Askey–Wilson algebras

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Abstract: A q-analogue of the Higgs algebra, which describes the symmetry properties of the harmonic oscillator on the 2-sphere, is obtained in the commutant of the $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2)$ subalgebra of $\mathfrak{o}_{q^{1/2}}(4)$ in the q-oscillator representation of the quantized universal enveloping algebra $U_q(\mathfrak{u}(4))$. This q-Higgs algebra is also found as a specialization of the Askey–Wilson algebra embedded in the tensor product $U_q(\mathfrak{su}(1,1)) \otimes U_q(\mathfrak{su}(1,1))$. The connection between these two approaches is established on the basis of the Howe duality of the pair $\Big(\mathfrak{o}_{q^{1/2}}(4), U_q(\mathfrak{su}(1,1))\Big)$.

5.1. Introduction

The Higgs algebra was first obtained by Higgs [1] as the algebra of the conserved quantities of the Coulomb problem and harmonic oscillator on the 2-sphere. Shown to be isomorphic to the Hahn algebra [2], it was also identified as the symmetry algebra of the Hartmann potential [3], of certain ring-shaped potentials [4] and of the singular oscillator in two dimensions [5, 6]. The Higgs algebra stands between Lie algebras and quantized universal enveloping algebras, as it can be viewed both as a deformation of the $\mathfrak{su}(2)$ Lie algebra [7] and a truncation of the $U_q(\mathfrak{sl}_2)$ quantum algebra [8]. It has been obtained as the quantum finite W-algebra $W(\mathfrak{sp}(4), 2\,\mathfrak{sl}(2))$ [9, 10] and has also appeared in the context of Heisenberg quantization of identical particles [11].

The Higgs algebra can be presented in the following form

$$
\begin{aligned}
[D, A_\pm] &= \pm 4 A_\pm, \\
[A_+, A_-] &= -D^3 + \alpha_1 D + \alpha_2,
\end{aligned}
\tag{5.1.1}
$$

where α_1, α_2 are central elements.

We here aim to construct a q-deformation of (5.1.1) that preserves the general algebraic underpinnings of this structure. This will lead to an algebra that differs from the one in [12] where a certain q-extension of the Higgs algebra was defined by simply replacing the cubic expression in D by one involving q-numbers (see (5.2.1)).

We propose to obtain a q-analogue of the Higgs algebra by following a commutant approach similar to [13] (see also [14, 15]), where the ordinary Higgs algebra was obtained in the commutant of the $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ subalgebra of $\mathfrak{o}(4)$ in the oscillator representation of $U(\mathfrak{u}(4))$. This characterization was shown to be in duality in the sense of Howe [16–19] with the well-established embedding of the Hahn algebra in $U(\mathfrak{su}(1,1)) \otimes U(\mathfrak{su}(1,1))$ [20, 21]. While Howe duality, sometimes called "complementarity", has not been thoroughly studied in the context of q-algebras (see for instance [22–26]), the results in [27] will provide appropriate background for our purposes. The merit of the approach we propose is that the q-Higgs algebra obtained in a commutant also appears in a dual fashion as a specialization of the Askey–Wilson algebra [28–30] in the tensor product $U_q(\mathfrak{su}(1,1))^{\otimes 2}$.

Let us now briefly present the contents of the paper. In Section 5.2, the q-deformations of $\mathfrak{su}(1,1)$ and $\mathfrak{o}(n)$ (respectively denoted $U_q(\mathfrak{su}(1,1))$ and $\mathfrak{o}_q(n)$) will be introduced along with their q-oscillator realizations. In Section 5.3, a q-deformation of the Higgs algebra will be obtained in a commutant of $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2) \subset \mathfrak{o}_{q^{1/2}}(4)$ in the q-oscillator realization of $U_q(\mathfrak{u}(4))$. The embedding of a special case of the Askey–Wilson algebra into $U_q(\mathfrak{su}(1,1))^{\otimes 2}$ will be presented in Section 5.4. As will be shown in Section 5.5, the q-Higgs algebra proves to be isomorphic to that specialization of the Askey–Wilson algebra, and this result will be explained by invoking the fact that the pair $\left(\mathfrak{o}_{q^{1/2}}(4), U_q(\mathfrak{su}(1,1))\right)$ behaves as a Howe dual pair in this context. Concluding remarks and perspectives will form the last section.

5.2. The $U_q(\mathfrak{su}(1,1))$, $\mathfrak{o}_q(n)$ algebras and their q-oscillators realizations

The duality connection that we shall invoke in our discussion involves the algebras $U_q(\mathfrak{su}(1,1))$ and $\mathfrak{o}_q(n)$. We shall thus begin by introducing these algebras and their q-oscillator realizations.

Let q be a complex number such that $|q| < 1$. One defines for any number x the following q-numbers:

$$
(x)_q := \frac{1 - q^x}{1 - q} \quad \text{and} \quad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}.
\tag{5.2.1}
$$

The same notation will be used for operators.

5.2.1. The $U_q(\mathfrak{su}(1,1))$ and $\mathfrak{o}_q(n)$ quantum algebras

$U_q(\mathfrak{sl}_2)$ [31, 32] is the quantized universal enveloping algebra with three generators j_0 and $j_\pm$ subjected to the relations

$$[j_0\,, j_\pm] = \pm j_\pm, \qquad [j_+\,, j_-] = [2j_0]_q. \tag{5.2.2}$$

It is endowed with a Hopf structure with coproduct $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$

$$\Delta(j_0) = j_0 \otimes 1 + 1 \otimes j_0, \qquad \Delta(j_+) = j_+ \otimes q^{2j_0} + 1 \otimes j_+, \qquad \Delta(j_-) = j_- \otimes 1 + q^{-2j_0} \otimes j_-. \tag{5.2.3}$$

We shall denote by $U_q(\mathfrak{su}(1,1))$ the non-compact real form of $U_q(\mathfrak{sl}_2)$ that has the three generators $J_\pm$ and J_0 obeying

$$[J_0\,, J_\pm] = \pm J_\pm, \qquad J_- J_+ - q^2 J_+ J_- = q^{2J_0}\,[2J_0]_q. \tag{5.2.4}$$

The coproduct $\Delta : U_q(\mathfrak{su}(1,1)) \to U_q(\mathfrak{su}(1,1)) \otimes U_q(\mathfrak{su}(1,1))$ will read

$$\Delta(J_0) = J_0 \otimes 1 + 1 \otimes J_0, \qquad \Delta(J_\pm) = J_\pm \otimes q^{2J_0} + 1 \otimes J_\pm. \tag{5.2.5}$$

The Casimir operator C of this algebra has the following expression

$$C = J_+ J_- q^{-2J_0+1} - \frac{q}{(q^2-1)^2}\left(q^{2J_0-1} + q^{-2J_0+1}\right) + \frac{q^2+1}{(q^2-1)^2}. \tag{5.2.6}$$

The coproduct being an algebra morphism, the relations (5.2.5) define an embedding of $U_q(\mathfrak{su}(1,1))$ into $U_q(\mathfrak{su}(1,1)) \otimes U_q(\mathfrak{su}(1,1))$.

Remark 5.1. *In the limit $q \to 1$, one recovers the usual $\mathfrak{su}(1,1)$ Lie algebra with Casimir operator $C = J_+ J_- - {J_0}^2 + J_0$. Moreover, the standard presentation of $U_q(\mathfrak{su}(1,1))$ [33] is recovered if one considers instead the generators $\widetilde{J}_0 = J_0$, $\widetilde{J}_+ = J_+ q^{-J_0}$ and $\widetilde{J}_- = q^{-J_0} J_-$, which satisfy the commutation relations $\left[\widetilde{J}_0\,, \widetilde{J}_\pm\right] = \pm \widetilde{J}_\pm$ and $\left[\widetilde{J}_-\,, \widetilde{J}_+\right] = \left[2\widetilde{J}_0\right]_q$ and have co-commutative coproduct.*

We introduce next the non-standard q-deformation $\mathfrak{o}_q(n)$ of $\mathfrak{o}(n)$ which is defined as the associative unital algebra with generators $L_{i,i+1}$ $(i = 1, \ldots, n-1)$ and relations

$$L_{i-1,i}\, {L_{i,i+1}}^2 - (q+q^{-1}) L_{i,i+1}\, L_{i-1,i}\, L_{i,i+1} + {L_{i,i+1}}^2\, L_{i-1,i} = -L_{i-1,i}, \tag{5.2.7a}$$

$$L_{i,i+1}\, {L_{i-1,i}}^2 - (q+q^{-1}) L_{i-1,i}\, L_{i,i+1}\, L_{i-1,i} + {L_{i-1,i}}^2\, L_{i,i+1} = -L_{i,i+1}, \tag{5.2.7b}$$

$$[L_{i,i+1}\,, L_{j,j+1}] = 0 \quad \text{for} \quad |i-j| > 1. \tag{5.2.7c}$$

In the literature, this non-standard deformation is often denoted $U'_q(\mathfrak{so}_n)$, see for instance [34–37]. It has been shown in [38] that $\mathfrak{o}_q(n)$ can be viewed as a q-analogue of the symmetric space based on the pair $(\mathfrak{gl}(n), \mathfrak{o}(n))$. Although it has no Hopf structure on its own, it is a coideal subalgebra of $U_q(\mathfrak{sl}(n))$ [38] and appears in many areas of mathematical physics [36].

The two cases where $n = 3$ and $n = 4$ are especially of interest to us.

Let us first note that it is possible to consider a so-called "Cartesian" presentation [39–41] of $U_q(\mathfrak{sl}_2)$, in which the three generators play an "equitable" role, and which corresponds to the non-standard deformation $\mathfrak{o}_q(3)$ (equivalently $U'_q(\mathfrak{so}_3)$ in refs. [40, 41]) of the universal enveloping algebra $U(\mathfrak{so}(3))$, obtained by modifying the defining relations for the skew-symmetric generators of $\mathfrak{so}(3)$.

It goes like this. With j_0, $j_\pm$, the $U_q(\mathfrak{sl}_2)$ generators, form the following elements:

$$\begin{aligned} j_1 &= ig\Big\{q^{\frac{1}{2}j_0}\,,\, j_+ + j_-\Big\}, \\ j_2 &= g\Big\{q^{-\frac{1}{2}j_0}\,,\, j_+ - j_-\Big\}, \end{aligned} \qquad g = \frac{1}{(q^{\frac{1}{4}} + q^{-\frac{1}{4}})(q^{\frac{1}{2}} + q^{-\frac{1}{2}})}, \tag{5.2.8}$$

where $\{a, b\} = ab + ba$ is the anticommutator and g is a normalization factor. Defining $j_3 \equiv [j_1, j_2]_q$, where $[a, b]_q := q^{\frac{1}{2}}ab - q^{-\frac{1}{2}}ba$ is the q-commutator, j_1, j_2 and j_3 then satisfy the "Cartesian" relations

$$[j_1, j_2]_q = j_3, \qquad [j_2, j_3]_q = j_1, \qquad [j_3, j_1]_q = j_2. \tag{5.2.9}$$

Upon identifying $L_{12} = j_1$, $L_{23} = j_2$, one finds that this corresponds precisely to the relations (6.3.1) for the algebra $\mathfrak{o}_q(3)$. Note that the relations (6.3.1c) do not exist in this case.

For what follows, it will also be useful to have the formulas for $\mathfrak{o}_q(4)$ in full. These relations read [42]

$$L_{12}\, {L_{23}}^2 - (q + q^{-1})L_{23}\, L_{12}\, L_{23} + {L_{23}}^2\, L_{12} = -L_{12}, \tag{5.2.10a}$$

$$L_{23}\, {L_{12}}^2 - (q + q^{-1})L_{12}\, L_{23}\, L_{12} + {L_{12}}^2\, L_{23} = -L_{23}, \tag{5.2.10b}$$

$$L_{23}\, {L_{34}}^2 - (q + q^{-1})L_{34}\, L_{23}\, L_{34} + {L_{34}}^2\, L_{23} = -L_{23}, \tag{5.2.10c}$$

$$L_{34}\, {L_{23}}^2 - (q + q^{-1})L_{23}\, L_{34}\, L_{23} + L_{23}^2\, L_{34} = -L_{34}, \tag{5.2.10d}$$

$$[L_{12}\,, L_{34}] = 0. \tag{5.2.10e}$$

It is immediate to see that L_{12}, L_{23} and L_{23}, L_{34} respectively generate two $\mathfrak{o}_q(3)$ subalgebras of $\mathfrak{o}_q(4)$, however they do not appear within a direct sum, in contrast to what happens with $\mathfrak{o}(4)$.

If one introduces the following elements:

$$L^\pm_{13} = [L_{12}\,, L_{23}]_{q^{\pm 1}}, \qquad L^\pm_{24} = [L_{23}\,, L_{34}]_{q^{\pm 1}}, \qquad L^\pm_{14} = [L^\pm_{13}\,, L_{34}]_{q^{\pm 1}}, \tag{5.2.11}$$

where $[a, b]_q$ is defined as above and $[a, b]_{q^{-1}} := q^{-\frac{1}{2}}ab - q^{\frac{1}{2}}ba$, the two independent Casimir operators of the algebra $\mathfrak{o}_q(4)$ are then given by [27, 34, 41]

$$C_4 = q^{-2}{L_{12}}^2 + {L_{23}}^2 + q^2{L_{34}}^2 + q^{-1}L^+_{13}\, L^-_{13} + qL^+_{24}\, L^-_{24} + L^+_{14}\, L^-_{14}, \tag{5.2.12a}$$

$$C'_4 = q^{-1}L_{12}\, L_{34} - L^+_{13}\, L^+_{24} + qL_{23}\, L^+_{14}. \tag{5.2.12b}$$

5.2.2. The q-oscillator algebras, Schwinger and metaplectic realizations

Let us now recall the properties of the q-oscillator operators that will be used to realize the algebras presented above. The q-oscillator algebra $\mathcal{A}_q(n)$ [43–45] is defined as the unital associative algebra over $\mathbb{C}$ generated by n independent sets of q-oscillators $\{A_i^{\pm},\, A_i^0\}$ verifying

$$[A_i^0\,, A_i^{\pm}] = \pm A_i^{\pm}, \qquad [A_i^-\,, A_i^+] = q^{A_i^0}, \qquad A_i^- A_i^+ - q A_i^+ A_i^- = 1, \qquad i = 1, \dots, n\,, \tag{5.2.13}$$

and such that the commutators between elements with different indices i are equal to zero. The last two relations allow one to express $N_i = A_i^+ A_i^-$ in terms of A_i^0:

$$N_i = A_i^+ A_i^- = \frac{1 - q^{A_i^0}}{1 - q} = (A_i^0)_q. \tag{5.2.14}$$

In the limit $q \to 1$, A_i^0 coincides with the usual number operator N_i.

The q-oscillator algebra has the following representation on the space spanned by the standard occupancy number states $|n_1, \cdots, n_n\rangle = |n_1\rangle \otimes \cdots \otimes |n_n\rangle$ ($n_i \in \mathbb{N}$):

$$A_i^0 |n_i\rangle = n_i |n_i\rangle, \qquad A_i^+ |n_i\rangle = \sqrt{\frac{1 - q^{n_i+1}}{1 - q}} |n_i + 1\rangle, \qquad A_i^- |n_i\rangle = \sqrt{\frac{1 - q^{n_i}}{1 - q}} |n_i - 1\rangle. \tag{5.2.15}$$

These commuting q-oscillators can now be used to build realizations of the algebras considered above.

Firstly, the algebra $\mathfrak{o}_q(3)$ can be realized à la Schwinger in terms of two q-oscillators. More precisely, using the homomorphism $\chi : U_q(\mathfrak{sl}_2) \to \mathcal{A}_q(2)$ given by

$$\chi(j_0) = \tfrac{1}{2}(A_1^0 - A_2^0), \qquad \chi(j_+) = q^{-\frac{1}{4}(A_1^0 + A_2^0 - 1)}\, A_1^+ A_2^-\,, \qquad \chi(j_-) = q^{-\frac{1}{4}(A_1^0 + A_2^0 - 1)}\, A_1^- A_2^+\,, \tag{5.2.16}$$

and the identification (5.2.8), the following realization of $\mathfrak{o}_q(3)$ is obtained:

$$\chi(j_1) = \frac{i q^{\frac{1}{4}}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}\, q^{-\frac{1}{2} A_2^0} \Big(q^{\frac{1}{4}} A_1^- A_2^+ + q^{-\frac{1}{4}} A_1^+ A_2^- \Big), \tag{5.2.17}$$

$$\chi(j_2) = \frac{q^{\frac{1}{4}}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}\, q^{-\frac{1}{2} A_1^0} \Big(q^{\frac{1}{4}} A_1^+ A_2^- - q^{-\frac{1}{4}} A_1^- A_2^+ \Big). \tag{5.2.18}$$

Another key ingredient is the metaplectic realization of $U_q(\mathfrak{su}(1,1))$, which is given by the homomorphism $\mu : U_q(\mathfrak{su}(1,1)) \to \mathcal{A}_q(1)$:

$$\mu(J_0) = \mathscr{J}_0 = \frac{1}{2}\left(A^0 + \frac{1}{2}\right), \qquad \mu(J_\pm) = \mathscr{J}_\pm = \frac{1}{[2]_{q^{1/2}}} (A^\pm)^2. \tag{5.2.19}$$

One sees immediately that it is a q-deformation of the usual metaplectic representation of $\mathfrak{su}(1,1)$.

Finally, we shall also use the realization of $\mathfrak{o}_{q^{1/2}}(4)$ in terms of 4 q-oscillators which is provided by:

$$\mathscr{L}_{i,i+1} = q^{-\frac{1}{2}A_i^0+\frac{1}{4}}\Big(q^{\frac{1}{4}}A_i^+A_{i+1}^- - q^{-\frac{1}{4}}A_i^-A_{i+1}^+\Big), \qquad i=1,2,3. \tag{5.2.20}$$

One checks that the $\mathscr{L}_{i,i+1}$ indeed verify relations of the form (5.2.10) but whose q's have been replaced by $q^{1/2}$'s. Furthermore, $\mathscr{L}_{12}$, $\mathscr{L}_{34}$ commute and hence generate a $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2)$ subalgebra of $\mathfrak{o}_{q^{1/2}}(4)$.

5.3. The commutant of $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2)$ in the q-oscillator realization of $U_q(\mathfrak{u}(4))$ and the q-Higgs algebra

It was shown in [13] that the Higgs algebra appears in the commutant of $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ in the universal enveloping algebra $U(\mathfrak{u}(4))$. This section aims to define the q-Higgs algebra through a q-analogue of this commutant picture.

We consider first the $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2)$ subalgebra of $\mathfrak{o}_{q^{1/2}}(4)$ generated by $\mathscr{L}_{12}$ and $\mathscr{L}_{34}$, and look for its commutant in $U_q(\mathfrak{u}(4))$.

Introduce the following three operators

$$M^+ = \left(q^{A_2^0+\frac{1}{2}}(A_1^+)^2 + (A_2^+)^2\right)\left(q^{A_4^0+\frac{1}{2}}(A_3^-)^2 + (A_4^-)^2\right), \tag{5.3.1a}$$

$$M^- = \left(q^{A_2^0+\frac{1}{2}}(A_1^-)^2 + (A_2^-)^2\right)\left(q^{A_4^0+\frac{1}{2}}(A_3^+)^2 + (A_4^+)^2\right), \tag{5.3.1b}$$

$$L = (A_1^0 + A_2^0) - (A_3^0 + A_4^0), \tag{5.3.1c}$$

which commute with the generators $\mathscr{L}_{12}$ and $\mathscr{L}_{34}$ (in the limit $q \to 1$, $\mathscr{L}_{12}$ and $\mathscr{L}_{34}$ correspond to rotations in the $(1,2)$ and $(3,4)$ planes).

One notes that each big parenthesis in the expression of the $M^\pm$ operators can actually be obtained by applying the coproduct of $U_q(\mathfrak{su}(1,1))$ to the $\mathscr{J}_\pm$ generators. Recalling that the bilinears of the form $E_{ij} = A_i^+A_j^-$, $i,j = 1,2,3,4$ realize the $U_q(\mathfrak{u}(4))$ algebra [46], it can be observed that $M^\pm$, L generate the non-trivial part of the commutant of $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2)$ in the q-oscillator realization of $U_q(\mathfrak{u}(4))$.

It is immediate to see that $M^\pm$ and L also commute with the central element

$$H = \sum_{i=1}^{4}(A_i^0 + \tfrac{1}{2}). \tag{5.3.2}$$

One could ask how were the expressions for L, $M^\pm$ obtained. First, the operator L obviously commutes with $\mathscr{L}_{12}$ and $\mathscr{L}_{34}$. Second, instead of obtaining the factors in $M^\pm$ from the coproduct one can look for elements $T^\pm$ in $\mathcal{A}_q(2)$ that commute with $\mathscr{L}_{12}$; this is most easily done "on-shell", that is, by solving $[\mathscr{L}_{12}, T^\pm]|n_1, n_2\rangle = 0$ for any two q-oscillator states. One thus arrives at

$$T^{\pm} = q^{\alpha(A_1^0+A_2^0)}\Big(q^{A_2^0+\frac{1}{2}}(A_1^{\pm})^2 + (A_2^{\pm})^2\Big), \qquad \alpha \in \mathbb{C}. \tag{5.3.3}$$

Since $A_1^0 + A_2^0 = \frac{1}{2}(L+H)$, only the second factor of $T^{\pm}$ is relevant. The same is done with $\mathscr{L}_{34}$ on the direct product states $|n_3, n_4\rangle$. It is then clear that the only combinations of the operators (5.3.3) and their $(3,4)$ analogues that will belong to $U_q(\mathfrak{u}(4))$ are those occurring in $M^{\pm}$.

It now remains to determine the algebra formed by the three generators $M^{\pm}$ and L.

Proposition 5.2. *The operators $M^{\pm}$ and L have the following commutators:*

$$\begin{aligned}
[L, M^{\pm}] &= \pm 4 M^{\pm},\\
[M^+, M^-] &= \frac{(1+q)}{q(1-q)^3}\, q^H\Big((q+q^{-1})(q^L - q^{-L}) - 2\Big(q^{\frac{1}{2}H} + q^{-\frac{1}{2}H}\Big)(q^{\frac{1}{2}L} - q^{-\frac{1}{2}L})\Big)\\
&\quad + \frac{(1+q)}{q^2(1-q)}\, q^H\Big(\Big(q^{-\frac{1}{2}H}{\mathscr{L}_{12}}^2 + q^{\frac{1}{2}H}{\mathscr{L}_{34}}^2\Big)q^{\frac{1}{2}L} - \Big(q^{\frac{1}{2}H}{\mathscr{L}_{12}}^2 + q^{-\frac{1}{2}H}{\mathscr{L}_{34}}^2\Big)q^{-\frac{1}{2}L}\Big).
\end{aligned} \tag{5.3.4a}$$

The elements $\mathscr{L}_{12}$, $\mathscr{L}_{34}$ and H, which are central, play the role of structure constants. We shall take these relations to define abstractly the (universal) q-Higgs algebra.

Remark 5.3. *Alternatively, if one considers the generator $q^{\frac{1}{2}L}$ instead of L, the first set of relations in* (5.3.4a) *becomes*

$$q^{\frac{1}{2}L} M^{\pm} = q^{\pm 2} M^{\pm} q^{\frac{1}{2}L}. \tag{5.3.4b}$$

PROOF. The first relations of (5.3.4a) are obvious. The last relation is obtained by a direct computation in the q-oscillator algebra. Starting with (5.3.1a)–(5.3.1b), and using the identity $[a_1^+a_2^-, a_1^-a_2^+] = [a_1^+, a_1^-]a_2^+a_2^- - a_1^+a_1^-[a_2^+, a_2^-]$ for $a_i^{\pm} = \Big(q^{A_{2i}^0+\frac{1}{2}}(A_{2i-1}^{\pm})^2 + (A_{2i}^{\pm})^2\Big)$, one gets

$$\begin{aligned}
&[M^+, M^-] =\\
&\Big[q^{2A_2^0+1}\Big[(A_1^+)^2, (A_1^-)^2\Big] + \Big[(A_2^+)^2, (A_2^-)^2\Big] + q^{A_2^0+\frac{1}{2}}(1-q^2)\Big((A_1^+)^2(A_2^-)^2 + q^{-2}(A_1^-)^2(A_2^+)^2\Big)\Big]\\
&\times\Big[q^{2A_4^0+1}(A_3^+)^2(A_3^-)^2 + (A_4^+)^2(A_4^-)^2 + q^{A_4^0+\frac{1}{2}}\Big((A_3^+)^2(A_4^-)^2 + q^{-2}(A_3^-)^2(A_4^+)^2\Big)\Big]\\
&-\Big[q^{2A_4^0+1}\Big[(A_3^+)^2, (A_3^-)^2\Big] + \Big[(A_4^+)^2, (A_4^-)^2\Big] + q^{A_4^0+\frac{1}{2}}(1-q^2)\Big((A_3^+)^2(A_4^-)^2 + q^{-2}(A_3^-)^2(A_4^+)^2\Big)\Big]\\
&\times\Big[q^{2A_2^0+1}(A_1^+)^2(A_1^-)^2 + (A_2^+)^2(A_2^-)^2 + q^{A_2^0+\frac{1}{2}}\Big((A_1^+)^2(A_2^-)^2 + q^{-2}(A_1^-)^2(A_2^+)^2\Big)\Big].
\end{aligned} \tag{5.3.5}$$

Now, from the expression (5.2.20), one obtains

$${\mathscr{L}_{12}}^2 = q^{-A_1^0+\frac{1}{2}}\Big(q(A_1^+)^2(A_2^-)^2 + q^{-1}(A_1^-)^2(A_2^+)^2 - q^{\frac{1}{2}}N_1 - q^{-\frac{1}{2}}N_2 - q(q^{\frac{1}{2}} + q^{-\frac{1}{2}})N_1N_2\Big) \tag{5.3.6}$$

and a similar expression for ${\mathscr{L}_{34}}^2$ with the replacement $A_1^{\bullet}, A_2^{\bullet} \to A_3^{\bullet}, A_4^{\bullet}$.

Using the relations

$$\Big[(A_i^+)^2,(A_i^-)^2\Big] = -(1+q)q^{A_i^0}\Big((q+q^{-1})N_i+1\Big) \tag{5.3.7}$$

and

$$q(A_i^+)^2(A_i^-)^2 = N_i^2 - N_i, \tag{5.3.8}$$

and after some algebra, one is left with the following equation

$$\begin{aligned}
&[M^+,M^-] = \\
&\Big[(1+q)q^{A_1^0+A_2^0-1}\Big((1-q)\mathscr{L}_{12}{}^2+\tfrac{q}{1-q}\left(q^{A_1^0+A_2^0}(1+q^2)-2\right)\Big)\Big]\Big[q^{A_3^0+A_4^0-1}\mathscr{L}_{34}{}^2+[(A_3^0+A_4^0)_q]^2\Big] \\
&-\Big[(1+q)q^{A_3^0+A_4^0-1}\Big((1-q)\mathscr{L}_{34}{}^2+\tfrac{q}{1-q}\left(q^{A_3^0+A_4^0}(1+q^2)-2\right)\Big)\Big]\Big[q^{A_1^0+A_2^0-1}\mathscr{L}_{12}{}^2+[(A_1^0+A_2^0)_q]^2\Big].
\end{aligned} \tag{5.3.9}$$

Expressing the A_i^0 generators in terms of L and H, one finally obtains the desired commutation relation.

Remark 5.4. *In the limit $q\to 1$, noting that*

$$\lim_{q\to1}\mathscr{L}_{12} = 2i\,\mathcal{L}_{12}, \qquad \lim_{q\to1}\mathscr{L}_{34} = 2i\,\mathcal{L}_{34}, \qquad \textit{with}\quad \mathcal{L}_{jk} = -\frac{i}{2}\left(x_j\frac{\partial}{\partial x_k}-x_k\frac{\partial}{\partial x_j}\right), \tag{5.3.10}$$

one easily recovers from (5.3.4a) *the commutation relations of the Higgs algebra* (5.1.1) *in the form:*

$$\begin{aligned}
[L,M^\pm] &= \pm 4M^\pm, \\
[M^+,M^-] &= -L^3+\alpha_1 L+\alpha_2,
\end{aligned} \tag{5.3.11}$$

where $\alpha_1 = H^2+8(\mathcal{L}_{12}{}^2+\mathcal{L}_{34}{}^2)-4$, and $\alpha_2 = -8H(\mathcal{L}_{12}{}^2-\mathcal{L}_{34}{}^2)$.

Hence, the relations (5.3.4) indeed define a q-deformation of the Higgs algebra.

5.4. The Askey–Wilson algebra and an embedding into $U_q(\mathfrak{su}(1,1))^{\otimes 2}$

We now indicate how (a special case of) the Askey–Wilson algebra can be embedded in the tensor product $U_q(\mathfrak{su}(1,1))\otimes U_q(\mathfrak{su}(1,1))$. With Δ the coproduct of $U_q(\mathfrak{su}(1,1))$ given in (5.2.5), we can take

$$K_1 = \frac{1}{4}\,\frac{1-q^{J_0}\otimes q^{-J_0}}{1-q}, \tag{5.4.1a}$$

$$\begin{aligned}
K_2 = \frac{1}{2}\,\Delta(C) = \frac{1}{2}\Big(&C\otimes q^{2J_0}+q^{-2J_0}\otimes C+J_+q^{-2J_0-1}\otimes J_-+J_-q^{-2J_0+1}\otimes J_+ \\
&+\frac{q^2+1}{(q^2-1)^2}\Big(q^{-2J_0}\otimes q^{2J_0}-1\otimes q^{2J_0}-q^{-2J_0}\otimes 1+1\otimes 1\Big)\Big),
\end{aligned} \tag{5.4.1b}$$

where C denotes the Casimir operator given in (6.2.7).

Defining $K_3 = [K_1, K_2]$, a direct calculation gives

$$K_3 = \frac{1}{8}\,(1+q^{-1})\Big(J_+ \otimes J_- - J_- \otimes J_+\Big)\Big(q^{-J_0} \otimes q^{-J_0}\Big). \tag{5.4.1c}$$

We now proceed to calculate the commutation relations of K_1, K_2, K_3. They are seen to take the form of the relations of the Askey–Wilson (AW) algebra which read

$$[K_1, K_2] = K_3, \tag{5.4.2a}$$

$$[K_2, K_3] = rK_2K_1K_2 + \xi_1\{K_1, K_2\} + \xi_2 K_2^2 + \xi_3 K_2 + \xi_4 K_1 + \xi_5, \tag{5.4.2b}$$

$$[K_3, K_1] = rK_1K_2K_1 + \xi_1 K_1^2 + \xi_2\{K_1, K_2\} + \xi_3 K_1 + \xi_6 K_2 + \xi_7, \tag{5.4.2c}$$

where r is as in (5.4.3) below and $\xi_1, \ldots, \xi_7$ are arbitrary in the generic AW situation.

After a rather cumbersome calculation, using the expressions (5.4.1) for the K_i's as well as the commutation relations (5.2.4), one finds that the K_i's indeed obey the relations (5.4.2) with the following specific expressions for the parameters:

$$\begin{gathered} r = -(q-q^{-1})^2, \qquad \xi_1 = \frac{1+q^{-2}}{2}, \qquad \xi_2 = \frac{(1+q)^2(1-q)}{4q^2}, \qquad \xi_3 = 4(q-1)\,\xi_7, \\ \xi_4 = 0, \qquad \xi_5 = -\frac{(1+q)(1+q^2)}{16q^3}(C^{(1)} - C^{(2)})\Big[J_0^{(12)}\Big]_q, \qquad \xi_6 = -\frac{(1+q)^2}{16q^2}, \\ \xi_7 = \frac{(1+q)^2}{32q^2}\left(C^{(1)}q^{J_0^{(12)}} + C^{(2)}q^{-J_0^{(12)}} - (1+q^{-2})\Big[\tfrac{1}{2}J_0^{(12)}\Big]_q^2\right), \end{gathered} \tag{5.4.3}$$

where $C^{(1)} = C \otimes 1$ and $C^{(2)} = 1 \otimes C$ are respectively the Casimir operators in the spaces 1 and 2 of the tensor product, and $J_0^{(12)} = \Delta(J_0)$. These quantities $C^{(i)}$ and $J_0^{(12)}$ commute with K_1, K_2 and K_3 and we hence have a version of (5.4.2) that is centrally extended.

Since there are only three independent quantities entering the ξ_i's (there are four in the general case), we conclude that the K_1, K_2, K_3 generate a specialization of the Askey–Wilson algebra. One checks that in the limit $q \to 1$, the parameters r, ξ_2, ξ_3 vanish and one recovers the Hahn algebra. The standard q-Hahn algebra is obtained from the Askey–Wilson algebra by setting for instance $\xi_1 = 0$ in (5.4.2). The algebra satisfied by K_1, K_2 and K_3 is actually isomorphic to the q-Hahn algebra as the standard form of the latter [28, 47] is obtained by taking $K_2 = \widetilde{K}_2 - \xi_1/r$. The limit $q \to 1$ is singular however if we adopt this presentation.

5.5. The q-Higgs algebra, the Askey–Wilson algebra, and the dual pair $\left(\mathfrak{o}_{q^{1/2}}(4)\,, U_q(\mathfrak{su}(1,1))\right)$

We shall explain in this section how the q-Higgs algebra obtained as a commutant and the specialized Askey–Wilson algebra found from the embedding just described are connected through Howe duality and are in fact isomorphic.

Take 4 metaplectic representations defined as in (5.2.19). We will add them first pairwise using the $U_q(\mathfrak{su}(1,1))$ coproduct (5.2.5):

$$\mathscr{J}_0^{(2i-1,2i)} = \tfrac{1}{2}\Big(A^0_{2i-1} + A^0_{2i} + 1\Big), \quad \mathscr{J}_\pm^{(2i-1,2i)} = \frac{1}{[2]_{q^{1/2}}}\Big(q^{A^0_{2i}+\frac{1}{2}}(A^+_{2i-1})^2 + (A^+_{2i})^2\Big), \quad i = 1, 2, \tag{5.5.1}$$

and then using the coproduct once more will form

$$\mathscr{J}_0^{(1234)} = \mathscr{J}_0^{(12)} + \mathscr{J}_0^{(34)} \qquad \text{and} \qquad \mathscr{J}_\pm^{(1234)} = \mathscr{J}_\pm^{(12)} q^{2\mathscr{J}_0^{(34)}} + \mathscr{J}_\pm^{(34)}. \tag{5.5.2}$$

Mindful of Section 5.3, it is immediate to check that

$$[\mathscr{L}_{i,i+1}, \mathscr{J}_\bullet^{(1234)}] = 0, \qquad i = 1, 2, 3, \tag{5.5.3}$$

where the $\mathscr{L}_{i,i+1}$ are defined as in (5.2.20). Let us stress that (5.5.3) makes the key statement that the algebras $U_q(\mathfrak{su}(1,1))$ and $\mathfrak{o}_{q^{1/2}}(4)$ are mutually commuting in the q-oscillator realization.

It has been shown [27] that $\mathfrak{o}_{q^{1/2}}(4)$ and $U_q(\mathfrak{su}(1,1))$ actually form a Howe dual pair. (They constitute precisely the quantum analogue of the classical pair $(\mathfrak{o}(4), \mathfrak{su}(1,1))$ which was used in the analysis of the Higgs and Hahn algebras [13].) This means that their representations can be connected through their Casimirs. We will now proceed to indicate explicitly how this is realized.

To that end, we first put the $\mathscr{J}_\bullet^{(2i-1,2i)}$ in correspondence with the $J_\bullet$ from Section 5.4. Let us focus on the coproduct embeddings (5.5.1). As each pairing of $U_q(\mathfrak{su}(1,1))$ in the spaces $(1,2)$ and $(3,4)$ gives a copy of $U_q(\mathfrak{su}(1,1))$, we can embed the specialization of the Askey–Wilson algebra of Section 5.4 into these two copies of $U_q(\mathfrak{su}(1,1))$.

Indeed, upon substitution of (5.5.1) into equations (5.4.1) for K_1, K_2, K_3, we obtain the following q-oscillator realization of the specialized Askey–Wilson algebra:

$$\mathscr{K}_1 = \frac{1}{4}\,\frac{1-q^{\frac{1}{2}L}}{1-q}, \tag{5.5.4a}$$

$$\begin{aligned}\mathscr{K}_2 = \frac{1}{2}\Delta^{(3)}(C) = \frac{1}{2}\Bigg(&\Big(\mathscr{C}^{(1)}q^{\frac{1}{2}H} + \mathscr{C}^{(2)}q^{-\frac{1}{2}H}\Big)q^{-\frac{1}{2}L} + (1+q^{-2})\,q^{-\frac{1}{2}L}\Big[\tfrac{L+H}{4}\Big]_q\Big[\tfrac{L-H}{4}\Big]_q \\ &+ \frac{q}{(1+q)^2}\Big(q^{-1}M_+ + qM_-\Big)q^{-\frac{1}{2}(H+L)}\Bigg),\end{aligned} \tag{5.5.4b}$$

$$\mathscr{K}_3 = \frac{1}{8(1+q)}\,\Big(M_+ - M_-\Big)q^{-\frac{1}{2}H}, \tag{5.5.4c}$$

where $\Delta^{(n)}(x) = (\mathrm{id}^{\otimes(n-1)} \otimes \Delta)\Delta^{(n-1)}(x)$, $\Delta^{(1)} = \Delta$, $\Delta^{(0)} = \mathrm{id}$; the generators $M_\pm$, L, H correspond to those given in (5.3.1) and (5.3.2) respectively and can alternatively be expressed as

$$M_{\pm} = ([2]_{q^{1/2}})^2 \mathscr{J}_{\pm}^{(12)} \mathscr{J}_{\mp}^{(34)}, \tag{5.5.5}$$

$$\tfrac{1}{2}L = \mathscr{J}_0^{(12)} - \mathscr{J}_0^{(34)}, \tag{5.5.6}$$

$$\tfrac{1}{2}H = \mathscr{J}_0^{(12)} + \mathscr{J}_0^{(34)} = \Delta^{(3)}(\mathscr{J}_0). \tag{5.5.7}$$

By construction the $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_3$ obey the relations of the specialized Askey–Wilson algebra (5.4.2) with the parameters (5.4.3).

Also note that the quantities $\mathscr{C}^{(1)}$ and $\mathscr{C}^{(2)}$ are the images of the Casimir operators $C^{(1)}$ and $C^{(2)}$ and that they are directly related to the $\mathscr{L}_{12}$ and $\mathscr{L}_{34}$ by

$$\mathscr{C}^{(1)} = \frac{1}{(1+q)^2}\left({\mathscr{L}_{12}}^2 + 1\right), \qquad \mathscr{C}^{(2)} = \frac{1}{(1+q)^2}\left({\mathscr{L}_{34}}^2 + 1\right). \tag{5.5.8}$$

In view of this, it is now evident that in the q-oscillator realization, the generators of the specialized Askey–Wilson algebra are expressible in terms of those of the q-Higgs algebra, and vice-versa. Hence these two algebras are isomorphic, as in the $q \to 1$ case.

To wrap things up, let us point out that the two Casimirs of $\mathfrak{o}_{q^{1/2}}(4)$ given in (5.2.12) have a direct interpretation in this q-oscillator framework.

The first Casimir of $\mathfrak{o}_{q^{1/2}}(4)$, denoted C_4, corresponds to the total Casimir of the quadruple tensor product of $U_q(\mathfrak{su}(1,1))$:

$$\begin{aligned} C_4 &= \left(q^{-1}{\mathscr{L}_{12}}^2 + {\mathscr{L}_{23}}^2 + q{\mathscr{L}_{34}}^2 + q^{-\frac{1}{2}}\mathscr{L}_{13}^{+}\mathscr{L}_{13}^{-} + q^{\frac{1}{2}}\mathscr{L}_{24}^{+}\mathscr{L}_{24}^{-} + \mathscr{L}_{14}^{+}\mathscr{L}_{14}^{-}\right) \\ &= \frac{(1+q)^2}{2}\Delta^{(3)}(C). \end{aligned} \tag{5.5.9}$$

This is precisely the pairing of the Casimirs of $\mathfrak{o}_{q^{1/2}}(4)$ and $U_q(\mathfrak{su}(1,1))$ that follows from the Howe duality.

The second Casimir of $\mathfrak{o}_{q^{1/2}}(4)$, denoted C_4', is identically zero in the q-oscillator realization:

$$C_4' = q^{-\frac{1}{2}}\mathscr{L}_{12}\mathscr{L}_{34} - \mathscr{L}_{13}^{+}\mathscr{L}_{24}^{+} + q^{\frac{1}{2}}\mathscr{L}_{23}\mathscr{L}_{14}^{+} = 0. \tag{5.5.10}$$

It can be seen as the q-analogue of the usual relation between the angular momenta, see for instance (4.1) in [48]: $M_{12}M_{34} + M_{13}M_{42} + M_{14}M_{23} = 0$.

Let us mention in closing this section that the $q \to 1$ limit of the above yields straightforwardly the duality presented in [13] between the Higgs or the Hahn algebras viewed as a commutant in $U(\mathfrak{u}(4))$ or embedded in $U(\mathfrak{su}(1,1)) \otimes U(\mathfrak{su}(1,1))$.

5.6. Conclusion

Summing up, we have introduced a q-analogue of the Higgs algebra by looking for the commutant of a $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2)$ subalgebra of $\mathfrak{o}_{q^{1/2}}(4)$ in the q-oscillator representation of

$U_q(\mathfrak{u}(4))$. This algebra was then seen to be isomorphic to a special case of the Askey–Wilson algebra (itself isomorphic to the standard q-deformation of the Hahn algebra) which has an embedding in $U_q(\mathfrak{su}(1,1)) \otimes U_q(\mathfrak{su}(1,1))$. The Howe dual pair $\left(\mathfrak{o}_{q^{1/2}}(4), U_q(\mathfrak{su}(1,1))\right)$ was then invoked as the reason behind this double picture.

The q-oscillator realization in which $\mathfrak{o}_{q^{1/2}}(4)$ and $U_q(\mathfrak{su}(1,1))$ commute can be generalized easily for $\mathfrak{o}_{q^{1/2}}(n)$ with n arbitrary. It is known that $\left(\mathfrak{o}_{q^{1/2}}(n), U_q(\mathfrak{su}(1,1))\right)$ is a dual pair [27]. This opens up the door to the study of the full Askey–Wilson algebra. We hypothesize that it should be possible to obtain this algebra in the commutant of a $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2)$ subalgebra of $\mathfrak{o}_{q^{1/2}}(6)$ in $\mathfrak{o}_{q^{1/2}}(6)$ in this q-oscillator representation. It would be also of high interest to see if the higher rank Askey–Wilson algebras [49, 50] could be obtained in a similar fashion.

It should moreover be mentioned that the dual pair $\left(\mathfrak{o}_{q^{1/2}}(n), U_q(\mathfrak{su}(1,1))\right)$ was analyzed in [27] in a q-commuting variable framework. It would be quite interesting to see if some sort of dimensional reduction in q-commuting variables could be performed to obtain a q-analogue of the superintegrable model on the n-sphere [51]. We hope to address all these questions in the near future.

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Chapitre 6

The dual pair $(U_q(\mathfrak{su}(1,1)), \mathfrak{o}_{q^{1/2}}(2n))$, q-oscillators and Askey–Wilson algebras

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Abstract: The universal Askey–Wilson algebra $AW(3)$ can be obtained in the commutant of $U_q(\mathfrak{su}(1,1))$ in $U_q(\mathfrak{su}(1,1))^{\otimes 3}$. We analyze the commutant of $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2)$ in q-oscillator representations of $\mathfrak{o}_{q^{1/2}}(6)$ and show that it also realizes $AW(3)$. These two pictures of $AW(3)$ are shown to be dual in the sense of Howe; this is made clear by highlighting the role of the intermediate Casimir elements of each member of the dual pair $\Big(U_q(\mathfrak{su}(1,1)), \mathfrak{o}_{q^{1/2}}(6)\Big)$. We also generalize these results. A higher rank extension of the Askey–Wilson algebra denoted $AW(n)$ can be defined in the commutant of $U_q(\mathfrak{su}(1,1))$ in $U_q(\mathfrak{su}(1,1))^{\otimes n}$ and a dual description of $AW(n)$ as the commutant of $\mathfrak{o}_{q^{1/2}}(2)^{\oplus n}$ in q-oscillator representations of $\mathfrak{o}_{q^{1/2}}(2n)$ is offered by calling upon the dual pair $\Big(U_q(\mathfrak{su}(1,1)), \mathfrak{o}_{q^{1/2}}(2n)\Big)$.

6.1. Introduction

The Askey–Wilson algebra encodes the bispectrality properties of the eponym polynomials [1]. It is finding applications in various areas such as integrable models [2–6], algebraic combinatorics [7–10], knot theory [11], double affine Hecke algebras and representation theory [12–14], etc. A universal extension is known to arise in an algebraic description of the Racah problem for $U_q(\mathfrak{sl}_2)$ [15, 16]. The goal of the present paper is to enlarge the fundamental understanding of this algebra by casting it in an alternate framework. We shall offer a picture of the universal Askey–Wilson algebra that is dual to the one which arises in the coupling of three representations of $U_q(\mathfrak{sl}_2)$ and that will be reviewed in Section 6.2. The presentation that is the object of this paper will be deemed dual, in the sense of Howe, to the $U_q(\mathfrak{sl}_2)$ tensorial approach because it will rely on the complementary member $\mathfrak{o}_{q^{1/2}}(2n)$ of the dual pair $\Big(U_q(\mathfrak{su}(1,1)), \mathfrak{o}_{q^{1/2}}(2n)\Big)$.

The concept of dual pairs has been introduced by Howe in [17, 18] and has been since connected to numerous physical models (for a non-exhaustive list see [19–21] and references therein). Let us recall the definition of the dual pairs in the context of Lie groups and Lie algebras [19]:

Let S be a Lie group and let G, G' be a pair of subgroups of S. We say (G, G') form a dual pair *of subgroups of S if G' is the full commutant of G in S, and vice versa. The pair $(\mathfrak{g}, \mathfrak{g}')$ of Lie algebras of (G, G') are a dual pair in the Lie algebra $\mathfrak{s}$ of S.*

When each subgroup of the pair is reductive (that is, completely reducible), the pair is referred to as a *reductive dual pair*. For the more technical details on the classification of these pairs, see for instance [17, 18, 22, 23]. If one of the members of the pair is a compact group, the following decomposition holds:

Consider a Hilbert space $\mathscr{H}$ which supports representations of S. Then, the actions of G and G' on $\mathscr{H}$ commute, the reductive dual pair (G, G') admits dual representations *on $\mathscr{H}$ and one obtains a multiplicity-free decomposition of the form:*

$$\mathscr{H} = \bigoplus_{\lambda} \Gamma^{(\lambda)} \otimes \Gamma'^{(\lambda)} \tag{6.1.1}$$

where Γ*'s and* Γ'*'s are irreducible modules of G and G' respectively.* (Note that a similar decomposition occurs in the context of the Schur–Weyl duality.) In simpler words, the irreps of each member of the pair are matched together. By virtue of the exponential mapping correspondence between Lie algebras and Lie groups, a decomposition of the form (6.1.1) also holds for irreducible modules of the Lie algebras $(\mathfrak{g}, \mathfrak{g}')$. We will be working at the algebra level in what follows.

An example of particular interest resides with the algebras $\mathfrak{sp}(2) \simeq \mathfrak{su}(1,1)$ and $\mathfrak{o}(2n)$ which form a dual pair in $\mathfrak{sp}(4n)$. This dual pair led to a novel interpretation of the Racah algebra $R(n)$ in the commutant of the $\mathfrak{o}(2)^{\oplus n}$ algebra in oscillator representations of $\mathfrak{o}(2n)$ [24, 25]. To this end, models of both the $\mathfrak{su}(1,1)^{\otimes 2n}$ and the $\mathfrak{o}(2n)$ algebras were constructed in terms of $2n$ oscillators. Due to the decomposition (6.1.1), the irreps could be paired. As expected by Schur's lemma, the pairing was expressed through the Casimirs which label the irreps. This was done as follows: The commutant of $\mathfrak{o}(2)^{\oplus n}$ in $\mathfrak{o}(2n)$ could be identified as the algebra generated by the quadratic Casimirs corresponding to $\mathfrak{o}(2m)$ embeddings in $\mathfrak{o}(2n)$ with $m = 1, \ldots, n$. In the oscillator model, all these quadratic Casimirs of $\mathfrak{o}(2m)$ were seen to be affinely related to some $\mathfrak{su}(1,1)$ Casimirs that arise from the recoupling of $2m$

copies of $\mathfrak{su}(1,1)$. But, the Racah algebra $R(n)$ obtained in the commutant of $\mathfrak{su}(1,1)$ in $U(\mathfrak{su}(1,1))^{\otimes n}$ is precisely generated by these $\mathfrak{su}(1,1)$ Casimirs occuring in the recoupling of $2m$ copies. Thus the two "dual" types of commutants give rise to the same Racah algebra. The pairing of the irreps (6.1.1) proves evident because the Casimirs of the two members of the pair are affinely related to each other, and it is this pairing that is fundamentally at the root of the dual descriptions of the Racah algebra $R(n)$.

The results on dual pairs that have been presented so far involve the so-called classical dual pairs. One may wonder what happens if q-deformations of the classical Lie algebras $\mathfrak{sl}_2$ and $\mathfrak{o}(2n)$ are considered, and if q-analogs of the dual pairs can be defined, while preserving a result analogous to (6.1.1) for the pairing of the irreps. Remarkably, a q-deformation of the pair $(\mathfrak{sl}_2, \mathfrak{o}(n))$ has been defined in [26]. The q-deformed dual pair $\big(U_q(\mathfrak{sl}_2), \mathfrak{o}_{q^{1/2}}(n)\big)$ will be used in an approach similar to the one employed for the Racah algebra to obtain dual pictures of the Askey–Wilson algebra. Note that we will actually restrict ourselves to the real form $U_q(\mathfrak{su}(1,1))$ of $U_q(\mathfrak{sl}_2)$ throughout this paper as this will allow to highlight more easily how the $q \to 1$ limit connects with the results in [24, 25].

Here is the outline of the rest of the paper. In Section 6.2, the (universal) Askey–Wilson algebra will be introduced along with its relation to $U_q(\mathfrak{su}(1,1))$. The q-oscillator algebra will be defined in Section 6.3 and then used to build realizations of the $U_q(\mathfrak{su}(1,1))$ and $\mathfrak{o}_{q^{1/2}}(m)$ algebras. In Section 6.4, the Askey–Wilson algebra $AW(3)$ will be obtained in the "dual" commutant and this result will then be generalized to $AW(n)$ in Section 6.5. Concluding remarks and opening questions will complete the paper.

6.2. A brief review of the Askey–Wilson algebra

6.2.1. The Askey–Wilson algebra $AW(3)$

The Askey–Wilson algebra was first introduced by Zhedanov in [1]. It can be presented in terms of two generators K_0, K_1 obeying the q-commutation relations

$$[K_0, K_1]_q = K_2, \qquad \begin{aligned} [K_1, K_2]_q &= bK_1 + c_0 K_0 + d_0, \\ [K_2, K_0]_q &= bK_0 + c_1 K_1 + d_1, \end{aligned} \tag{6.2.1}$$

where b, c_0, c_1, d_0, d_1 are (real) structure constants and $[A, B]_q = qAB - q^{-1}BA$. Throughout the paper, we consider the case where q is not a root of unity.

It is straightforward to reabsorb a few structure constants and to rescale the generators in order to arrive to the following $\mathbb{Z}_3$-symmetric presentation. Taking

$$
\begin{aligned}
K_A &= -\frac{q^2-q^{-2}}{\sqrt{c_1}}K_0, & \alpha &= \frac{d_0}{c_0\sqrt{c_1}}(q+q^{-1})^2(q-q^{-1}),\\
K_B &= -\frac{q^2-q^{-2}}{\sqrt{c_0}}K_1, & \beta &= \frac{d_1}{c_1\sqrt{c_0}}(q+q^{-1})^2(q-q^{-1}),\\
K_C &= -\frac{q^2-q^{-2}}{\sqrt{c_0\,c_1}}\left(K_2-\frac{b}{q-q^{-1}}\right), & \gamma &= \frac{b}{\sqrt{c_0\,c_1}}(q+q^{-1})^2,
\end{aligned}
\tag{6.2.2}
$$

relations (6.2.1) are rewritten as

$$
\begin{aligned}
\frac{[K_A,K_B]_q}{q^2-q^{-2}}+K_C &= \frac{\gamma}{q+q^{-1}},\\
\frac{[K_B,K_C]_q}{q^2-q^{-2}}+K_A &= \frac{\alpha}{q+q^{-1}},\\
\frac{[K_C,K_A]_q}{q^2-q^{-2}}+K_B &= \frac{\beta}{q+q^{-1}}.
\end{aligned}
\tag{6.2.3}
$$

The universal Askey–Wilson algebra [8] is defined by the relations (9.4.5) with α, β, γ being central elements. The universal Askey–Wilson algebra is the one that will be referred to in the remainder of this paper.

6.2.2. The $U_q(\mathfrak{su}(1,1))$ algebra and its Racah problem

We now review how the (universal) Askey–Wilson algebra appears in the context of the Racah problem of $U_q(\mathfrak{su}(1,1))$.

The $U_q(\mathfrak{su}(1,1))$ algebra has three generators, $J_\pm$ and J_0, obeying

$$
[J_0\,,J_\pm]=\pm J_\pm,\qquad J_-J_+-q^2J_+J_-=q^{2J_0}\,[2J_0]_q.
\tag{6.2.4}
$$

Here the notation $[x]_q$ stands for the q-number:

$$
[x]_q=\frac{q^x-q^{-x}}{q-q^{-1}}.
\tag{6.2.5}
$$

This algebra can be endowed with a Hopf structure; in particular it posesses a coproduct which is an algebra morphism that defines an embedding of $U_q(\mathfrak{su}(1,1))$ in $U_q(\mathfrak{su}(1,1))\otimes U_q(\mathfrak{su}(1,1))$:

$$
\Delta(J_0)=J_0\otimes 1+1\otimes J_0,\qquad \Delta(J_\pm)=J_\pm\otimes q^{2J_0}+1\otimes J_\pm.
\tag{6.2.6}
$$

The Casimir operator C of $U_q(\mathfrak{su}(1,1))$ has the following expression

$$
C=J_+J_-q^{-2J_0+1}+(J_0)_{q^2}(1-J_0)_{q^2},
\tag{6.2.7}
$$

where the notation $(x)_q$ stands for another type of q-number:

$$(x)_q = \frac{1-q^x}{1-q}. \tag{6.2.8}$$

Remark 6.1. *In the limit $q \to 1$, one recovers the usual $\mathfrak{su}(1,1)$ Lie algebra. The Casimir of $U_q(\mathfrak{su}(1,1))$ that is being used has been shifted by a constant with respect to the more conventional Casimir*

$$C = J_+J_-q^{-2J_0+1} - \frac{q}{(1-q^2)^2}\left(q^{2J_0-1} + q^{-2J_0+1}\right) \tag{6.2.9}$$

so as to make sure the $q \to 1$ limit is non-singular and yields the usual $\mathfrak{su}(1,1)$ Casimir $C = J_+J_- - {J_0}^2 + J_0$. Moreover, the standard presentation of $U_q(\mathfrak{su}(1,1))$ [27] is recovered from (6.2.4) *if one considers instead the generators $\widetilde{J}_0 = J_0$, $\widetilde{J}_+ = J_+q^{-J_0}$ and $\widetilde{J}_- = q^{-J_0}J_-$, which satisfy the commutation relations $[\widetilde{J}_0\,, \widetilde{J}_\pm] = \pm\widetilde{J}_\pm$, $[\widetilde{J}_-\,, \widetilde{J}_+] = [2\widetilde{J}_0]_q$.*

Let us now consider the addition of three irreducible representations of $U_q(\mathfrak{su}(1,1))$. Associated to each of those copies of $U_q(\mathfrak{su}(1,1))$ are the Casimirs

$$C^{(1)} = C \otimes 1 \otimes 1, \qquad C^{(2)} = 1 \otimes C \otimes 1, \qquad C^{(3)} = 1 \otimes 1 \otimes C.$$

The coassociativity of the coproduct ensures that the following two ways to pair the representations

$$(1 \oplus 2) \oplus 3 \simeq 1 \oplus (2 \oplus 3) \tag{6.2.10}$$

are equivalent.

In addition to the initial Casimirs $C^{(i)}$, there are the intermediate Casimirs associated to the different embeddings shown above in (6.2.10), $C^{(12)} = \Delta(C) \otimes 1$ and $C^{(23)} = 1 \otimes \Delta(C)$, as well as a total Casimir operator, $C^{(123)} = \Delta^{(2)}(C)$. Here, we use the notation $\Delta^{(N)} = (1^{\otimes(N-1)} \otimes \Delta)\Delta^{(N-1)}$, with $\Delta^{(0)} = 1$. The Racah problem of $U_q(\mathfrak{su}(1,1))$ is to find the overlap between the two different bases corresponding to (6.2.10), i.e. the one which diagonalizes $C^{(12)}$ and the other which diagonalizes $C^{(23)}$.

The connection with the Askey–Wilson algebra follows from the fact that the intermediate Casimirs of $U_q(\mathfrak{su}(1,1))$ realize it. In other words, the Askey–Wilson algebra can be described in the commutant of $U_q(\mathfrak{su}(1,1))$ in $U_q(\mathfrak{su}(1,1))^{\otimes 3}$. Indeed, performing the affine transformation

$$C_{\{\bullet\}} = -q(q-q^{-1})^2 C^{(\bullet)} + (q+q^{-1}), \tag{6.2.11}$$

one checks that relations (9.4.5) are verified for

$$\begin{aligned} K_A &= C_{\{1,2\}}, \qquad & \alpha &= C_{\{2\}}C_{\{3\}} + C_{\{1\}}C_{\{1,2,3\}}, \\ K_B &= C_{\{2,3\}}, & \beta &= C_{\{1\}}C_{\{3\}} + C_{\{2\}}C_{\{1,2,3\}}, \\ & & \gamma &= C_{\{1\}}C_{\{2\}} + C_{\{3\}}C_{\{1,2,3\}}. \end{aligned} \tag{6.2.12}$$

It is worth noting that instead of looking at the q-commutation relations of the intermediate Casimirs, one could have instead presented the relations using commutation relations. With $Q_1 = C^{(12)}$, $Q_2 = C^{(23)}$, one obtains the following relations:

$$\begin{aligned}
&[Q_1, Q_2] = Q_3, \\
&[Q_2, Q_3] = rQ_2Q_1Q_2 + \xi_1\{Q_1, Q_2\} + \xi_2 Q_2^2 + \xi_3 Q_2 + \xi_5, \\
&[Q_3, Q_1] = rQ_1Q_2Q_1 + \xi_1 Q_1^2 + \xi_2\{Q_1, Q_2\} + \xi_3 Q_1 + \xi_7,
\end{aligned} \tag{6.2.13}$$

where the elements r and ξ_i take the following values:

$$\begin{aligned}
r &= -(q - q^{-1})^2, \qquad \xi_1 = \xi_2 = (1 + q^{-2}), \\
\xi_3 &= -r\Big(C^{(1)}C^{(3)} + C^{(2)}C^{(123)}\Big) - \xi_1\left(C^{(1)} + C^{(2)} + C^{(3)} + C^{(123)}\right), \\
\xi_5 &= \xi_1\Big(C^{(2)} - C^{(3)}\Big)\Big(C^{(1)} - C^{(123)}\Big), \qquad \xi_7 = \xi_1\Big(C^{(2)} - C^{(1)}\Big)\Big(C^{(3)} - C^{(123)}\Big).
\end{aligned} \tag{6.2.14}$$

The $q \to 1$ limit of this algebra immediately leads to the (classical) Racah algebra.

6.3. q-oscillator realization of the dual pair $\Big(U_q(\mathfrak{su}(1,1)), \mathfrak{o}_{q^{1/2}}(2n)\Big)$

6.3.1. The $\mathfrak{o}_{q^{1/2}}(N)$ algebra

The non-standard q-deformation $\mathfrak{o}_{q^{1/2}}(N)$ of $\mathfrak{o}(N)$, often denoted $U'_{q^{1/2}}(\mathfrak{so}_N)$ in the literature [28–31], can be defined as the associative unital algebra with generators $L_{i,i+1}$ $(i = 1, \dots, N-1)$ obeying the relations

$$L_{i-1,i}\, L_{i,i+1}{}^2 - (q^{1/2} + q^{-1/2}) L_{i,i+1}\, L_{i-1,i}\, L_{i,i+1} + L_{i,i+1}{}^2\, L_{i-1,i} = -L_{i-1,i}, \tag{6.3.1a}$$

$$L_{i,i+1}\, L_{i-1,i}{}^2 - (q^{1/2} + q^{-1/2}) L_{i-1,i}\, L_{i,i+1}\, L_{i-1,i} + L_{i-1,i}{}^2\, L_{i,i+1} = -L_{i,i+1}, \tag{6.3.1b}$$

$$[L_{i,i+1}, L_{j,j+1}] = 0 \quad \text{for} \quad |i - j| > 1. \tag{6.3.1c}$$

This algebra possesses numerous properties of interest, among which: $\mathfrak{o}_q(N)$ can be viewed as a q-analogue of the symmetric space based on the pair $(\mathfrak{gl}(N), \mathfrak{o}(N))$ [32], it is a coideal subalgebra of $U_q(\mathfrak{sl}(N))$ [32, 33] and appears in various areas of mathematical physics [30].

The quadratic Casimir of $\mathfrak{o}_{q^{1/2}}(2n)$ is given in [26, 28]. We shall need the following elements recursively defined:

$$L^{\pm}_{ik} = [L^{\pm}_{ij}, L^{\pm}_{jk}]_{q^{\pm 1/4}} = q^{\pm 1/4} L^{\pm}_{ij} L^{\pm}_{jk} - q^{\mp 1/4} L^{\pm}_{jk} L^{\pm}_{ij}, \qquad \text{for any} \quad i < j < k \tag{6.3.2}$$

with $L^{\pm}_{i,i+1} = L_{i,i+1}$ by definition. The quadratic Casimir operator of the algebra $\mathfrak{o}_{q^{1/2}}(2n)$ then has the following expression:

$$\Lambda^{[2n]} = \sum_{1 \le i < j \le 2n} q^{\frac{-2n+i+j-1}{2}} L^{+}_{ij} L^{-}_{ij}. \tag{6.3.3}$$

6.3.2. The q-oscillator algebra

The q-oscillator algebra $\mathcal{A}_q(N)$ [27] is the unital associative algebra over $\mathbb{C}$ generated by N independent sets of q-oscillators $\{A_i^{\pm},\, A_i^0\}$ verifying

$$[A_i^0, A_i^{\pm}] = \pm A_i^{\pm}, \qquad [A_i^-, A_i^+] = q^{A_i^0}, \qquad A_i^- A_i^+ - q A_i^+ A_i^- = 1, \qquad i = 1, \ldots, N, \quad (6.3.4)$$

and such that the commutators between elements with distinct indices i are equal to zero. The last two relations lead to:

$$A_i^+ A_i^- = \frac{1 - q^{A_i^0}}{1-q} = (A_i^0)_q. \quad (6.3.5)$$

The q-oscillator algebra $\mathcal{A}_q(N)$ admits an irreducible representation bounded from below with orthonormal basis vectors $|n_1, \cdots, n_N\rangle = |n_1\rangle \otimes \cdots \otimes |n_N\rangle$ and with the operators A_i acting on the i'th factor of the states according to:

$$A_i^0 |n_i\rangle = n_i |n_i\rangle, \qquad A_i^+ |n_i\rangle = \sqrt{\frac{1-q^{n_i+1}}{1-q}}|n_i + 1\rangle, \qquad A_i^- |n_i\rangle = \sqrt{\frac{1-q^{n_i}}{1-q}}|n_i - 1\rangle. \quad (6.3.6)$$

These commuting q-oscillators can now be used to realize the algebras considered previously.

6.3.3. Dual realizations of the $\mathfrak{o}_{q^{1/2}}(2n)$ and $U_q(\mathfrak{su}(1,1))$ algebras

The algebras $U_q(\mathfrak{su}(1,1))$ and $\mathfrak{o}_{q^{1/2}}(2n)$ can be realized in terms of q-oscillators and shown to have commuting actions on the Hilbert space of q-oscillators.

To that end, let us first consider $2n$ copies of the q-deformation of the usual metaplectic representation [34] of $\mathfrak{su}(1,1)$, which is realized with $2n$ q-oscillators by taking

$$\mathscr{J}_0^i = \frac{1}{2}\left(A_i^0 + \frac{1}{2}\right), \qquad \mathscr{J}_\pm^i = \frac{1}{[2]_{q^{1/2}}}(A_i^{\pm})^2, \qquad i = 1, \ldots, 2n. \quad (6.3.7)$$

Owing to the fact that each set of $\mathscr{J}_0^i$, $\mathscr{J}_\pm^i$ acts only on the i'th factor and obeys the relations (6.2.4), (6.3.7) hence gives a realization of $U_q(\mathfrak{su}(1,1))^{\otimes 2n}$. It is then straightforward to embed $U_q(\mathfrak{su}(1,1))$ inside $\mathcal{A}_q(2n)$ by repeatedly making use of the coproduct (6.2.6):

$$\begin{aligned}
\mathscr{J}_0^{(2n)} &= \Delta^{(2n-1)}(\mathscr{J}_0) = \frac{1}{2}\sum_{i=1}^{2n}\left(A_i^0 + \frac{1}{2}\right), \\
\mathscr{J}_\pm^{(2n)} &= \Delta^{(2n-1)}(\mathscr{J}_\pm) = \frac{1}{[2]_{q^{1/2}}}\sum_{i=1}^{2n}\left((A_i^{\pm})^2 \prod_{j=i+1}^{2n} q^{A_j^0 + \frac{1}{2}}\right).
\end{aligned} \quad (6.3.8)$$

The algebra $\mathfrak{o}_{q^{1/2}}(2n)$ can also be realized in terms of $2n$ q-oscillators. The $2n-1$ generators take the form

$$\mathscr{L}_{i,i+1} = q^{-\frac{1}{2}(A_i^0 + \frac{1}{2})}\Big(q^{\frac{1}{4}} A_i^+ A_{i+1}^- - q^{-\frac{1}{4}} A_i^- A_{i+1}^+\Big), \qquad i = 1, \ldots, 2n-1. \quad (6.3.9)$$

A direct calculation shows that these $\mathscr{L}_{i,i+1}$'s verify the relations (6.3.1). All the other $\mathscr{L}_{ij}^{\pm}$'s can be obtained by (6.3.2).

Remark 6.2. *In this particular realization, the following q-analogs of the angular momenta relation $M_{12}M_{34} + M_{13}M_{42} + M_{14}M_{23} = 0$ in [35] hold. For $i < j < k < \ell$, one has:*

$$\begin{aligned} q^{-1/2}\mathscr{L}_{ij}^{+}\mathscr{L}_{k\ell}^{+} - \mathscr{L}_{ik}^{+}\mathscr{L}_{j\ell}^{+} + q^{+1/2}\mathscr{L}_{i\ell}^{+}\mathscr{L}_{jk}^{+} = 0,\\ q^{+1/2}\mathscr{L}_{ij}^{-}\mathscr{L}_{k\ell}^{-} - \mathscr{L}_{ik}^{-}\mathscr{L}_{j\ell}^{-} + q^{-1/2}\mathscr{L}_{i\ell}^{-}\mathscr{L}_{jk}^{-} = 0. \end{aligned} \tag{6.3.10}$$

It is easy to check that $[\mathscr{J}_0^{(2)}, \mathscr{L}_{12}] = [\mathscr{J}_{\pm}^{(2)}, \mathscr{L}_{12}] = 0$. A straightforward induction argument using the coproduct (6.2.6) and the form of the expression (6.3.9) leads to

$$[\mathscr{J}_0^{(2n)}, \mathscr{L}_{i,i+1}] = [\mathscr{J}_{\pm}^{(2n)}, \mathscr{L}_{i,i+1}] = 0, \qquad i = 1, \dots, 2n-1. \tag{6.3.11}$$

In other words, $U_q(\mathfrak{su}(1,1))$ and $\mathfrak{o}_{q^{1/2}}(2n)$ have commuting actions on the Hilbert space of $2n$ q-oscillators.

This feature precisely illustrates the Howe duality operating in this context and will be the key to obtaining the Askey–Wilson algebra of arbitrary rank as a "dual" commutant.

6.4. The Askey–Wilson algebra $AW(3)$ as a "dual" commutant

6.4.1. The commutant of $\mathfrak{o}_{q^{1/2}}(2)^{\oplus 3}$ in the q-oscillator realization of $\mathfrak{o}_{q^{1/2}}(6)$ and the Askey–Wilson algebra $AW(3)$

We now look for the commutant of the subalgebra $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2)$ of $\mathfrak{o}_{q^{1/2}}(6)$ generated by $\{\mathscr{L}_{12}, \mathscr{L}_{34}, \mathscr{L}_{56}\}$ in its q-oscillator realization. From the expressions of the quadratic Casimirs (6.3.3) it is easy to identify the following 6 independent elements:

$$\begin{aligned} \Lambda_1 &= {\mathscr{L}_{12}}^2, & \Lambda_{12} &= q^{-1}{\mathscr{L}_{12}}^2 + {\mathscr{L}_{23}}^2 + q{\mathscr{L}_{34}}^2\\ &&&\quad + q^{-1/2}\mathscr{L}_{13}^{+}\mathscr{L}_{13}^{-} + q^{1/2}\mathscr{L}_{24}^{+}\mathscr{L}_{24}^{-} + \mathscr{L}_{14}^{+}\mathscr{L}_{14}^{-},\\ \Lambda_2 &= {\mathscr{L}_{34}}^2, & \Lambda_{23} &= q^{-1}{\mathscr{L}_{34}}^2 + {\mathscr{L}_{45}}^2 + q{\mathscr{L}_{56}}^2\\ &&&\quad + q^{-1/2}\mathscr{L}_{35}^{+}\mathscr{L}_{35}^{-} + q^{1/2}\mathscr{L}_{46}^{+}\mathscr{L}_{46}^{-} + \mathscr{L}_{36}^{+}\mathscr{L}_{36}^{-},\\ \Lambda_3 &= {\mathscr{L}_{56}}^2, & \Lambda_{13} &= q^{-1}{\mathscr{L}_{12}}^2 + \mathscr{L}_{25}^{+}\mathscr{L}_{25}^{-} + q{\mathscr{L}_{56}}^2\\ &&&\quad + q^{-1/2}\mathscr{L}_{15}^{+}\mathscr{L}_{15}^{-} + q^{1/2}\mathscr{L}_{26}^{+}\mathscr{L}_{26}^{-} + \mathscr{L}_{16}^{+}\mathscr{L}_{16}^{-}. \end{aligned} \tag{6.4.1}$$

These 6 elements form a generating set for the non-trivial part of the commutant. Instead of using Λ_{13} as element of the generating set, one could alternatively take the element Λ_{123} which is a linear combination of the other $\Lambda_{\bullet}$'s above

$$\Lambda_{123} = q^{-1}\Lambda_{12} + \Lambda_{13} + q\Lambda_{23} - (q^{-1}\Lambda_1 + \Lambda_2 + q\Lambda_3) \tag{6.4.2}$$

and corresponds to the quadratic Casimir of $\mathfrak{o}_{q^{1/2}}(6)$ itself.

The algebraic relations obeyed by these $\Lambda_\bullet$'s correspond to those of the Askey–Wilson algebra. First make the affine transformation

$$\begin{aligned}
\widetilde{\Lambda}_i &= [2]_q - \left(\frac{q^{1/2}-q^{-1/2}}{1+q}\right)^2 (\Lambda_i + 1),\\
\widetilde{\Lambda}_{ij} &= [2]_q - \left(\frac{q^{1/2}-q^{-1/2}}{1+q}\right)^2 \Lambda_{ij},\\
\widetilde{\Lambda}_{123} &= [2]_q - \left(\frac{q^{1/2}-q^{-1/2}}{1+q}\right)^2 (\Lambda_{123} - [3]_{q^{1/2}}),
\end{aligned}\tag{6.4.3}$$

then take K_A and K_B to be

$$K_A = \widetilde{\Lambda}_{12}, \qquad K_B = \widetilde{\Lambda}_{23}.\tag{6.4.4}$$

A straightforward calculation shows that K_A and K_B obey the relations (9.4.5), with structure constants α, β, γ expressible in terms of Λ_1, Λ_2, Λ_3 and Λ_{123}.

$$\begin{aligned}
\alpha &= \widetilde{\Lambda}_2\widetilde{\Lambda}_3 + \widetilde{\Lambda}_1\widetilde{\Lambda}_{123},\\
\beta &= \widetilde{\Lambda}_3\widetilde{\Lambda}_1 + \widetilde{\Lambda}_2\widetilde{\Lambda}_{123},\\
\gamma &= \widetilde{\Lambda}_1\widetilde{\Lambda}_2 + \widetilde{\Lambda}_3\widetilde{\Lambda}_{123}.
\end{aligned}\tag{6.4.5}$$

6.4.2. The $U_q(\mathfrak{su}(1,1))$ and $\mathfrak{o}_{q^{1/2}}(6)$ descriptions of $AW(3)$ and Howe duality

In Section 6.2 the Askey–Wilson algebra was described in the commutant of $U_q(\mathfrak{su}(1,1))$ in $U_q(\mathfrak{su}(1,1))^{\otimes 3}$. That the AW algebra could also be described in the commutant of $\mathfrak{o}_{q^{1/2}}(2)^{\oplus 3}$ in q-oscillator representations of $\mathfrak{o}_{q^{1/2}}(6)$ is not a coincidence. We now make explicit the connection between these two approaches using Howe duality.

In [26] it was shown that $U_q(\mathfrak{su}(1,1))$ and $\mathfrak{o}_{q^{1/2}}(2m)$ are a dual pair in the sense of Howe. It follows from this that these two algebras have commuting actions on the Hilbert space of $2m$ q-oscillators (see (6.3.11)). This implies that their irreducible representations can be paired through the eigenvalues of the Casimirs that label them. We shall now indicate how the pairing occurs.

Take 6 q-oscillators, realizing 6 copies of $U_q(\mathfrak{su}(1,1))$. We first couple them pairwise and label each couple by $\bar{\imath} \equiv (2i-1, 2i)$ in order to obtain an embedding of $U_q(\mathfrak{su}(1,1))^{\otimes 3}$ in $U_q(\mathfrak{su}(1,1))^{\otimes 6}$:

$$\mathscr{J}_0^{\bar{\imath}} = \frac{1}{2}\left(A_{2i-1}^0 + A_{2i}^0 + 1\right), \qquad \mathscr{J}_\pm^{\bar{\imath}} = \frac{1}{[2]_{q^{1/2}}}\left((A_{2i-1}^\pm)^2 q^{A_{2i}^0 + \frac{1}{2}} + (A_{2i}^\pm)^2\right), \qquad i = 1, \ldots, 3.\tag{6.4.6}$$

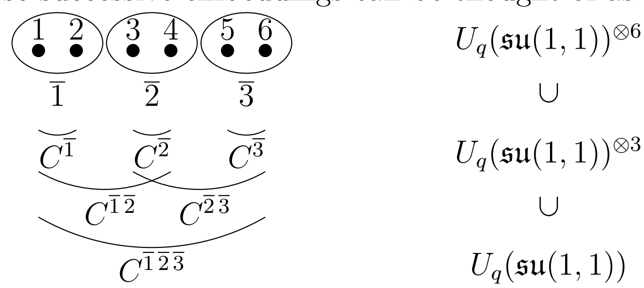
To each $\bar{\imath}$'th copy of $U_q(\mathfrak{su}(1,1))$ corresponds a Casimir $C^{\bar{\imath}}$ given by (6.2.7).

Three additional embeddings of $U_q(\mathfrak{su}(1,1))$ can be realized by repeatedly making use of the coproduct (6.2.6)

$$\begin{aligned}
\mathscr{J}_0^{\bar{1}\,\bar{2}} &= \mathscr{J}_0^{\bar{1}} + \mathscr{J}_0^{\bar{2}}, & \mathscr{J}_\pm^{\bar{1}\,\bar{2}} &= \mathscr{J}_\pm^{\bar{1}} q^{2\mathscr{J}_0^{\bar{2}}} + \mathscr{J}_\pm^{\bar{2}}, \\
\mathscr{J}_0^{\bar{2}\,\bar{3}} &= \mathscr{J}_0^{\bar{2}} + \mathscr{J}_0^{\bar{3}}, & \mathscr{J}_\pm^{\bar{2}\,\bar{3}} &= \mathscr{J}_\pm^{\bar{2}} q^{2\mathscr{J}_0^{\bar{3}}} + \mathscr{J}_\pm^{\bar{3}}, \\
\mathscr{J}_0^{\bar{1}\,\bar{2}\,\bar{3}} &= \mathscr{J}_0^{\bar{1}} + \mathscr{J}_0^{\bar{2}} + \mathscr{J}_0^{\bar{3}}, & \mathscr{J}_\pm^{\bar{1}\,\bar{2}\,\bar{3}} &= \mathscr{J}_\pm^{\bar{1}} q^{2\mathscr{J}_0^{\bar{2}}} q^{2\mathscr{J}_0^{\bar{3}}} + \mathscr{J}_\pm^{\bar{2}} q^{2\mathscr{J}_0^{\bar{3}}} + \mathscr{J}_\pm^{\bar{3}},
\end{aligned} \tag{6.4.7}$$

and the respective Casimirs associated to each of these embeddings, $C^{\bar{1}\,\bar{2}}$, $C^{\bar{2}\,\bar{3}}$, $C^{\bar{1}\,\bar{2}\,\bar{3}}$ can be obtained from (6.2.7).

Schematically, these successive embeddings can be thought of as:



Upon looking at the explicit expressions of these $C^{\bar{\imath}}$, $C^{\bar{\imath}\bar{\jmath}}$, $C^{\bar{1}\,\bar{2}\,\bar{3}}$ in terms of the q-oscillators, one finds (recall (6.4.1)) that

$$\begin{aligned}
C^{\bar{\imath}} &= \frac{1}{(1+q)^2}\left(\Lambda_i + 1\right), \\
C^{\bar{\imath}\bar{\jmath}} &= \frac{1}{(1+q)^2}\left(\Lambda_{ij}\right), \qquad \text{for consecutive } ij\text{'s} \\
C^{\bar{1}\,\bar{2}\,\bar{3}} &= \frac{1}{(1+q)^2}\left(\Lambda_{123} - [3]_{q^{1/2}}\right).
\end{aligned} \tag{6.4.8}$$

Those expressions show how the intermediate Casimirs of $U_q(\mathfrak{su}(1,1))$ and those of $\mathfrak{o}_{q^{1/2}}(6)$ are affinely related. Let us emphasize that owing to the Howe duality between $U_q(\mathfrak{su}(1,1))$ and $\mathfrak{o}_{q^{1/2}}(6)$, the multiplicity-free decomposition of the form (6.1.1) takes place; the relations (6.4.8) make this explicit keeping in mind the Schur's lemma.

Moreover, this pairing of the Casimirs is precisely what is behind the fact that the Askey–Wilson algebra, usually obtained from intermediate $U_q(\mathfrak{su}(1,1))$ Casimirs, is expressible in the commutant of the $\mathfrak{o}_{q^{1/2}}(2)^{\oplus 3}$ algebra in q-oscillator representations of $\mathfrak{o}_{q^{1/2}}(6)$. The duality of the two pictures is thus expressed in (6.4.8).

6.5. The case for general n and the algebra $AW(n)$

6.5.1. Towards the higher rank Askey–Wilson algebra

Let us first introduce the notation $[i;j]$ for sets of consecutive integers:

$$[i;j] = \begin{cases} \{i, i+1, \dots, j\} & j > i, \\ \{i\} & j = i, \\ \varnothing & j < i. \end{cases} \tag{6.5.1}$$

In [36], a higher rank extension of the Racah algebra $R(n)$ was realized in the algebra of the intermediate Casimir elements in $U(\mathfrak{su}(1,1))$ associated to embeddings (labelled by $A \subset [1;n]$) of $\mathfrak{su}(1,1)$ in its n-fold tensor product. A generating set for $R(n)$ is given by the intermediate Casimir operators related to consecutive tensor product space embeddings $[i;j]$, $1 \leq i \leq j \leq n$.

A similar story is emerging for the Askey–Wilson algebra. Studies towards a definition of a higher rank extension $AW(n)$ of $AW(3)$ have been based on a tensorial approach[37, 38] where one considers the algebra of the intermediate Casimir elements C_A of $U_q(\mathfrak{su}(1,1))$ associated to embeddings (labelled by $A \subset [1;n]$) of $U_q(\mathfrak{su}(1,1))$ in its n-fold tensor product. A generating set for these C_A's is given by all $C_{[i;j]}$'s, with $1 \leq i \leq j \leq n$. The $C_{[i;j]}$'s are obtained from the repeated action of the coproduct on the $U_q(\mathfrak{su}(1,1))$ Casimir elements.

The algebraic relations of $AW(4)$ are given in [37]. The full set of relations of $AW(n)$ is not known, however a large subset of those relations has been presented in [38]. Nevertheless, we here advance the understanding of these algebraic structures by establishing the dual connection between these intermediate Casimirs C_A in $U_q(\mathfrak{su}(1,1))^{\otimes n}$ and the generators of the commutant of a subalgebra of $\mathfrak{o}_{q^{1/2}}(2n)$.

6.5.2. The Howe duality in the $AW(n)$ case

We now proceed with this analysis of the higher rank case and look for the commutant of $\mathfrak{o}_{q^{1/2}}(2)^{\oplus n}$ in $\mathfrak{o}_{q^{1/2}}(2n)$. The algebra $\mathfrak{o}_{q^{1/2}}(2)^{\oplus n}$ is generated by the set $\{\mathscr{L}_{12}, \dots, \mathscr{L}_{2n-1,2n}\}$.

In view of the quadratic Casimirs (6.3.3), we examine the following $\binom{n+1}{2}$ elements:

$$\begin{aligned} \Lambda_i &= (\mathscr{L}_{2i-1,2i})^2, \\ \Lambda_{ij} &= q^{-1}\mathscr{L}_{2i-1,2i}{}^2 + \mathscr{L}^+_{2i,2j-1}\mathscr{L}^-_{2i,2j-1} + q\mathscr{L}_{2j-1,2j}{}^2 \qquad\qquad 1 \leq i < j \leq n. \\ &\quad + q^{-1/2}\mathscr{L}^+_{2i-1,2j-1}\mathscr{L}^-_{2i-1,2j-1} + q^{1/2}\mathscr{L}^+_{2i,2j}\mathscr{L}^-_{2i,2j} + \mathscr{L}^+_{2i-1,2j}\mathscr{L}^-_{2i-1,2j}, \end{aligned} \tag{6.5.2}$$

These elements all commute with $\mathfrak{o}_{q^{1/2}}(2)^{\oplus n}$:

$$[\Lambda_i, \mathscr{L}_{2k-1,2k}] = [\Lambda_{ij}, \mathscr{L}_{2k-1,2k}] = 0, \qquad i < j, \quad i,j,k = 1,\dots,n, \tag{6.5.3}$$

and they generate its commutant in the q-oscillator realization of $\mathfrak{o}_{q^{1/2}}(2n)$. We claim that the relations obeyed by these elements are precisely the relations of the higher rank Askey–Wilson algebra $AW(n)$ and that this follows from the Howe duality already observed in Section 6.4. We shall now explain how this conclusion is reached.

It will be useful to make the following linear transformation in order to work with elements Λ^A where $A \subseteq [1;n]$ is a set of consecutive indices. Form the $\Lambda^{[k;\ell]}$'s as follows:

$$\begin{aligned}
\Lambda^{[k;k]} &= \Lambda_k, \\
\Lambda^{[k;k+1]} &= \Lambda_{k,k+1}, \\
\Lambda^{[k;k+\ell-1]} &= \sum_{1\leq i<j\leq \ell} q^{i+j-(\ell+1)}\Lambda_{k-1+i,k-1+j} - [\ell-2]_{q^{1/2}} \sum_{i=1}^{\ell} q^{i-\frac{\ell+1}{2}}\Lambda_{k-1+i}, \qquad \ell \geq 3,
\end{aligned} \tag{6.5.4}$$

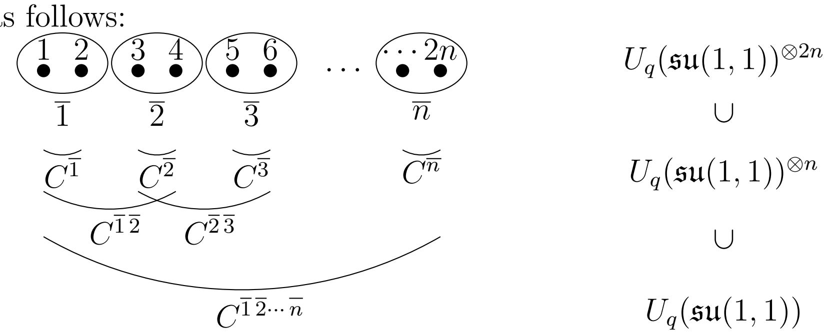
with $\Lambda^{\varnothing} = 0$ by convention. Note that there are still $\binom{n+1}{2}$ elements of the form $\Lambda^{[i;j]}$ since $\Big|\{[i;j] \mid [i;j] \subseteq [1;n]\}\Big| = \binom{n+1}{2}$.

For the sake of comprehensiveness, let us also give here the inverse change of basis:

$$\begin{aligned}
\Lambda_i &= \Lambda^{[i;i]}, \\
\Lambda_{ij} &= \Lambda^{[i;j]} + q^{-1}\Lambda^{[i;i]} + \Lambda^{[i+1;j-1]} + q\Lambda^{[j;j]} - q^{-1}\Lambda^{[i;j-1]} - q\Lambda^{[i+1;j]}.
\end{aligned} \tag{6.5.5}$$

We can now make use of the Howe duality observed in Section 6.4. Recall that the decomposition (6.1.1) implied that the quadratic Casimirs of $\mathfrak{o}_{q^{1/2}}(2m)$ were affinely related to intermediate Casimirs of $U_q(\mathfrak{su}(1,1))$ embeddings in $(2m)$ copies of itself. This still holds here.

Take $2n$ q-oscillators and couple them pairwise, with each couple labelled by $\bar{\imath} \equiv (2i-1, 2i)$ in order to obtain an embedding of $U_q(\mathfrak{su}(1,1))^{\otimes n}$ in $U_q(\mathfrak{su}(1,1))^{\otimes 2n}$. Nested embeddings then give rise to all the intermediate Casimirs needed to generate the $AW(n)$ algebra. This can be visualized as follows:



It remains to give the explicit correspondence between the paired Casimirs $C^{\bar{\imath}\cdots\bar{\jmath}}$ and the $\Lambda^{[i;j]}$'s (which will be the equivalent of (6.4.8)). We already know that they are affinely related, so we start by writing

$$C^{\bar{1}\,\bar{2}\cdots\,\bar{m}} = \Delta^{(2m-1)}(C) = \frac{1}{\beta_{2m}}\left(\Lambda^{[1;m]} + \alpha_{2m}\right). \tag{6.5.6}$$

After a quick look at the coproduct and the form of the $\Lambda^{[1;m]}$'s one convinces oneself that β_{2m} is constant, more precisely $\beta_{2m} = (1+q)^2$. It remains to evaluate the α_{2m}'s.

The α_{2m} can be obtained by acting with (6.5.6) on the ground state $|\vec{0}\,\rangle$ of $2m$ q-oscillators

$$\begin{aligned}\left(\Lambda^{[1;m]} + \alpha_{2m}\right)|\vec{0}\,\rangle &= (1+q)^2 C^{\overline{1}\,\overline{2}\cdots\overline{m}}|\vec{0}\,\rangle \\ &= (1+q)^2\Delta^{(2m-1)}\left(q^{-2\mathscr{J}_0+1}\mathscr{J}_+\mathscr{J}_- + (\mathscr{J}_0)_{q^2}(1-\mathscr{J}_0)_{q^2}\right)|\vec{0}\,\rangle\end{aligned} \tag{6.5.7}$$

where we recall the definition (6.2.7) for the $U_q(\mathfrak{su}(1,1))$ Casimir. Using (6.3.8) and recalling the action of the q-oscillators (6.3.6) from which the realizations are built, that is $\mathscr{L}^{\pm}_{ij}|\vec{0}\,\rangle = \mathscr{J}^{\,i}_{-}|\vec{0}\,\rangle = 0$, this is further simplified to

$$\left(\Lambda^{[1;m]} + \alpha_{2m}\right)|\vec{0}\,\rangle = \alpha_{2m}|\vec{0}\,\rangle = (1+q)^2\left((\textstyle\sum_{i=1}^{2m}\mathscr{J}^{\,i}_0)_{q^2}(1-\sum_{i=1}^{2m}\mathscr{J}^{\,i}_0)_{q^2}\right)|\vec{0}\,\rangle. \tag{6.5.8}$$

Furthermore, since $\mathscr{J}^{\,i}_0|\vec{0}\,\rangle = \frac{1}{4}|\vec{0}\,\rangle$, we have

$$\begin{aligned}\alpha_{2m}|\vec{0}\,\rangle &= (1+q)^2\left(({}^{m}\!/\!{}_{2})_{q^2}(1-{}^{m}\!/\!{}_{2})_{q^2}\right)|\vec{0}\,\rangle \\ &= -[m]_{q^{1/2}}[m-2]_{q^{1/2}}|\vec{0}\,\rangle,\end{aligned} \tag{6.5.9}$$

and we conclude that

$$C^{\overline{1}\,\overline{2}\cdots\overline{m}} = \frac{1}{(1+q)^2}\left(\Lambda^{[1;m]} - [m]_{q^{1/2}}[m-2]_{q^{1/2}}\right). \tag{6.5.10}$$

This leads to the desired generalization of (6.4.8):

$$C^{\overline{i}\cdots\overline{j}} = \frac{1}{(1+q)^2}\left(\Lambda^{[i;j]} - [j-i+1]_{q^{1/2}}[j-i-1]_{q^{1/2}}\right). \tag{6.5.11}$$

Upon shifting the Casimirs $C^{\overline{i}\cdots\overline{j}}$ using the procedure (6.2.11), one finally obtains the desired generating set for $AW(n)$

$$C_A = [2]_q - q(q-q^{-1})^2 C^{\overline{A}}, \tag{6.5.12}$$

where A is a consecutive set of indices. By virtue of the affine correspondence between the intermediate Casimirs of $U_q(\mathfrak{su}(1,1))$ and the $\mathfrak{o}_{q^{1/2}}(2m)$ quadratic Casimirs given in (6.5.11), the $AW(n)$ algebra generated by all C_A's therefore admits two dual descriptions.

Remark 6.3. *The $q \to 1$ limit of the expression for the pairing of the Casimirs in* (6.5.11) *coincides with the result obtained for the higher rank Racah algebra $R(n)$. Indeed, for $|A| = 2m$, one has*

$$C^A = -(J_0^A)^2 + J_+^A J_-^A + J_0^A = \frac{1}{4}\left(\sum_{\mu<\nu\in A} {L_{\mu\nu}}^2 - \frac{|A|(|A|-4)}{4}\right). \tag{6.5.13}$$

6.6. Conclusion

To sum up, we have used the Howe duality to provide two dual pictures of the Askey–Wilson algebra $AW(n)$. In addition to the description of $AW(n)$ in the commutant of

$U_q(\mathfrak{su}(1,1))$ in $U_q(\mathfrak{su}(1,1))^{\otimes n}$, we have also depicted the algebra in the commutant of $\mathfrak{o}_{q^{1/2}}(2)^{\oplus n}$ in q-oscillator representations of $\mathfrak{o}_{q^{1/2}}(2n)$. We have explained how the multiplicity-free decomposition of the modules of the joint action of the dual pair $\left(U_q(\mathfrak{su}(1,1)), \mathfrak{o}_{q^{1/2}}(2n)\right)$ given in (6.1.1) translates to an affine correspondence between Casimirs of $U_q(\mathfrak{su}(1,1))$ and $\mathfrak{o}_{q^{1/2}}(2m)$. This fact was then stressed to be the hallmark of the duality between the two pictures.

The $q \to 1$ limit is easily seen to give back results we have previously obtained on the higher rank Racah algebra $R(n)$. Note that in [25] we carried out the dimensional reduction corresponding to the imposition of the $\mathfrak{o}(2)^{\oplus n}$ invariance on the oscillator model and this had led us to the generic superintegrable model on the $(n-1)$-sphere [39, 40]. Such a dimensional reduction has not been performed here as the right q-analogues of polar coordinates are not known, but it would be an interesting question to examine in the future.

Another interesting limit is $q \to -1$. This limit yields the higher rank Bannai–Ito algebra $BI(n)$ if one starts from $AW(n)$. In [41] two dual pictures of $BI(n)$ were presented based on a Dirac model. An especially striking result is that the non-naive embeddings of $\mathfrak{osp}(1|2)$ associated to non-consecutive tensor product spaces could be explained in the context of the Dirac model by looking at the construction procedure of the higher dimension gamma matrices. It is still an open question to obtain an analogous explanation for $AW(n)$ and the corresponding $U_q(\mathfrak{su}(1,1))$ embeddings. We hope to return to this question soon.

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Chapitre 7

Superintegrability and the dual -1 Hahn algebra in superconformal quantum mechanics

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Abstract: A two-dimensional superintegrable system of singular oscillators with internal degrees of freedom is identified and exactly solved. Its symmetry algebra is seen to be the dual -1 Hahn algebra which describes the bispectral properties of the polynomials with the same name that are essentially the Clebsch–Gordan coefficients of the superconformal algebra $\mathfrak{osp}(1|2)$. It is also shown how this superintegrable model is obtained under dimensional reduction from a set of uncoupled harmonic oscillators in four dimensions.

7.1. Introduction

This paper introduces a simple superintegrable model in two-dimensions with internal degrees of freedom. Its Hamiltonian defined on $L^2(\mathbb{R}^2, \mathbb{C}^4)$ belongs to a realization of the superconformal algebra $\mathfrak{sl}(2|1)$ and reads

$$H = -\frac{1}{2}\left(\frac{\partial^2}{\partial\rho_1^2} + \frac{\partial^2}{\partial\rho_2^2}\right) + \frac{1}{2}\left(\rho_1^2 + \rho_2^2\right) + \frac{k_1(k_1 - \sigma_3 \otimes 1)}{2\rho_1^2} + \frac{k_2(k_2 - 1 \otimes \sigma_3)}{2\rho_2^2}, \qquad (7.1.1)$$

with $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ the standard Pauli matrix. The symmetry algebra generated by the constants of motion includes the not-so-familiar dual -1 Hahn algebra [1].

Superintegrable models in d dimensions have the property of admitting more than d (independent) constants of motion which hence form a non-Abelian symmetry algebra. In the scalar case, they are called maximally superintegrable if the number of these constants is equal to $2d-1$ when the Hamiltonian is included. They are typically exactly solvable. Interest in these systems is high. Some of the best known examples like the harmonic oscillator or the Kepler/Coulomb problem are discussed in textbooks and are of central use in Physics. Chiefly, these models are laboratories to study diverse expressions of extended symmetries. Therein lies the motivation for their systematic exploration and identification. The reader may consult [2] for a recent review.

One constructive approach to obtain superintegrable models that we shall use here combines one-dimensional systems equipped with ladder operators [3, 4]. Consider for concreteness the singular harmonic oscillator $\tilde{H} = p^2/(2m) + \omega\rho^2 + \lambda\rho^{-2}$ which can be cast as a generator in a realization of $\mathfrak{su}(1,1)$ and which hence exhibits conformal symmetry [5]. From the properties of the discrete series representations of $\mathfrak{su}(1,1)$, the spectrum of H is known to be equidistantly spaced and therefore linear in a quantum number n. Adding two such realizations say in the variable ρ_1 and ρ_2 to form a two-dimensional system leads to a Hamiltonian with manifest degeneracies. The corresponding constants of motion are readily constructed as products of the $\mathfrak{su}(1,1)$ raising and lowering operators from each of the two summands that leave the total energy unchanged. These constants are seen to generate [3, 6] the Higgs algebra [7–9] which is isomorphic to the dual Hahn algebra [10–12]. The latter is an example of the quadratic algebras [13] that are associated to the hypergeometric orthogonal polynomials of the Askey scheme [14]. These algebras are realized by the bispectral operators of the corresponding families of polynomials. In the case of the dual Hahn algebra, the associated dual Hahn polynomials are essentially the Clebsch–Gordan coefficients of $\mathfrak{su}(1,1)$.

We wish to examine if this approach extends fruitfully to the supersymmetric context. Can one combine one-dimensional superconformal Hamiltonian [15] with $\mathfrak{osp}(1|2)$ as dynamical algebra to obtain a superintegrable model? Positive indications arise from the fact that the representations of $\mathfrak{osp}(1|2)$ that belong to the discrete series imply a linear spectrum for the Cartan generator which leads, here also, to degenerate situations for this operator in combined representations. We shall call upon operators acting on vector-valued functions to provide the appropriate realizations. Interestingly, we shall thus find a model with internal degrees of freedom that has for symmetry algebra the one associated to the dual -1 Hahn polynomials. These polynomials have been singled out [16] as the $q \to -1$ limit of the dual q-Hahn polynomials [14] and shown to be basically the Clebsch–Gordan coefficients of $\mathfrak{osp}(1|2)$ [1].

The present study has similarities with the analysis of the Dunkl oscillator in the plane that also brings on supersymmetry [17–19]. In this case the relevant realizations of $\mathfrak{osp}(1|2)$ are the parabosonic [20, 21] ones constructed in terms of Dunkl and reflection operators [22]. A deformation of $\mathfrak{su}(2)$ obtained by extending the Schwinger construction to Dunkl creation and annihilation operators was identified as the symmetry algebra responsible for the superintegrability of the Dunkl oscillator in two dimensions. It was called the Schwinger–Dunkl algebra $sd(2)$ but it actually coincides with the dual -1 Hahn algebra.

The upshot of the present paper is that there is a simple superintegrable model with internal degrees of freedom that possesses the same dual -1 Hahn symmetry algebra and the attractive feature of not involving reflection operators.

A further observation is that this new model can be obtained as well through dimensional reduction. It was shown in [23] that harmonic oscillators in $2d$ dimensions can be reduced to maximally superintegrable systems of singular oscillators in d dimensions with integrals of motion inherited from those in the higher dimensions. This will be seen to prevail for the model with dual -1 Hahn symmetry upon projecting uncoupled harmonic oscillators with internal degrees of freedom from four to two dimensions.

The paper is structured as follows. Section 7.2 recalls facts about $\mathfrak{osp}(1|2)$ and its representations of the discrete series. Section 7.3 introduces the relevant realizations of $\mathfrak{osp}(1|2)$ in terms of matrix differential operators and the two-dimensional superconformal Hamiltonian of interest. This Hamiltonian is shown to be superintegrable in Section 7.4 and its symmetry algebra is identified as the dual -1 Hahn algebra. The wavefunctions separated in both Cartesian and polar coordinates are presented in Section 7.5. The total $\mathfrak{osp}(1|2)$ Casimir element, as one of the constants of motion, is also associated to separation in polar coordinates. Its eigenfunctions are obtained in Section 7.6 and their overlaps with the wavefunctions separated in Cartesian coordinates are shown to be given in terms of the dual -1 Hahn polynomials. Finally, how the superintegrable singular oscillator with internal degrees of freedom can be derived via dimensional reduction is the subject of Section 7.7. Concluding remarks are found in Section 13.6. Appendix 7.A gathers the main properties of the dual -1 Hahn polynomials and Appendix 7.B collects technical details on the solutions of relevant differential equations.

7.2. A review of $\mathfrak{osp}(1|2)$

The Lie superalgebra $\mathfrak{osp}(1|2)$ can be presented as the algebra with generators A_0, $A_\pm$ and an involution P encoding the $\mathbb{Z}_2$-grading of the superalgebra (P commutes with the even elements and anticommutes with the odd elements). These obey the relations

$$\{A_+, A_-\} = 2A_0, \qquad [A_0, A_\pm] = \pm A_\pm, \qquad [P, A_0] = 0, \qquad \{P, A_\pm\} = 0. \tag{7.2.1}$$

The sCasimir of $\mathfrak{osp}(1|2)$

$$S = A_+A_- - A_0 + \tfrac{1}{2} \tag{7.2.2}$$

belongs to the universal enveloping algebra of $\mathfrak{osp}(1|2)$ and satisfies the following relation

$$[S, A_0] = \{S, A_\pm\} = 0. \tag{7.2.3}$$

In other words, the sCasimir commutes with even generators and anticommutes with odd generators. One can then form a Casimir element by multiplying S with P:

$$Q = \left(A_+A_- - A_0 + \tfrac{1}{2}\right) P = \tfrac{1}{2} \left([A_+, A_-] + 1\right) P. \tag{7.2.4}$$

Positive infinite-dimensional discrete series representations are labelled by (μ, ϵ), with $\mu \geq 0$, $\epsilon = \pm 1$ and the actions of the generators on the associated basis vectors are given by

$$A_0|\mu, n, \epsilon\rangle = (n + \mu + \tfrac{1}{2})|\mu, n, \epsilon\rangle, \tag{7.2.5}$$

$$A_+|\mu, n, \epsilon\rangle = \sqrt{[n+1]_\mu}|\mu, n+1, \epsilon\rangle, \tag{7.2.6}$$

$$A_-|\mu, n, \epsilon\rangle = \sqrt{[n]_\mu}|\mu, n-1, \epsilon\rangle, \tag{7.2.7}$$

$$P|\mu, n, \epsilon\rangle = \epsilon(-1)^n|\mu, n, \epsilon\rangle, \tag{7.2.8}$$

with the *mu*-numbers $[n]_\mu$ defined as the following

$$[n]_\mu = n + \mu\left(1 - (-1)^n\right). \tag{7.2.9}$$

The Casimir element acts as a multiple of the identity on these irreps

$$Q|\mu, n, \epsilon\rangle = -\epsilon\mu|\mu, n, \epsilon\rangle. \tag{7.2.10}$$

In the realizations of $\mathfrak{osp}(1|2)$ that we shall soon provide, A_0 will be interpreted as the Hamiltonian.

7.2.1. The Clebsch–Gordan problem of $\mathfrak{osp}(1|2)$

We now turn to the study of the recoupling problem of two copies of $\mathfrak{osp}(1|2)$. We first introduce the coproduct $\Delta : \mathfrak{osp}(1|2) \to \mathfrak{osp}(1|2) \otimes \mathfrak{osp}(1|2)$, a coassociative algebra morphism that acts as follows on the generators:

$$\begin{aligned}
\Delta(A_0) &= A_0 \otimes 1 + 1 \otimes A_0 &&= A_0^{(1)} + A_0^{(2)}, \\
\Delta(A_\pm) &= A_\pm \otimes P + 1 \otimes A_\pm &&= A_\pm^{(1)}P^{(2)} + A_\pm^{(2)}, \\
\Delta(P) &= P \otimes P &&= P^{(1)}P^{(2)}.
\end{aligned} \tag{7.2.11}$$

As a result

$$\Delta(Q) = Q^{(12)} = (A_-^{(1)}A_+^{(2)} - A_+^{(1)}A_-^{(2)})P^{(1)} + Q^{(1)}P^{(2)} + Q^{(2)}P^{(1)} - \tfrac{1}{2}P^{(1)}P^{(2)}. \tag{7.2.12}$$

Let us now look at the recoupling of two irreducible representations of $\mathfrak{osp}(1|2)$ denoted (μ_1, ϵ_1) and (μ_2, ϵ_2), following [24]. There are two natural bases to consider.

To the direct product representation $(\mu_1,\epsilon_1)\otimes(\mu_2,\epsilon_2)$ are associated the basis vectors given by $|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle$ that diagonalize the Casimir elements $Q^{(1)}$ and $Q^{(2)}$, $A_0\otimes 1$, $1\otimes A_0$ and $P\otimes 1$, $1\otimes P$. It should be noted that it is equivalent to diagonalize $A_0\otimes 1$, $\Delta(A_0)$, $P\otimes 1$, $\Delta(P)$ instead of the previous four (this will be used later when we consider the Clebsch–Gordan algebra associated to $\mathfrak{osp}(1|2)$). All these elements act as follows on the basis vectors

$$\begin{aligned}
(Q\otimes 1)|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle &= -\epsilon_1\mu_1|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle,\\
(1\otimes Q)|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle &= -\epsilon_2\mu_2|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle,\\
(A_0\otimes 1)|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle &= (n_1+\mu_1+\tfrac{1}{2})|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle,\\
(1\otimes A_0)|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle &= (n_2+\mu_2+\tfrac{1}{2})|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle,\\
\Delta(A_0)|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle &= (n_1+n_2+\mu_1+\mu_2+1)|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle,\\
(P\otimes 1)|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle &= \epsilon_1(-1)^{n_1}|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle,\\
(1\otimes P)|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle &= \epsilon_2(-1)^{n_2}|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle,\\
\Delta(P)|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle &= \epsilon_1\epsilon_2(-1)^{n_1+n_2}|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle.
\end{aligned}\tag{7.2.13}$$

To the irreducible components (μ_{12},ϵ_{12}) of the decomposition of the direct product representation are attached the basis vectors $|\mu_{12},n_{12},\epsilon_{12}\rangle$. The diagonal operators are $\Delta(Q)$, $\Delta(A_0)$, $\Delta(P)$:

$$\begin{aligned}
\Delta(Q)|\mu_{12},n_{12},\epsilon_{12}\rangle &= -\epsilon_{12}\mu_{12}|\mu_{12},n_{12},\epsilon_{12}\rangle,\\
\Delta(A_0)|\mu_{12},n_{12},\epsilon_{12}\rangle &= (n_{12}+\mu_{12}+\tfrac{1}{2})|\mu_{12},n_{12},\epsilon_{12}\rangle,\\
\Delta(P)|\mu_{12},n_{12},\epsilon_{12}\rangle &= \epsilon_{12}(-1)^{n_{12}}|\mu_{12},n_{12},\epsilon_{12}\rangle.
\end{aligned}\tag{7.2.14}$$

Owing to the decomposition [25]

$$(\mu_1,\epsilon_1)\otimes(\mu_2,\epsilon_2)=\bigoplus_{j=0}^{\infty}(\mu_1+\mu_2+j+\tfrac{1}{2},(-1)^j\epsilon_1\epsilon_2)\tag{7.2.15}$$

one directly obtains that

$$\mu_{12}=\mu_1+\mu_2+j+\tfrac{1}{2},\qquad \epsilon_{12}=(-1)^j\epsilon_1\epsilon_2,\qquad j\in\mathbb{N}.\tag{7.2.16}$$

The Clebsch–Gordan coefficients are then defined as the expansion coefficients between these two bases

$$|\mu_{12},n_{12},\epsilon_{12}\rangle=\sum_{n_1,n_2}\mathcal{C}^{n_1,n_2}_{n_{12},j}|\mu_1,n_1,\epsilon_1\rangle\otimes|\mu_2,n_2,\epsilon_2\rangle.\tag{7.2.17}$$

It can be shown [1, 26] that the Clebsch–Gordan coefficients are expressible in terms of the dual -1 Hahn polynomials. These Clebsch–Gordan coefficients will appear in Section 7.6 as overlaps of the solutions of our spinorial superintegrable system separated in both the Cartesian and polar coordinates.

7.3. A spinorial realization of $\mathfrak{osp}(1|2)$

Dunkl realizations of $\mathfrak{osp}(1|2)$ have been studied extensively [17, 18, 20, 21]. These realizations are built with reflection operators and were taken to act on scalar wavefunctions. In this paper we shall focus instead on a realization with spin (internal) degrees of freedom which offers a valuable alternative perspective. The wavefunctions of the system are then given in terms of spinors. In the remainder of this paper, we will refer to this realization as the spinorial realization of $\mathfrak{osp}(1|2)$.

7.3.1. The spinorial model

Recall the usual Pauli matrices $\{\sigma_1, \sigma_2, \sigma_3\}$ and form

$$A_{\pm} = \frac{1}{\sqrt{2}}\left[\sigma_1\left(\rho \mp \frac{\partial}{\partial\rho}\right) \mp i\sigma_2\frac{k}{\rho}\right], \qquad A_0 = -\frac{1}{2}\frac{d^2}{d\rho^2} + \frac{\rho^2}{2} + \frac{k(k-\sigma_3)}{2\rho^2}. \tag{7.3.1}$$

Using $\sigma_a\sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c$, one easily checks that

$$\{A_+, A_-\} = 2A_0, \qquad [A_0, A_{\pm}] = \pm A_{\pm}. \tag{7.3.2}$$

We shall interpret $H = A_0$ as the Hamiltonian of this spinorial model. Note that parity $P : \rho \mapsto -\rho$ satisfies the remaining relations of $\mathfrak{osp}(1|2)$

$$[P, A_0] = \{P, A_{\pm}\} = 0. \tag{7.3.3}$$

Using the above, it is also seen that the Casimir element in this realization takes the form

$$Q = -k\sigma_3 P. \tag{7.3.4}$$

Recalling that Q has eigenvalue $-\epsilon\mu$ in a discrete series irrep of $\mathfrak{osp}(1|2)$ and taking $\mu = k$, we hence take

$$P\sigma_3 = \sigma_3 P = \epsilon \tag{7.3.5}$$

and it is seen that the identification $P = \epsilon\sigma_3$ in irreps is consistent with the relations (7.3.3).

What does this tell us about the wavefunctions? Recall that σ_3 has eigenvalue $+1$ for spin up ($\uparrow$) and -1 for spin down ($\downarrow$). On the other hand, P encodes the parity of the wavefunction: it has eigenvalue $+1$ for even wavefunctions and -1 for odd wavefunctions. The eigenvalues then combine like this:

$$\begin{array}{rclcl} \text{even wavefunction} & \& & \text{spin} \uparrow & \longrightarrow & \text{eigenvalue of } \sigma_3 P = +1 = \epsilon, \\ \text{odd wavefunction} & \& & \text{spin} \downarrow & \longrightarrow & \text{eigenvalue of } \sigma_3 P = +1 = \epsilon, \\ \text{even wavefunction} & \& & \text{spin} \downarrow & \longrightarrow & \text{eigenvalue of } \sigma_3 P = -1 = \epsilon, \\ \text{odd wavefunction} & \& & \text{spin} \uparrow & \longrightarrow & \text{eigenvalue of } \sigma_3 P = -1 = \epsilon. \end{array} \tag{7.3.6}$$

In other words, for our spinorial model and for a given k, the set of scenarios (7.3.6) provides us with an interpretation of the sign ϵ labelling the representations. Either (a) $\epsilon = +1$ and then there is a matching between the even wavefunctions and the spin up, and between the odd wavefunctions and the spin down, or (b) $\epsilon = -1$ and then there is a matching between the odd wavefunctions and the spin up, and between the even wavefunctions and the spin down.

This interpretation is in keeping with the fact that once we identify P with $\epsilon\sigma_3$ in irreps, we have from (7.2.8) that

$$P|\mu,n,\epsilon\rangle = \epsilon(-1)^n|\mu,n,\epsilon\rangle = \epsilon\sigma_3|\mu,n,\epsilon\rangle, \qquad \text{and hence} \qquad \sigma_3|\mu,n,\epsilon\rangle = (-1)^n|\mu,n,\epsilon\rangle. \tag{7.3.7}$$

This connects the spin with the energies which are known to depend on parity.

7.3.2. The $\mathfrak{osp}(2|2)$ dynamical algebra

The dynamical algebra of this spinorial model further realizes $\mathfrak{osp}(2|2) \simeq \mathfrak{sl}(2|1)$. This superalgebra can be presented [27] in terms of four bosonic generators $E^\pm$, $\bar{H}$, Z and two pairs of fermionic generators $F^\pm$, $\bar{F}^\pm$ obeying the relations

$$\begin{aligned}
&[\bar{H}, E^\pm] = \pm E^\pm, && [\bar{H}, F^\pm] = \pm\frac{1}{2}F^\pm,\\
&[E^+, E^-] = 2\bar{H}, && [Z, F^\pm] = \ \ \frac{1}{2}F^\pm,\\
&[Z, E^\pm] = 0 = [Z, \bar{H}], && [\bar{H}, \bar{F}^\pm] = \pm\frac{1}{2}\bar{F}^\pm,\\
&\{F^\pm, \bar{F}^\mp\} = Z \mp \bar{H}, && [Z, \bar{F}^\pm] = -\frac{1}{2}\bar{F}^\pm,\\
&\{F^\pm, \bar{F}^\pm\} = E^\pm, && [E^\pm, F^\mp] = -F^\pm,\\
&\{F^\pm, F^\pm\} = 0 = \{\bar{F}^\pm, \bar{F}^\pm\}, && [E^\pm, \bar{F}^\mp] = \ \ \bar{F}^\pm,\\
&\{F^\pm, F^\mp\} = 0 = \{\bar{F}^\pm, \bar{F}^\mp\}, && [E^\pm, F^\pm] = 0 = [E^\pm, \bar{F}^\pm].
\end{aligned} \tag{7.3.8}$$

Here is how the algebra is realized in our model. As previously defined in (7.3.1), the element A_0 is a bosonic generator and $A_\pm$ are fermionic generators. Introduce Y, which commutes with A_0 and is realized by

$$Y = \frac{\sigma_3}{2i}. \tag{7.3.9}$$

This element leads to another $\mathfrak{osp}(1|2)$ realization. Indeed, we can form supercharges tilde as follows

$$\tilde{A}_\pm = [A_\pm, Y] = \frac{1}{\sqrt{2}}\left[-\sigma_2\left(\rho \mp \frac{\partial}{\partial\rho}\right) \mp i\sigma_1\frac{k}{\rho}\right]. \tag{7.3.10}$$

Those supercharges also obey the $\mathfrak{osp}(1|2)$ defining relations:

$$\{\tilde{A}_+, \tilde{A}_-\} = 2A_0, \qquad [A_0, \tilde{A}_\pm] = \pm\tilde{A}_\pm, \qquad [P, A_0] = \{P, \tilde{A}_\pm\} = 0. \tag{7.3.11}$$

The dynamical algebra hence contains the two $\mathfrak{osp}(1|2)$ subalgebras mentioned above and can be identified as $\mathfrak{sl}(2|1)$ by mapping its generators in the following way:

$$\bar{H} = \frac{1}{2}A_0 = \frac{1}{4}\left(-\frac{\partial^2}{\partial\rho^2} + \frac{k(k-\sigma_3)}{\rho^2} + \rho^2\right), \tag{7.3.12}$$

$$Z = -\frac{1}{2}(k + iY) = -\frac{1}{2}\left(k + \frac{\sigma_3}{2}\right), \tag{7.3.13}$$

$$F^+ = \frac{(\tilde{A}_+ - iA_+)}{2\sqrt{2}} = \frac{i}{4}(\sigma_1 - i\sigma_2)\left(\frac{\partial}{\partial\rho} - \frac{k}{\rho} - \rho\right), \tag{7.3.14}$$

$$F^- = \frac{-i(\tilde{A}_- - iA_-)}{2\sqrt{2}} = -\frac{1}{4}(\sigma_1 - i\sigma_2)\left(\frac{\partial}{\partial\rho} - \frac{k}{\rho} + \rho\right), \tag{7.3.15}$$

$$\bar{F}^+ = \frac{i(\tilde{A}_+ + iA_+)}{2\sqrt{2}} = \frac{1}{4}(\sigma_1 + i\sigma_2)\left(\frac{\partial}{\partial\rho} + \frac{k}{\rho} - \rho\right), \tag{7.3.16}$$

$$\bar{F}^- = \frac{-\,(\tilde{A}_- + iA_-)}{2\sqrt{2}} = -\frac{i}{4}(\sigma_1 + i\sigma_2)\left(\frac{\partial}{\partial\rho} + \frac{k}{\rho} + \rho\right), \tag{7.3.17}$$

$$E^\pm = \{F^\pm, \bar{F}^\pm\} = -\frac{i}{4}\left[-\frac{\partial^2}{\partial\rho^2} + \frac{k(k-\sigma_3)}{\rho^2} - \rho^2 \pm \left(1 + 2\rho\frac{\partial}{\partial\rho}\right)\right]. \tag{7.3.18}$$

The defining relations (7.3.8) are checked directly.

In the following we shall focus only on the $\mathfrak{osp}(1|2)$ realization given in (7.3.1).

7.4. Constants of motion and the dual -1 Hahn algebra

In this Section we will show that the dual -1 Hahn algebra is the symmetry algebra that accounts for the superintegrability of the two-dimensional model with internal degrees of freedom (7.1.1).

7.4.1. Two-dimensional model

Using the $\mathfrak{osp}(1|2)$ coproduct, we can form the following two-dimensional realization:

$$\begin{aligned} A_\pm^{(12)} &= A_\pm \otimes P + 1 \otimes A_\pm \\ &= \epsilon_2 \frac{1}{\sqrt{2}}\left[(\sigma_1 \otimes \sigma_3)\left(\rho_1 \mp \frac{d}{d\rho_1}\right) \mp i(\sigma_2 \otimes \sigma_3)\frac{k_1}{\rho_1}\right] \\ &\quad + \frac{1}{\sqrt{2}}\left[\ (1 \otimes \sigma_1)\left(\rho_2 \mp \frac{d}{d\rho_2}\right) \mp i\ \ (1 \otimes \sigma_2)\frac{k_2}{\rho_2}\right] \end{aligned} \tag{7.4.1}$$

$$A_0^{(12)} = -\frac{1}{2}\left(\frac{d^2}{d\rho_1^2} + \frac{d^2}{d\rho_2^2}\right) + \frac{1}{2}(\rho_1^2 + \rho_2^2) + \frac{k_1(k_1 - \sigma_3 \otimes 1)}{2\rho_1^2} + \frac{k_2(k_2 - 1 \otimes \sigma_3)}{2\rho_2^2}. \tag{7.4.2}$$

The involution P is mapped to

$$P^{(12)} = P^{(1)}P^{(2)} = \epsilon_1\epsilon_2(\sigma_3 \otimes \sigma_3). \tag{7.4.3}$$

These elements still satisfy the $\mathfrak{osp}(1|2)$ algebra relations since they were obtained from the coproduct morphism.

Now introduce the gamma matrices

$$\gamma_1 = \sigma_1 \otimes \sigma_3, \qquad \gamma_2 = \sigma_2 \otimes \sigma_3, \qquad \gamma_3 = 1 \otimes \sigma_1, \qquad \gamma_4 = 1 \otimes \sigma_2. \tag{7.4.4}$$

These obey the Clifford algebra relations

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}. \tag{7.4.5}$$

Denoting $\Sigma_{ab} = i\gamma_a\gamma_b$, we form the two spin operators

$$\Sigma_{12} = -(\sigma_3 \otimes 1), \qquad \Sigma_{34} = -(1 \otimes \sigma_3), \tag{7.4.6}$$

and the expressions above can be rewritten as

$$A_0^{(12)} = -\frac{1}{2}\left(\frac{d^2}{d\rho_1^2} + \frac{d^2}{d\rho_2^2}\right) + \frac{1}{2}(\rho_1^2 + \rho_2^2) + \frac{k_1(k_1 + \Sigma_{12})}{2\rho_1^2} + \frac{k_2(k_2 + \Sigma_{34})}{2\rho_2^2}, \tag{7.4.7}$$

$$A_\pm^{(12)} = \epsilon_2 \frac{1}{\sqrt{2}}\left[\gamma_1\left(\rho_1 \mp \frac{d}{d\rho_1}\right) \mp i\gamma_2 \frac{k_1}{\rho_1}\right] + \frac{1}{\sqrt{2}}\left[\gamma_3\left(\rho_2 \mp \frac{d}{d\rho_2}\right) \mp i\gamma_4 \frac{k_2}{\rho_2}\right], \tag{7.4.8}$$

$$P^{(12)} = \epsilon_{12}\Sigma_{12}\Sigma_{34}. \tag{7.4.9}$$

We identify $A_0^{(12)}$ as the Hamiltonian of our two-dimensional spinorial model. Its energies are provided by the sum of two linear spectra:

$$n_1 + n_2 + \mu_1 + \mu_2 + 1. \tag{7.4.10}$$

This spectrum is degenerate. We will now explain the degeneracies from the symmetries of the Hamiltonian $H_{12} = A_0^{(12)}$ and the algebra they form.

7.4.2. Symmetries of $H_{12} = A_0^{(12)}$ and superintegrability

The total Hamiltonian of the two-dimensional system is

$$H_{12} = \Delta(A_0) = A_0^{(1)} + A_0^{(2)} \tag{7.4.11}$$

where

$$A_0^{(1)} = -\frac{1}{2}\left(\frac{d^2}{d\rho_1^2}\right) + \frac{1}{2}(\rho_1^2) + \frac{k_1(k_1 + \Sigma_{12})}{2\rho_1^2}, \tag{7.4.12}$$

$$A_0^{(2)} = -\frac{1}{2}\left(\frac{d^2}{d\rho_2^2}\right) + \frac{1}{2}(\rho_2^2) + \frac{k_2(k_2 + \Sigma_{34})}{2\rho_2^2}. \tag{7.4.13}$$

Given the ladder operators (7.2.6)–(7.2.7), combinations of the form $A^{(i)}_{\pm}A^{(j)}_{\mp}$ will be symmetries of the total Hamiltonian H_{12} (they are constants of motion that preserve the total energy of the system). We can form three independent such combinations (in addition to the Hamiltonian) that commute with H_{12}, thus showing that the model is superintegrable.

To identify the nature of the symmetry algebra, let

$$K_1 = \frac{1}{4}\left(\{A^{(1)}_+, A^{(1)}_-\} - \{A^{(1)}_+, A^{(1)}_-\}\right) = \frac{1}{2}\left(A^{(1)}_0 - A^{(2)}_0\right). \tag{7.4.14}$$

It is immediate to see that K_1 is a symmetry of H_{12}.

Another symmetry is $(A^{(1)}_- A^{(2)}_+ - A^{(1)}_+ A^{(2)}_-)$; in essence, this is the total Casimir $Q^{(12)} = \Delta(Q)$ presented in (8.4.9). This element obviously commutes with $H_{12} = \Delta(A_0)$ because the coproduct is an algebra homomorphism.

Note that both Σ_{12} and Σ_{34} are additional symmetries of H_{12}. A natural symmetry algebra generator is hence the total sCasimir $\Delta(S)$ which we shall denote by K_2:

$$K_2 = \epsilon_{12} Q^{(12)} \Sigma_{12}\Sigma_{34}. \tag{7.4.15}$$

The commutation relations obeyed by the generators of the symmetry algebra are then seen to be the defining relations of the dual -1 Hahn algebra [1]

$$\begin{aligned}
&[K_1, K_2] = K_3, \qquad [K_1, K_3] = K_2 - (\epsilon_1 Q^{(1)}\Sigma_{12} + \epsilon_2 Q^{(2)}\Sigma_{34} - \tfrac{1}{2}),\\
&[K_2, K_3] = 2K_3(\epsilon_1 Q^{(1)}\Sigma_{12} + \epsilon_2 Q^{(2)}\Sigma_{34}) - 4K_1(1 - \epsilon_1 Q^{(1)}\Sigma_{12} - \epsilon_2 Q^{(2)}\Sigma_{34})\\
&\qquad\qquad - 2H_{12}(\epsilon_1 Q^{(1)}\Sigma_{12} - \epsilon_2 Q^{(2)}\Sigma_{34}),\\
&\{K_2, \Sigma_{12}\} = 2(\epsilon_1 Q^{(1)}\Sigma_{12} + \epsilon_2 Q^{(2)}\Sigma_{34} + \tfrac{1}{2})\Sigma_{12},\\
&\{K_2, \Sigma_{34}\} = 2(\epsilon_1 Q^{(1)}\Sigma_{12} + \epsilon_2 Q^{(2)}\Sigma_{34} + \tfrac{1}{2})\Sigma_{34},\\
&0 = [K_1, \Sigma_{12}] = [K_1, \Sigma_{34}] = \{K_3, \Sigma_{12}\} = \{K_3, \Sigma_{34}\},
\end{aligned} \tag{7.4.16}$$

where $Q^{(1)}$, $Q^{(2)}$ and H_{12} are central elements. The algebra can also be recast in a more symmetric presentation by reabsorbing the ϵ_i's:

$$\begin{aligned}
&[K_1, K_2] = K_3, \qquad [K_1, K_3] = K_2 - (S^{(1)} + S^{(2)} - \tfrac{1}{2}),\\
&[K_2, K_3] = 2K_3(S^{(1)} + S^{(2)}) + 4K_1(S^{(1)} + S^{(2)} - 1) - 2H_{12}(S^{(1)} - S^{(2)}),\\
&\{K_2, P^{(1)}\} = 2(S^{(1)} + S^{(2)} + \tfrac{1}{2})P^{(1)},\\
&\{K_2, P^{(2)}\} = 2(S^{(1)} + S^{(2)} + \tfrac{1}{2})P^{(2)},\\
&0 = [K_1, P^{(1)}] = [K_1, P^{(2)}] = \{K_3, P^{(1)}\} = \{K_3, P^{(2)}\},
\end{aligned} \tag{7.4.17}$$

This presentation emphasizes the role of the sCasimirs $S^{(i)} = Q^{(i)}P^{(i)}$ and makes the correspondence with the presentation in [1] more explicit.

7.4.3. An embedding of the dual Hahn algebra

The dual Hahn algebra is connected with the Lie algebra $\mathfrak{su}(1,1)$, which is the even subalgebra of $\mathfrak{osp}(1|2)$. A natural question then arises: can the dual Hahn algebra be embedded in the -1 Hahn algebra? As will now be shown, the answer is affirmative.

$$J_{\pm} = \tfrac{1}{2}(A_{\pm})^2, \qquad J_0 = \tfrac{1}{2}A_0, \tag{7.4.18}$$

obey the defining relations of $\mathfrak{su}(1,1)$:

$$[J_0, J_{\pm}] = \pm J_{\pm}, \qquad [J_+, J_-] = -2J_0. \tag{7.4.19}$$

The Casimir element of $\mathfrak{su}(1,1)$ commutes with all J_0, $J_{\pm}$ generators and has the following expression in terms of the sCasimir $S = QP$ of $\mathfrak{osp}(1|2)$:

$$C = {J_0}^2 - J_+J_- - J_0 = \tfrac{1}{4}\left(S^2 + S - \tfrac{3}{4}\right). \tag{7.4.20}$$

The dual Hahn algebra appears when looking at the Clebsch–Gordan problem of $\mathfrak{su}(1,1)$. Here is how this is realized in terms of a two-dimensional model.

Consider the addition of the two realizations (7.3.1) of $\mathfrak{osp}(1|2)$ and the corresponding two-dimensional $\mathfrak{su}(1,1)$ model:

$$J_{\pm}^{(12)} = \frac{1}{2}(A_{\pm}^{(12)})^2 = \frac{1}{2}\left((A_{\pm}^{(1)})^2 + (A_{\pm}^{(2)})^2\right), \tag{7.4.21}$$

$$J_0^{(12)} = \frac{1}{2}(A_0^{(12)}) \ = \frac{1}{2}(A_0^{(1)} + A_0^{(2)}), \tag{7.4.22}$$

$$C^{(12)} = (J_0^{(12)})^2 - J_+^{(12)}J_-^{(12)} - J_0^{(12)}. \tag{7.4.23}$$

Now, form the following two quantities that commute with the total Hamiltonian $J_0^{(12)}$:

$$\begin{aligned} \mathcal{K}_1 &= \frac{1}{2}\left(J_0^{(1)} - J_0^{(2)}\right), \\ \mathcal{K}_2 &= C^{(12)} = (J_0^{(12)})^2 - J_+^{(12)}J_-^{(12)} - J_0^{(12)}, \end{aligned} \tag{7.4.24}$$

The relations obeyed by these elements are those of the dual Hahn algebra [10–12]

$$\begin{aligned} &[\mathcal{K}_1, \mathcal{K}_2] = \mathcal{K}_3, \\ &[\mathcal{K}_2, \mathcal{K}_3] = -2\{\mathcal{K}_1, \mathcal{K}_2\} + 4J_0^{(12)}(C^{(1)} - C^{(2)}), \\ &[\mathcal{K}_2, \mathcal{K}_3] = -2{\mathcal{K}_1}^2 - 4\mathcal{K}_2 + 2(J_0^{(12)})^2 + 4(C^{(1)} + C^{(2)}), \end{aligned} \tag{7.4.25}$$

with central elements $\delta_1 = 4J_0^{(12)}(C^{(1)} - C^{(2)})$ and $\delta_2 = 2(J_0^{(12)})^2 + 4(C^{(1)} + C^{(2)})$. This explicitly shows the embedding of the dual Hahn algebra in the dual -1 Hahn algebra.

7.5. Separated solutions

In this section we shall study the wavefunctions of the two-dimensional system described by the Hamiltonian

$$H_{12} = -\frac{1}{2}\left(\frac{d^2}{d\rho_1^2} + \frac{d^2}{d\rho_2^2}\right) + \frac{1}{2}(\rho_1^2 + \rho_2^2) + \frac{k_1(k_1 + \Sigma_{12})}{2\rho_1^2} + \frac{k_2(k_2 + \Sigma_{34})}{2\rho_2^2} \tag{7.5.1}$$

in two different coordinates system. In the next section, the knowledge of the symmetry algebra will allow to obtain the overlaps between the wavefunctions in these two coordinate systems.

7.5.1. Solutions in Cartesian coordinates

The one-dimensional system obeys the following Schrödinger equation

$$H\psi = \frac{1}{2}\left(-\frac{d^2}{d\rho^2} + \rho^2 + \frac{k(k-\sigma_3)}{\rho^2}\right)\psi = E\psi. \tag{7.5.2}$$

We give details in the Appendix 7.B on how this equation is solved.

The solutions $\psi_{m,k}(\rho)$ are more conveniently expressed in terms of the generalized Hermite polynomials $H_m^k(x)$ [20, 28, 29]. This family of polynomials is composed of two alternating sequences of generalized Laguerre polynomials:

$$H_{2n+p}^k(x) = (-1)^n\sqrt{\frac{n!}{\Gamma(n+p+k+\frac{1}{2})}}\; x^p\; L_n^{(k-\frac{1}{2}+p)}(x^2), \qquad p \in \{0,1\}. \tag{7.5.3}$$

Identifying p with $\frac{1-s}{2}$, where s is the eigenvalue of σ_3, the solutions are presented as

$$\begin{aligned} \psi_{m,k}(\rho) &= (-1)^{\lfloor \frac{m}{2} \rfloor}\mathsf{e}^{-\rho^2/2}\rho^k\, H_m^k(\rho), \qquad \text{with} \quad m = 2n+p, \\ E_m &= m + k + \tfrac{1}{2}, \end{aligned} \tag{7.5.4}$$

and $\lfloor x \rfloor$ is the floor function. In braket notation, the Schrödinger equation reads

$$H|m,k\rangle = (m + k + \tfrac{1}{2})|m,k\rangle, \tag{7.5.5}$$

with

$$|m,k\rangle = \begin{pmatrix}\psi_{m,k}\\ 0\end{pmatrix} \quad \text{if } m \text{ is even,} \qquad \text{and} \qquad |m,k\rangle = \begin{pmatrix}0\\ \psi_{m,k}\end{pmatrix} \quad \text{if } m \text{ is odd.} \tag{7.5.6}$$

One should note that we must have $k > -1/2$ in order for the solutions to be normalizable and the energies to be non-negative. The action of the parity involution $P : \rho \mapsto -\rho$ on these wavefunctions is

$$P\,\psi_{m,k}(\rho) = \psi_{m,k}(-\rho) = (-1)^{k+m}\psi_{m,k}(\rho). \tag{7.5.7}$$

Since P acts as $\epsilon(-1)^m$ on a given $\mathfrak{osp}(1|2)$ eigenvector $|m,k\rangle$, this means that we should choose

$$\epsilon = (-1)^k. \tag{7.5.8}$$

Since the positive-discrete series of $\mathfrak{osp}(1|2)$ in (7.2.5)–(7.2.8) is defined for $\epsilon = \pm 1$, we impose $k \in \mathbb{N}$.

It can be checked (using the Laguerre polynomials contiguity and recurrence relations [30]) that $A_\pm$ realized as (7.3.1) acts on the eigenstates $|m,k\rangle$ according to (7.2.6)–(7.2.7).

$$A_+|m,k\rangle = \sqrt{[m+1]_k}|m+1,k\rangle, \qquad A_-|m,k\rangle = \sqrt{[m]_k}|m-1,k\rangle. \tag{7.5.9}$$

We now look at the solutions of the two-dimensional system. The Cartesian solutions are obtained by combining two one-dimensional problems. Let us denote the coupled eigenstates

$$|m_1,k_1\rangle \otimes |m_2,k_2\rangle = |m_1,m_2,k_1,k_2\rangle, \qquad m_i = 2n_i + p_i. \tag{7.5.10}$$

These $|m_1,m_2,k_1,k_2\rangle$ are 4-component spinors, whose entries depend on the parity of both m_1 and m_2 following (7.5.6),

$$|2n_1,2n_2,k_1,k_2\rangle = \begin{pmatrix} \psi_{2n_1,2n_2,k_1,k_2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |2n_1,2n_2+1,k_1,k_2\rangle = \begin{pmatrix} 0 \\ \psi_{2n_1,2n_2+1,k_1,k_2} \\ 0 \\ 0 \end{pmatrix},$$
$$|2n_1+1,2n_2,k_1,k_2\rangle = \begin{pmatrix} 0 \\ 0 \\ \psi_{2n_1+1,2n_2,k_1,k_2} \\ 0 \end{pmatrix}, \qquad |2n_1+1,2n_2+1,k_1,k_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \psi_{2n_1+1,2n_2+1,k_1,k_2} \end{pmatrix}, \tag{7.5.11}$$

and the entries of these spinors are the product of two one-dimensional wavefunctions

$$\psi_{m_1,m_2,k_1,k_2}(\rho_1,\rho_2) = \psi_{m_1,k_1}(\rho_1)\psi_{m_2,k_2}(\rho_2). \tag{7.5.12}$$

The actions (7.2.5)–(7.2.8) extend naturally to the two-dimensional case. Moreover, from the action of $\Delta(P) = P_1P_2 : (\rho_1,\rho_2) \mapsto (-\rho_1,-\rho_2)$ on $\psi_{m_1,m_2,k_1,k_2}(\rho_1,\rho_2)$, it is checked that $\epsilon_{12} = \epsilon_1\epsilon_2$ as expected. The energies can be presented as

$$E_{m_1,m_2} = (m_1+m_2) + (k_1+k_2) + 1 \tag{7.5.13}$$

and have additional degeneracies, as is seen from

$$m_1 + m_2 = \begin{cases} 2(n_1+n_2) & \text{if } (p_1,p_2)=(0,0), \\ 2(n_1+n_2)+1 & \text{if } (p_1,p_2)=(0,1), \\ 2(n_1+n_2)+1 & \text{if } (p_1,p_2)=(1,0), \\ 2(n_1+n_2)+2 & \text{if } (p_1,p_2)=(1,1). \end{cases} \tag{7.5.14}$$

This two-fold degeneracy comes from the fact that the Hamiltonian (7.1.1) is invariant under the exchange of $\rho_1 \leftrightarrow \rho_2$, $k_1 \leftrightarrow k_2$ and the internal spaces $2 \leftrightarrow 3$.

7.5.2. Solutions in polar coordinates

Superintegrable systems typically admit separation of variable in more than one coordinate system [2]. This is the case here: one can also separate the solutions in polar coordinates. Write

$$\rho_1 = r\cos\phi, \qquad \rho_2 = r\sin\phi, \tag{7.5.15}$$

the Schrödinger equation takes the form

$$H\Psi = \frac{1}{2}\left[-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + r^2 - \frac{1}{r^2}\left(\frac{\partial^2}{\partial \phi^2} - \frac{k_1(k_1-\sigma_3\otimes 1)}{\cos^2\phi} - \frac{k_2(k_2 - 1\otimes\sigma_3)}{\sin^2\phi}\right)\right]\Psi = E\Psi. \tag{7.5.16}$$

To make the separation of variable in polar coordinates, write $\Psi(r,\phi) = R(r)\Phi(\phi)$. This yields the angular equation

$$\left[\frac{d^2}{d\phi^2} - \frac{k_1(k_1-\sigma_3\otimes 1)}{\cos^2\phi} - \frac{k_2(k_2-1\otimes\sigma_3)}{\sin^2\phi} + m^2\right]\Phi = 0 \tag{7.5.17}$$

and the radial equation

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(-r^2 - \frac{m^2}{r^2} + 2E\right)R = 0. \tag{7.5.18}$$

These equations are solved in Appendix 7.B.

The orthonormalized angular solution to (7.5.17) is given in terms of Jacobi polynomials $J_d^{(\alpha,\beta)}(\phi)$:

$$\begin{aligned} \Phi_{\ell,s_1,s_2}^{k_1,k_2}(\phi) = &\sqrt{\frac{2(2\ell+k_1+k_2)\ (\ell-\frac{2-s_1-s_2}{4})!\ \Gamma(\ell+k_1+k_2+\frac{2-s_1-s_2}{4})}{\Gamma(\ell+k_1+\frac{2-s_1+s_2}{4})\ \Gamma(\ell+k_2+\frac{2+s_1-s_2}{4})}} \\ &\times [\cos\phi]^{k_1+\frac{1-s_1}{2}}[\sin\phi]^{k_2+\frac{1-s_2}{2}}\ P_{\ell-\frac{2-s_1-s_2}{4}}^{(k_1-\frac{s_1}{2},k_2-\frac{s_2}{2})}(-\cos 2\phi), \end{aligned} \tag{7.5.19}$$

$$|m| = 2\ell + (k_1+k_2), \qquad s_1, s_2 \in \{\pm 1\}.$$

It is important to recall that in the above, $\ell \in \{0, 1, 2, \dots\}$ is a non-negative integer if $s_1 s_2 = 1$ and $\ell \in \{\frac{1}{2}, \frac{3}{2}, \dots\}$ is a half-integer if $s_1 s_2 = -1$. Also, the normalization condition is

$$\int_0^{\pi/2} d\phi \left[\Phi^{k_1,k_2}_{\ell,s_1,s_2}(\phi)\right]^2 = 1. \tag{7.5.20}$$

The solutions are defined in the first quadrant, which corresponds to the domain of definition of the Jacobi polynomials. In Section 7.7, it will be seen that it is natural for the solutions to be given only in the first quadrant. The wavefunctions can be extended straightforwardly to the four quadrants.

Next, the solutions of the radial equation are expressed in terms of Laguerre polynomials like in the one-dimensional case:

$$R_{N'}(r) = \sqrt{\frac{2(N!)}{\Gamma(N' + |m| + 1)}} r^{|m|} \mathsf{e}^{-r^2/2} L^{(|m|)}_{N'}(r^2). \tag{7.5.21}$$

The orthonormalized solutions of the two-dimensional system in polar coordinates are then given by

$$\psi^{k_1,k_2}_{\ell,N',s_1,s_2}(r,\phi) = \Phi^{k_1,k_2}_{\ell,s_1,s_2}(\phi) R_{N'}(r) \tag{7.5.22}$$

The total energy of the two-dimensional system is

$$E = 2N' + |m| + 1 = 2(N' + \ell) + (k_1 + k_2) + 1, \tag{7.5.23}$$

which matches what was obtained for the Cartesian solutions in (7.5.13) upon letting

$$2(N' + \ell) = m_1 + m_2. \tag{7.5.24}$$

Equation (7.5.16) can alternatively be presented in the form

$$H|\ell, N', s_1, s_2, k_1, k_2\rangle = E|\ell, N', s_1, s_2, k_1, k_2\rangle \tag{7.5.25}$$

where the $|\ell, N', s_1, s_2, k_1, k_2\rangle$ are 4-component spinors, whose entries depends on the value of s_1, s_2:

$$\begin{aligned}
|\ell, N', +, +, k_1, k_2\rangle &= \begin{pmatrix} \psi^{k_1,k_2}_{\ell,N',+,+} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |\ell, N', +, -, k_1, k_2\rangle = \begin{pmatrix} 0 \\ \psi^{k_1,k_2}_{\ell,N',+,-} \\ 0 \\ 0 \end{pmatrix}, \\
|\ell, N', -, +, k_1, k_2\rangle &= \begin{pmatrix} 0 \\ 0 \\ \psi^{k_1,k_2}_{\ell,N',-,+} \\ 0 \end{pmatrix}, \qquad |\ell, N', -, -, k_1, k_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \psi^{k_1,k_2}_{\ell,N',-,-} \end{pmatrix}.
\end{aligned} \tag{7.5.26}$$

7.6. Overlaps: the dual -1 Hahn polynomials

The goal is now to compute the overlaps between the wavefunctions arising from the separation of variables in Cartesian and polar coordinates and to relate these connection coefficients to the symmetry algebra exhibited previously in Section 7.4.

7.6.1. A basis diagonalizing $Q^{(12)}$

The separation of variables in polar coordinates amounted to the diagonalization of the operator

$$\mathbf{B}_{\phi} = -\frac{d^2}{d\phi^2} + \frac{k_1(k_1 - \sigma_3 \otimes 1)}{\cos^2\phi} + \frac{k_2(k_2 - 1 \otimes \sigma_3)}{\sin^2\phi} \tag{7.6.1}$$

We will now diagonalize $Q^{(12)}$. The reason for this choice is that the overlaps between the Cartesian eigenbasis and the eigenbasis of $Q^{(12)}$ are straightforward to obtain in light of our knowledge of the action of the ladder operators on an $\mathfrak{osp}(1|2)$ irrep (7.2.5)–(7.2.8) and of that fact that the eigenstates of $Q^{(12)}$ can be obtained as a linear combination of the polar eigenstates.

In our realization

$$Q^{(12)} = \left[-i\epsilon_2 \frac{d}{d\phi}\Sigma_{13} - \epsilon_2 k_1 \frac{\sin\phi}{\cos\phi}\Sigma_{23} + \epsilon_2 k_2 \frac{\cos\phi}{\sin\phi}\Sigma_{14} + k_1\Sigma_{12} + k_2\Sigma_{34} - \frac{1}{2}\right]\epsilon_{12}\Sigma_{12}\Sigma_{34}. \tag{7.6.2}$$

Let us now fix the eigenvalue of $\Sigma_{12}\Sigma_{34}$ to be $\delta = \pm 1$. We shall look for the eigenvectors of $Q^{(12)}$ such that

$$Q^{(12)}|F_{\delta}\rangle = q_{\delta}|F_{\delta}\rangle. \tag{7.6.3}$$

Looking at the eigenvalue of Σ_{34}, which is either ± 1, we write

$$|F_{\delta}\rangle = |f_{\delta}^{+}\rangle + |f_{\delta}^{-}\rangle, \tag{7.6.4}$$

with the labels $\pm$ chosen so that

$$\Sigma_{34}|f_{\delta}^{\pm}\rangle = \pm|f_{\delta}^{\pm}\rangle. \tag{7.6.5}$$

7.6.1.1. The case $\delta = +1$. In the case that $\delta = +1$ one has

$$|F_{+}\rangle = \begin{pmatrix} f_{+}^{-} \\ 0 \\ 0 \\ f_{+}^{+} \end{pmatrix} = |f_{+}^{-}\rangle + |f_{+}^{+}\rangle = f_{+}^{-}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + f_{+}^{+}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = f_{+}^{-}|+,-\rangle_{\Sigma} + f_{+}^{+}|+,+\rangle_{\Sigma}. \tag{7.6.6}$$

One sees that f_+^+ is related to f_+^- by virtue of (7.6.3). Now, explicitly writing down (7.6.2) gives rise to two equations that must be simultaneously satisfied

$$\epsilon_1\left(\quad\frac{d}{d\phi}-k_1\frac{\sin\phi}{\cos\phi}+k_2\frac{\cos\phi}{\sin\phi}\right)f_+^+-\epsilon_{12}\left(k_1+k_2+\tfrac{1}{2}\right)f_+^-=q_+f_+^-, \tag{7.6.7}$$

$$\epsilon_1\left(-\frac{d}{d\phi}-k_1\frac{\sin\phi}{\cos\phi}+k_2\frac{\cos\phi}{\sin\phi}\right)f_+^-+\epsilon_{12}\left(k_1+k_2-\tfrac{1}{2}\right)f_+^+=q_+f_+^+. \tag{7.6.8}$$

Inserting (7.6.7) into (7.6.8), denoting $\tilde{q}_+=q_++\frac{1}{2}\epsilon_{12}$ and recalling that $\epsilon_1^2=\epsilon_2^2=1$, we obtain the following second-order differential equation:

$$\frac{d^2f_+^-}{d\phi^2}-\left(\frac{k_1(k_1-1)}{\cos^2\phi}+\frac{k_2(k_2-1)}{\sin^2\phi}-\tilde{q}_+^2\right)f_+^-=0. \tag{7.6.9}$$

Comparing with (7.B.9), it is easily seen that the solutions are given in terms of Jacobi polynomials (the exact normalization will be given below). We have

$$\begin{aligned}f_+^-&=\sqrt{\frac{2\ (\ell!)\ (2\ell+k_1+k_2)\ \Gamma(\ell+k_1+k_2)}{\Gamma(\ell+k_1+\frac{1}{2})\ \Gamma(\ell+k_2+\frac{1}{2})}}(\cos\phi)^{k_1}(\sin\phi)^{k_2}P_\ell^{(k_1-\frac{1}{2},k_2-\frac{1}{2})}(-\cos2\phi)\\&=\Phi_{\ell,+,+}^{k_1,k_2}(\phi),\end{aligned} \tag{7.6.10}$$

and the eigenvalues $\tilde{q}_+$ are

$$\tilde{q}_+^2=(2\ell+k_1+k_2)^2,\qquad\Longrightarrow\qquad\tilde{q}_+=\pm(2\ell+k_1+k_2),\qquad\ell\in\mathbb{N}. \tag{7.6.11}$$

To obtain f_+^+, one could repeat what was done for f_+^- and solve the resulting second-order differential equation. This would yield, up to some normalization, $f_+^+=\Phi_{\ell,-,-}^{k_1,k_2}(\phi)$. Since the relative normalization between f_+^- and f_+^+ is crucial, we will use (7.6.8) instead. First suppose $\ell>0$. We note that

$$(\cos\phi)^{k_1}(\sin\phi)^{k_2}\frac{d}{d\phi}(\cos\phi)^{-k_1}(\sin\phi)^{-k_2}=\frac{d}{d\phi}+k_1\frac{\sin\phi}{\cos\phi}-k_2\frac{\cos\phi}{\sin\phi}. \tag{7.6.12}$$

Then, it is straightforward to obtain

$$\begin{aligned}f_+^+=&\frac{-\epsilon_1}{\tilde{q}_+-\epsilon_{12}(k_1+k_2)}\sqrt{\frac{2\ \ell!\ \Gamma(\ell+k_1+k_2)\ (2\ell+k_1+k_2)}{\Gamma(\ell+k_1+\frac{1}{2})\ \Gamma(\ell+k_2+\frac{1}{2})}}\\&\times(\cos\phi)^{k_1}(\sin\phi)^{k_2}\ \frac{d}{d\phi}P_\ell^{(k_1-\frac{1}{2},k_2-\frac{1}{2})}(-\cos2\phi),\end{aligned} \tag{7.6.13}$$

and using [30]

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x)=\frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{7.6.14}$$

one is led to

$$\begin{aligned} f_+^+ &= \frac{-2\epsilon_1\sqrt{\ell(\ell+k_1+k_2)}}{\tilde{q}_+-\epsilon_{12}(k_1+k_2)}\sqrt{\frac{2\ (\ell-1)!\ \Gamma(\ell+k_1+k_2+1)\ (2\ell+k_1+k_2)}{\Gamma(\ell+k_1+\frac{1}{2})\ \Gamma(\ell+k_2+\frac{1}{2})}} \\ &\quad\times(\cos\phi)^{k_1+1}(\sin\phi)^{k_2+1}\ P_{\ell-1}^{(k_1+\frac{1}{2},k_2+\frac{1}{2})}(-\cos 2\phi) \\ &= \frac{-2\epsilon_1\sqrt{\ell(\ell+k_1+k_2)}}{\tilde{q}_+-\epsilon_{12}(k_1+k_2)}\Phi_{\ell,-,-}^{k_1,k_2}(\phi). \end{aligned} \tag{7.6.15}$$

Recall that $\tilde{q}_+$ has two possible expressions $\pm(2\ell+k_1+k_2)$. Consider each case in turn. This gives two solutions, corresponding to each sign. The full orthonormalized solutions to (7.6.7)–(7.6.8) are then found to be:

$$|F_{\ell,+,+}\rangle = \sqrt{\frac{\ell+\frac{1-\epsilon_{12}}{2}(k_1+k_2)}{2\ell+k_1+k_2}}|\Phi_\ell;+,-\rangle - \epsilon_1\sqrt{\frac{\ell+\frac{1+\epsilon_{12}}{2}(k_1+k_2)}{2\ell+k_1+k_2}}|\Phi_\ell;+,+\rangle, \tag{7.6.16}$$

$$|F_{\ell,+,-}\rangle = \sqrt{\frac{\ell+\frac{1+\epsilon_{12}}{2}(k_1+k_2)}{2\ell+k_1+k_2}}|\Phi_\ell;+,-\rangle + \epsilon_1\sqrt{\frac{\ell+\frac{1-\epsilon_{12}}{2}(k_1+k_2)}{2\ell+k_1+k_2}}|\Phi_\ell;+,+\rangle, \tag{7.6.17}$$

with

$$|\Phi_\ell;+,-\rangle = \Phi_{\ell,+,+}^{k_1,k_2}(\phi)|+,-\rangle_\Sigma, \qquad |\Phi_\ell;+,+\rangle = \Phi_{\ell,-,-}^{k_1,k_2}(\phi)|+,+\rangle_\Sigma. \tag{7.6.18}$$

These solutions indeed diagonalize $Q^{(12)}$

$$Q^{(12)}|F_{\ell,+,\pm}\rangle = (\pm|\tilde{q}_+| - \tfrac{1}{2}\epsilon_{12})\ |F_{\ell,+,\pm}\rangle = \pm(2\ell+k_1+k_2\mp\tfrac{1}{2}\epsilon_{12})|F_{\ell,+,\pm}\rangle. \tag{7.6.19}$$

In the case where $\ell=0$, $\Phi_{0,-,-}^{k_1,k_2}(\phi)$ vanishes and there exists only a single eigenstate of $Q^{(12)}$, whose eigenvalue equation is

$$Q^{(12)}|F_{0,+,-\epsilon_{12}}\rangle = -\epsilon_{12}(k_1+k_2+\tfrac{1}{2}). \tag{7.6.20}$$

Let us now define the eigenvectors $|q_z\rangle$, with $z\in\mathbb{N}$:

$$|q_{2\ell}\rangle = |F_{\ell,+,-\epsilon_{12}}\rangle, \qquad |q_{2\ell-1}\rangle = |F_{\ell,+,\ \epsilon_{12}}\rangle, \qquad \ell\in\mathbb{N}. \tag{7.6.21}$$

The spectrum of $Q^{(12)}$ can then be repackaged in a single expression

$$Q^{(12)}|q_z\rangle = q_z|q_z\rangle, \qquad q_z = \epsilon_{12}(-1)^{z+1}(z+k_1+k_2+\tfrac{1}{2}), \qquad z\in\mathbb{N}. \tag{7.6.22}$$

7.6.1.2. The case $\delta=-1$. We repeat the analysis for the case $\delta=-1$. One looks for eigenvectors

$$|F_-\rangle = \begin{pmatrix}0\\ f_-^+\\ f_-^-\\ 0\end{pmatrix} = |f_-^-\rangle + |f_-^+\rangle = f_-^-\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + f_-^+\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = f_-^-|-,-\rangle_\Sigma + f_-^+|-,+\rangle_\Sigma. \tag{7.6.23}$$

Writing down (7.6.3) in the realization leads to the following equations:

$$\epsilon_1\left(\frac{d}{d\phi}-k_1\frac{\sin\phi}{\cos\phi}-k_2\frac{\cos\phi}{\sin\phi}\right)f_-^- +\epsilon_{12}\left(k_1-k_2+\tfrac{1}{2}\right)f_-^+=q_-f_-^+, \tag{7.6.24}$$

$$\epsilon_1\left(-\frac{d}{d\phi}-k_1\frac{\sin\phi}{\cos\phi}-k_2\frac{\cos\phi}{\sin\phi}\right)f_-^+ -\epsilon_{12}\left(k_1-k_2-\tfrac{1}{2}\right)f_-^-=q_-f_-^-. \tag{7.6.25}$$

Similarly, substituting (7.6.24) into (7.6.25), one obtains

$$\frac{d^2f_-^-}{d\phi^2}-\left(\frac{k_1(k_1+1)}{\cos^2\phi}+\frac{k_2(k_2-1)}{\sin^2\phi}-\tilde{q}_-^2\right)f_-^-=0, \tag{7.6.26}$$

with $\tilde{q}_-=q_--\frac{1}{2}\epsilon_{12}$. Comparing with (7.B.9), it is easily seen that the solutions are again given in terms of Jacobi polynomials:

$$f_-^-=\Phi^{k_1,k_2}_{\ell,-,+}(\phi),\qquad \ell\in\{\tfrac{1}{2},\tfrac{3}{2},\dots\} \tag{7.6.27}$$

and the eigenvalues are found to be

$$\tilde{q}_-^2=(2\ell+k_1+k_2)^2,\qquad\Longrightarrow\qquad \tilde{q}_-=\pm(2\ell+k_1+k_2). \tag{7.6.28}$$

We are interested in the relative normalization between f_-^- and f_-^+, so we use (7.6.24). It will be useful to call upon the identity

$$\frac{d}{dz}P_n^{(\alpha,\beta)}(z)=\frac{\alpha+n}{z-1}P_n^{(\alpha-1,\beta+1)}(z)-\frac{\alpha}{z-1}P_n^{(\alpha,\beta)}(z), \tag{7.6.29}$$

which is obtained by combining the Laguerre mixed relations [30, p. 264, equations (9)−(17)] with (7.6.14).

Treating the two possible values of $\tilde{q}_-$, we obtain two solutions. Finally, the orthonormalized basis is obtained:

$$|F_{\ell,-,+}\rangle=\sqrt{\frac{\ell+\frac{1-\epsilon_{12}}{2}k_1+\frac{1+\epsilon_{12}}{2}k_2}{2\ell+k_1+k_2}}|\Phi_\ell;-,-\rangle-\epsilon_1\sqrt{\frac{\ell+\frac{1+\epsilon_{12}}{2}k_1+\frac{1-\epsilon_{12}}{2}k_2}{2\ell+k_1+k_2}}|\Phi_\ell;-,+\rangle, \tag{7.6.30}$$

$$|F_{\ell,-,-}\rangle=\sqrt{\frac{\ell+\frac{1+\epsilon_{12}}{2}k_1+\frac{1-\epsilon_{12}}{2}k_2}{2\ell+k_1+k_2}}|\Phi_\ell;-,-\rangle+\epsilon_1\sqrt{\frac{\ell+\frac{1-\epsilon_{12}}{2}k_1+\frac{1+\epsilon_{12}}{2}k_2}{2\ell+k_1+k_2}}|\Phi_\ell;-,+\rangle, \tag{7.6.31}$$

with

$$|\Phi_\ell;-,-\rangle=\Phi^{k_1,k_2}_{\ell,-,+}(\phi)|-,-\rangle_\Sigma,\qquad |\Phi_\ell;-,+\rangle=\Phi^{k_1,k_2}_{\ell,+,-}(\phi)|-,+\rangle_\Sigma. \tag{7.6.32}$$

In this basis,

$$Q^{(12)}|F_{-,\pm}\rangle=\left(\pm|\tilde{q}_-|+\tfrac{1}{2}\epsilon_{12}\right)|F_{-,\pm}\rangle=\pm(2\ell+k_1+k_2\pm\tfrac{1}{2}\epsilon_{12})|F_{-,\pm}\rangle. \tag{7.6.33}$$

Defining the eigenvectors $|q_z\rangle$ for $z\in\mathbb{N}$ and $\ell\in\mathbb{N}$ as

$$|q_{2\ell}\rangle=|F_{\ell,+,-\epsilon_{12}}\rangle,\qquad |q_{2\ell+1}\rangle=|F_{\ell,-,\epsilon_{12}}\rangle,\qquad \ell\in\mathbb{N}, \tag{7.6.34}$$

the spectrum of $Q^{(12)}$ can be presented in a single expression

$$Q^{(12)}|q_z\rangle = q_z|q_z\rangle, \qquad q_z = \epsilon_{12}(-1)^{z+1}(z + k_1 + k_2 + \tfrac{1}{2}), \qquad z \in \mathbb{N}. \tag{7.6.35}$$

7.6.2. Overlaps with the Cartesian basis and the dual −1 Hahn polynomials

It is clear that the overlaps between eigenstates with different energies will vanish. Let us then consider cases where $m_1 + m_2 = 2(N' + \ell)$, i.e. cases where the energies of the eigenstates in Cartesian coordinates and polar coordinates are equal.

The overlaps between the eigenvectors $|q_z\rangle$ diagonalizing $Q^{(12)}$ and the Cartesian eigenvectors $|m_1, m_2, k_1, k_2\rangle \equiv |m_1; m_2\rangle$ are:

$$\langle q_z|Q^{(12)}|m_1; m_2\rangle = \langle q_z|(A_-^{(1)}A_+^{(2)} - A_+^{(1)}A_-^{(2)})P^{(1)} + Q^{(1)}P^{(2)} + Q^{(2)}P^{(1)} - \tfrac{1}{2}P^{(1)}P^{(2)}|m_1; m_2\rangle. \tag{7.6.36}$$

Using the actions given in (7.2.5)–(7.2.8) and defining $\langle q_z|m_1; m_2\rangle = M_{m_1,m_2}$ yields

$$\begin{aligned} q_z\ M_{m_1,m_2} &= -\epsilon_{12}\left(k_1(-1)^{m_2} + k_2(-1)^{m_1} + \tfrac{1}{2}(-1)^{m_1+m_2}\right)M_{m_1,m_2} \\ &+ \epsilon_1(-1)^{m_1}\left(\sqrt{[m_1]_{k_1}[m_2+1]_{k_2}}M_{m_1-1,m_2+1} - \sqrt{[m_1+1]_{k_1}[m_2]_{k_2}}M_{m_1+1,m_2-1}\right). \end{aligned} \tag{7.6.37}$$

Making the change of variables $N = m_1 + m_2, \quad m = m_1$ and writing

$$M_{m_1,m_2} = \left(\frac{\epsilon_2}{2}\right)^m (-1)^{\frac{m(m+2N+1)}{2}} \left(\prod_{p_1=1}^{m}\prod_{p_2=1}^{N-m}\sqrt{\frac{[p_2]_{k_2}}{[p_1]_{k_1}}}\right)\mathcal{N}_{0,N}\ \mathcal{N}_{m,N}, \tag{7.6.38}$$

we obtain a monic three term recurrence relation for the matrix elements:

$$\begin{aligned} (-1)^N[2\epsilon_{12}\, q_z]\mathcal{N}_{m,N} = \mathcal{N}_{m+1,N} &+ [2(-1)^{m+1}(k_1 + (-1)^N k_2) - 1]\mathcal{N}_{m,N} \\ &+ 4[m]_{k_1}[N-m+1]_{k_2}\ \mathcal{N}_{m-1,N}. \end{aligned} \tag{7.6.39}$$

A quick look at (8.A.1) shows us that these matrix elements $\mathcal{N}_{m,N}$ are dual −1 Hahn polynomials $P_m(x_z; k_1, k_2, N)$ in the variable

$$x_z = (-1)^{N+z+1}(2z + 2k_1 + 2k_2 + 1), \qquad \text{with} \quad z \in \mathbb{N}. \tag{7.6.40}$$

We then have the desired expression for $M_{m_1,m_2} = \langle q_z|m_1; m_2\rangle$ up to a normalization, which can be determined using the orthonormality of the two bases:

$$\delta_{m\bar{m}} = \sum_{q_z}\langle\bar{m}, N-\bar{m}|q_z\rangle\langle q_z|m, N-m\rangle. \tag{7.6.41}$$

If N is odd, (8.A.6) tells us that we have $x_z = x_s$ by taking $z = s \in \{0, 1, \ldots, N\}$ and it follows that

$$\langle q_z | m, N-m\rangle = P_m(x_z; k_1, k_2, N)\sqrt{\frac{w_z(k_1, k_2, N)}{\nu_m(k_1, k_2, N)}}. \tag{7.6.42}$$

If N is even, we recover $x_z = x_s$ by taking $z = N - s \in \{0, 1, \ldots, N\}$ and hence

$$\langle q_z | m, N-m\rangle = P_m(x_z; k_1, k_2, N)\sqrt{\frac{w_{N-z}(k_1, k_2, N)}{\nu_m(k_1, k_2, N)}}. \tag{7.6.43}$$

It is then simple to reexpress the eigenvectors $|q_z\rangle$ diagonalizing $Q^{(12)}$ in terms of the polar eigenvectors $|\ell, N', s_1, s_2, k_1, k_2\rangle$. Indeed, from the definitions of $|q_z\rangle$ in terms of $|\Phi_\ell; s_1, s_2\rangle$ in (7.6.21) and (7.6.34) as well as (7.5.26), the relations are easily inversed. From there, one can reexpress the overlaps between the polar and Cartesian eigenvectors as a linear combination of dual -1 Hahn polynomials following the results in (7.6.42)–(7.6.43).

7.7. Dimensional reduction

The spinorial model obtained in Section 7.3 can furthermore be derived through a dimensional reduction procedure.

We start with a system of 4 uncoupled standard harmonic oscillator acting on 4 dimensional space, described by the Hamiltonian

$$\tilde{H} = \sum_{i=1}^{4} a_i^\dagger a_i + 2\mathbb{I}_4, \tag{7.7.1}$$

where a_i, $a_i^\dagger$ are the usual annihilation/creation operators

$$a_i = \frac{1}{\sqrt{2}}\left(x_i + \frac{\partial}{\partial x_i}\right)\mathbb{I}_4, \qquad a_i^\dagger = \frac{1}{\sqrt{2}}\left(x_i - \frac{\partial}{\partial x_i}\right)\mathbb{I}_4, \qquad i = 1, \ldots, 4, \tag{7.7.2}$$

satisfying $[a_i^\dagger, a_j] = \delta_{ij}\mathbb{I}_4$ and $\mathbb{I}_4$ is the 4-dimensional identity matrix. One can now introduce the cylindrical coordinates

$$\begin{aligned} x_1 &= \rho_1 \cos\theta_1, \qquad & x_3 &= \rho_2 \cos\theta_2, \\ x_2 &= \rho_1 \sin\theta_1, \qquad & x_4 &= \rho_2 \sin\theta_2. \end{aligned} \tag{7.7.3}$$

The Hamiltonian (7.7.1) is rewritten as

$$\tilde{H} = \frac{1}{2}\sum_{i=1}^{2}\left(\rho_i^2 - \frac{\partial^2}{\partial\rho_i^2} - \frac{1}{\rho_i}\frac{\partial}{\partial\rho_i} - \frac{1}{\rho_i^2}\frac{\partial^2}{\partial\theta_i^2}\right)\mathbb{I}_4. \tag{7.7.4}$$

We can now effect the gauge transformation χ on the radii to get rid of the $\frac{1}{\rho_i}\frac{\partial}{\partial\rho_i}$ term:

$$\chi(\,\cdot\,) = (\rho_1\rho_2)^{1/2}(\,\cdot\,)(\rho_1\rho_2)^{-1/2}. \tag{7.7.5}$$

This yields

$$\chi(\tilde{H}) = H = \frac{1}{2}\sum_{i=1}^{2}\left(\rho_i^2 - \frac{\partial^2}{\partial\rho_i^2} - \frac{1/4}{\rho_i^2} - \frac{1}{\rho_i^2}\frac{\partial^2}{\partial\theta_i^2}\right)\mathbb{I}_4. \tag{7.7.6}$$

Owing to cylindrical symmetry, it is possible to set values for the spinorial angular momenta

$$\begin{aligned} J_{12} &= -i\frac{\partial}{\partial\theta_1} + \frac{1}{2}\Sigma_{12} = -k_1, \\ J_{34} &= -i\frac{\partial}{\partial\theta_2} + \frac{1}{2}\Sigma_{34} = -k_2, \end{aligned} \tag{7.7.7}$$

and the desired Hamiltonian H_{12} is obtained:

$$H_{12} = \frac{1}{2}\left[-\left(\frac{d^2}{d\rho_1^2} + \frac{d^2}{d\rho_2^2}\right) + (\rho_1^2 + \rho_2^2) + \frac{k_1(k_1+\Sigma_{12})}{\rho_1^2} + \frac{k_2(k_2+\Sigma_{34})}{\rho_2^2}\right]\mathbb{I}_4. \tag{7.7.8}$$

Recall that the solutions obtained in Section 7.5 were naturally defined for the first quadrant only, that is $\phi \in [0, \frac{\pi}{2}]$. It is now clear from the dimensional reduction procedure that this has to be the case. The reduced system coordinates ρ_1, ρ_2 are radial coordinates taking values in $\mathbb{R}^+$, and those positive coordinates are precisely those that define the first quadrant of the plane.

One may wonder if this dimensional reduction procedure can be carried out for all $\mathfrak{osp}(1|2)$ generators (that is, for the $A_\pm^{(12)}$ as well). The answer is affirmative, but there is some refinement needed: one needs to perform an additional gauge transformation on the spinorial space. This was carried out in [31, 32].

The reason for the need of an additional gauge transformation in spin space is that the gamma matrices appearing in the expressions (7.4.8) of the $A_\pm^{(12)}$ acquire an angular dependency when one passes to cylindrical coordinates. The spinorial gauge transformation is meant to "rotate out" the angular dependency so that the procedure described in (7.7.5)–(7.7.8) can be similarly applied.

The novel aspects of the superconformal system presented here come from the presence of internal degrees of freedom whose non-trivial ties arise in dimensional reduction by fixing the spinorial angular momentum. Comparison with the situation where only the spatial angular momentum is fixed is instructive. In case we set

$$\begin{aligned} L_{12} &= -i\frac{\partial}{\partial\theta_1} = -\kappa_1, \\ L_{34} &= -i\frac{\partial}{\partial\theta_2} = -\kappa_2, \end{aligned} \tag{7.7.9}$$

the Hamiltonian (7.7.6) reduces to

$$H'_{12} = \frac{1}{2}\left[-\left(\frac{d^2}{d\rho_1^2} + \frac{d^2}{d\rho_2^2}\right) + (\rho_1^2 + \rho_2^2) + \frac{\kappa_1^2 - \frac{1}{4}}{\rho_1^2} + \frac{\kappa_2^2 - \frac{1}{4}}{\rho_2^2}\right]\mathbb{I}_4. \tag{7.7.10}$$

This is in effect a system of two singular oscillators, and such oscillators are associated to the algebra $\mathfrak{su}(1,1)$ instead of $\mathfrak{osp}(1|2)$. These have been examined in detail and shown to be superintegrable in [23]. The fact that many four-dimensional oscillators were a priori considered is immaterial under this reduction process.

However, if one fixes the spinorial angular momentum (7.7.7), the reduction effectively couples the internal degrees of freedom. This is the origin of the relations (7.3.6), which connect the pure angular momentum (and thus the parity of the wavefunctions) with the spin (and thus the index of the components of the spinors).

7.8. Conclusion

This paper has introduced a superconformal system with internal degrees of freedom in two dimensions that is superintegrable and that has the dual -1 Hahn algebra as its symmetry algebra. This model has been obtained by combining two spinorial realizations of the superalgebra $\mathfrak{osp}(1|2)$ and identifying the Hamiltonian as the resulting Cartan generator.

What about combining more than two representations of $\mathfrak{osp}(1|2)$?

It is known [6, 18] that the generic superintegrable model on the two-sphere [33] is obtained in a similar spirit by combining three realizations of $\mathfrak{su}(1,1)$. In this case the Hamiltonian is taken to be the total Casimir element. A two-dimensional system is obtained because the norm of the radius vector is conserved. The constants of motion correspond to the intermediate Casimir operators which generate the symmetry algebra known under the name of Racah [6, 18, 34]. All other scalar second-order superintegrable models in two-dimensions can be obtained as special cases or contractions [35] of this generic model.

Three parabosonic realizations of $\mathfrak{osp}(1|2)$ have been similarly [36] combined to obtain a superintegrable model with reflections on the 2-sphere that has the Bannai–Ito algebra as symmetry algebra. Here again the Hamiltonian is related (quadratically) to the Casimir operator of the underlying superalgebra. It has been shown [36] that the superintegrable Dunkl oscillator in two dimensions can in fact be obtained as a contraction of this Bannai–Ito invariant model on S^2.

These observations suggest that it would be relevant to combine three spinorial representations of $\mathfrak{osp}(1|2)$ like the ones considered here to construct a model on S^2, without reflections, that should hence have by construction the Bannai–Ito algebra as its symmetry algebra. One would a priori expect the model presented here to be a contraction of the above. This raises interesting questions. One issue is that the number of degrees of freedom associated to combining two and three $\mathfrak{osp}(1|2)$ representations differs from the start; another has to do with the fact that the dual -1 Hahn algebra is known to be a contraction of the algebra of the complementary Bannai–Ito polynomials [37] which is quite different from the Bannai–Ito one. Sorting this out should prove enlightening.

Now adding three spinorial realizations of $\mathfrak{osp}(1|2)$ and taking as done here the Hamiltonian to be the total Cartan generator will yield a superintegrable singular oscillator with internal degrees of freedom in three dimensions. This has been performed with the parabosonic realizations to obtain the superintegrable Dunkl oscillator in three dimensions with an invariance algebra called the Schwinger-Dunkl algebra $sd(3)$ that extends $\mathfrak{su}(3)$. We may thus expect a similar outcome in the case with internal degrees of freedom. While this has not been established, we could anticipate that the symmetry algebra, likely $sd(3)$, is isomorphic to the algebra of the rank 2 dual -1 Hahn algebra. This would be the algebra associated to the bivariate or two-variable dual -1 Hahn polynomials that have not been characterized so far. While the bivariate Bannai–Ito polynomials have recently been introduced and studied [38], this is not the case for the bivariate complementary Bannai–Ito polynomials from which the bivariate dual -1 Hahn polynomials should descend. With respect to contractions, these three-dimensional singular oscillators should relate to systems on the three sphere obtained by considering the addition of four realizations of $\mathfrak{osp}(1|2)$ (see [39] in this connection).

In another register, we wish to point out that one may use the R-matrix approach to arrive [40] at the generic superintegrable model on S^2 and construct its constants of motion. One proceeds via dimensional reduction with a Lax matrix that involves three $\mathfrak{su}(1,1)$ elements. It has been shown recently [41] that the universal R-matrix of $\mathfrak{osp}(1|2)$ plays a central role in the description of the Bannai–Ito algebra. It should prove interesting to explore how this general formalism of integrable systems applies to the description of the superintegrable models with internal degrees of freedom that we have been discussing.

We thus observe that the superintegrable model introduced here presents itself as a nice basis to examine some of the various questions we have pointed out that pertains generally to the understanding of the algebras of Askey-Wilson type and their applications. We hope to follow up with these matters in the near future.

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7.A. The dual -1 Hahn polynomials

Here are a few useful definitions and properties of the dual -1 Hahn polynomials [1, 16], which have been introduced as a $q \to 1$ limit of the dual q-Hahn polynomials [14].

We denote the monic dual -1 Hahn polynomials $P_n(x;\xi,\zeta,N)$, where the parameters $\xi,\zeta > -\frac{1}{2}$ and N is an integer. These polynomials satisfy a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + [2(-1)^{n+1}(\xi + (-1)^N\zeta) - 1]\ P_n(x) + 4[n]_\xi[N-n+1]_\zeta\ P_{n-1}(x) \tag{7.A.1}$$

Note that the factors are chosen for consistency with the definitions in references [1, 16].

Recall that the hypergeometric series ${}_rF_s$ is defined by

$${}_rF_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!} \tag{7.A.2}$$

with $(c)_k = c(c+1)\cdots(c+k-1)$ the Pochhammer symbol. The dual -1 Hahn polynomials can be expressed as a generalized hypergeometric truncating series.

For N even, denote $\delta = -\frac{1}{2}(\xi+\zeta+N)$; the expressions are

$$P_{2n}(x) = 2^{4n}\left(-\tfrac{N}{2}\right)_n \left(\tfrac{1}{2} - \tfrac{N}{2} - \zeta\right)_n {}_3F_2\left(\begin{matrix} -n,\ \delta + \frac{1+x}{4},\ \delta - \frac{1+x}{4} \\ -\frac{N}{2},\ -\frac{N}{2} + \frac{1}{2} - \zeta \end{matrix}; 1\right), \tag{7.A.3}$$

$$P_{2n+1}(x) = (x + 2\xi + 2\zeta + 1)2^{4n}\left(1-\tfrac{N}{2}\right)_n \left(\tfrac{1}{2} - \tfrac{N}{2} - \zeta\right)_n {}_3F_2\left(\begin{matrix} -n,\ \delta + \frac{1+x}{4},\ \delta - \frac{1+x}{4} \\ 1-\frac{N}{2},\ -\frac{N}{2} + \frac{1}{2} - \zeta \end{matrix}; 1\right). \tag{7.A.4}$$

For N odd, denote $\eta = \frac{1}{2}(\xi+\zeta+1)$; the expressions are

$$P_{2n}(x) = 2^{4n}\left(\tfrac{1-N}{2}\right)_n \left(\xi + \tfrac{1}{2}\right)_n {}_3F_2\left(\begin{matrix} -n,\ \eta + \frac{1+x}{4},\ \eta - \frac{1+x}{4} \\ \frac{1-N}{2},\ \xi + \frac{1}{2} \end{matrix}; 1\right), \tag{7.A.5}$$

$$P_{2n+1}(x) = (x + 2\xi - 2\zeta + 1)\, 2^{4n}\left(\tfrac{1-N}{2}\right)_n \left(\xi + \tfrac{3}{2}\right)_n {}_3F_2\left(\begin{matrix} -n,\ \eta + \frac{1+x}{4},\ \eta - \frac{1+x}{4} \\ \frac{1-N}{2},\ \xi + \frac{3}{2} \end{matrix}; 1\right). \tag{7.A.6}$$

These polynomials obey an orthogonality relation of the form

$$\sum_{s=0}^{N} w_s(\xi,\zeta,N)P_n(x_s;\xi,\zeta,N)P_m(x_s;\xi,\zeta,N) = \nu_n(\xi,\zeta,N)\delta_{n,m} \tag{7.A.7}$$

on the grid points

$$x_s = \begin{cases} (-1)^s(2s - 2\xi - 2\zeta - 2N - 1) & N \text{ even}, \\ (-1)^s(2s + 2\xi + 2\zeta + 1) & N \text{ odd}. \end{cases} \tag{7.A.8}$$

The weights are given by

$$w_{2m+j}(\xi,\zeta,N)=\begin{cases}\frac{(-1)^m\left(-\frac{N}{2}\right)_{m+j}}{m!}\frac{\left(\frac{1-N}{2}-\zeta\right)_m}{\left(\frac{1-N}{2}-\xi\right)_m}\frac{(-N-\xi-\zeta)_m}{\left(-\frac{N}{2}-\xi-\zeta\right)_{m+j}} & N \text{ even},\\ \frac{(-1)^m\left(\frac{1-N}{2}\right)_m}{m!}\frac{\left(\xi+\frac{1}{2}\right)_{m+j}}{\left(\zeta+\frac{1}{2}\right)_{m+j}}\frac{(1+\xi+\zeta)_m}{\left(\frac{1}{2}(N+2\xi+2\zeta+3)\right)_m} & N \text{ odd},\end{cases} \tag{7.A.9}$$

and the normalizations are given by

$$v_{2m+j}(\xi,\zeta,N)=\begin{cases}(-1)^j 2^{4(2m+j)}m!\left(\xi+\frac{1}{2}\right)_{m+j}\left(\frac{1-N}{2}-\zeta\right)_m\left(-\frac{N}{2}\right)_{m+j}\frac{(-N-\xi-\zeta)_{N/2}}{\left(\frac{1-N}{2}-\xi\right)_{N/2}} & N \text{ even},\\ (-1)^j 2^{4(2m+j)}m!\left(\xi+\frac{1}{2}\right)_{m+j}\left(\frac{1-N}{2}\right)_m\left(-\zeta-\frac{N}{2}\right)_{m+j}\frac{(\xi+\zeta+1)_{(N+1)/2}}{\left(\zeta+\frac{1}{2}\right)_{(N+1)/2}} & N \text{ odd}.\end{cases} \tag{7.A.10}$$

with $j\in\{0,1\}$ and m an integer.

The dual -1 Hahn polynomials are bispectral, but they satisfy a five term difference relation [16] on the grid x_s, hence they fall outside the scope of Leonard duality.

7.B. Solutions of the differential equations

7.B.1. The one-dimensional Schrödinger equation

The Schrödinger equation of the one-dimensional system is

$$H\psi=\frac{1}{2}\left(-\frac{d^2}{d\rho^2}+\rho^2+\frac{k(k-\sigma_3)}{\rho^2}\right)\psi=E\psi. \tag{7.B.1}$$

Let

$$\psi=e^{-\rho^2/2}\rho^\alpha f, \tag{7.B.2}$$

where α remains to be fixed. Putting this back in (7.B.1) gives

$$\frac{d^2f}{d\rho^2}+2\left(-\rho+\frac{\alpha}{\rho}\right)\frac{df}{d\rho}+\frac{\alpha(\alpha-1)-k(k-\sigma_3)}{\rho^2}f+(2E-2\alpha-1)f=0. \tag{7.B.3}$$

The value of α is now chosen in order to cancel the term in ρ^{-2}, that is

$$\alpha=\begin{cases}k & \text{if } \sigma_3 \text{ has eigenvalue } s=+1,\\ k+1 & \text{if } \sigma_3 \text{ has eigenvalue } s=-1.\end{cases} \tag{7.B.4}$$

Effecting the change of variable $\rho=x^{1/2}$, this equation becomes

$$x\frac{d^2f}{dx^2}+\left(\alpha+\frac{1}{2}-x\right)\frac{df}{dx}+\frac{1}{4}\left(2E-2\alpha-1\right)f=0. \tag{7.B.5}$$

The solutions of this equation are identified as generalized Laguerre polynomials [14] $L_n^{(\beta)}(x)$ with parameter

$$\beta = \alpha - \tfrac{1}{2}. \tag{7.B.6}$$

The orthonormalized solutions of the one-dimensional system $\psi_{n,k,s}(\rho)$ and the energies E_n are then given by

$$\begin{gathered}\psi_{n,k,s}(\rho) = \langle \rho | n, k, s\rangle = \sqrt{\frac{n!}{\Gamma(n+k+1-s/2)}}\mathrm{e}^{-\rho^2/2}\rho^{k+\frac{1-s}{2}}\, L_n^{(k-\frac{s}{2})}(\rho^2), \\ E_n = 2n + k + 1 - s/2, \qquad s = \pm 1.\end{gathered} \tag{7.B.7}$$

Expressing these solutions in terms of the generalized Hermite polynomials $H_m^k(x)$ [20, 28, 29], one obtains (7.5.3).

7.B.2. Separation in polar coordinates

First start with the angular equation (7.5.17) and denote

$$\beta_1 = k_1(k_1 - s_1), \qquad \beta_2 = k_2(k_2 - s_2). \tag{7.B.8}$$

This yields

$$\left[\frac{d^2}{d\phi^2} - \frac{\beta_1}{\cos^2\phi} - \frac{\beta_2}{\sin^2\phi} + m^2\right]\Phi = 0. \tag{7.B.9}$$

Now take

$$\Phi = \sin^{\gamma}(\phi)\cos^{\delta}(\phi) f, \tag{7.B.10}$$

with γ and δ to be determined, and (7.B.9) becomes

$$\frac{d^2 f}{d\phi^2} + 2\left(\gamma\frac{\cos\phi}{\sin\phi} - \delta\frac{\sin\phi}{\cos\phi}\right)\frac{df}{d\phi} + \left[\frac{[\gamma(\gamma-1)-\beta_2]}{\sin^2\phi} + \frac{[\delta(\delta-1)-\beta_1]}{\cos^2\phi} - (\gamma+\delta)^2 + m^2\right] f = 0. \tag{7.B.11}$$

The terms in $\cos^{-2}\phi$ and $\sin^{-2}\phi$ are eliminated upon choosing

$$\delta(\delta - 1) = \beta_1, \qquad \gamma(\gamma - 1) = \beta_2. \tag{7.B.12}$$

Now introduce

$$x = -\cos 2\phi, \tag{7.B.13}$$

(7.B.11) is rewritten as the differential equation

$$(1-x^2)\frac{d^2 f}{dx^2} + [(\gamma-\delta) - (\gamma+\delta+1)x]\frac{df}{dx} + \frac{1}{4}\left(m^2 - (\gamma+\delta)^2\right) f = 0, \tag{7.B.14}$$

whose solutions are the Jacobi polynomials $P_d^{(\alpha,\beta)}(x)$ with parameters

$$\alpha = \delta - \frac{1}{2}, \qquad \beta = \gamma - \frac{1}{2} \tag{7.B.15}$$

and

$$|m| = 2d + \gamma + \delta. \tag{7.B.16}$$

Recalling (7.B.8), we finally obtain the orthonormalized angular solution to (7.B.9)

$$\begin{aligned}\Phi_{\ell,s_1,s_2}^{k_1,k_2}(\phi) = &\sqrt{\frac{2(2\ell+k_1+k_2)\ (\ell-\frac{2-s_1-s_2}{4})!\ \Gamma(\ell+k_1+k_2+\frac{2-s_1-s_2}{4})}{\Gamma(\ell+k_1+\frac{2-s_1+s_2}{4})\ \Gamma(\ell+k_2+\frac{2+s_1-s_2}{4})}} \\ &\times [\cos\phi]^{k_1+\frac{1-s_1}{2}}[\sin\phi]^{k_2+\frac{1-s_2}{2}}\ P_{\ell-\frac{2-s_1-s_2}{4}}^{(k_1-\frac{s_1}{2},k_2-\frac{s_2}{2})}(-\cos 2\phi),\end{aligned} \tag{7.B.17}$$

$$|m| = 2\ell + (k_1+k_2), \qquad s_1, s_2 \in \{\pm 1\}.$$

In the above, $\ell \in \{0,1,2,\dots\}$ is a non-negative integer if $s_1 s_2 = 1$ and $\ell \in \{\frac{1}{2},\frac{3}{2},\dots\}$ is a half-integer if $s_1 s_2 = -1$.

Next, the radial equation is

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(-r^2 - \frac{m^2}{r^2} + 2E\right) R = 0 \tag{7.B.18}$$

and its orthonormalized solutions are obtained like the one-dimensional system, see (7.B.2)–(7.B.7):

$$R = \sqrt{\frac{2(N!)}{\Gamma(N'+|m|+1)}}\, r^{|m|} e^{-r^2/2} L_{N'}^{(|m|)}(r^2). \tag{7.B.19}$$

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Chapitre 8

A Howe correspondence for the algebra of the $\mathfrak{osp}(1|2)$ Clebsch–Gordan coefficients

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Abstract: Two descriptions of the dual -1 Hahn algebra are presented and shown to be related under Howe duality. The dual pair involved is formed by the Lie algebra $\mathfrak{o}(4)$ and the Lie superalgebra $\mathfrak{osp}(1|2)$.

Keywords: Howe duality, dual -1 Hahn algebra, Schwinger–Dunkl algebra, spinor representation, oscillators.

8.1. Introduction

The dual -1 Hahn algebra [1] captures the bispectrality properties of the orthogonal polynomials bearing the same name. These polynomials were first obtained [2] as a $q = -1$ limit of the dual q-Hahn polynomials and were shown to essentially define the Clebsch–Gordan coefficients of the Lie superalgebra $\mathfrak{osp}(1|2)$. This paper provides two related pictures of the (centrally extended) dual -1 Hahn algebra and explains their connection on the basis of Howe duality.

The dual -1 Hahn algebra is an example of the quadratic algebras of Askey–Wilson type that are realized by the recurrence and the difference/differential equation operators of the hypergeometric orthogonal polynomials of the Askey tableaux that correspond to $q = 1$, q generic [3] but also to $q = -1$ in the case of the so-called Bannai–Ito scheme. Sitting at the top of each of these are respectively, the Racah, the Askey–Wilson and the Bannai–Ito [4] polynomials. It is known that these orthogonal polynomials are, in the order in which they are listed, the Racah coefficients of the Lie algebra $\mathfrak{su}(2)$ or $\mathfrak{su}(1,1)$, of the quantum algebra $U_q(\mathfrak{sl}(2))$ and of the superalgebra $\mathfrak{osp}(1|2)$. This hints to the proven fact that the algebras associated to each of these families of polynomials and called by their names are realized as centralizers of the diagonal action of either $\mathfrak{su}(1,1)$, $U_q(\mathfrak{sl}(2))$ or $\mathfrak{osp}(1|2)$ on their three-fold product, an observation which is paving the way to higher rank extensions [5–8].

This feature explains in part why these algebras of Askey–Wilson type have become preeminent in a number of areas in mathematics and physics such as representation theory [9–11], combinatorics [12], knot theory [13] and integrable models [14–16]. There is hence much interest in deepening their understanding. In this respect, the Racah, Bannai–Ito and Askey–Wilson algebras have been given complementary descriptions [17–20] as commutants of maximal Abelian subalgebras in the universal algebra $U(\mathfrak{o}(n))$ of the orthogonal algebra $\mathfrak{o}(n)$ and its non standard q-deformation denoted $\mathfrak{o}_q(n)$ or $U_q'(\mathfrak{o}(n))$. Furthermore, it has been observed that these alternative presentations are in Howe duality with the centralizer ones. For a recent review see [21].

Howe duality which was introduced in very influential papers [22] is an important concept in representation and classical invariant theories. It has found various applications in Physics that are reviewed in [23].

The dual Hahn polynomials, a limit of the Racah polynomials, enter in the Clebsch–Gordan coefficients of $\mathfrak{su}(1,1)$. Their algebra which is isomorphic to the Higgs algebra [24] arises in many contexts (see references in [25]) and in particular as a truncation of the reflection algebra [9]. This dual Hahn algebra also admits two presentations which are related through Howe duality based on the pair $(\mathfrak{o}(4), \mathfrak{su}(1,1))$ [25]: in the first one, it is in the commutant of $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ in the oscillator representation of $U(\mathfrak{u}(4))$ and in the second it is embedded in $U(\mathfrak{su}(1,1)) \otimes U(\mathfrak{su}(1,1))$. A q-deformation of this analysis was performed [26] to define a q-analog of the Higgs algebra and provide a two-fold description of the dual q-Hahn algebra with Howe duality resting in this case on the pair $(\mathfrak{o}_q(4), U_q(\mathfrak{su}(1,1))$. Hence at the second level of the (discrete) Askey tableaux, this leaves the case $q=-1$, that is the algebra of the dual -1 Hahn polynomials, as the only one for which a description in the framework of Howe duality has not been given. The purpose of the present paper is to fill this gap.

The dual -1 Hahn algebra has been characterized in [1]; it has been shown to arise in the studies [27, 28] of the Dunkl oscillator in the plane and quite recently as the symmetry algebra of a superintegrable two-dimensional singular oscillator with internal degrees of freedom [29]. The dual -1 Hahn polynomials form the Clebsch–Gordan coefficients of $\mathfrak{osp}(1|2)$. This superalgebra will be one element of the dual pair at play, the other will be $\mathfrak{o}(4)$. It will be seen that the dual -1 Hahn algebra is in the commutant of $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ in a spinorial representation of $\mathfrak{o}(4)$ given in terms of Bosonic and Fermionic oscillators. This picture will be shown to be in a duality relation with the embedding of the dual -1 Hahn algebra in $U(\mathfrak{osp}(1|2) \otimes U(\mathfrak{osp}(1|2)$ given in [1].

The paper will unfold as follows. Section 8.2 will recall the definition of the dual -1 Hahn algebra and introduce as well the Schwinger–Dunkl algebra $\mathfrak{sd}(2)$ which was identified [27, 28] as the symmetry algebra of the Dunkl oscillator in two dimensions. The nomenclature comes from the fact that $\mathfrak{sd}(2)$ is obtained when the raising operators in the Schwinger construction of $\mathfrak{su}(2)$ are replaced by creation and annihilation operators involving Dunkl operators instead of ordinary derivatives. Section 8.3 will confirm that the dual -1 Hahn algebra can be realized in a commutant in the fashion described above. We will initially identify $\mathfrak{sd}(2)$ and this is why the relation with the dual -1 Hahn algebra will have been established in the preceding section. Section 8.4 will recall how the dual -1 Hahn algebra is embedded in $U(\mathfrak{osp}(1|2) \otimes U(\mathfrak{osp}(1|2)$ in light of the fact that this dual -1 Hahn algebra characterizes the Clebsch–Gordan coefficients of $\mathfrak{osp}(1|2)$. That Howe duality connects the commutant and the embedding presentations of the dual -1 Hahn algebra will be the subject of Section 8.5 and concluding remarks will form Section 13.6.

8.2. The dual -1 Hahn algebra and the Schwinger-Dunkl algebra

We first introduce the two algebras of interest and show how they are closely related to each other.

8.2.1. The dual -1 Hahn algebra and polynomials

The dual -1 Hahn algebra is defined [1] by the generators $\mathbf{P}$, K_1, K_2, K_3 and the relations

$$\begin{gathered}
[K_1, K_2] = K_3, \qquad [K_1, K_3] = K_2 + \nu\mathbf{P} + \tfrac{1}{2}, \\
[K_2, K_3] = 4K_1(1 + \nu\mathbf{P}) - 2\nu K_3\mathbf{P} + \sigma\mathbf{P} + \rho, \\
[K_1, \mathbf{P}] = 0, \qquad \{K_2, \mathbf{P}\} = -\mathbf{P} - 2\nu, \qquad \{K_3, \mathbf{P}\} = 0,
\end{gathered} \tag{8.2.1}$$

where ν, σ, ρ are structure constants. By promoting the structure constants to central elements, one obtains what will be referred to as the centrally extended dual -1 Hahn algebra. Note that ρ can be reabsorbed in the generator K_1, which is equivalent to removing it from the rhs of the $[K_2, K_3]$ relation.

The dual -1 Hahn algebra captures the bispectral properties of the polynomials with the same name [2]. This algebra can be realized by taking $K_1 = \frac{1}{2}D$, where D is the dual -1 Hahn polynomials' 5-term difference equation, $K_2 = \frac{1}{2}x$, where x is the 3-term recurrence operator and $\mathbf{P} = R$, the parity involution of the polynomials. The exact expressions are detailed in the Appendix 11.A. In that realization, the parameters of the algebra take the following values:

$$\nu = \xi + (-1)^N \eta, \qquad \sigma = (-1)^N 2\eta - 2\xi(1+2N) - (1-(-1)^N)4\xi\eta, \qquad \rho = 2(\xi - \eta - N).$$

8.2.2. The Schwinger–Dunkl algebra

The Schwinger–Dunkl algebra is the symmetry algebra of a two-dimensional isotropic Dunkl oscillator in the plane. This system is described by the Hamiltonian

$$H_{12} = H_1 + H_2, \qquad H_i = -\tfrac{1}{2}(\mathcal{D}_{x_i}^{\mu_i})^2 + \tfrac{1}{2}{x_i}^2, \qquad \mathcal{D}_{x_i}^{\mu_i} = \partial_{x_i} + \frac{\mu_i}{x_i}(I - R_i), \tag{8.2.2}$$

where I is the identity operator and the R_i, $i \in \{1,2\}$ are the reflection operators

$$R_1 f(x_1,x_2) = f(-x_1,x_2), \qquad R_2 f(x_1,x_2) = f(x_1,-x_2). \tag{8.2.3}$$

The symmetry algebra of this system is obtained through the Schwinger construction. Form

$$\mathbf{a}_i^\dagger = \frac{x_i - \mathcal{D}_{x_i}^{\mu_i}}{\sqrt{2}}, \qquad \mathbf{a}_i = \frac{x_i + \mathcal{D}_{x_i}^{\mu_i}}{\sqrt{2}}, \tag{8.2.4}$$

the parabosonic creation and annihilation operators, whose commutation relations are

$$[\mathbf{a}_i, \mathbf{a}_j^\dagger] = (I + 2\mu_i R_i)\delta_{ij}. \tag{8.2.5}$$

Then the three quantities

$$J_1 = \tfrac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_2 + \mathbf{a}_1 \mathbf{a}_2^\dagger\right), \qquad J_2 = \tfrac{1}{2i}\left(\mathbf{a}_1^\dagger \mathbf{a}_2 - \mathbf{a}_1 \mathbf{a}_2^\dagger\right), \qquad J_3 = \tfrac{1}{2}\left(H_1 - H_2\right), \tag{8.2.6}$$

are symmetries of the Hamiltonian H_{12}, along with R_1, R_2. These elements J_1, J_2, J_3, R_1, R_2 obey the following commutation relations, which we will refer to as the relations of the Schwinger–Dunkl algebra $\mathfrak{sd}(2)$:

$$\begin{gathered} [J_2, J_3] = iJ_1, \qquad [J_3, J_1] = iJ_2, \\ [J_1, J_2] = i\left(J_3(1 + \mu_1 R_1 + \mu_2 R_2) - \tfrac{1}{2}H_{12}(\mu_1 R_1 - \mu_2 R_2)\right), \\ \{J_1, R_\alpha\} = 0, \qquad \{J_2, R_\alpha\} = 0, \qquad [J_3, R_\alpha] = 0, \qquad \alpha = 1,2. \end{gathered} \tag{8.2.7}$$

8.2.3. Connection between the two algebras

Starting from the generators $\mathbf{P}$, K_1, K_2, K_3, write

$$j_1 = \tfrac{i}{2}K_3, \qquad j_2 = -\tfrac{1}{2}(K_2 + \nu\mathbf{P} + \tfrac{1}{2}), \qquad j_3 = -K_1 - \frac{\rho}{4}, \tag{8.2.8}$$

the dual -1 Hahn algebra relations now take the form

$$\begin{gathered} [j_2, j_3] = ij_1, \qquad [j_3, j_1] = ij_2, \\ [j_1, j_2] = i\left(j_3(1 + \nu\mathbf{P}) + \tfrac{1}{4}(\nu\rho - \sigma)\mathbf{P}\right), \\ \{j_1, \mathbf{P}\} = 0, \qquad \{j_2, \mathbf{P}\} = 0, \qquad [j_3, \mathbf{P}] = 0. \end{gathered} \tag{8.2.9}$$

One then sees that the dual -1 Hahn algebra is indeed similar to the $\mathfrak{sd}(2)$ algebra. The difference is that $\mathfrak{sd}(2)$ has 2 reflection-type operators, R_1 and R_2, whilst the dual -1 Hahn algebra only has a single one, $\mathbf{P}$. It will thus prove more useful to work with R_1 and $R_{12} = R_1 R_2$ as the latter commutes with everything and can be viewed as a central element in $\mathfrak{sd}(2)$.

Then, the two algebras obey the same relations upon identifying

$$\begin{aligned} \mathbf{P} &= R_1, \\ \nu &= \mu_1 + \mu_2 R_{12}, \\ \rho &= 2H_{12}, \\ \sigma &= 2\mu_1\,\rho, \end{aligned} \tag{8.2.10}$$

where we recall that H_{12} and R_{12} are both central elements.

Hence, the Schwinger–Dunkl algebra $\mathfrak{sd}(2)$ is essentially the centrally extended dual -1 Hahn algebra. In the remainder of the paper, we will encounter instances of algebras presented in the form of this $\mathfrak{sd}(2)$ algebra.

8.3. The dual -1 Hahn algebra as a commutant

In this Section we will obtain the dual -1 Hahn algebra in the commutant of $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ in a spinorial realization.

8.3.1. The model

Consider the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{4}\{a_i^\dagger, a_i\}, \tag{8.3.1}$$

built from the standard Bosonic raising $a_i^\dagger$ and lowering a_i operators obeying

$$[a_i, a_j^\dagger] = \delta_{ij}, \qquad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0. \tag{8.3.2}$$

We also introduce Fermionic raising $b_i^\dagger$ and lowering b_i operators obeying

$$\{b_i, b_j^\dagger\} = \delta_{ij}, \qquad \{b_i, b_j\} = \{b_i^\dagger, b_j^\dagger\} = 0. \tag{8.3.3}$$

Above and below $i, j = 1, 2, 3, 4$. The Bosonic and Fermionic operators mutually commute with each other. One sees that the combinations

$$\gamma_i = b_i + b_i^\dagger \tag{8.3.4}$$

obey

$$\{\gamma_i, \gamma_j\} = \delta_{ij} \tag{8.3.5}$$

which are (up to a normalization) the relations of the Clifford algebra $C\ell_4$. These Clifford elements will be used as building blocks for a spinorial realization of $\mathfrak{o}(4)$.

The Lie algebra $\mathfrak{o}(4)$ is the algebra with 6 generators, $\ell_{\mu\nu}$, $1 \leq \mu < \nu \leq 4$, whose relations are given by

$$[\ell_{\mu\nu}, \ell_{\rho\sigma}] = -i\left(\delta_{\nu\rho}\ell_{\mu\sigma} - \delta_{\nu\sigma}\ell_{\mu\rho} - \delta_{\mu\rho}\ell_{\nu\sigma} + \delta_{\mu\sigma}\ell_{\nu\rho}\right). \tag{8.3.6}$$

Let us denote $L_{\mu\nu} = a_\mu^\dagger a_\nu - a_\mu a_\nu^\dagger$ and $\Sigma_{\mu\nu} = \frac{1}{2}\gamma_\mu\gamma_\nu$. Both the combinations $-iL_{\mu\nu}$ and $-i\Sigma_{\mu\nu}$ realize the $\mathfrak{o}(4)$ algebra. We now define the total angular momentum as the sum:

$$J_{\mu\nu} = -i(L_{\mu\nu} + \Sigma_{\mu\nu}). \tag{8.3.7}$$

These total angular momenta $J_{\mu\nu}$ realize again the $\mathfrak{o}(4)$ commutation relations.

8.3.2. The commutant

We look for the commutant of the $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ subalgebra of $\mathfrak{o}(4)$, that is, operators that commute with

$$\begin{aligned} J_{12} &= -i\left(a_1^\dagger a_2 - a_1 a_2^\dagger + \tfrac{1}{2}(b_1 + b_1^\dagger)(b_2 + b_2^\dagger)\right), \\ J_{34} &= -i\left(a_3^\dagger a_4 - a_3 a_4^\dagger + \tfrac{1}{2}(b_3 + b_3^\dagger)(b_4 + b_4^\dagger)\right). \end{aligned} \tag{8.3.8}$$

The combinations

$$\begin{aligned} \mathcal{K}_1 &= \tfrac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3 - a_4^\dagger a_4), \\ \mathcal{K}_2 &= 2(L_{12}\Sigma_{12} + L_{13}\Sigma_{13} + L_{14}\Sigma_{14} + L_{23}\Sigma_{23} + L_{24}\Sigma_{24} + L_{34}\Sigma_{34} - \tfrac{3}{4}), \\ r &= i(b_1 + b_1^\dagger)(b_2 + b_2^\dagger), \\ \mathcal{R} &= -(b_1 + b_1^\dagger)(b_2 + b_2^\dagger)(b_3 + b_3^\dagger)(b_4 + b_4^\dagger) \end{aligned} \tag{8.3.9}$$

commute with H and J_{12}, J_{34}. This is verified by a direct calculation. The algebra generated by the elements K_1, K_2, r closes onto the following form:

$$\begin{gathered} [\mathcal{K}_1, \mathcal{K}_2] = \mathcal{K}_3, \qquad [\mathcal{K}_1, \mathcal{K}_3] = \mathcal{K}_2 - (J_{12} + J_{34}\mathcal{R})r + \tfrac{1}{2}, \\ [\mathcal{K}_2, \mathcal{K}_3] = 4\mathcal{K}_1[(J_{12} + J_{34}\mathcal{R})r - 1] - 2\mathcal{K}_3(J_{12} + J_{34}\mathcal{R})r - 2H(J_{12} - J_{34}\mathcal{R})r, \\ [\mathcal{K}_1, r] = 0, \qquad \{\mathcal{K}_2, r\} = -r + 2(J_{12} + J_{34}\mathcal{R})r, \qquad \{\mathcal{K}_3, r\} = 0. \end{gathered} \tag{8.3.10}$$

Here H, J_{12}, J_{34} and $\mathcal{R}$ are central. These relations are identified with those in (8.2.1). Hence we have obtained the centrally extended dual -1 Hahn algebra, or equivalently, the Schwinger–Dunkl algebra $\mathfrak{sd}(2)$, in a commutant.

8.4. The algebra of the $\mathfrak{osp}(1|2)$ Clebsch–Gordan coefficients

In this Section, we first introduce the $\mathfrak{osp}(1|2)$ algebra, present its Clebsch–Gordan problem, and then show how it is connected to the dual -1 Hahn algebra.

8.4.1. The Lie superalgebra $\mathfrak{osp}(1|2)$

The $\mathfrak{osp}(1|2)$ algebra can be presented as the algebra with generators A_0, $A_\pm$ and an involution P encoding the $\mathbb{Z}_2$-grading of the superalgebra (P commutes with the even element A_0 and anticommutes with the odd elements $A_\pm$). The defining relations are

$$\{A_+, A_-\} = 2A_0, \qquad [A_0, A_\pm] = \pm A_\pm, \qquad [P, A_0] = 0, \qquad \{P, A_\pm\} = 0. \tag{8.4.1}$$

The algebra $\mathfrak{osp}(1|2)$ also possesses an sCasimir [30]

$$S = \tfrac{1}{2}\left([A_+, A_-] + 1\right) = A_+A_- - A_0 + \tfrac{1}{2} \tag{8.4.2}$$

which commutes with the even elements and anticommutes with the odd elements

$$[S, A_0] = \{S, A_\pm\} = 0. \tag{8.4.3}$$

Multiplying the sCasimir by the involution, we obtain a Casimir element for $\mathfrak{osp}(1|2)$

$$Q = \left(A_+A_- - A_0 + \tfrac{1}{2}\right) P. \tag{8.4.4}$$

This Casimir element commutes with all generators of $\mathfrak{osp}(1|2)$.

Positive infinite-dimensional discrete series representations for $\mathfrak{osp}(1|2)$ are labelled by (μ, ϵ), with $\mu \geq 0$, $\epsilon = \pm 1$. Let us denote by $|n, \mu, \epsilon\rangle$ the basis vectors associated to an irrep (μ, ϵ). The generators act as follows in this basis:

$$\begin{aligned} A_0|n,\mu,\epsilon\rangle &= (n + \mu + \tfrac{1}{2})|n,\mu,\epsilon\rangle, & A_+|n,\mu,\epsilon\rangle &= \sqrt{[n+1]_\mu}|n+1,\mu,\epsilon\rangle, \\ P|n,\mu,\epsilon\rangle &= \epsilon(-1)^n|n,\mu,\epsilon\rangle, & A_-|n,\mu,\epsilon\rangle &= \sqrt{[n]_\mu}|n-1,\mu,\epsilon\rangle, \end{aligned} \tag{8.4.5a}$$

where we define the *mu*-numbers $[n]_\mu$ as

$$[n]_\mu = n + \mu\left(1 - (-1)^n\right). \tag{8.4.5b}$$

By Schur's lemma, the Casimir element acts as a multiple of the identity on these irreps

$$Q|n,\mu,\epsilon\rangle = -\epsilon\mu|n,\mu,\epsilon\rangle. \tag{8.4.5c}$$

8.4.2. The Clebsch–Gordan problem of $\mathfrak{osp}(1|2)$

We shall now look at the recoupling of two representations (μ_1, ϵ_1) and (μ_2, ϵ_2) of $\mathfrak{osp}(1|2)$.

The direct product representation $(\mu_1, \epsilon_1) \otimes (\mu_2, \epsilon_2)$ has the associated basis vectors $|n_1, \mu_1, \epsilon_1\rangle \otimes |n_2, \mu_2, \epsilon_2\rangle$. The elements $A_0 \otimes 1$, $1 \otimes A_0$, $P \otimes 1$, $1 \otimes P$ as well as the Casimir elements $Q \otimes 1$, $1 \otimes Q$ are diagonal in this basis. Equivalently, we shall consider the elements $(A_0 \otimes 1 + 1 \otimes A_0)$, $(A_0 \otimes 1 - 1 \otimes A_0)$, $P \otimes 1$ and $P \otimes P$, also diagonal in this basis.

This direct product representation admits the following decomposition [31] in irreducibles:

$$(\mu_1, \epsilon_1) \otimes (\mu_2, \epsilon_2) = \bigoplus_{j=0}^{\infty} (\mu_{12}(j), \epsilon_{12}(j)), \tag{8.4.6}$$

where

$$\mu_{12}(j) = \mu_1 + \mu_2 + j + \tfrac{1}{2}, \qquad \epsilon_{12}(j) = (-1)^j \epsilon_1 \epsilon_2. \tag{8.4.7}$$

The coupled basis vectors associated to the irreducibles $(\mu_{12}, \epsilon_{12})$ are denoted $|n_{12}, \mu_{12}, \epsilon_{12}\rangle$. The Casimir elements $Q \otimes 1$, $1 \otimes Q$ are again diagonal in the coupled basis. The other diagonal elements can be obtained as follows:

Consider the coproduct map $\Delta : \mathfrak{osp}(1|2) \to \mathfrak{osp}(1|2) \otimes \mathfrak{osp}(1|2)$, which is a coassociative algebra morphism. The coproduct maps the $\mathfrak{osp}(1|2)$ generators as follows

$$\begin{aligned} \Delta(A_0) &= A_0^{(12)} = A_0 \otimes 1 + 1 \otimes A_0 &&= A_0^{(1)} + A_0^{(2)}, \\ \Delta(A_\pm) &= A_\pm^{(12)} = A_\pm \otimes P + 1 \otimes A_\pm &&= A_\pm^{(1)} P^{(2)} + A_\pm^{(2)}, \\ \Delta(P) &= P^{(12)} = P \otimes P &&= P^{(1)} P^{(2)}, \end{aligned} \tag{8.4.8}$$

and the Casimir element according to

$$\Delta(Q) = Q^{(12)} = (A_-^{(1)} A_+^{(2)} - A_+^{(1)} A_-^{(2)}) P^{(1)} + Q^{(1)} P^{(2)} + Q^{(2)} P^{(1)} - \tfrac{1}{2} P^{(1)} P^{(2)}, \tag{8.4.9}$$

where the superindex denotes on which factor of $\mathfrak{osp}(1|2)$ the operator is acting. The elements $\Delta(A_0) = A_0 \otimes 1 + 1 \otimes A_0$, $\Delta(P) = P \otimes P$ and $\Delta(Q)$ are diagonal in the coupled basis.

The decomposition (8.4.6) indicates that the vector spaces spanned by $|n_1, \mu_1, \epsilon_1\rangle \otimes |n_2, \mu_2, \epsilon_2\rangle$ and $|n_{12}, \mu_{12}, \epsilon_{12}\rangle$ are isomorphic; one defines the Clebsch–Gordan coefficients $\mathcal{C}_{n_{12},j}^{n_1,n_2}$ as the expansion coefficients between the two bases

$$|n_{12}, \mu_{12}(j), \epsilon_{12}(j)\rangle = \sum_{n_1,n_2} \mathcal{C}_{n_{12},j}^{n_1,n_2} |n_1, \mu_1, \epsilon_1\rangle \otimes |n_2, \mu_2, \epsilon_2\rangle. \tag{8.4.10}$$

The Clebsch–Gordan coefficients are characterized algebraically by the elements that are diagonalized in each of the bases. In particular, $P \otimes 1$, $(A_0 \otimes 1 - 1 \otimes A_0)$ and $\Delta(Q)$ are not diagonal in both bases, so they obey non-trivial commutation relations. The algebra formed by these elements determines the Clebsch–Gordan coefficients.

Write

$$\kappa_1 = \tfrac{1}{2}(A_0 \otimes 1 - 1 \otimes A_0), \qquad \kappa_2 = Q^{(12)} P^{(12)}, \qquad \text{and} \quad p = P \otimes 1, \tag{8.4.11}$$

a straightforward calculation in $\mathfrak{osp}(1|2)\otimes\mathfrak{osp}(1|2)$ yields

$$
\begin{gathered}
[\kappa_1,\kappa_2]=\kappa_3, \qquad\qquad [\kappa_1,\kappa_3]=\kappa_2-(Q^{(1)}+Q^{(2)}P^{(12)})p+\tfrac{1}{2},\\
[\kappa_3,\kappa_2]=4\kappa_1\Big(1-(Q^{(1)}+Q^{(2)}P^{(12)})p\Big)+2p\Big(\kappa_3(Q^{(1)}+Q^{(2)}P^{(12)})+A_0^{(12)}(Q^{(1)}-Q^{(2)}P^{(12)})\Big),\\
[\kappa_1,p]=0, \qquad \{\kappa_2,p\}=-p+2(Q^{(1)}+Q^{(2)}P^{(12)}), \qquad \{\kappa_3,p\}=0.
\end{gathered}
\tag{8.4.12}
$$

Keeping in mind that the elements $Q^{(1)}$, $Q^{(2)}$, $P^{(12)}$ and $A_0^{(12)}$ are central since they are diagonalized in both bases, one recognizes the defining relations of the (centrally extended) dual -1 Hahn algebra (8.2.1). This reveals that the Clebsch–Gordan coefficients of $\mathfrak{osp}(1|2)$ are (essentially) the dual -1 Hahn polynomials [1, 32, 33].

8.5. The Howe duality correspondence

In Sections 8.3 and 8.4, we have obtained the (centrally extended) dual -1 Hahn algebra in two different contexts. We will now reinterpret the contents of the last two Sections in order to display the two presentations of the algebra in a unified way.

8.5.1. Connecting the two approaches

Let us go back to our construction in Section 8.3 involving four Bosonic and Fermionic oscillators. We can form three copies of $\mathfrak{osp}(1|2)$ labelled by one of the sets $S\in\{\{1,2\},\{3,4\},\{1,2,3,4\}\}$ by introducing the operators:

$$
\mathcal{A}_-^S=\sum_{\mu\in S}a_\mu\gamma_\mu,\qquad \mathcal{A}_+^S=\sum_{\mu\in S}a_\mu^\dagger\gamma_\mu,\qquad \mathcal{A}_0^S=\frac{1}{2}\sum_{\mu\in S}\{a_\mu^\dagger,a_\mu\},\qquad \mathcal{P}^S=e^{i\pi|S|/4}\prod_{\mu\in S}\gamma_\mu,
\tag{8.5.1}
$$

which obey the defining relations of $\mathfrak{osp}(1|2)$ given in (8.4.1), (8.4.3). The Casimir element associated to each set S is given by

$$
\mathcal{Q}^S=(\mathcal{A}_+^S\mathcal{A}_-^S-\mathcal{A}_0^S+\tfrac{1}{2})\mathcal{P}^S.
\tag{8.5.2}
$$

Let us revisit the Clebsch–Gordan problem of $\mathfrak{osp}(1|2)$ in this framework. As seen in Section 8.4, the elements of the set

$$
\mathcal{E}=\Big\{(\mathcal{A}_0^{\{1,2\}}+\mathcal{A}_0^{\{3,4\}}),(\mathcal{A}_0^{\{1,2\}}-\mathcal{A}_0^{\{3,4\}}),\mathcal{P}^{\{1,2\}},\mathcal{P}^{\{1,2\}}\mathcal{P}^{\{3,4\}},\mathcal{Q}^{\{1,2\}},\mathcal{Q}^{\{3,4\}},\mathcal{Q}^{\{1,2,3,4\}}\Big\}
\tag{8.5.3}
$$

are diagonal in at least one of the bases. The elements that are diagonal in both bases commute with all others, and the remaining ones obey the commutation relations of the dual -1 Hahn algebra.

We now give the explicit form of all elements in $\mathcal{E}$ in the realization (8.5.1). It will appear that they can be matched with certain expressions given in Section 8.3, thus explaining why the dual -1 Hahn algebra appeared in two seemingly different situations. The expressions are the following ones when translated in terms of the a and b ladder operators:

$$\begin{aligned}
\mathcal{A}_0^{\{1,2\}} + \mathcal{A}_0^{\{3,4\}} &= H,\\
\mathcal{A}_0^{\{1,2\}} - \mathcal{A}_0^{\{3,4\}} &= 2K_1,\\
\mathcal{P}^{\{1,2\}} &= r,\\
\mathcal{P}^{\{1,2\}}\mathcal{P}^{\{3,4\}} &= P_{1234},\\
\mathcal{Q}^{\{1,2\}} &= J_{12},\\
\mathcal{Q}^{\{3,4\}} &= J_{34},\\
\mathcal{Q}^{\{1,2,3,4\}} &= K_2 P_{1234}.
\end{aligned} \tag{8.5.4}$$

In this framework, we easily see that having $\{\mathcal{Q}^{\{1,2\}}, \mathcal{Q}^{\{3,4\}}\}$ commute with all other generators implies that the commutant of $\{J_{12}, J_{34}\}$ contains the dual -1 Hahn algebra.

8.5.2. An instance of Howe duality

That the dual -1 Hahn algebra can be viewed on the one hand in the commutant of $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ in a spinorial representation of $\mathfrak{o}(4)$ and can be embedded in $U(\mathfrak{osp}(1|2)) \otimes U(\mathfrak{osp}(1|2))$ on the other hand can be attributed to Howe duality as we now explain.

It is known [34] that $\mathfrak{osp}(1|2)$ and $Pin(2n)$ have dual (commuting) actions on the space of polynomials $\mathcal{P}(\mathbb{R}^{2n}, \mathbb{S})$ defined in Euclidean space $\mathbb{R}^{2n}$ and taking values in a spinor space $\mathbb{S}$. In our situation, at the level of the algebras, this would correspond to the fact that the generators of $\mathfrak{osp}(1|2)$ and $\mathfrak{o}(2n)$ commute. A direct computation using the expressions (8.3.8) and (8.5.1) confirms that it is indeed the case:

$$[J_{12}, \mathcal{A}_\bullet^{\{1,2\}}] = [J_{34}, \mathcal{A}_\bullet^{\{3,4\}}] = 0, \tag{8.5.5}$$

$$[J_{ij}, \mathcal{A}_\bullet^{\{1,2,3,4\}}] = 0, \qquad 1 \leq i < j \leq 4, \tag{8.5.6}$$

where $\mathcal{A}_\bullet$ stands for any of $\mathcal{A}_0$, $\mathcal{A}_\pm$. As a byproduct, it can be shown that the Casimir elements of both algebras can be put in correspondance. Recall that the Casimir element of $\mathfrak{o}(2n)$ denoted $C^{\{1,\dots,2n\}}$ is given by

$$C^{\{1,\dots,2n\}} = \sum_{1\leq i<j\leq 2n} {J_{ij}}^2. \tag{8.5.7}$$

The Casimir elements of both $\mathfrak{osp}(1|2)$ and $\mathfrak{o}(2n)$ associated to each of the three copies labelled by S are related by

$$C^{\{1,2\}} = (\mathcal{Q}^{\{1,2\}})^2, \qquad C^{\{3,4\}} = (\mathcal{Q}^{\{3,4\}})^2, \qquad C^{\{1,2,3,4\}} = (\mathcal{Q}^{\{1,2,3,4\}})^2 - \tfrac{3}{4}. \tag{8.5.8}$$

Remark 8.1. *That the relation between the Casimir element of $\mathfrak{o}(2n)$ and the one of $\mathfrak{osp}(1|2)$ is quadratic does not come as a surprise. It is known [25] that in the context of Howe duality* *for the pair* $(\mathfrak{su}(1,1), \mathfrak{o}(2n))$*, the relation between the associated Casimir elements is linear, whilst the algebra* $\mathfrak{osp}(1|2)$ *can be seen as a "square root" of* $\mathfrak{su}(1,1)$.

8.6. Conclusion

To sum up, we have presented two frameworks that lead to the (centrally extended) dual -1 Hahn algebra: one in which we looked at the commutant of a spinorial realization of the $\mathfrak{o}(2)\oplus\mathfrak{o}(2)$ subalgebra of $\mathfrak{o}(4)$, and the other in which we looked at the algebra of the $\mathfrak{osp}(1|2)$ Clebsch–Gordan coefficients. We have explained how these two approaches are dual (in the sense of Howe) by considering representations that featured the dual pair $(\mathfrak{osp}(1|2), \mathfrak{o}(2n))$. We have also highlighted how the results presented in this report can be seen as a "square root" of those related to the dual pair $(\mathfrak{su}(1,1), \mathfrak{o}(2n))$ [25].

One should note that the construction presented here generalizes straightforwardly if one considers instead the commutant of the spinorial representation of the $\mathfrak{o}(m) \oplus \mathfrak{o}(m')$ subalgebra of $\mathfrak{o}(m+m')$; the (centrally extended) dual -1 Hahn algebra is still recovered and the Howe duality again operates.

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8.A. The dual -1 Hahn polynomials $P_n(x;\xi,\eta,N)$

The dual -1 Hahn polynomials can be obtained as a $q \to -1$ limit of the q-Hahn polynomials [1, 2]. These polynomials depend on three variables ξ, η and N, with ξ, $\eta > -\frac{1}{2}$ and N an integer that corresponds to the maximal degree of the family. We here regroup a few properties of the dual -1 Hahn polynomials with monic normalization, that is $P_n(x) = x^n + O(x^{n-1})$. To ease the notation, we shall omit the parameters: $P_n(x;\xi,\eta,N) \equiv P_n(x)$.

Firstly, the dual -1 Hahn polynomials satisfy a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + [2(-1)^{n+1}(\xi + (-1)^N\eta) - 1]\ P_n(x) + 4[n]_\xi[N-n+1]_\eta\ P_{n-1}(x), \tag{8.A.1}$$

where the *mu*-numbers $[n]_\mu$ are defined as $[n]_\mu = n + \mu(1-(-1)^n)$.

These polynomials admit an expression in terms of a truncating generalized hypergeometric series. Recall that the generalized hypergeometric series ${}_rF_s$ is defined by

$$ {}_rF_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!} \tag{8.A.2} $$

with $(c)_k = c(c+1)\cdots(c+k-1)$ the Pochhammer symbol.

The expressions depend on the parity of N and n. For N even, the expression is

$$ P_{2m+j}(x) = 2^{4m}(x+2\xi+2\eta+1)^j \left(j-\tfrac{N}{2}\right)_m \left(\tfrac{1-N}{2}-\eta\right)_m \; {}_3F_2\left(\begin{matrix} -m, \delta+\frac{1+x}{4}, \delta-\frac{1+x}{4} \\ j-\frac{N}{2}, \frac{1-N}{2}-\eta \end{matrix}; 1\right), \tag{8.A.3} $$

where we have defined $\delta = -\frac{1}{2}(\xi+\eta+N)$, m an integer and $j \in \{0,1\}$. For N odd the expression is

$$ P_{2m+j}(x) = 2^{4m}(x+2\xi-2\eta+1)^j \left(\tfrac{1-N}{2}\right)_m \left(\xi+j+\tfrac{1}{2}\right)_m \; {}_3F_2\left(\begin{matrix} -m, \gamma+\frac{1+x}{4}, \gamma-\frac{1+x}{4} \\ \frac{1-N}{2}, \xi+j+\frac{1}{2} \end{matrix}; 1\right), \tag{8.A.4} $$

where this time $\gamma = \frac{1}{2}(\xi+\eta+1)$.

These polynomials obey an orthogonality relation of the form

$$ \sum_{s=0}^{N} w_s(\xi,\eta,N) P_n(x_s) P_m(x_s) = \nu_n(\xi,\eta,N)\delta_{n,m} \tag{8.A.5} $$

on the grid points

$$ x_s = \begin{cases} (-1)^s(2s-2\xi-2\eta-2N-1) & N \text{ even}, \\ (-1)^s(2s+2\xi+2\eta+1) & N \text{ odd}. \end{cases} \tag{8.A.6} $$

The weights are given by

$$ w_{2m+j}(\xi,\eta,N) = \begin{cases} \frac{(-1)^m\left(-\frac{N}{2}\right)_{m+j}}{m!} \frac{\left(\frac{1-N}{2}-\eta\right)_m}{\left(\frac{1-N}{2}-\xi\right)_m} \frac{(-N-\xi-\eta)_m}{\left(-\frac{N}{2}-\xi-\eta\right)_{m+j}} & N \text{ even}, \\ \frac{(-1)^m\left(\frac{1-N}{2}\right)_m}{m!} \frac{\left(\xi+\frac{1}{2}\right)_{m+j}}{\left(\eta+\frac{1}{2}\right)_{m+j}} \frac{(1+\xi+\eta)_m}{\left(\frac{1}{2}(N+2\xi+2\eta+3)\right)_m} & N \text{ odd}, \end{cases} \tag{8.A.7} $$

and the normalizations are given by

$$ \upsilon_{2m+j}(\xi,\eta,N) = \begin{cases} (-1)^j 2^{4(2m+j)} m! \left(\xi+\frac{1}{2}\right)_{m+j} \left(\frac{1-N}{2}-\eta\right)_m \left(-\frac{N}{2}\right)_{m+j} \frac{(-N-\xi-\eta)_{N/2}}{\left(\frac{1-N}{2}-\xi\right)_{N/2}} & N \text{ even}, \\ (-1)^j 2^{4(2m+j)} m! \left(\xi+\frac{1}{2}\right)_{m+j} \left(\frac{1-N}{2}\right)_m \left(-\eta-\frac{N}{2}\right)_{m+j} \frac{(\xi+\eta+1)_{(N+1)/2}}{\left(\eta+\frac{1}{2}\right)_{(N+1)/2}} & N \text{ odd}. \end{cases} \tag{8.A.8} $$

The dual -1 Hahn polynomials are bispectral; in addition to their three-term recurrence relation, they satisfy a five-term difference relation [2]:

$$DP_n(x) = 2nP_n(x), \tag{8.A.9}$$

where D, the difference operator, has the following action on the polynomials:

$$\begin{aligned} DP_n(x) = {} & E_1(x)P_n(x+4) + E_2(x)P_n(x-4) + G_1(x)P_n(-x-2) + G_2(x)P_n(-x+2) \\ & - [E_1(x) + E_2(x) + G_1(x) + G_2(x)]P_n(x) \end{aligned} \tag{8.A.10}$$

and

$$\begin{aligned} E_1(x) &= \frac{(x+2\xi-(-1)^N 2\eta+3)(x+2\xi+(-1)^N 2\eta+1)(x-2\xi-2\eta-2N+1)}{4(x+1)(x+3)}, \\ E_2(x) &= -\frac{(x-2\xi-(-1)^N 2\eta-3)(x-2\xi+(-1)^N 2\eta-1)(x+2\xi+2\eta+2N+1)}{4(x-1)(x-3)}, \\ G_1(x) &= -\frac{4(x+2\xi+(-1)^N 2\eta+1)(\xi-(-1)^N\eta)(\xi+\eta+N+1)}{(x^2-1)(x+3)}, \\ G_1(x) &= -\frac{2(x-2\xi+(-1)^N 2\eta-1)(\xi+(-1)^N\eta)(x+2\xi+2\eta+2N+1)}{(x^2-1)(x-3)}. \end{aligned} \tag{8.A.11}$$

The polynomials also possess an operator R encoding their parity

$$RP_n(x) = (-1)^n P_n(x) \tag{8.A.12}$$

which acts as follows on polynomials:

$$RP_n(x) = P_n(-x-2) + 2\frac{\xi+(-1)^N\eta}{1+x}[P_n(-x-2) - P_n(x)]. \tag{8.A.13}$$

It is easily checked that this operator R is an involution.

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Chapitre 9

Howe duality and algebras of the Askey–Wilson type: an overview

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Abstract: The Askey–Wilson algebra and its relatives such as the Racah and Bannai–Ito algebras were initially introduced in connection with the eponym orthogonal polynomials. They have since proved ubiquitous. In particular they admit presentations in commutants that are related through Howe duality. This paper surveys these results.

Keywords: Howe duality, Racah, Bannai–Ito and Askey–Wilson algebras, commutants, reductive dual pairs.

9.1. Introduction

The quadratic algebras of Askey–Wilson type such as the Askey–Wilson algebra itself, the Racah and Bannai–Ito algebras and their specializations and contractions encode the bispectral properties of orthogonal polynomials that arise in recoupling coefficients such as the Clebsch–Gordan or Racah coefficients. It is therefore natural that these algebras be encountered in centralizers of the diagonal action of an algebra of interest $\mathfrak{g}'$ such as $\mathfrak{sl}(2)$, $\mathfrak{osp}(1|2)$ or $U_q(\mathfrak{sl}(2))$, on n-fold tensor products of representations of $\mathfrak{g}'$. Indeed, elements of these centralizers will be used as labelling operators to define bases whose overlaps will be expressed in terms of the corresponding orthogonal polynomials.

Often the algebra $\mathfrak{g}'$ forms a reductive pair with another algebra $\mathfrak{g}$ in which case the Howe duality operates in certain modules. This leads to alternative characterizations of the quadratic algebras that are in correspondance: on the one hand commutants in representations of the universal enveloping algebra $U(\mathfrak{g})$ and on the other hand, realizations of the type mentioned above as centralizers in recoupling problems for $\mathfrak{g}'$. This is the topic of this brief review which is organized as follows. Section 9.2 presents the general framework. Section 9.3 describes as illustration the dual commutant picture for the Racah algebra; this will involve the reductive pair $(\mathfrak{o}(6), \mathfrak{su}(1,1))$. Section 9.4 gives a summary of the different cases that have been analyzed and Section 13.6 provides a short outlook.

9.2. General Framework

We shall say following [1] that two algebras $\mathfrak{g}$ and $\mathfrak{g}'$ have dual representations on a Hilbert space $\mathcal{H}$ if (1) this space carries fully reducible representations of both $\mathfrak{g}$ and $\mathfrak{g}'$, (2) the action of $\mathfrak{g}$ and $\mathfrak{g}'$ commute, (3) the representation ρ of the direct sum $\mathfrak{g} \oplus \mathfrak{g}'$ defined by the actions of $\mathfrak{g}$ and $\mathfrak{g}'$ on $\mathcal{H}$ is multiplicity-free and (4) each irreducible representation of $\mathfrak{g}$ occurring in the decomposition of ρ is paired with a unique irreducible representation of $\mathfrak{g}'$ and vice-versa. This is the essence of Howe duality which can be proved in a number of situations. We shall consider such instances in this paper.

Consider now a setup with the representation of $\mathfrak{g}'$ in $\mathcal{H} = V^{\otimes 2n}$ given by

$$\bar{\sigma}^{\otimes 2n}[\Delta^{(2n-1)}(\mathfrak{g}')]$$

where $\bar{\sigma} : \mathfrak{g}' \to \mathit{End}\, V$ is a representation of $\mathfrak{g}'$ on the vector space V, $\Delta : \mathfrak{g}' \to \mathfrak{g}' \otimes \mathfrak{g}'$ is the coproduct and $\Delta^{(n)}$ is defined recursively by $\Delta^{(n)} = (\Delta \otimes 1^{\otimes(n-1)}) \circ \Delta^{(n-1)}$, with $\Delta^{(0)} = 1$. This symmetric situation makes it natural that there be an action of some other algebra $\mathfrak{g}$ on the carrier space $\mathcal{H}$ that commutes with the action of $\mathfrak{g}'$. Take the maximal Abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ to be $\mathfrak{h} \simeq \mathfrak{X}^{\oplus n}$ with $\mathfrak{X}$ one-dimensional. The pairing under Howe duality with the representations of $\mathfrak{X}^{\oplus n}$ implies that

$$\bar{\sigma}^{\otimes 2n}[\Delta^{(2n-1)}(\mathfrak{g}')] = \bar{\sigma}^{\otimes 2n}[\Delta^{\otimes n} \circ \Delta^{(n-1)}(\mathfrak{g}')]$$

decomposes into representations of the form

$$\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_n(\Delta^{(n-1)}(\mathfrak{g}'))$$

with the σ_i's being irreducible representations arising in the decomposition of $\bar{\sigma}^{\otimes 2}$. This quotienting by $\mathfrak{h}$ is a way of posing a generalized Racah problem for the recoupling of the n representations σ_i of $\mathfrak{g}'$.

We indicated in the introduction that the quadratic algebras $\mathcal{A}$ of Askey–Wilson type can be obtained as (subalgebras of) centralizers of diagonal actions in n-fold tensor products of representations. The intermediate Casimir elements in $\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_n$ manifestly centralize the action of $\mathfrak{g}'$ on $\mathcal{H} \mod \mathfrak{h}$. They are taken to realize the quadratic algebra of interest. This provides the first presentation of $\mathcal{A}$ in a commutant. The dual one is identified as follows in the present context. We know that $\mathfrak{g}$ is the commutant of $\mathfrak{g}'$ in $\mathcal{H}$. Moreover from the application of Howe duality, the generators of the representation $\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_n$ of $\mathfrak{g}'$ are known to commute with those that represent the subalgebra $\mathfrak{h} \simeq \mathfrak{X}^{\oplus n}$. The non-trivial part of the centralizer of $\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_n$ must therefore be obtained, in the given representation on $\mathcal{H} \mod \mathfrak{h}$, by those elements in the universal enveloping algebra of $\mathfrak{g}$ that commute with $\mathfrak{X}^{\oplus n}$. In other words, $\mathcal{A}$ can also be identified in the commutant of $\mathfrak{h} \subset \mathfrak{g}$ in $U(\mathfrak{g})$ as represented on $\mathcal{H}$.

There is an equivalent way of looking at this. The pairing of the representations of $\mathfrak{g}$ and $\mathfrak{g}'$ through Howe duality manifests itself in the fact that the Casimir elements of $\mathfrak{g}$ and $\mathfrak{g}'$ are affinely related. Let $\mathcal{C}$ be a Casimir element of $\mathfrak{g}'$. Consider for example the intermediate Casimir element given by

$$\bar{\sigma}^{\otimes 4}[((\Delta \otimes \Delta) \circ \Delta)(\mathcal{C})] \otimes 1^{\otimes(2n-4)}$$

corresponding to the embedding of $\mathfrak{g}'$ in the first four factors of ${\mathfrak{g}'}^{\otimes 2n}$. There will be a subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$ that will be dually related to $\mathfrak{g}'$ on the restriction of $\mathcal{H}$ to $V^{\otimes 4}$ so that its Casimir element will be essentially the one of $\mathfrak{g}'$. Next, looking at the intermediate Casimir element of $\mathfrak{g}'$ associated to a different embedding, for instance in the four last factors of ${\mathfrak{g}'}^{\otimes 2n}$, there will be a dual pairing with a different embedding in $\mathfrak{g}$ of the same subalgebra $\mathfrak{g}_1$ and again the two Casimir elements will basically coincide. These observations lead to the conclusion that the set of intermediate Casimir elements associated to the representation of $\mathfrak{g}'$ is algebraically identical to the set of Casimir elements of the subalgebras of $\mathfrak{g}$ that form dual pairs with $\mathfrak{g}'$ when intermediate representations of the latter are taken. It is not difficult to convince oneself that the set of invariants connected to the relevant subalgebras of $\mathfrak{g}$ consists in the commutant of the maximal Abelian subalgebra of $\mathfrak{g}$ as concluded differently before.

To summarize, in situations where Howe duality prevails with $(\mathfrak{g}, \mathfrak{g}')$ the pair of algebras that are dually represented on $\mathcal{H}$ and if the representation of $\mathfrak{g}'$ is of the form

$$\bar{\sigma}^{\otimes 2n}[\Delta^{(2n-1)}(\mathfrak{g}')],$$

the quadratic algebras $\mathcal{A}$ of Askey–Wilson type can be viewed on one hand in the commutant of this action of $\mathfrak{g}'$ on $\mathcal{H}$ and thus realized by the intermediate Casimir elements of $\mathfrak{g}'$, or on the other hand in the commutant of $\mathfrak{h} \subset \mathfrak{g}$ in the intervening representation of $U(\mathfrak{g})$. We shall present next an example of how this can be concretely realized.

9.3. The dual presentations of the Racah algebra

The Racah algebra $\mathcal{R}$ has three generators K_1, K_2, K_3 that are subjected to the relations [2]:

$$\begin{aligned} [K_1, K_2] &= K_3, \qquad & [K_2, K_3] &= {K_2}^2 + \{K_1, K_2\} + dK_2 + e_1, \\ & & [K_3, K_1] &= {K_1}^2 + \{K_1, K_2\} + dK_1 + e_2, \end{aligned} \tag{9.3.1}$$

where $[A, B] = AB - BA$, $\{A, B\} = AB + BA$ and d, e_1, e_2 are central.

We shall explain how dual presentations of the algebra $\mathcal{R}$ in a commutant are obtained in the fashion described in Section 2. The dual pair will be $(\mathfrak{o}(6), \mathfrak{su}(1,1))$ and the representation space $\mathcal{H}$ will be that of the state space of six quantum harmonic oscillators with annihilation and creation operators $a_\mu, a_\nu^\dagger,\ \mu, \nu = 1, \ldots, 6$ verifying

$$[a_\mu, a_\nu^\dagger] = \delta_{\mu\nu}.$$

The corresponding Hamiltonian

$$H = a_1^\dagger a_1 + \cdots + a_6^\dagger a_6$$

is manifestly invariant under the rotations in six dimensions. These are encoded in the Lie algebra $\mathfrak{o}(6)$, realized by the generators

$$L_{\mu\nu} = a_\mu^\dagger a_\nu - a_\mu a_\nu^\dagger$$

and possessing the Casimir element

$$\mathcal{C} = \sum_{\mu<\nu} {L_{\mu\nu}}^2.$$

The Lie algebra $\mathfrak{su}(1,1)$ has generators $J_0, J_\pm$ that obey the following commutation relations:

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -2J_0,$$

and its Casimir operator is given by

$$C = {J_0}^2 - J_+J_- - J_0.$$

The six harmonic oscillators also provide a realization of this algebra through the addition of six copies of the metaplectic representation of $\mathfrak{su}(1,1)$, for which the generators are mapped to:

$$J_0^{(\mu)} = \frac{1}{2}(a_\mu^\dagger a_\mu + \frac{1}{2}), \qquad J_+^{(\mu)} = \frac{1}{2}(a_\mu^\dagger)^2, \qquad J_-^{(\mu)} = \frac{1}{2}(a_\mu)^2, \qquad \mu = 1, \ldots, 6.$$

Note that the operators

$$\sum_{\mu=1}^{6} J_\bullet^{(\mu)}$$

are invariant under rotations. The space of state vectors $\mathcal{H}$ thus carries commuting representations of $\mathfrak{o}(6)$ and $\mathfrak{su}(1,1)$ and Howe duality takes place.

The maximal Abelian algebra of $\mathfrak{o}(6)$ is $\mathfrak{o}(2)\oplus\mathfrak{o}(2)\oplus\mathfrak{o}(2)$ and is generated by the set $\{L_{12},\, L_{34},\, L_{56}\}$. The non-abelian part of its commutant in the representation of $U(\mathfrak{o}(6))$ on $\mathcal{H}$ is generated by the two invariants

$$K_1 = \frac{1}{8}\Big({L_{12}}^2 + {L_{34}}^2 + {L_{13}}^2 + {L_{23}}^2 + {L_{14}}^2 + {L_{24}}^2\Big), \tag{9.3.2}$$

$$K_2 = \frac{1}{8}\Big({L_{34}}^2 + {L_{56}}^2 + {L_{35}}^2 + {L_{36}}^2 + {L_{45}}^2 + {L_{46}}^2\Big). \tag{9.3.3}$$

Define K_3 by $[K_1, K_2] = K_3$. Working out the commutation relations of K_3 with K_1 and K_2, it is found that they correspond to those (9.3.1) of the Racah algebra with the central parameters given by

$$\begin{aligned} d &= -\frac{1}{8}\Big(\mathcal{C} + {L_{12}}^2 + {L_{34}}^2 + {L_{56}}^2\Big),\\ e_1 &= -\frac{1}{64}\Big(\mathcal{C} - {L_{12}}^2 - 4\Big)\Big({L_{34}}^2 - {L_{56}}^2\Big),\\ e_2 &= -\frac{1}{64}\Big(\mathcal{C} - {L_{56}}^2 - 4\Big)\Big(L_{34}^2 - L_{12}^2\Big). \end{aligned}$$

For details see [3]. By abuse of notation we designate the abstract generators and their realizations by the same letter.

Regarding the $\mathfrak{su}(1,1)$ picture, let

$$J_\bullet^{(\mu,\nu,\rho,\lambda)} = J_\bullet^{(\mu)} + J_\bullet^{(\nu)} + J_\bullet^{(\rho)} + J_\bullet^{(\lambda)}$$

denote the addition of the four metaplectic representations labelled by the variables μ,ν,ρ,λ all assumed different. The corresponding Casimir operator is

$$C^{(\mu,\nu,\rho,\lambda)} = (J_0^{(\mu,\nu,\rho,\lambda)})^2 - J_+^{(\mu,\nu,\rho,\lambda)}J_-^{(\mu,\nu,\rho,\lambda)} - J_0^{(\mu,\nu,\rho,\lambda)}.$$

Quite clearly, these actions of $\mathfrak{su}(1,1)$ restricted to state vectors of four oscillators are paired with commuting actions of the Lie algebra $\mathfrak{o}(4)$ of rotations in the four dimensions labelled by μ,ν,ρ,λ. It is hence not surprising to find, owing to Howe duality, that

$$C^{(1234)} = -2K_1 \qquad \text{and} \qquad C^{(3456)} = -2K_2,$$

namely that the intermediate $\mathfrak{su}(1,1)$ Casimir operators corresponding to the recouplings of the first four and last four of the six metaplectic representations are equal (up to a factor) to the Casimir elements of the two corresponding $\mathfrak{o}(4)$ subalgebras of $\mathfrak{o}(6)$ which together generate as we observed the non-trivial part of the commutant of $\mathfrak{o}(2)\oplus\mathfrak{o}(2)\oplus\mathfrak{o}(2)$ in $U(\mathfrak{o}(6))$. This entails the description of the Racah algebra in the commutant in $U(\mathfrak{su}(1,1)^{\otimes 3})$ of the action of $\mathfrak{su}(1,1)$ on $\mathcal{H}$. Alternatively, picking the $\mathfrak{su}(1,1)$ representations associated to those of $\mathfrak{o}(2)\oplus\mathfrak{o}(2)\oplus\mathfrak{o}(2)$ under Howe duality yields the sum of three irreducible representations of $\mathfrak{su}(1,1)$ belonging to the discrete series; these are realized as dynamical algebras of three singular oscillators. Note that corresponding to the $\mathfrak{su}(1,1)$ representation

$$J_\bullet^{(\mu,\nu)} = J_\bullet^{(\mu)} + J_\bullet^{(\nu)}$$

is the Casimir

$$C^{(\mu\nu)} = -\frac{1}{4}\Big({L_{\mu\nu}}^2 + 1\Big).$$

With the dependance on the polar angles "rotated out", the total Casimir element $C^{(123456)}$ becomes the Hamiltonian of the generic superintegrable system on the two-sphere; the constants of motion are the quotiented intermediate Casimirs elements and the symmetry algebra that they generate is hence that of Racah.

9.4. More dual pictures – an overview

The main algebras of Askey–Wilson type have been studied recently from the commutant and Howe duality viewpoints. We summarize in the following the main results and give in particular the dualities that are involved.

9.4.1. The Racah family

The higher rank extension of the Racah algebra defined in the algebra generated by all the intermediate Casimir elements of

$$\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_n(\Delta^{(n-1)}(\mathfrak{su}(1,1)))$$

can be described in the framework of the preceding section with the help of the dual pair $(\mathfrak{o}(2n), \mathfrak{su}(1,1))$ using in this case the module formed by the state vectors of $2n$ harmonic oscillators. It is then seen to be dually in the commutant of $\mathfrak{o}(2)^{\oplus n}$ in the oscillator representation of $U(\mathfrak{o}(2n))$ [4].

The case $n = 2$ is special and of particular interest since it pertains to the Clebsch–Gordan problem for $\mathfrak{su}(1,1)$, that is, the recoupling of the two irreducible representations σ_1 and σ_2. There are no intermediate Casimirs here; the relevant operators associated to the direct product basis and the recoupled one are respectively

$$M_1 = \sigma_1(J_0) - \sigma_2(J_0)$$

and the total Casimir

$$M_2 = (\sigma_1 \otimes \sigma_2)\Delta(C).$$

These are seen to obey the commutation relations of the Hahn algebra [5]:

$$\begin{aligned} [M_1, M_2] = M_3, \qquad &[M_2, M_3] = -2\{M_1, M_2\} + \delta_1, \\ &[M_3, M_1] = -2{M_1}^2 - 4M_2 + \delta_2, \end{aligned} \tag{9.4.1}$$

where

$$\delta_1 = 4(\sigma_1(J_0) + \sigma_2(J_0))(\sigma_1(C) - \sigma_2(C)), \qquad \delta_2 = 2(\sigma_1(J_0) + \sigma_2(J_0))^2 + (\sigma_1(C) + \sigma_2(C))$$

are central. The name of the algebra comes from the fact that the $3j$-coefficients involve dual Hahn polynomials. In the setup with four harmonic oscillators, with $\mathcal{H}$ carrying the product of four metaplectic representations, Howe duality will imply that the total Casimir element $C^{(1234)}$ of $\mathfrak{su}(1,1)$ coincides with the Casimir of $\mathfrak{o}(4)$ – this is the same computation as the one described above. It is easily seen that $\sigma_1(J_0) - \sigma_2(J_0)$ is derived from $\frac{1}{2}(N_1 + N_2 - N_3 - N_4)$ under the quotient by $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ with $N_i = a_i^\dagger a_i$, $i = 1, \ldots, 4$. It can in fact be checked directly, again abusing notation, that

$$M_1 = \frac{1}{2}(N_1 + N_2 - N_3 - N_4), \qquad M_2 = -\frac{1}{4}\Big({L_{12}}^2 + {L_{34}}^2 + {L_{13}}^2 + {L_{23}}^2 + {L_{14}}^2 + {L_{24}}^2\Big)$$

satisfy the relations given in equation (9.4.1) with

$$\begin{aligned}
\delta_1 &= -\frac{1}{2}(N_1 + N_2 + N_3 + N_4 + 2)({L_{12}}^2 - {L_{34}}^2),\\
\delta_2 &= \frac{1}{2}(N_1 + N_2 + N_3 + N_4 + 2)^2 - ({L_{12}}^2 + {L_{34}}^2 + 2),
\end{aligned}$$

in correspondance with the preceding expressions for δ_1 and δ_2 in the realization $J_\bullet^{(1234)}$ of $\mathfrak{su}(1,1)$. From the expressions of these last M_1 and M_2, we can claim that the Hahn algebra is in the commutant of $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ in $U(\mathfrak{u}(4))$ represented on $\mathcal{H}$. Let us stress that it is the universal enveloping algebra of $\mathfrak{u}(4)$ that intervenes here.

9.4.2. The Bannai–Ito ensemble

The Bannai–Ito algebra [6] takes its name after the Bannai–Ito polynomials that enter in the Racah coefficients of the Lie superalgebra $\mathfrak{osp}(1|2)$. This algebra has three generators K_i, $i = 1, \ldots, 3$ that satisfy the relations

$$\{K_i, K_j\} = K_k + \omega_k, \qquad i \neq j \neq k \in \{1, 2, 3\} \tag{9.4.2}$$

with ω_i central and $\{X, Y\} = XY + YX$. The relevant reductive pair in this case is $(\mathfrak{o}(6), \mathfrak{osp}(1|2))$ and the representation space $\mathcal{H}$ is that of Dirac spinors in six dimensions with the Clifford algebra generated by the elements γ_μ verifying

$$\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}, \qquad \mu, \nu = 1, \ldots, 6.$$

That the pair $(\mathfrak{o}(6), \mathfrak{osp}(1|2))$ is dually represented on $\mathcal{H}$ is seen as follows:

The spinorial representation of $\mathfrak{o}(6)$ with generators

$$J_{\mu\nu} = -iL_{\mu\nu} + \Sigma_{\mu\nu}, \qquad L_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \qquad \Sigma_{\mu\nu} = \frac{i}{2}\gamma_\mu\gamma_\nu \tag{9.4.3}$$

leaves invariant the following operators:

$$J_- = -i \sum_{1\le\mu\le 6} \gamma_\mu \partial_\mu, \qquad J_+ = -i \sum_{1\le\mu\le 6} \gamma_\mu x_\mu, \qquad J_0 = \sum_{1\le\mu\le 6} x_\mu \partial_\mu, \tag{9.4.4}$$

which in turn realize the commutation relations of the Lie superalgebra $\mathfrak{osp}(1|2)$:

$$[J_0, J_\pm] = \pm J_\pm, \qquad \{J_+, J_-\} = -2J_0$$

with J_0 even and $J_\pm$ odd. Howe duality thus takes place. As a matter of fact, for any subset $A \subset \{1, \dots, 6\}$ of cardinality $|A|$ the operators

$$J_-^A = -i \sum_{\mu\in A} \gamma_\mu \partial_\mu, \qquad J_+^A = -i \sum_{\mu\in A} \gamma_\mu x_\mu, \qquad J_0^A = \tfrac{|A|}{2} + \sum_{\mu\in A} x_\mu \partial_\mu$$

realize $\mathfrak{osp}(1|2)$. The Casimir element of $\mathfrak{osp}(1|2)$ is given by

$$C = \frac{1}{2}([J_-, J_+] - 1)S$$

with S the grade involution obeying

$$S^2 = 1, \qquad [S, J_0] = 0, \qquad \{S, J_\pm\} = 0.$$

In the realizations at hand,

$$S^A = i^{|A|/2} \prod_{\mu\in A} \gamma_\mu$$

with $|A|$ even.

It can be checked that the operators

$$K_1 = M_1 + \tfrac{3}{2}\Sigma_{12}\Sigma_{34}, \qquad K_2 = M_2 + \tfrac{3}{2}\Sigma_{34}\Sigma_{56}, \qquad K_3 = M_3 + \tfrac{3}{2}\Sigma_{12}\Sigma_{56},$$

$$\begin{aligned}
M_1 &= (L_{12}\gamma_1\gamma_2 + L_{13}\gamma_1\gamma_3 + L_{14}\gamma_1\gamma_4 + L_{23}\gamma_2\gamma_3 + L_{24}\gamma_2\gamma_4 + L_{34}\gamma_3\gamma_4)\,\Sigma_{12}\Sigma_{34},\\
M_2 &= (L_{34}\gamma_3\gamma_4 + L_{35}\gamma_3\gamma_5 + L_{36}\gamma_3\gamma_6 + L_{45}\gamma_4\gamma_5 + L_{46}\gamma_4\gamma_6 + L_{56}\gamma_5\gamma_6)\,\Sigma_{34}\Sigma_{56},\\
M_3 &= (L_{12}\gamma_1\gamma_2 + L_{15}\gamma_1\gamma_5 + L_{16}\gamma_1\gamma_6 + L_{25}\gamma_2\gamma_5 + L_{26}\gamma_2\gamma_6 + L_{56}\gamma_5\gamma_6)\,\Sigma_{12}\Sigma_{56}
\end{aligned}$$

realize the relations (9.4.2) of the Bannai-Ito algebra upon taking the following:

$$\omega_{ij} = 2\Gamma_k\Gamma_{123} + 2\Gamma_i\Gamma_j,$$

where

$$\Gamma_1 = J_{12}, \qquad \Gamma_2 = J_{34}, \qquad \Gamma_3 = J_{56}, \qquad \Gamma_{123} = \Big(\tfrac{5}{2} - i \sum_{1\le\mu<\nu\le 6} L_{\mu\nu}\Sigma_{\mu\nu}\Big)\Sigma_{12}\Sigma_{34}\Sigma_{56}.$$

That these arise from dual pictures is explained as follows (see [7] for details). On the one hand, K_1, K_2, K_3 are observed to belong to the commutant in $U(\mathfrak{o}(6))$ of the $\mathfrak{o}(2)\oplus\mathfrak{o}(2)\oplus\mathfrak{o}(2)$ subalgebra of $\mathfrak{o}(6)$ spanned by $\{J_{12}, J_{34}, J_{56}\}$. On the other hand, considering the Casimir elements C^A of $\mathfrak{osp}(1|2)$ associated to the realization by the operators $\{J_0^A, J_\pm^A, S^A\}$, we find that

$$C^{(1234)} = K_1, \qquad C^{(3456)} = K_2, \qquad C^{(1256)} = K_3.$$

This confirms that the Bannai–Ito algebra can be dually presented either as in the commutant of $\mathfrak{o}(2) \oplus \mathfrak{o}(2) \oplus \mathfrak{o}(2)$ in the spinorial representation of $U(\mathfrak{o}(6))$ or in the centralizer of the action of $\mathfrak{osp}(1|2)$ on $\mathcal{H}$. These considerations can be extended to higher dimensions [7] so as to obtain analogously dual commutant pictures for the Bannai–Ito algebras of higher ranks [6].

9.4.3. The Askey–Wilson class

The Askey–Wilson algebra can be presented as follows:

$$\begin{aligned}\frac{[K_A, K_B]_q}{q^2 - q^{-2}} + K_C &= \frac{\gamma}{q+q^{-1}},\\ \frac{[K_B, K_C]_q}{q^2 - q^{-2}} + K_A &= \frac{\alpha}{q+q^{-1}},\\ \frac{[K_C, K_A]_q}{q^2 - q^{-2}} + K_B &= \frac{\beta}{q+q^{-1}},\end{aligned} \tag{9.4.5}$$

with $[A, B]_q = qAB - q^{-1}BA$ and α, β, γ central.

The $U_q(\mathfrak{su}(1,1))$ algebra has three generators, $J_\pm$ and J_0, obeying

$$[J_0\,, J_\pm] = \pm J_\pm, \qquad J_- J_+ - q^2 J_+ J_- = q^{2J_0}\,[2J_0]_q$$

with

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

Its coproduct is defined by

$$\Delta(J_0) = J_0 \otimes 1 + 1 \otimes J_0, \qquad \Delta(J_\pm) = J_\pm \otimes q^{2J_0} + 1 \otimes J_\pm.$$

The Casimir operator C of $U_q(\mathfrak{su}(1,1))$ is given by

$$C = J_+ J_- q^{-2J_0+1} - \frac{q}{(1-q^2)^2}\left(q^{2J_0-1} + q^{-2J_0+1}\right) + \frac{1+q^2}{(1-q^2)^2}.$$

The q-deformation $\mathfrak{o}_{q^{1/2}}(N)$ of $\mathfrak{o}(N)$ is defined as the algebra with generators $L_{i,i+1}$ ($i = 1, \ldots, N-1$) obeying the relations

$$\begin{aligned}&L_{i-1,i}\, {L_{i,i+1}}^2 - (q^{1/2} + q^{-1/2}) L_{i,i+1}\, L_{i-1,i}\, L_{i,i+1} + {L_{i,i+1}}^2\, L_{i-1,i} = -L_{i-1,i},\\ &L_{i,i+1}\, {L_{i-1,i}}^2 - (q^{1/2} + q^{-1/2}) L_{i-1,i}\, L_{i,i+1}\, L_{i-1,i} + {L_{i-1,i}}^2\, L_{i,i+1} = -L_{i,i+1},\\ &[L_{i,i+1}, L_{j,j+1}] = 0 \quad \text{for} \quad |i-j| > 1.\end{aligned}$$

We shall use the notation

$$L^\pm_{ik} = [L^\pm_{ij}\,, L^\pm_{jk}]_{q^{\pm 1/4}}$$

for any $i < j < k$, and by definition $L^{\pm}_{i,i+1} = L_{i,i+1}$.

The reductive pair $(\mathfrak{o}_{q^{1/2}}(6), U_q(\mathfrak{su}(1,1))$ is the one which is of relevance for the Askey–Wilson algebra. Let us indicate how $\mathfrak{o}_{q^{1/2}}(2n)$ and $U_q(\mathfrak{su}(1,1))$ are dually represented on the standard state space $\mathcal{H}$ of $2n$ independent q-oscillators described by operators $\{A_i^{\pm},\, A_i^0\}$ such that

$$[A_i^0, A_i^{\pm}] = \pm A_i^{\pm}, \qquad [A_i^-, A_i^+] = q^{A_i^0}, \qquad A_i^- A_i^+ - qA_i^+ A_i^- = 1, \qquad i = 1, \ldots, 2n.$$

The algebra $U_q(\mathfrak{su}(1,1))$ is represented on $\mathcal{H}$ by using the coproduct to embed it in the tensor product of $2n$ copies of the q-deformation of the metaplectic representation, this gives

$$\begin{aligned} J_0^{(2n)} &= \Delta^{(2n-1)}\Big(\tfrac{1}{2}\left(A_i^0 + \tfrac{1}{2}\right)\Big) = \frac{1}{2}\sum_{i=1}^{2n}\left(A_i^0 + \frac{1}{2}\right), \\ J_{\pm}^{(2n)} &= \Delta^{(2n-1)}\left(\frac{1}{[2]_{q^{1/2}}}(A_i^{\pm})^2\right) = \frac{1}{[2]_{q^{1/2}}}\sum_{i=1}^{2n}\left((A_i^{\pm})^2 \prod_{j=i+1}^{2n} q^{A_j^0+\frac{1}{2}}\right). \end{aligned} \tag{9.4.6}$$

The algebra $\mathfrak{o}_{q^{1/2}}(2n)$ can also be realized in terms of $2n$ q-oscillators. The $2n-1$ generators take the form

$$L_{i,i+1} = q^{-\frac{1}{2}(A_i^0+\frac{1}{2})}\Big(q^{\frac{1}{4}} A_i^+ A_{i+1}^- - q^{-\frac{1}{4}} A_i^- A_{i+1}^+\Big), \qquad i = 1, \ldots, 2n-1.$$

It can be checked that

$$[J_0^{(2n)}, L_{i,i+1}] = [J_{\pm}^{(2n)}, L_{i,i+1}] = 0, \qquad i = 1, \ldots, 2n-1,$$

in other words, that $U_q(\mathfrak{su}(1,1))$ and $\mathfrak{o}_{q^{1/2}}(2n)$ have commuting actions on the Hilbert space $\mathcal{H}$ of $2n$ q-oscillators. This sets the stage for Howe duality. In order to connect with the Askey–Wilson algebra we take $n = 3$. The expressions of the operators K_A and K_B acting on $\mathcal{H}$ that realize the relations (9.4.5) (together with the specific central elements) are rather involved and we shall refer the reader to [8] for the formulas. We shall only stress that these operators can be obtained in a dual way: They are affinely related to the generators of the commutant of $\mathfrak{o}_{q^{1/2}}(2)^{\oplus 3}$ in $\mathfrak{o}_{q^{1/2}}(6)$ as well as to the intermediate $U_q(\mathfrak{su}(1,1))$ Casimir elements

$$C^{(1234)} = \Delta^{(3)}(C) \otimes 1 \otimes 1 \qquad \text{and} \qquad C^{(3456)} = 1 \otimes 1 \otimes \Delta^{(3)}(C)$$

of the q-metaplectic representation (see (9.4.6)). This can be extended to higher ranks by letting n be arbitrary. For $n = 2$ we are looking at the Clebsch–Gordan problen for $U_q(\mathfrak{su}(1,1))$. The q-Hahn algebra that arises has two dual realizations [9]: one in the commutant of $\mathfrak{o}_{q^{1/2}}(2)^{\oplus 2}$ in $U_q(\mathfrak{u}(4))$ and the other in terms of the following two $U_q(\mathfrak{su}(1,1))$ operators, $(\Delta(J_0) \otimes 1 \otimes 1) - (1 \otimes 1 \otimes \Delta(J_0))$ and $\Delta^{(2)}(C)$ (the full Casimir element) in the q-metaplectic representation.

9.5. Conclusion

This paper has offered a summary of how the algebras of Racah, Hahn, Bannai–Ito, Askey–Wilson and q-Hahn types can be given dual descriptions in commutants of Lie algebras, superalgebras and quantum algebras. The connection between these dual pictures is rooted in Howe dualities whose various expressions have been stressed. The attentive reader will have noticed that the Clebsch–Gordan problem for $\mathfrak{osp}(1|2)$ has not been mentioned; this is because it has not been analyzed yet. We plan on adding this missing piece to complete the picture.

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Partie 2

Au-delà des algèbres de type Askey–Wilson : opérateurs de Sklyanin–Heun et dégénérations d'algèbres de Sklyanin

Introduction

La deuxième partie de cette thèse peut être approchée de plusieurs points de vue équivalents; en voici un. Cette deuxième partie généralise l'approche aux algèbres de type Askey–Wilson par les opérateurs de bispectralité qui a été développée par Zhedanov. Elle explore le cas où on part d'opérateurs encodant des propriétés des familles de polynômes plus élémentaires que la bispectralité, en un certain sens, et s'intéresse aux structures algébriques qu'ils génèrent.

Les trois résultats principaux dans cette partie sont les suivants :

(1) Les opérateurs de Sklyanin–Heun, définis comme une certaine classe d'opérateurs de Heun algébriques, sont introduits. Un sous-ensemble de 4 opérateurs de Sklyanin–Heun génère une algèbre qui s'avère être une dégénération de l'algèbre de Sklyanin.

(2) Ces 4 opérateurs peuvent être vus comme étant plus fondamentaux que les opérateurs de récurrence et différence donnant lieu aux algèbres de type Askey–Wilson. En ce sens, ces dégénérations de l'algèbre de Sklyanin peuvent être vues comme des structures algébriques plus fondamentales que les algèbres de type Askey–Wilson.

(3) Les opérateurs de Sklyanin–Heun offrent une interprétation algébrique de chacune des 4 familles de para-polynômes définies à ce jour : les familles de para-Krawtchouk, para-Racah, q-para-Krawtchouk et q-para-Racah. Les familles sont interprétées comme servant de base supportant des représentations irréductibles de dimension finie des dégénérations de l'algèbre de Sklyanin.

Les travaux de cette deuxième partie sont à l'intersection de plusieurs domaines. Mentionnons tout d'abord le concept d'opérateur de Heun venant du domaine de l'analyse de signaux. Un problème typique est la reconstruction d'un signal à partir d'un ensemble de mesures. En raison de contraintes évidentes, l'échantillon du signal qui peut être mesuré est limité au niveau du temps (on ne peut pas échantilloner le signal pendant une période de temps infinie) et des fréquences (les appareils de mesure utilisés ne peuvent capter qu'un certain intervalle de fréquences). Une reconstruction parfaite du signal observé est impossible sous ces conditions[1], et on cherche alors à approximer un signal de façon optimale, où la quantité à optimiser peut prendre plusieurs formes. Par exemple, on peut se demander comment obtenir une reconstruction qui approxime le mieux possible l'énergie du signal contenue dans l'intervalle de temps échantilloné.

La solution à ce problème nécessite d'introduire un opérateur intégral et de solutionner son problème aux valeurs propres, mais malheureusement cette procédure est difficile à implémenter numériquement en général, notamment en raison de la non-localité de l'opérateur

[1]Puisqu'un signal ne peut être limité au niveau du temps et des fréquences sans être trivial, on ne peut qu'approximer ce qu'on observe.

intégral. Dans les années 1960, Slepian, Pollak et Landau trouvent une solution « miraculeuse » [28–30]. Ils identifient un opérateur différentiel, un cas particulier de l'opérateur de Heun, qui commute avec l'opérateur intégral, possède donc les mêmes états propres mais dont le spectre est beaucoup plus facile à obtenir numériquement. La questions d'obtenir la meilleure approximation peut être ramenée à l'étude du problème aux valeurs propres de cet opérateur de Heun. L'existence de cet opérateur différentiel qui commute avec avec l'opérateur intégral reste un miracle inexpliqué, jusqu'en 1987, où Perline fournit une première piste d'explication en utilisant des propriétés de familles de polynômes orthogonaux classiques [31]. Une interprétation algébrique voit par la suite le jour en 2017 [32] et fournit une explication plus complète du phénomène. Le concept d'opérateur de Heun algébrique est introduit et connecte ces questions au domaine des polynômes orthogonaux du (q-)tableau d'Askey. Les opérateurs de Sklyanin–Heun introduits dans cette partie de la thèse peuvent être vus comme des blocs fondamentaux à partir desquels on peut reconstruire les opérateurs de Heun algébriques.

On trouve également dans cette intersection l'algèbre de Sklyanin (et ses dégénérations) qui appartient au domaine de l'intégrabilité. Dans le formalisme de la matrice de transfert et de l'équation de Yang–Baxter dans l'étude des systèmes intégrables, Sklyanin [3] étudie les conditions sous lesquelles l'équation $LLR = RLL$ est satisfaite pour le cas de la matrice R la plus générale (elliptique). Il en dérive une algèbre quadratique de 4 opérateurs, portant de nos jours le nom d'*algèbre de Sklyanin*. Les représentations de cette algèbre encodent des systèmes intégrables qui peuvent être résolus par la méthode de « scattering inverse ». Ces systèmes sont d'un grand intérêt, car ils peuvent être considérés comme des approximations sur des réseaux de divers systèmes intégrables continus. Diverses dégénérations de cette algèbre, telles que les dégénérations trigonométriques et rationnelles, existent et permettent de définir d'autres systèmes intégrables.

Ces algèbres sont quadratiques, riches en propriétés intéressantes pour les mathématiciens, tout en étant non-triviales [33]. Elles sont de plus en plus étudiées en classifiées par les algébristes et théoriciens des anneaux [34–36].

La façon dont les dégénérations de l'algèbre de Sklyanin sont obtenues dans cette partie de la thèse est nouvelle. Plutôt que de partir d'une matrice R, nous partons d'opérateurs de contiguïté et d'échelle associés à des familles de polynômes orthogonaux. En raison des liens étroits avec les polynômes orthogonaux dans cette approche, on peut s'attendre à ce que cette nouvelle approche permette d'obtenir des solutions explicites pour les nouveaux systèmes intégrables associés à ces dégénérations d'algèbres de Sklyanin.

Enfin, on trouve également le concept de para-polynômes dans cette intersection. Les para-polynômes ont été définis pour la première fois dans l'étude des problèmes de transfert parfait et de revitalisation fractionnelle dans les chaînes de spin [37]. La première famille identifiée est celle des para-Krawtchouk. Ces polynômes possèdent un spectre qui est une superposition de deux grilles linéaires. Ce spectre est analogue à celui d'un oscillateur parabosonique, d'où l'emploi de la terminologie « para-polynômes ». On a par la suite compris que les polynômes de para-Krawtchouk peuvent être obtenus comme une spécialisation des polynômes de Bannai–Ito complémentaires [38] ou bien à partir des polynômes continus de Hahn sous une procédure de troncation non-standard [39]. Par après, en utilisant des conditions de troncation analogues, des para-polynômes de type Racah, q-Krawtchouk et q-Racah ont pu être définis à partir des polynômes de Wilson, Big q-Jacobi et Askey–Wilson. Toutefois, aucune interprétation algébrique de ces familles de polynômes n'était connue jusqu'à tout récemment. Il était pourtant naturel de s'attendre à ce qu'il y en ait, car les familles du q-tableau d'Askey apparaissent dans divers contextes algébriques en théorie des représentations et il devrait en être de même pour leurs limites. Les travaux dans cette partie ont permis d'identifier ces para-polyômes comme des bases supportant des représentations irréductibles de dimension finie de diverses dégénérations de l'algèbre de Sklyanin.

Les opérateurs de Sklyanin–Heun sont définis en fonction d'une grille. La grille d'Askey–Wilson, associée aux familles de polynômes d'Askey–Wilson et de q-para-Racah, est étudiée au Chapitre 10. Puis la grille linéaire (associée aux familles des Hahn continus et de para-Krawtchouk) et la grille exponentielle[2] (associée aux familles des Big q-Jacobi et q-para-Krawtchouk) sont étudiées en Chapitre 11. Enfin, la grille quadratique (associée aux familles de Wilson et para-Racah) est étudiée au Chapitre 12. Des travaux subséquents sur les polynômes de para-Krawtchouk menant à une interprétation algébrique additionnelle de ceux-ci sont présentés au Chapitre 13 et concluent cette deuxième partie.

[2]ou q-linéaire

Chapitre 10

Degenerate Sklyanin algebras, Askey–Wilson polynomials and Heun operators

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Abstract: The q-difference equation, the shift and the contiguity relations of the Askey–Wilson polynomials are cast in the framework of the three and four-dimensional degenerate Sklyanin algebras $\mathfrak{ska}_3$ and $\mathfrak{ska}_4$. It is shown that the q-para Racah polynomials corresponding to a non-conventional truncation of the Askey–Wilson polynomials form a basis for a finite-dimensional representation of $\mathfrak{ska}_4$. The first order Heun operators defined by a degree raising condition on polynomials are shown to form a five-dimensional vector space that encompasses $\mathfrak{ska}_4$. The most general quadratic expression in the five basis operators and such that it raises degrees by no more than one is identified with the Heun–Askey–Wilson operator.

Keywords: Sklyanin algebras, Askey–Wilson operators and polynomials, q-para Racah polynomials, Heun operators.

10.1. Introduction

Quite some time ago, it was shown [1, 2] that the Askey–Wilson difference operator could be realized as a quadratic expression in the generators of the degenerate Sklyanin algebra (of dimension four). A little earlier Kalnins and Miller [3] used symmetry techniques to derive the orthogonality relation of the Askey–Wilson polynomials and identified to that end interesting ladder operators. Over the years the application of the factorization method [4] to these polynomials and the study of their structure relations [5] brought attention to related elements. More recently advances have been made in the elaboration of the theory of q-Heun operators and the Heun–Askey–Wilson [6] and rational [7] Heun operators have been identified by focusing on certain raising properties of their actions on appropriate spaces of functions. The purpose of this report is to stress the connections between these topics.

The paper will develop as follows. In Section 10.2 we shall introduce three operators involving q-shifts that realize a three-dimensional degenerate Sklyanin algebra $\mathfrak{ska}_3$. These operators will be ubiquitous; it will be observed that their linear combination is diagonal on special Askey–Wilson polynomials in base q and, following [1], that the most general quadratic expression formed with them yields the full Askey–Wilson operator in base q^2. The degenerate Sklyanin algebra $\mathfrak{ska}_4$ obtained by Gorsky and Zabrodin will also be introduced. It has $\mathfrak{ska}_3$ as a subalgebra and will be seen to admit a formal embedding of the Askey–Wilson algebra. In Section 10.3, it will be seen that the contiguity operators introduced by Kalnins and Miller in their treatment of the Askey–Wilson polynomials all belong to a model of the degenerate Sklyanin algebra $\mathfrak{ska}_4$. It will also be seen that the q-para Racah polynomials [8] support a finite-dimensional representation of the four-dimensional degenerate Sklyanin algebra. Section 10.4 will indicate how the Askey–Wilson bispectral operators emerge in this context. We shall consider first order q-difference operators and identify the conditions for such operators to raise by one the degree of polynomials in the symmetric variable $x = z+z^{-1}$. This will lead to a five dimensional vector space of operators. A basis will consist of one lowering operator, two that stabilize polynomials of a given degree and two that are raising this degree by one. The lowering and stabilizing operators will coincide with the operators realizing $\mathfrak{ska}_3$ introduced in Section 10.2. A combination involving the two raising operators will give the realization of the fourth generator of $\mathfrak{ska}_4$ beyond those of $\mathfrak{ska}_3$. The relations obeyed by these five operators will be found in an Appendix. It will further be seen that the most general quadratic operator in the five basis elements and not raising the degree by more than one is the Heun–Askey–Wilson operator [9]. This parallels the fact that in bispectral situations Heun operators could be defined equivalently as raising operators or as bilinear expressions in the bispectral operators. As will be indicated in the conclusion this approach is paving the way to the definition of Sklyanin-like Heun algebras associated to different degenerations of the Askey–Wilson grid.

10.2. Realizations of degenerate Sklyanin algebras and Askey–Wilson operators

It is well known that quantum algebras can be realized in terms of q-derivatives. In the case of $U_q(\mathfrak{su}(2))$ for example, the commutation relations

$$\begin{gathered}[\hat{B},\hat{C}] = \frac{\hat{A}^2-\hat{D}^2}{q-q^{-1}}, \qquad [\hat{A},\hat{D}]=0, \\ \hat{A}\hat{B}=q\hat{B}\hat{A}, \qquad \hat{B}\hat{D}=q\hat{D}\hat{B}, \qquad \hat{C}\hat{A}=q\hat{A}\hat{C}, \qquad \hat{D}\hat{C}=q\hat{C}\hat{D}\end{gathered} \tag{10.2.1}$$

are realized [10], [11] by taking

$$\begin{gathered}\hat{A}^{(\nu)} = q^{-\nu}T_+, \qquad \hat{B}^{(\nu)} = \frac{z}{2(q-q^{-1})}(q^{2\nu}T_- - q^{-2\nu}T_+), \\ \hat{C} = \frac{2}{(q-q^{-1})z}(T_+ - T_-), \qquad \hat{D}^{(\nu)} = q^{\nu}T_-,\end{gathered} \tag{10.2.2}$$

where in the case of finite dimensional representations ν is integer or half-integer (see below) and where T_+ and T_- are the q-shift operators that act as follows on functions of z:

$$T_+f(z) = f(qz), \qquad T_-f(z) = f(q^{-1}z). \tag{10.2.3}$$

We shall in the following look at models built with operators of the divided difference type.

10.2.1. The three-dimensional degenerate Sklyanin algebra $\mathfrak{skl}_3$

Let $p = (a,b,c,\alpha,\beta,\gamma)$ be a set of parameters. The generalized three-dimensional Sklyanin algebra $\hat{\mathcal{S}}^p$ as defined in [12] (see also [13]), is given by three generators u,v,y and the relations:

$$uv - avu - \alpha yy = 0, \qquad vy - byv - \beta uu, \qquad yu - cuy - \gamma vv = 0. \tag{10.2.4}$$

Consider the operators

$$\begin{aligned} Y &= \frac{1}{z-z^{-1}}(T_+ - T_-), \\ U &= \frac{1}{z-z^{-1}}(zT_+ - z^{-1}T_-), \\ V &= \frac{1}{z-z^{-1}}(zT_- - z^{-1}T_+). \end{aligned} \tag{10.2.5}$$

It is readily checked that they satisfy the following relations:

$$\begin{aligned} &VY - qYV = 0, \\ &YU - qUY = 0, \\ &[U,V] = (q-q^{-1})Y^2. \end{aligned} \tag{10.2.6}$$

Under the correspondence $\{y, u, v,\} \to \{Y, U, V\}$, it is seen that Y, U, V realize a special case of $\hat{\mathcal{S}}^p$ with

$$a = 1, \qquad b = c = q, \qquad \alpha = (q - q^{-1}), \qquad \beta = \gamma = 0. \tag{10.2.7}$$

We shall henceforth denote this algebra by $\mathfrak{ska}_3$. As shown in [12], it corresponds to one of the situations ($(a, b, c) \neq (0, 0, 0)$ and $\beta = \gamma = b - c = 0$)) for which the generalized Sklyanin algebra $\hat{\mathcal{S}}^p$ has a polynomial growth Hilbert series (PHS) and Koszul properties. The algebra $\mathfrak{ska}_3$ thus defined possesses a quadratic Casimir elements $\Omega^{(2)}$:

$$\Omega^{(2)} = uv + q^{-1} y^2 \tag{10.2.8}$$

that takes the value 1 in the realization (10.2.5) which implies that UV is related to Y^2.

10.2.2. The Askey–Wilson polynomials and algebra

Let us recall that the Askey–Wilson polynomials $p_n(x; a, b, c, d|q)$ defined by

$$\frac{a^n p_n(x; a, b, c, d|q)}{(ab, ac, ad; q)_n} = {}_4\phi_3\left[\begin{matrix} q^{-n}, & abcdq^{n-1}, & az, & az^{-1} \\ & ab, & ac, & ad \end{matrix}\middle| q; q\right] \tag{10.2.9}$$

with $x = z + z^{-1}$ are eigenfunctions of the operator $\mathcal{L}_q^{(a,b,c,d)}$ [14]

$$\mathcal{L}_q^{(a,b,c,d)} p_n(x; a, b, c, d|q) = \lambda_n p_n(x; a, b, c, d|q) \tag{10.2.10}$$

with eigenvalues

$$\lambda_n = q^{-n}(1 - q^n)(1 - abcdq^{n-1}). \tag{10.2.11}$$

We use standard notation for the basic hypergeometric functions and q-shifted factorials [14]. In base q^r, the Askey–Wilson operator reads

$$\mathcal{L}_{q^r}^{(a,b,c,d)} = A^{(r)}(z) T_+^r - [A^{(r)}(z) + A^{(r)}(z^{-1})]\mathcal{I} + A^{(r)}(z^{-1}) T_-^r \tag{10.2.12}$$

with

$$A^{(r)}(z) = \frac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{(1 - z^2)(1 - q^r z^2)} \tag{10.2.13}$$

and where $\mathcal{I}$ is the identity operator.

The Askey–Wilson algebra $AW(3)$ [15] that encodes the bispectrality of the polynomials p_n is realized by taking the generators $K_0 = \mathcal{L}_q^{(a,b,c,d)} + (1 + q^{-1}abcd)$ and $K_1 = x$ to find that the defining relations of $AW(3)$

$$[K_0, K_1]_q = K_2, \qquad \begin{aligned} [K_1, K_2]_q &= \mu K_1 + \nu_0 K_0 + \rho_0, \\ [K_2, K_0]_q &= \mu K_0 + \nu_1 K_1 + \rho_1, \end{aligned} \tag{10.2.14}$$

where $[A, B]_q = q^{1/2} AB - q^{-1/2} BA$, are verified with the parameters μ, ν and ρ related to those, a, b, c, d of the polynomials p_n (see for instance [16]).

Consider now the following general linear combination of Y, U and V:

$$\mathcal{M}^{(\alpha,\beta,\gamma)} = \alpha Y + \beta U + \gamma V. \tag{10.2.15}$$

Using (10.2.5), we see that

$$\mathcal{M}^{(\alpha,\beta,\gamma)} = F(z)T_+ + F(z^{-1})T_- \tag{10.2.16}$$

where

$$F(z) = \frac{\gamma(1-az)(1-bz)}{1-z^2} \tag{10.2.17}$$

with

$$\frac{\alpha}{\gamma} = (a+b), \qquad \frac{\beta}{\gamma} = -ab. \tag{10.2.18}$$

Since

$$F(z) + F(z^{-1}) = \gamma(1-ab), \tag{10.2.19}$$

we observe that

$$\mathcal{M}^{(\alpha,\beta,\gamma)} = \gamma[\mathcal{L}_q^{(a,b,q^{\frac{1}{2}},-q^{\frac{1}{2}})} + (1-ab)]. \tag{10.2.20}$$

It follows that the eigenfunctions of a linear combination of the operators Y, U, V such as $\mathcal{M}^{(\alpha,\beta,\gamma)}$ are special Askey–Wilson polynomials with the property of being “symmetric" when looked at from the dual perspective where variable and degree are exchanged; this is because the diagonal term in $\mathcal{M}^{(\alpha,\beta,\gamma)}$ is constant. Correspondingly, following [16], by taking

$$K_0 = \frac{1}{\gamma}\mathcal{M}^{(\alpha,\beta,\gamma)} \qquad \text{and} \qquad K_1 = x \tag{10.2.21}$$

we find that the Askey–Wilson algebra relations (10.2.14) are satisfied with

$$\begin{gathered} \mu = 0, \qquad \nu_0 = 1, \qquad \rho_0 = 0, \\ \nu_1 = -ab(q-q^{-1})^2, \qquad \rho_1 = (1-q^{-1})(a+b)(ab+q). \end{gathered} \tag{10.2.22}$$

We shall consider next quadratic expressions in the generators of $\mathfrak{ska}_3$.

10.2.3. The Askey–Wilson operator and $\mathfrak{ska}_3$

An important observation [1] comes from considering the most general quadratic expression in the operators $\{Y, U, V\}$ representing $\mathfrak{ska}_3$. Let us go over this. Define as before another general linear combination of these operators:

$$\mathcal{M}^{(\delta,\epsilon,\zeta)} = \delta Y + \epsilon U + \zeta V = G(z)T_+ + G(z^{-1})T_- \tag{10.2.23}$$

where

$$G(z) = \frac{\zeta(1-q^{-1}cz)(1-q^{-1}dz)}{1-z^2} \tag{10.2.24}$$

with

$$\frac{\delta}{\zeta} = q^{-1}(c+d), \qquad \frac{\epsilon}{\zeta} = -q^{-2}cd. \tag{10.2.25}$$

The product $\mathcal{M}^{(\alpha,\beta,\gamma)}\mathcal{M}^{(\delta,\epsilon,\zeta)}$ will take the form:

$$\begin{aligned}&\mathcal{M}^{(\alpha,\beta,\gamma)}\mathcal{M}^{(\delta,\epsilon,\zeta)}\\&\quad= F(z)G(qz)T_+^2 + [F(z)G(q^{-1}z^{-1}) + F(z^{-1})G(q^{-1}z)]\mathcal{I} + F(z^{-1})G(qz^{-1})T_-^2.\end{aligned} \tag{10.2.26}$$

A straightforward computation shows that for the specific functions $F(z)$ and $G(z)$ given in (10.2.17) and (10.2.25), the following identity holds:

$$F(z)G(q^{-1}z^{-1}) + F(z^{-1})G(q^{-1}z) = -F(z)G(qz) - F(z^{-1})G(qz^{-1}) + \Gamma \tag{10.2.27}$$

with Γ a constant given by

$$\Gamma = \gamma\zeta(abcdq^{-2} - ab - cdq^{-2} + 1). \tag{10.2.28}$$

Recalling the expression of $A^{(2)}(z)$ in (10.2.13), we see that

$$F(z)G(qz) = \gamma\zeta A^{(2)}(z) \tag{10.2.29}$$

and hence we write

$$\mathcal{M}^{(\alpha,\beta,\gamma)}\mathcal{M}^{(\delta,\epsilon,\zeta)} = \gamma\zeta\,[\mathcal{L}_{q^2}^{(a,b,c,d)} + (abcdq^{-2} - ab - cdq^{-2} + 1)\mathcal{I}]. \tag{10.2.30}$$

We have thus obtained a factorization of the Askey–Wilson operator $\mathcal{L}_{q^2}^{(a,b,c,d)}$ as a product of two linear combinations of the generators in the representation (10.2.5) of the special generalized Sklyanin algebra $\mathfrak{ska}_3$.

We also note that

$$\mathcal{M}^{(\alpha,\beta,\gamma)}\mathcal{M}^{(\delta,\epsilon,\zeta)} = (\alpha Y + \beta U + \gamma V)(\delta Y + \epsilon U + \zeta V) \tag{10.2.31}$$

provides the most general quadratic expression in the three generators $\{Y, U, V\}$. Taking into account the relations (10.2.5) between the generators and the expression of UV (10.2.45) (and VU) in terms of Y^2 provided by the value of the Casimir, the product $\mathcal{M}^{(\alpha,\beta,\gamma)}\mathcal{M}^{(\delta,\epsilon,\zeta)}$ can be reduced to:

$$\begin{aligned}&\mathcal{M}^{(\alpha,\beta,\gamma)}\mathcal{M}^{(\delta,\epsilon,\zeta)}\\&= \beta\epsilon U^2 + \gamma\zeta V^2 + (\alpha\delta - \beta\zeta q^{-1} - \gamma\epsilon q)Y^2 + (\alpha\epsilon q + \beta\delta)UY + (\alpha\zeta q^{-1} + \gamma\delta)VY + (\beta\zeta + \gamma\epsilon)\mathcal{I}.\end{aligned} \tag{10.2.32}$$

We thus recover (with a different parametrization) the result of Gorsky and Zabrodin [1] according to which the Askey–Wilson q-difference operator is a quadratic expression in the generators of $\mathfrak{ska}_3$. (As a matter of fact this result is presented in [1] in the context of the four-dimensional degenerate Sklyanin algebra to which we shall turn in a moment.)

Remark 10.1. *The idea of obtaining operators of interest, like the Askey–Wilson one, as quadratic expressions in the generators of fundamental algebras has precedents. Of note is the identification of the Askey–Wilson algebra as a coideal subalgebra of $U_q(\mathfrak{sl}(2))$ [17] and the* *use of the realization* (10.2.1) *to obtain the difference operator of the big q-Jacobi polynomials [18] as a generator in this embedding.*

Returning to the factorization formula, since γ and ζ only occur in the global factor, we may set $\gamma = \zeta = 1$. Summing up we thus have:

$$\mathcal{M}^{(\alpha,\beta,1)}\mathcal{M}^{(\delta,\epsilon,1)} = \mathcal{L}_{q^2}^{(a,b,c,d)} + (abcdq^{-2} - ab - cdq^{-2} + 1)\mathcal{I} \tag{10.2.33}$$

with

$$\begin{aligned} \alpha &= (a+b), & \beta &= -ab, \\ \delta &= q^{-1}(c+d), & \epsilon &= -q^{-2}cd. \end{aligned} \tag{10.2.34}$$

With the eigenvalues λ_n of $\mathcal{L}_{q^2}^{(a,b,c,d)}$ given by (10.2.11) with q replaced by q^2, it is straightforward to see that the Askey–Wilson polynomials with base q^2 correspondingly verify

$$\left[\mathcal{M}^{(\alpha,\beta,1)}\mathcal{M}^{(\delta,\epsilon,1)}\right] \, p_n(x;a,b,c,d|q^2) = \rho_n \, p_n(x;a,b,c,d|q^2) \tag{10.2.35}$$

with

$$\rho_n = q^{-2n}(1 - abq^{2n})(1 - cdq^{2n-2}). \tag{10.2.36}$$

Remark 10.2. *If we were to consider two linear combinations $\mathcal{M}^{(\bar{\alpha},\bar{\beta},1)}$ and $\mathcal{M}^{(\bar{\delta},\bar{\epsilon},1)}$ of Y, U and V where the roles of the pairs of parameters (a,b) and (c,d) are exchanged with respect to $\mathcal{M}^{(\alpha,\beta,1)}$ and $\mathcal{M}^{(\delta,\epsilon,1)}$, namely if we were to take*

$$\begin{aligned} \bar{\alpha} &= q^{-1}(a+b) = q^{-1}\alpha, & \bar{\beta} &= -q^{-2}ab = q^{-2}\beta, \\ \bar{\delta} &= (c+d) = q\delta, & \bar{\epsilon} &= -cd = q^2\epsilon, \end{aligned} \tag{10.2.37}$$

we would obtain again a factorization of the Askey–Wilson operator $\mathcal{L}_{q^2}^{(a,b,c,d)}$ of similar form

$$\mathcal{M}^{(\bar{\delta},\bar{\epsilon},1)}\mathcal{M}^{(\bar{\alpha},\bar{\beta},1)} = \mathcal{L}_{q^2}^{(a,b,c,d)} + (abcdq^{-2} - abq^{-2} - cd + 1)\mathcal{I}. \tag{10.2.38}$$

This is because $\mathcal{L}_{q^2}^{(a,b,c,d)}$ is invariant under the permutation of the parameters and the exchanges of the operators $\mathcal{M}$ with the q-shifts of the parameters given in (10.2.37) *simply amount to permuting the pairs (a,b) and (c,d) in the constant term of the rhs of* (10.2.33)*. This is in line with the fact that*

$$\mathcal{M}^{(\alpha,\beta,1)}\mathcal{M}^{(\delta,\epsilon,1)} - \mathcal{M}^{(\bar{\delta},\bar{\epsilon},1)}\mathcal{M}^{(\bar{\alpha},\bar{\beta},1)} = (-ab + cd)(1 - q^{-2})\mathcal{I} \tag{10.2.39}$$

as is easily checked using the relations (10.2.6) *as well as* (10.2.45)*. Conversely,*

$$\mathcal{M}^{(\alpha,\beta,1)}\mathcal{M}^{(\delta,\epsilon,1)} - \mathcal{M}^{(\bar{\delta},\bar{\epsilon},1)}\mathcal{M}^{(\bar{\alpha},\bar{\beta},1)} = (\beta - q^2\epsilon)(1 - q^{-2})\Omega^{(2)} \tag{10.2.40}$$

with the $\mathcal{M}$s taken as the linear combinations of the generators, can be seen to package (abstractly) the relations between y, u and v (10.2.4) *with parameters* (10.2.7)*.*

10.2.4. The four-dimensional degenerate Sklyanin algebra $\mathfrak{ska}_4$

The four-dimensional degenerate Sklyanin algebra $\mathfrak{ska}_4$ was obtained in [1] as a limit of the elliptic algebra originally introduced by Sklyanin [10] (see [19] for a mathematically oriented review). It is presented in terms of four generators A, B, C, D obeying the following homogeneous quadratic relations:

$$\begin{gathered} DC = qCD, \qquad CA = qAC, \qquad [A, D] = \frac{(q-q^{-1})^3}{4} C^2, \\ [B, C] = \frac{A^2 - D^2}{q - q^{-1}}, \\ AB - qBA = qDB - BD = -\frac{q^2 - q^{-2}}{4}(DC - CA). \end{gathered} \tag{10.2.41}$$

This algebra possesses two Casimir elements:

$$\Omega_0 = AD + \frac{(q-q^{-1})^2}{4q} C^2, \qquad \Omega_1 = \frac{q^{-1}A^2 + qD^2}{(q-q^{-1})^2} + BC + \frac{q+q^{-1}}{4} C^2. \tag{10.2.42}$$

We note that the subalgebra generated by $\{A, C, D\}$ is isomorphic to $\mathfrak{ska}_3$.

It was observed [1] that the degenerate Sklyanin algebra contracts to $\mathcal{U}_q(\mathfrak{su}(2))$; indeed, if one sets $A = \epsilon\hat{A}$, $B = \hat{B}$, $C = \epsilon^2\hat{C}$, $D = \epsilon\hat{D}$ and let ϵ go to zero we see that the relations (10.2.41) reduce to (10.2.1). In keeping with the representation theory [10] of the Sklyanin algebra, the finite dimensional representations of its degenerate version are characterized by an integer or half-integer ν and are of dimension $(2\nu + 1)$. We know from [1] that these can be realized by associating A, B, C, D to the following q-difference operators (we shall not distinguish here the abstract algebra element from its realization):

$$\begin{gathered} A = q^{-\nu}U, \qquad C = \frac{2}{(q-q^{-1})} Y, \qquad D = q^{\nu}V, \\ B = \frac{1}{2(q-q^{-1})(z-z^{-1})} \left[q^{2\nu}(z^2T_- - z^{-2}T_+) - q^{-2\nu}(z^2T_+ - z^{-2}T_-) - (q+q^{-1})(T_+ - T_-) \right], \end{gathered} \tag{10.2.43}$$

where U, V, Y are as in (10.2.5). In this realization the Casimir elements Ω_0 and Ω_1 take the following values:

$$\Omega_0 = 1, \qquad \Omega_1 = \frac{q^{2\nu+1} + q^{-2\nu-1}}{(q-q^{-1})^2}. \tag{10.2.44}$$

In light of (10.2.42) and (10.2.43), the former relation restates the already observed fact that UV is related to Y^2, namely that

$$UV = 1 - q^{-1}Y^2. \tag{10.2.45}$$

In the realization (10.2.43), the contraction from the degenerate Sklyanin algebra to $\mathcal{U}_q(\mathfrak{su}(2)$ amounts to taking z very large. It is quickly seen that in this limit the divided difference operators $\{A, B, C, D\}$ given above reduce to the $\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ of (10.2.2).

Now using the variable $x = z + z^{-1}$, it is readily found that B can be expressed as

$$B = \frac{1}{2(q - q^{-1})}\left[q^{-2\nu}(q^{-1}xU - Ux) + q^{2\nu}(qxV - Vx) - (q + q^{-1})Y\right] \tag{10.2.46}$$

in terms of the operators (10.2.5) realizing $\mathfrak{ska}_3$. We shall now indicate how x can be expressed as a formal power series in terms of A, B, C, D by inverting (10.2.46). In light of the commutation relation $[U, V] = (q - q^{-1})Y^2$ given in (10.2.5) and the Casimir relation (10.2.45) we have

$$UV = 1 - q^{-1}Y^2 \qquad \text{and} \qquad VU = 1 - qY^2. \tag{10.2.47}$$

It follows that V has an inverse V^{-1} given by the formal power series in Y expressed as follows:

$$V^{-1} = U(1 - qY^2)^{-1} = (1 - q^{-1}Y^2)^{-1}U. \tag{10.2.48}$$

Using the relations $Ux - qxU = -(q - q^{-1})Y$ and $xV - qVx = q(q - q^{-1})Y$, we arrive at

$$x = q^{-2\nu}\left[2B + \left(\frac{q + q^{-1}}{q - q^{-1}} + q^{2\nu} - q^{-2\nu}\right)Y\right][1 - q^{-4\nu}V^{-1}U]^{-1}V^{-1}. \tag{10.2.49}$$

As indicated before the Askey–Wilson operator $\mathcal{L}_{q^2}^{(a,b,c,d)}$ and x generate the Askey–Wilson algebra. Within the realization in terms of divided difference operators, we saw that $\mathcal{L}_{q^2}^{(a,b,c,d)}$ according to (10.2.33) is obtained as a quadratic expression in the generators of the subalgebra $\mathfrak{ska}_3$ of $\mathfrak{ska}_4$ and just found as per (10.2.49) that x is in the completion of the latter algebra. We can therefore assert that the Askey–Wilson algebra can be formally embedded in this realization of $\mathfrak{ska}_4$.

10.3. Contiguity operators of the Askey–Wilson polynomials and the degenerate Sklyanin algebra

In [3], Kalnins and Miller presented an elegant derivation of the weight function of the Askey–Wilson polynomials which is based on symmetry techniques. We here wish to point out that their approach can actually be cast in the framework of degenerate Sklyanin algebras. Central to the treatment in [3] are certain contiguity and ladder operators that will prove familiar. In order to facilitate comparison with the original reference we shall adopt essentially the same notation; we shall however use q^2 as the base.

Kalnins and Miller begin their considerations by observing that the Askey–Wilson polynomials satisfy the following contiguity relation

$$\mu^{(a,b,c,d)}p_n(x; a, b, c, d|q^2) = q^{-n}(1 - abq^{2n-2})p_n(x; aq^{-1}, bq^{-1}, cq, dq|q^2) \tag{10.3.1}$$

if $\mu^{(a,b,c,d)}$ is the following operator:

$$\mu^{(a,b,c,d)} = \frac{1}{(z-z^{-1})}\Big(-z^{-1}(1-aq^{-1}z)(1-bq^{-1}z)T_+ + z(1-aq^{-1}z^{-1})(1-bq^{-1}z^{-1})T_-\Big). \tag{10.3.2}$$

It is further observed that

$$\mu^{(cq,dq,aq^{-1},bq^{-1})}p_n(x;aq^{-1},bq^{-1},cq,dq|q^2) = q^{-n}(1-cdq^{2n})p_n(x;a,b,c,d|q^2). \tag{10.3.3}$$

We may proceed from here to derive the weight function by requesting that it be such that $\mu^{(cq,dq,aq^{-1},bq^{-1})}$ is the formal adjoint of $\mu^{(a,b,c,d)}$; this is done in [3]. Let us focus on the fact that in view of (10.3.1) and (10.3.3), the Askey–Wilson polynomials are eigenfunctions of $\mu^{(cq,dq,aq^{-1},bq^{-1})}\mu^{(a,b,c,d)}$, namely,

$$\Big[\mu^{(cq,dq,aq^{-1},bq^{-1})}\mu^{(a,b,c,d)}\Big]\, p_n(x;a,b,c,d|q^2) = \bar{\rho}_n\; p_n(x;a,b,c,d|q^2), \tag{10.3.4}$$

with

$$\bar{\rho}_n = q^{-2n}(1-cdq^{2n})(1-abq^{2n-2}). \tag{10.3.5}$$

Not surprisingly the factorization of the Askey–Wilson operator that this eigenvalue equation entails will coincide with the one described in the preceding section. This is readily established by recognizing that

$$\begin{aligned} \mu^{(a,b,c,d)} &= (aq^{-1}+bq^{-1})Y - abq^{-2}U + V = \mathcal{M}^{(\bar{\alpha},\bar{\beta},1)}, \\ \mu^{(cq,dq,aq^{-1},bq^{-1})} &= (c+d)Y - cdU + V \qquad\qquad = \mathcal{M}^{(\bar{\delta},\bar{\epsilon},1)}. \end{aligned} \tag{10.3.6}$$

These contiguity operators are thus found to belong to the realization (10.2.5) of the Sklyanin algebra $\mathfrak{sta}_3$ and we see that

$$\mu^{(cq,dq,aq^{-1},bq^{-1})}\mu^{(a,b,c,d)} = \mathcal{M}^{(\bar{\delta},\bar{\epsilon},1)}\mathcal{M}^{(\bar{\alpha},\bar{\beta},1)}, \tag{10.3.7}$$

with the connection with the Askey–Wilson operator provided by (10.2.38); we note moreover that the eigenvalue $\bar{\rho}_n$ in (10.3.5) coincides with the expression obtained from (10.2.36) under the exchange $(a,b) \leftrightarrow (c,d)$.

Kalnins and Miller consider in addition the lowering operator $\tau^{(a,b,c,d)}$:

$$\tau^{(a,b,c,d)} = \frac{1}{z-z^{-1}}(T_+ - T_-) = Y, \tag{10.3.8}$$

which is nothing else than our operator Y (or C). They proceed to find its adjoint $\tau^{(a,b,c,d)*}$ which reads:

$$\begin{aligned} \tau^{(a,b,c,d)*} = \frac{q^{-1}}{z-z^{-1}}\Big[&\frac{(1-az)(1-bz)(1-cz)(1-dz)}{z^2}T_+ \\ &- \frac{(1-az^{-1})(1-bz^{-1})(1-cz^{-1})(1-dz^{-1})}{z^{-2}}T_-\Big]. \end{aligned} \tag{10.3.9}$$

These operators act as follows on the Askey–Wilson polynomials:

$$\begin{aligned}&\tau^{(a,b,c,d)}p_n(x;a,b,c,d|q^2) = q^n(1-q^{-2n})(1-abcdq^{2n-2})p_{n-1}(x;aq,bq,cq,dq|q^2),\\&\tau^{(a,b,c,d)^*}p_{n-1}(x;aq,bq,cq,dq|q^2) = -q^{-n}p_n(x;a,b,c,d|q^2).\end{aligned}\tag{10.3.10}$$

The key point is that $\tau^{(a,b,c,d)^*}$ can be expressed as a linear combination of the generators A, B, C and D of the degenerate Sklyanin algebra $\mathfrak{ska}_4$. Let $e_1 = (a+b+c+d)$, $e_2 = (ab+ac+ad+bc+bd+cd)$, $e_3 = abc+abd+acd+bcd$ and $e_4 = abcd$ be the elementary symmetric functions in the parameters (a,b,c,d), one finds indeed that

$$\tau^{(a,b,c,d)^*}= q^{-1}\Big[-e_3(e_4)^{-\frac{1}{4}}A - 2(q-q^{-1})(e_4)^{\frac{1}{2}}B + \tfrac{(q-q^{-1})}{2}[e_2-(q+q^{-1})(e_4)^{\frac{1}{2}}]C + e_1(e_4)^{\frac{1}{4}}D\Big]\tag{10.3.11}$$

with

$$q^{-2\nu} = (abcd)^{\frac{1}{2}}.\tag{10.3.12}$$

We thus observe that the contiguity and raising operators $\mu^{(a,b,c,d)}, \tau^{(a,b,c,d)}$ and $\tau^{(a,b,c,d)^*}$ belong to the realization of the degenerate Sklyanin algebra which is hence represented on the Askey–Wilson polynomials. In general ν as given by the relation (10.3.12) above will not be an integer or half integer and the corresponding representation extends the finite-dimensional one discussed in Section 10.3 to an infinite-dimensional one.

Proposition 10.3. *The operator $\mu^{(a,b,c,d)}$, its adjoint $\mu^{(cq,dq,aq^{-1},bq^{-1})}$, $\tau^{(a,b,c,d)}$ and $\tau^{(a,b,c,d)^*}$ form a basis equivalent to the set $\{A,B,C,D\}$ as a representation of the degenerate Sklyanin algebra $\mathfrak{ska}_4$. Their action on the Askey–Wilson polynomials $p_n(x;a,b,c,d|q^2)$ is provided by* (10.3.1), (10.3.3) *and* (10.3.10) *respectively. The connection formula [20], [21] of Askey and Wilson can be used to express these formulae as combinations of polynomials $p_k(x;a,b,c,d|q^2)$, $k=0,1,\ldots$, with parameters a,b,c,d fixed, that span the representation space.*

Imposing that the representation be finite-dimensional amounts to enforcing the non-conventional truncation condition

$$(q^2)^{-N+1} = abcd, \qquad N = 2\nu+1\tag{10.3.13}$$

for the Askey–Wilson polynomials with base q^2. Quite strikingly this leads to polynomials called q-para Racah polynomials that have been recently characterized [8] and which are in particular orthogonal on a bilattice composed of two Askey–Wilson grids. We wish to stress this result.

Proposition 10.4. *The q-para Racah polynomials with base q^2 realize a basis for a representation of the degenerate Sklyanin algebra of dimension $N+1 = 2\nu+2$ with ν integer or half-integer.*

Remark 10.5. *Remarkably the operator $\tau^{(a,b,c,d)^*}$ also features centrally in Koornwinder's study [5] of the structure relations of the Askey–Wilson polynomials. These relations amount* *to raising and lowering relations where in contradistinction with the shift relations that we considered above (following Kalnins and Miller), the parameters are not affected. It is shown in [5] that such a structure relation is obtained when* $\tau^{(a,b,c,d)^*}$ *(denoted by L in [5] with the factor* q^{-1} *omitted) acts upon the Askey–Wilson polynomial* $p_n(x;a,b,c,d|q)$ *with base q. Note that the shift relation in* (10.3.10) *acts on polynomials with base* q^2*. It is also indicated in [5] that* $(q-1)\tau^{(a,b,c,d)^*} = [\mathcal{L}_q^{(a,b,c,d)}, x]$.

Remark 10.6. *It is further recognized in [5] on the basis of results of Rains [22] and Rosengren [23] that the operator* $\tau^{(a,b,c,d)^*}$ *generates a representation of the degenerate Sklyanin algebra* $\mathfrak{ska}_4$*. This is ascertained from the relation*

$$\tau^{(a,b,ce,de^{-1})^*}\tau^{(qa,qb,q^{-1}c,q^{-1}d)^*} = \tau^{(a,b,c,d)^*}\tau^{(qa,qb,q^{-1}ce,q^{-1}de^{-1})^*} \tag{10.3.14}$$

given in [5] and easily checked from (10.3.9)*. As observed by Koornwinder [5], it is the trigonometric specialization of a formula in [22] giving the defining relations of the Sklyanin algebra. We show below how* (10.3.14) *encapsulates the relations* (10.2.41) *of* $\mathfrak{ska}_4$*.*

Consider the expression (10.3.11) for $\tau^{(a,b,c,d)^*}$ as a linear combination of the operators A, B, C, D. Substituting (10.3.11) in (10.3.14) and multiplying by $(e_4)^{\frac{1}{4}}e(1-e)^{-1}(ce-d)^{-1}$, one arrives at

$$\begin{aligned}
0 = {}& (e_4)^{\frac{3}{4}}(q-q^{-1})(a+b)\left[A^2 - D^2 - (q-q^{-1})(BC-CB)\right] \\
& - 2(e_4)^{\frac{1}{2}}(q-q^{-1})ab(AB-qBA) \\
& - 2(e_4)(q-q^{-1})q^{-1}(BD-qDB) \\
& + (e_4)^{\frac{1}{4}}(a+b)(qab-q^{-1}cd)\left[(AD-DA) - \tfrac{1}{4}(q-q^{-1})^3C^2\right] \\
& - (e_4)^{\frac{1}{2}}\tfrac{(q-q^{-1})}{2}\left(((a+b)^2q^2-ab-cd)q^{-1} + (e_4)^{\frac{1}{2}}(1+q^{-2})\right)CD \\
& + (e_4)^{\frac{1}{2}}\tfrac{(q-q^{-1})}{2}\left(((a+b)^2q^2-abq^4-cd)q^{-2} + (e_4)^{\frac{1}{2}}(1+q^{-2})q\right)DC \\
& - \tfrac{(q-q^{-1})}{2}\left(ab(q+q^{-1})(e_4)^{\frac{1}{2}} + [e_4(2-q^{-2}) + (b^2cd+a^2cd-a^2b^2q^2)]\right)AC \\
& - \tfrac{(q-q^{-1})}{2}\left(-abq(q+q^{-1})(e_4)^{\frac{1}{2}} - q^{-1}[e_4(2-q^2) + (b^2cd+a^2cd-a^2b^2q^2)]\right)CA.
\end{aligned} \tag{10.3.15}$$

We shall illustrate how the defining relations of $\mathfrak{ska}_4$ can be obtained from (10.3.15). First choose $b=-a$ and $c=0$. The equality (10.3.15) implies

$$CA = qAC. \tag{10.3.16}$$

Substituting this back in (10.3.15) and multiplying by $(e_4)^{-\frac{1}{4}}$ yields

$$
\begin{aligned}
0 &= (e_4)^{\frac{1}{2}}(q-q^{-1})(a+b)\left[A^2-D^2-(q-q^{-1})(BC-CB)\right] \\
&\quad - 2(e_4)^{\frac{1}{4}}(q-q^{-1})ab(AB-qBA) \\
&\quad + (a+b)(qab-q^{-1}cd)\left[(AD-DA)-\tfrac{1}{4}(q-q^{-1})^3C^2\right] \\
&\quad - 2(e_4)^{\frac{3}{4}}(q-q^{-1})q^{-1}(BD-qDB) \\
&\quad - (e_4)^{\frac{1}{4}}\tfrac{(q-q^{-1})}{2}\left(((a+b)^2q^2-ab-cd)q^{-1}+(e_4)^{\frac{1}{2}}(1+q^{-2})\right)CD \\
&\quad + (e_4)^{\frac{1}{4}}\tfrac{(q-q^{-1})}{2}\left(((a+b)^2q^2-abq^4-cd)q^{-2}+(e_4)^{\frac{1}{2}}(1+q^{-2})q\right)DC \\
&\quad + \tfrac{(q-q^{-1})(q^2-q^{-2})}{2}((e_4)^{\frac{1}{4}}ab-q^{-1}(e_4)^{\frac{3}{4}})CA.
\end{aligned}
\tag{10.3.17}
$$

Once again, choose $c=0$ for instance. The equality (10.3.17) implies

$$
AD-DA=\frac{(q-q^{-1})^3}{4}C^2. \tag{10.3.18}
$$

Repeating the same kind of argument, one obtains the other relations (10.2.41) that define $\mathfrak{ska}_4$.

Through the realization that we have considered here, we have observed so far that the degenerate Sklyanin algebra $\mathfrak{ska}_4$ is a basic structure underneath the theory of Askey–Wilson polynomials. Much like a supersymmetric Hamiltonian is the "square" of supercharges, the Askey–Wilson operator is quadratic in generators realizing $\mathfrak{ska}_4$. We also saw that this is intimately connected to the application of Darboux transformations or of the factorization method [3], [4] to this operator. This approach as we know is based on the identification of raising operators. It has been realized recently that raising properties can provide a unifying principle in the theory of Heun operators [24]. We next take this angle to revisit the Heun–Askey–Wilson operator [6] and sort out the place occupied by the degenerate Sklyanin algebra in this Heun operator picture.

10.4. S-Heun operators and the Heun–Askey–Wilson operator

The standard Heun operator that defines the ordinary second order differential equation with four regular singularities [25] has the property of raising the degree of polynomials by one. It can also be obtained as a bilinear expression in the bispectral operators of the Jacobi polynomials, namely, multiplication by the variable and the hypergeometric operator [26]. Both viewpoints have been built upon to develop a broad perspective on operators of Heun type and the algebras they realize. The tridiagonalization method based on the hypergeometric operator has been generalized to any bispectral situation and the concept of algebraic Heun operator [27] has emerged in this fashion. In a nutshell this construct amounts to forming the generic bilinear expression in the bispectral operators. The raising property has been used to arrive at Heun operators defined on various lattices. In summary, one looks in this case for the most general second-order operator that raises by one the degree of polynomials on specified grids. Applied to the Askey–Wilson lattice or polynomials, both approaches have led equivalently to the Heun–Askey–Wilson operator [6]. (The Heun–Racah and Heun–Bannai–Ito operators have similarly been obtained [28].)

Let us mention that the Heun–Askey–Wilson operator has been shown [7] to arise as a degeneration of the one-variable Ruijsenaars-van Diejen Hamiltonian [29], [30], [31]. It has also been found that this operator can be diagonalized with the help of the algebraic Bethe ansatz [32]. We shall expand this by relating here the Heun–Askey–Wilson operators to our observations on Sklyanin algebras. To that end, we shall first focus on determining the most general first order operators acting on the Askey–Wilson grid that raise the degree of polynomials by one. We shall call them special Heun operators or S-Heun operators for short. These can be viewed as second order operators without diagonal terms. Indeed if the operator (11.2.1) given below is multiplied by T_+, we readily see that it takes the form of a first order operator $A_1(z)T_+^2 + A_2(z)$ on a grid with base q^2. Looking for S-Heun operators is in fact a more basic problem than searching for the generic second order operator with the raising property as a way of arriving directly at the Heun operator of Askey–Wilson type. It is hence not surprising that there will be factorization connotations. This undertaking will reveal that the S-Heun operators form a five-dimensional space that includes the operators (A, B, C, D) realizing $\mathfrak{ska}_4$. We shall further observe that the Heun–Askey–Wilson operator has a quadratic expression in terms of these S-Heun operators.

10.4.1. The S-Heun operators

Before we apply the raising condition to determine the S-Heun operators that act through q-differences on the symmetric variable $x = z + z^{-1}$, for reference, let us first go over the most simple case of first order differential operators that raise by one the degree of polynomials in the variable z. Consider the operator S

$$S = F(z)\frac{d}{dz} + G(z) \tag{10.4.1}$$

and demand that $Sp_n(z) = \tilde{p}_{n+1}(z)$ with p_n and $\tilde{p}_n$ polynomials of degree n. It is readily seen that the most general admissible functions $F(z)$ and $G(z)$ are

$$F(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2, \qquad G(z) = \beta_0 + \beta_1 z. \tag{10.4.2}$$

S therefore belongs to a 5-dimensional vector space with the following natural basis

$$L = \frac{d}{dz}, \qquad M_1 = 1, \qquad M_2 = z\frac{d}{dz}, \qquad R_1 = z, \qquad R_2 = z^2\frac{d}{dz}, \tag{10.4.3}$$

obtained by setting all coefficients α_i and β_i equal to zero except for one.

These operators can be combined to form the usual finite-dimensional differential realization of dimension $2j+1$ on monomials z^n, $n = 0, 1, \ldots$ of the Lie algebra $\mathfrak{sl}_2$, i.e.:

$$J_0 = z\frac{d}{dz} - j = M_2 - jM_1, \qquad J_+ = z^2\frac{d}{dz} - 2jz = R_2 - 2jR_1, \qquad J_- = \frac{d}{dz} = L. \tag{10.4.4}$$

This corresponds to the $q \to 1$ limit of the realization of $\mathcal{U}_q(\mathfrak{su}_2)$ given in Section 10.2.

Consider now the q-difference operator

$$S = A_1(z)T_+ + A_2(z)T_- \tag{10.4.5}$$

where $A_{1,2}(z)$ are functions of z. Note that these S-Heun operators can be viewed as "square roots" of the general (second order) Heun operators used in [6]. Impose again a raising condition on polynomials $P_n(x(z))$ of degree n in $x(z) = z + z^{-1}$:

$$SP_n(x(z)) = \tilde{P}_{n+1}(x(z)) \tag{10.4.6}$$

for all $n = 0, 1, 2, \ldots$.

It is sufficient to check property (11.2.3) for the elementary Askey–Wilson monomials

$$\chi_n(z) = z^n + z^{-n}, \tag{10.4.7}$$

that is to verify that

$$S\chi_n(z) = \sum_{k=0}^{n+1} a_{nk}\chi_k(z) \tag{10.4.8}$$

for some coefficients a_{nk}. Let us look at the action of S on the two Askey–Wilson monomials $\chi_n(x)$ of lowest degrees. For $n = 0$, the raising condition reads

$$A_1(z) + A_2(z) = a_{00} + a_{01}\chi_1(z) \tag{10.4.9}$$

and similarly, for $n = 1$ we have

$$A_1(z)(zq + z^{-1}q^{-1}) + A_2(z)(zq^{-1} + z^{-1}q) = a_{10} + a_{11}\chi_1(z) + a_{12}\chi_2(z) \tag{10.4.10}$$

where a_{00}, a_{01}, a_{10}, a_{11}, a_{12} are arbitrary parameters. Evaluating the action of S on the higher degree Askey–Wilson monomials does not give rise to new parameters: the higher coefficients a_{nk} with $n \geq 2$ are always expressed in terms of the a_{0k} and a_{1k}. Hence these 5 parameters account for all the degrees of freedom that the most general S-Heun operator defined on the Askey–Wilson grid possesses.

Combining (10.4.9) and (10.4.10), we find for $A_1(z)$

$$A_1(z) = \frac{\pi_4(z)}{z(1-z^2)(1-q^2)}, \tag{10.4.11}$$

where $\pi_4(z)$ is a polynomial of degree four:

$$\begin{aligned}\pi_4(z) = (a_{12}q - a_{01})z^4 + (qa_{11} - a_{00})z^3 - ((1+q^2)a_{01} - qa_{10})z^2 \\ + q(a_{11} - qa_{00})z + q(a_{12} - qa_{01}).\end{aligned} \tag{10.4.12}$$

From the observation that both the lhs and rhs of the system (10.4.9)–(10.4.10) are invariant under $z \to z^{-1}$, it follows that

$$A_2(z) = A_1(z^{-1}). \tag{10.4.13}$$

This leads to the following proposition.

Proposition 10.7. *The most general S-Heun operators on the Askey–Wilson grid which are required by definition to be of the form* (11.2.1) *and to raise by one the degrees of polynomials in $x = z + z^{-1}$ are specified by the functions $A_{1,2}(z)$ given in* (10.4.11)–(10.4.13)*.*

As the operator S depends on 5 free parameters, it gives rise as in the differential case to a 5-dimensional linear space of S-Heun operators. A natural basis for this space is formed by three sets which correspond respectively to lowering, stabilizing and raising operators:

(*i*) Taking $a_{10} = 1$ as the only non-zero parameter in (10.4.12) leads to the operator denoted L which decreases the degree of any polynomial in $x(z)$ by 1 and changes its parity.

(*ii*) Taking either $a_{00} = 1$ or $a_{11} = 1$ as the only non-zero parameter, one obtains stabilizing operators, denoted either M_1 or M_2. Both preserve the degree as well as the parity of any polynomial in $x(z)$.

(*iii*) The choice $a_{01} = 1$ and all other parameters equal to 0 leads to the raising operator R_1, while the choice $a_{01} = q$, $a_{12} = 1$ and all other parameters 0 yields the operator R_2. Both increase by one the degree of any polynomial in $x(z)$ and change parity.

For the sake of completeness, we give below the full expressions of these 5 operators

$$\begin{aligned}
L &= \frac{1}{q-q^{-1}}\frac{1}{z-z^{-1}}\left(T_+ - T_-\right), \\
M_1 &= \frac{1}{q-q^{-1}}\frac{1}{z-z^{-1}}\Big((qz+q^{-1}z^{-1})T_- - (q^{-1}z+qz^{-1})T_+\Big), \\
M_2 &= \frac{1}{q-q^{-1}}\frac{1}{z-z^{-1}}(z+z^{-1})(T_+ - T_-), \\
R_1 &= \frac{1}{q-q^{-1}}\frac{1}{z-z^{-1}}(z+z^{-1})\Big((qz+q^{-1}z^{-1})T_- - (q^{-1}z+qz^{-1})T_+\Big), \\
R_2 &= \frac{1}{q-q^{-1}}\left(\frac{q^2}{z-z^{-1}}(z+z^{-1})(zT_- - z^{-1}T_+) - (zT_- + z^{-1}T_+)\right).
\end{aligned} \tag{10.4.14}$$

Proposition 10.8. *The operators L, M_1, M_2, R_1, R_2 are linearly independent. They form a basis for the linear space of S-Heun operators.*

Note that the 3 operators L, M_1, M_2 span the 3-dimensional subspace of all "stabilizing" S-Heun operators. This means that any operator $S = \alpha_0 L + \alpha_1 M_1 + \alpha_2 M_2$ preserves the degree of any polynomial in $x(z)$, if at least one of α_1, α_2 is nonzero. Comparing (10.2.5) and (10.4.14), it is immediate to see that

$$Y = (q - q^{-1})L, \qquad U = M_1 + qM_2, \qquad V = M_1 + q^{-1}M_2, \tag{10.4.15}$$

and that the operators (L, M_1, M_2) equivalently realize $\mathfrak{ska}_3$. We can thus rephrase as follows the observations of Subsection 10.2.3 according to which the Askey–Wilson operator is given as a quadratic expression in the operators (Y, U, V) representing $\mathfrak{ska}_3$:

Proposition 10.9. *The Askey–Wilson operator can be given as the most general quadratic combination of the S-Heun operators L, M_1, M_2 that stabilize the degree of polynomials in $x(z)$.*

We know that the operators A, C, D in the realization (10.2.43) of $\mathfrak{ska}_4$ are proportional to U, Y, V respectively. It is not difficult to see that B in that same realization can be given as the following combination of L, R_1, R_2:

$$B = \frac{(q+q^{-1})[(q^{2\nu} - q^{-2\nu}) - (q - q^{-1})]}{2(q-q^{-1})} L + \frac{q^{1-2\nu}}{2} R_1 + \frac{(q^{2\nu-1} - q^{1-2\nu})}{2(q-q^{-1})} R_2. \tag{10.4.16}$$

We thus have:

Proposition 10.10. *The realization* (10.2.43) *of $\mathfrak{ska}_4$ is obtained from linear combinations of S-Heun operators on the Askey–Wilson grid.*

In addition, the operator x can be constructed as a quadratic polynomial in the the elementary S-Heun operators; we have indeed:

$$x = \frac{1}{q^2 - q^{-2}} \left[(1 + q^{-4})(qM_2R_2 - R_2M_2) + 2q^{-3}(qM_1R_2 - R_2M_1) \right]. \tag{10.4.17}$$

It follows that the Askey–Wilson algebra can be realized by combining quadratically the five basic S-Heun operators.

10.4.2. Heun–Askey–Wilson and S-Heun operators

We shall now obtain a formula for the Heun–Askey–Wilson operator in terms of S-Heun operators.

Consider the most general quadratic combination of the operators L, M_1, M_2, R_1, R_2 that raises the degree of polynomials in $x(z)$ by at most one. There should hence be no terms in ${R_1}^2$ and ${R_2}^2$. Using the relations in the Appendix 11.A, one can show that this combination may be written as follows

$$\begin{aligned} Q_{HAW} =& \alpha_1 L^2 + \alpha_2 LM_2 + \alpha_3 {M_1}^2 + \alpha_4 M_1M_2 + \alpha_5 M_2 L + \alpha_6 {M_2}^2 \\ &+ \beta_1 M_1R_1 + \beta_2 R_1M_1 + \beta_3 R_2M_2 \end{aligned} \tag{10.4.18}$$

where the γ_i's and δ_i's are arbitrary parameters. On functions $f(z)$ this operator takes the form:

$$Q_{HAW} f(z) = [A_1(z)T_+^2 + A_1(z^{-1})T_-^2 + A_0(z)\mathcal{I}]f(z), \tag{10.4.19}$$

with

$$A_1(z) = \frac{Q_6(z)}{z(1-z^2)(1-q^2z^2)}, \qquad A_0(z) = -(A_1(z) + A_1(z^{-1})) + p_1(x), \tag{10.4.20}$$

where $Q_6(z)$ is a generic polynomial of degree 6 in z and $p_1(x)$ is a generic polynomial of degree 1 in the variable $x = z + z^{-1}$. The exact parameters are expressible in terms of those of (10.4.18):

$$p_1(x) = \beta_2 x + \alpha_3, \qquad Q_6(z) = \frac{1}{q^2(q-q^{-1})^2}\sum_{k=0}^{6} r_k z^k, \tag{10.4.21}$$

$$\begin{gathered} r_0 = \beta_2 q^4 - \beta_3 q^4 + \beta_1 q^3 + \beta_3 q^2, \qquad r_1 = \alpha_3 q^4 - \alpha_4 q^3 + \alpha_6 q^2, \\ r_2 = -\beta_3 q^6 + \beta_1 q^5 + 2\beta_2 q^4 + (\alpha_5 + \beta_1) q^3 + (\alpha_2 + \beta_2 - \beta_3) q^2 + \beta_1 q, \\ r_3 = -\alpha_4 q^5 + (\alpha_3 + \alpha_6) q^4 + \alpha_1 q^3 + (\alpha_3 + \alpha_6) q^2 - \alpha_4 q, \\ r_4 = -\beta_3 q^6 + \beta_1 q^5 + (\alpha_2 + \beta_2 - \beta_3) q^4 + (\alpha_5 + \beta_1) q^3 + 2\beta_2 q^2 + \beta_1 q, \\ r_5 = \alpha_6 q^4 - \alpha_4 q^3 + \alpha_3 q^2, \qquad r_6 = \beta_1 q^3 + \beta_2 q^2. \end{gathered} \tag{10.4.22}$$

This operator is recognized as the Askey–Wilson Heun operator which has been identified and characterized in [6]. (See also [7] and [11].) It is immediately seen that the Askey–Wilson operator is recovered upon taking the β_i's equal to zero, which is equivalent to removing the terms involving raising S-Heun operators from Q_{HAW}.

This formula giving Q_{HAW} as the most general quadratic combination in the S-Heun operators on the Askey–Wilson grid provides a different characterization of the Heun–Askey–Wilson operator. As pointed out at the beginning of this section, this operator was identified in [6] on the one hand as the most general second order q-shift operator that raises by one the degree of polynomials on the Askey–Wilson grid and on the other hand, as the tridiagonalization of the Askey–Wilson operator as per the algebraic Heun construct. The presentation obtained here with the S-Heun operators as basic building blocks has the merit of providing, typically, a factorization of Q_{HAW}. Indeed it is seen that the Heun–Askey–Wilson operator can also be written generically in the form:

$$Q_{HAW} = (\xi_1 L + \xi_2 M_1 + \xi_3 M_2)(\eta_1 L + \eta_2 M_1 + \eta_3 M_2 + \eta_4 R_1 + \eta_5 R_2) + \kappa. \tag{10.4.23}$$

This formula for Q_{HAW} should be compared with equation (10.2.30) that provides the factorization of the Askey–Wilson operator as the product of two $\mathfrak{ska}_3$ elements. It is hence manifest from (10.4.23) that Q_{HAW} reduces to the Askey–Wilson operator when $\eta_4 = \eta_5 = 0$.

10.5. Conclusion

To conclude, let us first summarize our observations and second offer a brief outlook.

We have considered realizations of the Sklyanin algebras $\mathfrak{ska}_3$ and $\mathfrak{ska}_4$ in terms of q-difference operators and we determined the first order operators of that type – the S-Heun operators – that are the basic constituents of the most general degree raising operator in that class. Within these realizations, our salient observations are:

- The Askey–Wilson operator factorizes as the product of two linear combinations of elements in $\mathfrak{ska}_3$;
- In analogy with the dynamical enlargement of a symmetry algebra with the inclusion of ladder operators, the contiguity and shift operators of the Askey–Wilson polynomials have been shown to generate a realization of the degenerate Sklyanin algebra $\mathfrak{ska}_4$ which formally includes a realization of the Askey–Wilson algebra.
- The q-para Racah polynomials (with base q^2) have been identified as forming a basis for the finite-dimensional representations of the degenerate Sklyanin algebra $\mathfrak{ska}_4$.
- The set of S-Heun operators is five-dimensional and has a subset that realizes $\mathfrak{ska}_4$.
- The operator multiplication by x has a quadratic expression in terms of the S-Heun operators.
- The Heun–Askey–Wilson operator can also be written as a quadratic expression in the S-Heun operators.

With respect to these last two points, let us mention the following. We recall that the algebraic Heun construct gives the Heun–Askey–Wilson operator Q_{HAW} as a bilinear operator in x and the Askey–Wilson operator. In view of the first and next to last points, this implies that Q_{HAW} is quartic in the S-Heun operators, an expression that must be reducible to the quadratic formula (10.4.18) obtained here.

This study raises a number of questions. Let us mention two: (i) How does the examination of the S-Heun operators extend when the raising property is applied to rational functions as in [7]? (ii) What other algebraic structures akin to the degenerate Sklyanin algebras will emerge when the S-Heun operator approach is adapted to other lattices such as for example the quadratic one on which the Wilson polynomials are defined? We plan on addressing these and other related questions in the near future.

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10.A. Quadratic algebraic relations for L, M_1, M_2, R_1, R_2

The homogeneous quadratic algebraic relations between the five S-Heun operators L, M_1, M_2, R_1, R_2 are collected below:

$$\begin{aligned} [M_1, M_2] &= (q+q^{-1})^2 L^2, && (10.A.1)\\ M_1 L - (q+q^{-1}) L M_1 &= L M_2, && (10.A.2)\\ L M_1 + M_2 L &= 0, && (10.A.3)\\ {M_1}^2 + {M_2}^2 + (q+q^{-1}) M_2 M_1 &= 1, && (10.A.4) \end{aligned}$$

$$\begin{aligned} L R_1 &= 1 - {M_2}^2, && (10.A.5)\\ R_1 L &= 1 - {M_1}^2, && (10.A.6)\\ L R_2 &= -2L^2 + q^{-1} {M_2}^2 + M_1 M_2 + q, && (10.A.7)\\ R_2 L &= -2L^2 + q {M_2}^2 + q^2 M_2 M_1, && (10.A.8) \end{aligned}$$

$$\begin{aligned} R_1 M_2 + M_1 R_1 &= 0, && (10.A.9)\\ M_1 R_2 + R_2 M_2 &= 2(q+q^{-1}) M_2 L - (q+q^{-1})^2 L M_2, && (10.A.10)\\ q R_1 M_1 - M_1 R_1 &= R_2 M_1 + (q^2+q^{-2}) L M_1, && (10.A.11)\\ R_1 M_1 - (q+q^{-1}) M_1 R_1 &= M_2 R_1 - (q+q^{-1})(q-q^{-1})^2 M_2 L, && (10.A.12)\\ M_2 R_2 - (q+q^{-1}) R_2 M_2 &= R_2 M_1 + 2(q+q^{-1}) M_1 L - (2q^{-2}+1+q^4) L M_1, && (10.A.13) \end{aligned}$$

$$\begin{aligned} {R_2}^2 - q R_2 R_1 + q^{-1} R_1 R_2 &= -2(q+q^{-1})^2 M_1 M_2 - (q+q^{-1})^3 {M_2}^2 \\ &\quad + 2[(q^2+q^{-2}) - (q^2-q^{-2})^2] L^2 && (10.A.14) \end{aligned}$$

These relations are checked directly from the expressions of the operators in (10.4.14). They provide the necessary reorderings to reexpress the the most general quadratic combination of the 5 operators as in (10.4.18).

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Chapitre 11

Sklyanin-like algebras for (q-)linear grids and (q-)para-Krawtchouk polynomials

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Abstract: S-Heun operators on linear and q-linear grids are introduced. These operators are special cases of Heun operators and are related to Sklyanin-like algebras. The Continuous Hahn and Big q-Jacobi polynomials are functions on which these S-Heun operators have natural actions. We show that the S-Heun operators encompass both the bispectral operators and Kalnins and Miller's structure operators. These four structure operators realize special limit cases of the trigonometric degeneration of the original Sklyanin algebra. Finite-dimensional representations of these algebras are obtained from a truncation condition. The corresponding representation bases are finite families of polynomials: the para-Krawtchouk and q-para-Krawtchouk ones. A natural algebraic interpretation of these polynomials that had been missing is thus obtained. We also recover the Heun operators attached to the corresponding bispectral problems as quadratic combinations of the S-Heun operators.

Keywords: Sklyanin algebras, bispectral orthogonal polynomials, (q-)para-Krawtchouk polynomials, Heun operators.

11.1. Introduction

In the study of orthogonal polynomials (OPs), many of their properties are expressed as structure relations between family members with different parameters, arguments or degrees, examples are the three term recurrence relation, the differential/difference equation, the backward/forward relation, etc. As it turns out, the operators involved in these formulas realize algebras that synthesize much of the characterization of these polynomial ensembles. The present paper relates to this framework.

One such instance that has proven very fruitful is the (algebraic) study of the two bispectral operators associated to hypergeometric OPs. These operators are the recurrence and the differential/difference operators. Let us focus on the developments related to the Askey–Wilson polynomials; since these polynomials sit at the top of the Askey scheme, the gist of their description descends onto all the lower families in the scheme. The two bispectral operators for the Askey–Wilson polynomials do not commute: they form an algebra whose relations have been found by Zhedanov in [1] and it is usually referred to as the Askey–Wilson algebra.

This algebra has appeared in a great variety of contexts, such as knot theory [2], double affine Hecke algebras and representation theory [3–5], Howe duality [6, 7], integrable models [8–11], algebraic combinatorics [12–15], the Racah problem for $U_q(\mathfrak{sl}_2)$ [16, 17], etc. The abovementioned connections have some specializations for all entries of the Askey tableau.

The work of Kalnins and Miller [18–20] based on the use of four structure or contiguity operators is another approach that illustrates the use of symmetry techniques in the study of OPs. These operators that shall be referred to as structure operators in the following correspond to the backward and forward operators, as well as to another pair of operators that "factorize" [21] the differential/difference operator. It was recently observed [22] that for the Askey–Wilson polynomials, these operators realize the relations of the trigonometric degeneration [23] of the Sklyanin algebra [24]. To our knowledge, the Sklyanin-like algebras similarly connected to other families of OPs have not been described so far and will be the center of attention here.

The differential/difference operator of which the OPs are eigenfunctions belongs to the intersection of the sets of operators involved in the two pictures. A natural question is the following: *what is the most elementary set of operators that encompasses all operators in both of the approaches above?* In the case of the Askey–Wilson polynomials, this answer was given in [25]: it is the set of so-called S-Heun operators on the Askey–Wilson grid (these are special types of Heun operators that will be defined in the next section). Operators of the Heun type are related to the tridiagonalization procedure [26, 27] and have been given an algebraic formulation [28, 29]. They have been identified as Hamiltonians of quantum Euler–Arnold tops [30], they have been connected to band-time limiting [31, 32] and to the study of entanglement in spin chains [33, 34] and they have been studied quite a lot recently [35–44]. As will be shown below, the S-Heun operators allow a factorization of these Heun operators. Let us note that in addition to the unification of the two approaches described above, the S-Heun framework has also led to a novel algebraic interpretation of the q-para-Racah polynomials. The goal of the present paper is to look at the grids of linear type from the S-Heun operators point of view. As a byproduct, an algebraic interpretation of the para-Krawtchouk and q-para-Krawtchouk polynomials will be obtained. These polynomials were first identified in the context of perfect state transfer and fractional revival on quantum spin chains [45–48] and their algebraic interpretation was still lacking.

We will introduce the S-Heun operators on linear grids in Section 11.2. The simplest example of operators of this type will be worked out in Section 11.3 (this will involve differential operators, the Jacobi polynomials and the ordinary Heun operator). Section 11.4 will focus on the S-Heun operators on the discrete linear grid. A new degeneration of the Sklyanin algebra will be presented. Of relevance in this case, the Continuous Hahn polynomials will be seen to truncate to the para-Krawtchouk polynomials under a special condition and an algebraic interpretation of such a truncation will be given. The Heun operator on the uniform grid will also be recovered. The q-linear grid will be examined in Section 11.5 and the previous analysis will be repeated. The degeneration of the Sklyanin algebra that arises will be identified as $U_q(\mathfrak{sl}_2)$. The Big q-Jacobi polynomials will be involved, and they will be observed to reduce to the q-para-Krawtchouk polynomials under a certain condition. The Big q-Jacobi Heun operator will also be recovered as well. Connections between the three grids and the associated S-Heun operators and Sklyanin-type algebras will be presented in Section 11.6, followed by concluding remarks. The quadratic relations between the S-Heun operators for the three different types of grids are listed in Appendix 11.A.

11.2. S-Heun operators on linear-type grids

S-Heun operators are defined as the most general second order differential/difference operators without diagonal term that obey a degree raising condition. Like Heun operators, they can be defined on different grids. We now introduce the three linear grids that we will use and obtain the S-Heun operators associated to each.

11.2.1. The discrete linear grid

Consider the operator S

$$S = A_1T_+ + A_2T_- \tag{11.2.1}$$

where

$$T_+f(x) = f(x+1), \qquad T_-f(x) = f(x-1) \tag{11.2.2}$$

are shift operators, and $A_{1,2}$ are functions in the real variable x. Impose that S maps polynomials of degree n onto polynomials of degree no higher than $n+1$, namely,

$$SP_n(x) = \tilde{P}_{n+1}(x) \tag{11.2.3}$$

for all $n = 0, 1, 2, \ldots$. This defines the S-Heun operators on the discrete linear grid.

It is sufficient to enforce this raising condition on monomials x^n; for $n = 0$ and $n = 1$, it reads

$$A_1 + A_2 = a_{00} + a_{01}x, \tag{11.2.4a}$$

$$A_1(x+1) + A_2(x-1) = a_{10} + a_{11}x + a_{12}x^2, \tag{11.2.4b}$$

for some arbitrary parameters a_{ij}. This can be rewritten as

$$A_1 + A_2 = a_{00} + a_{01}x, \tag{11.2.5a}$$

$$A_1 - A_2 = a_{10} + (a_{11} - a_{00})x + (a_{12} - a_{01})x^2. \tag{11.2.5b}$$

Straightforward induction shows that in general one has

$$Sx^n = A_1(x+1)^n + A_2(x-1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k [A_1 + (-1)^{n-k} A_2], \tag{11.2.6}$$

which is a polynomial of degree $n+1$. Thus, the functions A_1, A_2

$$A_1 = \tfrac{1}{2}\left[(-a_{01} + a_{12})x^2 + (-a_{00} + a_{01} + a_{11})x + (a_{00} + a_{10})\right], \tag{11.2.7a}$$

$$A_2 = \tfrac{1}{2}\left[(+a_{01} - a_{12})x^2 + (+a_{00} + a_{01} - a_{11})x + (a_{00} - a_{10})\right] \tag{11.2.7b}$$

satisfy (11.2.5) and the operator (11.2.1) meets the degree raising condition.

Proposition 11.1. *With the functions A_1, A_2 given by* (11.2.7)*, the operator S in* (11.2.1) *is the most general S-Heun operator on the linear grid. S depends on* 5 *free parameters and spans a* 5*-dimensional linear space. The elements*

$$L = \tfrac{1}{2}\left[T_+ - T_-\right], \tag{11.2.8a}$$

$$M_1 = \tfrac{1}{2}\left[T_+ + T_-\right], \tag{11.2.8b}$$

$$M_2 = \tfrac{1}{2}x\left[T_+ - T_-\right], \tag{11.2.8c}$$

$$R_1 = \tfrac{1}{2}x\left[(1-2x)T_+ + (1+2x)T_-\right], \tag{11.2.8d}$$

$$R_2 = \tfrac{1}{2}x\left[T_+ + T_-\right]. \tag{11.2.8e}$$

form a basis for this space.

Using (11.2.6), one sees that the operator L is a lowering operator (it lowers by one the degree of polynomials in x), the operators M_1, M_2 are stabilizing operators (they do not change the degree) and the operators R_1, R_2 are raising operators (they raise it by one).

11.2.2. The q-linear grid

Condider now the q-linear grid $z = q^x$ (or exponential grid). The S-Heun operators $\hat{S}$ on that grid are of the form

$$\hat{S} = \hat{A}_1(z,q)\hat{T}_+ + \hat{A}_2(z,q)\hat{T}_-, \tag{11.2.9}$$

with

$$\hat{T}_\pm f(z) = f(q^{\pm 1} z), \tag{11.2.10}$$

and are taken to map polynomials in z onto polynomials of at most one degree higher: $\hat{S}P_n(z) = \tilde{P}_{n+1}(z)$. Imposing this degree raising condition on the first monomials 1 and z yields

$$\hat{A}_1(z,q) + \hat{A}_2(z,q) = a_{00} + a_{01}z, \tag{11.2.11a}$$

$$\hat{A}_1(z,q)q + \hat{A}_2(z,q)q^{-1} = a_{10}z^{-1} + a_{11} + a_{12}z. \tag{11.2.11b}$$

Straightforward induction shows that in general one has

$$\hat{S}z^n = (\hat{A}_1 q^n + \hat{A}_2 q^{-n})z^n = z^n[\hat{A}_1 q + \hat{A}_2 q^{-1}]\frac{q^n - q^{-n}}{q - q^{-1}} - z^n[\hat{A}_1 + \hat{A}_2]\frac{q^{n-1} - q^{1-n}}{q - q^{-1}}, \tag{11.2.12}$$

which is a polynomial of degree $n+1$ in z. Thus, an operator $\hat{S}$ with $\hat{A}_1(z,q)$ and $\hat{A}_2(z,q)$ that satisfies (11.2.11) will obey the degree raising condition on any monomial. We hence obtain:

$$\hat{A}_1(z,q) = \hat{A}_2(z,q^{-1}) = \frac{1}{(q-q^{-1})z}\left[a_{10} + (a_{11} - a_{00}q^{-1})z + (a_{12} - a_{01}q^{-1})z^2\right]. \tag{11.2.13}$$

Proposition 11.2. *With the functions $\hat{A}_1(z,q)$, $\hat{A}_2(z,q)$ given by* (11.2.13)*, the operator $\hat{S}$ in* (11.2.9) *is the most general S-Heun operator on the q-linear grid. $\hat{S}$ depends on* 5 *free parameters and spans a* 5*-dimensional linear space. The elements*

$$\hat{L} = \frac{1}{(q-q^{-1})} z^{-1}(\hat{T}_+ - \hat{T}_-), \tag{11.2.14a}$$

$$\hat{M}_1 = \frac{1}{(q-q^{-1})}(-q^{-1}\hat{T}_+ + q\hat{T}_-), \tag{11.2.14b}$$

$$\hat{M}_2 = \frac{1}{(q-q^{-1})}(\hat{T}_+ - \hat{T}_-), \tag{11.2.14c}$$

$$\hat{R}_1 = \frac{1}{(q-q^{-1})} z(-q^{-1}\hat{T}_+ + q\hat{T}_-), \tag{11.2.14d}$$

$$\hat{R}_2 = \frac{1}{(q-q^{-1})} z(\hat{T}_+ - \hat{T}_-). \tag{11.2.14e}$$

can be chosen as a basis for this space.

Looking at (11.2.12) and (11.2.13), one sees that the operator $\hat{L}$ lowers the degrees, and that the $\hat{M}_i$'s and the $\hat{R}_i$'s are respectively stabilizing and raising operators.

11.2.3. The simplest case: differential S-Heun operators

The definition of the S-Heun operators on the real line goes as follows. Consider the first-order differential operator

$$\bar{S} = \bar{A}_1(x)\frac{d}{dx} + \bar{A}_2(x) \tag{11.2.15}$$

and impose the raising condition $\bar{S}p_n(x) = \tilde{p}_{n+1}(x)$ which demands that $\bar{S}$ sends polynomials into polynomials of one degree higher. The general solution is given by

$$\bar{A}_1(x) = a_{10} + a_{11}x + a_{12}x^2, \qquad \bar{A}_2(x) = a_{20} + a_{21}x. \tag{11.2.16}$$

This leads to the following set of five linearly independent S-Heun operators [30]

$$\bar{L} = \frac{d}{dx}, \qquad \bar{M}_1 = 1, \qquad \bar{M}_2 = x\frac{d}{dx}, \qquad \bar{R}_1 = x, \qquad \bar{R}_2 = x^2\frac{d}{dx}, \tag{11.2.17}$$

which are once again labelled according to their property of lowering ($\bar{L}$), stabilizing ($\bar{M}$) or raising ($\bar{R}$) the degree of polynomials in the variable x.

These S-Heun operators can also be obtained as a $q \to 1$ limit of the ones defined on the q-linear grid. More precisely, writing $q = e^{\hbar}$ and letting $\hbar \to 0$, one obtains

$$\begin{gathered} \lim_{q\to 1} \hat{L} = \bar{L}, \qquad \lim_{q\to 1} \hat{M}_1 = \bar{M}_1 - \bar{M}_2, \qquad \lim_{q\to 1} \hat{M}_2 = \bar{M}_2, \\ \lim_{q\to 1} \hat{R}_1 = \bar{R}_1 - \bar{R}_2, \qquad \lim_{q\to 1} \hat{R}_2 = \bar{R}_2. \end{gathered} \tag{11.2.18}$$

This connects with the definition of the continuous S-Heun operators. These S-Heun operators will also be related to the ordinary Heun operator introduced in the next section.

11.3. The continuous case

The goal of this section is to revisit (mostly known) results with a point of view that will be adopted in the following sections. Here, we are interested in studying the OPs and algebras related to the set of the five S-Heun operators defined in Section 11.2.3.

11.3.1. The stabilizing subalgebra

We first study the subset $\{\bar{L}, \bar{M}_1, \bar{M}_2\}$ of S-Heun operators that stabilize the set of polynomials of a given degree. Let us denote by $\bar{Q}$ the most general quadratic combination of these operators. Using the relations of Appendix 11.A, it is always possible to reduce $\bar{Q}$ to an expression of the form

$$\bar{Q} = \alpha_1 \bar{L}^2 + \alpha_2 \bar{L}\bar{M}_1 + \alpha_3 \bar{L}\bar{M}_2 + \alpha_4 \bar{M}_1{}^2 + \alpha_5 \bar{M}_1 \bar{M}_2 + \alpha_6 \bar{M}_2^2. \tag{11.3.1}$$

Using the realizations (11.2.17), the eigenvalue equation for the second-order differential operator $\bar{Q}$ can be brought in the form

$$\begin{gathered}\bar{\mathcal{D}} P_n^{(\alpha,\beta)}(x) = n(n+\alpha+\beta+1) P_n^{(\alpha,\beta)}(x), \\ \bar{\mathcal{D}} = (x^2-1)\frac{d^2}{dx^2} + [(\alpha-\beta) + (\alpha+\beta+2)x]\frac{d}{dx},\end{gathered} \tag{11.3.2}$$

which is recognized as the differential equation satisfied by the Jacobi polynomials [49].

We have thus identified the family of OPs related to these (ordinary) S-Heun operators, and as will be seen in Subsection 11.3.2, certain combinations of these S-Heun operators provide the structure relations of these polynomials.

11.3.2. Jacobi polynomials and their structure relations

Consider the forward and backward operators for the Jacobi polynomials

$$\bar{\tau} = \bar{L}, \qquad \bar{\tau}^{(\alpha,\beta)^*} = -\bar{L} + (\alpha-\beta)\bar{M}_1 + (\alpha+\beta)\bar{R}_1 + \bar{R}_2. \tag{11.3.3a}$$

and the contiguity operators

$$\bar{\mu}^{(\alpha)} = -\bar{L} + \alpha \bar{M}_1 + \bar{M}_2, \qquad \bar{\mu}^{(\beta)^*} = \bar{L} + \beta\bar{M}_1 + \bar{M}_2. \tag{11.3.3b}$$

These four operators act very simply on the Jacobi polynomials:

$$\bar{\tau} P_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{11.3.4a}$$

$$\bar{\tau}^{(\alpha,\beta)^*} P_n^{(\alpha,\beta)}(x) = 2(n+1) P_{n+1}^{(\alpha-1,\beta-1)}(x), \tag{11.3.4b}$$

$$\bar{\mu}^{(\alpha)} P_n^{(\alpha,\beta)}(x) = (n+\alpha) P_n^{(\alpha-1,\beta+1)}(x), \tag{11.3.4c}$$

$$\bar{\mu}^{(\beta)^*} P_n^{(\alpha,\beta)}(x) = (n+\beta) P_n^{(\alpha+1,\beta-1)}(x). \tag{11.3.4d}$$

The operators $\bar{\mu}^{(\alpha)}$, $\bar{\mu}^{(\beta)^*}$, $\bar{\tau}$, $\bar{\tau}^{(\alpha,\beta)^*}$ built from linear combinations of S-Heun operators are of the type studied by Kalnins and Miller [19].

We have mentioned in the introduction that S-Heun operators encompass both the structure operators of Kalnins and Miller and the bispectral operators. Let us indicate how the latter operators appear in this context. First, as mentioned above, the Jacobi differential operator appears as a quadratic combination of the stabilizing generators. We can actually provide a factorization of this operator either as a product of two contiguous operators or as the product of the forward and backward operator:

$$\begin{aligned}\bar{\mathcal{D}} &= \bar{\mu}^{(\alpha+1)}\bar{\mu}^{(\beta)^*} - (\alpha+1)\beta \\ &= \bar{\mu}^{(\beta+1)^*}\bar{\mu}^{(\alpha)} - \alpha(\beta+1) \\ &= \bar{\tau}^{(\alpha+1,\beta+1)^*}\bar{\tau} \\ &= \bar{\tau}\bar{\tau}^{(\alpha,\beta)^*} - (\alpha+\beta).\end{aligned} \tag{11.3.5}$$

The other bispectral operator $\bar{X}$ is the multiplication by the variable x. It can be directly expressed as $\bar{R}_1$, but since it will appear as a quadratic combination of the S-Heun operators for other grids, we shall write it here as

$$\bar{X} = \bar{R}_1\bar{M}_1. \tag{11.3.6}$$

We have thus recovered the two bispectral operators as quadratic combinations in the S-Heun operators. This completes the observation that the S-Heun operators are the elementary blocks behind the two factorizations.

11.3.3. The Sklyanin-like algebra realized by the structure operators

We now focus on the algebras that are realized by these sets of operators. On the one hand the pair of bispectral Jacobi operators is known [50] to generate the Jacobi algebra that has been well studied [51]. On the other hand, the algebra formed by the 4 linear operators $\bar{\mu}^{(\alpha)}$, $\bar{\mu}^{(\beta)^*}$, $\bar{\tau}$, $\bar{\tau}^{(\alpha,\beta)^*}$ can be seen to be a degeneration of the Sklyanin algebra [24].

We now give a presentation of this algebra. Denote $\nu = -\frac{1}{2}(\alpha+\beta)$ and set

$$\bar{A} = \bar{M}_2 - \nu\bar{M}_1, \qquad \bar{B} = \bar{R}_2 - 2\nu\bar{R}_1, \qquad \bar{C} = \bar{L}, \qquad \bar{D} = \bar{M}_1. \tag{11.3.7}$$

These linear combinations of $\bar{\mu}^{(\alpha)}$, $\bar{\mu}^{(\beta)^*}$, $\bar{\tau}$, $\bar{\tau}^{(\alpha,\beta)^*}$ have been chosen in order to simplify the relations.

Proposition 11.3. *The operators $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ obey the homogeneous quadratic relations*

$$\begin{aligned}&[\bar{C},\bar{D}] = 0, \qquad [\bar{A},\bar{C}] = -\bar{C}\bar{D}, \qquad [\bar{A},\bar{D}] = 0, \\ &[\bar{B},\bar{C}] = -2\bar{A}\bar{D}, \qquad [\bar{A},\bar{B}] = \bar{B}\bar{D}, \qquad [\bar{B},\bar{D}] = 0.\end{aligned} \tag{11.3.8}$$

Remark 11.4. *One will notice that these relations are actually the relations of the $\mathfrak{sl}_2$ Lie algebra supplemented with a central element D (one recovers $U(\mathfrak{sl}_2)$ by quotienting the above algebra* (11.3.8) *by the additional relation $D = 1$). The reason why we wrote these in a quadratic fashion is to make easier the comparison with the other Sklyanin algebras that will be obtained later.*

One observes that if ν is an integer or half-integer, the realization (11.3.7) is associated to a finite dimensional representation of dimension $2\nu+1$.

11.3.4. Recovering the Heun operator

We now show how to recover the ordinary (differential) Heun operator from the knowledge of the S-Heun operators.

The generic Heun operator $\bar{W}$ can be expressed as the most general tridiagonalization of the hypergeometric operator [27]. It has been known to be

$$\bar{W} = Q_3(x)\frac{d^2}{dx^2} + Q_2(x)\frac{d}{dx} + Q_1(x), \tag{11.3.9}$$

where $Q_3(x)$, $Q_2(x)$ and $Q_1(x)$ are general polynomials of degree 3, 2 and 1 respectively.

Let us consider the most general quadratic combination of S-Heun operators that does not raise the degree of polynomials by more than one. Using the quadratic homogeneous relations of Appendix 11.A, it is always possible to simplify such an expression to

$$\begin{aligned} \bar{W} = {} & \alpha_1\bar{L}^2 + \alpha_2\bar{L}\bar{M}_1 + \alpha_3\bar{L}\bar{M}_2 + \alpha_4\bar{M}_1^2 + \alpha_5\bar{M}_1\bar{M}_2 + \alpha_6\bar{M}_2^2 \\ & + \beta_1\bar{M}_1\bar{R}_2 + \beta_2\bar{M}_2\bar{R}_1 + \beta_3\bar{M}_2\bar{R}_2. \end{aligned} \tag{11.3.10}$$

From the differential expressions of the generators we obtain

$$\begin{aligned} \bar{W} &= Q_3(x)\frac{d^2}{dx^2} + Q_2(x)\frac{d}{dx} + Q_1(x)\mathcal{I}, \\ Q_3(x) &= \alpha_1 + \alpha_3 x + \alpha_6 x^2 + \beta_3 x^3, \\ Q_2(x) &= (\alpha_2+\alpha_3) + (\alpha_5+\alpha_6)x + (\beta_1+\beta_2+2\beta_3)x^2, \\ Q_1(x) &= \alpha_4 + \beta_2 x, \end{aligned} \tag{11.3.11}$$

where $\mathcal{I}$ is the identity operator: $\mathcal{I}f(x) = f(x)$.

Proposition 11.5. *The generic Heun operator* (11.3.9) *can be obtained as the most general quadratic combination in the S-Heun generators* (11.2.17) *that does not raise the degree of polynomials by more than one.*

Calling upon the reordering relations of Appendix 11.A, it is seen that the Heun operator generically factorizes as the product of a general S-Heun operator with a stabilizing S-Heun operator:

$$\bar{W} = (\xi_1\bar{L} + \xi_2\bar{M}_1 + \xi_3\bar{M}_2)(\eta_1\bar{L} + \eta_2\bar{M}_1 + \eta_3\bar{M}_2 + \eta_4\bar{R}_1 + \eta_5\bar{R}_2) + \kappa. \tag{11.3.12}$$

11.4. S-Heun operators on the linear grid

We now come to one of the main topics of the paper, namely the S-Heun operators defined on the linear grid.

11.4.1. The stabilizing subset

The subset of S-Heun operators that stabilizes the polynomials of a given degree is $\{L, M_1, M_2\}$. The most general quadratic combination of these operators can always be reduced to an expression of the form

$$Q = \alpha_1 L^2 + \alpha_2 L M_1 + \alpha_3 L M_2 + \alpha_4 {M_1}^2 + \alpha_5 M_1 M_2 + \alpha_6 {M_2}^2 \tag{11.4.1}$$

using the relations of Appendix 11.A. Substituting the expressions (11.2.8), one sees that Q is a second-order difference operator. By straightforward manipulations, the eigenvalue equation for Q can be transformed into the difference equation of the Continuous Hahn polynomials [49]

$$\begin{aligned} \mathcal{D}P_n(\tilde{x}; a, b, c, d) &= n(n + a + b + c + d - 1)P_n(\tilde{x}; a, b, c, d), \\ \mathcal{D} &= B(\tilde{x})T_+^2 - [B(\tilde{x}) + D(\tilde{x})]\mathcal{I} + D(\tilde{x})T_-^2, \\ B(x) = (c - ix)(d - ix), &\qquad D(x) = (a + ix)(b + ix), \end{aligned} \tag{11.4.2}$$

with $\tilde{x} = i\frac{x}{2}$ and where a, b, c, d are given in terms of the α_i. From this, we recognize that the key family of OPs related to these S-Heun operators is the Continuous Hahn family.

11.4.2. Continuous Hahn polynomials and their structure relations

The following combinations of S-Heun operators

$$\tau = 2L, \tag{11.4.3a}$$

$$\tau^{(a,b,c,d)^*} = \mu_1 L + \mu_2 M_1 + \mu_3 M_2 + \mu_4 R_1 + \mu_5 R_2, \tag{11.4.3b}$$

with

$$\begin{aligned} \mu_1 &= \tfrac{1}{2}(1 - (a + b + c + d)) + (ab + cd), \\ \mu_2 &= \tfrac{1}{2}(a + b - c - d) - (ab - cd), \\ \mu_3 &= \tfrac{1}{2}(c + d - a - b), \\ \mu_4 &= -\tfrac{1}{4}, \\ \mu_5 &= \tfrac{1}{2}(a + b + c + d) - \tfrac{3}{4} \end{aligned} \tag{11.4.3c}$$

turn out to be the forward and backward operators, while

$$\mu^{(a,b,c,d)} = (d - a)L + (a + d - 1)M_1 + M_2, \tag{11.4.3d}$$

$$\mu^{(a,b,c,d)^*} = (c - b)L + (b + c - 1)M_1 + M_2, \tag{11.4.3e}$$

will act on polynomials as the contiguity relations. Indeed, these operators have the following actions on the Continuous Hahn polynomials:

$$\tau\ P_n\left(i\tfrac{x}{2},a,b,c,d\right)=i(n+a+b+c+d-1)P_{n-1}\left(i\tfrac{x}{2},a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right), \tag{11.4.4a}$$

$$\tau^{(a,b,c,d)^*}\ P_n\left(i\tfrac{x}{2},a,b,c,d\right)=-i(n+1)P_{n+1}\left(i\tfrac{x}{2},a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right), \tag{11.4.4b}$$

$$\mu^{(a,b,c,d)}\ P_n\left(i\tfrac{x}{2},a,b,c,d\right)=(n+a+d-1)P_n\left(i\tfrac{x}{2},a-\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d-\tfrac{1}{2}\right), \tag{11.4.4c}$$

$$\mu^{(a,b,c,d)^*}\ P_n\left(i\tfrac{x}{2},a,b,c,d\right)=(n+b+c-1)P_n\left(i\tfrac{x}{2},a+\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d+\tfrac{1}{2}\right). \tag{11.4.4d}$$

The 4 operators $\mu^{(a,b,c,d)}$, $\mu^{(a,b,c,d)^*}$, τ, $\tau^{(a,b,c,d)^*}$ have been studied by Kalnins and Miller in [19].

We now indicate how the two bispectral operators are formed from the S-Heun operators. As mentioned above, the Continuous Hahn difference operator can be formed by a quadratic combination of the stabilizing generators. Moreover, we can provide factorizations of this operator, either as a product of two contiguous operators or as the product of the backward and forward operators:

$$\begin{aligned}\mathcal{D}&=\mu^{(a+\frac{1}{2},b-\frac{1}{2},c-\frac{1}{2},d+\frac{1}{2})}\mu^{(a,b,c,d)^*}-(a+d)(b+c-1)\\&=\mu^{(a-\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d-\frac{1}{2})^*}\mu^{(a,b,c,d)}-(a+d-1)(b+c)\\&=\tau^{(a+\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d+\frac{1}{2})^*}\tau\\&=\tau\ \tau^{(a,b,c,d)^*}+2-(a+b+c+d).\end{aligned} \tag{11.4.5}$$

The remaining bispectral operator X is the multiplication by the variable x in this basis: $Xf(x)=xf(x)$. It appears as a quadratic combination in the S-Heun operators

$$X=[M_2,R_2]. \tag{11.4.6}$$

The framework of S-Heun operators presented here is thus seen to unite the symmetry techniques of Kalnins and Miller and the approach based on the bispectral operators (see [52] for more general context).

11.4.3. The Sklyanin-like algebra realized by the structure operators

Let us now look at the algebraic relations obeyed by these operators. On the one hand, the pair of bispectral Continuous Hahn operators realizes the Hahn algebra [39]. On the other hand, the algebra formed by the 4 linear operators $\mu^{(a,b,c,d)}$, $\mu^{(a,b,c,d)^*}$, τ, $\tau^{(a,b,c,d)^*}$ can be seen as a degeneration of the Sklyanin algebra.

This algebra can be presented as follows. Write $\nu = -\frac{1}{2}(a+b+c+d)$ and take

$$\begin{aligned} A &= 2(\nu+1)M_1 - 2M_2, \\ B &= \tfrac{1}{2}(2\nu+1)(2\nu+3)L - R_1 - (4\nu+3)R_2, \\ C &= L, \\ D &= M_1. \end{aligned} \tag{11.4.7}$$

These are linear combinations of $\mu^{(a,b,c,d)}$, $\mu^{(a,b,c,d)*}$, τ, $\tau^{(a,b,c,d)*}$ that have been chosen in order to simplify the relations.

Proposition 11.6. *The elements A, B, C, D obey the quadratic relations*

$$[C,D] = 0, \qquad [A,C] = \{C,D\}, \qquad [A,D] = \{C,C\}, \tag{11.4.8a}$$

$$[B,C] = \{D,A\}, \qquad [B,D] = \{C,A\}, \qquad [B,A] = \{B,D\}. \tag{11.4.8b}$$

We shall refer to these relations as those of the Skl_4 algebra.

The two quadratic Casimir elements are

$$\Omega_1 = D^2 - C^2, \qquad \Omega_2 = A^2 + D^2 - \{B,C\} \tag{11.4.9}$$

and they take the following values in the realization:

$$\Omega_1 = 1, \qquad \Omega_2 = (2\nu+3)^2. \tag{11.4.10}$$

Remark 11.7. *The stabilizing subalgebra of Skl_4* (11.4.8a)*, which we shall denote by Skl_3, has been identified in [53] as the algebra $T_7|_{(a,b)=(0,0)}$ whose relations are isomorphic to*

$$[x,y] = z^2, \qquad [y,z] = 0, \qquad [x,z] = zy. \tag{11.4.11}$$

It enjoys nice properties such as being Koszul, PBW, and being derived from a twisted potential. That the above algebra is Skl_3 is seen by setting $x = \frac{1}{2}A$, $y = D$, $z = C$.

We now explain that Skl_4 is a degeneration of the Sklyanin algebra. We rewrite the $\tau^{(a,b,c,d)*}$ in terms of A, B, C, D, using $e_1 = a+b+c+d$:

$$\begin{aligned} \tau^{(a,b,c,d)*} &= \tfrac{1}{4}(a+b-c-d)A + \tfrac{1}{4}B + \left[\tfrac{1}{8}(1-e_1)(1+e_1) + ab + cd\right] C \\ &\quad + \left[\tfrac{1}{4}e_1(a+b-c-d) - ab + cd\right] D. \end{aligned} \tag{11.4.12}$$

Two analogs of an identity due to Rains [54] can be obtained for $\tau^{(a,b,c,d)*}$. These are the quasi-commutation relations:

$$\tau^{(a+e,b,c,d-e)*}\tau^{(a-\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d-\frac{1}{2})*} = \tau^{(a,b,c,d)*}\tau^{(a-\frac{1}{2}+e,b+\frac{1}{2},c+\frac{1}{2},d-\frac{1}{2}-e)*}, \tag{11.4.13}$$

$$\tau^{(a,b+e,c-e,d)*}\tau^{(a+\frac{1}{2},b-\frac{1}{2},c-\frac{1}{2},d+\frac{1}{2})*} = \tau^{(a,b,c,d)*}\tau^{(a+\frac{1}{2},b-\frac{1}{2}+e,c-\frac{1}{2}-e,d+\frac{1}{2})*}. \tag{11.4.14}$$

Proposition 11.8. *Either of the quasi-commutation relation* (11.4.13), (11.4.14) *repackages the relations* (11.4.8) *of the Skl_4 algebra.*

Proof: Substituting the relation (11.4.12) into (11.4.13) and bringing all terms to the rhs, one obtains ($u = b - c$, $v = a - b - c + d$):

$$\begin{aligned}0 = \tfrac{e}{4}\Big\{ &\tfrac{1}{2}(AB - BA) + u(CB - BC) + \tfrac{1}{2}\left[(2-v)BD + vDB\right] \\ &+ u\left[(2-v)AD + vDA - 2(1-v)C^2\right] \\ &- \tfrac{1}{4}\left[(v^2 + 4u^2 - 4v + 3)AC - (v^2 + 4u^2 - 1)CA\right], \\ &+ \tfrac{1}{4}\left[v^3 - 4u^2 v + 8u^2 - 2v^2 - v + 2\right] CD - \tfrac{1}{4}\left[v^3 - 4u^2 v - 4v^2 + 3v\right] DC\Big\}.\end{aligned} \tag{11.4.15}$$

The dependence on the free parameter e factors out. Taking $v \to \infty$, one obtains immediately that

$$CD - DC = 0. \tag{11.4.16}$$

Also, taking $u \to 0$ and $v \to 0$, one gets

$$AB - BA = -2BD + \tfrac{3}{2}AC + \tfrac{1}{2}CA - CD. \tag{11.4.17}$$

Substituting these relations back in (11.4.15) leads to

$$\begin{aligned}0 = \tfrac{e}{4}\Big\{ &u(CB - BC) + \tfrac{v}{2}\left[DB - BD\right] + u\left[(2-v)AD + vDA - 2(1-v)C^2\right] \\ &- \tfrac{1}{4}\left[(v^2 + 4u^2 - 4v)AC - (v^2 + 4u^2)CA\right] + \tfrac{1}{4}\left[8u^2 + 2v^2 - 4v\right] CD\Big\}.\end{aligned} \tag{11.4.18}$$

Repeating a similar process, the remaining relations of (11.4.8) are found. A similar derivation starting from (11.4.14) instead yields the same relations. □

11.4.4. Finite-dimensional representations

It is known that finite-dimensional representations of the Hahn algebra relate to the Hahn polynomials [51]. We now wish to obtain finite-dimensional representations of the Skl_4 algebra; looking at (11.4.7), it is seen that one needs ν to be either an integer or half-integer. It will be shown that this corresponds in fact to a truncation of the Jacobi matrix of the Continuous Hahn polynomials.

Let us write the condition (ν is either an integer or half-integer) as

$$1 - (a + b + c + d) = N, \tag{11.4.19}$$

where N is a positive integer that corresponds to the maximal degree of the truncated family of polynomials.

This truncation condition is known [46] to be the one that takes the Wilson polynomials to the para-Racah polynomials. In the present case, we start from the Continuous Hahn OPs so the result of the truncation leads to a different family of para-polynomials.

Proposition 11.9. *The polynomials that arise from the truncation condition* (11.4.19) *form a basis that supports* $(N+1)$*-dimensional representations of the degenerate Sklyanin algebra* Skl_4 *and are identified as the para-Krawtchouk polynomials [45].*

We indicate below how the recurrence relation of the para-Krawtchouk polynomials is obtained from that of the Continuous Hahn polynomials by imposing (11.4.19).

11.4.4.1. $N = 2j+1$ odd. In the case where $N = 2j+1$ is odd (j is a non-negative integer), we parametrize the truncation condition as follows

$$c = -a - j + e_1 t, \qquad b = -d - j + e_2 t \tag{11.4.20}$$

and then take the limit $t \to 0$. We shall choose $e_1 = e_2$: this will lead to simpler expressions. The more general solutions corresponding to $e_1 \neq e_2$ can be recovered from the simpler solutions by the procedure of isospectral deformations, see for instance [55]. Using the chosen parametrization, the recurrence coefficients A_n, C_n appearing in the recurrence relation of the Continuous Hahn polynomials

$$\begin{gathered}(a+ix)P_n(x;a,b,c,d) = A_n P_{n+1}(x;a,b,c,d) + C_n P_{n-1}(x;a,b,c,d) - (A_n + C_n)P_n(x;a,b,c,d),\\ P_n(x;a,b,c,d) = \frac{n!}{i^n (a+c)_n (a+d)_n} p_n(x;a,b,c,d)\end{gathered} \tag{11.4.21}$$

become in the limit $t \to 0$:

$$A_n = -\frac{(n-N)(n+a+d)}{2(2n-N)}, \tag{11.4.22a}$$

$$C_n = +\frac{n(n-N-a-d)}{2(2n-N)}. \tag{11.4.22b}$$

Now take γ to be

$$\gamma = (b+c) - (a+d), \tag{11.4.23}$$

it follows that (11.4.22) can be rewritten in view of (11.4.19) as

$$A_n = -\frac{1}{2}\frac{(N-n)(N-1-2n+\gamma)}{2(2n-N)}, \tag{11.4.24a}$$

$$C_n = -\frac{1}{2}\frac{n(N+1-2n-\gamma)}{2(2n-N)}. \tag{11.4.24b}$$

These are recognized as the recurrence coefficients of the para-Krawtchouk polynomials in the variable $-\frac{x}{2}$ introduced in [45]. These polynomials are defined on the union of two linear lattices and the parameter γ describes the displacement of one lattice with respect to the other.

11.4.4.2. $N = 2j$ even. In the case where $N = 2j$ is even, we use the parametrization

$$c = -a - j + e_1 t, \qquad b = -d - j + e_1 t + 1 \tag{11.4.25}$$

and then take the limit $t \to 0$. The recurrence coefficients in the recurrence relation of the Continuous Hahn polynomials become

$$A_n = -\frac{(n-N)(n+a+d)}{2(2n-N+1)}, \tag{11.4.26a}$$

$$C_n = +\frac{n(n-N-a-d)}{2(2n-N-1)}, \tag{11.4.26b}$$

and upon writing

$$\gamma = 1 + (b+c) - (a+d), \tag{11.4.27}$$

we obtain

$$A_n = -\frac{1}{2}\frac{(N-n)(N-2-2n+\gamma)}{2(2n-N+1)}, \tag{11.4.28a}$$

$$C_n = -\frac{1}{2}\frac{n(N+2-2n-\gamma)}{2(2n-N-1)}. \tag{11.4.28b}$$

These are the recurrence coefficients of the para-Krawtchouk polynomials in the variable $-\frac{x}{2}$. The expressions for the monic polynomials are given in [46].

11.4.4.3. A remark on the truncation condition. It can be checked that in the realization (11.4.7), applying the truncation condition (11.4.19) seems to suggest that the raising operator B annihilates the monomial x^{N+1} and not x^N. A priori, this means that the truncation condition amounts to looking at $(N+2)$-dimensional representations of the algebra Skl_4, which would seem to contradict the fact that the para-Krawtchouk polynomials were truncated to have degrees at most N (and thus to span a space of dimension $N+1$).

Looking at the situation more closely, one observes that B indeed maps para-Krawtchouk polynomial of maximal degree N to a certain polynomial of degree $N+1$. But this polynomial of degree $N+1$ corresponds to the characteristic polynomial of the (upper block of the) truncated Jacobi matrix, hence it is null on the orthogonality grid points. Keeping in mind that the para-Krawtchouk polynomials are the basis vectors for the finite-dimensional representation of Skl_4, this characteristic polynomial thus corresponds to a null vector. Therefore the dimension of the space on which the representation of the Skl_4 algebra acts is indeed $N+1$.

11.4.5. Recovering the associated Heun operator

The Heun operator associated to the Continuous Hahn polynomials was implicitly defined in [39]. This operator W_{CH} is the most general second order operator that acts on the discrete linear grid and maps polynomials of degree n into polynomials of degree $n+1$. It can be expressed as

$$W_{CH} = \mathcal{A}_1 T_+ + \mathcal{A}_0 \mathcal{I} + \mathcal{A}_2 T_-, \tag{11.4.29}$$

where $\mathcal{A}_{1,2}$ are general polynomials of degree 3 with the same leading order coefficient, and $\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2 = \pi_1(x)$, with $\pi_1(x)$ a general polynomial of degree 1.

We now consider the most general quadratic combination of S-Heun operators that does not raise the degree of polynomials by more than one. Upon using the quadratic homogeneous relations of Appendix 11.A, this general combination can be brought into the form

$$\begin{aligned} W = {} & \alpha_1 L^2 + \alpha_2 L M_1 + \alpha_3 L M_2 + \alpha_4 {M_1}^2 + \alpha_5 M_1 M_2 + \alpha_6 {M_2}^2 \\ & + \beta_1 M_1 R_2 + \beta_2 M_2 R_1 + \beta_3 M_2 R_2. \end{aligned} \tag{11.4.30}$$

Substituting the expressions of the S-Heun basis operators (11.2.8), we obtain

$$\begin{aligned} W &= \mathcal{A}_1 T_+^2 + \mathcal{A}_0 \mathcal{I} + \mathcal{A}_2 T_-^2, \\ \mathcal{A}_1 &= \tfrac{1}{4}[-2\beta_2 x^3 + (\alpha_6 - 3\beta_2 + \beta_3)x^2 + (\alpha_3 + \alpha_5 + \alpha_6 + \beta_1 - \beta_2 + \beta_3)x \\ &\quad + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \beta_1)], \\ \mathcal{A}_2 &= \tfrac{1}{4}[-2\beta_2 x^3 + (\alpha_6 + 3\beta_2 - \beta_3)x^2 + (\alpha_3 - \alpha_5 - \alpha_6 + \beta_1 - \beta_2 + \beta_3)x \\ &\quad + (\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 + \alpha_5 - \beta_1)], \\ \mathcal{A}_0 &= (\beta_1 + \beta_2 + \beta_3)x + \alpha_4 - (\mathcal{A}_1 + \mathcal{A}_2). \end{aligned} \tag{11.4.31}$$

Proposition 11.10. *The generic Heun–Continuous Hahn operator* (11.4.29) *can be obtained as the most general quadratic combination in the S-Heun generators* (11.2.8) *that does not raise the degree of polynomials by more than one.*

Using the relations of Appendix 11.A, one can see that the Heun operator generically factorizes as the product of a general S-Heun operator with a stabilizing S-Heun operator:

$$W = (\xi_1 L + \xi_2 M_1 + \xi_3 M_2)(\eta_1 L + \eta_2 M_1 + \eta_3 M_2 + \eta_4 R_1 + \eta_5 R_2) + \kappa. \tag{11.4.32}$$

11.5. The case of the q-linear grid

We consider now the S-Heun operators associated to the q-linear (or exponential) grid.

11.5.1. The stabilizing subspace

The stabilizing subset of S-Heun operators is $\{\hat{L}, \hat{M}_1, \hat{M}_2\}$. Using the relations of Appendix 11.A, it is always possible to reduce the most general quadratic combination of these operators to

$$\hat{Q} = \alpha_1 \hat{L}^2 + \alpha_2 \hat{L}\hat{M}_1 + \alpha_3 \hat{L}\hat{M}_2 + \alpha_4 \hat{M}_1^2 + \alpha_5 \hat{M}_1 \hat{M}_2 + \alpha_6 \hat{M}_2^2. \tag{11.5.1}$$

Substituting the expressions (11.2.14), one recognizes $\hat{Q}$ as a second-order q-difference operator whose eigenvalue problem can be cast as the difference equation

$$\begin{aligned}\hat{\mathcal{D}}P_n(z;\alpha,\beta,\gamma;\tilde{q}) &= (\tilde{q}^{-n}-1)(1-\alpha\beta\tilde{q}^{n+1})P_n(z;\alpha,\beta,\gamma;\tilde{q}),\\ \hat{\mathcal{D}} &= B(z)\hat{T}_+^2-[B(z)+D(z)]\mathcal{I}+D(z)\hat{T}_-^2,\\ B(z) &= \frac{\alpha\tilde{q}(z-1)(\beta z-\gamma)}{z^2},\qquad D(z)=\frac{(z-\alpha\tilde{q})(z-\gamma\tilde{q})}{z^2}\end{aligned}\tag{11.5.2}$$

of the Big q-Jacobi polynomials [49] in base $\tilde{q}=q^2$, making those the OPs associated to S-Heun operators on the exponential lattice. We note that there is a duality between the Continuous Dual q-Hahn and the Big q-Jacobi polynomials [56] that can be pictured as follows: exchanging the degree with the variable in some way takes one family of polynomials into the other (with transformed parameters). Thus, if we were to write the S-Heun operators (11.2.14) by replacing the variable with the degree in the appropriate way, the Continuous Dual q-Hahn polynomials would arise instead.

11.5.2. Big q-Jacobi polynomials and their structure relations

Focusing on the structure and contiguity relations of the Big q-Jacobi polynomials, we shall show how the set of S-Heun operators spans a space that contains the relevant operators. Let

$$\hat{\tau}=(q-q^{-1})\hat{L},\tag{11.5.3a}$$

$$\hat{\tau}^{(a,b,c,d)*}=\mu_1\hat{L}+\mu_2\hat{M}_1+\mu_3\hat{M}_2+\mu_4\hat{R}_1+\mu_5\hat{R}_2,\tag{11.5.3b}$$

with

$$\begin{aligned}\mu_1&=-(q-q^{-1}),\\ \mu_2&=(a+b)q^{-1}-q(c^{-1}+d^{-1}),\\ \mu_3&=(a+b)-(c^{-1}+d^{-1}),\\ \mu_4&=-abq^{-2}+q^2c^{-1}d^{-1},\\ \mu_5&=-abq^{-1}+qc^{-1}d^{-1},\end{aligned}\tag{11.5.3c}$$

and

$$\hat{\mu}^{(a,b,c,d)}=(q-q^{-1})L-(aq^{-1}-qd^{-1})M_1-(a-d^{-1})M_2,\tag{11.5.3d}$$

$$\hat{\mu}^{(a,b,c,d)*}=(q-q^{-1})L-(bq^{-1}-qc^{-1})M_1-(b-c^{-1})M_2.\tag{11.5.3e}$$

The actions of these operators on the Big q-Jacobi polynomials $P_n(z;\alpha,\beta,\gamma;q^2)$ is best presented as follows. Let

$$\Phi_n^{(a,b,c,d)}(z;\tilde{q})=P_n(az;ac\tilde{q}^{-1},bd\tilde{q}^{-1},ad\tilde{q}^{-1};\tilde{q}).\tag{11.5.4}$$

It is clear that the parameter a is redundant. One has $\Phi_n^{(1,\beta/\gamma,\alpha\tilde{q},\gamma\tilde{q})}(z;\tilde{q}) = P_n(z;\alpha,\beta,\gamma;\tilde{q})$. It is seen that

$$\hat{\tau}\ \Phi_n^{(a,b,c,d)}(z;\tilde{q}) = \frac{aq(1-q^{-2n})(1-abcdq^{2n-2})}{(1-ad)(1-ac)}\Phi_{n-1}^{(aq,bq,cq,dq)}(z;\tilde{q}), \tag{11.5.5a}$$

$$\hat{\tau}^{(a,b,c,d)^*}\ \Phi_n^{(a,b,c,d)}(z;\tilde{q}) = \frac{(ac-q^2)(ad-q^2)}{acdq}\Phi_{n+1}^{(aq^{-1},bq^{-1},cq^{-1},dq^{-1})}(z;\tilde{q}), \tag{11.5.5b}$$

$$\hat{\mu}^{(a,b,c,d)}\ \Phi_n^{(a,b,c,d)}(z;\tilde{q}) = \frac{q}{d}(1-adq^{-2})\Phi_n^{(aq^{-1},bq,cq,dq^{-1})}(z;\tilde{q}), \tag{11.5.5c}$$

$$\hat{\mu}^{(a,b,c,d)^*}\ \Phi_n^{(a,b,c,d)}(z;\tilde{q}) = -\frac{q(ad-q^{-2n})(1-bcq^{2n-2})}{c(1-ad)}\Phi_n^{(aq,bq^{-1},cq^{-1},dq)}(z;\tilde{q}). \tag{11.5.5d}$$

The 4 operators $\hat{\mu}^{(a,b,c,d)}$, $\hat{\mu}^{(a,b,c,d)^*}$, $\hat{\tau}$, $\hat{\tau}^{(a,b,c,d)^*}$ built from linear combinations of S-Heun operators have been studied by Kalnins and Miller in [19].

Let us further indicate how the bispectral operators show up in this context. As mentioned above, the Big q-Jacobi difference operator appears as a quadratic combination of the stabilizing generators. Moreover, one can actually provide factorizations of this operator in terms of contiguity operators as well as backward and forward operators:

$$\begin{aligned}
\hat{\mathcal{D}} &= \alpha\gamma q^3 \mu^{(q,\frac{\beta}{\gamma q},\alpha q,\gamma q^3)}\mu^{(1,\frac{\beta}{\gamma},\alpha q^2,\gamma q^2)^*} - (1-\gamma q^2)(1-\tfrac{\alpha\beta}{\gamma}) \\
&= \alpha\gamma q^3 \mu^{(q^{-1},\frac{\beta q}{\gamma},\alpha q^3,\gamma q)^*}\mu^{(1,\frac{\beta}{\gamma},\alpha q^2,\gamma q^2)} - (1-\gamma)(1-\tfrac{\alpha\beta q^2}{\gamma}) \\
&= -\alpha\gamma q^3 \hat{\tau}^{(q,\frac{\beta q}{\gamma},\alpha q^3,\gamma q^3)^*}\tau \\
&= -\alpha\gamma q^3 \hat{\tau}\ \hat{\tau}^{(1,\frac{\beta}{\gamma},\alpha q^2,\beta q^2)^*} - (1-q^2)(1-\alpha\beta).
\end{aligned} \tag{11.5.6}$$

The second bispectral operator $\hat{X}$ is the multiplication by the variable z: $\hat{X}f(z) = zf(z)$. It also appears as the quadratic combination of S-Heun operators:

$$\hat{X} = \hat{M}_2\hat{R}_1 - \hat{M}_1\hat{R}_2. \tag{11.5.7}$$

The S-Heun operators thus underscore much of the characterization of the Big q-Jacobi operators.

11.5.3. The Sklyanin-type algebra realized by the structure operators

The pair of bispectral Big q-Jacobi operators is known to realize the Big q-Jacobi algebra [42, 57]. The algebra generated by the 4 linear operators $\hat{\mu}^{(a,b,c,d)}$, $\hat{\mu}^{(a,b,c,d)^*}$, $\hat{\tau}$, $\hat{\tau}^{(a,b,c,d)^*}$ is a familiar degeneration of the Sklyanin algebra [24].

Denote $q^{-\nu} = (abcd)^{\frac{1}{4}}$ and form

$$\begin{aligned}
\hat{A} &= q^{-\nu}(\hat{M}_1 + q\hat{M}_2),\\
\hat{B} &= \frac{1}{2(q-q^{-1})}[q^{2\nu}(\hat{R}_1 + q^{-1}\hat{R}_2) - q^{-2\nu}(\hat{R}_1 + q\hat{R}_2)],\\
\hat{C} &= 2\hat{L},\\
\hat{D} &= q^{\nu}(\hat{M}_1 + q^{-1}\hat{M}_2).
\end{aligned} \tag{11.5.8}$$

Proposition 11.11. *The operators $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ obey the quadratic relations*

$$\begin{gathered}
\hat{A}\hat{B} = q\hat{B}\hat{A}, \qquad \hat{B}\hat{D} = q\hat{D}\hat{B}, \qquad \hat{C}\hat{A} = q\hat{A}\hat{C}, \qquad \hat{D}\hat{C} = q\hat{C}\hat{D},\\
[\hat{B}, \hat{C}] = \frac{\hat{A}^2 - \hat{D}^2}{q - q^{-1}}, \qquad [\hat{A}, \hat{D}] = 0
\end{gathered} \tag{11.5.9a}$$

along with the additional relation

$$\hat{A}\hat{D} = \hat{D}\hat{A} = 1 \tag{11.5.9b}$$

which define $U_q(\mathfrak{sl}_2)$.

When ν is an integer or a half-integer, one obtains finite-dimensional representations of $U_q(\mathfrak{sl}_2)$ of dimension $2\nu + 1$. In that case, the maximal degree of the polynomials obtained from the action of the raising operator $\hat{B}$ is N.

Remark 11.12. *The $q \to 1$ limit of this realization yields the $\mathfrak{sl}_2$ commutation relations. In fact* (11.5.8) *tends to the differential Bargmann realization of $\mathfrak{sl}_2$. Under the limit, the q-linear grid becomes the continuum, and the above combinations of shift operators turn into differential operators.*

Remark 11.13. *The algebra* (11.3.8) *has been obtained in [58] as a so-called "homogenized $\mathfrak{sl}_2$ algebra" $H(\mathfrak{sl}_2)$. Many algebras of a similar type with 4 generators A, B, C, D, and D central, have been studied in [59]. A quantization of $H(\mathfrak{sl}_2)$ which is isomorphic to the algebra with relations* (11.5.9a) *and which can be seen as a homogenization of $U_q(\mathfrak{sl}_2)$ has been studied in [60].*

11.5.4. Finite-dimensional representations

We now wish to obtain finite-dimensional representations of $U_q(\mathfrak{sl}_2)$ corresponding to a particular truncation of the Jacobi matrix of the Big q-Jacobi polynomials. As mentioned previously, this can be accomplished by taking ν to be either an integer or a half-integer. In order to do so, we are led to take [61]

$$\sqrt{abcd} = q^{1-N}, \tag{11.5.10}$$

where N is a positive integer that corresponds to the maximal degree of the truncated family of polynomials.

Proposition 11.14. *The polynomials that arise from the truncation condition* (11.5.10) *form a basis that supports* $(N+1)$*-dimensional representations of* $U_q(\mathfrak{sl}_2)$ *in the realization* (11.5.8)*. The q-para-Krawtchouk polynomials [57] are the ones that arise from this truncation condition.*

We show below how their recurrence relation is obtained from the one of the Big q-Jacobi polynomials.

11.5.4.1. $N = 2j+1$ odd. In the case where $N = 2j+1$ is odd, we write

$$d = a^{-1}q^{-2j+e_1 t}, \qquad b = c^{-1}q^{-2j+e_1 t} \tag{11.5.11}$$

and then take the limit $t \to 0$. Using this parametrization, the recurrence relation of the Big q-Jacobi polynomials

$$zP_n(z;a,b,c;\tilde{q}) = A_n P_{n+1}(z;a,b,c;\tilde{q}) + C_n P_{n-1}(z;a,b,c;\tilde{q}) + [1-(A_n+C_n)]P_n(z;a,b,c;\tilde{q}) \tag{11.5.12}$$

has for coefficients

$$A_n = +\frac{(1-acq^{2n})(1-q^{2n-2N})}{(1+q^{2n-N+1})(1-q^{4n-2N})}, \tag{11.5.13a}$$

$$C_n = -\frac{q^{2n-N-1}(1-q^{2n})(ac-q^{2n-2N})}{(1+q^{2n-N-1})(1-q^{4n-2N})} \tag{11.5.13b}$$

after the use of (11.5.11) and the limit $t \to 0$. Now letting

$$ac = c_3 q^2 \tag{11.5.14}$$

it follows that (11.5.13) can be rewritten as

$$A_n = +\frac{(1-c_3q^{2n+2})(1-q^{2n-2N})}{(1+q^{2n-N+1})(1-q^{4n-2N})}, \tag{11.5.15a}$$

$$C_n = -\frac{q^{2n-N+1}(1-q^{2n})(c_3-q^{2n-2N-2})}{(1+q^{2n-N-1})(1-q^{4n-2N})}, \tag{11.5.15b}$$

and one recognizes the recurrence coefficients of the q-para-Krawtchouk polynomials in the base $\tilde{q} = q^2$ introduced in [57] when N is odd. These polynomials are defined on the union of two q-linear lattices and the parameter c_3 describes the shift of one lattice with respect to the other.

11.5.4.2. $N = 2j$ even. In the case where $N = 2j$ is even, we take

$$d = a^{-1}q^{-2j+e_1 t}, \qquad b = c^{-1}q^{-2j+e_2 t+2} \tag{11.5.16}$$

which ensures (11.5.10) in the limit $t \to 0$. Using this parametrization and after letting $t \to 0$, the recurrence coefficients of the Big q-Jacobi polynomials become

$$A_n = +\frac{(1-acq^{2n})(1-q^{2n-2N})}{(1+q^{2n-N})(1-q^{4n-2N+2})}, \tag{11.5.17a}$$

$$C_n = -\frac{q^{2n-N-2}(1-q^{2n})(ac-q^{2n-2N})}{(1+q^{2n-N})(1-q^{4n-2N-2})}, \tag{11.5.17b}$$

and upon letting

$$ac = c_3 q^2 \tag{11.5.18}$$

A_n and C_n can be rewritten as

$$A_n = +\frac{(1-c_3 q^{2n+2})(1-q^{2n-2N})}{(1+q^{2n-N})(1-q^{4n-2N+2})}, \tag{11.5.19a}$$

$$C_n = -\frac{q^{2n-N}(1-q^{2n})(c_3-q^{2n-2N-2})}{(1+q^{2n-N})(1-q^{4n-2N-2})}. \tag{11.5.19b}$$

These are the recurrence coefficients of the q-para-Krawtchouk polynomials in the base $\tilde{q} = q^2$ for N even. For more detail, see [57].

11.5.4.3. A remark on the truncation condition. There is once again an apparent mismatch in the dimensions of the representations of the algebra and those of the representation basis. The same remark as the one made in the preceding section applies here. It can be checked that in the realization (11.5.8), applying the truncation condition (11.5.10) seems to suggest that the raising operator $\hat{B}$ annihilates the monomial z^{N+1} and not z^N, which means that the truncation condition leads to representations of the algebra $U_q(\mathfrak{sl}_2)$ of dimension $N+2$. This would contradict the fact that the q-para-Krawtchouk polynomials were truncated to a maximal degree N (and thus span a space of dimension $N+1$).

It can be observed that $\hat{B}$ maps the q-para-Krawtchouk polynomial of degree N to a polynomial of degree $N+1$. The resulting polynomial is the characteristic polynomial of the (upper block of the) truncated Jacobi matrix, hence it is again null on the orthogonality grid points. In the representation basis with which we are working (i.e. where the q-para-Krawtchouk polynomials are the basis elements), this characteristic polynomial corresponds to a null vector. Hence, the dimension of the space on which the realization of the $U_q(\mathfrak{sl}_2)$ algebra acts is indeed $N+1$.

11.5.5. Recovering the related Heun operator

The Heun operator associated to the Big q-Jacobi polynomials is given in [42] and had also been introduced previously in [36]. This operator W_{BJ} is the most general second order q-difference operator that acts on the q-linear grid and maps polynomials of degree n into polynomials of degree $n+1$. Its expression is

$$W_{BJ} = \mathcal{A}_1 \hat{T}_+ + \mathcal{A}_0 \mathcal{I} + \mathcal{A}_2 \hat{T}_-, \tag{11.5.20}$$

where

$$\mathcal{A}_1 = \frac{\pi_3(z)}{z^2}, \qquad \mathcal{A}_2 = \frac{\tilde{q}\pi_3(z) + z\pi_2(z)}{z^2} \tag{11.5.21}$$

and $\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2 = \pi_1(z)$, with $\pi_k(z)$ a generic polynomial of degree k and $\tilde{q}$ the base.

Let us consider the most general quadratic combination of S-Heun operators that does not raise the degree of polynomials by more than one. Using the quadratic homogeneous relations of Appendix 11.A, we arrive at

$$\begin{aligned} W = {}& \alpha_1 \hat{L}^2 + \alpha_2 \hat{L}\hat{M}_1 + \alpha_3 \hat{L}\hat{M}_2 + \alpha_4 \hat{M}_1^2 + \alpha_5 \hat{M}_1\hat{M}_2 + \alpha_6 \hat{M}_2^2 \\ & + \beta_1 \hat{M}_1 \hat{R}_2 + \beta_2 \hat{M}_2 \hat{R}_1 + \beta_3 \hat{M}_2 \hat{R}_2. \end{aligned} \tag{11.5.22}$$

Substituting the expressions (11.2.14) for the generators we obtain

$$\begin{aligned} & W = \mathcal{A}_1 \hat{T}_+^2 + \mathcal{A}_0 \mathcal{I} + \mathcal{A}_2 \hat{T}_-^2, \\ & \mathcal{A}_1 = \frac{z^{-2}}{(1-q^2)^2}[(q\alpha_1) + (q^2\alpha_3 - q\alpha_2)z + (q^2\alpha_6 - q\alpha_5 + \alpha_4)z^2 + (q^3\beta_3 - q^2\beta_1 - q^2\beta_2)z^3], \\ & \mathcal{A}_2 = \frac{z^{-2}}{(1-q^2)^2}[(q^3\alpha_1) + (q^2\alpha_3 - q^3\alpha_2)z + (q^2\alpha_6 - q^3\alpha_5 + q^4\alpha_4)z^2 + (q\beta_3 - q^2\beta_1 - q^2\beta_2)z^3], \\ & \mathcal{A}_0 = \beta_2 z + \alpha_4 - (\mathcal{A}_1 + \mathcal{A}_2). \end{aligned} \tag{11.5.23}$$

Proposition 11.15. *The generic Heun–Big q-Jacobi operator* (11.5.20) *(with base q^2) can be obtained as the most general quadratic combination in the S-Heun generators* (11.2.14) *that does not raise the degree of polynomials by more than one.*

Moreover, using the relations of Appendix 11.A, we see that the Heun operator typically factorizes as the product of a raising S-Heun operator with a stabilizing S-Heun operator:

$$\hat{W} = (\xi_1 \hat{L} + \xi_2 \hat{M}_1 + \xi_3 \hat{M}_2)(\eta_1 \hat{L} + \eta_2 \hat{M}_1 + \eta_3 \hat{M}_2 + \eta_4 \hat{R}_1 + \eta_5 \hat{R}_2) + \kappa. \tag{11.5.24}$$

11.6. Connections between the different cases

It is well known that the three grids on which we have defined S-Heun operators can be obtained as limiting cases or contractions of the Askey–Wilson grid. We now observe that this translates into limits/contractions of the associated Sklyanin algebras.

Let us denote the points of the Askey–Wilson grid by

$$\lambda_s = z_s + z_s^{-1}, \qquad z_s = q^s. \tag{11.6.1}$$

The associated Sklyanin algebra was introduced in [23] as the trigonometric degeneration of the Sklyanin algebra [24] and was studied from the perspective of S-Heun operators in [25]. The defining relations read

$$\begin{gathered}\mathbf{DC} = q\mathbf{CD}, \qquad \mathbf{CA} = q\mathbf{AC}, \qquad [\mathbf{A},\mathbf{D}] = \frac{(q-q^{-1})^3}{4}\mathbf{C}^2,\\ [\mathbf{B},\mathbf{C}] = \frac{\mathbf{A}^2-\mathbf{D}^2}{q-q^{-1}},\\ \mathbf{AB}-q\mathbf{BA} = q\mathbf{DB}-\mathbf{BD} = -\frac{q^2-q^{-2}}{4}(\mathbf{DC}-\mathbf{CA}).\end{gathered} \tag{11.6.2}$$

The q-linear (or exponential) grid

$$\lambda_s = z_s, \qquad z_s = q^s \tag{11.6.3}$$

is obtained from the Askey–Wilson one in the asymptotic expansion $z_s \to \infty$ and the same limit takes the Askey–Wilson polynomials into the Big q-Jacobi OPs. At the level of the algebras, this corresponds to the following contraction. Writing

$$\mathbf{A} = \epsilon\hat{A}, \qquad \mathbf{B} = \hat{B}, \qquad \mathbf{C} = \epsilon^2\hat{C}, \qquad \mathbf{D} = \epsilon\hat{D} \tag{11.6.4}$$

and taking $\epsilon \to 0$, one recovers $U_q(\mathfrak{sl}_2)$:

$$\begin{gathered}\hat{A}\hat{B} = q\hat{B}\hat{A}, \qquad \hat{B}\hat{D} = q\hat{D}\hat{B}, \qquad \hat{C}\hat{A} = q\hat{A}\hat{C}, \qquad \hat{D}\hat{C} = q\hat{C}\hat{D},\\ [\hat{B},\hat{C}] = \frac{\hat{A}^2-\hat{D}^2}{q-q^{-1}}, \qquad [\hat{A},\hat{D}] = 0.\end{gathered} \tag{11.6.5}$$

We now compare the discrete linear grid to the continuum. A rescaling similar to the one discussed above takes this grid to the real line. This also takes the Continuous Hahn polynomials into the Jacobi ones. From the perspective of the algebras, (11.6.4) will relate one algebra to the other. The Sklyanin algebra (11.4.8) associated to the discrete grid is

$$\begin{gathered}[C,D] = 0, \qquad [A,C] = \{C,D\}, \qquad [A,D] = \{C,C\},\\ [B,C] = \{D,A\}, \qquad [B,D] = \{C,A\}, \qquad [B,A] = \{B,D\}\end{gathered} \tag{11.6.6}$$

and upon writing

$$A = \epsilon\bar{A}, \qquad B = \bar{B}, \qquad C = \epsilon^2\bar{C}, \qquad D = \epsilon\bar{D} \tag{11.6.7}$$

and taking $\epsilon \to 0$, we recover

$$\begin{gathered}[\bar{C},\bar{D}] = 0, \qquad [\bar{A},\bar{C}] = -\bar{C}\bar{D}, \qquad [\bar{A},\bar{D}] = 0,\\ [\bar{B},\bar{C}] = -2\bar{A}\bar{D}, \qquad [\bar{A},\bar{B}] = \bar{B}\bar{D}, \qquad [\bar{B},\bar{D}] = 0.\end{gathered} \tag{11.6.8}$$

We recall that the latter algebra is essentially the $\mathfrak{sl}_2$ Lie algebra with a central element D.

We have so far discussed the following contractions, denoted by full arrows:

$$\begin{tikzcd}
\boxed{\text{AW grid}} \arrow[r, dotted, "?"] \arrow[d] & \boxed{\text{discrete linear grid}} \arrow[d] \\
\boxed{q\text{-linear grid}} \arrow[r, dotted] & \boxed{\text{continuum}}
\end{tikzcd}$$

One could wonder if it is possible to complete the diagram with the dotted arrows. The bottom arrow is easy to add: this amounts to taking the limit $q \to 1$. This limit takes the q-linear grid to the continuum, the Big q-Jacobi polynomials to the Jacobi polynomials, and at the level of the algebra, it takes $U_q(\mathfrak{sl}_2)$ to $\mathfrak{sl}_2$.

The details corresponding to the upper arrow remain to be worked out. It is likely that an intermediary step related to the quadratic grid $\lambda_s = s^2$ should be required. Indeed, it is known that the $q \to 1$ limit of the Askey–Wilson grid leads to the quadratic grid. It should thus be possible to apply the S-Heun construction to the quadratic grid; the related polynomials should be those of Wilson, and the related Sklyanin algebra would stand in between the one of Askey–Wilson type (11.6.2) and the one of the discrete linear type (11.4.8).

11.7. Conclusion

The results of this paper are summarized as follows. We have introduced S-Heun operators on linear and q-linear grids. These operators are special cases of second order Heun operators with no diagonal term. On the real line and the discrete and q-linear grids, the sets of five S-Heun operators were constructed and shown to be related to the Jacobi, Continuous Hahn and Big q-Jacobi polynomials respectively. These S-Heun operators were also shown to encompass the bispectral and structure operators for each family of orthogonal polynomials. A presentation of the relations for the four structure operators of Kalnins and Miller was given in each case and identified as realizing degenerations, contractions or limits of the Sklyanin algebra. For the discrete and q-linear grids, the finite-dimensional representations of the Sklyanin-type algebras were obtained from a truncation condition on the Jacobi matrix of the associated polynomials; this yielded the para-Krawtchouk and q-para-Krawtchouk polynomials as bases of the finite representations and provided algebraic interpretations of these sets of OPs that had so far been missing.

The Sklyanin-like algebra related to the discrete linear grid (11.4.8) has a simple presentaton and a detailed study of its representation theory would be interesting. It would also be instructive to examine the types of Sklyanin algebra that the S-Heun operators on the quadratic grid would lead to. We plan on undertaking this in the near future. Note that we have restricted ourselves to Heun operators defined by actions on polynomials. The exploration of the generalizations that result from the extension to spaces of rational functions have been initiated in [41] and should be actively pursued in the S-Heun framework in particular.

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11.A. The homogeneous quadratic algebraic relations

The 14 quadratic homogeneous relations associated to all three sets of 5 S-Heun operators are collected here. One notes that all three sets of relations display a similar structure. These relations can be thought of as reordering relations and are especially useful when considering the most general quadratic combinations in the generators.

11.A.1. The continuum

The relations between the S-Heun operators $\bar{L}$, $\bar{M}_1$, $\bar{M}_2$, $\bar{R}_1$, $\bar{R}_2$ defined in (11.2.17) can be presented as the fourteen following relations:

$$\begin{aligned}
&\bar{M}_1\bar{L} = \bar{L}\bar{M}_1, && \bar{L}\bar{R}_1 = 1 + \bar{M}_1\bar{M}_2, && \bar{R}_1\bar{M}_1 = \bar{M}_2\bar{R}_1 - \bar{M}_1\bar{R}_2,\\
&\bar{M}_2\bar{L} = \bar{L}\bar{M}_2 - \bar{M}_1\bar{L}, && \bar{L}\bar{R}_2 = \bar{M}_2^2 + \bar{M}_1\bar{M}_2, && \bar{R}_2\bar{M}_1 = \bar{M}_1\bar{R}_2,\\
&\bar{M}_2\bar{M}_1 = \bar{M}_1\bar{M}_2, && \bar{R}_1\bar{L} = \bar{M}_1\bar{M}_2, && \bar{R}_1\bar{M}_2 = \bar{M}_1\bar{R}_2,\\
&\bar{M}_1^2 = 1, && \bar{R}_2\bar{L} = \bar{M}_2^2 - \bar{M}_1\bar{M}_2, && \bar{R}_2\bar{M}_2 = \bar{M}_2\bar{R}_2 - \bar{M}_1\bar{R}_2,\\
& && \bar{R}_2\bar{R}_1 = \bar{R}_1\bar{R}_2 + \bar{R}_1^2, && \bar{M}_1\bar{R}_1 = \bar{M}_2\bar{R}_1 - \bar{M}_1\bar{R}_2.
\end{aligned} \tag{11.A.1}$$

11.A.2. The discrete linear grid

Here are the relations between the S-Heun operators L, M_1, M_2, R_1, R_2 that have been defined in (11.2.8):

$$\begin{aligned} M_1L &= LM_1, & LR_1 &= 1 - 2{M_2}^2 - M_1M_2,\\ M_2L &= LM_2 - LM_1, & LR_2 &= 1 + M_1M_2,\\ M_2M_1 &= M_1M_2 - L^2, & R_1L &= 3M_1M_2 - 3L^2 - 2{M_2}^2,\\ {M_1}^2 &= 1 + L^2, & R_2L &= M_1M_2 - L^2, \end{aligned}$$

$$\begin{aligned} R_1M_1 &= 3M_1R_2 - 2M_2R_2 - 3LM_1,\\ R_1M_2 &= 2M_2R_2 - 3M_1R_2 + 3LM_2 + M_2R_1,\\ R_2M_1 &= M_1R_2 - LM_1,\\ R_2M_2 &= M_2R_2 - M_1R_2 + LM_2,\\ M_1R_1 &= 3M_1R_2 - 2M_2R_2 - 4LM_2,\\ R_2R_1 &= 2{R_2}^2 + R_1R_2 - 4{M_2}^2. \end{aligned} \tag{11.A.2}$$

11.A.3. The q-linear grid

We remind the reader that the q-number 2 is written as $[2]_q = q + q^{-1}$. The S-Heun operators $\hat{L}$, $\hat{M}_1$, $\hat{M}_2$, $\hat{R}_1$, $\hat{R}_2$ defined in (11.2.14) obey the fourteen quadratic relations:

$$\begin{aligned} \hat{M}_1\hat{L} &= [2]_q\hat{L}\hat{M}_1 + \hat{L}\hat{M}_2, & \hat{L}\hat{R}_1 &= 1 - \hat{M}_2^2,\\ \hat{M}_2\hat{L} &= -\hat{L}\hat{M}_1, & \hat{L}\hat{R}_2 &= [2]_q\hat{M}_2^2 + \hat{M}_1\hat{M}_2,\\ \hat{M}_2\hat{M}_1 &= \hat{M}_1\hat{M}_2, & \hat{R}_1L &= 1 - \hat{M}_1^2,\\ [2]_q\hat{M}_1\hat{M}_2 &= 1 - \hat{M}_1^2 - \hat{M}_2^2, & \hat{R}_2\hat{L} &= -\hat{M}_1\hat{M}_2, \end{aligned}$$

$$\begin{aligned} \hat{R}_1\hat{M}_1 &= -[2]_q^2\hat{M}_1\hat{R}_2 - [2]_q\hat{M}_2\hat{R}_2 + \hat{M}_2\hat{R}_1,\\ \hat{R}_1\hat{M}_2 &= [2]_q\hat{M}_1\hat{R}_2 + \hat{M}_2\hat{R}_2,\\ \hat{R}_2\hat{M}_1 &= [2]_q\hat{M}_1\hat{R}_2 + \hat{M}_2\hat{R}_2,\\ \hat{R}_2\hat{M}_2 &= -\hat{M}_1\hat{R}_2,\\ \hat{M}_1\hat{R}_1 &= -[2]_q\hat{M}_1\hat{R}_2 - \hat{M}_2\hat{R}_2,\\ [2]_q\hat{R}_1\hat{R}_2 &= -\hat{R}_1^2 - \hat{R}_2^2. \end{aligned} \tag{11.A.3}$$

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Chapitre 12

The rational Sklyanin algebra and the Wilson and para-Racah polynomials

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Abstract: The relation between Wilson and para-Racah polynomials and representations of the degenerate rational Sklyanin algebra is established. Second order Heun operators on quadratic grids with no diagonal terms are determined. These special or S–Heun operators lead to the rational degeneration of the Sklyanin algebra; they also entail the contiguity and structure operators of the Wilson polynomials. The finite-dimensional restriction yields a representation that acts on the para-Racah polynomials.

12.1. Introduction

This paper pursues the exploration of the links between Heun operators, Sklyanin algebras and orthogonal polynomials. Originally introduced in the context of quantum integrable systems [1], Sklyanin algebras are typically presented in terms of generators verifying homogeneous quadratic relations. These algebras have been the object of much attention from the perspective of algebraic geometry [2–4]. Classes of Heun operators can be defined [5] from the property that they increase by no more than one the degree of polynomials defined on certain continuous or discrete domains; they have been the focus of a continued research effort [6–12] with many applications [13–19]. A key observation for our purposes is that a special category of these operators, referred to as S–Heun operators, offers a path towards the identification of interesting Sklyanin-like algebras through the relations they realize. This connects with orthogonal polynomials as these concrete S–Heun operators are recognized as ladder and structure operators for families of bispectral polynomials belonging to the Askey scheme. It is thus observed that these sets of orthogonal polynomials form representation bases for Sklyanin algebras. Furthermore, the finite-dimensional representations of these Sklyanin algebras are found to provide the algebraic setting that had so far been lacking for the orthogonal polynomials of the so-called "para" type.

A first illustration of these connections was achieved in [20]. Building on results of Gorsky and Zabrodin [21] on the one hand and of Kalnins and Miller [22] on the other, this paper focused on S–Heun operators attached to the Askey–Wilson grid. The salient observations were: *i.* that a subset of the S–Heun operators realize the trigonometric degeneration of the original elliptic Sklyanin algebra and *ii.* that this Sklyanin algebra is a basic structure underneath the theory of Askey–Wilson polynomials. Indeed, as was stressed, the Askey–Wilson operator admits a factorization in terms of the S–Heun operators realizing this degenerate Sklyanin algebra and as was also pointed out, the ladder and structure operators for the Askey–Wilson polynomials obtained by Kalnins and Miller actually realize this degenerate algebra. In view of the fact that the Askey–Wilson algebra [23] accounts for the bispectrality of the eponym polynomials, a parallel was thus drawn with the dynamical extension of symmetry algebras by the inclusion of ladder operators in the set of generators. Finally, the q-para Racah polynomials were seen to form a basis for the finite-dimensional representation of the degenerate Sklyanin algebra. This set the course for the systematic examination of the Sklyanin-like operators formed by S–Heun operators on lattices admitting orthogonal polynomials.

The study of S–Heun operators on linear and exponential grid and of the Sklyanin algebras they realize was carried out in [24]. It allowed to tie the representations of these algebras to the continuous Hahn and big q-Jacobi polynomials and in finite dimensions to the para-Krawtchouk and q-para Krawtchouk polynomials. This analysis confirmed the important role that Sklyanin algebras play in the interpretation of hypergeometric orthogonal polynomials.

We here address the connection that the Wilson polynomials have with Sklyanin algebras. (We recall that these polynomials are at the top of the $q = 1$ part of the Askey scheme.) This will call for the determination of the S–Heun operators on quadratic grids. The rational degeneration of the Sklyanin algebra first found by Smirnov [25] will be seen to emerge and to be realized by the structure and ladder operators [26] of the Wilson polynomials. This will hence attach these polynomials to representations of the rational Sklyanin algebra. In keeping with preceding observations, the finite-dimensional restrictions of these representations will be seen to offer an algebraic interpretation of the para-Racah polynomials [27].

12.1.1. The Wilson polynomials and its truncations

As the Wilson polynomials will prove central in deriving subsequent results, some of their known properties are summarized here. The four-parameter Wilson polynomials [28] of degree n, denoted $W_n(x^2|a,b,c,d)$, are given by

$$W_n(x^2|a,b,c,d) = (a+b)_n(a+c)_n(a+d)_n\,{}_4F_3\left(\begin{matrix}-n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d\end{matrix}\middle| 1\right),$$

where $(a)_n = a(a+1)...(a+n-1)$ are the Pochhammer symbols and $0 < a,b,c,d \in \mathbb{R}$. These polynomials obey the orthogonality relation

$$\int_0^\infty W_n(x^2|a,b,c,d)W_m(x^2|a,b,c,d)\mathrm{d}\omega(x|a,b,c,d) = N_n(a,b,c,d)\delta_{n,m}. \tag{12.1.1}$$

The weight $\omega(x|a,b,c,d)$ and normalization $N_n(a,b,c,d)$ are given explicitly in [28]. For any admissible set of parameters, the Wilson polynomials form a basis of the space of polynomials on the support of $\omega(x|a,b,c,d)$. Belonging to the Askey–Wilson scheme, they are bispectral, that is, they diagonalize a three-term recurrence operator acting on the degree and a difference operator acting on the variable.

The Wilson polynomials form an infinite set of orthogonal polynomials that can be truncated [28] to a finite one by setting the parameters as follows

$$a = \tfrac{1}{2}(\gamma+\delta+1), \quad b = \tfrac{1}{2}(2\alpha-\gamma-\delta+1), \quad c = \tfrac{1}{2}(2\beta-\gamma+\delta+1), \quad d = \tfrac{1}{2}(\gamma-\delta+1),$$

and imposing any of the conditions

$$\alpha+1 = -N, \qquad \beta+\delta+1 = -N, \qquad \text{or} \qquad \gamma+1 = -N.$$

One thus obtains the Racah polynomials after taking

$$ix \longmapsto x + \tfrac{1}{2}(\gamma+\delta+1).$$

An additional truncation can be obtained [27] by imposing

$$a+b+c+d = -N+1. \tag{12.1.2}$$

Indeed, while one is at first sight led to singular expressions, well-defined orthogonal polynomials can nonetheless be obtained through the use of limits and the resulting polynomials, first introduced in [**lemay2016**], are the para-Racah polynomials. These polynomials form a three-parameter set of orthogonal polynomials $P_n(x^2|a,c,w)$ of maximal degree N. Explicit expressions can be found by setting $N = 2j+p$, where $j \in \mathbb{N}$ and $p = 0,1$, depending on the parity of N. The para-Racah polynomial $P_n(x^2|a,c,w)$ obtained from the truncation

(12.1.2) of the Wilson polynomial $W_n(x^2|a,b,c,d)$ is given by

$$P_n(x^2|a,c,w) = \eta_n \sum_{k=0}^{n} A_{n,k}\varphi_k(x^2), \qquad \varphi_k(x^2) \equiv (a-ix)_k(a+ix)_k, \tag{12.1.3}$$

where

$$A_{n,k} = \begin{cases} \frac{(-n)_k(n-N)_k}{(1)_k(-j)_k(a+c)_k(a-c-j+1-p)_k} & k \leq j, \\ \frac{w^{-1}(-n)_k(n-N)_{N-n}(1)_{n+k-1-N}}{(1)_k(-j)_j(1)_{k-j-1}(a+c)_k(a-c-j+1-p)_k} & k > j, \\ 0 & k > n, \end{cases} \tag{12.1.4}$$

with the normalization given by

$$\eta_n = \begin{cases} \frac{(1)_n(-j)_n(a+c)_n(a-c-j+1-p)_n}{(-n)_n(n-N)_n} & n \leq j, \\ \frac{w(1)_n(-j)_j(1)_{n-j-1}(a+c)_n(a-c-j+1-p)_n}{(-n)_n(n-N)_{N-n}(1)_{2n-1-N}} & n > j. \end{cases} \tag{12.1.5}$$

These polynomials are orthogonal on a discrete measure that has support on the zeros of the characteristic polynomial $P_{N+1}(x^2|a,c,w)$. The corresponding lattice is a quadratic bi-lattice given by

$$x_{2s+t} = \begin{cases} -(s+a)^2 & t=0,\ s=0,1,\ldots,j, \\ -(s+c)^2 & t=1,\ s=0,1,\ldots,j-1+p, \end{cases} \tag{12.1.6}$$

so that

$$\sum_{s=0}^{N} P_n(x^2|a,c,w)P_m(x^2|a,c,w)\bar{\omega}_s \propto \delta_{n,m}, \tag{12.1.7}$$

where the weight $\bar{\omega}_s$ is given explicitly in [27]. They also satisfy a three-term recurrence relation and a difference equation. However, they do not appear in classifications of classical orthogonal polynomials as their spectrum is doubly-degenerate.

12.1.2. Outline

The remainder of the paper is organized as follows. In section 12.2, the S–Heun operators are introduced and some of their properties are derived. The connection is made with the algebraic Heun operator of the Wilson/Racah type. Section 12.3 focuses on a subset of the S–Heun operators that preserves the degree of polynomials. A stabilizing algebra is defined from the quadratic relations they obey and its representations are constructed. This algebra is extended to a star algebra in section 12.4 for which a universal presentation is obtained; it is subsequently recognized as a Sklyanin-type algebra. Finally, section 12.5 provides a presentation of the rational degenerate Sklyanin algebra introduced in [25] and gives an isomorphism with the universal algebra of section 12.4. Using this isomorphism, representations of the rational degenerate Sklyanin algebra on the Wilson and para-Racah polynomials are constructed. A brief conclusion follows.

12.2. Sklyanin–Heun operators on a quadratic grid

The generic algebraic Heun operators on a domain λ have the property that, when acting on polynomials over λ, they raise the degree by at most one. The S–Heun operators are a specialization of these Heun operators without a diagonal term. In this section, we first identify the S–Heun operators on the quadratic grid and then proceed with a brief characterization.

12.2.1. Sklyanin–Heun operators

Let $\lambda = \lambda_x$ be a discrete grid indexed by x and define the shift operators $T^{\pm}$ acting on functions on λ as follows

$$T^{\pm} f(\lambda_x) \equiv f(\lambda_{x\pm1}).$$

Consider a second order operator S with no diagonal term

$$S = A_1(\lambda_x)T^{+} + A_2(\lambda_x)T^{-}, \tag{12.2.1}$$

where A_1 and A_2 are functions on λ_x. Demand that S satisfies the degree-raising property

$$S \cdot p_n(\lambda_x) = q_{n+1}(\lambda_x), \tag{12.2.2}$$

for p_n and q_{n+1} arbitrary polynomials of degree n and $n+1$, respectively. One can determine the coefficients A_1 and A_2 by acting on the first two monomials in λ_x as follows

$$S \cdot 1 = u_0 + u_1\lambda_x, \quad S \cdot \lambda_x = u_2 + u_3\lambda_x + u_4\lambda_x^2. \tag{12.2.3}$$

One finds

$$\begin{aligned} A_1(\lambda_x) &= \frac{u_2 + u_3\lambda_x + u_4\lambda_x^2 - u_0\lambda_{x-1} - u_1\lambda_{x-1}\lambda_x}{\lambda_{x+1} - \lambda_{x-1}}, \\ A_2(\lambda_x) &= -\frac{u_2 + u_3\lambda_x + u_4\lambda_x^2 - u_0\lambda_{x+1} - u_1\lambda_{x+1}\lambda_x}{\lambda_{x+1} - \lambda_{x-1}}. \end{aligned} \tag{12.2.4}$$

The S–Heun operators are defined as the set of operators of the form (12.2.1) with the coefficients given in (12.2.4). As these coefficients admit five independent parameters, the S–Heun operators form a five-dimensional vector space $\mathcal{SH}$ of operators on λ. A basis for this space can be chosen as follows

$$
\begin{aligned}
L &= \mathcal{N}(\lambda_x)[T^+ - T^-], \\
M_1 &= \mathcal{N}(\lambda_x)\left[(\lambda_x - \lambda_{x-1})T^+ + (\lambda_{x+1} - \lambda_x)T^-\right], \\
M_2 &= \mathcal{N}(\lambda_x)\left[(\lambda_x + \lambda_{x-1})T^+ - (\lambda_{x+1} + \lambda_x)T^-\right], \\
R_1 &= \mathcal{N}(\lambda_x)\lambda_x\left[(\lambda_x - \lambda_{x-1})T^+ + (\lambda_{x+1} - \lambda_x)T^-\right], \\
R_2 &= \mathcal{N}(\lambda_x)\lambda_x\left[(\lambda_x + \lambda_{x-1})T^+ - (\lambda_{x+1} + \lambda_x)T^-\right],
\end{aligned}
\tag{12.2.5}
$$

where

$$
\mathcal{N}(\lambda_x) \equiv [\lambda_{x+1} - \lambda_{x-1}]^{-1}.
$$

The naming conventions used in (12.2.5) will be explained in the next subsection.

Remark 12.1. *Acting on the left with T^+ for each of the operators in* (12.2.5)*, it can be seen that the set of operators $\mathcal{SH}$ can also be understood as the set of first order shift operators of step two over λ that satisfies the property* (12.2.2).

12.2.2. Sufficiency of the construction

As established above, for an operator S of the form (12.2.1) to satisfy the property (12.2.2), the expressions (12.2.4) are necessary conditions. The sufficiency of these conditions follows from the ensuing proposition.

Proposition 12.2. *A generic element $S \in \mathcal{SH}$ satisfies the property* (12.2.2) *if the grid λ_x is of one of the following forms*

$$
\lambda_x = \alpha q^x + \beta q^{-x} + \kappa, \qquad \lambda_x = \alpha x^2 + \beta x + \kappa, \qquad \textit{or} \qquad \lambda_x = (-1)^x(\alpha x + \beta) + \kappa,
\tag{12.2.6}
$$

for some constants α, β, κ.

Proof. An element $S \in \mathcal{SH}$ of the form (12.2.1) is specified by a set of parameters $\{u_i\}_{i=0,1,\cdots,4}$ (12.2.4). The action of S on a monomial in λ_x can be reduced by linearity to the five cases given by $u_i = \delta_{i,j}$ for $j = 0, 1, \ldots, 4$. Upon inspecting (12.2.4), one understands that only the operators defined by $u_i = \delta_{i,0}$ or $u_i = \delta_{i,2}$ need to be analyzed; those remaining amount to one of these two operators multiplied by some power of λ_x.

The first case we treat is $u_i = \delta_{i,2}$ and it corresponds to the operator we have denoted L. It can be seen from (12.2.4) that when $u_i = \delta_{i,3}$ or $u_i = \delta_{i,4}$, the corresponding operator is $\lambda_x L$ or ${\lambda_x}^2 L$, respectively. It follows that for S to satisfy property (12.2.2), one must have that L decreases the degree of polynomials in λ_x by one. Similarly, it follows from (12.2.4) that the case $u_i = \delta_{i,1}$ will satisfy (12.2.2) if the case of $u_i = \delta_{i,0}$, corresponding to the

S–Heun operator $\frac{1}{2}(M_1 - M_2)$, is an operator that stabilizes the set of polynomials of a given degree.

Thus, a generic element of the form (12.2.1) will satisfy (12.2.2) if the subset of operators generated by the cases $u_i = \delta_{i,j}$ for $j = 0, 2, 3$ preserves the degree of polynomials. As the generators of this subset are all tridiagonal operators, the proposition follows from the results in [29] which identifies (12.2.6) as the possible grids allowing second-order difference equations diagonalized by polynomials. □

On the quadratic grid, it can be shown that a generic element of the vector space spanned by (12.2.5) satisfies the property (12.2.2). Indeed, as derived above, the expressions (12.2.4) for the coefficients are necessary conditions. The derivations so far were grid-independent, but to proceed further, one needs to fix the grid. Let us consider the quadratic grid

$$\lambda_x = x^2. \tag{12.2.7}$$

For this choice of grid, one has that proposition 12.2 holds and the sufficiency of the construction is established.

The leading terms of the actions on monomials in λ_x are now computed for future reference. In the case of L, one obtains

$$L \cdot {\lambda_x}^n = \sum_{k=1}^{n} \binom{n}{k} \lambda_x^{n-k} \sum_{\substack{j \text{ odd} \\ 0\leq j\leq k}} \binom{k}{j} (4\,\lambda_x + p^2)^{\frac{j-1}{2}} = n\,{\lambda_x}^{n-1} + O(\lambda_x^{n-2}), \tag{12.2.8}$$

which is verified to be a degree lowering operator. Moreover, one finds that

$$\begin{aligned}\tfrac{1}{2}(M_1 - M_2) \cdot {\lambda_x}^n &= \sum_{\substack{k=0 \\ 0\leq j\leq k}}^{n} \binom{n}{k}\binom{k}{j} \lambda_x^{n-k} \left[\tfrac{1+(-1)^j}{2}(4\lambda_x + p^2)^{\frac{j}{2}} - \tfrac{1-(-1)^j}{2}(\lambda_x + 1)(4\lambda_x + p^2)^{\frac{j-1}{2}}\right] \\ &= (1-n){\lambda_x}^n + O({\lambda_x}^{n-1}),\end{aligned} \tag{12.2.9}$$

which preserves the degree of polynomials. The actions of the other generators follow from (12.2.8) and (12.2.9) by noting that

$$\frac{1}{2}(M_1 + M_2) = \lambda_x\,L, \qquad R_1 = \lambda_x M_1, \qquad R_2 = \lambda_x M_2. \tag{12.2.10}$$

With the above observations, it follows that a generic linear combination of the basis elements (12.2.5) displays the degree raising property (12.2.2). These calculations enable one to see that the choice of basis (12.2.5) decomposes the generic special Heun operator into operators that have a prescribed action on polynomials in λ_x. Indeed, L can be identified as a lowering operator, M_1 and M_2 as stabilizing operators while R_1 and R_2 are raising operators.

12.2.3. S–Heun operators of the Wilson type and the Heun–Racah operator

As the S–Heun operators are specialized algebraic Heun operators [5], they are related to the general algebraic Heun operators associated to the same grid. The Heun–Racah operator W on the quadratic grid introduced in [12] admits a quadratic embedding in the set $\mathcal{SH}$ of S–Heun operators on the quadratic grid. In view of Remark 12.1, it will come as no surprise that this embedding is obtained by first conjugating the Heun–Racah operator W by a scaling of the grid $\mu : x \to 2x$, such that the shift operators in W act with a step of two. One obtains

$$\begin{aligned} \mu^{-1} \circ W \circ \mu = R_1(a_1M_1 + a_2M_2 + a_3L) + R_2(a_4M_1 + a_5M_2 + a_6L) \\ + a_7LM_2 + a_8{M_2}^2 + a_9L^2, \end{aligned} \tag{12.2.11}$$

where the coefficients a_i, $i = 1, 2, \ldots, 9$ are given in terms of the parameters $t_0, t_1, u_0, u_1, u_2, v_0, v_1, v_2$ and v_3 of W in [12] as

$$\begin{gathered} a_1 = \tfrac{1}{4}(t_1 + u_2) - \tfrac{1}{16}v_3, \qquad a_2 = -\tfrac{1}{8}t_1 + 8u_0 + u_1 - 2v_1 + \tfrac{1}{16}v_3, \\ a_3 = \tfrac{1}{4}(-8t_0 - t_1 - 64u_0 - 3u_2 + 16v_1 + 2v_2), \\ a_4 = \tfrac{1}{4}u_2 - a_2, \qquad a_5 = \tfrac{1}{16}v_3, \qquad a_6 = a_3 - 2u_1, \qquad a_7 = 8u_0, \qquad a_8 = t_0, \\ a_9 = -t_0 - 24u_0 + 16v_0. \end{gathered} \tag{12.2.12}$$

The operator X that acts by multiplication by the grid variable λ_x can be written as a quadratic expression in terms of the S–Heun generators:

$$X \equiv x^2 = (R_1 + R_2)(M_1 - L) - \tfrac{1}{2}R_1M_2 - \tfrac{1}{2}R_2M_1. \tag{12.2.13}$$

12.3. The stabilizing subalgebra $\mathfrak{stab}$

By direct computations from the definitions (12.2.5), it can be seen that the S–Heun generators satisfy homogeneous quadratic relations, with the complete list given in the appendix 12.A. From these relations, it is observed that the subset of stabilizing S–Heun operators generated by L, M_1 and M_2 closes as a quadratic algebra to be called $\mathfrak{stab}$ whose relations are

$$[L, M_1] = 2L^2, \qquad [L, M_2] = \{M_1, L\}, \qquad [M_1, M_2] = \{M_2, L\} - 4L^2. \tag{12.3.1}$$

The Casimir element C is given by

$$C = {M_1}^2 - \{M_2, L\} + 3L^2, \tag{12.3.2}$$

and is equal to the identity in the realization (12.2.5) in terms of shift operators. It will prove fruitful to examine the stabilizing algebra (12.3.1) in this realization. Knowing that it stabilizes polynomials in λ_x of a given degree, one may set up an eigenvalue problem on this space.

12.3.1. Diagonalization of a generic linear element

Consider a generic linear combination of the operators L, M_1, M_2

$$P(s,t) = uL + vM_1 + wM_2, \tag{12.3.3}$$

parametrized as follows

$$u = \frac{(1+2s)(1+2t)-1}{4}, \qquad v = \frac{1}{2}(1+s+t), \qquad w = \frac{1}{2},$$

with $0 < s, t \in \mathbb{R}$ being arbitrary parameters. It is straightforward to show that, under the invertible transformation

$$\rho : x \mapsto -ix,$$

the operator P is given by

$$\tilde{P} \equiv \rho \circ P \circ \rho^{-1} = -\frac{1}{4ix}\left[(t-ix)(s-ix)\tilde{T}^{+} - (s+ix)(t+ix)\tilde{T}^{-}\right],$$

with $\tilde{T}^{\pm}$ defined by $\tilde{T}^{\pm} f(x) \mapsto f(x \pm i)$. Multiplying each term in the above by $(2ix \pm 1)/(2ix \pm 1)$, one recognizes the off-diagonal terms of the difference operator diagonalized by the continuous dual Hahn polynomials [28]. Denoting these polynomials as $S_n(x^2|1/2, s, t)$ one has

$$\tilde{P}\, S_n(x^2|1/2, s, t) = (n - (s+t)/2)\, S_n(x^2|1/2, s, t).$$

Once an element is specified by (12.3.3), this defines an eigenbasis in terms of the continuous dual Hahn polynomials. However, no meaningful action can be identified for the remaining elements in $\mathfrak{stab}$. We consider instead quadratic combinations in the elements of the algebra.

12.3.2. Action on Wilson polynomials

A natural action of the stabilizing algebra $\mathfrak{stab}$ on the Wilson polynomials arises from the realization (12.2.5). Indeed, defining the following pair of operators from (12.3.3)

$$\mu^{(a,b,c,d)} = P(2a-1, 2b-1), \qquad \mu^{*(a,b,c,d)} = P(2c, 2d), \tag{12.3.4}$$

such that manifestly

$$\mu^{*(a,b,c,d)} = \mu^{(c+1/2, d+1/2, a-1/2, b-1/2)},$$

one has the following proposition:

Proposition 12.3. *The quadratic element $Q \in \mathfrak{stab}$ defined by*

$$Q \equiv \mu^{*(a,b,c,d)}\mu^{(a,b,c,d)}, \tag{12.3.5}$$

where μ and μ^* are given by (12.3.4) *and with* $0 < a, b \in \mathbb{R}$ *and* $1/2 < c, d \in \mathbb{R}$ *is realized, up to a constant term, by the Wilson operator conjugated by the grid scaling*

$$\phi : x \mapsto -2ix. \tag{12.3.6}$$

PROOF. In the realization (12.2.5), conjugating Q by the scaling transformation (12.3.6), it can be seen by direct calculations that the transformed operator $\tilde{Q}$ is given by

$$\begin{aligned}
\tilde{Q} \equiv \phi \circ Q \circ \phi^{-1} = B(x)\tilde{T}^{+} + D(x)\tilde{T}^{-} - [B(x) + D(x)] + (a+b)(c+d-1),\\
B(x) = \frac{(a-ix)(b-ix)(c-ix)(d-ix)}{2ix(2ix-1)},\\
D(x) = \frac{(a+ix)(b+ix)(c+ix)(d+ix)}{2ix(2ix+1)}.
\end{aligned} \tag{12.3.7}$$

The above operator is identified as the Wilson operator [28], up to a constant term. □

Remark 12.4. *The operator X that acts by multiplication by the variable λ_x can be embedded* (12.2.13) *in the set $\mathcal{SH}$ of S–Heun operators. In addition, with the operator Q identified as the Wilson operator, the bispectral pair of operators that generates the Racah/Wilson algebra [30–32] admits an embedding in the set $\mathcal{SH}$ of S–Heun operators. Moreover, a quartic embedding of the Heun–Racah operator* (12.2.11) *is obtained from the construction of the Heun–Racah operator [12] by the tridiagonalization [33] of the Racah operator.*

The definition of Q in (12.3.5) naturally provides a factorization of the Wilson operator in terms of $\mu^{*(a,b,c,d)}$ and $\mu^{(a,b,c,d)}$. Moreover, it directly follows from proposition 12.3 that the operator $\tilde{Q}$ is diagonalized by the Wilson polynomials:

$$\tilde{Q}\, W_n(x^2|a,b,c,d) = [n(n+a+b+c+d-1) + (c+d)(a+b-1)]\; W_n(x^2|a,b,c,d).$$

Introducing a third operator $\tau^{(a,b,c,d)}$ defined by

$$\tau^{(a,b,c,d)} = 4L, \tag{12.3.8}$$

a presentation of $\mathfrak{stab}$ in terms of the generators $\mu^{(a,b,c,d)}$, $\mu^{*(a,b,c,d)}$ and $\tau^{(a,b,c,d)}$ can be given for generic values of the parameters a, b, c, d. This allows to construct representations of $\mathfrak{stab}$ on the Wilson polynomials.

Proposition 12.5. *A representation of* $\mathfrak{stab}$ *on the Wilson polynomials* $\tilde{W}$ *(see* (12.3.10)*) is given by the following actions*

$$\begin{aligned}
\mu^{(a,b,c,d)}\cdot \tilde{W}_n(x^2|a,b,c,d) &= -(n+a+b-1)\,\tilde{W}_n\left(x^2\middle|a-\tfrac{1}{2},b-\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right),\\
\tau^{(a,b,c,d)}\cdot \tilde{W}_n(x^2|a,b,c,d) &= n(n+a+b+c+d-1)\,\tilde{W}_{n-1}\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right),\\
\mu^{*(a,b,c,d)}\cdot \tilde{W}_n(x^2|a,b,c,d) &= -\sigma(n+a+b-1)\,\tilde{W}_n\left(x^2\middle|a-\tfrac{1}{2},b-\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right)\\
&\quad -(1-\sigma)(n+c+d-1)\,\tilde{W}_n\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right)\\
&\quad +[\sigma(ab-cd)-\tfrac{1}{2}(c+d)-\tfrac{1}{4}]n(n+e_1-1)\,\tilde{W}_{n-1}\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right),\\
\sigma \equiv (a+b-c-d)^{-1},&\qquad e_1\equiv a+b+c+d,
\end{aligned}\tag{12.3.9}$$

with

$$\tilde{W}_n(x^2|a,b,c,d)\equiv \phi^{-1}\cdot W_n(x^2|a,b,c,d) = W_n\Big(-\frac{x^2}{4}\Big|a,b,c,d\Big),\tag{12.3.10}$$

and ϕ *defined in* (12.3.6).

Proof. The conjugation of the three operators $\mu^{(a,b,c,d)}$, $\mu^{*(a,b,c,d)}$ and $\tau^{(a,b,c,d)}$ by the scaling map (12.3.6) yields operators that are identified as the structure and forward shift operators for the Wilson polynomials [28]. These structure operators have a known action on the Wilson polynomials [26]. Using the identity

$$\mu^{*(a,b,c,d)} = \sigma\,\mu^{(a,b,c,d)} + (1-\sigma)\mu^{(c,d,a,b)} + \left[\sigma(ab-cd)-\frac{1}{2}(c+d)-\frac{1}{4}\right]\tau^{(a,b,c,d)},\tag{12.3.11}$$

which is directly verified and applying the scaling (12.3.6) to the polynomials to get (12.3.10), one obtains the actions (12.3.9). As one can use the orthogonality relation (12.1.1) of the Wilson polynomials to express all polynomials with shifted parameters in (12.3.9) as sums of Wilson polynomials with the initial parameters, these actions define representations of the stabilizing algebra $\mathfrak{stab}$ on the Wilson polynomials. □

12.4. Extension of $\mathfrak{stab}$ to a star algebra

The construction laid out in the preceding section parallels the structural approach to orthogonal polynomials due to Kalnins and Miller [22, 26]. In particular, Miller derives in [26] the orthogonality (12.1.1) of the Wilson polynomials from the structural recurrence relations associated to $\mu^{(a,b,c,d)}$ and $\tau^{(a,b,c,d)}$ by identifying the operator $\mu^{*(a,b,c,d)}$ and deriving an inner product such that this operator is the adjoint of $\mu^{(a,b,c,d)}$. An operator $\tau^{*(a,b,c,d)}$ is then identified as the adjoint of $\tau^{(a,b,c,d)}$. A similar approach in the context of the S–Heun operators can be pursued at the algebraic level.

The representations defined through (12.3.9) are endowed with a natural inner product inherited from the orthogonality relation (12.1.1). This enables one to define a star operation, such that $\mu^{*(a,b,c,d)}$ is precisely the adjoint of $\mu^{(a,b,c,d)}$ under the inner product. It follows that $\tilde{Q}$ is a self-adjoint operator. However, the stabilizing algebra is not closed under the star operation. This can be seen by taking the adjoint of $\tau^{(a,b,c,d)}$, a lowering operator, which would involve raising operators that are not contained in the stabilizing algebra $\mathfrak{stab}$. We shall now extend $\mathfrak{stab}$ to its closure under the star operation.

12.4.1. Star operation

With the help of (12.1.1), one constructs an operator as the adjoint of the forward shift operator. This leads to the backward shift operator for the Wilson polynomials [28] with action given by

$$(\phi^{-1} \circ \tau^{*(a,b,c,d)} \circ \phi) \cdot W_n(x^2|a,b,c,d) = W_{n+1}\left(x^2 \middle| a-\tfrac{1}{2}, b-\tfrac{1}{2}, c-\tfrac{1}{2}, d-\tfrac{1}{2}\right). \tag{12.4.1}$$

The operator $\tau^{*(a,b,c,d)}$ can then be decomposed in terms of the S–Heun operators as follows

$$\tau^{*(a,b,c,d)} = a_1 L + a_2 M_1 + a_3 M_2 + a_4 R_1 + a_5 R_2, \tag{12.4.2}$$

with the coefficients given by

$$\begin{aligned} a_1 &= 4e_4 - e_3 + \frac{e_1 - 1}{4}, \qquad a_2 = e_3 - \frac{e_2}{2} + \frac{e_1}{8}, \\ a_3 &= \frac{e_2}{2} - \frac{5e_1}{8} + \frac{1}{2}, \qquad a_4 = \frac{e_1}{4} - \frac{3}{8}, \qquad a_5 = -\frac{1}{8}, \end{aligned} \tag{12.4.3}$$

where e_1, e_2, e_3 and e_4 are the elementary symmetric polynomials in the four parameters a, b, c and d:

$$\begin{aligned} e_1 &= a+b+c+d, & e_2 &= ab+ac+ad+bc+bd+cd, \\ e_3 &= abc+abd+acd+bcd, & e_4 &= abcd. \end{aligned} \tag{12.4.4}$$

Introducing $\tau^{*(a,b,c,d)}$ as a fourth generator together with those of the stabilizing algebra $\mathfrak{stab}$ leads to an algebra closed under the star operation.

Proposition 12.6. *The algebra $\mathfrak{stab}^*$ generated by $\mu^{(a,b,c,d)}$, $\mu^{*(a,b,c,d)}$, $\tau^{(a,b,c,d)}$ and $\tau^{*(a,b,c,d)}$, together with the relations induced from their definitions in terms of S–Heun operators given in* (12.3.4), (12.3.8) *and* (12.4.2) *admits the natural star map defined from its canonical action on the generators:*

$$* : \tau \longmapsto \tau^*, \tag{12.4.5}$$

$$\mu \longmapsto \mu^*. \tag{12.4.6}$$

Proof. The result follows from the results of [26] after conjugation of the generators by the scaling map (12.3.6). ☐

12.4.2. A universal presentation of $\mathfrak{stab}^*$

The algebra $\mathfrak{stab}^*$ can be presented in terms of quadratic relations by making use of the relations given in the appendix 12.A. However, such a presentation obfuscates the structure of the algebra because the parameters a, b, c and d of the Wilson polynomial appear explicitly in the relations. Thus, it does not define uniquely an algebra associated to the quadratic grid.

Recall that the normalized Wilson polynomials are known [26] to be fully symmetric under permutations of their four parameters. However, the definitions for the two stabilizing generators given in (12.3.4) do not make this symmetry manifest, because they contain the specific parameters of the representation. Nervertheless, the permutation symmetry of the polynomials can be made manifest at the level of the algebra to obtain a universal presentation.

Proposition 12.7. *The algebra $\mathfrak{stab}^*$ admits a presentation as a unital associative algebra with four generators U, V, Y and R obeying the following relations*

$$\begin{gathered}
[V,Y] = -\{U,Y\}, \qquad [U,Y] = -\{Y,Y\}, \qquad [U,V] = \{V,Y\} - 2\{Y,Y\}, \\
[R,Y] = \{U,U\} - \{U,V\} + \{V,Y\}, \qquad [R,V] = 2\{V,Y\} - \{Y,Y\} - \{V,V\} - \{U,R\}, \\
[R,U] = \{U,V\} + 2\{V,Y\} - 2\{U,Y\} - \{V,V\} - \{Y,Y\} - \{R,Y\}.
\end{gathered} \tag{12.4.7}$$

The two Casimir operators are given by

$$Q_1 = U^2 - \{V,Y\} + 3Y^2, \qquad Q_2 = U^2 + V^2 - \{U,V\} - \{U,Y\} - \{R,Y\}. \tag{12.4.8}$$

Proof. Consider the following generic linear combination of generators

$$u\,\mu^{(a,b,c,d)} + v\,\mu^{*(a,b,c,d)}.$$

Acting with the symmetric group S_4 on the parameters (a, b, c, d), one constructs a fully symmetric element in terms of the S–Heun operators as follows

$$\begin{aligned}
&\frac{1}{|S_4|}\sum_{\sigma\in S_4}\left[u\,\mu^{\sigma(a,b,c,d)} + v\,\mu^{*\,\sigma(a,b,c,d)}\right] = \\
&\qquad\qquad \frac{1}{2}[(u-v) - e_1(u+v)]M_1 - \frac{1}{2}(u+v)M_2 + \left[\frac{e_1}{2}(u-v) - \frac{2e_2}{3}(u+v)\right]L.
\end{aligned}$$

Setting $u = 1$ and either $u = v$ or $u = -v$ in the above yields two independent generators that are manifestly symmetric and can be used instead of μ and μ^* to obtain another presentation of $\mathfrak{stab}^*$. The relations in this new presentation now only involve the elementary symmetric polynomials (12.4.4). Subsequently, it becomes straightforward to eliminate all remaining parameters in the algebraic relations by further redefining the generators as

$$U = M_1 + e_1\, L, \qquad V = M_2 + e_1\, M_1 + \frac{1}{2}{e_1}^2 L, \qquad Y = L, \tag{12.4.9}$$

$$\begin{aligned} R = R_2 + (2e_1 - 3)R_1 + \frac{1}{2}(3{e_1}^2 - 10e_1 + 4)M_2 + \frac{1}{2}(e_1+1)({e_1}^2 - 4e_1 + 2)M_1 \\ + \frac{1}{8}({e_1}^4 - 4{e_1}^3 - 8{e_1}^2 + 24e_1 - 8)L. \end{aligned} \tag{12.4.10}$$

Using the quadratic relations of the S–Heun operators given in the appendix 12.A, the relations (12.4.7), as well as the centrality of the two operators in (12.4.8), are verified. □

In a realization in terms of S–Heun operators, the Casimir operators (12.4.8) are proportionnal to the identity and the coefficients are functions of the parameters of the polynomials. One has

$$Q_1 = 1, \qquad Q_2 = (e_1 - 2)(e_1 - 4), \tag{12.4.11}$$

where e_1 is given in (12.4.4).

Remark 12.8. *While a universal presentation of $\mathfrak{stab}^*$ has been given in proposition 12.7, the star structure is not universal and depends explicitly on the representation parameters. This is not surprising because the map* (12.4.5) *is constructed using the inner product* (12.1.1) *corresponding to a specific realization with fixed parameters. Nevertheless, one can work in a specific realization and write the generators in* (12.4.7) *in terms of the structural operators* (12.3.4), (12.3.8) *and* (12.4.2) *as follows*

$$\begin{aligned} &U = \frac{1}{|S_4|}\sum_{\sigma\in S_4}\Big[\mu^{\sigma(a,b,c,d)} - \mu^{*\,\sigma(a,b,c,d)}\Big], \qquad V = \frac{1}{|S_4|}\sum_{\sigma\in S_4}\Big[-\mu^{\sigma(a,b,c,d)} - \mu^{*\,\sigma(a,b,c,d)}\Big] + \alpha\, Y, \\ &Y = \frac{1}{4}\tau, \qquad R = 8\tau^{*(e_1,e_2,e_3,e_4)} - (2-3\alpha)V + (1-3\alpha+\beta)U + (1-\beta+\gamma)Y, \end{aligned} \tag{12.4.12}$$

where

$$\alpha = \frac{1}{2}{e_1}^2 - \frac{4}{3}e_2, \qquad \beta = -{e_1}^3 + 4e_2e_1 - 8e_3, \qquad \gamma = \frac{3}{4}\alpha\, {e_1}^2 - {e_1}^2 e_2 + 8e_1e_3 - 32e_4, \tag{12.4.13}$$

with e_1, e_2, e_3 and e_4 given in (12.4.4)*. With the above, one obtains*

$$\begin{aligned} U^* = -U, \qquad V^* = V + \alpha\,(Y^* - Y), \\ Y^* = \frac{1}{32}\left[R + (2-3\alpha)V - (1-3\alpha+\beta)U - (1-\beta+\gamma)Y\right], \\ R^* = \Big[32 + \alpha\,(2-3\alpha)\Big]Y - (2-3\alpha)V - (1-3\alpha+\beta)U + \Big[1-\beta+\gamma-\alpha\,(2-3\alpha)\Big]Y^*. \end{aligned} \tag{12.4.14}$$

12.4.3. The algebra $\mathfrak{stab}^*$ as a Sklyanin algebra

It can be seen from (12.4.9) and (12.4.10) that the generators of $\mathfrak{stab}^*$ only depend on the parameters a, b, c, d via the elementary symmetric polynomial $e_1(a, b, c, d)$. Thus, they will be invariant under a commensurate increase and decrease of any pair of parameters. A glance at (12.4.2) indicates that this will not be the case for $\tau^{*(a,b,c,d)}$. However, a pseudo-commutation relation similar to the one introduced by Rains in [34] is obtained.

Proposition 12.9. *In the realization* (12.2.5) *the identity*

$$\tau^{*(a,b,c+k,d-k)}\,\tau^{*(a+\frac{1}{2},b+\frac{1}{2},c-\frac{1}{2},d-\frac{1}{2})} = \tau^{*(a,b,c,d)}\,\tau^{*(a+\frac{1}{2},b+\frac{1}{2},c-\frac{1}{2}+k,d-\frac{1}{2}-k)}, \tag{12.4.15}$$

is satisfied. Moreover, at the abstract level (12.4.15) *encodes the algebraic relations of the* $\mathfrak{stab}^*$ *algebra* (12.4.7).

Proof. Using the definition (12.4.2), the identity (12.4.15) is readily verified. The second statement is demonstrated by using (12.4.12) to express $\tau^{*(a,b,c,d)}$ in terms of the generators (12.4.9) and (12.4.10) as

$$8\tau^{*(a,b,c,d)} = R + (2-3\alpha)V - (1-3\alpha+\beta)U - (1-\beta+\gamma)Y,$$

where α, β and γ are given in (12.4.13). Upon using the above in (12.4.15), one can pick any one of the parameters a, b, c, d and take the remaining ones to be vanishing. Equating the coefficients of each power of the remaining non-zero parameter in the left- and right-hand side of (12.4.15) yields a set of relations that is algebraically identical to the relations (12.4.7). □

That the relations of $\mathfrak{stab}^*$ are encoded in the identity (12.4.15) identifies the $\mathfrak{stab}^*$ algebra as a Sklyanin-type algebra [34].

12.5. The rational degenerate Sklyanin algebra

The rational degenerate Sklyanin algebra $\mathfrak{s}\mathfrak{K}_r$ is obtained in [25] from the Sklyanin algebra [1] and is associated to a rational degeneration of an elliptic R-matrix. A presentation can be given as a unital associative algebra generated by four elements S_0, S_3, S_+, S_- obeying the defining relations

$$\begin{gathered}
[S_0, S_-] = -2\{S_-, S_-\}, \qquad [S_0, S_+] = 16\{S_3, S_-\} - 16\{S_-, S_-\} + 2\{S_+, S_-\} - 4\{S_3, S_3\}, \\
[S_+, S_-] = 2\{S_0, S_3\}, \qquad [S_0, S_3] = 2\{S_3, S_-\} - 8\{S_-, S_-\}, \qquad [S_3, S_\pm] = \pm\{S_0, S_\pm\}.
\end{gathered} \tag{12.5.1}$$

The rational degenerate Sklyanin algebra admits two Casimir operators which are given in the above presentation by

$$C_1 = S_0^2 + S_3^2 + \frac{1}{2}\{S_+, S_-\}, \qquad C_2 = \frac{1}{2}\{S_+, S_-\} + 2\{S_-, S_3\} + S_3^2 - 6\{S_-, S_-\}. \quad (12.5.2)$$

The presentation (12.5.1) is recovered from the one in [25] upon setting the free parameter $\eta = 1$ and defining $S_\pm = S_1 \pm iS_2$. The following proposition identifies the $\mathfrak{stab}^*$ algebra with the rational degenerate Sklyanin algebra.

Proposition 12.10. *The $\mathfrak{sK}_r$ algebra defined in* (12.5.1) *is isomorphic to the $\mathfrak{stab}^*$ algebra defined in* (12.4.7).

Proof. The following map is readily verified to be an isomorphism of algebras.

$$S_0 = 4Y - 4U, \qquad S_3 = 4U - 2Y - 4V, \qquad S_+ = 16R - 14Y - 8U + 24V, \qquad S_- = -2Y. \quad (12.5.3)$$

□

12.5.1. A realization in terms of difference operators

A realization of the rational degenerate Sklyanin algebra in terms of difference operators is provided in [25]. The Casimir elements are realized as multiples of the identity and are given by

$$C_1 = 16(2s+1)^2 Id, \qquad C_2 = 64s(s+1)Id.$$

The generators thus represented can be written in terms of the S–Heun operators (12.2.5) as follows

$$S_0 = 4(2s-1)L - 4M_1, \qquad S_3 = -2(2s-1)^2 L + 4(2s-1)M_1 - 4M_2, \qquad S_1 - iS_2 = -2L,$$

$$\begin{aligned} S_1 + iS_2 = &- 2(4s^2-1)(4s^2-8s-1)L - 8(2s-1)(4s^2-4s-1)M_1 \\ &+ 8(2s-1)(6s+1)M_2 - 16(4s-1)R_1 + 16R_2. \end{aligned} \quad (12.5.4)$$

It is immediate from the above that the realization in terms of S–Heun operators of the $\mathfrak{sK}_r$ algebra involves coefficients that depend on the values of the Casimir operators. A similar observation could be made for the case of the $\mathfrak{stab}^*$ algebra in (12.4.9) and (12.4.10). It follows from proposition 12.10 that the parameters e_1 and s are related by

$$e_1 = 2 - 2s.$$

12.5.2. A family of representations

The identification of the rational degenerate Sklyanin algebra $\mathfrak{sK}_r$ with the $\mathfrak{stab}^*$ algebra directly leads to a family of representations of $\mathfrak{sK}_r$ on the Wilson polynomials.

Proposition 12.11. *A representation of the rational degenerate Sklyanin algebra* $\mathfrak{s}\mathfrak{K}_r$ (12.5.1) *on the Wilson polynomials is defined by the following actions*

$$\begin{aligned}S_0\cdot\tilde{W}_n\left(x^2\middle|a,b,c,d\right)&=4\sigma(n+c+d-1)\,\tilde{W}_n\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right)\\&+(4\sigma(ab-cd)-e_1)n(n+e_1-1)\,\tilde{W}_{n-1}\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right)\\&-4\sigma(n+a+b-1)\,\tilde{W}_n\left(x^2\middle|a-\tfrac{1}{2},b-\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right),\end{aligned}$$

$$\begin{aligned}S_3\cdot\tilde{W}_n\left(x^2\middle|a,b,c,d\right)&=-4(n+a+b-1)\,\tilde{W}_n\left(x^2\middle|a-\tfrac{1}{2},b-\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right)\\&+\frac{1}{2}(8(ab+cd)-{e_1}^2-1)n(n+e_1-1)\,\tilde{W}_{n-1}\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right)\\&-4(n+c+d-1)\,\tilde{W}_n\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right),\end{aligned}$$

$$S_-\cdot\tilde{W}_n\left(x^2\middle|a,b,c,d\right)=-\frac{1}{2}n(n+e_1-1)\,\tilde{W}_{n-1}\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right),$$

$$\begin{aligned}S_+\cdot\tilde{W}_n\left(x^2\middle|a,b,c,d\right)&=128\tilde{W}_{n+1}\left(x^2\middle|a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right)\\&+8(6\alpha-1+2\beta\sigma)(n+a+b-1)\,\tilde{W}_n\left(x^2\middle|a-\tfrac{1}{2},b-\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right)\\&+8(6\alpha-1-2\beta\sigma)(n+c+d-1)\,\tilde{W}_n\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right)\\&+8\left[(1-6\alpha)(ab+cd)-2\beta\sigma(ab-cd)+\xi\right]n(n+e_1-1)\\&\times\tilde{W}_{n-1}\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right),\end{aligned}$$

where α,β *and* γ *are defined in* (12.4.13) *and*

$$\xi\equiv\frac{1}{2}(1-2{e_1}^2+{e_1}^4-256e_4),$$

with e_1 *and* e_4 *defined in* (12.4.4).

Proof. One first derives the action of the symmetrized structure operators on the Wilson polynomials. It can be seen from (12.3.9) and (12.4.1) that the expressions in the case of τ and τ^* are fully symmetric under permutations of the parameters such that their actions are invariant under the symmetrization. To obtain similar expressions for μ and μ^*, one uses (12.3.11) to write

$$\begin{aligned}\mu^{(a,b,c,d)}\pm\mu^{*(a,b,c,d)}&=(\mu^{(a,b,c,d)}\pm\mu^{(c,d,a,b)})\mp\frac{1}{2}\Big(c+d+\frac{1}{2}\Big)\tau^{(a,b,c,d)}\\&\pm\sigma\left[\mu^{(a,b,c,d)}-\mu^{(c,d,a,b)}+(ab-cd)\tau^{(a,b,c,d)}\right].\end{aligned}\quad(12.5.5)$$

The last term in the right-hand side of (12.5.5) is independent of the parameters as

$$\sigma\left[\mu^{(a,b,c,d)}-\mu^{(c,d,a,b)}+(ab-cd)\tau^{(a,b,c,d)}\right]=Y-U, \tag{12.5.6}$$

and is thus invariant under the symmetrization. As it is verified that

$$\frac{1}{|S_4|}\sum_{\pi\in S_4}(\mu^{\pi(a,b,c,d)}-\mu^{\pi(c,d,a,b)})=0,$$

one can use the invariance of τ under permutations of the parameters to obtain from (12.5.5) using (12.5.6) that

$$\frac{1}{|S_4|}\sum_{\pi\in S_4}(\mu^{\pi(a,b,c,d)}-\mu^{*\,\pi(a,b,c,d)})=\sigma\mu^{(c,d,a,b)}-\sigma\mu^{(a,b,c,d)}+\left[\frac{1}{4}(e_1+1)-\sigma(ab-cd)\right]\tau^{(a,b,c,d)}.$$

Likewise, observing that $\mu^{(a,b,c,d)}+\mu^{(c,d,a,b)}+(ab+cd)\tau^{(a,b,c,d)}$, is symmetric under permutations of the parameters, one can use the invariance of τ and (12.5.6) in (12.5.5) to obtain

$$\begin{aligned}\frac{1}{|S_4|}\sum_{\pi\in S_4}(\mu^{\pi(a,b,c,d)}+\mu^{*\,\pi(a,b,c,d)})&=(1+\sigma)\mu^{(a,b,c,d)}+(1-\sigma)\mu^{(c,d,a,b)}\\&+\left[(ab+cd)+\sigma(ab-cd)-\frac{1}{3}e_2-\frac{1}{4}(e_1+1)\right]\tau^{(a,b,c,d)}.\end{aligned}$$

The actions on the scaled Wilson polynomials (12.3.10) of τ,τ^* and of the operators in (12.5.5) are obtained from (12.3.9) and found to be:

$$\begin{aligned}&\frac{1}{|S_4|}\sum_{\pi\in S_4}(\mu^{\pi(a,b,c,d)}-\mu^{*\,\pi(a,b,c,d)})\cdot\,\tilde{W}_n(x^2|a,b,c,d)=\\&\quad\left[\frac{1}{4}(e_1+1)-\sigma(ab-cd)\right]n(n+e_1-1)\tilde{W}_{n-1}\left(x^2|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right)\\&\qquad+\sigma(n+a+b-1)\tilde{W}_n\left(x^2|a-\tfrac{1}{2},b-\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right)\\&\qquad\quad-\sigma(n+c+d-1)\tilde{W}_n\left(x^2|a+\tfrac{1}{2},b+\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right),\end{aligned}$$

$$\begin{aligned}&\frac{1}{|S_4|}\sum_{\pi\in S_4}(\mu^{\pi(a,b,c,d)}+\mu^{*\,\pi(a,b,c,d)})\cdot\,\tilde{W}_n(x^2|a,b,c,d)=\\&\left[(ab+cd)+\sigma(ab-cd)-\frac{1}{3}e_2-\frac{1}{4}(e_1+1)\right]n(n+e_1-1)\tilde{W}_{n-1}\left(x^2|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right)\\&\qquad+(\sigma-1)(n+c+d-1)\tilde{W}_n\left(x^2|a+\tfrac{1}{2},b+\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right)\\&\qquad\quad-(\sigma+1)(n+a+b-1)\tilde{W}_n\left(x^2|a-\tfrac{1}{2},b-\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right)\end{aligned}$$

$$\frac{1}{|S_4|}\sum_{\sigma\in S_4}\tau^{\sigma(a,b,c,d)}\cdot\,\tilde{W}_n(x^2|a,b,c,d)=n(n+e_1-1)\,\tilde{W}_{n-1}\left(x^2\middle|a+\tfrac{1}{2},b+\tfrac{1}{2},c+\tfrac{1}{2},d+\tfrac{1}{2}\right),$$

$$\frac{1}{|S_4|}\sum_{\sigma\in S_4}\tau^{*\,\sigma(a,b,c,d)}\cdot\,\tilde{W}_n(x^2|a,b,c,d)=\tilde{W}_{n+1}\left(x^2\middle|a-\tfrac{1}{2},b-\tfrac{1}{2},c-\tfrac{1}{2},d-\tfrac{1}{2}\right).$$

Using (12.4.12) and the above, one can construct a representation of (12.4.7) on the Wilson polynomials (12.3.10). Proposition 12.11 then follows from proposition 12.10. □

Finite-dimensional representations can be obtained by truncating the representations of proposition 12.11.

Proposition 12.12. *The finite-dimensional representations obtained from truncations of the representations in proposition 12.11 act on the para-Racah polynomials.*

Proof. Looking at the content of proposition 12.11, it is seen that the only generator that raises the degree is S_+. Using (12.5.3), (12.4.12) and (12.4.10) this degree-raising action can be traced back to the following combination of S–Heun operators

$$R_2 + (2e_1 - 3)R_1.$$

With the help of (12.2.10),(12.2.9) and (12.2.8), one can obtain the leading term of the action of the above operator on a polynomial of degree N in λ_x:

$$\begin{aligned} &R_1 \cdot {\lambda_x}^N = {\lambda_x}^{N+1} + O({\lambda_x}^N), \qquad R_2 \cdot {\lambda_x}^N = (2N-1){\lambda_x}^{N+1} + O({\lambda_x}^N), \\ &[R_2 + (2e_1 - 3)R_1] \cdot {\lambda_x}^N = 2(N - 1 + e_1){\lambda_x}^{N+1} + O({\lambda_x}^N). \end{aligned} \tag{12.5.7}$$

Demanding that the leading term in the above vanishes is tantamount to truncating the representation of proposition 12.11 at the degree N. This truncation condition is precisely the one that leads to the para-Racah polynomials (12.1.2). Thus, the finite-dimensional representations of the rational degenerate Sklyanin algebra obtained under this truncation have for basis the para-Racah polynomials. □

The actions of the generators in these truncated representations are as given in proposition 12.11, although one has to carry the appropriate limiting process described in [27] to deal with the otherwise singular expressions. Proposition 12.12 provides for the algebraic interpretation of the para-Racah polynomials as the basis elements of the finite-dimensional representations of the rational degenerate Sklyanin algebra.

12.6. Conclusion

This paper has introduced the S–Heun operators associated to the quadratic grid as a special case of the algebraic Heun operator. These operators were shown to form a five-dimensional space. The subset of these operators which stabilizes the space of polynomials of a given degree was identified and the algebra that they realize was examined. The extension of this stabilizing algebra to a star algebra was identified as the rational degenerate Sklyanin algebra. This definition of the rational degenerate Sklyanin algebra through S–Heun operators directly led to the construction of infinite-dimensional representations on the Wilson polynomials as well as finite-dimensional representations on the para-Racah polynomials.

The rational degenerate Sklyanin algebra is known [25] to be a one parameter deformation of the Yangian $Y(\mathfrak{sl}_2)$. In the same way that the Yangian $Y(\mathfrak{sl}_2)$ is the quantum algebra that encodes the symmetry of integrable XXX spin-half chains associated with the ordinary rational R-matrix, the rational degenerate Sklyanin algebra can be understood as the symmetry algebra of a generalized XXX chain corresponding to a deformed rational R-matrix, a new integrable model. Thus, it would be of interest to use the representations introduced in section 12.5 to construct explicit realizations of this new integrable model in terms of finite and infinite spin chains. In the finite case, one would expect the para-Racah polynomials to appear as the basis of representations of the symmetry algebra. Interestingly, these para polynomials were first introduced in the context of perfect state transfer on spin chains [35] and the advances in this paper suggest they would also find applications as solutions to new integrable spin chain models.

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12.A. Quadratic relations of the S–Heun operators

The set of homogeneous quadratic algebraic relations satisfied by the S–Heun operators is given below for reference:

$$
\begin{aligned}
[L, M_1] = 2L^2, \qquad [L, M_2] &= \{M_1, L\}, \qquad [M_1, M_2] = \{M_2, L\} - 4L^2, \\
[L, R_1] &= {M_1}^2 + L^2 + \{M_1, L\} + \tfrac{1}{2}\{M_2, L\}, \\
[L, R_2] &= {M_1}^2 + L^2 + \{M_1, L\} + \tfrac{1}{2}\{M_2, L\} + \{M_1, M_2\}, \\
[M_1, R_1] &= 2{M_1}^2 - 3L^2 + \{M_1, M_2\} - \tfrac{1}{2}\{M_1, L\} - \{M_2, L\}, \\
[M_1, R_2] &= {M_1}^2 + {M_2}^2 + 7L^2 + 2\{R_2, L\} - \tfrac{5}{2}\{M_1, L\} - 5\{M_2, L\}, \\
[R_1, M_2] &= 3L^2 - {M_1}^2 - {M_2}^2 + 2\{R_1 + R_2, L\} - \{R_1, M_1\} \\
&\quad - \{M_1, M_2\} - 5\{M_1, L\} - \tfrac{9}{2}\{M_2, L\}, \\
[R_2, M_2] &= Y^2 - {M_1}^2 - {M_2}^2 + \{R_1, M_1 - M_2\} - \{M_1, M_2\} + \tfrac{1}{2}\{M_1, L\}, \\
[R_2, R_1] &= 2{R_1}^2 + {M_1}^2 + 2{M_2}^2 + 3L^2 + \tfrac{1}{2}\{R_2 - R_1, L\} \\
&\quad - \tfrac{3}{2}\{R_1 + R_2, M_2\} + \{M_1, M_2\} + \tfrac{3}{2}\{M_1, L\} - \tfrac{1}{2}\{M_2, Y\}, \\
{M_1}^2 - \{M_1, M_2\} + 3L^2 = 1, &\qquad \{R_1 - R_2, L\} + {M_2}^2 + \{M_2, L\} - 3L^2 = -3, \\
-2\{R_1, L\} - 3L^2 &+ \{M_1, M_2\} + 2\{M_1 + M_2, L\} = 4, \\
{M_1}^2 + \tfrac{1}{2}L^2 + \{R_1, M_1 - M_2\} &- \tfrac{5}{2}\{R_1, L\} - 2\{R_2, L\} + \{R_1 + R_2, M_1\} \\
&+ \tfrac{1}{4}\{M_1, M_2\} + 6\{M_1, L\} + 4\{M_2, L\} = 0.
\end{aligned}
\tag{12.A.1}
$$

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Chapitre 13

Orthogonal polynomials and the deformed Jordan plane

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Abstract: We consider the unital associative algebra $\mathcal{A}$ with two generators $\mathcal{X}$, $\mathcal{Z}$ obeying the defining relation $[\mathcal{Z},\mathcal{X}] = \mathcal{Z}^2 + \Delta$. We construct irreducible tridiagonal representations of $\mathcal{A}$. Depending on the value of the parameter Δ, these representations are associated to the Jacobi matrices of the para-Krawtchouk, continuous Hahn, Hahn or Jacobi polynomials.

Keywords: Para-Krawtchouk polynomials, deformed Jordan plane, tridiagonal representations.

13.1. Introduction

This paper is devoted to the study of irreducible tridiagonal representations of the two-generated algebra $\mathcal{A}$ which is a deformation of the Jordan plane. It is shown how the para-Krawtchouk polynomials appear quite naturally in this context, along with the other families of classical orthogonal polynomials (OPs) of the Jacobi, continuous Hahn and Hahn type.

The algebra $\mathcal{A}$ over $\mathbb{R}$, with generators $\mathcal{X}$, $\mathcal{Z}$ and satisfying

$$[\mathcal{Z},\mathcal{X}] = \mathcal{Z}^2 + \Delta \tag{13.1.1}$$

with Δ a parameter, is a special case of the most general two-generated quadratic algebra $\mathcal{Q}$ with defining relation

$$\alpha_1\mathcal{X}^2 + \alpha_2\mathcal{X}\mathcal{Z} + \alpha_3\mathcal{Z}\mathcal{X} + \alpha_4\mathcal{Z}^2 + \alpha_5\mathcal{X} + \alpha_6\mathcal{Z} + \alpha_7 = 0. \tag{13.1.2}$$

This algebra has been of interest to various communities. Ring theorists have provided classifications [1, 2] of the special cases it entails and studied its properties. The algebra $\mathcal{Q}$ has also been related to non-commutative probability theory [3] and is related to martingale polynomials associated to quadratic harnesses [4]. On the physics side, $\mathcal{Q}$ describes various 1D asymmetric exclusion models [5–7].

Recently, the last two authors have begun connecting $\mathcal{Q}$ and its various isomorphism classes to families of special functions. We recall that in the context of orthogonal polynomials, the coefficients appearing in their three-term recurrence relation can be presented in a tridiagonal matrix called the Jacobi matrix which has a diagonal action on the basis of the associated family of polynomials. The key observation in the paper will be that tridiagonal representations of the generators of the algebra $\mathcal{A}$ can be seen as the Jacobi matrices for a number of families of orthogonal polynomials. In [8], by studying the tridiagonal representations of the q-oscillator algebra $\mathcal{X}\mathcal{Z} - q\mathcal{Z}\mathcal{X} = 1$, the authors have identified how they encompass the recurrence relations of the big q-Jacobi, the q-Hahn and the q-para-Krawtchouk polynomials. The case of the q-Weyl algebra $\mathcal{X}\mathcal{Z} - q\mathcal{Z}\mathcal{X} = 0$ has also been studied in [9]. The present paper will add to this by considering an interesting special case of (13.1.2) and identifying how the orthogonal polynomials of Jacobi, Hahn, Continuous Hahn and para-Krawtchouk type are related to this algebra.

Since their introduction in [10], para-polynomials have been the object of growing interest. Four families have been defined and studied offering para-versions of the polynomials of Krawtchouk, q-Krawtchouk, Racah and q-Racah type. While they do not fall in the category of classical orthogonal polynomials [1], they are understood as non-standard truncations of infinite-dimensional families of classical OPs [11–13]. In addition to their natural occurence in the study of perfect state transfer and fractional revival in quantum spin chains [10, 14, 15], recent advances have identified these para-polynomials as the basis for finite-dimensional representations of degenerations of the Sklyanin algebra [16–18]. They have also appeared in the study of the Dunkl oscillator in the plane [19]. The main goal of this paper is to show that these para-Krawtchouk polynomials as well as the Jacobi, continuous Hahn and Hahn polynomials arise in representations of the two-generated algebra $\mathcal{A}$.

When $\Delta = 0$, $\mathcal{A}$ as defined in (13.1.1) is called the Jordan plane (with $\mathcal{X}$ and $\mathcal{Z}$ viewed as noncommutative coordinates). We refer to the general case (13.1.1) as the deformed Jordan plane. Three cases will be distinguished depending on whether $\Delta = 0$, $\Delta > 0$ or $\Delta < 0$.

[1] They obey a three term recurrence relation but a higher order difference equation.

These three cases will be studied separately and provide a complete picture of the connection between the algebra (13.1.1) and orthogonal polynomials.

The presentation is organized as follows. Section 13.2 will introduce the tridiagonal representations of the algebra $\mathcal{A}$ and the non-degeneracy condition. Standardized versions of $\mathcal{A}$ corresponding to $\Delta = 0$, $\Delta < 0$, $\Delta > 0$ will then be examined in the following sections. The case $\Delta = 0$ will be studied in section 13.3 and the Jacobi OPs will appear, while the case $\Delta > 0$ and the continuous Hahn polynomials will be the subject of section 13.4. Section 13.5 will focus on the case $\Delta < 0$ and will feature both the Hahn and the para-Krawtchouk polynomials. Some concluding remarks and perspectives will close the paper.

13.2. Tridiagonal representations of the algebra $\mathcal{A}$

Consider a tridiagonal representation of $\mathcal{A}$ where $\mathcal{X} \mapsto X$ and $\mathcal{Z} \mapsto Z$. The actions of X, Z on a semi-infinite orthonormal basis $|n\rangle, n = 0, 1, 2, \ldots$ are taken to be of the form

$$X|n\rangle = c_n|n-1\rangle + b_n|n\rangle + a_n|n+1\rangle, \tag{13.2.1a}$$

$$Z|n\rangle = u_n|n-1\rangle + v_n|n\rangle + w_n|n+1\rangle, \tag{13.2.1b}$$

with $c_0 = u_0 = 0$. To ensure that such a representation is irreducible we shall assume that the off-diagonal coefficients are non-zero for $n > 0$. Acting with (13.1.1) on the basis $|n\rangle$ and using the above definitions, one obtains

$$\begin{aligned}
&(ZX - XZ - Z^2 - \Delta)|n\rangle \\
&= (c_n u_{n-1} - c_{n-1}u_n - u_{n-1}u_n)|n-2\rangle \\
&+(b_n u_n - b_{n-1}u_n + c_n v_{n-1} - u_n v_{n-1} - c_n v_n - u_n v_n)|n-1\rangle \\
&+(-\Delta - a_{n-1}u_n + a_n u_{n+1} - v_n^2 + c_n w_{n-1} - u_n w_{n-1} - c_{n+1}w_n - u_{n+1}w_n)|n\rangle \\
&+(a_n v_{n+1} - a_n v_n + b_n w_n - b_{n+1}w_n - v_n w_n - v_{n+1}w_n)|n+1\rangle \\
&+(a_n w_{n+1} - a_{n+1}w_n - w_n w_{n+1})|n+2\rangle.
\end{aligned} \tag{13.2.2}$$

For the actions in (13.2.1) to define a representation of $\mathcal{A}$, each side of the above equation must vanish. As the basis vectors are orthonormal, one obtains the following conditions on the coefficients of (13.2.1) that define the representations:

$$0 = c_n u_{n-1} - c_{n-1}u_n - u_{n-1}u_n, \tag{13.2.3a}$$

$$0 = b_n u_n - b_{n-1}u_n + c_n v_{n-1} - u_n v_{n-1} - c_n v_n - u_n v_n, \tag{13.2.3b}$$

$$0 = -\Delta - a_{n-1}u_n + a_n u_{n+1} - v_n^2 + c_n w_{n-1} - u_n w_{n-1} - c_{n+1}w_n - u_{n+1}w_n, \tag{13.2.3c}$$

$$0 = a_n v_{n+1} - a_n v_n + b_n w_n - b_{n+1}w_n - v_n w_n - v_{n+1}w_n, \tag{13.2.3d}$$

$$0 = a_n w_{n+1} - a_{n+1}w_n - w_n w_{n+1}. \tag{13.2.3e}$$

13.2.1. General solutions to the recurrence relations

We now determine the general solutions to the above system of recurrence equations. Dividing (13.2.3a) by $u_n u_{n-1}$, one obtains

$$\frac{c_n}{u_n} - \frac{c_{n-1}}{u_{n-1}} = 1.$$

This implies

$$\phi_n = \phi_0 + n, \qquad \phi_n \equiv \frac{c_n}{u_n}. \tag{13.2.4}$$

Equation (13.2.3e) can be solved similarly. Dividing by $w_n w_{n+1}$, one has

$$\delta_n = \delta_0 - n, \qquad \delta_n \equiv \frac{a_n}{w_n}. \tag{13.2.5}$$

Rewriting (13.2.3b) and (13.2.3d) in terms of ϕ_n and δ_n and dividing by u_n or w_n, respectively, one obtains

$$b_{n-1} - b_n = (\phi_n - 1)v_{n-1} - (\phi_n + 1)v_n, \tag{13.2.6}$$

$$b_{n+1} - b_n = (\delta_n - 1)v_{n+1} - (\delta_n + 1)v_n. \tag{13.2.7}$$

To solve for v_n, shift the index of (13.2.6) and add (13.2.7) to find

$$0 = (\delta_n - \phi_{n+1} - 2)v_{n+1} - (\delta_n - \phi_{n+1} + 2)v_n. \tag{13.2.8}$$

Substituting the solutions (13.2.4) and (13.2.5) in (13.2.8) leads to

$$\begin{aligned} 0 &= (\delta_0 - \phi_0 - 2(n+2) + 1)v_{n+1} - (\delta_0 - \phi_0 - 2n + 1)v_n \\ &= \mu_{n+2}v_{n+1} - \mu_n v_n, \end{aligned} \tag{13.2.9}$$

with $\mu_n \equiv (\delta_0 - \phi_0 - 2n + 1)$. Multiplying the above by μ_{n+1} as an integrating factor, one can solve the recurrence to obtain

$$v_n = \frac{(\delta_0 - \phi_0 - 1)(\delta_0 - \phi_0 + 1)v_0}{(\delta_0 - \phi_0 - 2n + 1)(\delta_0 - \phi_0 - 2n - 1)}. \tag{13.2.10}$$

To find b_n, substract instead (13.2.6) with shifted index from (13.2.7) and get

$$b_{n+1} - b_n = \frac{1}{2}(\delta_n + \phi_n + 1)(v_{n+1} - v_n), \tag{13.2.11}$$

which, upon using (13.2.4) and (13.2.5), can be solved immediately and yields

$$b_n = \frac{1}{2}(\delta_0 + \phi_0 + 1)(v_n - v_0) + b_0. \tag{13.2.12}$$

Finally, (13.2.3c) is written as follows in terms of ϕ_n and δ_n using (13.2.4) and (13.2.5), as

$$\Delta + v_n^2 = (\delta_0 - \phi_0 - 2(n+1))\kappa_{n+1} - (\delta_0 - \phi_0 - 2(n-1))\kappa_n, \tag{13.2.13}$$

with

$$\kappa_n \equiv u_n w_{n-1}. \tag{13.2.14}$$

Multipliying both sides by $(\delta_0 - \phi_0 - 2n)$ as an integrating factor, one can reduce the above to

$$(\delta_0 - \phi_0 - 2n)(\delta_0 - \phi_0 - 2n + 2)\kappa_n = (\delta_0 - \phi_0)(\delta_0 - \phi_0 + 2)\kappa_0 + \sum_{k=0}^{n-1}(\Delta + v_k^2)(\delta_0 - \phi_0 - 2k). \tag{13.2.15}$$

The sum over k in (13.2.15) can be reexpressed[2] as

$$\sum_{k=0}^{n-1}(\Delta + v_k^2)(\delta_0 - \phi_0 - 2k) = \frac{n(\delta_0 - \phi_0 - n + 1)(\Delta(\delta_0 - \phi_0 - 2n + 1)^2 + v_0^2(\delta_0 - \phi_0 - 1)^2)}{(\delta_0 - \phi_0 - 2n + 1)^2}. \tag{13.2.16}$$

From (13.2.15) and (13.2.16), recalling that u_0 was required to vanish so that $\kappa_0 = u_0 w_{-1} = 0$, one has

$$\kappa_n = \frac{n(\delta_0 - \phi_0 - n + 1)(\Delta(\delta_0 - \phi_0 - 2n + 1)^2 + v_0^2(\delta_0 - \phi_0 - 1)^2)}{(\delta_0 - \phi_0 - 2n + 1)^2(\delta_0 - \phi_0 - 2n)(\delta_0 - \phi_0 - 2n + 2)}. \tag{13.2.17}$$

13.2.2. The linear pencil $\mathcal{X} + \mu\mathcal{Z}$

The algebra $\mathcal{A}$ is invariant under the affine transformation

$$\mathcal{X} \longmapsto \mathcal{X} + \mu\mathcal{Z}, \qquad \mu \in \mathbb{R}.$$

As a result, one expects the transformed solutions for the coefficients in (13.2.1) to be given by (13.2.4), (13.2.5) and (13.2.12) with modified parameters. Indeed one finds the parameters to be replaced by

$$\phi_0 \longmapsto \phi_0 - \mu, \qquad \delta_0 \longmapsto \delta_0 - \mu, \qquad b_0 \longmapsto b_0 - \mu v_0.$$

Thus, the diagonalization of the linear pencil $\mathcal{X} + \mu\mathcal{Z}$ amounts to the diagonalization of $\mathcal{X}$ up to a shift in the parameters.

13.2.3. Representations on polynomials

Denoting by $\langle x|$ the dual eigenvectors:

$$\langle x|X = x\langle x|,$$

one can look for the polynomials $q_n(x) \equiv \langle x|n\rangle$ that diagonalize X

$$Xq_n(x) \equiv x\, q_n(x) = c_n q_{n-1}(x) + b_n q_n(x) + a_n q_{n+1}(x). \tag{13.2.18}$$

[2]This is done by noticing the sum to be telescopic or via the polygamma function of the first order.

By appropriate renormalization, one obtains a monic recurrence relation

$$Xp_n(x) \equiv xp_n(x) = a_{n-1}c_n p_{n-1}(x) + b_n p_n(x) + p_{n+1}(x), \qquad p_n(x) = \left(\prod_{i=0}^{n-1} a_i\right) q_n(x). \tag{13.2.19}$$

The families of polynomials $p_n(x)$ that diagonalize X can be determined by identifying the coefficients $a_{n-1}c_n$ and b_n.

From (13.2.4), (13.2.5), (13.2.14) and (13.2.17), one has that

$$a_{n-1}c_n = (n+\phi_0)(n-\delta_0-1)\frac{n(n+\phi_0-\delta_0-1)(\Delta(2n+\phi_0-\delta_0-1)^2+v_0^2(\phi_0-\delta_0+1)^2)}{(2n+\phi_0-\delta_0-1)^2(2n+\phi_0-\delta_0)(2n+\phi_0-\delta_0-2)} \tag{13.2.20}$$

and from (13.2.12) and (13.2.10), that

$$b_n = \frac{1}{2}\frac{(\delta_0+\phi_0+1)(\phi_0-\delta_0+1)(\phi_0-\delta_0-1)v_0}{(2n+\phi_0-\delta_0-1)(2n+\phi_0-\delta_0+1)} + \tilde{b}_0, \qquad \tilde{b}_0 \equiv b_0 - \frac{1}{2}(\delta_0+\phi_0+1)v_0. \tag{13.2.21}$$

Finite-dimensional representations of dimension $N+1$ are obtained if $w_N = 0$ since it follows that $a_N = 0$ from (13.2.3e). This implies that $\kappa_{N+1} = 0$. From (13.2.17), we see that this is achieved for any value of Δ by

$$N = (\delta_0 - \phi_0). \tag{13.2.22}$$

If $\Delta \neq 0$, one finds an additional pair of solutions given by

$$N+1 = -\frac{1}{2}\left[\phi_0-\delta_0-1 \pm (\phi_0-\delta_0+1)v_0\sqrt{-\Delta^{-1}}\right]. \tag{13.2.23}$$

13.3. The case $\Delta = 0$: Jacobi polynomials

With Δ vanishing, the coefficient $a_{n-1}c_n$ (13.2.20) simplifies to

$$a_{n-1}c_n = \frac{n(n+\phi_0)(n-\delta_0-1)(n+\phi_0-\delta_0-1)(\phi_0-\delta_0+1)^2v_0^2}{(2n+\phi_0-\delta_0-1)^2(2n+\phi_0-\delta_0)(2n+\phi_0-\delta_0-2)}. \tag{13.3.1}$$

Setting $v_0 = 2(\phi_0-\delta_0+1)^{-1}$, one identifies the basis vector to be proportional to the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with parameters

$$\alpha = -\delta_0 - 1, \qquad \beta = \phi_0. \tag{13.3.2}$$

Indeed, it follows that $\tilde{b}_0 = 0$ and that the coefficient b_n of (13.2.21) is given by

$$b_n = \frac{(\beta^2+\alpha^2)}{(2n+\beta+\alpha)(2n+\beta+\alpha+2)}. \tag{13.3.3}$$

Comparing the expressions (13.3.1) and (13.3.3) for the coefficients using for instance [20], we conclude:

Proposition 13.1. *In the case $\Delta = 0$, the eigenfunctions $p_n(x)$ of X* (13.2.19) *are the monic Jacobi polynomials*

$$p_n^{(\alpha,\beta)}(x) = \frac{2^n n!}{(n+\alpha+\beta+1)_n} P_n^{(\alpha,\beta)}(x).$$

with parameters α, β given in (13.3.2).

The only truncation condition possible is (13.2.22). However, it yields singular expressions in (13.3.1) and (13.3.3) for $n \leq N$.

13.4. The case $\Delta > 0$: Continuous Hahn polynomials

If $\Delta \neq 0$, upon scaling the generators of the algebra according to

$$\tilde{\mathcal{X}} = \Omega \mathcal{X}, \qquad \tilde{\mathcal{Z}} = \Omega \mathcal{Z},$$

we obtain

$$[\tilde{\mathcal{Z}}, \tilde{\mathcal{X}}] = \tilde{\mathcal{Z}}^2 + \Omega^2 \Delta. \tag{13.4.1}$$

In view of (13.4.1), one can choose Ω so that $\Delta = \pm\frac{1}{4}$. In this section, we shall consider the case $\Delta = +\frac{1}{4}$. The coefficient $a_{n-1}c_n$ (13.2.20) is then given by

$$a_{n-1}c_n = (n+\phi_0)(n-\delta_0-1)\frac{n(n+\phi_0-\delta_0-1)((2n+\phi_0-\delta_0-1)^2/4 + v_0^2(\phi_0-\delta_0+1)^2)}{(2n+\phi_0-\delta_0-1)^2(2n+\phi_0-\delta_0)(2n+\phi_0-\delta_0-2)}. \tag{13.4.2}$$

Writing

$$\phi_0 + 1 = a + c, \qquad -\delta_0 = b + d, \qquad v_0 = -i\frac{(a-b-c+d)}{2(a+b+c+d)}, \tag{13.4.3}$$

one can factorize the term with v_0:

$$\tfrac{1}{4}(2n+\phi_0-\delta_0-1)^2 + v_0^2(\phi_0-\delta_0+1)^2 = (n+a+d-1)(n+b+c-1).$$

With (13.4.3) and the above, (13.4.2) becomes

$$\begin{aligned} a_{n-1}c_n &= (n+a+c-1)(n+b+d-1) \\ &\times \frac{n(n+a+b+c+d-2)(n+a+d-1)(n+b+c-1)}{(2n+a+b+c+d-1)(2n+a+b+c+d-2)^2(2n+a+b+c+d-3)}. \end{aligned} \tag{13.4.4}$$

Using (13.4.3) and taking $\tilde{b}_0 = \frac{i}{4}(a+b-c-d)$, the coefficient b_n (13.2.21) is found to be

$$\begin{aligned} b_n = i\Big[&-\frac{(n+a+b+c+d-1)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)} \\ &+ \frac{n(n+b+c-1)(n+b+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)} + a\Big]. \end{aligned} \tag{13.4.5}$$

The coefficients (13.4.4) and (13.4.5) can be identified in [20] and one arrives at:

Proposition 13.2. *In the case $\Delta > 0$, the eigenfunctions $p_n(x)$ of X (13.2.19) are the monic continuous Hahn polynomials $P_n^{(a,b,c,d)}(x)$ with parameters given in (13.4.3).*

13.4.1. Finite-dimensional representations and orthogonal polynomials

Using (13.4.3), condition (13.2.22) becomes

$$N - 1 = -a - c - b - d, \tag{13.4.6}$$

which leads to expressions for (13.4.2) and (13.4.5) that are ill-defined for $n < N$. However, this can be resolved using limits (see Section 13.5.2) and one thus obtains the para-Krawtchouk polynomials [10].

Condition (13.2.23) reads

$$N + 1 = -\frac{1}{2}\left[(a + b + c + d - 2) \pm (a - b - c + d)\right] = \begin{cases} -a - d + 1 \\ -b - c + 1 \end{cases} \tag{13.4.7}$$

and corresponds to the truncation of the continuous Hahn polynomials to Hahn polynomials.

However, for each of these truncations (13.4.6) and (13.4.7) to define real polynomials, the operators X and Z have to be scaled by an imaginary number: $X \to iX$, $Z \to iZ$. This is equivalent to setting $\Delta \to -\Delta$, which corresponds to the situation $\Delta < 0$ that is the subject of the next section.

13.5. The case $\Delta < 0$: Hahn and para-Krawtchouk polynomials

When $\Delta < 0$, polynomials of a real variable are obtained only if (13.2.22) or (13.2.23) are satisfied (see Section 13.4.1). We begin by treating the latter case.

13.5.1. Hahn polynomials

In view of (13.4.1), we may take $\Delta = -\frac{1}{4}$ without loss of generality. Expressing the parameters as follows

$$\phi_0 = \beta, \qquad -\delta_0 = \alpha + 1, \qquad v_0 = -\frac{(\alpha + \beta + 2N + 2)}{2(\alpha + \beta + 2)}, \qquad \tilde{b}_0 = \frac{1}{4}(2N - \alpha + \beta), \tag{13.5.1}$$

so that (13.2.23) is satisfied, one obtains

$$a_{n-1}c_n = \frac{n(n + \alpha)(n + \beta)(n + \alpha + \beta)(n + \alpha + \beta + N + 1)(N - n + 1)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)}, \tag{13.5.2}$$

as well as

$$b_n = \frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} + \frac{n(n+\alpha+\beta+N+1)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}. \tag{13.5.3}$$

The coefficients given by (13.5.2) and (13.5.3) are found in [20].

Proposition 13.3. *In the case $\Delta < 0$, the eigenfunctions $p_n(x)$ of X (13.2.19) related to the finite-dimensional representation condition (13.2.23) are given in terms of the monic Hahn polynomials $Q_n^{(\alpha,\beta)}(x)$ for the choice of parameters given in (13.5.1).*

As previously mentioned, these polynomials can also be obtained as a truncation of the recurrence defined by (13.4.2) and (13.4.5). Indeed, setting

$$\alpha = a + c - 1, \qquad \beta = b + d - 1, \tag{13.5.4}$$

with one of (13.4.7), the coefficients (13.4.2) and (13.4.5) become proportional to (13.5.2) and (13.5.3), respectively. Hence, the action of iX when $\Delta = +\frac{1}{4}$ also leads to the recurrence relation of the monic Hahn polynomials.

13.5.2. Para-Krawtchouk polynomials

We shall finally indicate how a family of finite-dimensional representations of $\mathcal{A}$ relates to para-Krawtchouk polynomials. Consider the condition (13.2.22). Although leading to singular expressions for certain values of n, well-defined polynomials are obtained by carefully taking limits. Mindful of (13.4.1), it is convenient in this case to take $\Delta = -1$. Let $N = 2j+p$ with j an integer and $p = 0, 1$ depending on the parity of N, and set

$$\phi_0 + 1 = e_1 t - j, \quad -\delta_0 = e_2 t - j + 1 - p, \quad v_0 = \frac{\gamma + p - 1}{e_1 t + e_2 t - 2j - p + 1}, \quad e_1 = e_2 = 1. \tag{13.5.5}$$

The parameters e_1 and e_2 are chosen equal in order to simplify the expressions. The more general solutions can be recovered using isospectral deformations [12, 21]. With the above parametrization, it can be seen that (13.2.22) is verified in the limit where $t \to 0$. With the parameters as in (13.5.5), the coefficient $a_{n-1}c_n$ (13.2.20) becomes

$$\begin{aligned} a_{n-1}c_n = {} & (n - j + t - 1)(n - j + t - p) \\ & \times \frac{n(n-2j+2t-p-1)(N-2n+p+\gamma)(N-2n-p+2-\gamma)}{(2n-2j+2t-p-1)^2(2n-2j+2t-p)(2n-2j+2t-p-2)}. \end{aligned} \tag{13.5.6}$$

Taking the limit $t \to 0$ and treating the cases for $p = 0, 1$ separately, one finds that the results can be combined as follows

$$\lim_{t\to 0} a_{n-1}c_n = \frac{n(N+1-n)(N-2n+p+\gamma)(N-2n-p+2-\gamma)}{4(2n-N+p-1)(2n-N-p-1)}. \tag{13.5.7}$$

For the coefficient b_n, setting $\tilde{b}_0 = \frac{1}{2}(N+\gamma-1)$ and inserting (13.5.5) in (13.2.21), one finds

$$b_n = \frac{1}{2}\frac{(p-1)(-2j-p+2t-1)(\gamma+p-1)}{(2n-2j-p+2t-1)(2n-2j-p+2t+1)} + \frac{1}{2}(N+\gamma-1). \tag{13.5.8}$$

Treating the cases $p=0$ or $p=1$ separately and taking the limit $t \to 0$, one sees that the results can be written jointly as

$$\lim_{t\to 0} b_n = -\frac{(N-n)(N-2n-2+p+\gamma)}{2(2n-N-p+1)} - \frac{n(N-2n+2-p-\gamma)}{2(2n-N+p-1)}. \tag{13.5.9}$$

The coefficients given by (13.5.7) and (13.5.9) are recognized in [12] as the coefficients for the recurrence relation of the monic para-Krawtchouk polynomials.

Proposition 13.4. *In the case $\Delta < 0$, the eigenfunctions $p_n(x)$ of X (13.2.19) in the finite-dimensional representation (13.2.22) of $\mathcal{A}$ are the monic para-Krawtchouk polynomials.*

13.6. Conclusion

We have studied tridiagonal representations of the algebra $\mathcal{A}$ with defining relation $[\mathcal{Z},\mathcal{X}] = \mathcal{Z}^2 + \Delta$. Depending on the value of Δ, in these representations, the linear pencil $X + \mu Z$ entailed the recurrence relations of the Jacobi ($\Delta = 0$), continuous Hahn ($\Delta > 0$), Hahn and para-Krawtchouk ($\Delta < 0$) polynomials.

In the wake of this work, two research avenues present themselves. One is the exploration of the tridiagonal representations of the algebra $[\mathcal{Z},\mathcal{X}] = \mathcal{Z}^2 + \alpha\mathcal{X}$, another class of the general quadratic algebra (13.1.2). It is expected that the tridiagonal representations will lead to the Wilson, Racah and para-Racah polynomials in a similar fashion.

Another related direction is the study of the so-called meta algebras, poised to describe both polynomial and rational functions of a given type, as shown in [22] for functions of the Hahn type. The meta-Hahn algebra is in fact obtained by adjoining to $\mathcal{A}$ an additional generator. As it turns out, the meta-algebra picture offers a rationale for considering tridiagonal representations; in the basis where the extra generator is diagonal, the generators of $\mathcal{A}$ are tridiagonal. These developments suggest in particular that the work on the q-oscillator algebra [8] should be revisited in order to bring to the fore the associated rational functions.

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Partie 3

Algèbres de type Askey–Wilson et centralisateurs

Introduction

Une question centrale qui résume bien les travaux dans cette troisième partie est :

Comment définir des analogues de plus haut rang des algèbres de type Askey–Wilson?

Les algèbres de type Askey–Wilson étant riches et apparaissant dans de nombreux contextes, il est assez naturel de vouloir les généraliser; on peut espérer que ces généralisations créent davantage de connexions entre des domaines variés tout en conservant un nombre de propriétés utiles et sans devenir trop compliquées.

À l'origine, l'algèbre d'Askey–Wilson a été obtenue à partir des opérateurs de bispectralité des polynômes portant le même nom. Cette procédure a par la suite été répétée pour les diverses familles de polynômes. Une possibilité de généralisation serait de regarder les opérateurs de bispectralité associés aux généralisations multivariées des diverses familles de polynômes orthogonaux et de calculer les relations algébriques auxquelles ils obéissent. Ce n'est pas une tâche facile et, historiquement, ce n'est pas l'avenue qui a été empruntée.

Tel que mentionné dans le Prologue et dans la première partie de la thèse, il existe une autre approche pour obtenir les algèbres de type Askey–Wilson qui revient à considérer les Casimir intermédiaires utilisés dans l'étude des problèmes de recouplement de représentations irréductibles des algèbres $\mathfrak{sl}_2$, $\mathfrak{osp}(1|2)$ et $U_q(\mathfrak{sl}_2)$. C'est cette approche qui a été généralisée, historiquement. Une algèbre de Racah de rang général $R(n)$ a pu être définie en étudiant les relations auxquelles obéissent les Casimir intermédiaires associés aux recouplements de n irreps de $\mathfrak{sl}_2$ [40]. (Il faut ici mentionner qu'à l'origine le chemin avait été tracé pour l'algèbre de Bannai–Ito [41], obtenue en étudiant les relations qu'obéissent les Casimir intermédiaires de $\mathfrak{osp}(1|2)$.) Voici de quoi a l'air l'algèbre $R(n)$ dans cette approche :

Soient e, f, h les générateurs de $\mathfrak{sl}_2$, obéissant

$$[h,e]=2e, \qquad [h,f]=-2f, \qquad [e,f]=h. \tag{13.6.1}$$

Le Casimir de $\mathfrak{sl}_2$ est donné par

$$C=\tfrac{1}{4}h^2-\tfrac{1}{2}h+ef. \tag{13.6.2}$$

Le coproduit Δ de $\mathfrak{sl}_2$ agit comme suit :

$$\Delta(g)=1\otimes g+g\otimes 1, \qquad g\in\{e,f,h\}. \tag{13.6.3}$$

Dénotons $\Delta(C)=C_{(1)}\otimes C_{(2)}$ en notation de Sweedler [42]. Introduisons maintenant les générateurs $C_i, C_{ij}\in U(\mathfrak{sl}_2)^{\otimes n}$

$$C_i=1^{\otimes(i-1)}\otimes C\otimes 1^{\otimes(n-i)}, \qquad C_{ij}=1^{\otimes(i-1)}\otimes C_{(1)}\otimes 1^{\otimes(j-i-1)}\otimes C_{(2)}\otimes 1^{\otimes(n-j)}. \tag{13.6.4}$$

On peut former les générateurs de $R(n)$

$$\mathsf{P}_{ii}=2C_i, \qquad \mathsf{P}_{ij}=C_{ij}-C_i-C_j=\mathsf{P}_{ji}. \tag{13.6.5}$$

L'algèbre $R(n)$, avec générateurs P_{ii}, P_{ij} obéit aux relations suivantes, pour des indices i, j, k, ℓ, $m \in \{1, \ldots, n\}$ et tous distincts :

$$\mathsf{P}_{ii} \quad \text{est central,} \tag{13.6.6a}$$

$$[\mathsf{P}_{ij}, \mathsf{P}_{k\ell}] = 0, \tag{13.6.6b}$$

$$[\mathsf{P}_{ij}, \mathsf{P}_{jk}] = 2\mathsf{F}_{ijk}, \tag{13.6.6c}$$

$$[\mathsf{P}_{jk}, \mathsf{F}_{ijk}] = \mathsf{P}_{ik}(\mathsf{P}_{jk} + \mathsf{P}_{jj}) - (\mathsf{P}_{jk} + \mathsf{P}_{kk})\mathsf{P}_{ij}, \tag{13.6.6d}$$

$$[\mathsf{P}_{k\ell}, \mathsf{F}_{ijk}] = \mathsf{P}_{ik}\mathsf{P}_{j\ell} - \mathsf{P}_{i\ell}\mathsf{P}_{jk}, \tag{13.6.6e}$$

$$[\mathsf{F}_{ijk}, \mathsf{F}_{jk\ell}] = -(\mathsf{F}_{ij\ell} + \mathsf{F}_{ik\ell})\mathsf{P}_{jk}, \tag{13.6.6f}$$

$$[\mathsf{F}_{ijk}, \mathsf{F}_{k\ell m}] = \mathsf{F}_{i\ell m}\mathsf{P}_{jk} - \mathsf{F}_{j\ell m}\mathsf{P}_{ik}, \tag{13.6.6g}$$

Ce qui est remarquable de cette algèbre $R(n)$ est que les relations peuvent être présentées de façon uniforme en termes d'indices. Autrement dit, tous les P_{ij} se comportent de façon uniforme. La raison derrière cela est que les générateurs de $\mathfrak{sl}_2$ sont des éléments primitifs. Autre remarque : un générateur P_{ij} (respectivement F_{ijk}) ne contient que des termes dans les espaces tensoriels i et j (respectivement i, j et k). L'identification des générateurs semble donc très naturelle car elle réflète bien leur contenu dans les espaces tensoriels et mène à une présentation symétrique des relations de l'algèbre.

Regardons maintenant ce qui arrive lorsqu'on q-déforme : on considère plutôt l'algèbre quantique $U_q(\mathfrak{sl}_2)$ et l'algèbre d'Askey–Wilson, mais beaucoup des remarques ci-haut ne tiennent plus. Une présentation symétrique des relations est possible dans le cas $n = 3$ pour l'algèbre $AW(3)$ dans la réalisation en termes de Casimir intermédiaires de $U_q(\mathfrak{sl}_2)$:

$$C_{12} + \frac{[C_{23}, C_{13}]_q}{q^2 - q^{-2}} = \frac{C_1C_2 + C_3C_{123}}{q + q^{-1}}, \tag{13.6.7a}$$

$$C_{23} + \frac{[C_{13}, C_{12}]_q}{q^2 - q^{-2}} = \frac{C_2C_3 + C_1C_{123}}{q + q^{-1}}, \tag{13.6.7b}$$

$$C_{13} + \frac{[C_{12}, C_{23}]_q}{q^2 - q^{-2}} = \frac{C_3C_1 + C_2C_{123}}{q + q^{-1}}, \tag{13.6.7c}$$

où les C_i, C_{12}, C_{23} et C_{123} sont les Casimir intermédiaires de $U_q(\mathfrak{sl}_2)$. Cependant, ici, le générateur C_{13} est difficile à interpréter et ne semble pas très naturel. Malgré le fait qu'il n'est étiqueté que par les indices 1 et 3, il contient en réalité des termes dans les espaces tensoriels 1, 2 et 3. Comment réconcilier cela avec la situation pour l'algèbre de Racah où tout semble clair?

Une première partie de réponse est donnée au chapitre 14. Dans cet article, on montre que l'élément C_{13} est obtenu de C_{12} par une conjugaison par la matrice R tressée de $U_q(\mathfrak{sl}_2)$. Ceci explique de façon simple les facteurs non-triviaux présents dans l'espace tensoriel 2.

Des travaux subséquents au chapitre 15 développent un calcul diagrammatique pour l'algèbre d'Askey–Wilson en exploitant sa ressemblance avec l'algèbre de skein du crochet de Kauffman pour une sphère à 4 trous. Cela permet notamment d'interpréter graphiquement la conjugaison par la matrice R tressée introduite au chapitre précédent ainsi que l'action du coproduit en terme de rotations de Dehn et de perforations de la sphère. Cela fournit également une autre perspective sur la présence de termes non-triviaux dans les espaces 1, 2 et 3 pour l'élément appelé C_{13}. On y formule également une conjecture que ces observations tiennent pour un rang arbitraire n. Cette conjecture semble naturelle et permet de rendre compte de tous les résultats obtenus à ce jour[3] sur les généralisations à plus haut rang de l'algèbre d'Askey–Wilson.

L'idée d'utiliser la matrice R tressée pour étudier les Casimir intermédiaires de $U_q(\mathfrak{sl}_2)$ est venue suite à l'observation que les Casimir intermédiaires appartiennent au centralisateur de $U_q(\mathfrak{sl}_2)$ dans $U_q(\mathfrak{sl}_2)^{\otimes 3}$: en utilisant les propriétés de la matrice R tressée et du coproduit, il est facile de montrer cela. L'algèbre d'Askey–Wilson est donc contenue dans le centralisateur. Mais est-ce le centralisateur complet, ou bien y a-t-il davantage de relations dans le centralisateur?

L'étude de cette question et l'utilisation de résultats classiques en théorie des invariants permet de répondre à cette question dans le cas non-déformé ($q = 1$). Au Chapitre 16, on montre qu'un quotient de l'algèbre de Racah de plus haut rang $R(n)$ est isomorphe au centralisateur $Z_n(\mathfrak{sl}_2)$ de $U(\mathfrak{sl}_2)$ dans $U(\mathfrak{sl}_2)^{\otimes n}$. Ce quotient est donné explicitement. Ceci boucle en quelque sorte la boucle entamée à la Partie 1, dans laquelle la théorie de la dualité de Howe, qui « transcend » la théorie classique des invariants était étudiée. Les résultats au dernier chapitre donnent un analogue non-commutatif des Premiers et Seconds Théorèmes Fondamentaux de théorie des invariants.

[3]au moment présent, lors de l'écriture de cette thèse...

Chapitre 14

Revisiting the Askey–Wilson algebra with the universal R-matrix of $U_q(\mathfrak{sl}_2)$

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Abstract: A description of the embedding of a centrally extended Askey–Wilson algebra, $AW(3)$, in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ is given in terms of the universal R-matrix of $U_q(\mathfrak{sl}_2)$. The generators of the centralizer of $U_q(\mathfrak{sl}_2)$ in its three-fold tensor product are naturally expressed through conjugations of Casimir elements with R. They are seen as the images of the generators of $AW(3)$ under the embedding map by showing that they obey the $AW(3)$ relations. This is achieved by introducing a natural coaction also constructed with the help of the R-matrix.

14.1. Introduction

This letter addresses a long-standing question regarding the intrinsic description of the generators of a centrally extended Askey–Wilson algebra in its embedding into $U_q(\mathfrak{sl}_2)^{\otimes 3}$. The answer will be shown to involve Casimir elements and the universal R-matrix of $U_q(\mathfrak{sl}_2)$.

The Askey–Wilson algebra can be defined with three generators and relations. It has first been introduced [1] as the algebra realized by the recurrence and q-difference operators intervening in the bispectral problem associated to the Askey–Wilson polynomials [2]. This explains the name. Since the structure relations are not affected by truncations, this algebra also encodes the properties of the q-Racah polynomials. Owing to the connection with these $6j$ or Racah coefficients for $U_q(\mathfrak{sl}_2)$ [3], a centrally extended Askey–Wilson algebra $AW(3)$

can be realized as the centralizer of the diagonal action of this quantum algebra in its threefold tensor product. Related are the references [4–7]. In this context, two generators of $AW(3)$ are naturally mapped under the coproduct onto the intermediate Casimir elements corresponding respectively to the recouplings of the first and last two factors in $U_q(\mathfrak{sl}_2)^{\otimes 3}$. A natural algebraic interpretation of the image of the third generator has however been lacking. This was circumvented so far by using one of the relations which gives the third generator as the q-commutator of the other two; while this allows the homomorphism from $AW(3)$ into $U_q(\mathfrak{sl}_2)^{\otimes 3}$ to be defined, the resulting expression for this third generator is far from illuminating. Note that all three generators are needed to provide a PBW basis for $AW(3)$. Besides the fact that this leaves a picture which is not fully satisfactory, this is a serious shortcoming in attempts to generalize $AW(3)$ to algebras of higher ranks. The natural approach - in fact the only one that has been conceived - is to define $AW(n)$ as the centralizer of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes n}$. Proceeding with such an extension calls for an algebraic understanding of all centralizing elements in the tensor product. Significant progress towards describing the algebras $AW(n)$ have been made nevertheless. The algebra $AW(4)$ has been explored in [8] by including generators defined through the q-commutators of coproduct images of the Casimir element, as done for $AW(3)$, and obtaining from there various structure relations. Meaningful results have thus been found. The identification of the general $AW(n)$ has been attacked and largely advanced in [9, 10]. Much has been achieved in this case by cleverly designing a coaction map that has been used to define the generators, starting from the Casimir element of $U_q(\mathfrak{sl}_2)$, so as to ensure that these generators obey a q-deformation of natural structure relations (i.e. those of the generalized Bannai–Ito algebra $BI(n)$ [11]) and by proving that this is so in many (but not all) cases. Still, in spite of this progress, an a priori algebraic description of the generators remained much desired.

We shall here settle this issue for $AW(3)$ by providing a simple expression for the image of its third generator in $U_q(\mathfrak{sl}_2)^{\otimes 3}$. The formula will involve conjugation with the universal R-matrix of $U_q(\mathfrak{sl}_2)$ and will be seen to explain the origin of the coaction introduced in [9]. Basic facts about $U_q(\mathfrak{sl}_2)$ and its universal R-matrix are collected in Section 14.2. Section 14.3 focuses on the centralizer of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$; it provides the algebraic description that was missing. An additional centralizing element, conjugated to the usual third generator of $AW(3)$ is also identified; this will be related to observations made in [8]. The universal R-matrix and the Yang–Baxter equation are central here. With the expressions for the generators (in $U_q(\mathfrak{sl}_2)^{\otimes 3}$) in hand, Section 15.2.1 looks at their products and recovers the $AW(3)$ relations. To that end, a map from $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ defined in terms of the R-matrix is introduced. It is pointed out that this map, once spelled out, coincides with the coaction used in [9]. The letter concludes with final remarks stressing the advantages of bringing the universal R-matrix in the description of the algebras $AW(n)$. As an illustration it is shown that a computation in $AW(4)$ can be performed with these tools in a comparatively much simpler way than otherwise.

14.2. $U_q(\mathfrak{sl}_2)$ and its universal R-matrix

In this section, we recall the definitions of the quantum algebra $U_q(\mathfrak{sl}_2)$ and of its universal R-matrix as well as some of their properties. This allows to fix the notations and to make this letter more self-contained.

The associative algebra $U_q(\mathfrak{sl}_2)$ is generated by E, F and q^H with the defining relations

$$q^H E = qEq^H, \qquad q^H F = q^{-1} F q^H \qquad \text{and} \qquad [E,F] = [2H]_q, \tag{14.2.1}$$

where $[X]_q = \frac{q^X - q^{-X}}{q - q^{-1}}$ and $q \neq \pm 1, \pm i$. The center of this algebra is generated by the following Casimir element

$$C = -\frac{(q-q^{-1})^2}{q+q^{-1}} \left(FE + \frac{qq^{2H} + q^{-1}q^{-2H}}{(q-q^{-1})^2} \right). \tag{14.2.2}$$

The normalization of the Casimir element C is irrelevant but chosen to yield computational simplifications. There exists a homomorphism $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$, called comultiplication, defined by

$$\Delta(E) = E \otimes q^{-H} + q^H \otimes E, \qquad \Delta(F) = F \otimes q^{-H} + q^H \otimes F \qquad \text{and} \qquad \Delta(q^H) = q^H \otimes q^H. \tag{14.2.3}$$

We recall that this comultiplication is coassociative

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta. \tag{14.2.4}$$

The quantum algebra $U_q(\mathfrak{sl}_2)$ is quasi-triangular because there exists a universal R-matrix $\mathcal{R} \in U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ which is invertible and satisfies

$$\Delta(x)\mathcal{R} = \mathcal{R}\Delta^{op}(x), \qquad \text{for } x \in U_q(\mathfrak{sl}_2), \tag{14.2.5}$$

where the opposite comultiplication $\Delta^{op}(x) = x_{(2)} \otimes x_{(1)}$ if $\Delta(x) = x_{(1)} \otimes x_{(2)}$ in the Sweedler notation, and

$$(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{12}\mathcal{R}_{13} \qquad \text{and} \qquad (\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{23}\mathcal{R}_{13}. \tag{14.2.6}$$

In the previous relation (14.2.6), we have used the usual notations $\mathcal{R}_{12} = \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha$ and $\mathcal{R}_{13} = \mathcal{R}^\alpha \otimes 1 \otimes \mathcal{R}_\alpha$ where $\mathcal{R} = \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha$ (the sum w.r.t. α is understood). We will also use the following element

$$\widetilde{\mathcal{R}} = \mathcal{R}_{21} = \mathcal{R}_\alpha \otimes \mathcal{R}^\alpha, \tag{14.2.7}$$

satisfying

$$\Delta^{op}(x)\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}\Delta(x), \qquad \text{for } x \in U_q(\mathfrak{sl}_2). \tag{14.2.8}$$

The universal R-matrix also satisfies the Yang–Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \tag{14.2.9}$$

and takes the following explicit form [12]

$$\mathcal{R} = q^{2(H\otimes H)} \sum_{n=0}^{\infty} \frac{(q-q^{-1})^n}{[n]_q!} q^{n(n-1)/2} (Eq^H \otimes q^{-H}F)^n, \tag{14.2.10}$$

where $[n]_q! = [n]_q[n-1]_q \ldots [2]_q[1]_q$ and, by convention, $[0]_q! = 1$. For future convenience, by using the commutation relations of $U_q(\mathfrak{sl}_2)$, we rewrite $\widetilde{\mathcal{R}}$ as follows

$$\widetilde{\mathcal{R}} = \sum_{n=0}^{\infty} \frac{(q-q^{-1})^n}{[n]_q!} q^{n(n-1)/2} (Fq^H \otimes q^{-H}E)^n \;\; q^{2(H\otimes H)} = \Theta \; q^{2(H\otimes H)}. \tag{14.2.11}$$

14.3. Centralizer of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$

In this section, we want to describe the centralizer $\mathfrak{C}_3$ of the diagonal action of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$:

$$\mathfrak{C}_3 = \left\{ X \in U_q(\mathfrak{sl}_2)^{\otimes 3} \;\middle|\; [(\Delta\otimes \mathrm{id})\Delta(x), X] = 0, \qquad \forall x \in U_q(\mathfrak{sl}_2) \right\}. \tag{14.3.1}$$

Let us define the so-called intermediate Casimir elements (in Sweedler's notation)

$$C_{12} = \Delta(C)\otimes 1 = C_{(1)}\otimes C_{(2)}\otimes 1 \qquad \text{and} \qquad C_{23} = 1\otimes\Delta(C) = 1\otimes C_{(1)}\otimes C_{(2)}, \tag{14.3.2}$$

and the total Casimir element

$$C_{123} = (\Delta\otimes \mathrm{id})\Delta(C). \tag{14.3.3}$$

We define also $C_1 = C\otimes 1\otimes 1$, $C_2 = 1\otimes C\otimes 1$ and $C_3 = 1\otimes 1\otimes C$. By using that the Casimir element C is central in $U_q(\mathfrak{sl}_2)$, we deduce for example that

$$[(\Delta\otimes \mathrm{id})\Delta(x), C_{12}] = 0 \qquad \text{and} \qquad [(\mathrm{id}\otimes\Delta)\Delta(x), C_{23}] = 0, \qquad \forall x \in U_q(\mathfrak{sl}_2). \tag{14.3.4}$$

By definition (14.3.1), C_1, C_2, C_3, C_{12}, C_{23} and C_{123} belong to the centralizer $\mathfrak{C}_3$ with C_1, C_2, C_3 and C_{123} belonging to the center of $\mathfrak{C}_3$. It is well-known that these elements satisfy the Askey–Wilson algebra [1]. We will come back to this point in Section 14.4.

In the case of $U(\mathfrak{sl}_2)$ (*i.e.* the limit $q\to 1$ of the case studied here), one can also prove that the intermediate Casimir $C_{13} = C_{(1)}\otimes 1\otimes C_{(2)}$ belongs to the centralizer. For $q\neq 1$, it is not the case and the main objective of this letter is to provide a definition of this element for the quantum algebra.

Theorem 14.1. *The following elements of* $U_q(\mathfrak{sl}_2)^{\otimes 3}$

$$C_{13}^{(0)} = \widetilde{\mathcal{R}}_{23}^{-1} C_{13} \widetilde{\mathcal{R}}_{23} = \mathcal{R}_{12} C_{13} \mathcal{R}_{12}^{-1}, \tag{14.3.5a}$$

$$C_{13}^{(1)} = \widetilde{\mathcal{R}}_{12}^{-1} C_{13} \widetilde{\mathcal{R}}_{12} = \mathcal{R}_{23} C_{13} \mathcal{R}_{23}^{-1}. \tag{14.3.5b}$$

are in the centralizer $\mathfrak{C}_3$*, where* $\mathcal{R}$ *and* $\widetilde{\mathcal{R}}$ *are defined in* (15.6.8) *and* (14.2.11) *and* $C_{13} = C_{(1)} \otimes 1 \otimes C_{(2)}$.

Proof. By using the coassociativity of the comultiplication (15.6.4) and by conjugating with $\mathcal{R}_{23}$, the first relation in (14.3.4) reads

$$[(\mathrm{id} \otimes \Delta^{op})\Delta(x), \mathcal{R}_{23}^{-1} C_{12} \mathcal{R}_{23}] = 0. \tag{14.3.6}$$

Finally, by exchanging the spaces 2 and 3, one gets that $C_{13}^{(0)}$ is in the centralizer

$$[(\mathrm{id} \otimes \Delta)\Delta(x), \underbrace{\widetilde{\mathcal{R}}_{23}^{-1} C_{13} \widetilde{\mathcal{R}}_{23}}_{=C_{13}^{(0)}}] = 0. \tag{14.3.7}$$

One proves similarly that $\mathcal{R}_{12} C_{13} \mathcal{R}_{12}^{-1}$, $\widetilde{\mathcal{R}}_{12}^{-1} C_{13} \widetilde{\mathcal{R}}_{12}$ and $\mathcal{R}_{23} C_{13} \mathcal{R}_{23}^{-1}$ are in the centralizer $\mathfrak{C}_3$.

We must prove also the equality between $\widetilde{\mathcal{R}}_{23}^{-1} C_{13} \widetilde{\mathcal{R}}_{23}$ and $\mathcal{R}_{12} C_{13} \mathcal{R}_{12}^{-1}$. One gets

$$C_{13}^{(0)} = \widetilde{\mathcal{R}}_{23}^{-1} C_{13} \widetilde{\mathcal{R}}_{23} = \widetilde{\mathcal{R}}_{23}^{-1} \left(C_{(1)} \otimes 1 \otimes C_{(2)} \right) \widetilde{\mathcal{R}}_{23} = \widetilde{\mathcal{R}}_{23}^{-1} \mathcal{R}_{13} \left(C_{(2)} \otimes 1 \otimes C_{(1)} \right) \mathcal{R}_{13}^{-1} \widetilde{\mathcal{R}}_{23}, \tag{14.3.8}$$

where we have used property (15.6.5). The Yang–Baxter equation (14.2.9) implies that

$$C_{13}^{(0)} = \mathcal{R}_{12} \mathcal{R}_{13} \widetilde{\mathcal{R}}_{23}^{-1} \mathcal{R}_{12}^{-1} \left(C_{(2)} \otimes 1 \otimes C_{(1)} \right) \mathcal{R}_{12} \widetilde{\mathcal{R}}_{23} \mathcal{R}_{13}^{-1} \mathcal{R}_{12}^{-1}. \tag{14.3.9}$$

Now, from (14.2.6), one deduces that $[\Delta(C) \otimes 1, (\Delta \otimes \mathrm{id})(\mathcal{R})] = [\Delta(C) \otimes 1, \mathcal{R}_{23}\mathcal{R}_{13}] = 0$ and that $[\left(C_{(2)} \otimes 1 \otimes C_{(1)} \right), \mathcal{R}_{12} \widetilde{\mathcal{R}}_{23}] = 0$. Then, one obtains

$$C_{13}^{(0)} = \mathcal{R}_{12} \mathcal{R}_{13} \left(C_{(2)} \otimes 1 \otimes C_{(1)} \right) \mathcal{R}_{13}^{-1} \mathcal{R}_{12}^{-1} = \mathcal{R}_{12} C_{13} \mathcal{R}_{12}^{-1}. \tag{14.3.10}$$

The equality between $\widetilde{\mathcal{R}}_{12}^{-1} C_{13} \widetilde{\mathcal{R}}_{12}$ and $\mathcal{R}_{23} C_{13} \mathcal{R}_{23}^{-1}$ is proven similarly. $\square$

From relations (14.3.5a) and (14.3.5b), one deduces that $C_{13}^{(0)}$ and $C_{13}^{(1)}$ are conjugated:

$$C_{13}^{(1)} = \mathcal{R}_{23} \widetilde{\mathcal{R}}_{23} C_{13}^{(0)} (\mathcal{R}_{23} \widetilde{\mathcal{R}}_{23})^{-1} = (\mathcal{R}_{12} \widetilde{\mathcal{R}}_{12})^{-1} C_{13}^{(0)} \mathcal{R}_{12} \widetilde{\mathcal{R}}_{12}. \tag{14.3.11}$$

14.4. The Askey–Wilson algebra $AW(3)$

In this section, we study the algebra satisfied by the intermediate Casimir elements introduced in the previous section and connect it with the central extension $AW(3)$ of the Askey–Wilson algebra introduced in [1]. We start by proving the following lemma.

Lemma 14.2. *The map defined by*

$$\begin{aligned}\tau\ :\ U_q(\mathfrak{sl}_2) &\to U_q(\mathfrak{sl}_2)\otimes U_q(\mathfrak{sl}_2)\\ x &\mapsto \widetilde{\mathcal{R}}^{-1}(1\otimes x)\widetilde{\mathcal{R}}\end{aligned} \tag{14.4.1}$$

yields the following explicit expressions when acting on the different elements of $U_q(\mathfrak{sl}_2)$ listed below:

$$\tau(C) = 1\otimes C, \tag{14.4.2a}$$

$$\tau(q^{-H}E) = q^{-2H}\otimes q^{-H}E, \tag{14.4.2b}$$

$$\tau(q^{-2H}) = 1\otimes q^{-2H} - (q-q^{-1})^2 q^{-H}F\otimes q^{-H}E, \tag{14.4.2c}$$

$$\tau(Fq^{-H}) = q^{2H}\otimes Fq^{-H} + q^{-1}(q+q^{-1})Fq^{H}\otimes(C+q^{-2H}) - (q-q^{-1})^2F^2\otimes q^{-H}E. \tag{14.4.2d}$$

Proof. We must prove that the map given in the theorem reproduces relations (14.4.2a)-(14.4.2d). For relation (14.4.2a), it is direct, knowing that C commutes with any element of $U_q(\mathfrak{sl}_2)$. To prove relation (14.4.2b), one computes (using the explicit form (14.2.11) of $\widetilde{\mathcal{R}}$)

$$\tau(q^{-H}E) = \widetilde{\mathcal{R}}^{-1}(1\otimes q^{-H}E)\Theta\ q^{2(H\otimes H)} = \widetilde{\mathcal{R}}^{-1}\Theta\ (1\otimes q^{-H}E)q^{2(H\otimes H)} = q^{-2H}\otimes q^{-H}E, \tag{14.4.3}$$

which reproduces (14.4.2b).

We want now to compute $\tau(q^{-2H})$:

$$\begin{aligned}\tau(q^{-2H}) &= \widetilde{\mathcal{R}}^{-1}(1\otimes q^{-2H})\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}^{-1}(1\otimes q^{-2H})q^{2(H\otimes H)}\sum_{n=0}^{\infty}a_n(q^{-H}F\otimes Eq^{H})^n\\ &= \widetilde{\mathcal{R}}^{-1}q^{2(H\otimes H)}\sum_{n=0}^{\infty}a_n q^{-2n}(q^{-H}F\otimes Eq^{H})^n\ (1\otimes q^{-2H}),\end{aligned} \tag{14.4.4}$$

where we have introduced the parameters

$$a_n = \frac{(q-q^{-1})^n}{[n]_q!}q^{n(n-1)/2}. \tag{14.4.5}$$

Remarking that

$$a_n q^{-2n} = a_n - a_n[n]_q q^{-n}(q-q^{-1}), \tag{14.4.6}$$

one gets

$$\tau(q^{-2H}) = \widetilde{\mathcal{R}}^{-1}\left(\widetilde{\mathcal{R}} - q^{2(H\otimes H)}\sum_{n=0}^{\infty}a_{n+1}[n+1]_q q^{-(n+1)}(q-q^{-1})(q^{-H}F\otimes Eq^{H})^{n+1}\right)(1\otimes q^{-2H}). \tag{14.4.7}$$

It is easy to show that the parameters a_n satisfy $a_{n+1}[n+1]_q = q^n(q-q^{-1})a_n$, which allows to recover (14.4.2c).

Similarly, to prove (14.4.2d), one computes

$$\begin{aligned}\tau(Fq^{-H}) &= \widetilde{\mathcal{R}}^{-1}(1\otimes Fq^{-H})\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}^{-1}(1\otimes Fq^{-H})q^{2(H\otimes H)}\sum_{n=0}^{\infty}a_n(q^{-H}F\otimes Eq^{H})^n \\ &= \widetilde{\mathcal{R}}^{-1}q^{2(H\otimes H)}(q^{2H}\otimes Fq^{-H})\sum_{n=0}^{\infty}a_n(q^{-H}F\otimes Eq^{H})^n.\end{aligned}\tag{14.4.8}$$

Then, the identity

$$[F,E^n] = \frac{[n]_q}{q-q^{-1}}(q^{n-1}q^{-2H} - q^{-(n-1)}q^{2H})E^{n-1}\tag{14.4.9}$$

can be used to write

$$\begin{aligned}\tau(Fq^{-H}) &= \widetilde{\mathcal{R}}^{-1}q^{2(H\otimes H)} \\ &\quad\times\sum_{n=0}^{\infty}a_n(q^{-H}F\otimes Eq^{H})^n\left(q^{-2n}q^{2H}\otimes Fq^{-H} + q^{-2}Fq^{H}\otimes(q^{-2n}q^{-2H}-q^{2H})\right).\end{aligned}\tag{14.4.10}$$

Using again relation (14.4.6), one finds

$$\begin{aligned}\tau(Fq^{-H}) &= q^{2H}\otimes Fq^{-H} - q^{-1}(q-q^{-1})^2Fq^{H}\otimes FE - Fq^{H}\otimes(q^{2H}-q^{-2H}) \\ &\quad - (q-q^{-1})^2F^2\otimes q^{-H}E.\end{aligned}\tag{14.4.11}$$

Finally, expressing FE in terms of C from definition (14.2.2), one recovers (14.4.2d). □

Using Lemma 14.2, we can rewrite $C_{13}^{(0)}$ (14.3.5a) as follows

$$\begin{aligned}C_{13}^{(0)} &= (1\otimes\tau)\Delta(C) &(14.4.12)\\ &= (q^{2H}+C)\otimes\tau(q^{-2H}) + q^{2H}\otimes\tau(C) \\ &\qquad - \frac{(q-q^{-1})^2}{q+q^{-1}}\left(q^{H}E\otimes\tau(Fq^{-H}) + Fq^{H}\otimes\tau(q^{-H}E)\right). &(14.4.13)\end{aligned}$$

Proposition 14.3. *The following relation*

$$\frac{1}{q-q^{-1}}[C_{12},C_{23}]_q = C_{13}^{(0)} + C_1C_3 + C_2C_{123}\tag{14.4.14}$$

holds in $U_q(\mathfrak{sl}_2)^{\otimes 3}$.

Proof. Using the expressions for the maps under τ given in Lemma 14.2, we obtain $C_{13}^{(0)}$ in terms of the generators of $U_q(\mathfrak{sl}_2)$. A direct computation using the commutation relations of $U_q(\mathfrak{sl}_2)$ proves the relation of the proposition. □

One of the advantages of the construction with the universal R-matrix is that we can deduce all the other defining relations of $AW(3)$ from (14.4.14) and some other relations.

Corollary 14.4. *The following relations*

$$\frac{1}{q-q^{-1}}[C_{13}^{(0)}, C_{12}]_q = C_{23} + C_2C_3 + C_1C_{123}, \tag{14.4.15a}$$

$$\frac{1}{q-q^{-1}}[C_{23}, C_{13}^{(0)}]_q = C_{12} + C_1C_2 + C_3C_{123}, \tag{14.4.15b}$$

$$\frac{1}{q-q^{-1}}[C_{23}, C_{12}]_q = C_{13}^{(1)} + C_1C_3 + C_2C_{123}, \tag{14.4.15c}$$

$$\frac{1}{q-q^{-1}}[C_{12}, C_{13}^{(1)}]_q = C_{23} + C_2C_3 + C_1C_{123}, \tag{14.4.15d}$$

$$\frac{1}{q-q^{-1}}[C_{13}^{(1)}, C_{23}]_q = C_{12} + C_1C_2 + C_3C_{123}, \tag{14.4.15e}$$

hold in $U_q(\mathfrak{sl}_2)^{\otimes 3}$.

Proof. We use the second relation in (14.3.5a) as well as the definitions (14.3.3) to write relation (14.4.14) as follows

$$\frac{1}{q-q^{-1}}[\Delta(C)\otimes 1, C_{23}]_q = \mathcal{R}_{12}C_{13}\mathcal{R}_{12}^{-1} + C_1C_3 + C_2(\Delta\otimes\mathrm{id})\Delta(C). \tag{14.4.16}$$

Exchanging the spaces 1 and 2, the previous relation becomes

$$\frac{1}{q-q^{-1}}[\Delta^{op}(C)\otimes 1, C_{13}]_q = \widetilde{\mathcal{R}}_{12}C_{23}\widetilde{\mathcal{R}}_{12}^{-1} + C_2C_3 + C_1(\Delta^{op}\otimes\mathrm{id})\Delta(C), \tag{14.4.17}$$

which leads to (14.4.15d) after conjugating by $\widetilde{\mathcal{R}}_{12}$ (using property (14.2.8)).

Then, one starts from the relation (14.4.15d) we have just proven, uses the second relation in (14.3.5b) to express $C_{13}^{(1)}$ and exchanges spaces 2 and 3 to write

$$\frac{1}{q-q^{-1}}[C_{13}, \widetilde{R}_{23}C_{12}\widetilde{R}_{23}^{-1}]_q = 1\otimes\Delta^{op}(C) + C_2C_3 + C_1(\mathrm{id}\otimes\Delta^{op})\Delta(C). \tag{14.4.18}$$

Conjugating with $\widetilde{R}_{23}$, one proves relation (14.4.15a). Performing the same two steps starting from (14.4.15a), one proves (14.4.15c) and (14.4.15b). Finally, the two same steps prove (14.4.15e) and give again the equation (14.4.14). □

We now have a number of remarks regarding the merits of the R-matrix approach developed above.

Remark 14.5. *Relations* (14.4.14), (14.4.15a) *and* (14.4.15b) *are the defining relations of central extension* $AW(3)$ *of the Askey–Wilson algebra introduced in [1]. Therefore, the results presented in this letter offer another proof that the intermediate Casimir elements of* $U_q(\mathfrak{sl}_2)$ *provide a realization of* $AW(3)$. *In previous works [1, 6, 8, 9, 13],* $C_{13}^{(0)}$ *was defined by relation* (14.4.14) *whereas in our approach, it is defined independently of the commutation relations via relation* (14.3.5a).

Remark 14.6. *The map* τ *with images given by* (14.4.2a)-(14.4.2d) *has in fact been introduced in [9, 13] so as to obtain* $C_{13}^{(0)}$ *as in relation* (14.4.12). *Our definition* (14.4.1) *gives a* *nice and powerful interpretation of this map. Let us remark that the comultiplication used in this letter is slightly different from the ones used in [9, 13]. In order to establish exactly the correspondence, the following transformation on our generators and Casimir element must be performed:* $q^{2H} \to K$, $E \to EK^{-1/2}$, $F \to K^{1/2}F$, $q \to Q$ *and* $C \to -\Lambda/(Q + Q^{-1})$.

Remark 14.7. *To illustrate the appropriateness and advantages of definition* (14.4.1) *of the map* τ, *we here prove its coaction property in a much simpler way than the direct calculation described in [9, 13]. Using relation* (14.2.6), *it is easy to compute, for* $x \in U_q(\mathfrak{sl}_2)$,

$$(\Delta \otimes \mathrm{id})\tau(x) = (\Delta \otimes \mathrm{id}) \left(\widetilde{\mathcal{R}}^{-1}(1 \otimes x)\widetilde{\mathcal{R}}\right) = \widetilde{\mathcal{R}}_{23}^{-1}\widetilde{\mathcal{R}}_{13}^{-1}(1 \otimes 1 \otimes x)\widetilde{\mathcal{R}}_{13}\widetilde{\mathcal{R}}_{23}, \tag{14.4.19}$$

and

$$(\mathrm{id} \otimes \tau)\tau(x) = (\mathrm{id} \otimes \tau) \left(\widetilde{\mathcal{R}}^{-1}(1 \otimes x)\widetilde{\mathcal{R}}\right) = \widetilde{\mathcal{R}}_{23}^{-1}\widetilde{\mathcal{R}}_{13}^{-1}(1 \otimes 1 \otimes x)\widetilde{\mathcal{R}}_{13}\widetilde{\mathcal{R}}_{23}. \tag{14.4.20}$$

This proves that $(\Delta \otimes \mathrm{id})\tau(x) = (\mathrm{id} \otimes \tau)\tau(x)$ *and thus that* τ *is a left coaction.*

Remark 14.8. *We can define also a right coaction* $\check{\tau}$ *given by*

$$\begin{aligned} \check{\tau} \ : \ U_q(\mathfrak{sl}(2)) &\to U_q(\mathfrak{sl}(2)) \otimes U_q(\mathfrak{sl}(2)) \\ x &\mapsto \mathcal{R}(x \otimes 1)\mathcal{R}^{-1}, \end{aligned} \tag{14.4.21}$$

satisfying

$$(\check{\tau} \otimes \mathrm{id})\check{\tau} = (\mathrm{id} \otimes \Delta)\check{\tau}. \tag{14.4.22}$$

We can show following steps similar to those of the proof of Lemma 14.2 that this right coaction coincides with the one introduced recently in [10] with the identifications: $q^{2H} \to K$, $E \to EK^{-1/2}$, $F \to K^{1/2}F$ *and* $C \to -\Lambda/(q + q^{-1})$.

Remark 14.9. *The element* $C_{13}^{(1)}$ *has been introduced previously in [8] (where it is called* $IQ^{(13)}$*) and defined by relation* (14.4.15c). *Our definition* (14.3.5b) *gives a new interpretation of this element.*

14.5. Conclusion and perspective

In this letter, we study the centralizer of the diagonal action of $U_q(\mathfrak{sl}_2)$ and its connection with the Askey–Wilson algebra $AW(3)$. In comparison with the previous approaches, we have emphasized the relevance of the universal R-matrix of $U_q(\mathfrak{sl}_2)$. We believe that its use offers a deeper understanding of the realization of the Askey–Wilson algebra in terms of the intermediate Casimir elements. It should moreover simplify the computations for further investigations. To illustrate this point, let us show how one computation can be simplified with this approach in the higher rank generalization $AW(4)$ of $AW(3)$ examined in [8]. The algebra $AW(4)$ can be embedded in $U_q(\mathfrak{sl}_2)^{\otimes 4}$ and, in particular, one defines

$$C_{13}^{(0)} = \widetilde{\mathcal{R}}_{23}^{-1} C_{13} \widetilde{\mathcal{R}}_{23} = \mathcal{R}_{12} C_{13} \mathcal{R}_{12}^{-1} \ , \tag{14.5.1a}$$

$$C_{24}^{(1)} = \widetilde{\mathcal{R}}_{23}^{-1} C_{24} \widetilde{\mathcal{R}}_{23} = \mathcal{R}_{34} C_{24} \mathcal{R}_{34}^{-1} \ . \tag{14.5.1b}$$

Looking at the commutation relations, we can prove that these elements correspond to $Q^{(13)}$ and $IQ^{(24)}$ of [8]. In the formalism introduced here, we see immediately that

$$[C_{13}^{(0)}, C_{24}^{(1)}] = 0, \tag{14.5.2}$$

whereas the proof without the use of the R-matrix presented in [8] is quite cumbersome. We believe that the R-matrix approach we have elaborated will prove quite helpful in the study of the higher rank generalizations of $AW(3)$. In a related series of papers [14, 15], the Temperley–Lieb algebra with $q = 1$, the Brauer algebra (and others) over 3 strands have been identified as quotients of the Racah [16] and Bannai–Ito [17] algebras of rank 1. The results reported here pave the way to the pursuit of this program for $AW(3)$ as well as in situations of higher ranks with an arbitrary number of strands. It is our intent to actively continue this research. Let us mention finally that, in a companion letter [18], we have provided a parallel description of the Bannai–Ito algebras using the universal R-matrix of the Lie superalgebra $\mathfrak{osp}(1|2)$.

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Chapitre 15

The Askey–Wilson algebra and its avatars

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Abstract: The original Askey–Wilson algebra introduced by Zhedanov encodes the bispectrality properties of the eponym polynomials. The name *Askey–Wilson algebra* is currently used to refer to a variety of related structures that appear in a large number of contexts. We review these versions, sort them out and establish the relations between them. We focus on two specific avatars. The first is a quotient of the original Zhedanov algebra; it is shown to be invariant under the Weyl group of type D_4 and to have a reflection algebra presentation. The second is a universal analogue of the first one; it is isomorphic to the Kauffman bracket skein algebra (KBSA) of the four-punctured sphere and to a subalgebra of the universal double affine Hecke algebra $(C_1^\vee, C_1)$. This second algebra emerges from the Racah problem of $U_q(\mathfrak{sl}_2)$ and is related via an injective homomorphism to the centralizer of $U_q(\mathfrak{sl}_2)$ in its threefold tensor product. How the Artin braid group acts on the incarnations of this second avatar through conjugation by R-matrices (in the Racah problem) or half Dehn twists (in the diagrammatic KBSA picture) is also highlighted. Attempts at defining higher rank Askey–Wilson algebras are briefly discussed and summarized in a diagrammatic fashion.

Keywords: Askey–Wilson algebra, Kauffman bracket skein algebra, $U_q(\mathfrak{sl}_2)$ algebra, double affine Hecke algebra, centralizer, universal R-matrix, $W(D_4)$ Weyl group, half Dehn twist.

15.1. Introduction

In order to provide an algebraic underpinning for the Askey–Wilson polynomials [1], Zhedanov introduced what he called the Askey–Wilson algebra [2]. We shall refer to it rather as the Zhedanov algebra. The Askey–Wilson polynomials sit at the top of the Askey classification scheme of the hypergeometric orthogonal polynomials [3] and are, consequently, of fundamental interest; their algebraic interpretation by Zhedanov hence bears commensurate importance. These q-polynomials are bispectral: in addition to verifying a three-term recurrence prescribed by Favard's theorem for any family of orthogonal polynomials [4], they are also eigenfunctions of a q-difference operator. The Zhedanov algebra was constructed by taking these two bispectral operators as generators and identifying the relations they obey. As sometimes happens with natural constructs, related structures have emerged in a variety of contexts and have typically all been called Askey–Wilson algebras. This propensity keeps rising and it is hence timely to review the topic. This paper will provide a taxonomy and a description of the algebras that loosely go under the name of *Askey–Wilson algebras* and will characterize in some depth two avatars of particular relevance. It will also set the stage for the exploration of generalizations.

The focus of this survey will be on algebraic aspects. Before we discuss the contents in more details, let us briefly go over some of the manifestations of these Askey–Wilson algebras and the advances they have generated. Grosso modo, they have had direct applications in physical models and have also been at the heart of mathematical developments establishing useful interconnections between fields. One occurrence is in the recoupling of three irreducible representations of $U_q(\mathfrak{sl}_2)$ which is called its Racah problem. It is known that the $6j$-symbols of this algebra are expressed in terms of q-Racah polynomials which are a finite truncation of the Askey–Wilson ones. As a rule, whenever the Askey–Wilson polynomials (or their truncated version) appear, the associated algebra will be present. In the case of the Racah problem, it is found that the intermediate Casimir elements verify Askey–Wilson relations [5, 6]. These polynomials and algebras appear in the study of the ASEP model with open boundaries [7], as martingale polynomials and quadratic harnesses in probabilistic models [8] and are connected to (a degeneration of) the Sklyanin algebra [9–11]. Quite generally, the Askey–Wilson algebras are present in the context of integrable models, through the Yang–Baxter and reflection equations [12–17], and can be viewed as truncations of the q-Onsager algebra [12]. Elements of representation theory have been investigated in [2, 6, 18–20] and another of its manifestations is as a coideal subalgebra of $U_q(\mathfrak{sl}_2)$ [21–23]. The Askey–Wilson algebras have also been cast in the framework of Howe duality using the pair $(U_q(\mathfrak{sl}_2), \mathfrak{o}_{q^{1/2}}(2n))$ [24–27]; they are special cases of the recently introduced Painlevé algebras [28] and belong to the Calabi–Yau class [29]. There is a significant connection to the field of algebraic combinatorics, as Askey–Wilson algebras are central in the classification of P-

and Q- polynomial association schemes and the study of Leonard pairs and triples [30–35]. The Askey–Wilson algebras have also been shown to offer a promising platform to extend the quantum Schur–Weyl duality to arbitrary representations and have been seen in that respect to admit the Temperley–Lieb and Birman–Murakami–Wenzl algebras [36] as quotients. Askey–Wilson algebras have moreover found their way in the general framework of knot theory through their identification with the Kauffman bracket skein algebras of the four-punctured sphere $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ and other elementary surfaces [37–39]. This is also closely connected to double affine Hecke algebras (DAHA) as the Askey–Wilson algebra is related to the spherical subalgebra of the DAHA of type $(C_1^\vee, C_1)$ [20, 28, 40–46].

This overview of the relevance of Askey–Wilson algebras in different domains motivates the present topical report. Let us make at this point a few additional remarks on the introduction of the algebra $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ in the Askey–Wilson picture to stress that this paper also features novel results relating the Askey–Wilson algebra, the Kauffman bracket skein algebra and the braid group.

Kauffman bracket skein algebras (KBSA) have been defined independently by Turaev [47] and Bullock and Przytycki [37] in the study of knot invariants and can be seen to encompass the celebrated Jones polynomial [48, 49]. Computations in the KBSA are done through diagrammatic manipulations given by a set of rules (the skein relations). It is appreciated that this $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ algebra is closely related to the centralizer of $U_q(\mathfrak{sl}_2)$ in its threefold tensor product. This ties in with the Temperley–Lieb algebra which admits a diagrammatic presentation [49–51] for generic q, is precisely the centralizer of $U_q(\mathfrak{sl}_2)$ in the threefold tensor product of the fundamental representations of $U_q(\mathfrak{sl}_2)$ [52] and, as already indicated, was found to be a quotient of the Askey–Wilson algebra [36].

A natural question that has arisen asks about higher rank extensions of Askey–Wilson algebras. In view of the ubiquity of the 3-generated Askey–Wilson algebras it is to be expected that such generalizations will prove quite fruitful. This question is non-trivial however since many avenues that are likely to yield different outcomes can be followed. Among those possibilities, one is to consider the algebra realized by the intermediate Casimir elements in multifold tensor products of $U_q(\mathfrak{sl}_2)$ [53–56], and another is to increase the rank of the algebra $U_q(\mathfrak{sl}_2)$ to, say, $U_q(\mathfrak{sl}_3)$ when studying the Racah problem. Augmenting the number of punctures of the sphere in the KBSA approach could also be envisaged. Making much sense is the idea to start from the multivariate Askey–Wilson polynomials [57], to work out the algebra formed by its bispectral operators [55, 58, 59] and to take things from there. This is after all how the story began. Steps have been taken in these directions but final conclusions have not been reached. Some authors have considered higher order truncations of the reflection algebra [60] understood as a quotient of the q-Onsager algebra (see also [61] for the classical limit of this result). The upshot is that there is currently no clear consensus on what the higher rank Askey–Wilson algebra is[1]. This is not too surprising since there are still a few loose ends in the rank one cases.

As a prelude to a solid understanding of the higher rank Askey–Wilson algebra, it is appropriate to clarify the picture for the ordinary Askey–Wilson algebras. Indeed, as these algebras have appeared in multiple instances in the literature, names, conventions and notations are quite diverse. We are here proposing a standardization and offering a number of new results. The paper will unfold as follows. The various Askey–Wilson avatars will be introduced in Section 15.2. They will be given names and defined in a comparative way. Emphasis will be put on two particular versions. The first is a quotient of the Zhedanov algebra which we will call the Special Zhedanov algebra. In Section 15.3, we will show that the Zhedanov algebra is obtained as the reflection algebra defined from particular R- and reflection matrices. In this formalism, the Special Zhedanov algebra corresponds to fixing the Sklyanin determinant to a certain value; the name *Special* is chosen in analogy with the nomenclature of Lie groups. A Weyl group $W(D_4)$ symmetry of the Special Zhedanov algebra will then be presented in Section 15.4, thus generalizing an analoguous result for the Racah algebra. The second avatar that will be closely looked at will be called the Special Askey–Wilson algebra. It can be seen as the equivalent of the Special Zhedanov algebra where the parameters are promoted to central elements in the algebra. That this algebra is isomorphic to the Kauffman bracket skein algebra of the four-punctured sphere $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ is the object of Section 15.5. In Section 15.6, the Special Askey–Wilson algebra will further be related to the algebra $\mathcal{A}_3$ associated to the Racah problem of $U_q(\mathfrak{sl}_2)$ and to the centralizer $\mathfrak{C}_3$ of $U_q(\mathfrak{sl}_2)$ in its threefold tensor product. An injective homomorphism of algebras between the latter two structures will be stated and its proof will be found in Appendix 15.A. The relation between the Special Askey–Wilson algebra and the universal double affine Hecke algebra (DAHA) of type $(C_1^\vee, C_1)$ will be discussed in Section 15.7. How the Artin braid group B_3 acts on both the $\mathcal{A}_3$ and $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ algebras, respectively through conjugation by braided R-matrices and through half Dehn twists will be highlighted in Section 15.8. The question of the possible higher-rank generalizations of the Askey–Wilson algebra will be addressed in Section 15.9. A crossing index will be introduced and used to summarize efficiently the main results of [53] and [56] and new relations for the higher rank analogues will be provided. Elements of interest for further study of the higher rank generalizations of the Special Askey–Wilson algebra will be offered in addition. Concluding remarks will end the paper.

15.2. Askey–Wilson algebras

[1]Remarkably, for the $q \to 1$ and $q \to -1$ limits of the Askey–Wilson algebra, higher rank extensions have been more successfully defined respectively in [62] for the Racah algebra and in [63] for the Bannai–Ito algebra.

15.2.1. A jungle of Askey–Wilson algebras

As mentioned in the above, the name *Askey–Wilson algebra* has appeared and been connected to diverse objects in a multitude of contexts. Therefore, the notations and appellations in the literature are sometimes confusing. For the sake of clarity, we start by presenting these different algebraic structures and give to them unambiguous names to distinguish them.

The Askey–Wilson algebra aw(3) is the unital associative algebra depending on the parameter q with generators C_{12}, C_{23}, C_{13} and central elements C_1, C_2, C_3, C_{123} obeying the $\mathbb{Z}_3$-symmetric relations

$$C_{12} + \frac{[C_{23}, C_{13}]_q}{q^2 - q^{-2}} = \frac{C_1C_2 + C_3C_{123}}{q + q^{-1}}, \tag{15.2.1a}$$

$$C_{23} + \frac{[C_{13}, C_{12}]_q}{q^2 - q^{-2}} = \frac{C_2C_3 + C_1C_{123}}{q + q^{-1}}, \tag{15.2.1b}$$

$$C_{13} + \frac{[C_{12}, C_{23}]_q}{q^2 - q^{-2}} = \frac{C_3C_1 + C_2C_{123}}{q + q^{-1}}, \tag{15.2.1c}$$

where the q-commutator is defined by $[A, B]_q = qAB - q^{-1}BA$. Throughout the paper, we suppose that $q \in \mathbb{C}$ is not a root of unity. The Casimir element of this algebra is

$$\begin{aligned} \Omega := qC_{12}C_{23}C_{13} + q^2{C_{12}}^2 + q^{-2}{C_{23}}^2 + q^2{C_{13}}^2 - qC_{12}(C_1C_2 + C_3C_{123}) \\ -q^{-1}C_{23}(C_2C_3 + C_1C_{123}) - qC_{13}(C_3C_1 + C_2C_{123}). \end{aligned} \tag{15.2.1d}$$

Let us emphasize that this algebra **aw**(3) is not the algebra called Askey–Wilson algebra by A. Zhedanov, and denoted $AW(3)$ in [2]. In the present paper, we call the latter the *Zhedanov algebra* (see below).

From the **aw**(3) algebra, we define multiple quotients or subalgebras which appear in different contexts; these justify the importance of this algebra.

The Special Askey–Wilson algebra saw(3) is the quotient of **aw**(3) by the supplementary relation

$$\Omega = (q + q^{-1})^2 - {C_{123}}^2 - {C_1}^2 - {C_2}^2 - {C_3}^2 - C_{123}C_1C_2C_3. \tag{15.2.2}$$

A justification of the adjective *special* is given in Section 15.3. This algebra is isomorphic to the Kauffman bracket skein module of the four-punctured sphere (see Section 15.5) and is directly associated to the centralizer of the diagonal action of $U_q(\mathfrak{sl}_2)$ in its threefold tensor product (see Section 15.6).

The universal Askey–Wilson algebra Δ_q defined in [33] is the subalgebra of **aw**(3) generated by C_{12}, C_{23}, C_{13} as well as the central elements $\alpha = C_1C_2 + C_3C_{123}$, $\beta = C_2C_3 +$

$C_1 C_{123}$ and $\gamma = C_3 C_1 + C_2 C_{123}$. The Casimir element of Δ_q becomes

$$\Omega = qC_{12}C_{23}C_{13} + q^2 {C_{12}}^2 + q^{-2}{C_{23}}^2 + q^2 {C_{13}}^2 - qC_{12}\alpha - q^{-1}C_{23}\beta - qC_{13}\gamma. \qquad (15.2.3)$$

An injective homomorphism of Δ_q into $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ has been studied in [6] and its finite irreducible representations have been classified in [19]. The universal Askey–Wilson algebra also intersects the theory of free Lie algebras, see e.g. [64] and [65].

The evaluated Askey–Wilson algebra $Z_q(m_1, m_2, m_3)$ is the quotient of $\mathbf{aw}(3)$ by the supplementary relations

$$C_i = q^{m_i} + q^{-m_i}, \qquad i = 1, 2, 3. \qquad (15.2.4)$$

It plays a central role in the study of the centralizer of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in the threefold tensor product of representations of $U_q(\mathfrak{sl}_2)$ [36].

The Zhedanov algebra $Zh_q(m_1, m_2, m_3, m_4)$ is the quotient of $\mathbf{aw}(3)$ by

$$C_i = q^{m_i} + q^{-m_i}, \qquad C_{123} = q^{m_4} + q^{-m_4}, \qquad i = 1, 2, 3, \qquad (15.2.5)$$

and was first introduced by Zhedanov as the algebra encoding the bispectrality of the Askey–Wilson polynomials [2]. To be precise, in [2], an alternative equivalent presentation recalled in (15.4.10a)–(15.4.10c), (15.4.10e)–(15.4.10g) has been given. The above $\mathbb{Z}_3$-symmetric presentation of $Zh_q(m_1, m_2, m_3, m_4)$ is introduced in [10]. This algebra appears to be also the proper algebraic setting to characterize the Leonard pairs [32].

The Special Zhedanov algebra $sZh_q(m_1, m_2, m_3, m_4)$ is obtained as the quotient of $\mathbf{saw}(3)$ by relations (15.2.5) (see (15.4.10a)–(15.4.10h) for an alternative presentation). It appears naturally as the commutation relations of the intermediate Casimir elements acting on the multiplicity space of the decomposition of the threefold tensor product of representations of $U_q(\mathfrak{sl}_2)$ (see Section 15.6.3).

15.2.2. Miscellaneous properties

PBW basis

The Askey–Wilson algebra $\mathbf{aw}(3)$ has a Poincaré–Birkhoff–Witt (PBW) basis given explicitly by the following elements

$${C_{12}}^i\, {C_{23}}^j\, {C_{13}}^k\, {C_1}^m\, {C_2}^n\, {C_3}^p\, {C_{123}}^q, \qquad i, j, k, m, n, p, q \in \mathbb{N}. \qquad (15.2.6)$$

The proof is a slight generalization of the proof of the PBW basis for the universal Askey–Wilson algebra Δ_q given in [33]. We can also obtain a PBW basis for the Special Askey–Wilson algebra $\mathbf{saw}(3)$ from the one of $\mathbf{aw}(3)$ by restricting the range of the exponent j to $\{0,1\}$ instead of $\mathbb{N}$.

Calabi–Yau algebra

The Zhedanov algebra $Zh_q(m_1,m_2,m_3,m_4)$ can be derived from a Calabi–Yau potential in the following sense [66]. Let $F=\mathbb{C}[x_1,x_2,x_3]$ be a free associative algebra and view F as a graded algebra such that $\deg(x_1)=d_1$, $deg(x_2)=d_2$ and $deg(x_3)=d_3$ (with $0<d_1\leq d_2\leq d_3$). We define $F_{cycl}=F/[F,F]$ and the map $\frac{\partial}{\partial x_j}:F_{cycl}\to F$ on cyclic words as follows

$$\frac{\partial[x_{i_1}x_{i_2}\dots x_{i_r}]}{\partial x_j}=\sum_{\{s|i_s=j\}}x_{i_s+1}x_{i_s+2}\dots x_{i_r}x_{i_1}x_{i_2}\dots x_{i_s-1} \tag{15.2.7}$$

and we extend it to F_{cycl} by linearity. Let $\Phi(x_1,x_2,x_3)\in F_{cycl}$ be a potential which can be decomposed as follows

$$\Phi(x_1,x_2,x_3)=\Phi^{(d)}(x_1,x_2,x_3)+\Phi^{<d}(x_1,x_2,x_3), \tag{15.2.8}$$

where $\Phi^{(d)}(x_1,x_2,x_3)$ is homogeneous of degree $d=d_1+d_2+d_3$ and $\Phi^{<d}(x_1,x_2,x_3)$ is composed of terms of degree strictly inferior to d. Then the algebra whose defining relations are given by

$$\frac{\partial\Phi}{\partial x_j}=0,\qquad j=1,2,3, \tag{15.2.9}$$

is a Calabi–Yau algebra [29].

Now, let $x_1=K_{12}$, $x_2=K_{23}$, $x_3=K_{13}$ and $deg(x_1)=deg(x_2)=2$, $deg(x_3)=3$. Consider the potential

$$\begin{aligned}
&\Phi^{(7)}(x_1,x_2,x_3)=q[x_1x_2x_3]-q^{-1}[x_1x_3x_2],\\
&\Phi^{<7}(x_1,x_2,x_3)=(q+q^{-1})([x_1{x_2}^2]+[{x_1}^2x_2])-\xi_4[x_1]-\xi_4'[x_2]-\tfrac{1}{2}[{x_3}^2]-\xi_2[x_1x_2].
\end{aligned} \tag{15.2.10}$$

It is easy to see that the defining relations of $Zh_q(m_1,m_2,m_3,m_4)$ presented in (15.4.10a)-(15.4.10c) are equivalent to imposing (15.2.9) for the potential (15.2.10). In other words, $Zh_q(m_1,m_2,m_3,m_4)$ derives from the Calabi–Yau potential Φ (15.2.10).

15.3. The Zhedanov algebra as a truncated reflection algebra

In this section, we recall [12] that the defining relations of the algebra $Zh_q(m_1,m_2,m_3,m_4)$ can be equivalently encoded in a reflection equation [67]. This realization of an algebra is usually called the FRT presentation, in honor of the authors of [68]. This presentation allows one to connect the Zhedanov algebra to the reflection algebra which is intensively studied in the context of quantum integrable systems. In addition, we show that the algebra $sZh_q(m_1, m_2, m_3, m_4)$ can be also obtained naturally by setting the Sklyanin determinant to a certain value; this justifies the appellation *special* for the quotiented algebra since it is obtained by fixing the value of a determinant, as in the definition of the Special Linear group SL_n.

The cornerstone of the FRT presentation is the R-matrix. For the case of the algebra $Zh_q(m_1, m_2, m_3, m_4)$, we start with the following R-matrix

$$R(u) = \begin{pmatrix} uq - \frac{1}{uq} & 0 & 0 & 0 \\ 0 & u - \frac{1}{u} & q - \frac{1}{q} & 0 \\ 0 & q - \frac{1}{q} & u - \frac{1}{u} & 0 \\ 0 & 0 & 0 & uq - \frac{1}{uq} \end{pmatrix}. \tag{15.3.1}$$

This R-matrix is associated to the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and is a solution of the Yang–Baxter equation

$$R_{12}({}^{u_1}/_{u_2})R_{13}({}^{u_1}/_{u_3})R_{23}({}^{u_2}/_{u_3}) = R_{23}({}^{u_2}/_{u_3})R_{13}({}^{u_1}/_{u_3})R_{12}({}^{u_1}/_{u_2}), \tag{15.3.2}$$

where $R_{12} = \sum_a R_a \otimes R^a \otimes \mathbb{1}$, $R_{23} = \sum_a \mathbb{1} \otimes R_a \otimes R^a$, $R_{13} = \sum_a R_a \otimes \mathbb{1} \otimes R^a$ if one writes $R = \sum_a R_a \otimes R^a$ and $\mathbb{1}$ as the 2×2 identity matrix. We define also the following truncated reflection matrix (see remark 15.2 below) given by

$$B(u) = \begin{pmatrix} uqC_{12} - \dfrac{C_{23}}{uq} + \dfrac{{}^{p_4}/_u + p_4' u}{u^2 - {}^1/_{u^2}} & qu^2 + \dfrac{1}{qu^2} - \dfrac{[C_{23}, C_{12}]_q}{q^2 - {}^1/_{q^2}} + \dfrac{p_4''}{q + {}^1/_q} \\ -qu^2 - \dfrac{1}{qu^2} + \dfrac{[C_{12}, C_{23}]_q}{q^2 - {}^1/_{q^2}} - \dfrac{p_4''}{q + {}^1/_q} & uqC_{23} - \dfrac{C_{12}}{uq} + \dfrac{p_4 u + {}^{p_4'}/_u}{u^2 - {}^1/_{u^2}} \end{pmatrix}, \tag{15.3.3}$$

where we refer to (15.4.7a)–(15.4.7c) for the definition of p_4, p_4' and p_4''.

Proposition 15.1. *[12] The set of relations obtained from the reflection equation*

$$R({}^u/_v)B_1(u)R(uv)B_2(v) = B_2(v)R(uv)B_1(u)R({}^u/_v), \tag{15.3.4}$$

where $B_1(u) = B(u) \otimes \mathbb{1}$ and $B_2(u) = \mathbb{1} \otimes B(u)$, is equivalent to the defining relations of $Zh_q(m_1, m_2, m_3, m_4)$.

Proof. We look at each matrix element of the reflection equation (15.3.4) and derive 16 relations. For each or them, we extract the different coefficients w.r.t. the parameter u; this provides relations between C_{12} and C_{23}. By direct investigation, we verify that all the obtained relations are equivalent to the defining relations of $Zh_q(m_1, m_2, m_3, m_4)$. □

Rephrasing this proposition, the Zhedanov algebra $Zh_q(m_1, m_2, m_3, m_4)$ is isomorphic to the truncated reflection algebra defined by the R-matrix (15.3.1) and the truncated reflection matrix (15.3.3).

Remark 15.2. *There exists a more general form for the reflection matrix, containing an infinite number of generators encompassed in formal series of u and $\frac{1}{u}$. The elements of the reflection matrix* (15.3.3) *can be obtained as a truncation of these formal series. The algebra defined by the general reflection matrix obeying the reflection equation* (15.3.4) *is isomorphic to the q-Onsager algebra [14]. Therefore, the Zhedanov algebra can also be seen as a quotient of the q-Onsager algebra.*

In the context of the reflection algebra it is well-known how to obtain central elements [67]. Indeed, let us define the Sklyanin determinant $\mathrm{sdet}B(u)$ as follows

$$\mathrm{sdet}B(u) := -\frac{1}{2} tr_{12} \left(R({}^1\!/_q) B_1({}^u\!/_q) R({}^{u^2}\!/_q) B_2(u) \right). \tag{15.3.5}$$

We can show that the coefficients of $\mathrm{sdet}B(u)$ commute with C_{12} and C_{23}. We recover in this way that the operator Ω given by (15.2.1d) commutes with C_{12} and C_{23}. The Sklyanin determinant gives solely Ω as a central element. Fixing the Sklyanin determinant to an appropriate value allows us to give a FRT presentation of $sZh_q(m_1, m_2, m_3, m_4)$:

Proposition 15.3. *The truncated reflection algebra defined by the R-matrix* (15.3.1)*, the truncated reflection matrix* (15.3.3) *and quotiented by the relation*

$$\begin{aligned} \mathrm{sdet}B(u) = q^2(1-q^4)^2 \, &(u^2 + q^{-m_2-m_4})(u^2 + q^{m_2+m_4})(u^2 + q^{m_4-m_2})(u^2 + q^{m_2-m_4}) \\ &\times (u^2 + q^{-m_1-m_3})(u^2 + q^{m_1+m_3})(u^2 + q^{m_3-m_1})(u^2 + q^{m_1-m_3}), \end{aligned} \tag{15.3.6}$$

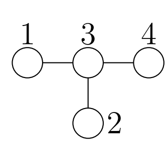
is isomorphic to $sZh_q(m_1, m_2, m_3, m_4)$.

Proof. By direct computations, we show that (15.3.6) is equivalent to imposing (15.2.2). □

The fact that $sZh_q(m_1, m_2, m_3, m_4)$ can be defined as a truncated reflection algebra was expected, but it is a surprise that the r.h.s. of (15.3.6) factorizes into such a simple form.

15.4. A W(D_4) symmetry

The algebra $sZh_q(m_1, m_2, m_3, m_4)$ has a remarkable symmetry based on the Weyl group $W(D_4)$ associated to the Lie algebra D_4. To describe it, let us introduce a root system of type D_4 and fix a set of simple roots α_1, α_2, α_3, α_4 with labeling according to the following Dynkin diagram:



The Weyl group $W(D_4)$ is generated by the reflections s_i associated to the simple roots α_i which satisfy, for $1 \le i, j \le 4$,

$$\begin{aligned} s_i^2 =& 1, \\ s_i s_j =& s_j s_i \qquad \text{if } i \text{ and } j \text{ are not connected in the Dynkin diagram,} \\ s_i s_j s_i =& s_j s_i s_j \qquad \text{if } i \text{ and } j \text{ are connected in the Dynkin diagram.} \end{aligned} \tag{15.4.1}$$

Its order is 192. Let us now associate the parameters m_1, m_2, m_3, m_4 with some of the roots as follows:

$$m_1 = \alpha_1, \qquad m_2 = \alpha_2, \qquad m_3 = \alpha_4, \qquad m_4 = \Theta, \tag{15.4.2}$$

where Θ is the longest positive root. The explicit expression of Θ is:

$$\alpha_3 = \frac{1}{2}(m_4 - m_1 - m_2 - m_3). \tag{15.4.3}$$

It is elementary to calculate the actions s_i expressed in terms of the parameters:

$$s_1 \,:\, m_1 \mapsto -m_1, \quad s_2 \,:\, m_2 \mapsto -m_2, \quad s_4 \,:\, m_3 \mapsto -m_3, \quad s_3 \,:\, \begin{cases} m_1 \mapsto m_1 + \alpha_3, \\ m_2 \mapsto m_2 + \alpha_3, \\ m_3 \mapsto m_3 + \alpha_3, \\ m_4 \mapsto m_4 - \alpha_3, \end{cases} \tag{15.4.4}$$

where the omitted actions are trivial and the explicit expression of α_3 is given above. The action of the Weyl group is extended to any function as follows:

$$(\sigma f)(m_1, m_2, m_3, m_4) = f(\sigma(m_1), \sigma(m_2), \sigma(m_3), \sigma(m_4)) \tag{15.4.5}$$

for $\sigma \in W(D_4)$.

Proposition 15.4. *The Weyl group $W(D_4)$ is a symmetry of $sZh_q(m_1, m_2, m_3, m_4)$ i.e.*

$$sZh_q(m_1, m_2, m_3, m_4) = sZh_q(\sigma(m_1), \sigma(m_2), \sigma(m_3), \sigma(m_4)), \tag{15.4.6}$$

for any $\sigma \in W(D_4)$.

Proof. In $sZh_q(m_1, m_2, m_3, m_4)$, we remark that the only functions of m_i which appear are

$$p_4 = \chi_{m_1}\chi_{m_2} + \chi_{m_3}\chi_{m_4}, \tag{15.4.7a}$$

$$p_4' = \chi_{m_2}\chi_{m_3} + \chi_{m_1}\chi_{m_4}, \tag{15.4.7b}$$

$$p_4'' = \chi_{m_1}\chi_{m_3} + \chi_{m_2}\chi_{m_4}, \tag{15.4.7c}$$

$$p_6 = {\chi_{m_1}}^2 + {\chi_{m_2}}^2 + {\chi_{m_3}}^2 + {\chi_{m_4}}^2 + \chi_{m_1}\chi_{m_2}\chi_{m_3}\chi_{m_4}, \tag{15.4.7d}$$

where $\chi_m = q^m + q^{-m}$. By direct computations, we can show that these functions are invariant by the transformations s_1, s_2, s_3 and s_4 given by (15.4.4), which concludes the proof since they generate $W(D_4)$. $\square$

In the study of the finite representations of the universal algebra Δ_q a $W(D_4)$ symmetry has been also investigated [69, 70].

15.4.1. Connection with the $W(D_4)$ symmetry in the Racah algebra

Let us perform the transformation

$$K_I = \frac{C_I - (q+q^{-1})}{(q-q^{-1})^2}, \tag{15.4.8}$$

with $I \in \{1, 2, 3, 123, 12, 23\}$. Note that 13 does not belong to this set. In the algebra $sZh_q(m_1, m_2, m_3, m_4)$, one gets, for $i = 1, 2, 3, 4$,

$$K_i = \frac{\chi_{m_i} - (q+q^{-1})}{(q-q^{-1})^2} = \left[\frac{m_i}{2}\right]_q^2 - \left[\frac{1}{2}\right]_q^2, \tag{15.4.9}$$

where the q-number is defined by $[m]_q = \frac{q^m - q^{-m}}{q-q^{-1}}$. The commutation relations of the algebra $sZh_q(m_1, m_2, m_3, m_4)$ become

$$[K_{12}, K_{23}]_q = K_{13}, \tag{15.4.10a}$$

$$[K_{23}, K_{13}]_q = (q+q^{-1})\Big(-\{K_{12}, K_{23}\} - {K_{23}}^2 + \xi_2 K_{23} + \xi_4\Big), \tag{15.4.10b}$$

$$[K_{13}, K_{12}]_q = (q+q^{-1})\Big(-\{K_{12}, K_{23}\} - {K_{12}}^2 + \xi_2 K_{12} + \xi_4'\Big), \tag{15.4.10c}$$

and the supplementary relation becomes

$$\begin{aligned}\frac{q^2}{(q+q^{-1})^2} {K_{13}}^2 - qK_{12}K_{23}K_{12} - q^{-1}K_{23}K_{12}K_{23} - q\frac{q-q^{-1}}{q+q^{-1}} K_{12}K_{23}K_{13} \\ + \left(\frac{\xi_2}{q+q^{-1}} - 1\right)\{K_{12}, K_{23}\} + q\xi_4 K_{12} + q^{-1}\xi_4' K_{23} \quad = \quad \xi_6 - \xi_4 - \xi_4' - \tfrac{1}{4}{\xi_2}^2,\end{aligned} \tag{15.4.10d}$$

with

$$\xi_2 = \frac{1}{[2]_q}\Big(2({M_1}^2 + {M_2}^2 + {M_3}^2 + {M_4}^2 - 1) + (q-q^{-1})^2({M_1}^2{M_3}^2 + {M_2}^2{M_4}^2)\Big), \tag{15.4.10e}$$

$$\xi_4 = ({M_1}^2 - {M_4}^2)({M_3}^2 - {M_2}^2), \tag{15.4.10f}$$

$$\xi_4' = ({M_1}^2 - {M_2}^2)({M_3}^2 - {M_4}^2), \tag{15.4.10g}$$

$$\xi_6 = ({M_1}^2{M_3}^2 - {M_2}^2{M_4}^2)({M_1}^2 - {M_2}^2 + {M_3}^2 - {M_4}^2) + \tfrac{1}{4}(q-q^{-1})^2({M_1}^2{M_3}^2 - {M_2}^2{M_4}^2)^2, \tag{15.4.10h}$$

where we use the notation $M_i = \left[\frac{m_i}{2}\right]_q$. As expected, we can check that the functions ξ_2, ξ_4, ξ_4' and ξ_6 are invariant under the action of the Weyl group $W(D_4)$.

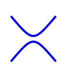
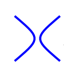
The advantage of this presentation of $sZh_q(m_1, m_2, m_3, m_4)$ is that the classical limit $q \to 1$ (see Appendix 15.A) is well-defined and provides straightforwardly the commutation relations of the Racah algebra. Thus, the description of the Weyl group $W(D_4)$ action also holds for the Racah algebra and we recover the results of [71].

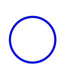
Remark 15.5. *In the classical limit $q \to 1$, the functions ξ_2, ξ_4, ξ_4' and ξ_6 form a basis for polynomials invariant under the action of $W(D_4)$, as expected. In the generic case ($q \in \mathbb{C}$, not a root of unity), one gets two different sets of invariant functions: $S_\xi = \{\xi_2, \xi_4, \xi_4', \xi_6\}$ on the one hand and $S_p = \{p_4, p_4', p_4'', p_6\}$ on the other hand. We have checked that there exists an invertible polynomial mapping between these two sets. However, only S_ξ admits a non-trivial classical limit.*

15.5. Kauffman bracket skein modules and algebras

Kauffman bracket skein module quantizations have been introduced in [37, 47] and further studied along our lines of interest for this paper in [39, 72, 73]. We will now recall some key definitions and results from these investigations. We shall work with an oriented 3-manifold $\mathcal{M}$ which is a thickened surface, that is $\mathcal{M} = \Sigma_{0,n} \times I$, where $I = [0, 1]$ and $\Sigma_{0,n}$ is the n-punctured sphere.

Definition 15.6. *The quantized skein module $\mathrm{Sk}_\theta(\mathcal{M})$ is the $\mathbb{C}[\theta^{\pm 1}]$-module spanned by framed and unoriented links in $\mathcal{M}$ modulo the Kauffman bracket skein relations that allow to "simplify the crossings":*

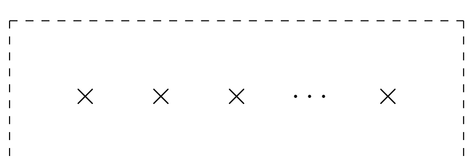
 $= \theta$  $+ \theta^{-1}$ , (15.5.1a)

 $= -(\theta^2 + \theta^{-2})$, (15.5.1b)

where $\theta \in \mathbb{C}$ is not a root of unity and in the framing relation (15.5.1b) *the link should not enclose a puncture. This defines an algebra, which we will denote $\mathrm{Sk}_\theta(\Sigma_{0,n})$, for which multiplication is given by stacking the links on top of each other in the I direction.*

We shall use diagrams that correspond to the projection of the links on the surface (all the while keeping the information about the relative "height" of the links in the I direction). Let us now establish the conventions for these drawings (framed links diagrams).

The n-punctured sphere $\Sigma_{0,n}$ is equivalent to the plane with $n-1$ punctures (denoted by the $(n-1)$ drawn $\times$'s):

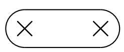


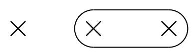
(15.5.2)

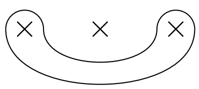
The dashed contour corresponds to the n^{th} puncture of the sphere. We will omit the contour in the subsequent diagrams but it is always understood to be there.

Framed links that enclose punctures are represented by loops drawn around the $\times$'s. We shall use the term "loops" to refer unambiguously to the framed links in the remainder of the paper. These loops can be homotopically deformed without crossing the holes (punctures). Remark that loops enclosing a single puncture are central elements in $\mathrm{Sk}_\theta(\Sigma_{0,n})$. This is also true for the n^{th} puncture, which amounts to saying that the loop enclosing the $(n-1)$ punctures $\times$ is also central.

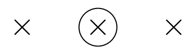
Let us now consider the surface $\Sigma_{0,4}$ and give names to a few loops:

 $= \mathbb{A}_{12}$

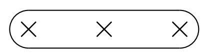
 $= \mathbb{A}_{23}$

 $= \mathbb{A}_{13}$

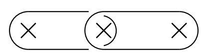
 $= \mathbb{A}_{1}$

 $= \mathbb{A}_{2}$

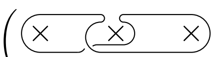
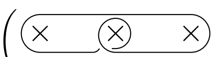
 $= \mathbb{A}_{3}$

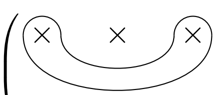
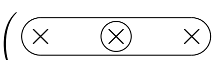
 $= \mathbb{A}_{123}$ (15.5.3)

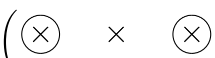
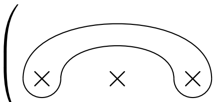
Following the definition, multiplication of two loops $X \cdot Y$ means putting Y *on top of* X, for example:

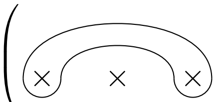
$\mathbb{A}_{12} \cdot \mathbb{A}_{23} =$  (15.5.4)

One would then proceed to use relations (15.5.1) to simplify the expressions:

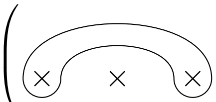
$\mathbb{A}_{12} \cdot \mathbb{A}_{23} = \theta \Big($  $\Big) - \theta^{-1} \Big($  $\Big)$

$= \theta^2 \Big($  $\Big) + \Big($  $\Big)$

$+ \Big($  $\Big) + \theta^{-2} \Big($  $\Big).$ (15.5.5)

$= \theta^2 \mathbb{A}_{13} + \mathbb{A}_2 \cdot \mathbb{A}_{123} + \mathbb{A}_1 \cdot \mathbb{A}_3 + \theta^{-2} \Big($  $\Big).$

Similarly, exchanging the order of multiplication, one obtains the same diagrams but with inverse coefficients:

$\mathbb{A}_{23} \cdot \mathbb{A}_{12} = \theta^{-2} \mathbb{A}_{13} + \mathbb{A}_2 \cdot \mathbb{A}_{123} + \mathbb{A}_1 \cdot \mathbb{A}_3 + \theta^{2} \Big($  $\Big).$ (15.5.6)

We see immediately that one gets

$$\theta^2 \mathbb{A}_{12} \cdot \mathbb{A}_{23} - \theta^{-2} \mathbb{A}_{23} \cdot \mathbb{A}_{12} = (\theta^4 - \theta^{-4}) \mathbb{A}_{13} + (\theta^2 - \theta^{-2})(\mathbb{A}_2 \cdot \mathbb{A}_{123} + \mathbb{A}_1 \cdot \mathbb{A}_3). \tag{15.5.7}$$

The skein algebra $Sk_\theta(\Sigma_{0,4})$ is directly linked to the Askey–Wilson algebra as stated in the following proposition:

Proposition 15.7. *The Special Askey–Wilson algebra* $\mathbf{saw}(3)$ *is isomorphic to the Kauffman bracket skein algebra* $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$*. The isomorphism is given by the following invertible map:*

$$C_I \mapsto \mathbb{A}_I, \tag{15.5.8}$$

for $I \in \{1, 2, 3, 123, 12, 23, 13\}$.

PROOF. The isomorphism is directly verified by comparing the relations of $\mathbf{saw}(3)$ and the ones of the Kauffman bracket skein algebra obtained in [37] (see also Proposition 3.1 of [73] and [38] for additional details). □

This proposition gives a diagrammatic approach to study the algebra $\mathbf{saw}(3)$.

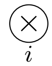
Let us emphasize that the previous isomorphism involves the Special Askey–Wilson algebra $\mathbf{saw}(3)$. If we replace $\mathbf{saw}(3)$ by $\mathbf{aw}(3)$ in the map of the proposition, the homomorphism would be not injective and if we instead replace $\mathbf{saw}(3)$ by Δ_q (as in [6, 74]), it would be not surjective.

One notes that the $\mathbb{Z}_3$-symmetry of the $\mathbf{saw}(3)$ relations is made manifest in terms of the framed links picture, as the punctures do not have fixed positions and can be switched around.

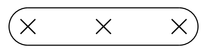
From now on we will unambiguously refer to the drawn loops identified as the generators of $Sk_{iq^{1/2}}(\Sigma_{0,4})$ directly as their C_I counterpart following (15.5.8). This correspondence (15.5.8) leads to a natural labeling of the punctures. Indeed, consider the generators given in (15.5.3): the punctures enclosed in a given loop correspond precisely to the set of indices I of the corresponding generator C_I if one labels the punctures consecutively as:

$$\underset{1}{\times} \quad \underset{2}{\times} \quad \underset{3}{\times} \tag{15.5.9}$$

Remark 15.8. *We recall that one arrives to the Special Zhedanov algebra* $sZh_q(m_1, m_2, m_3, m_4)$ *from the Special Askey–Wilson algebra* $\mathbf{saw}(3)$ *by attributing a value to the central elements* C_i*,* $i = 1, 2, 3, 123$*, see* (15.2.5)*. In the same way, starting from the Kauffman bracket skein algebra* $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$*, one can define an evaluated Kauffman bracket skein algebra, denoted* $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4}; m_1, m_2, m_3, m_4)$ *by attributing a value to the* *puncture-framing relations:*



$$= q^{m_i} + q^{-m_i}\,, \quad i = 1, 2, 3,$$



$$= q^{m_4} + q^{-m_4}. \tag{15.5.10}$$

Note that the last drawing corresponds in fact to a contour enclosing the fourth puncture on the sphere, see (15.5.2). *As a corollary of Proposition 15.7, the algebra* $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4}; m_1, m_2, m_3, m_4)$ *is isomorphic to the Special Zhedanov algebra* $sZh_q(m_1, m_2, m_3, m_4)$.

Relations (15.5.10) *with* $m_i = 1$ *already appear in the definition of the skein algebra of arcs and link introduced in [75], from where we borrowed the terminology 'puncture-framing'.*

15.6. $U_q(\mathfrak{sl}_2)$ and its centralizer in $U_q(\mathfrak{sl}_2)^{\otimes 3}$

The goal of this section is to discuss the notion of centralizer of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$, which we denote by $\mathfrak{C}_3$, and connect it with the Special Askey–Wilson algebra **saw**(3).

15.6.1. $U_q(\mathfrak{sl}_2)$ and its universal R-matrix

Let us fix the notation and conventions that will be used to perform the explicit calculations in $U_q(\mathfrak{sl}_2)$ (note that the results obtained will be independent of these conventions at the end). We shall first define the quasi-triangular Hopf algebra $U_q(\mathfrak{sl}_2)$, present its braided universal R-matrix and list some additional properties of interest.

$U_q(\mathfrak{sl}_2)$ is an associative algebra generated by E, F, q^H and q^{-H} obeying the defining relations

$$q^H q^{-H} = q^{-H} q^H = 1, \quad q^H E = qEq^H, \quad q^H F = q^{-1} F q^H \quad \text{and} \quad [E, F] = [2H]_q. \tag{15.6.1}$$

The center of this algebra is generated by the following Casimir element (denoted Λ in [54, 56])

$$Q = (q - q^{-1})^2 \left(FE + \frac{qq^{2H} + q^{-1}q^{-2H}}{(q - q^{-1})^2} \right). \tag{15.6.2}$$

The algebra $U_q(\mathfrak{sl}_2)$ can be endowed with a Hopf structure. In particular, its comultiplication (or coproduct) homomorphism $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ is given by

$$\Delta(E) = E \otimes q^{-H} + q^H \otimes E, \qquad \Delta(q^H) = q^H \otimes q^H, \tag{15.6.3a}$$

$$\Delta(F) = F \otimes q^{-H} + q^H \otimes F, \qquad \Delta(q^{-H}) = q^{-H} \otimes q^{-H}, \tag{15.6.3b}$$

and is coassociative

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta. \tag{15.6.4}$$

The quantum algebra $U_q(\mathfrak{sl}_2)$ is called quasi-triangular because in a completion of $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$, there exists a universal R-matrix $\mathcal{R}$ which is invertible and satisfies

$$\Delta(x)\mathcal{R} = \mathcal{R}\Delta^{op}(x) \qquad \text{for } x \in U_q(\mathfrak{sl}_2), \tag{15.6.5}$$

$$(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{12}\mathcal{R}_{13}, \tag{15.6.6}$$

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{23}\mathcal{R}_{13}, \tag{15.6.7}$$

where in the Sweedler notation we write the opposite comultiplication $\Delta^{op}(x) = x_{(2)} \otimes x_{(1)}$ if $\Delta(x) = x_{(1)} \otimes x_{(2)}$. In the previous relation, we have used the notations $\mathcal{R}_{12} = \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha$ and $\mathcal{R}_{13} = \mathcal{R}^\alpha \otimes 1 \otimes \mathcal{R}_\alpha$ where $\mathcal{R} = \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha$ (the sum over repeated indices α is understood). The universal R-matrix is given explicitly by [76]

$$\mathcal{R} = q^{2(H\otimes H)} \sum_{n=0}^{\infty} \frac{(q-q^{-1})^n}{[n]_q!} q^{n(n-1)/2} \Big(Eq^H \otimes q^{-H}F\Big)^n, \tag{15.6.8}$$

where $[n]_q! = [n]_q[n-1]_q \dots [2]_q[1]_q$ and, by convention, $[0]_q! = 1$.

One can also define the so-called braided universal R-matrix $\check{\mathcal{R}}$ by

$$\check{\mathcal{R}}_i = \mathcal{R}_{i,i+1}\sigma_{i,i+1} \tag{15.6.9}$$

where $\sigma_{i,i+1}$ acts on the i^{th} and $(i+1)^{\text{th}}$ factors of the tensor product as

$$\sigma_{i,i+1}(\dots \otimes x_i \otimes x_{i+1} \otimes \dots) = (\dots \otimes x_{i+1} \otimes x_i \otimes \dots)\sigma_{i,i+1}. \tag{15.6.10}$$

This braided universal R-matrix satisfies the braided Yang–Baxter equation

$$\check{\mathcal{R}}_i\check{\mathcal{R}}_{i+1}\check{\mathcal{R}}_i = \check{\mathcal{R}}_{i+1}\check{\mathcal{R}}_i\check{\mathcal{R}}_{i+1}. \tag{15.6.11}$$

15.6.2. An algebra generated by the intermediate Casimir elements

Let us define the following *intermediate* Casimir elements

$$\begin{gathered} Q_1 = Q \otimes 1 \otimes 1, \qquad Q_2 = 1 \otimes Q \otimes 1, \qquad Q_3 = 1 \otimes 1 \otimes Q, \\ Q_{12} = \Delta(Q) \otimes 1 = Q_{(1)} \otimes Q_{(2)} \otimes 1, \qquad Q_{23} = 1 \otimes \Delta(Q) = 1 \otimes Q_{(1)} \otimes Q_{(2)}, \\ Q_{123} = (\Delta \otimes \mathrm{id})\Delta(Q). \end{gathered} \tag{15.6.12}$$

The labeling of these intermediate Casimir elements is chosen so as to refer to the non-trivial factors in the tensor product $U_q(\mathfrak{sl}_2)^{\otimes 3}$.

Definition 15.9. *The algebra $\mathcal{A}_3$ is the subalgebra of $U_q(\mathfrak{sl}_2)^{\otimes 3}$ generated by the intermediate Casimir elements Q_1, Q_2, Q_3, Q_{12}, Q_{23} and Q_{123}.*

Let us define an additional intermediate Casimir element

$$Q_{13} = \check{\mathcal{R}}_2^{-1} Q_{12} \check{\mathcal{R}}_2 = \check{\mathcal{R}}_1 Q_{23} \check{\mathcal{R}}_1^{-1}. \tag{15.6.13}$$

It has been proven in [77] that this element is in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ (and not in its completion), that the second equality is compatible with the first one and that the following proposition holds:
Proposition 15.10. *The intermediate Casimir elements Q_1, Q_2, Q_3, Q_{123}, Q_{12}, Q_{23} and Q_{13} belong to the centralizer $\mathfrak{C}_3$ of the diagonal action of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ defined by*

$$\mathfrak{C}_3 = \Big\{X \in U_q(\mathfrak{sl}_2)^{\otimes 3} \;\Big|\; [(\Delta \otimes \mathrm{id})\Delta(x), X] = 0, \quad \forall x \in U_q(\mathfrak{sl}_2)\Big\}. \tag{15.6.14}$$

The precise links between the Askey–Wilson algebra, the centralizer and the algebra $\mathcal{A}_3$ generated by the intermediate Casimir elements are given in the following proposition.
Proposition 15.11. *The algebra* $\mathbf{saw}(3)$ *has an homomorphic injective image in $\mathfrak{C}_3$. The mapping is done as follows:*

$$C_I \mapsto Q_I, \qquad \textit{for} \qquad I \in \{1, 2, 3, 123, 12, 23, 13\}. \tag{15.6.15}$$

The algebra $\mathbf{saw}(3)$ *is isomorphic to $\mathcal{A}_3$.*

Proof. All the relations of $\mathbf{saw}(3)$ given by (15.2.1) and (15.2.2) are easily checked in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ upon rewriting the Q_I's in terms of the $U_q(\mathfrak{sl}_2)^{\otimes 3}$ generators. The proof of the injectivity is postponed to Appendix 15.A. The method used in [6] to prove the injectivity of Δ_q into $U_q(\mathfrak{sl}_2)^{\otimes 3}$ seems difficult to generalize to the case treated here and we propose an alternative method based on classical invariant theory. Since the algebra $\mathcal{A}_3$ is the image of the map (15.6.15), it follows that $\mathbf{saw}(3)$ is isomorphic to $\mathcal{A}_3$. ☐

This realization of the Askey–Wilson algebra in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ was the motivation for adding the relation (15.2.2) to the "intuitive" set of relations of $\mathbf{aw}(3)$. Indeed, since relation (15.2.2) is obeyed by the intermediate Casimir elements, it should also be included in the algebra encoding the properties of these Casimir elements.
Corollary 15.12. *The algebra $\mathcal{A}_3$ is isomorphic to the Kauffman bracket skein algebra of the four-punctured sphere $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$. The isomorphism is given by the following map:*

$$\phi : Q_I \mapsto \mathbb{A}_I, \qquad \textit{for} \quad I \in \{1, 2, 3, 123, 12, 23, 13\}. \tag{15.6.16}$$

Proof. A direct consequence of the Propositions 15.7 and 15.11.

15.6.3. Fundamental theorems of invariant theory

In the previous section, we introduced the centralizer $\mathfrak{C}_3$ of the diagonal action of $U_q(\mathfrak{sl}_2)$ in the threefold tensor product and showed its connection with the Askey–Wilson algebra $\mathbf{saw}(3)$. We now focus on similar objects in the case where we represent each factor $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ by a finite-dimensional irreducible representation.

The quantum algebra $U_q(\mathfrak{sl}_2)$ has finite irreducible representations of dimension $m = 2j+1$ that we will denote by $M(m)$, with $m \in \mathbb{Z}_{>0}$. The name "spin-j representation" is usually used to refer to $M(m = 2j+1)$. The representation map will be denoted by

$\pi_m : U_q(\mathfrak{sl}_2) \to \mathrm{End}(M(m))$. The representation of the Casimir element (15.6.2) in the space $M(m)$ is

$$\pi_m(Q) = \chi_m \mathbb{1}_m, \tag{15.6.17}$$

where $\chi_m = q^m + q^{-m}$ and $\mathbb{1}_m$ is the $m \times m$ identity matrix.

From now on, we fix three integers m_1, m_2 and m_3. The threefold tensor product of irreducible representations of $U_q(\mathfrak{sl}_2)$ decomposes into the following direct sum of irreducible representations

$$M(m_1) \otimes M(m_2) \otimes M(m_3) = \bigoplus_{m_4} M(m_4) \otimes V^{m_4}_{m_1,m_2,m_3}, \tag{15.6.18}$$

where $V^{m_4}_{m_1,m_2,m_3}$ is called the multiplicity space. We recall that we look at cases where q is not a root of unity otherwise the previous statement would be wrong.

We now fix four integers m_1, m_2, m_3, m_4 and denote by $\mathcal{Q}_I$ the image of Q_I in $V^{m_4}_{m_1,m_2,m_3}$ (for $I \in \{1, 2, 3, 123, 12, 23, 13\}$). We get $\mathcal{Q}_1 = \chi_{m_1}$, $\mathcal{Q}_2 = \chi_{m_2}$, $\mathcal{Q}_3 = \chi_{m_3}$ and $\mathcal{Q}_{123} = \chi_{m_4}$.

Proposition 15.13. *There exists a surjective algebra homomorphism from $sZh_q(m_1, m_2, m_3, m_4)$ to $End(V^{m_4}_{m_1,m_2,m_3})$ given by*

$$C_I \mapsto \mathcal{Q}_I, \qquad \text{for } I \in \{12, 23, 13\}. \tag{15.6.19}$$

This proposition which provides the generators for the centralizer of the diagonal action is sometimes called in invariant theory the "first fundamental theorem". The map in the previous proposition is not injective. The description of the kernel of this map is the subject of [36] (see also [78]) and is called the "second fundamental theorem".

We recall that the algebra $sZh_q(m_1, m_2, m_3, m_4)$ possesses a $W(D_4)$-symmetry. Let us remark that a similar Weyl group symmetry of type E_6 has been discovered recently [79] in the case of the centralizer of the diagonal embedding of $U(\mathfrak{sl}_3)$ in two copies of $U(\mathfrak{sl}_3)$.

15.7. The Double Affine Hecke Algebra $(C_1^\vee, C_1)$

Double affine Hecke algebras (DAHA) of type $(C_1^\vee, C_1)$ were introduced in [80] and their connections with Askey–Wilson polynomials were first explored in [18] and [40]. Universal analogues of these DAHA were later introduced and studied in [20, 43, 44].

In this section, we present another connection between the Special Askey–Wilson algebra **saw**(3) and a certain subalgebra of a universal DAHA of type $(C_1^\vee, C_1)$.

Definition 15.14. *We introduce the following algebras*

- *The universal Double Affine Hecke Algebra of type $(C_1^\vee, C_1)$ [43] is defined as the associative algebra $\widehat{H}_q$ with generators $\{t_i^{\pm 1}\,,\ i = 0, \dots, 3\}$ and relations:*

$$t_i t_i^{-1} = t_i^{-1} t_i = 1, \tag{15.7.1a}$$

$$t_i + t_i^{-1} \qquad \textit{is central,} \tag{15.7.1b}$$

$$t_0 t_1 t_2 t_3 = q^{-1}. \tag{15.7.1c}$$

The "usual" DAHA, denoted $H_q(k_0, k_1, k_2, k_3)$, is recovered when the central elements $t_i + t_i^{-1}$ have complex values $k_i + k_i^{-1}$, with $k_i \neq 0$.
- *The algebra Γ_q [44] is the subalgebra of $\widehat{H}_q$ commuting with the distinguished generator t_0 (Γ_q is the centralizer of t_0 in $\widehat{H}_q$):*

$$\Gamma_q = \{h \in \widehat{H}_q \mid [h, t_0] = 0\}. \tag{15.7.2}$$

- *Let $\mathbf{e}$ be the following idempotent of $H_q(k_0, k_1, k_2, k_3)$ [73]*

$$\mathbf{e} = \frac{t_0 - k_0}{k_0^{-1} - k_0}. \tag{15.7.3}$$

The spherical DAHA, denoted $SH_q(k_0, k_1, k_2, k_3)$ [41, 42], is defined as

$$SH_q(k_0, k_1, k_2, k_3) = \mathbf{e}\, H_q(k_0, k_1, k_2, k_3)\, \mathbf{e}. \tag{15.7.4}$$

The following theorems relate DAHA to the previously introduced algebraic structures.

Theorem 15.15. *[44] The map $\Theta :\ \mathbf{saw}(3) \to \Gamma_q$ defined by*

$$\begin{aligned} C_{12} &\mapsto t_1 t_0 + (t_1 t_0)^{-1}, \\ C_{23} &\mapsto t_3 t_0 + (t_3 t_0)^{-1}, \\ C_{13} &\mapsto t_2 t_0 + (t_2 t_0)^{-1}, \end{aligned} \qquad \begin{aligned} C_1 &\mapsto t_1 + t_1^{-1}, \\ C_2 &\mapsto t_2 + t_2^{-1}, \\ C_3 &\mapsto t_3 + t_3^{-1}, \\ C_{123} &\mapsto q^{-1} t_0 + q t_0^{-1}. \end{aligned} \tag{15.7.5}$$

is an injective algebra homomorphism.

Theorem 15.16. *(Theorem 3.2 in [42]) The Special Zhedanov algebra $sZh_q(m_1, m_2, m_3, m_4)$ is isomorphic to the spherical DAHA $SH_q(k_0, k_1, k_2, k_3)$.*

Remark 15.17. *Spherical DAHAs have also been connected to skein algebras of higher genus. The Kauffman bracket skein algebra of the once-punctured torus $\mathrm{Sk}_\theta(\Sigma_{1,1})$ is related to a (spherical) DAHA of type A_1 [37, 81] and the genus two skein algebra is related to a genus two spherical double affine Hecke algebra in [82].*

15.8. Actions of the braid group

In this section, we provide two actions of the braid group: the first one on the algebra $\mathcal{A}_3$ and the second one on the skein algebra $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$. Then, we show how these two actions are compatible and give a diagrammatic presentation of the intermediate Casimir elements of $U_q(\mathfrak{sl}_2)^{\otimes 3}$.

We recall that the braid group on n strands B_n is generated by the elements $s_1, \ldots, s_{n-1}$ as well as their inverses $s_1^{-1}, \ldots, s_{n-1}^{-1}$ satisfying

$$\begin{aligned} s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\ s_i s_j &= s_j s_i \qquad \text{if} \quad |i-j| \geq 2, \\ s_i^{-1} s_i &= s_i s_i^{-1} = 1. \end{aligned} \tag{15.8.1}$$

15.8.1. The braided universal R-matrix and a braid group action on $\mathcal{A}_3$

Let us recall that we define the generators Q_{13} as follows

$$Q_{13} = Q_{13d} = \check{\mathcal{R}}_2^{-1} Q_{12} \check{\mathcal{R}}_2 = \check{\mathcal{R}}_1 Q_{23} \check{\mathcal{R}}_1^{-1}. \tag{15.8.2}$$

From the result of Proposition 15.11, we know that Q_{13} satisfies

$$Q_{13} = \frac{Q_1 Q_3 + Q_2 Q_{123}}{q + q^{-1}} - \frac{[Q_{12}, Q_{23}]_q}{q^2 - q^{-2}}, \tag{15.8.3}$$

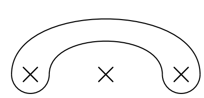
and is in the algebra $\mathcal{A}_3$ which is generated by Q_1, Q_2, Q_3, Q_{12}, Q_{23} and Q_{123}. Now from (15.8.2) it is natural to consider the following element which is analogous to Q_{13d}:

$$Q_{13u} = \check{\mathcal{R}}_2 Q_{12} \check{\mathcal{R}}_2^{-1} = \check{\mathcal{R}}_1^{-1} Q_{23} \check{\mathcal{R}}_1. \tag{15.8.4}$$

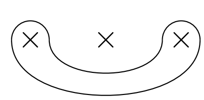
It has been shown in [77] that this element is also in $\mathcal{A}_3$ since it can be obtained as

$$Q_{13u} = \frac{Q_1 Q_3 + Q_2 Q_{123}}{q + q^{-1}} - \frac{[Q_{23}, Q_{12}]_q}{q^2 - q^{-2}}. \tag{15.8.5}$$

The labels u and d added on the Casimir elements Q_{13d} and Q_{13u} stand for *up* and *down*. These names come from the form of their image in $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ given in Corollary 15.12:



$$= \phi(Q_{13u}), \tag{15.8.6a}$$



$$= \phi(Q_{13d}). \tag{15.8.6b}$$

This procedure of obtaining additional elements of $\mathcal{A}_3$ by conjugations of braided R-matrices can be described by an automorphism action. Let us define the following automorphisms of $\mathcal{A}_3$ denoted Ψ_{s_i} and $\Psi_{s_i^{-1}}$ by

$$\Psi_{s_i}(X) = \check{\mathcal{R}}_i X \check{\mathcal{R}}_i^{-1} \qquad \text{and} \qquad \Psi_{s_i^{-1}}(X) = \check{\mathcal{R}}_i^{-1} X \check{\mathcal{R}}_i = \Psi_{s_i}^{-1}(X), \tag{15.8.7}$$

for $i = 1, 2$ and $X \in \mathcal{A}_3$, The previous maps are well-defined since the images of the generators of $\mathcal{A}_3$ are precisely in $\mathcal{A}_3$ (and not in its completion). Indeed, by direct computations making use of the explicit form (15.6.8) of the universal R-matrix and the commutation relations of $U_q(\mathfrak{sl}_2)$, one gets

$$\begin{gathered}\Psi_{s_1}(Q_1) = Q_2, \qquad \Psi_{s_1}(Q_2) = Q_1, \qquad \Psi_{s_1}(Q_3) = Q_3, \qquad \Psi_{s_1}(Q_{123}) = Q_{123}, \\ \Psi_{s_1}(Q_{12}) = Q_{12}, \qquad \Psi_{s_1}(Q_{23}) = Q_{13d}.\end{gathered} \tag{15.8.8}$$

and

$$\begin{gathered}\Psi_{s_2}(Q_1) = Q_1, \qquad \Psi_{s_2}(Q_2) = Q_3, \qquad \Psi_{s_2}(Q_3) = Q_2, \qquad \Psi_{s_2}(Q_{123}) = Q_{123}, \\ \Psi_{s_2}(Q_{12}) = Q_{13u}, \qquad \Psi_{s_2}(Q_{23}) = Q_{23}.\end{gathered} \tag{15.8.9}$$

We obtain similarly the actions of $\Psi_{s_i^{-1}}$ on the generators of $\mathcal{A}_3$.

Since the braided R-matrix satisfies the braided Yang–Baxter equation (15.6.11), we can show that the defining relations (15.8.1) of the braid group B_3 are reproduced

$$\Psi_{s_1} \circ \Psi_{s_2} \circ \Psi_{s_1} = \Psi_{s_2} \circ \Psi_{s_1} \circ \Psi_{s_2}, \tag{15.8.10a}$$

$$\Psi_{s_i} \circ \Psi_{s_i^{-1}} = \Psi_{s_i^{-1}} \circ \Psi_{s_i} = id. \tag{15.8.10b}$$

We extend the automorphisms Ψ_S to any $S \in B_3$ by

$$\Psi_S(X) = (\Psi_{g_1} \circ \Psi_{g_2} \circ \cdots \circ \Psi_{g_\ell})(X), \tag{15.8.11}$$

where S is decomposed as $S = g_1 g_2 \dots g_\ell$ and $g_i \in \{s_1, s_2, s_1^{-1}, s_2^{-1}\}$. Note that the map (15.8.11) does not depend on the choice of the decomposition of S due to (15.8.10).

Remark 15.18. *The realization of the braid group given by Ψ_S is not faithful. For example, one can verify that $\Psi_{(s_1 s_2)^3} = id$. This is checked to be true on the intermediate Casimir elements by making repeated use of* (15.8.8)-(15.8.9)*. It follows that it is also true for any polynomial in those elements. Moreover, some elements of $\mathcal{A}_3$ have additional stabilizers, e.g.*

$$\Psi_{s_1 s_1}(Q_1) = \check{\mathcal{R}}_1^{-1} \check{\mathcal{R}}_1^{-1} Q_1 \check{\mathcal{R}}_1 \check{\mathcal{R}}_1 = \check{\mathcal{R}}_1^{-1} Q_2 \check{\mathcal{R}}_1 = Q_1, \tag{15.8.12a}$$

$$\Psi_{s_2}(Q_{23}) = Q_{23}. \tag{15.8.12b}$$

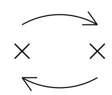
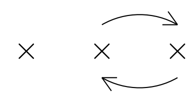
Identifying stabilizers of the braid group action on elements of $\mathcal{A}_3$ is easy to do but giving an exhaustive list is harder.

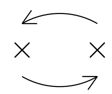
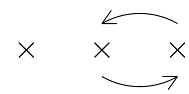
Remark 15.19. *It was shown in [83] how such a braid group action translates to the $q \to -1$ limit. This limit of the Askey–Wilson algebra is referred to as the Bannai–Ito algebra. In that case, the B_3 braid group action simplifies to an action of the S_3 symmetric group. It is possible to study more generally the action of the S_n symmetric group on the higher rank Bannai–Ito algebra $B(n)$.*

15.8.2. Half Dehn twists and the braid group action on $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$

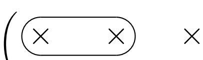
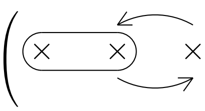
We now present a B_3 group action on the Kauffman bracket skein algebra $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$, denoted $\psi_S : \mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4}) \to \mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$, with $S \in B_3$. The braid group action rotates the placement of the punctures with respect to each other.

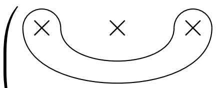
Here is how it goes. First, the actions ψ_{s_i} and $\psi_{s_i^{-1}}$ on $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ are defined by the so-called half Dehn twists [73, 84]. The four generators of B_3 act as

$\psi_{s_1} =$  , $\qquad \psi_{s_2} =$  ,

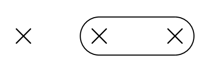
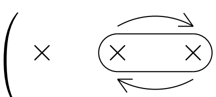
$\psi_{s_1^{-1}} =$  , $\qquad \psi_{s_2^{-1}} =$  , (15.8.13)

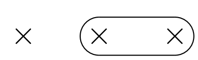
where any framed link gets deformed continuously without crossing the punctures as the rotations happen. For example, one gets

$\psi_{s_2^{-1}}(\mathbb{A}_{12}) = \psi_{s_2^{-1}}\Big($  $\Big) = \Big($  $\Big)$

$= \Big($  $\Big) = \mathbb{A}_{13},$ (15.8.14)

and

$\psi_{s_2}(\mathbb{A}_{23}) = \psi_{s_2}\Big($  $\Big) = \Big($  $\Big)$

$= \Big($  $\Big) = \mathbb{A}_{23}.$ (15.8.15)

Proposition 15.20. *The actions ψ_g for $g \in \{s_1, s_2, s_1^{-1}, s_2^{-1}\}$ are automorphisms of* $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$.

Proof. For any X, $Y \in \mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ and $g \in \{s_1, s_2, s_1^{-1}, s_2^{-1}\}$, one understands that

$$\psi_g(X \cdot Y) = \psi_g(X) \cdot \psi_g(Y). \tag{15.8.16}$$

Indeed, from the way they were defined, the rotations do not add or change crossings. Thus, the Kauffman bracket relations (15.5.1) that one makes use of to "simplify the crossings" of a given product are unchanged under these rotations. Since the rotations are also defined in order to avoid links crossing punctures, the topological properties (such as which punctures are circled by which links) are preserved. Hence the action ψ_g is a homomorphism. Moreover ψ_g is an endomorphism because links in $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ are mapped to other links in $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$, and it is invertible, as rotations can be inverted, thus ψ_g is an automorphism. $\square$

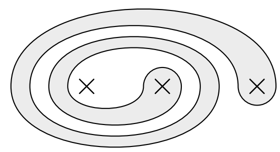
Let $S = g_1 g_2 \dots g_\ell \in B_3$ be a decomposition of an element of the braid group on three strands with $g_i \in \{s_1, s_2, s_1^{-1}, s_2^{-1}\}$. We define the automorphism ψ_S as follows:

$$\psi_S(X) = (\psi_{g_1} \circ \psi_{g_2} \circ \cdots \circ \psi_{g_\ell})(X). \tag{15.8.17}$$

We use also the definition $\psi_1 = id$. The previous map (15.8.17) does not depend on the choice of the decomposition of S. Indeed, it is straightforward to check that the defining relations of the braid group (15.8.1) are verified on the generators. By the homomorphism property (15.8.16), it follows that these braid relations are verified for any element of the Kauffman bracket skein module $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$.

Remark 15.21. *More visually complicated loops can always be created by further "twisting" the loops. For example,*

$$\psi_{(s_1^{-1})^4}(\mathbb{A}_{23}) = \tag{15.8.18}$$



is a more complicated analog of $\mathbb{A}_{23}$. The shadow filling the inside of the loop is there to guide the eyes of the reader. These have also been studied in [85].

Remark 15.22. *Let us remark that in [73], the author considers a similar braid group action by half Dehn twists on the Kauffman bracket skein algebra of the four-punctured sphere. In that paper, it is shown that the group $SL(2;\mathbb{Z})$ acts on the DAHA of type $(C_1^\vee C_1)$ through conjugations. Furthermore, the Artin braid group B_3 action on the Kauffman bracket skein algebra can be seen as a translation of this $SL(2;\mathbb{Z})$ action. We also note that Terwilliger had presented a B_3 action on both the universal Askey–Wilson algebra and the universal DAHA of type $(C_1^\vee, C_1)$ [44].*

15.8.3. Connection between both braid actions

The following proposition establishes the connections between both braid group actions presented above.

Proposition 15.23. *The following diagram of isomorphisms*

$$\begin{tikzcd}
\mathcal{A}_3 \arrow[r, "\Psi_S"] \arrow[d, "\phi"] & \mathcal{A}_3 \arrow[d, "\phi"] \\
\mathrm{Sk} \arrow[r, "\psi_S"] & \mathrm{Sk}
\end{tikzcd}$$

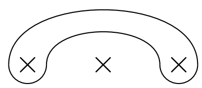
is commutative for any $S \in B_3$*. Here we used the shortened notation* $\mathrm{Sk} \equiv \mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$*. The isomorphisms* ϕ*,* Ψ_S *and* ψ_S *are given in* (15.6.16), (15.8.11) *and* (15.8.17)*, respectively.*

Proof. We can show that this diagram is commutative for all the generators of $\mathcal{A}$ and for any $S = s_i$ or $S = s_i^{-1}$. For example:

$$\phi \circ \Psi_{s_1}(Q_1) = \phi(Q_2) = \mathbb{A}_2 = \psi_{s_1}(\mathbb{A}_1) = \psi_{s_1} \circ \phi(Q_1). \tag{15.8.19}$$

A more complicated example is

$$\phi \circ \Psi_{s_2}(Q_{12}) = \phi(Q_{13u}) =$$



$$= \psi_{s_2}(\mathbb{A}_{12}) = \psi_{s_2} \circ \phi(Q_{12}). \tag{15.8.20}$$

Since all the maps of the diagram are homomorphisms, the commutativity of the diagram on the generators of $\mathcal{A}$ is enough to prove the proposition for any $S \in B_3$. $\square$

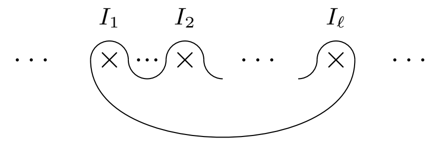
The commutativity of this diagram allows us to identify the conjugation by the braided R-matrix for $\mathcal{A}_3$ as half Dehn twists around the punctures of $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$. In addition, we can identify easily the elements of the algebra $\mathcal{A}_3$ obtained as an image by Ψ_S with a link of $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$.

15.9. Towards a higher rank saw(n) algebra

Some natural generalizations of the different algebras have previously been introduced and studied:

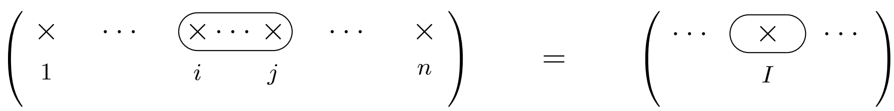
- the generalized Askey–Wilson algebra $\mathbf{aw}(n)$ is the algebra generated by $\{C_I \mid I \subset \{1, 2, \ldots, n\}\}$ subject to the relations introduced in Theorems 3.1 and 3.2 of [56];
- the algebra $\mathcal{A}_n$ is the subalgebra of $U_q(\mathfrak{sl}_2)^{\otimes n}$ generated by all the intermediate Casimir elements $\{Q_I \mid I \subset \{1, 2, \ldots, n\}\}$ obtained by the repeated action of the coproduct of $U_q(\mathfrak{sl}_2)$;
- the centralizer $\mathfrak{C}_n$ is defined by
$$\mathfrak{C}_n = \{X \in U_q(\mathfrak{sl}_2)^{\otimes n} \;\big|\; [\Delta^{(n-1)}(x), X] = 0\ , \quad \forall x \in U_q(\mathfrak{sl}_2)\} \tag{15.9.1}$$
where $\Delta^{(n)} = (\Delta^{(n-1)} \otimes id)\Delta$ and $\Delta^{(1)} = \Delta$;
- the algebra $Sk_\theta(\Sigma_{0,n+1})$ is the Kauffman bracket skein algebra associated to the $(n+1)$-punctured sphere $\Sigma_{0,n+1}$ [39]. Let us now associate to each set $I \subseteq [1; n] \equiv \{1, 2, \ldots, n\}$ a 'simple' loop $\mathbb{A}_I$ of $Sk_\theta(\Sigma_{0,n+1})$. We write a set I as $I = I_1 \cup I_2 \cup \cdots \cup I_\ell$,

where I_i are sets of consecutive integers and then we define the 'simple' loop $\mathbb{A}_I$ as:

$\mathbb{A}_I =$ 

(15.9.2)

These simple loops do not bend around, unlike (15.8.18). They are only extending in the lower half of the plane. In particular, for $I = \{i, i+1, \dots, j\}$, a set of consecutive integers, one gets

$\mathbb{A}_I =$ 

(15.9.3)

What is lacking in the previous list is the generalization $\mathbf{saw}(n)$ of the algebra $\mathbf{saw}(3)$. Such a generalization would provide a description of the algebra $\mathcal{A}_n$ in terms of generators and relation. We know that $\mathbf{saw}(n)$ will be a quotient of the algebra $\mathbf{aw}(n)$ by relation(s) of the type (15.2.2), with some Casimir elements to be determined. *We conjecture that the map ϕ_n from $\mathbf{saw}(n)$ to $Sk_\theta(\Sigma_{0,n+1})$ which sends Q_I to $\mathbb{A}_I$ is an isomorphism* [2].

Let us mention that there also exist generalizations in the non-deformed case ($q = 1$ and $q = -1$) of the Askey–Wilson algebra: these are respectively called the "higher rank Racah algebra" introduced in [62] as well as the "higher rank Bannai–Ito algebra" introduced in [63].

In the remainder, we give different indications regarding ways to define $\mathbf{saw}(n)$.

15.9.1. Punctures on a sphere and a coassociative homomorphism of Kauffman bracket skein modules

Recall we had highlighted that the punctures of the sphere were related to the tensor product factors. Additionally, a loop encircling a puncture is associated to some intermediate Casimir element with non-trivial factors in the tensor product factor corresponding to the puncture.

Further recall that the coproduct Δ acts as an algebra morphism from $U_q(\mathfrak{sl}_2)$ to $U_q(\mathfrak{sl}_2)^{\otimes 2}$. One can define an action of the coproduct on any i^{th} factor of a tensor product: for any $X \in U_q(\mathfrak{sl}_2)^{\otimes n}$, we define $\Delta_i : U_q(\mathfrak{sl}_2)^{\otimes n} \to U_q(\mathfrak{sl}_2)^{\otimes (n+1)}$ as:

$$\Delta_i(x) = \left(1^{\otimes(i-1)} \otimes \Delta \otimes 1^{\otimes(n-i)}\right)(X). \tag{15.9.4}$$

[2]During the preparation of this paper, the authors have been informed by J. Cooke that a similar idea was pursued in an upcoming publication [39].

Now in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ some intermediate Casimir elements are related to each other by the co-product, such as Q_1 and Q_{12}:

$$\Delta_1(Q_1) = (\Delta \otimes 1 \otimes 1)Q_1 = Q_{12} \otimes 1. \tag{15.9.5}$$

This relation between Q_1 and Q_{12} appears in the framed links picture as well.

More precisely, Δ_i has an analog, the δ_i morphism, which acts on a single puncture i by creating another puncture next to it. If the puncture i is enclosed in a loop, the created puncture is also enclosed in the same loop. The example (15.9.5) is illustrated as follows:

$$\begin{aligned}\delta_1\mathbb{A}_1 = \delta_1\left(\otimes \quad \times \quad \times\right) &= \delta_1\left(\boxed{\quad\times\quad} \quad \times \quad \times\right)\\ &= \left(\boxed{\times \quad \times} \quad \times \quad \times\right) = \mathbb{A}_{12} \in \mathrm{Sk}_\theta(\Sigma_{0,5})\end{aligned} \tag{15.9.6}$$

This δ_i is a Kauffman bracket skein module coassociative algebra homomorphism. It provides embeddings of $\mathrm{Sk}_\theta(\Sigma_{0,n}) \to \mathrm{Sk}_\theta(\Sigma_{0,n+1})$. This can be seen as the commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{A}_n \arrow[rr, "\Delta_i"] \arrow[d, "\phi_n"] & & \mathcal{A}_{n+1} \arrow[d, "\phi_{n+1}"] \\
\mathrm{Sk}_\theta(\Sigma_{0,n+1}) \arrow[rr, "\delta_i"] & & \mathrm{Sk}_\theta(\Sigma_{0,n+2})
\end{tikzcd}$$

15.9.2. A crossing index

The defining algebra relations of $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ (15.2.1)–(15.2.2) (see Proposition 15.7) can be classified in three types. The relations always involve two generators, whose product, commutator or q-commutator is reexpressed in terms of other generators. Now imagine we draw both generators simultaneously in a framed links diagram (as if we were to multiply them). Some crossings will appear if the two generators don't commute.

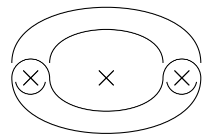
Definition 15.24. *The crossing index is defined as the minimal number of crossings that appear in a framed link diagram no matter how the generators are drawn.*

The relations (15.2.1)–(15.2.2) can be classified in terms of the crossing index as follows:

- If the generators can be drawn simultaneously in such a way that the loops have no crossings (crossing index of 0), they will commute (for example, this is the case for any central element Q_1, Q_2, Q_3, Q_{123} multiplied with any other generator).
- If the generators can be drawn in such a way that their minimum number of crossings is two (crossing index of 2), linear q-commutation relations of $\mathbf{aw}(3)$-type will be obtained, such as relations (15.2.1).

- If the generators have a crossing index of 4, such as

$$\phi(Q_{13u}Q_{13d}) =$$



$$, \qquad (15.9.7)$$

  higher order relations of the type (15.2.2) will be obtained.

This crossing index proves useful for the analysis of the higher rank generalizations of **saw**(3).

15.9.3. The algebras aw(n) and $Sk_\theta(\Sigma_{0,n+1})$

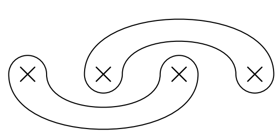
As mentioned previously, the algebra **aw**(n) is generated by C_I with $I \subseteq [1;n]$ and subject to the relations of Proposition 3.1 of [56]. We can show by using the action of the morphism δ_i that we have an homomorphism from **aw**(n) to $Sk_\theta(\Sigma_{0,n+1})$. Moreover, we can show that all the relations of Proposition 3.1 of [56] correspond to the product of two simple loops with crossing index 2. We believe that the relations in [56] exhaust all possibilities of relations involving the product of simple loops with crossing index 2. We conjecture also that the above mentioned homomorphism is surjective (but it is certainly not injective, even for the case $n = 3$). The description of the kernel would involve products of links with a crossing index strictly greater than 2. The complete description of this kernel would lead to the definition of **saw**(n) and give an algebraic description of $\mathcal{A}_n$ and $Sk_\theta(\Sigma_{0,n+1})$.

The study of **saw**(n) should be guided by the intuition gained from the framed links picture. To illustrate the type of insight we can gain, let us efficiently summarize some of the results of [53]. In this paper, the authors study the intermediate Casimir elements in $U_q(\mathfrak{sl}_2)^{\otimes 4}$ and introduce an involution I of the algebra as well as "involuted" generators IQ_{13} and IQ_{24} satisfying

$$[Q_{13}, IQ_{24}] = 0, \qquad \text{and} \qquad [IQ_{13}, Q_{24}] = 0. \qquad (15.9.8)$$

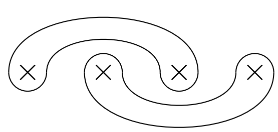
That these generators commute becomes evident when we rewrite (following our definitions) $IQ_{24} = Q_{24u}$, $IQ_{13} = Q_{13u}$, and then draw the corresponding links. Indeed, the products

$$\phi(Q_{13d}Q_{24u}) =$$



$$= \phi(Q_{24u}Q_{13d}), \qquad (15.9.9a)$$

$$\phi(Q_{13u}Q_{24d}) =$$

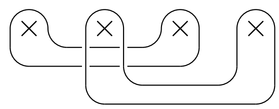


$$= \phi(Q_{24d}Q_{13u}), \qquad (15.9.9b)$$

have 0 crossing hence $[Q_{13d}, Q_{24u}] = 0$ and $[Q_{13u}, Q_{24d}] = 0$.

What about the product of terms like Q_{13d} and Q_{24d} ? This calculation has never appeared in the papers mentioned above because it has a crossing number of 4:

$$\phi(Q_{13d}Q_{24d}) = \tag{15.9.10}$$



Remarkably, this calculation can be effected in $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,5})$ using the conjectured morphism. One writes the Q_I in terms of $\mathbb{A}_I$, computes using the skein relations of $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,5})$, then reexpresses all $\mathbb{A}_I$ in terms of Q_I. This yields the following results

$$\begin{aligned} Q_{13}\,Q_{24} &= q^2 Q_{14}Q_{23} + q^{-2}Q_{12}Q_{34} + q(Q_{14}Q_2Q_3 + Q_{23}Q_1Q_4) + q^{-1}(Q_{12}Q_3Q_4 + Q_{34}Q_1Q_2) \\ &\quad + (q+q^{-1})Q_{1234} + Q_1Q_2Q_3Q_4 + Q_1Q_{234} + Q_2Q_{134} + Q_3Q_{124} + Q_4Q_{123} \end{aligned} \tag{15.9.11}$$

and

$$\begin{aligned} Q_{24}\,Q_{13} &= q^{-2} Q_{14}Q_{23} + q^{2}Q_{12}Q_{34} + q^{-1}(Q_{14}Q_2Q_3 + Q_{23}Q_1Q_4) + q(Q_{12}Q_3Q_4 + Q_{34}Q_1Q_2) \\ &\quad + (q+q^{-1})Q_{1234} + Q_1Q_2Q_3Q_4 + Q_1Q_{234} + Q_2Q_{134} + Q_3Q_{124} + Q_4Q_{123}. \end{aligned} \tag{15.9.12}$$

These have been checked to hold in $U_q(\mathfrak{sl}_2)^{\otimes 4}$.

Let us also mention that the action of the braid group B_3 can be generalized to the action of B_n on $Sk_\theta(\Sigma_{0,n+1})$ and $\mathcal{A}_n$. This might turn out useful for proving results in the future.

15.10. Conclusion

Three objectives were principally pursued in this paper. The first aimed to review the different avatars of the Askey–Wilson algebra and to clarify the relations between them. Among those algebras, we focused on two and presented novel results related to these cases; this was the second main goal. The Special Zhedanov algebra $sZh_q(m_1, m_2, m_3, m_4)$ was obtained from (a quotient of) the reflection algebra by setting the Sklyanin determinant to an appropriate value; its $W(D_4)$ symmetry was exhibited in addition. The Special Askey–Wilson algebra **saw**(3), a universal analogue of $sZh_q(m_1, m_2, m_3, m_4)$, was shown to be isomorphic to the algebra $\mathcal{A}_3$ that emerges from the Racah problem of $U_q(\mathfrak{sl}_2)$ and also to the Kauffman bracket skein algebra of the four-punctured sphere $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$. An injective homomorphism between $\mathcal{A}_3$ and the centralizer $\mathfrak{C}_3$ of $U_q(\mathfrak{sl}_2)$ in its threefold tensor product was stated and proved. Actions of the braid group on both $\mathrm{Sk}_{iq^{1/2}}(\Sigma_{0,4})$ (through half Dehn twists) and $\mathcal{A}_3$ (through conjugation by braided R-matrices) were illustrated and shown to be compatible. The third main objective was to discuss the generalization of **saw**(3) to **saw**(n). To that end, we emphasized the diagrammatic approach, defined a crossing index, and revisited the results of [53, 56] in a unified fashion.

Let us conclude with more remarks regarding generalizations of Askey–Wilson algebras. It would certainly be desirable to return to Zhedanov's original quest and to determine directly from the multivariate Askey–Wilson polynomials (of Tratnik type) [57] the algebra that encapsulates their bispectral properties. Steps have been carried out [55, 58, 59] but this should be completed. A definite higher rank generalization of the Zhedanov algebra will emerge, whose quotients and central extensions could then be examined and should connect to various fields in mathematics and physics. Considering higher rank Lie algebras $\mathfrak{g}$ instead of $\mathfrak{sl}_2$ is another avenue that should be explored. The centralizer of the diagonal action of $U_q(\mathfrak{g})$ in the n-fold tensor product $U_q(\mathfrak{g})^{\otimes n}$, or the algebra generated by all the intermediate Casimir elements of $\mathfrak{g}$ in the associated Racah problem should be studied. Connections with a generalization of $Sk_\theta(\Sigma_{0,n})$ to punctured manifolds of higher genera would be worth investigating (see also [82]). We may also wonder whether the braided universal R-matrix of $U_q(\mathfrak{g})$ plays a role in this context. Furthermore, the truncated reflection algebra presented in Section 15.3 provides a natural framework to obtain generalizations of Zhedanov algebras. Different possibilities are here conceivable. One could consider more general truncations of the reflection algebra. This type of generalization has been already studied in [60] and has been associated to quotients of q-Onsager algebras [3]. Connections with centralizers and/or skein algebras remain to be examined. Another possibility with respect to truncated reflection algebras is the following. Instead of using the R-matrix associated to quantum affine algebras, one could consider the R-matrix corresponding to Yangians. In this case, a particular truncation of the reflection algebra leads to the Hahn algebra, which is a specialization of the Zhedanov algebra, see [86]. Other truncations should provide interesting generalizations of this algebra. Finally, the FRT presentation of the reflection algebra associated to higher rank Lie algebras and superalgebras is well-known. For instance, the twisted Yangians $Y^{\mathrm{tw}}(\mathfrak{o}_n)$ and $Y^{\mathrm{tw}}(\mathfrak{sp}_n)$ [87] and the reflection algebra $\mathcal{B}(n,\ell)$ [88] correspond to subalgebras of the Yangian of $\mathfrak{sl}_n$. Some q-deformations of these structures have been also studied previously [89] and are related to the quantum affine algebra of $\mathfrak{sl}_n$. Their truncations have yet to be scrutinized and should possess interesting features [4]. These ideas that we plan on pursuing in the near future are indications that there is much lying ahead with respect to algebras of the Askey–Wilson type and what they will reveal and lead to.

[3]The classical limit $q \to 1$ leads to subalgebras of the loop algebra of $\mathfrak{sl}_2$ and to quotients of the Onsager algebra by Davis relations [61].

[4]Such an approach has been pursued in the classical limit $q \to 1$ [90] to obtain generalizations of the so-called classical Askey–Wilson algebra and are seen as subalgebras of the $\mathfrak{sl}_n$ Onsager algebra [91].

Acknowledgments

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15.A. Classical limit and injectivity

We provide an explicit description of the classical limit of the realization of $\mathbf{saw}(3)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ in terms of polarized traces, and use it to prove the injectivity of the map from $\mathbf{saw}(3)$ to the centralizer $\mathfrak{C}_3$. In this appendix, we will work with the formal series version of $U_q(\mathfrak{sl}_2)$ and reduce the proof of the injectivity statement to one in the universal enveloping algebra $U(sl_2)$, where we can use known results of classical invariant theory involving polarized traces.

15.A.1. Polarised traces in $U(\mathfrak{sl}_2)^{\otimes 3}$

The algebra $U(\mathfrak{sl}_2)$ is generated by elements e_{ij}, $i,j \in \{1,2\}$, with the defining relations $[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}$ and $e_{11} + e_{22} = 0$. To join up with the notations used in the paper for $U_q(\mathfrak{sl}_2)$, we set $E = e_{12}$, $F = e_{21}$ and $H = \frac{1}{2}(e_{11} + e_{22}) = e_{11} = -e_{22}$, and the relations become:

$$[H, E] = E, \qquad [H, F] = -F, \qquad [E, F] = 2H. \tag{15.A.1}$$

In a tensor product $U(\mathfrak{sl}_2)^{\otimes N}$, we denote the generators by $e_{ij}^{(a)}$, where $a \in \{1, \dots, N\}$ indicates the corresponding factor in the tensor product. The polarized traces are the following elements:

$$T^{(a_1,\dots,a_d)} = e_{i_2 i_1}^{(a_1)} e_{i_3 i_2}^{(a_2)} \dots e_{i_1 i_d}^{(a_d)}, \qquad a_1, \dots, a_d \in \{1, \dots, N\}, \tag{15.A.2}$$

where the summation over repeated indices is understood. The specific combinations of polarized traces that will appear are:

$$k_1 := T^{(1,1)}, \quad k_2 := T^{(2,2)}, \quad k_3 := T^{(3,3)}, \quad k_4 := k_1 + k_2 + k_3 + 2(T^{(1,2)} + T^{(2,3)} + T^{(1,3)}),$$
$$X := k_1 + k_2 + 2T^{(1,2)}, \qquad Y := k_2 + k_3 + 2T^{(2,3)}, \qquad Z := -8T^{(1,2,3)}. \tag{15.A.3}$$

15.A.2. The algebra $U_\alpha(\mathfrak{sl}_2)$

In this appendix, we will work with the formal series version of the quantum group $U_q(\mathfrak{sl}_2)$. We consider a formal parameter α. The algebra $U_\alpha(\mathfrak{sl}_2)$ is, as a vector space, the space $U(\mathfrak{sl}_2)[[\alpha]]$ of all formal power series in α with coefficients in $U(\mathfrak{sl}_2)$, and the multiplication is determined by the defining relations of $U_q(\mathfrak{sl}_2)$, see section 15.6.1, where q is replaced by e^α and q^H is replaced by $e^{\alpha H}$. This results in the following relations deforming (15.A.1):

$$[H, E] = E, \qquad [H, F] = -F, \qquad [E, F] = \frac{e^{2\alpha H} - e^{-2\alpha H}}{e^\alpha - e^{-\alpha}}. \tag{15.A.4}$$

Similarly, the algebra $U_\alpha(\mathfrak{sl}_2)^{\hat{\otimes} N}$ is the vector space $U(\mathfrak{sl}_2)^{\otimes N}[[\alpha]]$ of formal series with coefficients in $U(\mathfrak{sl}_2)^{\otimes N}$ and multiplication induced by the above relations in each factor. The comultiplication of $U_\alpha(\mathfrak{sl}_2)$ is naturally obtained from the comultiplication given for $U_q(\mathfrak{sl}_2)$.

Note that the limit $\alpha \to 0$ from $U_\alpha(\mathfrak{sl}_2)$ yields the algebra $U(\mathfrak{sl}_2)$ and the comultiplication becomes the diagonal embedding.

15.A.3. Reduction to $U(\mathfrak{sl}_2)$

We want to prove that the following elements

$$Q_{12}{}^i \, Q_{23}{}^j \, Q_{13}{}^k \, Q_1{}^m \, Q_2{}^n \, Q_3{}^p \, Q_{123}{}^q \qquad i, j, m, n, p, q \in \mathbb{N}, \qquad k \in \{0, 1\}, \tag{15.A.5}$$

are linearly independent in $U_\alpha(\mathfrak{sl}_2)^{\hat{\otimes} 3}$. First it is more convenient (and equivalent) to replace the generators Q_I by the modified versions introduced Section 15.4:

$$K_I = \frac{Q_I - (q + q^{-1})}{(q - q^{-1})^2}, \qquad I \in \{1, 2, 3, 123, 12, 23\}. \tag{15.A.6}$$

The index 13 does not belong to this set, and for this one, we set:

$$K_{13} = \frac{Q_{13} - (Q_1 + Q_2 + Q_3 + Q_{123} - Q_{12} - Q_{23}) + (q + q^{-1})}{(q - q^{-1})^3}. \tag{15.A.7}$$

Calculating explicitly the first terms in the expansions in α (up to order 3 for Q_{13} and up to order 2 for the others), we find that the new elements K_I are well-defined in $U_\alpha(\mathfrak{sl}_2)^{\hat{\otimes} 3}$, and moreover that their degree 0 coefficients are expressed in terms of polarized traces, using the notations in (15.A.3), as follows

$$K_i|_{\alpha=0} = \tfrac{1}{2}k_i \quad (i=1,2,3), \qquad K_{123}|_{\alpha=0} = \tfrac{1}{2}k_4, \tag{15.A.8}$$

$$K_{12}|_{\alpha=0} = \tfrac{1}{2}X, \qquad K_{23}|_{\alpha=0} = \tfrac{1}{2}Y, \qquad K_{13}|_{\alpha=0} = -\tfrac{1}{8}Z. \tag{15.A.9}$$

These are straightforward calculations, the one for K_{13} being a bit lengthy (for which one can use for example the explicit expression for Q_{13} using the R-matrix given in the paper).

Now, to prove that the elements of the set (15.A.5), with Q_I replaced by K_I, are linearly independent in $U_\alpha(\mathfrak{sl}_2)^{\hat{\otimes}3}$, it is enough to prove that their "classical limits" (the degree 0 coefficients) are linearly independent in $U(\mathfrak{sl}_2)^{\otimes 3}$. In view of the above calculations, it remains to show that the following set:

$${k_1}^i{k_2}^j{k_3}^k{k_4}^m X^n Y^p Z^q, \qquad\qquad i,j,k,m,n,p \in \mathbb{N}, \qquad q \in \{0,1\}, \tag{15.A.10}$$

is linearly independent in $U(\mathfrak{sl}_2)^{\otimes 3}$.

15.A.4. Racah algebra and diagonal centraliser in $U(\mathfrak{sl}_2)^{\otimes 3}$

To prove that the set (15.A.10) is linearly independent, we use the same line of arguments as the one used in the study of the recoupling of two copies of $\mathfrak{sl}(3)$. Thus we only give here a sketch and refer for more details to [79].

It is known from classical invariant theory [92, 93] that the centralizer of the diagonal embedding of $U(\mathfrak{sl}_2)$ in $U(\mathfrak{sl}_2)^{\otimes 3}$ is generated by the polarised traces $T^{(i,i)}$, $T^{(k,l)}$, $T^{(1,2,3)}$, with $i=1,2,3$ and $1 \leq k < l \leq 3$, and moreover that the Hilbert–Poincaré series of the centralizer is:

$$\frac{1-t^6}{(1-t^2)^6(1-t^3)}. \tag{15.A.11}$$

This series records the dimension for each degree of the centralizer, where the degree in $U(\mathfrak{sl}_2)^{\otimes 3}$ is defined by $deg(e_{ij}^{(a)}) = 1$. From this information, we extract at once that the set k_1, k_2, k_3, k_4, X, Y, Z generates the centralizer. Now, we have that these elements satisfy the classical Racah relations:

$$\begin{aligned}
&k_1, k_2, k_3, k_4 \quad \text{commute with all generators,}\\
&[X,Y] = Z,\\
&[X,Z] = 4\{X,Y\} + 4X^2 - 4(k_1+k_2+k_3+k_4)X + 4(k_1-k_2)(k_4-k_3),\\
&[Z,Y] = 4\{X,Y\} + 4Y^2 - 4(k_1+k_2+k_3+k_4)Y + 4(k_3-k_2)(k_4-k_1),
\end{aligned} \tag{15.A.12}$$

together with

$$\Gamma = 8(k_1-k_2+k_3-k_4)(k_1k_3-k_2k_4) - 32(k_1k_3+k_2k_4), \tag{15.A.13}$$

where the element Γ is

$$\begin{aligned}\Gamma := {} & Z^2 - 8(XYX + YXY) + 4(k_1 + k_2 + k_3 + k_4 - 4)\{X, Y\} \\ & - 8(k_1 - k_2)(k_4 - k_3)Y - 8(k_3 - k_2)(k_4 - k_1)X.\end{aligned} \tag{15.A.14}$$

The relations (15.A.12) allow to rewrite any product in terms of ordered monomials in the generators and (15.A.13) allows to rewrite Z^2. So we deduce easily that the set (15.A.10) is a spanning set for the centralizer. Finally, the comparison with the Hilbert–Poincaré series in (15.A.11) shows that this set must be linearly independent.

This concludes the proof of the injectivity of the map from $\mathbf{saw}(3)$ to $\mathfrak{C}_3 \subset U_\alpha(\mathfrak{sl}_2)^{\hat{\otimes}3}$.

Remark 15.25. *Specializing the central elements k_i to $\frac{m_i^2-1}{2}$, one finds that the relations (15.A.12)–(15.A.13) are expressed in terms of the polynomials:*

$$\sum_{i=1}^{4} {m_i}^2, \qquad \begin{cases} ({m_1}^2 - {m_2}^2)({m_4}^2 - {m_3}^2), \\ ({m_3}^2 - {m_2}^2)({m_4}^2 - {m_1}^2), \end{cases} \qquad ({m_1}^2{m_3}^2 - {m_2}^2{m_4}^2)({m_1}^2 - {m_2}^2 + {m_3}^2 - {m_4}^2). \tag{15.A.15}$$

These polynomials are invariant polynomials under the action of the Weyl group $W(D_4)$ of Section 15.4. This recovers explicitly the classical limit of the results in Section 15.4.

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Chapitre 16

Racah algebras, the centralizer $Z_n(\mathfrak{sl}_2)$ and its Hilbert–Poincaré series

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Abstract: The higher rank Racah algebra $R(n)$ introduced in [1] is recalled. A quotient of this algebra by central elements, which we call the special Racah algebra $sR(n)$, is then introduced. Using results from classical invariant theory, this $sR(n)$ algebra is shown to be isomorphic to the centralizer $Z_n(\mathfrak{sl}_2)$ of the diagonal embedding of $U(\mathfrak{sl}_2)$ in $U(\mathfrak{sl}_2)^{\otimes n}$. This leads to a first and novel presentation of the centralizer $Z_n(\mathfrak{sl}_2)$ in terms of generators and defining relations. An explicit formula of its Hilbert–Poincaré series is also obtained and studied. The extension of the results to the study of the special Askey–Wilson algebra and its higher rank generalizations is discussed.

Keywords: Racah algebra, centralizer, $U(\mathfrak{sl}_2)$, classical invariant theory, first and second fundamental theorems, Hilbert–Poincaré series.

16.1. Introduction

This paper clarifies the connection between the higher rank Racah algebra $R(n)$ and the centralizer $Z_n(\mathfrak{sl}_2)$ of the diagonal embedding of $U(\mathfrak{sl}_2)$ in its n-fold tensor product. The central result of the paper is:

A particular quotient of the Racah algebra $R(n)$, which we provide in (16.4.1), *is the centralizer.*

In the case $n = 3$, the centralizer $Z_3(\mathfrak{sl}_2)$ contains a number of subalgebras of interest for mathematics and physics. In representations, it is generated by the so-called "intermediate" or "quadratic" Casimir elements [2]. These elements realize the relations of the Racah algebra [3]. One may then wonder if the Racah algebra is the centralizer $Z_3(\mathfrak{sl}_2)$ or if more relations are needed in order to offer a full description of the centralizer by generators and relations.

A similar story repeats itself for the centralizer $Z_n(\mathfrak{sl}_2)$. One observes that the intermediate Casimir elements in the n-fold tensor product of $U(\mathfrak{sl}_2)$ realize the relations of the higher rank Racah algebra [1], but two questions remain: Is the centralizer generated by these intermediate Casimir elements, and are there additional relations needed in order to fully describe the centralizer?

As will be seen, these questions are quite close to questions that appear in classical invariant theory. The issue of finding a generating set (of polynomial functions, for example) is answered by the First Fundamental Theorem and the problem of obtaining defining relations between these generators is answered by the Second Fundamental Theorem.

In this regard, the results that we obtain in Corollary 16.22 and Theorem 16.23 can be seen as non-commutative analogues of the First and Second Fundamental Theorems.

16.1.1. On the Racah algebra

At this point it would be appropriate to provide some background on the Racah algebra, which arises in numerous areas of mathematics and physics.

A first approach to the Racah algebra is from the theory of orthogonal polynomials. The Racah polynomials are a family of bispectral classical orthogonal polynomials which are characterized by a difference and recurrence operator [4]. These operators obey the quadratic algebraic relations of the Racah algebra; this is actually how the algebra was originally introduced [5] and the reason why it inherited its name. Thus, the representation theory of the Racah algebra involves the eponym polynomials.

The problem of decomposing the tensor product of two irreducible representations of $\mathfrak{sl}_2$ in a direct sum of irreducible representations is known as the Clebsch–Gordan problem of $\mathfrak{sl}_2$. When the three-fold tensor product is considered, there exist two natural decompositions. The question of finding the overlaps between the two associated bases is called the Racah problem of $\mathfrak{sl}_2$. The intermediate Casimir elements labelling the two mentioned decompositions realize the Racah algebra [3], and thus the overlaps are found to be given in terms of Racah polynomials. There are also realizations of the Racah algebra in terms of $U(\mathfrak{sl}_2)$ [6–9].

The Racah algebra appears in various other mathematical contexts. For example, it arises in algebraic combinatorics where it plays an important role in the classification of P- and Q- polynomial association schemes [10]. The two operators satisfying the definition of a Leonard pair of Racah type obey also the Racah algebra relations and this allows one to connect the representation theory of the Racah algebra [11] with the classification of such pairs [12–14]. It is also deeply connected with other well-known algebras: the double affine Hecke algebras [15], the rational degeneration of the Sklyanin algebra [16] as well as the Temperley–Lieb and Brauer algebras [17].

The Racah algebra further shows up as an important object in physics. It is the symmetry algebra of the generic superintegrable system on the 2-sphere [18–21] from which all superintegrable systems in two dimensions with constants of motion of degree two or less in momenta can be obtained as limits or specializations of this generic system on the two-sphere [22]. It hence follows that the symmetry algebras of all such systems correspond to limits or contractions of the Racah algebra. The Racah algebra is also present in physical models in dimensions higher than two, including the isotropic oscillator in three dimensional space of constant positive curvature [23].

As is the case for various rich structures that have a lot of applications, the generalization of the Racah algebra is something desirable. A higher rank Racah algebra has been defined in [1] by looking at a 3D superintegrable system with the Racah algebra as its symmetry algebra and then generalizing this system to n dimensions. The algebraic relations between the constants of motion were then computed and used to define abstractly the higher rank Racah algebra $R(n)$. These relations of $R(n)$ are also verified by the intermediate Casimir elements that label various direct sum decompositions of the n-fold tensor product of $\mathfrak{sl}_2$ irreducible representations [24]. These higher rank Racah algebras have also been observed in physical models [25–29], interpreted in the framework of Howe duality [30], and the study of their relation to multivariate Racah polynomials has been initiated [31–33].

16.1.2. Outline

The paper is organized as follows. In section 16.2, the abstract $R(3)$ Racah algebra will be introduced and its quotient by a certain central element will be presented and named the special Racah algebra $sR(3)$. Then, the higher rank Racah algebra $R(n)$ will be presented in Section 16.3 and various properties will be highlighted. Section 16.4 will define the higher rank special Racah algebra $sR(n)$. This algebra is a quotient of $R(n)$ by a number of central elements which will be given precisely. We then come to the main results of the paper. It will be proven in Section 16.5 that the special Racah algebra is isomorphic to the diagonal centralizer $Z_n(\mathfrak{sl}_2)$ of $U(\mathfrak{sl}_2)$ in its n-fold tensor product. The Hilbert–Poincaré series of the centralizer will be obtained in Section 16.6 and its rich combinatorial properties will be examined. The PBW basis of the centralizer will then be given for the first few values of n. A conclusion offering some comments about the q-deformation of these results and the connection with the multivariate Racah polynomials will end the paper.

16.2. The Racah algebra (of rank 1) and a special quotient

We review the definition of the usual Racah algebra (of rank 1 in our terminology), along with the algebraic properties we need for the following. We then give the definition of the special Racah algebra of rank 1, to prepare for the generalization for any rank defined later in the paper.

Definition 16.1. *The Racah algebra* $R(3)$ *of rank* 1 *is the associative algebra with generators* P_{11}*,* P_{12}*,* P_{13}*,* P_{22}*,* P_{23}*,* P_{33}*,* F_{123}*, and with the following defining relations for indices* i, j, k *in* $\{1, 2, 3\}$ *and all distinct:*

$$P_{ii} \quad \textit{is central}, \tag{16.2.1a}$$

$$[P_{ij}, P_{jk}] = 2F_{ijk}, \tag{16.2.1b}$$

$$[P_{jk}, F_{ijk}] = P_{ik}(P_{jk} + P_{jj}) - (P_{jk} + P_{kk})P_{ij}. \tag{16.2.1c}$$

where $[A, B] = AB - BA$ *is the commutator, and* P_{ij} *and* F_{ijk} *are defined for any* $i, j, k \in \{1, 2, 3\}$ *by the requirements:*

$$P_{ij} = P_{ji} \qquad \textit{and} \qquad F_{ijk} = -F_{jik} = F_{jki} \qquad \textit{for any } i, j, k \in \{1, 2, 3\}. \tag{16.2.2}$$

We say that F_{ijk} is antisymmetric whereas P_{ij} is symmetric. Note that in view of (16.2.1b), the element F_{123} can be removed from the set of generators of $R(n)$ but it is more convenient to keep it.

Relation (16.2.1b) is given for any i, j, k. If i, j, k are not ordered, the symmetry properties of P and F are to be used. For example,

$$[P_{23}, P_{13}] = [P_{23}, P_{31}] = 2F_{231} = 2F_{123}, \tag{16.2.3}$$

where we used the symmetry of P, the antisymmetry of F and relation (16.2.1b) for $(i, j, k) = (2, 3, 1)$. Similar comments apply to relation (16.2.1c), which is given for any distinct i, j, k and not only the case $(i, j, k) = (1, 2, 3)$.

Let us introduce the notion of determinant for matrices with non-commuting entries. If A is a $n \times n$ matrix with entries $A_{i,j}$ $(1 \leq i, j \leq n)$, we define the symmetrized determinant of A as follows

$$\det A := \frac{1}{n!} \sum_{\rho, \sigma \in S_n} sgn(\rho) sgn(\sigma) A_{\rho(1),\sigma(1)} A_{\rho(2),\sigma(2)} \dots A_{\rho(n),\sigma(n)}, \tag{16.2.4}$$

where S_n is the permutation group of n elements and $sgn(\sigma)$ is the signature of σ. For commuting entries, it is the usual definition of the determinant of a matrix. We define also the following 3×3 matrix

$$P_{ijk}^{abc} = \begin{pmatrix} P_{ia} & P_{ib} & P_{ic} \\ P_{ja} & P_{jb} & P_{jc} \\ P_{ka} & P_{kb} & P_{kc} \end{pmatrix}. \tag{16.2.5}$$

There exist in addition to P_{ii}, other central elements in $R(3)$ given in the following proposition:

Proposition 16.2. *The following elements are central in* $R(3)$

$$Q_3 = P_{12} + P_{13} + P_{23}, \tag{16.2.6}$$

$$\begin{aligned} w_{ijk} :&= {F_{ijk}}^2 + \tfrac{1}{2}\det(P_{ijk}^{ijk}) \\ &\quad - \tfrac{1}{3}\left(\{P_{ij}, P_{ik}\} + \{P_{ij}, P_{jk}\} + \{P_{ik}P_{jk}\} + P_{ij}P_{kk} + P_{ik}P_{jj} + P_{jk}P_{ii}\right), \end{aligned} \tag{16.2.7}$$

for $1 \leq i, j, k \leq 3$ *distinct and where the anticommutator is defined as* $\{A, B\} = AB + BA$. *Moreover it is observed that* w_{ijk} *is symmetric i.e.*

$$w_{ijk} = w_{jik} = w_{jki}. \tag{16.2.8}$$

Proof. To show that an element is central, it is enough to show that it commutes with P_{12}, P_{13}, P_{23} (since the P_{ii}'s are central and $2F_{123} = [P_{12}, P_{23}]$). For Q_3, this is an easy verification.

The symmetry of w_{ijk} is immediate. So it remains to show the centrality of w_{123}. By symmetry of the algebra under renaming of the indices, only the commutation with one element, say P_{23}, needs to be checked. This is a direct calculation using the defining relations of the algebra. □

Remark 16.3. *It should be stressed that the element* w_{123} *is essentially the known Casimir element of the Racah algebra [5].*

Remark 16.4. *There exist other equivalent presentations of the Racah algebra* $R(3)$*. Indeed, introducing the following generators, for* $1 \leq i, j \leq 3$ *distinct,*

$$C_{ij} = P_{ij} + \frac{P_{ii} + P_{jj}}{2}, \tag{16.2.9}$$

the defining relations (16.2.1) *become*

$$P_{ii} \quad \textit{is central}, \tag{16.2.10a}$$

$$[C_{ij}, C_{jk}] = 2F_{ijk}, \tag{16.2.10b}$$

$$[C_{jk}, F_{ijk}] = C_{ik}C_{jk} - C_{jk}C_{ij} - \tfrac{1}{2}(P_{jj} - P_{kk})(C_{123} - \tfrac{1}{2}P_{ii}). \tag{16.2.10c}$$

with $C_{123} = C_{12} + C_{23} + C_{13} - \frac{1}{2}(P_{11} + P_{22} + P_{33})$ *(which is central by Proposition 16.2). This presentation is the one used for example in [11, 34].*

The special Racah algebra. The special Racah algebra $sR(3)$ is defined from the Racah algebra $R(3)$ by fixing the value of this non-trivial central element.

Definition 16.5. *The special Racah algebra $sR(3)$ of rank 1 is the quotient of $R(3)$ by*

$$w_{123} = 0. \tag{16.2.11}$$

Remark 16.6. *This relation* (16.2.11) *is akin to the relation that expresses the Casimir element of the Racah algebra in terms of its central elements, see equation (3.4) in [3] for instance.*

Remark 16.7. *It follows from* (16.2.8) *that w_{213} and other w_{ijk} obtained from permutations of the indices are null in $sR(3)$.*

The appellation "special" is used in the same way as in [35], where a quotient of the Askey–Wilson algebra by fixing the value of a central element expressed as a determinant was denoted as the "special Askey–Wilson algebra" (this was inspired by the nomenclature of Lie groups).

16.3. The higher rank Racah algebras

Following [1], we consider the following definition of the Racah algebra (of any rank). Note that we consider it as an abstract algebra defined by generators and relations. It will be clear that for $n = 3$ we recover Definition 16.1 for the rank one Racah algebra $R(3)$.

Definition 16.8. *The Racah algebra $R(n)$ of rank $n-2$ is the associative algebra with generators:*

$$\mathsf{P}_{ij},\ 1 \le i \le j \le n \qquad \text{and} \qquad \mathsf{F}_{ijk},\ 1 \le i < j < k \le n, \tag{16.3.1}$$

and the defining relations are, for all possible indices i, j, k, l, m in $\{1, \dots, n\}$:

$$\mathsf{P}_{ii} \quad \text{is central,} \tag{16.3.2a}$$

$$[\mathsf{P}_{ij}, \mathsf{P}_{k\ell}] = 0 \qquad\qquad \text{if both } i, j \text{ are distinct from } k, \ell, \tag{16.3.2b}$$

$$[\mathsf{P}_{ij}, \mathsf{P}_{jk}] = 2\mathsf{F}_{ijk}, \tag{16.3.2c}$$

$$[\mathsf{P}_{jk}, \mathsf{F}_{ijk}] = \mathsf{P}_{ik}(\mathsf{P}_{jk} + \mathsf{P}_{jj}) - (\mathsf{P}_{jk} + \mathsf{P}_{kk})\mathsf{P}_{ij}, \tag{16.3.2d}$$

$$[\mathsf{P}_{k\ell}, \mathsf{F}_{ijk}] = \mathsf{P}_{ik}\mathsf{P}_{j\ell} - \mathsf{P}_{i\ell}\mathsf{P}_{jk}, \tag{16.3.2e}$$

$$[\mathsf{F}_{ijk}, \mathsf{F}_{jk\ell}] = -(\mathsf{F}_{ij\ell} + \mathsf{F}_{ik\ell})\mathsf{P}_{jk}, \tag{16.3.2f}$$

$$[\mathsf{F}_{ijk}, \mathsf{F}_{k\ell m}] = \mathsf{F}_{i\ell m}\mathsf{P}_{jk} - \mathsf{F}_{j\ell m}\mathsf{P}_{ik}, \tag{16.3.2g}$$

where in each relation all indices involved are distinct and P_{ij} and F_{ijk} are defined by:

$$\mathsf{P}_{ij} = \mathsf{P}_{ji} \qquad \text{and} \qquad \mathsf{F}_{ijk} = -\mathsf{F}_{jik} = \mathsf{F}_{jki} \qquad \text{for any } i, j, k \in \{1, \dots, n\}. \tag{16.3.3}$$

We say that F_{ijk} (or simply F) is antisymmetric whereas P_{ij} is symmetric. The same comments as for the algebra $R(3)$ after Definition 16.1 apply here. In particular, in view of (16.3.2c), the elements F_{ijk} can be removed from the set of generators of $R(n)$ but it is more convenient to keep them. Note that to check that a certain element X is central in $R(n)$, it is enough to check that it commutes with all generators P_{ij}.

Remark 16.9. *The form of the defining relations above is very symmetrical, and this is quite useful in practice. Namely, for any permutation π of $\{1,\dots,n\}$, the corresponding renaming of the generators $(P_{ij}, F_{ijk}) \mapsto (P_{\pi(i),\pi(j)}, F_{\pi(i),\pi(j),\pi(k)})$ is an automorphism of the algebra. So when checking a relation in the algebra $R(n)$, it is enough to do it for a chosen set of indices. This property will be used in the proofs.*

Remark 16.10. *Relation* (16.3.2f) *seems different from the one provided in [1]. However, using relation* (16.3.18) *which is a consequence of the defining relations as proven in the following, relation* (16.3.2f) *can be transformed equivalently to the one of [1].*

Recall that P_{ii} is central in $R(n)$. It is easy to show using (16.3.2b)-(16.3.2c) that the following element is central in $R(n)$:

$$Q_n = \sum_{1\le i<j\le n} P_{ij} \ . \tag{16.3.4}$$

We now introduce some elements of $R(n)$ that will later play an important part. Recall the definition of $\det(P^{abc}_{ijk})$ (16.2.5) formulated in the preceding section. We define the following elements in $R(n)$:

$$\begin{aligned} w_{ijk} :=\ & {F_{ijk}}^2 + \tfrac{1}{2}\det(P^{ijk}_{ijk}) \\ & - \tfrac{1}{3}\left(\{P_{ij},P_{ik}\} + \{P_{ij},P_{jk}\} + \{P_{ik}P_{jk}\} + P_{ij}P_{kk} + P_{ik}P_{jj} + P_{jk}P_{ii}\right), \end{aligned} \tag{16.3.5}$$

$$x_{ijk\ell} := F_{ijk}F_{jk\ell} + \tfrac{1}{2}\det(P^{jk\ell}_{ijk}) + \tfrac{1}{2}(F_{ij\ell}+F_{ik\ell})P_{jk} - \tfrac{1}{3}(P_{ij}P_{k\ell} + P_{ik}P_{j\ell} + P_{i\ell}P_{jk}), \tag{16.3.6}$$

$$y_{ijk\ell m} := F_{ijk}F_{k\ell m} + \tfrac{1}{2}\det(P^{k\ell m}_{ijk}) + \tfrac{1}{2}(F_{ij\ell}P_{km} - F_{ijm}P_{k\ell}), \tag{16.3.7}$$

$$z_{ijk\ell mp} := F_{ijk}F_{\ell mp} + \tfrac{1}{2}\det(P^{\ell mp}_{ijk}), \tag{16.3.8}$$

where indices $i,j,k,\ell,m,p \in \{1,\dots,n\}$ are all distinct. Only the element w_{ijk} appears in the Racah algebra $R(3)$ since there are not enough different indices for the other elements.

16.3.1. The Racah algebra $R(4)$

The algebra $R(3)$ was previously studied in Section 16.2 so let us consider now the case $n=4$. According to the definition given above, there are 10 generators P_{ij} of $R(4)$ and 4 generators F_{ijk}. Of the relations (16.3.2b)–(16.3.2f) only those involving no more than 4 different indices are necessary here.

We already know that the elements P_{ii} and Q_4 given in (16.3.4) are central in $R(4)$. The following proposition gives less immediate consequences of the defining relations of $R(4)$, and in particular identifies the elements introduced in (16.3.5)–(16.3.6) as central elements.

Proposition 16.11. *The following assertions are true in $R(4)$:*

- *For $1 \leq a \leq 4$, the following relations hold:*

$$P_{a1}F_{234} - P_{a2}F_{134} + P_{a3}F_{124} - P_{a4}F_{123} = 0. \tag{16.3.9}$$

- *For distinct $i, j, k \in \{1, 2, 3, 4\}$, the elements w_{ijk} are symmetric ($w_{ijk} = w_{jik} = w_{jki}$) and are central in $R(4)$.*
- *For distinct $i, j, k, \ell \in \{1, 2, 3, 4\}$, the elements x_{ijkl} are symmetric ($x_{\sigma(i)\sigma(j)\sigma(k)\sigma(\ell)} = x_{ijk\ell}$ for $\sigma \in S_4$) and are central in $R(4)$.*

Proof. All these statements are proven by invoking the associativity of the algebra. Here is what is meant by that. Suppose that we have a word CBA, for A, B, C some generators, that we want to reorder into the form ABC. This is done by using the defining relations of the algebra (16.3.2). We decide, for example, to bring all P's to the left of the F's, and to order the F's and the P's between themselves in the lexicographical ordering of their indices. There are two ways to proceed: one can first start by reordering the pair (CB) into (BC)+some terms, or one could instead start by reordering the pair (BA) into (AB) + some other terms. We denote symbolically the difference at the end of these two computations by

$$\Big(C(BA) - (CB)A\Big), \tag{16.3.10}$$

and this must be identically 0 by the associativity of the algebra.

Let us first prove (16.3.9). In the present case, we shall look at the word $CBA = F_{ij\ell}P_{k\ell}P_{i\ell}$, for i, j, k, ℓ all distinct, and compute

$$\tfrac{1}{2}\Big((F_{ij\ell}P_{k\ell})P_{i\ell} - F_{ij\ell}(P_{k\ell}P_{i\ell})\Big). \tag{16.3.11}$$

Using relations (16.3.2), this can be brought to the form

$$P_{ii}F_{jk\ell} - P_{ij}F_{ik\ell} + P_{ik}F_{ij\ell} - P_{i\ell}F_{ijk}. \tag{16.3.12}$$

By the argument above, this expression has to be zero. Then, choosing $(i, j, k, \ell) = (1, 2, 3, 4)$, $(2, 3, 4, 1)$, $(3, 4, 1, 2)$ or $(4, 1, 2, 3)$, we recover (16.3.9) with $a = 1, 2, 3, 4$ respectively.

The proof that w_{ijk} is symmetric and that it commutes with all P_{ab} with $a, b \in \{i, j, k\}$ was already done in Section 16.2 for the algebra $R(3)$, and is still valid here. Using the symmetry of w_{ijk} and the symmetry of the algebra under renaming of the indices, to prove that w_{ijk} is central, it is enough to prove for example that $[P_{34}, w_{123}] = 0$.

This is done by making use of (16.3.9) and reducing the calculation to

$$[P_{34}, w_{123}] = \Big((F_{124} + 2F_{134})F_{123}\Big)P_{23} - (F_{124} + 2F_{134})\Big(F_{123}P_{23}\Big) \tag{16.3.13}$$

which is identically zero by the associativity of the algebra.

For the symmetry of $x_{ijk\ell}$, the particular case of $x_{ijk\ell} = x_{ikj\ell}$ is immediate using the symmetries of P and F. To complete the proof of the symmetry properties of x, it remains to show that $x_{jk\ell i} = x_{ijk\ell}$. Using the symmetry of the algebra $R(4)$ under renaming of the indices, it is enough to check that for example $x_{2341} = x_{1234}$. Substituting from the definition of $x_{ijk\ell}$, one has

$$\begin{aligned} x_{2341} - x_{1234} = (\mathsf{F}_{134} - \mathsf{F}_{123})\mathsf{F}_{234} + \tfrac{1}{2}\Big(\det(\mathsf{P}_{134}^{234}) - \det(\mathsf{P}_{123}^{234})\Big) \\ - \tfrac{1}{2}\Big(\mathsf{P}_{34}(\mathsf{F}_{123} + \mathsf{F}_{124}) + \mathsf{P}_{23}(\mathsf{F}_{134} + \mathsf{F}_{124})\Big). \end{aligned} \tag{16.3.14}$$

Now, looking at $CBA = \mathsf{F}_{234}\mathsf{F}_{124}\mathsf{P}_{34}$ and making use of (16.3.9), one computes

$$\begin{aligned} \tfrac{1}{2}\Big(\mathsf{F}_{234}(\mathsf{F}_{124}\mathsf{P}_{34}) - (\mathsf{F}_{234}\mathsf{F}_{124})\mathsf{P}_{34}\Big) = (\mathsf{F}_{134} - \mathsf{F}_{123})\mathsf{F}_{234} + \tfrac{1}{2}\Big(\det(\mathsf{P}_{134}^{234}) - \det(\mathsf{P}_{123}^{234})\Big) \\ - \tfrac{1}{2}\Big(\mathsf{P}_{34}(\mathsf{F}_{123} + \mathsf{F}_{124}) + \mathsf{P}_{23}(\mathsf{F}_{134} + \mathsf{F}_{124})\Big) \ . \end{aligned} \tag{16.3.15}$$

By the associativity of the algebra (see above), this expression has to be zero. This completes the proof of the symmetry of $x_{ijk\ell}$ since the right hand sides of (16.3.14) and (16.3.15) are the same.

Using the symmetry of $x_{ijk\ell}$ and the symmetry of the algebra under renaming of the indices, the proof that $x_{ijk\ell}$ is central reduces to proving that for example $[x_{1234}, \mathsf{P}_{23}] = 0$ which is also done by a direct computation using expression (16.3.17) of x_{1234}. □

Remark 16.12. *The elements w_{ijk} and $x_{ijk\ell}$ can be equivalently given by the following formulae*

$$w_{123} = {\mathsf{F}_{123}}^2 - \mathsf{F}_{123}\mathsf{P}_{13} - \mathsf{P}_{12}(\mathsf{P}_{13} + \mathsf{P}_{23} + \mathsf{P}_{33}) + \frac{1}{2}\sum_{\sigma\in S_3} sgn(\sigma)\mathsf{P}_{\sigma(1)1}\mathsf{P}_{\sigma(2)2}\mathsf{P}_{\sigma(3)3}, \tag{16.3.16}$$

and

$$x_{1234} = \mathsf{F}_{123}\mathsf{F}_{234} - \mathsf{F}_{123}\mathsf{P}_{24} + \mathsf{F}_{124}\mathsf{P}_{23} + \mathsf{F}_{134}\mathsf{P}_{23} - \mathsf{P}_{14}\mathsf{P}_{23} + \frac{1}{2}\sum_{\sigma\in S_3} sgn(\sigma)\mathsf{P}_{\sigma(1)2}\mathsf{P}_{\sigma(2)3}\mathsf{P}_{\sigma(3)4}. \tag{16.3.17}$$

16.3.2. The Racah algebra $R(n)$ for any n

Let now n be any positive integer. We already know that we have in $R(n)$ central elements P_{ii} and Q_n given in (16.3.4). Building upon all that has been proven up to now, we have the following final proposition about the Racah algebra $R(n)$.

Proposition 16.13. *The following assertions are true in $R(n)$:*

- *The relations below hold for $1 \le a \le n$ and $1 \le i < j < k < \ell \le n$:*

$$\mathsf{P}_{ai}\mathsf{F}_{jk\ell} - \mathsf{P}_{aj}\mathsf{F}_{ik\ell} + \mathsf{P}_{ak}\mathsf{F}_{ij\ell} - \mathsf{P}_{a\ell}\mathsf{F}_{ijk} = 0. \tag{16.3.18}$$

- *For distinct $i, j, k \in \{1, \ldots, n\}$, the elements w_{ijk} are symmetric and central in $R(n)$.*

- *For distinct $i,j,k,\ell \in \{1,\dots,n\}$, the elements $x_{ijk\ell}$ are symmetric and central in $R(n)$.*
- *For distinct $i,j,k,\ell,m \in \{1,\dots,n\}$, the elements $y_{ijk\ell m}$ are null in $R(n)$:*

$$y_{ijk\ell m} = 0 \qquad \text{for all distinct } i,j,k,\ell,m. \tag{16.3.19}$$

- *For distinct $i,j,k,\ell,m,p \in \{1,\dots,n\}$, the elements $z_{ijk\ell mp}$ are null in $R(n)$:*

$$z_{ijk\ell mp} = 0 \qquad \text{for all distinct } i,j,k,\ell,m,p. \tag{16.3.20}$$

PROOF. If a is equal to i,j,k or ℓ, relation (16.3.18) only involves 4 indices, and so its validity follows directly from the Proposition 16.11 concerning the algebra $R(4)$. To prove the case when a is different of i,j,k and ℓ, first compare (16.3.2g) for (i,j,k,ℓ,m) equals to $(1,2,5,3,4)$ and $(3,4,5,1,2)$. This leads to the identity

$$\mathsf{P}_{15}\mathsf{F}_{234} - \mathsf{P}_{25}\mathsf{F}_{134} + \mathsf{P}_{35}\mathsf{F}_{124} - \mathsf{P}_{45}\mathsf{F}_{123} = 0. \tag{16.3.21}$$

The symmetry of the algebra under renaming of the indices suffices to complete the proof of (16.3.18).

Regarding the statements about w_{ijk} and building on the verifications made in $R(3)$ and $R(4)$, it remains only to check that w_{ijk} commutes with $\mathsf{P}_{\ell m}$ when $\ell, m \notin \{i,j,k\}$. This is immediate since from the defining relations, any two elements (P's or F's) with no index in common commute.

Concerning $x_{ijk\ell}$, building on the proof of the preceding subsection, it remains to show, for example, that

$$[x_{1234}, \mathsf{P}_{45}] = 0. \tag{16.3.22}$$

This is shown using $y_{ijk\ell m} = 0$ (which is proven below) as well as (16.3.18). For $x_{ijk\ell}$, it remains only to check that it commutes with P_{mp} when $m,p \notin \{i,j,k,\ell\}$, and this is immediate.

To prove that $y_{ijk\ell m}$ is zero, we first look at the case $(i,j,k,\ell,m) = (1,2,3,4,5)$. Write

$$\tfrac{1}{2}\Big(\mathsf{F}_{245}(\mathsf{F}_{123}\mathsf{P}_{23}) - (\mathsf{F}_{245}\mathsf{F}_{123})\mathsf{P}_{23}\Big). \tag{16.3.23}$$

It is seen that this is equal to y_{12345}, by making use of (16.3.18). Invoking associativity, it follows that $y_{12345} = 0$. Since this can be repeated for all other combinations of distinct indices i,j,k,ℓ,m it is done.

For the nullity of $z_{ijk\ell mp}$, we invoke again the associativity of the algebra. Looking at

$$\tfrac{1}{2}\Big((\mathsf{F}_{356}\mathsf{F}_{234})\mathsf{P}_{12} - \mathsf{F}_{356}(\mathsf{F}_{234}\mathsf{P}_{12})\Big) \tag{16.3.24}$$

it is seen that this is equal to z_{123456}. Thus it follows that

$$z_{123456} = 0. \tag{16.3.25}$$

A similar reasoning can be repeated for different indices to complete the proof. $\square$

16.4. The special Racah algebra $sR(n)$

After the preliminary discussion of the Racah algebra $R(n)$, we are now ready to define the special Racah algebra for any n. This is a generalization to arbitrary rank of the special Racah algebra $sR(3)$ from Section 16.2.

Definition 16.14. *The special Racah algebra $sR(n)$ of rank $n-2$ is the quotient of $R(n)$ by all*

$$w_{ijk} = 0, \qquad x_{abcd} = 0, \tag{16.4.1}$$

such that $1 \leq i < j < k \leq n$ and $1 \leq a < b < c < d \leq n$.

Since $sR(n)$ is the algebra involved for the study of the centralizer in the next section, we collect here the generators and defining relations, to give an explicit definition without reference to the Racah algebra (we keep the same name for the generators, which is justified since this is a quotient, this should not lead to any ambiguity).

Definition 16.15 (Equivalent definition)**.** *The special Racah algebra $sR(n)$, of rank $n-2$ is the associative algebra with generators:*

$$\mathsf{P}_{ij},\ 1 \leq i \leq j \leq n \qquad \textit{and} \qquad \mathsf{F}_{ijk},\ 1 \leq i < j < k \leq n. \tag{16.4.2}$$

To give the defining relations, first we define P_{ij} and F_{ijk} by:

$$\mathsf{P}_{ij} = \mathsf{P}_{ji} \qquad \textit{and} \qquad \mathsf{F}_{ijk} = -\mathsf{F}_{jik} = \mathsf{F}_{jki} \qquad \textit{for any } i,j,k \in \{1,\ldots,n\}. \tag{16.4.3}$$

The defining relations are, for all possible indices i,j,k,ℓ,m in $\{1,\ldots,n\}$:

$$\mathsf{P}_{ii} \quad \textit{is central,} \tag{16.4.4a}$$

$$[\mathsf{P}_{ij}, \mathsf{P}_{k\ell}] = 0 \qquad \textit{if both } i,j \textit{ are distinct from } k,\ell, \tag{16.4.4b}$$

$$[\mathsf{P}_{ij}, \mathsf{P}_{jk}] = 2\mathsf{F}_{ijk}, \tag{16.4.4c}$$

$$[\mathsf{P}_{jk}, \mathsf{F}_{ijk}] = \mathsf{P}_{ik}(\mathsf{P}_{jk} + \mathsf{P}_{jj}) - (\mathsf{P}_{jk} + \mathsf{P}_{kk})\mathsf{P}_{ij}, \tag{16.4.4d}$$

$$[\mathsf{P}_{k\ell}, \mathsf{F}_{ijk}] = \mathsf{P}_{ik}\mathsf{P}_{j\ell} - \mathsf{P}_{i\ell}\mathsf{P}_{jk}, \tag{16.4.4e}$$

$$[\mathsf{F}_{ijk}, \mathsf{F}_{jk\ell}] = \mathsf{F}_{jk\ell}\mathsf{P}_{ij} - \mathsf{F}_{ijk}\mathsf{P}_{j\ell} - \mathsf{F}_{ik\ell}(\mathsf{P}_{jk} + \mathsf{P}_{jj}), \tag{16.4.4f}$$

$$[\mathsf{F}_{ijk}, \mathsf{F}_{k\ell m}] = \mathsf{F}_{i\ell m}\mathsf{P}_{jk} - \mathsf{F}_{j\ell m}\mathsf{P}_{ik}, \tag{16.4.4g}$$

along with

$$\mathsf{F}_{ijk}{}^2 + \tfrac{1}{2}\det(\mathsf{P}_{ijk}^{ijk}) = \tfrac{1}{3}\left(\{\mathsf{P}_{ij},\mathsf{P}_{ik}\} + \{\mathsf{P}_{ij},\mathsf{P}_{jk}\} + \{\mathsf{P}_{ik}\mathsf{P}_{jk}\} + \mathsf{P}_{ij}\mathsf{P}_{kk} + \mathsf{P}_{ik}\mathsf{P}_{jj} + \mathsf{P}_{jk}\mathsf{P}_{ii}\right), \tag{16.4.4h}$$

for $1 \le i < j < k \le n$, *and*

$$\mathsf{F}_{ijk}\mathsf{F}_{jk\ell} + \tfrac{1}{2}\det(\mathsf{P}_{ijk}^{jk\ell}) = -\tfrac{1}{2}(\mathsf{F}_{ij\ell} + \mathsf{F}_{ik\ell})\mathsf{P}_{jk} + \tfrac{1}{3}(\mathsf{P}_{ij}\mathsf{P}_{k\ell} + \mathsf{P}_{ik}\mathsf{P}_{j\ell} + \mathsf{P}_{i\ell}\mathsf{P}_{jk}), \tag{16.4.4i}$$

for $1 \le i < j < k < \ell \le n$.

Consequences of the relations. From the results of the preceding sections, we know that relations (16.4.4h)-(16.4.4i) for any distinct i, j, k, ℓ are automatically verified, along with the relations:

$$\mathsf{F}_{ijk}\mathsf{F}_{k\ell m} + \tfrac{1}{2}\det(\mathsf{P}_{ijk}^{k\ell m}) = \tfrac{1}{2}(\mathsf{F}_{ijm}\mathsf{P}_{k\ell} - \mathsf{F}_{ij\ell}\mathsf{P}_{km}) \tag{16.4.5a}$$

$$\mathsf{F}_{ijk}\mathsf{F}_{\ell mr} + \tfrac{1}{2}\det(\mathsf{P}_{ijk}^{\ell mr}) = 0, \tag{16.4.5b}$$

$$\mathsf{P}_{ai}\mathsf{F}_{jk\ell} - \mathsf{P}_{aj}\mathsf{F}_{ik\ell} + \mathsf{P}_{ak}\mathsf{F}_{ij\ell} - \mathsf{P}_{a\ell}\mathsf{F}_{ijk} = 0. \tag{16.4.5c}$$

for distinct $i, j, k, \ell, m, r \in \{1, \ldots, n\}$ and for any $a \in \{1, \ldots, n\}$. These relations, although satisfied, do not have to be included in the set of defining relations.

Example 16.16. *The special Racah algebra* $sR(4)$ *of rank* 2 *is the quotient of* $R(4)$ *by*

$$w_{123} = 0, \quad w_{124} = 0, \quad w_{134} = 0, \quad w_{234} = 0 \quad \textit{and} \quad x_{1234} = 0. \tag{16.4.6}$$

Remark 16.17. *Some analogues of the relations of* $sR(4)$ *(excluding the ones of the type* (16.4.5c)*) were obtained in a particular realization in [25]. For the higher rank case of* $sR(n)$*, analogous relations were also observed in a certain realization in [27], once again excluding the ones of the type* (16.4.5c).

16.5. Isomorphism between the centralizer $Z_n(\mathfrak{sl}_2)$ and the special Racah algebra $sR(n)$

The goal of this section is to connect the (higher rank) special Racah algebra introduced and characterized in the previous two sections with the centralizer $Z_n(\mathfrak{sl}_2)$ of the diagonal action of $U(\mathfrak{sl}_2)$ in $U(\mathfrak{sl}_2)^{\otimes n}$. This will provide an a posteriori justification for the quotient that was chosen to go from the Racah algebra $R(n)$ to the special Racah algebra $sR(n)$: as will be shown, this quotient is precisely the one that leads to an algebra isomorphic to the centralizer $Z_n(\mathfrak{sl}_2)$.

16.5.1. Centralizer $Z_n(\mathfrak{sl}_2)$ of the diagonal action $U(\mathfrak{sl}_2)$ into $U(\mathfrak{sl}_2)^{\otimes n}$ and the algebra of polarized traces

We here define the centralizer associated to the Lie algebra $\mathfrak{sl}_2$. The generators of $\mathfrak{sl}_2$ are e_{ij}, $i, j \in \{1, 2\}$ obeying the defining relations

$$[e_{ij}, e_{k\ell}] = \delta_{jk} e_{i\ell} - \delta_{\ell i} e_{kj}, \qquad e_{11} + e_{22} = 0. \tag{16.5.1}$$

We denote by $U(\mathfrak{sl}_2)$ the universal enveloping algebra of $\mathfrak{sl}_2$. Its Casimir element is given by

$$C = {e_{11}}^2 - e_{11} + e_{12}e_{21}. \tag{16.5.2}$$

Now consider the tensor product of n copies of $U(\mathfrak{sl}_2)$ and define the following notation for its generators

$$e_{ij}^{(a)} = 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(n-a)}. \tag{16.5.3}$$

The diagonal embedding of $U(\mathfrak{sl}_2)$ in its n-fold tensor product is given by

$$\begin{aligned} \delta :\ U(\mathfrak{sl}_2) &\to U(\mathfrak{sl}_2)^{\otimes n} \\ e_{ij} &\mapsto \sum_{a=1}^{n} e_{ij}^{(a)}. \end{aligned} \tag{16.5.4}$$

There is a natural degree-preserving action of $\mathfrak{sl}_2$ on $U(\mathfrak{sl}_2)^{\otimes n}$ given by composing the diagonal embedding δ followed by the adjoint action. On the generators, it is given by

$$e_{ij} \cdot e_{k\ell}^{(a)} = \delta_{jk} e_{i\ell}^{(a)} - \delta_{\ell i} e_{kj}^{(a)}. \tag{16.5.5}$$

We then define the centralizer $Z_n(\mathfrak{sl}_2)$ of the diagonal embedding of $U(\mathfrak{sl}_2)$ in $U(\mathfrak{sl}_2)^{\otimes n}$ as the kernel of this $\mathfrak{sl}_2$ action

$$Z_n(\mathfrak{sl}_2) = \Big\{ X \in U(\mathfrak{sl}_2)^{\otimes n} \ \Big|\ g \cdot X = [\delta(g), X] = 0 \quad \forall g \in U(sl_2) \Big\} \tag{16.5.6}$$

or in other words, as the set of elements in $U(\mathfrak{sl}_2)^{\otimes n}$ that commute with the diagonal embedding of $U(\mathfrak{sl}_2)$.

Let us also define the polarized traces (the summation convention is assumed):

$$T^{(a_1,\dots,a_d)} = e_{i_2 i_1}^{(a_1)} e_{i_3 i_2}^{(a_2)} \dots e_{i_1 i_d}^{(a_d)}, \qquad a_1, \dots, a_d \in \{1, \dots, n\}. \tag{16.5.7}$$

It is seen by a direct computation that these elements are in the centralizer $Z_n(\mathfrak{sl}_2)$.

Remark 16.18. *It is easily checked from the definition* (16.5.7) *that* $T^{(a_1,a_2)} = T^{(a_2,a_1)}$ *and that* $T^{(a_1,a_2,a_3)}$ *is antisymmetric in its indices* a_1, a_2, a_3*, i.e.* $T^{(a_1,a_2,a_3)} = T^{(a_2,a_3,a_1)} = -T^{(a_2,a_1,a_3)}$.

Remark 16.19. *A number of papers in the literature [3, 24] realize the Racah algebra with the so-called "intermediate Casimir" elements C_i, C_{ij}. These elements are given by*

$$C_i = 1^{\otimes(i-1)} \otimes C \otimes 1^{\otimes(n-i)}, \qquad C_{ij} = 1^{\otimes(i-1)} \otimes C_{(1)} \otimes 1^{\otimes(j-i-1)} \otimes C_{(2)} \otimes 1^{\otimes(n-j)}, \quad (16.5.8)$$

where $\Delta(e_{ij}) = e_{ij} \otimes 1 + 1 \otimes e_{ij}$ and we denote $\Delta(C) = C_{(1)} \otimes C_{(2)}$ in Sweedler's notation. Then the $T^{(i,i)}$ and $T^{(i,j)}$ can be expressed in terms of these intermediate Casimir elements as follows

$$T^{(i,i)} = 2C_i, \qquad T^{(i,j)} = C_{ij} - C_i - C_j. \quad (16.5.9)$$

16.5.2. Elements of classical invariant theory

We now present results from classical invariant theory about the algebra of polynomial functions on

$$\underbrace{\mathfrak{sl}_2 \times \cdots \times \mathfrak{sl}_2}_{n \text{ factors}} \equiv \mathfrak{sl}_2^n, \quad (16.5.10)$$

that are invariant under simultaneous conjugations by $SL(2)$. For G elements of $SL(2)$, these actions on a polynomial function of $\mathfrak{sl}_2^n$ are given by:

$$G \cdot f(M_1, \dots, M_n) = f(G^{-1}M_1G, \dots, G^{-1}M_nG), \quad (16.5.11)$$

for $M_i \in \mathfrak{sl}_2$. The first fundamental theorem of classical invariant theory states that:

Theorem 16.20 (see [36–38] or [39] and references therein)**.** *The algebra $\mathbb{C}[\mathfrak{sl}_2^n]^{inv}$ of polynomial functions on $\mathfrak{sl}_2^n$ that are invariant under simultaneous conjugations by $SL(2)$ elements is generated by the functions*

$$\mathfrak{T}^{(a_1,\dots,a_d)} \;:\; (M_1, \dots, M_n) \mapsto Tr(M_{a_1} \dots M_{a_d}) \quad (16.5.12)$$

for $M_i \in \mathfrak{sl}_2$, $d \geq 2$ and $a_1, \dots, a_d \in \{1, \dots, n\}$. Moreover, it is sufficient to take $\mathfrak{T}^{(i,j)}$ $(i \leq j)$ and $\mathfrak{T}^{(i,j,k)}$ $(i < j < k)$ to obtain a generating set.

The generating set of the invariant polynomial functions described in the preceding theorem (the ones of degrees 2 and 3) is not algebraically independent. A set of generators for their ideal of relations is given in the next theorem (second fundamental theorem on these invariants).

Theorem 16.21 (see [40], Theorem 2.3 (ii) or [39], Theorem 3.4 (ii)). *The defining relations for the algebra of polynomial invariant functions are*

$$\begin{aligned} &Tr([M_i, M_j]M_k)Tr([M_p, M_q]M_r) \\ &\quad + 2\begin{vmatrix} Tr(M_iM_p) & Tr(M_iM_q) & Tr(M_iM_r) \\ Tr(M_jM_p) & Tr(M_jM_q) & Tr(M_jM_r) \\ Tr(M_kM_p) & Tr(M_kM_q) & Tr(M_kM_r) \end{vmatrix} = 0, \end{aligned} \tag{16.5.13a}$$

$$\begin{aligned} &Tr([M_j, M_k]M_\ell)Tr(M_pM_i) - Tr([M_i, M_k]M_\ell)Tr(M_pM_j) \\ &\quad + Tr([M_i, M_j]M_\ell)Tr(M_pM_k) - Tr([M_i, M_j]M_k)Tr(M_pM_\ell) = 0, \end{aligned} \tag{16.5.13b}$$

with $i, j, k, \ell, m, n, p, q, r \in \{1, \dots, n\}$.

With the following reasoning that is adapted from [41], we now use Theorem 16.20 to extract information about the algebra of polarized traces and the centralizer.

- The algebra $U(\mathfrak{sl}_2)^{\otimes n}$ is filtered. Take the degree of all generators $e_{ij}^{(a)}$ to be 1, then the associated graded algebra is commutative. Recall the $\mathfrak{sl}_2$ action (16.5.5). This induces a natural action on $gr(U(\mathfrak{sl}_2)^{\otimes n})$. Denote the generators of the graded algebra by $\mathsf{e}_{ij}^{(a)}$. The induced $\mathfrak{sl}_2$-action is given as follows on the generators:

$$e_{ij} \cdot \mathsf{e}_{k\ell}^{(a)} = \delta_{jk}\mathsf{e}_{i\ell}^{(a)} - \delta_{\ell i}\mathsf{e}_{kj}^{(a)}. \tag{16.5.14}$$

- The algebra of polynomial functions on $\mathfrak{sl}_2^n$ is the algebra of polynomials $\mathbb{C}[x_{ij}^{(a)}]$, where $x_{ij}^{(a)}$ is the linear form giving the (i, j) coordinate of the a^{th} matrix in the product $\mathfrak{sl}_2^n$. The simultaneous conjugation action of an element G of $SL(2)$ on a polynomial function of $\mathfrak{sl}_2^n$ (16.5.11) is given infinitesimally by

$$\epsilon \sum_{k=1}^{n} f(M_1, \dots, [M_k, g], \dots, M_n) \tag{16.5.15}$$

for $G = e^{i\epsilon g}$ with $g \in \mathfrak{sl}_2$. Thus, on the generators of polynomials functions $x_{ij}^{(a)}$, this infinitesimal action is

$$e_{ij} \cdot x_{k\ell}^{(a)} = \delta_{j\ell}x_{ki}^{(a)} - \delta_{ik}x_{j\ell}^{(a)} \tag{16.5.16}$$

and can be identified with the induced $\mathfrak{sl}_2$ action on $gr(U(\mathfrak{sl}_2)^{\otimes n})$ through

$$\mathsf{e}_{ij}^{(a)} \leftrightarrow x_{ji}^{(a)}. \tag{16.5.17}$$

- It follows that under this identification, the invariant functions correspond to the elements of $gr(U(\mathfrak{sl}_2)^{\otimes n})$ in the kernel of the $\mathfrak{sl}_2$ action, or in other words to the image of the centralizer in $gr(U(\mathfrak{sl}_2)^{\otimes n})$. Moreover, again under this identification, the image in the graded algebra of the polarized trace $T^{(a_1,\dots,a_d)}$ defined in (16.5.12) is the polynomial function $Tr(M_{a_1} \dots M_{a_d})$.

16.5.3. The defining relations of $Z_n(\mathfrak{sl}_2)$

Knowing that the generators of the invariant functions (16.5.12) correspond to the polarized traces (16.5.7) (see Theorem 16.20), the image of the centralizer in $gr(U(\mathfrak{sl}_2)^{\otimes n})$ is therefore generated by the image of the polarized traces. Now consider an element of degree N in the centralizer. Up to terms of degree $N-1$, this element can be expressed as a polynomial in $T^{(a_1,\dots,a_d)}$. The same can then be argued for each of the remaining lower degree terms by induction, and thus any element in the centralizer can be expressed as a polynomial in the polarized traces. Since all polarized traces belong in the centralizer, the two algebras thus coincide. So we obtain the analogue of the first fundamental theorem:

Corollary 16.22. *It follows from Theorem 16.20 that the polarized traces $T^{(i,j)}$, $i \leq j$ and $T^{(i,j,k)}$, $i < j < k$ generate the centralizer $Z_n(\mathfrak{sl}_2)$ of the diagonal action of $U(\mathfrak{sl}_2)$ in $U(\mathfrak{sl}_2)^{\otimes n}$.*

Recall that relations (16.5.13) are a set of defining relations for the image of the centralizer in $gr(U(\mathfrak{sl}_2)^{\otimes n})$. We look for analogous defining relations for the centralizer in $U(\mathfrak{sl}_2)^{\otimes n}$. Once we find the deformations in $U(\mathfrak{sl}_2)^{\otimes n}$ of relations (16.5.13), we can prove their completeness using for the ideal of relations the same sort of reasoning as before Corollary 16.22. In fact the special Racah algebra was defined such that its set of defining relations gives precisely the complete set of relations for the centralizer.

Theorem 16.23. *With the following identification of the generators:*

$$\mathsf{P}_{ab} \mapsto T^{(a,b)} \qquad \text{and} \qquad \mathsf{F}_{ijk} \mapsto -T^{(i,j,k)} \tag{16.5.18}$$

for i, j, k all distinct, the centralizer $Z_n(\mathfrak{sl}_2)$ is isomorphic to the special Racah algebra $sR(n)$:

$$Z_n(\mathfrak{sl}_2) \cong sR(n). \tag{16.5.19}$$

In other words, a set of defining relations of $Z_n(\mathfrak{sl}_2)$ is given in (16.4.4)*, where P_{ab} and F_{ijk} are replaced by the corresponding polarized traces.*

Proof. The defining relations are verified to hold in $U(\mathfrak{sl}_2)^{\otimes n}$ by direct computations. Note that, due to the symmetry under renaming the indices, we only need to make calculations in $U(\mathfrak{sl}_2)^{\otimes n}$ for $n \leq 5$ in degrees less or equal to 6 in the generators.

Under the previous choice of degree for the generators of $U(\mathfrak{sl}_2)^{\otimes n}$ which was $deg(e_{ij}^{(a)}) = 1$, it follows that

$$deg(T^{(i,j)}) = 2, \qquad deg(T^{(i,j,k)}) = 3. \tag{16.5.20}$$

The same degrees are given to the generators of the special Racah algebra $sR(n)$: $deg(\mathsf{P}_{ij}) = 2$ and $deg(\mathsf{F}_{ijk}) = 3$. This makes it a filtered algebra, and it is straightforward to observe that its associated graded algebra is isomorphic to the algebra of polynomial invariants functions. Indeed, the first set of defining relations (16.3.2), or equivalently (16.4.4a)–(16.4.4g), ensures that the generators all commute in the graded algebra, and then the relations (16.4.4h)–(16.4.4i) and (16.4.5) are mapped to (16.5.13).

Therefore both algebras related by the morphism in (16.5.18) have the same associated graded algebras, and so in particular have the same dimensions in each component of the filtration (that is, in each degree). Moreover the morphism is surjective from Corollary 16.22. Consequently, an element in the kernel of the map (in other words, a relation in $Z_n(\mathfrak{sl}_2)$ not implied by the relations of the special Racah algebra), if non-zero, would contradict the equality of dimensions for some degree. □

Remark 16.24. *It is quite remarkable that we only need to quotient the Racah algebra $R(n)$ by the elements w_{ijk} and $x_{ijk\ell}$ in order to recover the centralizer for any value of n. Indeed, one could have expected that in order to recover the centralizer for increasing n, we would need to quotient by elements of increasing degree or spanning an increasing number of indices. That this is not the case is quite a surprising simplification.*

16.6. The Hilbert–Poincaré series of $Z_n(\mathfrak{sl}_2)$

For more information on Hilbert–Poincaré series of graded algebras, we refer to [42]. The Hilbert–Poincaré series contains useful information about a graded, or filtered, algebra. We will illustrate this for the diagonal centralizer $Z_n(\mathfrak{sl}_2)$. We will provide an explicit formula for its Hilbert–Poincaré series, and then use it in conjunction with the defining relations found in Theorem 16.23 to provide bases of $Z_n(\mathfrak{sl}_2)$ for small n.

16.6.1. An explicit formula

The commutative algebra $\mathbb{C}[\mathfrak{sl}_2^n]^{inv}$ of polynomial functions on $\mathfrak{sl}_2^n$ that are invariant under simultaneous conjugation by $SL(2)$ is a graded algebra: it is the direct sum of the subspaces $\mathbb{C}_k[\mathfrak{sl}_2^n]^{inv}$ of homogeneous invariant polynomial functions of degree k. The Hilbert–Poincaré series records the dimensions of all these subspaces:

$$F_n(t) = \sum_{k\geq 0} \dim\Big(\mathbb{C}_k[\mathfrak{sl}_2^n]^{inv}\Big)t^k. \tag{16.6.1}$$

The centralizer $Z_n(\mathfrak{sl}_2)$ inherits from $U(\mathfrak{sl}_2)^{\otimes n}$ the structure of a filtered algebra: it is the union of the increasing sequence (in k) of subspaces $Z_n(\mathfrak{sl}_2)_{\leq k}$ of elements of degree less or equal to k (the degree is in the generators of $U(\mathfrak{sl}_2)^{\otimes n}$). The Hilbert–Poincaré series of $Z_n(\mathfrak{sl}_2)$ records the dimensions of the homogeneous subspaces of the associated graded algebra:

$$F_n(t) = \sum_{k\geq 0} \dim\Big(Z_n(\mathfrak{sl}_2)_{\leq k}/Z_n(\mathfrak{sl}_2)_{<k}\Big)t^k, \tag{16.6.2}$$

and thus, from the discussion in Section 16.5, is the same as the Hilbert–Poincaré series of the invariant polynomial functions.

Several formulas, using various approaches, have been obtained for the Hilbert–Poincaré series $F_n(t)$ (see references in [39, 43, 44]). The formula presented below seems to be new.

Proposition 16.25. *Let $n \geq 2$ and recall that the rank r is defined by $r = n - 2$. The Hilbert–Poincaré series of $Z_n(\mathfrak{sl}_2)$ is:*

$$F_n(t) = \frac{P_r(t)}{(1-t^2)^{3(r+1)}}, \tag{16.6.3}$$

where the numerator is given by:

$$P_r(t) = (1+t)^r \sum_{k=0}^{2r} (-1)^k a_k t^k, \qquad \textit{where} \begin{cases} a_{2k} = \binom{r}{k}^2, \\ a_{2k+1} = \binom{r}{k}\binom{r}{k+1}. \end{cases} \tag{16.6.4}$$

Proof. We take a detour through the graded character of $SL(2)$ on the polynomial functions on $\mathfrak{sl}_2^n$. The character of $SL(2)$ for a finite-dimensional representation is seen as a Laurent polynomial in x, given by the trace of the action of the element $\mathrm{Diag}(x, x^{-1})$. For example, for the fundamental representation of $SL(2)$, it is $x + x^{-1}$. For the irreducible representation of dimension $d+1$, which is the d-symmetrized power of the fundamental representation, the character is thus $\frac{x^{d+1}-x^{-d-1}}{x-x^{-1}}$. Now, it is easy to check that, if we have the character $\chi(x)$ of an arbitrary finite-dimensional representation of $SL(2)$, then the formula:

$$\Big[(1-x^2)\chi(x)\Big]_0, \tag{16.6.5}$$

where $[\,\cdot\,]_0$ means taking the constant term of a Laurent polynomial, gives the multiplicity of the trivial representation.

After these classical preliminaries, note that the character of the adjoint representation of $SL(2)$ on $\mathfrak{sl}_2$ is $1+x^2+x^{-2}$. On the polynomial function on $\mathfrak{sl}_2$, the action of $SL(2)$ preserves the grading, and we record the character of the representation on its graded components as a formal power series in t (also called, the graded character). For each degree, the representation is a symmetrized power of the adjoint representation, so we find that the graded character is:

$$\frac{1}{(1-t)(1-tx^2)(1-tx^{-2})}. \tag{16.6.6}$$

Equivalently, this is the graded character for the adjoint action on $U(\mathfrak{sl}_2)$. On the polynomial functions $\mathfrak{sl}_2^n$ (or equivalently, on $U(sl_2)^{\otimes n}$), the graded character is thus:

$$\frac{1}{(1-t)^n(1-tx^2)^n(1-tx^{-2})^n}. \tag{16.6.7}$$

Now, in each degree, we look for the dimension of the invariant subspace for the action of $SL(2)$. In other words, we look for the multiplicity of the trivial representation. By what we have recalled above, we obtain that the Hilbert–Poincaré series of $Z_n(\mathfrak{sl}_2)$ is:

$$F_n(t) = \left[\frac{(1-x^2)}{(1-t)^n(1-tx^2)^n(1-tx^{-2})^n}\right]_0. \tag{16.6.8}$$

Using the expansion $(1-z)^{-n} = \sum_{k\geq 0}\binom{k+n-1}{k}z^k$ and straightforward manipulations, we obtain:

$$F_n(t) = \frac{1}{(1-t)^n}\sum_{k\geq 0}(-1)^k\tilde{a}_k t^k, \qquad \text{where } \begin{cases} \tilde{a}_{2k} = \binom{n+k-1}{k}^2, \\ \tilde{a}_{2k+1} = \binom{n+k-1}{k}\binom{n+k}{k+1}. \end{cases} \tag{16.6.9}$$

Thus the statement of the proposition reduces to the following equality of formal power series:

$$\frac{1}{(1-t)^{r+2}}\sum_{k\geq 0}(-1)^k\tilde{a}_k t^k = \frac{(1+t)^r}{(1-t^2)^{3(r+1)}}\sum_{k\geq 0}(-1)^k a_k t^k. \tag{16.6.10}$$

To prove this, we multiply the $\tilde{a}$-series by $(1+t)$ and the a-series by $(1-t)$, and after an application of Pascal rule for binomials, we reach the equivalent formula:

$$\sum_{k\geq 0}(-1)^k\tilde{a}'_k t^k = \frac{1}{(1-t^2)^{2(r+1)}}\sum_{k\geq 0}(-1)^k a'_k t^k, \tag{16.6.11}$$

where now we have $\begin{cases} \tilde{a}'_{2k} = \binom{r+k+1}{k}\binom{r+k}{k}, \\ \tilde{a}'_{2k+1} = \binom{r+k+1}{k}\binom{r+k+1}{k+1}, \end{cases}$ and $\begin{cases} a'_{2k} = \binom{r}{k}\binom{r+1}{k}, \\ a'_{2k+1} = \binom{r}{k}\binom{r+1}{k+1}. \end{cases}$

This last formula is verified by writing the expansion of the right hand side and checking the equality of the coefficients, making use of the following identity for binomial coefficients [45]:

$$\sum_i \binom{i+a+b}{i}\binom{b}{k-i}\binom{a}{k'-i} = \binom{k'+b}{k}\binom{k+a}{k'}. \tag{16.6.12}$$

This identity is valid for any a, b, k, k' and we use it for $a = r+1$, $b = r$ and $k' \in \{k, k+1\}$. $\square$

The exponent $3(r+1)$ appearing in the denominator of $F_n(t)$ is the Krull, or Gelfand–Kirillov, dimension of the algebra of invariant polynomial functions (see [46],[44]). Here it means that there is a set of $3(r+1)$ algebraically independent homogeneous elements (a system of parameters) $\theta_1, \ldots, \theta_{3(r+1)}$ such that the algebra is a free module of finite dimension over the polynomial subalgebra $\mathbb{C}[\theta_1, \ldots, \theta_{3(r+1)}]$. The freeness follows from the property called Cohen–Macaulay, which is ensured here by the Hochster–Roberts theorem from general invariant theory [47]. The form $F_n(t)$ above with the positivity of the numerator $P_r(t)$ (see below) suggests that it might be possible that a system of parameters consists of $3(r+1)$ elements of degrees 2. If it were to be the case, then $P_r(1)$ would be the dimension of the algebra over $\mathbb{C}[\theta_1, \ldots, \theta_{3(r+1)}]$. Moreover, the different monomials in $P_r(t)$ would indicate in which degrees the elements of a basis over $\mathbb{C}[\theta_1, \ldots, \theta_{3(r+1)}]$ would have to be found.

Finally, the palindromic property of the numerator $P_r(t)$ in the formula above shows that the Hilbert–Poincaré series satisfies the functional equation:

$$F_n(t^{-1}) = (-1)^{(n-1)} t^{3n} F_n(t). \tag{16.6.13}$$

This is well-known and related to a property, called being Gorenstein, for the algebra of invariant polynomial functions (see [39] and references therein).

Remark 16.26. *More generally, the Hilbert–Poincaré series of $Z_n(\mathfrak{sl}_2)$ can be defined as a power series in $t_1, \dots, t_n$ if we consider the filtration by the multidegree of $U(\mathfrak{sl}_2)^{\otimes n}$ (the degree of each component is recorded in an element of $\mathbb{N}^n$). Now, in the multigraded Hilbert–Poincaré series $F_n(t_1, \dots, t_n)$, the coefficient in front of $t_1^{k_1} \dots t_n^{k_n}$ is the dimension of $Z_n(\mathfrak{sl}_2)_{\leq \boldsymbol{k}}/Z_n(\mathfrak{sl}_2)_{<\boldsymbol{k}}$, where $\boldsymbol{k} = (k_1, \dots, k_n)$, and $\boldsymbol{l} \leq \boldsymbol{k}$ means that $l_i \leq k_i$ for all $i = 1, \dots, n$.*

A slight generalization of the part of the proof up to Formula (16.6.9) gives the multigraded version of this formula:

$$F_n(t_1, \dots, t_n) = \frac{1}{(1-t_1)\dots(1-t_n)} \Bigg(\sum_{\substack{\mu \models k \\ \nu \models k}} t_1^{\mu_1+\nu_1} \dots t_n^{\mu_n+\nu_n} - \sum_{\substack{\mu \models k \\ \nu \models k-1}} t_1^{\mu_1+\nu_1} \dots t_n^{\mu_n+\nu_n} \Bigg),$$

where $\mu \models k$ means that $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ such that $\mu_1 + \dots + \mu_n = k$. To recover Formula (16.6.9) from this, take $t_1 = \dots = t_n = t$ and use that the number of $\mu \models k$ is $\binom{k+n-1}{k}$.

Remark 16.27. *More generally, the Hilbert–Poincaré series of $Z_n(\mathfrak{sl}_2)$ can be defined as a power series in $t_1, \dots, t_n$ if we consider the gradation by the multidegree of $U(\mathfrak{sl}_2)^{\otimes n}$. A slight generalization of the part of the proof up to Formula (16.6.9) gives the multigraded version of this formula:*

$$F_n(t_1, \dots, t_n) = \frac{1}{(1-t_1)\dots(1-t_n)} \Bigg(\sum_{\substack{\mu \models k \\ \nu \models k}} t_1^{\mu_1+\nu_1} \dots t_n^{\mu_n+\nu_n} - \sum_{\substack{\mu \models k \\ \nu \models k-1}} t_1^{\mu_1+\nu_1} \dots t_n^{\mu_n+\nu_n} \Bigg), \tag{16.6.14}$$

where $\mu \models k$ means that $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ such that $\mu_1 + \dots + \mu_n = k$. To recover relation (16.6.9) from this, take $t_1 = \dots = t_n = t$ and use that the number of $\mu \models k$ is $\binom{k+n-1}{k}$.

16.6.2. Some related combinatorics

We have obtained an expression for the Hilbert–Poincaré series of $Z_n(\mathfrak{sl}_2)$ of the form:

$$F_n(t) = \frac{(1+t)^r Q_r(t)}{(1-t^2)^{3(r+1)}}, \qquad \text{where } Q_r(t) = \sum_{k=0}^{2r} (-1)^k a_k t^k, \tag{16.6.15}$$

and the coefficients a_k are given in the proposition above. It is perhaps not so surprising that the coefficients of the various polynomials involved show some connections with well-studied combinatorial objects of "Catalan" flavor.

The polynomial $Q_r(t)$. The coefficient a_k in the polynomial $Q_r(t)$ counts the number of symmetric Dyck paths of semi-length $2r+1$ with $k+1$ peaks (see A088855 in [48]). Their expression with binomial coefficients corresponds to choosing a certain number of peaks and troughs in the first r steps of the paths.

In fact, the polynomial $Q_r(t)$ is a t-deformation of the well-known Catalan number, that is, the value of $Q_r(t)$ at $t=1$ is the r-th Catalan number:

$$Q_r(1) = c_r = \binom{2r}{r} - \binom{2r}{r+1}. \tag{16.6.16}$$

Indeed it is not difficult to give a combinatorial proof that the alternating sum of the a_k's is equal to the Catalan number c_r (the number of Dyck paths of length $2r$). We can also see it as follows. The Catalan number is equal to the constant term of a Laurent polynomial in x:

$$c_r = \Big[(1-x^2)(x+x^{-1})^{2r}\Big]_0 = \Big[(1-x^2)(1+x^2)^r(1+x^{-2})^r\Big]_0. \tag{16.6.17}$$

Note that we keep the variable x^2 to stay coherent with the notation used during the proof above. In fact, the first equality expresses that the Catalan number c_r is the multiplicity of the trivial representation in $V^r \otimes (V^\star)^r$. The t-deformation is now immediate from this formula for c_r. Indeed, it follows from its explicit expression that the polynomial $Q_r(t)$ is given by:

$$Q_r(t) = \Big[(1-x^2)(1+tx^2)^r(1+tx^{-2})^r\Big]_0. \tag{16.6.18}$$

In this sense, the polynomial $Q_r(t)$ is a natural t-deformation of the r-th Catalan number.

The numerator $P_r(t)$. The numerator of the Hilbert–Poincaré series of $Z_n(\mathfrak{sl}_2)$ is $P_r(t) = (1+t)^r Q_r(t)$. We will show explicitly that its coefficients are positive.

First, from what we have said above about $Q_r(t)$, it follows that $P_r(t)$ is a t-deformation of the number $2^r c_r$, that is, its value at $t=1$ is $P_r(1) = 2^r c_r$. This number counts several classes of combinatorial objects (obtained from objects counted by the Catalan number, see A151374 in [48]). The t-deformation giving $P_r(t)$ can be expressed similarly as before as:

$$P_r(t) = \Big[(1-x^2)(1+t)^r(1+tx^2)^r(1+tx^{-2})^r\Big]_0. \tag{16.6.19}$$

Now regrouping the terms with an r-th power, this gives the following expression:

$$\begin{aligned} P_r(t) &= \Big[(1-x^2)\Big(1+t^3+(t+t^2)(1+x^2+x^{-2})\Big)^r\Big]_0 \\ &= \sum_{k=0}^{r} R_k \binom{r}{k} (1+t^3)^{r-k}(t+t^2)^k, \end{aligned} \tag{16.6.20}$$

where the positive integer R_k is the Riordan number, one of the closest relative of the Catalan number, which also admits many combinatorial interpretations (see A005043 in [48]). They are given by either one of the following equalities:

$$R_n = \Big[(1-x^2)(1+x^2+x^{-2})^n\Big]_0 = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}c_i. \tag{16.6.21}$$

We actually used the first one in the above calculation, while the second one shows that the Catalan sequence is the binomial transform of the Riordan sequence, and thus allows to recover that $P_r(1) = 2^r c_r$.

The formula (16.6.20) for $P_r(t)$ has the advantage to show explicitly that it has positive coefficients. So $P_r(t)$ is a t-deformation with positive coefficients of $2^r c_r$ and therefore should be given by an interesting statistics on a certain set of $2^r c_r$ objects.

16.6.3. PBW basis of $Z_n(\mathfrak{sl}_2)$ for small n

With the help of the Poincaré–Hilbert series obtained and discussed above, we here determine the PBW bases of $Z_n(\mathfrak{sl}_2)$ for $n = 2, 3, 4$. Using Theorem 16.23, we identify systematically $Z_n(\mathfrak{sl}_2)$ with the special Racah algebra, and use the generators and relations in Section 16.4. Note that from the discussion in Section 16.5.2, in order to show that a subset of $Z_n(\mathfrak{sl}_2)$ is a spanning set, it is enough to show that the images in the graded algebra of invariant polynomial functions is a spanning set.

The case $n = 2$. The Hilbert–Poincaré series of $Z_2(\mathfrak{sl}_2)$ is:

$$F_2(t) = \frac{1}{(1-t^2)^3}. \tag{16.6.22}$$

This allows easily to recover that the algebra of invariant polynomial functions, and $Z_2(\mathfrak{sl}_2)$, is a commutative polynomial algebra. A basis of $Z_2(\mathfrak{sl}_2)$ is $\mathsf{P}_{11}^a \mathsf{P}_{12}^b \mathsf{P}_{22}^c$ with $a, b, c \in \mathbb{Z}_{\geq 0}$. Indeed this set is obviously a spanning set, and moreover spans a subspace whose dimensions in each degree are given precisely by the series (16.6.22). So this set is also linearly independent, and is a basis.

The case $n = 3$. The Hilbert–Poincaré series of $Z_3(\mathfrak{sl}_2)$ is:

$$F_3(t) = \frac{1+t^3}{(1-t^2)^6}. \tag{16.6.23}$$

This allows to show that the following set is a basis of $Z_3(\mathfrak{sl}_2)$:

$$\{\mathsf{P}_{11}^a \mathsf{P}_{12}^b \mathsf{P}_{13}^c \mathsf{P}_{22}^d \mathsf{P}_{23}^e \mathsf{P}_{33}^f \mathsf{F}_{123}^\varepsilon\}, \qquad \text{where } a,b,c,d,e,f \in \mathbb{Z}_{\geq 0} \text{ and } \varepsilon \in \{0,1\}. \tag{16.6.24}$$

Indeed, such a set is a spanning set since ${\mathsf{F}_{123}}^2$ can be expressed in terms of the P's. Comparing with $F_3(t)$, we see directly that this set spans a subspace of the correct dimension in each degree. So this is a basis.

The case $n = 4$. The Hilbert–Poincaré series of $Z_4(\mathfrak{sl}_2)$ is:

$$F_4(t) = \frac{1 + t^2 + 4t^3 + t^4 + t^6}{(1 - t^2)^9}. \tag{16.6.25}$$

The four sets, for $a,b,c,d,e,f,g,h,i \in \mathbb{Z}_{\geq}0$,

$$\{\mathsf{F}_{123} \mathsf{P}_{11}^a \mathsf{P}_{12}^b \mathsf{P}_{13}^c \mathsf{P}_{14}^d \mathsf{P}_{22}^e \mathsf{P}_{23}^f \mathsf{P}_{24}^g \mathsf{P}_{33}^h \mathsf{P}_{34}^i\}, \tag{16.6.26a}$$

$$\{\mathsf{F}_{124} \mathsf{P}_{11}^a \mathsf{P}_{12}^b \mathsf{P}_{13}^c \mathsf{P}_{14}^d \mathsf{P}_{22}^e \mathsf{P}_{23}^f \mathsf{P}_{24}^g \mathsf{P}_{34}^h \mathsf{P}_{44}^i\}, \tag{16.6.26b}$$

$$\{\mathsf{F}_{134} \mathsf{P}_{11}^a \mathsf{P}_{12}^b \mathsf{P}_{13}^c \mathsf{P}_{14}^d \mathsf{P}_{23}^e \mathsf{P}_{24}^f \mathsf{P}_{33}^g \mathsf{P}_{34}^h \mathsf{P}_{44}^i\}, \tag{16.6.26c}$$

$$\{\mathsf{F}_{234} \mathsf{P}_{12}^a \mathsf{P}_{13}^b \mathsf{P}_{14}^c \mathsf{P}_{22}^d \mathsf{P}_{23}^e \mathsf{P}_{24}^f \mathsf{P}_{33}^g \mathsf{P}_{34}^h \mathsf{P}_{44}^i\}, \tag{16.6.26d}$$

and, for $a,b,c,d,e,f,g,h,i,j \in \mathbb{Z}_{\geq}0$ and $aehj = 0$,

$$\{\mathsf{P}_{11}^a \mathsf{P}_{12}^b \mathsf{P}_{13}^c \mathsf{P}_{14}^d \mathsf{P}_{22}^e \mathsf{P}_{23}^f \mathsf{P}_{24}^g \mathsf{P}_{33}^h \mathsf{P}_{34}^i \mathsf{P}_{44}^j\} \tag{16.6.26e}$$

form a basis of $Z_4(\mathfrak{sl}_2)$. To understand this, rewrite its Hilbert–Poincaré series as

$$F_4(t) = \frac{4t^3}{(1 - t^2)^9} + \frac{1 - t^8}{(1 - t^2)^{10}}. \tag{16.6.27}$$

The first term corresponds to the first 4 sets and the second term corresponds to the fifth one. These sets are spanning sets, since $\mathsf{F}_{ijk}\mathsf{F}_{mnp}$ can be expressed in terms linear in F by using the relations $w_{ijk} = 0$ and $x_{ijk\ell} = 0$, and moreover $\mathsf{F}_{ijk}\mathsf{P}_{\ell\ell}$ (i,j,k,ℓ pairwise distinct) can be expressed in terms of elements of the sets (16.6.26) by using (16.3.9). The condition $aehj = 0$ for the fifth set comes from the following fact. Let us define the following 2×2 and 4×4 matrices:

$$\mathsf{P}_{ij}^{ab} = \begin{pmatrix} \mathsf{P}_{ia} & \mathsf{P}_{ib} \\ \mathsf{P}_{ja} & \mathsf{P}_{jb} \end{pmatrix}, \qquad \mathsf{P}_{ijk\ell}^{abcd} = \begin{pmatrix} \mathsf{P}_{ia} & \mathsf{P}_{ib} & \mathsf{P}_{ic} & \mathsf{P}_{id} \\ \mathsf{P}_{ja} & \mathsf{P}_{jb} & \mathsf{P}_{jc} & \mathsf{P}_{jd} \\ \mathsf{P}_{ka} & \mathsf{P}_{kb} & \mathsf{P}_{kc} & \mathsf{P}_{kd} \\ \mathsf{P}_{\ell a} & \mathsf{P}_{\ell b} & \mathsf{P}_{\ell c} & \mathsf{P}_{\ell d} \end{pmatrix}. \tag{16.6.28}$$

Recall the definition of the symmetrized determinant (16.2.4). In $Z_4(\mathfrak{sl}_2)$, the following relation of degree 8 is satisfied by the generators:

$$\begin{aligned}\det(P^{1234}_{1234}) = &-\tfrac{1}{3}\Big(\det(P^{124}_{123}) - \det(P^{134}_{123}) + \det(P^{234}_{123}) + \det(P^{134}_{124}) - \det(P^{234}_{124}) + \det(P^{234}_{134})\Big)\\ &+ \tfrac{2}{3}\Big(P_{12}\det(P^{34}_{34}) + P_{13}\det(P^{24}_{24}) + P_{14}\det(P^{23}_{23})\\ &\qquad + P_{23}\det(P^{14}_{14}) + P_{24}\det(P^{13}_{13}) + P_{34}\det(P^{12}_{12})\Big)\,. \end{aligned} \tag{16.6.29}$$

The above relation is not a new relation (it is implied by the defining relations of $sR(4)$ given in (16.4.4)–(16.4.5)) and it allows one to express $P_{11}P_{22}P_{33}P_{44}$ in terms of the elements of the sets (16.6.26).

16.7. Conclusion

Classical results about the invariant theory of the polynomials on $\mathfrak{sl}_2^n$ has allowed to provide a description in terms of generators and relations of the diagonal centralizer of $\mathfrak{sl}_2$. A precise connection with the higher rank Racah algebra was given. Various questions arise and pave the way to different generalizations.

We would like to emphasize that the natural numbers appearing in the numerator of the Hilbert–Poincaré form very well-known series of integers that have numerous interpretations and appear already in the study of the representation theory of $\mathfrak{sl}_2$. This suggests that there should be a further understanding of these numbers.

The classification of the finite irreducible representations of the rank 1 Racah algebra has been done in [15]. Following this, it should be possible to study the finite-dimensional representations of the (special) higher rank Racah algebra $R(n)$ (resp. $sR(n)$). These representations must be closely related to the operators associated to the $(n-2)$-variable Racah polynomials. Indeed, the difference and recurrence operators characterizing the univariate Racah polynomials satisfy the relations of $R(3)$. The study of the generalization to $(n-2)$-variable polynomials has been initiated in [32, 33] and it would be interesting to verify if the operators used in this case realize $R(n)$ or $sR(n)$. The Racah algebra appears also as the symmetry algebra of some superintegrable models [18–22]. We trust that the observations and theorems of the present paper will lead to a deeper understanding of this symmetry.

We focused here on the diagonal centralizer of $\mathfrak{sl}_2$ in the n-fold tensor product of $\mathfrak{sl}_2$. Other cases where $\mathfrak{sl}_2$ is replaced by other algebras are also known. For example the diagonal centralizer of the oscillator algebra has been studied in [49], the diagonal centralizer of the super Lie algebra $\mathfrak{osp}(1|2)$ is known to be related to the Bannai–Ito algebra [50] and the centralizer of $\mathfrak{sl}_3$ in its twofold tensor product has been introduced in [41]. An important generalization concerns the quantum group $U_q(\mathfrak{sl}_2)$. Its diagonal centralizer in the 3-fold tensor product has been examined [51] and is associated to the Askey–Wilson algebra (see e.g. [35] for a review). Many attempts [52–54] to generalize this result to n-fold tensor products have yielded relations of this centralizer but certainly did not give all the defining relations. Looking ahead, we are planning to provide a complete set of defining relations, by using some deformation of the defining relations of the special Racah algebra given in this paper.

We have studied the centralizer $Z_n(\mathfrak{sl}_2)$ at an algebraic level. It is however equally important to study this centralizer when each factor in the n-fold tensor product is in a finite-dimensional irreducible representation of $\mathfrak{sl}_2$. In the case $n = 3$, a conjecture stating that the centralizer is a quotient of the Racah algebra $R(3)$ was given in [17] (see [55] for the q-deformed case and [56] for $\mathfrak{osp}(1|2)$). This quotient associates the Racah algebra with well-known algebras such as the Temperley–Lieb or Brauer algebras. The generalization of these results to the case of the n-fold tensor product is desirable and the results obtained in the present paper offer a nice starting point. As another follow-up, we plan on finding the explicit quotient that provides a description of these centralizers in representations and to compare them to the recent results reported in [57, 58].

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Conclusion

La recherche étant un processus de peaufinement sans fin, voici pour conclure un nombre de questions de recherche qui, à mon avis, mériteraient d'être explorées, en continuation des travaux de cette thèse.

Tout d'abord, dans la lignée des travaux sur la dualité de Howe, il devrait être possible d'exploiter le genre de construction qui a été développée ici afin d'étudier d'autres familles de paires duales réductives. Ceci permettrait d'obtenir des interprétations duales d'autres algèbres générées par des Casimirs intermédiaires.

Notons également que pour les travaux sur les cas q-déformés, aucune procédure de réduction dimensionnelle n'a été trouvée. Une telle procédure permettrait d'obtenir un modèle intégrable possédant l'algèbre d'Askey–Wilson comme algèbre de symétrie. Une piste de solution pour obtenir un tel modèle serait de travailler avec des coordonnées non-commutatives.

Le modèle en deux dimensions de mécanique quantique superconforme introduit au chapitre 7 pourrait aisément être généralisé à un modèle en 3, voire n dimensions. L'étude de son algèbre de symétrie serait intéressante et on peut espérer que l'algèbre associée aux polynômes de Bannai–Ito complémentaires fasse son apparition.

Tel que mentionné à plusieurs reprises déjà, les trois tableaux de polynômes orthogonaux étudiés ici sont le tableau d'Askey ainsi que le q-tableau et le -1-tableau d'Askey. Dans la foulée des résultats de la deuxième partie de la thèse, une question naturelle est de savoir s'il existe des opérateurs de Sklyanin–Heun associés à des familles -1 de polynômes orthogonaux, et quelles dégénérations de l'algèbre de Sklyanin sont associées à ceux-ci. Il s'agit d'une question non-triviale; en général, pour étudier des limites $q \to -1$ on ne peut pas tout simplement remplacer q par -1 et il y a de nombreuses façons de prendre la limite $q \to -1$. Des travaux préliminaires semblent indiquer que la notion d'opérateur de Sklyanin–Heun devra être étendue. On peut également se demander s'il existe une généralisation elliptique des opérateurs de Sklyanin–Heun. Celle-ci serait associée à des fonctions elliptiques et devrait permettre de retouver l'algèbre originale de Sklyanin [3]. Les fonctions spéciales servant de base pour des représentations irréductibles de dimension finie pourraient alors être considérées comme des généralisations des polynômes de q-para-Racah.

Dans un autre ordre d'idées, il serait intéressant d'explorer les liens entre équations de Painlevé et opérateurs de Heun algébriques. Dans le cas continu, il est connu que l'équation de Heun associée à l'opérateur de Heun continu ainsi que ses dégénérations sont reliées aux 6 équations de Painlevé par une procédure de « déquantification » [43]. Des versions discrètes, ultra-discrètes et q-déformées des équations de Painlevé existent et il serait intéressant de voir si des généralisations des opérateurs de Heun peuvent leur être associées. Des premiers pas ont été faits dans cette direction [44, 45] et donnent bon espoir que cela soit possible. Le rôle des opérateurs de Sklyanin–Heun dans ce contexte reste à être établi.

Les résultats du chapitre 13 font partie d'un programme de recherche plus général qui a pour but d'obtenir les diverses familles de polynômes du tableau d'Askey en considérant des représentations tridiagonales d'algèbres quadratiques à deux générateurs. Ces algèbres sont les diveres classes d'isomorphismes de l'algèbre quadratique à deux générateurs la plus générale. Cette algèbre quadratique à 2 générateurs est reliée aux modèles ASEP qui servent à modéliser une grande quantité de phénomènes tels que le traffic et la magnétophorèse [46]. L'étude des fonctions spéciales associées à cette algèbre pourrait trouver des applications dans ces problèmes. Il serait donc intéressant de compléter la classification. Cette étude s'inscrit également dans un autre programme de recherche qui a pour but d'identifier et classifier des familles de fonctions biorthogonales associées aux diverses familles de polynômes orthogonaux du q-tableau d'Askey. Des progrès ont déjà été faits pour les familles de Hahn [47] et Jacobi [48] mais il reste beaucoup à faire.

Plusieurs questions de recherche se présentent à nous suite aux travaux de la troisième partie. Tout d'abord, la connection entre l'algèbre des Casimir intermédiaires du problème de Racah de $U_q(\mathfrak{sl}_2)$ et l'algèbre de skein du crochet de Kauffman de la sphère à 4 trous semble un peu magique et sans explication. Il serait extrêmement intéressant de comprendre ce miracle : d'où vient cette correspondance aussi étroite entre deux algèbres qui semblent a priori définies dans des contexte complètements différents?

Les éléments de l'algèbre de skein du crochet de Kauffman de la sphère non-trouée sont les polynômes de Jones. Ceux-ci ont été identifiés dans les théories de Chern–Simon par Witten en 1989 [49]. Il serait particulièrement intéressant d'obtenir les algèbres d'Askey–Wilson dans des théories de Chern–Simons non-abéliennes. Ceci permettrait d'utiliser la machinerie des polynômes orthogonaux et fonctions spéciales dans ces théories qui trouvent de nombreuses applications.

Des conjectures sur les extensions à plus haut rang de l'algèbre d'Askey–Wilson ont été présentées au chapitre 15. Il serait très satisfaisant de réussir à les prouver. Également, on désirerait comparer les extensions proposées dans ces conjectures avec celles obtenues par l'approche par les opérateurs de bispectralité de polynômes multivariés. Il semble qu'une réponse définitive à la question d'identifier l'algèbre d'Askey–Wilson de plus haut rang devrait être en mesure d'unifier ces deux approches, tout comme ce qui a été accompli pour l'algèbre de Racah de plus haut rang.

Enfin, il serait intéressant d'obtenir une présentation du centralisateur de $U_q(\mathfrak{sl}_2)$ dans $U_q(\mathfrak{sl}_2)^{\otimes n}$ en termes de générateurs et relations définissantes. Ceci correspondrait à une q-généralisation des résultats au chapitre 16 et des Premiers et Seconds Théorèmes Fondamentaux de théorie des invariants.

Épilogue

Nous sommes en 2021.

Le protagoniste Alexei Zhedanov est maintenant un chercheur respecté, dont le nom est bien établi dans son domaine.

En raison de la pandémie COVID-19 qui sévit à travers le monde, cela fait maintenant plus d'un an qu'il est au Canada; un séjour de recherche d'un mois auprès de son collaborateur de longue date Luc Vinet s'est transformé en visite substantiellement plus longue.

Profitant du fait qu'ils habitent désormais sur le même fuseau horaire, un étudiant au doctorat en train d'écrire sa thèse et avec qui collabore le protagoniste lui demande comment ses idées de recherche lui sont venues à l'origine...

Et c'est ainsi que le Prologue de cette thèse voit le jour.

Références bibliographiques

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