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Domaines nodaux et points critiques de fonctions propres d'opérateurs de Schrödinger

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Résumé

La présente thèse porte sur les fonctions propres du laplacien et d'opérateurs de Schrödinger en dimension quelconque. Plus précisément, pour une variété (M,g) de dimension d et une fonction $V : M \to \mathbb{R}$, on considère les solutions de l'équation suivante:

$$(\Delta_g + V)\, f_\lambda = \lambda f_\lambda. \tag{0.0.1}$$

On appelle l'opérateur $\Delta_g + V$ un opérateur de Schrödinger et V le potentiel. Le cas le plus simple et le plus étudié est le laplacien (on pose $V \equiv 0$ sur M). Si M est compacte et sans bord, alors il existe une suite $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \nearrow +\infty$ qui forme le spectre de Δ_g et une suite de fonctions propres f_n qui satisfont à $\Delta_g f_n = \lambda_n f_n$. Cette propriété est aussi respectée pour beaucoup de potentiels et de variétés.

Premièrement, nous avons étudié le nombre de domaines nodaux des fonctions propres quand la valeur propre tend vers l'infini. Les domaines nodaux d'une fonction f sur M sont les composantes connexes de l'ensemble $M \backslash f^{-1}(0)$. Ils nous permettent de mesurer le caractère oscillatoire de f en comptant le nombre de fois où f change de signe. L'objectif principal de la thèse était de généraliser le théorème de Pleijel [**52**] sur le nombre de domaines nodaux des fonctions propres du laplacien à d'autre opérateurs de Schrödinger.

Dans l'article [**2**], nous avons montré que la borne du théorème de Pleijel s'applique aussi à l'oscillateur harmonique quantique dans $\mathbb{R}^d$. De plus, nous avons remarqué que cette borne pouvait être améliorée en fonction de la forme quadratique qui définit le potentiel.

Ensuite, dans l'article [**3**], nous avons généralisé le résultat obtenu dans [**2**] à une large classe de potentiels radiaux, incluant des potentiels qui tendent vers zéro à l'infini ou ayant une singularité à l'origine. Cela inclut le potentiel de Coulomb, qui modélise un atome d'hydrogène isolé dans l'espace. Pour ces potentiels, nous considérons les valeurs propres strictement inférieures au spectre essentiel.

Nous avons aussi étudié les points critiques des fonctions propres du laplacien. Jusqu'à tout récemment, il y avait seulement une borne inférieure sur le nombre de points critiques pour certaines variétés [**36**], mais il n'y avait pas de borne supérieure connue. En 2019, Buhovsky, Logunov et Sodin ont construit une métrique sur $\mathbb{T}^2$ et une suite de fonctions propres du laplacien qui ont toutes une infinité de points critiques. Dans l'article [**4**], nous utilisons une nouvelle méthode pour construire des métriques sur $\mathbb{T}^2$ et $\mathbb{S}^2$ et des fonctions propres pour ces métriques qui ont une infinité de points critiques. De plus, nous montrons que ces métriques peuvent être arbitrairement proches de la métrique plate sur $\mathbb{T}^2$ et de la métrique standard sur $\mathbb{S}^2$. Ces métriques donnent aussi des contre-exemples à la conjecture de Courant-Hermann sur le nombre de domaines nodaux des combinaisons linéaires de fonctions propres du laplacien.

Mots clés: Géométrie spectrale, laplacien, Schrödinger, domaines nodaux, points critiques, Courant, Pleijel.

Abstract

The theme of this thesis is the study of the eigenfunctions of the Laplacian and Schrödinger operators. Let (M,g) be a manifold and $V : M \to \mathbb{R}$. We are looking at solutions of the following equation:

$$(\Delta_g + V)\, f_\lambda = \lambda f_\lambda. \tag{0.0.2}$$

The operator $\Delta_g + V$ is called a Schrödinger operator and V is called the potential. The simplest and most studied example is the Laplacian (we put $V \equiv 0$ on M). If M is compact and without boundary, then there exists a sequence $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \nearrow +\infty$ that makes the spectrum of Δ_g and a sequence of eigenfunctions f_n such that $\Delta_g f_n = \lambda_n f_n$. This decomposition also holds for various potentials and manifolds.

Firstly, we studied the nodal domains of the eigenfunctions as the eigenvalues tend to infinity. The nodal domains of a function f on M are the connected components of $M \backslash f^{-1}(0)$. They can be used to understand the oscillatory character of eigenfunctions by counting the number of times that f changes sign. The principal goal of this thesis was to generalize Pleijel's nodal domain theorem [**52**] to other Schrödinger operators.

In the article [**2**], we showed that the upper bound in Pleijel's theorem also holds for the quantum harmonic oscillator. Furthermore, this bound can be improved depending on the quadratic form that defines the potential.

Afterwards, in the article [**3**], we generalized the result from [**2**] to a large class of radial potentials, including ones that tend to zero at infinity. These include the Coulomb potential, which modelizes the hydrogen atom in free space.

We also studied the number of critical points of Laplace eigenfunctions. Until recently, there were only known lower bounds for certain manifolds [**36**], but no upper bound was known. In 2019, Buhovsky, Logunov and Sodin [**18**] constructed a metric on $\mathbb{T}^2$ and a sequence of Laplace eigenfunctions which all have infinitely many critical points. In our article [**4**], we used a different method to create metrics on $\mathbb{T}^2$ and $\mathbb{S}^2$ and Laplace eigenfunctions for these metrics that have infinitely many critical points. Furthermore, these metrics can be taken arbitrarily close to the flat metric on $\mathbb{T}^2$ and the round metric on $\mathbb{S}^2$. These constructions also provide strong counterexamples to the Courant-Hermann conjecture on the number of nodal domains of linear combinations of Laplace eigenfunctions.

Keywords: Spectral geometry, Laplacian, Schrödinger, nodal domains, critical points, Courant, Pleijel.

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Liste des sigles et des abréviations

Δ_g	$-div_g\left(\nabla_g\right)$
$H^k(M)$	$\left\{f \mid \frac{\partial^{\lvert a\rvert}}{\partial_{x_1}^{a_1}\dots\partial_{x_j}^{a_j}} \in L^2(M)\right\}$ pour tous les multi-indices $a = (a_1, \dots, a_j)$ tels que $0 \leq \lvert a\rvert \leq k$. j est la dimension de M et les dérivées sont prises au sens faible.
$H_0^k(M)$	Complétion de l'ensemble des fonctions lisses à support compact dans $H^k(M)$.
J_a	Fonction de Bessel du premier type d'ordre a
j_a	Premier zéro positif de J_a
$\gamma(n)$	Constante de Pleijel $\frac{2^{n-2}n^2\Gamma(n/2)}{j_{n/2-1}^n}$

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Introduction

0.1. Présentation de la thèse

L'objectif du projet de doctorat était de mieux comprendre les domaines nodaux des fonctions propres d'opérateurs de Schrödinger pour une vaste classe de potentiels. Le principal résultat visé était une généralisation du résultat que nous avions obtenu lors de la maîtrise, qui était lui-même une généralisation du théorème de Pleijel pour le laplacien de Dirichlet. Nous nous sommes aussi intéressés aux points critiques des fonctions propres du laplacien.

Les résultats que nous avons obtenus durant la thèse ont été publiés dans trois articles ([**2**], [**3**], [**4**]). Dans l'introduction, nous ferons une présentation générale de la géométrie spectrale, puis nous introduirons les concepts spécifiques à notre recherche. Nous décrirons ensuite pour chaque concept les avancées que nous avons obtenues et l'approche générale que nous avons utilisée.

0.2. Présentation de la géométrie spectrale

La géométrie spectrale est l'étude du spectre d'opérateurs linéaires, principalement d'opérateurs différentiels ou pseudo-différentiels, agissant sur des variétés riemaniennes. Les origines de cette discipline remontent à l'étude des figures formées par du sable déposé sur une plaque vibrante. Ce phénomène avait été observé en 1680 par Robert Hooke, mais c'est Ernst Chladni qui a popularisé l'expérience en faisant résonner une plaque métallique avec un archet vers la fin des années 1700. Cela piqua l'intérêt de Napoléon Bonaparte, qui proposa avec l'Académie des Sciences un prix pour quiconque construirait une théorie des surfaces vibrantes. Sophie Germain remporta le prix en 1816 mais sa théorie restait incomplète. Elle fut rendue rigoureuse par Joseph Fourier en 1822 lors de l'étude de l'équation de la chaleur.

Voici une formulation moderne du problème: on représente une membrane vibrante par un ouvert $U \in \mathbb{R}^2$, et l'amplitude de la vibration au point x et au temps t par $A(x,t)$. Si on représente l'état initial de la membrane par $\bar{A}(x) = A(x,0)$ et la vitesse initiale $\frac{\partial A}{\partial t}(x,0) = B(x)$, et on impose que $A = 0$ au bord de U, alors la solution obéit à l'équation différentielle suivante:

$$\frac{\partial^2 A(x,t)}{\partial t^2} = \sum_{i=1}^{2} \frac{\partial^2 A(x,t)}{\partial x_i^2}\,. \tag{0.2.1}$$

Par séparation de variables, si on cherche des solutions de la forme $A(x,t) = T(t)f(x)$, on obtient les équations suivantes:

$$\begin{aligned} T''(t) &= -\lambda T(t), \\ \Delta f(x) &= -\lambda f(x), \\ f|_{\partial U} &= 0. \end{aligned} \tag{0.2.2}$$

En prenant une combinaison linéaire de ces solutions, on obtient la forme finale pour $A(x,t)$:

$$A(x,t) = \sum_{i\geq 1} (a_i \sin(t) + b_i \cos(t)) f_\lambda(x), \tag{0.2.3}$$

où f_λ obéit à l'équation $\Delta f_\lambda(x) = -\lambda f_\lambda(x)$. On utilise les conditions initiales $\bar{A}$ et B afin de déterminer les coefficients a_i et b_i.

On peut aussi utiliser cette méthode pour résoudre l'équation de la chaleur et l'équation de Schrödinger.

Pour que l'expression (0.2.3) des solutions de l'équation (0.2.1) nous soit utile, il faut vérifier plusieurs choses. Premièrement, il faut montrer que toute solution de l'équation (0.2.1) puisse être exprimée sous la forme (0.2.3). Ensuite, il faut trouver l'ensemble des valeurs propres de l'opérateur Δ agissant sur les fonctions s'annulant au bord du domaine. Finalement, il faut être capable d'évaluer les fonctions f_λ.

L'objectif de la géométrie spectrale est de répondre à ces trois questions dans un cadre plus général. Considérons une variété (M,g) de dimension n avec une métrique g et un opérateur linéaire $\bar{H} : C^\infty(M) \to C^\infty(M)$. On pose souvent des conditions additionnelles sur M (avec ou sans bord, compacte, etc.) ou sur les solutions (conditions au bord, etc.). Dans la majorité des cas, on utilise une extension H de $\bar{H}$ de telle façon que H soit auto-adjoint.

Voici quelques questions classiques en géométrie spectrale:

- Comment peut-on caractériser le spectre de H? Est-il discret ou continu?
- Peut-on lier le spectre de H aux propriétés géométriques de M, ou inversement, peut-on déduire des propriétés géométriques de M en sachant seulement le spectre de H?
- Si H possède des valeurs propres, quels liens existe-il entre les propriétés géométriques des fonctions propres de H et leur valeur propre associée?

Le problème le plus étudié est de trouver les fonctions propres de l'opérateur de Laplace-Beltrami (ou laplacien), Δ_g, sur une variété compacte (M,g). On cherche donc $\lambda \in \mathbb{R}$ et f_λ tels que $\Delta_g f_\lambda = \lambda f_\lambda$. Si M a un bord, on impose des conditions au bord parmi les deux types suivants: Dirichlet ($f_\lambda = 0$ sur ∂M) ou Neumann ($\frac{\partial f_\lambda}{\partial \nu} = 0$ sur ∂M, où ν est un vecteur normal pointant vers l'extérieur au bord). Afin d'étudier le spectre de Δ_g, on construit l'extension de Friedrichs pour rendre le laplacien auto-adjoint.

Un des résultats principaux caractérisant le spectre du laplacien est le suivant: si M est compacte, lisse et sans bord alors le spectre est discret, positif et les valeurs propres croissent vers l'infini. C'est aussi vrai pour des conditions de Dirichlet au bord ou des conditions de Neumann si le bord est Lipschitz. De plus, les fonctions propres forment une base de $L^2(M)$.

Nous dénoterons par $0 < \lambda_1 \leq \lambda_2 \leq \dots$ les valeurs propres du laplacien avec conditions de Dirichlet et par $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots$ dans le cas de Neumann ou si M est sans bord.

Dans la quasi-totalité des cas, il est impossible de calculer le spectre explicitement. Similairement, on ne connait pas précisément les fonctions propres du laplacien, sauf dans des cas très précis (carré, tore, sphères, boules). Cependant, beaucoup de propriétés du spectre et des fonctions propres peuvent être étudiées. Dans cette thèse, nous nous sommes concentrés sur les propriétés des domaines nodaux de fonctions propres d'opérateurs de Schrödinger et sur les points critiques de fonctions propres du laplacien sur des variétés compactes.

0.3. Forme variationnelle du spectre

Soit (M,g) une variété compacte avec ou sans bord et $f \in H^1(M)$. On définit le quotient de Rayeigh $R(f)$ pour le laplacien (si M a un bord, on pose des conditions de Dirichlet ou Neumann) sur M comme suit:

$$R_M(f) = \frac{< \Delta f, f >_{L^2(M)}}{< f,f >_{L^2(M)}} = \frac{< \nabla f, \nabla f >_{L^2(M)}}{< f,f >_{L^2(M)}}\,. \tag{0.3.1}$$

On utilise le théorème de la divergence et le fait que $f = 0$ ou $\partial f / \partial \nu = 0$ au bord.

Si M a un bord, on peut exprimer le spectre du laplacien avec conditions de Dirichlet comme suit:

$$\lambda_k = \inf_{\substack{V \subset H_0^1(M) \\ \dim V = k}} \sup_{\substack{f \in V \\ f \neq 0}} R_M(f)\,. \tag{0.3.2}$$

Si on pose des conditions de Neumann ou si M est sans bord, on peut exprimer le spectre comme suit:

$$\mu_k = \inf_{\substack{V \subset H^1(M) \\ \dim V = k}} \sup_{\substack{f \in V \\ f \neq 0 \\ <f,1>_{L^2(M)=0}}} .R_M(f) \tag{0.3.3}$$

On posera aussi $\mu_0 = 0$ et cette valeur propre sera associée à la fonction constante sur M.

Pour le laplacien de Dirichlet, la caractérisation variationnelle nous permet de déduire que la fonction propre associée à la valeur propre λ_1 ne change pas de signe sur M, et que λ_1 a une multiplicité 1.

Pour un opérateur de Schrödinger $H : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$, $Hf := \Delta f + Vf$, où V est une fonction réelle appelée "potentiel", on définit le quotient de Rayleigh comme suit:

$$R(f) = \frac{< Hf,f >_{L^2(\mathbb{R}^n)}}{< f,f >_{L^2(\mathbb{R}^n)}} = \frac{< \nabla f, \nabla f >_{L^2(\mathbb{R}^n)} + < Vf,f >_{L^2(\mathbb{R}^n)}}{< f,f >_{L^2(\mathbb{R}^n)}} . \tag{0.3.4}$$

La deuxième égalité est justifiée si $f \to 0$ suffisament rapidement à l'infini. On peut utiliser le principe du min-max 0.3.2 afin de définir le spectre de H agissant sur des fonctions s'annulant à l'infini (en général, on demande que les fonctions soient dans $L^2(\mathbb{R}^n)$).

Pour le laplacien et un opérateur de Schrödinger, l'avantage d'utiliser le quotient de Rayleigh est que $R_M(f_\lambda) = \lambda$ et $R(f_\lambda) = \lambda$. Cela nous permet aussi d'utiliser des fonctions tests afin de borner les valeurs propres et de comparer les valeurs propres sur différentes variétés ou avec des potentiels différents.

Dans les articles [**2**] et [**3**], nous avons dû montrer que le quotient de Rayleigh est bien défini pour les fonctions propres de l'oscillateur harmonique quantique, puis pour certains opérateurs de Schrödinger avec potentiels radiaux. En effet, on doit justifier le théorème de la divergence sur un ouvert non-borné, et il faut donc contrôler les dérivées de f_λ à l'infini.

0.4. Loi de Weyl

Un des résultats fondamentaux sur le spectre du laplacien sur une variété M de dimension n est la loi de Weyl, découverte vers 1911 [**68**]. On assume que M et ∂M soient suffisament lisses.

Soit $N(\lambda)$ le nombre de valeurs propres plus petites ou égales à λ. Alors,

$$N(\lambda) = \lambda^{n/2}(2\pi)^d \omega_d |M| + o(\lambda^{n/2}) . \tag{0.4.1}$$

Il existe une formule analogue pour $N(\lambda)$ pour certains opérateurs de Schrödinger. Si $H = \Delta + V$ sur $M \subset \mathbb{R}^n$, alors $N(\lambda)$ peut être exprimé de la façon suivante:

$$N(\lambda) = \int_M (\lambda - V)_+^{n/2} (1 + o_\lambda(1)) \ . \tag{0.4.2}$$

Pour que cet estimé soit vrai, on doit poser certaines conditions sur V, par exemple sur la croissance de V à l'infini, la régularité, la présence de singularités (voir par exemple [**3**], section 4). De plus, dans certains cas, il faut séparer le spectre discret du spectre essentiel si les deux sont présents.

Dans notre article [**3**], nous avons dû restreindre notre étude aux potentiels tels que l'équation 0.4.2 reste vraie.

0.5. Ensemble nodal

On définit l'ensemble nodal d'une fonction f par $Z_f := f^{-1}(0)$. Ce concept a été mis en évidence par Ernst Chladni dans les années 1780. En posant du sable sur une surface et en la faisant résonner avec un archet, le sable se déplace aux points où l'amplitude de vibration est minimale.

Beaucoup de recherches ont été effectuées sur l'ensemble nodal des fonctions propres du laplacien, notamment en lien avec la conjecture de Yau: si (M,g) est compacte et lisse, alors il existe $c,C > 0$ tels que $c\sqrt{\lambda} < Vol(Z_{f_\lambda}) < C\sqrt{\lambda}$. La conjecture a été prouvée si la métrique est analytique par Donnelly et Fefferman (voir [**26**]). La borne inférieure a été prouvée pour les variétés lisses et sans bord par Logunov (voir [**45**]), mais la borne supérieure n'a pas été prouvée. Le meilleur estimé connu aujourd'hui est une borne polynomiale de haut degré en dimensions 3 et plus (voir [**46**]), et de degré $3/4 - \epsilon$ en dimension 2.

0.6. Domaines nodaux

L'ensemble nodal nous permet de construire une partition de M. Si f_λ est une fonction propre du laplacien sur M, un domaine nodal Ω de f_λ est une composante connexe de $M \backslash Z_{f_\lambda}$. Cette décomposition de M est utile pour deux raisons. Premièrement, si on évalue le quotient de Rayleigh de f_λ restreinte à Ω, on obtient encore λ. Deuxièmement, la première valeur propre du laplacien de Dirichlet sur Ω est précisément λ, puisqu'il s'agit d'une fonction propre du laplacien qui ne change pas de signe.

0.7. Théorème de Courant

Pour une fonction propre f_λ, on dénote par $\mu(f_\lambda)$ le nombre de ses domaines nodaux.

Un des principaux résultats concernant les domaines nodaux est le théorème de Courant (voir [**22**]): si λ est la n-e valeur propre du laplacien de Dirichlet, alors $\mu(f_\lambda) \leq n$. Le résultat est aussi vrai dans le cas de Neumann ou si M est sans bord.

Ce théorème est une forme de généralisation du théorème de Sturm pour les fonctions propres d'opérateurs de Sturm-Liouville: la n-e fonction propre non-nulle de l'opérateur L, où $Lf = f'' + qf$ avec g une fonction positive, possède exactement n zéros. Cela reste aussi vrai si on considère des combinaisons linéaires de solutions: si $g = \sum_{i=j}^{l} a_i f_i$, où $a_i \in \mathbb{R}$ et f_i est la i-e fonction propre de L, alors g possède au minimum j et au maximum l zéros.

0.8. Conjecture de Courant-Herrmann

Le théorème de Sturm a mené Courant à formuler une conjecture: une combinaison linéaire des n premières fonctions propres du laplacien possède au maximum n domaines nodaux. On l'appelle aussi conjecture de Courant-Herrmann.

Cette conjecture a été démontrée fausse dans certains cas particuliers (voir entre autres [**9**], [**10**], [**11**]). En 2019, Buhovsky, Logunov et Sodin ont construit une métrique sur $\mathbb{T}^2$ et une infinité de combinaisons linéaires de fonctions propres d'indice fini avec une infinité de domaines nodaux [**18**]. La métrique sur $\mathbb{T}^2$ est lisse mais non analytique.

Dans notre article [**4**] avec Pierre Bérard et Bernard Helffer, nous avons fait plusieurs autres constructions de ce type.

La première utilise des triangles isocèles avec conditions de Neumann:

Proposition 0.8.1 ([**4**], Proposition 3.3). *Soit $T(b)$ le triangle avec sommets $(\sqrt{3},0)$, $(-\sqrt{3},0)$ et $(b,0)$. Si $0 < b < 1$, alors il existe $a \neq 0$ tel que pour la deuxième fonction propre de Neumann u_2, la fonction $u_2 - a$ possède 3 domaines nodaux. On considère u_1 comme la fonction constante.*

La deuxième utilise les N-gones réguliers avec conditions de Neumann:

Proposition 0.8.2 ([**4**], Proposition 4.1). *Soit $\mathcal{P}_N$ le N-gone régulier inscrit dans le disque unité. Alors, pour N assez grand, il existe $m \leq 6$, une fonction propre u_m associée à la m-ième fonction propre de Neumann sur $\mathcal{P}_N$ et $a > 0$ tels que $u_m - a$ possède $N+1$ domaines nodaux.*

La troisième utilise une métrique lisse sur $\mathbb{T}^2$:

Proposition 0.8.3 ([**4**], Proposition 5.2). *Il existe une métrique $g_Q = Q(x)\,(dx^2 + dy^2)$ sur $\mathbb{T}^2$ et une fonction propre Φ de Δ_Q, $\Delta_Q \Phi := Q^{-1}\,\Delta_0 \Phi$, telle que l'ensemble $\{\Phi > 1\}$ possède une infinité de composantes connexes.*

De plus, on peut prendre la fonction $Q(x)$ arbitrairement proche de 1 ([**4**], remarque 5.4).

La quatrième utilise une métrique analytique sur $\mathbb{T}^2$:

Proposition 0.8.4 ([**4**], Proposition 5.5). *Soit $n > 0$ n'importe quel entier. Il existe des métriques analytiques de la forme $g_Q = Q(x)\,(dx^2 + dy^2)$ sur $\mathbb{T}^2$ et une fonction propre Φ_Q de Δ_{g_Q} associée à la valeur propre 1 telle que l'ensemble $\{\Phi_Q > 1\}$ possède au moins n composantes connexes. La métrique g_Q peut être prise arbitrairement proche de la métrique plate g_0.*

De plus, si $n \geq 4$ et g_Q est assez proche de g_0, alors la valeur propre 1 est soit la deuxième, troisième ou quatrième de Δ_{g_Q}.

La dernière construction était une métrique lisse sur $\mathbb{S}^2$:

Proposition 0.8.5 ([**4**], Proposition 6.2)**.** *Il existe $M > 0$ tel que pour tout $m \geq M$, il existe des fonctions lisses Φ_m et G_m sur $\mathbb{S}^2$ avec les propriétés suivantes:*

(1) L'ensemble $\{\Phi_m > 1\}$ possède une infinité de composantes connexes.

(2) La fontion G_m est positive et définit une métrique $g_m = G_m\, g_0$ sur $\mathbb{S}^2$ et un opérateur $\Delta_{G_m} = G_m^{-1}\,\Delta_0$.

(3) Pour $m \geq M$, $\left(1 - \frac{2}{m}\right) \leq G_m \leq \left(1 + \frac{2}{m}\right)$, et

(4) $-\Delta_{G_m}\Phi_m = m(m+1)\,\Phi_m$.

Cela démontre que le nombre de domaines nodaux d'une combinaison linéaire de fonctions propres peut être extrêmement instable. En effet, comme les fonctions propres du laplacien sur $\mathbb{S}^2$ sont des polynômes dont le degré dépend de λ, et qu'on peut borner le nombre de domaines nodaux d'un polynôme sur une sphère en fonction du degré, alors le nombre de domaines nodaux d'une combinaison linéaire de fonctions propres sur $\mathbb{S}^2$ est borné (voir [**4**], corollaire 8.2).

Finalement, on peut voir les limitations de ces constructions: l'idée est de construire des fonctions propres avec une courbe de niveau contenant une infinité de composante connexes, puis d'ajouter une constante afin de ramener cette courbe de niveau à 0 pour obtenir un ensemble nodal. Cependant, on utilise le fait que sur des variétés fermées et pour le laplacien de Neumann, la fonction constante est une fonction propre. Nous ne savons toujours pas si on peut faire le même type de construction pour des combinaisons linéaires sans inclure la fonction constante.

0.9. Inégalite de Faber-Krahn

Il existe un lien entre la première valeur propre du laplacien de Dirichlet sur un ouvert $U \subset \mathbb{R}^n$: la boule minimise la première valeur propre entre tous les ouverts de même volume. Autrement dit, si $B_U \subset \mathbb{R}^n$ est la boule de même volume que U, alors $\lambda_1(B_U) \leq \lambda_1(U)$. De plus, l'égalité est seulement atteinte si U est une boule (modulo un ensemble de capacité 0). Cette propriété a été découverte indépendamment par Faber ([**27**]) en 1923 et Krahn ([**38**]) en 1925.

Dans le cas de la première valeur propre d'un opérateur de Schrödinger $H = \Delta + V$ sur un ouvert borné $U \subset \mathbb{R}^n$, la borne est plus difficile à démontrer. Dans notre article [**2**], nous avons utilisé la caractérisation variationnelle des valeurs propres pour borner inférieurement le volume de U en fonction de V et λ: si $Hf = \lambda f$ dans U et $f = 0$ sur ∂U, alors $|U| \geq C_n[\sup_{x\in U} \lambda - V(x)]^{-n/2}$, où C_n est la même constante que dans l'inégalité de Faber-Krahn. Des résultats plus généraux ont été prouvés pour ce type de problèmes ([**19**]).

0.10. Théorème de Pleijel pour le laplacien

Le théorème de Courant donne une borne maximale pour le nombre de domaines nodaux de toutes les fonctions propres du laplacien. Si on regarde seulement le comportement asymptotique lorsque λ tend vers l'infini, il est possible d'améliorer cette borne. Si M est une variété lisse de dimension n, le nombre de domaines nodaux d'une fonction propre f_k associée à la k-e valeur propre est borné par

$$\mu(f_k) \leq \frac{2^{n-2}n^2\Gamma(n/2)}{j^n_{n/2-1}}(k + o(k)), \tag{0.10.1}$$

où j_a est le premier zéro de la fonction de Bessel J_a. Ce théorème a été prouvé par Pleijel en 1956 ([**52**]) pour un ouvert de $\mathbb{R}^n$ avec conditions de Dirichlet, puis a été généralisé plusieurs fois:

(1) En 1978 par Bérard et Meyer ([**14**]) pour une variété M avec conditions de Dirichlet ou sans bord,
(2) en 2009 par Polterovich ([**53**]) avec conditions de Neumann pour les ouverts de $\mathbb{R}^2$ dont le bord est analytique par morceaux et
(3) en 2016 par Lena ([**40**]) avec conditions de Neumann ou Robin pour les variétés en dimension quelconque dont le bord est $C^{1,1}$.

Pour $n \geq 2$, $\frac{2^{n-2}n^2\Gamma(n/2)}{j^n_{n/2-1}}$ est strictement plus petit que 1, et décroît avec n. Donc, cet estimé implique que l'égalité est atteinte dans le théorème de Courant pour un nombre fini de fonctions propres seulement. Il est possible d'estimer le nombre maximal d'égalités pour la borne de Courant (voir par exemple [**12**]).

Nous dénoterons la constante $\frac{2^{n-2}n^2\Gamma(n/2)}{j^n_{n/2-1}}$ par $\gamma(n)$.

Pour prouver ce théorème, on divise le volume total de M par le volume minimal d'un domaine nodal donné par l'inégalité de Faber-Krahn. On utilise ensuite la loi de Weyl pour évaluer l'indice d'une valeur propre.

0.11. Théorème de Pleijel pour l'oscillateur harmonique quantique

L'oscillateur harmonique quantique en dimension n est défini comme H :, $Hf = \Delta f + |x|^2 f$. Cet opérateur est intéressant pour plusieurs raisons. Premièrement, il s'agit d'un des rares opérateurs de Schrödinger dont on connait les valeurs propres et les fonctions propres explicitement. Deuxièmement, les fonctions propres sont des polynômes multipliés par une fonction gaussienne, ce qui permet d'utiliser des outils de géométrie algébrique pour les étudier. Finalement, il peut approximer n'importe quel opérateur de Schrödinger au voisinage d'un minimum du potentiel.

Dans notre thèse de maîtrise, nous avons obtenu le résultat suivant:

Théorème 0.11.1 ([**1**]). *Soit $H = \Delta + V$ l'oscillateur harmonique quantique agissant sur $H^1(\mathbb{R}^n)$. Si on ordonne les valeurs propres λ_k en ordre croissant avec multiplicité, alors pour une fonction propre f_k associée à λ_k, on obtient la borne suivante:*

$$\mu(f_k) \leq \frac{2^{n-2} n^2 \Gamma(n/2)}{j^n_{n/2-1}} (k + o(k)) . \tag{0.11.1}$$

En effet, on obtient le même résultat que pour le laplacien de Dirichlet ou Neumann, et la constante est exactement la même. Nous utilisions directement la forme explicite des valeurs propres et le fait que les fonctions propres étaient des polynômes pour obtenir le résultat.

Dans notre premier article [**2**], nous avons étendu ce résultat à n'importe quel oscillateur harmonique quantique anisotrope:

Théorème 0.11.2. *Soit $H = \Delta + xAx$, où A est une forme quadratique définie positive, agissant sur $H^1(\mathbb{R}^n)$. Si au moins deux valeurs propres de A sont rationellement dépendantes, alors on obtient exactement la formule 0.11.1, tandis que si les valeurs propres sont toutes rationellement indépendantes, on obtient la borne suivante:*

$$\mu(f_k) \leq \frac{n!}{n^n} (k + o(k)) . \tag{0.11.2}$$

La constante dans la borne 0.11.2 est strictement plus petite que celle dans l'équation 0.11.1. Dans cette situation, les valeurs propres sont toutes simples et on peut donc calculer le nombre de domaines nodaux directement.

Dans la preuve originale du théorème de Pleijel, on divise le volume total de la variété par le volume minimal d'un domaine nodal pour obtenir la borne finale. Cependant, pour un opérateur de Schrödinger défini sur $\mathbb{R}^n$, on doit utiliser une autre approche. De plus, trouver une borne uniforme sur le volume de tous les domaines nodaux ne permet pas d'obtenir la bonne constante dans l'estimé 0.11.1.

Voici les nouvelles idées qui ont permis d'étendre la preuve du théorème de Pleijel pour le laplacien sur une variété bornée à l'oscillateur harmonique quantique:

(1) Montrer que tous les domaines nodaux d'une fonction propre f_λ intersectent la région $\{V \leq \lambda\}$.
(2) Créer une partition $0 = a_1 < a_2 < \ldots < a_{j(\lambda)} = \lambda$, avec $j(\lambda)$ à déterminer plus tard.
(3) Séparer les domaines nodaux en deux catégories: ceux qui intersectent au moins une hypersurface $\{V = a_i\}$ et ceux qui sont contenus dans une région $\{a_i < V < a_{i+1}\}$.
(4) Utiliser des éléments de géométrie algébrique et de théorie de Morse pour borner le nombre de domaines nodaux qui intersectent une hypersurface. Le résultat principal est le suivant: si $f : \mathbb{R}^n \to \mathbb{R}$ est un polynôme de degré k à n variables, alors la

restriction de f à $\mathbb{S}^{n-1}$ possède au plus $2^{2n+2}k^{n-1}$ domaines nodaux. Cet argument sera expliqué dans la prochaine section.

(5) Diviser le volume de chaque région $\{a_i < V < a_{i+1}\}$ par le volume minimal d'un domaine nodal obtenu par l'inégalité de Faber-Krahn.

(6) Choisir $j(\lambda)$ et la partition afin de balancer ces deux estimés et obtenir la borne finale.

Dans le cas de l'oscillateur anisotrope, on doit contrôler le degré maximal du polynôme définissant f_λ, et montrer que cela n'a pas d'impact sur la borne finale.

0.11.1. Éléments de topologie algébrique

Afin de borner le nombre de domaines nodaux de l'oscillateur harmonique quantique qui intersectent une sphère, nous avons utilisé un théorème de Milnor ([**48**], théorème 3):

Théorème 0.11.3. *Soit $V \subset \mathbb{R}^n$ une variété semi-algébrique définie par les équations $p_1 \geq 0$, $p_2 \geq 0 \ldots p_j \geq 0$. Soit d la somme des degrés des polynômes p_i. Alors, la somme des nombres de Betti dans la cohomologie de Cech est bornée par $\frac{1}{2}(d+2)(d+1)^{n-1}$.*

Afin d'utiliser ce théorème, il faut montrer que le nombre de composantes connexes d'une variété semi-algébrique est borné par la somme des nombres de Betti dans la cohomologie de Cech.

Premièrement, on sait que toute variété semi-algébrique est triangulable ([**47**]).

Deuxièmement, si un espace est homotope à un CW-complexe, alors sa cohomologie de Cech et sa cohomologie singulière sont isomorphes ([**31**], page 257).

Troisièmement, par le théorème des coefficients universels ([**31**], théorème 3.2, page 195), le 0-ème groupe de cohomologie sur $\mathbb{R}$ est isomorphe au groupe des $\mathbb{R}$-homomorphismes de modules du 0-ème groupe d'homologie singulière sur $\mathbb{R}$.

Quatrièmement, le 0-ème groupe d'homologie singulière sur $\mathbb{R}$ est isomorphe à $\mathbb{R}^k$, où k est le nombre de composantes connexes par arc ([**31**], théorème 2.7, page 109).

Finalement, comme le nombre de composantes connexes par arc est toujours supérieur ou égal au nombre de composantes connexes, le nombre de composantes connexes de V est borné par la somme de ses nombres de Betti.

Dans [**2**], on utilise ce résultat pour montrer que le nombre de composantes connexes de l'ensemble $\{p \geq 0 \cap B_r\}$, où p est un polynôme de degré d et B_r est la boule de rayon r dans $\mathbb{R}^n$. Par le théorème précédent, le nombre de composantes connexes est inférieur ou égal à $\frac{1}{2}(d+4)(d+3)^{n-1}$. Pour d assez grand, $\frac{1}{2}(d+4)(d+3)^{n-1} \leq (d+2)(d+1)^{n-1}$, et c'est l'estimé qui est utilisé dans l'article [**2**] à la section 2.4. Comme nous avons seulement besoin d'une borne de la forme Cd^n, la valeur de la constante C n'a pas d'influence sur notre résultat.

0.12. Théorème de Pleijel pour opérateurs de Schrödinger avec potentiels radiaux

La preuve du théorème de Pleijel pour l'oscillateur harmonique quantique reposait sur plusieurs propriétés spécifiques à cet opérateur. Cependant, on peut supposer que la borne sur le nombre de domaines nodaux est valide pour une large classe de potentiels. Soit $H = \Delta + V$ et Ω un domaine nodal de f_λ. L'inégalité de Faber-Krahn pour H peut être écrite comme suit:

$$|\Omega| \geq C_n \left[\sup_{\Omega} (\lambda - V)\right]^{-n/2} . \tag{0.12.1}$$

Supposons maintenant que le potentiel est presque constant à l'intérieur de Ω:

$$\sup_{x\in\Omega} V(x) \leq (1+\epsilon) \inf_{x\in\Omega} V(x) \tag{0.12.2}$$

pour ϵ suffisament petit.

On obtient alors l'approximation suivante:

$$1 \leq C_n \int_{\Omega} (1+\epsilon)^{n/2} (\lambda - V)_+^{n/2} . \tag{0.12.3}$$

Si on peut montrer que V ne varie pas trop pour la quasi-totalité des domaines nodaux, on obtient alors

$$\mu(f_\lambda) \leq C_n \int_{\mathbb{R}^n} (1+\epsilon)^{n/2} (\lambda - V)_+^{n/2} \left(1 + o_\lambda(1)\right) . \tag{0.12.4}$$

On voit que la partie de droite de l'inégalité est pratiquement identique à la loi de Weyl pour les opérateurs de Schrödinger 0.4.2. Si on peut montrer que $\epsilon \to 0$ quand λ tend vers l'infini, alors on obtient la borne finale en posant $\lambda = \lambda_n$.

Afin de construire une preuve de ce type, il faut montrer que la propriété 0.12.2 est vérifiée pour la majorité des domaines nodaux. Dans le cas de l'oscillateur harmonique quantique, on divisait l'ensemble $\{V \leq \lambda\}$ en régions où le potentiel varie moins. Il fallait ensuite montrer que le nombre de domaines nodaux passant d'une région à l'autre était négligeable. Cette étape cruciale de la preuve utilisait des éléments de géométrie algébrique. Il était seulement possible d'utiliser ces outils parce que les fonctions propres étaient des polynômes multipliés par une fonction strictement positive.

Dans notre article [**3**], nous avons généralisé cette méthode aux fonctions propres d'opérateurs de Schrödinger avec des potentiels radiaux. La première étape, qui permet de justifier l'utilisation de géométrie algébrique, était de réaliser qu'il existe une base de fonctions propres ayant les propriétés voulues (proposition 3.3): si $H = \Delta + V$ où V est une fonction radiale, alors il existe une base des fonctions propres de la forme $f(r)\Gamma(\theta)$, où $\Gamma(\theta)$ est une harmonique sphérique. De plus, on peut contrôler le degré du polynôme définissant $\Gamma(\theta)$ en fonction de la valeur propre (proposition 3.6). En utilisant cette décomposition, on peut calculer le nombre de domaines nodaux qui intersectent une hypersphère.

Pour des potentiels radiaux qui croissent à l'infini, on peut maintenant utiliser la construction pour l'oscillateur harmonique quantique. On construit une partition de $\{V < \lambda\}$ qui dépend de la croissance de V à l'infini, et le résultat est obtenu de la même façon.

Dans l'article, nous généralisons aussi la méthode à des potentiels tendant vers 0 à l'infini ou ayant une singularité à l'origine, par exemple le potentiel de Coulomb ($V(x) = |x|^{-1}$). Dans ce cas, on étudie seulement le spectre discret ($\lambda_n < 0$). La principale obstruction est la suivante: lorsqu'on estime l'aire minimale d'un domaine nodal, on utilise l'inégalité de Faber-Krahn sous la forme $|\Omega| \geq C_n[\sup_{x\in\Omega} \lambda - V(x)]^{-n/2}$. Cependant, comme $V(x)$ tend vers $-\infty$ à l'origine peu importe la partition utilisée, il restera une région où il sera impossible de borner inférieurement $|\Omega|$.

Pour contourner cette difficulté, nous avons démontré qu'un domaine nodal d'une fonction propre f_λ ne peut être inclus dans une boule de rayon $C_V\lambda^{-1/2}$ (propositions 3.4, 3.5). On partitionne ensuite l'ensemble $\{V < \lambda\} \cap B_{C_V\lambda^{-1/2}}(0)^c$ pour obtenir le résultat final.

Voici un exemple de restrictions que nous avons posées sur le potentiel afin d'utiliser la loi de Weyl:

Théorème 0.12.1 ([**56**], Theorem XIII.81)**.** *Soit V une fonction mesurable dans $\mathbb{R}^d$ ($d \geq 2$) satisfaisant aux conditions*

$$c_1\left(r^\beta - 1\right) \leq V(x) \leq c_2\left(r^\beta + 1\right), \tag{0.12.5}$$

$$|V(x) - V(y)| \leq c_3\left[\max\{|x|,|y|\}\right]^{\beta-1}|x-y|, \tag{0.12.6}$$

pour un certain $\beta > 1$ et des constantes $c_1, c_2, c_3 > 0$.
Alors,

$$N(\lambda) = \int_M (\lambda - V)_+^{n/2}\left(1 + o_\lambda(1)\right).$$

Le résultat final est le suivant:

Théorème 0.12.2 ([**3**], théorème 1.6)**.** *Soit $H = \Delta + V$ un opérateur de Schrödinger dans $\mathbb{R}^d$, $d \geq 2$, avec $V(x) = v(|x|)$ tel que v obéit aux quatre conditions suivantes:*

Monotonicité	*Il existe* $R_0 > 0$ *tel que pour* $r \geq R_0$, $v'(r) > 0$.
Différentiabilité	$v \in C^1(0, +\infty)$.
Conditions à l'origine	v *satisfait à une de ces deux conditions:* $v \in C^0([0, +\infty))$ *ou il existe* $s \in (0{,}2)$ *tel que* $v(r) \asymp -r^{-s}$ *quand* $r \to 0$.
Conditions à l'infini	v *satisfait à une de ces deux conditions: il existe* $m \in (-2{,}0)$ *tel que* $v(r) \asymp r^m$ *et* $v'(r) \asymp r^{m-1}$ *quand* $r \to \infty$ *ou il existe* $m > 1$ *tel que* $v(r) \asymp r^m$ *et* $v'(r) \asymp r^{m-1}$ *quand* $r \to \infty$.

Soient λ_n *les valeurs propres de* H *(strictement inférieures au spectre essentiel, s'il y en a un) et* ϕ_k *une base de fonctions propres associées. Alors, le nombre de domaines nodaux de* f_n *est borné par*

$$\limsup_{n \to +\infty} \frac{\mu(\phi_n)}{n} \leq \gamma(d)\,. \tag{0.12.7}$$

Parmi les exemples de potentiels qui vérifient le théorème, on peut retrouver le potentiel de Coulomb ($v(r) = -r^{-1}$ sur $\mathbb{R}^3$, l'oscillateur harmonique quantique isotrope ($v(r) = r^2$) ou la somme de ces deux potentiels.

0.13. Points critiques

Il existe une connection évidente entre les domaines nodaux et les points critiques d'une fonction propre du laplacien. En effet, comme une fonction atteint toujours son minimum et son maximum sur un ensemble compact, il y aura au moins un point critique par ensemble nodal. Si nous obtenons des bornes supérieures sur le nombre de points critiques d'une fonction propre, les mêmes bornes suivront pour le nombre de domaines nodaux.

Il existe plusieurs exemples de variétés compactes dont certaines fonctions propres ont très peu de points critiques. Dans [**36**], les auteurs construisent une métrique sur le tore telle qu'il existe une suite de fonctions propres avec au plus 16 points critiques. Ils démontrent ainsi qu'il n'existe pas de borne inférieure universelle sur le nombre de points critiques des fonctions propres du laplacien.

0.14. Construction de Buhovsky, Logunov et Sodin

Jusqu'à tout récemment, il n'existait pas de borne supérieure pour le nombre de points critiques. Cependant, en 2019, Buhovsky, Logunov et Sodin [**18**] ont construit une métrique sur $\mathbb{T}^2$ qui admet une suite de fonctions propres ayant une infinité de points critiques.

Leur idée était de construire une métrique de la forme $q(x)(dx^2+dy^2)$ en plusieurs étapes:

(1) Premièrement, on construit une fonction $a(x)$ oscillant infiniment souvent près de zéro, mais dont les oscillations décroissent exponentiellement rapidement.

(2) Deuxièmement, on crée une métrique de telle façon que la multiplicité de chaque valeur propre soit au moins 4.

(3) On utilise une combinaison linéaire f_λ de fonctions propres pour obtenir un point critique (x_0,y_0) dégénéré (on a $\frac{\partial^2 f_\lambda}{\partial x^2}(x_0,y_0) = \frac{\partial^2 f_\lambda}{\partial y^2}(x_0,y_0) = 0$). De plus, ces fonctions propres ont la forme $f_\lambda = g_m(x)\sin(my)$.

(4) Troisièmement, ils montrent qu'on peut combiner la métrique précédente et la fonction $a(x)$ pour créer une infinité d'oscillations à la fonction $g_m(x)$ près de x_0.

(5) Quatrièmement, ils utilisent un processus de diagonalisation pour construire une métrique et une suite de fonctions g_m qui ont ces propriétés.

Leur contre-exemple montrait qu'une borne supérieure universelle est impossible à trouver pour les fonctions propres du laplacien. Cependant, certaines propriétés spécifiques de leur construction rendaient la généralisation de leur résultat plus problématique. Premièrement, la métrique qu'ils avaient construite était lisse mais pas analytique. Deuxièmement, il n'est pas évident que leur méthode puisse être généralisée à des surfaces de genre différent.

0.15. Notre construction

Dans notre article [**4**], Pierre Bérard, Bernard Helffer et moi avons utilisé une méthode complètement différente. Au lieu de construire une métrique et de montrer qu'il existe des fonctions propres avec les propriétés voulues, nous avons commencé directement avec une fonction f_λ et nous avons construit une métrique g telle que f_λ est une fonction propre du laplacien avec valeur propre λ (voir section 5.2).

Nous utilisons la propriété suivante du laplacien sur les surfaces: si (M,g) est une surface avec métrique g et G est une fonction réelle strictement positive, alors $\Delta_{Gg} = G^{-1}\Delta_g$. Donc si pour une fonction f sur m, $\Delta_g f = \lambda(x) f$ et $\lambda(x) \approx \lambda$, alors la fonction f sera une fonction propre sur $(M,\tilde{g})$ avec $\tilde{g} = \lambda/\lambda(x) g$ (et valeur propre λ).

Pour créer nos métriques, nous avons débuté avec des fonctions propres f_λ pour la métrique plate sur $\mathbb{T}^2$ et la métrique ronde sur $\mathbb{S}^2$. Nous avons ensuite légèrement perturbé les fonctions pour ajouter des points critiques sans trop changer le quotient $(\Delta_g f_\lambda)/f_\lambda$.

Nous avons utilisé cette méthode pour les trois constructions sur $\mathbb{T}^2$ et $\mathbb{S}^2$ qui ont déjà été décrites à la section 0.8, propositions 0.8.3, 0.8.4 et 0.8.5. Voici une autre façon de caractériser ces métriques:

- Une métrique analytique sur $\mathbb{T}^2$ avec une fonction propre d'indice inférieur ou égal à 4 qui possède N points critiques, où N est un entier arbitraire fixé. Cette métrique peut être arbitrairement proche de la métrique plate.
- Une métrique lisse sur $\mathbb{T}^2$ avec une fonction propre d'indice fini qui possède une infinité de points critiques. Cette métrique peut aussi être arbitrairement proche de la métrique plate.
- Une suite de métriques lisses sur $\mathbb{S}^2$ qui convergent vers la métrique ronde et une suite de fonctions propres associées à chaque métrique ayant une infinité de fonctions propres.

La principale difficulté associée à ces constructions est de montrer que les métriques obtenues sont non-dégénérées. Il faut donc s'assurer que la perturbation de la fonction initiale soit assez fine.

0.16. Plan des chapitres

Le premier chapitre présente l'article *A Pleijel-type theorem for the quantum harmonic oscillator* [**2**], publié en décembre 2015 sur arXiv puis en 2018 dans le Journal of Spectral Theory. Le deuxième chapitre présente l'article *Pleijel's theorem for Schrödinger operators with radial potentials* [**3**], co-écrit avec Bernard Helffer et Thomas Hoffmann-Ostenhof, publié en avril 2016 sur arXiv puis en 2018 dans les Annales Mathématiques du Québec. Le troisième chapitre présente l'article *Non-boundedness of the number of super level domains of eigenfunctions* [**4**], co-écrit avec Pierre Bérard et Bernard Helffer, publié sur arXiv en juin 2019 sous le titre original *Non-boundedness of the number of nodal domains of a sum of eigenfunctions*, puis accepté en février 2020 au Journal d'Analyse Mathématique et en attente de publication.

Premier article.

A Pleijel-type theorem for the quantum harmonic oscillator

par

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Résumé. Nous prouvons un analogue du théorème de Pleijel pour le comportement asymptotique du nombre de domaines nodaux des fonctions propres de l'oscillateur harmonique quantique en dimension quelconque.

Mots clés : Théorème de Pleijel, oscillateur harmonique quantique, domaines nodaux, fonctions propres, opérateur de Schrödinger, Faber-Krahn

Abstract. We prove a version of Pleijel's theorem for the asymptotic behaviour of the number of nodal domains of the quantum harmonic dimension in any dimension.
Keywords: Pleijel theorem, quantum harmonic oscillator, nodal domains, eigenfunctions, Schrödinger operator, Faber–Krahn

1. Introduction and main results

1.1. Pleijel's nodal domain theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $\lambda_1 < \lambda_2 \leq \lambda_3 \ldots$ be the eigenvalues of the Dirichlet Laplacian in Ω and let $\{f_i\}_{i\geq 1}$ be an orthogonal basis of eigenfunctions associated with those eigenvalues.

Recall that a nodal domain of a function is a connected component of the complement of the zero-set of that function. Let $\mu(f)$ be the number of nodal domains of the function f.

Recall that Courant's nodal domain theorem states that $\mu(f_k) \leq k$. In 1956, Pleijel found a better estimate when eigenvalues tend to infinity. There exists a constant $\gamma(n) < 1$ that depends only on the dimension such that:

$$\limsup_{k\to\infty} \frac{\mu(f_k)}{k} \leq \gamma(n) = \frac{2^{n-2} n^2 \Gamma(n/2)^2}{(j_{\frac{n}{2}-1})^n}\,. \tag{1.1}$$

Here, $j_{\frac{n}{2}-1}$ is the first zero of the Bessel function of the first kind $J_{\frac{n}{2}-1}$.

This constant is strictly decreasing with n (see [**33**, p. 10]). Here are the first few values: $\gamma(2) = 0.69166, \gamma(3) = 0.455945, \gamma(4) = 0.296901, \gamma(5) = 0.19294$.

Remark 1.1. This result has been proved in the case of the Neumann Laplacian in dimension 2 for piecewise analytic domains in [**53**]. It is still unknown if the result holds in the Neumann case in higher dimensions. Recent efforts ([**17**], [**63**]) have been made to improve the estimate in dimension 2.

1.2. Quantum harmonic oscillator

Our goal is to study the nodal domains of eigenfunctions of the quantum harmonic oscillator.

The quantum harmonic oscillator is first defined on $S(\mathbb{R}^n)$ by:

$$H : S(\mathbb{R}^n) \to S(\mathbb{R}^n)\,,$$
$$Hf = -\Delta f + V(x) f\,. \tag{1.2}$$

Here, V is a positive-definite quadratic form and $S(\mathbb{R}^n)$ denotes the Schwartz space of rapidly decaying functions over $\mathbb{R}^n$.

There exists a unique self-adjoint extension of H over $L^2(\mathbb{R}^n)$, which will be denoted by $\mathbf{H}$. However, there exists a basis of $L^2(\mathbb{R}^n)$ consisting of eigenfunctions of $\mathbf{H}$ which are all in $S(\mathbb{R}^n)$.

The quantum harmonic oscillator can be viewed as a Schrödinger operator with potential $V(x)$. It has two properties that make it particularly interesting. Its spectrum is discrete since $\lim\limits_{|x|\to\infty} V(x) = +\infty$ (see [**62**]) and its eigenfunctions can be computed explicitly.

There exists an orthogonal basis $y_1, y_2, \dots, y_n$ of $\mathbb{R}^n$ and constants $a_1, a_2, \dots, a_n > 0$ such that $V(x) = \sum\limits_{i=1}^{n} a_i^2 y_i^2$. The Laplacian is invariant under orthogonal changes of the basis. Therefore, if we wish to study the nodal domains of the eigenfunctions of the harmonic oscillator, we can restrict ourselves to potentials of the following form:

$$V(x) = \sum_{i=1}^{n} a_i^2 x_i^2\,. \tag{1.3}$$

If all the coefficients a_i are equal, the quantum harmonic oscillator H is called *isotropic*.

A basis in $L^2(\mathbb{R}^n)$ of the eigenfunctions of H is given by

$$f_{k_1,\dots,k_n}(x) = \prod_{i=1}^{n} e^{\frac{-a_i x_i^2}{2}} H_{k_i}(\sqrt{a_i} x_i)\,. \tag{1.4}$$

Here, H_n denotes the n-th Hermite polynomial, see [**64**].

The corresponding eigenvalues are given by $\lambda_{k_1\dots k_n} = \sum\limits_{i=1}^{n} a_i(2k_i+1)$.

Note that Courant's theorem holds for H by a straightforward adaptation of the argument for the Laplacian. Two slightly improved results in the isotropic case can be found in [**13**] and [**43**].

1.3. Main result

The following result is the main result of this paper.

Theorem 1.2. *Let H be the quantum harmonic oscillator (1.2).*

The number $\mu(f_k)$ of nodal domains of the k-th eigenfunction of H satisfies:

$$\limsup_{k\to\infty} \frac{\mu(f_k)}{k} \leq \gamma(n)\,. \tag{1.5}$$

The constant $\gamma(n)$ is the same as in equation (1.1).

1.4. Eigenvalue multiplicities

If the coefficients in (1.3) are rationally independent, the eigenvalues of H are simple. Recall that $a_1, a_2, \ldots, a_n$ are rationally dependent if the only integers $k_1, k_2, \ldots, k_n$ that satisfy $a_1k_1 + a_2k_2 + \ldots + a_nk_n = 0$ are identically zero. In this case, we can compute the number of nodal domains of each eigenfunction since it is always a product of polynomials in one variable and obtain:

Theorem 1.3. *Let H be the quantum harmonic oscillator (1.2) with the coefficients $a_1, a_2, \ldots, a_n$ rationally independent.*

The number $\mu(f_k)$ of nodal domains of the k-th eigenfunction of H satisfies:

$$\limsup_{k\to\infty} \frac{\mu(f_k)}{k} = \frac{n!}{n^n}\,. \tag{1.6}$$

However, if some coefficients are rationally dependent, the eigenspace associated with an eigenvalue may have dimension greater than one and we need to deal with linear combinations of eigenfunctions.

For instance, in the isotropic case in $\mathbb{R}^n$, which is the most widely studied, the eigenvalues are $\lambda_k = 2j + n$ for all $k \in \left[\binom{n+j-1}{j-1} + 1, \binom{n+j}{j}\right]$.

Hence, the multiplicities grow to infinity. It is therefore hard to compute the number of nodal domains of the eigenfunctions directly. In this paper, we present a different approach that covers all cases.

1.5. Sketch of the proof of Theorem 1.2

When we analyse Pleijel's original proof of the theorem in the case of the Laplacian with Dirichlet boundary conditions on an Euclidian domain Ω, the main idea is to give a lower bound on the area of each nodal domain using Faber-Krahn's inequality. We then divide the area of Ω by this lower bound and apply Weyl's law to get the final inequality.

If we try to use the same argument for the quantum harmonic oscillator, there is an obstacle: we are considering functions over $\mathbb{R}^n$, which has infinite volume. We must therefore find a way to resolve this issue.

We first show that any nodal domain must intersect the classically allowed region $\{V(x) < \lambda\}$ (see [**30**]), which in our case is the interior of an ellipsoid.

We then divide this ellipsoid into regions called generalized annuli (see Definition 2.3). This is the main new idea, which lets us bound the number of nodal domains. We use a theorem of Milnor on the Betti numbers of sublevel sets of real polynomials in order to give an upper bound on the number of nodal domains that intersect more than one generalized annulus. Finally, we use Faber-Krahn's inequality to get lower bound on the area of each nodal domain located in each generalized annulus.

2. Proof of theorem 1.2

2.1. Eigenvalues and eigenfunctions of H

Recall that every eigenfunction of H is of the form

$$f = \prod_{i=1}^{n} e^{\frac{-a_i x_i^2}{2}} g(x),$$

where g is a polynomial. By slight abuse of notation, we define the degree of an eigenfunction f as the degree of its associated polynomial g.

Note that $f_{k_1 \dots k_n}$ is an eigenfunction of degree $k_1 + \dots + k_n$ from equation (1.4).

Remark 2.1. In the isotropic case, the eigenfunctions are ordered with their degrees as well as their eigenvalues. In the anisotropic case, the degrees of the eigenfunctions may not be strictly increasing.

We give upper bounds on the degree of f_k:

$$\deg(f_k) \leq \max_{\substack{k_1, k_2 \dots, k_n \in \mathbb{Z}^+ \\ \sum_{i=1}^{n} a_i(2k_i+1) \leq \lambda_k}} \sum_{i=1}^{n} k_i \,. \tag{2.1}$$

Take i such that $a_i = \min\{a_j, j = 1, \dots, n\}$. The maximum is obtained in the previous sum by putting $k_j = 0$ when $j \neq i$ and maximizing k_i, namely

$$\deg(f_k(x)) \leq \frac{\lambda_k - \frac{n}{2}}{2 \min_{i=1,\dots,n} a_i} \,. \tag{2.2}$$

Let $N(\lambda)$ be the number of eigenvalues of H that are not greater than λ. We have

$$N(\lambda) = \text{Card}\left(k_1, k_2 \ldots, k_n \in \mathbb{Z}^+ \;\middle|\; \sum_{i=1}^{n} a_i(2k_i+1) \leq \lambda \right).$$

Using the formula for the volume of an n-simplex, we obtain the following asymptotics when $\lambda \to +\infty$:

$$N(\lambda) = \lambda^n \left(\frac{1}{2^n n! \prod_{i=1}^{n} a_i} + o_\lambda(1) \right). \tag{2.3}$$

Also, if we put $\lambda = \lambda_k$ in (2.3), we get the following:

$$N(\lambda_k) = \lambda_k^n \left(\frac{1}{2^n n! \prod_{i=1}^{n} a_i} + o(1) \right).$$

We remark that $N(\lambda_k) \geq k$ since λ_k could have multiplicity greater than one. We can deduce the following:

$$\lambda_k^n \leq k \left(2^n n! \prod_{i=1}^{n} a_i + o(1) \right). \tag{2.4}$$

We can rewrite the previous equation the following way:

$$\lambda_k \leq k^{1/n} \left(\left(2^n n! \prod_{i=1}^{n} a_i \right)^{1/n} + o(1) \right). \tag{2.5}$$

Hence, from (2.2) and (2.5) we have the following inequality for the degree of f_k:

$$\deg(f_k(x)) \leq k^{1/n} \left(\frac{\left(2^n n! \prod_{i=1}^{n} a_i \right)^{1/n}}{2 \min\limits_{i=1\ldots n} a_i} + o_k(1) \right). \tag{2.6}$$

2.2. Unbounded nodal domains

Let Ω be an unbounded nodal domain of f_k. Since, for all k, $f_k \in S(\mathbb{R}^n)$, we have the following equality:

$$\lambda_k = \frac{\int_\Omega |\nabla f_k|^2 + \int_\Omega V(x) f_k^2}{\int_\Omega f_k^2} \,. \tag{2.7}$$

Lemma 2.2. *For each nodal domain Ω, there exists $x \in \Omega$ such that $V(x) \leq \lambda_k$*

Proof. If for all $x \in \Omega$, $V(x) > \lambda_k$, then

$$\lambda_k = \frac{\int_\Omega |\nabla f_k|^2 + \int_\Omega V(x) f_k^2}{\int_\Omega f_k^2} \geq \frac{\int_\Omega V(x) f_k^2}{\int_\Omega f_k^2} > \frac{\int_\Omega \lambda_k f_k^2}{\int_\Omega f_k^2} = \lambda_k \,, \tag{2.8}$$

hence a contradiction.

□

Therefore, every unbounded nodal domain intersects the following ellipsoid:

$$\{x \in \mathbb{R}^n \,|\, V(x) = \lambda_k\} \;.$$

2.3. Bounded nodal domains

Let us now study the bounded nodal domains. Since

$$f_k(x) = e^{-\frac{\sum_{i=1}^n a_i x_i^2}{2}} g_k(x) \,,$$

with $g_k(x)$ a polynomial, the nodal domains of f_k are the same as the nodal domains of g_k. First, let us define a specific subset of $\mathbb{R}^n$.

Definition 2.3. *Let $0 \leq b < B < +\infty$. We define a generalized annulus as*

$$\left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, b < \sum_{i=1}^n a_i x_i^2 < B \right\} \,. \tag{2.9}$$

We have just shown that every nodal domain intersect the interior of the ellipsoid described above. We divide this region in a given number of generalized annuli. The number of generalized annuli will depend on the eigenfunction. The number of generalized annuli is quite important since we count the number of nodal domains in two ways: those that are contained in one generalized annulus and those that intersect more than one generalized annulus. Having more generalized annuli will restrict the former and increase the latter, and conversely.

Let $M = M(\lambda_k)$ be the number of generalized annuli for a given eigenfunction. We will give an explicit formula for M later.

Now, let us define the following sets:

Definition 2.4.

$$A_i = \left\{ \Omega \;\middle|\; \forall x \in \Omega, \left(\frac{(i-1)}{M}\right)^{2/n} \lambda_k \leq V(x) < \left(\frac{i}{M}\right)^{2/n} \lambda_k \right\}.$$

Here, i can take the values 1,2, ..., M.

Definition 2.5.

$$B_j = \left\{ \Omega \;\middle|\; \Omega \cap \left\{ V(x) = \left(\frac{j}{M}\right)^{2/n} \lambda_k \right\} \neq \emptyset \right\}.$$

Again, j can take the values 1,2, ..., M.

In fact, every nodal domain, bounded or unbounded, is included in one of those sets. Indeed, as shown in Lemma 2.2, for each nodal domain Ω, there exists $x \in \Omega$ such that $V(x) \leq \lambda_k$. Hence, by the connectedness of each nodal domain, it belongs to one of the A_i or B_j.

2.4. Nodal domains intersecting more than one generalized annulus

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d in n variables. In this section, we will assume that d is large since we are concerned with asymptotics. We wish to give an upper bound on the number of nodal domains of f on the unit n-ball. Let $G(n,d) = (2+d)(1+d)^{n-1}$.

Let $F^+ = \{x \in B^n \mid f(x) > 0\}$. First, we show that the number of connected components of F^+ has an upper bound that depends only on the degree of f. We can find the following result in [**48**], theorem 3:

Theorem 2.6 (Milnor)**.** *Let f be a real polynomial of degree d in n variables, with d larger than some constant depending only on n*[1]*. We define P as follows:*

$$P = \{x \in B^n \mid f(x) \geq 0\}.$$

Then the zeroth[2] *Betti number of P is not greater than $G(n,d)$.*

Recall that the zeroth Betti number of a manifold is equal to the number of its connected components.

Remark 2.7. We could not find a similar result for the sum of the Betti numbers of $\{x \in B^n \mid f(x) > 0\}$ (the results are not immediately applicable to the strict inequality). Hence, we must add a few more arguments to complete the proof.

Let $P_m = \{x \in B^n \mid f(x) \geq 1/m\}$.

[1]This is a slight modification to the original article, where the condition on d was not specified, see section 0.11.1 of this thesis for more details.

[2]This corrects a typo in the original article, where "first Betti number" was used.

Then, the number of connected components of P_m is not greater than $G(n,d)$. Furthermore,

$$F^+ = \lim_{m\to\infty} P_m.$$

Lemma 2.8. *The number of connected components of F^+ is not greater than $G(n,d)$.*

PROOF. Suppose that F^+ has more than $G(n,d)$ connected components. Choose connected components $\{a_i\}, i = 1,2,\ldots,G(n,d)+1$ of F^+. Take $s_i \in a_i$ such that for all $x \in a_i$, $f(x) \le f(s_i)$. We can always find such s_i by the compactness of $\overline{a_i}$ and the continuity of f.

Now, define $S = \min\{f(s_i), i = 1,2,\ldots,G(n,d)+1\}$. There exists $m \in \mathbb{N}$ such that $1/m < S$. For each connected component a_i, there exists a connected component $b_i \subset P_m$ such that $b_i \subset a_i$. However, that would imply that P_m has at least $G(n,d)+1$ connected components, which would contradict Theorem 2.6.

□

We can now give an upper bound on the number of nodal domains of a polynomial on B^n.

Proposition 2.9. *Let $f : \mathbb{R}^n \to \mathbb{R}$ a polynomial of degree d with d larger than some constant $C(n)$* [3]*. The number of nodal domains of f in B^n is not greater than $2G(n,d)$.*

PROOF. Let $F^- = \{x \in B^n \mid f(x) < 0\}$. Clearly, F^+ and F^- are disjoint. By the same argument as before, the number of connected components of $F^+ \bigcup F^-$ is not greater than $2G(n,d)$. □

Now, let us find an upper bound on the number of nodal domains of the restriction of a polynomial in n variables to S^{n-1}.

Proposition 2.10. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d with dwith d larger than some constant $C(n)$* [4]*. Then, the number of nodal domains of the restriction of f to $\mathbb{S}^{n-1}$ is not greater than $2^{2n-1}d^{n-1}$.*

PROOF. On $\mathbb{S}^{n-1}$, we can use the relation

$$x_1^2 = 1 - \sum_{i=2}^{n} x_i^2.$$

We can then rewrite f in the following form:

$$f(x_1,x_2,\ldots,x_n) = g(x_2,x_3,\ldots,x_n) + x_1 \cdot h(x_2,x_3,\ldots,x_n)\,.$$

[3] This is a slight modification to the original article, where the condition on d was not specified, see section 0.11.1 of this thesis for more details.

[4] This is a slight modification to the original article, where the condition on d was not specified, see section 0.11.1 of this thesis for more details.

Here, g is a polynomial of degree at most d and h is a polynomial of degree at most $d-1$.

Now, define $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\bar{f}(x_1,x_2,\ldots,x_n) = g(x_2,x_3,\ldots,x_n) - x_1 \cdot h(x_2,x_3,\ldots,x_n)$$

On $\mathbb{S}^{n-1}$, we have the following:

$$f \cdot \bar{f} = g^2(x_2,x_3,\ldots,x_n) + (\sum_{i=2}^{n} x_i^2 - 1) \cdot h^2(x_2,x_3,\ldots,x_n)\,. \tag{2.10}$$

Hence, $f\bar{f}$ is a polynomial of degree $2d$ in only $n-1$ variables.

Define ϕ by:

$$\phi : B^{n-1} \to \left\{x \in \mathbb{S}^{n-1} \,|\, x_1 > 0\right\}\,,$$
$$\phi(x_2,\ldots,x_n) = (\sqrt{1 - \sum_{i=2}^{n} x_i^2}, x_2,\ldots,x_n)\,.$$

Let $\mathbf{f} : B^{n-1} \to \mathbb{R}$, $\tilde{f} = (f\bar{f}) \circ \phi$. It is the restriction of a polynomial of degree $2d$ in $n-1$ variables on the unit ball in $\mathbb{R}^{n-1}$. By Proposition 2.9, the number of nodal domains of $\mathbf{f}$ in B^{n-1} is not greater than $(2+2d)(1+2d)^{n-2}$.

We have the following for $d \geq 1$:

$$(2+2d)(1+2d)^{n-2} < 2^{2n-2}d^{n-1}\,. \tag{2.11}$$

The function ϕ projects the nodal domains of $\mathbf{f}$ onto $\mathbb{S}^{n-1}$. Hence, the number of nodal domains of $f\bar{f}$ in $\{x \in \mathbb{S}^{n-1} \,|\, x_1 > 0\}$ is not greater than $2^{2n-2}d^{n-1}$.

By the same argument, the number of nodal domains of $f\bar{f}$ in $\left\{x \in \mathbb{S}^{n-1} \,|\, x_1 < 0\right\}$ is not greater than $2^{2n-2}d^{n-1}$. Furthermore, each nodal domain is either located in the upper part of the n-sphere, the lower part of the n-sphere or both. Since the number of nodal domains of f is not greater than the number of nodal domains of $f\bar{f}$, we conclude the proof.

□

By rescaling variables, we can easily prove the following corollary.

Corollary 2.11. *Let $a \in \mathbb{R}, a > 0$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d with d larger than some constant $C(n)$* [5]*. Then, the number of nodal domains of the restriction of f on $\{V(x) = a\}$ is not greater than $2^{2n-1}d^{n-1}$.*

We can now give an upper bound on the number of nodal domains that intersect more than one generalized annulus.

Lemma 2.12. *There exists $C > 0$ such that for all k,*

$$\mathrm{Card}\left(\bigcup_{j=1}^{M} B_j\right) \leq CMk^{\frac{n-1}{n}} .$$

PROOF. Recall definition 2.5 for the sets B_j. By the Corollary 2.11, $\mathrm{Card}(B_j) \leq 2^{2n-1}\deg(f_k)^{n-1}$ for $1 \leq l \leq M$ if k is large enough. We now have the following inequality:

$$\mathrm{Card}\left(\bigcup_{j=1}^{M} B_j\right) \leq M2^{2n-1}(\deg(f_k))^{n-1} . \tag{2.12}$$

We replace $\deg(f_k)$ as in equation (2.6):

$$\mathrm{Card}\left(\bigcup_{j=1}^{M} B_j\right) \leq M2^{2n-1}[k^{1/n}\left(\frac{(2^n n! \prod_{i=1}^{n} a_i)^{1/n}}{2 \min\limits_{i=1 \dots n} a_i} + o_k(1)\right)]^{n-1} . \tag{2.13}$$

Here, the error term depends only on k (and not f_k) so there exists a constant $C > 0$ such that

$$\mathrm{Card}\left(\bigcup_{j=1}^{M} B_j\right) \leq CMk^{\frac{n-1}{n}} . \tag{2.14}$$

□

As a result of this, if we take M to grow slower than $k^{\frac{1}{n}}$, the last term will be negligible in our final estimate.

2.5. Nodal domains contained in a single generalized annulus

We now turn to the study of nodal domains strictly contained in a single generalized annulus. We first recall Faber-Krahn's inequality in dimension n. Let Ω be a bounded domain of $\mathbb{R}^n$. The first Dirichlet eigenvalue $\lambda_1(\Omega)$ satisfies the following:

[5] This is a slight modification to the original article, where the condition on d was not specified, see section 0.11.1 of this thesis for more details.

$$\lambda_1(\Omega) \geq \left(\frac{1}{|\Omega|}\right)^{\frac{2}{n}} \sigma_n{}^{\frac{2}{n}} (j_{\frac{n}{2}-1})^2 \,. \tag{2.15}$$

As before, $j_{\frac{n}{2}-1}$ is the first zero of the Bessel function of the first kind $J_{\frac{n}{2}-1}$ and σ_n is the volume of the unit ball in $\mathbb{R}^n$.

Now, let Ω be a bounded nodal domain of f_k. We have the following inequality:

$$\frac{\int_\Omega |\nabla f_k|^2}{\int_\Omega f_k^2} \geq \left(\frac{1}{|\Omega|}\right)^{\frac{2}{n}} \sigma_n{}^{\frac{2}{n}} (j_{\frac{n}{2}-1})^2 \,. \tag{2.16}$$

Recall definition 2.4 for the sets A_i, as well as equation (2.7). For each $\Omega \in A_i$,

$$\frac{\int_\Omega |\nabla f_k|^2}{\int_\Omega f_k^2} < \lambda_k - \left(\frac{i}{M}\right)^{\frac{2}{n}} \lambda_k \,. \tag{2.17}$$

Combining (2.16) and (2.17), we get:

$$|\Omega| \geq \frac{\sigma_n (j_{\frac{n}{2}-1})^n}{\left(\lambda_k - (\frac{i}{M})^{\frac{2}{n}} \lambda_k\right)^{\frac{n}{2}}} \,. \tag{2.18}$$

Let $w_n(x)$ denote the volume of an n-ball of radius x. The volume of the generalized annulus in which each element of A_i can be found is

$$\frac{1}{\prod_{i=1}^{n} a_i} \left(w_n\left(\left(\frac{i}{M}\right)^{\frac{1}{n}} \sqrt{\lambda_k}\right) - w_n\left(\left(\frac{i-1}{M}\right)^{\frac{1}{n}} \sqrt{\lambda_k}\right)\right)$$

$$= \frac{1}{M \prod_{i=1}^{n} a_i} \sigma_n \lambda_k^{\frac{n}{2}} \,. \tag{2.19}$$

Combining (2.18) and (2.19), we get the following:

$$\mathrm{Card}(A_i) \leq \frac{\lambda_k^n}{(j_{\frac{n}{2}-1})^n \prod_{i=1}^{n} a_i} \frac{(1 - (\frac{i}{M})^{\frac{2}{n}})^{\frac{n}{2}}}{M} \,. \tag{2.20}$$

Using the last inequality, we get the following inequality for the number of elements in every A_i:

$$\mathrm{Card}(\bigcup_{i=1}^{M} A_i) \leq \sum_{i=1}^{M} \frac{\lambda_k^n}{(j_{\frac{n}{2}-1})^n \prod_{i=1}^{n} a_i} \frac{(1-(\frac{i}{M})^{\frac{2}{n}})^{\frac{n}{2}}}{M} . \tag{2.21}$$

Here, the function $f(x) = (1 - x^{\frac{2}{n}})^{\frac{n}{2}}$ is integrable over [0,1], hence the Riemann sum with the partition $\{i/M\}, i = 0 \ldots M$ converges to the value of the integral when M goes to infinity.

Choose M such that M goes to infinity with k slower than $k^{\frac{1}{n}}$. Then,

$$\mathrm{Card}(\bigcup_{i=1}^{M} A_i) \leq \frac{\lambda_k^n}{(j_{\frac{n}{2}-1})^n \prod_{i=1}^{n} a_i} \left(\int_0^1 (1-x^{\frac{2}{n}})^{\frac{n}{2}} dx + o_k(1) \right) . \tag{2.22}$$

We can now compute the integral. Using the substitution $u = x^{\frac{2}{n}}$ (see for example [**29**]) gives us the following:

$$\int_0^1 (1-x^{\frac{2}{n}})^{\frac{n}{2}} dx = \frac{n}{2} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}+1)}{\Gamma(n+1)} = \frac{n^2\Gamma(n/2)^2}{2^2 n!} . \tag{2.23}$$

Using equation (2.5), we get:

$$\mathrm{Card}(\bigcup_{i=1}^{M} A_i) \leq \frac{k\left(2^n n! \prod_{i=1}^{n} a_i + o_k(1)\right)}{(j_{\frac{n}{2}-1})^n \prod_{i=1}^{n} a_i} \left(\frac{n^2\Gamma(n/2)^2}{2^2 n!} + o_k(1) \right) , \tag{2.24}$$

and finally:

$$\mathrm{Card}(\bigcup_{i=1}^{M} A_i) \leq k \left(\frac{2^{n-2} n^2 \Gamma(\frac{n}{2})^2}{(j_{\frac{n}{2}-1})^n} + o_k(1) \right) \tag{2.25}$$

Combining equation (2.25) and Lemma 2.12 and recalling the fact that we chose M to grow slower than $k^{\frac{1}{n}}$, we get the final inequality:

$$\limsup_{k\to\infty} \frac{N(f_k)}{k} \leq \frac{2^{n-2} n^2 \Gamma(\frac{n}{2})^2}{(j_{\frac{n}{2}-1})^n} , \tag{2.26}$$

which completes the proof of Theorem 1.2.

3. Proof of Theorem 1.3

Take the coefficients a_i to be rationally independent. Under this assumption, the eigenvalues of H are simple. We know that the n-th hermite polynomial has exactly n zeros. Hence, the eigenfunction $f_{k_1,\dots,k_n}(x)$ has exactly $\prod_{i=1}^{n}(k_i+1)$ nodal domains. We have the following expression for the maximal number of nodal domains of f_λ:

$$\mu(f_\lambda) = \sup_{\substack{k_1,\dots,k_n\in\mathbb{Z}^+\\ \sum_{i=1}^{n} a_i(2k_i+1)\le\lambda}} \prod_{i=1}^{n}(k_i+1)\,. \tag{3.1}$$

We can give an upper bound on $\mu(f_\lambda)$ in the following way:

$$\mu(f_\lambda) \le \sup_{\substack{k_1,\dots,k_n\in\mathbb{R}^+\\ \sum_{i=1}^{n} a_i(2k_i+1)\le\lambda}} \prod_{i=1}^{n}(k_i+1)\,. \tag{3.2}$$

We start by proving the following lemma.

Lemma 3.1. *Let $\lambda > 0, a_1, a_2, \dots, a_n \in \mathbb{R}^+$. We have the following:*

$$\sup_{\substack{k_1,\dots,k_n\in\mathbb{R}^+\\ \sum_{i=1}^{n} a_ik_i\le\lambda}} \prod_{i=1}^{n} k_i = \frac{\lambda^n}{n^n\prod_{i=1}^{n} a_i}\,. \tag{3.3}$$

Proof. We start by putting

$$\begin{aligned}
\sup_{\substack{k_1,\dots,k_n\in\mathbb{R}^+\\ \sum_{i=1}^{n} a_ik_i\le\lambda}} \prod_{i=1}^{n} k_i &= \frac{1}{\prod_{i=1}^{n} a_i}\sup_{\substack{k_1,\dots,k_n\in\mathbb{R}^+\\ \sum_{i=1}^{n} a_ik_i\le\lambda}} \prod_{i=1}^{n} a_ik_i\\
&= \frac{1}{\prod_{i=1}^{n} a_i}\sup_{\substack{k_1,\dots,k_n\in\mathbb{R}^+\\ \sum_{i=1}^{n} k_i\le\lambda}} \prod_{i=1}^{n} k_i\,.
\end{aligned}$$

We can use the fact that log is an increasing and concave function:

$$\sup_{\substack{k_1,\ldots,k_n \in \mathbb{R}^+ \\ \sum_{i=1}^n a_i k_i \le \lambda}} \prod_{i=1}^n k_i = \frac{1}{\prod_{i=1}^n a_i} \exp\left(\sup_{\substack{k_1,\ldots,k_n \in \mathbb{R}^+ \\ \sum_{i=1}^n k_i \le \lambda}} \sum_{i=1}^n \log k_i \right)$$

$$= \frac{1}{\prod_{i=1}^n a_i} \exp\left(\sum_{i=1}^n \log(\lambda/n) \right)$$

$$= \frac{\lambda^n}{n^n \prod_{i=1}^n a_i} .$$

□

Now, take $\lambda >> 0$. We can rewrite equation (3.2) in the following way:

$$\mu(f_\lambda) \le \sup_{\substack{k_1,\ldots,k_n \in \mathbb{R}^+ \\ \sum_{i=1}^n a_i(2k_i+1) \le \lambda}} \prod_{i=1}^n (k_i + 1)$$

$$= \sup_{\substack{k_1,\ldots,k_n \in \mathbb{R}^+ \\ 2\sum_{i=1}^n a_i k_i \le \lambda - 3\sum_{i=1}^n a_i}} \prod_{i=1}^n k_i .$$

By Lemma 3.1, we have the following estimate for $\mu(f_\lambda)$:

$$\mu(f_\lambda) \le \frac{\lambda^n}{2^n n^n \prod_{i=1}^n a_i} + o(\lambda^n) . \tag{3.4}$$

Combining (3.4) and (2.3), we obtain

$$\limsup_{\lambda \to \infty} \frac{\mu(f_\lambda)}{N(\lambda)} = U(n) ,$$

with $U(n) = \frac{n!}{n^n}$.

Now, let us check that this upper bound is attained by a sequence of eigenfunctions. First, we see that

$$\sup_{\substack{k_1,\ldots,k_n\in\mathbb{Z}^+\\ \sum_{i=1}^{n} a_i(2k_i+1)\le\lambda}} \prod_{i=1}^{n}(k_i+1) \geq \sup_{\substack{k_1,\ldots,k_n\in\mathbb{Z}^+\\ \sum_{i=1}^{n} a_i(2k_i+1)\le\lambda}} \prod_{i=1}^{n}(k_i)$$

$$= \sup_{\substack{k_1,\ldots,k_n\in\mathbb{R}^+\\ 2\sum_{i=1}^{n} a_ik_i\le\lambda-\sum_{i=1}^{n} a_i}} \prod_{i=1}^{n} k_i\,.$$

We use Lemma 3.1 to obtain:

$$\sup_{\substack{k_1,\ldots,k_n\in\mathbb{Z}^+\\ \sum_{i=1}^{n} a_i(2k_i+1)\le\lambda}} \prod_{i=1}^{n}(k_i+1) \geq \frac{\lambda^n}{2^n n^n \prod_{i=1}^{n} a_i} + o(\lambda^n)\,.$$

This means that for every $\lambda > 0$, there exists an eigenfunction f_λ such that

$$\mu(f_\lambda) \geq \frac{\lambda^n}{2^n n^n \prod_{i=1}^{n} a_i} + o(\lambda^n)\,.$$

We can then construct a sequence of eigenfunctions f_{n_k} such that

$$\limsup_{k\to\infty} \frac{\mu(f_{n_k})}{n_k} = U(n)\,.$$

This shows that $U(n)$ is indeed optimal, which completes the proof of theorem 1.3.

Let us compare $U(n)$ with $\gamma(n)$:

Proposition 3.2. *For all $n \geq 2$, $U(n) < \gamma(n)$. Furthermore,*

$$\frac{\gamma(n)}{U(n)} > 2^{n-\frac{5}{2}}\sqrt{\pi n}e^{-2\sqrt{n}}(1+o_n(1))$$

as n goes to infinity.

Therefore, $U(n)$ decays much faster than $\gamma(n)$ as n goes to infinity.

Proof. We start by putting

$$\frac{\gamma(n)}{U(n)} = \frac{2^{n-2}n^2\Gamma(\frac{n}{2})^2 n^n}{n!(j_{\frac{n}{2}-1})^n}\,.$$

If $n = 2k$, we have

$$\frac{\gamma(n)}{U(n)} = \frac{2^{2k-2}(2k)^2\Gamma(k)^2(2k)^{2k}}{(2k)!(j_{k-1})^{2k}}$$
$$= \frac{2^{4k}(k!)^2k^{2k}}{(2k)!(j_{k-1})^{2k}}.$$

It is shown in [**33**] that for $u > 0$, $\sqrt{u(u+2)} < j_u < \sqrt{u+1}(\sqrt{u+2}+1)$. Hence, for $u > 10$, $j_{u-1} < \sqrt{2}u$. Also, $(2k)! < 2^{3k}(k!)^2$ for $k \geq 1$. Combining those two facts with the previous equation, we get for $k > 10$

$$\frac{\gamma(n)}{U(n)} = \frac{2^{3k}(k!)^2}{(2k)!}\frac{(\sqrt{2}k)^{2k}}{(j_{k-1})^{2k}} > 1.$$

If $n = 2k+1$, we have

$$\frac{\gamma(n)}{U(n)} = \frac{2^{2k-1}(2k+1)^2\Gamma(k+1/2)^2(2k+1)^{2k+1}}{(2k+1)!(j_{k-1/2})^{2k+1}}$$

Using the identity $\Gamma(k+1/2) = \frac{(2k)!}{4^k k!}\sqrt{\pi}$, we get

$$\frac{\gamma(n)}{U(n)} = \frac{\pi 2^{2k-1}(2k+1)^2((2k)!)^2(2k+1)^{2k+1}}{2^{4k}(k!)^2(2k+1)!(j_{k-1/2})^{2k+1}}$$
$$= \frac{\pi(2k+1)!(2k+1)^{2k+1}}{(j_{k-1/2})^{2k+1}2^{2k+1}(k!)^2}.$$

We use the fact that $(2k+1)! > 2^{2k}(k!)^2$ and that $j_{u-1/2} < \sqrt{2}(u-1/2)$ for $u > 10$ to obtain for $k > 10$ that

$$\frac{\gamma(n)}{U(n)} = \frac{(2k+1)!}{2^{2k}(k!)^2}\frac{(2k+1)^{2k+1}}{(j_{k-1/2})^{2k+1}}\frac{\pi}{2} > 1.$$

We only need to check that $\gamma(n) > U(n)$ for $n = 1,2,\ldots,21$, which is done using Mathematica.

Now, using Stirling's formula and the estimate

$$j_{\frac{n}{2}-1} \leq \left(\sqrt{\frac{n}{2}} + \sqrt{\frac{1}{2}}\right)^2,$$

we have the following:

$$
\begin{aligned}
\frac{\gamma(n)}{U(n)} &> \frac{2^{n-2}\Gamma(\frac{n}{2}+1)^2 n^n}{\left(\sqrt{\frac{n}{2}}+\sqrt{\frac{1}{2}}\right)^2 n!} \\
&= \frac{2^{n-2}(\frac{n}{2e})^n \pi n(1+o_n(1))\sqrt{n}^{2n}}{\left(\sqrt{\frac{n}{2}}+\sqrt{\frac{1}{2}}\right)^{2n}(\frac{n}{e})^n\sqrt{2\pi n}(1+o_n(1))} \\
&= \frac{2^{n-\frac{5}{2}}\sqrt{\pi n}}{\left(1+\sqrt{\frac{1}{n}}\right)^{2n}}(1+o_n(1))
\end{aligned}
$$

Now, we use the fact that $\left(1+\sqrt{\frac{1}{n}}\right)^{\sqrt{n}} < e$ to obtain as n goes to infinity

$$
\frac{\gamma(n)}{U(n)} > 2^{n-\frac{5}{2}}\sqrt{\pi n}e^{-2\sqrt{n}}(1+o_n(1))\,.
$$

□

Remark 3.3. *It is clear that the constant $\gamma(n)$ can be improved for the quantum harmonic oscillator. It is still unknown if the constant $U(n)$ is the optimal constant in the general case. There is a similar question concerning Pleijel's theorem for the Dirichlet or Neumann Laplacian. In the case of an irrationnal rectangle, the constant $\gamma(n)$ can be lowered to $\frac{2}{\pi}$. It has been conjectured by I. Polterovich in* [**53**] *that $\frac{2}{\pi}$ is the optimal constant for any planar domain.*

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Deuxième article.

Pleijel's theorem for Schrödinger operators with radial potentials

par

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Les principales contributions de Philippe Charron à cet article sont présentées.
Les coauteurs ont contribué à l'article de façon égale.

Résumé. En 1956, il a été démontré par Pleijel que l'égalité dans le théorème de Courant ne peut être atteinte qu'un nombre fini de fois. Il s'agit d'une conséquence d'un théorème sur le comportement asymptotique du nombre de domaines nodaux lorsque la valeur propre tend vers l'infini. Cette propriété peut aussi être étudiée dans le cas des opérateurs de Schrödinger. Le premier résultat significatif lié à ce problème a été obtenu par le premier auteur dans le cas de l'oscillateur harmonique quantique. Nous généralisons ce théorème à une large classe de potentiels radiaux qui tendent vers $+\infty$ à l'infini ou vers 0 à l'infini. Si le potentiel est borné, nous étudierons seulement les valeurs propres inférieures au spectre essentiel.
Mots clés : Pleijel, Schrödinger, Domaines nodaux, Fonctions propres, Courant-sharp

Abstract. In 1956, *A.* Pleijel gave his celebrated theorem showing that the inequality in Courant's theorem on the number of nodal domains is strict for large eigenvalues of the Laplacian. This was a consequence of a stronger result giving an asymptotic upper bound for the number of nodal domains of the eigenfunction as the eigenvalue tends to $+\infty$. A similar question occurs naturally for the case of the Schrödinger operator. The first significant result has been obtained recently by the first author for the case of the harmonic oscilllator. The purpose of this paper is to consider more general potentials which are radial. We will analyze either the case when the potential tends to $+\infty$ or the case when the potential tends to zero. In the latter, we will consider eigenfunctions associated with eigenvalues below the essential spectrum.
Keywords: Pleijel, Schrödinger, Nodal domains, Eigenfunctions, Courant-sharp

1. Introduction

The goal of this paper is to extend Pleijel's theorem for the Dirichlet Laplacian $H(\Omega) = -\Delta$ in a bounded domain Ω to the case of the Schrödinger operator $H_V = -\Delta + V$ in $\mathbb{R}^d$. We are interested in counting the number of nodal domains of an eigenfunction and to relate this number with the labelling of the corresponding eigenvalue. Throughout this paper, for any function f defined over a domain $D \subset \mathbb{R}^d$, $\mu(f)$ will denote the number of nodal domains of f, namely the number of connected components of $D \setminus f^{-1}(0)$. The starting point of the analysis is Courant's Theorem (1923) [**22**].

Theorem 1.1 (Courant)**.** *If ϕ_n is an eigenfunction associated with the n-th eigenvalue λ_n of $H(\Omega)$ (ordered in non decreasing order and labelled with multiplicity), then*

$$\mu(\phi_n) \leq n\,. \tag{1.1}$$

Pleijel's theorem (1956) [**52**] says

Theorem 1.2 (Pleijel's weak theorem)**.** *If the dimension is ≥ 2, there is only a finite number of eigenvalues of the Dirichlet Laplacian in Ω for which we have equality in (1.1).*

Let us now give the strong form of Pleijel's theorem.

Theorem 1.3 (Pleijel's strong theorem)**.** *Let λ_n the non decreasing sequence of eigenvalues associated to the Dirichlet realization of the Laplacian. For any $d \geq 2$ for any orthonormal* *basis ϕ_n of eigenfunctions ϕ_n of $H(\Omega)$ (the Dirichlet Laplacian in Ω) associated with $\lambda_n(\Omega)$,*

$$\limsup_{n\to+\infty} \frac{\mu(\phi_n)}{n} \leq \gamma(d) = \frac{2^{d-2} d^2 \Gamma(d/2)^2}{(j_{\frac{d}{2}-1})^d}, \tag{1.2}$$

where j_ν denotes the first zero of the Bessel function J_ν.

The theorem was proved by Pleijel [**52**] for $d = 2$ and then extended by Peetre [**51**] and Bérard-Meyer [**14**]. We recall from [**14**, Lemma 9] that Pleijel's constant equals

$$\gamma(d) = w_d^{-1} \omega_d^{-1} \left(\lambda(B_d)\right)^{-d/2} < 1\,, \tag{1.3}$$

where

- w_d is the Weyl constant

$$w_d := (2\pi)^{-d} \omega_d\,, \tag{1.4}$$

-

$$\omega_d := |B_d|\,, \tag{1.5}$$

  where B_d is the unit ball in $\mathbb{R}^d$ and $|D|$ denotes for an open set $D \subset \mathbb{R}^d$ its volume;
- $\lambda(B_d)$ is the Dirichlet ground state energy of the Laplacian in B_d.

As $\gamma(d) < 1$, one recovers as a corollary that the inequality in Courant's theorem is strict for n large. The second point to notice is that the constant is independent of the open set. Complementary properties of $\gamma(d)$ have been obtained by B. Helffer and M. Persson-Sundqvist [**33**]. In particular $d \mapsto \gamma(d)$ is decreasing exponentially to 0. Finally note that this constant is not optimal (see [**17**], [**63**] and the discussion in [**32**]).

The original proof of Pleijel's theorem is based on a combination between the Weyl formula [**68**] and the Faber-Krahn inequality. Weyl's theorem reads, as $\lambda \to +\infty$,

$$N(\lambda) = |\Omega|\, w_d\, \lambda^{\frac{d}{2}} (1 + o(1))\,, \tag{1.6}$$

where

$$N(\lambda) := \#\{\lambda_j < \lambda\}\,. \tag{1.7}$$

The Faber-Krahn inequality [**27, 38**] reads

Theorem 1.4. *For any domain $D \subset \mathbb{R}^d$ ($d \geq 2$), we have*

$$|D|^{\frac{2}{d}}\, \lambda(D) \geq \omega_d^{\frac{2}{d}} \lambda(B_d)\,. \tag{1.8}$$

There are a lot of Weyl's formulas available in the context of the Schrödinger operator $H_V := -\Delta + V$. The use of the Faber-Krahn inequality is more problematic, except of course for the case of bounded domains with bounded potential which can be treated like the membrane case. In 1989 Leydold [**42**] obtained in his diploma thesis a weak Pleijel theorem for the isotropic harmonic oscillator (see also [**8**]). Two years ago Charron [**1**] in his master thesis proved Pleijel's strong theorem also for the harmonic oscillator:

Theorem 1.5 (Charron's theorem)**.** *Let $(\phi_n)_{n\in\mathbb{N}}$ be an orthogonal basis of eigenfunctions in $L^2(\mathbb{R}^d)$ of the harmonic oscillator associated with $(\lambda_n)_{n\in\mathbb{N}}$. Then*

$$\limsup_{n\to+\infty} \frac{\mu(\phi_n)}{n} \leq \gamma(d)\,. \tag{1.9}$$

The theorem is also proven in the case of the non-isotropic harmonic oscillator [**2**]. The interesting fact is that the potential $V(x) = \sum_i a_i x_i^2$ (with $a_i > 0$) does not appear on the right hand side of the upper bound. Note also that when there are no eigenvalue degeneracies a much stronger result is available in [**2**].

A natural question to ask is whether the theorem can be extended to more general Schrödinger operators. We will answer positively this question under the additional assumption that the potential is radial.
More precisely, we assume

$$d \geq 2$$

and we consider on $\mathbb{R}^d$ a Schrödinger operator $H_V = -\Delta + V$, where V is a radial potential:

$$V(x) = v(r)\,, \tag{1.10}$$

with $|x| = r$.
We will assume that

$$v \in C^1(0,+\infty)\,, \tag{1.11}$$

and that there exists $R_0 > 0$ such that

$$v'(r) > 0\,,\ \text{for } r \geq R_0\,. \tag{1.12}$$

In order to allow some singularity at the origin, we assume either

$$v \in C^0([0,+\infty))\,, \tag{1.13}$$

or that there exists $s \in (0,2)$ such that, as $r \to 0$,

$$v(r) \approx -r^{-s}\,. \tag{1.14}$$

Here, we say that $a \approx b$ is a/b and b/a are bounded, i.e. if there exists $C > 0$ such that

$$\frac{1}{C} \leq \frac{a}{b} \leq C\,.$$

We will study two cases according to the behavior of v at $+\infty$.
Case A: v tends to $+\infty$ as $r \to +\infty$.
More precisely, we assume (1.11), (1.12) and either (1.13) or (1.14) and that there exists[6]

[6]This condition appears when applying Weyl's formula given by Theorem 4.2 from [**56**]. At least under stronger assumptions on the regularity of v for $r \to +\infty$, it should be possible to assume $m > 0$.

$m > 1$ such that as $r \to +\infty$

$$v(r) \approx r^m \,, \tag{1.15}$$

and

$$v'(r) \approx r^{m-1} \,. \tag{1.16}$$

Case B: v tends to 0 as $r \to +\infty$.
More precisely, we assume (1.11), (1.12) and either (1.13) or (1.14) and that there exists $m \in (-2{,}0)$ such that

$$v(r) \approx -r^m \,, \tag{1.17}$$

and

$$v'(r) \approx r^{m-1} \,. \tag{1.18}$$

In the two cases there is a natural selfadjoint extension starting from $C_0^\infty(\mathbb{R}^d)$ (see Section 3). In Case A, the spectrum is discrete and consists of a non decreasing sequence of eigenvalues λ_n tending to $+\infty$. In Case B the spectrum is divided in two parts, the essential spectrum: $[0, +\infty)$ and the discrete spectrum, which consists of an infinite sequence of negative eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ tending to 0 (see for example Reed-Simon [**56**], Vol. IV, Theorem XIII.6). Associated with this sequence $(\lambda_n)_{n\in\mathbb{N}}$, we can consider an orthonormal sequence of eigenfunctions ϕ_n, where in Case A ϕ_n is an Hilbertian basis of $L^2(\mathbb{R}^d)$ and in Case B of the negative eigenspace.

Our analysis will contain two well-known potentials: the quantum harmonic oscillator (Case A) and the Coulomb potential (Case B). In both cases, we know the eigenvalues and an explicit basis of eigenfunctions but in the proof this property will not be used. Our aim is to prove the following result:

Theorem 1.6 (Pleijel's theorem for Schrödinger)**.** *In Cases A or B, if $(\phi_n)_{n\geq 1}$ is an orthogonal sequence of eigenfunctions of H_V associated with the above defined sequence λ_n, then*

$$\limsup_{n\to+\infty} \frac{\mu(\phi_n)}{n} \leq \gamma(d) \,. \tag{1.19}$$

The paper is organized as follows.
In Section 2, we discuss the general strategy and the methods used by Pleijel first and then by P. Charron. In Section 3, we review the general properties of the Schrödinger operator. In Section 4 we collect those Weyl-type results we need for the proof of Theorem 1.6.

In Section 5, we give the proof of our Pleijel's theorem in the two situations.

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2. About the methods

As recalled in the introduction, the original proof of . Pleijel was based on a tricky combination of Weyl's formula with the Faber-Krahn inequality. When considering the case of the Schrödinger operator in $\mathbb{R}^d$, Weyl's formula still exists but the use of Faber-Krahn is not easy: nodal domains could be unbounded and the variation of the potential inside a nodal domain could be very high. One has consequently to find an idea for proving that these two bad situations do not occur very often.

In the case of the harmonic oscillator Charron's proof relies on specific properties of the eigenfunctions and the potential. Namely, it used the fact that every eigenfunction is a linear combination of an exponential multiplied by polynomials whose degree can be controlled by a function of the labeling of the eigenvalue, that the hypersurfaces with constant potential are hyperspheres and the fact that the counting function $N(\lambda)$ behaves nicely as $\lambda \to +\infty$ (Weyl's law).

In addition, it also used that, every nodal domain of an eigenfunction of a Schrödinger operator intersects the classically allowed region associated with the eigenvalue λ, i.e

$$V^{(-1)}(-\infty,\lambda) := \{x \in \mathbb{R}^d \,|\, V(x) < \lambda\} . \tag{2.1}$$

This property is quite general and elementary.

The key was then to divide the classically allowed region in a finite number of annuli of the form $V^{(-1)}(a,b)$. Every nodal domain can either be contained in a single annulus or intersect more than one. To give an upper bound on the number of nodal domains not contained in one annulus, Charron uses properties of algebraic surfaces, as well as results arising in Morse theory adapted from Milnor [**48**].

Then, he used the Faber-Krahn inequality to give a lower bound on the volume of any nodal domain contained in a single annulus. Dividing the volume of each annulus by the volume of each nodal domain gave an upper bound on the number of nodal domains contained in that annulus.

The last step was to find an appropriate number of annuli to balance out both estimates.

To extend the methods of Charron's proof to more general potentials, we need to find Schrödinger operators such that:

(1) There are good lower bounds for the number of eigenvalues below any λ.
(2) We can count the number of nodal domains that intersect a given energy hypersurface $V^{(-1)}(b)$.
(3) We can give an upper bound on the number of nodal domains that are not contained in the classically allowed region.

In the case of the harmonic oscillator, the eigenvalues are known explicitly. However, for many potentials V, Weyl's law can be extended to the Schrödinger operator H_V for estimating the number of eigenvalues. Hence, we need to find a suitable class of potentials where this law holds.

So far, the only known method to give a suitable upper bound on the number of nodal domains that intersect an energy hypersurface are based on Milnor's theorem (see Subsection 3.7). Hence we need this hypersurface to be algebraic and the property that for any eigenvalue λ and any energy hypersurface (or at least a suitable family) the restriction of any associated eigenfunction u_λ equals the restriction of a polynomial to this hypersurface. This is why we focus in this paper on the study of radial potentials. In this case, the energy hypersurfaces are hyperspheres $\{r = \rho\}$ for some $\rho > 0$ and it can be shown that the restriction of an eigenfunction to a hypersphere is always a linear combination of hyperspherical harmonics, each one being the restriction to the hypersphere of a homogeneous harmonic polynomial. We will also have to control the degree of such polynomials by a function of λ or of its labelling. This last property will allow us to bound the number of nodal domains that are not contained in $V^{(-1)}(-\infty,\lambda)$.

Another problem might arise when estimating the number of nodal domains contained in one annulus. In the case of the harmonic oscillator, summing over all annuli gives us an expression which can be compared directly with an integral. The error term that arises becomes negligible as $\lambda \to +\infty$. It remains to show under which conditions on V the same method can be applied.

Finally, in the specific case of Coulomb-like potentials at the origin, we need to look at the behavior of the number of nodal domains near the origin.

3. On the spectral theory of the Schrödinger operators with radial potential

3.1. General theory

We first verify that our Schrödinger operator $H_V = -\Delta + V$ is well defined by a Friedrichs procedure starting from its sesquilinear form defined on $C_0^\infty(\mathbb{R}^d \setminus \{0\}) \times C_0^\infty(\mathbb{R}^d \setminus \{0\})$

$$(u,v) \mapsto a(u,v) := \int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla v(x)\, dx + \int_{\mathbb{R}^d} V(x)\, u(x)\, v(x)\, dx\, .$$

Note that the left term has a meaning as soon as $V \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$. In our case, this is a consequence of Assumption 1.11. Our operator, will be defined through a Friedrichs extension. This works as soon as $q(u) := a(u,u)$ is bounded from below by $-C\, ||u||^2_{L^2}$. It is consequently enough to control the integral $\int_{V<0} V(x)\, u(x)^2\, dx$ from below.

When $d \geq 3$, we use the standard Hardy inequality (see [**24**] and references therein)

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \geq \left(\frac{d-2}{2}\right)^2 \int_{\mathbb{R}^d} \frac{1}{r^2} |u(x)|^2 \, dx \,, \forall u \in C_0^\infty(\mathbb{R}^d) \,, \tag{3.1}$$

with $r = |x|$.
This inequality extends to $H^1(\mathbb{R}^d)$ by density.
For $d = 2$, we can use the modified Hardy inequality (see also [**24**]) in a disk $D(0,\check{R})$ which reads

$$\int_{\mathbb{R}^2} |\nabla u(x)|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{u(x)^2}{|x|^2 \log^2(|x|/\check{R})} \, dx \,, \forall u \in C_0^\infty(D(0,\check{R})) \,, \tag{3.2}$$

which also extends to $H_0^1(D(0,\check{R}))$.
Using these inequalities and a partition of unity, the semi-boundedness on $C_0^\infty(\mathbb{R}^d \setminus \{0\})$ follows immediately.

Let us now describe the form domain resulting from the Friedrichs extension procedure. We have:

Case A

$$Q_H = \{u \in H^1(\mathbb{R}^d) \,|\, \sqrt{V} u \in L^2(D(0,R_1)^c)\}$$

where R_1 is chosen such that $v(r) \geq 1$ for $r \in (R_1, +\infty)$,

Case B

$$Q_H = H^1(\mathbb{R}^d) \,.$$

We do not need to characterize the domain of the corresponding self-adjoint operator.

3.2. Nodal domains intersect the classically allowed region

We use a similar argument as in [**42**] and [**2**]. We assume that we are either in Case A or in Case B, but the result is much more general.

Proposition 3.1. *Let λ be an eigenvalue below the essential spectrum, u_λ be an eigenfunction of H_V associated with eigenvalue λ and Ω be a nodal domain of u_λ. Then*

$$\Omega \cap V^{(-1)}(-\infty,\lambda) \neq \emptyset \,.$$

Proof. If for all $x \in \Omega$, $V(x) > \lambda$, then

$$\lambda = \frac{\int_\Omega |\nabla u_\lambda(x)|^2 \, dx + \int_\Omega V(x) u_\lambda(x)^2 \, dx}{\int_\Omega u_\lambda(x)^2 \, dx} \geq \frac{\int_\Omega V(x) u_\lambda(x)^2 \, dx}{\int_\Omega u_\lambda(x)^2 \, dx} > \frac{\int_\Omega \lambda u_\lambda(x)^2 \, dx}{\int_\Omega u_\lambda(x)^2 \, dx} = \lambda \,, \quad (3.3)$$

hence a contradiction. □

Therefore, any nodal domain is either contained in the classically allowed region $\{V < \lambda\}$ or intersects the hypersurface $V^{(-1)}(\lambda)$.

3.3. The radial Schrödinger operator

Although the exposition there is limited to the case $d = 3$, one can refer to Reed-Simon [**56**], Vol. IV p. 90-91.
The Laplace operator $-\Delta$ can be written as

$$-\Delta = -\frac{\partial^2}{\partial r^2} - \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}(-\Delta_{\mathbb{S}^{d-1}}), \quad (3.4)$$

where $r = |x|$ is the radial variable and $\Delta_{\mathbb{S}^{d-1}}$ is the Laplace–Beltrami operator, acting in $L^2(\mathbb{S}^{d-1})$. The following proposition is standard (see for example [**60**], Theorem 22.1 and Corollary 22.1).

Proposition 3.2. *Assume that $d \geq 2$. The spectrum of $-\Delta_{\mathbb{S}^{d-1}}$ consists of eigenvalues*

$$\ell(\ell + d - 2) \,, \quad \ell \in \mathbb{N} \,.$$

The multiplicity of the eigenvalue $\ell(\ell + d - 2)$ is given by

$$\Lambda_{\ell,d} := \binom{\ell + d - 1}{d - 1} - \binom{\ell + d - 3}{d - 1} \,,$$

which coincides with the dimension of the space of homogeneous, harmonic polynomials of degree ℓ.

We denote by $\mathbb{S}^{d-1} \ni \omega \mapsto Y_{\ell,m}(\omega)$ an orthonormal basis of the $\Lambda_{\ell,d}$-dimensional eigenspace associated with $\ell(\ell + d - 2)$. We recall that each $Y_{\ell,m}$ is the restriction to $\mathbb{S}^{d-1}$ of a harmonic homogeneous polynomial of degree ℓ.
We now consider the Schrödinger operator H_V and assume

$$V(x) = v(r) \,. \quad (3.5)$$

In this case, one can determine the spectrum by using polar coordinates. In the spherical coordinates, we can determine the spectrum by considering the (closure of the) union of the spectra of the family (indexed by $\ell \in \mathbb{N}$) of Sturm Liouville operators $\mathcal{L}_\ell$ defined by

$$\mathcal{L}_\ell = -\frac{d^2}{dr^2} - \frac{d-1}{r}\frac{d}{dr} + \frac{\ell(\ell+d-2)}{r^2} + v(r)\,, \tag{3.6}$$

acting in $L^2((0,+\infty),r^{d-1}\,dr)$, with a suitable Dirichlet like condition at 0 (see Reed-Simon [**56**], p. 91, Proof of Lemma 1). Note that the "Dirichlet like" condition is expressed after the unitary transform $u \mapsto r^{\frac{d-1}{2}} u$ sending $L^2((0,+\infty);r^{d-1}dr)$ onto $L^2((0,+\infty);dr)$ and becomes the standard Dirichlet condition for $\ell = 0$. When $\ell > 0$, no condition is given. The new operator is then:

$$\widehat{\mathcal{L}}_\ell = -\frac{d^2}{dr^2} + \frac{(\ell+\frac{d-1}{2})(\ell+\frac{d-1}{2}-1)}{r^2} + v(r)\,.$$

Proposition 3.3. *Let $H_V = -\Delta + V$, where $V(x) = v(r)$ satisfies either Case A or Case B.*
Any eigenvalue of $-\Delta + V$ is of the form

$$\lambda = \lambda_{n,\ell}\,, \tag{3.7}$$

where $\lambda_{n,\ell}$ is the n-th eigenvalue of $\mathcal{L}_\ell$.
A corresponding basis of eigenfunctions has the form

$$u_{n,\ell,m}(r,\omega) = f_{n,\ell}(r) Y_{\ell,m}(\omega)\,, \tag{3.8}$$

where $Y_{\ell,m}(\omega)$ denotes an orthonormal basis of (hyper)spherical harmonics.

We recall that these functions form a basis of $L^2(\mathbb{R}^d)$ in Case A (see [**56**]) or a basis of the negative eigenspace in Case B.

3.4. Courant's nodal theorem and nodal behavior of eigenfunctions.

For the analysis of potentials with singularities it is worth to ask under which condition one can prove Courant's theorem or describe the local nodal structure of an eigenfunction. Under our assumptions the only point is the control at the origin. Outside the origin, (1.11) implies that the potential is C^1 and the structure of the nodal set is well known.

Looking at the proof of Courant's theorem, the only thing we need is the unique continuation theorem, i.e. we need to show that if an eigenfunction u_λ is identically 0 in a non empty open set ω then it is zero in $\mathbb{R}^d$. The argument clearly works if there is only a singularity at 0, because $\omega \setminus \{0\}$ is an open set where u_λ vanishes identically. See [**34**] for more properties. We will show in the next subsection that no nodal domain is contained in a sufficiently small ball around the origin. Hence the counting of nodal domains can start outside this ball.

3.5. No nodal domains in a small ball

In this subsection, we show that under our assumptions the nodal domain cannot be contained in a small neighborhood of the origin. We will start with Case B which is easier.

3.5.1. Case B. We have the following statement:

Proposition 3.4.

If $d \geq 2$ and in Case B there exists $r_d > 0$ such that, if $\lambda < 0$ is an eigenvalue and u_λ is a corresponding eigenfunction, there is no nodal domain ω of u_λ contained in $B(0,r_d)$.

Proof

We first deduce from Assumption (1.14), because $s < 2$, that:

- For $d \geq 3$ there exists r_d such that
$$v(r) + \frac{(d-2)^2}{4r^2} > 0\,, \tag{3.9}$$
for $r \in (0,r_d)$.
- For $d = 2$, there exists $r_2 > 0$ such that
$$v(r) + \frac{1}{4r^2 \ln^2(r/2r_2)} > 0\,, \tag{3.10}$$
for $r \in (0,r_2)$.

We now use the identity

$$\int_\omega |\nabla u_\lambda(x)|^2\, dx + \int_\omega V(x)|u_\lambda(x)|^2\, dx = \lambda \int_\omega |u_\lambda(x)|^2 dx\,,$$

and get because $\lambda < 0$

$$\int_\omega |\nabla u_\lambda(x)|^2\, dx + \int_\omega V(x)|u_\lambda(x)|^2\, dx < 0\,.$$

When $d \geq 3$, we use Hardy's inequality (3.1) (u_λ is extended in $\mathbb{R}^d$ by 0 outside ω and this extension is in $H^1(\mathbb{R}^d)$) and get that

$$\int_\omega \left(V(x) + \frac{(d-2)^2}{4r^2} \right) |u_\lambda(x)|^2 dx < 0\,.$$

This contradicts (3.9) if $\omega \subset B(0,r_d)$.

When $d = 2$, we use the modified Hardy inequality (3.2) with $\check{R} = 2r_2$, (u_λ is extended by 0 outside ω in $D(0,\check{R})$ and this extension is in $H_0^1(D(0,\check{R}))$) and get a contradiction with (3.10). $\square$

3.5.2. Case A. In Case A, with singularities, there is some difficulty because we consider λ large. When $d \geq 3$, the previous proposition will be true in a ball whose radius is

$r_d(\lambda) \approx \lambda^{-\frac{1}{2}}$. For $d = 2$, $r_2(\lambda)$ could be taken as $r_2(\lambda) \approx \lambda^{-\frac{1}{2}-\epsilon}$ for some $\epsilon > 0$. More precisely, we have

Proposition 3.5. *Under Assumption* (1.14) *and if* $d \geq 3$ *, there exists a constant* $c_d > 0$ *which depends on* V *and* d *only and* $\lambda_0 > 0$ *such that, for* $\lambda \geq \lambda_0$ *and if* u_λ *denotes an eigenfunction of* H_V *, there are no nodal domains of* u_λ *contained in* $B(0,c_d\lambda^{-1/2})$ *.*

If $d = 2$ *, for any* $\epsilon > 0$ *, there exists* $\lambda_\epsilon > 0$ *and* c_V *that depends only on* V *such that, for* $\lambda \geq \lambda_\epsilon$ *and if* u_λ *denotes an eigenfunction of* H_V*, there are no nodal domains of* u_λ *contained in* $B(0,c_V\lambda^{-1/2-\epsilon})$ *.*

Proof

By (1.14), there exists $C > 0$ and $r_0 > 0$, such that $V > -Cr^{-s}$ for $0 < r \leq r_0$. As $s \in (0,2)$, there exists λ_0 such that for $\lambda \geq \lambda_0$, there exists $r_d(\lambda) \sim \frac{d-2}{2}\lambda^{-\frac{1}{2}}$ such that

$$\frac{(d-2)^2}{4} r^{-2} - Cr^{-s} > \lambda \,, \forall r \in (0,r_d(\lambda)) \,.$$

This implies

$$\frac{(d-2)^2}{4} r^{-2} + v(r) - \lambda > 0 \,, \forall r \in (0,r_d(\lambda)) \text{ and } \lambda \geq \lambda_0 \,.$$

The proof is achieved by taking $0 < c < \frac{d-2}{2}$ in the statement of the proposiiton and using the Hardy inequality as in the second part of the proof of Proposition 3.4.

For the case $d = 2$, a not optimal $r_2(\lambda) = \lambda^{-\frac{1}{2}-\epsilon}$ (for some $\epsilon > 0$) together with the modified Hardy inequality do the job for λ_ϵ large enough.

3.6. Upper bound for the degree of the polynomials associated with the spherical harmonics

In this subsection, we prove the existence of a rather optimal upper bound on the degree ℓ of the polynomials associated with the spherical harmonics $Y_{\ell m}$ appearing in the decomposition of an eigenfunction u_λ.

Proposition 3.6.

In Cases A or B, if λ *is an eigenvalue of* H_V *such that* $\lambda < \liminf_{r\to+\infty} v$ *then there exists* p_λ *such that, for any associated eigenfunction* u_λ *and for any* τ *satisfying* $\inf v < \tau \leq \lambda$*, we can find a polynomial* $\mathcal{P}_{\tau,\lambda}$ *of* d *variables of degree at most* p_λ *such that on* $V^{(-1)}(\tau)$ *in* $\mathbb{R}^d$ *the restriction of* u_λ *is equal to the restriction of* $\mathcal{P}_{\tau,\lambda}$*.*

Moreover, p_λ *satisfies*

$$p_\lambda \leq \max\{\ell \,|\, \ell \geq 1 \text{ and } \inf_r \left(v(r) + \frac{\ell(\ell+d-2)}{r^2} \right) < \lambda \,\} \,. \tag{3.11}$$

PROOF. For given λ, u_λ has the form (see (3.8))

$$u_\lambda = \sum_{\lambda=\lambda_{n,\ell}} c_{n,\ell,m}\, u_{n,\ell,m}\,.$$

If we restrict u_λ to the hypersphere of radius $r_\tau = v^{(-1)}(\tau)$, we get

$$u_\lambda(r_\tau,\omega) = \sum_{\lambda=\lambda_{n,\ell}} d_{\lambda,\tau,\ell,m} Y_{\ell,m}(\omega)\,.$$

Considering the property of $Y_{\ell,m}$, we can choose for the proof of the proposition

$$\mathcal{P}_{\tau,\lambda} = \sum d_{\lambda,\tau,\ell,m} P_{\ell,m}\,, \tag{3.12}$$

where $P_{\ell,m}$ is the homogeneous harmonic polynomial of degree ℓ such that

$$(P_{\ell,m})_{/\{r=1\}} = Y_{\ell,m}\,.$$

It remains, in order to prove (3.11), to determine the highest $\ell \geq 1$ such that $\lambda_{n,\ell} = \lambda$. By the minimax principle, we have

$$\lambda = \lambda_{n,\ell} \geq m_\ell := \inf \left(v(r) + \frac{\ell(\ell+d-2)}{r^2} \right)\,.$$

We have indeed

$$\lambda_{n,\ell} = \int_0^{+\infty} |f'_{n,\ell}(r)|^2 r^{d-1} dr + \int_0^{+\infty} \left(v(r) + \frac{\ell(\ell+d-2)}{r^2} \right) |f_{n,\ell}(r)|^2 r^{d-1} dr\,.$$

□

The behavior of m_ℓ should be analyzed but note that our assumptions imply that $m_\ell > -\infty$. Furthermore $\ell \mapsto m_\ell$ is strictly increasing, so we can set $p_\lambda := [\check{p}_\lambda] + 1$, where $\check{p}_\lambda$ is the solution of $\lambda = m_{\check{p}_\lambda}$ and $[x]$ means the integer part of x.

Application: Determination of an upper bound of p_λ.

Case A

We can assume that $\ell \geq 1$. This simply implies later a choice of $p_\lambda \geq 1$ If we consider $v(r) = c\, r^m$ as a model case for $m > 1$ and $c > 0$, the infimum is obtained when

$$c\, m\, r^{m-1} - 2\frac{\ell(\ell+d-2)}{r^3} = 0\,,$$

i.e.

$$r = \left(\frac{2\ell(\ell+d-2)}{c\, m} \right)^{\frac{1}{m+2}}\,.$$

So we get

$$\begin{aligned} \inf \left(r^m + \tfrac{\ell(\ell+d-2)}{r^2} \right) &= \left(\tfrac{2\ell(\ell+d-2)}{c\,m} \right)^{\frac{m}{m+2}} + \ell(\ell+d-2) \left(\tfrac{2\ell(\ell+d-2)}{c\,m} \right)^{-\frac{2}{m+2}} \\ &= \left(\tfrac{2}{c\,m} \right)^{\frac{m}{m+2}} \tfrac{c\,m+2}{2} \left(\ell(\ell+d-2) \right)^{\frac{m}{m+2}}\,. \end{aligned}$$

This gives us

$$\check{p}_\lambda \sim a_m \lambda^{\frac{m+2}{2m}}\,, \tag{3.13}$$

with

$$a_m = (\frac{2}{c\,m})^{-\frac{1}{2}}\,(\frac{c\,m+2}{2})^{-\frac{2m}{m+2}}\,.$$

For $m=2$, we recover what we got for the harmonic oscillator by direct computation.

To treat the general case, we use the lower bound:

$$v(r) \geq c\,r^m - \frac{C}{r^s}\,. \tag{3.14}$$

We have to estimate

$$\inf\left(c\,r^m - \frac{C}{r^s} + \frac{\ell(\ell+d-2)}{r^2}\right)\,.$$

We observe that:

$$\begin{aligned}
&\inf\left(c\,r^m - \tfrac{C}{r^s} + \tfrac{\ell(\ell+d-2)}{r^2}\right)\\
&\quad \geq \inf\left(c\,r^m + \tfrac{\ell(\ell+d-2)}{2r^2}\right) + \inf\left(-\tfrac{C}{r^s} + \tfrac{\ell(\ell+d-2)}{2r^2}\right)\,.
\end{aligned}$$

But, there exists (see below the computation in (3.17) with $m=-s$) a constant $C_0>0$, such that, for all $\ell \geq 1$,

$$\inf\left(-\frac{C}{r^s} + \frac{\ell(\ell+d-2)}{2r^2}\right) \geq -\,C_0\,,$$

and we can use the lower bound of the model case above to get:

$$\inf_r\left(v(r) + \frac{\ell(\ell+d-2)}{r^2}\right) \geq \frac{1}{2}\left(\left(\frac{2}{c\,m}\right)^{\frac{m}{m+2}}\;\frac{c\,m+2}{2}\,(\ell(\ell+d-2))^{\frac{m}{m+2}}\right) -\; C_0\,.$$

Hence we obtain like for the model case:

Corollary 3.7. *In Case A, as* $\lambda \to +\infty$,

$$p_\lambda \approx \lambda^{\frac{m+2}{2m}}\,. \tag{3.15}$$

Case B

Let us now compute an example corresponding to Case B. If we take $v(r) = -r^m$ for $m \in (-2{,}0)$, the infimum is obtained when

$$mr^{m-1} + 2\frac{\ell(\ell+d-2)}{r^3} = 0\,.$$

i.e.

$$r = \left(\frac{2\ell(\ell+d-2)}{-m}\right)^{\frac{1}{m+2}}\,. \tag{3.16}$$

So we get

$$\begin{aligned}
\inf\left(-r^m + \tfrac{\ell(\ell+d-2)}{r^2}\right) &= -\left(\tfrac{2\ell(\ell+d-2)}{-m}\right)^{\frac{m}{m+2}} + \ell(\ell+d-2)\left(\tfrac{2\ell(\ell+d-2)}{-m}\right)^{-\frac{2}{m+2}}\\
&= -\left(\tfrac{2}{-m}\right)^{\frac{m}{m+2}}\;\tfrac{m+2}{2}\,(\ell(\ell+d-2))^{\frac{m}{m+2}}\,.
\end{aligned} \tag{3.17}$$

This gives us for $\lambda \to 0$, $\lambda > 0$,

$$\check{p}_\lambda \sim a_m (-\lambda)^{\frac{m+2}{2m}}\,, \tag{3.18}$$

with

$$a_m = (\frac{2}{-m})^{-\frac{1}{2}} (\frac{m+2}{2})^{-\frac{2m}{m+2}}\,. \tag{3.19}$$

In the Coulomb case $m = -1$ and $d = 3$, we get

$$\check{p}_\lambda \sim a_{-1} (-\lambda)^{-\frac{1}{2}}\,, \tag{3.20}$$

to compare with the direct computation which can be done for the Coulomb case.

In the general case, we can use

$$v(r) \geq -\,C\, r^{-s}\,, \forall r \in (0,R)$$

and

$$v(r) \geq -\,c\, r^{m}\,, \forall r \in (R, +\infty)\,.$$

We will use twice the analysis of the model, the first time with m replaced by $-s$. Let us first consider

$$\inf_{r\in(0,R)} \left(v(r) + \frac{\ell(\ell+d-2)}{r^2} \right) \geq \inf_{r\in(0,R)} \left(-C\, r^{-s} + \frac{\ell(\ell+d-2)}{r^2} \right)\,.$$

We observe (see (3.16) with $m = -s$) that for ℓ large enough the map $r \mapsto -C\, r^{-s} + \frac{\ell(\ell+d-2)}{r^2}$ is decreasing on $(0,R)$. Hence

$$\inf_{r\in(0,R)} \left(-C\, r^{-s} + \frac{\ell(\ell+d-2)}{r^2} \right) = \left(-C\, R^{-s} + \frac{\ell(\ell+d-2)}{R^2} \right) \geq -C_s\,.$$

For the second case, we can use

$$\inf_{r\in(R,+\infty)} \left(-\,C\, r^{m} + \frac{\ell(\ell+d-2)}{r^2} \right) \geq \inf_{r\in(0,+\infty)} \left(-C\, r^{m} + \frac{\ell(\ell+d-2)}{r^2} \right)\,,$$

and what we obtained for the homogeneous model.

Corollary 3.8. *In Case B,*

$$p_\lambda \approx (-\lambda)^{\frac{m+2}{2m}}\,. \tag{3.21}$$

3.7. Nodal domains on hyperspheres

Since the considered potentials V are radial, the energy hypersurfaces $\{V = \alpha\lambda\}$ are hyperspheres centered at 0. Also, the restriction of any eigenfunction u_λ of H_V to a hypersphere equals the restriction of some harmonic polynomial. We can use the following result proven in [**2**], which is based on [**48**]:

Proposition 3.9. *Let P be a polynomial of degree k with d variables. Then its restriction to the hypersphere $\mathbb{S}^{d-1}$ admits at most $2^{2d-1}k^{d-1}$ nodal domains.*

We will combine this with the previous estimates obtained in Corollaries 3.7 and 3.8 to obtain an upper bound on the number of nodal domains on any hypersphere.

4. Weyl's formula

4.1. Preliminaries

For Schrödinger operators, Weyl's formula takes (under of course suitable assumptions to be discussed below) the form

$$N(\lambda) \sim (2\pi)^{-d} \int_{\xi^2+V(x)\leq\lambda} dx d\xi \,. \tag{4.1}$$

After integration in the ξ variable, we get

$$N(\lambda) \sim W(\lambda)\,, \tag{4.2}$$

where

$$W(\lambda) := w_d \int (\lambda - V)_+^{\frac{d}{2}} dx\,, \tag{4.3}$$

with w_d defined in (1.4).

This formula makes sense in case A (as $\lambda \to +\infty$) and in case B (as $\lambda \to 0$ with $\lambda < 0$). Let us just compute the right hand side for the two toy models: the harmonic oscillator and the Schrödinger operator with Coulomb potential. For the harmonic oscillator, we simply get

$$W(\lambda) = w_d \int (\lambda - r^2)_+^{\frac{d}{2}} dx = h_d\, w_d\, \lambda^d\,, \tag{4.4}$$

with

$$h_d := \int (1 - r^2)_+^{\frac{d}{2}} dx > 0\,.$$

More generally, if $v(r) = r^m$ for $m > 0$, we obtain, as $\lambda \to +\infty$,

$$W(\lambda) = w_d\, h_{d,m}\, \lambda^{d(\frac{1}{2}+\frac{1}{m})}\,. \tag{4.5}$$

In the Coulomb case, we get, with $\lambda < 0$

$$W(\lambda) = w_d \int \left(\lambda + \frac{1}{r}\right)_+^{\frac{d}{2}} dx = e_d\, w_d\, (-\lambda)^{-\frac{d}{2}}\,, \tag{4.6}$$

with

$$e_d := \int \left(\frac{1}{|x|} - 1\right)_+^{\frac{d}{2}} dx = |\mathbb{S}^{d-1}| \int_0^1 (1-r)^{\frac{d}{2}}\, r^{\frac{d}{2}-1} dr < +\infty\,. \tag{4.7}$$

More generally, if $v(r) = -r^m$, for $m \in (-2,0)$, we obtain as $\lambda \to 0$ ($\lambda < 0$),

$$W(\lambda) = w_d h_{d,m} |\lambda|^{d(\frac{1}{2}+\frac{1}{m})} . \tag{4.8}$$

Observing that $N(\lambda_n) = n - 1$ if $\lambda_{n-1} < \lambda_n$, and assuming that the Weyl formula is proven (see below for the proof), we get conversly

$$\lambda_n \sim \check{w}_d \, n^{\frac{1}{d}} \quad \text{with } 1 = h_d \, w_d \, (\check{w}_d)^d , \tag{4.9}$$

in the case of the harmonic oscillator and

$$\lambda_n \sim -\check{v}_3 \, n^{-\frac{2}{3}} , \text{ with } \quad 1 = e_3 \, w_3 \, (\check{v}_3)^{-\frac{3}{2}} , \tag{4.10}$$

in the case of the Coulomb case.

More generally we have the proposition:

Proposition 4.1. *In Cases A or B*

$$W(\lambda) \approx |\lambda|^{d(\frac{1}{2}+\frac{1}{m})} , \tag{4.11}$$

where the asymptotics is as $\lambda \to +\infty$ in Case A and as $\lambda \to 0$ ($\lambda < 0$) in Case B.

Proof

Outside a ball we can use for estimating the integral defining $W(\lambda)$ the comparison of $v(r)$ with r^m and then use the previous computations for the models. The control of the integral in a ball will be done in Subsection 4.3. □

4.2. Weyl's formula under weak assumptions

There is vast literature on this subject: Reed-Simon [**56**] and references therein (for the historics), D. Robert [**59**], H. Tamura [**65**], Tulovski-Shubin [**61**], R. Beals [**7**], L. Hörmander [**35**], A. Mohamed [**50**]. In the recent contributions the goal is to control the remainder but this is not important in the applications considered here. Here, we prefer to work under weaker asssumptions and can use Theorem XIII.81 in Reed-Simon (Vol. 4) [**56**] for the case $V \to +\infty$ with a condition $d \geq 2$ and $m > 1$, and for the case $V \to 0$ as $|x| \to +\infty$, Theorem XIII.82. The treatment of the singularity is also explained (without detail) (see the discussion p. 277, lines -7 to -1, sending to Problem 132 therein). The idea there is to first prove a statement with V continuous and then to show that the addition of a potential W with compact support or in $L^{\frac{d}{2}}$ ($d \geq 3$) does not change the Weyl asymptotics.

Theorem XIII.81 in [**56**] reads:

Theorem 4.2. *Let V be a measurable function on $\mathbb{R}^d$ $(d \geq 2)$ obeying*

$$c_1\,(r^\beta - 1) \leq V(x) \leq c_2\,(r^\beta + 1)\,, \tag{4.12}$$

and

$$|V(x) - V(y)| \leq c_3\,[\max\{|x|,|y|\}]^{\beta-1}|x-y|\,, \tag{4.13}$$

for some $\beta > 1$ and suitable constants $c_1, c_2, c_3 > 0$.
Then

$$\lim_{\lambda\to+\infty} N(\lambda)/W(\lambda) = 1\,.$$

Remark 4.3. *The theorem is still true if we consider the Dirichlet problem for H_V in $\mathbb{R}^d \setminus B$, where B is a ball centered at 0. In Case A, the assumptions of the theorem are satisfied in $\mathbb{R}^d \setminus B$. This follows of our assumption (1.16) on v'.*

For Case B, Theorem XIII.82 in **[56]** reads:

Theorem 4.4. *Let V be a measurable function on $\mathbb{R}^d$ $(d \geq 2)$ obeying*

$$-\,c_1\,(r+1)^{-\beta} \leq V(x) \leq -c_2\,(r+1)^{-\beta}\,, \tag{4.14}$$

and

$$|V(x) - V(y)| \leq c_3\,[1 + \min\{|x|,|y|\}]^{-\beta-1}|x-y|\,, \tag{4.15}$$

for some $\beta < 2$ and suitable constants $c_1, c_2, c_3 > 0$.
Then

$$\lim_{\lambda\to+\infty} N(\lambda)/W(\lambda) = 1\,.$$

Remark 4.5. *The theorem is still true if we consider the Dirichlet problem for H_V in $\mathbb{R}^d \setminus B$, where B is a ball centered at 0. In Case B, the assumptions of the theorem are satisfied in $\mathbb{R}^d \setminus B$. This is a consequence of our assumption on v' in (1.18).*

4.3. Treatment of the singularity

To cover the question of the treatment of the singularity at the origin we could think of using (Problem 132 in **[56]**) to treat the singularity as a perturbation. Due to the use of the Cwickel-Lieb-Rozenblum inequality **[25]** in the argument, this approach works only under the condition $d \geq 3$. If we remember that we only need a lower bound for $N(\lambda)$ one can proceed for $d \geq 2$ in the following way. We can introduce a small ball $B = B(0,\epsilon)$ around the singularity and look at the Dirichlet problem in $\mathbb{R}^d \setminus B$. We denote by $N_B(\lambda)$ the corresponding counting function. Because the eigenvalues are greater than the initial problem by monotonicity of the domain, the estimate of the $N(\lambda)$ of the new problem will give the lower bound:

$$N_B(\lambda) \leq N(\lambda)\,.$$

The theorem in Reed-Simon [**56**] can be applied in $\mathbb{R}^d \setminus B$ (proof unchanged) and we get by Weyl's formula

$$N_B(\lambda) \sim W_B(\lambda)\,,$$

with

$$W_B(\lambda) = w_d \int_{r\geq\epsilon} (\lambda - V)_+^{\frac{d}{2}}\, dx\,.$$

It remains to compare $W_B(\lambda)$ and $W(\lambda)$ in our two cases.

Case B

Here $\lambda < 0$. It is enough to show that, for some $\epsilon > 0$,

$$\int_{B(0,\epsilon)} (\lambda - V)_+^{\frac{d}{2}}\, dx < +\infty\,.$$

We have

$$\int_{B(0,\epsilon)} (\lambda - V)_+^{\frac{d}{2}}\, dx \leq C \int_0^\epsilon r^{d-1-s\frac{d}{2}}\, dr < +\infty\,,$$

the finiteness resulting from the assumption $s < 2$.

Case A.

Here $\lambda \geq \lambda_0 > 0$. We will show that $\int_{B(0,\epsilon)} (\lambda - V)_+^{\frac{d}{2}}\, dx$ is relatively small in comparison with $N(\lambda)$. In Case A, we have seen that

$$W(\lambda) \approx \int (\lambda - r^m)_+^{\frac{d}{2}}\, r^{d-1} dr \approx \lambda^{\frac{d}{2}+\frac{d}{m}}\,, \tag{4.16}$$

We have

$$\int_{B(0,\epsilon)} (\lambda - V)_+^{\frac{d}{2}}\, dx \leq C \int_0^\epsilon (\lambda + r^{-s})^{\frac{d}{2}}\, r^{d-1} dr \leq \hat{C}\,(\hat{C} + \lambda^{\frac{d}{2}})\,.$$

This gives, as $\lambda \to +\infty$,

$$\int_{B(0,\epsilon)} (\lambda - V)_+^{\frac{d}{2}}\, dx / W(\lambda) = \mathcal{O}(\lambda^{-\frac{d}{m}})\,.$$

Hence in the two cases, we have shown that $W_A(\lambda) \sim W(\lambda)$. In conclusion, we have obtained the following theorem.

Theorem 4.6.

In Cases A or B, if $d \geq 2$, we have

$$N(\lambda) \geq W(\lambda)(1 + o(1))\,, \tag{4.17}$$

where the remainder $o(1)$ is as $\lambda \to +\infty$ in Case A and as $\lambda \to 0$ ($\lambda < 0$) in Case B.

5. Counting nodal domains

5.1. Preliminaries

We construct a radial partition of $\{V < \lambda\}$ of cardinality $\nu(\lambda)$ with $\nu(\lambda)$ to be defined later. When v is increasing on $(0, +\infty)$, $r_\lambda := v^{-1}(\lambda)$ is well defined in $(\inf v, +\infty)$ in Case A, and for any λ in $(-\infty,0)$ in Case B. But we will only be interested $\lambda \to +\infty$ in Case A, and in $\lambda \to 0$ $(\lambda < 0)$ in Case B.
Under the weaker Assumption (1.12), we can define in the two cases r_λ by

$$r_\lambda := \sup_{v(r)=\lambda} r\,,$$

and obtain the asymptotics

$$r_\lambda \approx |\lambda|^{\frac{1}{m}}\,.$$

Note that in both cases r_λ tends to $+\infty$.

5.2. Analysis of Case A

5.2.1. A first partition of the classical region. We recall that in this case, we assume $\lambda \geq \lambda_0 > 0$.

We introduce a partition of the classical region by introducing $\nu(\lambda)$ annuli, for $i = 1,\ldots,\nu(\lambda)$,

$$D_i(\lambda) := \left\{ x \in \mathbb{R}^d \,|\, \left(\frac{i-1}{\nu(\lambda)}\right)^{1/d} r_\lambda < r < \left(\frac{i}{\nu(\lambda)}\right)^{1/d} r_\lambda \right\}.$$

We note that each annulus has the same volume:

$$|D_i(\lambda)| = \omega_d \frac{1}{\nu(\lambda)} {r_\lambda}^d \approx \frac{\lambda^{\frac{d}{m}}}{\nu(\lambda)}\,. \tag{5.1}$$

The cardinality $\nu(\lambda)$ satisfies a priori the condition

$$\lim_{\lambda\to+\infty} \nu(\lambda) = +\infty\,, \tag{5.2}$$

but the condition

$$\lim_{\lambda\to+\infty} \nu(\lambda)^{-1/d} r_\lambda = +\infty\,, \tag{5.3}$$

will appear along the proof.

The determination of λ_0 "large enough" will be given during the proof. If u_λ denotes some eigenfunction, we denote by $\mathcal{D}(u_\lambda)$ the set of the nodal domains of u_λ. We now introduce in $\mathcal{D}(u_\lambda)$ the following subsets.

Definition 5.1.

$$A_i(u_\lambda) = \{\Omega \in \mathcal{D}(u_\lambda) \,|\, \Omega \subset D_i(\lambda)\}\,.$$

Here, i can take the values $1,2,\dots,\nu(\lambda)$.

Definition 5.2.

$$B_j(u_\lambda) = \left\{ \Omega \in \mathcal{D}(u_\lambda) \;\middle|\; \Omega \cap \left\{ r = \left(\frac{j}{\nu(\lambda)} \right)^{1/d} r_\lambda \right\} \neq \emptyset \right\} .$$

Here, j can take the values $1,2,\dots,\nu(\lambda)$.

From Subsection 3.2, we know that every nodal domain is contained in at least one of these sets.

Remark 5.3. *This partition will be refined by introducing, in the case of a singularity at the origin, a further partition of $A_1(u_\lambda)$.*

5.2.2. Counting the nodal domains contained in one annulus of the partition. We first count in each of the annuli $D_i(\lambda)$ for $i \geq 2$. Except if there are no singularity, the treatment of $A_1(u_\lambda)$ will be done separately. Hence we first prove the

Proposition 5.4.

In Case A, if $\nu(\lambda)$ satisfies (5.2) and (5.3), we have the following inequality, as $\lambda \to +\infty$,

$$\sum_{i=2}^{\nu(\lambda)} \# A_i(u_\lambda) \leq \gamma(d)\, W(\lambda) \left(1 + \mathcal{O}\left(\frac{1}{\nu(\lambda)} \right) \right) . \tag{5.4}$$

Proof

If Ω is a bounded nodal domain of u_λ, the Faber-Krahn inequality (see Theorem 1.4) gives:

$$\frac{\int_\Omega |\nabla u_\lambda(x)|^2\, dx}{\int_\Omega u_\lambda(x)^2\, dx} \geq \left(\frac{1}{|\Omega|} \right)^{\frac{2}{d}} \omega_d{}^{\frac{2}{d}} \lambda(B_d)\,, \tag{5.5}$$

where we recall that $|\Omega|$ denotes the volume of Ω.

We know that for a given bounded nodal domain Ω, we have

$$\lambda = \frac{\int_\Omega |\nabla u_\lambda|^2 dx + \int_\Omega V u_\lambda^2\, dx}{\int_\Omega u_\lambda^2\, dx} ,$$

which implies

$$\frac{\int_\Omega |\nabla u_\lambda|^2\, dx}{\int_\Omega u_\lambda^2\, dx} < \lambda - \inf_{x \in \Omega} V(x)\,. \tag{5.6}$$

For all $\Omega \in A_i(u_\lambda)$, we can, under Condition (5.3), use (1.12) to obtain that for λ large enough,

$$\frac{\int_\Omega |\nabla u_\lambda|^2\, dx}{\int_\Omega u_\lambda^2\, dx} < \lambda - v\left(\left(\frac{i-1}{\nu(\lambda)} \right)^{1/d} r_\lambda \right) , \tag{5.7}$$

for $i \geq 2$.
We can then combine (5.5) and (5.7) and obtain

$$|\Omega| \geq \frac{\omega_d \, \lambda(B_d)^{\frac{d}{2}}}{\left(\lambda - v\left(\left(\frac{i-1}{\nu(\lambda)}\right)^{1/d} r_\lambda\right)\right)^{\frac{d}{2}}} \,. \tag{5.8}$$

Observing that

$$\sum_{\Omega \in A_i(u_\lambda)} |\Omega| \leq |D_i(\lambda)| \,,$$

we obtain that

$$\# \, A_i(u_\lambda) \, \leq \frac{1}{\omega_d \lambda(B_d)^{\frac{d}{2}}} \left(|D_i(\lambda)| \left(\lambda - v\left(\left(\frac{i-1}{\nu(\lambda)}\right)^{1/d} r_\lambda\right)\right)^{\frac{d}{2}}\right) \,. \tag{5.9}$$

Summing up for $i = 2, \ldots, \nu(\lambda)$, we get

$$\sum_{i=2}^{\nu(\lambda)} \# \, A_i(u_\lambda) \, \leq \frac{1}{\omega_d \lambda(B_d)^{\frac{d}{2}}} \left(\sum_{i=2}^{\nu(\lambda)} |D_i(\lambda)| \left(\lambda - v\left(\left(\frac{i-1}{\nu(\lambda)}\right)^{1/d} r_\lambda\right)\right)^{\frac{d}{2}}\right) \,. \tag{5.10}$$

We recognize on the right hand side a Riemann sum for the function $x \mapsto (\lambda - V(x))^{\frac{d}{2}}$ in $D(0, r_\lambda) \setminus (D_{\nu(\lambda)}(\lambda) \cup D_1(\lambda))$.
More precisely, we can write, using the monotonicity of v,

$$\begin{aligned} \textstyle\sum_{i=2}^{\nu(\lambda)} |D_i(\lambda)| \left(\lambda - v\left(\left(\frac{i-1}{\nu(\lambda)}\right)^{1/d} r_\lambda\right)\right)^{\frac{d}{2}} \\ &\leq |D_2(\lambda)| \left(\lambda - v\left(\left(\frac{1}{\nu(\lambda)}\right)^{1/d} r_\lambda\right)\right)^{\frac{d}{2}} + \textstyle\int (\lambda - V(x))_+^{\frac{d}{2}} \, dx \\ &\leq |D_2(\lambda)| \, \lambda^{\frac{d}{2}} + \textstyle\int (\lambda - V(x))_+^{\frac{d}{2}} \, dx \,. \end{aligned}$$

Using the asymptotic behavior of v at $+\infty$ given in (1.17), the computation of $\gamma(d)$ in (1.3), the asymptotic behavior of $W(\lambda)$ given in (4.11), and (5.1), we achieve the proof of Proposition 5.4.

5.2.3. Counting the nodal sets meeting the boundary of the annuli. Let us now turn to the study of the sets $B_i(u_\lambda)$. We have shown in Corollary 3.7 that $p_\lambda \approx \lambda^{\frac{m+2}{2m}}$. Using Proposition 3.9, we obtain that the number of nodal domains in a given $B_i(u_\lambda)$ satisfies

$$\# B_i(u_\lambda) \leq 2^{2d-1} \, p_\lambda^{d-1} \leq C_d \, \lambda^{\frac{(d-1)(m+2)}{2m}} \,. \tag{5.11}$$

Comparing with (4.11), we get that, for some $C > 0$ and $\lambda \geq \lambda_0$,

$$\sum_i \# B_i(u_\lambda) \leq C \, \nu(\lambda) \, \lambda^{-\frac{m+2}{2m}} \, W(\lambda) \,.$$

If $\nu(\lambda)$ satisfies in addition,

$$\lim_{\lambda\to+\infty} \nu(\lambda)\,\lambda^{-\frac{m+2}{2m}} = 0\,, \tag{5.12}$$

we obtain

$$\lim_{\lambda\to+\infty} \frac{\sum_i \#B_i(u_\lambda)}{W(\lambda)} = 0\,. \tag{5.13}$$

5.2.4. Counting in $D_1(\lambda)$. We still have to consider when there is a singularity around 0. We treat first the case $d \geq 3$. We know from Proposition 3.5 that we can replace $D_1(\lambda)$ by the annulus $\widehat{D}_1(\lambda,C)$ defined by

$$\widehat{D}_1(\lambda,C) := \{x \in \mathbb{R}^d \,|\, \frac{1}{C}\lambda^{-\frac{1}{2}} < r < r_\lambda/\nu(\lambda)^{\frac{1}{d}}\}$$

for a sufficiently large C.
The number of nodal domains $\mu_{10}(\lambda)$ crossing the hypersphere $\{r = \frac{1}{C}\lambda^{-\frac{1}{2}}\}$ is controlled by (5.11):

$$\lim_{\lambda\to+\infty} \frac{\mu_{10}(\lambda)}{W(\lambda)} = 0\,. \tag{5.14}$$

To continue, we consider a partition of $\widehat{D}_1(\lambda)$ in two annuli:

$$D_{11}(\lambda) := \{x \in \mathbb{R}^d \,|\, \frac{1}{C}\lambda^{-\frac{1}{2}} < r < C\} \text{ and } D_{12}(\lambda) := \{x \in \mathbb{R}^d \,|\, C < r < r_\lambda/\nu(\lambda)^{\frac{1}{d}}\}\,,$$

where we keep the liberty to choose $C > 0$ larger than the previous one.
Again the number $\mu_{11}(\lambda)$ of nodal domains crossing the hypersphere $\{r = C\}$ is controlled by (5.11):

$$\lim_{\lambda\to+\infty} \frac{\mu_{11}(\lambda)}{W(\lambda)} = 0\,. \tag{5.15}$$

Control in $D_{12}(\lambda)$.
The treatment of $D_{12}(\lambda)$ can be done for C large enough (in order to have the monotonicity of v) like the analysis of the $D_i(\lambda)$ for $i \geq 1$. More precisely, we can replace (5.9) (for $i = 1$) by

$$\#\, A_{12}(u_\lambda) \,\leq\, \frac{1}{\omega_d \lambda(B_d)^{\frac{d}{2}}}\, |D_{12}(\lambda)|\, (\lambda - v(C))^{\frac{d}{2}}\,, \tag{5.16}$$

where

$$A_{12}(u_\lambda) := \{\Omega \in \mathcal{D}(u_\lambda) \,|\, \Omega \subset D_{12}(\lambda)\}\,.$$

Hence we get, for some constant $C_d > 0$,

$$\#\, A_{12}(u_\lambda) \,\leq C_d\, \nu(\lambda)^{-1}\, \lambda^{\frac{d}{2}+\frac{d}{m}}\,, \tag{5.17}$$

which implies

$$\lim_{\lambda\to+\infty} (\#\, A_{12}(u_\lambda)/W(\lambda)) = 0\,. \tag{5.18}$$

Control in $D_{11}(\lambda)$.
Note that in $D_{11}(\lambda)$, we have, for some constant $C_s > 0$,

$$V(x) \geq -C_s\, \lambda^{\frac{s}{2}}\,, \forall x \in D_{11}(\lambda)\,.$$

Hence, for $\lambda \geq \lambda_0$, we deduce from (5.6), that

$$\frac{\int_\Omega |\nabla u_\lambda(x)|^2\, dx}{\int_\Omega u_\lambda(x)^2\, dx} \leq 2\lambda\,.$$

As in the proof of (5.8) we obtain:

$$|\Omega| \geq \omega_d\, \lambda(B_d)^{\frac{d}{2}}\, 2^{-\frac{d}{2}}\, \lambda^{-\frac{d}{2}}\,. \tag{5.19}$$

Summing over the Ω's contained in $D_{11}(\lambda)$ and observing that the volume of $D_{11}(\lambda)$ is bounded, we get the existence of a constant $\hat{C}_d$ such that

$$\# A_{11}(u_\lambda) \leq \hat{C}_d\, \lambda^{\frac{d}{2}}\,,$$

where

$$A_{11}(u_\lambda) := \{\Omega \in \mathcal{D}(u_\lambda)\,|\, \Omega \subset D_{11}(\lambda)\}\,.$$

In particular we get

$$\lim_{\lambda \to +\infty} \frac{\# A_{11}(u_\lambda)}{W(\lambda)} = 0\,. \tag{5.20}$$

The case when $d = 2$ does not lead to new difficulties.

5.2.5. Conclusion for Case A. Summing all the upper bounds and having chosen $\nu(\lambda)$ satisfying (5.2), (5.3), and (5.12), we get, as $\lambda \to +\infty$,

$$\mu(u_\lambda) \leq \gamma(d)\, W(\lambda)\, (1 + o(1))\,. \tag{5.21}$$

Using the asymptotic upper bound (4.17), we get, as $\lambda \to +\infty$,

$$\mu(u_\lambda) \leq \gamma(d)\, N(\lambda)\, (1 + o(1))\,. \tag{5.22}$$

Observing that $N(\lambda_n) \leq n - 1$, we obtain Theorem 1.6.

5.3. Case B

The proof in case B follows the same lines. We define the sets $D_i(\lambda)$, $A_i(u_\lambda)$ and $B_i(u_\lambda)$ for $i = 2, \ldots, \nu(\lambda)$ as in case A, with $\nu(\lambda)$ satisfying conditions equivalent to (5.2), (5.3) and (5.12), replacing $\lambda \to \infty$ by $\lambda \to 0$ (with $\lambda < 0$). Hence, we assume that $\nu(\lambda)$ satisfies the conditions

$$\lim_{\lambda \to 0} \nu(\lambda) = +\infty\,, \tag{5.23}$$

$$\lim_{\lambda \to 0} \nu(\lambda)^{-1/d}\, r_\lambda = +\infty\,, \tag{5.24}$$

and

$$\lim_{\lambda \to +0} \nu(\lambda)\,|\lambda|^{-\frac{m+2}{2m}} = 0\,. \tag{5.25}$$

Proposition 5.5.

In Case B, if $\nu(\lambda)$ satisfies (5.23) and (5.24), then we have the following inequality, as $\lambda \to 0$ ($\lambda < 0$),

$$\sum_{i=2}^{\nu(\lambda)} \#A_i(u_\lambda) \le \gamma(d)\, W(\lambda)\,(1 + \mathcal{O}(\frac{1}{\nu(\lambda)}))\,. \tag{5.26}$$

The proof is the same as in case A. For $W(\lambda)$, we can use the asymptotics (4.11).

For the cardinalities of the sets $B_i(u_\lambda)$, (5.13) holds in case B under condition (5.25) and we can use Corollary 3.8 together with (4.11).

The treatment of the singularity is slightly easier in this case. We use Proposition 3.4 to make a partition of $D_1(\lambda)$ in two annuli:

$$D_{11}(\lambda{,}C) := \{x \in \mathbb{R}^d \,|\, r_d < r < C\}\,,$$

and

$$D_{12}(\lambda{,}C) := \{x \in \mathbb{R}^d \,|\, C < r < r_\lambda/\nu(\lambda)^{\frac{1}{d}}\}\,.$$

Again, we choose C such that v is strictly increasing for $|x| > C$. Since $\lambda < 0$, there exists $M > 0$ such that $\#A_{11}(u_\lambda) < M$. To give an upper bound on $\#A_{12}(u_\lambda)$, we follow the same steps as in case A.

The behavior of the number of eigenvalues below or equal to λ is given by (4.17), hence inequality (5.22) is verified. Since $N(\lambda_n) \le n-1$, we obtain Theorem 1.6 in this case as well.

Troisième article.

Non–boundedness of the number of super level domains of eigenfunctions

par

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Les principales contributions de Philippe Charron à cet article sont présentées.
Les coauteurs ont contribué à l'article de façon égale.

Résumé. Le théorème de Courant stipule que la n-e fonction propre du laplacien possède au maximum n domaines nodaux. Une généralisation, suggérée par Courant lui-même, serait de montrer qu'une combinaison linéaire des n premières fonctions propres possède au maximum n domaines nodaux (propriété de Courant étendue). Cette question est liée au nombre de composantes connexes des courbes de niveau des fonctions propres dans le cas de Neumann ou d'une variété sans bord. En effet, comme la fonction constante est une fonction propre dans ces cas, les courbes de niveau $\{f_k = a\}$ sont les mêmes que les lignes nodales de $f_k - a$.

Dans la première partie de l'article, nous montrons que la propriété de Courant étendue est fausse pour certains triangles isocèles et pour tous les N-gones avec les conditions de Neumann.

Dans la deuxième partie, nous construisons des métriques sur $\mathbb{T}^2$ et $\mathbb{S}^2$ ainsi qu'une fonction propre f_k pour chaque métrique telle que l'ensemble $\{f_k \neq 1\}$ possède une infinité de composantes connexes. De plus, nous montrons que ces métriques peuvent être arbitrairement proches de la métrique plate sur $\mathbb{T}^2$ ou la métrique ronde sur $\mathbb{S}^2$. Ces résultats sont inspirés de constructions récentes par Buhovsky, Logunov et Sodin.

Dans l'appendice B, nous montrons que la propriété de Courant étendue est vraie pour l'oscillateur harmonique quantique isotrope dans $\mathbb{R}^2$.
Mots clés : Fonctions propres, domaines nodaux, théorème de Courant

Abstract.

Generalizing Courant's nodal domain theorem, the "Extended Courant property" is the statement that a linear combination of the first n eigenfunctions has at most n nodal domains. A related question is to estimate the number of connected components of the (super) level sets of a Neumann eigenfunction u. Indeed, in this case, the first eigenfunction is constant, and looking at the level sets of u amounts to looking at the nodal sets $\{u - a = 0\}$, where a is a real constant. In the first part of the paper, we prove that the Extended Courant property is false for the subequilateral triangle and for regular N-gons (N large), with the Neumann boundary condition. More precisely, we prove that there exists a Neumann eigenfunction u_k of the N-gon, with labelling k, $4 \leq k \leq 6$, such that the set $\{u_k \neq 1\}$ has $(N+1)$ connected components.

In the second part, we prove that there exists a metric g on $\mathbb{T}^2$ (resp. on $\mathbb{S}^2$), which can be chosen arbitrarily close to the flat metric (resp. round metric), and an eigenfunction u of the associated Laplace-Beltrami operator, such that the set $\{u \neq 1\}$ has infinitely many connected components. In particular the Extended Courant property is false for these closed surfaces. These results are strongly motivated by a recent paper by Buhovsky, Logunov and Sodin. As for the positive direction, in Appendix B, we prove that the Extended Courant property is true for the isotropic quantum harmonic oscillator in $\mathbb{R}^2$.
Keywords: Eigenfunction, Nodal domain, Courant nodal domain theorem.

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2. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain (open connected set) with piecewise smooth boundary, or a compact Riemannian surface, with or without boundary, and let Δ be the Laplace-Beltrami operator. Consider the (real) eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega\,, \\ B(u) = 0 \text{ on } \partial\Omega\,, \end{cases} \tag{2.1}$$

where the boundary condition $B(u) = 0$ is either the Dirichlet or the Neumann boundary condition, $u = 0$ or $\partial_\nu u = 0$ on $\partial\Omega$, or the empty condition if $\partial\Omega$ is empty.

We arrange the eigenvalues of (2.1) in nondecreasing order, multiplicities taken into account,

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \tag{2.2}$$

The *nodal set* $\mathcal{Z}(u)$ of a (real) function u is defined to be

$$\mathcal{Z}(u) = \overline{\{x \in \Omega \mid u(x) = 0\}}\,. \tag{2.3}$$

The *nodal domains* of a function u are the connected components of $\Omega \backslash \mathcal{Z}(u)$. Call $\beta_0(u)$ the number of nodal domains of the function u.

The following classical theorem can be found in [**23**, Chap. VI.6].

Theorem 2.1 (Courant, 1923)**.** *An eigenfunction u, associated with the n-th eigenvalue λ_n of the eigenvalue problem (2.1), has at most n nodal domains, $\beta_0(u) \leq n$.*

For $n \geq 1$, denote by $\mathcal{L}_n(\Omega)$ the vector space of linear combinations of eigenfunctions of problem (2.1), associated with the n first eigenvalues, $\lambda_1, \ldots, \lambda_n$.

Conjecture 2.2 (Extended Courant Property)**.** *Let $w \in \mathcal{L}_n(\Omega)$ be a nontrivial linear combination of eigenfunctions associated with the n first eigenvalues of problem* (2.1)*. Then, $\beta_0(w) \leq n$.*

This conjecture is motivated by a statement made in a footnote[7] of Courant-Hilbert's book.

Conjecture 2.2 is known to be true in dimension 1 (Sturm, 1833). In higher dimensions, it was pointed out by V. Arnold (1973), in relation with Hilbert's 16th problem, see [**5**]. Arnold noted that the conjecture is true for $\mathbb{RP}^2$, the real projective space with the standard metric. It follows from [**44**] that Conjecture 2.2 is true when restricted to linear combinations of even (resp. odd) spherical harmonics on $\mathbb{S}^2$ equipped with the standard metric. Counterexamples

[7] p. 454 in [**23**].

to the conjecture were constructed by O. Viro (1979) for $\mathbb{R}\mathrm{P}^3$, see [**66**]. As far as we know, $\mathbb{R}\mathrm{P}^2$ is the only higher dimensional compact example for which Conjecture 2.2 is proven to be true. In Appendix 8.2, we prove that the conjecture is true for the isotropic quantum harmonic oscillator in $\mathbb{R}^2$ as well. Simple counterexamples to Conjecture 2.2 are given in [**9, 10, 11**]. They include smooth convex domains in $\mathbb{R}^2$, with Dirichlet or Neumann boundary conditions. A question related to the Extended Courant property is to estimate the number of connected components of the (super) level sets of a Neumann eigenfunction u. Indeed, in this case, the first eigenfunction is constant, and looking at the level sets of u amounts to looking at the nodal sets $\{u - a = 0\}$, where a is a real constant. Most counterexamples to the Extended Courant property, not all, are of this type. This is the case in the present paper[8] as well. Studying the topology of level sets of a Neumann eigenfunction is, in itself, an interesting question which is related to the "hot spots" conjecture, see [**6**].

Questions 2.3. *Natural questions, related to Conjecture 2.2, arise.*

(1) *Fix Ω as above, and $N \geq 2$. Can one bound $\beta_0(w)$, for $w \in \mathcal{L}_N(\Omega)$, in terms of N and geometric invariants of Ω?*
(2) *Assume that $\Omega \subset \mathbb{R}^2$ is a convex domain. Can one bound $\beta_0(w)$, for $w \in \mathcal{L}_N(\Omega)$, in terms of N, independently of Ω?*
(3) *Assume that Ω is a simply-connected closed surface. Can one bound $\beta_0(w)$, for $w \in \mathcal{L}_N(\Omega)$, in terms of N, independently of Ω?*

A negative answer to Question 2.3(1) for the 2-torus is given in [**18**]. In that paper, the authors construct a smooth metric g on $\mathbb{T}^2$, and a family of eigenfunctions ϕ_j with infinitely many isolated critical points. As a by-product of their construction, they prove that there exist a smooth metric g, a family of eigenfunctions ϕ_j, and a family of real numbers c_j such that $\beta_0(\phi_j - c_j) = +\infty$, see Proposition 5.1.

The main results of the present paper are as follows. In Section 3, we prove that Conjecture 2.2 is false for a subequilateral (to be defined later on) triangle with Neumann boundary condition, see Proposition 3.3. In Section 4, we prove that the regular N-gons, with Neumann boundary condition, provide negative answers to both Conjecture 2.2, and Question 2.3(2), at least for N large enough, see Proposition 4.1.

The second part of the paper, Sections 5 and 6, is strongly motivated by [**36, 18**]. We give a new proof that Conjecture 2.2 is false for the torus $\mathbb{T}^2$, and we prove that it is false for the sphere $\mathbb{S}^2$ as well. More precisely, we prove the existence of a smooth metric g on $\mathbb{T}^2$ (resp. $\mathbb{S}^2$), which can be chosen arbitrarily close to the flat metric (resp. round metric), and an eigenfunction Φ of the associated Laplace-Beltrami operator, such that the set $\{\Phi \neq 1\}$

[8]We changed the initial title of our paper (arXiv:1906.03668v2, June 20, 2019) to reflect this fact, as suggested by the referee.

has infinitely many connected components. We refer to Proposition 5.2 for the torus, and to Propositions 6.1 and 6.2 for the sphere.

In the case of $\mathbb{T}^2$, we also consider real *analytic* metrics. For such a metric, an eigenfunction can only have finitely many isolated critical points. In [**18**, Introduction], the authors ask whether, for analytic metrics, there exists an asymptotic upper bound for the number of critical points of an eigenfunction, in terms of the corresponding eigenvalue. Proposition 5.5 is related to this question. In Section 7, we make some final comments.

In Appendix 8.1, we prove the weaker result $\beta_0(w) \le 8\, d^2$ when w is the restriction to $\mathbb{S}^2$ of a polynomial of degree d in $\mathbb{R}^3$. This gives a partial answer to Question 2.3(3) in the case of the sphere. In Appendix 8.2, we prove that Conjecture 2.2 is true for the isotropic quantum harmonic oscillator in $\mathbb{R}^2$. Both appendices rely on [**2**].

3. Subequilateral triangle, Neumann boundary condition

Let $\mathcal{T}(b)$ denote the interior of the triangle with vertices $A = (\sqrt{3},0)$, $B = (0,b)$, and $C = (0,-b)$. When $b = 1$, $\mathcal{T}(1)$ is an equilateral triangle with sides of length 2. From now on, we assume that $0 < b < 1$. The angle at the vertex A is less than $\frac{\pi}{3}$, and we say that $T(b)$ is a *subequilateral* triangle, see Figure 1. Let $\mathcal{T}(b)_+ = \mathcal{T}(b) \cap \{y > 0\}$, and $\mathcal{T}(b)_- = \mathcal{T}(b) \cap \{y < 0\}$.

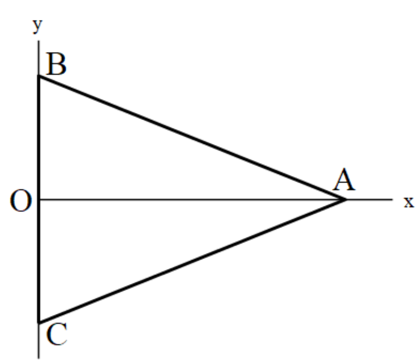

Fig. 1. Subequilateral triangle, $BC < AB = AC$

Call $\nu_i(\mathcal{T}(b))$ the Neumann eigenvalues of $\mathcal{T}(b)$, and write them in non-decreasing order, with multiplicities, starting from the labelling 1,

$$0 = \nu_1(\mathcal{T}(b)) < \nu_2(\mathcal{T}(b)) \le \nu_3(\mathcal{T}(b)) \le \cdots \tag{3.1}$$

We recall the following theorems.

Theorem 3.1 ([**39**], Theorem 3.1)**.** *Every second Neumann eigenfunction of a subequilateral triangle $\mathcal{T}(b)$ is even in y, $u(x,-y) = u(x,y)$.*

Theorem 3.2 ([**49**], Theorem B). *Let $\mathcal{T}(b)$ be a subequilateral triangle. Then, the eigenvalue $\nu_2(\mathcal{T}(b))$ is simple, and an associated eigenfunction u satisfies $u(O) \neq 0$, where O is the point $O = (0{,}0)$. Normalize u by assuming that $u(O) = 1$. Then, the following properties hold.*

(1) The partial derivative u_x is negative in $\overline{\mathcal{T}(b)} \setminus \big(\overline{BC} \cup \{A\}\big)$.

(2) The partial derivative u_y is positive in $\overline{\mathcal{T}(b)_+} \setminus \big(\overline{OA} \cup \{B\}\big)$, and negative in $\overline{\mathcal{T}(b)_-} \setminus \big(\overline{OA} \cup \{C\}\big)$.

(3) The function u has exactly four critical points O,A,B and C in $\overline{\mathcal{T}}$.

(4) The points B and C are the global maxima of u, and $u(B) = u(C) > u(O) > 0$.

(5) The point A is the global minimum of u, and $u(A) < 0$.

(6) The point O is the saddle point of u.

As a direct corollary of these theorems, we obtain the following result.

Proposition 3.3. *Let u be the second Neumann eigenfunction of the subequilateral triangle $\mathcal{T}(b)$, $0 < b < 1$, normalized so that,*

$$u(A) = \min u < 0 < u(O) = 1 < \max u = u(B) = u(C)\,.$$

For $a \in \mathbb{R}$, let $\beta_0(u - a)$ be the number of nodal domains of the function $u - a$ (equivalently the number of a-level domains of u). Then,

$$\begin{cases} \beta_0(u-a) = 1 & \textit{for} \quad a \leq \min u \ \textit{or} \ a \geq \max u\,, \\ \beta_0(u-a) = 2 & \textit{for} \quad \min u < a < 1\,, \\ \beta_0(u-a) = 3 & \textit{for} \quad 1 \leq a < \max u\,. \end{cases}$$

As a consequence, for $1 \leq a < \max u$, the linear combination $u - a$ provides a counterexample to Conjecture 2.2, see Figure 3.

Démonstration. Fix some $0 < b < 1$, denote $\mathcal{T}(b)$ by $\mathcal{T}$, and $\nu_2(\mathcal{T}(b))$ by ν_2 for simplicity. In the proof, we write $(A1)$ for Assertion (1) of Theorem 3.2, etc..

For $a \in \mathbb{R}$, call v_a the function $v_a := u - a$. This is a linear combination of a second and first Neumann eigenfunctions of $\mathcal{T}$. We shall now describe the nodal set of v_a carefully.

According to Theorem 3.1, for all a, the function v_a is even in y, so that it is sufficient to determine its nodal set in the triangle $\mathcal{T}_+ = OAB$, see Figure 2.

⋄ By (A4) and (A5), the nodal set $\mathcal{Z}(v_a)$ is nontrivial if and only if $u(A) < a < u(B)$.

⋄ By (A1) and (A2), the directional derivative of v_a in the direction of $\overrightarrow{BA}$ is negative in the open segment BA, so that $v_a|_{BA}$ is strictly decreasing from $v_a(B)$ to $v_a(A)$, and therefore vanishes at a unique point $Z_a = (\xi_a,\eta_a) \in BA$. We now consider three cases.

Case $u(A) < a < u(O)$**.**

⋄ By (A1), $v_a|_{OA}$ is strictly decreasing from $v_a(O)$ to $v_a(A)$, and therefore vanishes at a unique point $W_a = (\omega_a,0) \in OA$. By (A2), $\omega_a < \xi_a$.

⋄ By (A1) and (A2), it follows that the nodal set $\mathcal{Z}(v_a) \cap \mathcal{T}_+$ is contained in the rectangle $[\omega_a,\xi_a] \times [0,\eta_a]$, and that it is a smooth y-graph over $[\omega_a,\xi_a]$, and a smooth x-graph over $[0,\eta_a]$.

We have proved that v_a has exactly two nodal domains in $\mathcal{T}$.

Case $a_c = u(O)$**.**

The analysis is similar to the previous one, except that $\omega_{a_c} = 0$. As a consequence, v_{a_c} has exactly three nodal domains in $\mathcal{T}$.

Case $u(O) < a < u(B)$**.**

⋄ By (A2), $v_a|_{OB}$ is strictly increasing from $v_a(0)$ to $v_a(B)$, so that it vanishes at a unique point $V_a = (0,\zeta_a) \in OB$. From (A1), it follows that $\zeta_a < \eta_a$.

⋄ From (A1) and (A2), it follows that the nodal set $\mathcal{Z}(v_a) \cap \mathcal{T}_+$ is contained in the rectangle $[0,\xi_a] \times [\zeta_a,\eta_a]$, and that it is a smooth y-graph over $[0,\xi_a]$, and a smooth x-graph over $[\zeta_a,\eta_a]$.

It follows that $v_a < 0$ in $]-\zeta_a,\zeta_a[\times [0,\sqrt{3}] \cap \mathcal{T}$, and that v_a has precisely three nodal domains in $\mathcal{T}$. Proposition 3.3 is proved. ☐

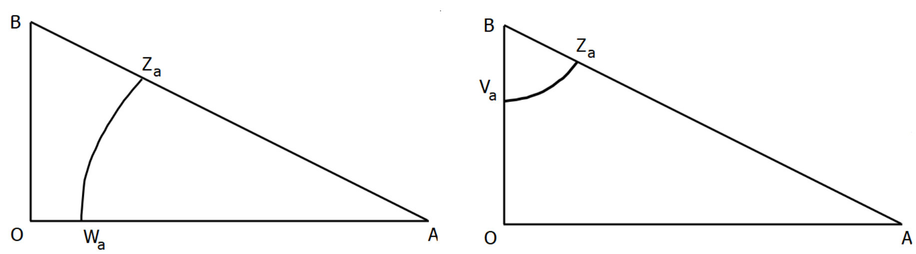

Fig. 2. Nodal behaviour of $u - a$ in the triangle OAB

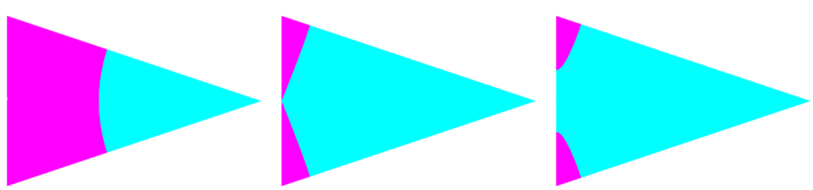

Fig. 3. Nodal domains of $u - a$ $(a < 1, a = 1, 1 < a)$

4. Regular N-gon, Neumann boundary condition

Proposition 4.1. *Let $\mathcal{P}_N$ denote the regular polygon with N sides, inscribed in the unit disk $\mathcal{D}$. Then, for N large enough, Conjecture 2.2 is false for $\mathcal{P}_N$, with the Neumann boundary condition. More precisely, there exist $m \leq 6$, an eigenfunction u_m associated with $\nu_m(\mathcal{P}_N)$, and a value a such that the function $u_m - a$ has $N + 1$ nodal domains.*

Démonstration. The general idea is to use the fact that a regular n-gon, $n \geq 7$, is made up of n copies of a subequilateral triangle, and to keep Figure 3 in mind. When N tends to infinity, the polygon $\mathcal{P}_N$ tends to the disk in the Hausdorff distance. According to [**41**, Remark 2, p. 206], it follows that, for all $j \geq 1$, the Neumann eigenvalue $\nu_j(\mathcal{P}_N)$ tends to the Neumann eigenvalue $\nu_j(\mathcal{D})$ of the unit disk. The Neumann eigenvalues of the unit disk satisfy

$$\nu_1(\mathcal{D}) < \nu_2(\mathcal{D}) = \nu_3(\mathcal{D}) < \nu_4(\mathcal{D}) = \nu_5(\mathcal{D}) < \nu_6(\mathcal{D}) < \nu_7(\mathcal{D}) \cdots \tag{4.1}$$

and are given respectively by the squares of the zeros of the derivatives of Bessel functions: $0 = j'_{0,1}, j'_{1,1}, j'_{2,1}, j'_{0,2}$, and $j'_{3,1}$. It follows that, for N large enough, the eigenvalue $\nu_6(\mathcal{P}_N)$ is simple.

From now on, we assume that N is sufficiently large to ensure that $\nu_6(\mathcal{P}_N)$ is a simple eigenvalue. Let u_6 be an associated eigenfunction.

Call $A_i, 1 \leq i \leq N$, the vertices of $\mathcal{P}_N$, so that the triangles OA_iA_{i+1} are subequilateral triangles with apex angle $\frac{2\pi}{N}$. Let $\mathcal{T}_N$ be the triangle OA_1A_2.

Call D_j the $2N$ lines of symmetry of $\mathcal{P}_N$. When $N = 2m$ is even, the lines of symmetry are the m diagonals joining opposite vertices, and the m lines joining the mid-points of opposite sides. When $N = 2m + 1$ is odd, the lines of symmetries are the N lines joining the vertex A_i to the mid-point of the opposite side. Call D_1 the line of symmetry passing through the first vertex. Call D_2 the line of symmetry such that the angle (D_1,D_2) is equal to π/N. Denote the corresponding mirror symmetries by D_1 and D_2 as well. The symmetry group of the regular N-gon is the dihedral group $\mathbb{D}_N$ with presentation,

$$\mathbb{D}_N = \left\{D_1,D_2 \mid D_1^2 = D_2^2 = 1, (D_2D_1)^N = 1\right\} . \tag{4.2}$$

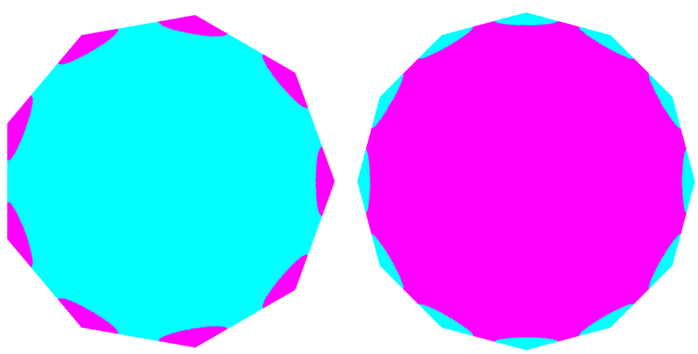

Fig. 4. $\mathcal{P}_9$ and $\mathcal{P}_{12}$, Neumann boundary condition

The group $\mathbb{D}_N$ acts on functions, and commutes with the Laplacian. It leaves the eigenspaces invariant, and we therefore have a representation of degree 1 in the eigenspace $\mathcal{E}(\nu_6)$. This representation must be equivalent to one of the irreducible representations of $\mathbb{D}_N$ of degree 1. When N is even, there are 4 such representations, ρ_{σ_1,σ_2} with $\sigma_1,\sigma_2 \in \{-1,1\}$, and such that $\rho_{\sigma_1,\sigma_2}(D_1) = \sigma_1$ and $\rho_{\sigma_1,\sigma_2}(D_2) = \sigma_2$. When N is odd, there are only 2 irreducible representations of degree 1, $\rho_{\sigma,\sigma}$, with $\sigma \in \{-1,1\}$. Eigenfunctions corresponding to simple eigenvalues must be invariant or anti-invariant under D_1 and D_2 depending on the signs of σ_1 and σ_2. Anti-invariant eigenfunctions must vanish on the corresponding line of symmetry. If $(\sigma_1,\sigma_2) \neq (1,1)$, the functions must have at least N nodal domains. For $N \geq 7$, this is not possible for $\mathcal{E}(\nu_6)$. An eigenfunction in $\mathcal{E}(\nu_6)$ must be D_1 and D_2 invariant, and must therefore correspond to an eigenfunction of $\mathcal{T}_N$ with Neumann boundary condition, and with eigenvalue $\nu_6 \geq \nu_2(\mathcal{T}_N)$. We can now apply Proposition 3.3. This is illustrated by Figure 4, keeping Figure 3 in mind. □

Remark 4.2. *The above proposition also shows that the regular N-gons, with the Neumann boundary condition, provides a counterexample to Conjecture 2.2 and to Question 2.3(2), when N is large enough.*

Remark 4.3. *As shown in* [**11**], *Conjecture 2.2 is false for the regular hexagon $\mathcal{P}_6$ with Neumann boundary condition. In this case, $\nu_6(\mathcal{P}_6) = \nu_7(\mathcal{P}_6)$, and has multiplicity* 2, *with two eigenfunctions associated with different irreducible representations of $\mathbb{D}_6$.*

Remark 4.4. *Numerical computations indicate that the first eight Neumann eigenvalues of $\mathcal{P}_7$ to $\mathcal{P}_{12}$ have the same multiplicities as the first eight eigenvalues of the disk and, in particular, that ν_6 is simple. Proposition 4.1 is probably true for all $N \geq 6$. Numerical computations also indicate that this proposition should be true for $\mathcal{P}_N$ with the Dirichlet boundary condition as well. The argument in the proof of Proposition 4.1 fails in the cases $N = 4$ and $N = 5$ which remain open.*

5. Counterexamples on $\mathbb{T}^2$

5.1. Previous results

This section is strongly motivated by the following result[9].

Proposition 5.1 ([**18**, Section 3]). *There exist a smooth metric g on the torus $\mathbb{T}^2$, in the form $g = Q(x)(dx^2 + dy^2)$, an infinite sequence ϕ_j of eigenfunctions of the Laplace-Beltrami operator Δ_g, and an infinite sequence c_j of real numbers, such that the level sets $\{(x,y) \mid \phi_j(x,y) = c_j\}$ have infinitely many connected components.*

In this section, we give an easy proof of Proposition 5.1, in the particular case of one eigenfunction only, avoiding the subtleties of [**18**]. This particular case is sufficient to prove that Conjecture 2.2 is false on $(\mathbb{T}^2,g)$ for some Liouville metrics which can be chosen arbitrarily close to the flat metric.

[9] The authors would like to thank I. Polterovich for pointing out [**18**, Section 3].

5.2. Metrics on $\mathbb{T}^2$ with a prescribed eigenfunction

As in [**18**], we use the approach of Jakobson and Nadirashvili [**36**]. We equip the torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ with a Liouville metric of the form $g_Q = Q(x)\, g_0$, where $g_0 = dx^2 + dy^2$ is the flat metric, and where Q is a positive C^∞ function on $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$. The respective Laplace-Beltrami operators are denoted $\Delta_0 = \partial_x^2 + \partial_y^2$, and $\Delta_Q = Q(x)^{-1}\, \Delta_0$.

Generally speaking, we identify functions on $\mathbb{T}^1$ (resp. $\mathbb{T}^2$) with periodic functions on $\mathbb{R}$ (resp. $\mathbb{R}^2$).

Given a positive function Q, a complete set of spectral pairs (λ,ϕ) for the eigenvalue problem

$$- \Delta_0\phi(x,y) = \lambda\, Q(x)\, \phi(x,y) \text{ on } \mathbb{T}^2\,, \tag{5.1}$$

is given by the pairs

$$\begin{cases} (\sigma_{m,j},F_{m,j}(x)\cos(my))\,, m \in \mathbb{N}, j \in \mathbf{N} \text{ and,} \\ (\sigma_{m,j},F_{m,j}(x)\sin(my))\,, m \in \mathbf{N}, j \in \mathbf{N}\,, \end{cases} \tag{5.2}$$

where, for a given $m \in \mathbb{N}$, and for $j \in \mathbf{N}$, the pair $(\sigma_{m,j},F_{m,j})$ is a spectral pair for the eigenvalue problem

$$- u''(x) + m^2 u(x) = \sigma\, Q(x)\, u(x) \text{ on } \mathbb{T}^1\,. \tag{5.3}$$

In order to prescribe an eigenfunction, we work the other way around. Choosing a positive C^∞ function F on $\mathbb{T}^1$, and an integer $m \in \mathbf{N}$, we define $\Phi(x,y) = F(x)\, \cos(my)$. Then, the function Φ is an eigenfunction of the eigenvalue problem (5.1), as soon as Q and λ satisfy,

$$Q(x) = \frac{m^2}{\lambda}\left(1 - \frac{1}{m^2}\,\frac{F''(x)}{F(x)}\right).$$

Since the nodal behaviour of eigenfunctions is not affected by rescaling of the metric, we may choose $\lambda = m^2$. In order to guarantee that the metric g_Q is well-defined, we need the function Q to be smooth and positive. Choosing F and m such that

$$\begin{cases} 0 < F(x) \text{ and,} \\ F''(x) < m^2\, F(x)\,, \quad \text{for all } x \in \mathbb{R}\,, \end{cases} \tag{5.4}$$

the function Q, given by

$$Q(x) = 1 - \frac{1}{m^2}\,\frac{F''(x)}{F(x)}\,, \tag{5.5}$$

defines a Liouville metric $g_Q = Q\, g_0$ on $\mathbb{T}^2$, for which

$$(\Delta_Q + m^2)\,(F(x)\, \cos(my)) = 0.$$

When m is large, the metric g_Q appears as a perturbation of the flat metric g_0.

In Subsections 5.3 and 5.4, we apply this idea to construct eigenfunctions with many level domains.

5.3. Example 1

In this subsection, we prove the following result by describing an explicit construction.

Proposition 5.2. *There exists a metric* $g_Q = Q(x)\,(dx^2+dy^2)$ *on the torus* $\mathbb{T}^2$*, and an eigenfunction* Φ *of the associated Laplace-Beltrami operator,* $\Delta_Q\Phi := Q^{-1}\,\Delta_0\Phi$*, such that the super-level set* $\{\Phi > 1\}$ *has infinitely many connected components. As a consequence, Conjecture 2.2 is false for* $(\mathbb{T}^2,g_Q)$*.*

Remark 5.3. *This proposition also implies that* Φ *has infinitely many isolated critical points, a particular case of* [**18**, Theorem 1].

Proof.

Step 1. Let $\phi : [-\pi,\pi] \to \mathbb{R}$ be a function such that

$$\begin{cases} 0 \le \phi(x) \le 1\,, \\ \mathrm{supp}(\phi) \subset [-\frac{\pi}{2},\frac{\pi}{2}]\,, \\ \phi \equiv 1 \text{ on } [-\frac{\pi}{3},\frac{\pi}{3}]\,. \end{cases} \tag{5.6}$$

Define the function $\psi_1 : [-\pi,\pi] \to \mathbb{R}$ by

$$\psi_1(x) = \phi(x)\,\exp\left(-\frac{1}{x^2}\right)\,\cos\left(\frac{1}{x^2}\right) + 1 - \phi(x)\,. \tag{5.7}$$

It is clear that ψ_1 satisfies

$$\begin{cases} |\psi_1(x)| \le 1\,, \\ |x| > \frac{\pi}{2} \Rightarrow \psi_1(x) = 1\,, \\ |x| > \frac{\pi}{3} \Rightarrow \psi_1(x) > 0\,. \end{cases} \tag{5.8}$$

It follows that ψ_1 can only vanish in $[-\frac{\pi}{3},\frac{\pi}{3}]$, with zero set $\mathcal{Z}(\psi_1) = \{x \mid \psi_1(x) = 0\}$ given by

$$\mathcal{Z}(\psi_1) = \{0\} \cup \left\{ \pm\frac{1}{\sqrt{\frac{\pi}{2}+k\pi}} \;\middle|\; k \in \mathbb{N} \right\}. \tag{5.9}$$

The zero set $\mathcal{Z}(\psi_1)$ is an infinite sequence with 0 as only accumulation point, and the function ψ_1 changes sign at each zero. The graph of ψ_1 over $[-\frac{\pi}{3},\frac{\pi}{3}]$ looks like the graph in the left part of Figure 5.

Step 2. Define ψ_0 to be the function ψ_1 extended as a 2π-periodic function on $\mathbb{R}$, and F to be $F := 1 + \frac{1}{2}\psi_0$. Given $m \in \mathbf{N}$, define the function $\Phi_m : \mathbb{T}^2 \to \mathbb{R}$ to be $\Phi_m(x,y) = F(x)\,\cos(my)$.

The functions F and Φ_m satisfy,

$$\begin{cases} F \in C^\infty(\mathbb{T}^1)\,, \\ F \geq \frac{1}{2}\,, \\ \{\psi_0 < 0\} \times \mathbb{T}^1 \subset \{\Phi_m < 1\}\ , \\ \{\Phi_m \geq 1\} \subset \{\psi_0 \geq 0\} \times \mathbb{T}^1\,, \\ \{\psi_0 \geq 0\} \times \{0\} \subset \{\Phi_m \geq 1\}\,. \end{cases} \tag{5.10}$$

It follows from (5.9) that $\{\psi_0 \geq 0\} \subset \mathbb{T}^1$ is the union of infinitely many pairwise disjoint closed intervals, $I_\ell, \ell \in \mathbb{Z}$. It follows from (5.10) that there is at least one connected component of the super-level set $\{\Phi_m > 1\}$ in each $I_\ell \times \mathbb{T}^1$. This construction is illustrated in Figure 5. The figure on the right displays the components of $\{\Phi_1 = 1\}$ (red curves), and the part of the graph of ψ_1 (black curve)[10] contained in $[0.1, 0.3] \times \mathbb{T}^1$. The number of connected components of $\{\Phi_1 = 1\}$ contained in $[\alpha, 0.3] \times \mathbb{T}^1$ tends to infinity as α tends to zero from above, and the components accumulate to (0,0).

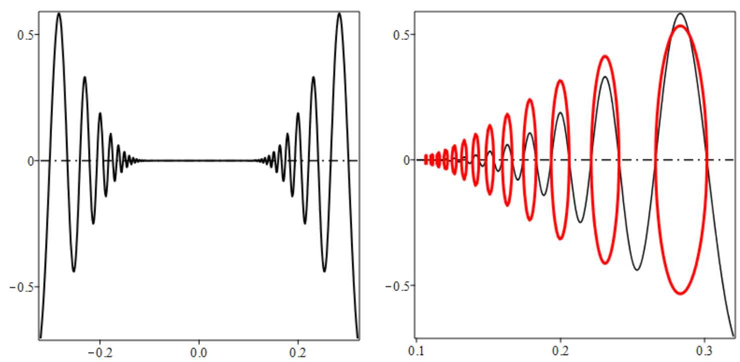

Fig. 5. Graph of ψ_1 near 0. Level set $\{\Phi = 1\}$

We have constructed a family of functions, $\Phi_m, m \in \mathbf{N}$, whose super-level sets $\{\Phi_m > 1\}$ have infinitely many connected components in $\mathbb{T}^2$.

Step 3. Since $F \in C^\infty(\mathbb{T}^1)$, and $F \geq \frac{1}{2}$, the function $\frac{F''}{F}$ is bounded from above. We choose m such that

$$m^2 > \sup_{x \in \mathbb{T}^1} \frac{F''(x)}{F(x)}\,, \tag{5.11}$$

and we define the function $Q_m : \mathbb{T}^1 \to \mathbb{R}$,

$$Q_m(x) = 1 - \frac{1}{m^2}\frac{F''(x)}{F(x)}\,. \tag{5.12}$$

According to Subsection 5.2, and under condition (5.11), the function Q_m defines a Liouville metric g_m on $\mathbb{T}^2$,

$$g_m = Q_m(x)\,\left(dx^2 + dy^2\right) \tag{5.13}$$

[10]As a matter of fact, we have used differently scaled functions in order to enhance the figures.

and this metric can be chosen arbitrarily close to the flat metric $dx^2 + dy^2$ as m goes to infinity. For the associated Laplace-Beltrami operator Δ_{g_m}, we have

$$- \Delta_{g_m} \Phi_m(x,y) = m^2 \Phi_m(x,y) \,, \tag{5.14}$$

so that the function Φ_m is an eigenfunction of Δ_{g_m}, with eigenvalue m^2. The super-level set $\{\Phi_m > 1\}$ has infinitely many connected components in $\mathbb{T}^2$. In particular, the function $\Phi_m - 1$ has infinitely many nodal domains. □

Remark 5.4. *One could slightly modify the above construction as follows. Replace the function ψ_1 by the function $\psi_{1,a} = a\,\psi_1$, where a is a (small) real parameter, and extend it to a periodic function $\psi_{0,a}$. The corresponding function F becomes $F_a = 1 + \frac{a}{2}\psi_{0,a}$, and the function*

$$Q_{m,a}(x) = 1 - \frac{a}{m^2}\,\frac{\psi_{0,a}''(x)}{2 + a\,\psi_{0,a}(x)}$$

defines a Liouville metric $g_{m,a}$ on $\mathbb{T}^2$, provided that a is small enough. The function

$$\Phi_{m,a}(x,y) = \Big(1 + \frac{a}{2}\,\psi_{0,a}(x)\Big)\,\cos(my)$$

is an eigenfunction of $-\Delta_{g_{m,a}}$, associated with the eigenvalue m^2. The super level set $\{\Phi_{m,a} > 1\}$ has infinitely many connected components. When m is fixed, and a tends to zero, the metric $g_{m,a}$ tends to the metric g_0, and the labelling of m^2, as eigenvalue of $-\Delta_{g_{m,a}}$, remains bounded.

5.4. Example 2

The metric constructed in Proposition 5.2 is smooth, not real analytic. In this subsection, we prove the following result in which we have a real analytic metric.

Proposition 5.5. *Let n be any given integer. Then, there exists a real analytic Liouville metric $g = Q(x)\,(dx^2 + dy^2)$ on $\mathbb{T}^2$, and an eigenfunction Φ of the associated Laplace-Beltrami operator, $-\Delta_g \Phi = \Phi$, with eigenvalue 1, such that the super-level set $\{\Phi > 1\}$ has at least n connected components. One can choose the metric g arbitrarily close to the flat metric g_0. Taking $n \geq 4$, and g close enough to g_0, the eigenvalue 1 is either the second, third or fourth eigenvalue of Δ_g.*

Remarks 5.6. *(i) It follows from the proposition that the function $\Phi - 1$ provides a counterexample to Conjecture 2.2 for $(\mathbb{T}^2,g)$.*

(ii) The proposition is related to a question raised in [**18**, Introduction]*: "For an analytic metric, does there exist an asymptotic upper bound for the number of critical points in terms of the corresponding eigenvalue". Indeed, given any $n \geq 4$, the function Φ given by the proposition is associated with the eigenvalue 1, whose labelling is at most 4, and Φ has at least n isolated critical points.*

Proof. Fix the integer n. For $0 \leq a < 1$, define the functions

$$\begin{cases} F_a(x) = 1 + a\,\cos(nx)\,, \\ \Phi_a(x,y) = F_a(x)\,\cos(y)\,. \end{cases} \tag{5.15}$$

For a small enough (depending on n), the function

$$Q_a(x) = 1 - \frac{F_a''(x)}{F_a(x)} = 1 + a\,n^2 \frac{\cos(nx)}{1 + a\,\cos(nx)}\,, \tag{5.16}$$

is positive. According to Subsection 5.2 (choosing $\psi_0(x) = a\,\cos(nx)$), the function Q_a defines a Liouville metric g_a on $\mathbb{T}^2$. The associated Laplace-Beltrami operator is $\Delta_a = (Q_a(x))^{-1}\,\Delta_0$, and we have,

$$\begin{cases} -\Delta_a F_a(x)\cos(y) = F_a(x)\cos(y)\,, \\ -\Delta_a F_a(x)\sin(y) = F_a(x)\sin(y)\,. \end{cases} \tag{5.17}$$

Call $\{\lambda_{a,j}, j \geq 1\}$ the eigenvalues of $-\Delta_a$, written in non-decreasing order, with multiplicities.

The eigenvalues of $-\Delta_0$ are given by

$$\begin{cases} \lambda_{0,1} &= 0\,, \\ \lambda_{0,j} &= 1 \text{ for } 2 \leq j \leq 5\,, \\ \lambda_{0,j} &\geq 2 \text{ for } j \geq 6. \end{cases} \tag{5.18}$$

For n fixed, and a small enough (depending on n), the eigenvalues $\lambda_{a,j}$ satisfy

$$\begin{cases} \lambda_{a,1} &= 0\,, \\ \lambda_{a,j} &\in \,]0.8,1.2[\text{ for } 2 \leq j \leq 5\,, \\ \lambda_{a,j} &\geq 1.8 \text{ for } j \geq 6. \end{cases} \tag{5.19}$$

We note that the metric g_a and the operators Δ_a are invariant under the symmetries $\Sigma_1 : (x,y) \to (-x,y)$ and $\Sigma_2 : (x,y) \to (x,-y)$, which commute. Consequently, the space $L^2(\mathbb{T}^2,g_a)$ decomposes into four orthogonal subspaces

$$\mathcal{S}_{\varepsilon_1,\varepsilon_2} = \left\{\phi \in L^2(\mathbb{T}^2) \mid \Sigma_1^*\phi = \varepsilon_1\phi,\ \Sigma_2^*\phi = \varepsilon_2\phi\right\}\,, \tag{5.20}$$

and the eigenvalue problem for Δ_a on $L^2(\mathbb{T}^2)$ splits into four independent problems by restriction to the subspaces $\mathcal{S}_{\varepsilon_1,\varepsilon_2}$, with $\varepsilon_1,\varepsilon_2 \in \{-,+\}$. The eigenvalue 0 is the first eigenvalue of $-\Delta_a|\mathcal{S}_{+,+}$.

When $a = 0$, the eigenvalue 1 arises with multiplicity 2 from $-\Delta_a|\mathcal{S}_{+,+}$ (the functions $\cos x$ and $\cos y$), with multiplicity 1 from $-\Delta_a|\mathcal{S}_{-,+}$ (the function $\sin x$), and multiplicity 1 from $-\Delta_a|\mathcal{S}_{+,-}$ (the function $\sin y$).

For a small enough, the same spaces yield the eigenvalues $\lambda_{a,j}, 2 \leq j \leq 5$. According to (5.17), the functions $F_a(x)\cos(y) \in \mathcal{S}_{+,+}$ and $F_a(x)\sin(y) \in \mathcal{S}_{+,-}$ correspond to the eigenvalue 1. In view of (5.19), there is another eigenvalues $\sigma(a)$ of $-\Delta_a|\mathcal{S}_{+,+}$, and another eigenvalue $\tau(a)$ of $-\Delta_a|\mathcal{S}_{-,+}$, with $\sigma(a), \tau(a) \in]0.8,1.2[$ (these eigenvalues could possibly be equal to 1). It follows that

$$\{\lambda_{a,j}, 2 \leq j \leq 5\} = \{1,1,\sigma(a),\tau(a)\} \ ,$$

so that the eigenvalue 1 of $-\Delta_a$ is either $\lambda_{a,2}$, $\lambda_{a,3}$, or $\lambda_{a,4}$. Note that, in view of (5.19), these eigenvalues are the smallest nonzero eigenvalues of $-\Delta_a$ restricted to the corresponding symmetry spaces $\mathcal{S}$.

Letting $\psi_0(x) = a\,\cos(nx)$, and choosing a small enough (depending on n), the functions F_a and Φ_a satisfy relations similar to the relations (5.10) of Subsection 5.3,

$$\begin{cases} F_a \in C^\infty(\mathbb{T}^1)\,, \\ F_a \geq \frac{1}{2}\,, \\ \{\psi_0 < 0\} \times \mathbb{T}^1 \subset \{\Phi_a < 1\}\ , \\ \{\Phi_a \geq 1\} \subset \{\psi_0 \geq 0\} \times \mathbb{T}^1\,, \\ \{\psi_0 \geq 0\} \times \{0\} \subset \{\Phi_a \geq 1\}\,. \end{cases} \tag{5.21}$$

These relations show that the super-level set $\{\Phi_a > 1\}$ has n connected components. This is illustrated in Figure 6 for $n = 5$. The black curve is the graph of $0.6\cos(nx)$. The red curves are components of the corresponding level set $\{\Phi_a = 1\}$ (in the figure, $a = 0.1$).

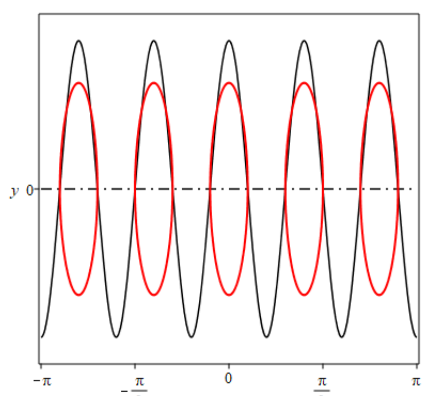

Fig. 6. Level sets $\{\Phi_a = 1\}$ for $n = 5$

It follows that the function $\Phi_a - 1$ has at least $n+1$ nodal domains. When $n \geq 4$, this also tells us that $\Phi_a - 1$ provides a counterexample to Conjecture 2.2. ☐

5.5. Perturbation theory

We use the same notation as in Subsection 5.4. Using perturbation theory, we now analyze the location of the eigenvalue 1 in the spectrum of the operator Δ_a, and refine Proposition 5.5. More precisely, we prove

Proposition 5.7. *For any given* $n \geq 3$*, and* a *small enough (depending on* n*), the eigenvalue* 1 *is the fourth eigenvalue of the operator* $-\Delta_a$ *associated with the metric* $g_a = Q_a(x)\,(dx^2 + dy^2)$*, where* $Q_a(x)$ *is defined in* (5.16)

DÉMONSTRATION. We have constructed the metrics g_a in such a way that 1 is always an eigenvalue of multiplicity at least 2 (see (5.17)). We may assume that a is small enough so that (5.19) holds.

The idea of the proof is to show that the eigenvalues $\sigma(a)$ and $\tau(a)$ are less than 1 by looking at their expansions[11] in powers of a. It will actually be sufficient to compute the first three terms of these expansions. As in the proof of Proposition 5.5, we use the symmetry properties of the metrics g_a.

From the proof of Proposition 5.5, we know that $\sigma(a)$ is an eigenvalue of $-\Delta_a|\mathcal{S}_{+,+}$ and $\tau(a)$ an eigenvalue of $-\Delta_a|\mathcal{S}_{-,+}$. It therefore suffices to look at eigenfunctions which are even in the variable y. Using Fourier cosine decomposition in the variable y, we reduce our problem to analyzing the family of eigenvalue problems,

$$-\,u''(x) + k^2 u(x) = \sigma\, Q_a(x)\, u(x) \text{ on } \mathbb{T}^1\,, \text{ for } k \in \mathbb{N}\,. \tag{5.22}$$

More precisely, to study $\sigma(a)$, we look at the family (5.22) restricted to *even* functions in the x-variable; to study $\tau(a)$, we look at the family (5.22) restricted to *odd* functions in the x-variable.

In the sequel, we use the notation $\langle f|g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt$ to denote the inner product of real functions in $L^2(\mathbb{T}^1)$.

Claim 1. For $n \geq 3$, and a small enough, $\sigma(a) < 1$.

Recall that to analyse $\sigma(a)$, we restrict (5.22) to *even functions* of x.

The eigenvalue 0 of $-\Delta_a$ appears as the first eigenvalue of (5.22), for $k = 0$. When $a = 0$, the eigenvalue 1 appears as the second eigenvalue of (5.22), for $k = 0$, and as the first eigenvalue of (5.22), for $k = 1$. When $a > 0$, the function F_a, defined in (5.15), is an eigenfunction of (5.22), for $k = 1$, and 1 is the first eigenvalue of this equation because F_a is positive. For a small enough, the second eigenvalue of this equation must be larger than 2. It follows from (5.19) that $\sigma(a)$ cannot be an eigenvalue of (5.22), for $k = 1$. By the

[11]For the existence of such expansions, we refer to **[57]** or **[37]**.

min-max, $\sigma(a)$ cannot either be an eigenvalue of (5.22), for $k \geq 2$. It again follows from (5.19) that $\sigma(a)$ must be the second eigenvalue of (5.22), for $k = 0$.

We rewrite (5.22), for $k = 0$, *restricted to even functions*, as

$$- u''(x) = \sigma\, Q_a(x)\, u(x) \text{ on } \mathbb{T}^1 ,\ u \text{ even.} \tag{5.23}$$

Since $\sigma(a)$ is a simple eigenvalue of (5.23), the perturbative analysis in a is easy. There exist expansions of the eigenvalue $\sigma(a)$ of (5.23), and of a corresponding eigenfunction $u(\cdot,a)$, in the form,

$$\sigma(a) = \sigma_0 + \sum_{j>0} \sigma_j\, a^j\, , u(\cdot,a) = u_0 + \sum_{j>0} u_j\, a^j\, ,$$

with $\sigma_0 = 1$, $u_0(x) = \cos(x)$ (respectively the unperturbed eigenvalue and eigenfunction), and with the additional orthogonality condition,

$$\langle u(\cdot,a) | u(\cdot,a) \rangle = \langle u_0 | u_0 \rangle \text{ for all } a\, . \tag{5.24}$$

In order to prove Claim 1, it suffices to show that $\sigma_1 = 0$ and $\sigma_2 < 0$. For this purpose, we now determine σ_1, σ_2, and u_1. Developing the left-hand side of (5.24) with respect to a, we find a series of orthogonality conditions on the functions u_j. In order to determine u_1, we only need the orthogonality condition,

$$\langle u_0 | u_1 \rangle = 0\, . \tag{5.25}$$

Develop the function Q_a in powers of a,

$$Q_a(x) = 1 + n^2 \sum_{j>0} (-1)^{j-1} \cos^j(nx)\, a^j\, . \tag{5.26}$$

Plugging the expansions of $\sigma(a), u(\cdot,a)$ and Q_a into equation (5.23), and equating the terms in $a^k, k \geq 0$, we find equations satisfied by the functions $u_k, k \geq 0$, in the form

$$- u_k'' = u_k + f_k \tag{5.27}$$

where $f_0 = 0$, and for $k \geq 1$, f_k depends on the σ_j, for $j \leq k$, and on the u_j, for $j \leq (k-1)$. Recall that the functions $u_k, k \geq 1$ are even and that they satisfy the orthogonality relations given by (5.24). In order to prove Claim 1, we only need to write equation (5.27) for u_1 and u_2, together with the associated parity and orthogonality conditions,

$$\begin{cases} -u_1''(x) = u_1(x) + \sigma_1 \cos(x) + n^2 \cos(x) \cos(nx)\, , \\ u_1 \text{ even,} \\ 0 = \langle u_1 | u_0 \rangle = \int_{-\pi}^{\pi} u_1(t) \cos(t)\, dt\, . \end{cases} \tag{5.28}$$

and

$$\begin{cases} -u_2''(x) = u_2(x) + \sigma_2 \cos(x) - n^2 \cos(x) \cos^2(nx) \\ \qquad +n^2 \cos(nx)\,(u_1(x) + \sigma_1 \cos(x)) + \sigma_1\, u_1(x)\,, \\ u_2 \text{ even}, \\ 0 = 2\langle u_2 | u_0 \rangle + \langle u_1 | u_1 \rangle\,. \end{cases} \tag{5.29}$$

Taking the L^2 inner product of the differential equation in (5.28) with $\cos x$, and for $n \geq 3$, we obtain that

$$\sigma_1 = 0\,. \tag{5.30}$$

Since $\sigma_1 = 0$, the differential equation satisfied by u_1 in (5.28) becomes,

$$-\, u_1''(x) = u_1(x) + n^2 \cos(nx) \cos x\,, \tag{5.31}$$

with u_1 even, satisfying (5.25). Writing

$$\cos(nx) \cos x = \tfrac{1}{2} \left\{ \cos((n+1)x) + \cos((n-1)x) \right\},$$

it is easy to check that the function $p(x)$ defined by

$$p(x) = \frac{n}{2} \left(\frac{\cos((n+1)x)}{n+2} + \frac{\cos((n-1)x)}{n-2} \right)\,, \tag{5.32}$$

is a particular solution of this differential equation. The general solution is given by $\alpha \cos(x) + \beta \sin(x) + p(x)$, and since u_1 is even and orthogonal to $\cos(x)$, we find that $u_1 = p$.

Taking the fact that $\sigma_1 = 0$ into account, the differential equation for u_2 in (5.29) becomes

$$-\, u_2''(x) = u_2(x) + n^2 \cos(nx) u_1(x) - n^2 \cos^2(nx) \cos x + \sigma_2 \cos x\,. \tag{5.33}$$

Taking the scalar product with $\cos x$, we obtain

$$n^2 \int \cos^2(nx) (\cos x)^2 dx - n^2 \int \cos(nx) u_1(x) \cos x\, dx = \sigma_2 \int (\cos x)^2 dx\,.$$

The sign of σ_2 is the same as the sign of

$$A_n := \int_{-\pi}^{+\pi} \cos(nx)^2 (\cos x)^2 dx - \int_{-\pi}^{+\pi} \cos(nx) u_1(x) \cos x\, dx\,.$$

Since $u_1 = p$, computing each term of the sum, we get

$$\int \cos(nx)^2 (\cos x)^2 dx = \frac{\pi}{2}\,,$$

and

$$\int \cos(nx) u_1(x) \cos x\, dx = \frac{\pi}{2} \frac{n^2}{n^2 - 4}$$

Finally,

$$\frac{2}{\pi} A_n = -\frac{4}{n^2 - 4} < 0\,.$$

Claim 1 is proved: $\sigma(a) < 1$ for a small enough.

Remark 5.8. *We could continue the construction at any order, but we do not need it for our purposes.*

Claim 2. For $n \geq 3$ and a small enough, $\tau(a) < 1$.

Recall, from the proof of Proposition 5.5, that $\tau(a)$ is the first eigenvalue of $-\Delta_a|\mathcal{S}_{-,+}$. In order to analyse $\tau(a)$, we restrict (5.22) to *odd functions* of x.

The eigenvalue $\tau(a)$ is actually the first eigenvalue of (5.22), for $k = 0$, restricted to odd functions of x. We rewrite (5.22), for $k = 0$, *restricted to odd functions*, as

$$- u''(x) = \sigma\, Q_a(x)\, u(x) \text{ on } \mathbb{T}^1\,,\ u \text{ odd.} \tag{5.34}$$

Since $\tau(a)$ is a simple eigenvalue of (5.34), the perturbative analysis in a is easy. There exist expansions of the eigenvalue $\tau(a)$ of (5.34), and of a corresponding eigenfunction $v(\cdot,a)$, in the form,

$$\tau(a) = \tau_0 + \sum_{j>0} \tau_j\, a^j\,, v(\cdot,a) = v_0 + \sum_{j>0} v_j\, a^j\,,$$

with $\tau_0 = 1$, $v_0(x) = \sin(x)$ (respectively the unperturbed eigenvalue and eigenfunction), and with the additional orthogonality condition,

$$\langle v(\cdot,a)|v(\cdot,a)\rangle = \langle v_0|v_0\rangle \text{ for all } a\,. \tag{5.35}$$

To prove Claim 2, it suffices to show that $\tau_1 = 0$ and $\tau_2 < 0$. For this purpose, it is sufficient to determine τ_1, τ_2, and v_1. Developing the left-hand side of (5.35) with respect to a, we find a series of orthogonality conditions on the functions v_j. In order to determine v_1, we only need the orthogonality condition,

$$\langle v_0|v_1\rangle = 0\,. \tag{5.36}$$

Plugging the expansions of $\tau(a), v(\cdot,a)$ and Q_a, see (5.26), into equation (5.34), and equating the terms in $a^k, k \geq 0$, we find equations satisfied by the functions $v_k, k \geq 0$, in the form

$$- v_k'' = v_k + h_k \tag{5.37}$$

where $h_0 = 0$, and for $k \geq 1$, h_k depends on the τ_j, for $j \leq k$, and on the v_j, for $j \leq (k-1)$. Recall that the functions $v_k, k \geq 1$ are odd and that they satisfy the orthogonality relations given by (5.35). In order to prove Claim 2, we only need to write equation (5.37) for v_1 and v_2, together with the associated parity and orthogonality conditions,

$$\begin{cases} -v_1''(x) = v_1(x) + \tau_1\, \sin(x) + n^2 \sin(x)\, \cos(nx)\,, \\ v_1 \text{ odd}, \\ 0 = \langle v_1|v_0\rangle = \int_{-\pi}^{\pi} v_1(t) \sin(t)\, dt\,. \end{cases} \tag{5.38}$$

and

$$\begin{cases} -v_2''(x) = v_2(x) + \tau_2 \sin(x) - n^2 \sin(x) \cos^2(nx) \\ \qquad +n^2 \cos(nx)\,(v_1(x) + \tau_1 \sin(x)) + \tau_1\, v_1(x)\,, \\ v_2 \text{ odd}, \\ 0 = 2\langle v_2 | v_0 \rangle + \langle v_1 | v_1 \rangle\,. \end{cases} \tag{5.39}$$

Taking the L^2 inner product of the differential equation in (5.38) with $\sin x$, and assuming that $n \geq 3$, we obtain that

$$\tau_1 = 0\,. \tag{5.40}$$

Since $\tau_1 = 0$, the differential equation satisfied by v_1 in (5.38) becomes,

$$-\, v_1''(x) = v_1(x) + n^2 \cos(nx) \sin(x)\,, \tag{5.41}$$

with v_1 odd, satisfying (5.36). Writing

$$\cos(nx) \sin x = \tfrac{1}{2} \left\{ \sin((n+1)x) - \sin((n-1)x) \right\},$$

it is easy to check that the function $q(x)$ defined by

$$q(x) = \frac{n}{2} \left(\frac{\sin((n+1)x)}{n+2} - \frac{\sin((n-1)x)}{n-2} \right)\,, \tag{5.42}$$

is a particular solution of this differential equation. The general solution is given by $\alpha \cos(x) + \beta \sin(x) + q(x)$, and since v_1 is odd and orthogonal to $\sin(x)$, we find that $v_1 = q$.

Taking the fact that $\tau_1 = 0$ into account, the differential equation for v_2 in (5.39) becomes

$$-\, v_2''(x) = v_2(x) + n^2 \cos(nx) v_1(x) - n^2 \cos^2(nx) \sin x + \tau_2 \sin x\,. \tag{5.43}$$

Taking the scalar product with $\sin x$, we obtain

$$n^2 \int \cos^2(nx) \sin^2(x)\, dx - n^2 \int \cos(nx) v_1(x) \sin x\, dx = \tau_2 \int \sin^2(x)\, dx\,.$$

The sign of τ_2 is the same as the sign of

$$B_n := \int_{-\pi}^{+\pi} \cos(nx)^2 (\sin x)^2 dx - \int_{-\pi}^{+\pi} \cos(nx) v_1(x) \sin x\, dx\,.$$

Computing each term of the sum, we get

$$\int \cos(nx)^2 (\sin x)^2 dx = \frac{\pi}{2}\,,$$

and

$$\int \cos(nx) v_1(x) \sin x\, dx = \frac{\pi}{2} \frac{n^2}{n^2 - 4}$$

Finally,

$$\frac{2}{\pi} B_n = -\frac{4}{n^2 - 4} < 0\,.$$

Claim 2 is proved: $\tau(a) < 1$ for a small enough.

It follows that 1 is an eigenvalue of $-\Delta_a$, with multiplicity 2 and least labelling 4. This proves Proposition 5.7. □

5.6. Comparison with a result of Gladwell and Zhu

The authors of **[28]** prove the following result for a bounded domain in $\mathbb{R}^d$.

Proposition 5.9. *Let $\Omega \subset \mathbb{R}^d$ be a connected bounded domain. Call (δ_j, u_j) the eigenpairs of the Dirichlet eigenvalue problem in Ω,*

$$\begin{cases} -\Delta u &= \delta\, u \text{ in } \Omega\,, \\ u &= 0 \text{ on } \partial\Omega\,, \end{cases} \tag{5.44}$$

where the eigenvalues $\delta_1 < \delta_2 \le \delta_3 \le \dots$ are listed in non-decreasing order, with multiplicities. Assume that the first eigenfunction u_1 is positive. For $n \ge 2$, let $v = u_n + cu_1$, for some positive constant c. Then, the function v has at most $(n-1)$ positive sign domains, i.e., the super-level set $\{v > 0\}$ has at most $(n-1)$ connected components.

The same result is true if instead of the Dirichlet boundary condition, one considers the Neumann boundary condition (assuming in this case that $\partial\Omega$ is smooth enough), or if one considers a closed real analytic Riemannian surface[12].

A more convenient formulation, is as follows. For a function w, and $\varepsilon \in \{-,+\}$, define $\beta_0^{\varepsilon}(w)$ to be the number of nodal domains of w, on which $\varepsilon\, w$ is positive. Proposition 5.9 can be restated as follows. For any $n \ge 2$, and any real nonzero constant c,

$$\beta_0^{\operatorname{sign}(c u_1)}(u_n + cu_1) \le (n-1)\,. \tag{5.45}$$

Proposition 5.9 is weaker than Conjecture 2.2. Indeed, it only gives control on the number of nodal domains where the function $u_n + cu_1$ has the sign of $\operatorname{sign}(cu_1)$. Propositions 5.2 and 5.5 show that one can a priori not control $\beta_0^{-\operatorname{sign}(cu_1)}(w)$, at least in the case of the Neumann (or empty) boundary condition. However, one can observe that, fixing n_0, it is easy to construct examples for which Conjecture 2.2 is true for all linear combinations of the n first eigenfunctions, $w \in \mathcal{L}_n$, with $n \le n_0$. Indeed, for L large, the rectangle $]0,1[\times]0,L[$ provides such an example for the Dirichlet or Neumann boundary conditions. More generally, one can consider manifolds which collapse on a lower dimensional manifold for which the Extended Courant property is true.

[12]It might be necessary to use a real analytic surface in order to apply Green's theorem to the nodal sets of a linear combination of eigenfunctions.

6. Counterexamples on $\mathbb{S}^2$

6.1. Results and general approach

In this section, we extend, to the case of the sphere, the construction made in Section 5. We prove the following results.

Proposition 6.1. *There exist C^∞ functions Φ and G on $\mathbb{S}^2$, with the following properties.*

1. *The super-level set $\{\Phi > 1\}$ has infinitely many connected components.*
2. *The function G is positive, and defines a conformal metric $g_G = G\, g_0$ on $\mathbb{S}^2$ with associated Laplace-Beltrami operator $\Delta_G = G^{-1}\,\Delta_0$.*
3. *$-\Delta_G \Phi = 2\,\Phi$.*
4. *The eigenvalue 2 of $-\Delta_G$ has labelling at most 4.*

Proposition 6.2. *There exists $M > 0$ such that, for any $m \geq M$, there exist C^∞ functions Φ_m and G_m on $\mathbb{S}^2$ with the following properties.*

1. *The super-level set $\{\Phi_m > 1\}$ has infinitely many connected components.*
2. *The function G_m is positive, and defines a conformal metric $g_m = G_m\, g_0$ on $\mathbb{S}^2$ with associated Laplace-Beltrami operator $\Delta_{G_m} = G_m^{-1}\,\Delta_0$.*
3. *For $m \geq M$, $\left(1 - \frac{2}{m}\right) \leq G_m \leq \left(1 + \frac{2}{m}\right)$, and*
4. *$-\Delta_{G_m} \Phi_m = m(m+1)\,\Phi_m$.*

These propositions provide counterexamples to Conjecture 2.2 and to Questions 2.3 on the sphere.

Remark 6.3. *The eigenfunctions on $\mathbb{S}^2$ constructed in the above propositions have infinitely many isolated critical points. For a similar result on $\mathbb{T}^2$, see Remark 5.3(ii) which is a particular case of* [**18**, Theorem 1].

The approach is inspired by Section 5, with the following steps.

1. Start from a special spherical harmonic Y of the standard sphere $(\mathbb{S}^2, g_0)$, with eigenvalue $m(m+1)$.
2. Modify Y into a smooth function Φ, whose super-level set $\{\Phi > 1\}$ has infinitely many connected components.
3. Construct a conformal metric $g_Q = Q\, g_0$ on $\mathbb{S}^2$, whose associated Laplace-Beltrami operator has Φ as eigenfunction, with eigenvalue $m(m+1)$.

The proof of Propositions 6.1 and 6.2, following the above steps, is split into the next subsections.

6.2. Metrics on $\mathbb{S}^2$ with a prescribed eigenfunction

Let g_0 be the standard metric on the sphere

$$\mathbb{S}^2 = \left\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\right\}.$$

The spherical coordinates are $(\theta,\phi) \mapsto \left(\sin\theta\,\cos\phi, \sin\theta\,\sin\phi, \cos\theta\right)$, with $(\theta,\phi) \in]0,\pi[\times]0,2\pi[$. In these coordinates,

$$g_0 = d\theta^2 + \sin^2\theta\, d\phi^2\,,$$

the associated measure is $\sin\theta\, d\theta\, d\phi$, and the Laplace-Beltrami operator of g_0 is given by

$$\Delta_0 = \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\big(\sin\theta\,\frac{\partial}{\partial\theta}\big) + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}\,.$$

We consider conformal metrics on $\mathbb{S}^2$, in the form $g_Q = Q\, g_0$, where Q is C^∞ and positive. We denote the associated Laplace-Beltrami operator by

$$\Delta_Q = Q^{-1}\,\Delta_0\,.$$

We assume that Q is invariant under the rotations with respect to the z-axis, i.e., that Q only depends on the variable θ.

Let Φ be a smooth function on $\mathbb{S}^2$, given in spherical coordinates by $\Phi(\theta,\phi) = T(\theta)P(\phi)$. If Φ is an eigenfunction of $-\Delta_Q$ associated with the eigenvalue λ, then the functions T and P satisfy the equations,

$$P''(\phi) + m^2 P(\phi) = 0\,, \tag{6.1}$$

$$\sin(\theta)\,(\sin(\theta)T'(\theta))' + \left(\lambda Q(\theta)\sin^2(\theta) - m^2\right)\,T(\theta) = 0\,, \tag{6.2}$$

where m is an integer. When $Q \equiv 1$, the solutions are the spherical harmonics of degree m, Y_m^k, $-m \le k \le m$ (as given for example in [**44**, p. 302]).

For $m \ge 1$, we consider the special spherical harmonic

$$Y_m^m(\theta,\phi) = \sin^m(\theta)\,\cos(m\phi)\,.$$

We could consider $\sin^m(\theta)\,\sin(m\phi)$ as well, since $m \ge 1$. For later purposes, we introduce the linear differential operator $\mathcal{K}_m$, defined by

$$\begin{aligned} T \mapsto (\mathcal{K}_m T)(\theta) \;&= \sin^2(\theta)\,T''(\theta) + \sin(\theta)\cos(\theta)\,T'(\theta) \\ &\quad + \left(m(m+1)\sin^2(\theta) - m^2\right)\,T(\theta)\,. \end{aligned} \tag{6.3}$$

In particular, we have

$$\mathcal{K}_m \sin^m(\cdot) = 0. \tag{6.4}$$

Given Q a smooth positive function, which only depends on θ, a necessary and sufficient condition for the function $\Phi(\theta,\phi) = T(\theta)\cos(m\phi)$ to satisfy $-\Delta_Q \Phi = m(m+1)\,\Phi$, is that

$$Q(\theta) = -\,\frac{\sin^2(\theta)T''(\theta) + \sin(\theta)\cos(\theta)T'(\theta) - m^2 T(\theta)}{m(m+1)\sin^2(\theta)T(\theta)}\,, \tag{6.5}$$

or, equivalently,

$$1 - Q(\theta) = \frac{(\mathcal{K}_m T)(\theta)}{m(m+1)\sin^2(\theta)T(\theta)} \tag{6.6}$$

In particular, taking $\Phi(\theta,\phi) = \sin^m(\theta)\cos(m\phi)$, we find that $Q \equiv 1$.

Remark 6.4. *As in Section 5, we work the other way around: we prescribe T, and look for a conformal metric on $\mathbb{S}^2$ admitting $\Phi(\theta,\phi) = T(\theta)\cos(m\phi)$ as eigenfunction. The main difficulty in prescribing the function T, is to show that the function Q defined by* (6.5) *is actually smooth and positive.*

6.3. Constructing perturbations of the function $\sin^m(\theta)$

In Section 5, we perturbed the eigenfunction $\cos(my)$ of the torus into the function $\Phi_m(x,y) = F(x)\cos(my)$, where F had rapidly decaying oscillations around $x = 0$. We do a similar construction here, with an extra flattening step.

Given $m \geq 1$, we start from the spherical harmonic $\sin^m(\theta)\cos(m\phi)$ in spherical coordinates. We first flatten the function $\sin^m(\theta)$ around $\theta = \pi/2$, before adding the rapidly decaying oscillations. More precisely, we look for functions Φ of the form $\Phi(\theta,\phi) = T(\theta)\cos(m\phi)$. To determine T, we construct a family $T_{m,n,\alpha}$ of perturbations of the function $\sin^m(\cdot)$, in the form,

$$T_{m,n,\alpha}(\theta) = \sin^m(\theta) + P_{m,n,\alpha}(\theta) + u_{m,n,\alpha}(\theta)\,, \tag{6.7}$$

with $n \in \mathbb{N}$ (to be chosen large), and $\alpha \in (0,\frac{1}{4}]$ (to be chosen small). The function $P_{m,n,\alpha}$ is constructed such that

$$\sin^m(\theta) + P_{m,n,\alpha}(\theta) \equiv 1 \tag{6.8}$$

in an interval around $\frac{\pi}{2}$, and $u_{m,n,\alpha}$ is a rapidly oscillating function in the same interval. They will both be designed in such a way that we can control the derivatives in equation (6.5). The construction of the family $T_{m,n,\alpha}$ is explained in the following paragraphs, and illustrated in Figure 7.

6.3.1. Construction of $P_{m,n,\alpha}$.

Proposition 6.5 (Construction of $P_{m,n,\alpha}$)**.** *For all $m \geq 1$ and $\alpha \in (0,\frac{1}{4}]$, there exist $N(m,\alpha) \in \mathbb{N}$, and a sequence of functions $(P_{m,n,\alpha})_{n\geq 1}$, $P_{m,n,\alpha} : [0,\pi] \to \mathbb{R}$, with the following properties for all $n \geq N(m,\alpha)$.*

(1) $P_{m,n,\alpha} \in C^\infty$ and $P_{m,n,\alpha}(\pi - \theta) = P_{m,n,\alpha}(\theta)$ for all $\theta \in [0,\pi]$;

(2) for $\theta \in [\frac{\pi}{2} + \frac{1}{mn}, \pi]$, $P_{m,n,\alpha}(\theta) = 0$;

(3) for $\theta \in [\frac{\pi}{2}, \frac{\pi}{2} + \frac{\alpha}{(mn)^2}]$, $P_{m,n,\alpha}(\theta) = 1 - \sin^m(\theta)$;

(4) for $\theta \in [\frac{\pi}{2}, \pi]$, $0 \leq P_{m,n,\alpha}(\theta) \leq \frac{2m}{(mn)^3}$, and $|P'_{m,n,\alpha}(\theta)| \leq \frac{2m}{(mn)^2}$;

(5) for $\theta \in [\frac{\pi}{2}, \pi]$, $-m(1 + 5\alpha) \leq P''_{m,n,\alpha}(\theta) \leq m(1 + \alpha)$.

The idea is to construct $P_{m,n,\alpha}$ as

$$P_{m,n,\alpha}(\theta) = \int_{\frac{\pi}{2}}^{\theta} R_{m,n,\alpha}(t)\, dt\,, \tag{6.9}$$

for $\theta \in [\frac{\pi}{2},\pi]$, and to extend it so that $P_{m,n,\alpha}(\pi - \theta) = P_{m,n,\alpha}(\theta)$. We first construct a sequence $S_{m,n,\alpha}$ (Lemma 6.6), and then a sequence $s_{m,n,\alpha}$, such that $R_{m,n,\alpha} = S_{m,n,\alpha} + s_{m,n,\alpha}$ (Lemma 6.7).

Lemma 6.6 (Construction of $S_{m,n,\alpha}$)**.** *For any $m \geq 1$, and any $\alpha \in (0,\frac{1}{4}]$, there exists a sequence of functions $(S_{m,n,\alpha})_{n\geq 1}$, $S_{m,n,\alpha} : [0,\pi] \to \mathbb{R}$, with the following properties for $n \geq 2$.*

(1) $S_{m,n,\alpha} \in C^\infty$ and $S_{m,n,\alpha}(\pi - \theta) = -S_{m,n,\alpha}(\theta)$ for all $\theta \in [0,\pi]$;

(2) for $\theta \in [\frac{\pi}{2} + \frac{1}{(mn)^2}, \pi]$, $S_{m,n,\alpha}(\theta) = 0$;

(3) for $\theta \in [\frac{\pi}{2}, \frac{\pi}{2} + \frac{\alpha}{(mn)^2}]$, $S_{m,n,\alpha}(\theta) = -m \cos(\theta) \sin^{m-1}(\theta)$;

(4) for $\theta \in [\frac{\pi}{2}, \pi]$, $0 \leq S_{m,n,\alpha}(\theta) \leq \frac{1}{mn^2}$;

(5) for $\theta \in [\frac{\pi}{2}, \pi]$, $-m(1 + 4\alpha) \leq S'_{m,n,\alpha}(\theta) \leq m$.

Proof of Lemma 6.6. We construct $S_{m,n,\alpha}$ on $[\frac{\pi}{2},\pi]$, and extend it to $[0,\pi]$ so that $S_{m,n,\alpha}(\pi-\theta) = -S_{m,n,\alpha}(\theta)$.

Choose a function $\chi_\alpha : \mathbb{R} \to [0,1]$, such that χ_α is smooth and even, $\chi_\alpha(t) = 1$ on $[-\alpha,\alpha]$, $\mathrm{supp}(\chi_\alpha) \subset [-1,1]$, and

$$-1 - 4\alpha \leq -\frac{1}{1 - 2\alpha} \leq \chi'_\alpha(t) \leq 0\,, \forall t \geq 0\,. \tag{6.10}$$

A natural Lipschitz candidate would be a piecewise linear function ξ_α which is equal to 1 in $[0,\alpha]$, and to $t \mapsto 1 - (t - \alpha)/(1 - \alpha)$ in $[\alpha,1]$. To get χ_α, we can regularize a function ξ_β, keeping the other properties at the price of a small loss in the control of the derivative in (6.10).

For $\theta \in [\frac{\pi}{2},\pi]$, we introduce $\hat{\theta} = \theta - \frac{\pi}{2}$.

We take $S_{m,n,\alpha}$ in the form

$$\begin{cases} S_{m,n,\alpha}(\theta) & = -m\, \chi_\alpha((mn)^2\, (\theta - \frac{\pi}{2}))\, \cos(\theta)\, \sin^{m-1}(\theta) \\ & = m\, \chi_\alpha((mn)^2\, \hat{\theta})\, \sin(\hat{\theta})\, \cos^{m-1}(\hat{\theta})\,. \end{cases} \tag{6.11}$$

Properties (1), (2) and (3) are clear. Property (4) follows from the inequality $|\sin(\hat{\theta})| \le |\hat{\theta}|$ and (2). To prove (5), we introduce

$$h(\theta) := -m\,\cos(\theta)\,\sin^{m-1}(\theta)\,.$$

For $m \ge 3$, we have

$$h'(\theta) = m^2\,\sin^{m-2}(\theta)\,\left(\frac{1}{m} - \sin^2(\hat{\theta})\right)\,,$$

and hence,

$$m\,\sin^{m-2}(\theta) \ge h'(\theta) \ge \frac{m}{2}\sin^{m-2}(\theta)\,,$$

in the set $\left\{\theta \mid 0 \le (mn)^2\hat{\theta} \le 1\right\}$, as soon as $n \ge 2$.
We have,

$$S'_{m,n,\alpha}(\theta) = \chi_\alpha((mn)^2\hat{\theta})\,h'(\theta) + (mn)^2\,\chi'_\alpha((mn)^2\hat{\theta})\,h(\theta)\,.$$

Using the inequality $|\cos\theta| = |\sin\hat{\theta}| \le |\hat{\theta}|$, and (6.10) for $0 \le \hat{\theta} \le \frac{1}{(mn)^2}$, we obtain

$$-\,m(1+4\alpha) \le S'_{m,n,\alpha}(\theta) \le m\,. \tag{6.12}$$

One can check directly that this inequality also holds for $m = 1$ and 2. Lemma 6.6 is proved.
□

Lemma 6.7 (Construction of $R_{m,n,\alpha}$)**.** *For all $m \ge 1$ and $\alpha \in (0,\frac{1}{4}]$, there exist $N(m,\alpha) \in \mathbb{N}$, and a sequence of functions $(R_{m,n,\alpha})_{n\ge1}$, $R_{m,n,\alpha} : [0,\pi] \to \mathbb{R}$ with the following properties for all $n \ge N(m,\alpha)$.*

(1) $R_{m,n,\alpha} \in C^\infty$ and $R_{m,n,\alpha}(\pi-\theta) = -R_{m,n,\alpha}(\theta)$ for all $\theta \in [0,\pi]$;
(2) for $\theta \in [\frac{\pi}{2}+\frac{1}{mn},\pi]$, $R_{m,n,\alpha}(\theta) = 0$;
(3) for $\theta \in [\frac{\pi}{2},\frac{\pi}{2}+\frac{\alpha}{(mn)^2}]$, $R_{m,n,\alpha}(\theta) = -m\,\cos(\theta)\,\sin^{m-1}(\theta)$;
(4) for $\theta \in [\frac{\pi}{2},\pi]$, $|R_{m,n,\alpha}(\theta)| \le \frac{2}{mn^2}$;
(5) for $\theta \in [\frac{\pi}{2},\pi]$, $-m(1+5\alpha) \le R'_{m,n,\alpha}(\theta) \le m(1+\alpha)$;
(6) $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\frac{1}{mn}} R_{m,n,\alpha}(\theta)d\theta = 0$;
(7) $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\frac{1}{mn}} |R_{m,n,\alpha}(\theta)|d\theta \le \frac{2m}{(mn)^3}$.

Proof of Lemma 6.7. We construct $R_{m,n,\alpha}$ on $[\frac{\pi}{2},\pi]$, and extend it to $[0,\pi]$ so that $R_{m,n,\alpha}(\pi-\theta) = -R_{m,n,\alpha}(\theta)$. Define

$$\beta_{m,n,\alpha} := \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\frac{1}{(mn)^2}} S_{m,n,\alpha}(t)\,dt. \tag{6.13}$$

Using (6.11) and a change of variable, we find that $\beta_{m,n,\alpha}$ satisfies,

$$\begin{cases} \beta_{m,n,\alpha} \geq m \int_0^{\frac{\alpha}{(mn)^2}} \sin(t)\, \cos^{m-1}(t)\, dt\,, \text{and} \\ \beta_{m,n,\alpha} \leq m \int_0^{\frac{1}{(mn)^2}} \sin(t)\, \cos^{m-1}(t)\, dt\,. \end{cases}$$

Using the inequalities $\frac{2}{\pi} t \leq \sin(t) \leq t$ for $t \in [0,\frac{\pi}{2}]$, we obtain,

$$\frac{\alpha^2}{\pi} \frac{m}{(mn)^4} \leq \frac{\alpha^2}{\pi} \frac{m}{(mn)^4} 2\cos^{m-1}(\frac{\alpha}{(mn)^2}) \leq \beta_{m,n,\alpha} \leq \frac{1}{2} \frac{m}{(mn)^4}\,, \tag{6.14}$$

where the first inequality holds provided that n is larger than some $N_1(m)$.

Choose a C^∞ function ξ, such that $0 \leq \xi \leq 1$, $\mathrm{supp}(\xi) \subset (\frac{1}{2},1)$, and $\int_{\mathbb{R}} \xi(t)\, dt = 1$. Note that for $n \geq 3$, $[0,\frac{1}{(mn)^2}] \cap [\frac{1}{2mn},\frac{1}{mn}] = \emptyset$.

Define $s_{m,n,\alpha}$ by,

$$s_{m,n,\alpha}(\theta) = -\, \gamma_{m,n,\alpha}\, \xi(mn\, \hat{\theta})\,, \tag{6.15}$$

where $\gamma_{m,n,\alpha}$ is a constant to be chosen later. Note that for $n \geq 3$, $\mathrm{supp}(S_{m,n,\alpha}) \cap \mathrm{supp}(s_{m,n,\alpha}) = \emptyset$.

Defining

$$R_{m,n,\alpha} = S_{m,n,\alpha} + s_{m,n,\alpha}\,, \tag{6.16}$$

Assertion (2) is satisfied. Choosing,

$$\gamma_{m,n,\alpha} = mn\, \beta_{m,n,\alpha}\,, \tag{6.17}$$

Assertion (6) is satisfied,

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\frac{1}{mn}} R_{n,\alpha}(t)\, dt = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\frac{1}{mn}} \left(S_{n,\alpha}(t) + s_{n,\alpha}(t)\right)\, dt = 0\,,$$

and the function $\int_{\frac{\pi}{2}}^{\theta} R_{n,\alpha}(t)\, dt$ vanishes for $\theta \geq \frac{\pi}{2} + \frac{1}{mn}$.

Using (6.14), we get

$$\frac{\alpha^2}{\pi} \frac{m}{(mn)^3} \leq \gamma_{m,n,\alpha} \leq \frac{1}{2} \frac{m}{(mn)^3},\ \text{for } n \geq N_1(m)\,. \tag{6.18}$$

Properties (1) and (3) are clear. Using the properties of $S_{m,n,\alpha}$ given by Lemma 6.6, inequality (6.18), and the fact that $\frac{1}{2} \leq mn\, \hat{\theta} \leq 1$ when $\xi' \neq 0$, we obtain

$$|R_{m,n,\alpha}(\theta)| \leq \frac{2}{mn^2}\,,$$

and

$$-m\left(1 + 4\alpha + \frac{1}{2(mn)^2}\, \|\xi'\|_\infty\right) \leq R'_{m,n,\alpha}(\theta) \leq m\left(1 + \frac{1}{2(mn)^2}\, \|\xi'\|_\infty\right)\,.$$

Assertions (4) and (5) follow by taking n larger than some $N(m,\alpha)$. Assertion (7) follows from Property (4). Lemma 6.7 is proved. □

Proof of Proposition 6.5. Recall that

$$P_{m,n,\alpha}(\theta) = \int_{\pi/2}^{\theta} R_{m,n,\alpha}(z)dz \ \geq 0\,,$$

for $\theta \geq \pi/2$, and that $P_{m,n,\alpha}$ is symmetric with respect to $\frac{\pi}{2}$. The properties of $P_{m,n,\alpha}$ follow from Lemma 6.7. For $|\theta - \frac{\pi}{2}| \leq \frac{\alpha}{(mn)^2}$, $P_{m,n,\alpha}(\theta) = 1 - \sin^m(\theta)$. We also note that $P_{m,n,\alpha}(\theta) = 0$ for $\theta \in (0, \frac{\pi}{2} - \frac{1}{mn}) \cup (\frac{\pi}{2} + \frac{1}{mn}, \pi)$. □

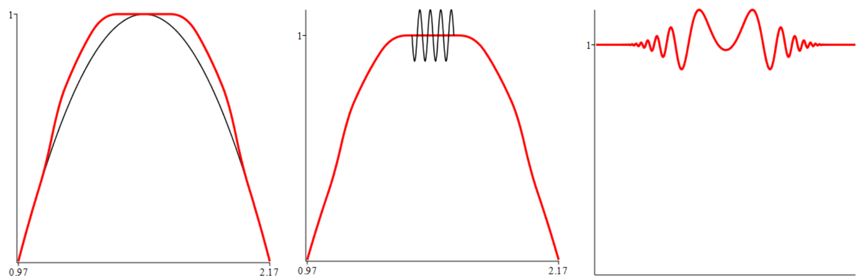

Fig. 7. Construction of a function $T_{m,n,\alpha}$

Figure 7 illustrates the construction of the functions $T_{m,n,\alpha}$.

- The left figure displays the graphs of the functions $\sin^m(\theta)$ (black) and $\sin^m(\theta) + P_{m,n,\alpha}(\theta)$ (red), for $\theta \in [\frac{\pi}{2} - 0.6, \frac{\pi}{2} + 0.6]$.
- The middle figure indicates (in black) where we will insert the rapidly oscillating perturbation $u_{m,n,\alpha}$ constructed in the next paragraph.
- The right figure displays a zoom on the function $u_{m,n,\alpha}$, whose support is contained in $\{\theta \mid \sin^m(\theta) + P_{m,n,\alpha}(\theta) = 1\}$.

6.3.2. Construction of $u_{m,n,\alpha}$. Define the function v as follows.

$$v(t) = \begin{cases} \exp\left(\frac{1}{(t+1)(t-1)}\right) \cos\left(\frac{1}{(t+1)(t-1)}\right) & \text{for } -1 < t < 1\,, \\ 0 \text{ elsewhere.} \end{cases} \tag{6.19}$$

This function is smooth, even, bounded, with bounded first and second derivatives. Define the family of functions $u_{m,n,\alpha} : [0,\pi] \to \mathbb{R}$, such that they are symmetric with respect to $\frac{\pi}{2}$, $u_{m,n,\alpha}(\pi - \theta) = u_{m,n,\alpha}(\theta)$ for all $\theta \in [0,\pi]$, and given by the formula

$$u_{m,n,\alpha}(\frac{\pi}{2} + \hat{\theta}) = a_{m,n,\alpha}\, v\left(\frac{(mn)^2\,\hat{\theta}}{\alpha}\right)\,, \tag{6.20}$$

where $\hat{\theta} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. The graph of $u_{m,n,\alpha}$ appears in Figure 7 (right). Note that $u_{m,n,\alpha}$ is supported in the set $\{\theta \mid \sin^m(\theta) + P_{m,n,\alpha}(\theta) = 1\}$. The constant $a_{m,n,\alpha}$ is chosen such that, for any $m,n \geq 1$, and $\alpha \in (0,\frac{1}{4}]$,

$$|u_{m,n,\alpha}| + |u'_{m,n,\alpha}| + |u''_{m,n,\alpha}| \leq \alpha\,. \tag{6.21}$$

6.3.3. Properties of $T_{m,n,\alpha}$.

From the construction, $T_{m,n,\alpha}(\theta) - 1$ changes sign infinitely many times on the interval $[\frac{\pi}{2} - \frac{\alpha}{(mn)^2}, \frac{\pi}{2} + \frac{\alpha}{(mn)^2}]$. Indeed, $\sin^m(\theta) + P_{m,n,\alpha}(\theta) = 1$ on that interval, and $u_{m,n,\alpha}$ changes sign infinitely often on the same interval. Also, since $\sin^m$, $P_{m,n,\alpha}$ and $u_{m,n,\alpha}$ are all smooth, $T_{m,n,\alpha}$ is smooth.

6.4. Non-degeneracy of the metric

We use Subsection 6.2. To the function $T_{m,n,\alpha}$ we associate the function $Q_{m,n,\alpha}$ through the relation (6.5). This function defines a conformal metric $g_{m,n,\alpha}$ on $\mathbb{S}^2$ provided that it is positive and smooth. Taking into account the relations (6.4) and (6.6). Write,

$$1 - Q_{m,n,\alpha}(\theta) = \frac{N(\theta)}{D(\theta)}\,, \tag{6.22}$$

where

$$\begin{cases} N(\theta) &= (\mathcal{K}_m P_{m,n,\alpha})(\theta) + (\mathcal{K}_m u_{m,n,\alpha})(\theta)\,, \\ D(\theta) &= m(m+1)\,\sin^2(\theta)\,T_{m,n,\alpha}(\theta)\,, \end{cases} \tag{6.23}$$

and recall that $\mathcal{K}_m(\sin^m) \equiv 0$. Because $P_{m,n,\alpha}$ and $u_{m,n,\alpha}$ are supported in $J_{mn} := [\frac{\pi}{2} - \frac{1}{mn}, \frac{\pi}{2} + \frac{1}{mn}]$, we have $Q_{m,n,\alpha} \equiv 1$ in $(0,\pi)\setminus J_{mn}$. It therefore suffices to study $Q_{m,n,\alpha}$ in the interval J_{mn} and, by symmetry with respect to $\frac{\pi}{2}$, in $J_{+,mn} := [\frac{\pi}{2}, \frac{\pi}{2} + \frac{1}{mn}]$. As above, we set $\hat{\theta} = \theta - \frac{\pi}{2}$.

From the definition of $T_{m,n,\alpha}$, Equation (6.7), we deduce

$$|T_{m,n,\alpha}(\theta) - 1| \leq 1 - \sin^m(\theta) + |P_{m,n,\alpha}(\theta)| + |u_{m,n,\alpha}(\theta)|\ .$$

Using (6.21) and Proposition 6.5(4), we obtain

$$|T_{m,n,\alpha}(\theta) - 1| \leq \frac{3m}{(mn)^2} + \alpha\,,\ \text{ for } n \geq N(m,\alpha),\ \theta \in J_{+,mn}\,.$$

It follows that for $0 < \alpha \leq \frac{1}{4}$, and $n \geq N(m,\alpha)$,

$$|T_{m,n,\alpha}(\theta) - 1| \leq \frac{1}{2} \text{ in } J_{+,mn}\,. \tag{6.24}$$

In particular, this inequality implies that for $0 < \alpha \leq \frac{1}{4}$, there exists some $C_\alpha > 0$ such that, for $\theta \in J_{+,mn}$ and for n large enough, $D(\theta) > C_\alpha$. It follows that $Q_{m,n,\alpha}$ is well-defined on $\mathbb{S}^2$ and C^∞ by equation (6.22).

Claim. For $0 < \alpha \leq \frac{1}{4}$, small enough, and for $n \geq N(m,\alpha)$ large enough, the function $Q_{m,n,\alpha}$ is close to 1.

Proof of the Claim. Since $Q_{m,n,\alpha} \equiv 1$ in $(0,\pi)\backslash J_{mn}$, and by symmetry around $\frac{\pi}{2}$, it suffices to consider $\theta \in J_{+,mn}$. In this interval, we have

$$\left| m(m+1)\sin^2(\theta) - m^2) \right| \leq m + \frac{1}{n^2} \,. \tag{6.25}$$

From (6.24), and for $n \geq N(m,\alpha)$ and $\theta \in J_{+,mn}$, we obtain

$$D(\theta) \geq m(m+1) \left(1 - \frac{5}{n^2} - \alpha\right) \geq m(m+1)(1-2\alpha) \,. \tag{6.26}$$

We estimate $N(\theta)$, using (6.23). From (6.3), (6.21), and (6.25), we obtain

$$\begin{aligned} |(\mathcal{K}_m u_{m,n,\alpha})(\theta)| \;&\leq |u''_{m,n,\alpha}(\theta)| + |u'_{m,n,\alpha}(\theta)| \\ &\quad + \big(m(m+1)\sin^2(\theta) - m^2)|u_{m,n,\alpha}(\theta)|\big) \\ &\leq (m+1)\alpha \,. \end{aligned}$$

We estimate $|\mathcal{K}_m P_{m,n,\alpha}|$ as follows.

$$\begin{aligned} |(\mathcal{K}_m P_{m,n,\alpha})(\theta)| \;&\leq |P''_{m,n,\alpha}(\theta)| + |P'_{m,n,\alpha}(\theta)| \\ &\quad + \big(m(m+1)\sin^2(\theta) - m^2)|P_{m,n,\alpha}(\theta)|\big) \,. \end{aligned}$$

Using the estimates in Proposition 6.5, and the fact that $|\sin(t)| \leq |t|$, we obtain the following inequalities for $n \geq N(m,\alpha)$ and $\theta \in J_{+,mn}$,

$$\begin{cases} |P''_{m,n,\alpha}(\theta)| & \leq m(1+5\alpha) \,, \\ |P'_{m,n,\alpha}(\theta)| & \leq \frac{2m}{(mn)^2} \,, \\ |P_{m,n,\alpha}(\theta)| & \leq \frac{2m}{(mn)^3} \,. \end{cases}$$

From these estimates and (6.25), we obtain

$$|(\mathcal{K}_m P_{m,n,\alpha})(\theta)| \leq m(1+5\alpha) + \frac{5}{n^2}$$

for $n \geq N(m,\alpha)$ and $\theta \in J_{+,mn}$.

Finally, for $\theta \in J_{+,mn}$ and $n \geq N(m,\alpha)$, we have

$$\begin{cases} m(m+1)(1-2\alpha) & \leq D(\theta) \,, \\ N(\theta) & \leq m\left(1 + \frac{5}{n^2} + 5\alpha\right) \,, \\ \left|1 - Q_{m,n,\alpha}(\theta)\right| & \leq \frac{1}{m+1}\left(1 + \frac{10}{n^2} + 10\alpha\right) \,. \end{cases}$$

The claim is proved. ∎

Combining the previous estimates, we obtain the main result of this section.

Proposition 6.8. *For any $m \geq 1$, and $\alpha \in (0,\frac{1}{24})$, there exists $N_1(m,\alpha)$ such that, for $n \geq N_1(m,\alpha)$,*

$$|Q_{m,n,\alpha}(\theta) - 1| \leq \frac{1}{m+1}(1 + 12\alpha)\,. \tag{6.27}$$

In particular, the metric $g_{m,n,\alpha} = Q_{m,n,\alpha}\, g_0$ is smooth, non-degenerate, and close to g_0 provided that m is large enough.

6.5. Proof of Propositions 6.1 and 6.2

We now apply the results obtained in Subsections 6.2, 6.3, and 6.4.

6.5.1. Proof of Proposition 6.1. Fix $m = 1$, and define the function Φ in spherical coordinates by,

$$\Phi(\theta,\phi) = T_{1,n,\alpha}(\theta)\, \cos(\phi)\,,$$

where $0 < \alpha \leq \frac{1}{24}$, and n is large enough, according to Subsection 6.4. The function Φ is clearly smooth away from the north and south poles of the sphere (θ away from 0 and π). Near the poles, Φ is equal to the spherical harmonic $\sin(\theta)\, \cos(\phi)$. It follows that Φ is smooth. By Proposition 6.8, the function $Q_{1,n,\alpha}$ associated with $T_{1,n,\alpha}$ by the relation (6.5) extends to a smooth positive function on $\mathbb{S}^2$. Choose $G = Q_{1,n,\alpha}\, g_0$, where g_0 is the standard round metric. Then, according to Subsection 6.2

$$-\,\Delta_G \Phi = 2\,\Phi. \tag{6.28}$$

This proves Assertions 6.1(2) and (3). Assertion 6.1(4) is a consequence of the min-max. Indeed, by Proposition 6.8,

$$\frac{1}{2} - 6\alpha \leq G \leq \frac{3}{2} + 6\alpha\,. \tag{6.29}$$

According to our choice of α, the left-hand side of this inequality is positive. Call R_G, resp. R_0, the Rayleigh quotient of $(\mathbb{S}^2,g_G)$, resp. on $(\mathbb{S}^2,g_0)$. Then, by (6.29),

$$\frac{2}{3}\,(1+4\alpha)^{-1}\, R_0(\psi) \leq R_G(\psi) \leq \frac{1}{2}\,(1-12\alpha)^{-1}\, R_0(\psi)\,,$$

for all $0 \neq \psi \in \mathbb{S}^2$. From the min-max, we conclude that

$$\frac{2}{3}\,(1+4\alpha)^{-1}\, \lambda_k(g_0) \leq \lambda_k(g_G) \leq 2\,(1-12\alpha)^{-1}\, \lambda_k(g_0)\,, \tag{6.30}$$

for all $k \geq 1$, where $\lambda_k(g)$ denotes the k-th eigenvalue of the Laplace-Beltrami operator for the metric g (eigenvalues arranged in nondecreasing order, starting from the labelling 1, with multiplicities accounted for).

We have $\lambda_1(g_0) = \lambda_1(g_G) = 0$. Since $\lambda_2(g_0) = \cdots = \lambda_4(g_0) = 2$, and $\lambda_5(g_0) = \cdots = \lambda_9(g_0) = 6$, we conclude from (6.30) and our choice of α, that

$$2 < 4\,(1+4\alpha)^{-1} \leq \lambda_5(g_G)\,. \tag{6.31}$$

From (6.28) we know that the eigenfunction Φ is associated with the eigenvalue 2 of $-\Delta_G$. It follows from (6.31) that this eigenvalue has labelling at most 4. This proves Assertion 6.1(4).

Assertion 6.1(1) is similar to Assertion (1) in Proposition 6.2. We defer its proof to Paragraph 6.5.3. Proposition 6.1 is proved. □

6.5.2. Proof of Proposition 6.2. According to Subsection 6.4, when $m \geq 1$, an appropriate choice of (α,n) yields a function

$$\Phi_m(\theta,\phi) = T_{m,n,\alpha}(\theta)\, \cos(m\phi)\,,$$

and a function $G_m = Q_{m,n,\alpha}$ satisfying (6.27), such that

$$-\Delta_{G_m}\Phi_m = m(m+1)\Phi_m.$$

Choosing m large enough, the metric $g_m = G_m\, g_0$ can be made as close as desired to the standard metric g_0, see however Remark 6.9. This proves Assertions 6.2(2)–(4).

6.5.3. Proof of Assertions 6.1(1) and 6.2(1) *(Nodal properties of the eigenfunction Φ_m).* For simplicity, denote the function $T_{m,n,\alpha}$ by T, so that

$$T(\theta) = \sin^m(\theta) + P_{m,n,\alpha}(\theta) + u_{m,n,\alpha}(\theta).$$

Let V denote the function

$$V(\theta) = \sin^m(\theta) + P_{m,n,\alpha}(\theta).$$

Taking Proposition 6.5 into account, we have the following properties for V: $V(\pi-\theta) = V(\theta)$, $V(\theta) = \sin^m(\theta)$ near 0 and π, and $V \equiv 1$ for $|\theta - \frac{\pi}{2}| \leq \frac{\alpha}{(mn)^2}$. In the interval $[\frac{\pi}{2},\pi]$, we have,

$$V'(\theta) = m\, \cos(\theta)\, \sin^{m-1}(\theta)\, \Big(1 - \chi_\alpha(m^2n^2\hat{\theta})\Big) - \gamma\, \xi(mn\,\hat{\theta})\,,$$

where $\gamma := \gamma_{m,n,\alpha}$, see (6.17). It follows that $V'(\theta) \leq 0$ in $[\frac{\pi}{2},\pi]$, so that $0 \leq V(\theta) \leq 1$ in $[0,\pi]$. With the notation $u = u_{m,n,\alpha}$, recalling that $V(\theta) \geq 0$, and that $\mathrm{supp}(u) \subset \{V = 1\}$, we conclude that

$$\begin{cases} \{u(\theta) < 0\} \times [0,2\pi] \subset \{\Phi_m < 1\}\,, \\ \{\Phi_m \geq 1\} \subset \{u(\theta) \geq 0\} \times [0,2\pi]\,, \\ \{u(\theta) > 0\} \times \{0\} \subset \{\Phi_m > 1\}\,. \end{cases} \tag{6.32}$$

This means that the set $\{\Phi_m > 1\}$ has at least one connected component in each band $\{u(\theta) > 0\} \times [0,2\pi]$.

The proof of Proposition 6.2 is now complete. □

Remark 6.9. *Note that, by* (6.8)*, for any n and α, $T_{m,n,\alpha}(\pi/2) = 1$, $T'_{m,n,\alpha}(\pi/2) = 0$, $T''_{m,n,\alpha}(\pi/2) = 0$, and hence, by the relation* (6.5)*, $Q_{m,n,\alpha}(\pi/2) = \frac{m}{m+1} \neq 1$. Therefore, it is* *impossible for each G_m to be arbitrarily close to the round metric, regardless of the choice of n and α. Proposition 6.2 merely states that we can find a sequence of metrics that converge in some sense to the round metric.*

Remark 6.10. *If we restrict the eigenfunction Φ (Proposition 6.1) or Φ_m (Proposition 6.2) to the hemisphere $\{(\theta,\phi) \in [0,\frac{\pi}{2}) \times [0,2\pi]\}$, we obtain counterexamples to Conjecture 2.2 for the hemisphere, with a metric conformal to the standard metric g_0, and Neumann boundary condition.*

7. Final comments

With respect to the Extended Courant property, we would like to point out that there are ways of counting nodal domains of sums of eigenfunctions which avoid the pathologies exhibited in the examples constructed in Sections 5 and 6. In the deterministic framework, we mention [**55, 54**] in which the nodal count involves some weights. In the probabilistic framework, the topological complexity of the nodal set of a random sum of eigenfunctions can be estimated. We refer to the recent thesis [**58**], and its bibliography.

With respect to [**18**], we would like to point out that although our starting point is the same (the idea to construct Liouville metrics with an oscillatory component), our goals and methods are different.

8. Appendix

8.1. Bounds on the number of nodal domains on $\mathbb{S}^2$ with the round metric

The following result can be found in [**2**]:

Proposition 8.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d. Then, the number of nodal domains of its restriction to $\mathbb{S}^{n-1}$ is bounded by $2^{2n-1}d^{n-1}$.*

In the case of $\mathbb{S}^2$ with the round metric, every eigenfunction is the restriction of a harmonic homogeneous polynomial to the sphere. Also, for such a polynomial of degree ℓ, its eigenvalue on the sphere is $\ell(\ell+1)$, with multiplicity $2\ell+1$. For a sum w of spherical harmonics of degree less than or equal to ℓ, Conjecture 2.2 would give

$$\beta_0(w) \leq 1 + \sum_{k=0}^{l-1}(2k+1) \leq \ell^2 + 1.$$

Using Proposition 8.1, we get the following weaker estimate.

Corollary 8.2. *Let g_0 be the round metric for $\mathbb{S}^2$. Then, the sum w of spherical harmonics of degree less than or equal to ℓ has at most $8\,\ell^2$ nodal domains.*

However, the direct nodal count is highly unstable in the case of C^∞ metrics, as we have shown in Section 6, see also Section 7.

8.2. Isotropic quantum harmonic oscillator in dimension 2

In this section, we will show that Conjecture 2.2 is true for the harmonic oscillator $H : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), H = -\Delta + x^2 + y^2$.

Proposition 8.3. *Let f_i be the eigenfunctions of H with eigenvalues ordered in increasing order with multiplicities. Then, for any linear combination $f = \sum_{i=1}^{n} a_i f_i$, we have $\beta_0(f) \leq n$.*

A basis f_n of eigenfunctions of H is given by

$$H_{a,b}(x,y) := e^{-\frac{x^2+y^2}{2}} H_a(x) H_b(y)\,,\ 0 \leq a,b \in \mathbb{N}\,,$$

where H_n refers to the n-th Hermite polynomial.
The associated eigenvalue is given by $2(a+b+1)$, with multiplicity $a+b+1$. Therefore, counting multiplicities, for each n in the interval $[\frac{k(k+1)}{2}+1, \frac{(k+1)(k+2)}{2}]$ for some positive integer k, f_n is a polynomial of degree k.

For a polynomial f of degree k in 2 variables, we have the following upper bound on the number of its nodal domains:

Lemma 8.4. *For any polynomial f of degree k in $\mathbb{R}^2$,*

$$\beta_0(f) \leq k(k+1)/2 + 1\,.$$

The upper bound is achieved by k non-parallel lines.

To prove this, we first note that the number of nodal domains is bounded from above by $U(f) + S(f) + 1$, where $U(f)$ is the number of connected components of the nodal set and $S(f) = \sum(s_i - 1)$, where the sum is taken over all singular points a_i and s_i is the order of the singularity at a_i (the lowest homogeneous order term in the Taylor expansion of f around a_i).

Now, we use classical theorems by Bézout and Harnack, see **[16]**. Recall that for a curve γ defined by $\gamma = F^{-1}(0)$ for some polynomial F, a singular point of degree d is a point x on γ such that all partial derivatives of F of order less than or equal to d vanish at x, but some derivative of order $d+1$ does not vanish.

Theorem 8.5 (Bézout's theorem)**.** *Let f and g be real algebraic curves of degree m and n. If the number of points in the intersection of f and g is infinite, then the polynomials defining f and g have a common divisor. If the number of points in the intersection of f and g is finite, then it is less than or equal to mn.*

Theorem 8.6 (Harnack's curve theorem)**.** *Let f be a real irreducible polynomial in two variables, of degree k. Let a_i be the singular points of the nodal set, with order s_i. We have the following inequality*[13] *for the number of connected components of its nodal set:*

$$U(f) \leq \frac{(k-1)k}{2} - \sum_i \frac{s_i(s_i-1)}{2} + 1\,.$$

Now, we proceed by induction. For $k = 1$, the lemma is trivial. Now, consider a polynomial f of degree $k > 1$. It can be either irreducible or the product of two smaller degree polynomials.

If f is irreducible, then by Harnack's theorem we have

$$\beta_0(f) \leq (k-1)k/2 + 2\,,$$

since for all $a \geq 1$, $a - 1 \leq a(a-1)/2$.

If $f = PQ$ with $\deg P = j$ and $\deg Q = k - j$, the number of nodal domains is bounded by $\beta_0(P) + \beta_0(Q) + j(n-j) - 1$. Indeed, every intersection between P and Q adds the same number of nodal domains as the degree of their intersection, and this number can be bounded by Bézout's theorem. We need to substract 1 to remove the initial original domain of $\mathbb{R}^2$ (otherwise, multiplying two linear functions would give 5 nodal domains.)

By induction, we have the following inequality:

$$\begin{aligned}\beta_0(f) &\leq \frac{j(j+1)}{2} + \frac{(n-j)(n-j+1)}{2} + j(n-j) + 1 \\ &\leq \frac{n(n+1)}{2} + 1\,.\end{aligned}$$

Now, since this was achieved by P and Q being the product of linear factors, then f is a product of linear factors. This proves lemma 8.4. □

We can now complete the proof of proposition 8.3.
Let $n \in [k(k+1)/2 + 1, (k+1)(k+2)/2]$. Then, any linear combination of $f_1, f_2, \ldots, f_n$ will be a polynomial of degree at most k. Any such polynomial has at most $k(k+1)/2 + 1$ nodal domains. Therefore, Conjecture 2.2 is true in the case of the isotropic two-dimensional quantum harmonic oscillator.

Remark 8.7. *It is still unclear if this upper bound can be reached for any $k > 2$.*

Remark 8.8. *Considering the results of this paper, it seems likely that a small perturbation of either the metric in $\mathbb{R}^2$ or the potential could break this upper bound.*

[13] In fact, the original theorem as stated in **[16]** deals with algebraic curves in $\mathbb{RP}^2$. However, it is easily adapted to $\mathbb{R}^2$ by adding at most k unbounded components.

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