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Algèbres de Temperley–Lieb, Birman–Murakami–Wenzl et Askey–Wilson, et autres centralisateurs de $U_q(\mathfrak{sl}_2)$

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Algèbres de Temperley–Lieb, Birman–Murakami–Wenzl et Askey–Wilson, et autres centralisateurs de $U_q(\mathfrak{sl}_2)$

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Résumé

Ce mémoire contient trois articles reliés par l'idée sous-jacente d'une généralisation de la dualité de Schur–Weyl. L'objectif principal est d'obtenir une description algébrique du centralisateur de l'image de l'action diagonale de $U_q(\mathfrak{sl}_2)$ dans le produit tensoriel de trois représentations irréductibles, lorsque q n'est pas une racine de l'unité. La relation entre une algèbre de Askey–Wilson étendue $AW(3)$ et ce centralisateur est examinée à cet effet. Dans le premier article, les éléments du centralisateur de l'action de $U_q(\mathfrak{sl}_2)$ dans son produit tensoriel triple sont définis à l'aide de la matrice R universelle de $U_q(\mathfrak{sl}_2)$. Il est montré que ces éléments respectent les relations définissantes de $AW(3)$. Dans le deuxième article, la matrice R universelle de la superalgèbre de Lie $\mathfrak{osp}(1|2)$ est utilisée de manière similaire avec l'algèbre de Bannai–Ito $BI(3)$. Dans ce cas, le formalisme de la matrice R permet de définir l'algèbre de Bannai–Ito de rang supérieur $BI(n)$ comme le centralisateur de l'action de $\mathfrak{osp}(1|2)$ dans son produit tensoriel n-fois. Le troisième article propose une conjecture qui établit un isomorphisme entre un quotient de $AW(3)$ et le centralisateur de l'image de l'action diagonale de $U_q(\mathfrak{sl}_2)$ dans le produit tensoriel de trois représentations irréductibles quelconques. La conjecture est prouvée pour plusieurs cas, et les algèbres de Temperley–Lieb, Birman–Murakami–Wenzl et Temperley–Lieb à une frontière sont retrouvées comme quotients de l'algèbre de Askey–Wilson.

Mots-clés : algèbre de Askey–Wilson, algèbre quantique $U_q(\mathfrak{sl}_2)$, algèbre de Temperley–Lieb, algèbre de Birman–Murakami–Wenzl, centralisateurs, représentations, dualité de Schur–Weyl, matrice R universelle, algèbre de Bannai–Ito, superalgèbre de Lie $\mathfrak{osp}(1|2)$.

Abstract

This master thesis contains three articles related by the underlying idea of a generalization of the Schur–Weyl duality. The main objective is to obtain an algebraic description of the centralizer of the image of the diagonal action of $U_q(\mathfrak{sl}_2)$ in the tensor product of three irreducible representations, when q is not a root of unity. The connection between a centrally extended Askey–Wilson algebra $AW(3)$ and this centralizer is examined for this purpose. In the first article, the elements of the centralizer of the action of $U_q(\mathfrak{sl}_2)$ in its threefold tensor product are defined with the help of the universal R-matrix of $U_q(\mathfrak{sl}_2)$. These elements are shown to satisfy the defining relations of $AW(3)$. In the second article, the universal R-matrix of the Lie superalgebra $\mathfrak{osp}(1|2)$ is used in a similar fashion with the Bannai–Ito algebra $BI(3)$. In this case, the formalism of the R-matrix allows to define the higher rank Bannai–Ito algebra $BI(n)$ as the centralizer of the action of $\mathfrak{osp}(1|2)$ in its n-fold tensor product. The third article proposes a conjecture that establishes an isomorphism between a quotient of $AW(3)$ and the centralizer of the image of the diagonal action of $U_q(\mathfrak{sl}_2)$ in the tensor product of any three irreducible representations. The conjecture is proved for several cases, and the Temperley–Lieb, Birman–Murakami–Wenzl and one-boundary Temperley–Lieb algebras are recovered as quotients of the Askey–Wilson algebra.

Keywords : Askey–Wilson algebra, quantum algebra $U_q(\mathfrak{sl}_2)$, Temperley–Lieb algebra, Birman–Murakami–Wenzl algebra, centralizers, representations, Schur–Weyl duality, universal R-matrix, Bannai–Ito algebra, Lie superalgebra $\mathfrak{osp}(1|2)$.

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Introduction

Dans la théorie de la représentation, la dualité de Schur–Weyl est un résultat central qui relie deux groupes fondamentaux en mathématiques, soient le groupe symétrique S_k des permutations sur k objets et le groupe général linéaire des matrices n par n inversibles GL_n. La dualité a été montrée par Schur [45] au début du 20e siècle et a été reprise par Weyl dans ses ouvrages [52, 53], entre autres pour son intérêt en physique. En effet, elle permet de décomposer le produit tensoriel multiple d'un espace vectoriel en espaces irréductibles (sous l'action du groupe général linéaire) caractérisés par leurs propriétés symétriques. C'est cette dualité qui explique pourquoi, par exemple, dans un système de particules identiques avec spin, il y a une correspondance entre le spin total des états couplés et la symétrie sous l'échange des particules. Des versions analogues de la dualité de Schur–Weyl ont été établies depuis sa première formulation. En particulier, il existe une dualité pour l'algèbre quantique $U_q(\mathfrak{sl}_2)$. L'idée à la base du travail présenté dans ce mémoire est une généralisation de la dualité de Schur–Weyl pour $U_q(\mathfrak{sl}_2)$, de laquelle découleront des relations entre l'algèbre de Askey–Wilson et les algèbres de Temperley–Lieb et de Birman–Murakami–Wenzl.

D'une part, l'algèbre de Askey–Wilson a été introduite [54] dans le but de décrire les propriétés des polynômes orthogonaux du même nom [4]. Ceux-ci se trouvent au sommet dans la classification du schéma de Askey [30]; tous les autres polynômes orthogonaux de ce schéma peuvent en être obtenus comme limite ou cas spécial. L'algèbre de Askey–Wilson est définie par trois générateurs et des relations de q-commutation. Elle est importante notamment dans la théorie des paires de Leonard [47, 48]. En physique, l'algèbre de Askey–Wilson apparaît dans l'étude de systèmes intégrables [2, 3, 5–8, 51]. Par exemple, dans [8], un opérateur de Heun–Askey–Wilson, défini par des éléments de l'algèbre de Askey–Wilson, est diagonalisé par la méthode de l'ansatz de Bethe. Cet opérateur peut être obtenu à partir de la matrice de transfert associée à certaines solutions de l'équation de réflexion. Un autre exemple est le processus d'exclusion simple asymétrique (ASEP) en une dimension avec des conditions aux bords ouvertes. Dans ce modèle, des particules sur un réseau linéaire peuvent se déplacer à gauche ou à droite selon une probabilité de saut asymétrique paramétrée par q, avec la restriction qu'un site ne peut être occupé par plus d'une particule. Quatre autres paramètres servent à encoder les taux d'entrée et de sortie des particules aux bords. Certaines quantités physiques du système dans l'état stationnaire peuvent être calculées exactement à l'aide de représentations matricielles tridiagonales qui font intervenir les polynômes de Askey–Wilson [50]; l'algèbre de Askey–Wilson est ici sous-jacente.

D'autre part, l'algèbre de Temperley–Lieb a fait surface dans le contexte de la mécanique statistique [46]. Notée $TL_n(q)$, où q est un paramètre et n est un entier positif, cette algèbre peut être définie par $n-1$ générateurs et des relations. Un exemple qui illustre la présence de cette algèbre en physique est le modèle de la chaîne de spins XXZ. Le Hamiltonien XXZ introduit dans [1], qui décrit une chaîne de n spins $\frac{1}{2}$ avec certaines conditions frontières, peut s'écrire comme [37, 42]

$$H_{\text{XXZ}} = -\sum_{i=1}^{n-1} E_i, \tag{0.0.1}$$

$$E_i = \frac{1}{2}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - \frac{1}{2}(q+q^{-1})(\sigma_i^z\sigma_{i+1}^z - 1) + \frac{1}{2}(q-q^{-1})(\sigma_i^z - \sigma_{i+1}^z)\right), \tag{0.0.2}$$

où σ_i^x,σ_i^y,σ_i^z sont des matrices de Pauli agissant sur le facteur i de l'espace tensoriel $(\mathbb{C}^2)^{\otimes n}$. Les matrices E_i vérifient les relations de $TL_n(q)$ et en constituent donc une représentation. D'autres modèles sur réseau, tels que celui de la percolation, peuvent être étudiés à l'aide d'opérateurs (comme des matrices de transfert) définis par des éléments de $TL_n(q)$, voir par exemple [19, 36, 38, 43]. De façon reliée, l'algèbre de Temperley–Lieb a aussi un intérêt notable dans la théorie des noeuds [17, 27]; elle est un quotient de l'algèbre du groupe des tresses et elle peut être représentée par des diagrammes. L'observation de la similarité entre les relations du groupe des tresses et l'équation de Yang–Baxter, qui est de grande importance dans la résolution des systèmes intégrables, a mené à une procédure de Baxterisation [28]. Celle-ci consiste à produire des solutions de l'équation de Yang–Baxter à l'aide des représentations du groupe des tresses. L'algèbre de Temperley–Lieb est dite Baxterisable, car elle permet de produire de telles solutions. De même, l'algèbre de Birman–Murakami–Wenzl [9, 39] est un quotient de l'algèbre du groupe des tresses et elle est aussi Baxterisable.

Ces algèbres de Askey–Wilson d'un côté, et de Temperley–Lieb et Birman–Murakami–Wenzl de l'autre, sont importantes dans leurs champs d'étude respectifs, mais n'ont jamais été directement connectées. Dans ce mémoire, un lien précis sera établi via l'algèbre quantique $U_q(\mathfrak{sl}_2)$. La théorie des groupes quantiques a été développée [18, 25] suite à son apparition dans l'étude de systèmes intégrables quantiques. Dans cette théorie, un paramètre complexe q est introduit de telle sorte que dans la limite $q \to 1$, on retrouve la situation classique; on parle alors de q-déformation. Ainsi, l'algèbre $U_q(\mathfrak{sl}_2)$ est la q-déformation de l'algèbre universelle enveloppante $U(\mathfrak{sl}_2)$. Le langage mathématique qui permet de comprendre les groupes quantiques est celui des algèbres de Hopf. Un concept important dans la structure d'une algèbre de Hopf $\mathcal{A}$ est celui du coproduit, qui est un homomorphisme $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. Pour les groupes quantiques, contrairement à leur analogue classique, le coproduit n'est pas cocommutatif. Si l'on dénote par τ l'application qui renverse les facteurs du produit tensoriel, c'est-à-dire $\tau(x \otimes y) = y \otimes x$, alors ceci signifie que $\Delta \neq \tau \circ \Delta$. Cette non-cocommutativité est encodée dans une matrice R universelle $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ qui est telle que $\Delta(x)\mathcal{R} = \mathcal{R}\tau(\Delta(x))$, pour tout $x \in \mathcal{A}$, et qui satisfait l'équation de Yang–Baxter; on parle alors d'algèbre de Hopf quasi-triangulaire. L'algèbre $U_q(\mathfrak{sl}_2)$ admet une telle structure. Par ailleurs, ses représentations irréductibles sont bien connues et correspondent à celles de l'algèbre de Lie $\mathfrak{sl}_2$, lorsque q n'est pas une racine de l'unité. (Pour plus de détails sur les groupes quantiques, voir [29] par exemple.) C'est par la combinaison de la structure algébrique et des représentations de $U_q(\mathfrak{sl}_2)$ qu'il sera possible de connecter les algèbres de Temperley–Lieb et Birman–Murakami–Wenzl à l'algèbre de Askey–Wilson; plus précisément, ces algèbres seront toutes reliées par l'étude des centralisateurs de $U_q(\mathfrak{sl}_2)$ dans le produit tensoriel de trois représentations irréductibles, dans le cadre d'une généralisation de la dualité de Schur–Weyl.

0.1. Dualité de Schur–Weyl et généralisations

Il sera utile pour la suite de présenter la version classique de la dualité de Schur–Weyl. Plusieurs références traîtent du sujet, par exemple voir [22, 44]. Il est commun d'énoncer la dualité en termes de l'algèbre de Lie $\mathfrak{gl}_n$ plutôt que de son groupe de Lie GL_n; l'application exponentielle permet de passer de l'algèbre au groupe.

On considère un espace vectoriel V de dimension finie n, que l'on prendra plus simplement ici $V = \mathbb{C}^n$. Cet espace V constitue la représentation dite définissante de $\mathfrak{gl}_n$ (sur le corps $\mathbb{C}$). Le produit tensoriel de rang k de cet espace V est dénoté par $V^{\otimes k}$. Un élément $X \in \mathfrak{gl}_n$ (vu ici comme une matrice) admet une action diagonale sur l'espace $V^{\otimes k}$ qui est définie par

$$\begin{aligned} X \cdot (v_1 \otimes v_2 \otimes \ldots \otimes v_k) = & \quad Xv_1 \otimes v_2 \otimes \ldots \otimes v_k \\ & + v_1 \otimes Xv_2 \otimes \ldots \otimes v_k \\ & + \ldots \\ & + v_1 \otimes v_2 \otimes \ldots \otimes Xv_k. \end{aligned} \tag{0.1.1}$$

D'autre part, un élément σ du groupe symétrique S_k agit naturellement sur l'espace $V^{\otimes k}$ par la permutation des facteurs du produit tensoriel

$$\sigma \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}. \tag{0.1.2}$$

On dénote par $\mathbb{C}S_k$ l'algèbre du groupe symétrique, dans laquelle il est permis de prendre des combinaisons linéaires d'éléments de S_k sur le corps $\mathbb{C}$, et par $U(\mathfrak{gl}_n)$ l'algèbre universelle enveloppante de l'algèbre de Lie $\mathfrak{gl}_n$, qui est générée par les polynômes en les éléments de $\mathfrak{gl}_n$. Les actions (0.1.1) et (0.1.2) permettent de représenter les éléments des algèbres $U(\mathfrak{gl}_n)$

et $\mathbb{C}S_k$ par des matrices $n^k \times n^k$ dans l'espace des endomorphismes de l'espace vectoriel $V^{\otimes k}$

$$\pi : U(\mathfrak{gl}_n) \to \mathrm{End}(V^{\otimes k}), \tag{0.1.3}$$

$$\rho : \mathbb{C}S_k \to \mathrm{End}(V^{\otimes k}). \tag{0.1.4}$$

Les actions (0.1.1) et (0.1.2) commutent, ce qui implique que les matrices de l'ensemble $\pi(U(\mathfrak{gl}_n))$ commutent avec les matrices de l'ensemble $\rho(\mathbb{C}S_k)$. La dualité de Schur–Weyl est l'affirmation plus forte que les images des algèbres $U(\mathfrak{gl}_n)$ et $\mathbb{C}S_k$ dans l'espace $\mathrm{End}(V^{\otimes k})$ de toutes les matrices agissant sur $V^{\otimes k}$ sont des centralisateurs mutuels complets :

$$\pi(U(\mathfrak{gl}_n)) = \{M \in \mathrm{End}(V^{\otimes k}) \mid [M,\rho(\sigma)] = 0, \quad \forall \sigma \in \mathbb{C}S_k\}, \tag{0.1.5}$$

$$\rho(\mathbb{C}S_k) = \{M \in \mathrm{End}(V^{\otimes k}) \mid [M,\pi(X)] = 0, \quad \forall X \in U(\mathfrak{gl}_n)\}, \tag{0.1.6}$$

où $[X,Y] = XY - YX$ est le commutateur. Une conséquence importante de ce résultat est que l'espace du produit tensoriel de V se décompose comme suit

$$V^{\otimes k} = \bigoplus_{\lambda} V_\lambda \otimes M_\lambda. \tag{0.1.7}$$

Dans l'équation précédente, la somme se fait sur les partitions régulières λ du nombre k d'au plus n parties, que l'on peut en fait associer à des diagrammes de Young. L'espace V_λ est une représentation irréductible de S_k correspondant à la partition λ, et M_λ est une représentation irréductible de $\mathfrak{gl}_n$. Puisque les représentations irréductibles de $\mathfrak{gl}_n$ restent irréductibles pour l'algèbre $\mathfrak{sl}_n$, celle-ci admet une dualité semblable.

L'équation (0.1.6) implique que le centralisateur de l'image de $U(\mathfrak{gl}_n)$ est généré par l'image de $\mathbb{C}S_k$; en d'autres mots, l'application ρ surjecte sur ce centralisateur. Par contre, l'application ρ est injective si et seulement si $n \geq k$. Dans le cas $n = 2$ et $k = 3$ par exemple, le centralisateur est un quotient (de dimension 5) de l'algèbre $\mathbb{C}S_3$; il est en fait isomorphe à l'algèbre de Temperley–Lieb $TL_3(1)$.

Il existe des versions analogues de la dualité de Schur–Weyl pour des algèbres autres que $\mathfrak{gl}_n$. L'algèbre de Brauer $B_k(\eta)$ a justement été introduite [10] dans le but d'étudier le centralisateur pour les groupes orthogonal et symplectique. Dans le cas de $\mathfrak{o}_n$, c'est l'algèbre $B_k(n)$ qui joue le rôle analogue à $\mathbb{C}S_k$, et dans le cas de $\mathfrak{sp}_n$ avec n pair, c'est $B_k(-n)$ qui joue ce rôle. La dualité de Schur–Weyl a aussi été généralisée à l'algèbre $U_q(\mathfrak{sl}_n)$ [26]. L'action diagonale (0.1.1) doit être q-déformée dans ce cas; en fait, elle se définit à l'aide du coproduit de $U_q(\mathfrak{sl}_n)$. Lorsque q n'est pas une racine de l'unité, c'est l'image de l'algèbre de Hecke $H_k(q)$, une q-déformation de $\mathbb{C}S_k$, qui est le centralisateur de l'image de $U_q(\mathfrak{sl}_n)$.

Dans toutes les versions de la dualité de Schur–Weyl présentées jusqu'ici, l'espace vectoriel V correspond à la représentation définissante de l'algèbre étudiée. Or, il est bien connu par exemple que l'algèbre de Lie $\mathfrak{sl}_2$ admet des représentations irréductibles finies de dimension $2j+1$ pour un spin $j = \frac{1}{2},1,\frac{3}{2},\ldots$ Il est donc possible de considérer l'action de $U(\mathfrak{sl}_2)$ sur le produit tensoriel $M_{j_1} \otimes M_{j_2} \otimes \cdots \otimes M_{j_k}$ avec M_{j_i} une représentation de spin j_i quelconque. Le cas du produit tensoriel triple ($k = 3$) a été étudié récemment dans [12]. Le point de départ consistait à remarquer que le centralisateur de l'image de $U(\mathfrak{sl}_2)$ dans le produit tensoriel de trois représentations irréductibles de spins j_1, j_2 et j_3 est généré par les matrices représentant des Casimirs intermédiaires, qui respectent l'algèbre de Racah $R(3)$ [20]. Ce fait est relié au problème de Racah de $\mathfrak{sl}_2$, dans lequel trois moments angulaires j_1, j_2 et j_3 sont couplés de trois manières intermédiaires différentes. Une conjecture d'un quotient de l'algèbre de Racah qui serait isomorphe au centralisateur pour tout choix de spins j_1,j_2,j_3 a été proposée dans [12] et prouvée pour plusieurs cas. Dans le cas de trois représentations fondamentales $j_1 = j_2 = j_3 = \frac{1}{2}$, l'algèbre de Temperley–Lieb $TL_3(1)$ a été retrouvée. Il est aussi connu [31, 32] que dans le cas $j_1 = j_2 = j_3 = 1$, le centralisateur est isomorphe à l'algèbre de Brauer $B_3(3)$, qui a aussi été retrouvée comme quotient de l'algèbre de Racah. Par ailleurs, un travail similaire a été effectué [11] pour la superalgèbre de Lie $\mathfrak{osp}(1|2)$; dans ce cas, c'est l'algèbre de Bannai–Ito $BI(3)$ [49] qui est utilisée pour décrire le centralisateur de l'image de $\mathfrak{osp}(1|2)$ dans le produit tensoriel de trois représentations irréductibles. L'algèbre de Brauer est retrouvée comme quotient de $BI(3)$ dans le cas de trois représentations fondamentales.

0.2. Objectifs et résultats

L'objectif principal de ce mémoire est d'offrir une description algébrique du centralisateur de l'image de l'action diagonale de $U_q(\mathfrak{sl}_2)$ dans le produit tensoriel de trois représentations irréductibles de spins j_1, j_2 et j_3 quelconques, généralisant par ainsi les résultats de [12].

L'algèbre de Askey–Wilson est une q-déformation de l'algèbre de Racah et elle est impliquée dans le problème de Racah pour $U_q(\mathfrak{sl}_2)$ [21, 23]. La première étape consiste donc à établir les liens précis entre une algèbre de Askey–Wilson étendue (avec éléments centraux), dénotée $AW(3)$, et le centralisateur $\mathcal{C}_{j_1,j_2,j_3}$ de l'image de l'action diagonale de $U_q(\mathfrak{sl}_2)$, pour tout choix de représentations j_1,j_2,j_3. L'objectif est de poser un homomorphisme $\phi : AW(3) \to \mathcal{C}_{j_1,j_2,j_3}$, et de montrer que son image génère le centralisateur complet. La stratégie qui sera suivie est illustrée par les flèches pleines du diagramme de la figure 0.1. L'action diagonale de $U_q(\mathfrak{sl}_2)$ dans son produit tensoriel triple $U_q(\mathfrak{sl}_2)^{\otimes 3}$ est obtenue par l'application $\Delta^{(2)} = (\Delta \otimes \mathrm{id})\Delta$, où Δ est le coproduit de $U_q(\mathfrak{sl}_2)$. Le centralisateur de cette action dans le produit tensoriel triple $U_q(\mathfrak{sl}_2)^{\otimes 3}$ est dénoté $\mathfrak{C}_3$. L'application π_{j_1,j_2,j_3} permet de passer aux représentations matricielles, là où se trouvent les éléments du centralisateur d'intérêt $\mathcal{C}_{j_1,j_2,j_3}$.

La deuxième étape consiste à déterminer les éléments qui font partie du noyau de l'application ϕ, de sorte à obtenir un isomorphisme entre un quotient $\overline{AW}(j_1,j_2,j_3)$ de $AW(3)$ et le centralisateur $\mathcal{C}_{j_1,j_2,j_3}$. Cet isomorphisme permettrait de décrire le centralisateur de l'image de $U_q(\mathfrak{sl}_2)$ en termes de générateurs et relations pour n'importe quel choix de spins j_1,j_2,j_3. Cette étape est illustrée par les flèches pointillées dans le diagramme de la figure 0.1.

Finalement, la dernière étape consiste à examiner en détails des cas particuliers pour les spins j_1,j_2,j_3. En effet, de façon analogue au cas classique, il est connu que le centralisateur de l'image de l'action diagonale de $U_q(\mathfrak{sl}_2)$ dans le produit tensoriel de trois représentations fondamentales (spin $\frac{1}{2}$) est isomorphe à l'algèbre de Temperley–Lieb $TL_3(q)$, qui est un quotient de l'algèbre de Hecke $H_3(q)$. Il est aussi connu [32] que dans le cas de trois représentations identiques de dimension 3 (spin 1), le centralisateur est isomorphe à une spécialisation de l'algèbre de Birman–Murakami–Wenzl, qui est une q-déformation de l'algèbre de Brauer; cette spécialisation sera dénotée pour l'instant par $BMW_3(q)$. Le but serait alors d'obtenir explicitement les algèbres $TL_3(q)$ et $BMW_3(q)$ comme quotients de l'algèbre de Askey–Wilson. La stratégie est illustrée dans la figure 0.2. Les algèbres $TL_3(q)$ et $BMW_3(q)$ y sont reliées comme quotients de l'algèbre du groupe des tresses sur trois brins $\mathcal{B}_3$. Les flèches pointillées représentent les isomorphismes à prouver.

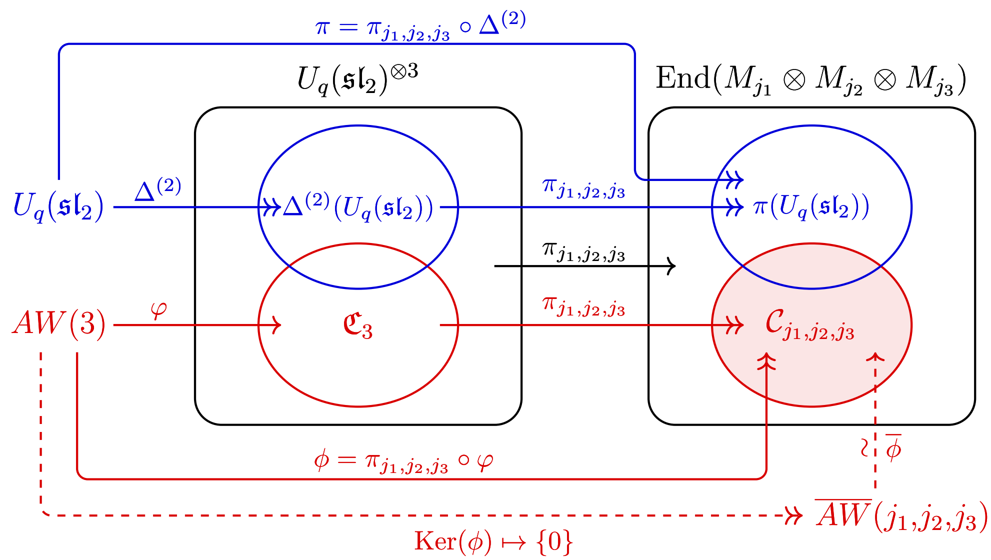

Fig. 0.1. Stratégie pour obtenir une description algébrique du centralisateur $\mathcal{C}_{j_1,j_2,j_3}$

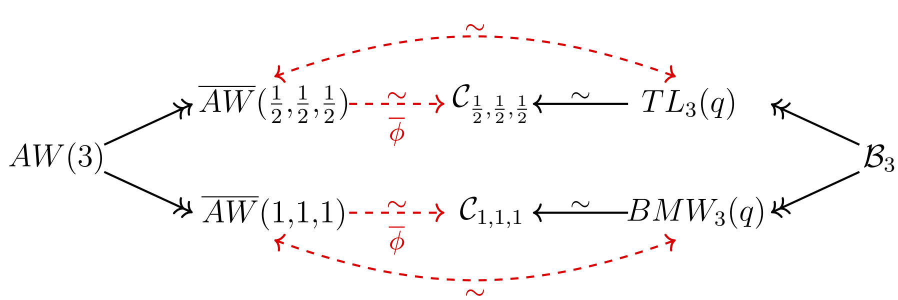

Fig. 0.2. Stratégie pour relier l'algèbre de Askey–Wilson aux algèbres de Temperley–Lieb et Birman–Murakami–Wenzl

Ce mémoire est divisé en trois chapitres qui contiennent chacun un article. Dans le premier article, la matrice R universelle $\mathcal{R}$ de $U_q(\mathfrak{sl}_2)$ est utilisée principalement dans le but d'obtenir une définition intrinsèque du Casimir intermédiaire de $U_q(\mathfrak{sl}_2)^{\otimes 3}$ associé au couplage des spins j_1 et j_3. En effet, pour des raisons liées à la non-trivialité du coproduit q-déformé, ce Casimir intermédiaire était typiquement défini en termes des deux autres, de façon à leur faire respecter l'algèbre de Askey–Wilson. Le Théorème 1.3.1 du premier article offre une définition du troisième Casimir intermédiaire en termes de conjugaisons par des matrices R. Ce résultat est d'importance capitale pour la suite, car il permet de déterminer automatiquement les valeurs propres de ce Casimir intermédiaire dans des représentations. Dans le Corollaire 1.4.1, la matrice R est aussi utilisée pour montrer plus simplement l'homomorphisme qui existe entre $AW(3)$ et le centralisateur de $U_q(\mathfrak{sl}_2)$.

Dans le deuxième article, le formalisme de la matrice R universelle est utilisé de façon analogue pour la superalgèbre de Lie $\mathfrak{osp}(1|2)$. Les Propositions 2.3.1 et 2.3.2 donnent une définition intrinsèque du troisième Casimir intermédiaire, et l'homomorphisme entre l'algèbre $BI(3)$ et le centralisateur est discuté dans la sous-section 2.3.2. Dans la section 2.4 (voir Proposition 2.4.1), le formalisme de la matrice R permet d'obtenir un homomorphisme entre l'algèbre de Bannai–Ito de rang supérieur $BI(k)$ et le centralisateur de $\mathfrak{osp}(1|2)$ dans $\mathfrak{osp}(1|2)^{\otimes k}$.

Finalement, le troisième chapitre contient l'article principal qui a pour but de généraliser la dualité de Schur–Weyl de $U_q(\mathfrak{sl}_2)$ à l'aide de l'algèbre $AW(3)$. Le centralisateur d'intérêt $\mathcal{C}_{j_1,j_2,j_3}$ est introduit à la sous-section 3.3.2. L'homomorphisme ϕ entre $AW(3)$ et ce centralisateur est posé dans la Proposition 3.4.1. La surjectivité est prouvée dans la Proposition 3.4.2. Un quotient $\overline{AW}(j_1,j_2,j_3)$ de $AW(3)$ est proposé dans la Définition 3.4.1; le résultat principal de l'article consiste à conjecturer que ce quotient est isomorphe au centralisateur $\mathcal{C}_{j_1,j_2,j_3}$ (Conjecture 3.4.1). Sans le travail effectué dans le premier article pour exprimer le troisième Casimir intermédiaire en termes de conjugaisons par des matrices R, il aurait été plus difficle de justifier cette conjecture, peut-être même impossible. Il est montré dans la Proposition 3.4.3 que la conjecture ne dépend pas de l'ordre des spins j_1,j_2,j_3. Dans la section 3.5, les représentations irréductibles du quotient $\overline{AW}(j_1,j_2,j_3)$ sont examinées dans le but de supporter la conjecture. Celle-ci est vérifiée pour plusieurs cas dans les sections qui suivent. Dans le cas $j_1 = j_2 = j_3 = \frac{1}{2}$, la conjecture est prouvée dans le Théorème 3.6.1, et l'algèbre de Temperley–Lieb est retrouvée dans le Théorème 3.6.2. Le cas $j_1 = j_2 = j_3 = 1$ est vérifié dans le Théorème 3.7.1, et l'algèbre de Birman–Murakami–Wenzl est obtenue comme quotient de $AW(3)$ dans le Théorème 3.7.2. La conjecture est aussi prouvée pour $j_1 = j_2 = j_3 = \frac{3}{2}$ dans le Théorème 3.8.1; dans ce cas, la description algébrique du centralisateur n'était pas connue. Finalement, le cas $j_1 = 1,\frac{3}{2},2,...$ et $j_2 = j_3 = \frac{1}{2}$ est vérifié dans le Théorème 3.9.1. Dans ce cas, il est montré que le centralisateur est isomorphe à une algèbre de Temperley–Lieb à une frontière (Théorème 3.9.2) [35, 38, 41]. Notons qu'il a aussi été vérifié tout au long de cet article que les résultats de [12] sont retrouvés dans la limite $q \to 1$.

0.3. Contributions de l'auteure

Le sujet du présent mémoire a été proposé par le directeur de recherche Luc Vinet, qui a de façon générale encadré le projet et contribué aux articles. L'idée était de faire suite aux résultats récents obtenus dans [11, 12] et de les généraliser. Les articles présentés dans ce mémoire ont tous été le fruit d'une collaboration avec Nicolas Crampé.

Nicolas Crampé a eu l'idée d'utiliser la matrice R universelle dans la définition des Casimirs intermédiaires. Il a rédigé en partie le premier article. L'auteure a contribué aux discussions qui ont mené à l'article, aux calculs et aux preuves qui y sont présentés et à sa rédaction. Luc Vinet a aussi été présent dans les discussions, a révisé l'article et a contribué à la rédaction de certaines parties. Julien Gaboriaud a contribué à améliorer la Proposition 1.4.1 et le Corollaire 1.4.1. Il a fait une révision de l'article et a fourni des commentaires.

Le deuxième article a été rédigé en grande partie par Nicolas Crampé et Luc Vinet. L'auteure a été impliquée dans les discussions, dans sa révision et sa correction.

Le troisième article a été principalement rédigé par l'auteure, qui a effectué la majorité des calculs et des preuves. Nicolas Crampé a notamment fourni de l'aide pour la preuve de la Proposition 3.4.2 et a proposé l'idée d'étudier les représentations irréductibles du quotient de $AW(3)$ (section 3.5). Plus généralement, Nicolas Crampé et Luc Vinet ont participé aux discussions, ont proposé des améliorations et ont révisé l'article.

Chapitre 1

Revisiting the Askey–Wilson algebra with the universal R-matrix of $U_q(\mathfrak{sl}_2)$

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Abstract. A description of the embedding of a centrally extended Askey–Wilson algebra, $AW(3)$, in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ is given in terms of the universal R-matrix of $U_q(\mathfrak{sl}_2)$. The generators of the centralizer of $U_q(\mathfrak{sl}_2)$ in its three-fold tensor product are naturally expressed through conjugations of Casimir elements with R. They are seen as the images of the generators of $AW(3)$ under the embedding map by showing that they obey the $AW(3)$ relations. This is achieved by introducing a natural coaction also constructed with the help of the R-matrix.

1.1. Introduction

This letter addresses a long-standing question regarding the intrinsic description of the generators of a centrally extended Askey–Wilson algebra in its embedding into $U_q(\mathfrak{sl}_2)^{\otimes 3}$. The answer will be shown to involve Casimir elements and the universal R-matrix of $U_q(\mathfrak{sl}_2)$.

The Askey–Wilson algebra can be defined with three generators and relations. It has first been introduced [18] as the algebra realized by the recurrence and q-difference operators intervening in the bispectral problem associated to the Askey–Wilson polynomials [12]. This explains the name. Since the structure relations are not affected by truncations, this algebra also encodes the properties of the q-Racah polynomials. Owing to the connection with these $6j$ or Racah coefficients for $U_q(\mathfrak{sl}_2)$ [10], a centrally extended Askey–Wilson algebra $AW(3)$ can be realized as the centralizer of the diagonal action of this quantum algebra in its three-fold tensor product. Related are the references [11,13,14,16]. In this context, two generators of $AW(3)$ are naturally mapped under the coproduct onto the intermediate Casimir elements corresponding respectively to the recouplings of the first and last two factors in $U_q(\mathfrak{sl}_2)^{\otimes 3}$.

A natural algebraic interpretation of the image of the third generator has however been lacking. This was circumvented so far by using one of the relations which gives the third generator as the q-commutator of the other two; while this allows the homomorphism from $AW(3)$ into $U_q(\mathfrak{sl}_2)^{\otimes 3}$ to be defined, the resulting expression for this third generator is far from illuminating. Note that all three generators are needed to provide a PBW basis for $AW(3)$. Besides the fact that this leaves a picture which is not fully satisfactory, this is a serious shortcoming in attempts to generalize $AW(3)$ to algebras of higher ranks. The natural approach - in fact the only one that has been conceived - is to define $AW(n)$ as the centralizer of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes n}$. Proceeding with such an extension calls for an algebraic understanding of all centralizing elements in the tensor product. Significant progress towards describing the algebras $AW(n)$ have been made nevertheless. The algebra $AW(4)$ has been explored in [**15**] by including generators defined through the q-commutators of coproduct images of the Casimir element, as done for $AW(3)$, and obtaining from there various structure relations. Meaningful results have thus been found. The identification of the general $AW(n)$ has been attacked and largely advanced in [**5**, **7**]. Much has been achieved in this case by cleverly designing a coaction map that has been used to define the generators, starting from the Casimir element of $U_q(\mathfrak{sl}_2)$, so as to ensure that these generators obey a q-deformation of natural structure relations (i.e. those of the generalized Bannai–Ito algebra $BI(n)$ [**4**]) and by proving that this is so in many (but not all) cases. Still, in spite of this progress, an a priori algebraic description of the generators remained much desired.

We shall here settle this issue for $AW(3)$ by providing a simple expression for the image of its third generator in $U_q(\mathfrak{sl}_2)^{\otimes 3}$. The formula will involve conjugation with the universal R-matrix of $U_q(\mathfrak{sl}_2)$ and will be seen to explain the origin of the coaction introduced in [**5**]. Basic facts about $U_q(\mathfrak{sl}_2)$ and its universal R-matrix are collected in Section 1.2. Section 1.3 focuses on the centralizer of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$; it provides the algebraic description that was missing. An additional centralizing element, conjugated to the usual third generator of $AW(3)$ is also identified; this will be related to observations made in [**15**]. The universal R-matrix and the Yang–Baxter equation are central here. With the expressions for the generators (in $U_q(\mathfrak{sl}_2)^{\otimes 3}$) in hand, Section 1.4 looks at their products and recovers the $AW(3)$ relations. To that end, a map from $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ defined in terms of the R-matrix is introduced. It is pointed out that this map, once spelled out, coincides with the coaction used in [**5**]. The letter concludes with final remarks stressing the advantages of bringing the universal R-matrix in the description of the algebras $AW(n)$. As an illustration it is shown that a computation in $AW(4)$ can be performed with these tools in a comparatively much simpler way than otherwise.

1.2. $U_q(\mathfrak{sl}_2)$ and its universal R-matrix

In this section, we recall the definitions of the quantum algebra $U_q(\mathfrak{sl}_2)$ and of its universal R-matrix as well as some of their properties. This allows to fix the notations and to make this letter more self-contained.

The associative algebra $U_q(\mathfrak{sl}_2)$ is generated by E, F and q^H with the defining relations

$$q^H E = qEq^H \quad , \qquad q^H F = q^{-1} F q^H \quad \text{and} \qquad [E,F] = [2H]_q \ , \tag{1.2.1}$$

where $[X]_q = \frac{q^X - q^{-X}}{q - q^{-1}}$ and $q \neq \pm 1, \pm\, i$. The center of this algebra is generated by the following Casimir element

$$C = -\frac{(q-q^{-1})^2}{q+q^{-1}} \left(FE + \frac{qq^{2H} + q^{-1}q^{-2H}}{(q-q^{-1})^2} \right) \ . \tag{1.2.2}$$

The normalization of the Casimir element C is irrelevant but chosen to yield computational simplifications. There exists a homomorphism $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$, called comultiplication, defined by

$$\Delta(E) = E \otimes q^{-H} + q^H \otimes E \quad , \qquad \Delta(F) = F \otimes q^{-H} + q^H \otimes F \quad \text{and} \qquad \Delta(q^H) = q^H \otimes q^H \ . \tag{1.2.3}$$

We recall that this comultiplication is coassociative

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta \ . \tag{1.2.4}$$

The quantum algebra $U_q(\mathfrak{sl}_2)$ is quasi-triangular because there exists a universal R-matrix $\mathcal{R} \in U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ which is invertible and satisfies

$$\Delta(x)\mathcal{R} = \mathcal{R}\Delta^{op}(x), \qquad \text{for } x \in U_q(\mathfrak{sl}_2) \ , \tag{1.2.5}$$

where the opposite comultiplication $\Delta^{op}(x) = x_{(2)} \otimes x_{(1)}$ if $\Delta(x) = x_{(1)} \otimes x_{(2)}$ in the Sweedler notation, and

$$(\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{12}\mathcal{R}_{13} \qquad \text{and} \qquad (\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{23}\mathcal{R}_{13} \ . \tag{1.2.6}$$

In the previous relation (1.2.6), we have used the usual notations $\mathcal{R}_{12} = \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha$ and $\mathcal{R}_{13} = \mathcal{R}^\alpha \otimes 1 \otimes \mathcal{R}_\alpha$ where $\mathcal{R} = \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha$ (the sum w.r.t. α is understood). We will also use the following element

$$\widetilde{\mathcal{R}} = \mathcal{R}_{21} = \mathcal{R}_\alpha \otimes \mathcal{R}^\alpha \ , \tag{1.2.7}$$

satisfying

$$\Delta^{op}(x)\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}\Delta(x), \qquad \text{for } x \in U_q(\mathfrak{sl}_2) \ . \tag{1.2.8}$$

The universal R-matrix also satisfies the Yang–Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \ , \tag{1.2.9}$$

and takes the following explicit form [8]

$$\mathcal{R} = q^{2(H\otimes H)} \sum_{n=0}^{\infty} \frac{(q-q^{-1})^n}{[n]_q!} q^{n(n-1)/2} (Eq^H \otimes q^{-H}F)^n \ , \tag{1.2.10}$$

where $[n]_q! = [n]_q[n-1]_q \dots [2]_q[1]_q$ and, by convention, $[0]_q! = 1$. For future convenience, by using the commutation relations of $U_q(\mathfrak{sl}_2)$, we rewrite $\widetilde{\mathcal{R}}$ as follows

$$\widetilde{\mathcal{R}} = \sum_{n=0}^{\infty} \frac{(q-q^{-1})^n}{[n]_q!} q^{n(n-1)/2} (Fq^H \otimes q^{-H}E)^n \ \ q^{2(H\otimes H)} = \Theta \ q^{2(H\otimes H)} \ . \tag{1.2.11}$$

1.3. Centralizer of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$

In this section, we want to describe the centralizer $\mathfrak{C}_3$ of the diagonal action of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$:

$$\mathfrak{C}_3 = \{X \in U_q(\mathfrak{sl}_2)^{\otimes 3} \ \big| \ [(\Delta \otimes \text{id})\Delta(x), X] = 0 \ , \ \ \forall x \in U_q(\mathfrak{sl}_2)\}. \tag{1.3.1}$$

Let us define the so-called intermediate Casimir elements (in Sweedler's notation)

$$C_{12} = \Delta(C) \otimes 1 = C_{(1)} \otimes C_{(2)} \otimes 1 \quad \text{and} \qquad C_{23} = 1 \otimes \Delta(C) = 1 \otimes C_{(1)} \otimes C_{(2)} \ , \tag{1.3.2}$$

and the total Casimir element

$$C_{123} = (\Delta \otimes \text{id})\Delta(C) \ . \tag{1.3.3}$$

We define also $C_1 = C \otimes 1 \otimes 1$, $C_2 = 1 \otimes C \otimes 1$ and $C_3 = 1 \otimes 1 \otimes C$. By using that the Casimir element C is central in $U_q(\mathfrak{sl}_2)$, we deduce for example that

$$[(\Delta \otimes \text{id})\Delta(x), C_{12}] = 0 \quad \text{and} \qquad [(\text{id} \otimes \Delta)\Delta(x), C_{23}] = 0 \ , \ \ \forall x \in U_q(\mathfrak{sl}_2) \ . \tag{1.3.4}$$

By definition (1.3.1), C_1, C_2, C_3, C_{12}, C_{23} and C_{123} belong to the centralizer $\mathfrak{C}_3$ with C_1, C_2, C_3 and C_{123} belonging to the center of $\mathfrak{C}_3$. It is well-known that these elements generate the Askey–Wilson algebra [18]. We will come back to this point in Section 1.4. Let us also remark that the connection between the Schur–Weyl duality and the Askey–Wilson algebra in the limit $q \to 1$ has been discussed in [2]. The crucial point is the knowledge of the minimal polynomials satisfied by these intermediate Casimir elements in some representations of sl_2. The case with q generic will be discussed in a forthcoming work.

In the case of $U(\mathfrak{sl}_2)$ (*i.e.* the limit $q \to 1$ of the case studied here), one can also prove that the intermediate Casimir $C_{13} = C_{(1)} \otimes 1 \otimes C_{(2)}$ belongs to the centralizer. For $q \neq 1$, it is not the case and the main objective of this letter is to provide a definition of this element for the quantum algebra.

Theorem 1.3.1. *The following elements of* $U_q(\mathfrak{sl}_2)^{\otimes 3}$

$$C_{13}^{(0)} = \widetilde{\mathcal{R}}_{23}^{-1} C_{13} \widetilde{\mathcal{R}}_{23} = \mathcal{R}_{12} C_{13} \mathcal{R}_{12}^{-1} \ , \tag{1.3.5}$$

$$C_{13}^{(1)} = \widetilde{\mathcal{R}}_{12}^{-1} C_{13} \widetilde{\mathcal{R}}_{12} = \mathcal{R}_{23} C_{13} \mathcal{R}_{23}^{-1} \ . \tag{1.3.6}$$

are in the centralizer $\mathfrak{C}_3$*, where* $\mathcal{R}$ *and* $\widetilde{\mathcal{R}}$ *are defined in* (1.2.10) *and* (1.2.11) *and* $C_{13} = C_{(1)} \otimes 1 \otimes C_{(2)}$.

Proof. By using the coassociativity of the comultiplication (1.2.4) and by conjugating with $\mathcal{R}_{23}$, the first relation in (1.3.4) reads

$$[(\mathrm{id} \otimes \Delta^{op})\Delta(x), \mathcal{R}_{23}^{-1} C_{12} \mathcal{R}_{23}] = 0 \ . \tag{1.3.7}$$

Finally, by exchanging the spaces 2 and 3, one gets that $C_{13}^{(0)}$ is in the centralizer

$$[(\mathrm{id} \otimes \Delta)\Delta(x), \underbrace{\widetilde{\mathcal{R}}_{23}^{-1} C_{13} \widetilde{\mathcal{R}}_{23}}_{=C_{13}^{(0)}}] = 0 \ . \tag{1.3.8}$$

One proves similarly that $\mathcal{R}_{12} C_{13} \mathcal{R}_{12}^{-1}$, $\widetilde{\mathcal{R}}_{12}^{-1} C_{13} \widetilde{\mathcal{R}}_{12}$ and $\mathcal{R}_{23} C_{13} \mathcal{R}_{23}^{-1}$ are in the centralizer $\mathfrak{C}_3$.

We must prove also the equality between $\widetilde{\mathcal{R}}_{23}^{-1} C_{13} \widetilde{\mathcal{R}}_{23}$ and $\mathcal{R}_{12} C_{13} \mathcal{R}_{12}^{-1}$. One gets

$$C_{13}^{(0)} = \widetilde{\mathcal{R}}_{23}^{-1} C_{13} \widetilde{\mathcal{R}}_{23} = \widetilde{\mathcal{R}}_{23}^{-1} \left(C_{(1)} \otimes 1 \otimes C_{(2)}\right) \widetilde{\mathcal{R}}_{23} = \widetilde{\mathcal{R}}_{23}^{-1} \mathcal{R}_{13} \left(C_{(2)} \otimes 1 \otimes C_{(1)}\right) \mathcal{R}_{13}^{-1} \widetilde{\mathcal{R}}_{23} \ , \tag{1.3.9}$$

where we have used property (1.2.5). The Yang–Baxter equation (1.2.9) implies that

$$C_{13}^{(0)} = \mathcal{R}_{12} \mathcal{R}_{13} \widetilde{\mathcal{R}}_{23}^{-1} \mathcal{R}_{12}^{-1} \left(C_{(2)} \otimes 1 \otimes C_{(1)}\right) \mathcal{R}_{12} \widetilde{\mathcal{R}}_{23} \mathcal{R}_{13}^{-1} \mathcal{R}_{12}^{-1} \ . \tag{1.3.10}$$

Now, from (1.2.6), one deduces that $[\Delta(C) \otimes 1, (\Delta \otimes \mathrm{id})(\mathcal{R})] = [\Delta(C) \otimes 1, \mathcal{R}_{23} \mathcal{R}_{13}] = 0$ and that $[\left(C_{(2)} \otimes 1 \otimes C_{(1)}\right), \mathcal{R}_{12} \widetilde{\mathcal{R}}_{23}] = 0$. Then, one obtains

$$C_{13}^{(0)} = \mathcal{R}_{12} \mathcal{R}_{13} \left(C_{(2)} \otimes 1 \otimes C_{(1)}\right) \mathcal{R}_{13}^{-1} \mathcal{R}_{12}^{-1} = \mathcal{R}_{12} C_{13} \mathcal{R}_{12}^{-1} \ . \tag{1.3.11}$$

The equality between $\widetilde{\mathcal{R}}_{12}^{-1} C_{13} \widetilde{\mathcal{R}}_{12}$ and $\mathcal{R}_{23} C_{13} \mathcal{R}_{23}^{-1}$ is proven similarly. □

From relations (1.3.5) and (1.3.6), one deduces that $C_{13}^{(0)}$ and $C_{13}^{(1)}$ are conjugated:

$$C_{13}^{(1)} = \mathcal{R}_{23} \widetilde{\mathcal{R}}_{23} C_{13}^{(0)} (\mathcal{R}_{23} \widetilde{\mathcal{R}}_{23})^{-1} = (\mathcal{R}_{12} \widetilde{\mathcal{R}}_{12})^{-1} C_{13}^{(0)} \mathcal{R}_{12} \widetilde{\mathcal{R}}_{12} \ . \tag{1.3.12}$$

1.4. The Askey–Wilson algebra $AW(3)$

In this section, we study the algebra satisfied by the intermediate Casimir elements introduced in the previous section and connect it with the central extension $AW(3)$ of the Askey–Wilson algebra introduced in [18]. We start by proving the following lemma.

Lemma 1.4.1. *The map defined by*

$$\tau \ : \ U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \qquad (1.4.1)$$
$$x \mapsto \widetilde{\mathcal{R}}^{-1}(1 \otimes x)\widetilde{\mathcal{R}}$$

yields the following explicit expressions when acting on the different elements of $U_q(\mathfrak{sl}_2)$ listed below:

$$\tau(C) = 1 \otimes C \ , \qquad (1.4.2)$$
$$\tau(q^{-H}E) = q^{-2H} \otimes q^{-H}E \ , \qquad (1.4.3)$$
$$\tau(q^{-2H}) = 1 \otimes q^{-2H} - (q-q^{-1})^2 q^{-H}F \otimes q^{-H}E \ , \qquad (1.4.4)$$
$$\tau(Fq^{-H}) = q^{2H} \otimes Fq^{-H} + q^{-1}(q+q^{-1})Fq^{H} \otimes (C + q^{-2H}) - (q-q^{-1})^2 F^2 \otimes q^{-H}E \ . \qquad (1.4.5)$$

Proof. We must prove that the map given in the theorem reproduces relations (1.4.2)-(1.4.5). For relation (1.4.2), it is direct, knowing that C commutes with any element of $U_q(\mathfrak{sl}_2)$. To prove relation (1.4.3), one computes (using the explicit form (1.2.11) of $\widetilde{\mathcal{R}}$)

$$\tau(q^{-H}E) = \widetilde{\mathcal{R}}^{-1}(1 \otimes q^{-H}E)\Theta\, q^{2(H\otimes H)} = \widetilde{\mathcal{R}}^{-1}\Theta\, (1 \otimes q^{-H}E)q^{2(H\otimes H)} = q^{-2H} \otimes q^{-H}E \ , \qquad (1.4.6)$$

which reproduces (1.4.3).

We want now to compute $\tau(q^{-2H})$:

$$\tau(q^{-2H}) = \widetilde{\mathcal{R}}^{-1}(1 \otimes q^{-2H})\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}^{-1}(1 \otimes q^{-2H})q^{2(H\otimes H)} \sum_{n=0}^{\infty} a_n (q^{-H}F \otimes Eq^{H})^n \qquad (1.4.7)$$
$$= \widetilde{\mathcal{R}}^{-1} q^{2(H\otimes H)} \sum_{n=0}^{\infty} a_n q^{-2n} (q^{-H}F \otimes Eq^{H})^n \ (1 \otimes q^{-2H}) \ , \qquad (1.4.8)$$

where we have introduced the parameters

$$a_n = \frac{(q-q^{-1})^n}{[n]_q!} q^{n(n-1)/2} \ . \qquad (1.4.9)$$

Remarking that

$$a_n q^{-2n} = a_n - a_n [n]_q q^{-n}(q-q^{-1}) \ , \qquad (1.4.10)$$

one gets

$$\tau(q^{-2H}) = \widetilde{\mathcal{R}}^{-1} \left(\widetilde{\mathcal{R}} - q^{2(H\otimes H)} \sum_{n=0}^{\infty} a_{n+1}[n+1]_q q^{-(n+1)}(q-q^{-1})(q^{-H}F \otimes Eq^{H})^{n+1} \right) (1 \otimes q^{-2H}) \ . \qquad (1.4.11)$$

It is easy to show that the parameters a_n satisfy $a_{n+1}[n+1]_q = q^n(q-q^{-1})a_n$, which allows to recover (1.4.4).

Similarly, to prove (1.4.5), one computes

$$\tau(Fq^{-H}) = \widetilde{\mathcal{R}}^{-1}(1 \otimes Fq^{-H})\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}^{-1}(1 \otimes Fq^{-H})q^{2(H\otimes H)} \sum_{n=0}^{\infty} a_n (q^{-H}F \otimes Eq^{H})^n \tag{1.4.12}$$

$$= \widetilde{\mathcal{R}}^{-1} q^{2(H\otimes H)}(q^{2H} \otimes Fq^{-H}) \sum_{n=0}^{\infty} a_n (q^{-H}F \otimes Eq^{H})^n \ . \tag{1.4.13}$$

Then, the identity

$$[F,E^n] = \frac{[n]_q}{q-q^{-1}}(q^{n-1}q^{-2H} - q^{-(n-1)}q^{2H})E^{n-1} \tag{1.4.14}$$

can be used to write

$$\tau(Fq^{-H}) = \widetilde{\mathcal{R}}^{-1} q^{2(H\otimes H)} \sum_{n=0}^{\infty} a_n (q^{-H}F \otimes Eq^{H})^n \left(q^{-2n}q^{2H} \otimes Fq^{-H} + q^{-2}Fq^{H} \otimes (q^{-2n}q^{-2H} - q^{2H})\right) \ . \tag{1.4.15}$$

Using again relation (1.4.10), one finds

$$\tau(Fq^{-H}) = q^{2H} \otimes Fq^{-H} - q^{-1}(q-q^{-1})^2 Fq^{H} \otimes FE - Fq^{H} \otimes (q^{2H} - q^{-2H}) - (q-q^{-1})^2 F^2 \otimes q^{-H}E \ . \tag{1.4.16}$$

Finally, expressing FE in terms of C from definition (1.2.2), one recovers (1.4.5). □

Using Lemma 1.4.1, we can rewrite $C_{13}^{(0)}$ (1.3.5) as follows

$$C_{13}^{(0)} = (1 \otimes \tau)\Delta(C) \tag{1.4.17}$$

$$= (q^{2H} + C) \otimes \tau(q^{-2H}) + q^{2H} \otimes \tau(C) - \frac{(q-q^{-1})^2}{q+q^{-1}} \left(q^{H}E \otimes \tau(Fq^{-H}) + Fq^{H} \otimes \tau(q^{-H}E)\right). \tag{1.4.18}$$

Proposition 1.4.1. *The following relation*

$$\frac{1}{q-q^{-1}}[C_{12},C_{23}]_q = C_{13}^{(0)} + C_1C_3 + C_2C_{123} \tag{1.4.19}$$

holds in $U_q(\mathfrak{sl}_2)^{\otimes 3}$.

Proof. Using the expressions for the maps under τ given in Lemma 1.4.1, we obtain $C_{13}^{(0)}$ in terms of the generators of $U_q(\mathfrak{sl}_2)$. A direct computation using the commutation relations of $U_q(\mathfrak{sl}_2)$ proves the relation of the proposition. □

One of the advantages of the construction with the universal R-matrix is that we can deduce all the other defining relations of $AW(3)$ from (1.4.19) and some other relations.

Corollary 1.4.1. *The following relations*

$$\frac{1}{q-q^{-1}}[C_{13}^{(0)},C_{12}]_q = C_{23} + C_2C_3 + C_1C_{123}, \tag{1.4.20}$$

$$\frac{1}{q-q^{-1}}[C_{23},C_{13}^{(0)}]_q = C_{12} + C_1C_2 + C_3C_{123}, \tag{1.4.21}$$

$$\frac{1}{q-q^{-1}}[C_{23},C_{12}]_q = C_{13}^{(1)} + C_1C_3 + C_2C_{123}, \tag{1.4.22}$$

$$\frac{1}{q-q^{-1}}[C_{12},C_{13}^{(1)}]_q = C_{23} + C_2C_3 + C_1C_{123}, \tag{1.4.23}$$

$$\frac{1}{q-q^{-1}}[C_{13}^{(1)},C_{23}]_q = C_{12} + C_1C_2 + C_3C_{123}, \tag{1.4.24}$$

hold in $U_q(\mathfrak{sl}_2)^{\otimes 3}$.

Proof. We use the second relation in (1.3.5) as well as the definitions (1.3.3) to write relation (1.4.19) as follows

$$\frac{1}{q-q^{-1}}[\Delta(C)\otimes 1,C_{23}]_q = \mathcal{R}_{12}C_{13}\mathcal{R}_{12}^{-1} + C_1C_3 + C_2(\Delta\otimes \mathrm{id})\Delta(C)\ . \tag{1.4.25}$$

Exchanging the spaces 1 and 2, the previous relation becomes

$$\frac{1}{q-q^{-1}}[\Delta^{op}(C)\otimes 1,C_{13}]_q = \widetilde{\mathcal{R}}_{12}C_{23}\widetilde{\mathcal{R}}_{12}^{-1} + C_2C_3 + C_1(\Delta^{op}\otimes \mathrm{id})\Delta(C)\ , \tag{1.4.26}$$

which leads to (1.4.23) after conjugating by $\widetilde{\mathcal{R}}_{12}$ (using property (1.2.8)).

Then, one starts from the relation (1.4.23) we have just proven, uses the second relation in (1.3.6) to express $C_{13}^{(1)}$ and exchanges spaces 2 and 3 to write

$$\frac{1}{q-q^{-1}}[C_{13},\widetilde{R}_{23}C_{12}\widetilde{R}_{23}^{-1}]_q = 1\otimes\Delta^{op}(C) + C_2C_3 + C_1(\mathrm{id}\otimes\Delta^{op})\Delta(C)\ . \tag{1.4.27}$$

Conjugating with $\widetilde{R}_{23}$, one proves relation (1.4.20). Performing the same two steps starting from (1.4.20), one proves (1.4.22) and (1.4.21). Finally, the two same steps prove (1.4.24) and give again the equation (1.4.19). □

We now have a number of remarks regarding the merits of the R-matrix approach developed above.

Remark 1.4.1. *Relations* (1.4.19), (1.4.20) *and* (1.4.21) *are the defining relations of central extension $AW(3)$ of the Askey–Wilson algebra introduced in [**18**]. Therefore, the results presented in this letter offer another proof that the intermediate Casimir elements of $U_q(\mathfrak{sl}_2)$ provide a realization of $AW(3)$. In previous works [**5**, **6**, **15**, **16**, **18**], $C_{13}^{(0)}$ was defined by relation* (1.4.19) *whereas in our approach, it is defined independently of the commutation relations via relation* (1.3.5).

Remark 1.4.2. *The map τ with images given by* (1.4.2)*-*(1.4.5) *has in fact been introduced in [**5**, **6**] so as to obtain $C_{13}^{(0)}$ as in relation* (1.4.17)*. Our definition* (1.4.1) *gives a nice* *and powerful interpretation of this map. Let us remark that the comultiplication used in this letter is slightly different from the ones used in [**5**, **6**]. In order to establish exactly the correspondence, the following transformation on our generators and Casimir element must be performed:* $q^{2H} \to K$, $E \to EK^{-1/2}$, $F \to K^{1/2}F$, $q \to Q$ *and* $C \to -\Lambda/(Q + Q^{-1})$.

Remark 1.4.3. *To illustrate the appropriateness and advantages of definition* (1.4.1) *of the map* τ*, we here prove its coaction property in a much simpler way than the direct calculation described in [**5**, **6**]. Using relation* (1.2.6)*, it is easy to compute, for* $x \in U_q(\mathfrak{sl}_2)$,

$$(\Delta \otimes id)\tau(x) = (\Delta \otimes id)\left(\widetilde{\mathcal{R}}^{-1}(1 \otimes x)\widetilde{\mathcal{R}}\right) = \widetilde{\mathcal{R}}_{23}^{-1}\widetilde{\mathcal{R}}_{13}^{-1}(1 \otimes 1 \otimes x)\widetilde{\mathcal{R}}_{13}\widetilde{\mathcal{R}}_{23} \ , \tag{1.4.28}$$

and

$$(id \otimes \tau)\tau(x) = (id \otimes \tau)\left(\widetilde{\mathcal{R}}^{-1}(1 \otimes x)\widetilde{\mathcal{R}}\right) = \widetilde{\mathcal{R}}_{23}^{-1}\widetilde{\mathcal{R}}_{13}^{-1}(1 \otimes 1 \otimes x)\widetilde{\mathcal{R}}_{13}\widetilde{\mathcal{R}}_{23} \ . \tag{1.4.29}$$

This proves that $(\Delta \otimes id)\tau(x) = (id \otimes \tau)\tau(x)$ *and thus that* τ *is a left coaction.*

Remark 1.4.4. *We can define also a right coaction* $\check{\tau}$ *given by*

$$\begin{aligned} \check{\tau} \ : \ U_q(\mathfrak{sl}(2)) &\to U_q(\mathfrak{sl}(2)) \otimes U_q(\mathfrak{sl}(2)) \\ x &\mapsto \mathcal{R}(x \otimes 1)\mathcal{R}^{-1} \ , \end{aligned} \tag{1.4.30}$$

satisfying

$$(\check{\tau} \otimes id)\check{\tau} = (id \otimes \Delta)\check{\tau} \ . \tag{1.4.31}$$

*We can show following steps similar to those of the proof of Lemma 1.4.1 that this right coaction coincides with the one introduced recently in [**7**] with the identifications:* $q^{2H} \to K$, $E \to EK^{-1/2}$, $F \to K^{1/2}F$ *and* $C \to -\Lambda/(q + q^{-1})$.

Remark 1.4.5. *The element* $C_{13}^{(1)}$ *has been introduced previously in [**15**] (where it is called* $IQ^{(13)}$*) and defined by relation* (1.4.22)*. Our definition* (1.3.6) *gives a new interpretation of this element.*

1.5. Conclusion and perspective

In this letter, we study the centralizer of the diagonal action of $U_q(\mathfrak{sl}_2)$ and its connection with the Askey–Wilson algebra $AW(3)$. In comparison with the previous approaches, we have emphasized the relevance of the universal R-matrix of $U_q(\mathfrak{sl}_2)$. We believe that its use offers a deeper understanding of the realization of the Askey–Wilson algebra in terms of the intermediate Casimir elements. It should moreover simplify the computations for further investigations. To illustrate this point, let us show how one computation can be simplified with this approach in the higher rank generalization $AW(4)$ of $AW(3)$ examined in [15].

The algebra $AW(4)$ can be embedded in $U_q(\mathfrak{sl}_2)^{\otimes 4}$ and, in particular, one defines

$$C_{13}^{(0)} = \widetilde{\mathcal{R}}_{23}^{-1} C_{13} \widetilde{\mathcal{R}}_{23} = \mathcal{R}_{12} C_{13} \mathcal{R}_{12}^{-1} \ , \tag{1.5.1}$$

$$C_{24}^{(1)} = \widetilde{\mathcal{R}}_{23}^{-1} C_{24} \widetilde{\mathcal{R}}_{23} = \mathcal{R}_{34} C_{24} \mathcal{R}_{34}^{-1} \ . \tag{1.5.2}$$

Looking at the commutation relations, we can prove that these elements correspond to $Q^{(13)}$ and $IQ^{(24)}$ of [**15**]. In the formalism introduced here, we see immediately that

$$[C_{13}^{(0)}, C_{24}^{(1)}] = 0 \ , \tag{1.5.3}$$

whereas the proof without the use of the R-matrix presented in [**15**] is quite cumbersome. We believe that the R-matrix approach we have elaborated will prove quite helpful in the study of the higher rank generalizations of $AW(3)$. In a related series of papers [**2**], [**1**], the Temperley–Lieb algebra with $q = 1$, the Brauer algebra (and others) over 3 strands have been identified as quotients of the Racah [**9**] and Bannai–Ito [**17**] algebras of rank 1. The results reported here pave the way to the pursuit of this program for $AW(3)$ as well as in situations of higher ranks with an arbitrary number of strands. It is our intent to actively continue this research. Let us mention finally that, in a companion letter [**3**], we have provided a parallel description of the Bannai–Ito algebras using the universal R-matrix of the Lie superalgebra $\mathfrak{osp}(1|2)$.

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Chapitre 2

Bannai–Ito algebras and the universal R-matrix of $\mathfrak{osp}(1|2)$

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Abstract. The Bannai–Ito algebra $BI(n)$ is viewed as the centralizer of the action of $\mathfrak{osp}(1|2)$ in the n-fold tensor product of the universal algebra of this Lie superalgebra. The generators of this centralizer are constructed with the help of the universal R-matrix of $\mathfrak{osp}(1|2)$. The specific structure of the $\mathfrak{osp}(1|2)$ embeddings to which the centralizing elements are attached as Casimir elements is explained. With the generators defined, the structure relations of $BI(n)$ are derived from those of $BI(3)$ by repeated action of the coproduct and using properties of the R-matrix and of the generators of the symmetric group $\mathfrak{S}_n$.

2.1. Introduction

This letter explains the essential role that the universal R-matrix of $\mathfrak{osp}(1|2)$ plays in the algebraic underpinnings of the Bannai–Ito algebra $BI(3)$ and its higher rank generalization $BI(n)$.

The universal Bannai–Ito algebra $BI(3)$ is generated by the central elements $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_{123}$ and three generators $\mathcal{C}_{12}$, $\mathcal{C}_{23}$ and $\mathcal{C}_{13}$ satisfying the defining relations [4]

$$\{\mathcal{C}_{12},\mathcal{C}_{23}\} = 2(-\mathcal{C}_{13} + \mathcal{C}_1\mathcal{C}_3 + \mathcal{C}_2\mathcal{C}_{123})\,, \tag{2.1.1a}$$

$$\{\mathcal{C}_{12},\mathcal{C}_{13}\} = 2(-\mathcal{C}_{23} + \mathcal{C}_2\mathcal{C}_3 + \mathcal{C}_1\mathcal{C}_{123})\,, \tag{2.1.1b}$$

$$\{\mathcal{C}_{23},\mathcal{C}_{13}\} = 2(-\mathcal{C}_{12} + \mathcal{C}_1\mathcal{C}_2 + \mathcal{C}_3\mathcal{C}_{123})\,, \tag{2.1.1c}$$

where $\{X,Y\} = XY + YX$.

The algebra $BI(3)$ was first introduced in [15] as an encoding of the bispectral properties of the eponym orthogonal polynomials [1]. In this context the generators $\mathcal{C}_{12}$ and $\mathcal{C}_{23}$ are realized by the Dunkl shift operators of which the Bannai–Ito polynomials are eigenfunctions and the operator multiplication by the argument of those polynomials. In this representation the central terms $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{123})$ become constants related to the four parameters of the polynomials.

The centrally extended $BI(3)$ was subsequently defined in [9] following the observation that the Bannai–Ito polynomials are essentially the Racah coefficients of the Lie superalgebra $\mathfrak{osp}(1|2)$. This casts $BI(3)$ as the centralizer of the action of $\mathfrak{osp}(1|2)$ in the three-fold tensor product $U(\mathfrak{osp}(1|2))^{\otimes 3}$ where $U(\mathfrak{osp}(1|2))$ stands for the universal enveloping algebra of $\mathfrak{osp}(1|2)$. The generators $\mathcal{C}_{12}, \mathcal{C}_{23}, \mathcal{C}_{13}$ are then mapped to the Casimir elements attached to embeddings of $\mathfrak{osp}(1|2)$ into $\mathfrak{osp}(1|2)^{\otimes 3}$ which are indexed by the 2-element subsets of $\{1, 2, 3\}$. This paved the way to the construction of the extension $BI(n)$ of arbitrary rank as the centralizer of the action of $\mathfrak{osp}(1|2)$ in the n-fold product $U(\mathfrak{osp}(1|2))^{\otimes n}$ with the generators identified as the Casimir elements associated to $\mathfrak{osp}(1|2)$ embeddings now labeled by subsets A of $[n] = \{1, 2, ..., n\}$. This was actually achieved using models of $\mathfrak{osp}(1|2)$ given in terms of Dirac-Dunkl operators [6], [7]. For reviews of these algebras and some of their applications see [4], [5].

A notable feature of these tensorial constructs is the fact that the embeddings involved do not all correspond to the simple ones where only the factors of the n-fold product that are enumerated by the elements the sets A enter non-trivially. The proper Casimir elements are in some cases associated to modified embeddings. Sorting this out is addressed here. It will be shown that conjugations of the simple embeddings by the universal R-matrix will in general be required to ensure that the attached Casimir elements belong to the centralizer.

Throughout this paper, we shall use a presentation of $\mathfrak{osp}(1|2)$ that calls upon a grading involution P. This P is group-like under the coproduct and when this framework is used, it enters in the formula for the universal R-matrix. In a separate study [8], expressions for the centralizing elements of $\mathfrak{osp}(1|2)$ have been provided in situations where the grade involution admits a refinement as a product of supplementary involutions. This is manifestly the case under embeddings in tensor products. The centralizing elements thus given have been shown in [8] to coincide with the Casimirs of the modified embeddings. That this should be so will be made clear in the following.

The description of the Bannai–Ito algebra in the framework of the universal R-matrix of $\mathfrak{osp}(1|2)$ has the striking benefit of allowing to fully characterize abstractly $BI(n)$ for arbitrary n (in the centralizer view) without recourse to any model. As shall be shown, the centralizing elements associated to subsets A of $[n] = \{1, 2, ..., n\}$ are given through repeated action of the coproduct on $\mathfrak{osp}(1|2)$ Casimir elements and conjugation by products of braided universal R-matrices. With these generators in hand, the structure relations that they verify can be inferred consistently from those of $BI(3)$ (i.e. (2.1.1a)-(2.1.1c)) by exploiting properties of the R-matrix and of the permutations of the symmetric group $\mathfrak{S}_n$. A definite picture for the generalized Bannai–Ito algebra $BI(n)$ as the centralizer of $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes n}$ is thus obtained. This approach based on the universal R-matrix has already contributed to the understanding of the Askey–Wilson algebra of rank 1 [?] and the advances presented here in the description of the Bannai–Ito algebra for $n > 3$ should show the way towards a complete picture of the higher ranks Askey-Wilson algebras.

This letter will proceed as follows. Section 2.2 will offer a short review of $\mathfrak{osp}(1|2)$ and will focus on the universal R-matrix of this Lie superalgebra. In Section 2.3, the centralizing elements of $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes 3}$ will be given in terms of Casimir elements and the universal R-matrix will be shown to play a key role. The connection between that centralizer and $BI(3)$ will moreover be made. Section 2.4 will extend the results to $n > 3$ and derive the algebra homomorphism $BI(n) \to U(\mathfrak{osp}(1|2))^{\otimes n}$ making essential use of the universal R-matrix formalism. Short concluding remarks will follow in Section 2.5.

2.2. Properties of the Lie superalgebra $\mathfrak{osp}(1|2)$

2.2.1. The Lie superalgebra $\mathfrak{osp}(1|2)$

The superalgebra $\mathfrak{osp}(1|2)$ has two odd generators $F^{\pm}$ and three even generators H, $E^{\pm}$ satisfying the following (anti-)commutation relations [11]

$$[H,E^{\pm}] = \pm E^{\pm}\,, \qquad [E^{+},E^{-}] = 2H\,, \tag{2.2.1}$$

$$[H,F^{\pm}] = \pm\frac{1}{2}F^{\pm}\,, \quad \{F^{+},F^{-}\} = \frac{1}{2}H\,, \tag{2.2.2}$$

$$[E^{\pm},F^{\mp}] = -F^{\pm}\,, \quad \{F^{\pm},F^{\pm}\} = \pm\frac{1}{2}E^{\pm}\,. \tag{2.2.3}$$

The $\mathbb{Z}_2$-grading of $\mathfrak{osp}(1|2)$ can be encoded by the grading involution P satisfying

$$[P,E^{\pm}] = 0\,, \qquad [P,H] = 0\,, \qquad \{P,F^{\pm}\} = 0 \qquad \text{and} \qquad P^2 = 1\,. \tag{2.2.4}$$

One defines the central element C of $U(\mathfrak{osp}(1|2))$ by [12, 13]

$$C = 8[F^{+},F^{-}]P + P\,. \tag{2.2.5}$$

The $U(\mathfrak{osp}(1|2))$ algebra is endowed with a coproduct Δ defined as the algebra homomorphism satisfying

$$\Delta(E^{\pm}) = E^{\pm}\otimes 1 + 1\otimes E^{\pm}\,, \qquad \Delta(H) = H\otimes 1 + 1\otimes H\,, \tag{2.2.6}$$

$$\Delta(F^{\pm}) = F^{\pm}\otimes P + 1\otimes F^{\pm}\,, \qquad \Delta(P) = P\otimes P\,. \tag{2.2.7}$$

We recall that this comultiplication is coassociative

$$(\Delta\otimes \mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta\,. \tag{2.2.8}$$

2.2.2. The universal R-matrix of $\mathfrak{osp}(1|2)$

The universal R-matrix of $\mathfrak{osp}(1|2)$ is given by

$$\mathcal{R} = \frac{1}{2}(1\otimes 1 + P\otimes 1 + 1\otimes P - P\otimes P) \ . \tag{2.2.9}$$

For $x \in U(\mathfrak{osp}(1|2))$, it satisfies

$$\Delta(x)\mathcal{R} = \mathcal{R}\Delta^{op}(x) \ , \tag{2.2.10}$$

where the opposite comultiplication $\Delta^{op}(x) = x^{(2)}\otimes x^{(1)}$ if $\Delta(x) = x^{(1)}\otimes x^{(2)}$ in the Sweedler's notation. Let us note that

$$\mathcal{R}^2 = 1\otimes 1 \ , \qquad \mathcal{R}_{21} = \mathcal{R} \ . \tag{2.2.11}$$

The universal R-matrix (2.2.9) satisfies

$$(\mathrm{id}\otimes\Delta)\mathcal{R} = \mathcal{R}_{12}\mathcal{R}_{13} \quad \text{and} \quad (\Delta\otimes\mathrm{id})\mathcal{R} = \mathcal{R}_{23}\mathcal{R}_{13} \ . \tag{2.2.12}$$

It also satisfies the Yang–Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \ . \tag{2.2.13}$$

We remark that in the case of $\mathfrak{osp}(1|2)$, the universal R-matrix satisfies $[\mathcal{R}_{12},\mathcal{R}_{13}] = 0$. However, we shall not use this property in the following so as to keep the computations performed in this letter more generic and applicable to situations involving algebras other than $\mathfrak{osp}(1|2)$.

2.3. The Bannai–Ito algebra as the centralizer of $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes 3}$

2.3.1. Centralizer of the diagonal action of $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes 3}$

To identify $BI(3)$ as the centralizer of $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes 3}$, it is appropriate to first look for the centralizing elements $X \in U(\mathfrak{osp}(1|2))^{\otimes 3}$ such that

$$[(\Delta\otimes\mathrm{id})\Delta(x), X] = 0 \qquad \text{for } x \in \mathfrak{osp}(1|2). \tag{2.3.1}$$

It is straightforward to observe that the elements

$$C_1 = C\otimes 1\otimes 1 \ , \qquad C_2 = 1\otimes C\otimes 1 \ , \qquad C_3 = 1\otimes 1\otimes C \ , \tag{2.3.2}$$

$$C_{12} = \Delta(C)\otimes 1 \ , \qquad C_{23} = 1\otimes\Delta(C) \ , \qquad C_{123} = (\Delta\otimes\mathrm{id})\Delta(C) \ . \tag{2.3.3}$$

will be centralizing. Now let $\Delta(C) = C^{(1)}\otimes C^{(2)}$ in the Sweedler's notation and write

$$\overline{C}_{13} = C^{(1)}\otimes 1\otimes C^{(2)}. \tag{2.3.4}$$

At first glance one might think that $\overline{C}_{13}$ also belongs to the centralizer. It is the Casimir element corresponding to the simple homomorphism

$$\mathfrak{osp}(1|2) \to \mathfrak{osp}(1|2)^{\otimes 3}$$
$$x \mapsto x^{(1)} \otimes 1 \otimes x^{(2)}$$

with $\Delta(x) = x^{(1)} \otimes x^{(2)}$. This however is not true and is where the universal R-matrix comes in.

Proposition 2.3.1. *The element*[1]

$$C_{13} = \mathcal{R}_{32}^{-1}\overline{C}_{13}\mathcal{R}_{32} \tag{2.3.5}$$

belongs to the centralizer of $\mathfrak{osp}(1|2)$ *in* $U(\mathfrak{osp}(1|2))^{\otimes 3}$*.*

Proof. Since the Casimir element is central, we have for $x \in \mathfrak{osp}(1|2)$,

$$[(\Delta \otimes \mathrm{id})\Delta(x), \Delta(C) \otimes 1] = 0\ . \tag{2.3.6}$$

Using the coassociativity of the comultiplication (2.2.8) and conjugating by $\mathcal{R}_{23}$, transforms the previous relation into

$$[(\mathrm{id} \otimes \Delta^{op})\Delta(x), \mathcal{R}_{23}^{-1}(\Delta(C) \otimes 1)\mathcal{R}_{23}] = 0\ . \tag{2.3.7}$$

Finally, exchanging the spaces 2 and 3, one gets that C_{13} is in the centralizer

$$[(\mathrm{id} \otimes \Delta)\Delta(x), \mathcal{R}_{32}^{-1}\overline{C}_{13}\mathcal{R}_{32}] = [(\mathrm{id} \otimes \Delta)\Delta(x), C_{13}] = 0\ . \tag{2.3.8}$$

□

Let us emphasize that C_{13} is in the centralizer whereas $\overline{C}_{13}$ is not. In particular, for $x = C$ in the previous relation, we get

$$[C_{123}, C_{13}] = 0\ . \tag{2.3.9}$$

There is the following alternative formula for C_{13}.

Proposition 2.3.2. *The element* C_{13} *is also given by*

$$C_{13} = \mathcal{R}_{12}\overline{C}_{13}\mathcal{R}_{12}^{-1}. \tag{2.3.10}$$

Proof. From property (2.2.10), one gets

$$C_{13} = \mathcal{R}_{23}^{-1}\mathcal{R}_{13}\left(C^{(2)} \otimes 1 \otimes C^{(1)}\right)\mathcal{R}_{13}^{-1}\mathcal{R}_{23}\ . \tag{2.3.11}$$

Using the Yang–Baxter equation (2.2.13) and equality (2.2.11), this relation becomes

$$C_{13} = \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}^{-1}\mathcal{R}_{12}^{-1}\left(C^{(2)} \otimes 1 \otimes C^{(1)}\right)\mathcal{R}_{12}\mathcal{R}_{23}\mathcal{R}_{13}^{-1}\mathcal{R}_{12}^{-1}\ . \tag{2.3.12}$$

[1] In what follows we shall keep using the inverse of $\mathcal{R}$ even though $\mathcal{R}^{-1} = \mathcal{R}$ (for $\mathfrak{osp}(1|2)$) to make clear that conjugations are involved.

Now, from (2.2.12), one deduces that $[\Delta(C)\otimes 1, (\Delta\otimes \mathrm{id})(\mathcal{R})] = [\Delta(C)\otimes 1, \mathcal{R}_{23}\mathcal{R}_{13}] = 0$ and that $[\left(C^{(2)}\otimes 1\otimes C^{(1)}\right), \mathcal{R}_{12}\mathcal{R}_{23}] = 0$. One then obtains

$$C_{13} = \mathcal{R}_{12}\mathcal{R}_{13}\left(C^{(2)}\otimes 1\otimes C^{(1)}\right)\mathcal{R}_{13}^{-1}\mathcal{R}_{12}^{-1}\ , \tag{2.3.13}$$

which after using (2.2.10) again yields the desired result. □

At this point we can introduce two maps $\hat{\tau}$ and $\check{\tau}$ from $U(\mathfrak{osp}(1|2))$ to $U(\mathfrak{osp}(1|2))^{\otimes 2}$ by

$$\hat{\tau}(x) = \mathcal{R}^{-1}(1\otimes x)\mathcal{R} \quad \text{and} \qquad \check{\tau}(x) = \mathcal{R}^{-1}(x\otimes 1)\mathcal{R}\ . \tag{2.3.14}$$

Corollary 2.3.1. *The following relations hold in* $U(\mathfrak{osp}(1|2))^{\otimes 3}$

$$(id\otimes\hat{\tau})\Delta(C) = C_{13} \quad \textit{and} \qquad (\check{\tau}\otimes id)\Delta(C) = C_{13} \tag{2.3.15}$$

Proof. These results follow directly from Propositions 2.3.1 and 2.3.2 and the fact that $\mathcal{R} = \mathcal{R}^{-1}$. □

Using the definitions (2.3.14) and the universal R-matrix (2.2.9), one gets

$$\hat{\tau}(P) = 1\otimes P\ , \quad \hat{\tau}(F^{\pm}) = P\otimes F^{\pm}\ , \quad \hat{\tau}(E^{\pm}) = I\otimes E^{\pm}\ , \quad \hat{\tau}(H) = I\otimes H\ , \tag{2.3.16}$$

$$\check{\tau}(P) = P\otimes 1\ , \quad \check{\tau}(F^{\pm}) = F^{\pm}\otimes P\ , \quad \check{\tau}(E^{\pm}) = E^{\pm}\otimes I\ , \quad \check{\tau}(H) = H\otimes I\ . \tag{2.3.17}$$

Either more abstractly with the help of eqs. (2.2.12) or using the formulas above, one readily observes that $\hat{\tau}$ and $\check{\tau}$ define coactions, that is verify

$$(\mathrm{id}\otimes\hat{\tau})\hat{\tau} = (\Delta\otimes\mathrm{id})\hat{\tau} \qquad (\check{\tau}\otimes\mathrm{id})\check{\tau} = (\mathrm{id}\otimes\Delta)\check{\tau}\ . \tag{2.3.18}$$

It hence follows that $(\mathrm{id}\otimes\hat{\tau})\Delta$ and $(\check{\tau}\otimes\mathrm{id})\Delta$ define two different homomorphisms of $U(\mathfrak{osp}(1|2))$ into $U(\mathfrak{osp}(1|2))^{\otimes 3}$ which yield for C the same image, namely:

$$C_{13} = \left(8[F^{+}\otimes P\otimes P + 1\otimes 1\otimes F^{+}, F^{-}\otimes P\otimes P + 1\otimes 1\otimes F^{-}] + 1\right)P\otimes 1\otimes P\ . \tag{2.3.19}$$

This can be checked directly by applying both $(\mathrm{id}\otimes\hat{\tau})$ and $(\check{\tau}\otimes\mathrm{id})$ to

$$\Delta(C) = 8\left([F^{+}\otimes P + 1\otimes F^{+}, F^{-}\otimes P + 1\otimes F^{-}] + 1\right)P\otimes P \tag{2.3.20}$$

$$= 16\left(F^{-}\otimes F^{+} - F^{+}\otimes F^{-}\right)\ (P\otimes 1) + 8C\otimes P + P\otimes C - P\otimes P\ . \tag{2.3.21}$$

Note that

$$\hat{\tau}(C) = 1\otimes C \qquad \text{and} \qquad \check{\tau}(C) = C\otimes 1. \tag{2.3.22}$$

We may hence pick the homomorphism given by $(\check{\tau}\otimes\mathrm{id})\Delta$ and identify the three embeddings labelled by the pairs (1,2), (2,3) and (1,3) (see also [8]):

$$\begin{aligned} &H_{ij} = H_i + H_j\ , \quad E^{\pm}_{ij} = E^{\pm}_i + E^{\pm}_j\ , \qquad i,j = 1,2,3,\\ &F^{\pm}_{12} = F^{\pm}_1 P_2 + F^{\pm}_2\ , \quad F^{\pm}_{23} = F^{\pm}_2 P_3 + F^{\pm}_3\ , \quad F^{\pm}_{13} = F^{\pm}_1 P_2 P_3 + F^{\pm}_3\ ,\\ &P_{12} = P_1 P_2\ , \qquad P_{23} = P_2 P_3\ , \qquad P_{13} = P_1 P_3\ , \end{aligned} \tag{2.3.23}$$

with the subscripts denoting (as on the R-matrix) the factor in the tensor product where the element appears. The centralizing elements C_{ij} are then simply the Casimir element given by

$$C_{ij} = \Big(8[F_{ij}^+, F_{ij}^-] + 1\Big)P_{ij} \tag{2.3.24}$$

as is manifest in particular from (2.3.19) and we now understand the reasons for the choice of the (13) embedding. In this notation we have

$$C_i = \Big(8[F_i^+, F_i^-] + 1\Big)P_i, \qquad i = 1,2,3 \tag{2.3.25}$$

$$\text{and} \quad C_{123} = \Big(8[F_{123}^+, F_{123}^-] + 1\Big)P_1P_2P_3 \tag{2.3.26}$$

$$\text{with} \quad F_{123}^{\pm} = F_1^{\pm}P_2P_3 + F_2^{\pm}P_3 + F_3^{\pm}. \tag{2.3.27}$$

2.3.2. The image of $BI(3)$ in $U(\mathfrak{osp}(1|2))^{\otimes 3}$

We wish to identify the Bannai–Ito algebra $BI(3)$ with relations (2.1.1a)-(2.1.1c) by mapping its generators $\mathcal{C}$ with one, two and three indices onto the corresponding C. To that end we need to obtain the relations between the Casimir elements C. Using the formulas (2.3.24), (2.3.25), (2.3.26), relation (2.1.1a) is readily verified under $\mathcal{C} \to C$.

Note that C_{13} could have been taken to be defined by (2.1.1a) assuming that the Bannai–Ito relations will be realized. (This is typically the approach.) Given that C_{12} and C_{23} are centralizing $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes 3}$, it then follows that C_{13} must also be in the centralizer. We have here adopted the view point of first identifying the centralizing elements and hence of first defining C_{13}, before obtaining the relations between the generators of the centralizers. Since the tensorial embedding is so far the only approach that has been designed to obtain the higher rank generalization of the Bannai–Ito algebra, having these definitions of the centralizing elements proves essential in this respect.

Given the definitions of C_{12}, C_{23} and C_{13}, as already said, one directly checks that (2.1.1a) is satisfied. It is then seen, remarkably, that the remaining defining relations of the Bannai–Ito algebra are implied. One has

$$\{C_{12}, C_{23}\} = 2(-\mathcal{R}_{12}\overline{C}_{13}\mathcal{R}_{12}^{-1} + C_1C_3 + C_2C_{123}). \tag{2.3.28}$$

Interchanging the factors 1 and 2 yields

$$\{C_{21}, \overline{C}_{13}\} = 2(-\mathcal{R}_{21}C_{23}\mathcal{R}_{21}^{-1} + C_2C_3 + C_1C_{213}). \tag{2.3.29}$$

Mindful that $C_{21} = \Delta^{op}(C) \otimes 1$ and that $C_{213} = (\Delta^{op} \otimes 1)\Delta(C)$, upon conjugating with $\mathcal{R}_{21}^{-1} = \mathcal{R}_{12}$, we find

$$\{C_{12}, \mathcal{R}_{12}\overline{C}_{13}\mathcal{R}_{12}^{-1}\} = 2(-C_{23} + C_2C_3 + C_1C_{123}) \tag{2.3.30}$$

given that $\mathcal{R}\Delta^{op} = \Delta\mathcal{R}$. We thus recover (2.1.1b) from (2.1.1a). The defining relation (2.1.1c) is also obtained from (2.1.1a) in a similar fashion. In this case one interchanges the factors 2 and 3 and makes use of the other expression for C_{13} namely, $C_{13} = \mathcal{R}_{23}^{-1}\overline{C}_{13}\mathcal{R}_{23}$.

In conclusion, given C_{12}, C_{23} and once C_{13} has been defined with the help of the universal R-matrix, it is a matter of calculation to obtain one relation between these centralizing elements and one sees thereafter that the other two defining relations of the Bannai–Ito algebra follow simply from the first one in light of the properties of the generators and their connection to the R-matrix.

2.4. The higher rank Bannai–Ito algebras

In this section, we shall take n be any positive integer and $[n] = \{1,2,\dots,n\}$. The higher rank universal Bannai–Ito algebra $BI(n)$ is generated by $\mathcal{C}_A$ for $A \subset [n]$ (by convention $\mathcal{C}_\emptyset = 1$) and the following defining relations [7], for $A,B \subset [n]$,

$$\{\mathcal{C}_A,\mathcal{C}_B\} = 2(-\mathcal{C}_{(A\cup B)\setminus(A\cap B)} + \mathcal{C}_{A\setminus(A\cap B)}\mathcal{C}_{B\setminus(A\cap B)} + \mathcal{C}_{A\cap B}\mathcal{C}_{A\cup B})\,. \tag{2.4.1}$$

Let us remark that there is a factor (-2) between the generators used here and the ones of [7] which explains the apparent discrepancy between the defining relations. We shall give an image of $BI(n)$ in $U(\mathfrak{osp}(1|2))^{\otimes n}$. For that, we follow the same logic as before and study the centralizer of $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes n}$.

We define by induction $\Delta^{(k)} = (\mathrm{id}\otimes\Delta^{(k-1)})\Delta$ with $\Delta^{(0)} = \mathrm{id}$ which allows to define, for $1 \le k \le \ell \le n$,

$$C_{k,k+1,\dots\ell} = 1^{\otimes(k-1)} \otimes \Delta^{(\ell-k)}(C) \otimes 1^{\otimes(n-\ell)}\ . \tag{2.4.2}$$

These elements commute with $\Delta^{(n-1)}(x)$ for $x \in \mathfrak{osp}(1|2)$. We thus obtain elements of the centralizer associated to each subset $K \subset [n]$ with successive integers. We want to find centralizing elements associated to each subset $A \subset [n]$ without restriction. Let $\mathfrak{S}_n$ be the permutation group of n objects generated by the transpositions $s_1,s_2,\dots,s_{n-1}$. For $s = s_{i_1}s_{i_2}\dots s_{i_p}$ some permutation of $\mathfrak{S}_n$ (we recall that any permutation can be written as a product of transpositions), we define the action γ_s on $X \in U(\mathfrak{osp}(1|2))^{\otimes n}$ by

$$\gamma_s(X) = \check{\mathcal{R}}_{i_1}\check{\mathcal{R}}_{i_2}\dots\check{\mathcal{R}}_{i_p}X(\check{\mathcal{R}}_{i_1}\check{\mathcal{R}}_{i_2}\dots\check{\mathcal{R}}_{i_p})^{-1}, \tag{2.4.3}$$

where

$$\check{\mathcal{R}}_i = \mathcal{R}_{i,i+1}\sigma_{i,i+1} \tag{2.4.4}$$

and $\sigma_{i,i+1}(x_1\otimes\cdots\otimes x_i\otimes x_{i+1}\otimes\cdots\otimes x_n) = (x_1\otimes\cdots\otimes x_{i+1}\otimes x_i\otimes\cdots\otimes x_n)\sigma_{i,i+1}$. Such a $\check{\mathcal{R}}_i$ is called braided universal R-matrix. It satisfies

$$\Delta(x)\check{\mathcal{R}} = \check{\mathcal{R}}\Delta(x)\ , \tag{2.4.5}$$

and the braided Yang–Baxter equation

$$\check{\mathcal{R}}_i\check{\mathcal{R}}_{i+1}\check{\mathcal{R}}_i=\check{\mathcal{R}}_{i+1}\check{\mathcal{R}}_i\check{\mathcal{R}}_{i+1}\ . \tag{2.4.6}$$

Let us emphasize that the definition of γ_s does not depend on the choice of the decomposition of the permutation s in terms of the transpositions since the $\check{R}_i$ and the s_i satisfy the same algebra.

We define the intermediate Casimir element associated to any subset $A\subset[n]$ as follows

$$C_A=C_{s(K)}=\gamma_s(C_K) \tag{2.4.7}$$

where $K\subset[n]$ with successive integers, C_K is defined by (2.4.2) and $s\in\mathfrak{S}_n$ is chosen such that

$$s(K)=s(\{K_1,K_2,\dots,K_k\})=\{s(K_1),s(K_2),\dots,s(K_k)\}=A. \tag{2.4.8}$$

We remark that if the permutation s leaves the subset K invariant one gets $\gamma_s(C_K)=C_K$. It is easy to show using (2.4.5) that C_A is in the centralizer given that C_K is in the centralizer as already proved.

The following example shows that there are different ways to compute C_A depending on the set K we start with.

Example 2.4.1. *From $s_1(\{2,3\})=\{1,3\}$ or $s_2(\{1,2\})=\{1,3\}$, the definition (2.4.7) gives for C_{13}*

$$C_{13}=\gamma_{s_1}(C_{23})=\check{\mathcal{R}}_1C_{23}\check{\mathcal{R}}_1^{-1}=\mathcal{R}_{12}\sigma_{12}C_{23}\sigma_{12}^{-1}\mathcal{R}_{12}^{-1}=\mathcal{R}_{12}\overline{C}_{13}\mathcal{R}_{12}^{-1} \tag{2.4.9}$$

$$=\gamma_{s_2}(C_{12})=\check{\mathcal{R}}_2C_{12}\check{\mathcal{R}}_2^{-1}=\mathcal{R}_{23}\sigma_{23}C_{12}\sigma_{23}^{-1}\mathcal{R}_{23}^{-1}=\mathcal{R}_{23}\overline{C}_{13}\mathcal{R}_{23}^{-1}\ . \tag{2.4.10}$$

We recover the equivalent expressions (2.3.5) or (2.3.10) of C_{13} given in the previous section (we recall that $\mathcal{R}_{12}=\mathcal{R}_{21}=\mathcal{R}_{12}^{-1}$).

To have a well-posed definition of C_A, such different paths must lead to the same result. To confirm that, we must prove that for two subsets $K,L\subset[n]$ of successive integers defined by (2.4.2) the following relation holds

$$C_K=\gamma_s(C_L)\ , \tag{2.4.11}$$

where $s(L)=K$. It is sufficient to prove (2.4.11) for the sets $L=\{1,2,\dots,\ell\}$ and $K=\{k+1,\dots,k+\ell\}$ to prove it in general. The following permutation

$$s=(s_ks_{k+1}\dots s_{k+\ell-1})\dots(s_2s_3\dots s_{\ell+1})(s_1s_2\dots s_\ell) \tag{2.4.12}$$

satisfies $s(L) = K$. Then, from definition (2.4.3), one gets

$$\gamma_s(C_L) = (\check{\mathcal{R}}_k\check{\mathcal{R}}_{k+1}\dots\check{\mathcal{R}}_{k+\ell-1})\dots(\check{\mathcal{R}}_1\check{\mathcal{R}}_2\dots\check{\mathcal{R}}_\ell)C_L(\check{\mathcal{R}}_\ell\dots\check{\mathcal{R}}_2\check{\mathcal{R}}_1)\dots(\check{\mathcal{R}}_{k+\ell-1}\dots\check{\mathcal{R}}_{k+1}\check{\mathcal{R}}_k) \tag{2.4.13}$$

$$= (\mathcal{R}_{k,k+1}\mathcal{R}_{k,k+2}\dots\mathcal{R}_{k,k+\ell})\dots(\mathcal{R}_{1,k+1}\mathcal{R}_{1,k+2}\dots\mathcal{R}_{1,k+\ell})$$
$$C_K(\mathcal{R}_{1,k+\ell}\dots\mathcal{R}_{1,k+2}\mathcal{R}_{1,k+1})\dots(\mathcal{R}_{k,k+\ell}\dots\mathcal{R}_{k,k+2}\mathcal{R}_{k,k+1})\ . \tag{2.4.14}$$

The last relation has been obtained using the definition of $\check{\mathcal{R}}$ and the properties of $\sigma_{i,i+1}$. Then, noticing that from relation (2.2.12) one gets $(\mathrm{id}^{\otimes k}\otimes\Delta^{(\ell-1)})(\mathcal{R}_{i,k+1}) = \mathcal{R}_{i,k+1}\mathcal{R}_{i,k+2}\dots\mathcal{R}_{i,k+\ell}$ (for $1\leq i\leq k$) and $(\mathrm{id}^{\otimes k}\otimes\Delta^{(\ell-1)})(C_{k+1}) = C_K$, one obtains $[\mathcal{R}_{i,k+1}\mathcal{R}_{i,k+2}\dots\mathcal{R}_{i,k+\ell},C_K] = 0$ which proves (2.4.11) in view of (2.4.14).

We are now ready to present the main result of this section.

Proposition 2.4.1. *The map*

$$BI(n)\to U(\mathfrak{osp}(1|2))^{\otimes n} \tag{2.4.15}$$
$$\mathcal{C}_A\mapsto C_A$$

is an algebra homomorphism.

Proof. We must prove that the centralizing elements C_A satisfy the relations (2.4.1). We know from the previous section that one has

$$\{C_{12},C_{23}\} = 2(-\mathcal{R}_{23}\overline{C}_{13}\mathcal{R}_{23}^{-1} + C_1C_3 + C_2C_{123}) \tag{2.4.16}$$

which can be transformed as

$$\{C_{12},C_{23}\} = 2(-\gamma_s(C_{12}) + C_1C_3 + C_2C_{123}) \qquad \text{with} \quad s(\{1,2\}) = \{1,3\}. \tag{2.4.17}$$

By acting with the coproduct on the second space of relation (2.4.16), one gets

$$\{C_{123},C_{234}\} = 2(-\mathcal{R}_{34}\mathcal{R}_{24}\overline{C}_{14}\mathcal{R}_{24}^{-1}\mathcal{R}_{34}^{-1} + C_1C_4 + C_{23}C_{1234}) \tag{2.4.18}$$

which becomes

$$\{C_{123},C_{234}\} = 2(-\gamma_s(C_{12}) + C_1C_4 + C_{23}C_{1234}) \qquad \text{with} \quad s(\{1,2\}) = \{1,4\}. \tag{2.4.19}$$

Similarly, by acting with the coproduct successively on the second space of relation (2.4.16), one gets, for $L = \{2,\dots,\ell+1\}$,

$$\{C_{1,L},C_{L,\ell+2}\} = 2(-\mathcal{R}_{\ell+1,\ell+2}\dots\mathcal{R}_{2,\ell+2}\overline{C}_{1,\ell+2}\mathcal{R}_{2,\ell+2}^{-1}\dots\mathcal{R}_{\ell+1,\ell+2}^{-1} + C_1C_{\ell+2} + C_LC_{1,L,\ell+2}) \tag{2.4.20}$$

which becomes

$$\{C_{1,L},C_{L,\ell+2}\} = 2(-\gamma_s(C_{12}) + C_1C_{\ell+2} + C_LC_{1,L,\ell+2}) \qquad \text{with} \quad s(\{1,2\}) = \{1,\ell+2\}. \tag{2.4.21}$$

Finally, by acting with the coproduct successively on the first and third spaces on relation (2.4.16), one can prove

$$\{C_{KL},C_{LM}\} = 2(-\gamma_s(C_{K,k+1,k+2,\ldots,k+m}) + C_K C_M + C_L C_{KLM}) \tag{2.4.22}$$

where $s(K,k+1,k+2,\ldots,k+m) = KM$ and $K = \{1,\ldots k\}$, $L = \{k+1,\ldots k+\ell\}$ and $M = \{k+\ell+1,\ldots k+\ell+m\}$. This proves that for the sets K, L and M given above, the $BI(n)$ relations (2.4.1) are satisfied by the corresponding centralizing elements. We can similarly see relation (2.4.1) to hold when K, L or M are chosen empty. Let $s \in \mathfrak{S}_n$. Using the definition (2.4.3), one gets

$$\gamma_s(XX') = \gamma_s(X)\gamma_s(X'). \tag{2.4.23}$$

Then, we have

$$\{C_{s(KL)},C_{s(LM)}\} = 2(-C_{s(KM)} + C_{s(K)}C_{s(M)} + C_{s(L)}C_{s(KLM)}). \tag{2.4.24}$$

We conclude the proof by remarking that $s(KM) = (s(KL) \cup s(LM)) \setminus (s(KL) \cap s(LM))$, $s(K) = s(KL)\setminus(s(KL)\cap s(LM))$, $s(M) = s(LM)\setminus(s(KL)\cap s(LM))$, $s(L) = s(KL)\cap s(LM)$ and $s(KLM) = s(KL)\cup s(LM)$ and by noting that there exist K, L and M and s such that $s(KL) = A$ and $s(LM) = B$ for any $A,B \subset [n]$. □

2.5. Conclusions

This letter has offered a complete description of the Bannai–Ito algebras as centralizers of the diagonal action of $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes n}$ by bringing the universal R-matrix to bear on the topic. This has proved most appropriate. In addition to the elegance it confers to the presentation, this approach gave answers to questions that had so far been unsettled. It provided an intrinsic algebraic definition of all centralizing elements independently of the defining relations. It also shed light on the specific form of the intermediate embeddings of $\mathfrak{osp}(1|2)$ in $\mathfrak{osp}(1|2)^{\otimes n}$ that yield the generators through the associated Casimir elements. Importantly, it has entailed a simple constructive derivation of the structure relations of $BI(n)$ satisfied by these generators through bootstrapping from the relations of $BI(3)$. Another possible merit is that casting Bannai–Ito algebras in this framework might bring experts familiar with universal R-matrices to contribute further to the field and its applications.

This universal R-matrix approach has already been applied to the study of the Askey-Wilson algebra $AW(3)$ [10] as the centralizer of the diagonal action of $U_q(\mathfrak{sl}(2))$ into its threefold product and has also been seen to hold promises for advancing the understanding of the higher rank $AW(n)$ where one is looking at the centralizer of $U_q(\mathfrak{sl}(2))$ in $U_q(\mathfrak{sl}(2))^{\otimes n}$ [?]. While advances have been made on this last front [3], [14], a complete description of $AW(n)$ is still lacking. We trust that the treatment given here of the Bannai–Ito algebra $BI(n)$ using the universal R-matrix might hold the clues towards bringing this quest to a satisfactory conclusion. We hope to report on this in the near future.

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Chapitre 3

Temperley–Lieb, Birman–Murakami–Wenzl and Askey–Wilson algebras and other centralizers of $U_q(\mathfrak{sl}_2)$

Par Nicolas Crampé, Luc Vinet et Meri Zaimi.
Soumis dans *Compositio Mathematica*.

Abstract. The centralizer of the image of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in the tensor product of three irreducible representations is examined in a Schur–Weyl duality spirit. The aim is to offer a description in terms of generators and relations. A conjecture in this respect is offered with the centralizers presented as quotients of the Askey–Wilson algebra. Support for the conjecture is provided by an examination of the representations of the quotients. The conjecture is also shown to be true in a number of cases thereby exhibiting in particular the Temperley–Lieb, Birman–Murakami–Wenzl and one-boundary Temperley–Lieb algebras as quotients of the Askey–Wilson algebra.

3.1. Introduction

The objective of this paper is to establish precisely the connections between the Askey–Wilson algebra and the centralizers of the quantum algebra $U_q(\mathfrak{sl}_2)$ such as the Temperley–Lieb and Birman–Murakami–Wenzl algebras.

In previous works, the connections between the Racah, Temperley–Lieb and Brauer algebras and other centralizers of $\mathfrak{sl}_2$ were studied in the spirit of the Schur–Weyl duality [4]. In a similar fashion, the Bannai-Ito algebra was connected to the centralizers of the superalgebra $\mathfrak{osp}(1|2)$, and in particular to the Brauer algebra [1]. The present paper generalizes the results of [4] by examining their q-deformation.

The Askey–Wislon algebra was first introduced in [22] and is defined by three generators satisfying some q-commutation relations. This algebra encodes the properties of the Askey–Wilson polynomials [13] and is related to the Racah problem for $U_q(\mathfrak{sl}_2)$ [8]. Due to this connection, a centrally extended Askey–Wilson algebra can be mapped to the centralizer of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ into $U_q(\mathfrak{sl}_2)^{\otimes 3}$ [9,21]. In the q-deformation of the universal enveloping algebra of $\mathfrak{sl}_2$ to the quantum algebra $U_q(\mathfrak{sl}_2)$, the Askey–Wilson algebra plays a role analogous to that of the Racah algebra.

From the Schur–Weyl duality, the centralizer of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in the tensor product of its fundamental representation is connected to the Hecke algebra. In the case of the threefold tensor product, the centralizer is known [12] to be isomorphic to the Temperley–Lieb algebra [20], which is a quotient of the Hecke algebra. In fact, the algebra $U_q(\mathfrak{sl}_2)$ has infinitely many finite irreducible representations, labeled by a half-integer or integer spin j. In the case of the tensor product of three spin-1 representations, it is also known [16] that the centralizer of $U_q(\mathfrak{sl}_2)$ is isomorphic to the Birman–Murakami–Wenzl algebra [11], which is a q-deformation of the Brauer algebra. However, in the general case of three irreducible representations of spins j_1,j_2 and j_3, an algebraic description of the centralizer is not known.

The present paper provides an attempt to describe the centralizer of the image of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in the tensor product of any three irreducible representations in terms of generators and relations, by using the connections with the Askey–Wilson algebra. It is first shown that there is a surjective map (between generators) from the Askey–Wilson algebra to the centralizer. This statement corresponds in invariant theory to the first fundamental theorem [14]. A conjecture is then proposed in order to obtain an isomorphism between a quotient (given in terms of relations) of the Askey–Wilson algebra and the centralizer of $U_q(\mathfrak{sl}_2)$ – this relates to the second fundamental theorem in invariant theory [14]. The conjecture is proved for three spin-$\frac{1}{2}$ representations, in which case the Temperley–Lieb algebra is obtained explicitly as a quotient of the Askey–Wilson algebra. Similarly, for three spin-1 representations, it is shown that the conjecture holds and that the Birman–Murakami–Wenzl algebra is isomorphic to a quotient of the Askey–Wilson algebra. The conjecture is also verified for three spin-$\frac{3}{2}$ representations, and for one spin-j and two spin-$\frac{1}{2}$ representations, for j any spin greater than $\frac{1}{2}$. In the latter case, it is shown that the centralizer is isomorphic to the one-boundary Tempereley–Lieb algebra [17–19].

The plan of this paper is as follows. Section 3.2 gives the precise connection between the centralizer of $U_q(\mathfrak{sl}_2)$ and the Askey–Wilson algebra. Subsection 3.2.1 presents the quantum algebra $U_q(\mathfrak{sl}_2)$ and its properties. The centralizer of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ and the intermediate Casimirs are defined in Subsection 3.2.2. A homomorphism between the centrally extended Askey–Wilson algebra $AW(3)$ and this centralizer is given in Subsection 3.2.3. Section 3.3 is concerned with the representations of elements in $U_q(\mathfrak{sl}_2)^{\otimes 3}$. The finite irreducible representations of $U_q(\mathfrak{sl}_2)$ and their tensor product decomposition rules are recalled in Subsection 3.3.1. Subsection 3.3.2 introduces the object of main interest, that is the centralizer of the image of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in the tensor product of three irreducible representations. Section 3.4 aims to describe this centralizer in terms of generators and relations. Subsection 3.4.1 maps $AW(3)$ to this centralizer, and this is shown to be a surjection in Subsection 3.4.2. The kernel of this map is discussed in Subsection 3.4.3, and a conjecture proposing that a quotient of $AW(3)$ is isomorphic to the centralizer is formulated. Subsection 3.4.4 contains the proof that the conjecture does not depend on the ordering of the three spins j_1,j_2,j_3. In order to support the conjecture, Section 3.5 studies the finite irreducible representations of the quotient of $AW(3)$. The remaining sections contain the proofs of the conjecture for some particular cases. Section 3.6 focuses on the case $j_1 = j_2 = j_3 = \frac{1}{2}$. It is shown in Subsection 3.6.1 that the conjecture holds in this case, and the precise connection with the Temperley–Lieb algebra is given in Subsection 3.6.2. Section 3.7 considers the case $j_1 = j_2 = j_3 = 1$. The proof of the conjecture is given in Subsection 3.7.1. An isomorphism between the quotient of $AW(3)$ and the Birman–Murakami–Wenzl algebra is obtained in Subsection 3.7.2. The conjecture for the case $j_1 = j_2 = j_3 = \frac{3}{2}$ is proved in Section 3.8. Finally, Section 3.9 studies the case $j_1 = j$ for $j = 1,\frac{3}{2},\dots$ and $j_2 = j_3 = \frac{1}{2}$. The conjecture is verified in Subsection 3.9.1, and the connection with the one-boundary Temperley–Lieb algebra is described in Subsection 3.9.2.

3.2. Centralizer of $U_q(\mathfrak{sl}_2)$ and Askey–Wilson algebra

In this section, we recall well-known properties of the quantum algebra $U_q(\mathfrak{sl}_2)$ to fix the notations. Then, the definition of the centralizer of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ is recalled and its homomorphism with the centrally extended Askey–Wilson algebra $AW(3)$ is presented.

3.2.1. $U_q(\mathfrak{sl}_2)$ algebra

The associative algebra $U_q(\mathfrak{sl}_2)$ is generated by E, F and q^H with the defining relations

$$q^H E = qEq^H \ , \quad q^H F = q^{-1} F q^H \text{ and } \quad [E,F] = [2H]_q \ , \tag{3.2.1}$$

where $[X]_q = \frac{q^X - q^{-X}}{q - q^{-1}}$. Throughout this paper, q is a complex number not root of unity. There is a central element in $U_q(\mathfrak{sl}_2)$, called quadratic Casimir element, given by

$$\Gamma = (q - q^{-1})^2 FE + qq^{2H} + q^{-1}q^{-2H} \ . \tag{3.2.2}$$

There exists also an algebra homomorphism $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$, called comultiplication, defined on the generators by

$$\Delta(E) = E \otimes q^{-H} + q^H \otimes E \ , \quad \Delta(F) = F \otimes q^{-H} + q^H \otimes F \quad \text{and} \quad \Delta(q^H) = q^H \otimes q^H \ . \tag{3.2.3}$$

This comultiplication is coassociative

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta =: \Delta^{(2)} \ . \tag{3.2.4}$$

We define the opposite comultiplication $\Delta^{op} = \sigma \circ \Delta$, where $\sigma(x \otimes y) = y \otimes x$, for $x,y \in U_q(\mathfrak{sl}_2)$. It is a homomorphism from $U_q(\mathfrak{sl}_2)$ to $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ different from Δ. Both are related by the universal R-matrix $\mathcal{R} \in U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ satisfying

$$\Delta(x)\mathcal{R} = \mathcal{R}\Delta^{op}(x) \quad \text{for } x \in U_q(\mathfrak{sl}_2) \ . \tag{3.2.5}$$

The universal R-matrix also satisfies the Yang–Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \ . \tag{3.2.6}$$

We have used the usual notations: if $\mathcal{R} = \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha$, then $\mathcal{R}_{12} = \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha$ and $\mathcal{R}_{13} = \mathcal{R}^\alpha \otimes 1 \otimes \mathcal{R}_\alpha$ (the sum w.r.t. α is understood).

3.2.2. Centralizer of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$

The centralizer $\mathfrak{C}_3$ of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ is

$$\mathfrak{C}_3 = \{X \in U_q(\mathfrak{sl}_2)^{\otimes 3} \ \Big| \ [\Delta^{(2)}(x), X] = 0 \ , \ \ \forall x \in U_q(\mathfrak{sl}_2)\}. \tag{3.2.7}$$

This centralizer is a subalgebra of $U_q(\mathfrak{sl}_2)^{\otimes 3}$ and we want to describe this subalgebra with some generators and defining relations. Let us first give some elements of $\mathfrak{C}_3$ by using the Casimir element Γ which is central in $U_q(\mathfrak{sl}_2)$. We define the following Casimir elements of $U_q(\mathfrak{sl}_2)^{\otimes 3}$

$$\Gamma_1 = \Gamma \otimes 1 \otimes 1 \ , \qquad \Gamma_2 = 1 \otimes \Gamma \otimes 1 \ , \qquad \Gamma_3 = 1 \otimes 1 \otimes \Gamma \ . \tag{3.2.8}$$

These elements are central in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ and thus belong to $\mathfrak{C}_3$. We also define the total Casimir

$$\Gamma_{123} = \Delta^{(2)}(\Gamma) \ . \tag{3.2.9}$$

This element belongs to $\mathfrak{C}_3$ because $[\Delta^{(2)}(\Gamma),\Delta^{(2)}(x)] = \Delta^{(2)}([\Gamma,x]) = 0$ for all $x \in U_q(\mathfrak{sl}_2)$. Let us notice that Γ_{123} is central in $\mathfrak{C}_3$ since it is also an element of the diagonal embedding of $U_q(\mathfrak{sl}_2)$.

We then define the intermediate Casimirs associated to the recoupling of the two first or the two last factors of $U_q(\mathfrak{sl}_2)^{\otimes 3}$

$$\Gamma_{12} = \Delta(\Gamma) \otimes 1 \quad \text{and} \qquad \Gamma_{23} = 1 \otimes \Delta(\Gamma) \ . \tag{3.2.10}$$

One uses the properties of the comultiplication to show that Γ_{12} and Γ_{23} are in $\mathfrak{C}_3$; indeed, for all $x \in U_q(\mathfrak{sl}_2)$,

$$[\Gamma_{12},\Delta^{(2)}(x)] = [\Delta(\Gamma) \otimes 1,(\Delta \otimes \mathrm{id})\Delta(x)] = (\Delta \otimes \mathrm{id})[\Gamma \otimes 1,\Delta(x)] = 0 \ , \tag{3.2.11}$$

$$[\Gamma_{23},\Delta^{(2)}(x)] = [1 \otimes \Delta(\Gamma),(\mathrm{id} \otimes \Delta)\Delta(x)] = (\mathrm{id} \otimes \Delta)[1 \otimes \Gamma,\Delta(x)] = 0 \ . \tag{3.2.12}$$

In the limit $q \to 1$, it can be shown that the element

$$\Gamma_{13} = \sum_{\alpha} \Gamma_\alpha \otimes 1 \otimes \Gamma^\alpha \ , \tag{3.2.13}$$

where $\Delta(\Gamma) = \sum_\alpha \Gamma_\alpha \otimes \Gamma^\alpha$, belongs to the centralizer of $U(\mathfrak{sl}_2)$ in $U(\mathfrak{sl}_2)^{\otimes 3}$. However, this is not the case for the quantum algebra $U_q(\mathfrak{sl}_2)$. This difficulty that arises in the q-deformation of the algebra $U(\mathfrak{sl}_2)$ was addressed in [2] where a definition of the third intermediate Casimir element of $U_q(\mathfrak{sl}_2)$ is provided with the help of the universal R-matrix. It is shown in [2] that the following elements

$$\Gamma_{13}^{(0)} = \mathcal{R}_{12}\Gamma_{13}\mathcal{R}_{12}^{-1} = \mathcal{R}_{32}^{-1}\Gamma_{13}\mathcal{R}_{32} \ , \tag{3.2.14}$$

$$\Gamma_{13}^{(1)} = \mathcal{R}_{23}\Gamma_{13}\mathcal{R}_{23}^{-1} = \mathcal{R}_{21}^{-1}\Gamma_{13}\mathcal{R}_{21} \ , \tag{3.2.15}$$

are in the centralizer $\mathfrak{C}_3$.

3.2.3. Connection with the Askey–Wilson algebra

The intermediate Casimir elements Γ_{12}, Γ_{23}, $\Gamma_{13}^{(0)}$ and $\Gamma_{13}^{(1)}$ do not commute pairwise but satisfy certain relations which are identified as those of the Askey–Wilson algebra $AW(3)$.

Definition 3.2.1. *The centrally extended Askey–Wilson algebra $AW(3)$ is generated by A, B, D and central elements α_1, α_2, α_3 and K subject to the following defining relations*

$$A + \frac{[B,D]_q}{q^2 - q^{-2}} = \frac{\alpha_1\alpha_2 + \alpha_3 K}{q + q^{-1}} \ , \tag{3.2.16}$$

$$B + \frac{[D,A]_q}{q^2 - q^{-2}} = \frac{\alpha_2\alpha_3 + \alpha_1 K}{q + q^{-1}} \ , \tag{3.2.17}$$

$$D + \frac{[A,B]_q}{q^2 - q^{-2}} = \frac{\alpha_1\alpha_3 + \alpha_2 K}{q + q^{-1}} \ , \tag{3.2.18}$$

where $[X,Y]_q = qXY - q^{-1}YX$. We also define the element $D' \in AW(3)$ by the following relation

$$D' + \frac{[B,A]_q}{q^2 - q^{-2}} = \frac{\alpha_1\alpha_3 + \alpha_2 K}{q + q^{-1}} \ . \tag{3.2.19}$$

The algebra $AW(3)$ has a Casimir element given by

$$\Omega = qA(\alpha_1\alpha_2+\alpha_3K)+q^{-1}B(\alpha_2\alpha_3+\alpha_1K)+qD(\alpha_1\alpha_3+\alpha_2K)-q^2A^2-q^{-2}B^2-q^2D^2-qABD \ . \tag{3.2.20}$$

The connection between the centralizer $\mathfrak{C}_3$ defined by (3.2.7) and the Askey–Wilson algebra is given in the following proposition.

Proposition 3.2.1. *The map $\varphi : AW(3) \to \mathfrak{C}_3$ defined by*

$$\varphi(\alpha_i) = \Gamma_i \ , \ \varphi(A) = \Gamma_{12} \ , \ \varphi(B) = \Gamma_{23} \ , \ \varphi(K) = \Gamma_{123} \ , \tag{3.2.21}$$

is an algebra homomorphism. We deduce that

$$\varphi(D) = \Gamma_{13}^{(0)} \ , \ \varphi(D') = \Gamma_{13}^{(1)} \ . \tag{3.2.22}$$

The homomorphism has been proved in [8]; a direct computation shows that the intermediate Casimir elements satisfy all the relations of $AW(3)$. Relations (3.2.22) have been proved more recently in [2] and a simpler proof of the homomorphism using the universal R-matrix has also been given. Let us remark that a similar proof has also been simplified in the case of the Bannai–Ito algebra and the centralizer for the super Lie algebra $\mathfrak{osp}(1|2)$ [6].

Using (3.2.18) to replace D in (3.2.16) and (3.2.17), one shows that the following relations provide an equivalent presentation of $AW(3)$ which will be useful for later computations

$$\frac{[B,[A,B]_q]_q}{(q-q^{-1})^2} =(q+q^{-1})^2A + (\alpha_1\alpha_3+\alpha_2K)B - (q+q^{-1})(\alpha_1\alpha_2+\alpha_3K) \ , \tag{3.2.23}$$

$$\frac{[[A,B]_q,A]_q}{(q-q^{-1})^2} =(q+q^{-1})^2B + (\alpha_1\alpha_3+\alpha_2K)A - (q+q^{-1})(\alpha_2\alpha_3+\alpha_1K) \ . \tag{3.2.24}$$

Furthermore, noticing that $[X,[Y,X]_q]_q = [[X,Y]_q,X]_q$ and using the element D' defined in (3.2.19), one finds that (3.2.23) and (3.2.24) imply

$$A + \frac{[D',B]_q}{q^2-q^{-2}} = \frac{\alpha_1\alpha_2+\alpha_3K}{q+q^{-1}} \ , \tag{3.2.25}$$

$$B + \frac{[A,D']_q}{q^2-q^{-2}} = \frac{\alpha_2\alpha_3+\alpha_1K}{q+q^{-1}} \ . \tag{3.2.26}$$

Relations (3.2.19), (3.2.25) and (3.2.26) provide another $\mathbb{Z}_3$ symmetric presentation of $AW(3)$.

Remark 3.2.1. *Upon performing the affine transformation* $X = (q-q^{-1})^2\tilde{X} + q + q^{-1}$ *on the elements* $X = A,B,D,D',\alpha_i,K$ *of* $AW(3)$*, one sees that relations* (3.2.23)*–*(3.2.24) *can be written as*

$$\begin{aligned}[\tilde{B},[\tilde{A},\tilde{B}]_q]_q =&(q+q^{-1})\left(-\tilde{B}^2 - \{\tilde{A},\tilde{B}\} + (\tilde{K}+\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3)\tilde{B} + (\tilde{\alpha}_1-\tilde{K})(\tilde{\alpha}_3-\tilde{\alpha}_2)\right)\\ &+(q-q^{-1})^2(\tilde{\alpha}_1\tilde{\alpha}_3+\tilde{\alpha}_2\tilde{K})\tilde{B} \ ,\end{aligned} \tag{3.2.27}$$

$$\begin{aligned}[[\tilde{A},\tilde{B}]_q,\tilde{A}]_q =&(q+q^{-1})\left(-\tilde{A}^2 - \{\tilde{A},\tilde{B}\} + (\tilde{K}+\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3)\tilde{A} + (\tilde{\alpha}_3-\tilde{K})(\tilde{\alpha}_1-\tilde{\alpha}_2)\right)\\ &+(q-q^{-1})^2(\tilde{\alpha}_1\tilde{\alpha}_3+\tilde{\alpha}_2\tilde{K})\tilde{A} \ ,\end{aligned} \tag{3.2.28}$$

where $\{X,Y\} = XY + YX$*. By taking the limit* $q \to 1$ *of* (3.2.27) *and* (3.2.28)*, one recovers the defining relations of the Racah algebra used in [4]. Relations* (3.2.18) *and* (3.2.19) *are* *transformed into*

$$\frac{[\tilde{A},\tilde{B}]_q}{q-q^{-1}} = \tilde{\alpha}_1\tilde{\alpha}_3 + \tilde{\alpha}_2\tilde{K} + \frac{q+q^{-1}}{(q-q^{-1})^2}(\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3+\tilde{K}-\tilde{A}-\tilde{B}-\tilde{D}) \ , \tag{3.2.29}$$

$$\frac{[\tilde{B},\tilde{A}]_q}{q-q^{-1}} = \tilde{\alpha}_1\tilde{\alpha}_3 + \tilde{\alpha}_2\tilde{K} + \frac{q+q^{-1}}{(q-q^{-1})^2}(\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3+\tilde{K}-\tilde{A}-\tilde{B}-\tilde{D}') \ . \tag{3.2.30}$$

In the limit $q \to 1$, the elements $\tilde{D}$ and $\tilde{D}'$ are equal, and the images by φ of (3.2.29) and (3.2.30) both reduce to the well-known linear relation $\tilde{\Gamma}_1+\tilde{\Gamma}_2+\tilde{\Gamma}_3+\tilde{\Gamma}_{123}-\tilde{\Gamma}_{12}-\tilde{\Gamma}_{23}-\tilde{\Gamma}_{13}=0$ that holds in $U(\mathfrak{sl}_2)^{\otimes 3}$.

3.3. Decomposition of tensor product of representations and centralizer

In the previous section, we introduced the centralizer $\mathfrak{C}_3$ of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ and showed the connection of this subalgebra of $U_q(\mathfrak{sl}_2)^{\otimes 3}$ with the Askey–Wilson algebra $AW(3)$. We now focus on the corresponding objects when each $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ is taken in a finite irreducible representation.

3.3.1. Finite irreducible representations of $U_q(\mathfrak{sl}_2)$

The quantum algebra $U_q(\mathfrak{sl}_2)$ has finite irreducible representations of dimension $2j+1$ that we will denote by M_j, with $j = 0,\frac{1}{2},1,\frac{3}{2},...$ The representation map will be denoted by $\pi_j : U_q(\mathfrak{sl}_2) \to \text{End}(M_j)$. We will use the name spin-j representation to refer to M_j. The representation of the Casimir element (3.2.2) in the space M_j is

$$\pi_j(\Gamma) = \chi_j \mathbb{I}_{2j+1} \quad \text{where } \chi_j = q^{2j+1} + q^{-2j-1}, \tag{3.3.1}$$

and $\mathbb{I}_{2j+1}$ is the $2j+1$ by $2j+1$ identity matrix. We define the following sets, for three half-integers or integers j_1, j_2 and j_3

$$\mathcal{J}(j_1,j_2) = \{|j_1-j_2|,|j_1-j_2|+1,...,j_1+j_2\} \ , \tag{3.3.2}$$

$$\mathcal{J}(j_1,j_2,j_3) = \bigcup_{j\in\mathcal{J}(j_1,j_2)} \mathcal{J}(j,j_3) \ . \tag{3.3.3}$$

Notice that there are no repeated numbers in $\mathcal{J}(j_1,j_2,j_3)$, and this set is invariant under any permutation of j_1, j_2 and j_3.

For q not a root of unity, the tensor product of two irreducible representations of $U_q(\mathfrak{sl}_2)$ decomposes into the following direct sum of irreducible representations

$$M_{j_1} \otimes M_{j_2} = \bigoplus_{j\in\mathcal{J}(j_1,j_2)} M_j \ . \tag{3.3.4}$$

Similarly, the threefold tensor product of irreducible representations of $U_q(\mathfrak{sl}_2)$ decomposes into the following direct sum

$$M_{j_1} \otimes M_{j_2} \otimes M_{j_3} = \bigoplus_{j \in \mathcal{J}(j_1,j_2,j_3)} d_j M_j \ , \tag{3.3.5}$$

where $d_j \in \mathbb{Z}_{>0}$ is the degeneracy of M_j and is referred to as the Littlewood–Richardson coefficient.

3.3.2. Centralizer of $U_q(\mathfrak{sl}_2)$ in End$(M_{j_1} \otimes M_{j_2} \otimes M_{j_3})$

From now on, we fix three half-integers or integers j_1, j_2 and j_3. The centralizer $\mathcal{C}_{j_1,j_2,j_3}$ of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in End$(M_{j_1} \otimes M_{j_2} \otimes M_{j_3})$ is

$$\mathcal{C}_{j_1,j_2,j_3} = \{ m \in \text{End}(M_{j_1} \otimes M_{j_2} \otimes M_{j_3}) \ \Big| \ [\pi_{j_1,j_2,j_3}(\Delta^{(2)}(x)), m] = 0, \forall x \in U_q(\mathfrak{sl}_2) \} \ , \tag{3.3.6}$$

where we have used the shortened notation $\pi_{j_1,j_2,j_3} = \pi_{j_1} \otimes \pi_{j_2} \otimes \pi_{j_3}$. This centralizer as a subalgebra of End$(M_{j_1} \otimes M_{j_2} \otimes M_{j_3})$ is the object of interest of this paper. In the next section, we conjecture a presentation of this centralizer in terms of generators and relations by using the connections with the Askey–Wilson algebra $AW(3)$. Let us first recall some known properties of this centralizer.

The knowledge of the centralizer permits to write the decomposition rule (3.3.5) as follows

$$M_{j_1} \otimes M_{j_2} \otimes M_{j_3} = \bigoplus_{j \in \mathcal{J}(j_1,j_2,j_3)} M_j \otimes V_j \ , \tag{3.3.7}$$

where V_j is a finite irreducible representation of dimension d_j of $\mathcal{C}_{j_1,j_2,j_3}$. The set $\{V_j \mid j \in \mathcal{J}(j_1,j_2,j_3)\}$ is the complete set of non-equivalent irreducible representations of $\mathcal{C}_{j_1,j_2,j_3}$. In particular, one deduces that the dimension of the centralizer is

$$\dim(\mathcal{C}_{j_1,j_2,j_3}) = \sum_{j \in \mathcal{J}(j_1,j_2,j_3)} d_j^2 \ . \tag{3.3.8}$$

These representations V_j are explicitly given in Subsection 3.4.2.

We now define the images of the centralizing elements (3.2.8)–(3.2.10) and (3.2.14)–(3.2.15) of $U_q(\mathfrak{sl}_2)^{\otimes 3}$ in the representation End$(M_{j_1} \otimes M_{j_2} \otimes M_{j_3})$ as follows

$$\begin{array}{rccc} \pi_{j_1,j_2,j_3} : & \mathcal{C}_3 & \to & \mathcal{C}_{j_1,j_2,j_3} \\ & \Gamma_i, \Gamma_{ij}, \Gamma_{123} & \mapsto & C_i, C_{ij}, C_{123} \ . \end{array} \tag{3.3.9}$$

Therefore, $\mathcal{C}_{j_1,j_2,j_3}$ contains the elements C_i, C_{ij} and C_{123}. According to (3.3.1), the elements C_i are simply constant matrices of value χ_{j_i} for $i = 1,2,3$. The intermediate Casimirs C_{12}, C_{23}, $C_{13}^{(0)}$ and $C_{13}^{(1)}$ and the total Casimir C_{123} of $\mathcal{C}_{j_1,j_2,j_3}$ can be diagonalized if q is not a root of unity.

Since C_{12} is the Casimir associated to the recoupling of the two first factors of the threefold tensor product of $U_q(\mathfrak{sl}_2)$, one finds (using the decomposition rule (3.3.4)) that its eigenvalues are χ_j for $j \in \mathcal{J}(j_1,j_2)$. Similarly, the eigenvalues of C_{23} (resp. C_{123}) are χ_j for $j \in \mathcal{J}(j_2,j_3)$ (resp. $\mathcal{J}(j_1,j_2,j_3)$). The same argument cannot be applied directly to the intermediate Casimirs $C_{13}^{(0)}$ and $C_{13}^{(1)}$ since they are not trivial in the space 2. However, the element C_{13} defined in (3.2.13) only couples the spaces 1 and 3 such that its eigenvalues are χ_j for $j \in \mathcal{J}(j_1,j_3)$. From the definitions (3.2.14) and (3.2.15), we see that $C_{13}^{(0)}$ and $C_{13}^{(1)}$ are both conjugations of C_{13} by an R-matrix. Hence, their eigenvalues are the same as those of C_{13}. The previous discussion implies that the minimal polynomials of the intermediate Casimirs and the total Casimir take the following form

$$\prod_{j\in\mathcal{J}(j_1,j_2)} (C_{12} - \chi_j) = 0 \ , \quad \prod_{j\in\mathcal{J}(j_2,j_3} (C_{23} - \chi_j) = 0 \ , \quad \prod_{j\in\mathcal{J}(j_1,j_2,j_3)} (C_{123} - \chi_j) = 0 \ , \tag{3.3.10}$$

$$\prod_{j\in\mathcal{J}(j_1,j_3)} (C_{13}^{(0)} - \chi_j) = 0 \ , \quad \prod_{j\in\mathcal{J}(j_1,j_3)} (C_{13}^{(1)} - \chi_j) = 0 \ . \tag{3.3.11}$$

Because C_{123} is central in $\mathcal{C}_{j_1,j_2,j_3}$, it can be diagonalized simultaneously with C_{12}, C_{23}, $C_{13}^{(0)}$ or $C_{13}^{(1)}$. Therefore, one gets the following minimal polynomials

$$\prod_{m\in\mathcal{M}(j_1,j_2,j_3)} (C_{123} - C_{12} - m) = 0 \ , \quad \prod_{m\in\mathcal{M}(j_2,j_3,j_1)} (C_{123} - C_{23} - m) = 0 \ , \tag{3.3.12}$$

$$\prod_{m\in\mathcal{M}(j_1,j_3,j_2)} (C_{123} - C_{13}^{(0)} - m) = 0 \ , \quad \prod_{m\in\mathcal{M}(j_1,j_3,j_2)} (C_{123} - C_{13}^{(1)} - m) = 0 \ , \tag{3.3.13}$$

where

$$\mathcal{M}(j_a,j_b,j_c) = \bigcup_{j\in\mathcal{J}(j_a,j_b)} \{\chi_\ell - \chi_j \Big| \ell \in \mathcal{J}(j,j_c)\} \ . \tag{3.3.14}$$

In the previous set $\mathcal{M}(j_a,j_b,j_c)$, there are no repeated numbers.

Before concluding this section, let us notice that if one performs the transformation given in Remark 3.2.1 on the Casimir element Γ of $U_q(\mathfrak{sl}_2)$, its value in the representation $\mathrm{End}(M_j)$ is $\tilde{\chi}_j = [j]_q[j+1]_q$. By construction, similar results hold for the eigenvalues of the transformed elements $\tilde{C}_i$,$\tilde{C}_{ij}$ and $\tilde{C}_{123}$. In the limit $q \to 1$, the minimal polynomials of these transformed elements thus reduce to the ones discussed in [4].

3.4. Algebraic description of the centralizer $\mathcal{C}_{j_1,j_2,j_3}$

Take j_1, j_2 and j_3 to be three fixed half-integers or integers. This section contains an attempt to give a definition of the centralizer $\mathcal{C}_{j_1,j_2,j_3}$ in terms of generators and relations. We rely on the connection with the Askey–Wilson algebra $AW(3)$.

3.4.1. Homomorphism with $AW(3)$

The intermediate Casimir elements C_i, C_{ij} and C_{123} belonging to $\mathcal{C}_{j_1,j_2,j_3}$ satisfy the defining relations of the Askey–Wilson algebra as stated precisely in the following proposition.

Proposition 3.4.1. *The map* $\phi : AW(3) \to \mathcal{C}_{j_1,j_2,j_3}$ *defined by*

$$\phi(\alpha_i) = C_i \ , \ \phi(A) = C_{12} \ , \ \phi(B) = C_{23} \ , \ \phi(K) = C_{123} \ , \tag{3.4.1}$$

is an algebra homomorphism.

Proof. The result follows from the fact that ϕ is the composition of two homomorphisms $\phi = \pi_{j_1,j_2,j_3} \circ \varphi$, where φ is defined in Proposition 3.2.1. □

We recall that C_i is $\chi_{j_i} = q^{2j_i+1} + q^{-2j_i-1}$ times the identity matrix. Therefore, it can be identified as the number χ_{j_i}. Let us also emphasize that $\phi(D) = C_{13}^{(0)}$ and $\phi(D') = C_{13}^{(1)}$. Moreover, the image by ϕ of the Casimir element Ω of $AW(3)$ defined in (3.2.20) is equal to an expression involving only central elements [8] :

$$\phi(\Omega) = C_1^2 + C_2^2 + C_3^2 + C_{123}^2 + C_1C_2C_3C_{123} - (q+q^{-1})^2. \tag{3.4.2}$$

3.4.2. Surjectivity

We now show that the intermediate Casimir elements C_i, C_{12}, C_{23} and C_{123} generate the whole centralizer $\mathcal{C}_{j_1,j_2,j_3}$.

Proposition 3.4.2. *The map* $\phi : AW(3) \to \mathcal{C}_{j_1,j_2,j_3}$ *is surjective.*

Proof. To reach that conclusion, we prove that the dimension of the image of ϕ is at least $\sum_{\ell \in \mathcal{J}(j_1,j_2,j_3)} d_\ell^2$, the dimension of $\mathcal{C}_{j_1,j_2,j_3}$. Let $\ell \in \mathcal{J}(j_1,j_2,j_3)$ and

$$\mathcal{S}^\ell(j_1,j_2,j_3) = \{ \ j \in \mathcal{J}(j_1,j_2) \mid \ell \in \mathcal{J}(j,j_3) \ \} \ . \tag{3.4.3}$$

From the definition (3.3.2), we deduce that $\mathcal{S}^\ell(j_1,j_2,j_3) = \{j_{\min}, j_{\min}+1, \dots, j_{\max}\}$ with

$$j_{\min} = \max(|j_1 - j_2|, |j_3 - \ell|) \quad \text{and} \qquad j_{\max} = \min(j_1 + j_2, j_3 + \ell) \ . \tag{3.4.4}$$

The cardinality of this set is $d_\ell = j_{\max} - j_{\min} + 1$. We denote by M_ℓ^+ the vector space spanned by the highest weight vectors of the representations M_ℓ in the decomposition (3.3.5). The dimension of M_ℓ^+ is d_ℓ and we can choose d_ℓ independent vectors $v_j \in M_\ell^+$ with $j \in \mathcal{S}^\ell(j_1,j_2,j_3)$ such that

$$\pi_{j_1,j_2,j_3}(\Delta^{(2)}(E))v_j = 0 \ , \quad \pi_{j_1,j_2,j_3}(\Delta^{(2)}(q^H))v_j = q^\ell v_j \ , \quad C_{123}v_j = \chi_\ell v_j \ , \quad C_{12}v_j = \chi_j v_j \ , \tag{3.4.5}$$

and

$$C_{23}v_j = \sum_{k \in \mathcal{S}^\ell(j_1,j_2,j_3)} \alpha_{j,k} v_k \ , \tag{3.4.6}$$

where $\alpha_{j,k}$ are complex numbers. The elements C_{12} and C_{23} are the images by ϕ of A and B. Therefore, they satisfy the Askey–Wilson algebra. It is enough to determine the constants $\alpha_{j,k}$ as shown previously in [22]. We reproduce this computation in the particular case needed here. We define the constants

$$\mathfrak{a} = \chi_{j_1}\chi_{j_2} + \chi_{j_3}\chi_\ell \ , \quad \mathfrak{b} = \chi_{j_2}\chi_{j_3} + \chi_{j_1}\chi_\ell \quad \text{and} \quad \mathfrak{c} = \chi_{j_1}\chi_{j_3} + \chi_{j_2}\chi_\ell \ . \tag{3.4.7}$$

We act with relation (3.2.24) on the vector v_j (for $j \in \mathcal{S}^\ell(j_1,j_2,j_3)$) and project the result on v_k with $k \neq j$ and on v_j. We get

$$\Big([j+k+2]_q[j+k]_q[k-j-1]_q[k-j+1]_q\Big)\alpha_{j,k} = 0 \ , \tag{3.4.8}$$

$$\alpha_{j,j} = \frac{\mathfrak{c}\chi_j - \mathfrak{b}\chi_0}{\chi_j^2 - \chi_0^2} \qquad \text{for} \quad j \neq 0 \ . \tag{3.4.9}$$

The projection on v_j is trivial if $j = 0$. From relation (3.4.8), we deduce that $\alpha_{j,k} = 0$ for $j \in \mathcal{S}^\ell(j_1,j_2,j_3)$ and $k \neq j+1, j-1, j$.

Then, we act with relation (3.2.23) on the vector v_j and project the result on $v_{j-2}, v_{j-1}, \dots, v_{j+2}$. The projections are trivial except the one on v_j which gives the following relation

$$[2j+3]_q\alpha_{j,j+1}\alpha_{j+1,j} - [2j-1]_q\alpha_{j-1,j}\alpha_{j,j-1} = \frac{1}{\chi_0}(\mathfrak{c} - \chi_j\alpha_{j,j})\alpha_{j,j} + \chi_0\chi_j - \mathfrak{a} \ , \tag{3.4.10}$$

with the boundary conditions $\alpha_{j_{\min},j_{\min}-1} = 0$ and $\alpha_{j_{\max},j_{\max}+1} = 0$. By using (3.4.9), one can show that the recurrence relation (3.4.10) and the boundary condition $\alpha_{j_{\max},j_{\max}+1} = 0$ imply

$$\alpha_{j-1,j}\alpha_{j,j-1} = \frac{\prod_{i=1}^4([j-r_i]_q[j+r_i]_q)}{[2j-1]_q[2j]_q^2[2j+1]_q}(q-q^{-1})^4 \qquad \text{for} \quad j \neq 0 \ , \tag{3.4.11}$$

where $r_1 = j_1 - j_2$, $r_2 = j_3 - \ell$, $r_3 = j_1 + j_2 + 1$ and $r_4 = \ell + j_3 + 1$. We see from (3.4.4) that the second boundary condition $\alpha_{j_{\min},j_{\min}-1} = 0$ is automatically satisfied if $j_{\min} > 0$. In the case where $j_{\min} = 0$ (which only happens if $j_1 = j_2$ and $\ell = j_3$), the limit $j \to 0$ of (3.4.11) vanishes. Moreover, we can deduce from (3.4.10) that $\alpha_{0,0} = \chi_{j_1}\chi_{j_3}/\chi_0$, which is the limit $j \to 0$ of (3.4.9).

To conclude the proof, we notice that equation (3.4.4) implies that the R.H.S. of relation (3.4.11) is never zero for $j_{\min} < j \leq j_{\max}$, and that the eigenvalues of C_{12} are pairwise distinct. Therefore, for a given $\ell \in \mathcal{J}(j_1,j_2,j_3)$, C_{12} and C_{23} generate a vector space of dimension d_ℓ^2. □

3.4.3. Kernel

The map ϕ defined in the Proposition 3.4.1 is not injective since there are non-trivial elements of $AW(3)$ that are mapped to zero, as seen from the results (3.3.10)–(3.3.13). We want to provide a description of the kernel of the map ϕ in order to find a quotient of $AW(3)$ that is isomorphic to the centralizer $\mathcal{C}_{j_1,j_2,j_3}$. Let us first define a quotient of $AW(3)$.

Definition 3.4.1. *The algebra* $\overline{AW}(j_1,j_2,j_3)$ *is the quotient of the centrally extended Askey–Wilson algebra* $AW(3)$ *by the following relations*

$$\alpha_i = \chi_{j_i} \ , \tag{3.4.12}$$

$$\prod_{j\in\mathcal{J}(j_1,j_2)} (A-\chi_j) = 0 \ , \quad \prod_{j\in\mathcal{J}(j_2,j_3)} (B-\chi_j) = 0 \ , \quad \prod_{j\in\mathcal{J}(j_1,j_2,j_3)} (K-\chi_j) = 0 \ , \tag{3.4.13}$$

$$\prod_{j\in\mathcal{J}(j_1,j_3)} (D-\chi_j) = 0 \ , \quad \prod_{j\in\mathcal{J}(j_1,j_3)} (D'-\chi_j) = 0 \ , \tag{3.4.14}$$

$$\prod_{m\in\mathcal{M}(j_1,j_2,j_3)} (K-A-m) = 0 \ , \qquad \prod_{m\in\mathcal{M}(j_2,j_3,j_1)} (K-B-m) = 0 \ , \tag{3.4.15}$$

$$\prod_{m\in\mathcal{M}(j_1,j_3,j_2)} (K-D-m) = 0 \ , \qquad \prod_{m\in\mathcal{M}(j_1,j_3,j_2)} (K-D'-m) = 0 \ , \tag{3.4.16}$$

$$\Omega = \chi_{j_1}^2 + \chi_{j_2}^2 + \chi_{j_3}^2 + K^2 + \chi_{j_1}\chi_{j_2}\chi_{j_3}K - (q+q^{-1})^2 \ , \tag{3.4.17}$$

where we recall that D *and* D' *are defined through* (3.2.18)–(3.2.19)*, and* Ω *is defined in* (3.2.20)*.*

Let us emphasize that all the relations (3.4.12)–(3.4.17) are in the kernel of the map ϕ in view of the results of Subsections 3.3.2 and 3.4.1. We are now in position to state a conjecture that proposes an algebraic description of $\mathcal{C}_{j_1,j_2,j_3}$.

Conjecture 3.4.1. *The map* $\overline{\phi} : \overline{AW}(j_1,j_2,j_3) \to \mathcal{C}_{j_1,j_2,j_3}$ *given by*

$$\overline{\phi}(A) = C_{12} \ , \ \overline{\phi}(B) = C_{23} \ , \ \overline{\phi}(K) = C_{123} \ , \tag{3.4.18}$$

is an algebra isomorphism.

To support this conjecture, we remark that by taking the limit $q \to 1$ (as described in Remark 3.2.1) of relations (3.4.12)–(3.4.16), we recover the conjecture proposed in [4] and proved in numerous cases. Let us notice that in this limit, the two relations in (3.4.14) reduce to only one relation, and similarly for the two relations in (3.4.16). The relation (3.4.17) involving the Casimir element of $AW(3)$ is new and will be useful when we discuss the representations of $\overline{AW}(j_1,j_2,j_3)$ in Section 3.5.

From the previous results, we know that $\overline{\phi}$ is a surjective homomorphism. It remains to prove that it is injective, which can be done by demonstrating that

$$\dim(\overline{AW}(j_1,j_2,j_3)) \leq \sum_{j\in\mathcal{J}(j_1,j_2,j_3)} d_j^2 = \dim(\mathcal{C}_{j_1,j_2,j_3}) \ . \tag{3.4.19}$$

To simplify the demonstration of (3.4.19), we can decompose $\overline{AW}(j_1,j_2,j_3)$ into a direct sum of simpler algebras. Indeed, let us introduce the following central idempotents, for $k \in \mathcal{J}(j_1,j_2,j_3)$,

$$\mathcal{K}_k = \prod_{\substack{r\in\mathcal{J}(j_1,j_2,j_3)\\ r\neq k}} \frac{K-\chi_r}{\chi_k-\chi_r} \ , \tag{3.4.20}$$

which satisfy $\mathcal{K}_k\mathcal{K}_\ell = \delta_{k,\ell}\mathcal{K}_k$, $\sum_{k\in\mathcal{J}(j_1,j_2,j_3)} \mathcal{K}_k = 1$ and $K\mathcal{K}_k = \mathcal{K}_k K = \chi_k\mathcal{K}_k$. We deduce that

$$\overline{AW}(j_1,j_2,j_3) = \bigoplus_{k\in\mathcal{J}(j_1,j_2,j_3)} \mathcal{K}_k\overline{AW}(j_1,j_2,j_3)\mathcal{K}_k \ . \tag{3.4.21}$$

Then, confirming inequality (3.4.19) amounts to proving the following inequalities, for $k \in \mathcal{J}(j_1,j_2,j_3)$,

$$\dim(\mathcal{K}_k\overline{AW}(j_1,j_2,j_3)\mathcal{K}_k) \leq d_k^2 \ . \tag{3.4.22}$$

The algebras $\mathcal{K}_k\overline{AW}(j_1,j_2,j_3)\mathcal{K}_k$ are simpler to study than $\overline{AW}(j_1,j_2,j_3)$. Roughly speaking, they correspond to replacing in the defining relations of $\overline{AW}(j_1,j_2,j_3)$ the central elements α_i (resp. K) by χ_{j_i} (resp. χ_k). One thus gets two annihilating polynomials for A (similarly for B, D and D')

$$\prod_{j\in\mathcal{J}(j_1,j_2)} (A - \chi_j) = 0 \ , \qquad \prod_{m\in\mathcal{M}(j_1,j_2,j_3)} (A - \chi_k + m) = 0 \ , \tag{3.4.23}$$

which reduce to only one, *i.e.*

$$\prod_{j\in\mathcal{J}^k(j_1,j_2,j_3)} (A - \chi_j) = 0 \ , \tag{3.4.24}$$

with $\mathcal{J}^k(j_a,j_b,j_c) = \{j \in \mathcal{J}(j_a,j_b) \mid \chi_j \in \{\chi_k - m \mid m \in \mathcal{M}(j_a,j_b,j_c)\}\}$.

In fact, in the quotient of $\mathcal{C}_{j_1,j_2,j_3}$ where $C_{123} = \chi_k$, the minimal polynomial of C_{12} is

$$\prod_{j\in\mathcal{S}^k(j_1,j_2,j_3)} (C_{12} - \chi_j) = 0 \ , \tag{3.4.25}$$

where we recall that (see proof of Proposition 3.4.2)

$$\mathcal{S}^k(j_a,j_b,j_c) = \{ \ j \in \mathcal{J}(j_a, j_b) \mid k \in \mathcal{J}(j,j_c) \ \} \ . \tag{3.4.26}$$

Similar results hold for C_{23}, $C_{13}^{(0)}$ and $C_{13}^{(1)}$. Let us emphasize that

$$\mathcal{S}^k(j_a,j_b,j_c) \subseteq \mathcal{J}^k(j_a,j_b,j_c) \ , \tag{3.4.27}$$

and that the cardinality of $\mathcal{S}^k(j_a,j_b,j_c)$ is equal to d_k and does not depend on the ordering of j_a,j_b,j_c. This discussion suggests the definition of another quotient of $AW(3)$.

Definition 3.4.2. *The algebra* $\overline{AW}^k(j_1,j_2,j_3)$*, where* $k \in \mathcal{J}(j_1,j_2,j_3)$*, is the quotient of the centrally extended Askey–Wilson algebra* $AW(3)$ *by* $\alpha_i = \chi_{j_i}$ *and the following relations*

$$K = \chi_k \ , \tag{3.4.28}$$

$$\prod_{j\in\mathcal{S}^k(j_1,j_2,j_3)} (A - \chi_j) = 0 \ , \quad \prod_{j\in\mathcal{S}^k(j_2,j_3,j_1)} (B - \chi_j) = 0 \ , \tag{3.4.29}$$

$$\prod_{j\in\mathcal{S}^k(j_1,j_3,j_2)} (D - \chi_j) = 0 \ , \quad \prod_{j\in\mathcal{S}^k(j_1,j_3,j_2)} (D' - \chi_j) = 0 \ . \tag{3.4.30}$$

Let us remark that the four annihilating polynomials (3.4.29)–(3.4.30) are of degree d_k. These quotients lead to another conjecture.

Conjecture 3.4.2. *The direct sum*

$$\widetilde{AW}(j_1,j_2,j_3) = \bigoplus_{k\in\mathcal{J}(j_1,j_2,j_3)} \overline{AW}^k(j_1,j_2,j_3) \tag{3.4.31}$$

is isomorphic to $\mathcal{C}_{j_1,j_2,j_3}$.

As for Conjecture 3.4.1, the proof of this conjecture reduces to showing that

$$\dim\left(\overline{AW}^k(j_1,j_2,j_3)\right) \leq d_k^2 \ . \tag{3.4.32}$$

In view of (3.4.27), we see that Conjecture 3.4.2 is true if Conjecture 3.4.1 is. Moreover, in this case, $\overline{AW}^k(j_1,j_2,j_3)$ is isomorphic to $\mathcal{K}_k\overline{AW}(j_1,j_2,j_3)\mathcal{K}_k$.

To conclude this section, let us emphasize that both conjectures presented above would provide an algebraic description of the centralizer $\mathcal{C}_{j_1,j_2,j_3}$. A strategy to prove these conjectures would be to establish inequalities (3.4.32) and then to derive the isomorphism between $\overline{AW}^k(j_1,j_2,j_3)$ and $\mathcal{K}_k\overline{AW}(j_1,j_2,j_3)\mathcal{K}_k$.

3.4.4. Invariance under permutations of $\{j_1,j_2,j_3\}$

The algebras involved in Conjecture 3.4.1 depend on the choice of three spins j_1,j_2 and j_3. We now show that it is sufficient to check the conjecture for only one ordering for the spins j_1,j_2 and j_3.

Proposition 3.4.3. *Let j_1, j_2 and j_3 be three positive half-integers or integers. If Conjecture 4.1 is true for the sequence of spins $\{j_1,j_2,j_3\}$, then it is also true for every permutation of j_1,j_2,j_3.*

Proof. For any two representation maps π_{j_1} and π_{j_2} of $U_q(\mathfrak{sl}_2)$, it is known that there exists an invertible matrix P such that for all $x \in U_q(\mathfrak{sl}_2)$ we have $(\pi_{j_2}\otimes\pi_{j_1})(\Delta(x)) = P^{-1}(\pi_{j_1}\otimes\pi_{j_2})(\Delta(x))P$. Therefore, from the definition of the centralizer (3.3.6) and the coassociativity of the comultiplication (3.2.4), we deduce that for any permutation σ of the symmetric group $\mathfrak{S}_3$, $\mathcal{C}_{j_1,j_2,j_3}$ is isomorphic to $\mathcal{C}_{j_{\sigma(1)},j_{\sigma(2)},j_{\sigma(3)}}$.

We must now show that the quotiented Askey–Wilson algebra $\overline{AW}(j_{\sigma(1)},j_{\sigma(2)},j_{\sigma(3)})$ is isomorphic to $\overline{AW}(j_1,j_2,j_3)$ for any permutation $\sigma\in\mathfrak{S}_3$. Since $\mathfrak{S}_3$ is generated by the transpositions (1,2) and (1,3), it suffices to prove the isomorphism for these two transformations.

The following maps are algebra isomorphisms

$$\phi_1 : \overline{AW}(j_3,j_2,j_1) \to \overline{AW}(j_1,j_2,j_3)$$

$$\phi_1(\alpha_1)=\alpha_3,\ \phi_1(\alpha_2)=\alpha_2,\ \phi_1(\alpha_3)=\alpha_1,\ \phi_1(A)=B,\ \phi_1(B)=A,\ \phi_1(K)=K\ , \tag{3.4.33}$$

$$\phi_2 : \overline{AW}(j_2,j_1,j_3) \to \overline{AW}(j_1,j_2,j_3)$$
$$\phi_2(\alpha_1) = \alpha_2,\ \phi_2(\alpha_2) = \alpha_1,\ \phi_2(\alpha_3) = \alpha_3,\ \phi_2(A) = A,\ \phi_2(B) = D',\ \phi_2(K) = K\ . \quad (3.4.34)$$

To see the homomorphism for the defining relations of $AW(3)$, it is easier to work with the symmetric presentations (3.2.16)–(3.2.19) and (3.2.25)–(3.2.26). By noticing that $\phi_1(D) = D'$ and $\phi_2(D) = B$, the homomorphism immediately follows. In order to preserve relation (3.4.12), the central elements α_i have to be permuted in the same way as the spins j_i, which is the case for the two maps given above. For the quotiented relations (3.4.13)–(3.4.16), the homomorphism is checked by observing that $\mathcal{J}(j_a,j_b) = \mathcal{J}(j_b,j_a)$, $\mathcal{M}(j_a,j_b,j_c) = \mathcal{M}(j_b,j_a,j_c)$ and that $\mathcal{J}(j_a,j_b,j_c)$ is invariant under any permutation of its entries. The R.H.S. of relation (3.4.17) is invariant under any permutation of the elements α_i. By using relations (3.2.16)–(3.2.19) and (3.2.25)–(3.2.26), it is straightforward to show that $\phi_1(\Omega) = \Omega$ and $\phi_2(\Omega) = \Omega$, which proves the homomorphism for relation (3.4.17). Finally, since the maps ϕ_1 and ϕ_2 are surjective and invertible, they are bijective. □

3.5. Finite irreducible representations of $\overline{AW}(j_1,j_2,j_3)$

To support Conjecture 3.4.1, we want to show that the sum of the squares of the dimensions of all the finite irreducible representations of $\overline{AW}(j_1,j_2,j_3)$ is equal to the dimension of the centralizer. This result implies that Conjecture 3.4.1 is true if and only if $\overline{AW}(j_1,j_2,j_3)$ is semisimple. Moreover, if $\overline{AW}(j_1,j_2,j_3)$ is not semisimple, the previous result proves that the missing relations (if there are any) in the kernel of ϕ are in a nilpotent radical of $\overline{AW}(j_1,j_2,j_3)$.

To identify all the finite irreducible representations of $\overline{AW}(j_1,j_2,j_3)$, we use the classification of the representations of the universal Askey–Wilson algebra given in [**10**] and look for the ones where the different relations of the quotient are satisfied. The universal Askey–Wilson algebra Δ_q, introduced in [**21**], is generated by three elements A,B,C and has three central elements α, β, γ. There is a surjective algebra homomorphism from Δ_q to the quotient of $AW(3)$ by $\alpha_i = \chi_{j_i}$, with the following mappings

$$A \mapsto A\ , \quad B \mapsto B\ , \quad C \mapsto D\ , \tag{3.5.1}$$
$$\alpha \mapsto \chi_{j_1}\chi_{j_2} + \chi_{j_3}K\ , \quad \beta \mapsto \chi_{j_2}\chi_{j_3} + \chi_{j_1}K, \quad \gamma \mapsto \chi_{j_1}\chi_{j_3} + \chi_{j_2}K\ . \tag{3.5.2}$$

We deduce that the quotient of $AW(3)$ by $\alpha_i = \chi_{j_i}$ is isomorphic to the quotient of Δ_q by the relations $(\alpha - \chi_{j_1}\chi_{j_2})/\chi_{j_3} = (\beta - \chi_{j_2}\chi_{j_3})/\chi_{j_1} = (\gamma - \chi_{j_1}\chi_{j_3})/\chi_{j_2}$. We can therefore use the representation theory of Δ_q in order to determine all the finite irreducible representations of $\overline{AW}(j_1,j_2,j_3)$.

The finite irreducible modules of the universal Askey–Wilson algebra Δ_q for q not a root of unity are classified in [**10**]. They are given by the isomorphism classes of the $n+1$-dimensional modules $V_n(a,b,c)$ defined in [**10**], for $n \geq 0$, under certain conditions on a,b,c (see Theorem 4.7 in [**10**]). In the representation $V_n(a,b,c)$, the central elements α,β and γ of

Δ_q take the following values

$$\alpha = (q^{n+1} + q^{-n-1})(a + a^{-1}) + (b + b^{-1})(c + c^{-1}) \ , \tag{3.5.3}$$

$$\beta = (q^{n+1} + q^{-n-1})(b + b^{-1}) + (c + c^{-1})(a + a^{-1}) \ , \tag{3.5.4}$$

$$\gamma = (q^{n+1} + q^{-n-1})(c + c^{-1}) + (a + a^{-1})(b + b^{-1}) \ . \tag{3.5.5}$$

The characteristic polynomials of $A,B,C \in \Delta_q$ in this representation are (see Lemma 4.3 of [10]) $K_a(X),K_b(X),K_c(X)$, with

$$K_x(X) = \prod_{i=0}^{n}(X - (q^{2i-n}x + q^{n-2i}x^{-1})) \ . \tag{3.5.6}$$

The Casimir element of the algebra Δ_q is given by

$$qA\alpha + q^{-1}B\beta + qC\gamma - q^2A^2 - q^{-2}B^2 - q^2C^2 - qABC \ . \tag{3.5.7}$$

It is straightforward to compute the value ω of this element in the representation $V_n(a,b,c)$ by using the representation matrices given in [10]. One gets

$$\begin{aligned}\omega = &(q^{n+1} + q^{-n-1})^2 + (a + a^{-1})^2 + (b + b^{-1})^2 + (c + c^{-1})^2 \\ &+ (q^{n+1} + q^{-n-1})(a + a^{-1})(b + b^{-1})(c + c^{-1}) - (q + q^{-1})^2 \ .\end{aligned} \tag{3.5.8}$$

We want to find all the irreducible representations of Δ_q that pass to the quotient $\overline{AW}(j_1,j_2,j_3)$. By comparing the annihilating polynomials of the elements $A,B,D \in \overline{AW}(j_1,j_2,j_3)$ given in (3.4.13)–(3.4.14) with the characteristic polynomials (3.5.6), we get the following restrictions for the inequivalent representations $V_n(a,b,c)$ to pass to the quotient:

$$0 \leq n \leq \min\{j_1 + j_2 - |j_1 - j_2|, j_2 + j_3 - |j_2 - j_3|, j_1 + j_3 - |j_1 - j_3|\} \ , \tag{3.5.9}$$

$$a = q^{2x+n+1} \ , \quad b = q^{2y+n+1} \ , \quad c = q^{2z+n+1} \ , \quad \text{for } x,y,z \text{ integers or half-integers,} \tag{3.5.10}$$

$$|j_1 - j_2| \leq x \leq j_1 + j_2 - n \ , \quad |j_2 - j_3| \leq y \leq j_2 + j_3 - n \ , \quad |j_1 - j_3| \leq z \leq j_1 + j_3 - n \ . \tag{3.5.11}$$

We recall that K is a central element with the annihilating polynomial given in (3.4.13). Therefore, K has to be a constant equal to χ_ℓ for some $\ell \in \mathcal{J}(j_1,j_2,j_3)$ in any irreducible representation of $\overline{AW}(j_1,j_2,j_3)$. We also recall that the Casimir element Ω of $AW(3)$ satisfies relation (3.4.17) in the quotient $\overline{AW}(j_1,j_2,j_3)$. From this discussion and from the results (3.5.2)–(3.5.5), (3.5.7), (3.5.8) and (3.5.10), we deduce that the following equations must hold so that the representation $V_n(a,b,c)$ passes to the quotient $\overline{AW}(j_1,j_2,j_3)$:

$$\chi_{\frac{n}{2}}\chi_{x+\frac{n}{2}} + \chi_{y+\frac{n}{2}}\chi_{z+\frac{n}{2}} = \chi_{j_1}\chi_{j_2} + \chi_{j_3}\chi_\ell \ , \tag{3.5.12}$$

$$\chi_{\frac{n}{2}}\chi_{y+\frac{n}{2}} + \chi_{z+\frac{n}{2}}\chi_{x+\frac{n}{2}} = \chi_{j_2}\chi_{j_3} + \chi_{j_1}\chi_\ell \ , \tag{3.5.13}$$

$$\chi_{\frac{n}{2}}\chi_{z+\frac{n}{2}} + \chi_{x+\frac{n}{2}}\chi_{y+\frac{n}{2}} = \chi_{j_1}\chi_{j_3} + \chi_{j_2}\chi_\ell \ , \tag{3.5.14}$$

$$\chi^2_{\frac{n}{2}} + \chi^2_{x+\frac{n}{2}} + \chi^2_{y+\frac{n}{2}} + \chi^2_{z+\frac{n}{2}} + \chi_{\frac{n}{2}}\chi_{x+\frac{n}{2}}\chi_{y+\frac{n}{2}}\chi_{z+\frac{n}{2}} = \chi^2_{j_1} + \chi^2_{j_2} + \chi^2_{j_3} + \chi^2_\ell + \chi_{j_1}\chi_{j_2}\chi_{j_3}\chi_\ell \ . \tag{3.5.15}$$

In the case of three identical spins $j_1 = j_2 = j_3 = s$, we find by using mathematical software that there are only 192 possible solutions for x,y,z,n to the system of equations (3.5.12)–(3.5.15). The only solutions respecting conditions (3.5.9)–(3.5.11) and corresponding to inequivalent representations $V_n(a,b,c)$ are

$$n = 2\ell \ , \qquad x = y = z = s - \ell \ , \qquad \text{if} \quad \ell \leq s \ , \tag{3.5.16}$$

$$n = 3s - \ell \ , \qquad x = y = z = \ell - s \ , \qquad \text{if} \quad \ell > s \ , \tag{3.5.17}$$

$$n = s + \ell \ , \qquad x = y = 0, z = s - \ell \ , \qquad \text{if} \quad \ell < s \ , \tag{3.5.18}$$

$$n = s - \ell - 1 \ , \qquad x = y = 0 \ , z = s + l + 1 \ , \qquad \text{if} \quad \ell \leq s - 1 \ , \tag{3.5.19}$$

and any permutation of x,y,z in the previous equations is also a solution.

Since $K = \chi_\ell$ for $\ell \in \mathcal{J}(j_1,j_2,j_3)$ in some irreducible representation, the annihilating polynomial of $K - A$ given in (3.4.15) implies that the annihilating polynomial of A reduces to the relation (3.4.24) in this representation. If the set $\mathcal{J}^\ell(j_1,j_2,j_3)$ is equal to the set $\mathcal{S}^\ell(j_1,j_2,j_3)$, then this reduced annihilating polynomial for A leads to the constraint

$$\max(|j_1 - j_2|, |j_3 - \ell|) \leq x \leq \min(j_1 + j_2, j_3 + \ell) - n \ . \tag{3.5.20}$$

Similar results hold for the annihilating polynomials of B (resp. D) and the constraints on y (resp. z). In the case of identical spins $j_1 = j_2 = j_3 = s$, this implies that $s - \ell \leq x,y,z \leq s + \ell - n$ for $\ell \leq s$, and $\ell - s \leq x,y,z \leq 2s - n$ for $\ell > s$. The only solutions remaining are (3.5.16) and (3.5.17). For s half-integer, we do not find any cases where $\mathcal{J}^\ell(s,s,s) \neq \mathcal{S}^\ell(s,s,s)$. For s integer, as a consequence of the fact that $0 \in \mathcal{M}(s,s,s)$, we find $\mathcal{J}^\ell(s,s,s) \setminus \mathcal{S}^\ell(s,s,s) = \{\ell\}$ if $\ell < s/2$, and otherwise the previous set is empty. We have verified numerically the sets $\mathcal{J}^\ell(s,s,s) \setminus \mathcal{S}^\ell(s,s,s)$ given above for at least $s = \frac{1}{2},1,...,10$. In any case, the upper bound on the values of x,y,z remains the same, and we still conclude that the only solutions are (3.5.16) and (3.5.17). Therefore, the sum of the squares of the dimensions $n+1$ of all the irreducible modules of $\overline{AW}(s,s,s)$ is

$$\sum_{\substack{\ell \in \mathcal{J}(s,s,s) \\ \ell \leq s}} (2\ell + 1)^2 + \sum_{\substack{\ell \in \mathcal{J}(s,s,s) \\ \ell > s}} (3s - \ell + 1)^2 = \frac{1}{2}(2s+1)((2s+1)^2 + 1) \ , \tag{3.5.21}$$

which is equal to the dimension of the centralizer $\dim(\mathcal{C}_{s,s,s}) = \sum_{\ell \in \mathcal{J}(s,s,s)} d_\ell^2$.

Let us remark that for $j_1 = j_2 = j_3 = \frac{1}{2},1,\dots,\frac{17}{2}$, we used mathematical software to test all the possible integer values for x,y,z,n such that the restrictions (3.5.9) and (3.5.11) and the three equations (3.5.12)–(3.5.14) are respected. The only solutions we found are those given in (3.5.16)–(3.5.19). Hence, equation (3.5.15) is perhaps not necessary if one wants to find all solutions for x,y,z,n integers.

Let us also notice that in the general case where j_1,j_2,j_3 are any three fixed integers or half-integers, we find at least 192 solutions to the system of equations (3.5.12)–(3.5.14) with n integer and x,y,z integers or half-integers. If these are the only such solutions, then it is possible to argue (in a similar manner as for the case of identical spins) that the sum of the squares of the dimensions of the irreducible representations that pass to the quotient $\overline{AW}(j_1,j_2,j_3)$ is also equal to the dimension of the centralizer $\mathcal{C}_{j_1,j_2,j_3}$.

Finally, we notice that in the representations $V_n(a,b,c)$, the element D' of $AW(3)$ has the same characteristic polynomial $K_c(X)$ as the element D. Therefore, the second relations in (3.4.14) and (3.4.16) do not provide any additional constraint on the values of n,a,b,c.

3.6. Quotient $\overline{AW}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and Temperley–Lieb algebra

In this section, we consider the case $j_1 = j_2 = j_3 = \frac{1}{2}$ and show that the quotient $\overline{AW}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ is isomorphic to the centralizer $\mathcal{C}_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}$, which is known to be the Temperley–Lieb algebra. We give an explicit isomorphism between $\overline{AW}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and the Temperley–Lieb algebra.

3.6.1. $\overline{AW}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ algebra

From the definitions (3.3.2)–(3.3.3) and (3.3.14), we find the sets

$$\mathcal{J}\left(\frac{1}{2},\frac{1}{2}\right) = \{0,1\}\ , \quad \mathcal{J}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = \left\{\frac{1}{2},\frac{3}{2}\right\}, \tag{3.6.1}$$

$$\mathcal{M}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = \{\chi_{\frac{3}{2}} - \chi_1,\chi_{\frac{1}{2}} - \chi_1,\chi_{\frac{1}{2}} - \chi_0\}\ . \tag{3.6.2}$$

The degeneracies are $d_{\frac{1}{2}} = 2$ and $d_{\frac{3}{2}} = 1$. We find from (3.3.8) that $\dim\left(\mathcal{C}_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\right) = 5$. The central elements α_i of $\overline{AW}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ can all be replaced by the constant $\chi_{\frac{1}{2}}$. For computational convenience, we perform the transformation $X = (q-q^{-1})^2\tilde{X} + q + q^{-1}$ on the elements $X = A,B,D,D',K$, as in Remark 3.2.1, and we define the shifted central element $\tilde{G} = \tilde{K} + [1/2]_q^2$. By using the sets given in (3.6.1) and (3.6.2), one finds that the defining relations of $\overline{AW}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ are

$$[\tilde{B},[\tilde{A},\tilde{B}]_q]_q = [2]_q\left(-\tilde{B}^2 - \{\tilde{A},\tilde{B}\}\right) + (q^2+q^{-2})\tilde{G}\tilde{B} + [2]_q^2\tilde{B} \ , \tag{3.6.3}$$

$$[[\tilde{A},\tilde{B}]_q,\tilde{A}]_q = [2]_q\left(-\tilde{A}^2 - \{\tilde{A},\tilde{B}\}\right) + (q^2+q^{-2})\tilde{G}\tilde{A} + [2]_q^2\tilde{A} \ , \tag{3.6.4}$$

$$\tilde{A}(\tilde{A}-[2]_q) = 0 \ , \quad \tilde{B}(\tilde{B}-[2]_q) = 0 \ , \quad (\tilde{G}-1)(\tilde{G}-[2]_q^2) = 0 \ , \tag{3.6.5}$$

$$\tilde{D}(\tilde{D}-[2]_q) = 0 \ , \quad \tilde{D}'(\tilde{D}'-[2]_q) = 0 \ , \tag{3.6.6}$$

$$(\tilde{G}-\tilde{A}+[2]_q-[2]_q^2)(\tilde{G}-\tilde{A}+[2]_q-1)(\tilde{G}-\tilde{A}-1) = 0 \ , \tag{3.6.7}$$

$$(\tilde{G}-\tilde{B}+[2]_q-[2]_q^2)(\tilde{G}-\tilde{B}+[2]_q-1)(\tilde{G}-\tilde{B}-1) = 0 \ , \tag{3.6.8}$$

$$(\tilde{G}-\tilde{D}+[2]_q-[2]_q^2)(\tilde{G}-\tilde{D}+[2]_q-1)(\tilde{G}-\tilde{D}-1) = 0 \ , \tag{3.6.9}$$

$$(\tilde{G}-\tilde{D}'+[2]_q-[2]_q^2)(\tilde{G}-\tilde{D}'+[2]_q-1)(\tilde{G}-\tilde{D}'-1) = 0 \ , \tag{3.6.10}$$

$$\begin{aligned}&(q^2+q^{-2})(q-q^{-1})(q\tilde{A}+q^{-1}\tilde{B}+q\tilde{D})\tilde{G} - [2]_q(([2]_q-q^3)(\tilde{A}+\tilde{D})+q^{-3}\tilde{B})\\ &-(q-q^{-1})(q^2\tilde{A}^2+q^{-2}\tilde{B}^2+q^2\tilde{D}^2) - q[2]_q(q-q^{-1})(\tilde{A}\tilde{B}+\tilde{A}\tilde{D}+\tilde{B}\tilde{D}) - q(q-q^{-1})^3\tilde{A}\tilde{B}\tilde{D}\\ &= (q-q^{-1})\tilde{G}^2 + (q^5-q^{-5}-q^2[2]_q)\tilde{G} - q^{-1}[2]_q^2 \ ,\end{aligned} \tag{3.6.11}$$

where

$$\tilde{D} = [2]_q + (q^2+q^{-2})\tilde{G} - \tilde{A} - \tilde{B} - \frac{q-q^{-1}}{q+q^{-1}}[\tilde{A},\tilde{B}]_q \ , \tag{3.6.12}$$

$$\tilde{D}' = [2]_q + (q^2+q^{-2})\tilde{G} - \tilde{A} - \tilde{B} - \frac{q-q^{-1}}{q+q^{-1}}[\tilde{B},\tilde{A}]_q \ . \tag{3.6.13}$$

We want to show that $\overline{AW}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ is isomorphic to the centralizer $\mathcal{C}_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}$.

Proposition 3.6.1. *The relations defining the quotient* $\overline{AW}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ *can be given as follows*

$$\tilde{A}^2 = [2]_q\tilde{A} \ , \quad \tilde{B}^2 = [2]_q\tilde{B} \ , \tag{3.6.14}$$

$$\tilde{A}\tilde{B}\tilde{A} = [2]_q\{\tilde{A},\tilde{B}\} - [3]_q\tilde{A} - [2]_q^2\tilde{B} + [2]_q[3]_q \ , \tag{3.6.15}$$

$$\tilde{B}\tilde{A}\tilde{B} = [2]_q\{\tilde{A},\tilde{B}\} - [3]_q\tilde{B} - [2]_q^2\tilde{A} + [2]_q[3]_q \ . \tag{3.6.16}$$

Proof. The two first relations in (3.6.5) directly lead to (3.6.14). The third relation in (3.6.5) implies

$$\tilde{G}^2 = ([2]_q^2+1)\tilde{G} - [2]_q^2 \ . \tag{3.6.17}$$

Developing (3.6.3) and (3.6.4) and using (3.6.14), one gets

$$\tilde{B}\tilde{A}\tilde{B} = \tilde{G}\tilde{B} \ , \quad \tilde{A}\tilde{B}\tilde{A} = \tilde{G}\tilde{A} \ . \tag{3.6.18}$$

Expanding (3.6.7) and (3.6.8) and simplifying with the help of (3.6.14) and (3.6.17), one gets

$$\tilde{G}\tilde{A} = [2]_q\tilde{G} + \tilde{A} - [2]_q \ , \quad \tilde{G}\tilde{B} = [2]_q\tilde{G} + \tilde{B} - [2]_q \ , \tag{3.6.19}$$

which implies

$$\tilde{G}\tilde{A}\tilde{B} = [2]_q^2\tilde{G} + \tilde{A}\tilde{B} - [2]_q^2 \ , \quad \tilde{G}\tilde{B}\tilde{A} = [2]_q^2\tilde{G} + \tilde{B}\tilde{A} - [2]_q^2 \ . \tag{3.6.20}$$

Equations (3.6.6) and (3.6.9)–(3.6.11) can be simplified using the previous relations, and they lead to

$$\tilde{G} = -[2]_q(\tilde{A} + \tilde{B}) + \{\tilde{A},\tilde{B}\} + [2]_q^2 \ . \tag{3.6.21}$$

Substituting (3.6.21) in (3.6.19) and (3.6.20), one finds

$$\tilde{G}\tilde{A} = [2]_q\{\tilde{A},\tilde{B}\} - [3]_q\tilde{A} - [2]_q^2\tilde{B} + [2]_q[3]_q \ , \tag{3.6.22}$$

$$\tilde{G}\tilde{B} = [2]_q\{\tilde{A},\tilde{B}\} - [3]_q\tilde{B} - [2]_q^2\tilde{A} + [2]_q[3]_q \ , \tag{3.6.23}$$

$$\tilde{G}\tilde{A}\tilde{B} = [2]_q^2\tilde{B}\tilde{A} + ([2]_q^2 + 1)\tilde{A}\tilde{B} - [2]_q^3(\tilde{A} + \tilde{B}) + [2]_q^2[3]_q \ , \tag{3.6.24}$$

$$\tilde{G}\tilde{B}\tilde{A} = [2]_q^2\tilde{A}\tilde{B} + ([2]_q^2 + 1)\tilde{B}\tilde{A} - [2]_q^3(\tilde{A} + \tilde{B}) + [2]_q^2[3]_q \ . \tag{3.6.25}$$

Equations (3.6.18) and (3.6.22)–(3.6.23) imply the relations (3.6.15)–(3.6.16) of the proposition.

It remains to show that the generator $\tilde{G}$ can be suppressed from the presentation, or in other words that (3.6.17) and (3.6.22)–(3.6.25) are implied from the relations of the proposition. Suppose that relations (3.6.14)–(3.6.16) are true and let $\tilde{G} = -[2]_q(\tilde{A}+\tilde{B})+\{\tilde{A},\tilde{B}\}+[2]_q^2$. Multiplying the expression of $\tilde{G}$ on the left and on the right by $\tilde{A}$ and $\tilde{B}$, one finds

$$\tilde{G}\tilde{A} = \tilde{A}\tilde{G} = \tilde{A}\tilde{B}\tilde{A} \ , \quad \tilde{G}\tilde{B} = \tilde{B}\tilde{G} = \tilde{B}\tilde{A}\tilde{B} \ . \tag{3.6.26}$$

Using (3.6.15) and (3.6.16), equations (3.6.22) and (3.6.23) are recovered. Multiplying (3.6.15) on the right by $\tilde{B}$ and (3.6.16) on the right by $\tilde{A}$, one finds

$$\tilde{G}\tilde{A}\tilde{B} = \tilde{A}\tilde{B}\tilde{A}\tilde{B} = \tilde{A}\tilde{B} - [2]_q\tilde{B} + [2]_q\tilde{B}\tilde{A}\tilde{B} \ , \tag{3.6.27}$$

$$\tilde{G}\tilde{B}\tilde{A} = \tilde{B}\tilde{A}\tilde{B}\tilde{A} = \tilde{B}\tilde{A} - [2]_q\tilde{A} + [2]_q\tilde{A}\tilde{B}\tilde{A} \ , \tag{3.6.28}$$

from which one easily recovers (3.6.24) and (3.6.25). Finally, it is straightforward to arrive at

$$\tilde{G}^2 = -[2]_q^3(\tilde{A} + \tilde{B}) + [2]_q^2\{\tilde{A},\tilde{B}\} - [2]_q(\tilde{A}\tilde{B}\tilde{A} + \tilde{B}\tilde{A}\tilde{B}) + \tilde{A}\tilde{B}\tilde{A}\tilde{B} + \tilde{B}\tilde{A}\tilde{B}\tilde{A} + [2]_q^4 \tag{3.6.29}$$

and to use the results (3.6.27) and (3.6.28) to recover (3.6.17). □

Theorem 3.6.1. *Conjecture 3.4.1 is verified for* $j_1 = j_2 = j_3 = \frac{1}{2}$.

Proof. We already know from proposition 3.4.2 that the map $\overline{\phi}$ is surjective. From the previous proposition, it is easy to show that $\{1,\tilde{A},\tilde{B},\tilde{A}\tilde{B},\tilde{B}\tilde{A}\}$ is a linearly generating set of $\overline{AW}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Since $\dim\left(\mathcal{C}_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\right) = 5$, this shows the injectivity of the map $\overline{\phi}$. □

3.6.2. Connection with the Temperley–Lieb algebra

It is known that the Temperley–Lieb algebra is isomorphic to the centralizer of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in the tensor product of three fundamental representations [12]. Hence, from the results of the previous subsection, the quotiented Askey–Wilson algebra $\overline{AW}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ is isomorphic to the Temperley–Lieb algebra.

Definition 3.6.1. *[20] The Temperley–Lieb algebra $TL_3(q)$ is generated by σ_1 and σ_2 with the following defining relations*

$$\sigma_1^2 = (q+q^{-1})\sigma_1 \ , \quad \sigma_2^2 = (q+q^{-1})\sigma_2 \ , \tag{3.6.30}$$

$$\sigma_1\sigma_2\sigma_1 = \sigma_1 \ , \quad \sigma_2\sigma_1\sigma_2 = \sigma_2 \ . \tag{3.6.31}$$

Theorem 3.6.2. *The quotiented Askey–Wilson algebra $\overline{AW}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ is isomorphic to the Temperley–Lieb algebra $TL_3(q)$. This isomorphism is given explicitly by*

$$\overline{AW}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \to TL_3(q)$$

$$\tilde{A} \mapsto (q+q^{-1}) - \sigma_1 \ , \tag{3.6.32}$$

$$\tilde{B} \mapsto (q+q^{-1}) - \sigma_2 \ . \tag{3.6.33}$$

Proof. It is straightforward to show that the defining relations (3.6.30) and (3.6.31) of $TL_3(q)$ are equivalent to the relations (3.6.14)–(3.6.16) of $\overline{AW}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. □

3.7. Quotient $\overline{AW}(1,1,1)$ and Birman–Murakami–Wenzl algebra

In this section, we choose $j_1 = j_2 = j_3 = 1$ and prove that the quotient $\overline{AW}(1,1,1)$ is isomorphic to the centralizer $\mathcal{C}_{1,1,1}$. In this case, $\mathcal{C}_{1,1,1}$ is known to be connected to the Birman–Murakami–Wenzl algebra. We give an explicit isomorphism between $\overline{AW}(1,1,1)$ and a specialization of the BMW algebra.

3.7.1. $\overline{AW}(1,1,1)$ algebra

We have the following sets

$$\mathcal{J}(1,1) = \{0,1,2\} \ , \quad \mathcal{J}(1,1,1) = \{0,1,2,3\} \ , \tag{3.7.1}$$

$$\mathcal{M}(1,1,1) = \{\chi_1-\chi_2,\chi_0-\chi_1,0,\chi_1-\chi_0,\chi_2-\chi_1,\chi_3-\chi_2\} \ . \tag{3.7.2}$$

The degeneracies are $d_0 = d_3 = 1$, $d_1 = 3$ and $d_2 = 2$, and the dimension of the centralizer is $\dim(C_{1,1,1}) = 15$. The central elements α_i of $\overline{AW}(1,1,1)$ can all be replaced by the constant χ_1. For computational convenience again, we perform the transformation $X = (q-q^{-1})^2\tilde{X} + q+q^{-1}$ on the elements $X = A,B,D,D',K$ (see Remark 3.2.1). We recall that the eigenvalues χ_j are transformed to $\tilde{\chi}_j = [j]_q[j+1]_q$, and we define the constants $m_1 = \tilde{\chi}_1-\tilde{\chi}_0$, $m_2 = \tilde{\chi}_2-\tilde{\chi}_1$

and $m_3 = \tilde{\chi}_3 - \tilde{\chi}_2$. The defining relations (3.4.13)–(3.4.16) of $\overline{AW}(1,1,1)$ are written as

$$(q^2+q^{-2})\tilde{B}\tilde{A}\tilde{B} = \tilde{A}\tilde{B}^2 + \tilde{B}^2\tilde{A} - [2]_q\tilde{B}^2 - [2]_q\{\tilde{A},\tilde{B}\} + [2]_q([2]_q^2-3)\tilde{K}\tilde{B} + [2]_q^2[3]_q\tilde{B} \ , \tag{3.7.3}$$

$$(q^2+q^{-2})\tilde{A}\tilde{B}\tilde{A} = \tilde{B}\tilde{A}^2 + \tilde{A}^2\tilde{B} - [2]_q\tilde{A}^2 - [2]_q\{\tilde{A},\tilde{B}\} + [2]_q([2]_q^2-3)\tilde{K}\tilde{A} + [2]_q^2[3]_q\tilde{A} \ , \tag{3.7.4}$$

$$\tilde{A}(\tilde{A}-\tilde{\chi}_1)(\tilde{A}-\tilde{\chi}_2) = 0 \ , \quad \tilde{B}(\tilde{B}-\tilde{\chi}_1)(\tilde{B}-\tilde{\chi}_2) = 0 \ , \tag{3.7.5}$$

$$\tilde{K}(\tilde{K}-\tilde{\chi}_1)(\tilde{K}-\tilde{\chi}_2)(\tilde{K}-\tilde{\chi}_3) = 0 \ , \tag{3.7.6}$$

$$\tilde{D}(\tilde{D}-\tilde{\chi}_1)(\tilde{D}-\tilde{\chi}_2) = 0 \ , \quad \tilde{D}'(\tilde{D}'-\tilde{\chi}_1)(\tilde{D}'-\tilde{\chi}_2) = 0 \ , \tag{3.7.7}$$

$$(\tilde{K}-\tilde{A}+m_2)(\tilde{K}-\tilde{A}+m_1)(\tilde{K}-\tilde{A})(\tilde{K}-\tilde{A}-m_1)(\tilde{K}-\tilde{A}-m_2)(\tilde{K}-\tilde{A}-m_3) = 0 \ , \tag{3.7.8}$$

$$(\tilde{K}-\tilde{B}+m_2)(\tilde{K}-\tilde{B}+m_1)(\tilde{K}-\tilde{B})(\tilde{K}-\tilde{B}-m_1)(\tilde{K}-\tilde{B}-m_2)(\tilde{K}-\tilde{B}-m_3) = 0 \ , \tag{3.7.9}$$

$$(\tilde{K}-\tilde{D}+m_2)(\tilde{K}-\tilde{D}+m_1)(\tilde{K}-\tilde{D})(\tilde{K}-\tilde{D}-m_1)(\tilde{K}-\tilde{D}-m_2)(\tilde{K}-\tilde{D}-m_3) = 0 \ , \tag{3.7.10}$$

$$(\tilde{K}-\tilde{D}'+m_2)(\tilde{K}-\tilde{D}'+m_1)(\tilde{K}-\tilde{D}')(\tilde{K}-\tilde{D}'-m_1)(\tilde{K}-\tilde{D}'-m_2)(\tilde{K}-\tilde{D}'-m_3) = 0 \ , \tag{3.7.11}$$

where

$$\tilde{D} = [2]_q[3]_q + ([2]_q^2-3)\tilde{K} - \frac{(q-q^{-1})}{[2]_q}[\tilde{A},\tilde{B}]_q - \tilde{A} - \tilde{B} \ , \tag{3.7.12}$$

$$\tilde{D}' = [2]_q[3]_q + ([2]_q^2-3)\tilde{K} - \frac{(q-q^{-1})}{[2]_q}[\tilde{B},\tilde{A}]_q - \tilde{A} - \tilde{B} \ . \tag{3.7.13}$$

Theorem 3.7.1. *Conjecture 4.1 is verified for $j_1 = j_2 = j_3 = 1$.*

Proof. We already know from proposition 3.4.2 that the map $\overline{\phi}$ is surjective. We only need to prove that it is injective in this case.

We define the set

$$\mathcal{S} = \{1,\tilde{A},\tilde{B},\tilde{A}^2,\tilde{A}\tilde{B},\tilde{B}\tilde{A},\tilde{B}^2,\tilde{A}^2\tilde{B},\tilde{A}\tilde{B}\tilde{A},\tilde{A}\tilde{B}^2,\tilde{B}\tilde{A}^2,\tilde{B}\tilde{A}\tilde{B},\tilde{A}^2\tilde{B}^2,\tilde{A}\tilde{B}\tilde{A}\tilde{B},\tilde{B}\tilde{A}\tilde{B}\tilde{A}\} \ . \tag{3.7.14}$$

Using relations (3.7.3)–(3.7.6), it can be shown that $\mathcal{S}_r = \mathcal{S} \cup \tilde{K}\mathcal{S} \cup \tilde{K}^2\mathcal{S} \cup \tilde{K}^3\mathcal{S}$ is a linearly generating set for $\overline{AW}(1,1,1)$. We can construct the 60 by 60 matrices $\tilde{A}_r$, $\tilde{B}_r$ and $\tilde{K}_r$ corresponding to the regular actions of $\tilde{A}$, $\tilde{B}$ and $\tilde{K}$ on the set $\mathcal{S}_r$. Knowing that $\tilde{A}_r$, $\tilde{B}_r$ and $\tilde{K}_r$ have to satisfy (3.7.3)–(3.7.6) and the first relation of (3.7.7), we find 32 independant relations between the elements of $\mathcal{S}_r$ and we can reduce the generating set to

$$\mathcal{S}'_r = \mathcal{S} \cup \tilde{K}\{1,\tilde{B},\tilde{A}^2,\tilde{A}\tilde{B},\tilde{B}^2,\tilde{A}^2\tilde{B},\tilde{A}\tilde{B}\tilde{A},\tilde{A}\tilde{B}^2,\tilde{B}\tilde{A}\tilde{B},\tilde{A}^2\tilde{B}^2,\tilde{A}\tilde{B}\tilde{A}\tilde{B}\} \cup \tilde{K}^2\{\tilde{A}^2,\tilde{A}^2\tilde{B}\} \ . \tag{3.7.15}$$

We repeat the procedure and construct 28 by 28 matrices corresponding to the regular actions on $\mathcal{S}'_r$. Only using again (3.7.3)–(3.7.6) and the first of (3.7.7), we can reduce the generating set to

$$\mathcal{S}''_r = \mathcal{S} \cup \{\tilde{K},\tilde{K}\tilde{B},\tilde{K}\tilde{A}^2,\tilde{K}\tilde{A}\tilde{B},\tilde{K}\tilde{B}^2,\tilde{K}\tilde{A}^2\tilde{B},\tilde{K}\tilde{A}\tilde{B}\tilde{A},\tilde{K}\tilde{A}\tilde{B}^2,\tilde{K}\tilde{B}\tilde{A}\tilde{B}\} \ . \tag{3.7.16}$$

We repeat and construct 24 by 24 matrices. At this point, relations (3.7.3)–(3.7.7) are already satisfied. We must use relations (3.7.8)–(3.7.10) to reduce the generating set to

$$\mathcal{S}'''_r = \mathcal{S} \cup \{\tilde{K},\tilde{K}\tilde{A}^2,\tilde{K}\tilde{B}^2\} \ . \tag{3.7.17}$$

We repeat one last time by constructing 18 by 18 matrices and we use (3.7.3)–(3.7.11) to find 3 independant relations which allow to reduce the generating set to $\mathcal{S}$. It can also be verified that the matrices of the regular action satisfy the defining relation (3.4.17) involving the Casimir element Ω. We made the previous computations by using a formal mathematical software.

From these results, we have that $\mathcal{S}$ is a linearly generating set for $\overline{AW}(1,1,1)$ with 15 elements. Since $\dim(\mathcal{C}_{1,1,1}) = 15$, we conclude that $\overline{\phi}$ is injective. □

3.7.2. Connection with the Birman–Murakami–Wenzl algebra

It is known [16] that the Birman–Murakami–Wenzl algebra is isomorphic to the centralizer of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in the tensor product of three spin-1 representations. Hence, from the previous theorem, the quotiented Askey–Wilson algebra $\overline{AW}(1,1,1)$ is isomorphic to the BMW algebra.

Definition 3.7.1. *[11] The Birman–Murakami–Wenzl algebra $BMW_3(Q,\mu)$ is generated by invertible elements s_1 and s_2 with the following defining relations*

$$s_1 s_2 s_1 = s_2 s_1 s_2 \ , \tag{3.7.18}$$

$$e_1 s_1 = s_1 e_1 = \mu^{-1} e_1 \ , \quad e_2 s_2 = s_2 e_2 = \mu^{-1} e_2 \ , \tag{3.7.19}$$

$$e_1 s_2^{\epsilon} e_1 = \mu^{\epsilon} e_1 \ , \quad e_2 s_1^{\epsilon} e_2 = \mu^{\epsilon} e_2 \ , \quad \epsilon = \pm 1 \ , \tag{3.7.20}$$

$$e_i = 1 - \frac{s_i - s_i^{-1}}{Q - Q^{-1}} \ , \quad i = 1,2 \ . \tag{3.7.21}$$

Theorem 3.7.2. *The quotiented Askey–Wilson algebra $\overline{AW}(1,1,1)$ is isomorphic to the Birman–Murakami–Wenzl algebra $BMW_3(q^2,q^4)$. This isomorphism is given explicitly by*

$$\overline{AW}(1,1,1) \to BMW_3(q^2,q^4)$$

$$\tilde{A} \mapsto (q+q^{-1})(s_1 - q^{-2} e_1) + (q+q^{-1})^2 q^{-1} \ , \tag{3.7.22}$$

$$\tilde{B} \mapsto (q+q^{-1})(s_2 - q^{-2} e_2) + (q+q^{-1})^2 q^{-1} \ . \tag{3.7.23}$$

Proof. The algebras $\overline{AW}(1,1,1)$ and $BMW_3(q^2,q^4)$ are both isomorphic to $\mathcal{C}_{1,1,1}$, hence they are isomorphic to each other. It can be verified that the image of $\tilde{A}$ (resp. $\tilde{B}$) in $\mathrm{End}(M_1^{\otimes 3})$ is equal to the image of the R.H.S. of (3.7.22) (resp. (3.7.23)), which justifies the explicit mapping. The inverse map is given by

$$s_1 \mapsto q^{-2}(q+q^{-1})^{-2}\tilde{A}^2 - q^{-2}(q+q^{-1})^{-1}(2+q^{-2})\tilde{A} + q^{-4} \ , \tag{3.7.24}$$

$$s_2 \mapsto q^{-2}(q+q^{-1})^{-2}\tilde{B}^2 - q^{-2}(q+q^{-1})^{-1}(2+q^{-2})\tilde{B} + q^{-4} \ . \tag{3.7.25}$$

□

3.8. Quotient $\overline{AW}(\frac{3}{2},\frac{3}{2},\frac{3}{2})$

In this section, we take $j_1 = j_2 = j_3 = \frac{3}{2}$ and show that the $\overline{AW}(\frac{3}{2},\frac{3}{2},\frac{3}{2})$ algebra is isomorphic to the centralizer $\mathcal{C}_{\frac{3}{2},\frac{3}{2},\frac{3}{2}}$.

From the decomposition rules of the tensor product, we find the sets

$$\mathcal{J}\left(\frac{3}{2},\frac{3}{2}\right) = \{0,1,2,3\} \ , \quad \mathcal{J}\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right) = \left\{\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2},\frac{9}{2}\right\} \ , \tag{3.8.1}$$

$$\begin{aligned}\mathcal{M}\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right) = \{&\chi_{\frac{9}{2}} - \chi_3, \chi_{\frac{7}{2}} - \chi_3, \chi_{\frac{7}{2}} - \chi_2, \chi_{\frac{5}{2}} - \chi_3, \chi_{\frac{5}{2}} - \chi_2, \chi_{\frac{5}{2}} - \chi_1, \\ &\chi_{\frac{3}{2}} - \chi_3, \chi_{\frac{3}{2}} - \chi_2, \chi_{\frac{3}{2}} - \chi_1, \chi_{\frac{3}{2}} - \chi_0, \chi_{\frac{1}{2}} - \chi_2, \chi_{\frac{1}{2}} - \chi_1\} \ .\end{aligned} \tag{3.8.2}$$

The degeneracies are $d_{\frac{9}{2}} = 1$, $d_{\frac{7}{2}} = d_{\frac{1}{2}} = 2$, $d_{\frac{5}{2}} = 3$ and $d_{\frac{3}{2}} = 4$, and the dimension of the centralizer is $\dim\left(\mathcal{C}_{\frac{3}{2},\frac{3}{2},\frac{3}{2}}\right) = 34$. The central elements α_i of $\overline{AW}(\frac{3}{2},\frac{3}{2},\frac{3}{2})$ are each equal to the constant $\chi_{\frac{3}{2}}$.

In order to prove the injectivity of the map $\overline{\phi}$ in this case, we will use the strategy described in Subsection 4.3 and show that

$$\dim\left(\mathcal{K}_k\overline{AW}\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right)\mathcal{K}_k\right) \leq d_k^2 \qquad \forall k \in \mathcal{J}\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right) \ . \tag{3.8.3}$$

We recall that for each $k \in \mathcal{J}\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right)$, the central element K is replaced by the constant χ_k in the algebra $\mathcal{K}_k\overline{AW}\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right)\mathcal{K}_k$, and the annihilating polynomials for A (similarly for B, D and D') reduce to

$$\prod_{j\in\mathcal{J}^k\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right)} (A - \chi_j) = 0 \ , \tag{3.8.4}$$

where $\mathcal{J}^k\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right) = \{j \in \mathcal{J}\left(\frac{3}{2},\frac{3}{2}\right) \mid \chi_j \in \{\chi_k - m \mid m \in \mathcal{M}\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right)\}\}$. Once again, we perform the transformation $X = (q-q^{-1})^2\tilde{X} + q + q^{-1}$ on the elements $X = A,B,D,D',K$ (see Remark 3.2.1). Therefore, one finds that the following relations hold in the algebras $\mathcal{K}_k\overline{AW}\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right)\mathcal{K}_k$:

- $k = \frac{9}{2}$

$$\tilde{A} = \tilde{B} = \chi_3 \ . \tag{3.8.5}$$

- $k = \frac{7}{2}$

$$\tilde{B}\tilde{A}\tilde{B} = [2]_q^3\{\tilde{A},\tilde{B}\} + [3]_q^2(-2[2]_q^2\tilde{A} + (q^4+q^{-4}-1)\tilde{B} + [2]_q^3) \ , \tag{3.8.6}$$

$$\tilde{A}\tilde{B}\tilde{A} = [2]_q^3\{\tilde{A},\tilde{B}\} + [3]_q^2(-2[2]_q^2\tilde{B} + (q^4+q^{-4}-1)\tilde{A} + [2]_q^3) \ , \tag{3.8.7}$$

$$(\tilde{A}-\tilde{\chi}_2)(\tilde{A}-\tilde{\chi}_3) = 0 \ , \quad (\tilde{B}-\tilde{\chi}_2)(\tilde{B}-\tilde{\chi}_3) = 0 \ , \tag{3.8.8}$$

$$(\tilde{D}-\tilde{\chi}_2)(\tilde{D}-\tilde{\chi}_3) = 0 \ , \quad (\tilde{D}'-\tilde{\chi}_2)(\tilde{D}'-\tilde{\chi}_3) = 0 \ . \tag{3.8.9}$$

- $k = \frac{5}{2}$

$$(q^2+q^{-2})\tilde{B}\tilde{A}\tilde{B} = \tilde{A}\tilde{B}^2 + \tilde{B}^2\tilde{A} - [2]_q\tilde{B}^2 - [2]_q\{\tilde{A},\tilde{B}\} + (2[2]_q\tilde{\chi}_{\frac{3}{2}} + (q^4+q^{-4})(\tilde{\chi}_{\frac{3}{2}}+\tilde{\chi}_{\frac{5}{2}}))\tilde{B} \ , \tag{3.8.10}$$

$$(q^2+q^{-2})\tilde{A}\tilde{B}\tilde{A} = \tilde{B}\tilde{A}^2 + \tilde{A}^2\tilde{B} - [2]_q\tilde{A}^2 - [2]_q\{\tilde{A},\tilde{B}\} + (2[2]_q\tilde{\chi}_{\frac{3}{2}} + (q^4+q^{-4})(\tilde{\chi}_{\frac{3}{2}}+\tilde{\chi}_{\frac{5}{2}}))\tilde{A} \ , \tag{3.8.11}$$

$$(\tilde{A}-\tilde{\chi}_1)(\tilde{A}-\tilde{\chi}_2)(\tilde{A}-\tilde{\chi}_3) = 0 \ , \quad (\tilde{B}-\tilde{\chi}_1)(\tilde{B}-\tilde{\chi}_2)(\tilde{B}-\tilde{\chi}_3) = 0 \ , \tag{3.8.12}$$

$$(\tilde{D}-\tilde{\chi}_1)(\tilde{D}-\tilde{\chi}_2)(\tilde{D}-\tilde{\chi}_3) = 0 \ , \quad (\tilde{D}'-\tilde{\chi}_1)(\tilde{D}'-\tilde{\chi}_2)(\tilde{D}'-\tilde{\chi}_3) = 0 \ . \tag{3.8.13}$$

- $k = \frac{3}{2}$

$$(q^2+q^{-2})\tilde{B}\tilde{A}\tilde{B} = \tilde{A}\tilde{B}^2 + \tilde{B}^2\tilde{A} - [2]_q\tilde{B}^2 - [2]_q\{\tilde{A},\tilde{B}\} + 2([2]_q + q^4 + q^{-4})\tilde{\chi}_{\frac{3}{2}}\tilde{B} \ , \tag{3.8.14}$$

$$(q^2+q^{-2})\tilde{A}\tilde{B}\tilde{A} = \tilde{B}\tilde{A}^2 + \tilde{A}^2\tilde{B} - [2]_q\tilde{A}^2 - [2]_q\{\tilde{A},\tilde{B}\} + 2([2]_q + q^4 + q^{-4})\tilde{\chi}_{\frac{3}{2}}\tilde{A} \ , \tag{3.8.15}$$

$$\tilde{A}(\tilde{A}-\tilde{\chi}_1)(\tilde{A}-\tilde{\chi}_2)(\tilde{A}-\tilde{\chi}_3) = 0 \ , \quad \tilde{B}(\tilde{B}-\tilde{\chi}_1)(\tilde{B}-\tilde{\chi}_2)(\tilde{B}-\tilde{\chi}_3) = 0 \ , \tag{3.8.16}$$

$$\tilde{D}(\tilde{D}-\tilde{\chi}_1)(\tilde{D}-\tilde{\chi}_2)(\tilde{D}-\tilde{\chi}_3) = 0 \ , \quad \tilde{D}'(\tilde{D}'-\tilde{\chi}_1)(\tilde{D}'-\tilde{\chi}_2)(\tilde{D}'-\tilde{\chi}_3) = 0 \ . \tag{3.8.17}$$

- $k = \frac{1}{2}$

$$(q^2+q^{-2})\tilde{B}\tilde{A}\tilde{B} = [3]_q([2]_q\{\tilde{A},\tilde{B}\} - 2[2]_q^2\tilde{A} + [2]_q^3) + [2]_q^2(q^4+q^{-4})\tilde{B} \ , \tag{3.8.18}$$

$$(q^2+q^{-2})\tilde{A}\tilde{B}\tilde{A} = [3]_q([2]_q\{\tilde{A},\tilde{B}\} - 2[2]_q^2\tilde{B} + [2]_q^3) + [2]_q^2(q^4+q^{-4})\tilde{A} \ , \tag{3.8.19}$$

$$(\tilde{A}-\tilde{\chi}_1)(\tilde{A}-\tilde{\chi}_2) = 0 \ , \quad (\tilde{B}-\tilde{\chi}_1)(\tilde{B}-\tilde{\chi}_2) = 0 \ , \tag{3.8.20}$$

$$(\tilde{D}-\tilde{\chi}_1)(\tilde{D}-\tilde{\chi}_2) = 0 \ , \quad (\tilde{D}'-\tilde{\chi}_1)(\tilde{D}'-\tilde{\chi}_2) = 0 \ . \tag{3.8.21}$$

For each value of k, the elements $\tilde{D}$ and $\tilde{D}'$ are given by

$$\tilde{D} = \frac{(q^4+q^{-4})}{[2]_q}(\tilde{\chi}_{\frac{3}{2}} + \tilde{\chi}_k) + 2\tilde{\chi}_{\frac{3}{2}} - \frac{(q-q^{-1})}{[2]_q}[\tilde{A},\tilde{B}]_q - \tilde{A} - \tilde{B} \ , \tag{3.8.22}$$

$$\tilde{D}' = \frac{(q^4+q^{-4})}{[2]_q}(\tilde{\chi}_{\frac{3}{2}} + \tilde{\chi}_k) + 2\tilde{\chi}_{\frac{3}{2}} - \frac{(q-q^{-1})}{[2]_q}[\tilde{B},\tilde{A}]_q - \tilde{A} - \tilde{B} \ . \tag{3.8.23}$$

Theorem 3.8.1. *Conjecture 4.1 is verified for $j_1 = j_2 = j_3 = \frac{3}{2}$.*

Proof. Since we already know that the map $\overline{\phi}$ is surjective, we only need to prove (3.8.3). For the case $k = \frac{9}{2}$, all the elements are constants and $\dim(\mathcal{K}_{\frac{9}{2}}\overline{AW}\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right)\mathcal{K}_{\frac{9}{2}}) = 1$. For the case $k=\frac{7}{2}$ (resp. $k=\frac{1}{2}$), one uses (3.8.22) in the first relation of (3.8.9) (resp. (3.8.21)) to find (after some simplifications using the defining relations) $\tilde{B}\tilde{A}=-\tilde{A}\tilde{B}+x_1(\tilde{A}+\tilde{B})+x_2$, for some constants x_1 and x_2 that can be computed. Therefore, in both cases we see that a linearly generating set is given by $\{1,\tilde{A},\tilde{B},\tilde{A}\tilde{B}\}$. For the case $k=\frac{5}{2}$, we used formal mathematical software to show that $\{1,\tilde{A},\tilde{B},\tilde{A}^2,\tilde{A}\tilde{B},\tilde{B}\tilde{A},\tilde{B}^2,\tilde{A}\tilde{B}^2,\tilde{B}\tilde{A}^2\}$ is a generating set. Similarly, for the case $k=\frac{3}{2}$, a generating set is given by $\{1,\tilde{A},\tilde{B},\tilde{A}^2,\tilde{A}\tilde{B},\tilde{B}\tilde{A},\tilde{B}^2,\tilde{A}^3,\tilde{A}\tilde{B}\tilde{A},\tilde{A}\tilde{B}^2,\tilde{B}\tilde{A}^2,\tilde{B}\tilde{A}\tilde{B},\tilde{B}^3,\tilde{A}^3\tilde{B},\tilde{A}^2\tilde{B}^2,\tilde{A}\tilde{B}^3\}$. From these results and the degeneracies d_k given at the beginning of the section, we see that (3.8.3) holds, which concludes the proof. □

Let us notice that the defining relation (3.4.17) of $\overline{AW}(j_1,j_2,j_3)$ which involves the Casimir element Ω of $AW(3)$ has not been called upon in the previous proof. It is straightforward to verify that this relation is satisfied in each of the algebras $\mathcal{K}_k\overline{AW}\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right)\mathcal{K}_k$ by using the relations given above.

3.9. Quotient $\overline{AW}(j,\frac{1}{2},\frac{1}{2})$ and one-boundary Temperley–Lieb algebra

In this section, we consider the case $j_1=j$, for $j=1,\frac{3}{2},2,...$, and $j_2=j_3=\frac{1}{2}$. We show that the algebra $\overline{AW}(j,\frac{1}{2},\frac{1}{2})$ is isomorphic to the centralizer $\mathcal{C}_{j,\frac{1}{2},\frac{1}{2}}$. We also find an explicit isomorphism between this quotient of the Askey–Wilson algebra and a specialization of the one-boundary Temperley–Lieb algebra.

3.9.1. $\overline{AW}(j,\frac{1}{2},\frac{1}{2})$ algebra

From the tensor decomposition rules, we find the sets

$$\mathcal{J}\left(j,\frac{1}{2}\right)=\left\{j-\frac{1}{2},j+\frac{1}{2}\right\}\ ,\quad \mathcal{J}\left(\frac{1}{2},\frac{1}{2}\right)=\{0,1\}\ ,\quad \mathcal{J}\left(j,\frac{1}{2},\frac{1}{2}\right)=\{j-1,j,j+1\}\ ,$$
$$\mathcal{M}\left(j,\frac{1}{2},\frac{1}{2}\right)=\{\chi_{j+1}-\chi_{j+\frac{1}{2}},\chi_j-\chi_{j+\frac{1}{2}},\chi_{j+1}-\chi_{j-\frac{1}{2}},\chi_{j-1}-\chi_{j-\frac{1}{2}}\}\equiv\{m_1,m_2,m_3,m_4\}\ ,$$
$$\mathcal{M}\left(\frac{1}{2},\frac{1}{2},j\right)=\{\chi_{j+1}-\chi_1,\chi_j-\chi_1,\chi_j-\chi_0,\chi_{j-1}-\chi_1\}\ .$$

The degeneracies are $d_{j-1}=d_{j+1}=1$ and $d_j=2$, and the dimension of the centralizer is $\dim(\mathcal{C}_{j,\frac{1}{2},\frac{1}{2}})=6$. The central elements α_i take the values $\alpha_1=\chi_j$ and $\alpha_2=\alpha_3=\chi_{\frac{1}{2}}$ in the quotient $\overline{AW}(j,\frac{1}{2},\frac{1}{2})$. As in the previous sections, we perform the transformation $X=(q-q^{-1})^2\tilde{X}+q+q^{-1}$ on the generators $X=A,B,D,D',K$ (see Remark 3.2.1). The defining relations of $\overline{AW}(j,\frac{1}{2},\frac{1}{2})$ can be written as follows

$$\tilde{B}\tilde{A}\tilde{B} = \left(\tilde{\chi}_j - [2]_q[1/2]_q^2\right)\tilde{B} + \tilde{K}\tilde{B} \ , \tag{3.9.1}$$

$$\tilde{A}\tilde{B}\tilde{A} = a_1\{\tilde{A},\tilde{B}\} - a_2\tilde{B} + a_3\tilde{A} + \tilde{K}(\tilde{A}-a_1) + a_4 \ , \tag{3.9.2}$$

$$(\tilde{A}-\tilde{\chi}_{j-\frac{1}{2}})(\tilde{A}-\tilde{\chi}_{j+\frac{1}{2}}) = 0 \ , \quad \tilde{B}(\tilde{B}-[2]_q) = 0 \ , \quad (\tilde{K}-\tilde{\chi}_{j-1})(\tilde{K}-\tilde{\chi}_j)(\tilde{K}-\tilde{\chi}_{j+1}) = 0 \ , \tag{3.9.3}$$

$$(\tilde{D}-\tilde{\chi}_{j-\frac{1}{2}})(\tilde{D}-\tilde{\chi}_{j+\frac{1}{2}}) = 0 \ , \quad (\tilde{D}'-\tilde{\chi}_{j-\frac{1}{2}})(\tilde{D}'-\chi_{j+\frac{1}{2}}) = 0 \ , \tag{3.9.4}$$

$$(\tilde{K}-\tilde{B}-\tilde{\chi}_{j+1}+\tilde{\chi}_1)(\tilde{K}-\tilde{B}-\tilde{\chi}_j+\tilde{\chi}_1)(\tilde{K}-\tilde{B}-\tilde{\chi}_j)(\tilde{K}-\tilde{B}-\tilde{\chi}_{j-1}+\tilde{\chi}_1) = 0 \ , \tag{3.9.5}$$

$$\prod_{i=1}^{4}(\tilde{K}-\tilde{A}-m_i) = 0 \ , \quad \prod_{i=1}^{4}(\tilde{K}-\tilde{D}-m_i) = 0 \ , \quad \prod_{i=1}^{4}(\tilde{K}-\tilde{D}'-m_i) = 0 \ , \tag{3.9.6}$$

$$\Omega = \chi_j^2 + 2\chi_{\frac{1}{2}}^2 + K^2 + \chi_j\chi_{\frac{1}{2}}^2 K - \chi_0^2 \ , \tag{3.9.7}$$

where

$$\tilde{D} = \frac{(q^2+q^{-2})}{[2]_q}(\tilde{K}+\tilde{\chi}_j) + 2\tilde{\chi}_{\frac{1}{2}} - \frac{(q-q^{-1})}{[2]_q}[\tilde{A},\tilde{B}]_q - \tilde{A} - \tilde{B} \ , \tag{3.9.8}$$

$$\tilde{D}' = \frac{(q^2+q^{-2})}{[2]_q}(\tilde{K}+\tilde{\chi}_j) + 2\tilde{\chi}_{\frac{1}{2}} - \frac{(q-q^{-1})}{[2]_q}[\tilde{B},\tilde{A}]_q - \tilde{A} - \tilde{B} \ , \tag{3.9.9}$$

$$\Omega = q(A+D)\chi_{\frac{1}{2}}(\chi_j+K) + q^{-1}B(\chi_{\frac{1}{2}}^2+\chi_jK) - q^2A^2 - q^{-2}B^2 - q^2D^2 - qABD \ , \tag{3.9.10}$$

and where we have used the following constants

$$a_1 = \frac{[2]_q[j-\frac{1}{2}]_q[j+\frac{3}{2}]_q}{q^2+q^{-2}} \ , \quad a_2 = 2\frac{\tilde{\chi}_{j-\frac{1}{2}}\tilde{\chi}_{j+\frac{1}{2}}}{q^2+q^{-2}} \ ,$$

$$a_3 = \frac{[2]_q}{q^2+q^{-2}}(2\tilde{\chi}_{\frac{1}{2}} - [2]_q[j+\tfrac{1}{2}]_q^2) + \tilde{\chi}_j \ , \quad a_4 = a_1([j+\tfrac{1}{2}]_q^2 + \tilde{\chi}_{\frac{1}{2}}) \ ,$$

$$m_1 = \tilde{\chi}_{j+1} - \tilde{\chi}_{j+\frac{1}{2}} \ , \quad m_2 = \tilde{\chi}_j - \tilde{\chi}_{j+\frac{1}{2}} \ , \quad m_3 = \tilde{\chi}_{j+1} - \tilde{\chi}_{j-\frac{1}{2}} \ , \quad m_4 = \tilde{\chi}_{j-1} - \tilde{\chi}_{j-\frac{1}{2}} \ .$$

Proposition 3.9.1. *The quotient $\overline{AW}(j,\frac{1}{2},\frac{1}{2})$ can be presented with the following relations*

$$\tilde{A}^2 = (\tilde{\chi}_{j-\frac{1}{2}} + \tilde{\chi}_{j+\frac{1}{2}})\tilde{A} - \tilde{\chi}_{j-\frac{1}{2}}\tilde{\chi}_{j+\frac{1}{2}} \ , \quad \tilde{B}^2 = [2]_q\tilde{B} \ , \tag{3.9.11}$$

$$\tilde{B}\tilde{A}\tilde{B} = [2]_q\{\tilde{A},\tilde{B}\} - [2]_q^2\tilde{A} - ([j+\tfrac{3}{2}]_q^2 + [j-\tfrac{1}{2}]_q^2 - 1)(\tilde{B}-[2]_q) \ . \tag{3.9.12}$$

Proof. We first show that the relations (3.9.1)–(3.9.6) imply the relations of the proposition. The two equations in (3.9.11) follow directly from (3.9.3). We also deduce from the first relation of (3.9.3) that

$$\left(\tilde{A}-a_1\right)\left(\tilde{A}-\tilde{\chi}_{j-\frac{1}{2}}-\tilde{\chi}_{j+\frac{1}{2}}+a_1\right) = \frac{[2j-1]_q[2j+3]_q}{q^2+q^{-2}} \ . \tag{3.9.13}$$

Since the R.H.S. of the previous relation does not vanish for $j > \frac{1}{2}$, it can be used in (3.9.2) to find

$$\tilde{K} = \{\tilde{A},\tilde{B}\} - (\tilde{\chi}_{j-\frac{1}{2}} + \tilde{\chi}_{j+\frac{1}{2}})\tilde{B} - [2]_q\tilde{A} + (1 + [2]_q)([2j + \tfrac{3}{2}]_q[\tfrac{1}{2}]_q + \tilde{\chi}_{j-\frac{1}{2}}) \ . \tag{3.9.14}$$

Using (3.9.14) in (3.9.8) and (3.9.9), and then substituting in (3.9.4), one obtains expressions for $\tilde{A}\tilde{B}\tilde{A}\tilde{B}$ and $\tilde{B}\tilde{A}\tilde{B}\tilde{A}$ in terms of the elements $1,\tilde{A},\tilde{B},\tilde{A}\tilde{B},\tilde{B}\tilde{A},\tilde{A}\tilde{B}\tilde{A}$ and $\tilde{B}\tilde{A}\tilde{B}$. By using (3.9.14) and the expressions for $\tilde{A}\tilde{B}\tilde{A}\tilde{B}$ and $\tilde{B}\tilde{A}\tilde{B}\tilde{A}$ in the third relation of (3.9.3), one gets (3.9.12).

Finally, we want to show that (3.9.11) and (3.9.12) imply the defining relations of $\overline{AW}(j,\frac{1}{2},\frac{1}{2})$. To that end, we suppose that the relations of the proposition are true and we define the element $\tilde{K}$ as in (3.9.14). It is then straightforward to verify that $\tilde{K}$ is central and that (3.9.1)–(3.9.7) hold. ☐

Theorem 3.9.1. *Conjecture 4.1 is verified for $j_1 = j$ and $j_2 = j_3 = \frac{3}{2}$, where $j = 1,\frac{3}{2},2,...$*

Proof. We already know that the map $\overline{\phi}$ is surjective. From the previous proposition, we conclude that $\{1,\tilde{A},\tilde{B},\tilde{A}\tilde{B},\tilde{B}\tilde{A},\tilde{A}\tilde{B}\tilde{A}\}$ is a generating set for $\overline{AW}(j,\frac{1}{2},\frac{1}{2})$. Therefore, $\dim\left(\overline{AW}(j,\frac{1}{2},\frac{1}{2})\right) \leq \dim\left(\mathcal{C}_{j,\frac{1}{2},\frac{1}{2}}\right) = 6$, which shows the injectivity of $\overline{\phi}$. ☐

3.9.2. Connection with the one-boundary Temperley–Lieb algebra

On the basis of the findings for the limit $q \to 1$ [4], one might expect that the centralizer in the case of one spin-j and two spin-$\frac{1}{2}$ will be isomorphic to the one-boundary Temperley–Lieb algebra. We can indeed confirm that this algebra is recovered as a quotient of $AW(3)$.

Definition 3.9.1. *[17–19] The one-boundary Temperley–Lieb algebra $1bTL_2(q,\omega)$ is generated by σ_0 and σ_1 with the following defining relations*

$$\sigma_0^2 = \frac{[\omega]_q}{[\omega-1]_q}\sigma_0 \ , \quad \sigma_1^2 = (q+q^{-1})\sigma_1 \ , \quad \sigma_1\sigma_0\sigma_1 = \sigma_1 \ . \tag{3.9.15}$$

Theorem 3.9.2. *The quotiented Askey–Wilson algebra $\overline{AW}(j,\frac{1}{2},\frac{1}{2})$, for $j = 1,\frac{3}{2},2...$, is isomorphic to the one-boundary Temperley–Lieb algebra $1bTL_2(q,2j+1)$. This isomorphism is given explicitly by*

$$\overline{AW}\left(j,\frac{1}{2},\frac{1}{2}\right) \to 1bTL_2(q,2j+1)$$

$$\tilde{A} \mapsto \tilde{\chi}_{j+\frac{1}{2}} - [2j]_q\sigma_0 \ , \tag{3.9.16}$$

$$\tilde{B} \mapsto [2]_q - \sigma_1 \ . \tag{3.9.17}$$

Proof. It is easy to see that the map φ is bijective. The homomorphism can be directly verified from the relations of the proposition 3.9.1. ☐

3.10. Conclusion and perspectives

Summing up, we have offered a conjecture according to which a quotient of the Askey–Wilson algebra is isomorphic to the centralizer of the image of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in the tensor product of any three irreducible representations. It has been proved in several cases, and we thus obtained the Temperley–Lieb, Birman–Murakami–Wenzl and one-boundary Temperley–Lieb algebras as quotients of the Askey–Wilson algebra. In the limit $q \to 1$, the results of the paper [4] are recovered. We have provided further evidence in support of the conjecture by studying the finite irreducible representations of the quotient of the Askey–Wilson algebra, more particularly in the case of three identical spins.

Proving the conjecture in the case of three arbitrary spins j_1,j_2,j_3 would be an obvious continuation of the work presented here. If true, this conjecture would provide a presentation of the centralizer of $U_q(\mathfrak{sl}_2)$ in terms of generators and relations for any three irreducible representations.

We could first consider, more simply, the case of three identical spins $j_1 = j_2 = j_3 = s$. As for the Temperley–Lieb ($s = \frac{1}{2}$) and the Birman–Murakami–Wenzl ($s = 1$) algebras, we expect that the centralizer for any spin s will be linked to a quotient of the braid group algebra.

In [3], a diagrammatic description of the centralizers of $U_q(gl_n)$ has been proposed. It is based on the notion of fused Hecke algebras. Developing a connection between this diagrammatic approach and the Askey–Wilson algebra could prove fruitful.

Throughout the present paper, we assume q to be not a root of unity. This choice allows to decompose the tensor product of irreducible representations of $U_q(\mathfrak{sl}_2)$ into a direct sum of irreducible representations (see Subsection 3.3.1). As a consequence, the matrices C_i,C_{ij},C_{123} are diagonalizable and their minimal polynomials are those discussed in Subsection 3.3.2. It could be interesting to study the centralizer when q is a root of unity and to examine how the quotient of $AW(3)$ is affected.

Another generalization of the results presented here would be to consider the n-fold tensor product of irreducible representations of $U_q(\mathfrak{sl}_2)$ and to connect the centralizer to a higher rank Askey–Wilson algebra $AW(n)$. The approach using the R-matrix proposed in [2] should be helpful for this purpose. In fact, a simpler starting point could be to generalize either the conjecture given in [4] by studying the connection between the centralizers of $\mathfrak{sl}_2$ and a higher rank Racah algebra, or the one given in [1] by examining how centralizers of $\mathfrak{osp}(1|2)$ relate to the higher rank Bannai-Ito algebra $BI(n)$ (see [6]).

Yet another direction to generalize the results of this paper would be to study the centralizer of the diagonal embedding of $\mathfrak{g}$ or $U_q(\mathfrak{g})$ with $\mathfrak{g}$ a higher rank Lie algebra. A first step in this direction was made recently in [5] where the centralizer $Z_2(sl_3)$ of the diagonal embedding of sl_3 in the twofold tensor product of sl_3 has been identified. A proposition similar to Proposition 3.4.2 has also been proved in that case. A quotient of the algebra $Z_2(sl_3)$ that describes the centralizer for any representations of sl_3 has still to be investigated. We hope to report on some of this issues in the future.

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Conclusion

En résumé, ce mémoire a exploré l'idée d'une généralisation de la dualité de Schur–Weyl, avec pour objet d'étude principal le centralisateur de l'image de l'action diagonale de $U_q(\mathfrak{sl}_2)$ dans le produit tensoriel de trois représentations irréductibles quelconques. Les deux premiers articles ont révélé l'importance de la matrice R universelle dans la définition des Casimirs intermédiaires de l'algèbre quantique $U_q(\mathfrak{sl}_2)$ et de la superalgèbre de Lie $\mathfrak{osp}(1|2)$, qui ont toutes deux un coproduit non-trivial. Les relations de l'algèbre de Askey–Wilson dans le premier cas, et de l'algèbre de Bannai–Ito dans le deuxième, ont été prouvées à l'aide du formalisme de la matrice R. Dans le troisième article, un isomorphisme entre un quotient de l'algèbre de Askey–Wilson et le centralisateur de l'image de l'action diagonale de $U_q(\mathfrak{sl}_2)$ dans le produit tensoriel de trois représentations irréductibles a été posé en conjecture et prouvé pour plusieurs cas. Ceci a permis de retrouver les algèbres de Temperley–Lieb et Birman–Murakami–Wenzl comme quotients de l'algèbre de Askey–Wilson; ce lien n'était pas connu auparavant. Il a aussi été montré qu'une algèbre de Temperley–Lieb à une frontière est isomorphe au centralisateur dans le cas d'une représentation de spin $j = 1, \frac{3}{2}, 2...$ et de deux représentations de spin $\frac{1}{2}$. Dans la limite $q \to 1$, les résultats de [12] ont été retrouvés, comme il se doit pour une q-déformation. L'intérêt de la conjecture du troisième article, si elle vraie, est de présenter les centralisateurs de $U_q(\mathfrak{sl}_2)$ en termes de générateurs et relations, pour tout choix de représentations.

Il serait bien entendu plus satisfaisant de prouver la conjecture de façon générale. De nouvelles stratégies (comparativement à celles utilisées dans le troisième article) devraient probablement être envisagées à cet effet.

Il serait intéressant d'étudier en détails le centralisateur pour trois représentations irréductibles identiques de spin j quelconque ($j_1 = j_2 = j_3 = j$). On s'attend à ce que ce centralisateur soit isomorphe à un quotient de l'algèbre du groupe des tresses sur trois brins, comme c'est le cas pour les algèbres de Temperley–Lieb et Birman–Murakami–Wenzl de rang 3 (revoir la figure 0.2). L'un des intérêts à étudier ces centralisateurs en particulier est qu'ils pourraient se prêter à la procédure de Baxterisation et ainsi permettre de produire des solutions à l'équation de Yang–Baxter.

Un aspect qui n'a pas été considéré dans le travail présenté dans ce mémoire est l'étude du cas où le paramètre de déformation q est une racine de l'unité. Certains arguments qui ont été utilisés dans le troisième article du mémoire ne sont plus valides dans ce cas, car les représentations de $U_q(\mathfrak{sl}_2)$ diffèrent. En particulier, les polynômes minimaux des matrices représentant les Casimirs intermédiaires de $U_q(\mathfrak{sl}_2)^{\otimes 3}$ doivent possiblement être modifiés, ce qui pourrait avoir des conséquences sur la conjecture proposée. L'étude détaillée du cas où q est une racine de l'unité pourrait être d'intérêt par exemple pour les modèles de Potts en mécanique statistique, qui généralisent le modèle de Ising. En effet, l'algèbre de Temperley–Lieb admet des représentations dans les modèles de Potts à deux, trois et quatre états lorsque q prend les valeurs $e^{i\pi/4}$, $e^{i\pi/6}$ et 1, respectivement (voir par exemple [**40**] pour plus de détails).

Les résultats obtenus dans les deux premiers articles de ce mémoire ouvrent la porte à l'étude des centralisateurs dans un produit tensoriel de rang supérieur. En effet, le formalisme de la matrice R a permis de construire tous les Casimirs intermédiaires du centralisateur de $\mathfrak{osp}(1|2)$ dans $\mathfrak{osp}(1|2)^{\otimes n}$, et il a permis de montrer que ces éléments respectent l'algèbre de Bannai–Ito de rang supérieur $BI(n)$. Il serait intéressant d'effectuer un travail semblable pour les Casimirs intermédiaires du centralisateur de $U_q(\mathfrak{sl}_2)$ dans $U_q(\mathfrak{sl}_2)^{\otimes n}$ et de définir l'algèbre de Askey–Wilson de rang supérieur $AW(n)$ à l'aide des relations de q-commutation de ces Casimirs intermédiaires. Ceci constituerait un début à l'étude du centralisateur de $U_q(\mathfrak{sl}_2)$ dans le produit tensoriel de n représentations irréductibles quelconques, ce qui serait une généralisation du troisième article de ce mémoire. Une approche plus simple pourrait être de commencer par examiner la limite $q \to 1$ et de tenter de décrire le centralisateur de $\mathfrak{sl}_2$ dans le produit tensoriel de n représentations irréductibles quelconques à l'aide de l'algèbre de Racah de rang supérieur $R(n)$, ce qui généraliserait [**12**]. Similairement, le centralisateur de $\mathfrak{osp}(1|2)$ dans le produit tensoriel de n représentations irréductibles pourrait être étudié avec l'aide de l'algèbre de Bannai–Ito de rang supérieur $BI(n)$, ce qui généraliserait [**11**].

Ces algèbres de Bannai–Ito, Racah et Askey–Wilson de rang supérieur sont associées à des polynômes orthogonaux éponymes à plusieurs variables et apparaissent comme algèbres de symétrie de systèmes quantiques superintégrables de dimension supérieure [**13–16**, **24**], d'où leur intérêt pour la physique. L'étude des généralisations de la dualité de Schur–Weyl de $\mathfrak{sl}_2$, $U_q(\mathfrak{sl}_2)$ et $\mathfrak{osp}(1|2)$ pourrait permettre d'établir des liens précis entre ces algèbres de rang supérieur et des algèbres reliées au groupe des tresses sur n brins. Par exemple, il est connu de façon plus générale que l'algèbre de Temperley–Lieb $TL_n(q)$, qui est un quotient de l'algèbre de Hecke $H_n(q)$, est isomorphe au centralisateur de l'image de $U_q(\mathfrak{sl}_2)$ dans le produit tensoriel de n représentations irréductibles fondamentales (spin $\frac{1}{2}$) [**33**]. On peut alors se demander s'il est possible de retrouver plus généralement l'algèbre $TL_n(q)$ comme quotient de l'algèbre de Askey–Wilson de rang supérieur $AW(n)$, comme cela a été fait dans ce mémoire pour le cas $n = 3$. Ceci enlargirait le potentiel d'application aux systèmes physiques de plus grande taille ($n > 3$) qui font intervenir l'algèbre $TL_n(q)$, comme la chaîne de spins XXZ à n sites.

Les aspects discutés ci-dessus pourraient tous être le sujet de projets futurs faisant directement suite au travail présenté dans ce mémoire.

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