Université de Montréal

Axiomatic Approach to Cellular Algebras

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Mémoire présenté à la Faculté des études supérieures et postdoctorales

en vue de l'obtention du grade de

Maître ès sciences (M.Sc.)

en Mathématiques

January 6, 2020

Université de Montréal

Faculté des études supérieures et postdoctorales

Ce mémoire intitulé

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January 6, 2020

Résumé

Les algèbres cellulaires furent introduite par J.J. Graham et G.I. Lehrer en 1996. Elles forment une famille d'algèbres associatives de dimension finie définies en termes de « données cellulaires » satisfaisant certains axiomes. Ces données cellulaires, lorsqu'elles sont identifiées pour une certaine algèbre, permettent une construction explicite de tous ses modules simples, à isomorphisme près, et de leurs couvertures projectives. Dans ce mémoire, nous définissons ces algèbres cellulaires en introduisant progressivement chacun des éléments constitutifs d'une façon axiomatique.

Deux autres familles d'algèbres associatives sont discutées, à savoir les algèbres quasi-héréditaires et celles dont les modules forment une catégorie de plus haut poids. Ces familles furent introduites durant la même période de temps, au tournant des années quatre-vingt-dix. La relation entre ces deux familles ainsi que celle entre elles et les algèbres cellulaires sont prouvées.

Mots clés: algèbres cellulaires, catégorie de plus haut poids, algèbre quasi-héréditaire, matrice de Cartan, groupe de Grothendieck, algèbre associative de dimension finie, théorie des modules.

Abstract

Cellular algebras were introduced by J.J. Graham and G.I. Lehrer in 1996. They are a class of finite-dimensional associative algebras defined in terms of a "cellular datum" satisfying some axioms. This cellular datum, when made explicit for a given associative algebra, allows for the explicit construction of all its simple modules, up to isomorphism, and of their projective covers. In this work, we define these cellular algebras by introducing each building block of the cellular datum in a fairly axiomatic fashion.

Two other families of associative algebras are discussed, namely the quasi-hereditary algebras and those whose modules form a highest weight category. These families were introduced at about the same period. The relationships between these two, and between them and the cellular ones, are made explicit.

Key words: cellular algebra, highest weight category, quasi-hereditary algebra, Cartan matrix, Grothendieck group, finite-dimensional assosiative algebra, module theory.

Acknowledgement

First of all, I would like to show my appreciation to Yvan Saint-Aubin, my thesis supervisor. Yvan, you spent your time during weekends to read and find all the mistakes and errors in my dissertation. You have challenged and enriched my ideas. Also, we had our weekly meetings, and you spent your time, patience, and energy, helping me to find the solutions of my questions. You always behaved good-temperedly and patiently with me. Also as friend, you invited me several times to your house. Thank you Yvan for everything.

This work has been written during my stay at the Mathematics Department of the Université de Montréal. I would like to thank the Département de mathématiques et de statistique for providing excellent working conditions and for financial support. I gratefully acknowledge the help of Marlène Frigon (current chair of the Département) who extended my M.Sc. period thus allowing me to conclude my dissertation. Thank you Marlène. Also, I would like to give special thanks to Marie-Claude Turmel for her support and careful attention in my registration status, my boursary, and my administrational affairs.

I would like to recognize the help of Sabin Lessard (former chair of the Département), Lise Imbeault, and Anne-Marie Dupuis. You genereously welcomed me right upon my arrival in Summer 2017.

The thesis has also benefited from suggestions and comments made by Mme.Christiane Rousseau and M.Abraham Broer who have read through the dissertation. I would like to appreciate them. Finally, a warm word for my friends who have helped me during preparation of my dissertation and have always been a major source of support in discussion. I would like to thank Pooyan, Khashayar, Majid, Théo, and Alexis for being generous with their time, discussing math and sharing ideas with me. I also wish to thank Pooyan for spending his time to help me in typing this work with LaTeX. Thank you for sharing your time and your friendship.

Introduction

This thesis establishes the ties between three families of associative algebras that were introduced in the last quarter-century: the quasi-hereditary algebras [9], those whose category of modules is a highest weight category [7] and the cellular algebras [12]. I was inspired to understand these ties by the works [11] and [5].

In chapter 1, we give the algebraic preliminaries which are required in working with cellular algebras. The background concepts are those of finite-dimensional associative algebras, the theory of their representation and category theory. The references [13] and [25] cover most of these algebraic preliminaries.

In chapter 2, we give the definition of quasi-hereditary algebras and highest weight category. We also give the fundamental theorem which relates them. The notion of highest weight module originates from the representation theory of Lie algebras (see for example [14] and [4]), but the highest weight categories that characterize some associative algebras emerged more recently in the work of Cline, Parshall and Scott [7]. Reference [16] gives pure ideas and excellent motivations about the origin of highest weight category. The definition of quasi-hereditary algebras has a more complex history. Hereditary algebras were defined a long time ago and are now covered in basic courses on algebras and their modules. But quasi-hereditary algebras appeared first in the same work of Parshall *et al* [21], but also in [20] and [10].

We devote chapter 3 to cellular algebras. They were first defined by Graham and Lehrer [12] (see also [19]). We do it by introducing one after the other the building blocks constituting the "cellular datum". By doing it in this progressive fashion, we hope to reveal the role of each of these ingredients. Their introduction in [12] was triggered, in part, by the special bases for Hecke algebras observed by Kazhdan and Lusztig [18]. Their crucial feature is indeed the existence of a special basis that reveals their filtered structure. We will also show that, for a cellular algebra $\mathcal{A}$, the category mod-$\mathcal{A}$ is a highest weight category and, consequently, cellular algebras are quasi-hereditary algebras. Finally, we study in details the main tools such as the Grothendieck group, the decomposition and Cartan matrices of a cellular algebra.

Contents

Chapter 1

Preliminaries

In this preliminary chapter we summarize results with which the reader should be familiar, coupled with some explicit references to the textbook literature.

1.1. Relations

1.1.1. Equivalence Relations

Let X and Y be two non-empty sets. A **binary relation** or **relation**, from X to Y, $\mathcal{R}$ is an arbitrary non-empty subset $\mathcal{R}$ of $\mathrm{X} \times \mathrm{Y}$, the cartesian product of X and Y. When $\mathrm{X} = \mathrm{Y}$, we say $\mathcal{R}$ is a relation on X. We usually write $x\mathcal{R}y$ when $(x, y) \in \mathcal{R}$. Now, suppose $\mathcal{R}$ and $\mathcal{S}$ are two relations on X. We define $\mathcal{R} \bigcup \mathcal{S}$ to be **the relation generated by $\mathcal{R}$ and $\mathcal{S}$**,that is, $x\Big(\mathcal{R} \bigcup \mathcal{S}\Big)y$ if and only if $x\mathcal{R}y$ or $x\mathcal{S}y$, for $x, y \in \mathrm{X}$.
A relation $\mathcal{R}$ on X is called **reflexive** if $x\mathcal{R}x$ for all $x \in \mathrm{X}$. It is called **symmetric** if $x\mathcal{R}y$ implies $y\mathcal{R}x$. Also, it is called **antisymmetric** if $x\mathcal{R}y$ and $y\mathcal{R}x$ then $x = y$. Finally, it is called **transitive** if $x\mathcal{R}y$ and $y\mathcal{R}z$ implies $x\mathcal{R}z$.
A relation $\mathcal{R}$ on X is called an **equivalence relation** if $\mathcal{R}$ is reflexive, symmetric, and transitive. In this case, we define the **equivalence class** of x by

$$\mathrm{C}_x := \Big\{y \in \mathrm{X} : x\mathcal{R}y\Big\}$$

for every $x \in \mathrm{X}$. We call $\mathrm{X}/\mathcal{R} := \Big\{\mathrm{C}_x : x \in \mathrm{X}\Big\}$ the **quotient space of** X by $\mathcal{R}$.
Now, suppose X,Y, and Z are three non-empty sets, and suppose $\mathcal{R}$ is a relation from from X to Y and $\mathcal{S}$ is a relation from Y to Z. Then $\mathcal{R}$ and $\mathcal{S}$ give rise to a relation from X to Z,

denoted by $\mathcal{R} \circ \mathcal{S}$, and it is called the **composition** of $\mathcal{R}$ and $\mathcal{S}$, defined as follows:

$$\mathcal{R} \circ \mathcal{S} := \Big\{(x, z) \in X \times Z : \exists\, y \in Y \text{ for which } x\mathcal{R}y \text{ and } y\mathcal{S}z\Big\}.$$

When X = Y = Z, we write $\mathcal{R}^2 := \mathcal{R} \circ \mathcal{R}$, and we define inductively $\mathcal{R}^n := \mathcal{R}^{n-1} \circ \mathcal{R}$, for $n \geqslant 2, n \in \mathbb{N}$. For example, $x\mathcal{R}^m y$ if there are $x_2, x_3, \ldots, x_{m-1}, x_m \in X$ such that

$$x\mathcal{R}x_2\ ,\ x_2\mathcal{R}x_3\ ,\ \ldots\ ,\ x_{m-1}\mathcal{R}x_m\ ,\ x_m\mathcal{R}y.$$

Definition 1.1.1. Let X be a non-empty set, and $\mathcal{R}$ be a relation on X. We define $\mathrm{T}(\mathcal{R}) := \bigcup\limits_{n \geqslant 1} \mathcal{R}^n$ and call it the **transitive closure** of $\mathcal{R}$. It can be verified that $\mathrm{T}(\mathcal{R})$ is the smallest transitive relation on X that contains $\mathcal{R}$, smallest with respect to inclusion. It means that if $\mathcal{S}$ is a transitive relation on X and $\mathcal{R} \subset \mathcal{S}$, then $\mathrm{T}(\mathcal{R}) \subseteq \mathcal{S}$.

Remark 1.1.2. $\mathrm{T}(\mathcal{R})$ is a relation on X and for every $x, y \in$ X, we have:

$$\begin{aligned} x\mathrm{T}(\mathcal{R})y &\Longleftrightarrow (x, y) \in \mathrm{T}(\mathcal{R}) \\ &\Longleftrightarrow \text{there is } m \in \mathbb{N} \text{ and } x_2, x_3, \ldots, x_{m-1}, x_m \in X \text{ such that} \\ &\qquad x\mathcal{R}x_2\ ,\ x_2\mathcal{R}x_3\ ,\ \ldots\ ,\ x_{m-1}\mathcal{R}x_m\ ,\ x_m\mathcal{R}y. \end{aligned}$$

1.1.2. Ordered Sets

Suppose $\mathcal{R}$ is a relation on X. It is called a **pre-order** on X if it is reflexive and transitive. In this case, we say $(\mathrm{X}, \mathcal{R})$ is a **pre-ordered** set. Also, it is called a **partial order** on X if it is reflexive, antisymmetric, and transitive. We say $(\mathrm{X}, \mathcal{R})$ is a **partially-ordered set** when $\mathcal{R}$ is a partial order. Also, it is called a **totally-ordered set** if for every two elements $x, y \in$ X we have $x\mathcal{R}y$ or $y\mathcal{R}x$. A pre-ordering or a partial ordering is frequently denoted by $\preceq$.

Now, suppose $\mathcal{R}$ and $\mathcal{S}$ are two pre-orders on X. Let τ be the binary relation for $x, y \in$ X defined by:

$$x\tau y \text{ if and only if } x\mathcal{R}y \text{ or } x\mathcal{S}y.$$

The transitive closure of τ is a pre-order and is noted $\mathcal{R} \bigcup \mathcal{S}$. It is the pre-order generated by $\mathcal{R}$ and $\mathcal{S}$.

Example 1.1.3. Suppose $\mathcal{R}$ is a reflexive relation on a non-empty set X. It can be verified that $\mathcal{R}^n$ is a reflexive relation on X, for all $n \in \mathbb{N}$. So, $\mathrm{T}(\mathcal{R})$ is a reflexive relation on X and transitive, by its definition. Hence, $\mathrm{T}(\mathcal{R})$ is a pre-ordering on X.

Now suppose $(\mathrm{X}, \preceq')$ is a pre-ordered set. We want to build a partial order on X by $\preceq$. We define a new relation $\mathcal{R}$ on X by:

$$x\mathcal{R}y \Longleftrightarrow x \preceq' y \text{ and } y \preceq' x.$$

It can be verified that $\mathcal{R}$ is an equivalence relation on X, and for every $x \in \mathrm{X}$,

$$\begin{aligned} \mathrm{C}_x &= \big\{y \in \mathrm{X} : x\mathcal{R}y\big\} \\ &= \big\{y \in \mathrm{X} : x \preceq' y, y \preceq' x\big\} \end{aligned}$$

We now show that the quotient space $\mathrm{X}/\mathcal{R}$ is a partially-ordered set. Now consider the quotient space $\mathrm{X}/\mathcal{R} = \Big\{\mathrm{C}_x : x \in \mathrm{X}\Big\}$; we define $\preceq$ on $\mathrm{X}/\mathcal{R}$ by:

$$\mathrm{C}_x \preceq \mathrm{C}_y \Longleftrightarrow x \preceq' y$$

for $\mathrm{C}_x, \mathrm{C}_y \in \mathrm{X}/\mathcal{R}$. We show that $\preceq$ on $\mathrm{X}/\mathcal{R}$ is well-defined. Let $x' \in \mathrm{C}_x$, then $x' \preceq' x$. If $\mathrm{C}_x \preceq \mathrm{C}_y$ then $x' \preceq' x \preceq' y$ and by transitivity $x' \preceq' y$. Moreover, if $x' \preceq' y$, then $x \preceq' x' \preceq' y$ and $x \preceq' y$ and thus $\mathrm{C}_x \preceq \mathrm{C}_y$. Thus $\preceq$ is well-defined. It is obvious that $\preceq$ is reflexive and transitive, since $\preceq'$ is. Now, suppose $\mathrm{C}_x \preceq \mathrm{C}_y$ and $\mathrm{C}_y \preceq \mathrm{C}_x$, so we have $x \preceq' y$ and $y \preceq' x$. So, $x\mathcal{R}y$ which automatically implies $\mathrm{C}_x = \mathrm{C}_y$. Hence $\preceq$ is a partial order. We proved the following crucial theorem:

Theorem 1.1.4. [1] *Suppose* $(\mathrm{X}, \preceq')$ *is a pre-ordered set. We define the new relation* $\mathcal{R}$ *on* X *by:*

$$x\mathcal{R}y \Longleftrightarrow x \preceq' y \textit{ and } y \preceq' x$$

for every $x, y \in \mathrm{X}$. *Then* $\mathcal{R}$ *is an equivalence relation on* X .
On the quotient space $\mathrm{X}/\mathcal{R}$, *define* $\preceq$ *as*

$$\mathrm{C}_x \preceq \mathrm{C}_y \Longleftrightarrow x \preceq' y$$

for any $\mathrm{C}_x, \mathrm{C}_y \in \mathrm{X}/\mathcal{R}$. *Then* $(\mathrm{X}/\mathcal{R}, \preceq)$ *is a partially-ordered set.*

Remark 1.1.5. Let $(\mathrm{X}, \preceq)$ be a partially-ordered finite set of cardinal m. Since it is finite, We can form a total order $\preceq''$ inductively which is compatible with the partial order $\preceq$. In order to do it, we know that X has at least one maximal element. We call it y_m; now consider the resulting partially-ordered set $\mathrm{X} \setminus \{y_m\}$. It has at least one maximal element y_{m-1}. We

[1] The theorem plays a crucial role in the structure of cells in cellular algebra, see 1.2.70

define the order $\preceq''$ by: $y_{m-1} \preceq'' y_m$. We remove y_{m-1} from the set $\mathrm{X} \setminus \{y_m\}$. Now consider the set

$$(\mathrm{X} \setminus \{y_m\}) \setminus \{y_{m-1}\} = \mathrm{X} \setminus \{y_m, y_{m-1}\}.$$

It has at least one maximal element, call it y_{m-2}. Now we define $y_{m-2} \preceq'' y_{m-1}$. By continuing inductively the process we will arrive at the following chain

$$y_1 \preceq'' y_2 \preceq'' \ldots \preceq'' y_{m-1} \preceq'' y_m.$$

So the new total order $\preceq''$ on X is compatible with the natural order on $\{1, \ldots, m\}$, that is

$$i \leqslant j \Longrightarrow y_i \preceq'' y_j.$$

Since $1 \leqslant \cdots \leqslant m$, so we have $y_1 \preceq'' \cdots \preceq'' y_m$. Hence, every finite set of cardinal m can be totally ordered with its order compatible with the natural order on $\{1, \ldots, m\}$. Such a total order $\preceq''$ is constructed from the partial order $\preceq$ by giving a filtration

$$0 = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_m = \Lambda$$

where the subset Λ_i is obtained from Λ_{i+1} by deleting one of its maximal elements. So $\Lambda_i \setminus \Lambda_{i-1}$ contains a single element y_i.

1.2. Algebras

1.2.1. Basics on $\mathbb{K}$-algebras

Definition 1.2.1. [$\mathbb{K}$-algebras]
Let $\mathbb{K}$ be a field. A $\mathbb{K}$-**algebra** is a $\mathbb{K}$-vector space $(\mathcal{A}, +_{\mathcal{A}})$ along with a bilinear map $\cdot_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, which is usually called multiplicative mapping or law. We denote it by $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$, when $\mathcal{A}$ is a $\mathbb{K}$-algebra. We can restrict algebraic operations of $\mathcal{A}$ to any non-empty subset of it. A non-empty subset $\mathcal{C} \subseteq \mathcal{A}$ is called a $\mathbb{K}$-**subalgebra** of $\mathcal{A}$ if $\mathcal{C}$ is a $\mathbb{K}$-algebra under the restriction of $+_{\mathcal{A}}$ and $\cdot_{\mathcal{A}}$ on itself. We say that $\mathbb{K}$-algebra $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ is **associative** if the multiplication law is associative. In this text, all $\mathbb{K}$-algebras are associative and finite-dimensional unless the contrary is stated.

Definition 1.2.2. A $\mathbb{K}$-algebra $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ is called **unital algebra** if it has an element which is usually denoted by $1_{\mathcal{A}}$ and called the identity, such that for all $a \in \mathcal{A}$:

$$a \cdot_{\mathcal{A}} 1_{\mathcal{A}} = 1_{\mathcal{A}} \cdot_{\mathcal{A}} a = a.$$

Definition 1.2.3. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ and $(\mathcal{B}, +_{\mathcal{B}}, \cdot_{\mathcal{B}})$ be two $\mathbb{K}$-algebras. A map $f : \mathcal{A} \to \mathcal{B}$ between two $\mathbb{K}$-algebras is called an **algebra homomorphism** if it has the following properties:

(1) f is a $\mathbb{K}$-linear mapping between two $\mathbb{K}$-vector spaces $(\mathcal{A}, +_{\mathcal{A}})$ and $(\mathcal{B}, +_{\mathcal{B}})$.

(2) $f(a_1 \cdot_{\mathcal{A}} a_2) = f(a_1) \cdot_{\mathcal{B}} f(a_2)$, for all $a_1, a_2 \in \mathcal{A}$.

Definition 1.2.4. [Ideals] Let $\mathcal{A}$ be a $\mathbb{K}$-algebra and $I \subseteq \mathcal{A}$ be a $\mathbb{K}$-subalgebra.We call I a **left ideal** if $\mathcal{A} \cdot_{\mathcal{A}} I \subseteq I$, a **right ideal** if $I \cdot_{\mathcal{A}} \mathcal{A} \subseteq I$, and a **two-sided ideal** if it is both a left and right ideal. A proper (not equal to $\mathcal{A}$) left (right, two-sided) ideal $\mathcal{M}$ in $\mathcal{A}$ is called **left maximal** (**right maximal**, **two-sided maximal**) if it is not contained strictly in any other left (right, two-sided) ideal other than $\mathcal{A}$ itself.

Definition 1.2.5. [Orthogonal and Primitive Idempotent]

Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be a unital $\mathbb{K}$-algebra. An element $e \in \mathcal{A}$ is called an **idempotent** if $e^2 = e$. Two idempotents e and f are called **orthogonal** if $e \cdot_{\mathcal{A}} f = f \cdot_{\mathcal{A}} e = 0$. An equality $1_{\mathcal{A}} = e_1 +_{\mathcal{A}} e_2 +_{\mathcal{A}} \cdots +_{\mathcal{A}} e_n$, where $e_1, e_2, \cdots, e_n$ are pairwise orthogonal idempotents, will be called a **decomposition of the identity** of the $\mathbb{K}$-algebra $\mathcal{A}$. An idempotent $e \in \mathcal{A}$ is said to be **primitive** if e has no decomposition into a sum of nonzero orthogonal idempotents $e = e_1 + e_2$ in $\mathcal{A}$.

1.2.2. Radical of an Algebra

Definition 1.2.6. Let $\mathcal{A}$ be a $\mathbb{K}$-algebra. The **Jacobson radical** or **radical** of $\mathcal{A}$ denoted by $\mathfrak{J}(\mathcal{A})$ or $Rad(\mathcal{A})$, is the intersection of all maximal left ideals of $\mathcal{A}$.

Proposition 1.2.7. *[Properties of Jacobson radical] Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be a unital .$\mathbb{K}$-algebra and $Rad(\mathcal{A})$ be its radical. It has the following properties:*

(1) $Rad(\mathcal{A})$ is the intersection of all maximal right ideals of $\mathcal{A}$.

(2) $Rad(\mathcal{A})$ is a two-sided ideal of $\mathcal{A}$ and $Rad(\mathcal{A}/Rad(\mathcal{A})) = 0$.

(3) $Rad(\mathcal{A}) = \big\{r \in \mathcal{A} : 1 - r \cdot_{\mathcal{A}} a$ is right invertible for all $a \in \mathcal{A}\big\}$.

(4) $Rad(\mathcal{A}) = \big\{t \in \mathcal{A} : 1 - a \cdot_{\mathcal{A}} t$ is left invertible for all $a \in \mathcal{A}\big\}$.

(5) $Rad(\mathcal{A}) = \big\{u \in \mathcal{A} : 1 - a \cdot_{\mathcal{A}} u \cdot_{\mathcal{A}} b$ is invertible for all $a, b \in \mathcal{A}\big\}$.

(6) $Rad(e \cdot_{\mathcal{A}} \mathcal{A} \cdot_{\mathcal{A}} e) = e \cdot_{\mathcal{A}} Rad(\mathcal{A}) \cdot_{\mathcal{A}} e$, for any idempotent $e \in \mathcal{A}$.

(7) The $\mathbb{K}$-algebra $e \cdot_{\mathcal{A}} \mathcal{A} \cdot_{\mathcal{A}} e$ has no nontrivial idempotents, for any idempotent $e \in \mathcal{A}$.

PROOF. (1) See proposition 3.4.7 in [13].

(2) See corollary 1.4 in [1].

(3) See proposition 3.4.5 in [13].

(4) See corollary 1.4 in [1].

(5) See proposition 3.4.6 in [13].

(6) See proposition 3.4.8 in [13].

(7) See proposition 2.4.4 in [13]. ■

1.2.3. Modules over algebras

Definition 1.2.8. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be a unital $\mathbb{K}$-algebra. A **left module** over $\mathcal{A}$ or simply a **left $\mathcal{A}$-module** is an additive abelian group M together with a map $\cdot_M : \mathcal{A} \times M \to M$ that has the following properties:

(1) $a \cdot_M (m_1 +_M m_2) = (a \cdot_M m_1) +_M (a \cdot_M m_2)$

(2) $(a_1 +_{\mathcal{A}} a_2) \cdot_M m = (a_1 \cdot_M m) +_M (a_2 \cdot_M m)$

(3) $(a_1 {\cdot_{\mathcal{A}}} a_2) \cdot_M m = a_1 \cdot_M (a_2 \cdot_M m)$

(4) $1_A \cdot_M m = m$

for any $m, m_1, m_2 \in M$ and any $a, a_1, a_2 \in \mathcal{A}$.

In a similar way, one can define the notion of a right $\mathcal{A}$-module. We shall sometimes write $M_{\mathcal{A}} := M$ to emphasize the right action of $\mathcal{A}$ on M. M is called **two-sided $\mathcal{A}$-module** if it is both left and right $\mathcal{A}$-module.

Definition 1.2.9. Let M and N be two left $\mathcal{A}$-modules. A *linear mapping* ($\mathcal{A}$-linear mapping or $\mathcal{A}$-homomorphism) of M into N is any mapping $T : M \to N$ such that:

(1) $T(x +_M y) = T(x) +_N T(y)$, for all $x \in M$, $y \in M$.

(2) $T(a \cdot_M x) = a \cdot_N T(x)$, for all $x \in M$, $a \in \mathcal{A}$.

The set of all such $\mathcal{A}$-homomorphisms f is denoted by $Hom_{\mathcal{A}}(M, N)$. It has a natural $\mathbb{K}$-module structure.

Definition 1.2.10. Let M be a left $\mathcal{A}$-module and N be any non-empty subset of M. We say N is a left $\mathcal{A}$-submodule of M if:

(1) $(N, +|_N)$ is an Abelian group.

(2) The restriction of $\cdot_M$ to N takes its values in N and satisfies the properties 1,2,3, and 4 of 1.2.8.

Definition 1.2.11. Let M and N be two left $\mathcal{A}$-modules and $T : M \to N$ be an $\mathcal{A}$-linear mapping. We define

$$Ker(T) := \{x \in M : T(x) = 0\}$$
$$Im(T) := \{T(x) : x \in M\}$$
$$Coker(T) := N/Im(T)$$

It can be verified that $Ker(T)$ and $Im(T)$ are two $\mathcal{A}$ - submodules of M and N, respectively.

Theorem 1.2.12. *Let M and N be two left $\mathcal{A}$-modules and $T : M \to N$ be an $\mathcal{A}$-linear mapping. We have:*

(i) T is injective (monomorphism) $\Longleftrightarrow Ker(T) = \{0_M\}$

(ii) T is surjective (epimorphism) $\Longleftrightarrow Im(T) = N$ or $Coker(T) = 0$.

(iii) T is bijective (isomorphism) $\Longleftrightarrow$ T is injective and surjective.

When two $\mathcal{A}$-modules M and N are isomorphic, we write $M \underset{\mathcal{A}}{\simeq} N$. In the case of vector spaces, $\mathcal{A} = \mathbb{K}$, we write $M \underset{\mathbb{K}}{\simeq} N$. For every idempotent $e \in \mathcal{A}$, we define $e\mathcal{A} = \big\{ea : a \in \mathcal{A}\big\}$. The operations $ex + ey = e(x + y)$, $(ex)a = e(x.a)$ provide a right $\mathcal{A}$-module structure on $e\mathcal{A}$.

Theorem 1.2.13. *Let e and f be two idempotents of a unital $\mathbb{K}$-algebra $\mathcal{A}$. Then*

$Hom_{\mathcal{A}}(e\mathcal{A}, f\mathcal{A}) \underset{\mathbb{K}}{\simeq} f\mathcal{A}e$. Moreover, $End_{\mathcal{A}}(e\mathcal{A}) \simeq e\mathcal{A}e$ and $\mathcal{A} \simeq End_{\mathcal{A}}(\mathcal{A})$, as $\mathbb{K}$-algebra.

Proof. See page 31 of [13] ■

Definition 1.2.14. Suppose M, N, and P are three $\mathcal{A}$-modules (all left or right modules), and $T : M \to N$ and $S : N \to P$ are two $\mathcal{A}$-linear mappings. We say the ordered pair (T, S) is an **exact sequence** if we have: $Ker(S) = Im(T)$. We then say that

$$M \xrightarrow{T} N \xrightarrow{S} P$$

is exact or exact at N.

Proposition 1.2.15. *Let M, N,and P be three $\mathcal{A}$-modules, and $T : M \to N$ be an A-linear mapping.*

(1) The necessary and sufficient condition for T to be an $\mathcal{A}$-linear injective mapping is that the the sequence $0 \longrightarrow M \xrightarrow{T} N$ be exact.

(2) The necessary and sufficient condition for T to be a surjective $\mathcal{A}$-linear mapping is that the sequence $M \xrightarrow{T} N \longrightarrow 0$ *be exact.*

(3) The necessary and sufficient condition for that T to be $\mathcal{A}$-linear isomorphism is that the sequence $0 \longrightarrow M \xrightarrow{T} N \longrightarrow 0$ *be exact sequence at M and N.*

Definition 1.2.16. Let M,M',N and N' be $\mathcal{A}$-modules, $u : M' \to M$ and $v : N \to N'$ be two $\mathcal{A}$-linear mappings. Then for any $\mathcal{A}$-linear mapping $T : M \to N$ we can assign the $\mathcal{A}$-linear mapping $v \circ T \circ u : M' \to N'$, and we denote it by $Hom_{\mathcal{A}}(u, v)(T)$.

$$\begin{tikzcd} M \arrow[r, "T"] & N \arrow[d, "v"] \\ M' \arrow[u, "u"] \arrow[r, "Hom_{\mathcal{A}}(u,v)(T)"] & N' \end{tikzcd}$$

Indeed, $Hom_{\mathcal{A}}(u, v)$ is an $\mathcal{A}$-linear mapping from $Hom_{\mathcal{A}}(M, N)$ to $Hom_{\mathcal{A}}(M', N')$.

Proposition 1.2.17. *Let $\mathcal{A}$ be an algebra, M,M',M'' three $\mathcal{A}$-modules, and $u : M' \to M$, $v : M \to M''$ be two $\mathcal{A}$-linear mappings. For the sequence* $M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$ *to be exact, it is necessary and sufficient that the sequence of vector spaces*

$$0 \longrightarrow Hom_{\mathcal{A}}(M'', F) \xrightarrow{Hom_{\mathcal{A}}(v,1_F)} Hom_{\mathcal{A}}(M, F) \xrightarrow{Hom_{\mathcal{A}}(u,1_F)} Hom_{\mathcal{A}}(M', F)$$

is an exact sequence of $\mathbb{K}$-modules for any $\mathcal{A}$-module F.

Proof. See page 229 of [6]. ■

Theorem 1.2.18. *Let $\mathcal{A}$ be a $\mathbb{K}$-algebra, M be a $\mathbb{K}$-vector space, and N, L be two $\mathcal{A}$-modules. Then $Hom_{\mathbb{K}}(M, N)$ is an $\mathcal{A}$-module and we have*

$$Hom_{\mathcal{A}}(L, Hom_{\mathbb{K}}(M, N)) \underset{\mathbb{K}}{\cong} Hom_{\mathbb{K}}(M, Hom_{\mathcal{A}}(L, N))$$

Proof. We define

$$\cdot : \mathcal{A} \times Hom_{\mathbb{K}}(M, N) \longrightarrow Hom_{\mathbb{K}}(M, N)$$

$$(a, f) \longrightarrow a \cdot f$$

by $(a \cdot f)(m) = a(f(m))$, for all $a \in \mathcal{A}$, $m \in M$. It can be verified that $Hom_{\mathbb{K}}(M, N)$ is an $\mathcal{A}$-module with this multiplication. Also, We define

$$\phi : Hom_{\mathcal{A}}(L, Hom_{\mathbb{K}}(M, N)) \longrightarrow Hom_{\mathbb{K}}(M, Hom_{\mathcal{A}}(L, N))$$
$$f \longmapsto \widehat{f}$$

where $\widehat{f}(m)(l) = f(l)(m)$. It can be verified that ϕ is an $\mathcal{A}$-module isomorphism. ■

1.2.4. The Jordan-Hölder Theorem

Let $\mathcal{A}$ be a $\mathbb{K}$-algebra, and M be an $\mathcal{A}$-module. We say that M satisfies the **descending chain condition** (or d.c.c.) if, for every descending chain of $\mathcal{A}$-submodules of M :

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$$

there exists an integer n such that $M_n = M_{n+1} = M_{n+2} = \cdots$. Similarly, an $\mathcal{A}$-module M satisfies the **ascending chain condition** (or a.c.c.) if, for every ascending chain of $\mathcal{A}$-submodules of M :

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

there exists an integer n such that $M_n = M_{n+1} = M_{n+2} = \cdots$.

Definition 1.2.19. An $\mathcal{A}$-module M is called **Artinian** (**Noetherian**) if it satisfies the d.c.c (a.c.c.) condition.

Definition 1.2.20. A nonzero $\mathcal{A}$-module M is called **simple (or irreducible)** if it has only 0 and M as $\mathcal{A}$-submodules. A module M is called **semisimple (or completely reducible)** if it is isomorphic to a direct sum of simple $\mathcal{A}$-modules.

Lemma 1.2.21. *[Schur's Lemma]*

Let $\mathbb{K}$ be an algebraically closed field,that is, such that every non-constant polynomial with coefficients in $\mathbb{K}$ has a root in $\mathbb{K}$. Let $\mathcal{A}$ be a $\mathbb{K}$-algebra, M and N be two finite dimensional simple $\mathcal{A}$-modules. Then, every non-zero $\mathcal{A}$-module homomorphism $f : M \longrightarrow N$ is an $\mathcal{A}$-module isomorphism. Consequently, $Hom_{\mathcal{A}}(M, N) \cong \mathbb{K}$.

Proof. The simplicity of M and 1.2.12(1) implies that f is monomorphism. Also, the simplicity of N and 1.2.12(2) implies that f is epimorphism. Now, we define the natural

$\mathbb{K}$-homomorphism

$$\Phi : \mathbb{K} \longrightarrow Hom_{\mathcal{A}}(M, M)$$

$$\lambda \longmapsto \lambda Id_M$$

It is clear that Φ is injective if $\lambda \neq 0$, since $\mathbb{K}$ is a field. Also for every $f \in Hom_{\mathcal{A}}(M, M)$, f has an eigenvalue, since $\mathbb{K}$ is closed. Hence, $f - \lambda Id_M$ is zero, which proves our assertion. ■

Proposition 1.2.22. *Let $\mathcal{A}$ be a $\mathbb{K}$-algebra, M be a semisimple $\mathcal{A}$-module. The following conditions are equivalent:*

(i) M is Artinian;

(ii) M is Noetherian;

(iii) M is a direct sum of a finite number of simple $\mathcal{A}$-modules.

Proof. see page 63 of [13] ■

Definition 1.2.23. Let $\mathcal{A}$ be a $\mathbb{K}$-algebra. A **finite chain of submodules** or **series** of an $\mathcal{A}$-module M is a sequence of $\mathcal{A}$-submodules $\big(0 = M_0, M_1, \cdots, M_n = M\big)$ of M such that $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$. The chain is called a **composition series** of M if all the quotient $\mathcal{A}$-modules $\frac{M_{i+1}}{M_i}$ are simple, for $i = 0, 1, \cdots, n-1$. The quotient modules M_{i+1}/M_i are called the **composition factors** of this series and the number n is called the length of the series, and it is often denoted by $l(M)$.

Definition 1.2.24. Let S be a simple $\mathcal{A}$-module and let M be a $\mathcal{A}$-module with composition series $(M_i)_{i=1}^{l(M)}$ with $l(M) < \infty$. Then the number of elements of the set $\Big\{i : 0 < i \leq l(M), M_i/M_{i-1} \underset{\mathcal{A}}{\simeq} S\Big\}$ is called the S**-length of** M. We denote the S-length of M by $l_S(M)$ and we will say that S is a composition factor if $l_S(M) \geqslant 1$.

Theorem 1.2.25. *Suppose that M is an $\mathcal{A}$-module which has a composition series. Then any finite chain of $\mathcal{A}$-submodules of M can be included in a composition series. The lengths of any two composition series of the module M are equal and between the composition factors of these series one can establish a bijection in such a way that the corresponding factors are isomorphic.*

Proof. See page 65 of [13]. ■

Theorem 1.2.26. *Suppose M is a finite-length $\mathcal{A}$-module and S be a simple $\mathcal{A}$-submodule of M, then S is a composition factor of M.*

PROOF. We know that M/S is an $\mathcal{A}$-module with finite length, so it has a composition series like $0 = S/S \subseteq M_1/S \subset M_2/S \subseteq M_3/S \cdots \subseteq M/S$. So the sequence

$$S \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \subseteq M$$

is a series and $\dfrac{M_i/S}{M_j/S} \simeq M_i/M_j$ implies that it is a composition series. ■

1.2.5. Supplementary Sub-Module and Split Module

Definition 1.2.27. Let E be an $\mathcal{A}$-module, two submodules M and N of E are called **suplementary submodules** if E is a direct sum of M and N.

Definition 1.2.28. A submodule M of an $\mathcal{A}$-module E is called a direct factor of E if it has a supplementary submodule in E.

Proposition 1.2.29. *Suppose that E, F, and G are three $\mathcal{A}$-modules and the sequence*

$$0 \longrightarrow E \xrightarrow{f} F \xrightarrow{g} G \longrightarrow 0$$

is exact. Then the following conditions are equivalent:

(1) The submodule $f(E)$ of F is a direct factor.

(2) There exists an $\mathcal{A}$-linear mapping $r : F \to E$ such that $r \circ f = 1_E$.

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "f"] \arrow[d, "1_E"'] & F \arrow[r, "g"] \arrow[dl, dashed, "r"'] & G \arrow[r] & 0 \\ & E & & & \end{tikzcd}$$

*and then r is called a **retraction** for f.*

(3) There exists an $\mathcal{A}$-linear mapping $s : G \to F$ such that $g \circ s = 1_G$:

$$\begin{tikzcd} & & & G \arrow[d, "1_G"] \arrow[dl, dashed, "s"'] & \\ 0 \arrow[r] & E \arrow[r, "f"] & F \arrow[r, "g"] & G \arrow[r] & 0 \end{tikzcd}$$

*and then s is called a **section** for g. When the conditions hold, $f + s : E \bigoplus G \to F$ is an isomorphism.*

PROOF. See page 211 of [6]. ■

Definition 1.2.30. Let E,F, and G be three $\mathcal{A}$-modules, and the sequence

$$0 \longrightarrow E \xrightarrow{f} F \xrightarrow{g} G \longrightarrow 0$$

be exact. If the sequence satisfies one of the conditions of 1.2.29, then we say that the sequence **splits** or (F, f, g) is a **trivial extension** of G by E.

1.2.6. Projective Modules

Definition 1.2.31. An $\mathcal{A}$-module P is called **projective** if for every exact sequence

$$F'' \xrightarrow{u} F \xrightarrow{v} F'$$

of $\mathcal{A}$-modules and $\mathcal{A}$-linear mappings, the sequence of vector spaces

$$Hom_{\mathcal{A}}(P, F'') \xrightarrow{Hom_{\mathcal{A}}(1_P,u)} Hom_{\mathcal{A}}(P, F) \xrightarrow{Hom_{\mathcal{A}}(1_P,v)} Hom_{\mathcal{A}}(P, F')$$

is exact.

Proposition 1.2.32. *Let P be an $\mathcal{A}$-module. Then the following properties are equivalent:*

(1) P is projective.

(2) For every exact sequence $0 \longrightarrow F'' \xrightarrow{u} F \xrightarrow{v} F' \longrightarrow 0$ *of $\mathcal{A}$-modules and $\mathcal{A}$-linear mappings, the following sequence*

$$0 \longrightarrow Hom_{\mathcal{A}}(P, F'') \xrightarrow{Hom_{\mathcal{A}}(1_P,u)} Hom_{\mathcal{A}}(P, F) \xrightarrow{Hom_{\mathcal{A}}(1_P,v)} Hom_{\mathcal{A}}(P, F') \longrightarrow 0$$

is exact.

(3) For every surjective $\mathcal{A}$-linear mapping $u : E \to E''$ and every $\mathcal{A}$-linear mapping $f : P \to E''$, there exists an $\mathcal{A}$-linear mapping $g : P \to E$ such that $f = u \circ g$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists g"'] \arrow[d, "f"] & \\ E \arrow[r, "u"] & E'' \arrow[r] & 0 \end{tikzcd}$$

(4) Every exact sequence $0 \longrightarrow E' \longrightarrow E \longrightarrow P \longrightarrow 0$ *of $\mathcal{A}$-linear mappings splits and therefore P is isomorphic to a direct factor of E.*

Proof. See page 231 of [6] ■

1.2.7. Tensor Product

In the case of $\mathbb{K}$-vector spaces, the construction of tensor product solves the universal mapping property of $\mathbb{K}$-bilinear forms in Vect($\mathbb{K}$), the category of all $\mathbb{K}$-vector spaces (See [23] Chapter 14). Similarly, we know that the construction of tensor product over a commutative algebra $\mathcal{A}$ solves the universal mapping property of $\mathbb{A}$-bilinear forms in the category $\mathcal{A}$-algebras,(See [26]). The difficult situation appears when the $\mathbb{K}$-algebra $\mathcal{A}$ is not commutative, because in this case, being left $\mathcal{A}$-module is not equal with being right $\mathcal{A}$-module. In this case, the tensor product is constructed as $\mathbb{Z}$-module, or as left or right $\mathbb{K}$-modules. Then it is endowed with a left and right module structure which does not solve the universal mapping property in the category Mod-$\mathcal{A}$, when $\mathcal{A}$ is not commutative. In this section, we solve this problem. In this part, when we write $X \times Y$ for two algebraic structures X and Y, it means only we consider the set as cartesian product, without any algebraic structure.

Definition 1.2.33. [$(\mathcal{A}, \mathcal{B})$-bimodule]
Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{K}$-algebras. An Abelian group M is called an $(\mathcal{A}, \mathcal{B})$-bimodule if M is both a left $\mathcal{A}$-module and a right $\mathcal{B}$-module such that

$$(a \cdot m) \cdot b = a \cdot (m \cdot b)$$

for all $a \in \mathcal{A}, m \in M,$ and $b \in \mathcal{B}$. In this case M is denoted by ${}_{\mathcal{A}}M_{\mathcal{B}}$.

Definition 1.2.34. [$(\mathcal{A})$-Balanced form]
Let $\mathcal{A}$ be a $\mathbb{K}$-algebra, M be a right $\mathcal{A}$-module, N be a left $\mathcal{A}$-module, and S be a non-empty set. A function $f : M \times N \to S$ is called **balanced on** $\mathcal{A}$ or $\mathcal{A}$-balanced if we have:

$$f(ma, n) = f(m, an)$$

for any $m \in M, n \in N, a \in \mathcal{A}$.

Definition 1.2.35. [$(\mathcal{A}, \mathcal{B})$- Bilinear forms]
Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{K}$-algebras, M be a left $\mathcal{A}$-module, N be a right $\mathcal{B}$-bimodule, and G be a $(\mathcal{A}, \mathcal{B})$-bimodule. An **$(\mathcal{A}, \mathcal{B})$-bilinear form** is a function $g : M \times N \to G$ which satisfies the following conditions:

(1) g is **$\mathcal{A}$-linear on its first component**:

$$g(m_1 + m_2, n) = g(m_1, n) + g(m_2, n) \quad , \quad g(a.m, n) = a.g(m, n)$$

(2) g is **$\mathcal{B}$-linear on its second component**:

$$g(m, n_1 + n_2) = g(m, n_1) + g(m, n_2) \quad , \quad g(m, n.b) = g(m, n).b$$

for any $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, $a \in \mathcal{A}$, and $b \in \mathcal{B}$.

We shall denote the set of all $(\mathcal{A}, \mathcal{B})$-bilinear forms of $M \times N$ into G by $Bil_{(\mathcal{A},\mathcal{B})}(M \times N, G)$.

Definition 1.2.36. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be three $\mathbb{K}$-algebras, M be a $(\mathcal{A}, \mathcal{B})$-bimodule, N be a $(\mathcal{B}, \mathcal{C})$-bimodule, and G be a $(\mathcal{A}, \mathcal{C})$ -bimodule. A function $h : M \times N \to G$ is called **$(\mathcal{A}, \mathcal{B}, \mathcal{C})$-balanced bilinear form** if h is $(\mathcal{A}, \mathcal{C})$-bilinear form and $\mathcal{B}$-balanced.

We shall denote by $BBil_{(\mathcal{A},\mathcal{B},\mathcal{C})}(M \times N, G)$ the set of all $\mathcal{B}$-balanced $(\mathcal{A}, \mathcal{C})$-bilinear forms of $M \times N$ into G.

Definition 1.2.37. [$(\mathcal{A}, \mathcal{B})$-Homomorphisms] Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{K}$-algebras and M, N be two $(\mathcal{A}, \mathcal{B})$-bimodules. An **$(\mathcal{A}, \mathcal{B})$-homomorphism** is a function $\phi : M \to N$ such that:

$$\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2) \quad , \quad \phi(amb) = a\phi(m)b$$

for any $m, m_1, m_2 \in M, a \in \mathcal{A}, b \in \mathcal{B}$.

We denote the set of all $(\mathcal{A}, \mathcal{B})$-homomorphisms of M into N by $Hom_{(\mathcal{A},\mathcal{B})}(M, N)$. Analogously for bimodules one can introduce all other concepts which were introduced for modules: isomorphism, subbimodule, quotient bimodule, direct sum, etc.

Definition 1.2.38. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{K}$-algebras and M be a left $\mathcal{A}$-module, and N be a right $\mathcal{B}$-module, and $X \subseteq M$, $Y \subseteq N$ be two subsets of M and N.

We define $Sp_{(\mathcal{A},\mathcal{B})}(X \times Y) :=$

$$\Big\{ \sum_{i,j=1}^{n,m} (a_i x_i, y_j b_j) : a_i \in \mathcal{A}, x_i \in X, b_j \in \mathcal{B}, y_j \in Y, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m \quad n, m \in \mathbb{N} \Big\}.$$

Also, we define the $(\mathcal{A}, \mathcal{B})$-action on $Sp_{(\mathcal{A},\mathcal{B})}(X \times Y)$ by $a \cdot (x, y) \cdot b = (ax, yb)$, for any $x \in X, y \in Y, a \in \mathcal{A}, b \in \mathcal{B}$.

Theorem 1.2.39. [Universal property of $Sp_{(\mathcal{A},\mathcal{B})}(X \times Y)$]

Consider $\mathcal{A}, \mathcal{B}, X, Y$ the same as 1.2.38, G be an $(\mathcal{A}, \mathcal{B})$-bimodule, and $f : X \times Y \to G$ be a $(\mathcal{A}, \mathcal{B})$-bilinear form. Then

(1) $Sp_{(\mathcal{A},\mathcal{B})}(X \times Y)$ has $(\mathcal{A}, \mathcal{B})$-bimodule property.

(2) There is a unique $(\mathcal{A}, \mathcal{B})$-homomorphism $\overline{f} : Sp_{(\mathcal{A},\mathcal{B})}(X \times Y) \to G$ which commutes the following:

$$\begin{tikzcd} X \times Y \arrow[r, "f"] \arrow[d, hook, "i"'] & G \\ Sp_{(\mathcal{A},\mathcal{B})}(X \times Y) \arrow[ur, dotted, "\exists ! \overline{f}"'] & \end{tikzcd}$$

Proof. (1) Just definition 1.2.38.

(2) One can check that $\overline{f}$ defined by $\overline{f}(\sum_{i,j=1}^{n,m}(a_i x_i, y_j b_j)) = \sum_{i,j=1}^{n,m} f(a_i x_i, y_j b_j)$ satisfies all the requested properties.

$\blacksquare$

Definition 1.2.40. Let $\mathcal{A}$,$\mathcal{B}$, and $\mathcal{C}$ be three $\mathbb{K}$-algebras and M, N be two $(\mathcal{A},\mathcal{C})$ and $(\mathcal{C},\mathcal{B})$ bimodules, respectively. We consider $K_{\mathcal{C}} \subset Sp_{(\mathcal{A},\mathcal{B})}(M \times N)$ as $Sp_{(\mathcal{A},\mathcal{B})}$ of the following elements:

$$(x_1 + x_2, y) - (x_1, y) - (x_2, y)$$
$$(x, y_1 + y_2) - (x, y_1) - (x, y_2)$$
$$(xc, y) - (x, cy)$$
$$(ax, y) - a(x, y)$$
$$(x, yb) - (x, y)b$$

for any $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$, $a \in \mathcal{A}, b \in \mathcal{B}, c \in \mathcal{C}$.

We define $M \underset{(\mathcal{A},\mathcal{C},\mathcal{B})}{\otimes} N := \frac{Sp_{(\mathcal{A},\mathcal{B})}(M \times N)}{K_{\mathcal{C}}}$ and call it the **$(\mathcal{A},\mathcal{C},\mathcal{B})$-tensor product of M and** N.

Proposition 1.2.41. *[Fundamental Properties of $(\mathcal{A},\mathcal{B})$-tensor product]*

Let $\mathcal{A},\mathcal{B},\mathcal{C}$ be three $\mathbb{K}$-algebras and M, N, G be three $(\mathcal{A},\mathcal{C})$-, $(\mathcal{C},\mathcal{B})$-, $(\mathcal{A},\mathcal{B})$- bimodules, respectively. We have the following properties:

(1) $M \underset{(\mathcal{A},\mathcal{C},\mathcal{B})}{\otimes} N$ has $(\mathcal{A},\mathcal{B})$-bimodule structure.

(2) $Hom_{(\mathcal{A},\mathcal{B})}(M \underset{(\mathcal{A},\mathcal{C},\mathcal{B})}{\otimes} N, G) \underset{(\mathcal{A},\mathcal{B})}{\simeq} BBil_{(\mathcal{A},\mathcal{C},\mathcal{B})}(M \times N, G)$

Proof. (1) It is clear by 1.2.40.

(2)

$$\begin{aligned}
Hom_{(\mathcal{A},\mathcal{B})}(M \underset{(\mathcal{A},\mathcal{C},\mathcal{B})}{\otimes} N, G) &= Hom_{(\mathcal{A},\mathcal{B})}\Big(\frac{Sp_{(\mathcal{A},\mathcal{B})}(M \times N)}{K_{\mathcal{C}}}, G\Big) \\
&= \Big\{\phi \in Hom_{(\mathcal{A},\mathcal{B})}\big(Sp_{(\mathcal{A},\mathcal{B})}(M \times N), G\big) : \phi \mid_{K_{\mathcal{C}}} = 0\Big\} \\
&\underset{(\mathcal{A},\mathcal{B})}{\simeq} Bil_{(\mathcal{A},\mathcal{C},\mathcal{B})}(M \times N, G)
\end{aligned}$$

■

Example 1.2.42. [Classical definition of tensor product]

Let $\mathcal{C}$ be a $\mathbb{K}$-algebra, $\mathcal{A} = \mathbb{K} = \mathcal{B}$, and M be a right $\mathcal{C}$-module and N be a left $\mathcal{C}$-module. In this case, we have:

$$M \underset{(\mathcal{A},\mathcal{C},\mathcal{B})}{\otimes} N = M \underset{(\mathbb{K},\mathcal{C},\mathbb{K})}{\otimes} N = M \underset{\mathcal{C}}{\otimes} N$$

which gives the clasical definition of tensor product of M and N.

Definition 1.2.43. Let $\mathcal{A}$ be a $\mathbb{K}$-algebra. The left $\mathcal{A}$-module M is **finitely generated** if there exist $m_1, m_2, \ldots, m_n$ in M such that for any m in M, there exist $a_1, a_2, \ldots, a_n$ in $\mathcal{A}$ with $m = a_1m_1 + a_2m_2 + ... + a_nm_n$.

Theorem 1.2.44. *Let $\mathcal{A}$ be a $\mathbb{K}$-algebra, M be a finite dimensional $\mathbb{K}$-module, N be a left $\mathcal{A}$-module. Then $N \underset{\mathbb{K}}{\otimes} M \cong N^{dim_{\mathbb{K}}M}$ as left modules.*

PROOF. Suppose $\Big\{a_i : 1 \leq i \leq dim_{\mathbb{K}}M\Big\}$ be a basis of M. So we have:

$$\begin{aligned}
N \underset{\mathbb{K}}{\otimes} M &= \sum_{i=1}^{dim_{\mathbb{K}}M} N \otimes a_i \\
&= \bigoplus_{i=1}^{dim_{\mathbb{K}}M} N \otimes a_i \\
&\cong \bigoplus_{i=1}^{dim_{\mathbb{K}}M} N \\
&= N^{\oplus\ dim_{\mathbb{K}}M}
\end{aligned}$$

■

Theorem 1.2.45. *Let $\mathcal{A}$ be a $\mathbb{K}$-algebra, M be a $\mathbb{K}$-vector space, P be a right $\mathcal{A}$-module, and N be a left $\mathcal{A}$-modules. Then $Hom_{\mathbb{K}}(N, M)$ is a right $\mathcal{A}$-module and we have*

$$Hom_{\mathbb{K}}(P \underset{\mathcal{A}}{\otimes} N, M) \underset{\mathbb{K}}{\cong} Hom_{\mathcal{A}}(P, Hom_{\mathbb{K}}(N, M)).$$

PROOF. The isomorphism is given by

$$Hom_{\mathbb{K}}(P \underset{\mathcal{A}}{\otimes} N, M) \longrightarrow Hom_{\mathcal{A}}(P, Hom_{\mathbb{K}}(N, M))$$

$$\phi \longrightarrow \widehat{\phi}$$

where $\widehat{\phi}(p)(n) = \phi(p \underset{\mathcal{A}}{\otimes} n)$, for all $p \in P, n \in N$. See page 267 of [6] for more details. ∎

1.2.8. Radical of a Module

For a given $\mathcal{A}$-module M, we shall introduce "a measure of how far it is from being semisimple": the set of all elements $m \in M$ such that $f(m) = 0$ for any homomorphism f of M to a simple module. Evidently, these elements form a submodule of M, which will be called the radical of the module M and be denoted by $Rad_{\mathcal{A}} M$.

A proper (not equal to M) non-zero $\mathcal{A}$-submodule N of a left $\mathcal{A}$-module M is called **left maximal** if it is not contained in any other left $\mathcal{A}$-submodule of M.

Definition 1.2.46. Let M be an arbitrary left $\mathcal{A}$-module. The **Jacobson radical** or **radical** of M is the intersection of all its maximal left $\mathcal{A}$-submodules, and it is denoted by $Rad_{\mathcal{A}}(M)$.

Proposition 1.2.47. *[Properties of Jacobson radical of a module] Let M be an arbitrary left $\mathcal{A}$-module. Then we have the following properties:*

(1) $Rad_{\mathcal{A}}(M) = \bigcap_{\phi \in Simp(\mathcal{A})} Ker\phi$, where $Simp(\mathcal{A})$ is the set of all $\mathcal{A}-$homomorphisms from M to a simple $\mathcal{A}$-module.

(2) If $\mathcal{A}$ is finite dimensional, then $Rad_{\mathcal{A}}(M) = Rad(\mathcal{A})M$.

(3) The $\mathcal{A}$-module $M/Rad_{\mathcal{A}}(M)$ is semisimple and it is a module over the $\mathbb{K}$-algebra $\mathcal{A}/Rad(\mathcal{A})$.

PROOF. (1) See proposition 3.4.1 in [13].

(2) See proposition 3.7 in [1].

(3) See corollary 3.8 in [1]. ∎

Definition 1.2.48. [Top]

Let M be an $\mathcal{A}$-module. We define $Top(M) := M/Rad_{\mathcal{A}}(M)$. By 1.2.47.3, $Top(M)$ is a $\mathcal{A}/Rad(\mathcal{A})$ - module.

Theorem 1.2.49. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be a $\mathbb{K}$-algebra, and M be an $\mathcal{A}$-module, and S be a simple $\mathcal{A}$-module. Then the following hold:*

(1) $Hom_{\mathcal{A}}(\mathcal{A}, S) \neq 0$

(2) Every nonzero $\mathcal{A}$-homomorphism $\phi : M \longrightarrow S$ is an $\mathcal{A}$-epimorphism.

(3) If $Top(M)$ is a simple $\mathcal{A}$-module, then $Rad_{\mathcal{A}}(M) = Ker\phi$, $\forall\ 0 \neq \phi \in Hom_{\mathcal{A}}(M, S)$.

Proof. (1) Since S is a simple $\mathcal{A}$-module, so S contains a nonzero element $s_0 \in S$. We define $\phi : \mathcal{A} \longrightarrow S$ by: $\phi(a) = as_0$. So $Hom_{\mathcal{A}}(\mathcal{A}, S) \neq 0$.

(2) $\forall\ 0 \neq \psi \in Hom_{\mathcal{A}}(\mathcal{A}, S)$, $Im(\psi)$ is an $\mathcal{A}$- submodule of L and simplicity of S implies $Im(\psi) = S$. So ψ is an $\mathcal{A}$-epimorphism.

(3) Since $Top(M)$ is a simple $\mathcal{A}$-module, so $Rad_{\mathcal{A}}(M)$ is a maximal $\mathcal{A}$-submodule of M, also by 1.2.47(1) we have $Rad_{\mathcal{A}}(M) \subseteq Ker(\phi)$, $\forall \phi : M \longrightarrow L$, since $Rad_{\mathcal{A}}(M)$ is maximal, so we must have $Ker(\phi) = Rad_{\mathcal{A}}(M)$.

■

1.2.9. Projectives by Means of Idempotents

Proposition 1.2.50. *Let L be a right ideal of a unitary $\mathbb{K}$-algebra $\mathcal{A}$ generated by an idempotent e, $L = e\mathcal{A}$. We define $\mathcal{O}(\mathcal{A})$ the set of all pairwise orthogonal idempotents of $\mathcal{A}$,*

$$\mathcal{D}(e) = \Big\{(e_1, \ldots, e_r) \in \mathcal{O}(\mathcal{A}) : e_1 + \cdots + e_r = e, \Big\}$$

$$\mathcal{D}(L) = \Big\{(L_1, \ldots, L_r) : L = L_1 \oplus \ldots \oplus L_r,\ L_i \textit{ is a right ideal of } \mathcal{A}\Big\}$$

Now, consider the map

$$\phi : \mathcal{D}(e) \to \mathcal{D}(L)$$

$$(e_1, \cdots, e_r) \to (e_1\mathcal{A}, \cdots, e_r\mathcal{A})$$

Then this map is bijective.

Proof. See page 16 in [25]. ■

Proposition 1.2.51. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be a $\mathbb{K}$-algebra. For any nonzero idempotent $e \in \mathcal{A}$ the following conditions are equivalent:*

(1) $e \cdot_{\mathcal{A}} \mathcal{A}$ is indecomposable as a right $\mathcal{A}$-module.

(2) $\mathcal{A} \cdot_{\mathcal{A}} e$ is indecomposable as a left $\mathcal{A}$-module.

(3) The idempotent e is primitive.

Proof. See page 51 of [13]. ■

Lemma 1.2.52. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be a $\mathbb{K}$-algebra and I be an ideal of $\mathcal{A}$. Then I is an idempotent ideal, $I^2 = I$, if and only if $I = \mathcal{A} \cdot_{\mathcal{A}} e \cdot_{\mathcal{A}} \mathcal{A}$, for some idempotent element $e \in \mathcal{A}$.*

Proof. See page 700 in [8]. ■

Lemma 1.2.53. *Let* $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ *be a* $\mathbb{K}$*-algebra and* e *be an idempotent in* $\mathcal{A}$*. Then* $Rad(e \cdot_{\mathcal{A}} \mathcal{A} \cdot_{\mathcal{A}} e) = e \cdot_{\mathcal{A}} Rad(\mathcal{A}) \cdot_{\mathcal{A}} e$

Proof. See Proposition 3.4.8, page 70 in [13]. ■

The next lemma plays a crucial role in the understanding of the notion of quasi-hereditary algebra.

Lemma 1.2.54. *Let* $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ *be a* $\mathbb{K}$*-algebra and* I *be an idempotent ideal in* $\mathcal{A}$ *such that* $I = \mathcal{A} \cdot_{\mathcal{A}} e \cdot_{\mathcal{A}} \mathcal{A}$ *with* e *an idempotent. Then* $e \cdot_{\mathcal{A}} \mathcal{A} \cdot_{\mathcal{A}} e$ *is semi-simple if and only if* $I \cdot_{\mathcal{A}} Rad(\mathcal{A}) \cdot_{\mathcal{A}} I = 0$.

Proof.

$$\begin{aligned}
I \cdot_{\mathcal{A}} Rad(\mathcal{A}) \cdot_{\mathcal{A}} I = 0 &\Longleftrightarrow (\mathcal{A} \cdot_{\mathcal{A}} e \cdot_{\mathcal{A}} \mathcal{A}) \cdot_{\mathcal{A}} Rad(\mathcal{A}) \cdot_{\mathcal{A}} (\mathcal{A} \cdot_{\mathcal{A}} e \cdot_{\mathcal{A}} \mathcal{A}) = 0 \\
&\overset{1.2.13(ii)}{\Longleftrightarrow} \mathcal{A} \cdot_{\mathcal{A}} e \cdot_{\mathcal{A}} Rad(\mathcal{A}) \cdot_{\mathcal{A}} e \cdot_{\mathcal{A}} \mathcal{A} = 0 \\
&\overset{1.2.53}{\Longleftrightarrow} \mathcal{A} \cdot_{\mathcal{A}} (Rad(e \cdot_{\mathcal{A}} \mathcal{A} \cdot_{\mathcal{A}} e)) \cdot_{\mathcal{A}} \mathcal{A} = 0 \\
&\overset{1.2.13(ii)}{\Longleftrightarrow} Rad(e \cdot_{\mathcal{A}} \mathcal{A} \cdot_{\mathcal{A}} e) = 0 \\
&\Longleftrightarrow e \cdot_{\mathcal{A}} \mathcal{A} \cdot_{\mathcal{A}} e \quad \text{is semi-simple.}
\end{aligned}$$

■

Now, we want to identify the structure of a projective $\mathcal{A}$-module, when $\mathcal{A}$ is finite-dimensional

Theorem 1.2.55. *Let* $\mathcal{A}$ *be a unital* $\mathbb{K}$*-algebra,* $(e_1, \ldots, e_r)$ *be a family of pairwise orthogonal idempotents in* $\mathcal{A}$ *such that* $e_1 + \cdots + e_r = 1_{\mathcal{A}}$*. Let* $\mathcal{A}_{\mathcal{A}} = e_1\mathcal{A} \oplus e_2\mathcal{A} \oplus \cdots \oplus e_r\mathcal{A}$*. Then, any right projective* $\mathcal{A}$*-module* P *can be written as*

$$P = P_1 \oplus P_2 \oplus \cdots \oplus P_s$$

where every summand P_i *is indecomposable and isomorphic to some* $e_{i'}\mathcal{A}$*, for some* $1 \leq i' \leq r$.

Proof. See page 26 of [1] ■

Definition 1.2.56. [Essential Epimorphism]

Let M and N be two $\mathcal{A}$-modules. An epimorphism $f : M \to N$ is called **essential** if $f(L) \neq N$ for any proper submodule $L \subsetneq M$.

It is possible to verify that an epimorphism $f : M \to N$ is essential if for every $\mathcal{A}$-module Q and every homomorphism $g : Q \to M$, the surjectivity $f \circ g$ implies that g is surjective.

Definition 1.2.57. [Projective Cover]

A **projective cover** of an $\mathcal{A}$-module M is an essential $\mathcal{A}$-module homomorphism $f : P \to M$ where P is projective.

Remark 1.2.58. It can be proved that projective cover is unique up to $\mathcal{A}$-module isomorphism.

Now, we want to identify the structure of a projective cover in the case of a finite-dimensional algebra.

Theorem 1.2.59. *Let $\mathcal{A}$ be a finite-dimensional K-algebra, $\{e_1, \ldots, e_n\}$ be a complete family of primitive pairwise othogonal idempotents of $\mathcal{A}$ such that $\mathcal{A}_{\mathcal{A}} = e_1\mathcal{A} \oplus \ldots \oplus e_n\mathcal{A}$. Then for any $\mathcal{A}$-module M, there exists a projective cover P_M, $P_M \xrightarrow{h} M \to 0$, where $P_M = (e_1\mathcal{A})^{s_1} \oplus \cdots \oplus (e_n\mathcal{A})^{s_n}$ and $s_1 \geq 0, \cdots, s_n \geq 0$. The epimorphism h induces an isomorphism $\overline{h} : P_M/Rad_{\mathcal{A}}P_M \to M/Rad_{\mathcal{A}}M$.*

PROOF. See page 29 of [1]. ■

Remark 1.2.60. One can show that a projective cover is unique up to an $\mathcal{A}$-module isomorphism.

Theorem 1.2.61. *Suppose that $\mathcal{A}_{\mathcal{A}} = e_1\mathcal{A} \oplus \ldots \oplus e_n\mathcal{A}$ is a decomposition of $\mathcal{A}$ into indecomposable submodules. Then*

(1) Every indecomposable projective right $\mathcal{A}$-module is isomorphic to one of the modules

$$P(1) = e_1\mathcal{A}, P(2) = e_2\mathcal{A}, \ldots, P(n) = e_n\mathcal{A}$$

(2) Every simple right $\mathcal{A}$-module is isomorphic to one of the modules:

$$S(1) = Top(P(1)), \ldots, S(n) = Top(P(n))$$

Moreover, $P(i) \simeq P(j)$ if and only if $S(i) \simeq S(j)$.

PROOF. See 5.17, page 32 in [1]. ■

Now, we give a crucial theorem which will be used in chapter 3.

Theorem 1.2.62. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be a finite-dimensional $\mathbb{K}$-algebra and P be a left projective $\mathcal{A}$-module, then the following are equivalent:*

(1) P is indecomposable.

(2) $Rad_{\mathcal{A}}(P)$ is the unique maximal submodule of P.

PROOF. See page 15 in [2] ■

1.2.10. Cells in Algebras

Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be a finite-dimensional $\mathbb{K}$-algebra and $\mathrm{B} = \{b_i : 1 \leqslant i \leqslant n = dim_{\mathbb{K}}\mathcal{A}\}$ be a $\mathbb{K}$-basis for $\mathcal{A}$, that is every element of $\mathcal{A}$ is a finite linear combination of elements in B with coefficients in $\mathbb{K}$; since $\mathcal{A}$ is finite-dimensional we can suppose that $\mathrm{B} = \big\{b_1, \cdots, b_n\big\}$. For convenience, we define $\Lambda = \{1, 2, \ldots, n\}$. So for every two elements $b_i, b_j \in \mathrm{B}$ we have $b_i \cdot_{\mathcal{A}} b_j = \sum\limits_{r=1}^{n} \alpha_{b_i b_j b_r} b_r$. We say $\alpha_{b_i b_j b_r}$ is the coefficient of b_r in the expansion of $b_i \cdot_{\mathcal{A}} b_j$, for any $i, j \in \Lambda$.

Definition 1.2.63. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be an $\mathbb{K}$-algebra, $\mathrm{B} = \big\{b_1, \cdots, b_n\big\}$ be a $\mathbb{K}$-basis for $\mathcal{A}$, and $\Lambda = \big\{1, \cdots, n\big\}$ be the index set for B. We define the relation $\xleftarrow{L}$ on B as follows[2]:
$b_i \xleftarrow{L} b_j$ if there is $a \in \mathcal{A}$ such that the coefficient of b_i in the expansion of $a \cdot_{\mathcal{A}} b_j$ is non-zero, $\alpha_{a b_j b_i} \neq 0$.

Remark 1.2.64. For every $b \in \mathrm{B}$ we have $1_{\mathcal{A}} \cdot_{\mathcal{A}} b = b = 1_{\mathbb{K}} b$. Hence, $\xleftarrow{L}$ is reflexive. Now consider its transitive closure, $\mathrm{T}(\xleftarrow{L}) = \bigcup\limits_{m \geqslant 1} (\xleftarrow{L})^m$. By 1.1.3, $\mathrm{T}(\xleftarrow{L})$ is a pre-ordering on B. By 1.1.2 we have:

$$\begin{aligned} b_i \mathrm{T}(\xleftarrow{L}) b_j &\Longleftrightarrow (b_i, b_j) \in \mathrm{T}(\xleftarrow{L}) \\ &\Longleftrightarrow \exists\ k \in \mathbb{N}\ ,\ b_i (\xleftarrow{L})^k b_j \\ &\Longleftrightarrow b_i \xleftarrow{L} b_{k_2}\ ,\ b_{k_2} \xleftarrow{L} b_{k_3}\ ,\ \ldots\ ,\ b_{k_{m-1}} \xleftarrow{L} b_{k_m}\ ,\ b_{k_m} \xleftarrow{L} b_j \\ &\Longleftrightarrow \alpha_{b_{k_2'} b_{k_2} b_i} \neq 0, \alpha_{b_{k_3'} b_{k_3} b_{k_2}} \neq 0\ ,\ \ldots\ ,\ \alpha_{b_{k_m'} b_{k_m} b_j \neq 0} \end{aligned}$$

for some $b_{k_2}, b_{k_3}, \ldots, b_{k_{m-1}}, b_{k_m}, b_{k_2'}, b_{k_3'}, \ldots, b_{k_{m-1}'}, b_{k_m'} \in \mathrm{B}$.
For simplicity, we denote $\mathrm{T}(\xleftarrow{L})$ by $\preceq_L^t$, by which we mean

$$a \mathrm{T}(\xleftarrow{L}) b \Longleftrightarrow b \preceq_L^t a$$

so $(\mathrm{B}, \preceq_L^t)$ is a pre-ordered set.

Definition 1.2.65. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be an $\mathbb{K}$-algebra and $\mathrm{B} := \big\{b_1, \cdots, b_n\big\}$ be a $\mathbb{K}$-basis for $\mathcal{A}$. We define the relation $\xleftarrow{R}$ [3] on B as follows:

[2] L stands for the left multiplication.

[3] R stands for the right multiplication.

$b_1 \overset{R}{\leftarrow} b_2$ if there is $b_3 \in \mathrm{B}$ such that the coefficient of b_1 in the expansion of $b_2 \cdot_{\mathcal{A}} b_3$ is non-zero, $\alpha_{b_2 b_3 b_1} \neq 0$.

Remark 1.2.66. For every $b \in \mathrm{B}$ we have $b \cdot_{\mathcal{A}} 1_{\mathcal{A}} = b = 1_{\mathbb{K}} b$. Hence, $\overset{R}{\leftarrow}$ is reflexive. Now consider its transitive closure, $\mathrm{T}(\overset{R}{\leftarrow}) = \bigcup\limits_{n \geqslant 1} (\overset{R}{\leftarrow})^n$. By 1.1.3, $\mathrm{T}(\overset{R}{\leftarrow})$ is a pre-ordering on B. By 1.1.2 we have:

$$\begin{aligned} b\mathrm{T}(\overset{R}{\leftarrow})b' &\Longleftrightarrow (b, b') \in \mathrm{T}(\overset{R}{\leftarrow}) \\ &\Longleftrightarrow \exists\ m \in \mathbb{N}\ ,\ b(\overset{R}{\leftarrow})^m b' \\ &\Longleftrightarrow b \overset{R}{\leftarrow} b_2\ ,\ b_2 \overset{R}{\leftarrow} b_3\ ,\ \ldots\ ,\ b_{m-1} \overset{R}{\leftarrow} b_m\ ,\ b_m \overset{R}{\leftarrow} b' \\ &\Longleftrightarrow \alpha_{b_2 b_2' b} \neq 0, \alpha_{b_3 b_3' b_2} \neq 0,\ ,\ \ldots\ ,\ \alpha_{b_m b_m' b' \neq 0} \end{aligned}$$

for some $b_2, b_3, \ldots, b_{m-1}, b_m, b_2', b_3', \ldots, b_{m-1}', b_m' \in \mathrm{B}$.

For simplicity, we denote $\mathrm{T}(\overset{R}{\leftarrow})$ by $\preceq_R^t$, by which mean

$$a\mathrm{T}(\overset{R}{\leftarrow})b \Longleftrightarrow b \preceq_R^t a$$

so $(\mathrm{B}, \preceq_R^t)$ is a pre-ordered set.

Definition 1.2.67. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be an $\mathbb{K}$-algebra and $\mathrm{B} := \{b_1, \cdots, b_n\}$ be a $\mathbb{K}$-basis for $\mathcal{A}$. We define the pre-order $\preceq_{LR}^t$ on B by:

$$\preceq_{LR}^t := \preceq_L^t \cup \preceq_R^t$$

that is the pre-order generated by $\preceq_L^t$ and $\preceq_R^t$.

Remark 1.2.68. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be an $\mathbb{K}$-algebra and $\mathrm{B} := \{b_1, \cdots, b_n\}$ be a $\mathbb{K}$-basis for $\mathcal{A}$. By 1.2.67, $b \preceq_{LR}^t b' \Longleftrightarrow b \preceq_L^t b'$ or $b \preceq_R^t b'$

$$b \preceq_L^t b' \overset{1.2.64}{\Longleftrightarrow} \alpha_{b_2' b_2 b} \neq 0, \alpha_{b_3' b_3 b_2} \neq 0,\ ,\ \ldots\ ,\ \alpha_{b_m' b_m b' \neq 0}$$

for some $b_2, b_3, \ldots, b_{m-1}, b_m, b_2', b_3', \ldots, b_{m-1}', b_m' \in \mathrm{B}$ and $m \in \mathbb{N}$.

$$b \preceq_R^t b' \overset{1.2.66}{\Longleftrightarrow} \alpha_{c_2 c_2' b} \neq 0, \alpha_{c_3 c_3' b_2} \neq 0,\ ,\ \ldots\ ,\ \alpha_{c_n c_n' b' \neq 0}$$

for some $c_2, c_3, \ldots, c_{n-1}, c_n, c_2', c_3', \ldots, c_{n-1}', c_n' \in \mathrm{B}$ and $n \in \mathbb{N}$.

Notation 1.2.69. We introduced the three pre-orders $\preceq_L^t$, $\preceq_R^t$, and $\preceq_{LR}^t$ on B. Instead of repeating a property for these three pre-orders, we write the property with $\preceq_?^t$, $? \in \{L, R, LR\}$.

Now suppose that $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ is a $\mathbb{K}$-algebra and $\mathrm{B} := \{b_1, \cdots, b_n\}$ be a $\mathbb{K}$-basis for $\mathcal{A}$. By 1.2.63,1.2.65, and 1.2.67, $(\mathrm{B}, \preceq^t_?)$ is a pre-ordered set for $? \in \{L, R, LR\}$, respectively. We define the relation $\sim_?$ on B by

$$b \sim_? b' \Longleftrightarrow b \preceq^t_? b' \text{ and } b' \preceq^t_? b$$

for every $b, b' \in \mathrm{B}$ and $? \in \{L, R, LR\}$. It is clear that $\mathcal{R}_?$ is an equivalence relation on B. So, for all $b \in \mathrm{B}$, we have $\mathrm{C}^b_? := \{b' \in \mathrm{B} : b \preceq^t_? b' \text{ and } b' \preceq^t_? b\}$, for $? \in \{L, R, LR\}$.
Now consider $\mathrm{B}/\sim_? = \{\mathrm{C}^b_? : b \in \mathrm{B}\}$, define the relation $\preceq_?$ on $\mathrm{B}/\sim_?$ by:

$$\mathrm{C}_b \preceq_? \mathrm{C}_{b'} \Longleftrightarrow b \preceq^t_? b'$$

for every $\mathrm{C}^b_?, \mathrm{C}^{b'}_? \in \mathrm{B}/\sim_?$. Then by 1.1.4 $(\mathrm{B}/\sim_?, \preceq_?)$ is a partially-ordered set.

Definition 1.2.70. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be an $\mathbb{K}$-algebra and $\mathrm{B} := \{b : b \in \mathcal{A}\}$ be a $\mathbb{K}$-basis for $\mathcal{A}$. $(\mathrm{B}, \preceq^t_?)$ is a pre-ordered set, and $\sim_?$ is an equivalence relation on B. The equivalence class $\mathrm{C}^b_?$ is called the **left, right or two-sided cell**, according to whether ?= L,R or LR.

1.3. Category

1.3.1. Basic notions in Categories

Definition 1.3.1. A **category** $\mathscr{C}$ consists of the following:

(i) A class $\mathrm{Ob}\mathscr{C}$ whose elements are called the objects. [4]

(ii) For each (not necessarily distinct) pair of objects A and $B \in \mathscr{C}$, a set $Hom_{\mathscr{C}}(A, B)$ called the **Hom-set**[5] for the pair (A, B). The elements of $Hom_{\mathscr{C}}(A, B)$ are called **morphisms**, **maps** or **arrows** from A to B. If $f \in Hom_{\mathscr{C}}(A, B)$, we also write $f : A \longrightarrow B$ or f_{AB}.

(iii) For $f \in Hom_{\mathscr{C}}(A, B)$ and $g \in Hom_{\mathscr{C}}(B, C)$ there is a morphism $g \circ f \in Hom_{\mathscr{C}}(A, C)$, called the **composition of** g **with** f. Moreover, composition is associative: $(f \circ g) \circ h = f \circ (g \circ h)$ whenever the compositions are defined.

[4]It is customary to write $A \in \mathscr{C}$ instead of $A \in \mathrm{Ob}\mathscr{C}$.

[5]Distinct hom-sets are disjoint, that is, $Hom_{\mathscr{C}}(A, B)$ and $Hom_{\mathscr{C}}(C, D)$ are disjoint unless $A = C$ and $B = D$.

(iv) For each object $A \in \mathscr{C}$ there is a morphism $1_A \in Hom_{\mathscr{C}}(A, A)$, called the **identity morphism** for A, with the property that if $f_{AB} \in Hom_{\mathscr{C}}(A, B)$ then $1_B \circ f_{AB} = f_{AB}$ and $f_{AB} = f_{AB} \circ 1_A$. The class of all morphisms of $\mathscr{C}$ is denoted by $\mathrm{Mor}\mathscr{C}$.

Example 1.3.2. (1) The Category **Set** of Sets: Ob(**Set**) is the class of all sets. $Hom(A, B)$ is the set of all functions from A to B.

(2) The Category **Mon** of Monoids: Ob(**Set**) is the class of all monoids. $Hom(A, B)$ is the set of all monoid homomorphisms from A to B.

(3) The Category **Grp** of Groups: Ob(**Grp**) is the class of all groups. $Hom(A, B)$ is the set of all group homomorphisms from A to B.

(4) The Category **AbGrp** of Abelian Groups: Ob(**AbGrp**) is the class of all abelian groups. $Hom(A, B)$ is the set of all group homomorphisms from A to B.

(5) The Category **Mod-**R of R-modules, where R is a ring. Ob(**Mod**R) is the class of all R-modules. $Hom(A, B)$ is the set of all R-modules homomorphisms from A to B.

(6) The Category **Vect(**$\mathbb{K}$**)** of Vector Spaces over a Field $\mathbb{K}$: Ob(**Vect**($\mathbb{K}$)) is the class of all vector spaces over $\mathbb{K}$. $Hom(A, B)$ is the set of all $\mathbb{K}$-linear transformations from A to B.

(7) The Category **Rng** of Rings: Ob(**Rng**) is the class of all rings (with unit). $Hom(A, B)$ is the set of all ring homomorphisms from A to B.

(8) The Category **CRng** of Commutative Rings with identity: Ob(**CRng**) is the class of all commutative rings with identity. $Hom(A, B)$ is the set of all ring homomorphisms from A to B.

(9) The Category **Poset** of Partially-Ordered Sets: Ob(**Poset**) is the class of all partially-ordered sets. $Hom(A, B)$ is the set of all monotone functions from A to B, that is functions $f : A \to B$ satisfying:

$$a_1 \preceq_A a_2 \Longrightarrow f(a_1) \preceq_B f(a_2)$$

Definition 1.3.3. Let $\mathscr{C}$ be a category and $f, g : A \longrightarrow B$ be two morphisms in $\mathscr{C}$.

- A **monomorphism** is a morphism $h : B \longrightarrow C$ such that

$$h \circ g = h \circ f \Longrightarrow g = f$$

  We usually show it by $h : B \hookrightarrow C$ in diagrams.

- An **epimorphism** is a morphism $h : C \longrightarrow A$ such that

$$g \circ h = f \circ h \Longrightarrow g = f$$

We usually show it by $h : C \twoheadrightarrow A$ in diagrams.

- A **section** is a morphism $f : A \longrightarrow B$ such that there is a morphism $r : B \longrightarrow A$ such that: $r \circ f = 1_A$.
- A **retraction** is a morphism $f : A \longrightarrow B$ such that there is a morphism $s : B \longrightarrow A$ such that: $f \circ s = 1_B$.
- An **isomorphism** is a morphism $f : A \longrightarrow B$ such that there is a morphism $g : B \longrightarrow A$ such that $f \circ g = 1_A$, $g \circ f = 1_B$.

  We usually show it by $f : A \cong B$ in diagrams.

Remark 1.3.4. Isomorphisms are always retractions and sections; sections are always monomorphisms and retractions are always epimorphisms, so isomorphisms are always monomorphisms and epimorphisms.

Definition 1.3.5. A category $\mathscr{C}$ is called **small** if $\mathrm{Ob}\mathscr{C}$ the class of objects of $\mathscr{C}$, and $\mathrm{Mor}\mathscr{C}$ the class of morphisms of $\mathscr{C}$ are sets. Otherwise, $\mathscr{C}$ is called a **large** category.

Example 1.3.6. The category **Set**, **Mon**, **Grp**, **AbGrp**, **Vect**($\mathbb{K}$), **Mod**R, and **Poset** are some common examples of small categories.

Definition 1.3.7. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A **functor** $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}$ is a pair of functions:

(1) The object part of the functor

$$\mathcal{F} : \mathrm{Ob}\mathscr{C} \longrightarrow \mathrm{Ob}\mathscr{C}$$

maps objects in $\mathscr{C}$ to objects in $\mathscr{D}$.

(2) The arrow part $\mathcal{F} : \mathrm{Mor}(\mathscr{C}) \to \mathrm{Mor}(\mathscr{D})$ maps morphisms in $\mathscr{C}$ to morphisms in $\mathscr{C}$ as follows:

(2.1) For a **covariant functor**, $\mathcal{F}(Hom_{\mathscr{C}}(A, B)) \subset Hom_{\mathscr{D}}(\mathcal{F}(A), \mathcal{F}(B))$ for all $A, B \in \mathscr{C}$, that is, $\mathcal{F}$ maps a morphism $f : A \to B$ in $\mathscr{C}$ to a morphism $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ in $\mathscr{D}$.

(2.2) For a **contravariant functor**, $\mathcal{F}(Hom_{\mathscr{C}}(A, B)) \subset Hom_{\mathscr{D}}(\mathcal{F}(B), \mathcal{F}(A))$. for all $A, B \in \mathscr{C}$, that is, $\mathcal{F}$ maps a morphism $f : A \to B$ in $\mathscr{C}$ to a morphism $\mathcal{F}(f) : \mathcal{F}(B) \to \mathcal{F}(A)$ in $\mathscr{D}$.[6]

(3) $\mathcal{F}$ preserves composition, that is,

[6] Note the reversal of direction.

for a covariant functor, $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$.

for a contravariant functor, $\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$.

(4) $\mathcal{F}$ preserves identity morphisms: $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$, for every object $A \in \mathscr{C}$.

Definition 1.3.8. A **forgetful** functor is a functor $\mathcal{F}$ from a small category $\mathscr{C}$ to the category set **Set**,$\mathcal{F} : \mathscr{C} \longrightarrow$ **Set**, which sends objects in $\mathrm{Ob}(\mathscr{C})$ to their underlying sets and morphisms to their underlying functions, *forgetting* any additional structure.

Example 1.3.9. For any object A in a small category $\mathscr{C}$ we can define a functor $Hom(A, -) : \mathscr{C} \longrightarrow$ **Set** by sending B in $\mathscr{C}$ to $\mathcal{F}(B) = Hom_{\mathbf{Set}}(A, B)$, and sending $f : B \longrightarrow C$ to $\mathcal{F}(f) : Hom_{\mathbf{Set}}(A, B) \longrightarrow Hom_{\mathbf{Set}}(A, C)$ by defining $\mathcal{F}(f)(g) = f \circ g$.

Definition 1.3.10. [Terminal, Initial, and Zero Objects]

Let $\mathscr{C}$ be a category. An object T in $\mathscr{C}$ is called **terminal** if, for any object A, there is exactly one morphism $f : A \to T$.

An object S in $\mathscr{C}$ is called **initial** if, for any object A, there is exactly one morphism $f : S \to A$. An object 0 in $\mathscr{C}$ is called a **zero object** if it is both initial and terminal. In a category with a zero object we write $0_{AB} : A \to B$, or just 0 if it is unambiguous, for the unique morphism $A \to 0 \to B$.

Terminal, initial, and therefore zero objects are unique up to an isomorphism in a category.

Definition 1.3.11. A category $\mathscr{C}$ is called a **linear category** if:

i) $Hom_{\mathscr{C}}(A, B)$ is an abelian group, for all objects A and B in $\mathrm{Ob}\mathscr{C}$.

ii) $\circ : Hom_{\mathscr{C}}(B, C) \times Hom_{\mathscr{C}}(A, B) \longrightarrow Hom_{\mathscr{C}}(A, C)$ is bilinear that is

$$g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2 \ , \ (g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$$

for all $f, f_1, f_2 \in Hom_{\mathscr{C}}(A, B)$ and $g, g_1, g_2 \in Hom_{\mathscr{C}}(B, C)$.

Definition 1.3.12. Let $f : A \to B$ be a morphism in a category $\mathscr{C}$ with a zero object. We define a **kernel** of f to be a map $k : K \to A$ such that $f \circ k = 0_{KB}$; and for any other morphism $c : C \to A$ where $f \circ c = 0_{CB}$, there exists a unique morphism $g : C \to K$ such that the following diagram commutes:

$$
\begin{tikzcd}
& B & \\
A \arrow[ur, "f"] & K \arrow[l, "k"] \arrow[u, "0_{KB}"'] & \\
& & C \arrow[uul, bend right, "0_{CB}"'] \arrow[ull, bend left, "c"] \arrow[ul, dotted, "\exists! g"']
\end{tikzcd}
$$

If such a k exists, it is unique up to isomorphism and we write $k = ker f$.

Definition 1.3.13. Let $f : A \to B$ be a morphism in a category $\mathscr{C}$ with a zero object. We define a **cokernel** of f to be the map $c : B \to C$ such that $c \circ f = 0_{AC}$; and for any other morphism $q : B \to Q$ where $q \circ f = 0_{AQ}$, there exists a unique morphism $g : C \to Q$ such that the following diagram commutes:

$$
\begin{tikzcd}
& B \arrow[d, "c"] \arrow[ddr, bend left, "q"] & \\
A \arrow[ur, "f"] \arrow[r, "0_{AC}"'] \arrow[drr, bend right, "0_{AQ}"'] & C \arrow[dr, dotted, "\exists! g"] & \\
& & Q
\end{tikzcd}
$$

If such a c exists, it is unique up to isomorphism and we write $c = coker f$.

Definition 1.3.14. We say a category is **abelian** if:

i) It is a linear category.

ii) It has a zero object.

iii) It has all binary products and coproducts.

iv) It has all kernels and cokernels.

v) Any monomorphism is the kernel of a morphism, and any epimorphism is a cokernel of a morphism.

Alternatively, an abelian category is a category whose Hom-sets are abelian groups, that has kernels and cokernels for all concrete morphism, finite products and coproducts.

Definition 1.3.15. Let $\mathscr{C}$ be an abelian category, and let A be an object in $\mathscr{C}$. Then we define a **subobject** of A to be an equivalence class of monomorphisms $\{u : S \to A\}$

under the equivalence relation $\big(u : S \to A\big) \equiv \big(v : T \to A\big)$ if there exists an isomorphism $w : S \to T$ such that $u = v \circ w$.

$$\begin{tikzcd}
S \arrow[rr, dotted, "\exists w"] \arrow[dr, "u"'] & & T \arrow[dl, "v"] \\
& A &
\end{tikzcd}$$

We denote the collection of subobjects of A as $\mathbf{sub}A$. Often we abuse notation and write $B \subseteq A$ for the subobject $\big\{f : B \to A\big\}$.

Definition 1.3.16. Let $\mathscr{C}$ be an abelian category, and let A be an object in $\mathscr{C}$. Then we define a **quotient object** of A to be an equivalence class of epimorphisms $\big\{u : A \to S\big\}$ under the equivalence relation $\big(u : A \to S\big) \equiv \big(v : A \to T\big)$ if there exists an isomorphism $w : S \to T$ such that $v = w \circ u$.

$$\begin{tikzcd}
S \arrow[rr, dotted, "\exists w"] & & T \\
& A \arrow[ul, "u"] \arrow[ur, "v"'] &
\end{tikzcd}$$

We denote the collection of quotient objects of A as **quotA**.

1.3.2. Grothendieck Group

An abelian monoid is a non-empty set M with a binary operation $\diamond$ which has the following properties:

- $(M, \diamond)$ is closed.
- $(M, \diamond)$ is associative.
- $(M, \diamond)$ is commutative.
- $(M, \diamond)$ has a neutral element.

When we talk about a monoid, we often show its operation by $+$. A homomorphism between two monoid $(M, +_M)$ and $(N, +_N)$ is a function $T : M \to N$ such that

$$f(m +_M m') = f(m) +_N f(m').$$

Suppose that $(M, +)$ is an abelian monoid. Our goal in this section is to associate to every monoid M an abelian group $S(M)$ and a monoid homomorphism $s : M \to S(M)$ which is as small as possible in the following sense: for every abelian group G and monoid homomorphism

$f : M \to G$, there exists a unique group homomorphism $\overline{f} : S(M) \to G$ such that $\overline{f} \circ s = f$ as shown in the following diagram:

$$\begin{tikzcd} M \arrow[rr, "s"] \arrow[dr, "f"'] & & S(M) \arrow[dl, "\overline{f}"] \\ & G & \end{tikzcd}$$

For every $m \in M$, we define $\widehat{m} : M \to \mathbb{Z}$ by

$$\widehat{m}(x) = \begin{cases} 1 & : x = m \\ 0 & : x \neq m \end{cases} \tag{1}$$

We define

$$\mathcal{F}(M) := \mathbb{Z}\text{-module generated by} \Big\{ \widehat{m} : m \in M \Big\} \tag{2}$$

$$\mathcal{N}_M := \mathbb{Z}\text{-module generated by} \Big\{ \widehat{m +_M n} - \widehat{m} - \widehat{n} : m, n \in M \Big\} \tag{3}$$

Since $\mathcal{F}(M)$ is a free abelian group, $\mathcal{N}_M$ is a normal subgroup of $\mathcal{F}(M)$, hence we can form the quotient group $\frac{\mathcal{F}(M)}{\mathcal{N}_M}$ and we denote it by $S(M)$. We define $s : M \to S(M)$ by $s(m) := \widehat{m} + \mathcal{N}_M$. We have the following theorem:

Theorem 1.3.17. *Let M be an abelian monoid, G be an abelian group, and $f : M \to G$ be a monoid homomorphism. Then there is a unique group homomorphism $\overline{f} : S(M) \to G$ such that $\overline{f} \circ s = f$.*

$$\begin{tikzcd} M \arrow[rr, "s"] \arrow[dr, "f"'] & & S(M) \arrow[dl, "\overline{f}"] \\ & G & \end{tikzcd}$$

Proof. We know that every element in $S(M)$ is of the form $\sum\limits_{i \in I} \alpha_i s(m_i)$, where I is an arbitrary finite set. With this in mind, we define $\overline{f}(\sum\limits_{i \in I} \alpha_i s(m_i)) = \sum\limits_{i \in I} \alpha_i f(m_i)$. To show that this is well-defined, one has to prove that if $s(m) = s(n)$ then $f(m) = f(n)$. By definition, $s(m) = s(n)$ if and only if there exist $\mu, \sigma, \rho \in M$ such that $\mu = \sigma +_M \rho$ (that is $\widehat{\mu} - \widehat{\sigma} - \widehat{\rho} \in \mathcal{N}_M$)

and $m +_M \sigma +_M \rho = n +_M \mu$. Note that $\mu = \sigma +_M \rho$ implies $f(\mu) - f(\sigma) - f(\rho) = 0$, then

$$\begin{aligned} f(m) &= f(m) + f(\sigma) + f(\rho) - f(\sigma) - f(\rho) \\ &= f(m +_M \sigma +_M +\rho) - f(\sigma) - f(\rho) \\ &= f(m +_M \mu) - f(\sigma) - f(\rho) \\ &= f(n) + f(\mu) - f(\sigma) - f(\rho) = f(n) \end{aligned}$$

and $f(m) = f(n)$ as required. Also, it can be verified that $\overline{f} \circ s = f$. We show now that $\overline{f}$ is unique. Suppose that $g : S(M) \to G$ is a group homorphism such that $g \circ s = f$. We show that $g = \overline{f}$. We have:

$$\begin{aligned} g(\sum_{i\in I} \alpha_i s(m_i)) &= \sum_{i\in I} g(\alpha_i s(m_i)) \\ &= \sum_{i\in I} \alpha_i g(s(m_i)) \\ &= \sum_{i\in I} \alpha_i g \circ s(m_i) \\ &= \sum_{i\in I} \alpha_i f(m_i) \\ &= \sum_{i\in I} \alpha_i \overline{f}(s(m_i)) \\ &= \sum_{i\in I} \overline{f}(\alpha_i s(m_i)) \\ &= \overline{f}(\sum_{i\in I} \alpha_i s(m_i)) \end{aligned}$$

So we have $\overline{f} = g$. ■

Consider the cartesian product $M \times M$ of an abelian monoid $(M, +)$. We define the following equivalence relation $\sim$ on $M \times M$ by:

$$(m, n) \sim (m', n') \longleftrightarrow \exists p \in M \;\text{ such that }\; m + n' + p = m' + n + p.$$

It can be verified that $\sim$ is an equivalence relationship on $M \times M$. So we form the quotient space $M \times M/\sim := \Big\{[(m, n)] : (m, n) \in M \times M\Big\}$. We define the quotient map $\pi : M \to M \times M$ by: $\pi(m) = [(m, 0)]$.

Theorem 1.3.18. *For every abelian monoid M, $M \times M/\sim$ is an abelian group. Also, for every abelian group G and every monoid homomorphism $f : M \to G$, there exists a unique* *group homomorphism $\overline{f} : M \times M/\sim \to G$, such that $\overline{f} \circ \pi = f$, or equivalently the following diagram is commutative:*

$$\begin{tikzcd} M \arrow[rr, "\pi"] \arrow[dr, "f"'] & & M \times M/\sim \arrow[dl, "\overline{f}"] \\ & G & \end{tikzcd}$$

Proof. We define $[(m,n)] +_{(M\times M/\sim)} [(m',n')] := [(m +_M m', n +_M n')]$. We show that it is well-defined. Suppose

$$[(m_1,n_1)] = [(m_2,n_2)]$$
$$[(m'_1,n'_1)] = [(m'_2,n'_2)]$$

so there are $p, q \in M$ such that

$$m_1 + n_2 + p = m_2 + n_1 + p$$
$$m'_1 + n'_2 + q = m'_2 + n'_1 + q$$

which implies that

$$m_1 + m'_1 + n_2 + n'_2 + p + q = m_2 + m'_2 + n_1 + n'_1 + p + q$$

so we have

$$[(m_1,n_1)] +_{(M\times M/\sim)} [(m'_1,n'_1)] = [(m_2,n_2)] +_{(M\times M/\sim)} [(m'_2,n'_2)].$$

Also, it can be verified that $(M \times M/\sim, +_{M\times M/\sim})$ is a monoid with the neutral element $[(0,0)]$. Also the inverse element of $[(m,n)]$ is $[(n,m)]$, for all $m,n \in M$. Now for every abelian group G and a monoid homomorphism $f : M \to G$, we define $\overline{f} : M \times M/\sim \to G$ by $\overline{f}[(m,n)] = f(m) - f(n)$ so we have $\overline{f} \circ \pi = f$. We want to show that $\overline{f}$ is unique. Consider a group homomorphism $g : M \times M/\sim \to G$ such that $g \circ \pi = f$. We have

$$\begin{aligned} g([(m,n)]) &= g([(m,0)]) +_G g([(0,n)] \\ &= g([(m,0)]) +_G g(-[(n,0)]) \\ &= g([(m,0)]) - g([(n,0)]) \\ &= g \circ \pi(m) - g \circ \pi(n) \\ &= f(m) - f(n) = \overline{f}[(m,n)] \end{aligned}$$

so we have $g = \overline{f}$. ■

By 1.3.17, 1.3.18, and the universal mapping property of $M \times M/\sim$ and $S(M)$, we have $S(M) \underset{\mathbb{Z}}{\simeq} M \times M/\sim$, so we have the following definition:

Definition 1.3.19. Let M be a monoid, we define[7] $\mathcal{K}(M) := \mathcal{F}(M)/\mathcal{N}_M \underset{\mathbb{Z}}{\simeq} M \times M/\sim$ where $\mathcal{F}(M)$ and $\mathcal{N}_M$ are defined in (2) and (3), and call it the **Grothendieck Group** of M.

One of the fundamental example occurs in the case of an additive category. Suppose $\mathscr{C}$ is an additive category. For every $B \in Ob(\mathscr{C})$, we define the **isomorphism class of B** to be the set of all objects of $\mathcal{C}$ which are isomorphic to B. We denote it by $[B]$. Set $M := \Big\{[B] : B \in Ob(\mathscr{C})\Big\}$. We want to endow M with a monoid structure. In order to do it, we define $+ : M \times M \to M$ by $[B] + [B'] := [B \sqcup B']$, where $\sqcup$ means the direct sum of objects.

1.3.3. Cartan Matrix

Theorem 1.3.20. *Let $\mathcal{A}$ be a $\mathbb{K}$-algebra, $\mathcal{S} = \big\{S_i : 1 \leqslant i \leqslant n\big\}$ be the isomorphism classes of all simple $\mathcal{A}$-modules, and $\mathscr{C}_{fl}$ be the category of finite-length $\mathcal{A}$-modules. Then $\mathcal{K}(\mathscr{C}_{fl})$ is a free abelian group with basis $\big\{[S_i] : 1 \leqslant i \leqslant n\big\}$ and for each finite length $\mathcal{A}$-module M we have that $[M] = \sum_{i=1}^{n} l_{S_i}(M)[S_i]$ in $\mathcal{K}(\mathscr{C}_{fl})$, where $l_{S_i}(M)$ was defined by 1.2.24.*

Proof. See Theorem 1.7 in [2]. ■

So this theorem shows that if $\{S_i\}_{1 \leq i \leq n}$ is the set of isomorphism classes of simple $\mathcal{A}$-modules in $\mathscr{C}_{fl}$, then the Grothendieck group $\mathcal{K}(\mathscr{C}_{fl})$ is a free abelian group with basis elements

[7] the letter "K" comes from the German word "Klassen".

$[S_i]_{1\leq i\leq n}$. So for every $M \in Ob((\mathscr{C})_{fl})$, we can write $[M]$ as a linear combination of $[S_i]$'s, that is

$$[M] = \sum_{1\leq i\leq n} m_i(M, S_i)[S_i]$$

where the coefficients $m_i(M, S_i)$ are usually denoted by $[M : S_i]$. Each $[M : S_i]$ is the multiplicity of S_i in some composition series of M. The $[M : S_i]$ are well-defined by Jordan-Hölder theorem. This structure leads to an invariant of finite-dimensional associative $\mathbb{K}$-algebras: the Cartan Matrix.

Definition 1.3.21. The **Cartan matrix** of $\mathcal{A}$ is the $n \times n$ matrix $C = ([P_i : S_j])_{i,j=1}^n$ where $S_1, \cdots, S_n$ are the simple $\mathcal{A}$-modules and P_i is the projective cover of S_i.

Chapter 2

Quasi-Hereditary Algebras, Highest Weight Category

2.1. Quasi-Hereditary Algebra

Quasi-hereditary algebras are a class of finite-dimensional algebras which were first introduced by E. Cline, B. Parshall and L. Scott in order to deal with highest weight categories arising from the representations of semi-simple complex Lie algebras and algebraic groups. In the defnition below, we shall only consider $\mathbb{K}$-algebras $\mathcal{A}$ which are finitely generated as $\mathbb{K}$-modules,i.e. $\mathbb{K}$-finite algebras. We have two ways to define quasi-hereditary algebras. The first definition is based on the notion of hereditary ideals, and the second one is based on the crucial notion of standard modules which leads to the notion of highest weight category. Both definitions are equivalent [7, 10]. The reference [8] introduces quasi-hereditary algebras.

We first introduce the concept of hereditary ideal. It can be defined by three distinct definitions, whose equivalence is also shown in [8].

Definition 2.1.1. A non-zero (two-sided) ideal J in a $\mathbb{K}$-algebra $\mathcal{A}$ is called a **hereditary ideal** of $\mathcal{A}$ if the three following conditions hold:

(i) J is an idempotent ideal, that is, $\mathcal{A}$ possesses an idempotent e such that $J = \mathcal{A}e\mathcal{A}$.

(ii) J is projective as a left $\mathcal{A}$-module.

(iii) The $\mathbb{K}$-algebra $End_{\mathcal{A}}({}_{\mathcal{A}}J)$ is semi-simple.

Definition 2.1.2. A non-zero (two-sided) ideal J in a $\mathbb{K}$-algebra $\mathcal{A}$ is called a **hereditary ideal** of $\mathcal{A}$ if the three following conditions hold:

(i) J is projective as a left $\mathcal{A}$-module.

(ii') J is idempotent, that is, $\mathcal{A}$ possesses an idempotent e such that $J = \mathcal{A}e\mathcal{A}$.

(ii) $J(Rad\mathcal{A})J = 0$.

Definition 2.1.3. A non-zero (two-sided) ideal J in a $\mathbb{K}$-algebra $\mathcal{A}$ is called a **hereditary ideal** of $\mathcal{A}$ if the three following conditions hold:

(i') The surjective multiplication map $\mathcal{A}e \underset{e\mathcal{A}e}{\otimes} e\mathcal{A} \longrightarrow J$ is bijective for any idempotent e satisfying $\mathcal{A}e\mathcal{A} = J$.

(ii') J is idempotent, that is, $\mathcal{A}$ possesses an idempotent e such that $J = \mathcal{A}e\mathcal{A}$.

(iii) $J(Rad\mathcal{A})J = 0$.

Here is now the first definition of a quasi-hereditary algebra.

Definition 2.1.4. The finite-dimensional unital $\mathbb{K}$-algebra $\mathcal{A}$ is called **quasi-hereditary** if there is a chain of (two-sided) ideals of $\mathcal{A}$

$$0 = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n = \mathcal{A}$$

such that for any $m \in \{1, \cdots, n\}$, J_m/J_{m-1} is a hereditary ideal of $\mathcal{A}/J_{m-1}$.

Remark 2.1.5. Here is a quick history of quasi-hereditary algebras. The original inspiration for quasi hereditary algebras comes from algebraic geometry. The category $\mathcal{P}$ of perverse sheaves on a stratified topological space X is an abelian subcategory of a suitable derived category of constructible sheaves on X. As discussed in [3], $\mathcal{P}$ admits a recursive construction in it which is built up from constant sheaves on a strata of X. Similarly, a quasi-hereditary algebra is a finite-dimensional algebra $\mathcal{A}$ built up from semisimple algebras. We do not pursue the perverse sheaves approach here, but it can be shown that $\mathcal{P} \simeq \mathcal{A}$-mod, for a quasi-hereditary algebra $\mathcal{A}$. For more details and information, see [20, 21].

Remark 2.1.6. The word "inherit" means to receive properties, and in the case of quasi-hereditary algebras, it means to pass down the property of projectiveness from an algebra to its hereditary ideals.

Now we give a second definition of quasi-hereditary algebra whose equivalence with the first is proved [7, 10].

Definition 2.1.7. Let $\mathcal{A}$ be a finite-dimensional $\mathbb{K}$-algebra, $(I, \leqslant)$ a partially-ordered set indexing all non-isomorphic classes $\left\{S_i \, : \, i \in I\right\}$ of simple $\mathcal{A}$-modules. Then $(\mathcal{A}, I, \leqslant)$ is called **quasi-hereditary algebra** if the following assertions hold:

(i) For each $i \in I$, there exists a finite-dimensional $\mathcal{A}$-module $\Delta(i)$ with an epimorphism $\Delta(i) \to S(i)$ such that the composition factors of the kernel satisfy $j \lneq i$.

(ii) For each $i \in I$, a projective cover $P(i)$ of $S(i)$ maps onto $\Delta(i)$ such that the kernels has a finite filtration with factors $\Delta(j)$ satisfying $j \gneq i$.

The module $\Delta(i)$ is called **standard module** of index i.

2.2. Highest Weight Categories

Definition 2.2.1. Given an object A in an abelian category, one can define a partial order on the collection of subobjects of A, $subA$. For $u, v \in subA$ define $u \preceq v$ if there exists a morphism w such that $u = v \circ w$. Then $\preceq$ defines a partial order structure on $subA$, which we will call the **subobject lattice** of A.

Definition 2.2.2. Let $\mathscr{C}$ be a category that has a zero object 0, and A be a non-zero object in $\mathscr{C}$. We call A a **simple object** if its only subobjects are 0 or A up to isomorphisms. Additionally, we say that a subobject is simple if its domain is simple and, similarly, that a quotient object is simple if its codomain is.

Definition 2.2.3. For an object A in an abelian category $\mathscr{C}$ with a zero object, a composition series of A is a finite sequence of subobjects:

$$A = A_n \supsetneq A_{n-1} \supsetneq \cdots \supsetneq A_1 = 0$$

such that each $coker(A_i \overset{J_i}{\hookrightarrow} A_{i+1}) = A_{i+1}/A_i$, called a composition factor, is simple. If an object A has a composition series we say it is of finite length.

Definition 2.2.4. If S is a simple object and A is an object of finite length, we define the **composition multiplicity**, or simply **multiplicity**, $[A : S]$ to be the number of composition factors of A that are isomorphic to S. (Due to the Jordan-Hölder theorem, this is well-defined.)

Definition 2.2.5. For a given category $\mathscr{C}$, we call $P \in \mathscr{C}$ a **projective object** if, for every epimorphism $\phi : B \twoheadrightarrow A$ and morphism $f : P \to A$, there exists $g : P \to B$ such that $\phi \circ g = f$, as shown in the following commutative diagram:

$$\begin{tikzcd}
A & P \arrow[l, "f"'] \arrow[dl, dotted, "g"] \\
B \arrow[u, two heads, "\phi"] &
\end{tikzcd}$$

Definition 2.2.6. We call a monomorphism $f : A \to B$ **essential** if, for any $g : B \to C$, $g \circ f$ is a monomorphism only if g is.

Definition 2.2.7. [$\mathbb{K}$-linear categories]

We say a category $\mathscr{C}$ is $\mathbb{K}$-linear for a field $\mathbb{K}$ if: For every $A, B \in \mathscr{C}$, $Hom_{\mathscr{C}}(A, B)$ has the structure of a vector space over $\mathbb{K}$; and if composition of morphisms $\circ : Hom_{\mathscr{C}}(B, C) \times Hom_{\mathscr{C}}(A, B) \to Hom_{\mathscr{C}}(A, C)$ is a bilinear mapping.

Definition 2.2.8. Let $\mathscr{C}$ be a finite $\mathbb{K}$-linear category and $\{\Delta(\lambda), \lambda \in \Lambda\}$ a set of objects of $\mathscr{C}$ indexed by Λ. An object $M \in Ob(\mathscr{C})$ has a Δ-filtration if it has a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$$

such that each quotient $M_i/M_{i-1}, 1 \leq i \leq n$, is isomorphic to one of the objects $\Delta(\lambda)$.

We now give the definition of highest weight category as it appears in [**11**].

Definition 2.2.9. A **highest weight category** $\mathscr{C}$ is a finite $\mathbb{K}$-linear category defined on a weight partially-ordered set $(\Lambda, \leq)$ such that $\mathscr{C}$ is satisfying:

(HWC)(1) The non-isomorphic simple objects in $\mathscr{C}$ are indexed as $L(\lambda)$, $\lambda \in \Lambda$.

(HWC)(2) For any $\lambda \in \Lambda$, there is a *standard* object $\Delta(\lambda) \in \mathscr{C}$ such as $Top(\Delta(\lambda)) = L(\lambda)$, and whose composition factors are of the form $L(\mu)$ with $\mu \leq \lambda$, and moreover, $L(\lambda)$ occurs with multiplicity one.

(HWC)(3) The projective cover $P(\lambda)$ of $L(\lambda)$ has a Δ-filtration and $\Delta(\lambda)$ occurs with multiplicity one.

Now, we give a very fundamental theorem which states the relation between highest weight categories and quasi-hereditary algebras. The proof is that of Parshall [**22**].

Theorem 2.2.10. *Suppose $\mathcal{A}$ is a $\mathbb{K}$-algebra. Then $\mathcal{A}$ is a quasi-hereditary algebra if and only if Mod-$\mathcal{A}$ is a highest weight category relative to some partial order $\preceq$ on Λ and choice of standard objects.*

Proof. Suppose $(\mathcal{A}, \Lambda, \leq)$ is a quasi-hereditary algebra with the sequence $0 = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n = \mathcal{A}$. For each i, consider the distinct indecomposable summands of the projective $\mathcal{A}/J_{i-1}$-module J_i/J_{i-1}. By theorem 1.2.61 any such projective module is identified by its top. Let $\Lambda_i \subset \Lambda$ be the set of λ's such that $\Delta(\lambda)$ is an indecomposable module of J_i/J_{i-1}. If $i \neq j$, we claim that $\Lambda_i \cap \Lambda_j = \emptyset$. In order to prove it, suppose that $i \lneq j$ and $\lambda \in \Lambda_i \subset \Lambda_j$. Because $\mathcal{A}/J_{j-1}$ is a homomorphic image of $\mathcal{A}/J_{i-1}$, and $Top(\Delta_j(\lambda)) = L(\lambda)$, so we deduce that $\Delta_j(\lambda)$ is a homomorphic image of the projective $\mathcal{A}/J_{i-1}$-module $\Delta_i(\lambda)$. So, there exists a nonzero $\mathcal{A}/J_{i-1}$-homomorphism $J_i/J_{i-1} \to J_j/J_{j-1}$, which is a contradiction, because J_i/J_{i-1} is an idempotent ideal in $\mathcal{A}/J_{i-1}$ and $J_i \subset J_{j-1}$. It proves our claim. Also, any irreducible $\mathcal{A}$-module $L(\lambda)$ is a homomorphic image of some J_i/J_{i-1}, so is a homomorphic image of some J_i/J_{i-1}, hence of some $\Delta_i(\lambda)$. Thus Λ is a disjoint union of the Λ_i. So we proved that for every $\lambda \in \Lambda$, there is a unique i such that $\lambda \in \Lambda_i$ and we put $\Delta(\lambda) = \Delta(\lambda_i)$. So we can continue this process to find all the standards objects in Mod-$\mathcal{A}$. For every λ and μ in Λ, we define the partial order $\lambda \leq \mu$ if and only if $\lambda \in \Lambda_i$ and $\mu \in \Lambda_j$, for some $j \leq i$. We want to verify all the axioms in the definition of a highest weight category. First, by theorem 1.2.61 $Top(\Delta(\lambda)) = L(\lambda)$, for $\lambda \in \Lambda_j$. Now, suppose that $S(\mu)$ is a composition factor of $Rad_{\mathcal{A}}(\Delta(\lambda))$. With the same reasoning, if $i \lneq j$, there exists a nonzero $\mathcal{A}/J_{i-1}$-homomorphism $J_i/J_{i-1} \to J_j/J_{j-1}$ by the projectivity of the $\mathcal{A}/J_{i-1}$-module J_i/J_{i-1}. As we showed above, the existence of such a homomorphism is impossible. Thus, we must have $j \leq i$. So, we have two cases: $i = j$ or $i > j$. If $i = j$, there is a nonzero morphism $J_i/J_{i-1} \xrightarrow{f} Rad_{\mathcal{A}/J_{i-1}}(J_i/J_{i-1})$. Hence, we have

$$f(J_i/J_{i-1}) = f((J_i/J_{i-1})^2) = f\big(J_i/J_{i-1}\big)(J_i/J_{i-1}) \subseteq Rad_{\mathcal{A}/J_{i-1}}(J_i/J_{i-1})(J_i/J_{i-1}) \overset{1.2.47(2)}{=} (J_i/J_{i-1})Rad(\mathcal{A})(J_i/J_{i-1}) \overset{2.1.2(ii)}{=} 0$$

which is a contradiction, so we have $i > j$. Now, we want to show that every projective cover $P(\lambda)$ has a Δ-filtration. By 1.2.61(2), there is some primitive idempotent $e \in \mathcal{A}$ such that $P(\lambda) = e\mathcal{A}$. For every i, we have $(J_i/J_{i-1})e \simeq J_ie/J_{i-1}e$ is a direct sum of various $\Delta(\nu)$, for $\nu \in \Lambda_i$. Now suppose that $\lambda \in \Lambda_i$. We know that, by definition, $\Delta(\lambda)$ is an $\mathcal{A}/J_{i-1}$-direct summand of J_i/J_{i-1}, so there is a surjective $\mathcal{A}$-module epimorphism $J_i \to \Delta(\lambda) \to 0$. Since J_i is an idempotent ideal, we have $\Delta(\lambda)J_i = \Delta(\lambda)$, so $L(\lambda)J_i = L(\lambda)$. Hence, the projective covering epimorphism $P(\lambda) \to L(\lambda)$ can be restricted to an epimorphism $P(\lambda)J_i \to L(\lambda)$, which means that $P(\lambda)J_i = P(\lambda)$. On the other hand, if $P(\lambda)J_{i-1} = P(\lambda)$, then $L(\lambda)J_{i-1} = L(\lambda)$ and so $Hom_{\mathcal{A}}(J_{i-1}, L(\lambda)) \neq 0$, and so $\lambda \in \Lambda_i \cap \Lambda_j$, which is a contradiction. Thus $P(\lambda)J_{i-1}$ is a proper submodule of $P(\lambda)$, and so $Top(P(\lambda)/P(\lambda)J_{i-1}) = L(\lambda)$. It follows that $P(\lambda)/P(\lambda)J_{i-1} \simeq P(\lambda)J_i/P(\lambda)J_{i-1} \simeq \Delta(\lambda)$. Thus, the sequence

$$0 = P(\lambda)J_0 \subseteq P(\lambda)J_1 \subseteq P(\lambda)J_2 \subseteq \cdots \subseteq P(\lambda)J_n = P(\lambda)\mathcal{A} = P(\lambda)$$

defines a Δ-filtration of $P(\lambda)$ with $Top(P(\lambda)) = \Delta(\lambda)$, and $P(\lambda)J_j \simeq \Delta(\mu)$ with $j \leq i-1$, for some $\mu \in \Lambda_1 \cup \ldots \Lambda_{i-1}$, that is with $\mu > \lambda$. We have shown that mod-$\mathcal{A}$ is a highest weight category with the partially-ordered set $(\Lambda, \leq)$.

Conversely, suppose that mod-$\mathcal{A}$ is a highest weight category, for the partially-ordered set $(\Lambda, \leq)$. Let $\lambda \in \Lambda$ be a maximal element, thus $\Delta(\lambda) = P(\lambda)$ is a projective $\mathcal{A}$-module. Put $J_1 := \sum\limits_{\phi} \phi(\Delta(\lambda))$ where the sum runs over all $\phi \in Hom_{\mathcal{A}}(\Delta(\lambda), \mathcal{A})$. In the category mod-$\mathcal{A}$ we have

$$\mathcal{A} \simeq \bigoplus_{\mu \in \Lambda} P(\mu)^{\oplus \ dim \ L(\mu)}$$

Since $\Delta(\lambda)$ is a projective module, the filtration of $P(\mu)$ with $\mu \neq \lambda$

$$0 = P_0^{\mu} \subseteq P_1^{\mu} \subseteq P_2^{\mu} \subseteq \cdots \subseteq P_n^{\mu} = P(\mu)$$

that exists by 2.2.9(3) can be chosen such that all the quotients P_i^{μ}/P_{i-1}^{μ} isomorphic to $\Delta(\lambda)$ appear at the bottom of the filtration. In other words, there exists an index i_0 (depending on μ) such that $P_{i_0}^{\mu}$ is a direct sum of copies of $\Delta(\lambda)$ and $\Delta(\lambda)$ does not appear as a quotient in the filtration induced on $P(\mu)/P_{i_0}^{\mu}$ by the main filtration. Now 2.2.9(1) implies that $Hom_{\mathcal{A}}(\Delta(\lambda), \Delta(\mu)) = 0$ for $\mu \neq \lambda$. Therefore, $J_1 \cong \Delta^{\oplus m}$ for some integer m. Hence, J_1 is a projective left $\mathcal{A}$-module. Because $\Delta(\lambda)$ is a projective indecomposable $\mathcal{A}$-module, there is a primitive idempotent $e \in \mathcal{A}$ such that $\mathcal{A}e \cong \Delta(\lambda)$. This implies that $f \to f(e) = ef(e)$ defines an isomorphism $Hom_{\mathcal{A}}(\mathcal{A}e, \mathcal{A}) \cong e\mathcal{A}$ of vector space. Thus, $J_1 \cong \mathcal{A}e\mathcal{A}$, so $J_1^2 = J_1$. Finally, $e\mathcal{A}e \cong Hom_{\mathcal{A}}(\mathcal{A}e, \mathcal{A}e)^{op} \cong End_{\mathcal{A}}(L(\lambda))^{op}$ is semi-simple. Therefore, J_1 is a hereditary ideal. Thus, the $\Delta(\mu)$ with $\lambda \neq \mu$ can be regarded as $\mathcal{A}/J_1$-module. Similarly, the $L(\mu)$, $\mu \neq \lambda$, identify precisely with the irreducible $\mathcal{A}/J_1$-modules. If the filtration of $P(\mu)$ is adjusted as in the previous paragraph, then $J_1P(\mu)$ is that term of the filtration whose quotients are those precisely isomorphic to $\Delta(\lambda)$. (Namely, we have that $P_{i_0}^{\mu} = J_1P(\mu)$ in the notation of the previous paragraph.) Thus, $P(\mu)/J_1P(\mu)$ has a filtration with top section $\Delta(\mu)$ and lower sections $\Delta(\tau)$ for $\tau > u$ and $\tau \neq \lambda$. Also, it is immediate that $P(\mu)/J_1P(\mu)$ is the projective cover of $L(\mu)$ in $\mathcal{A}/J_1$-mod, We have therefore verified that $\mathcal{A}/J_1$-mod is a highest weight category with poset $\Lambda \setminus \{\lambda\}$ and standard objects $\Delta(\mu)$, $\mu \neq \lambda$. By induction on $\mid \Lambda \mid$, $\mathcal{A}/J_1$ is a quasi-hereditary algebra. If

$$0 \subseteq \overline{J}_2 \subseteq \cdots$$

is a defining sequence for $\mathcal{A}/J_1$, let J_i be the inverse images of the ideal $\overline{J}_i$ under the canonical quotient map $\mathcal{A} \longrightarrow \mathcal{A}/J_1$, for $i \geq 2$. Then the sequence

$$0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_t = \mathcal{A}$$

is a defining sequence for $\mathcal{A}$, proving that $\mathcal{A}$ is quasi-hereditary algebra. $\blacksquare$

This theorem leaves open the question of whether a highest weight category is always the module category of some quasi-hereditary algebra.

Chapter 3

Cellular Algebras

One of the central problems in the representation theory of finite groups and finite-dimensional algebras is to determine the non-isomorphic simple modules. One of the strengths of the theory of cellular algebras is that it provides a complete list of their absolutely irreducible modules over a field. In this chapter all the algebras are finite-dimensional unital algebras over a field. The definitions and main results on cellular algebras are due to Graham and Lehrer [12].

3.1. Abstract Cellular Algebra

From what follows, we denote $Sp_{\mathbb{K}}M$ to be the $\mathbb{K}$-linear space generated by the set M.

Definition 3.1.1. Suppose $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ is a $\mathbb{K}$-algebra, Λ is a finite indexed set, and $\mathcal{B} = \{B_\lambda : \lambda \in \Lambda\}$ is a family of disjoint subsets of $\mathcal{A}$ indexed by Λ such that $\bigsqcup\limits_{\lambda\in\Lambda} B_\lambda$ is a basis of $\mathcal{A}$, that is $\mathcal{A} = \bigoplus\limits_{\lambda\in\Lambda} Sp_{\mathbb{K}}B_\lambda$. For every subset Γ of Λ, we define $\mathcal{A}(\Gamma) := Sp_{\mathbb{K}}\big(\bigsqcup\limits_{\gamma\in\Gamma} B_\gamma\big) := \bigoplus\limits_{\gamma\in\Gamma} Sp_{\mathbb{K}}B_\gamma$, and we define for convenience $\mathcal{A}(\varnothing) := 0$, and we say $\mathcal{A}(\Gamma)$ is **fibered by** $\{B_\gamma : \gamma \in \Gamma\}$ and that $\mathcal{A}$ has a **fibered basis $\mathcal{B}$ over** Λ.

Definition 3.1.2. Let $(\Lambda, \leq)$ be a partially-ordered set. For every $\lambda \in \Lambda$, we define

(1) $I_{\leq\lambda} = \{\mu \in \Lambda : \mu \leq \lambda\}$

(2) $I_{<\lambda} = \{\mu \in \Lambda : \mu < \lambda\}$

Notation 3.1.3. We note

(1) $\mathcal{A}(\leq \lambda) := \mathcal{A}(I_{\leq\lambda}) = \bigoplus\limits_{\mu\in I_{\leq\lambda}} Sp_{\mathbb{K}}B_\mu = \bigoplus\limits_{\mu\leq\lambda} Sp_{\mathbb{K}}B_\mu$

(2) $\mathcal{A}(< \lambda) := \mathcal{A}(I_{<\lambda}) = \bigoplus\limits_{\mu\in I_{<\lambda}} Sp_{\mathbb{K}}B_\mu = \bigoplus\limits_{\mu<\lambda} Sp_{\mathbb{K}}B_\mu$

Definition 3.1.4. Let $(\Lambda, \leq)$ be a partially-ordered set. A subset $\Gamma \subset \Lambda$ is called an **ideal** of Λ if $\Gamma = \bigcup\limits_{\lambda \in \Gamma} I_{\leq \lambda}$.

Theorem 3.1.5. *Suppose $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ is a $\mathbb{K}$-algebra, $(\Lambda, \leq)$ is a partially-ordered finite indexed set, I and J are two subsets of Λ, and $\mathcal{B} = \{B_\lambda : \lambda \in \Lambda\}$ is a family of subsets of $\mathcal{A}$ indexed by Λ. Then*

(1) $\mathcal{A}(I \cup J) = \mathcal{A}(I) +_{\mathcal{A}} \mathcal{A}(J)$.

(2) $\mathcal{A}(I \cap J) = \mathcal{A}(I) \cap \mathcal{A}(J)$.

(3) If $J \subseteq I$, then $\mathcal{A}(I \setminus J) \underset{\mathbb{K}}{\simeq} \dfrac{\mathcal{A}(I)}{\mathcal{A}(J)}$.

(4) $\mathcal{A}(\lambda) \underset{\mathbb{K}}{\simeq} \dfrac{\mathcal{A}(\leq \lambda)}{\mathcal{A}(< \lambda)}$, for every $\lambda \in \Lambda$.

(5) If $J \subset I$, then $\mathcal{A}(J) \subset \mathcal{A}(I)$.

Proof. (1) We know that $\bigsqcup\limits_{\lambda \in I \cup J} B_\lambda = \bigsqcup\limits_{\lambda \in I} B_\lambda \cup \bigsqcup\limits_{\lambda \in J} B_\lambda$, so

$$Sp_{\mathbb{K}}\Big(\bigsqcup_{\lambda \in I \cup J} B_\lambda\Big) = Sp_{\mathbb{K}}\Big(\bigsqcup_{\lambda \in I} B_\lambda \cup \bigsqcup_{\lambda \in J} B_\lambda\Big) = Sp_{\mathbb{K}}\Big(\bigsqcup_{\lambda \in I} B_\lambda\Big) +_{\mathcal{A}} Sp_{\mathbb{K}}\Big(\bigsqcup_{\lambda \in J} B_\lambda\Big)$$

which proves (1).

(2) We know that $\bigsqcup\limits_{\lambda \in I \cap J} B_\lambda = \bigsqcup\limits_{\lambda \in I} B_\lambda \cap \bigsqcup\limits_{\lambda \in J} B_\lambda$ So

$$Sp_{\mathbb{K}}\Big(\bigsqcup_{\lambda \in I \cap J} B_\lambda\Big) = Sp_{\mathbb{K}}\Big(\bigsqcup_{\lambda \in I} B_\lambda \cap \bigsqcup_{\lambda \in J} B_\lambda\Big) = Sp_{\mathbb{K}}\Big(\bigsqcup_{\lambda \in I} B_\lambda\Big) \cap Sp_{\mathbb{K}}\Big(\bigsqcup_{\lambda \in J} B_\lambda\Big)$$

which proves (2).

(3) We know that $J \cup (I \setminus J) = I$, so $\mathcal{A}(I) = \mathcal{A}(J \cup I \setminus J)$ and :

$$\mathcal{A}(I) = \mathcal{A}(J \cup I \setminus J) \overset{(1)}{=} \mathcal{A}(J) +_{\mathcal{A}} \mathcal{A}(I \setminus J)$$

$$\mathcal{A}(J) \cap \mathcal{A}(I \setminus J) \overset{(2)}{=} \mathcal{A}\Big(J \cap I \setminus J\Big) = \mathcal{A}(\varnothing) = 0_{\mathcal{A}}$$

Hence,

$$\frac{\mathcal{A}(I)}{\mathcal{A}(J)} = \frac{\mathcal{A}(J) +_{\mathcal{A}} \mathcal{A}(I \setminus J)}{\mathcal{A}(J)} \underset{\mathbb{K}}{\simeq} \frac{\mathcal{A}(I \setminus J)}{\mathcal{A}(J) \cap \mathcal{A}(I \setminus J)} \underset{\mathbb{K}}{\simeq} \mathcal{A}(I \setminus J)$$

.

(4) Choose $I = I_{\leq \lambda}$ and $J = I_{< \lambda}$ and apply (3).

(5) It is easily concluded from the definition 3.1.1.

■

Definition 3.1.6. Let $(\Lambda, \leq)$ be a partially-ordered set and $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be a fibered basis $\mathbb{K}$-algebra with fibered basis $\mathcal{B} := \left\{B_\lambda : \lambda \in \Lambda\right\}$ over Λ. We say that $\mathcal{A}$ is an **abstract cellular algebra** if for every $\lambda \in \Lambda$ we have:

$$\cdot_{\mathcal{A}} : \mathcal{A} \times B_\lambda \to \mathcal{A}(\leq \lambda) \quad , \quad \cdot_{\mathcal{A}} : B_\lambda \times \mathcal{A} \to \mathcal{A}(\leq \lambda)$$

The partial order $\leq$ on Λ is used to define a pre-order $\leq^t_?$, for any $? \in \{L, R, LR\}$, on elements of $B = \sqcup B_\lambda$. This is done by the technique described in section 1.2.10. Even though the symbol $\leq^t_?$ is used, it is crucial to remember that this is not in general a partial order, but only a pre-order.

Theorem 3.1.7. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be an abstract cellular algebra over the partially-ordered set $(\Lambda, \leq)$ with fibered basis $\mathcal{B} := \left\{B_\lambda : \lambda \in \Lambda\right\}$. Then for every $b \in B_\lambda, b' \in B_\mu$, $? \in \left\{L, R, LR\right\}$,we have :*
(1) $b \leqslant_? b'$ implies $\mu \leq \lambda$.
(2) $b \leqslant^t_? b'$ implies $\mu \leq \lambda$.
(3) $b \sim_? b' \iff \lambda = \mu$.
(4) $\mathrm{C}^b_L = \left\{b' \in B_\lambda : b \sim_L b'\right\}$
(5) $\mathrm{C}^b_R = \left\{b' \in B_\lambda : b \sim_R b'\right\}$
(6) $\mathrm{C}^b_{LR} = \left\{b' \in B_\lambda : b \sim_{LR} b'\right\}$
(7) $B_\lambda = \bigcup\limits_{b \in B_\lambda} \mathrm{C}^b_{LR}$.

Proof. (1). We give the proof for ? = L, that is $b \leq_L b'$

$$b \leq_L b' \iff \exists b'' \in \mathcal{B} : \alpha_{b''bb'} \neq 0$$

By 3.1.6, we have : $b''b \in \mathcal{A}(\leq \lambda)$, that is, $b''b$ can be expanded as a linear combination of the elements in B_β, for $\beta \leq \lambda$. Hence, the condition $\alpha_{b''bb'} \neq 0$ states that the coefficient of the component in $b' \in B_\mu$ in the expansion is nonzero which implies that $\mu \leq \lambda$.

(2). A straight application of (1) in 1.2.64,1.2.66,1.2.68 implies (2), for $? \in \left\{L, R, LR\right\}$, respectively.

(3). (2) implies (3).

(4). (2) and (3) implies (4) for ? = L.

(5). (2) and (3) implies (5) for ? = R.

(6). (2) and (3) implies (6) for ? = LR.

(7). (4),(5), and (6) imply (7), for $? \in \Big\{L, R, LR\Big\}$.

■

Question 3.1.8. *Recall that 3.1.1 says that for every subset* Γ *of* Λ*,* $\mathcal{A}(\Gamma)$ *is fibered by* $\big\{B_\gamma : \gamma \in \Gamma\big\}$*, and for every* B_γ *we know by 3.1.6* $\mathcal{A} \cdot_{\mathcal{A}} B_\gamma \subseteq \mathcal{A}(\leq \gamma)$*. So we can extend the conclusion for the multiplication* $\cdot_{\mathcal{A}}$ *from a single set* B_γ *into that for elements of* $\bigsqcup_{\gamma \in \Gamma} B_\gamma$ *just by the definition of direct sum, that is* $a \cdot_{\mathcal{A}} \sum_{\gamma \in \Lambda} b_\gamma = \sum_{\gamma \in \Lambda} a \cdot_{\mathcal{A}} b_\gamma$ *. We know that* $a \cdot_{\mathcal{A}} b_\gamma \in \mathcal{A}(\leq \gamma)$*, for every* $\gamma \in \Gamma$*. The question which comes to mind is what can be said of* $a \cdot_{\mathcal{A}} b_\gamma$ *for* $\gamma \in \Gamma$*? For example,* $\mathcal{A} \cdot_{\mathcal{A}} \mathcal{A}(\Gamma) \subseteq \mathcal{A}(\Gamma)$*? and* $\mathcal{A}(\Gamma) \cdot_{\mathcal{A}} \mathcal{A} \subseteq \mathcal{A}(\Gamma)$*?*
The following theorem answers our question.

Theorem 3.1.9. *Let* $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ *be an abstract cellular algebra over the partially-ordered set* $(\Lambda, \leq)$ *with fibered basis* $\mathcal{B} := \big\{B_\lambda : \lambda \in \Lambda\big\}$*. Then for every subset* $\Gamma \subseteq \Lambda$*, if* Γ *is an ideal of* Λ*, then* $\mathcal{A}(\Gamma)$ *is an ideal of* $\mathcal{A}$*.* [1]

Proof. Suppose Γ is an ideal of Λ, that is, $\Gamma = \bigcup_{\gamma \in \Gamma} I_{\leq \gamma}$. So, for every $\gamma \in \Gamma$ we have $I_{\leq \gamma} \subseteq \Gamma$ which implies that $\mathcal{A}(\leq \gamma) \subseteq \mathcal{A}(\Gamma)$. So we have

$$\cdot_{\mathcal{A}} : \mathcal{A} \times \mathcal{A}(\leq \gamma) \to \mathcal{A}(\leq \gamma) \subset \mathcal{A}(\Gamma).$$

and similarly for the right multiplication. Hence, $\mathcal{A}(\Gamma)$ is an ideal in $\mathcal{A}$. ■

3.2. Anti-Involutive Abstract Cellular Algebra

Definition 3.2.1. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be an $\mathbb{K}$-algebra, and $f : \mathcal{A} \longrightarrow \mathcal{A}$ such that $f^2 = f$. Then a non-empty subset X of $\mathcal{A}$ is called **f-invariant** if $f(X) = X$ or equivalently, $f(X) \subseteq X$.

Definition 3.2.2. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be an $\mathbb{K}$-algebra. We say that $\mathfrak{i} : \mathcal{A} \to \mathcal{A}$ is an *involution* if

1. $\mathfrak{i}(\mathfrak{i}(a)) = a$, for all $a \in \mathcal{A}$.
2. $\mathfrak{i}(a +_{\mathcal{A}} b) = \mathfrak{i}(a) +_{\mathcal{A}} \mathfrak{i}(b)$ for all $a \in \mathcal{A}, b \in \mathcal{A}$.
3. $\mathfrak{i}(\lambda a) = \lambda \mathfrak{i}(a)$ for all $a \in \mathcal{A}, \lambda \in \mathbb{K}$.
4. $\mathfrak{i}(a \cdot_{\mathcal{A}} b) = \mathfrak{i}(a) \cdot_{\mathcal{A}} \mathfrak{i}(b)$ for all $a \in \mathcal{A}, b \in \mathcal{A}$.

If $\mathcal{A}$ admits such an involution, we say that $\mathcal{A}$ is an **involutive** $\mathbb{K}$-algebra.

[1]This theorem clarifies the origin of the definition of an ideal in a partially-ordered set with the notion of ideal in an algebra.

Definition 3.2.3. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be an $\mathbb{K}$-algebra. We say that $\mathfrak{i} : \mathcal{A} \to \mathcal{A}$ is an *anti-involution* if

(1) $\mathfrak{i}(\mathfrak{i}(a)) = a$, for all $a \in \mathcal{A}$.

(2) $\mathfrak{i}(a +_{\mathcal{A}} b) = \mathfrak{i}(a) +_{\mathcal{A}} \mathfrak{i}(b)$ for all $a \in \mathcal{A}, b \in \mathcal{A}$.

(3) $\mathfrak{i}(\lambda a) = \lambda \mathfrak{i}(a)$ for all $a \in \mathcal{A}, \lambda \in \mathbb{K}$.

(4) $\mathfrak{i}(a \cdot_{\mathcal{A}} b) = \mathfrak{i}(b) \cdot_{\mathcal{A}} \mathfrak{i}(a)$ for all $a \in \mathcal{A}, b \in \mathcal{A}$.

If $\mathcal{A}$ admits an anti-involution, we say that $\mathcal{A}$ is an anti-involutive $\mathbb{K}$-algebra.

We often use $*_{\mathcal{A}}$ instead of i when $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ is an i anti-involution on a $\mathbb{K}$-algebra $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$, and we denote it by $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}})$.

Definition 3.2.4. Suppose $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}})$ is an anti-involutive $\mathbb{K}$-algebra and $\mathcal{I}$ is a non-empty subset of $\mathcal{A}$. We say $\mathcal{I}$ is **anti-involutive invariant** if it is $*_{\mathcal{A}}-$**invariant**.

If $\mathcal{A}$ is an abstract cellular algebra over the partial order set $(\Lambda, \leq)$, we know that, when $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}})$ is an anti-involutive algebra, we have $a \cdot_{\mathcal{A}} b = (b^* \cdot_{\mathcal{A}} a^*)^*$, so we can choose to define of an abstract cellular algebra as:

Definition 3.2.5. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be an abstract cellular $\mathbb{K}$-algebra with fibered basis, $\mathcal{B} := \{B_\lambda : \lambda \in \Lambda\}$ over the partially-ordered set Λ. We say that $\mathcal{A}$ is an **anti-involutive abstract cellular algebra** if $\mathcal{A}(\leq \lambda)$ is $*_{\mathcal{A}}$-invariant, for every $\lambda \in \Lambda$.

Remark 3.2.6. If $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}})$ is an anti-involutive algebra, then $\mathcal{A}(\leq \lambda)$ is a two-sided ideal of $\mathcal{A}$, for every $\lambda \in \Lambda$. Hence, we can reduce the two properties

$$\cdot_{\mathcal{A}} : \mathcal{A} \times B_\lambda \to \mathcal{A}(\leq \lambda) \quad , \quad \cdot_{\mathcal{A}} : B_\lambda \times \mathcal{A} \to \mathcal{A}(\leq \lambda)$$

to the statement that $\mathcal{A}(\leq \lambda)$ is a two-sided ideal.

3.3. Standard Cellular Algebra

Now, the long-awaited time has come. Theorem 3.1.7 was the most powerful tool to understand the structure of $B = \bigsqcup_{\lambda \in \Lambda} B_\lambda$. We want to descend gradually from abstract cellular algebras to the land of standard cellular algebras. Our goal is to come nearer as far as possible to the fibered basis $\mathcal{B}$ as a means of achieving sufficient information about the algebra $\mathcal{A}$.

Definition 3.3.1. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be a $\mathbb{K}$-algebra with fibered basis $\mathcal{B} := \{B_\lambda : \lambda \in \Lambda\}$ over the partially-ordered set $(\Lambda, \leq)$. We say that the fibered basis $\mathcal{B}$ is **standard** if the following conditions hold:

(1) For any $\lambda \in \Lambda$ there are finite index sets I_λ and J_λ such that

$$B_\lambda = \{a_{ij}^\lambda : (i,j) \in I(\lambda) \times J(\lambda)\}$$

(2) For every $\lambda \in \Lambda$, $a \in \mathcal{A}$ and $a_{ij}^\lambda \in B_\lambda$, there exist $l_{a,\lambda}(i',i)$ and $r_{\lambda,a}(j',j) \in \mathbb{K}$ such that

$$a \cdot_{\mathcal{A}} a_{ij}^\lambda \overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{i' \in I(\lambda)} l_{a,\lambda}(i',i) a_{i'j}^\lambda \ ,$$

$$a_{ij}^\lambda \cdot_{\mathcal{A}} a \overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{j' \in J(\lambda)} r_{\lambda,a}(j',j) a_{ij'}^\lambda$$

Definition 3.3.2. A **standard cellular algebra** is a unital anti-involutive abstract cellular algebra $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}})$ with a standard fibered basis $\mathcal{B} := \{B_\lambda : \lambda \in \Lambda\}$ over the partially-ordered set Λ, where $B_\lambda := \{a_{ij}^\lambda : (i,j) \in I(\lambda) \times J(\lambda)\}$ and the set $\{a_{ij}^\lambda : (i,j) \in I(\lambda) \times J(\lambda)\}$ is a $\mathbb{K}$-basis for B_λ, for any $\lambda \in \Lambda$. We denote it for convenience by $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$.

Remark 3.3.3. Consider all the conditions of definition 3.3.2. Then for every $\lambda \in \Lambda$,

$$a_{ij}^\lambda = 1_{\mathcal{A}} \cdot_{\mathcal{A}} a_{ij}^\lambda \overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{i' \in I(\lambda)} l_{1_{\mathcal{A}},\lambda}(i',i) a_{i'j}^\lambda$$

which implies

$$l_{1_{\mathcal{A}},\lambda}(i',i) = \begin{cases} 1_{\mathbb{K}} & : i' = i \\ 0 & : i' \neq i \end{cases}$$

Similarly by 3.3.1(1)

$$r_{\lambda,1_{\mathcal{A}}}(j,j') = \begin{cases} 1_{\mathbb{K}} & : j' = j \\ 0 & : j' \neq j \end{cases}$$

Theorem 3.3.4. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra.Then we have:*

(1) $a_{ij}^\lambda \leqslant_L a_{rs}^\lambda \Longleftrightarrow j = s, \exists a \in \mathcal{A}; l_{a,\lambda}(r,i) \neq 0$

(2) $a_{ij}^\lambda \leqslant_L^t a_{rs}^\lambda \Longleftrightarrow j = s, \exists n \in \mathbb{N}, \exists a_1, \ldots, a_n \in \mathcal{A}, \exists i_1, \ldots, i_n \in I(\lambda)$ *such that*

$$l_{a_1,\lambda}(i_1,i) \neq 0, l_{a_2,\lambda}(i_2,i_1) \neq 0, \ldots, l_{a_n,\lambda}(r,i_n) \neq 0$$

(3) $a_{ij}^\lambda \leqslant_R a_{rs}^\lambda \Longleftrightarrow i = r, \exists a \in \mathcal{A}$ *such that* $r_{\lambda,a}(s,j) \neq 0$

(4) $a_{ij}^\lambda \leqslant_R^t a_{rs}^\lambda \Longleftrightarrow i = r, \exists m \in \mathbb{N}, \exists a_1, \ldots, a_m \in \mathcal{A}, \exists j_1, \ldots, j_m \in J(\lambda)$ *such that*

$$r_{\lambda,a_1}(j_1,j) \neq 0, r_{\lambda,a_2}(j_2,j_1) \neq 0, \ldots, r_{\lambda,a_m}(s,j_m) \neq 0$$

Proof. Because of the similarity between the proofs of (3) and (4) with those of (1) and (2), we prove only (1) and (2). For every $a' \in \mathcal{A}$, we know that

$$a' \cdot_{\mathcal{A}} a^{\lambda}_{ij} = \sum_{\substack{(r',s') \in I(\mu) \times J(\mu) \\ \mu \in \Lambda}} \alpha_{a' a^{\lambda}_{ij}}(r', s') a^{\mu}_{r's'}$$

By 3.1.6, we know that $a' \cdot_{\mathcal{A}} a^{\lambda}_{ij} \in \mathcal{A}(\leq \lambda)$. Hence, the above equation becomes:

$$a' \cdot_{\mathcal{A}} a^{\lambda}_{ij} = \sum_{\substack{(r',s') \in I(\mu) \times J(\mu) \\ \mu \leqslant \lambda}} \alpha_{a' a^{\lambda}_{ij}}(r', s') a^{\mu}_{r's'}$$

so in this case

$$a' \cdot_{\mathcal{A}} a^{\lambda}_{ij} \overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{(r',s') \in I(\lambda) \times J(\lambda)} \alpha_{a' a^{\lambda}_{ij}}(r', s') a^{\lambda}_{r's'}$$

So our hypothesis $a^{\lambda}_{ij} \leq_L a^{\lambda}_{rs}$ implies that there is $a \in \mathcal{A}$ such that $\alpha_{a a^{\lambda}_{ij}}(r, s) \neq 0$. Also by 3.3.2(2), we have

$$a \cdot_{\mathcal{A}} a^{\lambda}_{ij} \overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) a^{\lambda}_{i'j}$$

.

So combining these implies that

$$\sum_{(r',s') \in I(\lambda) \times J(\lambda)} \alpha_{a a^{\lambda}_{ij}}(r', s') a^{\lambda}_{r's'} \overset{mod(\mathcal{A}<\lambda)}{\equiv} a \cdot_{\mathcal{A}} a^{\lambda}_{ij} \overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) a^{\lambda}_{i'j}$$

Therefore,

$$\sum_{(r',s') \in I(\lambda) \times J(\lambda)} \alpha_{a' a^{\lambda}_{ij}}(r', s') a^{\lambda}_{r's'} \overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) a^{\lambda}_{i'j}$$

Since $\left\{ a^{\lambda}_{pq} : (p, q) \in I(\lambda) \times J(\lambda) \right\}$ is a $\mathbb{K}$-basis, so we have $s = s' = j$ and $l_{a,\lambda}(i', i) = \alpha_{a' a^{\lambda}_{ij}}(i', s)$ for every $i' \in I(\lambda)$. So for $i' = r$, we have $l_{a,\lambda}(r, i) \neq 0$.
The second part (2), is a direct application of (1) in 1.2.64. ■

Lemma 3.3.5. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. Then we have:*
(1) If $a^{\lambda}_{ij} \cdot_{\mathcal{A}} a^{\mu}_{rs} \overset{mod(\mathcal{A}<\mu)}{\not\equiv} 0$, then $\mu \leq \lambda$.
(2) If $a^{\lambda}_{ij} \cdot_{\mathcal{A}} a^{\mu}_{rs} \overset{mod(\mathcal{A}<\mu)}{\not\equiv} 0$, $a^{\lambda}_{ij} \cdot_{\mathcal{A}} a^{\mu}_{rs} \overset{mod(\mathcal{A}<\lambda)}{\not\equiv} 0$, then $\lambda = \mu$.

Proof. (1) We have:

$$a^{\lambda}_{ij} \cdot_{\mathcal{A}} a^{\mu}_{rs} \overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{j' \in J(\lambda)} r_{\lambda, a^{\lambda}_{rs}}(j'j) a^{\lambda}_{ij'}$$

or equivalenty

$$a^\lambda_{ij} \cdot_{\mathcal{A}} a^\mu_{rs} = \sum_{j' \in J(\lambda)} r_{\lambda, a^\lambda_{rs}}(j', j) a^\lambda_{ij'} +_{\mathcal{A}} \sum_{\substack{(s't') \in I(\nu) \times J(\nu) \\ \nu < \lambda}} \eta_{(s',t')} a^\nu_{s't'}. \tag{4}$$

We denote

$$\nu_\lambda = \sum_{j' \in J(\lambda)} r_{\lambda, a^\lambda_{rs}}(j', j) a^\lambda_{i,j'}$$

$$\nu_{<\lambda} = \sum_{\substack{(s',t') \in I(\nu) \times J(\nu) \\ \nu < \lambda}} \eta_{(s',t')} a^\nu_{s't'}.$$

So we have:

$$a^\lambda_{ij} \cdot_{\mathcal{A}} a^\mu_{rs} = \nu_\lambda + \nu_{<\lambda}$$

Hence if $\lambda < \mu$, then $\nu_\lambda + \nu_{<\lambda} \in \mathcal{A}(< \mu)$ which implies that

$$a^\lambda_{ij} \cdot_{\mathcal{A}} a^\mu_{rs} = \nu_\lambda + \nu_{<\lambda} \overset{mod(\mathcal{A}<\mu)}{\equiv} 0$$

which is a contradiction. So we must have $\mu \leq \lambda$.

(2) It follows from (1).

$\blacksquare$

Theorem 3.3.6. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. Then the following holds:*

(1) If $a^\lambda_{ij} \sim_L a^\mu_{rs}$, then $\lambda = \mu$, $j = s$.

(2) If $a^\lambda_{ij} \sim_R a^\mu_{rs}$, then $\lambda = \mu$, $i = r$.

Proof. (1) By 3.1.7(3), $\lambda = \mu$, by 3.3.4(2), we have $j = s$.

(2) By 3.1.7(3), $\lambda = \mu$, by 3.3.4(4), we have $i = r$.

$\blacksquare$

Remark 3.3.7. Now, we can describe precisely $\mathrm{C}_L^{a^\lambda_{ij}}$. We have

$$a^\lambda_{ij} \sim_L a^\lambda_{i'j} \longleftrightarrow a^\lambda_{ij} \leqslant^t_L a^\lambda_{i'j} \text{ and } a^\lambda_{i'j} \leqslant^t_L a^\lambda_{ij}$$

$$a^\lambda_{ij} \leqslant^t_L a^\lambda_{i'j} \overset{3.3.4(2)}{\longleftrightarrow} \exists n \in \mathbb{N}, a_1, \ldots, a_n \in \mathcal{A}, i_1, \ldots, i_n \in I(\lambda)$$

$$\text{such that } l_{a_1,\lambda}(i_1, i) \neq 0, l_{a_2,\lambda}(i_2, i_1) \neq 0, \ldots, l_{a_n,\lambda}(i', i_n) \neq 0$$

Similarly,

$$a^{\lambda}_{i'j} \leqslant^{t}_{L} a^{\lambda}_{ij} \overset{3.3.4(4)}{\Longleftrightarrow} \exists m \in \mathbb{N}, a'_1, \ldots, a'_m \in \mathcal{A}, i'_1, \ldots, i'_m \in I(\lambda)$$
$$\text{such that } l_{a'_1,\lambda}(i'_1, r) \neq 0, l_{a'_2,\lambda}(i'_2, i'_1) \neq 0, \ldots, l_{a'_m,\lambda}(i, i'_m) \neq 0$$

Hence,

$$\begin{aligned} \mathrm{C}_L^{a^{\lambda}_{ij}} = \Big\{ a^{\lambda}_{i'j} : i' \in I(\lambda), \exists n, m \in \mathbb{N}, \exists a_1, \ldots, a_n, a'_1, \ldots, a'_m \in \mathcal{A}, \exists i_1, \ldots, i_n, i'_1, \ldots, i'_m \in I(\lambda) \\ \text{such that } l_{a_1,\lambda}(i_1, i) \neq 0, l_{a_2,\lambda}(i_2, i_1) \neq 0, \ldots, l_{a_n,\lambda}(i', i_n) \neq 0, \\ l_{a'_1\lambda}(i'_1, r) \neq 0, l_{a'_2,\lambda}(i'_2, i'_1) \neq 0, \ldots, l_{a'_m,\lambda}(i, i'_m) \neq 0 \Big\} \end{aligned}$$

With the similar arguments for $\mathrm{C}_R^{a^{\lambda}_{ij}}$, we have:

$$\begin{aligned} \mathrm{C}_R^{a^{\lambda}_{ij}} = \Big\{ a^{\lambda}_{ij'} : j' \in J(\lambda), \exists t, s \in \mathbb{N}, \exists b_1, \ldots, b_t, b'_1, \ldots, b'_s \in \mathcal{A}, \exists j_1, \ldots, j_t, j'_1, \ldots, j'_s \in J(\lambda) \\ \text{such that } r_{\lambda,b_1}(j_1, j) \neq 0, r_{\lambda,b_2}(j_2, j_1) \neq 0, \ldots, r_{\lambda,b_t}(j', j_t) \neq 0, \\ r_{\lambda,b'_1}(j'_1, j') \neq 0, r_{\lambda,b'_2}(j'_2, j'_1) \neq 0, \ldots, r_{\lambda,b_t}(j, j'_s) \neq 0 \Big\} \end{aligned}$$

Finally, we know that $\mathrm{C}_{LR}^{a^{\lambda}_{ij}} = \mathrm{C}_L^{a^{\lambda}_{ij}} \cup \mathrm{C}_R^{a^{\lambda}_{ij}}$.

Proposition 3.3.8. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. We define*

$$\mathcal{B}_{I(\lambda),j} := \Big\{ a^{\lambda}_{i'j} : i' \in I(\lambda) \Big\}$$

$$\mathcal{B}_{i,J(\lambda)} := \Big\{ a^{\lambda}_{ij'} : j' \in J(\lambda) \Big\}$$

Then, we have:

(1) $\mathcal{B}_{I(\lambda),j} = \bigcup_{i' \in I(\lambda)} \mathrm{C}_L^{a^{\lambda}_{i'j}}$

(2) $\mathcal{B}_{i,J(\lambda)} = \bigcup_{j' \in J(\lambda)} \mathrm{C}_R^{a^{\lambda}_{ij'}}$

(3) $\mathcal{B}_{\lambda} = \bigcup_{i \in I(\lambda)} \mathcal{B}_{i,J(\lambda)} = \bigcup_{j \in J(\lambda)} \mathcal{B}_{I(\lambda),j}$.

Proof. (1) By 3.3.6(1), 1.2.66, we have $a^{\lambda}_{i'j} \sim_L a^{\lambda}_{i'j}$ so $a^{\lambda}_{i'j} \in \mathrm{C}_L^{a^{\lambda}_{i'j}}$. Hence, $\mathcal{B}_{I(\lambda),j} \subseteq \mathrm{C}_L^{a^{\lambda}_{i'j}}$

Obviously, $\mathrm{C}_L^{a^{\lambda}_{i'j}} \subseteq \mathcal{B}_{I(\lambda),j}$, this proves (1).

(2) The proof of (2) is analogous to (1).

(3) It follows from the definition of $\mathcal{B}_{\lambda}, \mathcal{B}_{i,J(\lambda)}, \mathcal{B}_{I(\lambda),j}$.

■

Theorem 3.3.9. *[Graham-Lehrer's Cellular Algebra]*

Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard fibered basis algebra. Also suppose that for every $\lambda \in \Lambda$, we have $I(\lambda) = J(\lambda)$. If there is an anti-involution $*_{\mathcal{A}}$ on $\mathcal{A}$ such that

$$*_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$$
$$\sum_{(i,j)\in I(\lambda)\times I(\lambda)} \alpha_{ij} a^{\lambda}_{ij} \longmapsto \sum_{(i,j)\in I(\lambda)\times I(\lambda)} \alpha_{ij} a^{\lambda}_{ji}$$

for all $\lambda \in \Lambda$. Then we have:

(1) $\mathcal{A}(\leq \lambda)$ is $*_{\mathcal{A}}$-invariant.

(2) $l_{a,\lambda}(i', i) = r_{\lambda,a^*}(i, i')$ for all $a \in \mathcal{A}$, $i, i' \in I(\lambda)$.

(3) $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ is a standard cellular algebra.

*In this case, the algebra $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}})$ is called a **Graham-Lehrer cellular algebra**.*

Proof. (1) Using (1) in 3.1.3 implies:

$$\mathcal{A}(\leq \lambda) = \bigoplus_{\mu\leq\lambda} Sp_{\mathbb{K}} B_{\mu} = \bigoplus_{\mu\leq\lambda} Sp_{\mathbb{K}} \Big\{ a^{\lambda}_{ij} : (i,j) \in I(\lambda) \times I(\lambda) \Big\}$$

so we have:

$$\begin{aligned} \Big(\mathcal{A}(\leq \lambda)\Big)^{*_{\mathcal{A}}} &= Sp_{\mathbb{K}} \Big\{ a^{\lambda}_{ij} : (i,j) \in I(\lambda) \times I(\lambda) \Big\}^{*_{\mathcal{A}}} \\ &= Sp_{\mathbb{K}} \Big\{ a^{\lambda}_{ji} : (i,j) \in I(\lambda) \times I(\lambda) \Big\} \\ &= \mathcal{A}(\leq \lambda) \end{aligned}$$

(2) We know that

$$a \cdot_{\mathcal{A}} a^{\lambda}_{ij} = (a^{\lambda}_{ji} \cdot_{\mathcal{A}} a^*)^* \overset{mod(\mathcal{A}<\lambda)}{\equiv} \Big(\sum_{j'\in J(\lambda)} r_{\lambda,a^*}(j', i) a^{\lambda}_{jj'} \Big)^* = \sum_{j'\in J(\lambda)} r_{\lambda,a^*}(j', i) a^{\lambda}_{j'j}$$

Also

$$a \cdot_{\mathcal{A}} a^{\lambda}_{ij} \overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{i'\in I(\lambda)} l_{a,\lambda}(i', i) a^{\lambda}_{i'j}$$

So

$$\sum_{i'\in I(\lambda)} r_{\lambda,a^*}(i', i) a^{\lambda}_{i'j} \overset{mod(\mathcal{A}<\lambda)}{\equiv} a \cdot_{\mathcal{A}} a^{\lambda}_{ij} \overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{i'\in I(\lambda)} l_{a,\lambda}(i', i) a^{\lambda}_{i'j}$$

Since $\big\{ a^{\lambda}_{ij} : (i,j) \in I(\lambda) \times I(\lambda) \big\}$ is a $\mathbb{K}$-basis of B_{λ}, so we must have: $l_{a,\lambda}(i', i) = r_{\lambda,a^*}(i, i')$.

∎

Theorem 3.3.10. *Let* $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ *be a Graham-Lehrer standard cellular algebra. Then we have:*

(1) $a_{ij}^{\lambda} \leqslant_L a_{rj}^{\lambda} \Longleftrightarrow a_{ji}^{\lambda} \leqslant_R a_{jr}^{\lambda}$

(2) $a_{ij}^{\lambda} \leqslant_L^t a_{rj}^{\lambda} \Longleftrightarrow a_{ji}^{\lambda} \leqslant_R^t a_{jr}^{\lambda}$

(3) $a_{ij}^{\lambda} \sim_L a_{rj}^{\lambda} \Longleftrightarrow a_{ji}^{\lambda} \sim_R a_{jr}^{\lambda}$

(4) $a_{ij}^{\lambda} \sim_{LR} a_{rj}^{\lambda} \Longleftrightarrow a_{ji}^{\lambda} \sim_{LR} a_{jr}^{\lambda}$

(5) $\mathrm{C}_L^{a_{ij}^{\lambda}} = (\mathrm{C}_R^{a_{ji}^{\lambda}})^*$

(6) $\mathrm{C}_R^{a_{ij}^{\lambda}} = (\mathrm{C}_L^{a_{ji}^{\lambda}})^*$

(7) $\mathrm{C}_{LR}^{a_{ij}^{\lambda}} = (\mathrm{C}_{LR}^{a_{ji}^{\lambda}})^*$

(8) $\mathcal{B}_{i,I(\lambda)} = (\mathcal{B}_{I(\lambda),i})^*$, *for all* $i \in I(\lambda)$.

PROOF. (1)

$$\begin{aligned} a_{ij}^{\lambda} \leqslant_L a_{rj}^{\lambda} &\Longleftrightarrow \exists a \in \mathcal{A}; l_{a,\lambda}(r,i) \neq 0 \\ &\overset{3.3.9(2)}{\Longleftrightarrow} \exists a \in \mathcal{A}; r_{\lambda,a^*}(r,i) \neq 0 \\ &\Longleftrightarrow a_{ji}^{\lambda} \leqslant_R a_{jr}^{\lambda} \end{aligned}$$

(2) Apply (1) in 3.3.4(2).

(3) It is implied by (2).

(4) It is obviously carried by (3).

(5) It is obviously carried by (3).

(6) It is obviously carried by (3).

(7) It is obviously carried by (4).

(8) $\mathcal{B}_{i,I(\lambda)} \overset{3.3.8(2)}{=} \bigcup_{i' \in I(\lambda)} \mathrm{C}_R^{a_{ii'}^{\lambda}} \overset{3.3.10(5)}{=} \bigcup_{i' \in I(\lambda)} (\mathrm{C}_L^{a_{i'i}^{\lambda}})^* = (\bigcup_{i' \in I(\lambda)} \mathrm{C}_L^{a_{i'i}^{\lambda}})^* \overset{3.3.8(1)}{=} (\mathcal{B}_{I(\lambda),i})^*$

■

Suppose $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}})$ is a standard cellular algebra. For any $\lambda \in \Lambda$, $a \in \mathcal{A}$, $a_{rs}^{\lambda}, a_{ij}^{\lambda} \in B_{\lambda}$ we have:

$$a_{ij}^{\lambda} \cdot_{\mathcal{A}} (a \cdot_{\mathcal{A}} a_{rs}^{\lambda}) = (a_{ij}^{\lambda} \cdot_{\mathcal{A}} a) \cdot_{\mathcal{A}} a_{rs}^{\lambda}$$

Let us compute the left and right sides of the above equation:

$$
\begin{aligned}
a_{ij}^{\lambda} \cdot_{\mathcal{A}} (a \cdot_{\mathcal{A}} a_{rs}^{\lambda}) &\overset{mod(\mathcal{A}<\lambda)}{\equiv} a_{ij}^{\lambda} \cdot_{\mathcal{A}} \Big(\sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) a_{r's}^{\lambda} \Big) \\
&\overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) \big(a_{ij}^{\lambda} \cdot_{\mathcal{A}} a_{r's}^{\lambda} \big) \\
&\overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) \Big(\sum_{j' \in J(\lambda)} r_{\lambda, a_{r's}^{\lambda}}(j', j) \Big) a_{ij'}^{\lambda} \\
&\overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{r' \in I(\lambda), j' \in J(\lambda)} l_{a,\lambda}(r', r) r_{\lambda, a_{r's}^{\lambda}}(j', j) a_{ij'}^{\lambda}
\end{aligned}
$$

Also we know that

$$
\big(a_{ij}^{\lambda} \cdot_{\mathcal{A}} a_{r's}^{\lambda} \big) \overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{r'' \in I(\lambda)} l_{a_{ij}^{\lambda},\lambda}(r'', r') a_{r''s}^{\lambda}
$$

Hence,

$$
\begin{aligned}
a_{ij}^{\lambda} \cdot_{\mathcal{A}} (a \cdot_{\mathcal{A}} a_{rs}^{\lambda}) &\overset{mod(\mathcal{A}<\lambda)}{\equiv} a_{ij}^{\lambda} \cdot_{\mathcal{A}} \Big(\sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) a_{r's}^{\lambda} \Big) \\
&\overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) \big(a_{ij}^{\lambda} \cdot_{\mathcal{A}} a_{r's}^{\lambda} \big) \\
&\overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) \Big(\sum_{r'' \in I(\lambda)} l_{a_{ij}^{\lambda},\lambda}(r'', r') a_{r''s}^{\lambda} \Big) \\
&\overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{r', r'' \in I(\lambda)} l_{a,\lambda}(r', r) l_{a_{ij}^{\lambda},\lambda}(r'', r') a_{r''s}^{\lambda}
\end{aligned}
$$

Hence

$$
\sum_{r' \in I(\lambda), j' \in J(\lambda)} l_{a,\lambda}(r', r) r_{\lambda, a_{r's}^{\lambda}}(j', j) a_{ij'}^{\lambda} \overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{r', r'' \in I(\lambda)} l_{a,\lambda}(r', r) l_{a_{ij}^{\lambda},\lambda}(r'', r') a_{r''s}^{\lambda}
$$

which implies $i = r''$, $j' = s$, and $\sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) r_{\lambda, a_{r's}^{\lambda}}(s, j) = \sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) l_{a_{ij}^{\lambda},\lambda}(i, r')$.

So the left side of the last equation is dependent on r, j, s and the right side on r, j, i. Since i and s are independent on each other, both sides are dependent only on j, r. We thus define

$$
\phi_a(j, r) := \sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) r_{\lambda, a_{r's}^{\lambda}}(j, s) = \sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) l_{a_{ij}^{\lambda},\lambda}(i, r')
$$

and we have

$$
a_{ij}^{\lambda} \cdot_{\mathcal{A}} a \cdot_{\mathcal{A}} a_{rs}^{\lambda} \overset{mod(\mathcal{A}<\lambda)}{\equiv} \phi_a(j, r) a_{is}^{\lambda}.
$$

We have proved the following important proposition:

Proposition 3.3.11. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. Then, for any $a \in \mathcal{A}, \lambda \in \Lambda$:*

(1) there is a function $\phi_a^\lambda : J(\lambda) \times I(\lambda) \to \mathbb{K}$ such that

$$a_{ij}^\lambda \cdot_{\mathcal{A}} a \cdot_{\mathcal{A}} a_{rs}^\lambda \overset{mod(\mathcal{A}<\lambda)}{\equiv} \phi_a^\lambda(j,r) a_{is}^\lambda$$

for any $a_{ij}^\lambda, a_{rs}^\lambda \in B_\lambda$.

(2) $\sum_{r' \in I(\lambda)} l_{a,\lambda}(r',r) r_{\lambda, a_{r's}^\lambda}(s,j) = \sum_{r' \in I(\lambda)} l_{a,\lambda}(r',r) l_{a_{ij}^\lambda,\lambda}(i,r')$, for all i and s.

Corollary 3.3.12. *In a special case of 3.3.11, when $a = 1_{\mathcal{A}}$, we have:*

$$a_{ij}^\lambda \cdot_{\mathcal{A}} a_{rs}^\lambda \overset{mod(\mathcal{A}<\lambda)}{\equiv} \phi_{1_{\mathcal{A}}}^\lambda(j,r) a_{is}^\lambda = \sum_{r' \in I(\lambda)} l_{1_{\mathcal{A}},\lambda}(r',r) r_{\lambda, a_{r's}^\lambda}(s,j) a_{is}^\lambda \overset{3.3.3}{=} r_{\lambda, a_{rs}^\lambda}(s,j) a_{is}^\lambda$$

$$a_{ij}^\lambda \cdot_{\mathcal{A}} a_{rs}^\lambda \overset{mod(\mathcal{A}<\lambda)}{\equiv} \phi_{1_{\mathcal{A}}}^\lambda(j,r) a_{is}^\lambda = \sum_{r' \in I(\lambda)} l_{1_{\mathcal{A}},\lambda}(r',r) l_{a_{ij}^\lambda,\lambda}(i,r') a_{is}^\lambda \overset{3.3.3}{=} l_{a_{ij}^\lambda,\lambda}(i,r) a_{is}^\lambda$$

So we have:

$$a_{ij}^\lambda \cdot_{\mathcal{A}} a_{rs}^\lambda \overset{mod(\mathcal{A}<\lambda)}{\equiv} \phi_{1_{\mathcal{A}}}^\lambda(j,r) a_{is}^\lambda = r_{\lambda, a_{rs}^\lambda}(s,j) a_{is}^\lambda = l_{a_{ij}^\lambda,\lambda}(i,r) a_{is}^\lambda.$$

Therefore:

$$\phi_{1_{\mathcal{A}}}^\lambda(j,r) = r_{\lambda, a_{rs}^\lambda}(s,j) = l_{a_{ij}^\lambda,\lambda}(i,r).$$

Corollary 3.3.13. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. Then for any $i, k \in I(\lambda), j \in J(\lambda)$ we have*

$$l_{a_{ij}^\lambda,\lambda}(k',k) = \begin{cases} 0 & : k' \neq i \\ \phi_{1_{\mathcal{A}}}(j,k) & : k' = i \end{cases}$$

Proof. For every $s \in J(\lambda)$, we have:

$$a_{ij}^\lambda \cdot_{\mathcal{A}} a_{ks}^\lambda \overset{mod(\mathcal{A}<\lambda)}{\equiv} \phi_{1_{\mathcal{A}}}(j,k) a_{is}^\lambda$$

$$a_{ij}^\lambda \cdot_{\mathcal{A}} a_{ks}^\lambda \overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{k' \in I(\lambda)} l_{a_{ij}^\lambda,\lambda}(k',k) a_{k's}^\lambda$$

So we must have:

$$\phi_{1_{\mathcal{A}}}(j,k) a_{is}^\lambda = \sum_{k' \in I(\lambda)} l_{a_{ij}^\lambda,\lambda}(k',k) a_{k's}^\lambda.$$

Since $\left\{a_{ij}^{\lambda} : (i,j) \in I(\lambda) \times J(\lambda)\right\}$ is a $\mathbb{K}$-basis for B_{λ}, so we must have:

$$l_{a_{ij}^{\lambda},\lambda}(k',k) = \begin{cases} 0 & : k' \neq i \\ \phi_{1_{\mathcal{A}}}(j,k) & : k' = i. \end{cases}$$

■

The mapping $\phi_a^{\lambda} : J(\lambda) \times I(\lambda) \to \mathbb{K}$ will be used several times in the definition of other mappings. So, it is reasonable to find out its main properties. We recall that for any $\lambda \in \Lambda$, that both $\mathcal{A}(\leqslant \lambda)$ and $\mathcal{A}(< \lambda)$ are ideals of $\mathcal{A}$. So their quotient $\mathcal{A}(\lambda) = \mathcal{A}(\leqslant \lambda)/\mathcal{A}(< \lambda)$ is a $(\mathcal{A}, \mathcal{A})$-bimodule. Also we can view $\mathcal{A}$ as a $\mathbb{K}$-submodule of $\mathcal{A}$ spanned by B_{λ}.

Theorem 3.3.14. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. We define*

$$\begin{aligned} \gamma_{\lambda} : \mathcal{A}(\lambda) \times \mathcal{A}(\lambda) &\longrightarrow \mathbb{K} \\ \Big(\sum_{\substack{i\in I(\lambda)\\ j\in J(\lambda)}} \alpha_{ij} a_{ij}^{\lambda}, \sum_{\substack{r\in I(\lambda)\\ s\in J(\lambda)}} \beta_{rs} a_{rs}^{\lambda}\Big) &\longmapsto \sum_{\substack{(i,j)\in I(\lambda)\times J(\lambda)\\ (r,s)\in I(\lambda)\times J(\lambda)}} \alpha_{ij}\beta_{rs}\phi_{1_{\mathcal{A}}}^{\lambda}(j,r) \end{aligned}$$

Then γ_{λ} has the following properties:

(1) γ_{λ} is $(\mathbb{K}, \mathcal{A}, \mathbb{K})$-balanced bilinear form.

(2) γ_{λ} induces a $\mathbb{K}$-linear form $\overline{\gamma} : \mathcal{A}(\lambda) \underset{\mathcal{A}}{\otimes} \mathcal{A}(\lambda) \to \mathbb{K}$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{A}(\lambda) \times \mathcal{A}(\lambda) \arrow[r, "\gamma_{\lambda}"] \arrow[d, "\underset{\mathcal{A}}{\otimes}"'] & \mathbb{K} \\ \mathcal{A}(\lambda) \underset{\mathcal{A}}{\otimes} \mathcal{A}(\lambda) \arrow[ur, dashed, "\overline{\gamma}_{\lambda}"'] & \end{tikzcd}$$

Proof. (1) It is clear by its definition that γ_{λ} is $(\mathbb{K}, \mathbb{K})$- bilinear form. We show that it is $\mathcal{A}$-balanced. It is sufficient by its definition to verify it only on the basis elements of $\mathcal{A}$, that is, $\gamma_{\lambda}(a_{ij}^{\lambda} \cdot_{\mathcal{A}} a, a_{rs}^{\lambda}) = \gamma_{\lambda}(a_{ij}^{\lambda}, a \cdot_{\mathcal{A}} a_{rs}^{\lambda})$ we have

$$a_{ij}^{\lambda} \cdot_{\mathcal{A}} (a \cdot_{\mathcal{A}} a_{rs}^{\lambda}) \overset{mod(\mathcal{A}<\lambda)}{\underset{3.3.12}{\equiv}} \sum_{r'\in I(\lambda)} l_{a,\lambda}(r',r)\phi_{1_{\mathcal{A}}}^{\lambda}(j,r')a_{is}^{\lambda}$$

$$(a_{ij}^{\lambda} \cdot_{\mathcal{A}} a) \cdot_{\mathcal{A}} a_{rs}^{\lambda} \overset{mod(\mathcal{A}<\lambda)}{\underset{3.3.12}{\equiv}} \sum_{j'\in J(\lambda)} r_{\lambda,a}(j',j)\phi_{1_{\mathcal{A}}}^{\lambda}(j',r)a_{is}^{\lambda}$$

$$
\begin{aligned}
a_{ij}^{\lambda} \cdot_{\mathcal{A}} (a \cdot_{\mathcal{A}} a_{rs}^{\lambda}) &\overset{mod(\mathcal{A}<\lambda)}{\equiv} a_{ij}^{\lambda} \cdot_{\mathcal{A}} \Big(\sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) a_{r's}^{\lambda} \Big) \\
&\overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) \big(a_{ij}^{\lambda} \cdot_{\mathcal{A}} a_{r's}^{\lambda} \big) \\
&\underset{3.3.12}{\overset{mod(\mathcal{A}<\lambda)}{\equiv}} \sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) \phi_{1_{\mathcal{A}}}^{\lambda}(j, r') a_{is}^{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
(a_{ij}^{\lambda} \cdot_{\mathcal{A}} a) \cdot_{\mathcal{A}} a_{rs}^{\lambda} &\overset{mod(\mathcal{A}<\lambda)}{\equiv} \Big(\sum_{j' \in J(\lambda)} r_{\lambda,a}(j', j) a_{ij'}^{\lambda} \cdot_{\mathcal{A}} \Big) a_{rs}^{\lambda} \\
&\overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{j' \in J(\lambda)} r_{\lambda,a}(j', j) \big(a_{ij'}^{\lambda} \cdot_{\mathcal{A}} a_{rs}^{\lambda} \big) \\
&\underset{3.3.12}{\overset{mod(\mathcal{A}<\lambda)}{\equiv}} \sum_{j' \in J(\lambda)} r_{\lambda,a}(j', j) \phi_{1_{\mathcal{A}}}^{\lambda}(j', r) a_{is}^{\lambda}
\end{aligned}
$$

so we have

$$
\sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) \phi_{1_{\mathcal{A}}}^{\lambda}(j, r') a_{is}^{\lambda} \overset{mod(\mathcal{A}<\lambda)}{\equiv} a_{ij}^{\lambda} \cdot_{\mathcal{A}} a \cdot_{\mathcal{A}} a_{rs}^{\lambda} \overset{mod(\mathcal{A}<\lambda)}{\equiv} \sum_{j' \in J(\lambda)} r_{\lambda,a}(j', j) \phi_{1_{\mathcal{A}}}^{\lambda}(j', r) a_{is}^{\lambda}.
$$

Since $\left\{ a_{ij}^{\lambda} : (i, j) \in I(\lambda) \times J(\lambda) \right\}$ is a $\mathbb{K}$-basis, so we must have

$$
\sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) \phi_{1_{\mathcal{A}}}^{\lambda}(j, r') = \sum_{j' \in J(\lambda)} r_{\lambda,a}(j', j) \phi_{1_{\mathcal{A}}}^{\lambda}(j', r) \tag{5}
$$

$$
\begin{aligned}
\gamma_{\lambda}(a_{ij}^{\lambda} \cdot_{\mathcal{A}} a, a_{rs}^{\lambda}) &= \gamma_{\lambda}\big(\sum_{j' \in J(\lambda)} r_{\lambda,a}(j', j) a_{ij'}^{\lambda}, a_{rs}^{\lambda} \big) \\
&= \sum_{j' \in J(\lambda)} r_{\lambda,a}(j', j) \gamma_{\lambda}\big(a_{ij'}^{\lambda}, a_{rs}^{\lambda} \big) \\
&= \sum_{j' \in J(\lambda)} r_{\lambda,a}(j', j) \phi_{1_{\mathcal{A}}}^{\lambda}(j', r) \\
&\overset{(5)}{=} \sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) \phi_{1_{\mathcal{A}}}^{\lambda}(j, r') \\
&= \sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) \gamma_{\lambda}\big(a_{ij}^{\lambda} \cdot_{\mathcal{A}} a_{r's}^{\lambda} \big) \\
&= \gamma_{\lambda}\big(a_{ij}^{\lambda}, \sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) a_{r's}^{\lambda} \big) \\
&= \gamma_{\lambda}(a_{ij}^{\lambda}, a \cdot_{\mathcal{A}} a_{rs}^{\lambda})
\end{aligned}
$$

So γ_λ is $\mathcal{A}$-balanced.

(2) The existence of $\overline{\gamma}_\lambda$ is guaranteed by 1.2.41(2).

■

Theorem 3.3.15. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. We define*

$$\begin{aligned} m_\lambda : \mathcal{A}(\lambda) \times \mathcal{A}(\lambda) &\longrightarrow \mathcal{A}(\lambda) \\ \Big(\sum_{\substack{i \in I(\lambda) \\ j \in J(\lambda)}} \alpha_{ij} a^\lambda_{ij}, \sum_{\substack{r \in I(\lambda) \\ s \in J(\lambda)}} \beta_{rs} a^\lambda_{rs} \Big) &\longmapsto \sum_{\substack{(i,j) \in I(\lambda) \times J(\lambda) \\ (r,s) \in I(\lambda) \times J(\lambda)}} \alpha_{ij} \beta_{rs} \phi^\lambda_{1_{\mathcal{A}}}(j, r) a^\lambda_{is} \end{aligned}$$

Then, the following hold:

(1) m_λ is an $(\mathcal{A}, \mathcal{A}, \mathcal{A})$-balanced bilinear mapping.

(2) m_λ induces an $\mathcal{A}$-linear mapping $\overline{m}_\lambda$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{A}(\lambda) \times \mathcal{A}(\lambda) \arrow[r, "m_\lambda"] \arrow[d, "\bigotimes_{\mathcal{A}}"'] & \mathcal{A}(\lambda) \\ \mathcal{A}(\lambda) \bigotimes_{\mathcal{A}} \mathcal{A}(\lambda) \arrow[ur, dashed, "\overline{m}_\lambda"'] & \end{tikzcd}$$

Proof. (1) We show that $m_\lambda(a \cdot_{\mathcal{A}} a^\lambda_{ij}, a^\lambda_{rs}) = a \cdot_{\mathcal{A}} m_\lambda(a^\lambda_{ij}, a^\lambda_{rs})$. We have

$$\begin{aligned} m_\lambda(a \cdot_{\mathcal{A}} a^\lambda_{ij}, a^\lambda_{rs}) &= m_\lambda\Big(\sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) a^\lambda_{i'j}, a^\lambda_{rs} \Big) \\ &= \sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) m_\lambda(a^\lambda_{i'j}, a^\lambda_{rs}) \\ &= \sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) \phi^\lambda_{1_{\mathcal{A}}}(j, r) a^\lambda_{i's} \end{aligned}$$

Besides,

$$\begin{aligned} a \cdot_{\mathcal{A}} m_\lambda(a^\lambda_{ij}, a^\lambda_{rs}) &= a \cdot_{\mathcal{A}} \left(\phi^\lambda_{1_{\mathcal{A}}}(j, r) a^\lambda_{is} \right) \\ &= \phi^\lambda_{1_{\mathcal{A}}}(j, r)(a \cdot_{\mathcal{A}} a^\lambda_{is}) \\ &= \phi^\lambda_{1_{\mathcal{A}}}(j, r) \Big(\sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) a^\lambda_{i's} \Big) \\ &= \sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) \phi^\lambda_{1_{\mathcal{A}}}(i', r) a^\lambda_{i's} \end{aligned}$$

So we have:

$$m_\lambda(a \cdot_{\mathcal{A}} a^\lambda_{ij}, a^\lambda_{rs}) = a \cdot_{\mathcal{A}} m_\lambda(a^\lambda_{ij}, a^\lambda_{rs})$$

Similarly, we can show that

$$m_\lambda(a^\lambda_{ij}, a^\lambda_{rs} \cdot_{\mathcal{A}} a) = m_\lambda(a^\lambda_{ij}, a^\lambda_{rs}) \cdot_{\mathcal{A}} a$$

We show that m_λ is $\mathcal{A}$-balanced.

$$\begin{aligned}
m_\lambda(a^\lambda_{ij} \cdot_{\mathcal{A}} a, a^\lambda_{rs}) &= m_\lambda\big(\sum_{j' \in J(\lambda)} r_{\lambda,a}(j', j) a^\lambda_{ij'}, a^\lambda_{rs}\big) \\
&= \sum_{j' \in J(\lambda)} r_{\lambda,a}(j', j) m_\lambda\big(a^\lambda_{ij'}, a^\lambda_{rs}\big) \\
&= \sum_{j' \in J(\lambda)} r_{\lambda,a}(j', j) \phi^\lambda_{1_{\mathcal{A}}}(j', r) a^\lambda_{is} \\
&\overset{(5)}{=} \sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) \phi^\lambda_{1_{\mathcal{A}}}(j, r') a^\lambda_{is} \\
&= \sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) m_\lambda\big(a^\lambda_{ij}, a^\lambda_{r's}\big) \\
&= m_\lambda\big(a^\lambda_{ij}, \sum_{r' \in I(\lambda)} l_{a,\lambda}(r', r) a^\lambda_{r's}\big) \\
&= m_\lambda(a^\lambda_{ij}, a \cdot_{\mathcal{A}} a^\lambda_{rs})
\end{aligned}$$

So, m_λ is $\mathcal{A}$-balanced.

(2) By 1.2.41(2), m_λ induced an $\mathcal{A}$-homorphism $\overline{m}_\lambda$ which commutes the diagram.

■

Now, we want to talk about the injectivity and surjectivity of m_λ. Suppose that there are $i_0, i_1 \in I(\lambda), j_0, j_1 \in J(\lambda)$ such that $\phi^\lambda_{1_{\mathcal{A}}}(j_0, i_0) = \phi^\lambda_{1_{\mathcal{A}}}(j_1, i_1) = 1_{\mathbb{K}}$, hence: $m_\lambda(a^\lambda_{ij_0}, a^\lambda_{i_0 j}) = a^\lambda_{ij} = m_\lambda(a^\lambda_{ij_1}, a^\lambda_{i_1 j})$, but we can choose $a^\lambda_{ij_0}$ and $a^\lambda_{ij_1}$ such that

$$a^\lambda_{ij_0} \neq a^\lambda_{ij_1} \tag{6}$$

or similarly, we can choose $a^\lambda_{i_0 j}$ and $a^\lambda_{i_1 j}$ such that

$$a^\lambda_{i_0 j} \neq a^\lambda_{i_1 j} \tag{7}$$

Therefore, m_λ would not be injective. If we look carefully at both of the inequalities, (6) states that the first component of the index is fixed and the second component of the index runs through in $J(\lambda)$. Also in the case of (7), the inequality states that the second component of the index is fixed and the first component of the index tuns through in $I(\lambda)$. So, the first thing which comes to mind is to think of the notions of left and right cells. So we could define

$$
\begin{aligned}
m'_\lambda : \mathcal{B}_{I(\lambda),j} \times \mathcal{B}_{i,J(\lambda)} &\longrightarrow \mathcal{A}(\lambda) \\
\big(\mathrm{C}_L^{a^\lambda_{i'j}}, \mathrm{C}_R^{a^\lambda_{ij'}}\big) &\longmapsto \phi^\lambda_{1_\mathcal{A}}(j,i) a^\lambda_{i'j'}
\end{aligned}
$$

Unfortunately, if

$$\mathrm{C}_L^{a^\lambda_{i'j}} = \mathrm{C}_L^{a^\lambda_{i''j}} \quad , \quad \mathrm{C}_R^{a^\lambda_{ij'}} = \mathrm{C}_R^{a^\lambda_{ij''}}$$

then we would have

$$m'_\lambda\big(\mathrm{C}_L^{a^\lambda_{i'j}}, \mathrm{C}_R^{a^\lambda_{ij'}}\big) = \phi^\lambda_{1_\mathcal{A}}(j,i) a^\lambda_{i'j'} \quad , \quad m'_\lambda\big(\mathrm{C}_L^{a^\lambda_{i''j}}, \mathrm{C}_R^{a^\lambda_{ij''}}\big) = \phi^\lambda_{1_\mathcal{A}}(j,i) a^\lambda_{i''j''}$$

and m'_λ would not be well-defined. Also, we could exchange the place of $\mathcal{B}_{I(\lambda),j}$ and $\mathcal{B}_{i,J(\lambda)}$, but it does not solve our issue, because

$$m'_\lambda\big(\mathrm{C}_R^{a^\lambda_{ij'}}, \mathrm{C}_L^{a^\lambda_{i'j}}\big) = \phi^\lambda_{1_\mathcal{A}}(j',i') a^\lambda_{ij} \quad , \quad m'_\lambda\big(\mathrm{C}_R^{a^\lambda_{ij''}}, \mathrm{C}_L^{a^\lambda_{i''j}}\big) = \phi^\lambda_{1_\mathcal{A}}(j'',i'') a^\lambda_{ij}$$

In order to find a solution to our problem, we introduce a left $\mathcal{A}$-module $\Delta(I(\lambda), j)$ and a right $\mathcal{A}$-module $\Delta(i, J(\lambda))$.

Definition 3.3.16. Let $(\mathcal{A}, +_\mathcal{A}, \cdot_\mathcal{A}, *_\mathcal{A}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. We define

$$
\begin{aligned}
\Delta(i,\lambda) &:= Sp_\mathbb{K}\Big\{\mathcal{B}_{i,J(\lambda)}\Big\} \\
\Delta(\lambda,j) &:= Sp_\mathbb{K}\Big\{\mathcal{B}_{I(\lambda),j}\Big\}
\end{aligned}
$$

$\forall i \in I(\lambda), \forall j \in J(\lambda)$.

Theorem 3.3.17. *Let $(\mathcal{A}, +_\mathcal{A}, \cdot_\mathcal{A}, *_\mathcal{A}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. Then the following hold:*

(1) $\Delta(i,\lambda)$ is a right $\mathcal{A}$-module, for all $i \in I(\lambda)$.

(2) $\Delta(\lambda,j)$ is a left $\mathcal{A}$-module, for all $j \in J(\lambda)$.

(3) $\Delta(i_1,\lambda) \simeq \Delta(i_2,\lambda)$, as right $\mathcal{A}$-modules, for all $i_1, i_2 \in I(\lambda)$.

(4) $\Delta(\lambda,j_1) \simeq \Delta(\lambda,j_2)$, as left $\mathcal{A}$-modules, for all $j_1, j_2 \in J(\lambda)$.

Proof. (1) By 3.3.1(2), $\Delta(i,\lambda)$ is a right $\mathcal{A}$-module, for all $i \in I(\lambda)$.

(2) By 3.3.1(2), $\Delta(\lambda,j)$ is a left $\mathcal{A}$-module, for all $j \in J(\lambda)$.

(3) We define $\phi_{i_2,i_1} : \Delta(i_1,\lambda) \longrightarrow \Delta(i_2,\lambda)$, by $\phi_{i_2,i_1}\big(\sum\limits_{j\in J(\lambda)} \alpha_j a^\lambda_{i_1 j}\big) = \sum\limits_{j\in J(\lambda)} \alpha_j a^\lambda_{i_2 j}$. It is obvious that ϕ_{i_2,i_1} is $\mathbb{K}$-isomorphism. It remains to show that ϕ_{i_2,i_1} is $\mathcal{A}$-linear. We have:

$$\begin{aligned}
\phi_{i_2,i_1}(a^\lambda_{i_1 j} \cdot_{\mathcal{A}} a) &\overset{mod(\mathcal{A}\prec\lambda)}{\equiv} \phi_{i_2,i_1}\big(\sum_{j'\in J(\lambda)} r_{\lambda,a}(j',j) a^\lambda_{i_1 j'}\big) \\
&= \sum_{j'\in J(\lambda)} r_{\lambda,a}(j',j) a^\lambda_{i_2 j'} \\
&\overset{mod(\mathcal{A}\prec\lambda)}{\equiv} a^\lambda_{i_2 j} \cdot_{\mathcal{A}} a \\
&= \phi_{i_2,i_1}(a^\lambda_{i_2 j}) \cdot_{\mathcal{A}} a
\end{aligned}$$

So ϕ_{i_2,i_1} is $\mathcal{A}$-isomorphism.

(4) The proof of (4) is similar to (3).

■

As shown in the following theorem, $\Delta(i,\lambda)$ and $\Delta(\lambda,j)$ solve the problem of non-injectivity of m_λ.

Theorem 3.3.18. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. Suppose there are $i_0 \in I(\lambda)$, $j_0 \in J(\lambda)$ such that $\phi^\lambda_{1_{\mathcal{A}}}(j_0, i_0) \neq 0$.*

We define:

$$\begin{aligned}
m_\lambda : \Delta(\lambda, j_0) \times \Delta(i_0, \lambda) &\longrightarrow \mathcal{A}(\lambda) \\
\Big(\sum_{i\in I(\lambda)} \alpha_i a^\lambda_{i j_0}, \sum_{j\in J(\lambda)} \beta_j a^\lambda_{i_0 j}\Big) &\longmapsto \sum_{(i,j)\in I(\lambda)\times J(\lambda)} \alpha_i \beta_j \phi^\lambda_{1_{\mathcal{A}}}(j_0, i_0) a^\lambda_{ij}
\end{aligned}$$

Then, the following are hold:

(1) m_λ is an $(\mathcal{A}, \mathbb{K}, \mathcal{A})$-balanced bilinear mapping.

(2) m_λ induces an $(\mathcal{A}, \mathcal{A})$-isomorphism $\overline{m}_\lambda$ such that the following diagram commutes:

$$\begin{tikzcd}
\Delta(\lambda, j_0) \times \Delta(i_0, \lambda) \arrow[r, "m_\lambda"] \arrow[d, "\bigotimes_{\mathbb{K}}"'] & \mathcal{A}(\lambda) \\
\Delta(\lambda, j_0) \bigotimes_{\mathbb{K}} \Delta(i_0, \lambda) \arrow[ur, dashed, "\overline{m}_\lambda"'] &
\end{tikzcd}$$

Proof. (1) We show that $m_\lambda(a \cdot_{\mathcal{A}} a^\lambda_{ij}, a^\lambda_{rs}) = a \cdot_{\mathcal{A}} m_\lambda(a^\lambda_{ij}, a^\lambda_{rs})$. We have

$$
\begin{aligned}
m_\lambda(a \cdot_{\mathcal{A}} a^\lambda_{ij}, a^\lambda_{rs}) \quad &= m_\lambda\Big(\sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) a^\lambda_{i'j}, a^\lambda_{rs}\Big) \\
&= \sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) m_\lambda(a^\lambda_{i'j}, a^\lambda_{rs}) \\
&= \sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) \phi^\lambda_{1_{\mathcal{A}}}(j, r) a^\lambda_{i's}
\end{aligned}
$$

Besides,

$$
\begin{aligned}
a \cdot_{\mathcal{A}} m_\lambda(a^\lambda_{ij}, a^\lambda_{rs}) \quad &= a \cdot_{\mathcal{A}} \big(\phi^\lambda_{1_{\mathcal{A}}}(j, r) a^\lambda_{is}\big) \\
&= \phi^\lambda_{1_{\mathcal{A}}}(j, r)(a \cdot_{\mathcal{A}} a^\lambda_{is}) \\
&= \phi^\lambda_{1_{\mathcal{A}}}(j, r)\Big(\sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) a^\lambda_{i's}\Big) \\
&= \sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) \phi^\lambda_{1_{\mathcal{A}}}(i', r) a^\lambda_{i's}
\end{aligned}
$$

So we have:

$$
m_\lambda(a \cdot_{\mathcal{A}} a^\lambda_{ij}, a^\lambda_{rs}) = a \cdot_{\mathcal{A}} m_\lambda(a^\lambda_{ij}, a^\lambda_{rs})
$$

Similarly, we can show that

$$
m_\lambda(a^\lambda_{ij}, a^\lambda_{rs} \cdot_{\mathcal{A}} a) = m_\lambda(a^\lambda_{ij}, a^\lambda_{rs}) \cdot_{\mathcal{A}} a.
$$

Also it is obvious that m_λ is $\mathbb{K}$-balanced which proves (1).

(2) Since there are $i_0 \in I(\lambda)$, $j_0 \in J(\lambda)$ such that $\phi^\lambda_{1_{\mathcal{A}}}(j_0, i_0) \neq 0$, hence m_λ is surjective. We show that $\overline{m}_\lambda$ is injective. Suppose that

$$
\overline{m}_\lambda\Big(\sum_{i \in I(\lambda)} \alpha_i a^\lambda_{ij_0} \underset{\mathbb{K}}{\otimes} \sum_{j \in J(\lambda)} \beta_j a^\lambda_{i_0 j}\Big) = \overline{m}_\lambda\Big(\sum_{i \in I(\lambda)} c_i a^\lambda_{ij_0} \underset{\mathbb{K}}{\otimes} \sum_{j \in J(\lambda)} d_j a^\lambda_{i_0 j}\Big)
$$

We have:

$$\begin{aligned}
\overline{m}_\lambda\Big(\sum_{i\in I(\lambda)} \alpha_i a^\lambda_{ij_0} \underset{\mathbb{K}}{\otimes} \sum_{j\in J(\lambda)} \beta_j a^\lambda_{i_0 j}\Big) &= \overline{m}_\lambda\Big(\sum_{\substack{i\in I(\lambda)\\ j\in J(\lambda)}} \alpha_i\beta_j a^\lambda_{ij_0} \underset{\mathbb{K}}{\otimes} a^\lambda_{i_0 j}\Big) \\
&= \sum_{\substack{i\in I(\lambda)\\ j\in J(\lambda)}} \alpha_i\beta_j \overline{m}_\lambda(a^\lambda_{ij_0} \underset{\mathbb{K}}{\otimes} a^\lambda_{i_0 j}) \\
&= \sum_{\substack{i\in I(\lambda)\\ j\in J(\lambda)}} \alpha_i\beta_j \phi^\lambda_{1_\mathcal{A}}(j_0, i_0) a^\lambda_{ij}.
\end{aligned}$$

Similarly,

$$\overline{m}_\lambda\Big(\sum_{i\in I(\lambda)} c_i a^\lambda_{ij_0} \underset{\mathbb{K}}{\otimes} \sum_{j\in J(\lambda)} d_j a^\lambda_{i_0 j}\Big) = \sum_{\substack{i\in I(\lambda)\\ j\in J(\lambda)}} c_i d_j \phi^\lambda_{1_\mathcal{A}}(j_0, i_0) a^\lambda_{ij}.$$

Hence,

$$\begin{aligned}
\overline{m}_\lambda\Big(\sum_{i\in I(\lambda)} \alpha_i a^\lambda_{ij_0} \underset{\mathbb{K}}{\otimes} \sum_{j\in J(\lambda)} \beta_j a^\lambda_{i_0 j}\Big) &= \overline{m}_\lambda\Big(\sum_{i\in I(\lambda)} c_i a^\lambda_{ij_0} \underset{\mathbb{K}}{\otimes} \sum_{j\in J(\lambda)} d_j a^\lambda_{i_0 j}\Big) \\
&\iff \alpha_i\beta_j = c_i d_j \quad \forall i \in I(\lambda), \forall j \in J(\lambda).
\end{aligned}$$

So we have:

$$\begin{aligned}
\sum_{i\in I(\lambda)} \alpha_i a^\lambda_{ij_0} \underset{\mathbb{K}}{\otimes} \sum_{j\in J(\lambda)} \beta_j a^\lambda_{i_0 j} &= \sum_{\substack{i\in I(\lambda)\\ j\in J(\lambda)}} \alpha_i\beta_j a^\lambda_{ij_0} \underset{\mathbb{K}}{\otimes} a^\lambda_{i_0 j} \\
&= \sum_{\substack{i\in I(\lambda)\\ j\in J(\lambda)}} c_i d_j a^\lambda_{ij_0} \underset{\mathbb{K}}{\otimes} a^\lambda_{i_0 j} \\
&= \sum_{i\in I(\lambda)} c_i a^\lambda_{ij_0} \underset{\mathbb{K}}{\otimes} \sum_{j\in J(\lambda)} d_j a^\lambda_{i_0 j}
\end{aligned}$$

Hence, $\overline{m}_\lambda$ is injective. ∎

The modules $\Delta(i,\lambda)$ and $\Delta(\lambda,j)$ solved the problem of injectivity of $\overline{m}_\lambda$. So it is reasonable to study them more precisely. Theorem 3.3.17 tells us for all $i_1,\dots,i_n \in I_\lambda$, $j_1,\dots,j_m \in J_\lambda$ we have:

$$\Delta(i_1,\lambda) \underset{\mathcal{A}}{\simeq} \Delta(i_2,\lambda) \underset{\mathcal{A}}{\simeq} \dots \underset{\mathcal{A}}{\simeq} \Delta(i_n,\lambda)$$

$$\Delta(\lambda,j_1) \underset{\mathcal{A}}{\simeq} \Delta(\lambda,j_2) \underset{\mathcal{A}}{\simeq} \dots \underset{\mathcal{A}}{\simeq} \Delta(\lambda,j_m)$$

Also in the product

$$a \cdot_{\mathcal{A}} a^{\lambda}_{ij} \overset{mod(\mathcal{A}\prec\lambda)}{\equiv} \sum_{i'\in I(\lambda)} l_{a,\lambda}(i',i)a^{\lambda}_{i'j}$$

the coefficients $l_{a,\lambda}(i',i)$ are independent of $J(\lambda)$. Indeed, for $j_1,\ldots,j_m \in J_\lambda$, we have:

$$\begin{aligned}
a \cdot_{\mathcal{A}} a^{\lambda}_{ij_1} &\overset{mod(\mathcal{A}\prec\lambda)}{\equiv} \sum_{i'\in I(\lambda)} l_{a,\lambda}(i',i)a^{\lambda}_{i'j_1}\\
a \cdot_{\mathcal{A}} a^{\lambda}_{ij_2} &\overset{mod(\mathcal{A}\prec\lambda)}{\equiv} \sum_{i'\in I(\lambda)} l_{a,\lambda}(i',i)a^{\lambda}_{i'j_2}\\
\vdots \qquad & \quad \vdots \qquad\qquad \vdots\\
a \cdot_{\mathcal{A}} a^{\lambda}_{ij_n} &\overset{mod(\mathcal{A}\prec\lambda)}{\equiv} \sum_{i'\in I(\lambda)} l_{a,\lambda}(i',i)a^{\lambda}_{i'j_n}
\end{aligned}$$

This leads us to close our eyes on the second component of the index a^{λ}_{ij}, so for every $\square \in J(\lambda)$, we have

$$a \cdot_{\mathcal{A}} a^{\lambda}_{i\square} \overset{mod(\mathcal{A}\prec\lambda)}{\equiv} \sum_{i'\in I(\lambda)} l_{a,\lambda}(i',i)a^{\lambda}_{i'\square}$$

Similarly, we can close our eyes on the first component of the index a^{λ}_{ij}, so for every $\square \in I(\lambda)$, we have

$$a^{\lambda}_{\square j} \cdot_{\mathcal{A}} a \overset{mod(\mathcal{A}\prec\lambda)}{\equiv} \sum_{j'\in J(\lambda)} r_{\lambda,a}(j',j)a^{\lambda}_{\square j'}$$

Definition 3.3.19. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. We define

$$\Delta(\lambda) := Sp_{\mathbb{K}}\Big\{a^{\lambda}_{i\square} : i \in I(\lambda)\Big\}$$

with left $\mathcal{A}$-module action $\cdot_L$, defined by:

$$\begin{aligned}
\cdot_L : \mathcal{A} \times \Delta(\lambda) &\longrightarrow \Delta(\lambda)\\
(a, a^{\lambda}_{i\square}) &\longmapsto \sum_{i'\in I(\lambda)} l_{a,\lambda}(i',i)a^{\lambda}_{i'\square}
\end{aligned}$$

The module $\Delta(\lambda)$ is called **standard module.** Also we define

$$\Delta^{op}(\lambda) := Sp_{\mathbb{K}}\Big\{a^{\lambda}_{\square j} : j \in J(\lambda)\Big\}$$

with right $\mathcal{A}$-module action defined by:

$$\begin{aligned} \cdot_R : \Delta^{op}(\lambda) \times \mathcal{A} &\longrightarrow \Delta^{op}(\lambda) \\ (a^{\lambda}_{\square}j, a) &\longmapsto \sum_{j' \in J(\lambda)} r_{\lambda,a}(j', j) a^{\lambda}_{\square j'} \end{aligned}$$

for all $\lambda \in \Lambda$. Finally, we define $\nabla(\lambda) := Hom_{\mathcal{A}}(\Delta(\lambda), \mathbb{K})$. The module $\nabla(\lambda)$ is called **co-standard module.**

Theorem 3.3.20. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. We define*

$$\begin{aligned} M_{\lambda} : \Delta(\lambda) \times \Delta^{op}(\lambda) &\longrightarrow \mathcal{A} \\ \Big(\sum_{i \in I(\lambda)} \beta_i a^{\lambda}_{i\square} \ , \ \sum_{j \in J(\lambda)} \alpha_j a^{\lambda}_{\square j} \Big) &\longmapsto \sum_{\substack{i \in I(\lambda) \\ j \in J(\lambda)}} \beta_i \alpha_j a^{\lambda}_{ij} \end{aligned}$$

Then we have:

(1) M_{λ} is $(\mathcal{A}, \mathbb{K}, \mathcal{A})$ - balanced bilinear mapping.

(2) M_{λ} induces an $(\mathcal{A}, \mathcal{A})$-bilinear isomorphism $\overline{M}_{\lambda}$ which makes the following diagram commutes:

$$\begin{tikzcd} \Delta(\lambda) \times \Delta^{op}(\lambda) \arrow[r, "M_{\lambda}"] \arrow[d, "\bigotimes_{\mathbb{K}}"'] & \mathcal{A}(\lambda) \\ \Delta(\lambda) \bigotimes_{\mathbb{K}} \Delta^{op}(\lambda) \arrow[ur, dashed, "\overline{M}_{\lambda}"'] & \end{tikzcd}$$

(3) $\bigoplus_{j \in J(\lambda)} \Delta(\lambda) \underset{\mathbb{K}}{\cong} \mathcal{A}(\lambda) \underset{\mathbb{K}}{\cong} \bigoplus_{i \in I(\lambda)} \Delta^{op}(\lambda)$, that is, $\mathcal{A}(\lambda)$ is isomorphic to a direct sum of $\mid I \mid$ copies of $\Delta^{op}(\lambda)$ and to a direct sum of $\mid J \mid$ copies of $\Delta(\lambda)$.

Proof. (1) We show that $M_{\lambda}(a \cdot_{\mathcal{A}} a^{\lambda}_{i\square}, a^{\lambda}_{\square j}) = a \cdot_{\mathcal{A}} M_{\lambda}(a^{\lambda}_{i\square}, a^{\lambda}_{\square j})$ we have

$$\begin{aligned} M_{\lambda}(a \cdot_{\mathcal{A}} a^{\lambda}_{i\square}, a^{\lambda}_{\square j}) &= M_{\lambda}\Big(\sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) a^{\lambda}_{i'\square}, a^{\lambda}_{\square j} \Big) \\ &= \sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) M_{\lambda}(a^{\lambda}_{i'\square}, a^{\lambda}_{\square j}) \\ &= \sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) a^{\lambda}_{i'j} \end{aligned}$$

Besides,

$$\begin{aligned} a \cdot_{\mathcal{A}} M_\lambda(a^\lambda_{i\square}, a^\lambda_{\square j}) &= a \cdot_{\mathcal{A}} a^\lambda_{ij} \\ &= (a \cdot_{\mathcal{A}} a^\lambda_{ij}) \\ &= \sum_{i' \in I(\lambda)} l_{a,\lambda}(i', i) a^\lambda_{i'j} \end{aligned}$$

So we have:

$$M_\lambda(a \cdot_{\mathcal{A}} a^\lambda_{i\square}, a^\lambda_{\square j}) = a \cdot_{\mathcal{A}} M_\lambda(a^\lambda_{i\square}, a^\lambda_{\square j})$$

Similarly, we can show that

$$M_\lambda(a^\lambda_{i\square}, a^\lambda_{\square j} \cdot_{\mathcal{A}} a) = M_\lambda(a^\lambda_{i\square}, a^\lambda_{j\square}) \cdot_{\mathcal{A}} a$$

Also, it is clear that M_λ is $\mathbb{K}$-balanced.

(2) By 1.2.41(2), M_λ induced an $(\mathcal{A}, \mathcal{A})$-bilinear homomorphism $\overline{M}_\lambda$ which makes the diagram commute. We show that $\overline{M}_\lambda$ is injective. Suppose that

$$\overline{M}_\lambda\Big(\sum_{i \in I(\lambda)} \alpha_i a^\lambda_{i\square} \underset{\mathbb{K}}{\otimes} \sum_{j \in J(\lambda)} \beta_j a^\lambda_{\square j}\Big) = \overline{M}_\lambda\Big(\sum_{i \in I(\lambda)} c_i a^\lambda_{i\square} \underset{\mathbb{K}}{\otimes} \sum_{j \in J(\lambda)} d_j a^\lambda_{\square j}\Big)$$

We have:

$$\begin{aligned} \overline{M}_\lambda\Big(\sum_{i \in I(\lambda)} \alpha_i a^\lambda_{i\square} \underset{\mathbb{K}}{\otimes} \sum_{j \in J(\lambda)} \beta_j a^\lambda_{\square j}\Big) &= \overline{M}_\lambda\Big(\sum_{\substack{i \in I(\lambda) \\ j \in J(\lambda)}} \alpha_i \beta_j a^\lambda_{i\square} \underset{\mathbb{K}}{\otimes} a^\lambda_{\square j}\Big) \\ &= \sum_{\substack{i \in I(\lambda) \\ j \in J(\lambda)}} \alpha_i \beta_j \overline{m}_\lambda(a^\lambda_{i\square} \underset{\mathbb{K}}{\otimes} a^\lambda_{\square j}) \\ &= \sum_{\substack{i \in I(\lambda) \\ j \in J(\lambda)}} \alpha_i \beta_j a^\lambda_{ij} \end{aligned}$$

Similarly,

$$\overline{M}_\lambda\Big(\sum_{i \in I(\lambda)} c_i a^\lambda_{i\square} \underset{\mathbb{K}}{\otimes} \sum_{j \in J(\lambda)} d_j a^\lambda_{\square j}\Big) = \sum_{\substack{i \in I(\lambda) \\ j \in J(\lambda)}} c_i d_j a^\lambda_{ij}$$

Hence,

$$\overline{M}_\lambda\Big(\sum_{i\in I(\lambda)} \alpha_i a^\lambda_{i\square} \underset{\mathbb{K}}{\otimes} \sum_{j\in J(\lambda)} \beta_j a^\lambda_{\square j}\Big) \;=\; \overline{M}_\lambda\Big(\sum_{i\in I(\lambda)} c_i a^\lambda_{i\square} \underset{\mathbb{K}}{\otimes} \sum_{j\in J(\lambda)} d_j a^\lambda_{\square j}\Big)$$
$$\iff \alpha_i\beta_j = c_i d_j \quad \forall i \in I(\lambda), \forall j \in J(\lambda)$$

So we have:

$$\begin{aligned}\sum_{i\in I(\lambda)} \alpha_i a^\lambda_{i\square} \underset{\mathbb{K}}{\otimes} \sum_{j\in J(\lambda)} \beta_j a^\lambda_{\square j} &= \sum_{\substack{i\in I(\lambda)\\ j\in J(\lambda)}} \alpha_i\beta_j a^\lambda_{i\square} \underset{\mathbb{K}}{\otimes} a^\lambda_{\square j}\\ &= \sum_{\substack{i\in I(\lambda)\\ j\in J(\lambda)}} c_i d_j a^\lambda_{i\square} \underset{\mathbb{K}}{\otimes} a^\lambda_{\square j}\\ &= \sum_{i\in I(\lambda)} c_i a^\lambda_{i\square} \underset{\mathbb{K}}{\otimes} \sum_{j\in J(\lambda)} d_j a^\lambda_{\square j}\end{aligned}$$

Hence, $\overline{M}_\lambda$ is injective, and $\overline{M}_\lambda$ is obviously surjective by its definition.

(3) By the proof of 1.2.44, we know that the vector spaces $\Delta(\lambda)\underset{\mathbb{K}}{\otimes}\Delta^{op}(\lambda)$ and $\sum_{j\in J(\lambda)} \Delta(\lambda)\underset{\mathbb{K}}{\otimes} a_{\square j}$ are isomorphic as $\mathbb{K}$-vector spaces, also the vector spaces $\sum_{j\in J(\lambda)} \Delta(\lambda)\underset{\mathbb{K}}{\otimes} a_{\square j}$ and $\bigoplus_{j\in J(\lambda)} \Delta(\lambda)\underset{\mathbb{K}}{\otimes} a_{\square j}$ are isomorphic as $\mathbb{K}$-vector spaces. Besides, by (2) we have

$$\begin{aligned}\mathcal{A}(\lambda) &\cong \Delta(\lambda) \underset{\mathbb{K}}{\otimes} \Delta^{op}(\lambda)\\ &\underset{\mathbb{K}}{\cong} \sum_{j\in J(\lambda)} \Delta(\lambda) \underset{\mathbb{K}}{\otimes} a_{\square j}\\ &\underset{\mathbb{K}}{\cong} \bigoplus_{j\in J(\lambda)} \Delta(\lambda) \underset{\mathbb{K}}{\otimes} a_{\square j}\\ &\underset{\mathbb{K}}{\cong} \bigoplus_{j\in J(\lambda)} \Delta(\lambda)\end{aligned}$$

Similarly, $\mathcal{A}(\lambda) \underset{\mathbb{K}}{\cong} \bigoplus_{i\in I(\lambda)} \Delta^{op}(\lambda)$.

■

Lemma 3.3.21. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. If $a^\lambda_{rs} \cdot_L a^\mu_{i\square} \neq 0$, for some λ, μ, r, i, j, then $\mu \leq \lambda$.*

Proof. We know that

$$a^\lambda_{rs} \cdot_L a^\mu_{i\square} = \sum_{i'\in I(\mu)} l_{a^\lambda_{rs},\lambda}(i', i) a^\lambda_{i'\square}$$

So

$$
\begin{aligned}
a^{\lambda}_{rs}\cdot_L a^{\mu}_{i\square} &\neq 0 \\
&\Longleftrightarrow \ \exists\, i'' \in I(\mu); l_{a^{\lambda}_{rs},\mu}(i'',i)\neq 0 \\
&\Longleftrightarrow \ a^{\lambda}_{rs}\cdot_L a^{\mu}_{ij} \overset{mod(\mathcal{A}\prec\mu)}{\not\equiv} 0 \ , \forall j\in J(\lambda) \\
&\overset{3.3.5(1)}{\Longrightarrow} \ \mu\leq\lambda
\end{aligned}
$$

■

Definition 3.3.22. Let $(\mathcal{A},+_{\mathcal{A}},\cdot_{\mathcal{A}},*_{\mathcal{A}},\mathcal{B},\Lambda)$ be a standard cellular algebra. Then for every $\lambda\in\Lambda$, we define

$$
\begin{aligned}
\xi_\lambda : \Delta(\lambda)^{op}\times\Delta(\lambda) &\longrightarrow \mathbb{K} \\
\Big(\sum_{j\in J(\lambda)}\alpha_j a^{\lambda}_{\square j}, \sum_{i\in I(\lambda)}\beta_i a^{\lambda}_{i\square}\Big) &\longmapsto \sum_{(j,i)\in J(\lambda)\times I(\lambda)}\alpha_j\beta_i\phi^{\lambda}_{1_{\mathcal{A}}}(j,i)
\end{aligned}
$$

Theorem 3.3.23. *Let $(\mathcal{A},+_{\mathcal{A}},\cdot_{\mathcal{A}},*_{\mathcal{A}},\mathcal{B},\Lambda)$ be a standard cellular algebra. Then for evey $\lambda\in\Lambda$, ξ_λ has the following properties:*

(1) ξ_λ is $(\mathbb{K},\mathcal{A},\mathbb{K})$ balanced bilinear form.

(2) $\forall x,z\in\Delta(\lambda),\forall y\in\Delta(\lambda)^{op}$: $\overline{M}(x\underset{\mathcal{A}}{\otimes}y)\cdot_L z=\xi_\lambda(y,z)x$

(3) For every $x\in\Delta(\lambda)$, we put $I^{\lambda}_x:=\Big\{\overline{\xi}_\lambda(y,x) : y\in\Delta(\lambda)^{op}\Big\}$; then $I^{\lambda}_x\Delta(\lambda)=\mathcal{A}(\lambda)\cdot_L x$, and specifically, if $I_x\neq 0$, then $\Delta(\lambda)$ is $\mathcal{A}$-cyclic, that is, $\Delta(\lambda)=\mathcal{A}\cdot_L x$.

Proof. (1) ξ_λ is $(\mathbb{K},\mathbb{K})$-bilinear form by its definition, so it remains to show that it is $\mathcal{A}$-balanced.

$$\begin{aligned}
\xi_\lambda(a^\lambda_{\square j}\cdot_R a, a^\lambda_{r\square}) &= \xi_\lambda\big(\sum_{j'\in J(\lambda)} r_{\lambda,a}(j',j)a^\lambda_{\square j'}, a^\lambda_{r\square}\big)\\
&= \sum_{j'\in J(\lambda)} r_{\lambda,a}(j',j)\xi_\lambda\big(a^\lambda_{\square j'}, a^\lambda_{r\square}\big)\\
&= \sum_{j'\in J(\lambda)} r_{\lambda,a}(j',j)\phi^\lambda_{1_\mathcal{A}}(j',r)\\
&\overset{(5)}{=} \sum_{r'\in I(\lambda)} l_{a,\lambda}(r',r)\phi^\lambda_{1_\mathcal{A}}(j,r')\\
&= \sum_{r'\in I(\lambda)} l_{a,\lambda}(r',r)\xi_\lambda\big(a^\lambda_{\square j}\cdot_\mathcal{A} a^\lambda_{r'\square}\big)\\
&= \xi_\lambda\big(a^\lambda_{ij}, \sum_{r'\in I(\lambda)} l_{a,\lambda}(r',r)a^\lambda_{r'\square}\big)\\
&= \xi_\lambda(a^\lambda_{\square j}, a\cdot_L a^\lambda_{r\square})
\end{aligned}$$

(2) It is sufficient to check the property on a generating set:

$$\begin{aligned}
\overline{M}_\lambda(a^\lambda_{i\square}\underset{\mathcal{A}}{\otimes} a^\lambda_{\square j})\cdot_L a_{k\square} &= a^\lambda_{ij}\cdot_L a_{k\square}\\
&= \sum_{k'\in I(\lambda)} l_{a^\lambda_{ij},\lambda}(k',k)a^\lambda_{k'\square}\\
&\overset{3.3.13}{=} \phi_{1_\mathcal{A}}(j,k)a^\lambda_{i\square}\\
&= \xi_\lambda(a^\lambda_{\square j}, a^\lambda_{k\square})a^\lambda_{i\square}
\end{aligned}$$

(3) We have:

$$\begin{aligned}
I^\lambda_x\Delta(\lambda) &= \Big\{\xi_\lambda(y,x)z : y\in\Delta^{op}(\lambda), z\in\Delta(\lambda)\Big\}\\
&\overset{(2)}{=} \Big\{\overline{M}_\lambda(z\underset{\mathcal{A}}{\otimes} y)\cdot_L x : y\in\Delta^{op}(\lambda), z\in\Delta(\lambda)\Big\}\\
&= \mathcal{A}(\lambda)\cdot_L x\subseteq\mathcal{A}(x)\subseteq\Delta(\lambda).
\end{aligned}$$

Since $\mathbb{K}$ is a field, if $I^\lambda_x\neq 0$, then $I^\lambda_x=\mathbb{K}$. In this case

$$\Delta(\lambda)=I^\lambda_x\Delta(\lambda)=\mathcal{A}(\lambda)\cdot_L x.$$

So $\Delta(\lambda)$ is $\mathcal{A}$-cyclic.

■

Theorem 3.3.24. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. Suppose that there is $x_0 \in \Delta(\lambda)$ such that $I^{\lambda}_{x_0} \neq 0$. Then*

(1) $Hom_{\mathcal{A}}(\Delta(\lambda), \Delta(\mu)) = 0$ unless, $\mu \leq \lambda$.

(2) $Hom_{\mathcal{A}}(\Delta(\lambda), \Delta(\lambda)/M) \underset{\mathbb{K}}{\simeq} \mathbb{K}$, for any left $\mathcal{A}$-submodule M of $\Delta(\lambda)$.

(3) $Hom_{\mathcal{A}}(\Delta(\lambda), \Delta(\lambda)) \underset{\mathbb{K}}{\simeq} \mathbb{K}$.

PROOF. (1) Suppose that $Hom_{\mathcal{A}}(\Delta(\lambda), \Delta(\mu)) \neq 0$, then there is $f \in Hom_{\mathcal{A}}(\Delta(\lambda), \Delta(\mu))$ such that $f \neq 0$. Since $I_i = \mathbb{K}$, so we have $\Delta(\lambda) \overset{3.3.23(5)}{=} \mathcal{A}(\lambda) \cdot_L a^{\lambda}_{i\square}$. Hence, there is $a \in \mathcal{A}(\lambda)$ such that $f(a \cdot_L a^{\lambda}_{i\square}) = (a \cdot_L f(a^{\lambda}_{i\square})) \neq 0$. Since $f(a \cdot_L a^{\lambda}_{i\square}) \in \Delta(\mu)$, so the condition $f(a \cdot_L a^{\lambda}_{i\square}) = (a \cdot_L f(a^{\lambda}_{i\square})) \neq 0$ implies that $f(a \cdot_L a^{\lambda}_{i\square}) = (a \cdot_L f(a^{\lambda}_{i\square})) \overset{mod(\mathcal{A}\leq\mu)}{\not\equiv} 0$. So we must have $\mu \leq \lambda$, by 3.3.21.

(2) Since $\Delta(\lambda) = \mathcal{A}(\lambda) \cdot_L x_0$ so we consider $f(x_0) = a_0 \cdot_L x_0 + M$. For every $0 \neq f \in Hom_{\mathcal{A}}(\Delta(\lambda), \Delta(\lambda)/M)$, we have $f(x_0) \neq 0_{\Delta(\lambda)/M}$, because if $f(x_0) = 0_{\Delta(\lambda)/M}$, then for every $a \in \mathcal{A}(\lambda)$ we have $f(a \cdot_L x_0) = a \cdot_L f(x_0) = 0_{\Delta(\lambda)/M}$, which is a contradiction with $f \neq 0$. The condition $I^{\lambda}_{x_0} \neq 0$ guarantees that there is $y_0 \in \Delta^{op}(\lambda)$ such that $\xi_{\lambda}(y_0, x_0) = 1_{\mathbb{K}}$. We have:

$$\begin{aligned}
f(x_0) &= f(1_{\mathbb{K}} x_0) \\
&= f(\xi_{\lambda}(y_0, x_0) x_0) \\
&\overset{3.3.23(2)}{=} f(\overline{M}_{\lambda}(x_0 \underset{\mathcal{A}}{\otimes} y_0) x_0) \\
&= \overline{M}_{\lambda}(x_0 \underset{\mathcal{A}}{\otimes} y_0) f(x_0) \\
&= \overline{M}_{\lambda}(x_0 \underset{\mathcal{A}}{\otimes} y_0) \cdot_L (a_0 \cdot_L x_0) + M \\
&\overset{3.3.23(2)}{=} \xi_{\lambda}(y_0, a_0 \cdot_L x_0) x_0
\end{aligned}$$

Hence,

$$f(x_0) \neq 0 \Longleftrightarrow \xi_{\lambda}(y_0, a_0 \cdot_L x_0) \neq 0, \tag{8}$$

which implies that f is an $\mathcal{A}$-isomorphism, and moreover it gives us a two-sided correspondence between $Hom_{\mathcal{A}}(\Delta(\lambda), \Delta(\lambda))$ and $\mathbb{K}$:

$$f \neq 0 \Longleftrightarrow \xi_{\lambda}(y_0, a_0 \cdot_L x_0) \neq 0 \tag{9}$$

Hence, $Hom_{\mathcal{A}}(\Delta(\lambda), \Delta(\lambda)/M) \underset{\mathbb{K}}{\cong} \mathbb{K}$.

(3) Put $M = 0$ in (2).

$\blacksquare$

3.4. Standard Cellular Algebras and Highest-Weight Category

Definition 3.4.1. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. We define

$$\Lambda_1 := \{\lambda \in \Lambda : \xi_\lambda \neq 0\}.$$

Proposition 3.4.2. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. Then for every $\lambda \in \Lambda_1$, we define:*

$$R(\lambda) := \Big\{v \in \Delta(\lambda) : \xi_\lambda(y, v) = 0, \forall y \in \Delta^{op}(\lambda)\Big\}.$$

Then, the following hold:

(1) $R(\lambda)$ is a left $\mathcal{A}$-submodule of $\Delta(\lambda)$.

(2) $R(\lambda) = Rad_{\mathcal{A}}(\Delta(\lambda))$.

(3) $L(\lambda) := \dfrac{\Delta(\lambda)}{R(\lambda)}$ is a simple $\mathcal{A}$-module.

(4) If $L(\lambda)$ is a composition factor of $\Delta(\mu)$, then $\mu \leq \lambda$. Moreover, $[\Delta(\lambda) : L(\lambda)] = 1$.

(5) $L(\lambda)$ is absolutely irreducible, that is, $Hom_{\mathcal{A}}(L(\lambda), L(\lambda)) \underset{\mathbb{K}}{\simeq} \mathbb{K}$.

(6) $\Big\{L(\lambda) : \lambda \in \Lambda_1\Big\}$ is a complete set of all non-isomorphic simple $\mathcal{A}$-modules.

Proof. (1) For any $x \in R(\lambda)$ and $a \in \mathcal{A}$, we have

$$\xi_\lambda(y \cdot_R a, x) = 0, \forall y \in \Delta^{op}(\lambda).$$

We show that $a \cdot_L x \in R(\lambda)$. We have

$$\begin{aligned} a \cdot_L x \in R(\lambda) &\iff \xi_\lambda(y, a \cdot_L x) = 0, \forall y \in \Delta^{op}(\lambda) \\ &\overset{3.3.23(1)}{\iff} \xi_\lambda(y \cdot_R a, x) = 0, \forall y \in \Delta^{op}(\lambda) \end{aligned}$$

which proves (1).

(2) Since $\lambda \in \Lambda_1$, so $R(\lambda) \neq \Delta(\lambda)$. Thus for any non-zero $z \in \Delta(\lambda)/R(\lambda)$, we can write $z = z_1 + R(\lambda)$. Since $z_1 \notin R(\lambda)$, there exists $y_1 \in \Delta^{op}(\lambda)$ such that $\xi_\lambda(y_1, z_1) = 1_{\mathbb{K}}$. So, for every $x \in \Delta(\lambda)$, we have:

$$x = 1_{\mathbb{K}} x = \xi_\lambda(y_1, z_1) x = \overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1) \cdot_L z_1$$

Therefore $\Delta(\lambda) \subseteq \mathcal{A}(\lambda)(z_1) \subseteq \mathcal{A}(z_1) \subseteq \Delta(\lambda)$. So we have $\Delta(\lambda) = \mathcal{A}(z_1)$ and $\Delta(\lambda)/R(\lambda) \underset{\mathcal{A}}{\cong} \mathcal{A}(z)$. So, $\Delta(\lambda)/R(\lambda)$ is a simple left $\mathcal{A}$-module. Hence, $Rad_{\mathcal{A}}(\Delta(\lambda) \subseteq R(\lambda)$. It remains to show that $R(\lambda) \subseteq Rad_{\mathcal{A}}(\Delta(\lambda))$. It is obvious that if for any simple $\mathcal{A}$-module $M/Rad_{\mathcal{A}}(\Delta(\lambda))$ where $M \subseteq \Delta(\lambda)$, and every $\mathcal{A}$-epimorphism $\phi : \Delta(\lambda) \longrightarrow M/Rad_{\mathcal{A}}(\Delta(\lambda))$, we have $\phi(R(\lambda)) = Rad_{\mathcal{A}}(\Delta(\lambda))$, then $R(\lambda) = Rad_{\mathcal{A}}(\Delta(\lambda))$. Now suppose that $R(\lambda) \nsubseteq Rad_{\mathcal{A}}(\Delta(\lambda))$, so $R(\lambda) \neq Rad_{\mathcal{A}}(\Delta(\lambda))$, hence there are a simple $\mathcal{A}$-module $L_0/Rad_{\mathcal{A}}(\Delta(\lambda))$ and an $\mathcal{A}$-epimorphism $\phi_0 : \Delta(\lambda) \longrightarrow L_0/Rad_{\mathcal{A}}(\Delta(\lambda))$ such that

$$\phi_0(R(\lambda)) \neq \underbrace{Rad_{\mathcal{A}}(\Delta(\lambda))}_{0_{L_0/Rad_{\mathcal{A}}\Delta(\lambda)}}$$

Since $L_0/Rad_{\mathcal{A}}\Delta(\lambda)$ is simple, so we have

$$\phi_0(R(\lambda)) = L_0/Rad_{\mathcal{A}}(\Delta(\lambda))$$

Also from its definition we have:

$$\phi_0(\Delta(\lambda)) = L_0/Rad_{\mathcal{A}}(\Delta(\lambda))$$

so

$$\phi_0(\Delta(\lambda)) = \phi_0(R(\lambda))$$

We choose $z_1 \in \Delta(\lambda) \setminus R(\lambda)$, so there is $u \in R(\lambda)$ such that

$$\phi_0(z_1) = \phi_0(u) \tag{10}$$

and $\xi_\lambda(y_1, u) = 0$, therefore for every $x \in \Delta(\lambda)$, we have:

$$\begin{aligned}
\phi_0(x) &= \phi_0(1_{\mathbb{K}} x) \\
&= \phi_0(\xi_\lambda(y_1, z_1)x) \\
&= \phi_0(\overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1) \cdot_L z_1) \\
&= \overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1) \cdot_L \phi_0(z_1) \\
&= \xi_\lambda(y_1, \phi_0(z_1))x \\
&\overset{(10)}{=} \xi_\lambda(y_1, \phi_0(u))x \\
&= \overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1) \cdot_L \phi_0(u) \\
&= \phi_0(\overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1) \cdot_L u) \\
&= \phi_0(\xi_\lambda(y_1, u)x) \\
&= \xi_\lambda(y_1, u)\phi_0(x) \\
&= 0_{\mathbb{K}}\phi_0(x) \\
&= 0_{L_0/Rad_{\mathcal{A}}\Delta(\lambda)}
\end{aligned}$$

which implies $\phi_0 = 0_{L_0/Rad_{\mathcal{A}}\Delta(\lambda)}$, and it is a contradiction. So we must have

$$R(\lambda) \subseteq Rad_{\mathcal{A}}(A)$$

consequently, $R(\lambda) = Rad_{\mathcal{A}}(\Delta(\lambda))$.

(3) Suppose that $L(\lambda)$ is a composition factor of $\Delta(\mu)$, so there is a composition series of length m such that

$$\{0\} = N_0^\mu \subseteq N_1^\mu \subseteq \ldots \subseteq N_m^\mu = \Delta(\mu)$$

such that $\dfrac{N_{i+1}^\mu}{N_i^\mu} \simeq L(\lambda)$, for some $0 \leqslant i \leqslant m-1$. We have

$$\Delta(\lambda) \longrightarrow \frac{\Delta(\lambda)}{Rad_{\mathcal{A}}(\Delta(\lambda))} = L(\lambda) \overset{\simeq}{\longrightarrow} \frac{N_{i+1}^\mu}{N_i^\mu} \hookrightarrow \frac{\Delta(\mu)}{N_i^\mu}$$

and the composition of these mappings allows us to define the nonzero mapping

$$\Psi : \Delta(\lambda) \longrightarrow \frac{\Delta(\mu)}{N_i^\mu}$$

Obviously, we have $Im(\Psi) \simeq L(\lambda)$. Also, suppose that $\Psi(z_1) = w + N_i^{\mu}$, for some nonzero $w \in \Delta(\mu)$ and $y \in \Delta^{op}(\lambda)$ such that $\xi_\lambda(y_1, z_1) = 1_{\mathbb{K}}$. For every $x \in \Delta(\lambda)$, we have:

$$\begin{aligned}
\Psi(x) &= \Psi(1_{\mathbb{K}}x) \\
&= \Psi(\xi_\lambda(y_1, z_1)x) \\
&= \Psi(\overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1) \cdot_L z_1) \\
&= \overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1) \cdot_L \Psi(z_1) \\
&= \overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1) \cdot_L (w + N_i^{\mu}) \\
&= (\overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1) \cdot_L w) + N_i^{\mu}
\end{aligned}$$

Since $\overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1) \in \mathcal{A}(\lambda)$ and $w \in \Delta(\mu)$, so by 3.3.21 the condition $(\overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1) \cdot_L w) \neq 0$ implies that $\mu \leq \lambda$.

Now, we want to prove the second part. By 1.2.47(1), we have $Rad_{\mathcal{A}}\Delta(\lambda) \subseteq Ker(\Psi)$, so Ψ induces $\overline{\Psi}$ such that the following diagram is commutative:

$$\begin{tikzcd}
\Delta(\lambda) \arrow[rr, "\Psi"] \arrow[d, "\pi_\lambda"'] & & \dfrac{\Delta(\lambda)}{N_i^{\mu}} \\
\Delta(\lambda)/Rad_{\mathcal{A}}(\Delta(\lambda)) \arrow[urr, dashed, "\overline{\Psi}"'] & &
\end{tikzcd}$$

Now consider $\overline{\Psi} : \Delta(\lambda)/Rad_{\mathcal{A}}(\Delta(\lambda)) \longrightarrow \dfrac{\Delta(\lambda)}{N_i^{\mu}}$, and suppose that

$$\overline{\Psi}(z_1 + Rad_{\mathcal{A}}(\Delta(\lambda))) = w_1 + N_i^{\mu}$$

for some $w_1 \in \Delta(\lambda)$. Now for every $x + Rad_{\mathcal{A}}(\Delta(\lambda)) \in \dfrac{\Delta(\lambda)}{Rad_{\mathcal{A}}(\Delta(\lambda))}$, we have:

$$\begin{aligned}
\overline{\Psi}(x + Rad_{\mathcal{A}}(\Delta(\lambda))) &= \overline{\Psi}(1_{\mathbb{K}}x + Rad_{\mathcal{A}}(\Delta(\lambda))) \\
&= \overline{\Psi}(\xi_\lambda(y_1, z_1)x + Rad_{\mathcal{A}}(\Delta(\lambda))) \\
&= \overline{\Psi}(\overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1) \cdot_L z_1 + Rad_{\mathcal{A}}(\Delta(\lambda))) \\
&= \overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1)\overline{\Psi}(z_1 + Rad_{\mathcal{A}}(\Delta(\lambda))) \\
&= \overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1)w_1 + N_i^{\mu} \\
&= \xi_\lambda(y_1, w_1)x + N_i^{\mu}
\end{aligned}$$

which states that $\overline{\Psi}$ is surjective. So we have $Im(\Psi) \simeq \dfrac{\Delta(\lambda)}{N_i^{\mu}}$. Also, we know that $Im(\Psi) \simeq L(\lambda)$, so $L(\lambda) \simeq \dfrac{\Delta(\lambda)}{N_i^{\mu}}$ which implies that N_i^{μ} is a maximal $\mathcal{A}$-submodule of $\Delta(\lambda)$. Therefore, we have $Rad_{\mathcal{A}}(\Delta(\lambda)) \subseteq N_i^{\mu}$. By (3), $L(\lambda) = Top(\Delta(\lambda))$ is a simple $\mathcal{A}$-module, so $Rad_{\mathcal{A}}(\Delta(\lambda))$ is a maximal $\mathcal{A}$-submodule of $\Delta(\lambda)$ which automatically forces $Rad_{\mathcal{A}}(\Delta(\lambda)) = N_i^{\mu}$.

It remains to show that $[\Delta(\lambda) : L(\lambda)] = 1$. Let

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{k-2} \subseteq M_{k-1} = Rad_{\mathcal{A}}(\Delta(\lambda)) \subseteq M_k = \Delta(\lambda)$$

be a composition series with $M_k/M_{k-1} \cong L(\lambda)$. Suppose moreover that there is another $i < k$ such that $M_i/M_{i-1} \cong L(\lambda)$. Consider then $\Delta(\lambda)/M_{i-1}$. Since $M_i/M_{i-1} \cong L(\lambda)$, this means that $Top(M_i)$ has $L(\lambda)$ as one of its direct summands. But then there are at least two linearly independent $\mathcal{A}$-module morphisms

$$\phi_1 : \Delta(\lambda) \longrightarrow \Delta(\lambda)/M_{i-1}$$

and

$$\phi_2 : \Delta(\lambda) \longrightarrow \Delta(\lambda)/M_{i-1}$$

such that ϕ_1 is the natural projection with $Ker(\phi_1) = M_1$, and ϕ_2 is a projection that sends $Top(\Delta(\lambda))$ to $L(\lambda)$ with $Ker(\phi_2) \cong Rad_{\mathcal{A}}\Delta(\lambda)$. Since $M_{i-1} \ncong Rad_{\mathcal{A}}\Delta(\lambda)$, these are clearly non-zero and distinct. Thus $\mathbb{K}^2 \subseteq Hom_{\mathcal{A}}(\Delta(\lambda), \Delta(\lambda)/M_{i-1})$ contradicting 3.3.24(2). It follows that $[\Delta(\lambda) : L(\lambda)] = 1$.

(4) Suppose that there are $\lambda, \mu \in \Lambda_1$ such that $L(\lambda) \simeq L(\mu)$. Then there is an $\mathcal{A}$-isomorphism

$$\varrho : \frac{\Delta(\lambda)}{Rad_{\mathcal{A}}(\Delta(\lambda))} \longrightarrow \frac{\Delta(\mu)}{Rad_{\mathcal{A}}(\Delta(\mu))}$$

where $\varrho(z_1 + Rad_{\mathcal{A}}(\Delta(\mu))) = w_\varrho + Rad_{\mathcal{A}}(\Delta(\mu))$.

So for every $x + Rad_{\mathcal{A}}(\Delta(\lambda))$ we have,

$$\begin{aligned}\varrho(x + Rad_{\mathcal{A}}(\Delta(\lambda))) &= \varrho(1_{\mathbb{K}}x + Rad_{\mathcal{A}}(\Delta(\mu)))\\ &= \varrho(\xi_\lambda(y_1, z_1)x + Rad_{\mathcal{A}}(\Delta(\mu)))\\ &= \varrho(\overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1) \cdot_L z_1) + Rad_{\mathcal{A}}(\Delta(\mu)))\\ &= \overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1)\varrho(z_1 + Rad_{\mathcal{A}}(\Delta(\mu)))\\ &= \overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1) \cdot_L w_\varrho + Rad_{\mathcal{A}}(\Delta(\mu)))\end{aligned}$$

Since $\varrho \neq 0$, we must have: $\overline{M}_\lambda(x \underset{\mathcal{A}}{\otimes} y_1) \cdot_L w_\varrho \neq 0$, so by 3.3.21, we have: $\mu \leq \lambda$, similarly, we can do the same process for

$$\varrho^{-1} : \frac{\Delta(\mu)}{Rad_{\mathcal{A}}(\Delta(\mu))} \longrightarrow \frac{\Delta(\lambda)}{Rad_{\mathcal{A}}(\Delta(\lambda))}$$

and conclude that $\lambda \leq \mu$, so we have $\lambda = \mu$.

(5) Follows from the proof of 3.3.24(3) and the simplicity of $\Delta(\lambda)/Rad_{\mathcal{A}}(\Delta(\lambda))$.

(6) We show that $\left\{L(\lambda) : \lambda \in \Lambda_1\right\}$ is a complete set of non-isomorphic simple $\mathcal{A}$-modules. Since Λ is a partially-ordered finite set, then by 1.1.5 there exists a filtration of Λ by maximal subsets $\emptyset = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_n = \Lambda$ such that $\Lambda_i \setminus \Lambda_{i-1}$ contains a single element λ_i. This, in turn, gives the following filtration of $\mathcal{A}$:

$$0 \subseteq \mathcal{A}(\Lambda_1) \subseteq \mathcal{A}(\Lambda_2) \subseteq \cdots \subseteq \mathcal{A}(\Lambda_n) = \mathcal{A} \tag{11}$$

Now, consider a simple $\mathcal{A}$-module L. By 1.2.49(1) $Hom_{\mathcal{A}}(\mathcal{A}, L) \neq 0$, so there is $0 \neq f \in Hom_{\mathcal{A}}(\mathcal{A}, L) \neq 0$ and by 1.2.49(2) f is an epimorphism. If for every $1 \leqslant i \leqslant n$, $f \mid_{\mathcal{A}(\Lambda_i)} = 0$, then we must have $f = 0$, which is a contradiction. So at least one of these restrictions is non-zero. Let j be the smallest integer such that $f \mid_{\mathcal{A}(\Lambda_j)} \neq 0$. Therefore $f \mid_{\mathcal{A}(\Lambda_i)} = 0$ for all $i < j$ and

$$\mathcal{A}(\Lambda_j)\underbrace{\hookrightarrow \mathcal{A} \xrightarrow{f}}_{\overline{f}}L$$

with $\overline{f} \neq 0$. Let λ_j be the single element in $\Lambda_j \setminus \Lambda_{j-1}$. Then

$$\frac{\mathcal{A}(\Lambda_j)}{\mathcal{A}(\Lambda_{j-1})} \underset{3.1.5(3)}{\cong} \mathcal{A}(\lambda_j) \underset{3.3.20(2)}{\cong} \Delta(\lambda_j) \underset{\mathbb{K}}{\otimes} \Delta^{op}(\lambda_j) \underset{3.3.20(3)}{\cong} \bigoplus_{j \in J(\lambda)} \Delta(\lambda)$$

So there exists $\widehat{f}$ which makes the following diagram commute:

$$\begin{tikzcd}
\mathcal{A}(\Lambda_j) \arrow[r, "\overline{f}"] \arrow[d, "\pi_{\overline{f}}"'] & L \\
\mathcal{A}(\Lambda_j)/\mathcal{A}(\Lambda_{j-1}) \cong \mathcal{A}(\lambda_j) \arrow[ur, dashed, "\widehat{f}"'] &
\end{tikzcd}$$

because $\mathcal{A}(\Lambda_{j-1}) \subseteq Ker(\overline{f})$. Since

$$Top(\mathcal{A}(\lambda_j)) \cong Top(\bigoplus_{j\in J(\lambda)} \Delta(\lambda)) \cong \bigoplus_{j\in J(\lambda)} Top(\Delta(\lambda)) \underset{3.4.2(3)}{\cong} \bigoplus_{j\in J(\lambda)} L(\lambda),$$

then $L \cong L(\lambda)$.

■

Definition 3.4.3. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra, Λ_1 and Λ_2 be two ideals of Λ such that $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda$, and M be a left $\mathcal{A}$-module. Then we define

$$M(\Lambda_2/\Lambda_1) := \mathcal{A}(\Lambda_2/\Lambda_1) \underset{\mathcal{A}}{\otimes} M$$

Remark 3.4.4. There is an important specific case of definition 3.4.3. For every $\lambda \in \Lambda$, consider $\Lambda_2 = I_{\leq\lambda}, \Lambda_1 = I_{<\lambda}$, we have

$$M(\lambda) = \mathcal{A}(\lambda) \underset{\mathcal{A}}{\otimes} M.$$

Proposition 3.4.5. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra, P be an indecomposable projective left $\mathcal{A}$-module, and Ω be the ideal of Λ generated by $\Big\{\lambda \in \Lambda : P(\lambda) \neq 0\Big\}$. If λ_0 is a maximal element of Ω, then*

(1) $\xi_{\lambda_0} \neq 0$.

(2) λ_0 is the unique maximal element of Ω and $\Omega = I_{\leq\lambda_0}$. [2]

(3) P is the projective cover of $L(\lambda_0)$ and $P(\leq \lambda_0) \simeq \mathcal{A}(\leq \lambda_0)P = P$.

Proof. (1) Since $\lambda_0 \in \Omega$, so $\mathcal{A}(\lambda_0) \underset{\mathcal{A}}{\otimes} P \neq 0$, also by 3.3.20(2) we have : $(\Delta(\lambda_0) \underset{\mathbb{K}}{\otimes} \Delta^{op}(\lambda_0)) \underset{\mathcal{A}}{\otimes} P \neq 0$ which implies

$$\Delta^{op}(\lambda_0) \underset{\mathcal{A}}{\otimes} P \neq 0 \tag{12}$$

[2]See definition 3.1.2(1).

$$
\begin{aligned}
Hom_{\mathcal{A}}(P(\lambda_0),\Delta(\lambda_0)) &= Hom_{\mathcal{A}}(\mathcal{A}(\lambda_0)\underset{\mathcal{A}}{\otimes} P,\Delta(\lambda_0))\\
&\cong Hom_{\mathcal{A}}((\Delta(\lambda_0)\underset{\mathbb{K}}{\otimes}\Delta^{op}(\lambda_0))\underset{\mathcal{A}}{\otimes} P,\Delta(\lambda_0))\\
&\cong Hom_{\mathbb{K}}(\Delta^{op}(\lambda_0)\underset{\mathcal{A}}{\otimes} P, Hom_{\mathcal{A}}(\Delta(\lambda_0),\Delta(\lambda_0)))\\
&\underset{3.3.24(3)}{\cong} Hom_{\mathbb{K}}(\Delta^{op}(\lambda_0)\underset{\mathcal{A}}{\otimes} P,\mathbb{K})\\
&\cong \Delta^{op}(\lambda_0)\underset{\mathcal{A}}{\otimes} P \overset{(12)}{\neq} 0
\end{aligned}
$$

So $P(\lambda_0)$ has a quotient isomorphic to $L(\lambda_0)$. Since P is an indecomposable projective left $\mathcal{A}$-module, then by 1.2.62(2), $Top(P)$ is simple, therefore by 3.4.2(6) there is $\mu\in\Lambda_1$ such that $Top(P)=L(\mu)$. So $L(\mu)$ is a composition factor of $\Delta(\lambda_0)$ by 1.2.26. By 3.4.2(4), we have $\lambda_0\leq\mu$. However, since $\Delta(\mu)$ is a homonorphic image of P and $\Delta(\mu)=\mathcal{A}(\mu)\cdot_L\Delta(\mu)$, it follows that $\mathcal{A}(\mu)P\neq 0$. Therefore $\mu\leq\lambda_0$, and hence by maximality of λ_0, we must have $\lambda_0=\mu\in\Lambda_1$ which means that $\xi_{\lambda_0}\neq 0$ proving (1). So λ_0 is the unique maximal element of Ω which proves (2).

(3) We have:

$$
\begin{tikzcd}
\mathcal{A}(\leq\lambda_0)\times P \arrow[r, "\cdot_P"] \arrow[d, "\underset{\mathcal{A}}{\otimes}"'] & P \\
\mathcal{A}(\leq\lambda_0)\underset{\mathcal{A}}{\otimes} P \arrow[ur, dashed, "\cdot_P"'] &
\end{tikzcd}
$$

Also by the definition of a $\mathcal{A}$-module structure on P, we have:

$$
\begin{tikzcd}
\mathcal{A}(\leq\lambda_0)\times P \arrow[r, "\cdot_P"] \arrow[d, "\cdot_P"'] & P \\
\mathcal{A}(\leq\lambda_0)\cdot_P P \arrow[ur, dashed, "\cdot_P"'] &
\end{tikzcd}
$$

so the map

$$
\mathcal{A}(\leq\lambda_0)\underset{\mathcal{A}}{\otimes} P\longrightarrow \mathcal{A}(\leq\lambda_0)\cdot_P P
$$

$$
a\underset{\mathcal{A}}{\otimes} p\longmapsto a\cdot_P p
$$

defines an $\mathcal{A}$-module isomorphism $\mathcal{A}(\leq \lambda_0) \cdot_P P \simeq \mathcal{A}(\leq \lambda_0) \underset{\mathcal{A}}{\otimes} P$. Since $\xi_{\lambda_0} \neq 0$, so we have $\mathcal{A}(\leq \lambda_0)L(\lambda_0) \neq 0$. So $\mathcal{A}(\leq \lambda_0)L(\lambda_0)$ is a non-zero $\mathcal{A}$-submodule of the simple $\mathcal{A}$-module $L(\lambda_0)$. Hence we have: $\mathcal{A}(\leq \lambda_0)L(\lambda_0) = L(\lambda_0)$. By (2), consider the canonical projection $P \longrightarrow \dfrac{P}{Rad_{\mathcal{A}}(P)} \simeq L(\lambda_0)$. Now suppose that M is a $\mathcal{A}$-submodule of P which covers $L(\lambda_0)$ by the canonical projection. We know that every submodule of a finitely generated module is contained in a maximal submodule. In this case, by 1.2.62(2) $Rad_{\mathcal{A}}(P)$ is the unique maximal ideal of P, and by 1.2.47(1) $Rad_{\mathcal{A}}(P)$ equals the kernel of the canonical projection. So, every non-zero $\mathcal{A}$-submodule of P is contained in $Rad_{\mathcal{A}}(P)$, and therefore mapped onto 0 by the canonical projection. So P is the projective cover of $L(\lambda_0)$. Now we want to show that $P(\lambda_0)$ is the projective cover of $L(\lambda_0)$. Since $\lambda_0 \in \Lambda_1$, so $A(\leq \lambda_0)P \neq 0$. Now consider the mapping $A(\leq \lambda_0)P \longrightarrow A(\leq \lambda_0)L(\lambda_0) = L(\lambda_0)$. So $A(\leq \lambda_0)P$ covers $L(\lambda_0)$, by 1.2.58 we have $A(\leq \lambda_0)P = P$, so $P(\lambda_0)$ is the projective cover of $L(\lambda_0)$.

∎

From now on, we will denote by $P(\lambda)$ the projective cover of $L(\lambda)$.

Theorem 3.4.6. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. Let λ be an element of Λ_1. Then*

(1) The projective cover $P(\lambda)$ has Δ-filtration.

(2) If $[P(\lambda) : \Delta(\mu)]$ denotes the number of quotients isomorphic to $\Delta(\mu)$ in such a filtration, then $[P(\lambda) : \Delta(\mu)] \neq 0$ implies $\mu \leq \lambda$.

Proof. (1) Since Λ is a partially-ordered finite set, then by 1.1.5 it can be totally ordered, $\lambda_1 \preceq'' \cdots \preceq'' \lambda_n$, so consider the sequence till λ,

$$\lambda_1 \preceq'' \cdots \preceq'' \lambda_m = \lambda$$

which gives us the following filtration

$$0 \subseteq \mathcal{A}(\preceq'' \lambda_1) \subseteq \mathcal{A}(\preceq'' \lambda_2) \subseteq \cdots \subseteq \mathcal{A}(\preceq'' \lambda_m). \tag{13}$$

Now consider $0 \leqslant i \leqslant m-1$, so we have $\emptyset \subset I_{\preceq'' \lambda_i} \subseteq I_{\preceq'' \lambda_{i+1}}$, which implies that the following short sequence of $(\mathcal{A}, \mathcal{A})$-bimodule

$$0 = \mathcal{A}(\emptyset) \longrightarrow \mathcal{A}(\preceq'' \lambda_i) \longrightarrow \mathcal{A}(\preceq'' \lambda_{i+1}) \longrightarrow \frac{\mathcal{A}(\preceq'' \lambda_{i+1})}{\mathcal{A}(\preceq'' \lambda_i)} \longrightarrow 0$$

is exact. Also by 3.1.5(4), we have

$$0 \longrightarrow \mathcal{A}(\preceq'' \lambda_i) \longrightarrow \mathcal{A}(\preceq'' \lambda_{i+1}) \longrightarrow \mathcal{A}(\lambda_{i+1}) \longrightarrow 0$$

Since $P(\lambda)$ is a projective left $\mathcal{A}$-module, then the functor $- \underset{\mathcal{A}}{\otimes} P(\lambda)$ is an exact functor, so we have

$$0 \longrightarrow \mathcal{A}(\preceq'' \lambda_i) \underset{\mathcal{A}}{\otimes} P(\lambda) \longrightarrow \mathcal{A}(\preceq'' \lambda_{i+1}) \underset{\mathcal{A}}{\otimes} P(\lambda) \longrightarrow \mathcal{A}(\lambda_{i+1}) \underset{\mathcal{A}}{\otimes} P(\lambda) \longrightarrow 0.$$

We denote

$$P(\preceq'' \lambda_i) := \mathcal{A}(\preceq'' \lambda_i) \underset{\mathcal{A}}{\otimes} P(\lambda) \ , \ P(\lambda_i) := \mathcal{A}(\lambda_i) \underset{\mathcal{A}}{\otimes} P(\lambda)$$

so we have:

$$0 \longrightarrow P(\preceq'' \lambda_i) \longrightarrow P(\preceq'' \lambda_{i+1}) \longrightarrow P(\lambda_{i+1}) \longrightarrow 0.$$

Also, the filtration

$$0 \subseteq \mathcal{A}(\preceq'' \lambda_1) \subseteq \mathcal{A}(\preceq'' \lambda_2) \subseteq \cdots \subseteq \mathcal{A}(\preceq'' \lambda_m) \tag{14}$$

gives us the filtration

$$0 \subseteq P(\preceq'' \lambda_1) \subseteq P(\preceq'' \lambda_2) \subseteq \cdots \subseteq P(\preceq'' \lambda_m) = P(\lambda). \tag{15}$$

For every $1 \leq k \leq m$, we have:

$$\begin{array}{rcl} \dfrac{P(\preceq'' \lambda_k)}{P(\preceq'' \lambda_{k-1})} & = & P(\lambda_k) \\ & = & \mathcal{A}(\lambda_k) \underset{\mathcal{A}}{\otimes} P(\lambda) \\ & \underset{3.3.20(2)}{\cong} & \Big(\Delta(\lambda_k) \underset{\mathbb{K}}{\otimes} \Delta^{op}(\lambda_k)\Big) \underset{\mathcal{A}}{\otimes} P(\lambda) \\ & = & \Delta(\lambda_k) \underset{\mathbb{K}}{\otimes} \Big(\Delta^{op}(\lambda_k) \underset{\mathcal{A}}{\otimes} P(\lambda)\Big) \\ & \underset{1.2.44}{\cong} & \Delta(\lambda_k)^{\oplus \ dim_{\mathbb{K}}(\Delta^{op}(\lambda_k) \underset{\mathcal{A}}{\otimes} P(\lambda))} \end{array}$$

as left $\mathcal{A}$-modules. Thus $P(\lambda)$ has a Δ-filtration. In particular,

$$\mathcal{A}(\lambda) \underset{\mathcal{A}}{\otimes} P(\lambda) \cong \Delta(\lambda_k)^{\oplus \ dim_{\mathbb{K}}(\Delta^{op}(\lambda) \underset{\mathcal{A}}{\otimes} P(\lambda))}.$$

But $P(\lambda)$ is a projective cover of $L(\lambda)$ and its Δ-filtration may contain only a single copy of $\Delta(\lambda)$, otherwise its top would be semisimple (and not simple). Thus $\mathcal{A}(\lambda) \underset{\mathcal{A}}{\otimes} P(\lambda) \cong \Delta(\lambda)$.

(2) By (1), the only $\Delta(\mu)$ that may appear as quotients in the Δ-filtration (15) are those with $\mu \in \Omega$, that is $[P(\lambda) : \Delta(\mu)] \neq 0$ implies $\mu \leq \lambda$.

■

Corollary 3.4.7. *Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. Then the $\mathbb{K}$-finite category mod-$\mathcal{A}$ is a highest weight category, consequently $\mathcal{A}$ is a quasi-hereditary algebra.*

Proof. Consider a standard $\mathcal{A}$-module $\Delta(\lambda)$, by 3.4.2(1) $Top(\Delta(\lambda)) = L(\lambda)$ and all of its compositions factors are of form $L(\mu)$ with $\mu \leq \lambda$ by 3.4.2(4). So, it satisfies 2.2.9(2). By 3.4.2(3), $L(\lambda)$ is a simple $\mathcal{A}$-module, and by 3.4.2(6) $\left\{L(\lambda) : \lambda \in \Lambda_1\right\}$ is a complete set of non-isomorphic simple objects in mod-$\mathcal{A}$. So, it satisfies 2.2.9(1). The projective cover $P(\lambda)$ of $L(\lambda)$ has Δ-filtration by 3.4.6(1), and $\Delta(\lambda)$ occurs with multiplicity 1 by 3.4.6(2). So, mod-$\mathcal{A}$ satisfies 2.2.9(3). Hence, mod-$\mathcal{A}$ is a highest weight category. Consequently, $\mathcal{A}$ is quasi-hereditary algebra by 2.2.10. ■

3.5. Cartan Matrix of a Cellular Algebra

In this section, we compute the Cartan matrix of a cellular algebra.

Definition 3.5.1. Let $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}}, *_{\mathcal{A}}, \mathcal{B}, \Lambda)$ be a standard cellular algebra. We define $D = (d_{\lambda\mu})_{(\lambda,\mu)\in\Lambda\times\Lambda_1}$, where $d_{\lambda\mu} := [\Delta(\lambda) : L(\mu)]$. It is called the **decomposition matrix**.

Lemma 3.5.2. *Let $\mathbb{K}$ be a closed field, $(\mathcal{A}, +_{\mathcal{A}}, \cdot_{\mathcal{A}})$ be a $\mathbb{K}$-algebra, M be an $\mathcal{A}$-module, and P be an indecomposable projective $\mathcal{A}$-module. Then*

$$[M : Top(P)] = dim_{\mathbb{K}} Hom_{\mathcal{A}}(P, M)$$

Proof. We know that

$$Hom_{\mathcal{A}}(P, Top(P)) \cong End_{\mathcal{A}}(Top(P)) \cong \mathbb{K}$$

Also by 1.2.21 for any simple $\mathcal{A}$-module S we have:

$$Hom_{\mathcal{A}}(P, S) = \begin{cases} \mathbb{K} & : S \simeq Top(P) \\ 0 & : S \not\simeq Top(P) \end{cases}$$

Now, if M is simple, then the result follows obviously. If M is not simple, then there is a maximal $\mathcal{A}$-submodule M' of M such that M/M' is simple. So we have the following short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0$$

and, since P is a projective module, so the functor $Hom_{\mathcal{A}}(P,-)$ is exact, so we have:

$$0 \longrightarrow Hom_{\mathcal{A}}(P,M') \longrightarrow Hom_{\mathcal{A}}(P,M) \longrightarrow Hom_{\mathcal{A}}(P,M/M') \longrightarrow 0$$

and

$$dim_{\mathbb{K}} Hom_{\mathcal{A}}(P,M) = dim_{\mathbb{K}} Hom_{\mathcal{A}}(P,M') + dim_{\mathbb{K}} Hom_{\mathcal{A}}(P,M/M')$$

Now, if M' is simple, it is done, otherwise by doing induction on it, the result follows. ■

Theorem 3.5.3. *Let $(\mathcal{A},+_{\mathcal{A}},\cdot_{\mathcal{A}},*_{\mathcal{A}},\mathcal{B},\Lambda)$ be a Graham-Lehrer cellular algebra. Then we have:*

(1) $d_{\lambda\lambda}=1$ and the matrix D is uppertriangular, that is, if $\mu<\lambda$, then $d_{\lambda\mu}=0$.
(2) $d_{\nu\lambda}=dim_{\mathbb{K}} Hom_{\mathcal{A}}(P(\lambda),\Delta(\nu))=dim_{\mathbb{K}}\Delta^{op}(\nu)\underset{\mathcal{A}}{\otimes}P(\lambda)$.
(3) $C=D^tD$.

Proof. (1) By 3.4.2(3), $d_{\lambda\lambda}=1$. Suppose $\mu<\lambda$, then by the contraposition of 3.4.2(3), $L(\mu)$ is not a composition factor of $\Delta(\lambda)$, so we have $d_{\lambda\mu}=0$.
(2) $d_{\nu\lambda}=[\Delta(\nu),Top(P(\lambda))] \overset{1.2.61(2)}{=} [\Delta(\nu),L(\lambda)] \overset{3.5.2}{=} dim_{\mathbb{K}} Hom_{\mathcal{A}}(P(\lambda),\Delta(\nu))$. Now, we show the second equality of statement (2). Since the standard cellular algebra is a Graham-Lehrer one, the sets $I(\lambda)$ and $J(\lambda)$ coincide and there is a natural isomorphism between $Hom_{\mathbb{K}}(\Delta^{op}(\nu),\mathbb{K})$ and $\Delta(\nu)$ as left $\mathcal{A}$-modules. Thus, as $\mathbb{K}$-vector spaces

$$\begin{aligned}\Delta^{op}(\nu)\underset{\mathcal{A}}{\otimes}P(\lambda) &\cong Hom_{\mathbb{K}}(\Delta^{op}(\nu)\underset{\mathcal{A}}{\otimes}P(\lambda),\mathbb{K})\\ &\underset{1.2.45}{\cong} Hom_{\mathcal{A}}(P(\lambda),Hom_{\mathbb{K}}(\Delta^{op}(\nu),\mathbb{K}))\\ &\cong Hom_{\mathcal{A}}((P(\lambda),\Delta(\nu))\end{aligned}$$

and thus, $dim_{\mathbb{K}} Hom_{\mathcal{A}}(P(\lambda),\Delta(\nu))=dim_{\mathbb{K}}\Delta^{op}(\nu)\underset{\mathcal{A}}{\otimes}P(\lambda)$.

(3) We know that the Cartan matrix of $\mathcal{A}$ is $C = ([P(\lambda) : L(\mu)])_{(\lambda,\mu)\in\Lambda_1\times\Lambda_1}$. So we have

$$\begin{aligned}
[P(\lambda) : L(\mu)] &= [\mathcal{A} \underset{\mathcal{A}}{\otimes} P(\lambda) : L(\mu)] \\
&= [\mathcal{A}(\Lambda) \underset{\mathcal{A}}{\otimes} P(\lambda) : L(\mu)] \\
&= \Big[\sum_{\nu\in\Lambda} \mathcal{A}(\nu) \underset{\mathcal{A}}{\otimes} P(\lambda) : L(\mu)\Big] \\
&= \sum_{\nu\in\Lambda} [\mathcal{A}(\nu) \underset{\mathcal{A}}{\otimes} P(\lambda) : L(\mu)] \\
&\overset{3.3.20(2)}{=} \sum_{\nu\in\Lambda} [(\Delta(\nu) \underset{\mathbb{K}}{\otimes} \Delta^{op}(\nu)) \underset{\mathcal{A}}{\otimes} P(\lambda)) : L(\mu)] \\
&= \sum_{\nu\in\Lambda} [\Delta(\nu) \underset{\mathbb{K}}{\otimes} (\Delta^{op}(\nu) \underset{\mathcal{A}}{\otimes} P(\lambda)) : L(\mu)] \\
&\overset{1.2.44}{=} \sum_{\nu\in\Lambda} [\Delta(\nu)^{dim_{\mathbb{K}} \Delta^{op}(\nu) \underset{\mathcal{A}}{\otimes} P(\lambda)} : L(\mu)] \\
&= \sum_{\nu\in\Lambda} dim_{\mathbb{K}} \Delta^{op}(\nu) \underset{\mathcal{A}}{\otimes} P(\lambda) [\Delta(\nu) : L(\mu)] \\
&\overset{3.5.3(2)}{=} \sum_{\nu\in\Lambda} d_{\nu\lambda} [\Delta(\nu) : L(\mu)] \\
&= \sum_{\nu\in\Lambda} d_{\nu\lambda} d_{\nu\mu}.
\end{aligned}$$

So

$$C = ([P(\lambda) : L(\mu)])_{(\lambda,\mu)\in\Lambda_1\times\Lambda_1} = (\sum_{\nu\in\Lambda} d_{\nu\lambda} d_{\nu\mu})_{(\lambda,\mu)\in\Lambda_1\times\Lambda_1} = D^t D.$$

■

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