Université de Montréal

Polynômes Orthogonaux : Processus limites et modèles exactement résolubles

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Thèse présentée à la Faculté des études supérieures et postdoctorales
en vue de l'obtention du grade de
Philosophiæ Doctor (Ph.D.)
en physique

juin 2019

Université de Montréal

Faculté des études supérieures et postdoctorales

Cette thèse intitulée

Polynômes Orthogonaux : Processus limites et modèles exactement résolubles

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Thèse acceptée le :
23 septembre 2019

Sommaire

Cette thèse porte sur l'étude des familles de polynômes orthogonaux et leurs liens avec les modèles exactement résolubles. Elle se décline en deux parties. Dans la première, on caractérise quatre nouvelles familles de polynômes orthogonaux à l'aide de processus limites appliqués à des familles appartenant aux schéma d'Askey et de Bannai-Ito. Des troncations singulières des polynômes de Wilson et d'Askey-Wilson sont considérées. Deux premières extensions bivariées de polynômes appartenant au tableau de Bannai-Ito sont également introduites. La deuxième partie présente quatre modèles exactement résolubles en lien avec la théorie des polynômes orthogonaux. Les propriétés de transfert parfait d'information quantique et de partage d'intrication d'un modèle de chaîne de spin XX dont les couplage sont liés aux polynômes de para-Racah sont examinées. Deux modèles superintégrables contenant des opérateurs de réflexions sont proposés. Leurs solutions sont obtenues et leurs symétries s'encodent respectivement dans l'algèbre de Bannai-Ito de rang deux et de rang arbitraire ce qui mène à conjecturer l'apparition des polynômes de Bannai-Ito multivariés comme coefficients de connection. Finalement, par la théorie des représentations de la superalgèbre $\mathfrak{osp}(1|2)$, deux identités de convolution pour des familles de polynômes du tableau de Bannai-Ito sont offertes. Une réalisation en termes d'opérateurs de Dunkl conduit à une fonction génératrice bilinéaire pour les polynômes de Big -1 Jacobi.

Mots-clés

- Polynômes orthogonaux
- Modèles exactement résolubles
- Tableau de Bannai-Ito
- Opérateurs de Dunkl
- Superalgèbres de Lie

Summary

This thesis is concerned with the study of families of orthogonal polynomials and their connection to exactly solvable models. It comprises two parts. In the first one, four novel families of orthogonal polynomials are caracterized through limit processes applied to families belonging to the Askey and Bannai-Ito schemes. Singular truncations of the Wilson and Askey-Wilson polynomials are considered. The first two bivariate extensions of families of the Bannai-Ito tableau are also introduced. The second part presents four exactly solvable models connected to the theory of orthogonal polynomials. The perfect transfer of quantum information and entanglement generation properties of an XX spin chain model whose couplings are linked to the para-Racah polynomials are examined. Two superintegrable models containing reflexion operators are proposed. Their solutions are obtained and their symmetries are encoded respectively in the rank two and arbitrary rank Bannai-Ito algebra which leads to conjecture the apparition of multivariate Bannai-Ito polynomials as overlaps. Finally, via the representation theory of the $\mathfrak{osp}(1|2)$ Lie superalgebra, two convolution identities for families of orthogonal polynomials of the Bannai-Ito tableau are offered. Realizations in terms of Dunkl operators lead to a bilinear generating function for the Big -1 Jacobi polynomials.

Keywords

- Orthogonal polynomials
- Exactly solvable models
- Bannai-Ito scheme
- Dunkl operators
- Lie superlagebras

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Remerciements

Le doctorat a été d'abord et avant tout une expérience humaine extrêmement enrichissante. L'espace manque pour souligner les contributions uniques de chacun. Je tiens à remercier Luc en premier. Travailler sous sa supervision depuis les tout débuts de mon parcours universitaire a été un privilège extraordinaire. Je remercie mon ami Charles, mon Jed, qui m'a accompagné de sa présence inspirante depuis le début de cette aventure. Merci à Yvan, pédagogue hors pair, qui m'a embarqué dans de nombreux projets fascinants. Merci à Julien et Geoffroy; sans nos discussions interminables, mon doctorat n'aurait pas été le même! Un merci spécial à Ninon qui m'a fait confiance dès le début et qui m'a permis de partager ma passion par l'enseignement. Merci à ma famille pour leur support et leur amour inconditionnel. Merci à Marilou qui m'a accompagné pour un long bout de chemin. Un merci très spécial à Sandrine et ses encouragements intarissables. Finalement, un merci à tous mes amis dont les apports sont inestimables.

Introduction

Dans son manuel *Real and Complex analysis* [1], notoire avec plus de 15 000 citations, W. Rudin[1] débute son prologue avec « This is the most important function in mathematics ». Il fait référence à la fonction exponentielle dont l'ubiquité en mathématiques et en physique n'est pas à démontrer. Ses applications vont de sujets aussi élémentaires que la notation scientifique des nombres qu'à des sujets avancés comme les algèbres de Lie. Lorsqu'une fonction apparaît dans plusieurs problèmes différents, il devient naturel de s'intéresser à la nature de celle-ci et d'en faire un sujet d'étude. C'est en quelque sorte l'objet de la théorie des fonctions spéciales. L'exponentielle en est probablement l'exemple le plus connu. Il faut toutefois mentionner que les contours de ce domaine d'étude sont difficiles à cerner. Cette idée de cataloguer les propriétés d'objets mathématiques utiles n'est pas l'apanage des fonctions spéciales, mais plutôt un héritage qui remonte au moins aussi loin qu'aux Babyloniens qui utilisaient des tables d'inverses multiplicatifs et de carrés pour réduire des problèmes de multiplications et des divisions à des additions et des soustractions. L'essentiel est de réutiliser des unités de calculs pour résoudre de nouveaux problèmes. La théorie des fonctions spéciales s'inscrit dans cette tradition et bien qu'il n'existe pas de définition précise de ce qui constitue une fonction spéciale, il y a plutôt des familles de fonctions reconnues comme telles dont les propriétés sont étudiées et cataloguées [2]. Pour une introduction au problème d'une définition rigoureuse ainsi qu'un bref survol de l'histoire des fonctions spéciales, le lecteur peut se référer à [3].

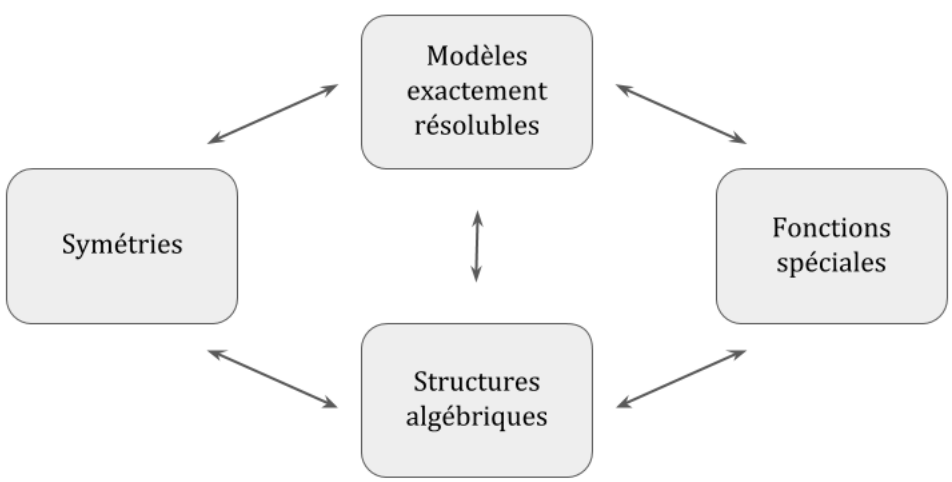
La théorie des fonctions spéciales est également intimement liée à la physique théorique. Les fonctions spéciales forment la base du langage dans lequel les solutions aux modèles exactement résolubles s'expriment, ces modèles jouant eux-mêmes un rôle important dans l'élaboration de nouvelles théories. En effet, la présence de solutions exactes permet la validation de principes théoriques fondamentaux ainsi que la formulation d'hypothèses très précises qui peuvent être éprouvées en laboratoire. De plus, l'exploration analytique de ces modèles permet la découverte de phénomènes

[1]Rudin a été le superviseur de C. Dunkl dont les travaux sont à l'origine des opérateurs de Dunkl qui jouent un rôle important dans cette thèse.

physiques intéressants sans avoir recours à des expérimentations parfois laborieuses et coûteuses. Un exemple très pertinent est l'élaboration de chaînes de spin basées sur les polynômes de Krawtchouk permettant le transport d'information quantique. En effet, le transport d'un qubit d'un bout à l'autre de la chaîne repose sur un ensemble de couplages très précis qui proviennent de la théorie de ces polynômes [4]. Des validations expérimentales dans des réseaux de fibres optiques ont même été réalisées [5, 6]. La robustesse de ces modèles a également pu être démontrée mathématiquement à l'aide d'équations diophantiennes [7]. De surcroît, l'exploration d'interactions au-delà des proches voisins a permis de prédire de nouvelles phénoménologies [8]. Les modèles exactement résolubles sont aussi une plateforme de choix pour développer une intuition et une compréhension plus fine du contenu physique d'une théorie. Ils occupent ainsi une place très importante dans l'enseignement de la physique. Mentionnons finalement que les techniques et les outils utilisés dans la résolution d'un problème sont souvent source d'inspiration au développement de nouveaux modèles plus complexes. À ce titre, on peut penser à toutes les variantes de modèles basés sur l'oscillateur harmonique [9, 10, 11, 12, 13, 14] : le couplage d'oscillateurs pour réaliser des modèles de plus haute dimension, le modèle d'oscillateur fini, la déformation du potentiel pour avoir un oscillateur singulier, l'oscillateur parabosonique avec l'ajout d'opérateurs de réflexions, etc.

La notion de résolubilité exacte signifie qu'on peut exprimer explicitement les quantités d'intérêt d'un problème en terme d'objets mathématiques connus, généralement des fonctions spéciales. Le lien se fait par une structure algébrique qui encode d'un point de vue mathématique les propriétés des fonctions spéciales en jeu et d'un point de vue physique les symétries du modèle étudié. Cette interaction s'illustre bien par le diagramme suivant :

Fig. 0.1. Interaction entre les modèles exactement résolubles et les fonctions spéciales



Dans ce diagramme, une avancée dans une des boîtes peut avoir des répercussions sur toutes les autres. C'est un terrain de recherche où les mathématiques et la physique se développent en symbiose. Bien qu'ils soient tous envisageables, on considère habituellement deux chemins dans la figure précédente. Le premier débute avec un modèle physique dont on souhaite obtenir les solutions. On s'affaire alors à identifier ses symétries et à les caractériser en termes d'une structure algébrique. La théorie des représentations de cette structure nous guide alors dans la connection entre les fonctions spéciales qui interviennent et les solutions du modèle. L'autre chemin vise plutôt l'interprétation physique d'une fonction spéciale. Dans ce cas, on cherche à décrire la structure algébrique qui encode les propriétés de cette fonction. Une fois ciblée, une réalisation en terme d'opérateurs (typiquement différentiels) de celle-ci permet d'extraire un modèle physique dont les quantités d'intérêt s'exprimeront en terme de la fonction spéciale initiale.

C'est à cette interface entre la physique et les mathématiques que se situent les travaux de recherche présentés dans cette thèse. D'une part, l'étude de fonctions spéciales et de leurs propriétés. D'autre part, leurs liens avec les modèles exactement résolubles. Ceci se reflète à travers les deux parties de cette thèse, chacune comprenant quatre articles. Le leitmotiv qui les unit est une classe particulière de fonctions spéciales : les polynômes orthogonaux. Ces objets jouissent d'une riche théorie [15] qui a débuté dans les travaux de Legendre sur les orbites planétaires et qui bénéficie aujourd'hui d'une classification très fructueuse. Les principales structures sont les deux tableaux du schéma d'Askey [16] ainsi que le récent tableau de Bannai-Ito qui est toujours en développement [17]. Les contributions apportées dans cette thèse touchent aux trois.

La partie I se concentre sur les processus limites pour introduire quatre nouvelles familles de polynômes orthogonaux. Le premier chapitre porte sur une troncation singulière des polynômes de Wilson. À travers une limite finement paramétrisée, on obtient une famille originale nommée polynômes de para-Racah. On déforme ensuite par un paramètre cette famille dans le chapitre 2. La troncation singulière des polynômes d'Askey-Wilson conduit alors aux q-polynômes de para-Racah. Les deux chapitres suivants portent sur l'extension à deux variables de familles de polynômes orthogonaux du tableau de Bannai-Ito. Il s'agit des premières familles multivariées de ce tableau d'organisation. Par des limites $q \to -1$, on obtient des familles bivariées de polynômes de Big -1 Jacobi dans le chapitre 3 et de polynômes de Bannai-Ito dans le chapitre 4. On montre également que ces deux familles sont bispectrales.

La partie II étudie des modèles où ces polynômes et des familles proches interviennent. Dans le chapitre 5, un modèle de chaînes de spins dont les couplages sont donnés par les coefficients de récurrence des polynômes de para-Racah est introduit. On montre qu'il possède les propriétés de transfert parfait d'information quantique ainsi que de revitalisation fractionnelle. Les trois derniers chapitres portent tous sur l'étude des représentations de la superalgèbre $\mathfrak{osp}(1|2)$. Dans les chapitres 6 et 7, on prend une réalisation en termes d'opérateurs de Dunkl de cette superalgèbre pour introduire des modèles superintégrables avec réflexions respectivement sur la 3-sphère et la n-sphère. À l'aide de la décomposition de Fischer et du théorème d'extension de Cauchy-Kovalevskaia, une base de fonction propres pour ces modèles est obtenue en terme des polynômes de Jacobi. On conjecture l'apparition des polynômes de Bannai-Ito multivariés comme coefficients de connection entre des bases associées à différentes sous-algèbres abéliennes maximales. Finalement, on développe dans le chapitre 8 deux identités de convolution pour des polynômes du tableau de Bannai-Ito. Ces résultats reposent sur les coefficients de Clebsch-Gordan et de Racah pour la superalgèbre $\mathfrak{osp}(1|2)$.

Partie 1

Processus limites et polynômes orthogonaux

Introduction

L'ensemble de la recherche qui a conduit à cette thèse repose sur l'importante classe de fonctions spéciales que constituent les polynômes orthogonaux. Leur pertinence tient à leurs multiples domaines d'application : probabilités, processus stochastiques, théorie de l'approximation, problèmes spectraux inverses, systèmes superintégrables, matrices aléatoires, combinatoire algébrique, etc. Une suite de polynômes à coefficients réels $\{P_n(x)\}_{n=0}^{\infty}$ où $P_n(x)$ est un polynôme de degré n dans la variable x continue ou discrète est dite orthogonale s'il existe une fonctionnelle linéaire $\mathcal{L}$ telle que pour tous entiers $n, m \geq 0$ on a

$$\mathcal{L}[P_m(x)P_n(x)] = 0 \quad \text{si} \quad m \neq n \quad \text{et} \quad \mathcal{L}[P_n^2(x)] \neq 0.$$

Une de leurs propriétés caractéristiques est de satisfaire une équation de récurrence à trois termes [15]. Sans perte de généralité, celle-ci peut s'écrire sous la forme

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x)$$

avec les conditions initiales $P_{-1}(x) = 0, P_0(x) = 1$ et où les coefficients de récurrence b_n sont des nombres réels pour tout $n \in \mathbb{N}$ et $u_n > 0$ pour $n \geq 1$. Les polynômes orthogonaux que l'on retrouve le plus souvent en physique mathématique ont également la propriété d'être hypergéométriques. C'est-à-dire qu'ils s'expriment en terme de la série hypergéométrique généralisée [2] définie par

$${}_pF_q\left[\begin{matrix} a_1, & \dots, & a_p \\ b_1, & \dots, & b_q \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!},$$

avec $(a)_k = a(a+1)(a+2)\dots(a+k-1)$ qui dénote le symbole de Pochhammer, ou encore en terme de sa q-déformation [18]

$${}_r\phi_s\left[\begin{matrix} a_1, & \dots, & a_r \\ b_1, & \dots, & b_s \end{matrix}\middle| q,\ z\right] = \sum_{k=0}^{\infty} \frac{(a_1;q)_k \dots (a_r;q)_k}{(b_1;q)_k \dots (b_s;q)_k} (-1)^{(1+s-r)k} q^{(1+s-r)\binom{2}{k}} \frac{z^k}{(q;q)_k}$$

avec $(a;q)_k = (1-a)(1-aq)\dots(1-aq^{k-1})$ qui dénote le q-symbole de Pochhammer. Ces familles ont la particularité d'être bispectrales. C'est-à-dire qu'en plus de la relation de récurrence à trois termes, ils solutionnent également une équation différentielle ou aux différences.

Les polynômes orthogonaux hypergéométriques sont regroupés dans une hiérarchie qu'on nomme schéma d'Askey [16]. Celui-ci se décline en deux tableaux : Au sommet du premier trônent les familles des polynômes de Wilson et de Racah qui dépendent de quatre paramètres. Le second est gouverné par les polynômes d'Askey-Wilson et de q-Racah qui contiennent un cinquième paramètre q dit de déformation. Toutes les autres familles s'obtiennent par spécialisation des paramètres ou par processus limite. En particulier, lorsque le paramètre $q \to 1$, on parle de « limite classique » et les familles du second tableau sont en correspondance avec celles du premier.

En 1984, Bannai et Ito ont identifié une nouvelle famille de polynômes bispectraux qui porte aujourd'hui leurs noms [19]. Celle-ci apparaissait comme limite $q \to -1$ des polynômes de q-Racah dans la classification de certains schémas d'associations. La différence entre ces polynômes et ceux du schéma d'Askey est que l'équation aux différences qu'ils satisfont contient également des opérateurs de réflexion définis par $Rf(x) = f(-x)$ [20]. Cette découverte a conduit au développement qui se poursuit toujours d'une hiérarchie complémentaire de polynômes orthogonaux bispectraux : le tableau de Bannai-Ito. La famille des polynômes de Bannai-Ito et leurs partenaires, nommés Bannai-Ito complémentaires, sont perchés au haut de ce tableau et plusieurs de leurs descendants correspondent également à des limites $q \to -1$ de familles appartenant au q-tableau d'Askey [17, 20, 21, 22, 23, 24].

Soulignons un détail important : les limites entre familles de polynômes orthogonaux s'accompagnent généralement d'une reparamétrisation complète des polynômes. Bien qu'on utilise généralement des expressions comme « limite $q \to -1$ » d'une famille de polynômes, celle-ci n'a de sens qu'avec la paramétrisation qui l'accompagne. À titre d'exemple, les deux familles en haut du tableau de Bannai-Ito peuvent chacune être obtenue à partir d'une limite $q \to -1$ des polynômes d'Askey-Wilson. Plus précisement, en réexprimant les paramètres q, a, b, c, d ainsi que la variable x des polynômes d'Askey-Wilson à l'aide des relations

$$q = -e^t, a = -ie^{t\alpha}, b = -ie^{t\beta}, c = ie^{t\gamma}, d = ie^{t\delta}, x = ie^{-2ty},$$

la limite $t \to 0$ conduit à des polynômes de Bannai-Ito dans la variable y qui dépendront des paramètres α, β, γ et δ. Toutefois, si on considère plutôt la paramétrisation

$$q = -e^t, a = ie^{t\alpha}, b = -ie^{t\beta}, c = ie^{t\gamma}, d = ie^{t\delta}, x = ie^{-2ty}$$

la limite $t \to 0$ des polynômes d'Askey-Wilson conduit alors à la famille des Bannai-Ito complémentaires. Cet exemple illustre bien la subtilité des processus limites entre familles de polynômes orthogonaux.

Une extension naturelle à la théorie des polynômes orthogonaux est la recherche de familles multivariées. L'étude générale de cette question se complique rapidement par le fait que la mesure d'orthogonalité ne caractérise plus uniquement les polynômes associés [25]. On retrouve par exemple dans le chapitre 8 des exemples de familles bivariées orthogonales par rapport à une même mesure. Soulignons deux approches qui ont fait des apports considérables à ce sujet. La première concerne la théorie des fonctions symétriques et associe des familles de polynômes orthogonaux multivariées à des systèmes de racines [26]. Ces principaux acteurs sont les familles des polynômes de Macdonald et de Koornwinder. La seconde vise à généraliser directement le schéma d'Askey par produit de familles univariées avec un entrelacement des paramètres. La clé de cette construction est de choisir les paramètres de telle sorte que l'orthogonalité des polynômes découle par induction de l'orthogonalité dans le cas univarié. Les premières familles sont dues aux travaux de Griffiths [27, 28] ainsi que ceux de Karlin et Mcgregor [29]. Toutefois, c'est à Tratnik qu'est due la généralisation à plusieurs variables du tableau $q = 1$ du schéma d'Askey [30, 31]. Par la suite, c'est Gasper et Rahman qui ont complété le portrait avec le q-tableau [32, 33]. Mentionnons également les contributions de Koornwinder sur les polynômes bivariés [34].

Cette première partie de la thèse porte sur les processus limites de familles de polynômes orthogonaux. Par cette approche, on introduit quatre nouvelles familles. La première concerne une troncation singulière des polynômes de Wilson qui se trouvent en haut du tableau $q = 1$ d'Askey en compagnie des polynômes de Racah. La principale différence entre ces deux familles est que la première est orthogonale par rapport à une mesure continue, tandis que la seconde l'est pour une mesure discrète. Il est possible de passer des polynômes de Wilson à la famille des Racah par une troncation qui consiste à spécialiser les paramètres de telle sorte que les coefficients de récurrence soient nuls pour un entier choisi. Dans ce cas particulier, sept troncations sont possibles et six sont bien connues; la dernière étant souvent écartée parce qu'elle introduit une singularité dans les coefficients de récurrence. Les travaux du premier chapitre montrent qu'il est possible de résoudre cette singularité à travers un processus limite bien choisi. Cela conduit à l'obtention d'une nouvelle famille de polynômes orthogonaux qu'on s'applique ensuite à caractériser. On nomme cette famille les polynômes de para-Racah. Le second chapitre présente une adaptation pour le q-tableau d'Askey de cette technique. On obtient alors des q-polynômes de para-Racah par une troncation singulière de la famille d'Askey-Wilson.

Les deux chapitres suivants introduisent les premiers polynômes orthogonaux en deux variables appartenant au tableau de Bannai-Ito. Les extensions bivariées de type Tratnik des familles de Big -1 Jacobi et de Bannai-Ito sont définies à l'aide de processus limites $q \to -1$. Dans chaque cas, on montre qu'elles sont bispectrales. Les relations de récurrences à trois et neuf termes sont obtenues ainsi que les opérateurs de type Dunkl que ces polynômes diagonalisent.

Chapitre 1

The para-Racah polynomials

J.M. Lemay, L. Vinet et A. Zhedanov (2016). The para-Racah polynomials. *Journal of Mathematical Analysis and Applications 438 (2), 565-577*

Abstract. New bispectral polynomials orthogonal on a quadratic bi-lattice are obtained from a truncation of Wilson polynomials. Recurrence relation and difference equation are provided. The recurrence coefficients can be encoded in a perturbed persymmetric Jacobi matrix. The orthogonality relation and an explicit expression in terms of hypergeometric functions are also given. Special cases and connections with the para-Krawtchouk polynomials and the dual-Hahn polynomials are also discussed.

1.1. Introduction

Hypergeometric orthogonal polynomials have numerous applications. We shall be concerned with polynomials that have $q = 1$ as base. The Askey tableau presents a hierarchical organization of these special functions [1]. It is comprised of a continuous part and of a discrete one. At the top of the continuous part are the Wilson polynomials expressed in terms of ${}_4F_3$ generalized hypergeometric series. A standard truncation condition on the parameters of the Wilson polynomials leads to the Racah polynomials which are orthogonal over a finite set of real points that form a quadratic lattice. The simplest limiting case of the Racah polynomials is that of the Krawtchouk polynomials orthogonal on the linear lattice.

Two of us have identified a family of orthogonal polynomials that fall outside the Askey scheme [2]. These para-Krawtchouk, as they were called, proved orthogonal over a linear bi-lattice formed by superimposing two linear lattices shifted one with respect to the other. The para-Krawtchouk polynomials naturally arise in quantum transport problems over spin chains [2, 3].

We here identify the polynomials that are orthogonal with respect to quadratic bi-lattices. They are obtained from the Wilson polynomials through a novel truncation condition. They have the dual-Hahn polynomials as a special case and the para-Krawtchouk polynomials as a special limit.

Consider the Wilson polynomials with parameters a, b, c, d denoted by $W_n(x^2; a, b, c, d)$. They obey the recurrence relation [1]

$$-(a^2 + x^2)\tilde{W}_n(x^2) = A_n\tilde{W}_{n+1}(x^2) - (A_n + C_n)\tilde{W}_n(x^2) + C_n\tilde{W}_{n-1}(x^2) \tag{1.1}$$

where

$$\tilde{W}_n(x^2) = \frac{W_n(x^2; a, b, c, d)}{(a+b)_n(a+c)_n(a+d)_n} \tag{1.2}$$

with $(a)_k = a(a+1)\ldots(a+k-1)$ the usual Pochhammer symbol and

$$A_n = \frac{(n+a+b+c+d-1)(n+a+b)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)} \tag{1.3}$$

$$C_n = \frac{n(n+b+c-1)(n+b+d-1)(n+c+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}. \tag{1.4}$$

They also satisfy the difference equation

$$\begin{aligned} n(n+a+b+c+d-1)\tilde{W}_n(x^2) = {} & \overline{\mathcal{D}(x)}\tilde{W}_n((x+i)^2) - (\overline{\mathcal{D}(x)} + \mathcal{D}(x))\tilde{W}_n(x^2) \\ & + \mathcal{D}(x)\tilde{W}_n((x-i)^2), \end{aligned} \tag{1.5}$$

where $\overline{\mathcal{D}(x)}$ is the complex conjugate of $\mathcal{D}(x)$

$$\mathcal{D}(x) = \frac{(a+ix)(b+ix)(c+ix)(d+ix)}{(2ix)(2ix+1)}. \tag{1.6}$$

The Wilson polynomials with parameters a, b, c, d, admit an explicit expression given by

$$\tilde{W}_n(x^2; a, b, c, d) = {}_4F_3\left[\begin{matrix} -n,\ n+a+b+c+d-1,\ a-ix,\ a+ix \\ a+b,\ a+c,\ a+d \end{matrix}; 1\right] \equiv \sum_k A_{n,k}\Phi_k(x^2) \tag{1.7}$$

where

$$A_{n,k} = \frac{(-n)_k(n+a+b+c+d-1)_k}{(1)_k(a+b)_k(a+c)_k(a+d)_k}, \quad \Phi_k(x^2) = (a-ix)_k(a+ix)_k. \tag{1.8}$$

It is well-known [1] that the Wilson polynomials can be reduced to a finite set of $N+1$ orthogonal polynomials if

$$A_N C_{N+1} = 0. \tag{1.9}$$

This can be achieved by setting $a+b$, $a+c$, $a+d$, $b+c$, $b+d$ or $c+d$ equal to $-N$. This leads to the Racah polynomials. Another possibility is to take $a+b+c+d-1 = -N$, but this introduces a singularity in the denominator of the recurrence coefficients. However, one can get around this problem with the use of limits and obtain new orthogonal polynomials. Our goal here is to study and characterize these polynomials which we shall call para-Racah polynomials.

This new truncation of Wilson polynomials will be presented in section 2 for odd values of N and the corresponding recurrence relation and difference equation will be obtained. In section 3, we derive an explicit expression for the para-Racah polynomials in terms of hypergeometric functions and compute their weights. The para-Racah polynomials with N even are presented in section 4. In section 5, we discuss special cases and the connections with the para-Krawtchouk polynomials and the dual-Hahn polynomials.

1.2. Recurrence relation and difference equation for N odd

1.2.1. Recurrence relation

Let $N = 2j+1$ be an odd integer. The truncation condition (1.9) with $a+b+c+d-1 = -N$ can be achieved by setting

$$b = -a - j + e_1 t, \quad d = -c - j + e_2 t, \tag{1.10}$$

and taking the limit $t \to 0$. Here, a, c are free and e_1, e_2 are deformation parameters. Inserting (1.10) in the recurrence coefficients (1.3), it is straightforward to verify that only A_j and C_{j+1} depend on e_1 and e_2 in the limit $t \to 0$. Indeed, one finds :

$$A_n = \begin{cases} \dfrac{(n-N)(n+a+c)(n+a-c-j)}{2(2n-N)} & \text{if } n \neq j, \\[2ex] \dfrac{e_1}{e_1+e_2}(j+1)(j+a+c)(a-c) & \text{if } n = j, \end{cases} \tag{1.11}$$

$$C_n = \begin{cases} \dfrac{n(n-j-1-a+c)(n-N-a-c)(n-j-1)}{(2n-1-N)(2n-N)} & \text{if } n \neq j+1, \\[2ex] \dfrac{e_2}{e_1+e_2}(j+1)(j+a+c)(a-c) & \text{if } n = j+1. \end{cases} \tag{1.12}$$

Notice that these new recurrence coefficients are now regular for all n. It can be seen from these expressions that the combinations of deformation parameters e_1, e_2 that occur are not independent and lead to a single deformation parameter α defined by

$$\frac{e_1}{e_1+e_2} = \alpha, \quad \frac{e_2}{e_1+e_2} = 1-\alpha. \tag{1.13}$$

Hence, the new recurrence coefficients are essentially those of the Wilson polynomials, except for A_j and C_{j+1} which involve the coefficient α :

$$A_j = \alpha(j+1)(j+a+c)(a-c) \tag{1.14}$$

$$C_{j+1} = (1-\alpha)(j+1)(j+a+c)(a-c). \tag{1.15}$$

It is manifest from (1.11) that the truncation condition $A_N C_{N+1} = 0$ is achieved. The resulting recurrence coefficients give rise to a finite set of polynomials $\tilde{P}_n(x^2)$ that are orthogonal with respect to a discrete measure. Consider the monic version of these polynomials where the leading coefficient is equal to 1 :

$$P_n(x^2) = (-1)^n A_1 A_2 \dots A_{n-1} \tilde{P}_n(x^2) = (x^2)^n + O((x^2)^{n-1}). \tag{1.16}$$

We shall refer to them as the (normalized) para-Racah polynomials and will denote them by $P_n(x^2; N, a, c, \alpha)$ in general or simply by $P_n(x^2)$ when no confusion can arise. They obey the recurrence relation

$$\begin{aligned} x^2 P_n(x^2) &= P_{n+1}(x^2) + (A_n + C_n - a^2) P_n(x^2) + A_{n-1} C_n P_{n-1}(x^2) \\ &\equiv P_{n+1}(x^2) + b_n P_n(x^2) + u_n P_{n-1}(x^2). \end{aligned} \tag{1.17}$$

Using (1.11) and (1.12), the recurrence coefficients can be expressed as

$$b_n = \begin{cases} -\frac{1}{2}[a(a+j) + c(c+j) + n(N-n)] & \text{if } n \neq j,\ j+1, \\ -a^2 - \frac{1}{2}j(1+a-c)(1+a+c+j) + \alpha(a-c)(1+j)(a+c+j) & \text{if } n = j, \\ -a^2 - \frac{1}{2}j(1+a-c)(1+a+c+j) + (1-\alpha)(a-c)(1+j)(a+c+j) & \text{if } n = j+1, \end{cases} \tag{1.18a}$$

$$u_n = \begin{cases} \frac{n(N+1-n)(N-n+a+c)(n-1+a+c)\left((n-j-1)^2-(a-c)^2\right)}{4(N-2n)(N-2n+2)} & \text{if } n \neq j+1, \\ \alpha(1-\alpha)(a-c)^2(1+j)^2(a+c+j)^2 & \text{if } n = j+1. \end{cases} \tag{1.18b}$$

In order to respect the general positivity condition $u_n > 0$ for $n = 1, 2, \dots, N$, one must choose one of the two following set of restrictions on a, c and α :

$$\begin{cases} -j-1 < a+c < -j+1, \\ |c-a| > j, \\ 0 < \alpha < 1, \end{cases} \quad \text{or} \quad \begin{cases} a+c < -N+1 \text{ or } 0 < a+c, \\ |c-a| < 1, \\ 0 < \alpha < 1. \end{cases} \tag{1.19}$$

When $\alpha = 1/2$, the coefficients are mirror-symmetric

$$\begin{aligned} b_n &= b_{N-n} && \text{for } n = 0, 1, \dots, N \\ u_n &= u_{N-n+1} && \text{for } n = 1, 2, \dots, N \end{aligned} \tag{1.20}$$

and for other values $0 \leq \alpha \leq 1$, the middle coefficients b_j and b_{j+1} are perturbed and no longer equal. The properties of the para-Racah polynomials can be encoded in the following tri-diagonal or Jacobi matrix with the recurrence coefficients as elements :

$$J = \begin{pmatrix} b_0 & \sqrt{u_1} & & & \\ \sqrt{u_1} & b_1 & \sqrt{u_2} & & \\ & \sqrt{u_2} & b_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{u_N} \\ & & & \sqrt{u_N} & b_N \end{pmatrix}. \tag{1.21}$$

Its action on the canonical basis $|e_n\rangle$ is

$$J|e_n\rangle = \sqrt{u_{n+1}}|e_{n+1}\rangle + b_n|e_n\rangle + \sqrt{u_n}|e_{n-1}\rangle \tag{1.22}$$

assuming $u_0 = u_{N+1} = 0$. Furthermore, the Jacobi matrix is clearly Hermitian and its eigenvalues define the set of points on which the para-Racah polynomials are orthogonal. One can introduce the eigenbasis

$$J|s\rangle = x_s|s\rangle \tag{1.23}$$

where the eigenvalues are chosen in increasing order $x_0 < x_1 < \cdots < x_N$. It is well known and straightforward to see that the eigenbasis and the canonical one are connected as follows (see e.g. [4])

$$|s\rangle = \sum_{n=0}^{N} \frac{\sqrt{w_s}\ P_n(x_s)}{\sqrt{u_1 \ldots u_n}}|e_n\rangle \tag{1.24}$$

where P_n denotes the para-Racah polynomials and w_s their weights. The mirror-symmetry (1.20) implies that the matrix J is persymmetric which means that it is symmetric with respect to the main anti-diagonal:

$$RJR = J \tag{1.25}$$

with

$$R = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}. \tag{1.26}$$

This persymmetry property has been widely studied especially in the context of inverse spectral problems [5, 6, 7, 8].

1.2.2. Difference equation

By inserting the parametrization (1.10) in the difference equation of the Wilson polynomials (1.5), it is easily seen that the limit $t \to 0$ is trivial since there are no parameters in the denominator. Hence, the para-Racah polynomials obey the same difference equation as the Wilson polynomials :

$$-n(N-n)P_n(x^2) = \overline{\mathcal{D}(x)}P_n((x+i)^2) - (\overline{\mathcal{D}(x)} + \mathcal{D}(x))P_n(x^2) + \mathcal{D}(x)P_n((x-i)^2) \tag{1.27}$$

with

$$\mathcal{D}(x) = \frac{(a+ix)(-a-j+ix)(c+ix)(-c-j+ix)}{(2ix)(2ix+1)}. \tag{1.28}$$

Remark that the spectrum is doubly-degenerate since P_n and P_{N-n} are eigenfunctions with the same eigenvalue $-n(N-n)$. These polynomials are hence bispectral but they do not belong to classical orthogonal polynomials in the usual sense.

1.3. Explicit expression and orthogonality relation for N odd

1.3.1. Explicit expression

The explicit expression for the para-Racah polynomials can be found via a limit of the Wilson polynomials. Using the parametrization (1.10) in the coefficients (1.8) yields

$$A_{n,k} = \frac{(-n)_k(n-N+(e_1+e_2)t)_k}{(1)_k(-j+e_1t)_k(a+c)_k(a-c-j+e_2t)_k}. \tag{1.29}$$

The monic para-Racah polynomials can be defined as

$$P_n(x^2) = \eta_n \sum_k \lim_{t\to 0} A_{n,k}\Phi_k(x^2) \tag{1.30}$$

where η_n simply is a normalization factor to ensure monicity. Using (1.13) and (1.29), a simple calculation gives

$$\lim_{t\to 0} A_{n,k} = \begin{cases} \dfrac{(-n)_k(n-N)_k}{(1)_k(-j)_k(a+c)_k(a-j-c)_k} & \text{if } k \leq j \text{ and } k \leq n, \\ \dfrac{\alpha^{-1}(-n)_k(n-N)_{N-n}(1)_{k-1+n-N}}{(1)_k(-j)_j(1)_{k-j-1}(a+c)_k(a-j-c)_k} & \text{if } k > j \text{ and } k \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{1.31}$$

For $n < j$, the sum (1.30) corresponds to the hypergeometric function

$$P_n(x^2) = \eta_n {}_4F_3\left[\begin{matrix} -n,\ n-N,\ a-ix,\ a+ix \\ -j,\ a+c,\ a-c-j \end{matrix}; 1\right]. \tag{1.32}$$

For $n = j$, one obtains

$$P_j(x^2) = \eta_n \sum_{k=0}^{j} \frac{(-j-1)_k (a-ix)_k (a+ix)_k}{(1)_k (a+c)_k (a-c-j)_k}. \tag{1.33}$$

This corresponds to (1.32) with $n = j$, but without the $k = j+1$ term in the summation. This discrepancy is due to the simplification of the parameter $-n$ with $-j$ when $n = j$ in the hypergeometric function, which would be responsible for the truncation at the proper degree. A similar problem occurs for $n = j+1$ if we want to express it in terms of hypergeometric functions. Instead, we write

$$P_{j+1}(x^2) = \eta_n \sum_{k=0}^{j} \frac{(-j-1)_k (a-ix)_k (a+ix)_k}{(1)_k (a+c)_k (a-c-j)_k} + \eta_n \frac{(-1)^j (a+ix)_{j+1} (a-ix)_{j+1}}{\alpha (a+c)_{j+1} (a-c-j)_{j+1}}. \tag{1.34}$$

Finally, for $n > j+1$, the sum splits in two hypergeometric functions for $k \leq j$ and $k > j$ respectively :

$$\begin{aligned} P_n(x^2) = \eta_n {}_4F_3 &\left[\begin{matrix} -n,\ n-N,\ a-ix,\ a+ix \\ -j,\ a+c,\ a-c-j \end{matrix} ; 1 \right] \\ &+ \eta_n \frac{(-n)_{j+1} (n-N)_{N-n} (a-ix)_{j+1} (a+ix)_{j+1} (1)_{n-j-1}}{\alpha (1)_{j+1} (-j)_j (a+c)_{j+1} (a-c-j)_{j+1}} \\ &\times {}_4F_3 \left[\begin{matrix} -n+j+1,\ n-j,\ a+j+1-ix,\ a+j+1+ix \\ j+2,\ a+c+j+1,\ a-c+1 \end{matrix} ; 1 \right]. \end{aligned} \tag{1.35}$$

(Take note that even though there is a negative integer $-j$ in the bottom parameters of the hypergeometric functions, the truncation of the series occurs with $-n$ for $n \leq j$ and with $n - N$ for $n > j$ before a zero appears in the denominator). The normalization factor is given by

$$\eta_n = \begin{cases} \dfrac{(1)_n (-j)_n (a+c)_n (a-c-j)_n}{(-n)_n (n-N)_n} & \text{if } n \leq j, \\[2ex] \dfrac{\alpha (1)_n (-j)_j (1)_{n-j-1} (a+c)_n (a-c-j)_n}{(-n)_n (n-N)_{N-n} (1)_{2n-1-N}} & \text{if } n > j. \end{cases} \tag{1.36}$$

1.3.2. Orthogonality relation

For a finite set of $(N+1)$ orthogonal polynomials of degree $n = 0, \ldots, N$, the zeros of the characteristic polynomial P_{N+1} of degree $N+1$ define the grid on which the polynomials are orthogonal. Here, it can be computed via

$$P_{N+1}(x^2) = \eta_{N+1} \sum_k \lim_{t \to 0} t A_{N+1,k} \Phi_k(x^2), \tag{1.37}$$

where an additionnal t is necessary to obtain non-zero coefficients. The sum can be carried with the use of the Saalschütz summation formula [1] to get

$$P_{N+1}(x^2) = \prod_{s=0}^{j} \left((a+s)^2 + x^2 \right) \left((c+s)^2 + x^2 \right). \tag{1.38}$$

Hence, the para-Racah polynomials are orthogonal on the quadratic bi-lattice

$$\begin{aligned} x_{2s} &= -(s+a)^2, \quad s = 0,\ldots,j, \\ x_{2s+1} &= -(s+c)^2, \quad s = 0,\ldots,j. \end{aligned} \tag{1.39}$$

From the standard theory of orthogonal polynomials [9], the discrete weights can be obtained via the formula

$$w_s = \frac{u_1 \ldots u_N}{P_N(x_s)P'_{N+1}(x_s)}, \quad s = 0, 1, \ldots, N \tag{1.40}$$

and the orthogonality relation is

$$\sum_{s=0}^{N} P_n(x_s)P_m(x_s)w_s = u_1 \ldots u_n \delta_{nm}. \tag{1.41}$$

There is however a simpler procedure to compute the weights that has been explained in [10]. Recall that the persymmetry property (1.20) is observed when $\alpha = 1/2$. It thus follows that in this case, the polynomial $P_N(x^2)$ takes the following values at the spectral points [4, 6] :

$$P_N(x_s) = \sqrt{u_1 \ldots u_N}(-1)^{s+1}. \tag{1.42}$$

formula (1.40) then reduces to

$$\tilde{w}_s = \frac{\sqrt{u_1 \ldots u_N}}{|P'_{N+1}(x_s)|}. \tag{1.43}$$

The corresponding positive weights can straightforwardly be computed

$$\begin{aligned} \tilde{w}_{2s} &= \frac{\kappa_N(-j)_s(2a)_s(a+1)_s(a-c-j)_s(a+c)_s}{(a+c)_{j+1}(c-a)_{j+1}(2a+1)_j s!(a)_s(2a+1+j)_s(a-c+1)_s(a+c+j+1)_s}, \\ \tilde{w}_{2s+1} &= \frac{-\kappa_N(-j)_s(2c)_s(c+1)_s(c-a-j)_s(a+c)_s}{(a+c)_{j+1}(a-c)_{j+1}(2c+1)_j s!(c)_s(2c+1+j)_s(c-a+1)_s(a+c+j+1)_s} \end{aligned} \tag{1.44}$$

with

$$\kappa_N = \frac{(a-c-j)_N(a+c)_N}{2(-1)^{j+1}\binom{2j}{j}j!}. \tag{1.45}$$

It has also been shown in [10] that the weights for a general α are related to those of the mirror-symmetric case by a simple multiplicative factor

$$w_s = const(1+\beta(-1)^s)\tilde{w}_s \tag{1.46}$$

where β is a real parameter. It is a simple matter to identify this parameter by comparing (1.40) and (1.44) for a fixed value of N. One easily verifies that $\beta = 1-2\alpha$. The general weights are then given by

$$\begin{aligned} w_{2s} &= 2(1-\alpha)\tilde{w}_{2s}, \\ w_{2s+1} &= 2\alpha\tilde{w}_{2s+1}. \end{aligned} \tag{1.47}$$

The weights have the special property that

$$\sum_{s=0}^{j} w_{2s} = 1 - \alpha, \qquad \sum_{s=0}^{j} w_{2s+1} = \alpha, \tag{1.48}$$

which generalizes a known result when $\alpha = 1/2$ for persymmetric Jacobi matrices [10].

1.4. Even N case

We have so far only considered the truncation of the Wilson polynomials for odd N. The even case is treated analogously. We shall thus only summarize how this is done and provide the results when $N = 2j$. First of all, the truncation condition (1.9) is now achieved by the parametrization

$$b = -a - j + e_1 t, \quad d = 1 - c - j + e_2 t, \tag{1.49}$$

where, as before, the limit $t \to 0$ is to be taken. With the help of (1.49) and (1.13), the recurrence coefficients (1.3) become

$$A_n = \begin{cases} \dfrac{(n-N)(n+a+c)(n+a-c-j+1)}{2(2n+1-N)} & \text{if } n \neq j, \\ \alpha j(j+a+c)(c-a-1) & \text{if } n = j, \end{cases} \tag{1.50}$$

$$C_n = \begin{cases} \dfrac{n(n-N-a-c)(n-j-1+c-a)}{2(2n-1-N)} & \text{if } n \neq j, \\ (1-\alpha)j(j+a+c)(c-a-1) & \text{if } n = j. \end{cases} \tag{1.51}$$

The recurrence relation of the para-Racah polynomials is (1.17) with

$$b_n = \frac{(n-N)(n+a+c)(n+a-c-j+1)}{2(2n+1-N)} + \frac{n(n-N-a-c)(n-j-1+c-a)}{2(2n-1-N)} - a^2 \tag{1.52a}$$

and

$$u_n = \begin{cases} \frac{n(2j-n+1)(a+c+n-1)(a-c+j-n+1)(-a+c+j-n)(a+c+2j-n)}{4(N-2n+1)^2} & \text{if } n \neq j,\ j+1, \\ -\frac{1}{2}(1-\alpha)j(j+1)(a-c)(a-c+1)(a+c+j-1)(a+c+j) & \text{if } n = j, \\ -\frac{1}{2}\alpha j(j+1)(a-c)(a-c+1)(a+c+j-1)(a+c+j) & \text{if } n = j+1. \end{cases} \tag{1.52b}$$

The positivity condition $u_n > 0$ for $n = 1, 2, \ldots, N$ now requires one of the following set of constraints on a, c and α :

$$\begin{cases} -j < a + c < -j + 1, \\ a - c < -j \text{ or } c - a < -j + 1, \\ 0 \leq \alpha \leq 1, \end{cases} \quad \text{or} \quad \begin{cases} a + c < -N + 1 \text{ or } 0 < a + c, \\ 0 < c - a < 1, \\ 0 \leq \alpha \leq 1. \end{cases} \tag{1.53}$$

The coefficients are again mirror-symmetric for $\alpha = 1/2$, but in this case, it is u_j and u_{j+1} that are perturbed for other values of α with $0 \leq \alpha \leq 1$.

Again, the difference equation for the Wilson polynomials does not change under the parametrization (1.49) and can be written as

$$-n(N-n)P_n(x^2) = \overline{\mathcal{D}(x)}P_n((x+i)^2) - (\overline{\mathcal{D}(x)} + \mathcal{D}(x))P_n(x^2) + \mathcal{D}(x)P_n((x-i)^2) \tag{1.54}$$

with

$$\mathcal{D}(x) = \frac{(a+ix)(-a-j+ix)(c+ix)(1-c-j+ix)}{(2ix)(2ix+1)}. \tag{1.55}$$

The spectrum of the corresponding Jacobi matrix is the same for N odd and N even and each eigenvalue is doubly-degenerate, except for the level $n = j$ which is non-degenerate when $N = 2j$. The para-Racah polynomials can be expressed as a sum of the form (1.30) with

$$\lim_{t \to 0} A_{n,k} = \begin{cases} \dfrac{(-n)_k(n-N)_k}{(1)_k(-j)_k(a+c)_k(a-j-c+1)_k} & \text{if } k \leq j \text{ and } k \leq n, \\ \dfrac{\alpha^{-1}(-n)_k(n-N)_{N-n}(1)_{k-1+n-N}}{(1)_k(-j)_j(1)_{k-j-1}(a+c)_k(a-j-c+1)_k} & \text{if } k > j \text{ and } k \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{1.56}$$

For $n \leq j$, the sum corresponds to the hypergeometric function

$$P_n(x^2) = \eta_{n4}F_3\left[\begin{matrix} -n, \ n-N, \ a-ix, \ a+ix \\ -j, \ a+c, \ a-c-j+1 \end{matrix}; 1\right], \tag{1.57}$$

whereas for $n > j$, the sum splits in two hypergeometric functions for $k \leq j$ and $k > j$ respectively :

$$\begin{aligned} P_n(x^2) = \eta_{n4}F_3&\left[\begin{matrix} -n, \ n-N, \ a-ix, \ a+ix \\ -j, \ a+c, \ a-c-j+1 \end{matrix}; 1\right] \\ &+ \eta_n \frac{(-n)_{j+1}(n-N)_{N-n}(a-ix)_{j+1}(a+ix)_{j+1}(1)_{n-j}}{\alpha(1)_{j+1}(-j)_j(a+c)_{j+1}(a-c-j+1)_{j+1}} \\ &\times {}_4F_3\left[\begin{matrix} -n+j+1, \ n-j+1, \ a+j+1-ix, \ a+j+1+ix \\ j+2, \ a+c+j+1, \ a-c+2 \end{matrix}; 1\right]. \end{aligned} \tag{1.58}$$

(As for the N odd case, the truncation of the hypergeometric series occurs before the negative integer $-j$ in the bottom parameters produces a zero). The normalization factor is given by

$$\eta_n = \begin{cases} \dfrac{(1)_n(-j)_n(a+c)_n(a-c-j+1)_n}{(-n)_n(n-N)_n} & \text{if } n \leq j, \\ \dfrac{\alpha(1)_n(-j)_j(1)_{n-j-1}(a+c)_n(a-c-j+1)_n}{(-n)_n(n-N)_{N-n}(1)_{2n-1-N}} & \text{if } n > j. \end{cases} \tag{1.59}$$

Finally, the orthogonality relation for $N = 2j$ is again (1.41) on the bi-lattice

$$\begin{aligned} x_{2s} &= -(s+a)^2, \quad s = 0, \dots, j, \\ x_{2s+1} &= -(s+c)^2, \quad s = 0, \dots, j-1, \end{aligned} \tag{1.60}$$

with the weights now given by

$$\begin{aligned} w_{2s} &= \frac{2(1-\alpha)\kappa_N(-j)_s(2a)_s(a+1)_s(a-c-j+1)_s(a+c)_s}{j!(a+c)_j(c-a)_j(2a+1)_j(a)_s s!(2a+1+j)_s(a-c+1)_s(a+c+j)_s}, \\ w_{2s+1} &= \frac{-2\alpha\kappa_N(-j+1)_s(2c)_s(c+1)_s(c-a-j)_s(a+c)_s}{(j-1)!(a+c)_{j+1}(a-c)_{j+1}(2c+1)_{j-1}(c)_s s!(2c+j)_s(c-a+1)_s(a+c+j+1)_s}, \end{aligned} \tag{1.61}$$

with

$$\kappa_N = \frac{(a-c-j+1)_N(a+c)_N}{(-1)^j\binom{2j}{j}}. \tag{1.62}$$

The above weights satisfy

$$\sum_{s=0}^{j} w_{2s} = 1-\alpha, \qquad \sum_{s=0}^{j-1} w_{2s+1} = \alpha, \tag{1.63}$$

that compares to (1.48) with a different range in the second sum.

1.5. Special cases

1.5.1. Connection with the para-Krawtchouk polynomials

The para-Racah polynomials that we have introduced here are orthogonal on a quadratic bi-lattice given by (1.39) for N odd and (1.60) for N even. It is possible to further deform these bi-lattices into a linear bi-lattice. Take a and c as follows

$$a(\theta) = \frac{\theta - \Delta}{2}, \qquad c(\theta) = \frac{\theta + \Delta}{2}, \tag{1.64}$$

it is straightforward to verify that

$$\lim_{\theta\to\infty} \frac{-2(x_{2s} + a(\theta)^2)}{\theta} = 2s, \quad \lim_{\theta\to\infty} \frac{-2(x_{2s+1} + a(\theta)^2)}{\theta} = 2s + 2\Delta, \tag{1.65}$$

where x_i is given by (1.39) or (1.60). This linear bi-lattice corresponds to the orthogonality set of the para-Krawtchouk polynomials [2] with parameter 2Δ. The same procedure can be applied to

(1.17) to obtain the recurrence relation of the para-Krawtchouk polynomials. Indeed, consider the polynomials $Q_n(y)$ related to the para-Racah polynomials $P_n(x^2; N, a, c, \alpha)$ by

$$x^2 = -\frac{\theta}{2}y - \left(\frac{\theta-\Delta}{2}\right)^2, \quad Q_n(y) = \left(-\frac{\theta}{2}\right)^{-n} P_n\left(x^2; N, \frac{\theta-\Delta}{2}, \frac{\theta+\Delta}{2}, \frac{1}{2}\right). \tag{1.66}$$

Inserting this in the recurrence relation (1.17) yields

$$yQ_n(y) = Q_{n+1}(y) - \frac{2b_n + 2a(\theta)^2}{\theta}\, Q_n(y) + \frac{4u_n}{\theta^2}\, Q_{n-1}(y). \tag{1.67}$$

Taking the limit $\theta \to \infty$ with the recurrence coefficients b_n and u_n given by (1.18) gives for N odd

$$\begin{aligned} &\lim_{\theta\to\infty} -\frac{2b_n + 2a(\theta)^2}{\theta} = \frac{N-1+2\Delta}{2}, \\ &\lim_{\theta\to\infty} \frac{4u_n}{\theta^2} = \frac{n(N+1-n)(2n-N-1-2\Delta)(2n-N-1+2\Delta)}{4(2n-N)(2n-N-2)}, \end{aligned} \tag{1.68}$$

and using instead (1.52) for N even, one obtains the recurrence coefficients

$$\begin{aligned} &\lim_{\theta\to\infty} -\frac{2b_n + 2a(\theta)^2}{\theta} = \frac{N-1+2\Delta}{2} + \frac{(2\Delta-1)(N+1)}{4}\left(\frac{1}{2n-N-1} - \frac{1}{2n-N+1}\right), \\ &\lim_{\theta\to\infty} \frac{4u_n}{\theta^2} = \frac{n(N+1-n)(2n-N-2\Delta)(2n-N-2+2\Delta)}{4(2n-N-1)^2}. \end{aligned} \tag{1.69}$$

The weights of the para-Krawtchouk polynomials are similarily obtained from those of the para-Racah polynomials. Substituting (1.64) in (1.44) and (1.61) and taking the limit $\theta \to \infty$ gives

$$w_{2s} = \frac{(-\Delta-j)_N(-j)_s(-\Delta-j)_s}{2(-1)^{j+1}\binom{2j}{j}s!j!(\Delta)_{j+1}(1-\Delta)_s}, \qquad w_{2s+1} = \frac{(-\Delta-j)_N(-j)_s(\Delta-j)_s}{2(-1)^{j}\binom{2j}{j}s!j!(-\Delta)_{j+1}(1+\Delta)_s},$$

for $N = 2j+1$ and

$$w_{2s} = \frac{(-\Delta-j+1)_N(-j)_s(-\Delta-j+1)_s}{(-1)^{j}\binom{2j}{j}s!j!(\Delta)_{j}(1-\Delta)_s}, \qquad w_{2s+1} = \frac{(-\Delta-j+1)_N(-j+1)_s(\Delta-j)_s}{(-1)^{j+1}\binom{2j}{j}s!(j-1)!(-\Delta)_{j+1}(1+\Delta)_s},$$

for $N = 2j$. This corresponds to the features of the para-Krawtchouk polynomials defined in [2] with parameter 2Δ. We have thus given a limiting relation between the para-Racah polynomials and the para-Krawtchouk polynomials.

1.5.2. Connection with the dual-Hahn polynomials

If one sets $c = a + 1/2$, it can be seen from (1.39) and (1.60) that the orthogonality points now form a single quadratic lattice of the form

$$x_s = -\left(\frac{s}{2} + a\right)^2, \qquad s = 0, 1, \ldots, N. \tag{1.70}$$

Furthermore, it can be shown that the para-Racah polynomials with $c = a + 1/2$ and $\alpha = 1/2$ reduce to dual-Hahn polynomials. It is straightforward to verify from the recurrence relation that

$$\left(-\frac{1}{4}\right)^n P_n\left(-\frac{1}{4}y - a^2; N, a, a+\frac{1}{2}, \frac{1}{2}\right) = r_n\left(y; N, \frac{4a-1}{2}, \frac{4a-1}{2}\right) \tag{1.71}$$

where $r_n(y; N, \gamma, \delta)$ denotes monic dual-Hahn polynomials [1].

1.5.3. Cases $\alpha = 0$ and $\alpha = 1$

The para-Racah polynomials have special properties for $\alpha = 0$ and $\alpha = 1$. When $N = 2j + 1$, the recurrence coefficient u_{j+1} becomes zero for both special values of α. In view of (1.21), we can see that the Jacobi matrix J now splits into two $(j + 1) \times (j + 1)$ tri-diagonal blocks:

$$J = \begin{pmatrix} b_0 & \sqrt{u_1} & 0 & 0 & & & & \\ \sqrt{u_1} & \ddots & \ddots & 0 & & & & \\ 0 & \ddots & \ddots & \sqrt{u_j} & & & & \\ 0 & 0 & \sqrt{u_j} & b_j & & & & \\ 1 & & & & b_{j+1} & \sqrt{u_{j+2}} & 0 & 0 \\ & & & & \sqrt{u_{j+2}} & \ddots & \ddots & 0 \\ & & & & 0 & \ddots & \ddots & \sqrt{u_N} \\ & & & & 0 & 0 & \sqrt{u_N} & b_N \end{pmatrix}. \tag{1.72}$$

This means that the recurrence relation for the para-Racah polynomials splits into two and defines independent sets of orthogonal polynomials. The first set has polynomials from degree 0 to j and the second set has polynomials from degree $j + 1$ to N. Furthermore, it can be seen from (1.47) that upon setting $\alpha = 0$ or $\alpha = 1$, half of the weights vanishes and the orthogonality grid thus reduce to a single quadratic lattice.

When $N = 2j$, similar properties arise. For $\alpha = 0$, we have $u_{j+1} = 0$ and the Jacobi matrix takes the form (1.72) with two blocks of dimension $(j + 1) \times (j + 1)$ and $(j) \times (j)$ respectively. Similarily, for $\alpha = 1$, the Jacobi matrix becomes

$$J = \begin{pmatrix} b_0 & \sqrt{u_1} & 0 & 0 & & & & \\ \sqrt{u_1} & \ddots & \ddots & 0 & & & & \\ 0 & \ddots & \ddots & \sqrt{u_{j-1}} & & & & \\ 0 & 0 & \sqrt{u_{j-1}} & b_{j-1} & & & & \\ 1 & & & & b_j & \sqrt{u_{j+1}} & 0 & 0 \\ & & & & \sqrt{u_{j+1}} & \ddots & \ddots & 0 \\ & & & & 0 & \ddots & \ddots & \sqrt{u_N} \\ & & & & 0 & 0 & \sqrt{u_N} & b_N \end{pmatrix} \tag{1.73}$$

where the first block has dimension $(j) \times (j)$ and the second block has dimension $(j+1) \times (j+1)$. Again, this means that the para-Racah polynomials splits into two sets of mutually orthogonal polynomials on a single quadratic bi-lattice.

1.6. Conclusion

To sum up, the para-Racah polynomials have been introduced and characterized. These are polynomials of a discrete variable that are orthogonal on finite quadratic bi-lattices. They are obtained from a novel truncation of the Wilson polynomials. The explicit expression, orthogonality property, recurrence relation and difference equation have been provided. The cases of odd and even numbers of cardinalities of the finite sets of polynomials must be distinguished. The para-Racah polynomials have the dual-Hahn polynomials as a special case and the para-Krawtchouk polynomials as a limiting case. (The latter are orthogonal on linear bi-lattices). Looking forward, we plan to obtain the q-generalization of the para-Racah polynomials by following an approach similar to the one adopted in this paper.

In closing, we would like to point out that the para-Racah polynomials have already found applications in the general framework of quantum information [11]. Indeed, they have been used in the design of a spin chain that can perform the perfect transfer of quantum states and generate maximally entangled states. We trust that the para-Racah polynomials will prove to have many other uses.

Acknowledgments

The authors would like to thank S. Tsujimoto for stimulating discussions. J.M.L. holds a scholarship from the Fonds de recherche du Québec – Nature et technologies (FRQNT). The research of L.V. is supported in part by NSERC. A. Z. wishes to thank the Centre de recherches mathématiques (CRM) for its hospitality.

References

[1] R. Koekoek, P. A. Lesky, R. F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, Springer, 2010.

[2] L. Vinet, A. Zhedanov, Para-Krawtchouk polynomials on a bi-lattice and a quantum spin chain with perfect state transfer, Journal of Physics A: Mathematical and Theoretical 45 (26) (2012) 265304.

[3] V. X. Genest, L. Vinet, A. Zhedanov, Quantum spin chains with fractional revival, Annals of Physics 371 (2016) 348–367.

[4] L. Vinet, A. Zhedanov, How to construct spin chains with perfect state transfer, Physical Review A 85 (2012) 012323.

[5] G. M. L. Gladwell, Inverse Problems in Vibration. Solid Mechanics and Its Applications, 2nd Edition, Springer, 2004.

[6] C. de Boor, G. H. Golub, The numerically stable reconstruction of a jacobi matrix from spectral data, Linear Algebra and its Applications 21 (3) (1978) 245 – 260.

[7] G. H. Golub, C. F. Van Loan (Eds.), Matrix Computations, The John Hopkins University Press, 1983.

[8] D. Boley, G. H. Golub, A survey of matrix inverse eigenvalue problems, Inverse Problems 3 (4) (1987) 595.

[9] T. S. Chihara, An introduction to orthogonal polynomials, Gordon and Breach, 1978.

[10] V. X. Genest, S. Tsujimoto, L. Vinet, A. Zhedanov, Persymmetric matrices, isospectral deformations and orthogonal polynomials, In preparation.

[11] J.-M. Lemay, L. Vinet, A. Zhedanov, An analytic spin chain model with fractional revival, Journal of Physics A: Mathematical and Theoretical 49 (33) (2016) 335302.

Chapitre 2

A q-generalization of the para-Racah polynomials

J.M. Lemay, L. Vinet et A. Zhedanov (2018). A q-generalization of the para-Racah polynomials. *Journal of Mathematical Analysis and Applications 462 (1), 323-336*

Abstract. New bispectral orthogonal polynomials are obtained from an unconventional truncation of the Askey-Wilson polynomials. In the limit $q \to 1$, they reduce to the para-Racah polynomials which are orthogonal with respect to a quadratic bi-lattice. The three term recurrence relation and q-difference equation are obtained through limits of those of the Askey-Wilson polynomials. An explicit expression in terms of hypergeometric series and the orthogonality relation are provided. A q-generalization of the para-Krawtchouk polynomials is obtained as a special case. Connections with the q-Racah and dual-Hahn polynomials are also presented.

2.1. Introduction

Persymmetric Jacobi matrices are invariant under reflections with respect to the anti-diagonal [1, 2, 3, 4]. Recent studies found applications for these matrices in transfer of quantum information along Heisenberg spin chains [5, 6, 7, 8, 9, 10, 11]. Necessary and sufficient conditions for achieving perfect end-to-end transfer of a qubit have been framed as properties of Jacobi matrices and their spectra [9]. In addition to persymmetry, differences of consecutive eigenvalues must satisfy certain relations. This suggested to study Jacobi matrices whose spectrum are the superposition of two lattices (bi-lattices). Interestingly, this idea led to the characterization of two novel sets of orthogonal polynomials : the para-Krawtchouk polynomials [10] which are orthogonal with respect to a linear bi-lattice and the para-Racah polynomials [12] which are orthogonal with respect to a quadratic bi-lattice.

It turns out that the para-Racah polynomials can be seen to arise from a singular truncation of the Wilson polynomials which sits atop the Askey scheme [13]. They are however not classical in the usual sense since they satisfy the same difference equation as the Wilson polynomials but with degenerate eigenvalues. In this setting, the corresponding Jacobi matrix is not persymmetric, but rather corresponds to an one-parameter isospectral deformation of the persymmetric one. Hence, the general para-Racah polynomials depend on three real parameters a, c, α and the Jacobi matrix is persymmetric only when $\alpha = 1/2$. Additionally, the para-Krawtchouk polynomials can be recovered from the para-Racah polynomials by an appropriate limit.

Our goal here is to report on a q-generalization of these results. More precisely, we will construct new orthogonal polynomials via a singular truncation of the Askey-Wilson polynomials. We name them the q-para-Racah polynomials since they reduce to the para-Racah ones in the limit $q \to 1$. A study of the most general tridiagonal representations of the q-oscillator algebra $AB - qBA = 1$ has given hints of their existence and their connection to the q-para-Krawtchouk polynomials [14].

The paper will unfold as follows. In section 2, we review some basic properties of the Askey-Wilson polynomials. We then proceed with the construction of a set of $N+1$ q-para-Racah polynomials. We start in section 3 with the case of $N = 2j+1$. The three-term recurrence relation, the q-difference equation and an explicit expression in terms of hypergeometric series are obtained from a singular truncation of the Askey-Wilson OPs. The general orthogonality relation is obtained by making use of persymmetry features that are observed when $\alpha = 1/2$. Section 4 is dedicated to the case $N = 2j$ and the corresponding formulas. A q-generalization of the para-Krawtchouk polynomials is obtained as a special case in section 5. A connection to the q-Racah and dual-Hahn polynomials is also presented and a short conclusion follows.

2.2. Askey-Wilson polynomials

Let us first review some properties of the Askey-Wilson polynomials. They are sitting atop the continuous part of the q-Askey scheme and depend on four parameters a, b, c, d. We shall denote them by $W_n(x; a, b, c, d|q)$ or simply $W_n(x)$ when the parameters need not be explicit. They admit a simple expression in terms of the hypergeometric function

$$W_n(x; a, b, c, d|q) = {}_4\phi_3\left[\begin{matrix} q^{-n},\ abcdq^{n-1},\ ae^{i\theta},\ ae^{-i\theta} \\ ab,\ ac,\ ad \end{matrix}\middle| q; q\right] \tag{2.1}$$

in the variable $x = \cos(\theta)$. For future considerations, it will be useful to write this hypergeometric series explicitly

$$W_n(x; a, b, c, d|q) \equiv \sum_{k=0}^{\infty} A_{n,k}\Phi_k(x) \tag{2.2}$$

where

$$A_{n,k} = \frac{(q^{-n}, abcdq^{n-1}; q)_k}{(q, ab, ac, ad; q)_k}, \quad \Phi_k(x) = (ae^{i\theta}, ae^{-i\theta}; q)_k, \tag{2.3}$$

with the standard notation for the q-Pochhammer symbol $(a;q)_k = (1-a)(1-aq)\dots(1-aq^{k-1})$ and its products $(a_1, a_2, \dots, a_n; q)_k = (a_1;q)_k(a_2;q)_k \dots (a_n;q)_k$. The Askey-Wilson polynomials satisfy a q-difference equation

$$q^{-n}(1-q^n)(1-abcdq^{n-1})W_n(x) = A(\theta)T_+W_n(x) - [A(\theta)+\overline{A}(\theta)]W_n(x) + \overline{A}(\theta)T_-W_n(x) \tag{2.4}$$

where the $T_\pm$ are the following q-shift operators

$$\begin{aligned} T_+e^{i\theta} &= qe^{i\theta} \qquad & T_-e^{i\theta} &= q^{-1}e^{i\theta} \\ T_+e^{-i\theta} &= q^{-1}e^{-i\theta} \qquad & T_-e^{-i\theta} &= qe^{-i\theta} \end{aligned} \tag{2.5}$$

and the coefficients $A(\theta)$ and its complex conjugate $\overline{A}(\theta)$ are defined by

$$A(\theta) = \frac{(1-ae^{i\theta})(1-be^{i\theta})(1-ce^{i\theta})(1-de^{i\theta})}{(1-e^{2i\theta})(1-qe^{2i\theta})}. \tag{2.6}$$

Moreover, the $W_n(x)$ obey the three-term recurrence relation

$$2xW_n(x) = A_nW_{n+1}(x) + (a + a^{-1} - A_n - C_n)W_n(x) + C_nW_{n-1}(x) \tag{2.7}$$

with recurrence coefficients

$$\begin{aligned} A_n =& \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})}, \\ C_n =& \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}. \end{aligned} \tag{2.8}$$

It is sometimes more convenient to work with the monic Askey-Wilson polynomials $\tilde{W}_n(x)$. They are given by

$$W_n(x) = \frac{2^n}{A_1A_2\dots A_{n-1}}\tilde{W}_n(x) \tag{2.9}$$

and satisfy the recurrence relation

$$x\tilde{W}_n(x) = \tilde{W}_{n+1}(x) + \tfrac{1}{2}(a + a^{-1} - A_n - C_n)\tilde{W}_n(x) + \tfrac{1}{4}A_{n-1}C_n\tilde{W}_{n-1}(x). \tag{2.10}$$

The orthogonality relation and many more properties on the Askey-Wilson polynomials can be found in [**15**, **16**]. It is well-known that the Askey-Wilson polynomials can be reduced to a finite set of $N+1$ orthogonal polynomials if

$$A_NC_{N+1} = 0. \tag{2.11}$$

It is easy to see that there are multiple ways to achieve this condition by looking at the recurrence coefficients (2.8). Many of these possibilities lead to the q-Racah polynomials. More precisely, if one chooses the parameters so that either $(1-abq^N),(1-acq^N),(1-adq^N),(1-bcq^N),(1-bdq^N)$ or $(1-cdq^N)$ is equal to zero, then the corresponding polynomials will be the q-Racah polynomials. There is however another possibility which consists in choosing

$$1 - abcdq^{N-1} = 0, \tag{2.12}$$

but this introduces a singularity in the denominators of A_n and C_n for $n \sim N/2$. The present article deals with this option and we show that under an appropriate parametrization, it is possible to recover a set of orthogonal polynomials distinct from the q-Racah polynomials that has not been studied before as far as we know. In [13], we presented a similar argument for the case $q = 1$.

2.3. Odd case : $N = 2j + 1$

We aim to study the singular truncation (2.12) and to characterize the corresponding orthogonal polynomials. Remark first that depending on the parity of N, the singular factors in the denominators of A_n and C_n given by (2.8) change. This implies that both cases must be treated separately. This section will deal with the case N odd and we relegate the case N even to the next section.

Now, fix $N = 2j + 1$ and let

$$b = a^{-1}q^{-j+e_1 t}, \qquad d = c^{-1}q^{-j+e_2 t}, \qquad t \to 0. \tag{2.13}$$

Notice that in the limit $t \to 0$, this parametrization realizes the truncation condition (2.12).

2.3.1. Recurrence relation

Injecting the formulas (2.13) in the recurrence coefficients (2.8) and taking the limit $t \to 0$, one can check that the only coefficients depending on e_1 and e_2 are

$$\lim_{t\to 0} A_j = \left(\frac{e_1}{e_1+e_2}\right) \frac{(1-acq^j)(c-a)(1-q^{-j-1})}{ac(1-q^{-1})}, \tag{2.14}$$

$$\lim_{t\to 0} C_{j+1} = \left(\frac{e_2}{e_1+e_2}\right) \frac{(1-q^{j+1})(c-a)(ac-q^{-j})}{ac(1-q)}. \tag{2.15}$$

Remark that the two occurences of the parameters e_1 and e_2 are not independent since they sum to one. They can hence be combined into a single deformation parameter α :

$$\frac{e_1}{e_1+e_2} = \alpha, \qquad \frac{e_2}{e_1+e_2} = 1-\alpha. \tag{2.16}$$

With this notation, the limiting recurrence coefficients can be expressed as

$$\lim_{t\to 0} A_n = \begin{cases} \dfrac{(1-acq^n)(c-aq^{n-j})(1-q^{n-2j-1})}{ac(1-q^{2n-2j-1})(1+q^{n-j})} & n \neq j, \\ \dfrac{\alpha(1-acq^j)(c-a)(1-q^{-j-1})}{ac(1-q^{-1})} & n = j, \end{cases} \tag{2.17}$$

$$\lim_{t\to 0} C_n = \begin{cases} \dfrac{(1-q^n)(a-cq^{n-j-1})(ac-q^{n-2j-1})}{ac(1+q^{n-j-1})(1-q^{2n-2j-1})} & n \neq j+1, \\ \dfrac{(1-\alpha)(1-q^{j+1})(a-c)(ac-q^{-j})}{ac(1-q)} & n = j+1. \end{cases} \tag{2.18}$$

These new recurrence coefficients verify $A_N C_{N+1} = 0$ with $N = 2j+1$ and thus provides a finite set of polynomials. We define the (monic) q-para-Racah polynomials $R_n(x; a, c, \alpha|q)$ (or $R_n(x)$ for short) via the three-term recurrence relation

$$xR_n(x) = R_{n+1}(x) + b_n R_n(x) + u_n R_{n-1}(x) \tag{2.19}$$

with initial conditions $R_{-1}(x) = 0, R_0(x) = 1$ and where the recurrence coefficients are those of the monic Askey-Wilson polynomials (2.10) under the limit given by (2.13)

$$b_n = \lim_{t\to 0} \tfrac{1}{2}(a + a^{-1} - A_n - C_n), \tag{2.20}$$

$$u_n = \lim_{t\to 0} \tfrac{1}{4} A_{n-1} C_n. \tag{2.21}$$

Using formulas (2.17) and (2.18), they can be written as

$$b_n = \begin{cases} \dfrac{(a+c)\left(q^{j+1}+1\right)q^n\left(acq^j+1\right)}{2ac\left(q^j+q^n\right)\left(q^{j+1}+q^n\right)} & n \neq j, j+1, \\ \dfrac{a+a^{-1}}{2} + \dfrac{\alpha(c-a)\left(q^{j+1}-1\right)q^{-j}\left(acq^j-1\right)}{2ac(q-1)} - \dfrac{\left(q^j-1\right)q^{-j}(c-aq)\left(acq^{j+1}-1\right)}{2ac\left(q^2-1\right)} & n = j, \\ \dfrac{a+a^{-1}}{2} + \dfrac{(1-\alpha)(c-a)\left(q^{j+1}-1\right)q^{-j}\left(acq^j-1\right)}{2ac(q-1)} - \dfrac{\left(q^j-1\right)q^{-j}(c-aq)\left(acq^{j+1}-1\right)}{2ac\left(q^2-1\right)} & n = j+1, \end{cases} \tag{2.22}$$

$$u_n = \begin{cases} \dfrac{(q^n-1)\left(q^n-q^{2j+2}\right)(acq^n-q)\left(q^n-acq^{2j+1}\right)\left(aq^n-cq^{j+1}\right)\left(cq^n-aq^{j+1}\right)}{4a^2c^2\left(q^{j+1}+q^n\right)^2\left(q^{2n}-q^{2j+1}\right)\left(q^{2n}-q^{2j+3}\right)} & n \neq j+1, \\ \dfrac{(1-\alpha)\alpha(c-a)^2 q^{-2j}\left(q^{j+1}-1\right)^2\left(acq^j-1\right)^2}{4a^2c^2(q-1)^2} & n = j+1. \end{cases} \tag{2.23}$$

Remark : It is straightforward to verify that these coefficients are persymmetric when $\alpha = 1/2$, i.e. that the following relations hold :

$$\begin{aligned} b_n &= b_{N-n}, \qquad n = 0, 1, \ldots, N, \\ u_n &= u_{N-n+1}, \quad n = 1, 2, \ldots, N. \end{aligned} \tag{2.24}$$

For $\alpha \neq 1/2$, the coefficients b_j and b_{j+1} are perturbed and no longer equal.

Furthermore, owing to Favard's theorem, these polynomials will be orthogonal if they satisfy $u_n > 0$ for $n = 1, \ldots, N$. With some easy computation, this is seen to be tantamount to the following conditions on the parameters

$$\begin{aligned} &0 < q < 1, \qquad 0 < \alpha < 1, \qquad c \neq a, \\ &q < \tfrac{a}{c} < q^{-1}, \qquad ac < 1 \quad \text{or} \quad ac > q^{1-N}. \end{aligned} \tag{2.25}$$

2.3.2. q-Difference equation

Exploiting again the limit procedure given by (2.13), it is possible to recover a q-difference equation for the q-para-Racah polynomials from that of the Askey-Wilson polynomials. Indeed, this procedure is trivial since the q-difference equation (2.4) contains no parameters in the denominator. Thus, the $R_n(x)$ will satisfy

$$q^{-n}(1-q^n)(1-q^{n-N})R_n(x) = A(\theta)T_+R_n(x) - [A(\theta)+\overline{A}(\theta)]R_n(x) + \overline{A}(\theta)T_-R_n(x) \tag{2.26}$$

where the $T_\pm$ are again given by (2.5) and the coefficients $A(\theta)$ become

$$A(\theta) = \frac{(1-ae^{i\theta})(1-a^{-1}q^{-j}e^{i\theta})(1-ce^{i\theta})(1-c^{-1}q^{-j}e^{i\theta})}{(1-e^{2i\theta})(1-qe^{2i\theta})} \tag{2.27}$$

with $\overline{A}(\theta)$ obtained by complex conjugation. The q-para-Racah polynomials are thus bispectral, but we remark that upon scaling the polynomials by a factor q^{-n}, the eigenvalues from equation (2.26) are degenerate in contrast to the usual classical orthogonal polynomials.

2.3.3. Explicit expression

It is possible to obtain an explicit expression for the q-para-Racah polynomials from the hypergeometric expression of the Askey-Wilson polynomials. Consider the series expansion (2.2) and use the parametrization (2.13). In the limit $t \to 0$, the coefficients (2.3) reduces to

$$\lim_{t\to 0} A_{n,k} = \begin{cases} \dfrac{(q^{-n}, q^{n-2j-1}; q)_k q^k}{(q^{-j}, \frac{a}{c}q^{-j}, ac, q; q)_k} & k \leq j \text{ and } k \leq n, \\[2ex] \dfrac{(q^{n-2j-1}; q)_{2j+1-n}(q;q)_{n+k-2j-2}(q^{-n};q)_k q^k}{\alpha(q^{-j};q)_j(q;q)_{k-j-1}(\frac{a}{c}q^{-j}, ac, q; q)_k} & k > j \text{ and } k \leq n, \\[2ex] 0 & \text{otherwise.} \end{cases} \tag{2.28}$$

This allows us to express the $R_n(x)$ as

$$R_n(x) = \eta_n \sum_k \left(\lim_{t\to 0} A_{n,k}\right) \Phi_k(x) \tag{2.29}$$

where η_n is a normalization factor to ensure the polynomials are monic. With the help of some well-known identities for q-Pochhammer symbols, the following expressions can be obtained :
If $n < j$,

$$R_n(x) = \eta_n {}_4\phi_3\left[\begin{matrix} q^{-n},\ q^{n-2j-1},\ ae^{i\theta},\ ae^{-i\theta} \\ q^{-j},\ ac,\ ac^{-1}q^{-j} \end{matrix}\middle| q; q\right]. \tag{2.30}$$

If $n = j$,

$$R_j(x) = \eta_j \sum_{k=0}^{j} \frac{(q^{-j-1}, ae^{i\theta}, ae^{-i\theta}; q)_k q^k}{(q, ac, ac^{-1}q^{-j}; q)_k}. \tag{2.31}$$

If $n = j+1$,

$$R_{j+1}(x) = \eta_{j+1} \sum_{k=0}^{j} \frac{(q^{-j-1}, ae^{i\theta}, ae^{-i\theta}; q)_k q^k}{(q, ac, ac^{-1}q^{-j}; q)_k} + \eta_{j+1} \frac{(q^{-j-1}, ae^{i\theta}, ae^{-i\theta}; q)_{j+1} q^{j+1}}{\alpha(q, ac, ac^{-1}q^{-j}; q)_{j+1}}. \tag{2.32}$$

If $j+1 < n \leq N$,

$$\begin{aligned} R_n(x) = \eta_n {}_4\phi_3&\left[\begin{matrix} q^{-n},\ q^{n-2j-1},\ ae^{i\theta},\ ae^{-i\theta} \\ q^{-j},\ ac,\ ac^{-1}q^{-j} \end{matrix}\middle| q; q\right] \\ &+ \eta_n \frac{(q^{n-2j-1}; q)_{2j+1-n}(q^{-n}, ae^{i\theta}, ae^{-i\theta}; q)_{j+1}(q; q)_{n-j-1} q^{j+1}}{\alpha(q^{-j}; q)_j (q, ac, ac^{-1}q^{-j}; q)_{j+1}} \\ &\times {}_4\phi_3\left[\begin{matrix} q^{j+1-n},\ q^{n-j},\ aq^{j+1}e^{i\theta},\ aq^{j+1}e^{-i\theta} \\ q^{j+2},\ acq^{j+1},\ ac^{-1}q \end{matrix}\middle| q; q\right]. \end{aligned} \tag{2.33}$$

The normalization η_n is given by

$$\eta_n = \begin{cases} \dfrac{(q, q^{-j}, ac^{-1}q^{-j}, ac; q)_n}{(q^{n-2j-1}, q^{-n}; q)_n (-2a)^n q^{n(n+1)/2}} & n \leq j, \\ \dfrac{\alpha(q^{-j}; q)_j (q; q)_{n-j-1} (ac^{-1}q^{-j}, ac, q; q)_n}{(q^{n-2j-1}; q)_{2j+1-n} (q; q)_{2n-2j-2} (q^{-n}; q)_n (-2a)^n q^{n(n+1)/2}} & n > j. \end{cases} \tag{2.34}$$

The q-para-Racah polynomials thus generally admit an explicit expression as a linear combination of two basic hypergeometric functions. However, due to some cancellations between parameters in the numerator and the denominator of the hypergeometric function when $n = j$ and $n = j+1$, the polynomials $R_j(x)$ and $R_{j+1}(x)$ have to be expressed as a sum which corresponds to a "truncated" hypergeometric series.

Moreover, the non-monic q-para-Racah polynomials given by $R_n(x; a, c, \alpha|q)/\eta_n$ reduces to the (non-monic) para-Racah polynomials described in [13] upon substituting

$$a \to q^a, \quad c \to q^c, \quad e^{i\theta} \to q^{ix} \tag{2.35}$$

and taking the limit $q \to 1$. Akin to the connection between the Askey-Wilson and the Wilson polynomials, the $q \to 1$ limit is taken in the explicit expressions of the polynomials instead of the recurrence relation.

2.3.4. Orthogonality relation

In order to obtain the orthogonality relation of the q-para-Racah polynomials, we begin by computing the characteristic polynomial which gives the spectrum of the Jacobi matrix or, equivalently, the orthogonality lattice. In a similar fashion to the derivation of the $R_n(x)$, consider

$$\lim_{t\to 0}(1-q^{(e_1+e_2)t})A_{N+1,k} = \begin{cases} \dfrac{(q^{-N-1};q)_k q^k}{\alpha(ac,ac^{-1}q^{-j};q)_k(q^{-j};q)_j(q;q)_{k-j-1}} & j+1 \leq k \leq N+1, \\ 0 & \text{otherwise,} \end{cases} \tag{2.36}$$

which can be summed with the $\Phi_k(x)$ to obtain

$$R_{N+1}(x) \propto (ae^{i\theta}, ae^{-i\theta};q)_{j+1} \times {}_3\phi_2\left[\begin{matrix} q^{-j-1}, \ aq^{j+1}e^{i\theta}, \ aq^{j+1}e^{-i\theta} \\ acq^{j+1}, \ ac^{-1}q \end{matrix}\middle| q;q\right]. \tag{2.37}$$

Note that it is possible to neglect the normalization constant since we are only interested in the zeros of $R_{N+1}(x)$. Now, using the Saalschutz q-summation formula, (2.37) can be factorized as

$$R_{N+1}(x) \propto (ae^{i\theta}, ae^{-i\theta};q)_{j+1}(ce^{i\theta}, ce^{-i\theta};q)_{j+1}. \tag{2.38}$$

The orthogonality lattice will correspond to the zeros of (2.38) :

$$x_{2s} = \tfrac{1}{2}(a^{-1}q^{-s} + aq^s) \qquad s = 0,1,\dots,j, \tag{2.39}$$

$$x_{2s+1} = \tfrac{1}{2}(c^{-1}q^{-s} + cq^s) \qquad s = 0,1,\dots,j. \tag{2.40}$$

Hence, the q-para-Racah polynomials will obey an orthogonality relation of the form

$$\sum_{s=0}^{N} w_s R_n(x_s) R_m(x_s) = u_1 u_2 \dots u_n \delta_{nm} \tag{2.41}$$

where the x_s are given by (2.39) and the normalization constants are given by the recurrence coefficients (2.23). A standard formula from the theory of orthogonal polynomials explicitly gives the weights [17] :

$$w_s = \frac{u_1 u_2 \dots u_N}{R_n(x_s) R'_{N+1}(x_s)}, \quad s = 0,1,\dots,N. \tag{2.42}$$

However, due to the involved nature of $R_N(x)$ given by (2.33), we will use a simpler procedure which exploits the persymmetry that arises when $\alpha = 1/2$ [18]. In this case, when the polynomial $R_N(x)$ is evaluated at the zeros of the characteristic polynomial x_s, one obtains a simple expression which is due to the interlacing properties of their zeros :

$$R_N(x_s) = \sqrt{u_1 u_2 \dots u_N}(-1)^{N+s}. \tag{2.43}$$

Combining (2.42) and (2.43), it is easy to compute the weights for $\alpha = 1/2$ which we shall denote by $\tilde{w}_s$. The weights for general α have been shown to be related to the $\tilde{w}_s$ by a simple multiplicative factor in [18] :

$$w_s \propto (1 + \beta(-1)^s)\tilde{w}_s \tag{2.44}$$

where β is a real parameter independent of N. One can easily obtain β by comparing (2.42) and (2.44) for a fixed value of N (e.g. $N = 3$ for simplicity). Here, one obtains $\beta = 1 - 2\alpha$. Carrying through the calculation, one readily obtains

$$w_{2s} = -\frac{2(1-\alpha)K_N 2^{2j+1} a^j c^{j+1} q^{(2j+1)s+(j+1)j}\left(1-a^2q^{2s}\right)\left(a^2;q\right)_s\left(q^{-j};q\right)_s(ac;q)_s\left(\frac{aq^{-j}}{c};q\right)_s}{(q;q)_j\left(a^2q;q\right)_j\left(\frac{c}{a};q\right)_{j+1}(ac;q)_{j+1}\left(1-a^2\right)(q;q)_s\left(\frac{aq}{c};q\right)_s\left(a^2q^{j+1};q\right)_s\left(acq^{j+1};q\right)_s} \tag{2.45}$$

$$w_{2s+1} = \frac{2\alpha K_N 2^{2j+1} c^j a^{j+1} q^{(2j+1)s+(j+1)j}\left(1-c^2q^{2s}\right)\left(c^2;q\right)_s\left(q^{-j};q\right)_s(ac;q)_s\left(\frac{cq^{-j}}{a};q\right)_s}{(q;q)_j\left(c^2q;q\right)_j\left(\frac{a}{c};q\right)_{j+1}(ac;q)_{j+1}\left(1-c^2\right)(q;q)_s\left(\frac{cq}{a};q\right)_s\left(c^2q^{j+1};q\right)_s\left(acq^{j+1};q\right)_s} \tag{2.46}$$

where K_N is a normalization constant arising in the persymmetric case $\alpha = 1/2$ and given by

$$K_N = \sqrt{u_1 u_2 \dots u_N}. \tag{2.47}$$

It can easily be computed by using the persymmetry of the u_n (2.24) :

$$K_N = \frac{(a-c)q^{-j}\left(q^{j+1}-1\right)\left(acq^j-1\right)\left(q^{-j};q\right)_j^2\left(q^{-2j-1};q\right)_j(q;q)_j(ac;q)_j\left(\frac{aq^{-j}}{c};q\right)_j\left(\frac{cq^{-j}}{a};q\right)_j\left(\frac{q^{-2j}}{ac};q\right)_j}{ac(q-1)2^{2j+2}\left(q^{-2j-1};q^2\right)_j\left(q^{-2j};q^2\right)_j^2\left(q^{1-2j};q^2\right)_j}.$$

The weights are normalized to verify

$$\sum_{s=0}^{j} w_{2s} = 1 - \alpha \qquad \sum_{s=0}^{j} w_{2s+1} = \alpha \tag{2.48}$$

which generalizes a known result for persymmetric Jacobi matrices.

When $c = a$ the spectrum becomes doubly degenerate, i.e. $x_{2s} = x_{2s+1}$. This degeneracy is related with the degeneracy of the recurrence coefficient $u_{j+1} = 0$ as seen from (2.23). This means that the corresponding Hermitian Jacobi (tridiagonal) matrix of the recurrence coefficients b_n and u_n becomes reducible: it can be decomposed into a direct sum of two independent Jacobi matrices each having the same (now simple) spectrum x_{2s}.

2.4. Even case : $N = 2j$

The construction of the q-para-Racah polynomials for even values of N is similar to the N odd case. We review quickly the procedure in this section and give the corresponding results.

Let $N = 2j$. The singular truncation (2.12) is achieved via the parametrization and limit

$$b = a^{-1}q^{-j+e_1t}, \qquad d = c^{-1}q^{-j+1+e_2t}, \qquad t \to 0. \tag{2.49}$$

As before, the parameters e_1 and e_2 are not independent and can be encoded in a single deformation parameter α by

$$\frac{e_1}{e_1+e_2} = \alpha \qquad \frac{e_2}{e_1+e_2} = 1-\alpha. \tag{2.50}$$

2.4.1. Recurrence relation

The q-para-Racah polynomials $R_n(x;a,c,\alpha|q)$, or $R_n(x)$ for short, are defined by the recurrence relation

$$xR_n(x) = R_{n+1}(x) + b_n R_n(x) + u_n R_{n-1}(x) \tag{2.51}$$

with initial conditions $R_{-1}(x) = 0, R_0(x) = 1$ and with the coefficients given by

$$b_n = \lim_{t\to 0} \tfrac{1}{2}(a + a^{-1} - A_n - C_n), \tag{2.52}$$

$$u_n = \lim_{t\to 0} \tfrac{1}{4} A_{n-1}C_n. \tag{2.53}$$

where A_n and C_n are the recurrence coefficients of the Askey-Wilson polynomials (2.8) in which we substituted parametrization (2.49). A straightforward calculation yields

$$b_n = \frac{a+a^{-1}}{2} + \frac{(q^n-1)\left(acq^{2j}-q^n\right)\left(aq^{j+1}-cq^n\right)}{2ac\left(q^j+q^n\right)\left(q^{2j+1}-q^{2n}\right)} + \frac{\left(q^{2j}-q^n\right)\left(acq^n-1\right)\left(cq^j-aq^{n+1}\right)}{2ac\left(q^j+q^n\right)\left(q^{2j}-q^{2n+1}\right)}, \tag{2.54}$$

$$u_n = \begin{cases} \dfrac{(q^n-1)\left(q^n-q^{2j+1}\right)\left(acq^n-q\right)\left(q^n-acq^{2j}\right)\left(aq^n-cq^j\right)\left(cq^n-aq^{j+1}\right)}{4a^2c^2\left(q^j+q^n\right)\left(q^{j+1}+q^n\right)\left(q^{2j+1}-q^{2n}\right)^2} & n \neq j, j+1, \\ \dfrac{(1-\alpha)(c-a)q^{-2j}\left(q^j-1\right)\left(q^{j+1}-1\right)\left(aq-c\right)\left(acq^j-1\right)\left(acq^j-q\right)}{4a^2c^2(q-1)^2(q+1)} & n = j, \\ \dfrac{\alpha(c-a)q^{-2j}\left(q^j-1\right)\left(q^{j+1}-1\right)\left(aq-c\right)\left(acq^j-1\right)\left(acq^j-q\right)}{4a^2c^2(q-1)^2(q+1)} & n = j+1. \end{cases} \tag{2.55}$$

The recurrence coefficients are also persymmetric, i.e. satisfy (2.24), when $\alpha = 1/2$. The positivity conditions $u_n > 0$ for $n = 1, \ldots, N$ are verified when the parameters obey

$$\begin{gathered} 0<q<1, \qquad 0<\alpha<1, \qquad c\neq a, \\ q<\tfrac{a}{c}<q^{-1}, \qquad ac<1 \quad \text{or} \quad ac>q^{1-N}. \end{gathered} \tag{2.56}$$

2.4.2. q-Difference equation

The q-difference equation for $N = 2j$ is obtained by inserting the limit procedure given by (2.49) in (2.4). In this case, the $R_n(x)$ satisfy

$$q^{-n}(1-q^n)(1-q^{n-N})R_n(x) = A(\theta)T_+R_n(x) - [A(\theta)+\overline{A}(\theta)]R_n(x) + \overline{A}(\theta)T_-R_n(x) \tag{2.57}$$

where the $T_\pm$ are given by (2.5) and the coefficients by

$$A(\theta) = \frac{(1-ae^{i\theta})(1-a^{-1}q^{-j}e^{i\theta})(1-ce^{i\theta})(1-c^{-1}q^{-j+1}e^{i\theta})}{(1-e^{2i\theta})(1-qe^{2i\theta})} \tag{2.58}$$

and its complex conjugate. Again, the q-para-Racah polynomials are bispectral, but each eigenvalues is degenerate upon rescaling the polynomials by q^{-n}.

2.4.3. Explicit expression

An explicit expression for the $R_n(x)$ can readily be obtained by inserting the parametrization (2.49) in the series expansion (2.2). Summing the resulting terms, one obtains for $n \leq j$:

$$R_n(x) = \eta_n {}_4\phi_3\left[\begin{matrix} q^{-n},\ q^{n-2j},\ ae^{i\theta},\ ae^{-i\theta} \\ q^{-j},\ ac,\ ac^{-1}q^{-j+1} \end{matrix}\middle| q; q\right], \tag{2.59}$$

and for $j+1 \leq n \leq N$:

$$\begin{aligned} R_n(x) = {} & \eta_n {}_4\phi_3\left[\begin{matrix} q^{-n},\ q^{n-2j},\ ae^{i\theta},\ ae^{-i\theta} \\ q^{-j},\ ac,\ ac^{-1}q^{-j+1} \end{matrix}\middle| q; q\right] \\ & + \eta_n \frac{(q^{n-2j};q)_{2j-n}(q^{-n}, ae^{i\theta}, ae^{-i\theta};q)_{j+1}(q;q)_{n-j}q^{j+1}}{\alpha(q^{-j};q)_j(q, ac, ac^{-1}q^{-j+1};q)_{j+1}} \\ & \times {}_4\phi_3\left[\begin{matrix} q^{j+1-n},\ q^{n-j+1},\ aq^{j+1}e^{i\theta},\ aq^{j+1}e^{-i\theta} \\ q^{j+2},\ acq^{j+1},\ ac^{-1}q^{2} \end{matrix}\middle| q; q\right] \end{aligned} \tag{2.60}$$

with monicity ensured by the normalization

$$\eta_n = \begin{cases} \dfrac{(q, q^{-j}, ac^{-1}q^{-j+1}, ac;q)_n}{(q^{n-2j}, q^{-n};q)_n(-2a)^n q^{n(n+1)/2}} & n \leq j, \\ \dfrac{\alpha(q^{-j};q)_j(q;q)_{n-j-1}(ac^{-1}q^{-j+1}, ac, q;q)_n}{(q^{n-2j};q)_{2j-n}(q;q)_{2n-2j-1}(q^{-n};q)_n(-2a)^n q^{n(n+1)/2}} & n > j. \end{cases} \tag{2.61}$$

The polynomials of degree j and $j+1$ need not be distinguished when $N = 2j$ because the simplification of the parameters in the hypergeometric function does not change where the series truncate in constrast with the N odd case.

2.4.4. Orthogonality relation

The characteristic polynomial can once more be computed via

$$R_{N+1}(x) = \eta_{N+1} \sum_{k=0}^{\infty} \lim_{t\to 0}(1-q^{(e_1+e_2)t})A_{N+1,k}\Phi_k(x). \tag{2.62}$$

Carrying through the computation and using the Saalschutz q-summation formula, it can be expressed in factorized form as

$$R_{N+1}(x) \propto (ae^{i\theta}, ae^{-i\theta};q)_{j+1}(ce^{i\theta}, ce^{-i\theta};q)_j. \tag{2.63}$$

The orthogonality grid is again has the form of a biexponential bi-lattice :

$$x_{2s} = \tfrac{1}{2}(a^{-1}q^{-s} + aq^s) \qquad s = 0, 1, \ldots, j, \tag{2.64}$$

$$x_{2s+1} = \tfrac{1}{2}(c^{-1}q^{-s} + cq^s) \qquad s = 0, 1, \ldots, j-1, \tag{2.65}$$

and the orthogonality relation is

$$\sum_{s=0}^{N} w_s R_n(x_s) R_m(x_s) = u_1 u_2 \ldots u_n \delta_{nm}. \tag{2.66}$$

As in the previous section, one can compute the general weights by using the persymmetry when $\alpha = 1/2$. The result is

$$w_{2s} = \frac{(1-\alpha)K_N a^j c^j q^{(2j)s}\left(1-a^2q^{2s}\right)\left(a^2;q\right)_s\left(q^{-j};q\right)_s (ac;q)_s \left(\frac{aq^{-j+1}}{c};q\right)_s}{(q;q)_j\left(a^2q;q\right)_j\left(\frac{c}{a};q\right)_{j+1}(ac;q)_j\left(1-a^2\right)(q;q)_s\left(\frac{aq}{c};q\right)_s\left(a^2q^{j+1};q\right)_s\left(acq^j;q\right)_s}, \tag{2.67}$$

$$w_{2s+1} = \frac{-\alpha K_N a^{j+1}c^{j-1}q^{(2j)s}\left(1-c^2q^{2s}\right)\left(c^2;q\right)_s\left(q^{-j+1};q\right)_s (ac;q)_s\left(\frac{cq^{-j}}{a};q\right)_s}{(q;q)_{j-1}(c^2q;q)_{j-1}\left(\frac{a}{c};q\right)_{j+1}(ac;q)_{j+1}(1-c^2)(q;q)_s\left(\frac{cq}{a};q\right)_s(c^2q^j;q)_s(acq^{j+1};q)_s}, \tag{2.68}$$

with

$$K_N = \frac{(c-a)(1+q^j)(1-q^{j+1})(1-q^{2j+1})(c-aq)(q;q)_j(q^{-2j-1};q)_j(\frac{ac}{q};q)_{j+2}(\frac{cq^{-j-1}}{a};q)_j(\frac{q^{-2j}}{ac};q)_j(\frac{aq^{-j}}{c};q)_j}{q^{-2j^2-1}(1-q)^2(ac-q)(q^{-2j-1};q^2)_j^2(-q;q)_j^2(a-cq^j)(1-acq^{2j})(c-aq^{j+1})}.$$

The weights also satisfy the relations

$$\sum_{s=0}^{j} w_{2s} = 1-\alpha, \qquad \sum_{s=0}^{j-1} w_{2s+1} = \alpha. \tag{2.69}$$

2.5. Special cases

2.5.1. q-para-Krawtchouk

Under an appropriate reparametrization, it is possible to reduce the $R_n(x)$ to polynomials orthogonal with respect to an exponential bi-lattice instead of a biexponential bi-lattice. We call the corresponding polynomials the q-para-Krawtchouk polynomials because they reduce to the para-Krawtchouk polynomials when $q \to 1$. The persymmetric case $\alpha = 1/2$ for $N = 2j+1$ has been briefly mentionned in [14]. We here obtain the general q-para-Krawtchouk polynomials for general N and general α. To this end, let us first rewrite the parameters a and c in terms of new parameters θ and Δ given by

$$\theta = ac, \qquad \Delta = \frac{a}{c}, \tag{2.70}$$

$$a^2 = \theta\Delta, \qquad c^2 = \frac{\theta}{\Delta}. \tag{2.71}$$

Rescaling the lattice x_s and taking the limit $\theta \to \infty$, one obtains an exponential bi-lattice y_s described by

$$y_{2s} = \lim_{\theta\to\infty} \frac{2a}{\theta} x_{2s} = \Delta q^s, \tag{2.72}$$

$$y_{2s+1} = \lim_{\theta\to\infty} \frac{2a}{\theta} x_{2s+1} = q^s. \tag{2.73}$$

The q-para-Krawtchouk polynomials $Q_n(y)$ can be obtained by taking a similarity transformation

$$x = \frac{\theta}{2a} y, \qquad Q_n(y) = \left(\frac{\theta}{2a}\right)^{-n} R_n(x). \tag{2.74}$$

Under this transformation, the recurrence relations (2.19) and (2.51) of the q-para-Racah becomes

$$yQ_n(y) = Q_{n+1} + \tilde{b}_n Q_n(y) + \tilde{u}_n Q_{n-1}(y) \tag{2.75}$$

where

$$\tilde{b}_n = \lim_{\theta\to\infty} \frac{2a}{\theta} b_n, \qquad \tilde{u}_n = \lim_{\theta\to\infty} \frac{4a^2}{\theta^2} u_n. \tag{2.76}$$

Inserting the recurrence coefficients (2.22) and (2.23) in the previous formula gives

$$\tilde{b}_n = \begin{cases} \frac{q^{n+j}(1+q^{j+1})(1+\Delta)}{(q^j+q^n)(q^{j+1}+q^n)} & n \neq j, j+1, \\ \Delta - \frac{\alpha(1-q^{j+1})(\Delta-1)}{1-q} + \frac{q(1-q^j)(\Delta q-1)}{(1-q^2)} & n = j, \\ \Delta - \frac{(1-\alpha)(1-q^{j+1})(\Delta-1)}{1-q} + \frac{q(1-q^j)(\Delta q-1)}{(1-q^2)} & n = j+1, \end{cases} \tag{2.77}$$

$$\tilde{u}_n = \begin{cases} \frac{q^{2j+1+n}(1-q^n)(q^{2j+2}-q^n)(q^n-\Delta q^{j+1})(q^{j+1}-\Delta q^n)}{(q^{j+1}+q^n)^2(q^{2j+1}-q^{2n})(q^{2j+3}-q^{2n})} & n \neq j+1, \\ \frac{\alpha(1-\alpha)(\Delta-1)^2(1-q^{j+1})^2}{(1-q)^2} & n = j+1 \end{cases} \tag{2.78}$$

for $N = 2j+1$. Using instead (2.54) and (2.55) gives

$$\tilde{b}_n = \Delta - \frac{q^{2j}(q^n-1)(q^n-\Delta q^{j+1})}{(q^j+q^n)(q^{2j+1}-q^{2n})} + \frac{q^n(q^{2j}-q^n)(q^j-\Delta q^{n+1})}{(q^j+q^n)(q^{2j}-q^{2n+1})}, \tag{2.79}$$

$$\tilde{u}_n = \begin{cases} \frac{q^{2j+n}(q^n-1)(q^{2j+1}-q^n)(q^n-\Delta q^{j+1})(\Delta q^n-q^j)}{(q^j+q^n)(q^{j+1}+q^n)(q^{2j+1}-q^{2n})^2} & n \neq j, j+1, \\ \frac{(1-\alpha)(1-q^j)(1-q^{j+1})(\Delta-1)(1-q\Delta)}{(1-q)^2(1+q)} & n = j, \\ \frac{\alpha(1-q^j)(1-q^{j+1})(\Delta-1)(1-q\Delta)}{(1-q)^2(1+q)} & n = j+1 \end{cases} \tag{2.80}$$

for $N = 2j$. In the limit $q \to 1$, this recurrence relation reduces to the one of the para-Krawtchouk polynomials up to another similarity transformation. Remark that these coefficients are also persymmetric when $\alpha = 1/2$. The orthogonality relation for the $Q_n(y)$ can be obtained by the same procedure used in section 3 and 4 to obtain the orthogonality relation for the q-para-Racah polynomials. Omitting the details, one readily finds

$$\sum_{s=0}^{N} w_s Q_n(y_s) Q_m(y_s) = \tilde{u}_1 \tilde{u}_2 \ldots \tilde{u}_n \delta_{nm} \tag{2.81}$$

where the weights are given by

$$\begin{aligned} w_{2s} &= K_N \frac{(1-\alpha)\left(1-\frac{1}{\Delta}\right) q^s \left(\Delta q^{-j};q\right)_j \left(\frac{q^{-j}}{\Delta};q\right)_j \left(q^{-j};q\right)_s \left(\Delta q^{-j};q\right)_s}{(q;q)_s \left(\frac{1}{\Delta};q\right)_{j+1} \Delta^j (\Delta q;q)_s} \\ w_{2s+1} &= K_N \frac{\alpha(1-\Delta)\Delta^j q^s \left(\frac{q^{-j}}{\Delta};q\right)_j \left(\Delta q^{-j};q\right)_j \left(q^{-j};q\right)_s \left(\frac{q^{-j}}{\Delta};q\right)_s}{(q;q)_s (\Delta;q)_{j+1} \left(\frac{q}{\Delta};q\right)_s} \\ K_N &= \frac{(-1)^j q^{j(j-1)} \left(1-q^{2j+1}\right)}{(1-q)(-q;q)_j \left(q^{-2j-1};q^2\right)_j}. \end{aligned} \tag{2.82}$$

for $N = 2j+1$ and by

$$\begin{aligned} w_{2s} &= K_N \frac{(1-\alpha) q^s \left(q^{-j};q\right)_s \left(\Delta q^{1-j};q\right)_s}{\Delta^j (q;q)_s \left(\frac{q}{\Delta};q\right)_j (\Delta q;q)_s} \\ w_{2s+1} &= K_N \frac{\alpha \Delta^{j-1} \left(1-q^j\right) q^s \left(q^{1-j};q\right)_s \left(\frac{q^{-j}}{\Delta};q\right)_s}{\left(1-\frac{q^j}{\Delta}\right) (q;q)_s (\Delta q;q)_j \left(\frac{q}{\Delta};q\right)_s} \\ K_N &= \frac{(-1)^j q^{\frac{3}{2}j(j-1)} \left(q^j+1\right)\left(1-q^{j+1}\right)\left(1-q^{2j+1}\right)\left(q^{-2j-1};q\right)_j (1-\Delta q) \left(\frac{q^{-j-1}}{\Delta};q\right)_j \left(\Delta q^{-j};q\right)_j}{\left(1-\Delta q^{j+1}\right)(1-q)^2 \left(q^{-2j-1};q^2\right)_j^2 (-q;q)_j^2} \end{aligned} \tag{2.83}$$

for $N = 2j$.

2.5.2. Reduction to a single lattice

Consider the persymmetric case $\alpha = \frac{1}{2}$. For $c = aq^{\frac{1}{2}}$, the orthogonality lattice reduces to a single biexponential lattice of the form

$$x_s = \frac{1}{2}\left(a^{-1} q^{-\frac{s}{2}} + a q^{\frac{s}{2}}\right) \quad s = 0, 1, 2, \ldots, N. \tag{2.84}$$

In this setting, the q-para-Racah polynomials connect with the q-Racah polynomials in base $q^{\frac{1}{2}}$. More precisely, the following relation holds for any N :

$$R_n(x; a, aq^{\frac{1}{2}}, \tfrac{1}{2}|q) = (2a)^{-n} p_n(2ax; aq^{-\frac{1}{4}}, -a^{-1}q^{-j-\frac{3}{4}}, -aq^{-\frac{1}{4}}, -aq^{-\frac{1}{4}}|q^{\frac{1}{2}}) \tag{2.85}$$

where $p_n(y; \alpha, \beta, \gamma, \delta|q) \equiv p_n(y)$ are the monic q-Racah polynomials defined in [15]. This can be checked by substituting directly (2.85) in the recurrence relation (2.19) for N odd and (2.51) for N even and comparing the coefficients with those of the monic q-Racah polynomials found in [15].

In addition, this special instance of q-para-Racah polynomials bears a connection with the dual-Hahn polynomials in the limit $q \to 1$. To see this, consider the recurrence coefficients A_n and C_n given in (2.17) and (2.18) and let $\alpha = \frac{1}{2}$ and $c = aq^{\frac{1}{2}}$. Now, substitute $a \to q^a$ and compute the limits

$$\begin{aligned}
\lim_{q\to 1} \frac{A_n}{(1-q^{\frac{1}{2}})^2} &= (n + \tfrac{4a-1}{2} + 1)(n-N), \\
\lim_{q\to 1} \frac{C_n}{(1-q^{\frac{1}{2}})^2} &= n(n - \tfrac{4a-1}{2} - N - 1).
\end{aligned} \tag{2.86}$$

These results corresponds precisely to the recurrence coefficients (also denoted by A_n and C_n) of the dual-Hahn polynomials given in [15] with parameters $\gamma = \delta = \frac{4a-1}{2}$. It is a trivial matter to verify that the same result holds for even values of N. This is in perfect correspondance with the special case of the para-Racah polynomials that reduces to the dual-Hahn polynomials with parameters $\gamma = \delta = \frac{4a-1}{2}$ when the orthogonality bi-lattice of the former reduces to a single lattice [13].

2.6. Conclusion

To summarize, we constructed new orthogonal polynomials from a singular truncation of the Askey-Wilson polynomials. They have been called the q-para-Racah polynomials because their construction is parallel to the one of the para-Racah polynomials starting from the Wilson polynomials. Furthermore, they can be connected by a $q \to 1$ limit in their (unnormalized) explicit expression in a similar fashion to the connection between the Askey-Wilson and the Wilson polynomials. A three-term recurrence relation, a q-difference equation, an explicit expression and the orthogonality relation have been obtained both for sets containing an even or odd numbers of polynomials. We further characterized the q-para-Krawtchouk polynomials as a special case of the q-para-Racah polynomials. This is also of interest because these last polynomials had never been much characterized in the literature before. This specialization occurs in a limit where the orthogonality grid reduces from a biexponential bi-lattice to an exponential bi-lattice. A connection to the q-Racah and dual-Hahn polynomials has also been presented in the special case where the bi-lattice reduces to a single lattice.

The q-para-Racah and the q-para-Krawtchouk are both associated to an isospectral deformation of persymmetric Jacobi matrices. Specifically, their Jacobi matrices are persymmetric only when the parameter $\alpha = 1/2$. This is an interesting feature which we hope could see them arise in future applications. An idea is to interpret their Jacobi matrices as the restriction to the one-excitation sector of an Heisenberg spin chain Hamiltonian and to study their ability to produce transfer of quantum information or generate entangled pairs. Another direction would be to study their bispectrality. Although the q-para-Racah polynomials possess the bispectrality property, the spectrum of the corresponding q-difference operators in (2.26) and (2.57) is doubly degenerate. This means that the q-para-Racah polynomials do not belong to the category of "classical" orthogonal polynomials with the Leonard duality property. It would be interesting to find an appropriate algebraic description of these polynomials. We hope to report on these questions in the near future.

Acknowledgments

JML holds an Alexander-Graham-Bell PhD fellowship from the Natural Science and Engineering Research Council (NSERC) of Canada. LV is grateful to NSERC for support through a discovery grant.

References

[1] C. de Boor, G. H. Golub, The numerically stable reconstruction of a jacobi matrix from spectral data, Linear Algebra and its Applications 21 (3) (1978) 245 – 260.

[2] D. Boley, G. H. Golub, A survey of matrix inverse eigenvalue problems, Inverse Problems 3 (4) (1987) 595.

[3] C. de Boor, E. B. Saff, Finite sequences of orthogonal polynomials connected by a jacobi matrix, Linear algebra and its applications 75 (1986) 43–55.

[4] G. M. L. Gladwell, Inverse Problems in Vibration. Solid Mechanics and Its Applications, 2nd Edition, Springer, 2004.

[5] T. Shi, Y. Li, Z. Song, C.-P. Sun, Quantum-state transfer via the ferromagnetic chain in a spatially modulated field, Physical Review A 71 (2005) 032309.

[6] R. Chakrabarti, J. Van der Jeugt, Quantum communication through a spin chain with interaction determined by a Jacobi matrix, Journal of Physics A: Mathematical and General 43 (2010) 085302.

[7] E. I. Jafarov, J. Van der Jeugt, Quantum state transfer in spin chains with q-deformed interaction terms, Journal of Physics A: Mathematical and General 43 (2010) 405301.

[8] N. I. Stoilova, J. Van der Jeugt, An exactly solvable spin chain related to Hahn polynomials, SIGMA 7 (33-45).

[9] L. Vinet, A. Zhedanov, How to construct spin chains with perfect state transfer, Physical Review A 85 (2012) 012323.

[10] L. Vinet, A. Zhedanov, Para-Krawtchouk polynomials on a bi-lattice and a quantum spin chain with perfect state transfer, Journal of Physics A: Mathematical and Theoretical 45 (26) (2012) 265304.

[11] L. Vinet, A. Zhedanov, Dual -1 Hahn polynomials and perfect state transfer, Journal of Physics: Conference Series 343 (2012) 012125.

[12] J.-M. Lemay, L. Vinet, A. Zhedanov, An analytic spin chain model with fractional revival, Journal of Physics A: Mathematical and Theoretical 49 (33) (2016) 335302.

[13] J.-M. Lemay, L. Vinet, A. Zhedanov, The para-Racah polynomials, Journal of Mathematical Analysis and Applications 438 (2) (2016) 565–577.

[14] S. Tsujimoto, L. Vinet, A. Zhedanov, Tridiagonal representations of the q-oscillator algebra and askey–wilson polynomials, Journal of Physics A: Mathematical and Theoretical 50 (23) (2017) 235202.

[15] R. Koekoek, P. A. Lesky, R. F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, Springer, 2010.

[16] M. E. H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, Vol. 13 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2009.

[17] T. S. Chihara, An introduction to orthogonal polynomials, Gordon and Breach, 1978.

[18] V. X. Genest, S. Tsujimoto, L. Vinet, A. Zhedanov, Persymmetric jacobi matrices, isospectral deformations and orthogonal polynomials, Journal of Mathematical Analysis and Applications 450 (2) (2017) 915–928.

Chapitre 3

Two-variable -1 Jacobi polynomials

V.X. Genest, J.M. Lemay, L. Vinet et A. Zhedanov (2015). Two-variable -1 Jacobi polynomials. *Integral Transforms and Special Functions 26 (6), 411-425*

Abstract. A two-variable generalization of the Big -1 Jacobi polynomials is introduced and characterized. These bivariate polynomials are constructed as a coupled product of two univariate Big -1 Jacobi polynomials. Their orthogonality measure is obtained. Their bispectral properties (eigenvalue equations and recurrence relations) are determined through a limiting process from the two-variable Big q-Jacobi polynomials of Lewanowicz and Woźny. An alternative derivation of the weight function using Pearson-type equations is presented.

3.1. Introduction

The purpose of this paper is to introduce and study a family of bivariate Big -1 Jacobi polynomials. These two-variable polynomials, which shall be denoted by $\mathcal{J}_{n,k}(x,y)$, depend on four real parameters $\alpha, \beta, \gamma, \delta$ such that $\alpha, \beta, \gamma > -1$, $\delta \neq 1$ and are defined as

$$\mathcal{J}_{n,k}(x,y) = J_{n-k}\left(y; \alpha, 2k+\beta+\gamma+1, (-1)^k\delta\right)\, \rho_k(y)\, J_k\left(\frac{x}{y}; \gamma, \beta, \frac{\delta}{y}\right), \tag{3.1}$$

with $k = 0, 1, \ldots$ and $n = k, k+1, \ldots$, where

$$\rho_k(y) = \begin{cases} y^k \left(1 - \frac{\delta^2}{y^2}\right)^{\frac{k}{2}}, & k \text{ even}, \\ y^k \left(1 - \frac{\delta^2}{y^2}\right)^{\frac{k-1}{2}} \left(1 + \frac{\delta}{y}\right), & k \text{ odd}, \end{cases}$$

and where $J_n(x; a, b, c)$ denotes the one-variable Big -1 Jacobi polynomials [1] (see section 1.1). It will be shown that these polynomials are orthogonal with respect to a positive measure defined on the disjoint union of four triangular domains in the real plane. The polynomials $\mathcal{J}_{n,k}(x,y)$ will also be identified as a $q \to -1$ limit of the two-variable Big q-Jacobi polynomials introduced by

Lewanowicz and Woźny in [2], which generalize the bivariate little q-Jacobi polynomials introduced by Dunkl in [3]. The bispectral properties of the Big -1 Jacobi polynomials will be determined from this identification. The polynomials $\mathcal{J}_{n,k}(x,y)$ will be shown to satisfy an explicit vector-type three term recurrence relation and it will be seen that they are the joint eigenfunctions of a pair of commuting first order differential operators involving reflections. By solving the Pearson-type system of equations arising from the symmetrization of these differential/difference operators, the weight function for the polynomials $\mathcal{J}_{n,k}(x,y)$ will be recovered.

The defining formula of the two-variable Big -1 Jacobi polynomials (3.1) is reminiscent of the expressions found in [4] for the Krall-Sheffer polynomials [5], which, as shown in [6], are directly related to two-dimensional superintegrable systems on spaces with constants curvature (see [7] for a review of superintegrable systems). The polynomials $\mathcal{J}_{n,k}(x,y)$ do not belong to the Krall-Sheffer classification, as they will be seen to obey *first order* differential equations with reflections. The results of [6] however suggest that the polynomials $\mathcal{J}_{n,k}(x,y)$ could be related to two-dimensional integrable systems with reflections such as the ones recently considered in [8, 9, 10, 11]. This fact motivates our examination of the polynomials $\mathcal{J}_{n,k}(x,y)$.

3.1.1. The Big -1 Jacobi polynomials

Let us now review some of the properties of the Big -1 Jacobi polynomials which shall be needed in the following. The Big -1 Jacobi polynomials, denoted by $J_n(x;a,b,c)$, were introduced in [1] as a $q=-1$ limit of the Big q-Jacobi polynomials [12]. They are part of the Bannai-Ito scheme of -1 orthogonal polynomials [13, 14, 15, 16]. They are defined by

$$J_n(x;a,b,c)=\begin{cases} {}_2F_1\left[\begin{matrix}-\frac{n}{2}, \frac{n+a+b+2}{2} \\ \frac{a+1}{2}\end{matrix}; \frac{1-x^2}{1-c^2}\right] + \frac{n(1-x)}{(1+c)(a+1)}\, {}_2F_1\left[\begin{matrix}1-\frac{n}{2}, \frac{n+a+b+2}{2} \\ \frac{a+3}{2}\end{matrix}; \frac{1-x^2}{1-c^2}\right], & n \text{ even},\\ {}_2F_1\left[\begin{matrix}-\frac{n-1}{2}, \frac{n+a+b+1}{2} \\ \frac{a+1}{2}\end{matrix}; \frac{1-x^2}{1-c^2}\right] - \frac{(n+a+b+1)(1-x)}{(1+c)(a+1)}\, {}_2F_1\left[\begin{matrix}-\frac{n-1}{2}, \frac{n+a+b+3}{2} \\ \frac{a+3}{2}\end{matrix}; \frac{1-x^2}{1-c^2}\right], & n \text{ odd},\end{cases} \tag{3.2}$$

where ${}_2F_1$ is the standard Gauss hypergeometric function [17]; when no confusion can arise, we shall simply write $J_n(x)$ instead of $J_n(x;a,b,c)$. The polynomials (3.2) satisfy the recurrence relation

$$x\,J_n(x)=A_n\,J_{n+1}(x)+(1-A_n-C_n)\,J_n(x)+C_n\,J_{n-1}(x),$$

with coefficients

$$A_n=\begin{cases}\frac{(n+a+1)(c+1)}{2n+a+b+2}, & n \text{ even},\\ \frac{(1-c)(n+a+b+1)}{2n+a+b+2}, & n \text{ odd},\end{cases} \qquad C_n=\begin{cases}\frac{n(1-c)}{2n+a+b}, & n \text{ even},\\ \frac{(n+b)(1+c)}{2n+a+b}, & n \text{ odd}.\end{cases}$$

It can be seen that for $a,b>-1$ and $|c|\neq 1$ the polynomials $J_n(x)$ are positive-definite. The Big -1 Jacobi polynomials satisfy the eigenvalue equation

$$\mathcal{L}J_n(x)=\{(-1)^n\,(n+a/2+b/2+1/2)\}\,J_n(x), \tag{3.3}$$

where $\mathcal{L}$ is the most general first-order differential operator with reflection preserving the space of polynomials of a given degree. This operator has the expression

$$\mathcal{L}=\left[\frac{(x+c)(x-1)}{x}\right]\partial_x R+\left[\frac{c}{2x^2}+\frac{ca-b}{2x}\right](R-\mathbb{I})+\left[\frac{a+b+1}{2}\right]R,$$

where R is the reflection operator, i.e. $Rf(x)=f(-x)$, and $\mathbb{I}$ stands for the identity. The orthogonality relation of the Big -1 Jacobi polynomials is as follows. For $|c|<1$, one has

$$\int_{\mathcal{C}} J_n(x;a,b,c)\,J_m(x;a,b,c)\,\omega(x;a,b,c)\,\mathrm{d}x=\left[\frac{(1-c^2)^{\frac{a+b+2}{2}}}{(1+c)}\right]h_n(a,b)\,\delta_{nm}, \tag{3.4}$$

where the interval is $\mathcal{C}=[-1,-|c|]\cup[|c|,1]$ and the weight function reads

$$\omega(x;a,b,c)=\theta(x)\,(1+x)\,(x-c)\,(x^2-c^2)^{\frac{b-1}{2}}\,(1-x^2)^{\frac{a-1}{2}}, \tag{3.5}$$

with $\theta(x)$ is the sign function. The normalization factor h_n is given by

$$h_n(a,b)=\begin{cases}\dfrac{2\,\Gamma\left(\frac{n+b+1}{2}\right)\Gamma\left(\frac{n+a+3}{2}\right)\left(\frac{n}{2}\right)!}{(n+a+1)\,\Gamma\left(\frac{n+a+b+2}{2}\right)\left(\frac{a+1}{2}\right)_{\frac{n}{2}}^2}, & n \text{ even},\\[2ex] \dfrac{(n+a+b+1)\,\Gamma\left(\frac{n+b+2}{2}\right)\Gamma\left(\frac{n+a+2}{2}\right)\left(\frac{n-1}{2}\right)!}{2\,\Gamma\left(\frac{n+a+b+3}{2}\right)\left(\frac{a+1}{2}\right)_{\frac{n+1}{2}}^2}, & n \text{ odd},\end{cases} \tag{3.6}$$

where $(a)_n$ stands for the Pochhammer symbol [17]. For $|c|>1$, one has

$$\int_{\widetilde{\mathcal{C}}} J_n(x;a,b,c)\,J_m(x;a,b,c)\,\widetilde{\omega}(x;a,b,c)\,\mathrm{d}x=\left[\frac{\theta(c)(c^2-1)^{\frac{a+b+2}{2}}}{1+c}\right]\widetilde{h}_n(a,b)\,\delta_{nm}, \tag{3.7}$$

where the interval is $\widetilde{\mathcal{C}}=[-|c|,-1]\cup[1,|c|]$ and the weight function reads

$$\widetilde{\omega}(x;a,b,c)=\theta(c\,x)\,(1+x)\,(c-x)\,(c^2-x^2)^{\frac{b-1}{2}}\,(x^2-1)^{\frac{a-1}{2}}. \tag{3.8}$$

In this case the normalization factor has the expression

$$\widetilde{h}_n(a,b)=\begin{cases}\dfrac{2\,\Gamma\left(\frac{n+b+1}{2}\right)\Gamma\left(\frac{n+a+3}{2}\right)\left(\frac{n}{2}\right)!}{(n+a+1)\Gamma\left(\frac{n+a+b+2}{2}\right)\left(\frac{a+1}{2}\right)_{\frac{n}{2}}^2}, & n \text{ even},\\[2ex] \dfrac{(n+a+b+1)\,\Gamma\left(\frac{n+b+2}{2}\right)\Gamma\left(\frac{n+a+2}{2}\right)\left(\frac{n-1}{2}\right)!}{2\,\Gamma\left(\frac{n+a+b+3}{2}\right)\left(\frac{a+1}{2}\right)_{\frac{n+1}{2}}^2} & n \text{ odd}.\end{cases} \tag{3.9}$$

The normalization factors h_n and $\widetilde{h}_n$ were not derived in [1]. They have been obtained here using the orthogonality relation for the Chihara polynomials provided in [14] and the fact that the Big -1 Jacobi polynomials are related to the latter by a Christoffel transformation. The details of this derivation are presented in appendix A.

3.2. Orthogonality of the two-variable Big -1 Jacobi polynomials

We now prove the orthogonality property of the two-variable Big -1 Jacobi polynomials.

Proposition 3.1. *Let $\alpha,\beta,\gamma > -1$ and $|\delta| < 1$. The two-variable Big -1 Jacobi polynomials defined by (3.1) satisfy the orthogonality relation*

$$\int_{D_y}\int_{D_x} \mathcal{J}_{n,k}(x,y)\ \mathcal{J}_{m,\ell}(x,y)\ W(x,y)\ \mathrm{d}x\ \mathrm{d}y = H_{nk}\ \delta_{k\ell}\delta_{nm}, \tag{3.10}$$

with respect to the weight function

$$W(x,y) = \theta(x\,y)|y|^{\beta+\gamma}(1+y)\left(1+\frac{x}{y}\right)\left(\frac{x-\delta}{y}\right)(1-y^2)^{\frac{\alpha-1}{2}}\left(1-\frac{x^2}{y^2}\right)^{\frac{\gamma-1}{2}}\left(\frac{x^2-\delta^2}{y^2}\right)^{\frac{\beta-1}{2}}. \tag{3.11}$$

The integration domain is prescribed by

$$D_x = [-|y|,-|\delta|]\cup[|\delta|,|y|], \qquad D_y = [-1,-|\delta|]\cup[|\delta|,1], \tag{3.12}$$

and the normalization factor H_{nk} has the expression

$$H_{nk} = \left[\frac{(1-\delta^2)^{\frac{2k+\alpha+\beta+\gamma+3}{2}}}{(1+(-1)^k\delta)}\right]\ h_k(\gamma,\beta)\ h_{n-k}(\alpha,2k+\gamma+\beta+1),$$

where $h_n(a,b)$ is given by (3.6).

Proof. We proceed by a direct calculation. We denote the orthogonality integral by

$$I = \int_{D_y}\int_{D_x} \mathcal{J}_{n,k}(x,y)\ \mathcal{J}_{m,\ell}(x,y)\ W(x,y)\ \mathrm{d}x\ \mathrm{d}y.$$

Upon using the expressions (3.1) and (3.11) in the above, one writes

$$\begin{aligned} I = \int_{D_y} & J_{n-k}(y;\alpha,2k+\gamma+\beta+1,(-1)^k\delta)\ \ J_{m-\ell}(y,\alpha,2k+\gamma+\beta+1,(-1)^k\delta) \\ & \times\left[\rho_k(y)\rho_\ell(y)|y|^{\beta+\gamma}(1+y)(1-y^2)^{\frac{\alpha-1}{2}}\right]\ \mathrm{d}y \\ \times \int_{D_x} & J_k\left(\frac{x}{y};\gamma,\beta,\frac{\delta}{y}\right) J_\ell\left(\frac{x}{y};\gamma,\beta,\frac{\delta}{y}\right)\left[\theta\left(\frac{x}{y}\right)\left(1+\frac{x}{y}\right)\left(\frac{x-\delta}{y}\right)\left(1-\frac{x^2}{y^2}\right)^{\frac{\gamma-1}{2}}\left(\frac{x^2-\delta^2}{y^2}\right)^{\frac{\beta-1}{2}}\right]\mathrm{d}x. \end{aligned}$$

The integral over D_x is directly evaluated using the change of variables $u = x/y$ and comparing with the orthogonality relation (3.4). The result is thus

$$\begin{aligned} I = h_k(\gamma,\beta)\ \delta_{k\ell}\times\int_{D_y} & J_{n-k}(y;\alpha,2k+\gamma+\beta+1,(-1)^k\delta)\ J_{m-\ell}(y,\alpha,2k+\gamma+\beta+1,(-1)^k\delta) \\ & \times\rho_k(y)\ \rho_\ell(y)\left[\theta(y)\ (1+y)\ (1-y^2)^{\frac{\alpha-1}{2}}\ \frac{(y^2-\delta^2)^{\frac{2+\gamma+\beta}{2}}}{(y+\delta)}\right]\ \mathrm{d}y. \end{aligned}$$

Assuming that $k=\ell$ is an even integer, the integral takes the form

$$I = h_k(\gamma,\beta)\delta_{k\ell} \times \int_{D_y} J_{n-k}(y;\alpha,2k+\gamma+\beta+1,\delta)\;\; J_{m-\ell}(y,\alpha,2k+\gamma+\beta+1,\delta)$$
$$\times (y^2-\delta^2)^k \left[\theta(y)\;(1+y)\;(1-y^2)^{\frac{\alpha-1}{2}}\;\frac{(y^2-\delta^2)^{\frac{2+\gamma+\beta}{2}}}{(y+\delta)}\right]\;\mathrm{d}y,$$

which in view of (3.4) yields

$$I = h_k(\gamma,\beta)h_{n-k}(\alpha,2k+\gamma+\beta+1)\left[\frac{(1-\delta^2)^{\frac{2k+\alpha+\beta+\gamma+3}{2}}}{1+\delta}\right]\delta_{k\ell}\delta_{mn}.$$

Assuming that $k=\ell$ is an odd integer, the integral takes the form

$$I = h_k(\gamma,\beta)\delta_{k\ell} \times \int_{D_y} J_{n-k}(y;\alpha,2k+\gamma+\beta+1,-\delta)\;\; J_{m-\ell}(y,\alpha,2k+\gamma+\beta+1,-\delta)$$
$$\times (y^2-\delta^2)^{k-1}(y+\delta)^2 \left[\theta(y)\;(1+y)\;(1-y^2)^{\frac{\alpha-1}{2}}\;\frac{(y^2-\delta^2)^{\frac{2+\gamma+\beta}{2}}}{(y+\delta)}\right]\;\mathrm{d}y,$$

which given (3.4) gives

$$I = h_k(\gamma,\beta)h_{n-k}(\alpha,2k+\gamma+\beta+1)\left[\frac{(1-\delta^2)^{\frac{2k+\alpha+\beta+\gamma+3}{2}}}{1-\delta}\right]\delta_{k\ell}\delta_{mn}.$$

Upon combining the k even and k odd cases, one finds (3.10). This completes the proof. □

It is not difficult to see that the region (3.12) corresponds to the disjoint union of four triangular domains. The $|\delta|=1/5$ case is illustrated in the following figure.

For $\alpha,\beta,\gamma>-1$ and $|\delta|<1$, it can be verified that the weight function (3.11) is positive on (3.12). The orthogonality relation for $|\delta|>1$ can be obtained in a similar fashion. The result is as follows.

Proposition 3.2. *Let $\alpha,\beta,\gamma>-1$ and $|\delta|>1$. The two-variable Big -1 Jacobi polynomials defined by (3.1) satisfy the orthogonality relation*

$$\int_{\widetilde{D}_y}\int_{\widetilde{D}_x} \mathcal{J}_{n,k}(x,y)\;\mathcal{J}_{m,\ell}(x,y)\;\widetilde{W}(x,y)\;\mathrm{d}x\;\mathrm{d}y = \widetilde{H}_{nk}\;\delta_{k\ell}\delta_{nm},$$

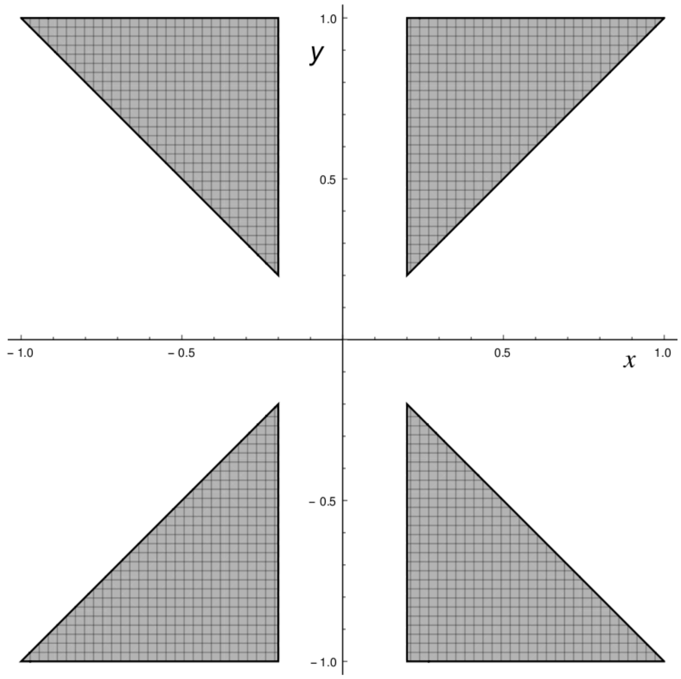
with respect to the weight function

$$\widetilde{W}(x,y) = \theta(\delta\,x\,y)|y|^{\gamma+\beta}(1+y)\left(1+\frac{x}{y}\right)\left(\frac{\delta-x}{y}\right)(y^2-1)^{\frac{\alpha-1}{2}}\left(\frac{x^2}{y^2}-1\right)^{\frac{\gamma-1}{2}}\left(\frac{\delta^2-x^2}{y^2}\right)^{\frac{\beta-1}{2}}. \tag{3.13}$$

The integration domain is

$$\widetilde{D}_x = [-|\delta|,-|y|]\cup[|y|,|\delta|],\qquad \widetilde{D}_y = [-|\delta|,1]\cup[1,|\delta|], \tag{3.14}$$

Fig. 3.1. Orthogonality region for $|\delta| = 1/5$.



and the normalization factor is of the form

$$\widetilde{H}_{nk} = (-1)^k \theta(\delta) \left[\frac{(\delta^2-1)^{\frac{2k+\alpha+\beta+\gamma+3}{2}}}{1+(-1)^k\delta} \right] \widetilde{h}_k(\gamma,\beta)\, \widetilde{h}_{n-k}(\alpha, 2k+\beta+\gamma+1),$$

where $\widetilde{h}_n(a,b)$ *is given by* (3.9).

Proof. Similar to proposition 2.1 using instead (3.7), (3.8) and (3.9). □

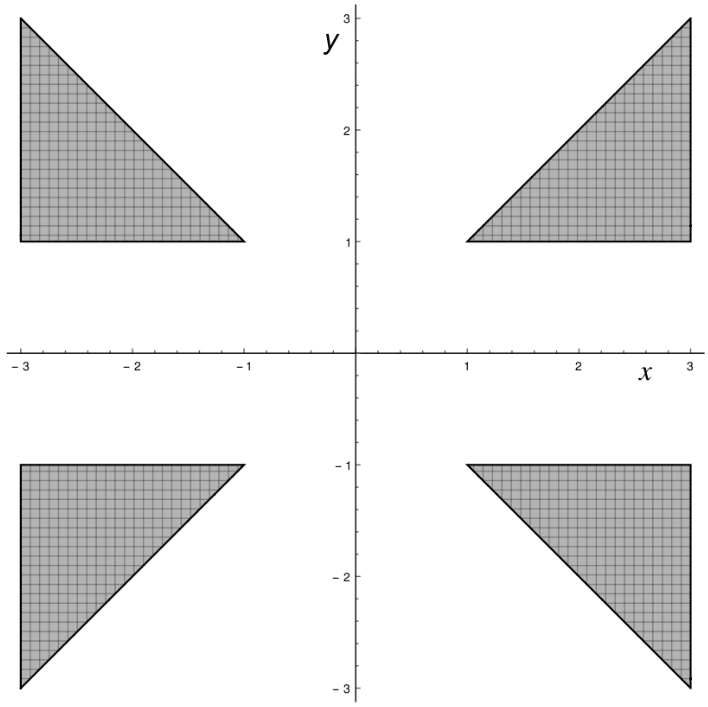
It can again be seen that the weight function (3.13) is positive on the domain (3.14) provided that $\alpha, \beta, \gamma > -1$ and $|\delta| > 1$. The orthogonality region defined by (3.14) again corresponds to the disjoint union of four triangular domains, as illustrated by the next figure for the case $|\delta| = 3$.

3.2.1. A special case: the bivariate Little −1 Jacobi polynomials

When $c = 0$, the Big −1 Jacobi polynomials $J_n(x; a, b, c)$ defined by (3.2) reduce to the so-called Little −1 Jacobi polynomials $j_n(x; a, b)$ introduced in [16]. These polynomials have the hypergeometric representation

$$j_n(x;a,b) = \begin{cases} {}_2F_1\left[\begin{matrix} -\frac{n}{2}, \frac{n+a+b+2}{2} \\ \frac{a+1}{2} \end{matrix}; 1-x^2\right] + \frac{n(1-x)}{(a+1)}\, {}_2F_1\left[\begin{matrix} 1-\frac{n}{2}, \frac{n+a+b+2}{2} \\ \frac{a+3}{2} \end{matrix}; 1-x^2\right], & n \text{ even}, \\ {}_2F_1\left[\begin{matrix} -\frac{n-1}{2}, \frac{n+a+b+1}{2} \\ \frac{a+1}{2} \end{matrix}; 1-x^2\right] - \frac{(n+a+b+1)(1-x)}{(a+1)}\, {}_2F_1\left[\begin{matrix} -\frac{n-1}{2}, \frac{n+a+b+3}{2} \\ \frac{a+3}{2} \end{matrix}; 1-x^2\right], & n \text{ odd}. \end{cases} \tag{3.15}$$

Fig. 3.2. Orthogonality region for $|\delta| = 3$.



Taking $\delta = 0$ in (3.1) leads to the following definition for the two-variable Little -1 Jacobi polynomials:

$$q_{n,k}(x,y) = j_{n-k}(y;\alpha,2k+\beta+\gamma+1)\; y^k\; j_k\left(\frac{x}{y};\gamma,\beta\right), \quad k=0,1,2\ldots, \quad n=k,k+1,\ldots. \tag{3.16}$$

It is seen from (3.16) that the two-variable Little -1 Jacobi polynomials have the structure corresponding to one of the methods to construct bivariate orthogonal polynomials systems proposed by Koornwinder in [18]. For the polynomials (3.16), the weight function, which can be obtained by taking $\delta = 0$ in either (3.11) or (3.13), can also be recovered using the general scheme given in [18]. For $\delta = 0$, the region (3.12) reduces to two vertically opposite triangles.

3.3. Bispectrality of the bivariate Big -1 Jacobi polynomials

In this section, the two-variable Big -1 Jacobi polynomials $\mathcal{J}_{n,k}(x,y)$ are shown be the joint eigenfunctions of a pair of first-order differential operators involving reflections. Their recurrence relations are also derived. The results are obtained through a limiting process from the corresponding properties of the two-variable Big q-Jacobi polynomials introduced by Lewanowicz and Woźny [2].

3.3.1. Bivariate Big q-Jacobi polynomials

Let us review some of the properties of the bivariate q-polynomials introduced in [2]. The two-variable Big q-Jacobi polynomials, denoted $\mathcal{P}_{n,k}(x,y;a,b,c,d;q)$ are defined as

$$\mathcal{P}_{n,k}(x,y;a,b,c,d;q) = P_{n-k}(y;a,bcq^{2k+1},dq^k;q)\; y^k\; \left(\frac{dq}{y};q\right)_k\; P_k\left(\frac{x}{y};c,b,\frac{d}{y};q\right), \tag{3.17}$$

where $(a;q)_n$ stands for the q-Pochhammer symbol [19] and where $P_n(x;a,b,c;q)$ are the Big q-Jacobi polynomials [12]. The two-variable Big q-Jacobi polynomials satisfy the eigenvalue equation [2]

$$\Omega\,\mathcal{P}_{n,k}(x,y) = \left[\frac{q^{1-n}(q^n-1)(abcq^{n+2}-1)}{(q-1)^2}\right]\mathcal{P}_{n,k}(x,y), \tag{3.18}$$

where Ω is the q-difference operator

$$\begin{aligned}\Omega = (x-dq)(x-acq^2)\mathbf{D}_{q,x}\mathbf{D}_{q^{-1},x} &+ (y-aq)(y-dq)\mathbf{D}_{q,y}\mathbf{D}_{q^{-1},y}\\ + q^{-1}(x-dq)(y-aq)\mathbf{D}_{q^{-1},x}\mathbf{D}_{q^{-1},y} &+ acq^3(bx-d)(y-1)\mathbf{D}_{q,x}\mathbf{D}_{q,y}\\ + \tfrac{(abcq^3-1)(x-1)-(acq^2-1)(dq-1)}{q-1}\mathbf{D}_{q,x} &+ \tfrac{(abcq^3-1)(y-1)-(aq-1)(dq-1)}{q-1}\mathbf{D}_{q,y},\end{aligned}$$

and where $\mathbf{D}_{q,x}$ stands for the q-derivative

$$\mathbf{D}_{q,x}f(x,y) = \frac{f(qx,y)-f(x,y)}{x(q-1)}.$$

The bivariate Big q-Jacobi polynomials also satisfy the pair of recurrence relations [2]

$$\begin{aligned} y\,\mathcal{P}_{n,k}(x,y) &= a_{nk}\,\mathcal{P}_{n+1,k}(x,y) + b_{nk}\,\mathcal{P}_{n,k}(x,y) + c_{nk}\,\mathcal{P}_{n-1,k}(x,y),\\ x\,\mathcal{P}_{n,k}(x,y) &= e_{nk}\,\mathcal{P}_{n+1,k-1}(x,y) + f_{nk}\,\mathcal{P}_{n+1,k}(x,y) + g_{nk}\,\mathcal{P}_{n+1,k+1}(x,y)\\ &\quad + r_{nk}\,\mathcal{P}_{n,k-1}(x,y) + s_{nk}\,\mathcal{P}_{n,k}(x,y) + t_{nk}\,\mathcal{P}_{n,k+1}(x,y)\\ &\quad + u_{nk}\,\mathcal{P}_{n-1,k-1}(x,y) + v_{nk}\,\mathcal{P}_{n-1,k}(x,y) + w_{nk}\,\mathcal{P}_{n-1,k+1}(x,y),\end{aligned} \tag{3.19}$$

where the recurrence coefficients read

$$\begin{aligned} a_{nk} &= \frac{(1-aq^{n-k+1})(1-abcq^{n+k+2})(1-dq^{n+1})}{(abcq^{2n+2};q)_2}, & b_{nk} &= 1-a_{nk}-c_{nk},\\ w_{nk} &= \frac{abc\sigma_k q^{n+2k+3}(abcq^{n+1}-d)(q^{n-k-1};q)_2}{(1-dq^{k+1})(abcq^{2n+1};q)_2}, & f_{nk} &= a_{nk}(bcq^k\tau_k-\sigma_k+1),\\ e_{nk} &= \frac{\tau_k bcq^k(dq^k-1)(1-dq^{n+1})(aq^{n-k+1};q)_2}{(abcq^{2n+2};q)_2}, & v_{nk} &= c_{nk}(bcq^k\tau_k-\sigma_k+1),\\ g_{nk} &= \frac{\sigma_k(1-dq^{n+1})(abcq^{n+k+2};q)_2}{(1-dq^{k+1})(abcq^{2n+2};q)_2}, & s_{nk} &= b_{nk}(bcq^k\tau_k-\sigma_k+1)+d(q^{k+1}\sigma_k-\tau_k),\end{aligned}$$

with

$$t_{nk} = \frac{q^{k+1}\sigma_k z_n(1-q^{n-k})(1-abcq^{n+k+2})}{1-dq^{k+1}}, \quad c_{nk} = \frac{adq^{n+1}(q^{n-k}-1)(1-bcq^{n+k+1})(1-abcd^{-1}q^{n+1})}{(abcq^{2n+1};q)_2},$$

$$r_{nk} = \tau_k z_n(dq^k-1)(1-aq^{n-k+1})(1-bcq^{n+k+1}), \; u_{nk} = \frac{\tau_k aq^{n-k+1}(dq^k-1)(abcq^{n+1}-d)(bcq^{n+k};q)_2}{(abcq^{2n+1};q)_2},$$

and where σ_k, τ_k and z_n are given by

$$\sigma_k = \frac{(1-cq^{k+1})(1-bcq^{k+1})}{(bcq^{2k+1};q)_2}, \quad z_n = \frac{abcq^{n+1}(1+q-dq^{n+1})-d}{(1-abcq^{2n+1})(1-abcq^{2n+3})},$$
$$\tau_k = -\frac{cq^{k+1}(1-q^k)(1-bq^k)}{(bcq^{2k};q)_2}.$$

3.3.2. $\mathcal{J}_{n,k}(x,y)$ as a $q \to -1$ limit of $\mathcal{P}_{n,k}(x,y)$

The two-variable Big -1 Jacobi polynomials $\mathcal{J}_{n,k}(x,y)$ can be obtained from the bivariate Big q-Jacobi by taking $q \to -1$. Indeed, a direct calculation using the expression (3.17) shows that

$$\lim_{\epsilon\to 0} \mathcal{P}_{n,k}(x,y;-e^{\epsilon\alpha},-e^{\epsilon\beta},-e^{\epsilon\gamma},\delta,-e^{\epsilon}) = \mathcal{J}_{n,k}(x,y;\alpha,\beta,\gamma,\delta), \tag{3.20}$$

where we have used the notation $\mathcal{J}_{n,k}(x,y;\alpha,\beta,\gamma,\delta)$ to exhibit the parameters appearing in the Big -1 Jacobi polynomials defined in (3.1). A similar limit was considered in [1] to obtain the univariate Big -1 Jacobi polynomials in terms of the Big q-Jacobi polynomials.

3.3.3. Eigenvalue equation for the Big -1 Jacobi polynomials

The eigenvalue equation (3.18) for the Big q-Jacobi polynomials and the relation (3.20) between the Big q-Jacobi polynomials and the Big -1 Jacobi polynomials can be used to obtain an eigenvalue equation for the latter.

Proposition 3.3. *Let L_1 be the first-order differential/difference operator*

$$\begin{aligned} L_1 = {} & G_5(x,y)\,R_xR_y\partial_y + G_6(x,y)\,R_y\partial_y + G_7(x,y)\,R_xR_y\partial_x + G_8(x,y)\,R_x\partial_x \\ & + G_1(x,y)\,R_xR_y + G_2(x,y)\,R_x + G_3(x,y)\,R_y - (G_1(x,y)+G_2(x,y)+G_3(x,y))\,\mathbb{I}, \end{aligned} \tag{3.21}$$

where R_x, R_y are reflection operators and where the coefficients read

$$G_1(x,y) = \frac{x[1+\beta+\gamma-y(\alpha+\beta+\gamma+2)]-\delta[y(\alpha+\gamma+1)-\gamma]}{4xy}, \tag{3.22a}$$

$$G_2(x,y) = -\frac{x[x(\beta+\gamma+1)-\beta y]+\delta(y+\gamma x)}{4x^2y}, \qquad G_6(x,y) = \frac{\delta(x-y)(y-1)}{2xy}, \tag{3.22b}$$

$$G_3(x,y) = -\delta\left(\frac{x+\alpha xy-y[y(\alpha+\gamma+1)-\gamma]}{4xy^2}\right), \qquad G_7(x,y) = \frac{(\delta+x)(y-1)}{2y}, \tag{3.22c}$$

$$G_5(x,y) = \frac{(\delta+x)(y-1)}{2x}, \qquad G_8(x,y) = \frac{(\delta+x)(x-y)}{2xy}. \tag{3.22d}$$

The Big -1 Jacobi polynomials satisfy the eigenvalue equation

$$L_1\, \mathcal{P}_{n,k}(x,y) = \mu_n\, \mathcal{P}_{n,k}(x,y), \qquad \mu_n = \begin{cases} -\frac{n}{2}, & n \text{ even}, \\ \frac{n+\alpha+\beta+\gamma+2}{2}, & n \text{ odd}. \end{cases}$$

Furthermore, let L_2 be the differential/difference operator

$$L_2 = \frac{2(y-x)(x+\delta)}{x} R_x \partial_x + \frac{(\gamma+\beta+1)x^2 + (\delta\gamma - \beta y)x + \delta y}{x^2}(R_x - \mathbb{I}),$$

The Big -1 Jacobi polynomials satisfy the eigenvalue equation

$$L_2\, \mathcal{P}_{n,k}(x,y) = \nu_k\, \mathcal{P}_{n,k}(x,y), \qquad \nu_k = \begin{cases} 2k, & k \text{ even}, \\ -2(k+\beta+\gamma+1), & k \text{ odd}. \end{cases}$$

Proof. The eigenvalue equation with respect to L_1 is obtained by dividing both sides of (3.18) by $(1+q)$ and taking the $q \to -1$ limit according to (3.20). The eigenvalue equation with respect to L_2 is obtained by combining (3.1), (3.2) and (3.3). □

The two-variable Big -1 Jacobi polynomials are thus the joint eigenfunctions of the first order differential operators with reflections L_1 and L_2. It is directly verified that these operators commute with one another, as should be.

The $q \to -1$ limit (3.20) of the recurrence relations (3.19) can also be taken to obtain the recurrence relations satisfied by the Big -1 Jacobi polynomials. The result is as follows.

Proposition 3.4. *The Big -1 Jacobi polynomials satisfy the recurrence relations*

$$\begin{aligned} y\, \mathcal{J}_{n,k}(x,y) &= \widetilde{a}_{nk}\, \mathcal{J}_{n+1,k}(x,y) + \widetilde{b}_{nk}\, \mathcal{J}_{n,k}(x,y) + \widetilde{c}_{nk}\, \mathcal{J}_{n-1,k}(x,y), \\ x\, \mathcal{J}_{n,k}(x,y) &= \widetilde{e}_{nk}\, \mathcal{J}_{n+1,k-1}(x,y) + \widetilde{f}_{nk}\, \mathcal{J}_{n+1,k}(x,y) + \widetilde{g}_{nk}\, \mathcal{J}_{n+1,k+1}(x,y) \\ &\quad + \widetilde{r}_{nk}\, \mathcal{J}_{n,k-1}(x,y) + \widetilde{s}_{nk}\, \mathcal{J}_{n,k}(x,y) + \widetilde{t}_{nk}\, \mathcal{J}_{n,k+1}(x,y) \\ &\quad + \widetilde{u}_{nk}\, \mathcal{J}_{n-1,k-1}(x,y) + \widetilde{v}_{nk}\, \mathcal{J}_{n-1,k}(x,y) + \widetilde{w}_{nk}\, \mathcal{J}_{n-1,k+1}(x,y). \end{aligned}$$

With

$$\widetilde{\tau}_k = \frac{k+\beta\phi_k}{2k+\beta+\gamma}, \quad \widetilde{\sigma}_k = \frac{k+\beta\phi_k+\gamma+1}{2k+\beta+\gamma+2}, \quad \widetilde{z}_n = \frac{(-1)^n - \delta(2n+\alpha+\beta+\gamma+2)}{(2n+\alpha+\beta+\gamma+1)(2n+\alpha+\beta+\gamma+3)}$$

where $\phi_k = (1-(-1)^k)/2$ is the characteristic function for odd numbers, the recurrence coefficients read

$$\widetilde{a}_{n,k} = \frac{1+\delta_n}{2n+\alpha+\beta+\gamma+3} \times \begin{cases} n-k+\alpha+1, & n+k \text{ even}, \\ n+k+\alpha+\beta+\gamma+2, & n+k \text{ odd}, \end{cases}$$

$$\widetilde{c}_{n,k} = \frac{1+\delta_{n+1}}{2n+\alpha+\beta+\gamma+1} \times \begin{cases} n-k, & n+k \ \textit{even}, \\ n+k+\beta+\gamma+1, & n+k \ \textit{odd}, \end{cases}$$

$$\widetilde{e}_{n,k} = \frac{\widetilde{\tau}_k(1-\delta_k)(1+\delta_n)}{2n+\alpha+\beta+\gamma+3} \times \begin{cases} n-k+\alpha+1, & n+k \ \textit{even}, \\ n-k+\alpha+2, & n+k \ \textit{odd}, \end{cases}$$

$$\widetilde{g}_{n,k} = \frac{\widetilde{\sigma}_k(1+\delta_n)}{(1+\delta_k)(2n+\alpha+\beta+\gamma+3)} \times \begin{cases} n+k+\alpha+\beta+\gamma+3, & n+k \ \textit{even}, \\ n+k+\alpha+\beta+\gamma+2, & n+k \ \textit{odd}, \end{cases}$$

$$\widetilde{r}_{n,k} = 2\widetilde{\tau}_k\widetilde{z}_n((-1)^k-\delta) \times \begin{cases} n-k+\alpha+1 & n+k \ \textit{even} \\ n+k+\beta+\gamma+1 & n+k \ \textit{odd} \end{cases}$$

$$\widetilde{t}_{n,k} = \frac{2(-1)^{k+1}\widetilde{\sigma}_k\widetilde{z}_n}{1+\delta_k} \times \begin{cases} n-k, & n+k \ \textit{even}, \\ n+k+\alpha+\beta+\gamma+2, & n+k \ \textit{odd}, \end{cases}$$

$$\widetilde{u}_{nk} = \frac{\widetilde{\tau}_k(1-\delta_k)(1-\delta_n)}{(2n+\alpha+\beta+\gamma+1)} \times \begin{cases} n+k+\beta+\gamma, & n+k \ \textit{even}, \\ n+k+\beta+\gamma+1, & n+k \ \textit{odd}, \end{cases}$$

$$\widetilde{w}_{n,k} = \frac{\widetilde{\sigma}_k(1-\delta_n)}{(1+\delta_k)(2n+\alpha+\beta+\gamma+1)} \times \begin{cases} n-k, & n+k \ \textit{even}, \\ n-k-1, & n+k \ \textit{odd}, \end{cases}$$

with $\delta_n = (-1)^n\delta$ *and*

$$\widetilde{b}_{n,k} = 1-\widetilde{a}_{n,k}-\widetilde{c}_{n,k}, \qquad \widetilde{s}_{n,k} = \widetilde{b}_{n,k}(1-\widetilde{\sigma}_k-\widetilde{\tau}_k)-\delta_k(\widetilde{\sigma}_k-\widetilde{\tau}_k),$$

$$\widetilde{f}_{n,k} = \widetilde{a}_{n,k}(1-\widetilde{\sigma}_k-\widetilde{\tau}_{n,k}), \qquad \widetilde{v}_{n,k} = \widetilde{c}_{n,k}(1-\widetilde{\sigma}_k-\widetilde{\tau}_k).$$

Proof. The result is obtained by applying the limit (3.20) to the recurrence relations (3.19). □

3.4. A Pearson-type system for Big -1 Jacobi

Let us now show how the weight function $W(x,y)$ for the polynomials $\mathcal{J}_{n,k}(x,y)$ can be recovered as the symmetry factor for the operator L_1 given in (3.21). The symmetrization condition for L_1 is

$$(W(x,y)L_1)^* = W(x,y)L_1, \tag{3.23}$$

where M^* denotes the Lagrange adjoint. For an operator of the form

$$M = \sum_{\mu,\nu,k,j} A_{k,j}(x,y)\,\partial_x^k\,\partial_y^j\,R_x^{\mu}\,R_y^{\nu},$$

for $\mu, \nu \in \{0, 1\}$ and $k, j = 0, 1, 2, \ldots$, the Lagrange adjoint reads

$$M^* = \sum_{\mu,\nu,k,j} (-1)^{k+j} \, R_x^{\mu} R_y^{\nu} \, \partial_y^j \, \partial_x^k \, A_{k,j}(x, y),$$

where we have assumed that $W(x, y)$ is defined on a symmetric region with respect to R_x and R_y. Imposing the condition (3.23), one finds the following system of Pearson-type equations:

$$W(x, y)\, G_8(x, y) = W(-x, y)\, G_8(-x, y), \tag{3.24a}$$

$$W(x, y)\, G_7(x, y) = W(-x, -y)\, G_7(-x, -y), \tag{3.24b}$$

$$W(x, y)\, G_6(x, y) = W(x, -y)\, G_6(x, -y), \tag{3.24c}$$

$$W(x, y)\, G_5(x, y) = W(-x, -y)\, G_5(-x, -y) \tag{3.24d}$$

$$W(x, y)\, G_3(x, y) = W(x, -y)\, G_3(x, -y) - \partial_y(W(x, -y)\, G_6(x, -y)), \tag{3.24e}$$

$$W(x, y)\, G_2(x, y) = W(-x, y)\, G_2(-x, y) - \partial_x(W(-x, y)\, G_8(-x, y)), \tag{3.24f}$$

$$\begin{aligned} W(x, y)\, G_1(x, y) = {} & W(-x, -y)\, G_1(-x, -y) \\ & - \partial_x(W(-x, -y)\, G_7(-x, -y)) - \partial_y(W(-x, -y)\, G_5(-x, -y)), \end{aligned} \tag{3.24g}$$

where the functions $G_i(x, y)$, $i = 1, \ldots, 8$, are given by (3.22). We assume that $W(x, y) > 0$ and moreover that $|\delta| \leqslant |x| \leqslant y \leqslant 1$. Upon substituting (3.22) in (3.24a), one finds

$$(x + \delta)\, (x - y)\, W(x, y) = -(x - \delta)\, (x + y)\, W(-x, y),$$

for which the general solution is of the form

$$W(x, y) = \theta(x)\, (x - \delta)\, (x + y)\, f_1(x^2, y), \tag{3.25}$$

where f_1 is an arbitrary function. Using (3.25) in (3.24b) yields

$$(y - 1)\, f_1(x^2, y) = (y + 1)\, f_1(x^2, -y),$$

which has the general solution

$$f_1(x^2, y) = \theta(y)\, (y + 1)\, f_2(x^2, y^2),$$

where f_2 is an arbitrary function. The function $W(x, y)$ is thus of the general form

$$W(x, y) = \theta(x)\theta(y)\, (x - \delta)\, (x + y)\, (y + 1)\, f_2(x^2, y^2). \tag{3.26}$$

Upon substituting the above expression in (3.24f), one finds after simplifications

$$\left[\frac{\beta - 1}{x^2 - \delta^2} + \frac{\gamma - 1}{x^2 - y^2}\right] x\, f_2(x^2, y^2) = \partial_x\, f_2(x^2, y^2).$$

After separation of variable, the result is

$$f_2(x^2, y^2) = (x^2 - \delta^2)^{\frac{\beta-1}{2}} (y^2 - x^2)^{\frac{\gamma-1}{2}} f_3(y^2).$$

Finally, upon substituting the above equation in (3.24e) one finds

$$y(\alpha - 1) f_3(y^2) = (y^2 - 1)\partial_y f_3(y^2),$$

which gives $f_3 = (1 - y^2)^{\frac{\alpha-1}{2}}$ and thus

$$f_2(x^2, y^2) = (x^2 - \delta^2)^{\frac{\beta-1}{2}} (y^2 - x^2)^{\frac{\gamma-1}{2}} (1 - y^2)^{\frac{\alpha-1}{2}}.$$

Upon combining the above expression with (3.26), we find

$$W(x, y) = \theta(x\, y)|y|^{\beta+\gamma}(1 + y)\left(1 + \frac{x}{y}\right)\left(\frac{x - \delta}{y}\right)(1 - y^2)^{\frac{\alpha-1}{2}}\left(1 - \frac{x^2}{y^2}\right)^{\frac{\gamma-1}{2}}\left(\frac{x^2 - \delta^2}{y^2}\right)^{\frac{\beta-1}{2}},$$

which indeed corresponds to the weight function of proposition 2.1. The weight function $W(x, y)$ for the two-variable Big -1 Jacobi polynomials thus corresponds to the symmetry factor for L_1.

3.5. Conclusion

In this paper, we have introduced and characterized a new family of two-variable orthogonal polynomials that generalize of the Big -1 Jacobi polynomials. We have constructed their orthogonality measure and we have derived explicitly their bispectral properties. We have furthermore shown that the weight function for these two-variable polynomials can be recovered by symmetrization of the first-order differential operator with reflections that these polynomials diagonalize. The two-variable orthogonal polynomials introduced here are the first example of the multivariate generalization of the Bannai-Ito scheme. It would be of great interest to construct multivariate extensions of the other families of polynomials of this scheme.

Acknowledgments

VXG holds an Alexander-Graham-Bell fellowship from the Natural Science and Engineering Research Council of Canada (NSERC). JML holds a scholarship from NSERC. The research of LV is supported in part by NSERC. AZ would like to thank the Centre de recherches mathématiques for its hospitality.

Appendix : Normalization coefficients for Big -1 Jacobi polynomials

In this appendix, we present a derivation of the normalization coefficients (3.6) and (3.9) appearing in the orthogonality relation of the univariate Big -1 Jacobi polynomials. The result is obtained by using their kernel partners, the Chihara polynomials. These polynomials, denoted by $C_n(x;\alpha,\beta,\gamma)$, have the expression [14]

$$C_{2n}(x;\alpha,\beta,\gamma) = (-1)^n \frac{(\alpha+1)_n}{(n+\alpha+\beta+1)_n}\, {}_2F_1\left[\begin{matrix} -n, & n+\alpha+\beta+1 \\ & \alpha+1 \end{matrix}; x^2-\gamma^2\right],$$
$$C_{2n+1}(x;\alpha,\beta,\gamma) = (-1)^n \frac{(\alpha+2)_n}{(n+\alpha+\beta+2)_n}(x-\gamma)\, {}_2F_1\left[\begin{matrix} -n, & n+\alpha+\beta+2 \\ & \alpha+2 \end{matrix}; x^2-\gamma^2\right].$$

For $\alpha,\beta > -1$, they satisfy the orthogonality relation

$$\int_{\mathcal{E}} C_n(x;\alpha,\beta,\gamma)\, C_m(x;\alpha,\beta,\gamma)\, \theta(x)(x+\gamma)(x^2-\gamma^2)^{\alpha}(1+\gamma^2-x^2)^{\beta}\, \mathrm{d}x = \eta_n \delta_{nm}, \tag{3.27}$$

on the interval $\mathcal{E} = [-\sqrt{1+\gamma^2}, -|\gamma|] \cup [|\gamma|, \sqrt{1+\gamma^2}]$ and their normalization coefficients read

$$\eta_{2n} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} \frac{n!}{(2n+\alpha+\beta+1)[(n+\alpha+\beta+1)_n]^2},$$
$$\eta_{2n+1} = \frac{\Gamma(n+\alpha+2)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \frac{n!}{(2n+\alpha+\beta+2)[(n+\alpha+\beta+2)_n]^2}.$$

Let $\widehat{J}_n(x)$ be the monic Big -1 Jacobi polynomials:

$$\widehat{J}_n(x) = \kappa_n J_n(x) = x^n + \mathcal{O}(x^{n-1}),$$

where κ_n is given by

$$\kappa_n(a,b,c) = \begin{cases} \dfrac{(1-c^2)^{\frac{n}{2}}\left(\frac{a+1}{2}\right)_{\frac{n}{2}}}{\left(\frac{n+a+b+2}{2}\right)_{\frac{n}{2}}}, & n \text{ even}, \\[2ex] \dfrac{(1+c)(1-c^2)^{\frac{n-1}{2}}\left(\frac{a+1}{2}\right)_{\frac{n+1}{2}}}{\left(\frac{n+a+b+1}{2}\right)_{\frac{n+1}{2}}}, & n \text{ odd}. \end{cases}$$

The kernel polynomials $\widetilde{K}_n(x;a,b,c)$ associated to $\widehat{J}_n(x;a,b,c)$ are defined through the Christoffel transformation [20]:

$$\widetilde{K}_n(x;a,b,c) = \frac{\widehat{J}_{n+1}(x) - \frac{\widehat{J}_{n+1}(\nu)}{\widehat{J}_n(\nu)}\widehat{J}_n(x)}{x-\nu}. \tag{3.28}$$

For $\nu = 1$, the monic polynomials $\widetilde{K}_n(x;a,b,c)$ can be expressed in terms of the Chihara polynomials. Indeed, one can show that [14]

$$\widetilde{K}_n(x;a,b,c) = (\sqrt{1-c^2})^n\, C_n\left(\frac{x}{\sqrt{1-c^2}}; \frac{b-1}{2}, \frac{a+1}{2}, \frac{-c}{\sqrt{1-c^2}}\right). \tag{3.29}$$

Let $\mathcal{M}$ be the orthogonality functional for the Big -1 Jacobi polynomials. In view of the relation (3.28), one can write [20]

$$\mathcal{M}[(x-\nu)\;\widetilde{K}_n(x;a,b,c)\;x^k] = -\frac{\widehat{J}_{n+1}(\nu)}{\widehat{J}_n(\nu)}\mathcal{M}[\widehat{J}_n^2(x;a,b,c)]\,\delta_{kn}.$$

By linearity, one thus finds

$$\widetilde{\eta}_n = -\frac{\widehat{J}_{n+1}(\nu)}{\widehat{J}_n(\nu)}\hat{h}_n.$$

where $\widetilde{\eta}_n = \mathcal{M}[(x-\nu)\;\widetilde{K}_n^2(x)]$ and $\hat{h}_n = \mathcal{M}[\widehat{P}_n^2(x)]$. For $\nu = 1$, the value of $\widetilde{\eta}_n$ is easily computed from (3.27) and (3.29). The above relation then gives the desired coefficients.

References

[1] L. Vinet, A. Zhedanov, A limit $q = -1$ for the big q-Jacobi polynomials, Trans. Am. Math. Soc. 364 (10) (2012) 5491–5507.

[2] S. Lewanowicz, P. Woźny, Two-variable orthogonal polynomials of big q-Jacobi type, J. Comput. Appl. Math. 233 (6) (2010) 1554–1561.

[3] C. F. Dunkl, Orthogonal Polynomials in Two Variables of q-Hahn and q-Jacobi Type, SIAM J. Algebr. Discret. Methods 1 (2) (1980) 137–151.

[4] K. H. Kwon, J. K. Lee, L. L. Littlejohn, Orthogonal polynomial eigenfunctions of second-order partial differerential equations, Trans. Am. Math. Soc. 353 (09) (2001) 3629–3648.

[5] H. L. Krall, I. M. Sheffer, Orthogonal polynomials in two variables, Ann. di Mat. Pura ed Appl. Ser. 4 76 (1) (1967) 325–376.

[6] J. Harnad, L. Vinet, O. Yermolayeva, A. Zhedanov, Two-dimensional Krall-Sheffer polynomials and integrable systems, J. Phys. A. Math. Gen. 34 (48) (2001) 10619–10625.

[7] W. Miller, S. Post, P. Winternitz, Classical and quantum superintegrability with applications, J. Phys. A Math. Theor. 46 (42) (2013) 423001.

[8] V. X. Genest, M. E. Ismail, L. Vinet, A. Zhedanov, The Dunkl oscillator in the plane: I. Superintegrability, separated wavefunctions and overlap coefficients, J. Phys. A Math. Theor. 46 (14) (2013) 145201.

[9] V. X. Genest, L. Vinet, A. Zhedanov, The singular and the 2:1 anisotropic Dunkl oscillators in the plane, J. Phys. A Math. Theor. 46 (32) (2013) 325201.

[10] V. X. Genest, M. E. Ismail, L. Vinet, A. Zhedanov, The Dunkl Oscillator in the Plane II: Representations of the Symmetry Algebra, Commun. Math. Phys. 329 (3) (2014) 999–1029.

[11] V. X. Genest, L. Vinet, A. Zhedanov, A Laplace-Dunkl Equation on S2 and the Bannai–Ito Algebra, Commun. Math. Phys. 336 (1) (2015) 243–259.

[12] R. Koekoek, P. A. Lesky, R. F. Swarttouw, Hypergeometric Orthogonal Polynomials, Springer, 2010.

[13] V. X. Genest, L. Vinet, A. Zhedanov, Bispectrality of the complementary Bannai-Ito polynomials, Symmetry, Integr. Geom. Methods Appl. 9 (2013) 18.

[14] V. X. Genest, L. Vinet, A. Zhedanov, A continuous limit of the Complementary Bannai-Ito polynomials : Chihara polynomials, Symmetry, Integr. Geom. Methods Appl. 10 (0) (2014) 18–38.

[15] S. Tsujimoto, L. Vinet, A. Zhedanov, Dunkl shift operators and Bannai-Ito polynomials, Adv. Math. (N. Y). 229 (4) (2012) 2123–2158.

[16] L. Vinet, A. Zhedanov, A 'missing' family of classical orthogonal polynomials, J. Phys. A Math. Theor. 44 (8) (2011) 085201.

[17] G. E. Andrews, R. Askey, R. Roy, Special functions, Vol. 71, Cambridge university press, 2000.

[18] T. Koornwinder, Two-Variable Analogues of the Classical Orthogonal Polynomials, Theory Appl. Spec. Funct. (1975) 435–495.

[19] G. Gasper, M. Rahman, G. George, Basic hypergeometric series, Vol. 96, Cambridge university press, 2004.

[20] T. S. Chihara, An introduction to orthogonal polynomials, Gordon and Breach, 1978.

Chapitre 4

Bivariate Bannai-Ito polynomials

J.M. Lemay et L. Vinet (2018). Bivariate Bannai-Ito polynomials. *Journal of Mathematical Physics 59 (12), 121703*

Abstract. A two-variable extension of the Bannai-Ito polynomials is presented. They are obtained via $q \to -1$ limits of the bivariate q-Racah and Askey-Wilson orthogonal polynomials introduced by Gasper and Rahman. Their orthogonality relation is obtained. These new polynomials are also shown to be bispectral. Two Dunkl shift operators are seen to be diagonalized by the bivariate Bannai-Ito polynomials and 3- and 9-terms recurrence relations are provided.

4.1. Introduction

In their classification of P- and Q-polynomial association schemes [1], Bannai and Ito identified a new 4-parameter family of orthogonal polynomials that now bear their names. They provided the explicit expressions of these polynomials and observed that they correspond to a $q \to -1$ limit of the q-Racah polynomials. The understanding of the Bannai-Ito polynomials has considerably increased in recent years. Of particular relevance to the present study is the fact that they have been shown [2] to also arise as $q \to -1$ limits of the Askey-Wilson polynomials. The Bannai-Ito polynomials are now known to be bispectral : they are eigenfunctions of the most general first order shift operator of Dunkl type that preserves the space of polynomials of a given degree [2]. They have been identified with the non-symmetric Wilson polynomials [3] and are essentially the Racah coefficients of the Lie superalgebra $\mathfrak{osp}(1|2)$ [4]. They have moreover found various applications beyond algebraic combinatorics especially in the context of superintegrable and exactly solvable models [5, 6, 7]. The Bannai-Ito polynomials and their kernel partners, the complementary Bannai-Ito polynomials admit various bispectral families of orthogonal polynomials as descendants and special cases and thus sit at the top of a $q = -1$ analog of the Askey-scheme [8, 9, 10, 11, 2, 12, 13].

The extension to many variables of the theory of univariate orthogonal polynomials is obviously of great interest. There are two major directions in this broad topic (see [14, 15] for instance). One involves the theory of symmetric functions [16] and has the Macdonald and Koornwinder polynomials associated to root systems as main characters. The other works through the coupling of univariate polynomials and features the multivariable extension of the Racah and Wilson polynomials and their descendants introduced by Tratnik [17, 18]. We shall focus on the latter area in the following. A key feature of Tratnik's construction is that the multivariate orthogonality relation is obtained by induction on the univariate one. Iliev and Geronimo have shown that these Tratnik polynomials are multispectral [19, 20]. Their q-generalizations have been discovered by Gasper and Rahman who thus provided multivariable extension of the q-Racah and Askey-Wilson polynomials and in so doing of the entire q-scheme [21, 22].

The goal of the present paper is to initiate a multivariable extension for the $q = -1$ scheme. Specifically, we introduce a bivariate extension of the Bannai-Ito polynomials and provide various structure relations. These polynomials are defined in formula (4.21).

The paper will be comprised of three main sections. We begin with a review of the Bannai-Ito polynomials and their structure relations. In section 2, we define the bivariate Bannai-Ito polynomials from a $q \to -1$ limit of Gasper and Rahman's two-variables q-Racah polynomials. The truncation conditions are examined and the orthogonality relation is obtained. In section 3, we obtain an untruncated definition for the bivariate Bannai-Ito polynomials from a $q \to -1$ limit of two-variable Askey-Wilson polynomials. This has the benefit of expressing the bispectrality relations in terms of operators which act directly on the variables instead of the orthogonality grid. A connection with the first definition is established and bispectrality relations for the polynomials are derived. Remarks and open questions are discussed in the conclusion.

4.2. Univariate Bannai-Ito polynomials

The monic Bannai-Ito polynomials $B_n(x; \rho_1, \rho_2, r_1, r_2)$, or $B_n(x)$ for short, depend on 4 parameters ρ_1, ρ_2, r_1, r_2 and are symmetric with respect to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group transformations generated by $\rho_1 \leftrightarrow \rho_2$ and $r_1 \leftrightarrow r_2$. Explicitly, this means that the BI polynomials verify

$$B_n(x; \rho_1, \rho_2, r_1, r_2) = B_n(x; \rho_2, \rho_1, r_1, r_2) = B_n(x; \rho_1, \rho_2, r_2, r_1) = B_n(x; \rho_2, \rho_1, r_2, r_1). \qquad (4.1)$$

We denote by g the combination of parameters

$$g = \rho_1 + \rho_2 - r_1 - r_2. \qquad (4.2)$$

Throughout this section, it will be convenient to write integers as follows

$$n = 2n_e + n_p, \qquad n_p \in \{0, 1\}, \quad n \in \mathbb{N}. \tag{4.3}$$

With these notations, the Bannai-Ito polynomials can be expressed in terms of two hypergeometric functions

$$\frac{1}{\eta_n} B_n(x; \rho_1, \rho_2, r_1, r_2) = {}_4F_3\left[\begin{matrix} -n_e,\ n_e+g+1,\ x-r_1+\frac{1}{2},\ -x-r_1+\frac{1}{2} \\ 1-r_1-r_2,\ \rho_1-r_1+\frac{1}{2},\ \rho_2-r_1+\frac{1}{2} \end{matrix}; 1\right] \tag{4.4}$$
$$+ \frac{(-1)^n (n_e+n_p+gn_p)\big(x-r_1+\frac{1}{2}\big)}{\big(\rho_1-r_1+\frac{1}{2}\big)\big(\rho_2-r_1+\frac{1}{2}\big)} {}_4F_3\left[\begin{matrix} -n_e-n_p+1,\ n_e+n_p+g+1,\ x-r_1+\frac{3}{2},\ -x-r_1+\frac{1}{2} \\ 1-r_1-r_2,\ \rho_1-r_1+\frac{3}{2},\ \rho_2-r_1+\frac{3}{2} \end{matrix}; 1\right]$$

where the normalization coefficient is given by

$$\eta_n = (-1)^n \frac{(\rho_1 - r_1 + \frac{1}{2})_{n_e+n_p} (\rho_2 - r_1 + \frac{1}{2})_{n_e+n_p} (1 - r_1 - r_2)_{n_e}}{(n_e + g + 1)_{n_e+n_p}}. \tag{4.5}$$

The expression (4.4) can be obtained from a $q \to -1$ limit of the q-Racah polynomials [1] and also from a $q \to -1$ limit of the Askey-Wilson polynomials [2]. Note that the two hypergeometric functions appearing in (4.4) are almost identical except for two $+1$ shifts in the upper parameter row and two in the lower row.

The $B_n(x)$ satisfy the three-term recurrence relation

$$xB_n(x) = B_{n+1}(x) + (\rho_1 - A_n - C_n)B_n(x) + A_{n-1}C_nB_{n-1}(x), \tag{4.6}$$

with the initial conditions $B_{-1}(x) = 0$ and $B_0(x) = 1$. The recurrence coefficients A_n and C_n are given by

$$A_n = \begin{cases} \dfrac{(n + 2\rho_1 - 2r_1 + 1)(n + 2\rho_1 - 2r_2 + 1)}{4(n + g + 1)}, & n \text{ even}, \\ \dfrac{(n + 2g + 1)(n + 2\rho_1 + 2\rho_2 + 1)}{4(n + g + 1)}, & n \text{ odd}, \end{cases}$$
$$C_n = \begin{cases} -\dfrac{n(n - 2r_1 - 2r_2)}{4(n + g)}, & n \text{ even}, \\ -\dfrac{(n + 2\rho_2 - 2r_2)(n + 2\rho_2 - 2r_1)}{4(n + g)}, & n \text{ odd}. \end{cases} \tag{4.7}$$

It can be seen from the above relations that the positivity conditions $u_n = A_{n-1}C_n > 0$ cannot be satisfied for all $n \in \mathbb{N}$. This comes from the fact that C_n becomes negative for large n. It follows that the Bannai-Ito polynomials can only form a finite set of orthogonal polynomials for which the conditions $u_n > 0$, $n = 1, 2, \ldots, N$ are verified. This requires that the parameters realize a truncation condition for which

$$u_0 = u_{N+1} = 0. \tag{4.8}$$

We call the integer N the truncation parameter.

If these conditions are fulfilled, the BI polynomials $B_n(x)$ satisfy the discrete orthogonality relation

$$\sum_{k=0}^{N} w_k B_n(x_k) B_m(x_k) = h_n \delta_{nm}, \tag{4.9}$$

with respect to a positive set of weights w_k. The orthogonality grid x_k corresponds to the simple roots of the polynomial $B_{N+1}(x)$. The explicit formulas for the weight function w_k and the grid points x_k depend on the parity of N and more explicitly on the realization of the truncation condition $u_{N+1} = 0$.

If N is even, it follows from (4.7) that the condition $u_{N+1} = 0$ is tantamount to one of the following requirements associated to all possible values of j and ℓ :

$$i)\ r_j - \rho_\ell = \frac{N+1}{2}, \qquad j, \ell \in \{1, 2\}. \tag{4.10}$$

Note that the four possibilities coming from the choices of j and ℓ are equivalent since the polynomials $B_n(x)$ are invariant under the exchanges $\rho_1 \leftrightarrow \rho_2$ and $r_1 \leftrightarrow r_2$. To make the formulas explicit, fix $j = \ell = 1$. Then the grid points have the expression

$$x_k = (-1)^k (k/2 + \rho_1 + 1/4) - 1/4, \tag{4.11}$$

for $k = 0, \ldots, N$ and using (4.3) the weights take the form

$$w_k = \frac{(-1)^k}{k_e!} \frac{(\rho_1 - r_1 + 1/2)_{k_e+k_p} (\rho_1 - r_2 + 1/2)_{k_e+k_p} (\rho_1 + \rho_2 + 1)_{k_e} (2\rho_1 + 1)_{k_e}}{(\rho_1 + r_1 + 1/2)_{k_e+k_p} (\rho_1 + r_2 + 1/2)_{k_e+k_p} (\rho_1 - \rho_2 + 1)_{k_e}}, \tag{4.12}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol. The normalization factors are

$$h_n = \frac{n_e! N_e! (1+2\rho_1)_{N_e} (1+\rho_1+\rho_2)_{n_e} (1+n_e+g)_{N_e-n_e} (\frac{1}{2}+\rho_1-r_2)_{n_e+n_p} (\frac{1}{2}+\rho_2-r_2)_{n_e+n_p}}{(N_e-n_e-n_p)! (\frac{1}{2}+\rho_1+r_2)_{N_e-n_e} (\frac{1}{2}+n_e+n_p+\rho_2-r_1)_{N_e-n_e-n_p} (1+n+g)^2_{n_e+n_p}}. \tag{4.13}$$

The formulas for other values of j and ℓ can be obtained by using the appropriate substitutions $\rho_1 \leftrightarrow \rho_2$ and $r_1 \leftrightarrow r_2$ in (4.10)–(4.13).

If N is odd, it follows from (4.7) that the condition $u_{N+1} = 0$ is equivalent to one of the following restrictions:

$$ii)\ \rho_1 + \rho_2 = -\frac{N+1}{2}, \qquad iii)\ r_1 + r_2 = \frac{N+1}{2}, \qquad iv)\ \rho_1 + \rho_2 - r_1 - r_2 = -\frac{N+1}{2}. \tag{4.14}$$

We refer to the possible truncation conditions as type $i)$ to type $iv)$. Note however that type $iv)$ leads to a singularity in u_n when $n = (N+1)/2$ so it is not admissible[1]. For type $ii)$, the

[1]It might be possible to absorb this singularity with some fine-tuning of the parameters as has been done for the Racah and q-Racah polynomials [23, 24] but this has not been explored yet and goes beyond the scope of this paper.

formulas (4.11) and (4.12) hold and the normalization factors are given by

$$h_n = \frac{n_e!N_e!(1+2\rho_1)_{N_e+1}(1-r_1-r_2)_{n_e}(1+n_e+g)_{N_e+1-n_e}(\frac{1}{2}+\rho_1-r_1)_{n_e+n_p}(\frac{1}{2}+\rho_1-r_2)_{n_e+n_p}}{(N_e-n_e)!(\frac{1}{2}+\rho_1+r_1)_{N_e+1-n_e-n_p}(\frac{1}{2}+n_e+n_p+\rho_2-r_2)_{N_e+1-n_e-n_p}(1+n+g)^2_{n_e+n_p}}. \tag{4.15}$$

For type *iii*), the spectral points are given by

$$x_k = (-1)^k(r_1 - k/2 - 1/4) - 1/4, \tag{4.16}$$

for $k = 0, \ldots, N$, the weight function is given by the formula (4.12) with the substitutions $(\rho_1, \rho_2, r_1, r_2) \to -(r_1, r_2, \rho_1, \rho_2)$ and the normalization factors are

$$h_n = \frac{n_e!N_e!(2r_2-N)_{N_e+1}(\rho_1+\rho_2-N_e)_{N_e+1+n_e}(\rho_1+\rho_2-N_e)_{n_e}(r_2+\rho_1-\frac{1}{2}-N_e)_{n_e+n_p}(r_2+\rho_2-\frac{1}{2}-N_e)_{n_e+n_p}}{(N_e-n_e)!(\rho_1+\rho_2-N_e)^2_n(r_2-\rho_1-\frac{1}{2}-N_e)_{N_e+1-n_e-n_p}(r_2-\rho_2+\frac{1}{2}+n_e+n_p)_{N_e+1-n_e-n_p}}. \tag{4.17}$$

To construct a bivariate extension of the Bannai-Ito polynomials, the different truncation conditions for different parities of N will play an important role.

The BI polynomials also verify a difference equation :

$$\mathcal{L}B_n(x) = \lambda_n B_n(x) \tag{4.18}$$

with

$$\mathcal{L} = \frac{(x-\rho_1)(x-\rho_2)}{2x}(1-R_x) + \frac{(x-r_1+\frac{1}{2})(x-r_2+\frac{1}{2})}{2x+1}(T_x^1 R_x - 1) \tag{4.19}$$

where 1 is the identity operator, $R_x f(x) = f(-x)$ denotes the reflexion operator and $T_x^m f(x) = f(x+m)$ is a shift operator. The eigenvalues are given by

$$\lambda_n = \begin{cases} \frac{n}{2} & n \text{ even,} \\ r_1 + r_2 - \rho_1 - \rho_2 - \frac{n+1}{2} & n \text{ odd.} \end{cases} \tag{4.20}$$

It was shown in [2] that $\mathcal{L}$ is in fact the most general first order Dunkl difference operator with orthogonal polynomials as eigenfunctions.

4.3. Limit from bivariate q-Racah OPs

4.3.1. Defining the bivariate Bannai-Ito polynomials

Definition 4.1. *The bivariate Bannai-Ito polynomials are defined by*

$$B_{n_1,n_2}(z_1,z_2) = B_{n_1}\left(z_1 - \tfrac{1}{4}; \rho_1^{(1)}, \rho_2^{(1)}, r_1^{(1)}, r_2^{(1)}\right) B_{n_2}\left((-1)^{n_1} z_2 - \tfrac{1}{4}; \rho_1^{(2)}, \rho_2^{(2)}, r_1^{(2)}, r_2^{(2)}\right) \tag{4.21}$$

where the B_n and η_n are as in (4.4) and (4.5) and the parameters are given by

$$\begin{aligned} \rho_1^{(1)} &= c - p_1 + \tfrac{1}{2}, & r_1^{(1)} &= \tfrac{1}{2} - p_1, \\ \rho_2^{(1)} &= z_2 + p_2 - \tfrac{1}{4}, & r_2^{(1)} &= z_2 - p_2 + \tfrac{1}{4} \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} \rho_1^{(2)} &= \tfrac{n_1+1}{2} + c + p_2 - p_1, & r_1^{(2)} &= \tfrac{1-n_1}{2} - p_1 - p_2, \\ \rho_2^{(2)} &= p_3 - (-1)^{n_1+N}(\tfrac{N}{2} + p_1 + p_2 + p_3), & r_2^{(2)} &= -p_3 - (-1)^{n_1+N}(\tfrac{N}{2} + p_1 + p_2 + p_3). \end{aligned} \tag{4.23}$$

It is useful to denote the first and second BI polynomial in the definition by $B_{n_1}^{(1)}(z_1)$ and $B_{n_2}^{(2)}(z_2)$. Note that $B_{n_1}^{(1)}(z_1)$ contains the variable z_2, while $B_{n_2}^{(2)}(z_2)$ contains the degree n_1 in its parameters. We shall see that the $B_{n_1,n_2}(z_1, z_2)$ are orthogonal polynomials of degree $n_1 + n_2 \leq N$ in the variables z_1 and z_2 which depend on the four parameters p_1, p_2, p_3 and c.

Let us motivate this definition. In a spirit similar to the one that led to the discovery of the Bannai-Ito polynomials, we look at a $q \to -1$ limit of q-Racah polynomials in two-variable introduced by Gasper and Rahman in [21] as a q-generalization of Tratnik's multivariable Racah polynomials. We start here by specializing their q-Racah polynomials to two variables. Consider the product of q-Racah polynomials $R_{n_1}^{(1)} \times R_{n_2}^{(2)}$ depending on four parameters a_1, a_2, a_3 and b where $R_{n_1}^{(1)}$ and $R_{n_2}^{(2)}$ are defined by

$$\begin{aligned} R_{n_1}^{(1)} &= {}_4\phi_3\left[\begin{matrix} q^{-n_1},\ ba_2q^{n_1},\ q^{-x_1},\ a_1q^{x_1} \\ bq,\ a_1a_2q^{x_2},\ q^{-x_2} \end{matrix}\,\middle|\, q; q\right], \\ R_{n_2}^{(2)} &= {}_4\phi_3\left[\begin{matrix} q^{-n_2},\ ba_2a_3q^{2n_1+n_2},\ q^{n_1-x_2},\ a_1a_2q^{n_1+x_2} \\ ba_2q^{2n_1+1},\ a_1a_2a_3q^{N+n_1},\ q^{n_1-N} \end{matrix}\,\middle|\, q; q\right] \end{aligned} \tag{4.24}$$

in terms of the usual basic hypergeometric function ${}_r\phi_s$ (see e.g. [25]). Up to normalization of the polynomials, those corresponds to the q-Racah polynomials introduced by Gasper and Rahman [21]. Our goal is to take a $q \to -1$ limit of these polynomials in such a way that each hypergeometric function reduces to a Bannai-Ito polynomials and that each parameter survives the limit. Let us first write the hypergeometric functions as series :

$$R_{n_1}^{(1)} = \sum_{k=0}^{\infty} A_k^{(1)} q^k, \qquad R_{n_2}^{(2)} = \sum_{k=0}^{\infty} A_k^{(2)} q^k \tag{4.25}$$

where the coefficients are given by

$$\begin{aligned} A_k^{(1)} &= \prod_{i=0}^{k-1} \frac{(1-q^{-n_1+i})(1-ba_2q^{n_1+i})(1-q^{-x_1+i})(1-a_1q^{x_1+i})}{(1-q^{1+i})(1-bq^{1+i})(1-a_1a_2q^{x_2+i})(1-q^{-x_2+i})}, \\ A_k^{(2)} &= \prod_{i=0}^{k-1} \frac{(1-q^{-n_2+i})(1-ba_2a_3q^{2n_1+n_2+i})(1-q^{n_1-x_2+i})(1-a_1a_2q^{n_1+x_2+i})}{(1-q^{1+i})(1-ba_2q^{2n_1+1+i})(1-a_1a_2a_3q^{N+n_1+i})(1-q^{n_1-N+i})}. \end{aligned} \tag{4.26}$$

Note that the coefficients of an hypergeometric series are usually written in terms of Pochhammer symbols, but for our purpose, it is essential to expand them as products because the parity of the dummy index i will play an important role. Now, achieve the $q \to -1$ limit with the following parametrization

$$\begin{aligned} &q \to -e^t, \qquad t \to 0, \qquad q^{x_1} \to (-1)^{\frac{s_1}{2}} e^{ty_1}, \quad q^{x_2} \to (-1)^{\frac{s_2}{2}} e^{ty_2}, \\ &a_1 \to (-1)^{\frac{s_3}{2}} e^{t\alpha_1}, \quad a_2 \to (-1)^{\frac{s_4}{2}} e^{t\alpha_2}, \quad a_3 \to (-1)^{\frac{s_5}{2}} e^{t\alpha_3}, \quad b \to (-1)^{\frac{s_6}{2}} e^{t\beta} \end{aligned} \tag{4.27}$$

where the $s_i \in \{0, 1\}$, $i = 1, 2, \ldots, 6$ are integers to be determined. The precise formulas to select the s_i are a bit cumbersome. Hence, for clarity, let us simply sketch how one chooses a proper set of s_i. First, insert the parametrization (4.27) in the coefficients (4.26) to obtain

$$\begin{aligned} A_k^{(1)} &= \prod_{i=0}^{k-1} \frac{(1-(-1)^{b_1+i}e^{tB_1})(1-(-1)^{b_2+i}e^{tB_2})(1-(-1)^{b_3+i}e^{tB_3})(1-(-1)^{b_4+i}e^{tB_4})}{(1-(-1)^{b_5+i}e^{tB_5})(1-(-1)^{b_6+i}e^{tB_6})(1-(-1)^{b_7+i}e^{tB_7})(1-(-1)^{b_8+i}e^{tB_8})}, \\ A_k^{(2)} &= \prod_{i=0}^{k-1} \frac{(1-(-1)^{b_9+i}e^{tB_9})(1-(-1)^{b_{10}+i}e^{tB_{10}})(1-(-1)^{b_{11}+i}e^{tB_{11}})(1-(-1)^{b_{12}+i}e^{tB_{12}})}{(1-(-1)^{b_{13}+i}e^{tB_{13}})(1-(-1)^{b_{14}+i}e^{tB_{14}})(1-(-1)^{b_{15}+i}e^{tB_{15}})(1-(-1)^{b_{16}+i}e^{tB_{16}})} \end{aligned} \tag{4.28}$$

where the b_j are linear combinations of n_1, n_2, N and the s_i, while the B_j are linear combinations of the dummy index i, the degrees n_1, n_2, N, the variables y_1, y_2 and the parameters $\alpha_1, \alpha_2, \alpha_3, \beta$. The choice of s_i should be such that the b_j are integers. Thus, depending on the parity of the b_j, each factor in the limit $t \to 0$ will alternate between 0 and 2 for incrementing values of i. It is straightforward to see that the parities of the b_j must be chosen in such a way that there are the same number of zeroes and two in the numerator and in the denominator. Otherwise, the limit would diverge or become zero. Now, ratios of 2 will simply cancel out while ratios of 0 will give a non-trivial limit :

$$\frac{1+e^{tB_l}}{1+e^{tB_m}} \xrightarrow{t\to 0} 1, \qquad \frac{1-e^{tB_l}}{1-e^{tB_m}} \xrightarrow{t\to 0} \frac{B_l}{B_m}. \tag{4.29}$$

The key to obtaining Bannai-Ito polynomials in the limit $t \to 0$ is to chose the s_i in such a way that, for each i, there are always 2 zeroes and 2 twos in both numerators and denominators. Such a choice is not unique, but it can be verified by enumeration that all possible choices yield results that are equivalent under affine transformations of the parameters. This implies that (4.28) will reduce to

$$\begin{aligned} A_k^{(1)} &\xrightarrow{t\to 0} \prod_{i \text{ even}} \frac{B_{j_1}B_{j_2}}{B_{j_5}B_{j_6}} \times \prod_{i \text{ odd}} \frac{B_{j_3}B_{j_4}}{B_{j_7}B_{j_8}}, \\ A_k^{(2)} &\xrightarrow{t\to 0} \prod_{i \text{ even}} \frac{B_{j_9}B_{j_{10}}}{B_{j_{13}}B_{j_{14}}} \times \prod_{i \text{ odd}} \frac{B_{j_{11}}B_{j_{12}}}{B_{j_{15}}B_{j_{16}}} \end{aligned} \tag{4.30}$$

for some permutation $\pi \in S_{16}$ of the integers $j_k = \pi(k)$, $k = 1, \ldots, 16$ depending on the choice of the s_i. The explicit computation of the limit requires to consider separately all possible parities of the degrees n_1, n_2, N and also of the dummy indices k and i. Some notable features arise : First, the products in (4.30) can be written in terms of Pochhammer symbols. However, the products over even values of i will get additional factors when k is odd. Thus, the sums in (4.25) must be split between even and odd values of k. Each sum can be expressed as an hypergeometric ${}_4F_3$, but the additional factors for k odd have to be pulled in front. One obtains a linear combination of two similar ${}_4F_3$ with some $+1$ shifts. It is then possible to compare the result with (4.4) to express the result in terms of Bannai-Ito polynomials.

Let us consider without loss of generality one possible parametrization for the limit $q \to -1$:

$$\begin{aligned} &q \to -e^t, \qquad t \to 0, \qquad q^{x_1} \to e^{ty_1}, \quad q^{x_2} \to e^{ty_2}, \\ &a_1 \to -e^{t\alpha_1}, \quad a_2 \to e^{t\alpha_2}, \quad a_3 \to e^{t\alpha_3}, \quad b \to -e^{t\beta}. \end{aligned} \tag{4.31}$$

For convenience, we also use a different set of parameters :

$$\begin{aligned} &\alpha_1 = 4p_1 - 1, \quad \alpha_2 = 4p_2, \quad \alpha_3 = 4p_3 + 1, \\ &\beta = 2c, \quad y_1 = \tfrac{1}{2} - 2z_1, \quad y_2 = \tfrac{1}{2} - 2z_2 - 2p_1 - 2p_2. \end{aligned} \tag{4.32}$$

Using (4.31) and (4.32), a straightforward computation yields

$$R^{(1)}_{n_1} \to \frac{1}{\eta_{n_1}} B_{n_1}\left(z_1 - \tfrac{1}{4}; \rho_1^{(1)}, \rho_2^{(1)}, r_1^{(1)}, r_2^{(1)}\right) \qquad R^{(2)}_{n_2} \to \frac{1}{\eta_{n_2}} B_{n_2}\left((-1)^{n_1} z_2 - \tfrac{1}{4}; \rho_1^{(2)}, \rho_2^{(2)}, r_1^{(2)}, r_2^{(2)}\right)$$

where the parameters are given by (4.22) and (4.23). Omitting the normalization factors, this corresponds to Definition 1 of the bivariate Bannai-Ito polynomials given above. There are two reasons for removing the factors η_{n_i} : It is more natural to define the bivariate polynomials as a product of two monic Bannai-Ito polynomials and more importantly, the normalization factor η_{n_1} being a rational function in z_2 would break the polynomial structure.

4.3.2. Truncation conditions and orthogonality relation

Given our definition of the bivariate Bannai-Ito polynomials, the most important property to verify is orthogonality. We begin by stating the result.

Proposition 4.1. *The bivariate Bannai-Ito polynomials defined in (4.21) satisfy the orthogonality relation*

$$\sum_{s=0}^{N}\sum_{r=0}^{N} w^{(1)}_{r,N-s} w^{(2)}_{s,N} B_{n_1,n_2}(z_1(r), z_2(s)) B_{m_1,m_2}(z_1(r), z_2(s)) = H_{n_1,n_2,N}\delta_{n_1,m_1}\delta_{n_2,m_2} \tag{4.33}$$

where the grids are given by

$$z_1(r) = \tfrac{1}{2}\left[(-1)^{r+s+N}(r + s - N - 2p_1 + \tfrac{1}{2})\right] \qquad r = 0, \dots, N \tag{4.34}$$

$$z_2(s) = \tfrac{1}{2}\left[(-1)^{s+N}(s - N - 2p_1 - 2p_2 + \tfrac{1}{2})\right] \qquad s = 0, \dots, N \tag{4.35}$$

the weights by

$$
\begin{aligned}
w^{(1)}_{2r,2\tilde{s}} &= \frac{(2p_2)_r(-\tilde{s})_r(1-2p_1-2\tilde{s})_r(\frac{3}{2}+c-2p_1-\tilde{s})_r}{r!(\frac{1}{2}-c-\tilde{s})_r(1-2p_1-\tilde{s})_r(1-2p_1-2p_2-2\tilde{s})_r}\\
w^{(1)}_{2r+1,2\tilde{s}} &= -\frac{(2p_2)_{r+1}(-\tilde{s})_{r+1}(1-2p_1-2\tilde{s})_r(\frac{3}{2}+c-2p_1-\tilde{s})_r}{r!(\frac{1}{2}-c-\tilde{s})_r(1-2p_1-\tilde{s})_{r+1}(1-2p_1-2p_2-2\tilde{s})_{r+1}}\\
w^{(1)}_{2r,2\tilde{s}+1} &= \frac{(2p_2)_r(-\tilde{s})_r(-2p_1-2\tilde{s})_r(\frac{1}{2}+c-2p_1-\tilde{s})_r}{r!(-\frac{1}{2}-c-\tilde{s})_r(1-2p_1-\tilde{s})_r(-2p_1-2p_2-2\tilde{s})_r}\\
w^{(1)}_{2r+1,2\tilde{s}+1} &= -\frac{(2p_2)_{r+1}(-\tilde{s})_r(-2p_1-2\tilde{s})_r(\frac{1}{2}+c-2p_1-\tilde{s})_{r+1}}{r!(-\frac{1}{2}-c-\tilde{s})_{r+1}(1-2p_1-\tilde{s})_r(-2p_1-2p_2-2\tilde{s})_{r+1}}
\end{aligned}
\tag{4.36}
$$

and

$$
\begin{aligned}
w^{(2)}_{2s,2N} &= \frac{(-1)^s\binom{N}{s}(\frac{3}{2}+c-2p_1-N)_s(1-2p_1-2p_2-2N)_s(\frac{1}{2}+2p_3)_s}{N!(\frac{1}{2}-c-N)_s(1-2p_1-2p_2-2N)_{2s}(\frac{1}{2}-2p_1-2p_2-2p_3-2N)_s(1-2p_1-2N+2s)_{N-s}}\\
w^{(2)}_{2s+1,2N} &= \frac{(-1)^s\binom{N-1}{s}(\frac{3}{2}+c-2p_1-N)_s(1-2p_1-2p_2-2N)_s(\frac{1}{2}+2p_3)_{s+1}}{(N-1)!(\frac{1}{2}-c-N)_s(1-2p_1-2p_2-2N)_{2s+1}(\frac{1}{2}-2p_1-2p_2-2p_3-2N)_{s+1}(2-2p_1-2N+2s)_{N-s}}\\
w^{(2)}_{2s,2N+1} &= \frac{(-1)^s\binom{N}{s}(\frac{1}{2}+c-2p_1-N)_s(-2p_1-2p_2-2N)_s(\frac{1}{2}+2p_3)_s}{N!(-\frac{1}{2}-c-N)_s(-2p_1-2p_2-2N)_{2s}(-\frac{1}{2}-2p_1-2p_2-2p_3-2N)_s(-2p_1-2N+2s)_{N+1-s}}\\
w^{(2)}_{2s+1,2N+1} &= \frac{(-1)^{s+1}\binom{N}{s}(\frac{1}{2}+c-2p_1-N)_{s+1}(-2p_1-2p_2-2N)_s(\frac{1}{2}+2p_3)_{s+1}}{N!(-\frac{1}{2}-c-N)_{s+1}(-2p_1-2p_2-2N)_{2s+1}(-\frac{1}{2}-2p_1-2p_2-2p_3-2N)_{s+1}(1-2p_1-2N+2s)_{N-s}}
\end{aligned}
\tag{4.37}
$$

and the normalization coefficients $H_{n_1,n_2,N}$ are given in the appendix.

Proof. Our main tool will be orthogonality relation for the univariate BI polynomials which depends on the truncation conditions (4.10) and (4.14).

Notice that definition (4.21) involves a truncation parameter N inherited from the q-Racah polynomials in the limit process and which appears in the parameters (4.23). This implies that the definition comes with truncation conditions that are already built-in. Indeed, one can easily check that $B^{(2)}_{n_2}(z_2)$ satisfies a mixture of type *i*) (4.10) and type *iii*) (4.14) truncation conditions :

$$
\begin{cases}
r_1^{(2)} - \rho_2^{(2)} = \frac{N-n_1+1}{2} & \text{if } N-n_1 \text{ even,}\\
r_1^{(2)} + r_2^{(2)} = \frac{N-n_1+1}{2} & \text{if } N-n_1 \text{ odd}
\end{cases}
\tag{4.38}
$$

with truncation parameters $N-n_1$. Both conditions impose grid points for the variable z_2 :

$$
(-1)^{n_1}z_2 - \tfrac{1}{4} = x_s \tag{4.39}
$$

where x_s is given by (4.11) when $N+n_1$ is even and by (4.16) when $N+n_1$ is odd. Inspecting this equation for each possible parities of n_1 and N, one obtains (4.35). Substituting this relation for

z_2 in the parameters (4.22), one can now check the truncation conditions satisfied by B_{n_1} :

$$\begin{cases} r_1^{(1)} - \rho_2^{(1)} = \frac{N-s+1}{2} & \text{if } N-s \text{ even,} \\ r_1^{(1)} + r_2^{(1)} = \frac{N-s+1}{2} & \text{if } N-s \text{ odd.} \end{cases} \tag{4.40}$$

Thus, the polynomial B_{n_1} also satisfies the mixed type $i)$ and type $iii)$ truncation conditions with parameter $N-s$. Again, both conditions impose grid points for the variable z_1 :

$$z_1 - \tfrac{1}{4} = x_r \tag{4.41}$$

with x_r given by (4.11) for $N-s$ even and (4.16) for $N-s$ odd. This amounts to (4.34). Now, we are looking for an orthogonality relation of the form

$$\sum_{r,s} w_{r,N-s}^{(1)} w_{s,N}^{(2)} B_{n_1,n_2}(z_1(r), z_2(s)) B_{m_1,m_2}(z_1(r), z_2(s)) = H_{n_1,n_2,N}\delta_{n_1,m_1}\delta_{n_2,m_2} \tag{4.42}$$

In view of (4.40), the polynomial $B_{n_1}^{(1)}(z_1)$ satisfies the orthogonality relation

$$\sum_{r=0}^{N-s} w_{r,N-s}^{(1)} B_{n_1}^{(1)}(z_1(r)) B_{m_1}^{(1)}(z_1(r)) = h_{n_1,N-s}^{(1)}\delta_{n_1,m_1}. \tag{4.43}$$

Using (4.12), one obtains the weights (4.36). The normalization coefficients are retrieved from (4.13) and (4.17) :

$$\begin{aligned}
h_{2n_1,2\tilde{s}}^{(1)} &= \frac{\tilde{s}!\; n_1!(2p_2)_{n_1}(\frac{1}{2}+c+n_1+2p_2)_{\tilde{s}-n_1}(\frac{1}{2}+c+2p_2+\tilde{s})_{n_1}(\frac{3}{2}+c-2p_1-\tilde{s})_{n_1}(1-2p_1-2\tilde{s})_{\tilde{s}}}{(\tilde{s}-n_1)!(\frac{1}{2}+c+n_1)_{\tilde{s}-n_1}(\frac{1}{2}+c+n_1+2p_2)^2_{n_1}(1-2p_1-2p_2-2\tilde{s})_{\tilde{s}-n_1}}, \\
h_{2n_1+1,2\tilde{s}}^{(1)} &= \frac{\tilde{s}!\; n_1!(2p_2)_{n_1+1}(\frac{1}{2}+c+n_1+2p_2)_{\tilde{s}-n_1}(\frac{1}{2}+c+2p_2+\tilde{s})_{n_1+1}(\frac{3}{2}+c-2p_1-\tilde{s})_{n_1}(1-2p_1-2\tilde{s})_{\tilde{s}}}{(\tilde{s}-n_1-1)!(\frac{1}{2}+c+n_1+1)_{\tilde{s}-n_1-1}(\frac{1}{2}+c+n_1+2p_2)^2_{n_1+1}(1-2p_1-2p_2-2\tilde{s})_{\tilde{s}-n_1}}, \\
h_{2n_1,2\tilde{s}+1}^{(1)} &= \frac{\tilde{s}!\; n_1!(2p_2)_{n_1}(\frac{1}{2}+c+n_1+2p_2)_{\tilde{s}+1-n_1}(\frac{3}{2}+c+2p_2+\tilde{s})_{n_1}(\frac{1}{2}+c-2p_1-\tilde{s})_{n_1}(-2p_1-2\tilde{s})_{\tilde{s}+1}}{(\tilde{s}-n_1)!(\frac{1}{2}+c+n_1)_{\tilde{s}+1-n_1}(\frac{1}{2}+c+n_1+2p_2)^2_{n_1}(-2p_1-2p_2-2\tilde{s})_{\tilde{s}+1-n_1}}, \\
h_{2n_1+1,2\tilde{s}+1}^{(1)} &= \frac{\tilde{s}!\; n_1!(2p_2)_{n_1+1}(\frac{1}{2}+c+n_1+2p_2)_{\tilde{s}+1-n_1}(\frac{1}{2}+c+2p_2+\tilde{s})_{n_1+1}(\frac{1}{2}+c-2p_1-\tilde{s})_{n_1}(-2p_1-2\tilde{s})_{\tilde{s}+1}}{(\tilde{s}-n_1)!(\frac{3}{2}+c+n_1)_{\tilde{s}-n_1}(\frac{1}{2}+c+n_1+2p_2)^2_{n_1+1}(-2p_1-2p_2-2\tilde{s})_{\tilde{s}-n_1}}.
\end{aligned} \tag{4.44}$$

Using (4.43) in (4.42), one gets that

$$\sum_{s} h_{n_1,N-s}^{(1)} w_{s,N}^{(2)} B_{n_2}^{(2)}(z_2(s)) B_{m_2}^{(2)}(z_2(s)) = H_{n_1,n_2,N}\delta_{n_2,m_2} \tag{4.45}$$

should be the orthogonality relation satisfied by the univariate BI polynomials $B_{n_2}^{(2)}(z_2)$. Indeed, using (4.38) and the corresponding weights from section 1, one readily checks that the $w_{s,N}^{(2)}$ are given by (4.37) and the normalization coefficients are given in the appendix.

Hence the bivariate Bannai-Ito polynomials obey the orthogonality relation

$$\sum_{s=0}^{N-n_1}\sum_{r=0}^{N-s} w_{r,N-s}^{(1)} w_{s,N}^{(2)} B_{n_1,n_2}(z_1(r), z_2(s)) B_{m_1,m_2}(z_1(r), z_2(s)) = H_{n_1,n_2,N}\delta_{n_1,m_1}\delta_{n_2,m_2} \tag{4.46}$$

with the weights, the grids and the normalization coefficients given above. It is not hard to verify that both sums can be extended from 0 to N without changing the results. Indeed, one can check that all the extra terms are in fact zero because of the weights. □

4.4. Limit from bivariate Askey-Wilson OPs

In this section, a different definition for the bivariate Bannai-Ito polynomials via a $q \to -1$ limit of the Askey-Wilson polynomials is investigated. While very similar to the approach from q-Racah polynomials, the main difference lies in the fact that the Askey-Wilson polynomials do not have truncation conditions. Hence, no truncation parameter N is carried through the limit and a definition for untruncated bivariate Bannai-Ito polynomials is obtained. This definition has the advantage that its bispectrality relations can be expressed in terms of operators acting directly on the variables instead of acting on the orthogonality grids. The connection between both approaches is established.

4.4.1. Untruncated bivariate Bannai-Ito polynomials

Definition 4.2. *The untruncated bivariate Bannai-Ito polynomials are defined by*

$$B_{n_1,n_2}(z_1,z_2) = B_{n_1}\left(z_1 - \tfrac{1}{4};\ \beta, z_2 + \epsilon - \tfrac{1}{4}, \alpha, z_2 - \epsilon + \tfrac{1}{4}\right) \qquad (4.47)$$
$$\times B_{n_2}\left((-1)^{n_1} z_2 - \tfrac{1}{4};\ \beta + \epsilon + \tfrac{n_1}{2}, (1-\pi_{n_1})\gamma + \pi_{n_1}\delta, \alpha - \epsilon - \tfrac{n_1}{2}, (\pi_{n_1} - 1)\delta - \pi_{n_1}\gamma\right)$$

in terms of the monic BI polynomials $B_n(x)$ and where

$$\pi_n = \frac{1+(-1)^n}{2} = \begin{cases} 1 & n \text{ even}, \\ 0 & n \text{ odd}, \end{cases} \qquad (4.48)$$

is the indicator function of even numbers.

Note that Definition 2 reduces to Definition 1 (4.21) of section 2 if we let

$$\alpha \to \tfrac{1}{2} - p_1, \qquad \beta \to \tfrac{1}{2} + c - p_1, \qquad \epsilon \to p_2,$$
$$\gamma = \begin{cases} -\frac{N}{2} - p_1 - p_2 & N \text{ even}, \\ \frac{N-1}{2} + p_1 + p_2 + p_3 & N \text{ odd}, \end{cases} \qquad \delta = \begin{cases} \frac{N-1}{2} + p_1 + p_2 + p_3 & N \text{ even}, \\ -\frac{N}{2} - p_1 - p_2 & N \text{ odd}. \end{cases} \qquad (4.49)$$

To motivate Definition 2, let us first consider the Askey-Wilson polynomials

$$\hat{p}_n(x;a,b,c,d) = {}_4\phi_3\left[\begin{matrix} q^{-n},\ abcdq^{n-1},\ az,\ az^{-1} \\ ab,\ ac,\ ad \end{matrix}\middle| q;q\right] \qquad (4.50)$$

in the variable $x = \frac{1}{2}(z + z^{-1})$. The bivariate Askey-Wilson polynomials depending on five parameters a, b, c, d, a_2 as introduced by Gasper and Rahman in [22] are

$$\hat{P}_{n_1,n_2}(x_1, x_2) = \hat{p}_{n_1}(x_1; a, b, a_2 z_2, a_2 z_2^{-1})\hat{p}_{n_2}(x_2; a a_2 q^{n_1}, b a_2 q^{n_1}, c, d). \tag{4.51}$$

The limiting procedure is the same as in the previous section. Briefly, the choice of parametrization amounts to a selection of phases in front of each parameter defined as exponentials. Expanding the hypergeometric functions as series and expanding the Pochhammer symbols as products, one obtains an expression of the form (4.28) and must select the phases in such a way that there are always 2 zeroes and 2 twos in both numerators and denominators for each value of the dummy index i. Again, this choice is not unique, but all possibilities can again be shown to yield equivalent Bannai-Ito polynomials under affine transformations of the parameters.

We take

$$\begin{aligned} &q \to -e^{t}, \quad t \to 0, \qquad z_1 \to e^{t y_1}, \quad z_2 \to e^{t y_2}, \\ &a \to i e^{t\tilde{a}}, \quad b \to -i e^{t\tilde{b}}, \quad c \to i e^{t\tilde{c}}, \quad d \to -i e^{t\tilde{d}}, \quad a_2 \to e^{t\tilde{a_2}} \end{aligned} \tag{4.52}$$

with the reparametrization

$$\begin{aligned} &\tilde{a} = -2\alpha + \tfrac{1}{2}, \quad \tilde{b} = 2\beta + \tfrac{1}{2}, \quad \tilde{c} = 2\gamma + \tfrac{1}{2}, \quad \tilde{d} = 2\delta + \tfrac{1}{2}, \\ &y_1 = -2z_1, \quad y_2 = -2z_2, \quad \tilde{a_2} = 2\epsilon. \end{aligned} \tag{4.53}$$

This gives

$$P_{n_1}^{(1)}(\cos\theta_1) \to \frac{1}{\eta_{n_1}} B_{n_1}\left(z_1 - \tfrac{1}{4};\ \beta, z_2 + \epsilon - \tfrac{1}{4}, \alpha, z_2 - \epsilon + \tfrac{1}{4}\right) \tag{4.54}$$

$$P_{n_2}^{(2)}(\cos\theta_2) \to \frac{1}{\eta_{n_2}} B_{n_2}\left((-1)^{n_1} z_2 - \tfrac{1}{4};\ \beta + \epsilon + \tfrac{n_1}{2}, (1-\pi_{n_1})\gamma + \pi_{n_1}\delta, \alpha - \epsilon - \tfrac{n_1}{2}, (\pi_{n_1}-1)\delta - \pi_{n_1}\gamma\right).$$

The definition for the untruncated bivariate Bannai-Ito polynomials is thus obtained by taking the product of the corresponding monic Bannai-Ito polynomials, again dropping the normalization factors.

4.4.2. Multispectrality of the bivariate BI polynomials

Iliev demonstrated the bispectrality of the bivariate Askey-Wilson polynomials (4.51) (with a different normalization) in [20]. This section examines how the bispectrality relations of these polynomials are carried in the $q \to -1$ limit. Recalling the definition of the shift operator T_x^m and the reflection operator R_x given after (4.19), we have the following equations in the variables z_1 and z_2.

Proposition 4.2. *The untruncated bivariate Bannai-Ito polynomials* (4.47) *obey the difference equations*

$$L_1 B_{n_1,n_2}(z_1,z_2) = \mu_{n_1} B_{n_1,n_2}(z_1,z_2) \tag{4.55}$$

$$L_2 B_{n_1,n_2}(z_1,z_2) = \lambda_{n_1,n_2} B_{n_1,n_2}(z_1,z_2) \tag{4.56}$$

with the operators

$$L_1 = \frac{(\epsilon - z_1 + z_2)(z_1 - \beta - \frac{1}{4})}{2(z_1 - \frac{1}{4})}(T_{z_1}^{-1/2}R_{z_1} - 1) + \frac{(\epsilon + z_1 - z_2)(z_1 - \alpha + \frac{1}{4})}{2(z_1 + \frac{1}{4})}(T_{z_1}^{1/2}R_{z_1} - 1) \tag{4.57}$$

and

$$L_2 = \sum_{i,j=-1}^{1} c_{i,j} T_{z_1}^{i/2} R_{z_1}^{i} T_{z_2}^{j/2} R_{z_2}^{j} \tag{4.58}$$

with coefficients

$$\begin{aligned}
c_{-1,-1} &= \frac{(z_1 - \alpha + \frac{1}{4})(z_2 + \gamma + \frac{1}{4})(\epsilon + z_1 + z_2 + \frac{1}{2})}{4(z_1 + \frac{1}{4})(z_2 + \frac{1}{4})} \\
c_{-1,0} &= \frac{(z_1 - \alpha + \frac{1}{4})(\epsilon + z_1 - z_2)(\delta(z_2 + \frac{1}{4}) - \gamma(z_2 - \frac{1}{4}))}{4(z_1 + \frac{1}{4})(z_2 - \frac{1}{4})(z_2 + \frac{1}{4})} \\
c_{-1,1} &= \frac{(z_1 - \alpha + \frac{1}{4})(z_2 - \delta - \frac{1}{4})(\epsilon + z_1 - z_2)}{4(z_1 + \frac{1}{4})(z_2 - \frac{1}{4})} \\
c_{0,-1} &= \frac{(z_2 + \gamma + \frac{1}{4})(\epsilon - z_1 + z_2)(\alpha(z_1 - \frac{1}{4}) + \beta(z_1 + \frac{1}{4}))}{4(z_1 - \frac{1}{4})(z_1 + \frac{1}{4})(z_2 + \frac{1}{4})} \\
c_{0,0} &= \alpha(\gamma - \delta + \tfrac{1}{2}) + \beta(\gamma - \delta - \tfrac{1}{2}) - \tfrac{1}{2}(\gamma + \delta + \tfrac{1}{2}) - \epsilon \\
&\quad + \frac{(\epsilon + 4z_1 z_2 - \frac{1}{4})(\alpha(z_1 - \frac{1}{4}) + \beta(z_1 + \frac{1}{4}))(\delta(z_2 + \frac{1}{4}) - \gamma(z_2 - \frac{1}{4}))}{4(z_1 - \frac{1}{4})(z_1 + \frac{1}{4})(z_2 - \frac{1}{4})(z_2 + \frac{1}{4})} \\
c_{0,1} &= \frac{(z_2 - \delta - \frac{1}{4})(\epsilon + z_1 - z_2)(\alpha(z_1 - \frac{1}{4}) + \beta(z_1 + \frac{1}{4}))}{4(z_1 - \frac{1}{4})(z_1 + \frac{1}{4})(z_2 - \frac{1}{4})} \\
c_{1,-1} &= \frac{(z_1 - \beta - \frac{1}{4})(z_2 + \gamma + \frac{1}{4})(\epsilon - z_1 + z_2)}{4(z_1 - \frac{1}{4})(z_2 + \frac{1}{4})} \\
c_{1,0} &= \frac{(z_1 - \beta - \frac{1}{4})(\epsilon - z_1 + z_2)(\delta(z_2 + \frac{1}{4}) - \gamma(z_2 - \frac{1}{4}))}{4(z_1 - \frac{1}{4})(z_2 - \frac{1}{4})(z_2 + \frac{1}{4})} \\
c_{1,1} &= \frac{(z_1 - \beta - \frac{1}{4})(z_2 - \delta - \frac{1}{4})(\epsilon - z_1 - z_2 + \frac{1}{2})}{4(z_1 - \frac{1}{4})(z_2 - \frac{1}{4})}.
\end{aligned} \tag{4.59}$$

The eigenvalues are given by

$$\mu_{n_1} = \begin{cases} \frac{n_1}{2} & n_1 \text{ even}, \\ -\frac{n_1}{2} + \alpha - \beta - 2\epsilon & n_1 \text{ odd}, \end{cases} \tag{4.60}$$

and

$$\lambda_{n_1,n_2} = \begin{cases} \frac{n_1+n_2}{2} & n_1+n_2 \text{ even}, \\ \frac{n_1+n_2+1}{2} - \alpha + \beta + \gamma + \delta + 2\epsilon & n_1+n_2 \text{ odd}. \end{cases} \tag{4.61}$$

Proof. Consider the renormalized Askey-Wilson polynomials

$$p_n(x; a, b, c, d) = \xi_n(a, b, c, d)\hat{p}_n(x; a, b, c, d) \tag{4.62}$$

where

$$\xi_n(a, b, c, d) = \frac{(ab, ac, ad; q)_n}{a^n} \tag{4.63}$$

and their corresponding bivariate extension

$$P_{n_1,n_2}(x_1, x_2) = \zeta_{n_1,n_2} p_{n_1}(x_1; a, b, a_2 z_2, a_2 z_2^{-1}) p_{n_2}(x_2; a a_2 q^{n_1}, b a_2 q^{n_1}, c, d) \tag{4.64}$$

with normalization

$$\zeta_{n_1,n_2} = \frac{c^{n_1+n_2} a_2^{n_1}}{(a_2^2; q)_{n_1} (aca_2; q)_{n_1+n_2} (bca_2; q)_{n_1+n_2} (cd; q)_{n_2}}. \tag{4.65}$$

They obey the q-difference equation [20]

$$\mathcal{L} P_{n_1,n_2}(x_1, x_2) = \Lambda_{n_1,n_2} P_{n_1,n_2}(x_1, x_2) \tag{4.66}$$

where

$$\mathcal{L} = \sum_{i,j=-1}^{1} C_{i,j} E_{q,z_1}^i E_{q,z_2}^j \tag{4.67}$$

in terms of the shifts operators $E_{q,z}$ which send $z \to qz$. The explicit expression for the coefficients $C_{i,j}$ can be found in the appendix and the eigenvalues are given by

$$\Lambda_{n_1,n_2} = (q^{-n_1-n_2} - 1)\left(1 - a a_2^2 bcd q^{n_1+n_2-1}\right). \tag{4.68}$$

The difference equation for the bivariate Bannai-Ito polynomials is found as a limit of this relation. The operator $\mathcal{L}$ will correspond to the operator L_2 given by (4.58) in the limit (4.52) with the reparametrization (4.53). The coefficients $c_{i,j}$ are obtained by the limits

$$c_{i,j} = \lim_{q \to -1} \frac{C_{i,j}}{4(1+q)} \tag{4.69}$$

and the eigenvalues by

$$\lambda_{n_1,n_2} = \lim_{q\to -1} \frac{\Lambda_{n_1,n_2}}{4(1+q)} = \begin{cases} \frac{n_1+n_2}{2} & n_1+n_2 \text{ even}, \\ \frac{n_1+n_2+1}{2} - \alpha+\beta+\gamma+\delta+2\epsilon & n_1+n_2 \text{ odd}. \end{cases} \tag{4.70}$$

The factor $\frac{1}{4}$ is just for convenience. The bivariate BI polynomials (4.47) will thus satisfy the difference equation (4.56).

The difference equation (4.55) follows directly from the univariate Dunkl difference equation given by (4.18), (4.19) and (4.20). It is also possible to obtain it from a $q \to -1$ limit of the bivariate Askey-Wilson polynomials second q-difference equation [20]. □

Let us now turn to the recurrence relations.

Proposition 4.3. *The untruncated bivariate Bannai-Ito polynomials $B_{n_1,n_2}(z_1,z_2)$ defined in (4.47) verify the 3-term recurrence relation*

$$\begin{aligned}\left((-1)^{n_1} z_2 - \tfrac{1}{4}\right) B_{n_1,n_2}(z_1,z_2) = B_{n_1,n_2+1}(z_1,z_2) + (\beta+\epsilon+\tfrac{n_1}{2} - A_{n_2} - C_{n_2}) B_{n_1,n_2}(z_1,z_2) \\ + A_{n_2-1}C_{n_2}B_{n_1,n_2-1}(z_1,z_2)\end{aligned} \tag{4.71}$$

where the coefficients A_{n_2} and C_{n_2} are given by (4.7) with the parameters ρ_1,ρ_2,r_1,r_2 being those of the second BI polynomial of (4.47).

They also satisfy the 9-term recurrence relation

$$\begin{aligned}(z_1 - \alpha^2+\beta^2)B_{n_1,n_2}(z_1,z_2) = \ & \theta^{(1)}_{n_1,n_2} B_{n_1+1,n_2} + \theta^{(2)}_{n_1,n_2} B_{n_1+1,n_2-1} + \theta^{(3)}_{n_1,n_2} B_{n_1+1,n_2-2} \\ & + \theta^{(4)}_{n_1,n_2} B_{n_1,n_2+1} + \theta^{(5)}_{n_1,n_2} B_{n_1,n_2} + \theta^{(6)}_{n_1,n_2} B_{n_1,n_2-1} \\ & + \theta^{(7)}_{n_1,n_2} B_{n_1-1,n_2+2} + \theta^{(8)}_{n_1,n_2} B_{n_1-1,n_2+1} + \theta^{(9)}_{n_1,n_2} B_{n_1-1,n_2}\end{aligned} \tag{4.72}$$

where the explicit expression for the coefficients $\theta^{(i)}_{n_1,n_2}$ are given in the appendix.

Proof. The polynomials $P_{n_1,n_2}(x_1,x_2)$ verify [20]

$$\begin{aligned}ca_2\left[\tfrac{(a+b)(ab+q)}{ab(1+q)} - z_1 - z_1^{-1}\right] P_{n_1,n_2}(x_1,x_2) = \ & \tau^{(1)}_{n_1,n_2} P_{n_1+1,n_2} + \tau^{(2)}_{n_1,n_2} P_{n_1+1,n_2-1} + \tau^{(3)}_{n_1,n_2} P_{n_1+1,n_2-2} \\ & + \tau^{(4)}_{n_1,n_2} P_{n_1,n_2+1} + \tau^{(5)}_{n_1,n_2} P_{n_1,n_2} + \tau^{(6)}_{n_1,n_2} P_{n_1,n_2-1} \\ & + \tau^{(7)}_{n_1,n_2} P_{n_1-1,n_2+2} + \tau^{(8)}_{n_1,n_2} P_{n_1-1,n_2+1} + \tau^{(9)}_{n_1,n_2} P_{n_1-1,n_2}.\end{aligned} \tag{4.73}$$

The expression for the coefficients $\tau^{(i)}_{n_1,n_2}$ can be found in the appendix. This will become a 9-term recurrence relation for the bivariate Bannai-Ito polynomials in the $q \to -1$ limit. The only tricky part is to keep track of all the changes in normalization of the various polynomials in play. Denote by

$$\mathcal{N}_{n_1,n_2} = \zeta_{n_1,n_2}\xi_{n_1}(x_1; a, b, a_2z_2, a_2z_2^{-1})\xi_{n_2}(x_2; aa_2q^{n_1}, ba_2q^{n_1}, c, d) \tag{4.74}$$

the normalization factors that appear in (4.64) and

$$\mathcal{M}_{n_1,n_2} = \eta^{(1)}_{n_1}\eta^{(2)}_{n_2} \tag{4.75}$$

the normalization coefficients in the monic BI OPs (4.47) given by (4.5). Now, the recurrence coefficients are obtained by the following limits :

$$\theta^{(1)}_{n_1,n_2} = \frac{\mathcal{M}_{n_1,n_2}}{\mathcal{M}_{n_1+1,n_2}} \lim_{q\to-1} \frac{\mathcal{N}_{n_1+1,n_2}}{\mathcal{N}_{n_1,n_2}} \frac{\tau^{(1)}_{n_1,n_2}}{4(1+q)}, \qquad \theta^{(6)}_{n_1,n_2} = \frac{\mathcal{M}_{n_1,n_2}}{\mathcal{M}_{n_1,n_2-1}} \lim_{q\to-1} \frac{\mathcal{N}_{n_1,n_2-1}}{\mathcal{N}_{n_1,n_2}} \frac{\tau^{(6)}_{n_1,n_2}}{4(1+q)},$$

$$\theta^{(2)}_{n_1,n_2} = \frac{\mathcal{M}_{n_1,n_2}}{\mathcal{M}_{n_1+1,n_2-1}} \lim_{q\to-1} \frac{\mathcal{N}_{n_1+1,n_2-1}}{\mathcal{N}_{n_1,n_2}} \frac{\tau^{(2)}_{n_1,n_2}}{4(1+q)}, \qquad \theta^{(7)}_{n_1,n_2} = \frac{\mathcal{M}_{n_1,n_2}}{\mathcal{M}_{n_1-1,n_2+2}} \lim_{q\to-1} \frac{\mathcal{N}_{n_1-1,n_2+2}}{\mathcal{N}_{n_1,n_2}} \frac{\tau^{(7)}_{n_1,n_2}}{4(1+q)},$$

$$\theta^{(3)}_{n_1,n_2} = \frac{\mathcal{M}_{n_1,n_2}}{\mathcal{M}_{n_1+1,n_2-2}} \lim_{q\to-1} \frac{\mathcal{N}_{n_1+1,n_2-2}}{\mathcal{N}_{n_1,n_2}} \frac{\tau^{(3)}_{n_1,n_2}}{4(1+q)}, \qquad \theta^{(8)}_{n_1,n_2} = \frac{\mathcal{M}_{n_1,n_2}}{\mathcal{M}_{n_1-1,n_2+1}} \lim_{q\to-1} \frac{\mathcal{N}_{n_1-1,n_2+1}}{\mathcal{N}_{n_1,n_2}} \frac{\tau^{(8)}_{n_1,n_2}}{4(1+q)},$$

$$\theta^{(4)}_{n_1,n_2} = \frac{\mathcal{M}_{n_1,n_2}}{\mathcal{M}_{n_1,n_2+1}} \lim_{q\to-1} \frac{\mathcal{N}_{n_1,n_2+1}}{\mathcal{N}_{n_1,n_2}} \frac{\tau^{(4)}_{n_1,n_2}}{4(1+q)}, \qquad \theta^{(9)}_{n_1,n_2} = \frac{\mathcal{M}_{n_1,n_2}}{\mathcal{M}_{n_1-1,n_2}} \lim_{q\to-1} \frac{\mathcal{N}_{n_1-1,n_2}}{\mathcal{N}_{n_1,n_2}} \frac{\tau^{(9)}_{n_1,n_2}}{4(1+q)},$$

$$\theta^{(5)}_{n_1,n_2} = \lim_{q\to-1} \frac{\tau^{(5)}_{n_1,n_2}}{4(1+q)}.$$

The limits are assumed to be parametrized by (4.52) and (4.53). The results of these limits can be found in the appendix. Moreover, the recurrence relation operator is obtained via

$$\lim_{q\to-1} \frac{ca_2}{4(1+q)}\left[\tfrac{(a+b)(ab+q)}{ab(1+q)} - z_1 - z_1^{-1}\right] = z_1 - \alpha^2 + \beta^2. \tag{4.76}$$

Combining all these results, the recurrence relation (4.73) reduces to the desired 9-term recurrence relation for the bivariate Bannai-Ito polynomials.

The 3-term recurrence relation simply follows from the the recurrence relation of the univariate Bannai-Ito polynomials (4.6) applied to the second polynomial of (4.47). □

These two propositions establish the full bispectrality of the bivariate Bannai-Ito polynomials. Importantly, the recurrence relations prove that the B_{n_1,n_2} are polynomials and not simply rational functions of z_1 and z_2.

4.5. Conclusion

This paper has enlarged the catalogue of orthogonal polynomials in two variables with the construction of bivariate polynomials of Bannai-Ito type. Their identification and characterization made us of the $q \to -1$ limits of both the bivariate q-Racah and Askey-Wilson polynomials of Gasper and Rahman. The first instance led to a truncated version (Definition 1) equipped with a set of positive-definite weights on a two-dimensionnal lattice against which the BI polynomials are orthogonal. The $q \to -1$ limit of the bivariate Askey-Wilson polynomials yielded untruncated

Bannai-Ito polynomials in two variables (Definition 2) out of which the finite ones (Definition 1) can be obtained by the choice of parameters (4.49). This latter approach allowed for the identification of the difference equations and recurrence relations obeyed by the resulting functions showing in particular that they are indeed polynomials. Let us remark that the finite bivariate Bannai-Ito polynomials that have been found do only make use of the truncation conditions $i)$ and $iii)$ that the univariate polynomials admit. The question of whether there are other bivariate extensions that rely on different reduction mixtures and in particular condition $ii)$ is open and certainly worth exploring.

In another vein, one may wonder if there are natural multivariate generalizations of the Bannai-Ito polynomials along the symmetric function direction. In this respect, the examination of the $q \to -1$ limit of the Koornwinder polynomials of BC_2 type could prove illuminating and is envisaged.

We have initiated this exploration of the Bannai-Ito polynomials in many variables within the Tratnik framework because of the expected occurence of extensions of that type in the representation theory of the higher rank Bannai-Ito algebra [26] as well as in certain superintegrable models that have been constructed [27, 28]. Let us mention the following to be concrete. A Hamiltonian system on the 3-sphere whose symmetries realize the Bannai-Ito algebra of rank 2 has been constructed in [27] and various bases of wavefunctions have been explicitly obtained using the Cauchy-Kovalevskaia extension theorem. It is expected that bivariate Bannai-Ito polynomials arise in the interbasis connection coefficients. Do these overlaps coincide with the two-variable polynomials constructed here or do they belong to another extension yet to be found. We plan on looking into this in the near future. Another related question is to determine the algebra underscoring the multispectrality of the two variable BI polynomials we have defined, that is the algebra generated by L_1, L_2, x_1 and x_2. How does the resulting algebra compare with the rank 2 Bannai-Ito algebra? We hope to report on some of these questions soon.

Acknowledgments

JML holds an Alexander-Graham-Bell PhD fellowship from the Natural Science and Engineering Research Council (NSERC) of Canada. LV is grateful to NSERC for support through a discovery grant.

Appendix

To make the article more reader friendly, some cumbersome formulas have been omitted from the text. We give in this appendix their explicit expressions.

The normalization coefficients appearing in orthogonality relation (4.33) are given by

$$H_{2n_1,2n_2,2N} = \frac{n_1!n_2!(2p_2)_{n_1}(2p_3+\frac{1}{2})_{n_2}(c+n_1+2p_2+\frac{1}{2})_N(c-N-2p_1+\frac{3}{2})_{n_1+n_2}}{(N-n_1-n_2)!(c+n_1+\frac{1}{2})_{N-n_1}(c+n_1+2p_2+\frac{1}{2})^2_{n_1}(c+2n_1+n_2+2p_2+\frac{1}{2})_{N-n_1-n_2}}$$
$$\times \frac{(c+N+n_1+2p_2+2p_3+1)_{n_2}(c+2n_1+n_2+2p_2+2p_3+1)_{N-n_1-n_2}}{(c+2n_1+n_2+2p_2+2p_3+1)^2_{n_2}(-2N-2p_1-2p_2-2p_3+\frac{1}{2})_{N-n_1-n_2}}$$

$$H_{2n_1+1,2n_2,2N} = \frac{n_1!n_2!(2p_2)_{n_1+1}(2p_3+\frac{1}{2})_{n_2}(c+n_1+2p_2+\frac{1}{2})_N(c-N-2p_1+\frac{3}{2})_{n_1+n_2}}{(N-n_1-n_2-1)!(c+n_1+\frac{3}{2})_{N-n_1-1}(c+n_1+2p_2+\frac{1}{2})^2_{n_1+1}(c+2n_1+n_2+2p_2+\frac{3}{2})_{N-n_1-n_2-1}}$$
$$\times \frac{(c+N+n_1+2p_2+2p_3+2)_{n_2}(c+2n_1+n_2+2p_2+2p_3+2)_{N-n_1-n_2}}{(c+2n_1+n_2+2p_2+2p_3+2)^2_{n_2}(-2N-2p_1-2p_2-2p_3+\frac{1}{2})_{N-n_1-n_2}}$$

$$H_{2n_1,2n_2+1,2N} = \frac{n_1!n_2!(2p_2)_{n_1}(2p_3+\frac{1}{2})_{n_2+1}(c+n_1+2p_2+\frac{1}{2})_N(c-N-2p_1+\frac{3}{2})_{n_1+n_2}}{(N-n_1-n_2-1)!(c+n_1+\frac{1}{2})_{N-n_1}(c+n_1+2p_2+\frac{1}{2})^2_{n_1}(c+2n_1+n_2+2p_2+\frac{3}{2})_{N-n_1-n_2-1}}$$
$$\times \frac{(c+N+n_1+2p_2+2p_3+1)_{n_2+1}(c+2n_1+n_2+2p_2+2p_3+1)_{N-n_1-n_2}}{(c+2n_1+n_2+2p_2+2p_3+1)^2_{n_2+1}(-2N-2p_1-2p_2-2p_3+\frac{1}{2})_{N-n_1-n_2}}$$

$$H_{2n_1+1,2n_2+1,2N} = \frac{n_1!n_2!(2p_2)_{n_1+1}(2p_3+\frac{1}{2})_{n_2+1}(c+n_1+2p_2+\frac{1}{2})_N(c-N-2p_1+\frac{3}{2})_{n_1+n_2+1}}{(N-n_1-n_2-1)!(c+n_1+\frac{3}{2})_{N-n_1-1}(c+n_1+2p_2+\frac{1}{2})^2_{n_1+1}(c+2n_1+n_2+2p_2+\frac{5}{2})_{N-n_1-n_2-2}}$$
$$\times \frac{(c+N+n_1+2p_2+2p_3+2)_{n_2}(c+2n_1+n_2+2p_2+2p_3+2)_{N-n_1-n_2}}{(c+2n_1+n_2+2p_2+2p_3+2)^2_{n_2+1}(-2N-2p_1-2p_2-2p_3+\frac{1}{2})_{N-n_1-n_2-1}}$$

$$H_{2n_1,2n_2,2N+1} = \frac{n_1!n_2!(2p_2)_{n_1}(2p_3+\frac{1}{2})_{n_2}(c+n_1+2p_2+\frac{1}{2})_N(c-N-2p_1+\frac{1}{2})_{n_1+n_2}}{(N-n_1-n_2)!(c+n_1+\frac{1}{2})_{N-n_1+1}(c+n_1+2p_2+\frac{1}{2})^2_{n_1}(c+2n_1+n_2+2p_2+\frac{1}{2})_{N-n_1-n_2}}$$
$$\times \frac{(c+N+n_1+2p_2+2p_3+2)_{n_2}(c+2n_1+n_2+2p_2+2p_3+1)_{N-n_1-n_2+1}}{(c+2n_1+n_2+2p_2+2p_3+1)^2_{n_2}(-2N-2p_1-2p_2-2p_3-\frac{1}{2})_{N-n_1-n_2+1}}$$

$$H_{2n_1+1,2n_2,2N+1} = \frac{n_1!n_2!(2p_2)_{n_1+1}(2p_3+\frac{1}{2})_{n_2}(c+n_1+2p_2+\frac{1}{2})_N(c-N-2p_1+\frac{1}{2})_{n_1+n_2+1}}{(N-n_1-n_2)!(c+n_1+\frac{3}{2})_{N-n_1}(c+n_1+2p_2+\frac{1}{2})^2_{n_1+1}(c+2n_1+n_2+2p_2+\frac{3}{2})_{N-n_1-n_2-1}}$$
$$\times \frac{(c+N+n_1+2p_2+2p_3+2)_{n_2}(c+2n_1+n_2+2p_2+2p_3+2)_{N-n_1-n_2}}{(c+2n_1+n_2+2p_2+2p_3+2)^2_{n_2}(-2N-2p_1-2p_2-2p_3-\frac{1}{2})_{N-n_1-n_2}}$$

$$H_{2n_1,2n_2+1,2N+1} = \frac{n_1!n_2!(2p_2)_{n_1}(2p_3+\frac{1}{2})_{n_2+1}(c+n_1+2p_2+\frac{1}{2})_N(c-N-2p_1+\frac{1}{2})_{n_1+n_2+1}}{(N-n_1-n_2)!(c+n_1+\frac{1}{2})_{N-n_1+1}(c+n_1+2p_2+\frac{1}{2})^2_{n_1}(c+2n_1+n_2+2p_2+\frac{3}{2})_{N-n_1-n_2-1}}$$
$$\times \frac{(c+N+n_1+2p_2+2p_3+1)_{n_2+1}(c+2n_1+n_2+2p_2+2p_3+1)_{N-n_1-n_2}}{(c+2n_1+n_2+2p_2+2p_3+1)^2_{n_2+1}(-2N-2p_1-2p_2-2p_3-\frac{1}{2})_{N-n_1-n_2}}$$

$$H_{2n_1+1,2n_2+1,2N+1} = \frac{n_1!n_2!(2p_2)_{n_1+1}(2p_3+\frac{1}{2})_{n_2+1}(c+n_1+2p_2+\frac{1}{2})_N(c-N-2p_1+\frac{1}{2})_{n_1+n_2+1}}{(N-n_1-n_2-1)!(c+n_1+\frac{3}{2})_{N-n_1}(c+n_1+2p_2+\frac{1}{2})^2_{n_1+1}(c+2n_1+n_2+2p_2+\frac{5}{2})_{N-n_1-n_2-2}}$$
$$\times \frac{(c+N+n_1+2p_2+2p_3+2)_{n_2+1}(c+2n_1+n_2+2p_2+2p_3+2)_{N-n_1-n_2}}{(c+2n_1+n_2+2p_2+2p_3+2)^2_{n_2+1}(-2N-2p_1-2p_2-2p_3-\frac{1}{2})_{N-n_1-n_2}}$$

The coefficients of the differential operator $\mathcal{L}$ (4.67) of section 3 are given by

$$C_{-1,-1} = -\frac{(z_1-a)(z_1-b)(z_2-c)(z_2-d)(z_1z_2-a_2)(z_1z_2-a_2q)}{(z_1^2-1)(z_2^2-1)(z_1^2-q)(q-z_2^2)}$$

$$C_{-1,0} = \frac{z_2q(q+1)(z_1-a)(z_1-b)(z_1z_2-a_2)(z_1-a_2z_2)\left(1+\frac{cd}{q}-\frac{(z_2^2+1)(c+d)}{(q+1)z_2}\right)}{(z_1^2-1)(z_1^2-q)(q-z_2^2)(qz_2^2-1)}$$

$$C_{-1,1} = \frac{(a-z_1)(z_1-b)(cz_2-1)(1-dz_2)(z_1-a_2z_2)(z_1-a_2qz_2)}{(z_1^2-1)(z_2^2-1)(z_1^2-q)(qz_2^2-1)}$$

$$C_{0,-1} = \frac{z_1q(q+1)(z_2-c)(z_2-d)(z_1z_2-a_2)(z_2-a_2z_1)\left(1+\frac{ab}{q}-\frac{(z_1^2+1)(a+b)}{z_1(q+1)}\right)}{(z_2^2-1)(z_1^2-q)(1-qz_1^2)(z_2^2-q)}$$

$$C_{0,0} = -1+\frac{a_2(a+b)(c+d)}{q+1}-\frac{aa_2^2bcd}{q}$$
$$+\frac{z_1^2z_2^2q^2(q+1)^2\left(1+\frac{ab}{q}-\frac{(z_1^2+1)(a+b)}{z_1(q+1)}\right)\left(1+\frac{a_2^2}{q}-\frac{a_2(z_1^2+1)(z_2^2+1)}{z_1z_2(q+1)}\right)\left(1+\frac{cd}{q}-\frac{(z_2^2+1)(c+d)}{z_2(q+1)}\right)}{(z_1^2-q)(1-qz_1^2)(z_2^2-q)(1-qz_2^2)}$$

$$C_{0,1} = \frac{z_1q(q+1)(1-cz_2)(1-dz_2)(z_1-a_2z_2)(1-a_2z_1z_2)\left(1+\frac{ab}{q}-\frac{(z_1^2+1)(a+b)}{z_1(q+1)}\right)}{(1-z_2^2)(z_1^2-q)(1-qz_1^2)(1-qz_2^2)}$$

$$C_{1,-1} = \frac{(az_1-1)(bz_1-1)(c-z_2)(z_2-d)(a_2z_1-z_2)(a_2qz_1-z_2)}{(z_1^2-1)(z_2^2-1)(qz_1^2-1)(q-z_2^2)}$$

$$C_{1,0} = \frac{z_2q(q+1)(1-az_1)(1-bz_1)(z_2-a_2z_1)(1-a_2z_1z_2)\left(1+\frac{cd}{q}-\frac{(z_2^2+1)(c+d)}{(q+1)z_2}\right)}{(z_1^2-1)(qz_1^2-1)(q-z_2^2)(qz_2^2-1)}$$

$$C_{1,1} = -\frac{(az_1-1)(bz_1-1)(cz_2-1)(1-dz_2)(a_2z_1z_2-1)(a_2qz_1z_2-1)}{(z_1^2-1)(z_2^2-1)(qz_1^2-1)(qz_2^2-1)}$$

The recurrence coefficients for the Askey-Wilson polynomials appearing in (4.73) have the following expressions

$$\tau^{(1)}_{n_1,n_2} = -\frac{\left(a_2^2q^{n_1}-1\right)\left(aa_2^2bq^{n_1}-q\right)\left(aa_2cq^{n_1+n_2}-1\right)\left(a_2bcq^{n_1+n_2}-1\right)\left(aa_2^2bcdq^{2n_1+n_2}-1\right)\left(aa_2^2bcdq^{2n_1+n_2}-q\right)}{\left(aa_2^2bq^{2n_1}-1\right)\left(aa_2^2bq^{2n_1}-q\right)\left(aa_2^2bcdq^{2(n_1+n_2)}-1\right)\left(aa_2^2bcdq^{2(n_1+n_2)}-q\right)}$$

$$\tau^{(2)}_{n_1,n_2} = \frac{a_2cq^{n_1}\left(q^{n_2}-1\right)\left(a_2^2q^{n_1}-1\right)\left(aa_2^2bq^{n_1}-q\right)\left(aa_2^2bcdq^{2n_1+n_2}-q\right)\left((a+b)(q+aa_2^2bcdq^{2(n_1+n_2)})-aa_2b(q+1)(c+d)q^{n_1+n_2}\right)}{\left(q-aa_2^2bq^{2n_1}\right)\left(aa_2^2bq^{2n_1}-1\right)\left(q^2-aa_2^2bcdq^{2(n_1+n_2)}\right)\left(aa_2^2bcdq^{2(n_1+n_2)}-1\right)}$$

$$\tau^{(3)}_{n_1,n_2} = -\frac{aa_2^2bc^2q^{2n_1}\left(q^{n_2}-1\right)\left(q^{n_2}-q\right)\left(a_2^2q^{n_1}-1\right)\left(aa_2^2bq^{n_1}-q\right)\left(aa_2dq^{n_1+n_2}-q\right)\left(a_2bdq^{n_1+n_2}-q\right)}{\left(aa_2^2bq^{2n_1}-1\right)\left(aa_2^2bq^{2n_1}-q\right)\left(aa_2^2bcdq^{2(n_1+n_2)}-q\right)\left(aa_2^2bcdq^{2(n_1+n_2)}-q^2\right)}$$

$$\tau^{(4)}_{n_1,n_2} = -\frac{q(q+1)\left(1-cdq^{n_2}\right)\left(1-aa_2cq^{n_1+n_2}\right)\left(1-a_2bcq^{n_1+n_2}\right)\left(1-aa_2^2bcdq^{2n_1+n_2-1}\right)\left(1+\frac{a_2^2}{q}-\frac{(a^2b+aq)(a^2a_2^2bq^{2n_1}+aq)}{a^3bq^{n_1+1}(q+1)}\right)}{\left(1-\frac{q^{2-2n_1}}{aa_2^2b}\right)\left(1-aa_2^2bq^{2n_1}\right)\left(1-aa_2^2bcdq^{2(n_1+n_2)}\right)\left(1-aa_2^2bcdq^{2n_1+2n_2-1}\right)}$$

$$\tau^{(5)}_{n_1,n_2} = -\frac{q^2(q+1)^2\left(1+\frac{a_2^2}{q}-\frac{(ab+q)(q+aa_2^2bq^{2n_1})}{abq^{n_1+1}(q+1)}\right)\left(1+\frac{c}{d}-\frac{(a+b)(q+aa_2^2bcdq^{2(n_1+n_2)})}{aa_2bdq^{n_1+n_2}(q+1)}\right)\left(1+\frac{cd}{q}-\frac{(q+aa_2^2bq^{2n_1})(q+aa_2^2bcdq^{2(n_1+n_2)})}{aa_2^2bq^{1+2n_1+n_2}(q+1)}\right)}{\left(1-\frac{q^{2-2n_1}}{aa_2^2b}\right)\left(1-aa_2^2bq^{2n_1}\right)\left(1-\frac{q^{-2(n_1+n_2-1)}}{aa_2^2bcd}\right)\left(1-aa_2^2bcdq^{2(n_1+n_2)}\right)}$$

$$\tau^{(6)}_{n_1,n_2} = \frac{a_2^2c^2q^{n_1}\left(q^{n_2}-1\right)\left(q-aa_2^2bq^{2n_1+n_2}\right)\left(q-aa_2dq^{n_1+n_2}\right)\left(q-a_2bdq^{n_1+n_2}\right)\left(abq^{n_1}(1+q)(a_2^2+q)-(ab+q)(q+aa_2^2bq^{2n_1})\right)}{\left(q^2-aa_2^2bq^{2n_1}\right)\left(aa_2^2bq^{2n_1}-1\right)\left(q-aa_2^2bcdq^{2(n_1+n_2)}\right)\left(q^2-aa_2^2bcdq^{2(n_1+n_2)}\right)}$$

$$\tau^{(7)}_{n_1,n_2} = \frac{aa_2^4bq^{2n_1+1}\left(1-q^{n_1}\right)\left(q-abq^{n_1}\right)\left(1-cdq^{n_2}\right)\left(1-cdq^{n_2+1}\right)\left(1-aa_2cq^{n_1+n_2}\right)\left(1-a_2bcq^{n_1+n_2}\right)}{\left(q-aa_2^2bq^{2n_1}\right)\left(q^2-aa_2^2bq^{2n_1}\right)\left(q-aa_2^2bcdq^{2(n_1+n_2)}\right)\left(aa_2^2bcdq^{2(n_1+n_2)}-1\right)}$$

$$\tau^{(8)}_{n_1,n_2} = \frac{a_2^3cq^{n_1}\left(q^{n_1}-1\right)\left(q-abq^{n_1}\right)\left(cdq^{n_2}-1\right)\left(q-aa_2^2bq^{2n_1+n_2}\right)\left((a+b)(q+aa_2^2bcdq^{2(n_1+n_2)})-aa_2b(c+d)(1+q)q^{n_1+n_2}\right)}{\left(q-aa_2^2bq^{2n_1}\right)\left(q^2-aa_2^2bq^{2n_1}\right)\left(q^2-aa_2^2bcdq^{2(n_1+n_2)}\right)\left(1-aa_2^2bcdq^{2(n_1+n_2)}\right)}$$

$$\tau^{(9)}_{n_1,n_2} = -\frac{a_2^2c^2\left(q^{n_1}-1\right)\left(abq^{n_1}-q\right)\left(aa_2^2bq^{2n_1+n_2}-q\right)\left(aa_2^2bq^{2n_1+n_2}-q^2\right)\left(aa_2dq^{n_1+n_2}-q\right)\left(a_2bdq^{n_1+n_2}-q\right)}{\left(q-aa_2^2bq^{2n_1}\right)\left(q^2-aa_2^2bq^{2n_1}\right)\left(q-aa_2^2bcdq^{2(n_1+n_2)}\right)\left(q^2-aa_2^2bcdq^{2(n_1+n_2)}\right)}$$

The coefficients for the 9-term recurrence relation (4.72) satisfied by the bivariate Bannai-Ito polynomials are

$$\theta^{(1)}_{n_1,n_2} = (-1)^{n_2}$$

$$\theta^{(2)}_{n_1,n_2} = \begin{cases} \frac{n_2}{4}\left(\frac{2\beta+2\gamma+2\epsilon+n_1+n_2+1}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2+1}-\frac{-2\alpha+2\gamma+2\epsilon+n_1+n_2}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2}\right) & n_1 \text{ even, } n_2 \text{ even,} \\ \frac{1}{4}(2\gamma+2\delta+n_2)\left(\frac{-2\alpha+2\gamma+2\epsilon+n_1+n_2+1}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2+1}-\frac{2\beta+2\gamma+2\epsilon+n_1+n_2}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2}\right) & n_1 \text{ even, } n_2 \text{ odd,} \\ \frac{n_2}{4}\left(\frac{2\beta+2\gamma+2\epsilon+n_1+n_2}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2}-\frac{-2\alpha+2\gamma+2\epsilon+n_1+n_2+1}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2+1}\right) & n_1 \text{ odd, } n_2 \text{ even,} \\ \frac{1}{4}(2\gamma+2\delta+n_2)\left(\frac{-2\alpha+2\gamma+2\epsilon+n_1+n_2}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2}-\frac{2\beta+2\gamma+2\epsilon+n_1+n_2+1}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2+1}\right) & n_1 \text{ odd, } n_2 \text{ odd,} \end{cases}$$

$$\theta^{(3)}_{n_1,n_2} = \begin{cases} -\frac{n_2(2\gamma+2\delta+n_2-1)(-2\alpha+2\gamma+2\epsilon+n_1+n_2)(2\beta+2\delta+2\epsilon+n_1+n_2)}{16(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2)^2} & n_1 \text{ even, } n_2 \text{ even,} \\ \frac{(n_2-1)(2\gamma+2\delta+n_2)(-2\alpha+2\delta+2\epsilon+n_1+n_2)(2\beta+2\gamma+2\epsilon+n_1+n_2)}{16(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2)^2} & n_1 \text{ even, } n_2 \text{ odd,} \\ -\frac{n_2(2\gamma+2\delta+n_2-1)(-2\alpha+2\delta+2\epsilon+n_1+n_2)(2\beta+2\gamma+2\epsilon+n_1+n_2)}{16(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2)^2} & n_1 \text{ odd, } n_2 \text{ even,} \\ \frac{(n_2-1)(2\gamma+2\delta+n_2)(-2\alpha+2\gamma+2\epsilon+n_1+n_2)(2\beta+2\delta+2\epsilon+n_1+n_2)}{16(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2)^2} & n_1 \text{ odd, } n_2 \text{ odd,} \end{cases}$$

$$\theta^{(4)}_{n_1,n_2} = \begin{cases} 1-\frac{2\epsilon+\frac{n_1}{2}}{-\alpha+\beta+2\epsilon+n_1+\frac{1}{2}}-\frac{\frac{n_1}{2}}{-\alpha+\beta+2\epsilon+n_1-\frac{1}{2}} & n_1 \text{ even,} \\ 1-\frac{2\epsilon+\frac{n_1-1}{2}}{-\alpha+\beta+2\epsilon+n_1-\frac{1}{2}}-\frac{\frac{n_1+1}{2}}{-\alpha+\beta+2\epsilon+n_1+\frac{1}{2}} & n_1 \text{ odd,} \end{cases}$$

$$
\theta^{(5)}_{n_1,n_2} = \begin{cases}
-\frac{1}{4}\left(\frac{n_1}{-2\alpha+2\beta+4\epsilon+2n_1-1}+\frac{4\epsilon+n_1}{-2\alpha+2\beta+4\epsilon+2n_1+1}-1\right)\left(2\alpha+2\beta-\frac{2\beta+2\gamma+2\epsilon+n_1+n_2+1}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2+1}+1\right) & \\
\times\ \left(-2\alpha+2\beta-\frac{n_2}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2}+4\epsilon+2n_1+1\right) & n_1 \text{ even}, n_2 \text{ even}, \\
-\frac{1}{4}\left(\frac{n_1}{-2\alpha+2\beta+4\epsilon+2n_1-1}+\frac{4\epsilon+n_1}{-2\alpha+2\beta+4\epsilon+2n_1+1}-1\right)\left(2\alpha+2\beta+\frac{2(\beta+\gamma)+2\epsilon+n_1+n_2}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2}-1\right) & \\
\times\ \left(-2\alpha+2\beta+\frac{n_2+1}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2+1}+4\epsilon+2n_1-1\right) & n_1 \text{ even}, n_2 \text{ odd}, \\
-\frac{1}{4}\left(\frac{n_1+1}{-2\alpha+2\beta+4\epsilon+2n_1+1}+\frac{4\epsilon+n_1-1}{-2\alpha+2\beta+4\epsilon+2n_1-1}-1\right)\left(2\alpha+2\beta+\frac{-2\alpha+2\gamma+2\epsilon+n_1+n_2+1}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2+1}-1\right) & \\
\times\ \left(-2\alpha+2\beta-\frac{n_2}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2}+4\epsilon+2n_1+1\right) & n_1 \text{ odd}, n_2 \text{ even}, \\
-\frac{1}{4}\left(\frac{n_1+1}{-2\alpha+2\beta+4\epsilon+2n_1+1}+\frac{4\epsilon+n_1-1}{-2\alpha+2\beta+4\epsilon+2n_1-1}-1\right)\left(2\alpha+2\beta-\frac{-2\alpha+2\gamma+2\epsilon+n_1+n_2}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2}+1\right) & \\
\times\ \left(-2\alpha+2\beta+\frac{n_2+1}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2+1}+4\epsilon+2n_1-1\right) & n_1 \text{ odd}, n_2 \text{ odd},
\end{cases}
$$

$$
\theta^{(6)}_{n_1,n_2} = \begin{cases}
\frac{n_2\left(2\alpha-2\beta+\frac{2n_1}{-2\alpha+2\beta+4\epsilon+2n_1-1}-1\right)\left(-\alpha+\gamma+\epsilon+\frac{n_1+n_2}{2}\right)\left(\beta+\delta+\epsilon+\frac{n_1+n_2}{2}\right)\left(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+\frac{n_2}{2}\right)}{2(-2\alpha+2\beta+4\epsilon+2n_1+1)(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2)^2} & \\
 & n_1 \text{ even}, n_2 \text{ even}, \\
\frac{\left(\gamma+\delta+\frac{n_2}{2}\right)\left(2\alpha-2\beta+\frac{2n_1}{-2\alpha+2\beta+4\epsilon+2n_1-1}-1\right)\left(-\alpha+\beta+2\epsilon+n_1+\frac{n_2}{2}\right)\left(-\alpha+\delta+\epsilon+\frac{n_1+n_2}{2}\right)\left(\beta+\gamma+\epsilon+\frac{n_1+n_2}{2}\right)}{(-2\alpha+2\beta+4\epsilon+2n_1+1)(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2)^2} & \\
 & n_1 \text{ even}, n_2 \text{ odd}, \\
\frac{n_2\left(2\alpha-2\beta+\frac{8\epsilon+2n_1-2}{-2\alpha+2\beta+4\epsilon+2n_1-1}-1\right)\left(-\alpha+\delta+\epsilon+\frac{n_1+n_2}{2}\right)\left(\beta+\gamma+\epsilon+\frac{n_1+n_2}{2}\right)\left(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+\frac{n_2}{2}\right)}{2(-2\alpha+2\beta+4\epsilon+2n_1+1)(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2)^2} & \\
 & n_1 \text{ odd}, n_2 \text{ even}, \\
\frac{\left(\gamma+\delta+\frac{n_2}{2}\right)\left(2\alpha-2\beta+\frac{8\epsilon+2n_1-2}{-2\alpha+2\beta+4\epsilon+2n_1-1}-1\right)\left(-\alpha+\beta+2\epsilon+n_1+\frac{n_2}{2}\right)\left(-\alpha+\gamma+\epsilon+\frac{n_1+n_2}{2}\right)\left(\beta+\delta+\epsilon+\frac{n_1+n_2}{2}\right)}{(-2\alpha+2\beta+4\epsilon+2n_1+1)(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2)^2} & \\
 & n_1 \text{ odd}, n_2 \text{ odd},
\end{cases}
$$

$$
\theta^{(7)}_{n_1,n_2} = \begin{cases}
(-1)^{n_2}\frac{n_1(-2\alpha+2\beta+4\epsilon+n_1-1)}{(-2\alpha+2\beta+4\epsilon+2n_1-1)^2} & n_1 \text{ even}, \\
(-1)^{n_2}\frac{(4\epsilon+n_1-1)(-2\alpha+2\beta+n_1)}{(-2\alpha+2\beta+4\epsilon+2n_1-1)^2} & n_1 \text{ odd},
\end{cases}
$$

$$\theta^{(8)}_{n_1,n_2} = \begin{cases} -\dfrac{\frac{n_1}{2}(-2\alpha+2\beta+4\epsilon+n_1-1)\left(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+\frac{n_2}{2}\right)\left(2\alpha+2\beta-\frac{-2\alpha+2\gamma+2\epsilon+n_1+n_2}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2}+1\right)}{(-2\alpha+2\beta+4\epsilon+2n_1-1)^2(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2+1)} & n_1 \text{ even, } n_2 \text{ even,} \\ \dfrac{\frac{n_1}{2}(-2\alpha+2\beta+4\epsilon+n_1-1)\left(-\alpha+\beta+2\epsilon+n_1+\frac{n_2}{2}\right)\left(2\alpha+2\beta+\frac{-2\alpha+2\gamma+2\epsilon+n_1+n_2+1}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2+1}-1\right)}{(-2\alpha+2\beta+4\epsilon+2n_1-1)^2(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2)} & n_1 \text{ even, } n_2 \text{ odd,} \\ \dfrac{\left(2\epsilon+\frac{n_1-1}{2}\right)(2\alpha-2\beta-n_1)\left(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+\frac{n_2}{2}\right)\left(2\alpha+2\beta+\frac{-2\alpha+2\gamma+2\epsilon+n_1+n_2+1}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2+1}-1\right)}{(-2\alpha+2\beta+4\epsilon+2n_1-1)^2(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2)} & n_1 \text{ odd, } n_2 \text{ even,} \\ -\dfrac{\left(2\epsilon+\frac{n_1-1}{2}\right)(2\alpha-2\beta-n_1)\left(-\alpha+\beta+2\epsilon+n_1+\frac{n_2}{2}\right)\left(2\alpha+2\beta-\frac{-2\alpha+2\gamma+2\epsilon+n_1+n_2}{-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2}+1\right)}{(-2\alpha+2\beta+4\epsilon+2n_1-1)^2(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2+1)} & n_1 \text{ odd, } n_2 \text{ odd,} \end{cases}$$

$$\theta^{(9)}_{n_1,n_2} = \begin{cases} \dfrac{\frac{n_1}{2}(-2\alpha+2\beta+4\epsilon+n_1-1)(-2\alpha+2\beta+4\epsilon+2n_1+n_2-1)\left(-\alpha+\gamma+\epsilon+\frac{n_1+n_2}{2}\right)\left(\beta+\delta+\epsilon+\frac{n_1+n_2}{2}\right)\left(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+\frac{n_2}{2}\right)}{(-1)^{n_2+1}(-2\alpha+2\beta+4\epsilon+2n_1-1)^2(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2)^2} \\ \qquad n_1 \text{ even, } n_2 \text{ even,} \\ \dfrac{\frac{n_1}{2}(-2\alpha+2\beta+4\epsilon+n_1-1)(-2\alpha+2\beta+4\epsilon+2n_1+n_2)\left(-\alpha+\delta+\epsilon+\frac{n_1+n_2}{2}\right)\left(\beta+\gamma+\epsilon+\frac{n_1+n_2}{2}\right)\left(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+\frac{n_2-1}{2}\right)}{(-1)^{n_2+1}(-2\alpha+2\beta+4\epsilon+2n_1-1)^2(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2)^2} \\ \qquad n_1 \text{ even, } n_2 \text{ odd,} \\ \dfrac{\left(2\epsilon+\frac{n_1-1}{2}\right)(-2\alpha+2\beta+n_1)(-2\alpha+2\beta+4\epsilon+2n_1+n_2-1)\left(-\alpha+\delta+\epsilon+\frac{n_1+n_2}{2}\right)\left(\beta+\gamma+\epsilon+\frac{n_1+n_2}{2}\right)\left(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+\frac{n_2}{2}\right)}{(-1)^{n_2+1}(-2\alpha+2\beta+4\epsilon+2n_1-1)^2(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2)^2} \\ \qquad n_1 \text{ odd, } n_2 \text{ even,} \\ \dfrac{\left(2\epsilon+\frac{n_1-1}{2}\right)(-2\alpha+2\beta+n_1)(-2\alpha+2\beta+4\epsilon+2n_1+n_2)\left(-\alpha+\gamma+\epsilon+\frac{n_1+n_2}{2}\right)\left(\beta+\delta+\epsilon+\frac{n_1+n_2}{2}\right)\left(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+\frac{n_2-1}{2}\right)}{(-1)^{n_2+1}(-2\alpha+2\beta+4\epsilon+2n_1-1)^2(-\alpha+\beta+\gamma+\delta+2\epsilon+n_1+n_2)^2} \\ \qquad n_1 \text{ odd, } n_2 \text{ odd.} \end{cases}$$

References

[1] E. Bannai, T. Ito, Algebraic combinatorics, Benjamin/Cummings Menlo Park, 1984.

[2] S. Tsujimoto, L. Vinet, A. Zhedanov, Dunkl shift operators and Bannai–Ito polynomials, Advances in Mathematics 229 (4) (2012) 2123–2158.

[3] V. Genest, L. Vinet, A. Zhedanov, The non-symmetric Wilson polynomials are the Bannai–Ito polynomials, Proceedings of the American Mathematical Society 144 (12) (2016) 5217–5226.

[4] V. Genest, L. Vinet, A. Zhedanov, The Bannai-Ito polynomials as Racah coefficients of the $sl_{-1}(2)$ algebra, Proceedings of the American Mathematical Society 142 (5) (2014) 1545–1560.

[5] V. X. Genest, L. Vinet, A. Zhedanov, A Laplace-Dunkl equation on S^2 and the Bannai–Ito algebra, Communications in Mathematical Physics 336 (1) (2015) 243–259.

[6] H. De Bie, V. X. Genest, L. Vinet, A Dirac–Dunkl equation on S^2 and the Bannai–Ito algebra, Communications in Mathematical Physics 344 (2) (2016) 447–464.

[7] V. X. Genest, L. Vinet, A. Zhedanov, The Bannai–Ito algebra and a superintegrable system with reflections on the two-sphere, Journal of Physics A: Mathematical and Theoretical 47 (20) (2014) 205202.

[8] L. Vinet, A. Zhedanov, A missing family of classical orthogonal polynomials, Journal of Physics A: Mathematical and Theoretical 44 (8) (2011) 085201.

[9] L. Vinet, A. Zhedanov, A limit $q = -1$ for the big q-Jacobi polynomials, Transactions of the American Mathematical Society 364 (10) (2012) 5491–5507.

[10] L. Vinet, A. Zhedanov, et al., A Bochner theorem for Dunkl polynomials, SIGMA. Symmetry, Integrability and Geometry: Methods and Applications 7 (2011) 020.

[11] S. Tsujimoto, L. Vinet, A. Zhedanov, Dual -1 Hahn polynomials: "classical" polynomials beyond the Leonard duality, Proceedings of the American Mathematical Society 141 (3) (2013) 959–970.

[12] V. X. Genest, L. Vinet, A. Zhedanov, et al., Bispectrality of the complementary Bannai-Ito polynomials, SIGMA. Symmetry, Integrability and Geometry: Methods and Applications 9 (2013) 018.

[13] V. X. Genest, L. Vinet, A. S. Zhedanov, A "continuous" limit of the Complementary Bannai–Ito polynomials: Chihara polynomials, Symmetry, Integrability and Geometry: Methods and Applications 10 (0) (2014) 38–18.

[14] C. F. Dunkl, Y. Xu, Orthogonal polynomials of several variables, no. 155, Cambridge University Press, 2014.

[15] L. Fernández, F. Marcellán, T. E. Pérez, M. A. Pinar, Recent trends on two variable orthogonal polynomials, in: Differential algebra, complex analysis and orthogonal polynomials, Vol. 509, The American Mathematical Society Providence, RI, 2010, pp. 59–86.

[16] I. G. Macdonald, et al., Affine Hecke algebras and orthogonal polynomials, Vol. 157, Cambridge University Press, 2003.

[17] M. Tratnik, Some multivariable orthogonal polynomials of the askey tableau-discrete families, Journal of mathematical physics 32 (9) (1991) 2337–2342.

[18] M. Tratnik, Some multivariable orthogonal polynomials of the askey tableau-continuous families, Journal of mathematical physics 32 (8) (1991) 2065–2073.

[19] J. S. Geronimo, P. Iliev, Bispectrality of multivariable racah–wilson polynomials, Constructive Approximation 31 (3) (2010) 417–457.

[20] P. Iliev, Bispectral commuting difference operators for multivariable askey-wilson polynomials, Transactions of the American Mathematical Society 363 (3) (2011) 1577–1598.

[21] G. Gasper, M. Rahman, Some systems of multivariable orthogonal q-racah polynomials, The Ramanujan Journal 13 (1-3) (2007) 389–405.

[22] G. Gasper, M. Rahman, Some systems of multivariable orthogonal askey-wilson polynomials, in: Theory and applications of special functions, Springer, 2005, pp. 209–219.

[23] J.-M. Lemay, L. Vinet, A. Zhedanov, The para-Racah polynomials, Journal of Mathematical Analysis and Applications 438 (2) (2016) 565–577.

[24] J.-M. Lemay, L. Vinet, A. Zhedanov, A q-generalization of the para-Racah polynomials, Journal of Mathematical Analysis and Applications 462 (1) (2018) 323–336.

[25] R. Koekoek, P. A. Lesky, R. F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, Springer Science & Business Media, 2010.

[26] H. De Bie, V. X. Genest, L. Vinet, The Z_2^n Dirac–Dunkl operator and a higher rank Bannai-Ito algebra, Advances in Mathematics 303 (2016) 390–414.

[27] H. De Bie, V. X. Genest, J.-M. Lemay, L. Vinet, A superintegrable model with reflections on S^3 and the rank two Bannai-Ito algebra, Acta Polytechnica 56 (2016) 166–172.

[28] H. De Bie, V. X. Genest, J.-M. Lemay, L. Vinet, A superintegrable model with reflections on S^{n-1} and the higher rank Bannai-Ito algebra, Journal of Physics A: Mathematical and Theoretical 50 (19) (2017) 195202.

Partie 2

Modèles exactement résolubles

Introduction

Dans cette deuxième partie de la thèse on se penche sur des modèles où les polynômes orthogonaux jouent un rôle important. Le premier présente une application directe des polynômes de para-Racah introduits au premier chapitre de cette thèse et témoigne de travaux qui ont été menés parallèlement au développement de cette famille de fonctions spéciales. Le modèle consiste en une chaîne de spin de Heisenberg de type XX. La première tâche d'intérêt consiste à transporter un qubit, ou de façon équivalente un état de spin, d'un bout à l'autre de la chaîne dans un temps donné. On nomme cette tâche le transfert parfait. Vinet et Zhedanov ont montré que le choix de couplages permettant le transfert parfait se réduit à un problème spectral inverse en correspondance directe avec des familles de polynômes orthogonaux [35]. Le modèle le plus connu est certainement celui associé aux polynômes de Krawtchouk [4]. Une deuxième tâche aux applications prometteuses est la génération et le partage d'états intriqués. Ceci peut être réalisé par la revitalisation fractionnelle où une excitation initialement localisée sur le premier site de la chaîne se retrouverait éventuellement aux deux extrémités. L'investigation de chaînes XX dont les couplages réaliseraient cette tâche ont conduit naturellement à la découverte des polynômes de para-Krawtchouk qui sont orthogonaux sur la superposition de deux réseaux linéaires [36]. Le modèle qu'on présente au chapitre 5 poursuit cette idée et introduit un second modèle réalisant de la revitalisation fractionnelle en utilisant des couplages associés aux coefficients de récurrence polynômes de para-Racah qui sont orthogonaux sur un bi-réseau quadratique.

Dans leur classification de tous les systèmes superintégrables d'ordre deux en deux dimensions [37], Miller, Post et Winternitz ont identifié un modèle général duquel tous les autres peuvent être obtenus par contractions ou spécialisations. Les symétries de ce *système générique à trois paramètres sur la sphère* s'encodent dans l'algèbre de Racah. Cette structure algébrique a également la propriété de caractériser les polynômes de Racah et de Wilson au sommet du tableau $q = 1$ du schéma d'Askey [38]. Il en découle la remarquable propriété que les systèmes superintégrables de cette classification sont en correspondance avec les membres de cette hiérarchie de polynômes orthogonaux [39]. Une extension à trois dimensions de ce modèle a été introduite [40]. Une des motivations derrière la caractérisation de ce nouveau système superintégrable repose dans ses symétries. En effet, l'approche algébrique qui a mené à celui-ci a permis de montrer que ses symétries correspondent à l'algèbre de Racah de rang deux [41]. Celle-ci encode maintenant les propriétés de bispectralité de la famille de Racah des polynômes bivariés de type Tratnik. Ces polynômes apparaissent alors comme coefficients de recouplement dans le problème de Racah pour l'algèbre qui sous-tend ce système. De façon équivalente, ces polynômes agissent comme coefficients de connexion entre différentes bases de fonctions propres du système. Dans une autre direction, une généralisation du système générique à trois paramètres sur la sphère à l'aide d'opérateurs de Dunkl a conduit à une déformation avec des opérateurs de réflexions de ce modèle [42]. Les symétries sont alors encodées par l'algèbre de Bannai-Ito et les polynômes du même nom apparaissent comme coefficients de changement de base. Les chapitres 6 et 7 de cette thèse développent les modèles analogues de plus haute dimension avec réflexions. On considère le modèle à trois dimensions dans le chapitre 6 et la généralisation à n variables dans le chapitre suivant. En particulier, on conjecture l'apparition des polynômes de Bannai-Ito multivariés dans ces systèmes. Cette conjecture a maintenant été démontrée par un des coauteurs dans [43]. Les solutions du modèle sont également obtenues en terme des polynômes de Jacobi à l'aide de l'extension de Cauchy-Kovalevskaia et de la décomposition de Fischer.

Finalement, le dernier chapitre de cette thèse porte sur la théorie des représentations de la superalgèbre de Lie $\mathfrak{osp}(1|2)$. C'est la même structure algébrique qui sous-tend la construction des modèles des chapitres 6 et 7. Ce travail reprend une construction de Van der Jeugt et Koelink pour l'algèbre $\mathfrak{su}(1,1)$ [44], elle même basée sur des idées de Granovskii et Zhedanov [45], dans le cadre de la superalgèbre. On se concentre sur un élément dont les représentations sont tridiagonales et démontre sa connection avec une famille de polynômes appartenant au tableau de Bannai-Ito. Dans le produit tensoriel de représentations, on utilise les coefficients de Clebsch-Gordan et de Racah pour construire deux identités de convolution où quatre familles de ce même tableau interviennent. Le modèle introduit agit donc comme vitrine pour ces familles de polynômes. De plus, à l'aide d'une réalisation en termes d'opérateurs de Dunkl, une fonction génératrice bilinéaire pour les polynômes de Big -1 Jacobi est introduite. De surcroît, une interprétation des polynômes qui interviennent dans le modèle comme familles bivariés de type Tratnik est présentée.

Chapitre 5

An analytic spin chain model with fractional revival

J.M. Lemay, L. Vinet et A. Zhedanov (2016). An analytic spin chain model with fractional revival. *Journal of Physics A: Mathematical and Theoretical 49 (33), 335302*

Abstract. New analytic spin chains with fractional revival are introduced. Their nearest-neighbor couplings and local magnetic fields correspond to the recurrence coefficients of para-Racah polynomials which are orthogonal on quadratic bi-lattices. These models generalize the spin chain associated to the dual-Hahn polynomials. Instances where perfect state transfer also occurs are identified.

5.1. Introduction

Fractional revival (FR) is observed when clones of a wave packet reproduce with periodicities in a localized fashion [1]. This phenomenon has been shown to be possible in XX spin chains with pre-engineered couplings. Specifically, in such instances, a state with a spin up initially at one end of the chain evolves after some time T into a state for which the amplitude to find the spin up at a given site is non zero only for the two ends of the chain [2, 3, 4, 5, 6, 7]. The special case when the spin up localizes exclusively at the end of the chain at some time T is referred to as perfect state transfer (PST). (See [8] and references therein.) The wave packet splitting realized via FR provides, like PST, new tools for applications in quantum interference. As indicated in [5], in the multiple particle context, FR can generate Hanbury Brown and Twiss correlations, Hong-Ou-Mandel bunching effects and it can also lead to quantum interference patterns known as quantum carpets [9]. The possibility to enact FR (as well as PST) in spin chains has made these systems very attractive for the design of wires that can transport quantum information with high fidelity, generate entanglement or provide remote gates. The interest in these questions is generating an abundant literature (See [10, 4, 11] for reviews and further references, see also [12, 13, 14, 15, 16, 17]). A key advantage of these models is that the tasks are performed through the chain dynamics without the need for external control operations beyond the input/output interventions.

This motivates the identification of the XX spin chains that will exhibit FR. A systematic analysis of FR at two sites in such models as XX spin chains has been carried in [7]. In general, FR is essentially described by two parameters that can be tuned independently. The first is connected to a prescribed one-parameter deformation [6]. The second comes from a remarkable analytic model. Let us briefly explain its features.

For the purpose of studying fractional revival it suffices to consider states with only one spin up. When restricted to the one-excitation sector, the XX spin chain Hamiltonians with nearest-neighbor couplings become tridiagonal matrices J that are diagonalized by polynomials orthogonal on the finite set of points formed by the eigenvalues of J. One necessary condition for PST is that the matrices J possess a special property called mirror-symmetry [18]. Looking for fractional revival in systems with special Hamiltonians whose restrictions J are mirror-symmetric, one analytic model has been found where the couplings and magnetic fields are exactly given by the recurrence coefficients of polynomials that are orthogonal on linear bi-lattices [7]. By a linear bi-lattice we mean the set of points obtained by shifting two equally-spaced linear lattices with respect to one another:

$$x_s = x_0 + s + \frac{1}{2}(\delta - 1)(1 - (-1)^s) \quad s = 0, \ldots, N. \tag{5.1}$$

We assume that the chains have $N+1$ sites and thus N links. Interestingly, the associated orthogonal polynomials have only been discovered recently in the context of PST studies [19]. They have been called para-Krawtchouk polynomials. The corresponding FR parameter is related to the relative shift δ. To our knowledge this model is so far the only known mirror-symmetric analytic Hamiltonian with FR.

Now there is a procedure known as spectral surgery [8] that allows to remove spectral points while preserving mirror-symmetry. In principle this permits to obtain any prescribed spectrum for J and the corresponding mirror-symmetric couplings and magnetic fields by removing in this way the appropriate elements from the linear bi-lattice set. As explained in [7], upon performing the isospectral deformation (mentioned above) of surgered models, one could construct additional XX spin chains with FR.

The usefulness of exactly solvable models does not need to be stressed. They make possible the analytic exploration of the dynamics and are rooted in a secular tradition in theoretical physics. They also offer very useful benchmarks for experimental testing. Now it is a fact that the repeated removal of energy levels will yield expressions that are more and more complicated for the chain data and will thus have the effect of obscuring the analytic properties of the system. There is hence much interest in finding directly other manifestly analytic models with FR. The purpose of the paper is to introduce a second analytic mirror-symmetric XX spin chain that possesses fractional revival (Note that it is not so clear and at this moment unknown, how this new model could be recovered from the para-Krawtchouk system by spectral surgeries). This new model has more parameters than the para-Krawtchouk one. This could prove useful in an experimental implementation; these parameters must however satisfy certain constraints for FR to occur.

The discovery of the para-Krawtchouk polynomials orthogonal on linear bi-lattices has prompted the search for polynomials orthogonal on quadratic bi-lattices. These functions have been found very recently [20]. They have been called the para-Racah polynomials and are denoted by $P_n(x^2; N; a, c, \alpha)$. As the notation suggests, they are polynomials of degree n in the variable x^2 (like the Wilson polynomials [21]) and they depend on 3 parameters, two of which being related to the definition of the grid or bi-lattice, in addition to the natural number N.

We shall here discuss the fractional revival properties of the XX spin chains associated to these orthogonal polynomials. Mirror-symmetry requires $\alpha = \frac{1}{2}$. When this is so, FR will occur provided a and c are expressed in terms of solutions of a quadratic Diophantine equation. PST will happen for a subset of the values of the parameters for which FR is realized. When $c = a + \frac{1}{2}$, the models will be seen to reduce to the ones associated to the dual-Hahn polynomials - a paradigm example of chains exhibiting PST [18]. When an isospectral transformation is applied, mirror-symmetry is broken and more general models with FR are obtained and found to correspond to para-Racah polynomials now with an arbitrary α.

There has recently been remarkable experimental verifications of PST in optical arrays [22, 23, 24]. The XX spin chain dynamics is then reproduced by the propagation of photons in arrays of evanescently coupled waveguides that are arranged side by side. The proper engineering of the interactions is realized by adjusting the distances between the fibers. The specifications of the chain/array are chosen so as to optimize the experimental conditions. It would now be of great interest to also obtain experimental observations of FR in photonic lattices. The para-Racah model that will be described in this paper has features similar to those of the para-Krawtchouk system. In particular, both models exhibit irregularities in the distribution of the couplings around the middle of the chains. This is not really worrisome in a photonic implementation since it just requires fixing the distances one by one accordingly. The fact that the para-Racah model has more parameters could however prove advantageous in giving some flexibility to minimize undesirable experimental features such as propagation losses, array inhomogeneity and non-nearest-neighbour couplings. This adds motivation to the search for other analytic models beyond the general interest in enlarging the class of analytic models with FR.

The paper will unfold as follows. In Section 2, relevant aspects of fractional revival in XX spin chains will be reviewed. The models with mirror-symmetric couplings and magnetic fields corresponding to the recurrence coefficients of the para-Racah polynomials with $\alpha = \frac{1}{2}$ will be introduced in Section 3 for N odd, that is for an even number of sites. The conditions (on the parameters) for fractional revival to take place will be determined and the situations of PST will also be identified. The presentation of the models for N even will be carried out via the spectral surgery procedure. This last point will be the object of Section 3 where we shall give the couplings and magnetic fields that result when the last level of the chains with N odd is removed. That the models reduce to the one associated with the dual-Hahn polynomials will be discussed in Section 5. The shape of the couplings and magnetic fields will be depicted in plots exhibiting the differences between the odd and even N cases as well as with the dual-Hahn polynomials situation. The isospectral deformation of the mirror-symmetric chains considered up to that point will be carried out in Section 6 to find analytic models with FR corresponding to the general para-Racah polynomials. The classification will be done in full generality for spin chains with $N > 4$. We shall sum up to conclude.

5.2. Fractional Revival and Orthogonal Polynomials

We shall consider XX spin chains with $N+1$ sites and nearest-neighbor interactions governed by Hamiltonians H of the form

$$H = \frac{1}{2}\sum_{n=0}^{N-1} J_{n+1}(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y) + \frac{1}{2}\sum_{n=0}^{N} B_n(\sigma_n^z + 1). \tag{5.2}$$

J_n is the coupling constant between the sites $n-1$ and n with $n = 1, \ldots, N-1$ and B_n is the magnetic field strength at the site n with $n = 0, 1, \ldots, N$. The symbols σ_n^x, σ_n^y, σ_n^z stand for the Pauli matrices with the index n indicating on which $\mathbb{C}^2$ copy of $(\mathbb{C}^2)^{\otimes N+1}$ they act. It is easy to see that the Hamiltonian H is invariant under rotation around the z-axis:

$$\left[H, \frac{1}{2}\sum_{n=0}^{N}(\sigma_n^z + 1)\right] = 0. \tag{5.3}$$

This implies that the eigenstates of H split in subspaces labeled by the number of spins over the chain that are up, i.e. that are eigenstates of σ^z with eigenvalue $+1$. To study fractional revival it suffices to focus on the restriction J of H to the one-excitation sector. The states of that subspace are naturally described by the canonical basis vectors of $\mathbb{Z}^{\otimes N+1}$

$$|n\rangle = (0, 0, \ldots, 1, \ldots, 0)^{\mathsf{T}} \tag{5.4}$$

with the single 1 in the n^{th} position corresponding to the single spin up at the n^{th} site. The action of J in that basis follows from (5.2) and is given by

$$J|n\rangle = J_{n+1}|n+1\rangle + B_n|n\rangle + J_n|n-1\rangle \tag{5.5}$$

where $J_0 = J_{N+1} = 0$ is assumed. That is, J is the following Jacobi matrix

$$J = \begin{pmatrix} B_0 & J_1 & & & \\ J_1 & B_1 & J_2 & & \\ & J_2 & B_2 & \ddots & \\ & & \ddots & \ddots & J_N \\ & & & J_N & B_N \end{pmatrix}. \tag{5.6}$$

It will be said to be mirror-symmetric with respect to the anti-diagonal or persymmetric if

$$RJR = J \tag{5.7}$$

with

$$R = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}. \tag{5.8}$$

In terms of the couplings and magnetic field strengths, this amounts to

$$J_n = J_{N+1-n}, \quad B_n = B_{N-n}. \tag{5.9}$$

Since $B_n, J_n \in \mathbb{R}$, J is clearly hermitian and has real eigenvalues. We can introduce the eigenbasis of J :

$$J|x_s\rangle = x_s|x_s\rangle, \tag{5.10}$$

where the eigenvalues x_s are assumed to be non-degenerate and are taken to be in increasing order $x_0 < x_1 < \cdots < x_N$. Then, in view of (5.5), it is easy to show that we have the following expansions

$$|x_s\rangle = \sum_{n=0}^{N} \sqrt{w_s}\chi_n(x_s)|n\rangle, \quad |n\rangle = \sum_{s=0}^{N} \sqrt{w_s}\chi_n(x_s)|x_s\rangle, \tag{5.11}$$

where $\chi_n(x)$ are orthonormal polynomials on the finite set of spectral points x_s that obey the recurrence relation

$$x\chi_n(x) = J_{n+1}\chi_{n+1}(x) + B_n\chi_n(x) + J_n\chi_{n-1}(x). \tag{5.12}$$

It can be shown [8] that J is persymmetric if and only if

$$\chi_N(x_s) = (-1)^{N+s}, \quad s = 0, 1, \ldots, N. \tag{5.13}$$

Let us now come to fractional revival at two sites. A wave packet initially localized at the site 0 will be revived at sites 0 and N after time T if

$$e^{-iTJ}|0\rangle = \xi|0\rangle + \eta|N\rangle \tag{5.14}$$

with $|\xi|^2 + |\eta|^2 = 1$. Given the normalization condition, ξ and η parametrize the points of a 3-sphere and thus amount to 3 angles. Moreover since the overall phase of the state on the rhs of (5.14) is not physically meaningful and is one of the 3 angles, we see that fractional revival at two sites is essentially characterized by two real angles. Observe that we are in a situation of perfect state transfer when $\xi = 0$:

$$e^{-iTJ}|0\rangle = e^{i\phi'}|N\rangle.$$

With the help of expansion (5.11), it is immediate to see that condition (5.14) translates into

$$e^{-iTx_s} = \xi + \eta\chi_N(x_s). \tag{5.15}$$

Note that for the right hand side of (5.15) to have modulus 1, we must have

$$\chi_N^2(x_s) + 2\frac{\mathrm{Re}(\xi\eta^*)}{|\eta|^2} = 1. \tag{5.16}$$

The implications of (5.15) have been examined in [7] to obtain a characterization of the chains with FR. The analysis proceeds in two steps. Condition (5.15) is first enforced when ξ and η involve only one of the two essential angles (apart from the global phase ϕ) and are expressed like this :

$$\xi = e^{i\phi}\sin 2\theta, \quad \eta = ie^{i\phi}\cos 2\theta. \tag{5.17}$$

The additional parameter (second essential angle) is then introduced by subsequently performing an isospectral deformation of the Jacobi matrix obtained as a result of the first step. This course will be followed here. In what immediately follows, the parametrization with the single angle θ (apart from ϕ) will be used. How the second angle σ is brought in the analysis will be explained in Section 6.

When ξ and η take the special form (5.17), we see from (5.16) that $\chi_N^2(x_s) = 1$. Simple considerations (explained fully in [7]) lead one to conclude that condition (5.13) must then be obeyed. In other words, the fractional revival condition (5.15) requires that J be persymmetric when ξ and η are as in (5.17). Moreover, with these expressions for ξ and η, the real and imaginary parts of (5.15) yield the following relations that the eigenvalues x_s of J must satisfy :

$$\cos(Tx_s + \phi) = \cos\left(\frac{\pi}{2} - 2\theta\right), \qquad \sin(Tx_s + \phi) = (-1)^{N+s+1}\sin\left(\frac{\pi}{2} - 2\theta\right). \tag{5.18}$$

The determination of J and of H as a result, is henceforth framed as an inverse spectral problem that can be solved using the theory of orthogonal polynomials. The eigenvalues x_s found to verify the FR conditions (5.18) determine the characteristic polynomial of degree $N+1$ and the knowledge of χ_N at the $N+1$ points x_s, as prescribed by (5.13), completely specifies this polynomial also. All the other polynomials $\chi_n(x)$ can be constructed from these two by using the Euclidian algorithm (see [8]) and this gives J and H.

The generic set of eigenvalues satisfying conditions (5.18) is a linear bi-lattice and the algorithm we just explained leads in this case to the para-Krawtchouk polynomials [19]. The specifications of the chain are then provided by the recurrence coefficients.

In the following, we shall not adopt this deductive approach which is explained fully in [7]. We shall rather identify a new analytic model by proceeding in the reverse. We shall first provide a set of mirror-symmetric couplings and magnetic fields known to form the recurrence coefficients of polynomials that are orthogonal with respect to quadratic bi-lattices. We shall then determine for what values of the grid parameters are the FR conditions (5.18) satisfied. The isospectral deformations that allow to relax the condition that J is persymmetric will be presented in the last section.

5.3. A model with fractional revival based on the para-Racah polynomials for N odd

The analytic model introduced here is based on the para-Racah polynomials $P_n(x^2; N; a, c, \alpha)$. These polynomials have only been identified recently [20]. For now, take N to be odd and write

$$N = 2j + 1. \tag{5.19}$$

When $\alpha = \frac{1}{2}$, the recurrence coefficients of the para-Racah polynomials provide the following explicit expressions for couplings and magnetic fields :

$$\begin{aligned} B_n &= \frac{1}{2}\left[a(a+j) + c(c+j) + n(N-n)\right], \\ J_n &= \left[\frac{n(N+1-n)(N-n+a+c)(n-1+a+c)(n-j-1)^2 - (a-c)^2}{4(N-2n)(N-2n+2)}\right]^{\frac{1}{2}}, \end{aligned} \tag{5.20}$$

where the parameters a and c are such that $a > -\frac{1}{2}$ and $|a| < c < |a+1|$. It can be checked directly that these J_n and B_n satisfy (5.9) and that they hence form a persymmetric matrix.

The para-Racah polynomials are known [20] to satisfy a discrete orthogonality relation on the points of the quadratic bi-lattice defined by

$$\begin{aligned} x_{2s} &= (s+a)^2, \quad s = 0, \dots, j, \\ x_{2s+1} &= (s+c)^2, \quad s = 0, \dots, j. \end{aligned} \tag{5.21}$$

Now, FR will be observed when such eigenvalues obey conditions (5.18). Splitting the even and odd cases and using (5.19), FR requires that

$$\begin{aligned} \cos(T(s+a)^2+\phi) &= \cos\left(\frac{\pi}{2}-2\theta\right), \qquad \cos(T(s+c)^2+\phi) = \cos\left(-\frac{\pi}{2}+2\theta\right), \\ \sin(T(s+a)^2+\phi) &= \sin\left(\frac{\pi}{2}-2\theta\right), \qquad \sin(T(s+c)^2+\phi) = \sin\left(-\frac{\pi}{2}+2\theta\right), \end{aligned} \tag{5.22}$$

for $s = 0, 1, \dots, j$. This implies that

$$T(s+a)^2 = \frac{\pi}{2} - 2\theta - \phi + 2\pi M_s, \qquad T(s+c)^2 = -\frac{\pi}{2} + 2\theta - \phi + 2\pi L_s, \tag{5.23}$$

where M_s and L_s are arbitrary sequences of integers. Now, since the LHS of (5.23) are quadratic functions of s, the RHS must also be quadratic functions of s. Hence, M_s and L_s take the general form

$$M_s = A_1 s^2 + C_1 s + \gamma_1, \qquad L_s = A_2 s^2 + C_2 s + \gamma_2. \tag{5.24}$$

For $j > 1$, the sequences M_s and L_s will take integer values for all s if and only if γ_i is an arbitrary integer and A_i, C_i are both simultaneously either integers or half-integers for $i = 1$ and $i = 2$. The proof is elementary and we omit it. We shall henceforth assume that the above restriction ($j > 1$) is verified and shall thus determine the FR conditions for generic chains with $N > 3$. For very small chains, there are additional special possibilities that we shall not spell out here. With this understanding, we can cast the A_i and C_i as

$$A_1 = \frac{\alpha_1}{2}, \quad C_1 = \frac{\beta_1}{2}, \quad A_2 = \frac{\alpha_2}{2}, \quad C_2 = \frac{\beta_2}{2}, \tag{5.25}$$

where α_1 and β_1 are integers with the same parity and α_2 and β_2 are also integers with the same parity. With (5.24) and (5.25), condition (5.23) becomes

$$\begin{aligned} 0 =& (T - \pi\alpha_1)s^2 + (2aT - \pi\beta_1)s + (a^2T - \frac{\pi}{2} + 2\theta + \phi - 2\pi\gamma_1), \\ 0 =& (T - \pi\alpha_2)s^2 + (2cT - \pi\beta_2)s + (c^2T + \frac{\pi}{2} - 2\theta + \phi - 2\pi\gamma_2). \end{aligned} \tag{5.26}$$

Since these relations must hold for $s = 0, 1, \dots, j$, again with $j > 1$, each coefficient must vanish. First, the coefficients of s^2 yield

$$T = \pi\alpha_1 = \pi\alpha_2, \tag{5.27}$$

which says that $\alpha_1 = \alpha_2$ is a positive integer, that α_1, β_1 and β_2 must share the same parity and that the time for FR to occur is a multiple of π. Second, in view of (5.27), equating the coefficients of s in eq.(5.26) gives

$$a = \frac{\beta_1}{2\alpha_1}, \quad c = \frac{\beta_2}{2\alpha_1}. \tag{5.28}$$

The parameters a and c are thus rational numbers. Recall that $a > -\frac{1}{2}$ and $|a| < c < |a+1|$ implying that

$$\beta_1 > -\alpha_1, \quad |\beta_1| < \beta_2 < |\beta_1 + 2\alpha_1|. \tag{5.29}$$

Third, equating the constant terms provides

$$a^2 T = \frac{\pi}{2} - \phi - 2\theta + 2\pi\gamma_1, \quad c^2 T = -\frac{\pi}{2} - \phi + 2\theta + 2\pi\gamma_2. \tag{5.30}$$

It is helpful to consider the sum and difference of these two equations:

$$T(c^2 - a^2) = -\pi + 4\theta + 2\pi(\gamma_2 - \gamma_1), \tag{5.31a}$$

$$T(c^2 + a^2) = -2\phi + 2\pi(\gamma_1 + \gamma_2). \tag{5.31b}$$

Using (5.27) and (5.28), equation (5.31a) provides a condition on the parameters β_1,β_2 and θ:

$$\frac{(\beta_2 - \beta_1)(\beta_1 + \beta_2)}{4} = \left[2(\gamma_2 - \gamma_1) - 1 + \frac{4\theta}{\pi}\right]\alpha_1, \tag{5.32}$$

where $\beta_1, \beta_2, \gamma_1, \gamma_2$ are integers and α_1 is a positive integer. Note that since β_1 and β_2 share the same parity, the LHS of (5.32) is an integer. This implies that θ must be a multiple of $\frac{\pi}{4\alpha_1}$:

$$\theta = \frac{\pi p}{4q} \tag{5.33}$$

where $\frac{p}{q}$ is an irreducible fraction and q divides α_1. Finally, upon substituting (5.27) and (5.28) in (5.31b), one obtains that the global phase ϕ is a fraction of π. Hence, the Hamiltonian (5.2) with couplings and magnetic fields given by (5.20) will admit fractional revival if the parameters a, c are of the form (5.28) with $\beta_1, \beta_2, \alpha_1$ solutions of (5.32) respecting (5.29). An example of solution is : $\beta_1 = 14$, $\beta_2 = 16$, $\alpha_1 = 6$, $\theta = \frac{3\pi}{8}$ and $\gamma_2 - \gamma_1 = 1$ which gives $a = \frac{7}{6}$ and $c = \frac{8}{6}$. Mathematically, one could say that (5.32) yields all the possible values of θ (guaranteed to be of the form (5.33)) when the integers $\alpha_1, \beta_1, \beta_2, \gamma_1$ and γ_2 run over their admissible values. In practice however one might wish to pick a value of θ, for instance the value $\frac{\pi}{8}$ that gives balanced FR, and find the set of parameters that will give this θ.

Mirror-symmetry which is realized in the models we have been discussing so far is a necessary condition for PST. It is hence of interest to enquire if PST can also be observed in spin chains that have been found to exhibit FR. Once one has (5.14) with ξ and η given by (5.17), it can be shown [7] that one has also

$$e^{-iMTJ}|0\rangle = e^{i\phi}\Big[\cos M\left(\frac{\pi}{2}-2\theta\right)|0\rangle + i\sin M\left(\frac{\pi}{2}-2\theta\right)|N\rangle\Big] \tag{5.34}$$

with M an integer. This simply follows from formulas for e^{-iTJ} that will be given for N odd and N even at the beginning of section 6. Perfect state transfer will then occur at time MT if

$$M\left(\frac{\pi}{2}-2\theta\right) = M_1\left(\frac{\pi}{2}\right) \tag{5.35}$$

with M_1 an arbitrary odd number. Recall that (5.32) requires (5.33). For fixed p and q, given a set of parameters solving (5.32), we see that the corresponding model will possess PST in addition if

$$\frac{M_1}{M} = \frac{q-p}{q}. \tag{5.36}$$

This requires[1] that p and q have opposite parities and $M = q$. This is ensured by the irreducibility of $\frac{p}{q}$ for q even, but not for q odd. Thus, spin chains with FR at time T will also show PST at time qT if q is even but when q is odd, PST will only happen at time qT if p is even.

It turns out that the solutions with PST can be identified with the models presented by Albanese and Lawi in an unpublished paper [25] although the authors have assumed not quite correctly that the underlying polynomials are special cases of the Racah polynomials.

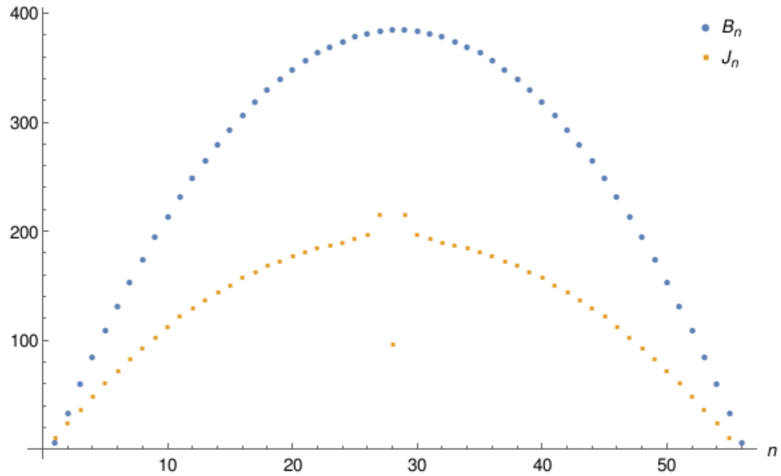
A particular feature of the para-Racah model is the bump in the couplings J_n in the middle of the of chain. The figure 5.1 shows the strengths of the couplings along the chain.

5.4. Spectral Surgery and the case N even

A procedure called spectral surgery and described in [8] allows to modify the spectrum of a Jacobi matrix while preserving its mirror-symmetry. It is hence possible to relate a chain with N sites to one with $N-1$ sites by removing the last eigenvalue. That the modified chain will still enact fractional revival is obvious since the conditions (5.18) will remain satisfied for $s = 0, 1, \ldots, N-1$.

[1] The case $p = 0$ is also possible with $M = M_1$ but in this instance there will just be replications of PST at times M_1T.

Fig. 5.1. Couplings strenghts for $N = 55$, $a = 1/8$ and $c = 3/8$.



This technique hence permit to define our models for even N. The new couplings $\hat{B}_n, \hat{J}_n$ are related to (5.20) by

$$\hat{B}_n = B_{n+1} + A_n N_{n+1} - A_n, \quad \hat{J}_n = J_n \left[\frac{A_n}{A_{n-1}} \right]^{\frac{1}{2}}, \tag{5.37}$$

where $A_n = \frac{P_{n+1}(-x_N; N; a, c, \frac{1}{2})}{P_n(-x_N; N; a, c, \frac{1}{2})}$ is given by

$$A_n = \frac{(N-n)(n-j+a-c)(N-n-1+a+c)}{2(2n-N)}. \tag{5.38}$$

Upon changing $N \to N+1$, a direct computation yields the recurrence coefficients for $N = 2j$:

$$\begin{aligned} \hat{B}_n = & \tfrac{1}{2}(a^2 + c^2 + n - n^2) + \tfrac{1}{4}(2n + a + c)(N-1) \\ & + \tfrac{(n+1)(n+a+c)(1+2a-2c)}{4(1+2n-N)} + \tfrac{n(n-1+a+c)(1+2a-2c)}{4(1-2n+N)} \end{aligned} \tag{5.39a}$$

and

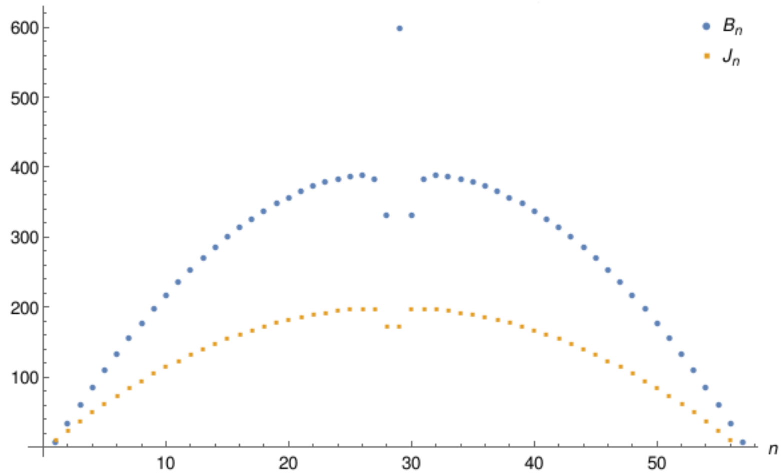
$$\hat{J}_n = \left[\tfrac{n(N+1-n)(n-1+a+c)(N-n+a+c)(n-j+a-c)(n-j+c-a-1)}{4(N-2n+1)^2} \right]^{\frac{1}{2}}. \tag{5.39b}$$

It can be directly checked that the persymmetry condition (5.9) still holds while the spectrum of J is now

$$\begin{aligned} x_{2s} = & (s+a)^2, \quad s = 0, \ldots, j, \\ x_{2s+1} = & (s+c)^2, \quad s = 0, \ldots, j-1. \end{aligned} \tag{5.40}$$

Fractional revival will occur for a and c again given by (5.28) and (5.32). Here, in view of (5.40) and of the remarks after equations (5.24), we need to assume that $j > 2$. Our analysis thus provide a full classification of the FR conditions for chains with $N > 4$. The plot showing the strengths of the couplings along the chains exhibit a different behavior for N even; here, both the couplings J_n and the magnetic fields B_n have a bump in the middle of the chains :

Fig. 5.2. Couplings strenghts for $N = 56$, $a = 1/8$ and $c = 3/8$.



The presence of irregularities in the distribution of the couplings and magnetic fields (when N is even) around the middle of the chain is a particular feature of the para-Racah model that is shared by the para-Krawtchouk chain. This is not deemed to be especially problematic for applications in systems that are relatively small. Note from formulas (5.20) and (5.39) that these variations are modulated by the values of a and c and that they become more important as N grows.

5.5. Special case $c = a + \frac{1}{2}$

The models presented here encompass simpler known models. Note that upon setting $c = a + \frac{1}{2}$, the bi-lattice (5.21) reduces to a single quadratic lattice of the form

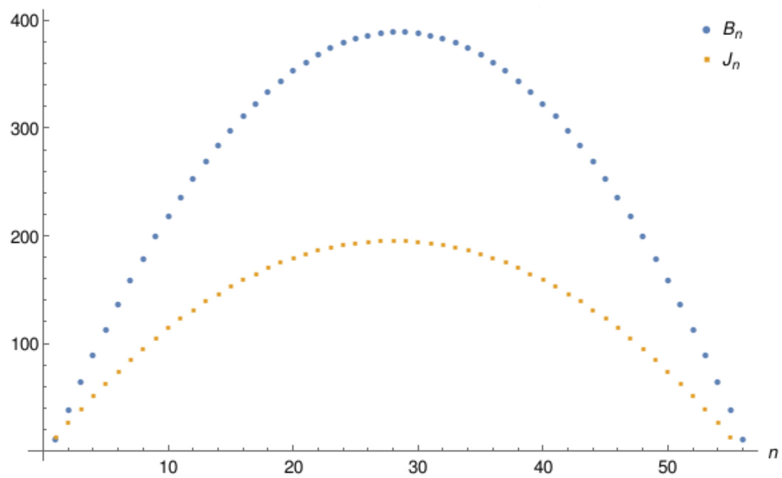
$$x_s = \left(\frac{s}{2} + a\right)^2. \tag{5.41}$$

In this case, the recurrence coefficients for odd N (5.20) and even N (5.39) need not be distinguished and are given by

$$B_n = N + 4N(a+n) + a^2 - \frac{n^2}{2},$$
$$J_n = \left[\frac{n(n+2a-\frac{1}{2})(N-n+2a+\frac{1}{2})(N+1-n)}{16}\right]^{\frac{1}{2}}.$$

This correspond to the analytic spin chain models connected to the dual-Hahn polynomials that were identified by Albanese *et al.* in [18]. Note that our analysis shows that these simpler models also exhibit FR for some values of the parameters. In this case, the bump in the middle couplings disappear and the patterns are those of smooth parabolas :

Fig. 5.3. Couplings strenghts for $N = 55$, $a = 1/6$ and $c = 4/6$.



5.6. Isospectral deformations

We shall now indicate how a second parameter in addition to θ can be introduced in the amplitudes of the two revived clones by means of an isospectral deformation.

Let us first point out that the conditions (5.18) do not only imply (5.14) with ξ and η as in (5.17) but more generally that

$$e^{-iTJ} = e^{i\phi} \begin{pmatrix} \sin 2\theta & & & & & i\cos 2\theta \\ & \ddots & & & \iddots & \\ & & \sin 2\theta & i\cos 2\theta & & \\ & & i\cos 2\theta & \sin 2\theta & & \\ & \iddots & & & \ddots & \\ i\cos 2\theta & & & & & \sin 2\theta \end{pmatrix}$$

for N odd and

$$e^{-iTJ} = e^{i\phi} \begin{pmatrix} \sin 2\theta & & & & & & i\cos 2\theta \\ & \ddots & & & & \iddots & \\ & & \sin 2\theta & 0 & i\cos 2\theta & & \\ & & 0 & e^{i(\frac{\pi}{2}-2\theta)} & 0 & & \\ & & i\cos 2\theta & 0 & \sin 2\theta & & \\ & \iddots & & & & \ddots & \\ i\cos 2\theta & & & & & & \sin 2\theta \end{pmatrix}$$

for N even. This is not difficult to show and has been derived in [7].

Now introduce the symetric matrix

$$V = \begin{pmatrix} \sin\sigma & & & & & \cos\sigma \\ & \ddots & & & \iddots & \\ & & \sin\sigma & \cos\sigma & & \\ & & \cos\sigma & -\sin\sigma & & \\ & \iddots & & & \ddots & \\ \cos\sigma & & & & & -\sin\sigma \end{pmatrix}, \tag{5.42}$$

for N odd and

$$V = \begin{pmatrix} \sin\sigma & & & & & & \cos\sigma \\ & \ddots & & & & \iddots & \\ & & \sin\sigma & 0 & \cos\sigma & & \\ & & 0 & 1 & 0 & & \\ & & \cos\sigma & 0 & -\sin\sigma & & \\ & \iddots & & & & \ddots & \\ \cos\sigma & & & & & & -\sin\sigma \end{pmatrix}, \tag{5.43}$$

for N even. It is easy to verify that $V^2 = 1$. Now let J be a persymmetric Jacobi matrix and define $\tilde{J}$ via the conjugation

$$\tilde{J} = VJV. \tag{5.44}$$

It is clear that this provides an isospectral deformation of J. $\tilde{J}$ will no longer be persymmetric but will satisfy

$$\tilde{J} = Q\tilde{J}Q \tag{5.45}$$

with $Q = VRV$. A direct computation shows that $e^{-iT\tilde{J}}$ acting on $|0\rangle$ gives

$$e^{-iT\tilde{J}}|0\rangle = Ve^{-iTJ}V|0\rangle = \xi|0\rangle + \eta|N\rangle \tag{5.46}$$

for both N odd and N even with

$$\xi = e^{i\phi}(\sin 2\theta + 2i\cos 2\theta \cos\sigma \sin\sigma) \tag{5.47a}$$

$$\eta = ie^{i\phi}\cos 2\theta(\cos^2\sigma - \sin^2\sigma). \tag{5.47b}$$

An additional angle σ has thus been introduced in the parameterization of ξ and η and it is easily verified that the normalization condition $|\xi|^2 + |\eta|^2 = 1$ is still satisfied. It is observed that $\tilde{J}$ is also tridiagonal, it thus provides the couplings and magnetic fields of an XX spin chain that exhibits again fractional revival at two sites. Remarkably, most of the coupling constants and magnetic fields of J remains unchanged. Carrying out the transformation (5.44), it is seen that the only entries of $\tilde{J}$ that differ from those of J are

$$\begin{aligned} \widetilde{J}_{\frac{N+1}{2}} &= J_{\frac{N+1}{2}}\cos 2\sigma, \\ \widetilde{B}_{\frac{N\mp1}{2}} &= B_{\frac{N-1}{2}} \pm J_{\frac{N+1}{2}}\sin 2\sigma, \end{aligned} \tag{5.48a}$$

for N odd and

$$\begin{aligned} \widetilde{J}_{\frac{N}{2}} &= J_{\frac{N}{2}}(\cos\sigma - \sin\sigma), \\ \widetilde{J}_{\frac{N}{2}+1} &= J_{\frac{N}{2}}(\cos\sigma + \sin\sigma), \end{aligned} \tag{5.48b}$$

for N even. Remembering the expressions (5.20) and (5.39) for the entries of J, it is seen that the elements of $\tilde{J}$ ($\tilde{J}_n$ and $\tilde{B}_n$) correspond to the recurrence coefficients of the para-Racah polynomials $P_n(x^2; N; a, c, \alpha)$ [20] with $\sin(2\sigma) = 1 - 2\alpha$ for N odd and $\sin\sigma = \frac{\sqrt{\alpha}-\sqrt{1-\alpha}}{\sqrt{2}}$ for N even. We thus observe that the general para-Racah polynomials are associated with XX spin chains with generic fractional revival described by two parameters provided that a and c remain given by (5.28) in terms of solutions of (5.32).

5.7. Conclusion

Let us summarize our findings and make a few additional remarks to conclude. We have introduced a novel analytic XX spin chain with fractional revival. It depends on 3 parameters a, c, α in addition to the number of sites $N + 1$. Its nearest-neighbor couplings and local magnetic field strengths are provided by the recurrence coefficients of the recently identified para-Racah polynomials $P_n(x^2; N; a, c, \alpha)$ which are orthogonal on the quadratic bi-lattice (5.21) characterized by a and c. When $c = a + \frac{1}{2}$, these P_n reduce to the dual-Hahn polynomials. There are constraints on the parameters for fractional revival to take place after time T. One must have according to (5.27) and (5.28) :

$$a = \beta_1 \frac{\pi}{2T}, \quad c = \beta_2 \frac{\pi}{2T}, \tag{5.49}$$

where β_1 and β_2 are integers that solve the Diophantine equation (5.32). (There might be additional special solutions when $N \leq 4$). There are two angles θ and σ that determine the FR amplitudes ξ and η in (5.14) as per the parameterization (5.47). The first, θ, is determined by (5.32) and forced to take the restricted values specified by $\theta = \frac{\pi p}{4q}$ with p, q co-primes and q dividing T/π. The second, σ, is directly related to α in a way that depends on whether N is odd or even (see the formulas in the paragraph after eqs.(5.48)). When $\sigma = 0$, that is when $\alpha = \frac{1}{2}$, the recurrence coefficients J_n and B_n form a mirror-symmetric matrix. A necessary condition for perfect state transfer is then realized and it has been indicated that if FR happens at time T, PST will occur at time qT when the parameters a and c lead to an angle of the form $\theta = \frac{\pi p}{4q}$ where q and p have different parities. Solutions of (5.32) with $\theta = 0$ provide models that exhibit PST but not FR.

The study presented here complements and extends the analysis [7] of the transport properties of the spin chains connected to the para-Krawtchouk polynomials [19] which are orthogonal on the linear bi-lattice (5.1) instead of the quadratic bi-lattice (5.21). It is informative to compare the features of the latter model with those of the para-Racah system that we have examined so far in this paper.

To focus on the model with FR which is based on the para-Krawtchouk polynomials per se, we must set the angle σ equal to zero. We are thus in a mirror-symmetric situation. For the para-Krawtchouk model, the FR angle θ is related in a simple way to the bi-lattice parameter δ in (5.1), one has

$$\delta = 1 \pm \frac{4\theta}{\pi}. \tag{5.50}$$

There are no further conditions to have FR. However, in order for PST to manifest itself, we must have that

$$\delta = \frac{M_1}{M} \tag{5.51}$$

where M_1 and M are positive co-prime integers and M_1 is odd.

The para-Krawtchouk spin chains also possess another interesting property. When δ is an irrational number, even though state transfer is no longer perfect, it can be approached in finite time to any level of precision. It is indeed possible to show [26] using classical theorems of Diophantine approximations, that there exists a sequence of times t_n, $n = 0, 1, 2, \dots$, such that

$$|\langle N|e^{-iJt_n}|0\rangle| \to 1 \tag{5.52}$$

as $n \to \infty$. It follows that we have in these models, manifestations of what is called almost perfect state transfer (APST).

These last comments suggest directions for future investigations. A first question is : under what circumstances would the mirror-symmetric spin chains associated to the para-Racah polynomials exhibit APST? Other questions relate to FR. We have been dealing in this paper with perfect fractional revival. Let $||\psi\rangle|$ denote the norm of $|\psi\rangle$ and ϵ be a small positive real number. It would be of interest to determine the conditions for almost perfect fractional revival where

$$|e^{-iTJ}|0\rangle - (\xi|0\rangle + \eta|N\rangle)| < \epsilon \tag{5.53}$$

or where, in other words, $e^{-iTJ}|0\rangle$ can be as close as desired to the state with two localized clones at $|0\rangle$ and $|N\rangle$. It would seem relevant to probe the para-Krawtchouk and para-Racah chains in this respect.

Acknowledgments

The authors wishes to acknowledge stimulating conversations with Leonardo Banchi and Vincent X. Genest. J.M.L. holds a scholarship from the Fonds de recherche du Québec – Nature et technologies (FRQNT). The research of L.V. is supported in part by NSERC. A. Z. wishes to thank the Centre de Recherches Mathématiques (CRM) for its hospitality.

References

[1] R. W. Robinett, Quantum wave packet revivals, Physics Reports 392 (1-2) (2004) 1–119.

[2] B. Chen, Z. Song, C.-P. Sun, Fractional revivals of the quantum state in a tight-binding chain, Physical Review A 75 (2007) 012113.

[3] L. Dai, Y. Feng, L. Kwek, Engineering quantum cloning through maximal entanglement between boundary qubits in an open spin chain, Journal of Physics A: Mathematical and Theoretical 43 (3) (2010) 035302.

[4] A. Kay, A Review of Perfect State Transfer and its Application as a Constructive Tool, International Journal of Quantum Information 8 (2010) 641–676.

[5] L. Banchi, E. Compagno, S. Bose, Perfect wave-packet splitting and reconstruction in a one-dimensional lattice, Physical Review A 91 (2015) 052323.

[6] V. X. Genest, L. Vinet, A. Zhedanov, Exact fractional revival in spin chains, Modern Physics Letters B 30 (26) (2016) 1650315.

[7] V. X. Genest, L. Vinet, A. Zhedanov, Quantum spin chains with fractional revival, Annals of Physics 371 (2016) 348–367.

[8] L. Vinet, A. Zhedanov, How to construct spin chains with perfect state transfer, Physical Review A 85 (2012) 012323.

[9] M. Berry, I. Marzoli, W. Schleich, Quantum carpets, carpets of light, Physics World 14 (6) (2001) 39.

[10] S. Bose, Quantum communication through spin chain dynamics: an introductory overview, Contemporary Physics 48 (2007) 13–30.

[11] G. M. Nikolopoulos, I. Jex, Quantum State Transfer and Network Engineering, Springer, 2014.

[12] R. Chakrabarti, J. Van der Jeugt, Quantum communication through a spin chain with interaction determined by a Jacobi matrix, Journal of Physics A: Mathematical and General 43 (2010) 085302.

[13] N. I. Stoilova, J. Van der Jeugt, An Exactly Solvable Spin Chain Related to Hahn Polynomials, SIGMA 7 (33-45).

[14] K. R. K. Rao, T. S. Mahesh, A. Kumar, Efficient simulation of unitary operators by combining two numerical algorithms: An NMR simulation of the mirror-inversion propagator of an XY spin chain, Phys. Rev. A 90 (2014) 012306.

[15] T. Shi, Y. Li, Z. Song, C.-P. Sun, Quantum-state transfer via the ferromagnetic chain in a spatially modulated field, Physical Review A 71 (2005) 032309.

[16] C. Godsil, S. Kirkland, S. Severini, J. Smith, Number-Theoretic Nature of Communication in Quantum Spin Systems, Physical Review Letters 109 (2012) 050502.

[17] M.-H. Yung, S. Bose, Perfect state transfer, effective gates, and entanglement generation in engineered bosonic and fermionic networks, Physical Review A 71 (2005) 032310.

[18] C. Albanese, M. Christandl, N. Datta, A. Ekert, Mirror inversion of quantum states in linear registers, Physical Review Letters 93 (2004) 230502.

[19] L. Vinet, A. Zhedanov, Para-Krawtchouk polynomials on a bi-lattice and a quantum spin chain with perfect state transfer, Journal of Physics A: Mathematical and Theoretical 45 (26) (2012) 265304.

[20] J.-M. Lemay, L. Vinet, A. Zhedanov, The para-racah polynomials, Journal of Mathematical Analysis and Applications 438 (2) (2016) 565–577.

[21] R. Koekoek, P. A. Lesky, R. F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, Springer, 2010.

[22] M. Bellec, G. M. Nikolopoulos, S. Tzortzakis, Faithful communication hamiltonian in photonic lattices, Optics letters 37 (21) (2012) 4504–4506.

[23] A. Perez-Leija, R. Keil, A. Kay, H. Moya-Cessa, S. Nolte, L.-C. Kwek, B. M. Rodríguez-Lara, A. Szameit, D. N. Christodoulides, Coherent quantum transport in photonic lattices, Physical Review A 87 (1) (2013) 012309.

[24] R. J. Chapman, M. Santandrea, Z. Huang, G. Corrielli, A. Crespi, M.-H. Yung, R. Osellame, A. Peruzzo, Experimental perfect state transfer of an entangled photonic qubit, Nature communications 7 (2016) 11339.

[25] C. Albanese, S. Lawi, Classification of solvable mirror-periodic quantum spin chains.
URL `http://quant.hk/papers/racahtransfer.pdf`

[26] L. Vinet, A. Zhedanov, Almost perfect state transfer in quantum spin chains, Physical Review A 86 (2012) 052319.

Chapitre 6

A superintegrable model with reflections on S^3 and the rank two Bannai-Ito algebra

H. de Bie, V.X. Genest, J.M. Lemay et L. Vinet (2016). A superintegrable model with reflections on S^3 and the rank two Bannai-Ito algebra. *Acta Polytechnica, 56(3), 166-172*

Abstract. A quantum superintegrable model with reflections on the three-sphere is presented. Its symmetry algebra is identified with the rank-two Bannai-Ito algebra. It is shown that the Hamiltonian of the system can be constructed from the tensor product of four representations of the superalgebra $\mathfrak{osp}(1|2)$ and that the superintegrability is naturally understood in that setting. The exact separated solutions are obtained through the Fischer decomposition and a Cauchy-Kovalevskaia extension theorem.

This paper is dedicated with admiration and gratitude to Jiří Patera and Pavel Winternitz on the occasion of their 80th birthdays.

6.1. Introduction

Superintegrability shares an intimate connection with exact solvability. For classical systems, this connection is fully understood while it remains an empirical observation for general quantum systems. The study of superintegrable models has proved fruitful in understanding symmetries and their algebraic description, and has also contributed to the theory of special functions. A quantum system in n dimensions with Hamiltonian H is said to be maximally superintegrable if it possesses $2n-1$ algebraically independent constants of motion $c_1, c_2, \ldots, c_{2n-1}$ commuting with H, that is $[H, c_i] = 0$ for $i = 1, \ldots, 2n-1$, where one of these constants is the Hamiltonian itself. Such a system is further said to be superintegrable of order l if the maximum order in momenta of the constants of motion (except H) is l.

One of the important quantum superintegrable models is the so-called generic three-parameter system on the two-sphere [1], whose symmetries generate the Racah algebra which characterizes the Wilson and Racah polynomials sitting atop the Askey scheme [2]. All two-dimensional second order superintegrable models of the form $H = \Delta + V$ where Δ denotes the Laplace-Beltrami operator have been classified [1] and can be obtained from the generic three-parameter model through contractions and specializations [3]. A similar model with four parameters defined on the three-sphere has also been introduced and its connection to bivariate Wilson and Racah polynomials has been established [4].

Recently, superintegrable models defined by Hamiltonians involving reflection operators have been the subject of several investigations [5, 6, 7, 8, 9]. One of the interesting features of these models is their connection to less known bispectral orthogonal polynomials referred to as -1 polynomials. Many efforts have been deployed to characterize these polynomials, which can be organized in a tableau similar to the Askey one [10, 11, 12, 13, 14, 15, 16]. Of particular relevance to the present paper is the Laplace-Dunkl equation on the two-sphere studied in [17, 18], which has the rank-one Bannai-Ito algebra as its symmetry algebra [14]. This Bannai-Ito algebra encodes the bispectrality of the Bannai-Ito polynomials which depend on four parameters and stand at the highest level of the hierarchy of -1 orthogonal polynomials. As such, this Laplace-Dunkl system on the two sphere can be thought of as a generalization with reflection operators of the generic three-parameter model (without reflections) on the two-sphere which is recovered when wavefunctions with definite parities are considered. The goal of this paper is to introduce a novel quantum superintegrable model with reflections on the three-sphere which similarly embodies the generic four-parameter model introduced and studied in [4].

The paper is divided as follows. In section 2, we introduce a superintegrable model with four-parameters on the three-sphere and exhibit its symmetries explicitly. In section 3, it is shown how the Hamiltonian of the model can be constructed from four realizations of the superalgebra $\mathfrak{osp}(1|2)$. Moreover, the symmetry algebra is characterized and is seen to correspond to a rank-two generalization of the Bannai-Ito algebra. In section 4, the structure of the space of polynomial solutions is exhibited using a Fischer decomposition and an explicit basis for the eigenfunctions is constructed with the help of a Cauchy-Kovalevskaia extension theorem. Some concluding remarks are offered in section 6.

6.2. A superintegrable model on S^3

Let s_1, s_2, s_3, s_4 be the Cartesian coordinates of a four-dimensional Euclidian space and take the restriction to the embedded three-sphere: $s_1^2 + s_2^2 + s_3^2 + s_4^2 = 1$. Consider the system with four parameters $\mu_1, \mu_2, \mu_3, \mu_4$ with $\mu_i \geq 0$ for $i = 1, 2, 3, 4$ governed by the Hamiltonian

$$H = \sum_{1 \leq i < j \leq 4} J_{ij}^2 + \sum_{i=1}^{4} \frac{\mu_i}{s_i^2}(\mu_i - R_i), \tag{6.1}$$

where

$$J_{ij} = \frac{1}{i}(s_i \partial_{s_j} - s_j \partial_{s_i}), \qquad R_i f(s_i) = f(-s_i), \tag{6.2}$$

are the angular momentum operators and reflection operators, respectively. The six quantities

$$L_{jk} = \left(\frac{1}{2} + \mu_j R_j + \mu_k R_k + \left(iJ_{jk} + \mu_j \frac{s_k}{s_j} R_j - \mu_k \frac{s_j}{s_k} R_k \right) \prod_{l=j+1}^{k} R_l \right) R_j R_k, \quad 1 \leq j < k \leq 4, \tag{6.3}$$

can easily be verified to commute with H on the 3-sphere and are thus conserved. It can be shown that any four of the L_{jk} are algebraically independent. Hence H defines a maximally superintegrable system of first order. There are also four more conserved quantities of the form

$$M_A = \left(1 + \sum_{i \in A} \mu_i R_i + \sum_{\substack{j<k \\ j,k \in A}} \left(iJ_{jk} + s_k \frac{\mu_j}{s_j} R_j - s_j \frac{\mu_k}{s_k} R_k \right) \prod_{l=j+1}^{k} R_l \right) \prod_{i \in A} R_i, \tag{6.4}$$

where $A = \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}$ or $\{2, 3, 4\}$. Furthermore, a direct computation yields

$$[H, R_i] = 0, \qquad i = 1, 2, 3, 4. \tag{6.5}$$

The reflections are thus discrete symmetries of the system.

6.3. Algebraic construction from $\mathfrak{osp}(1|2)$

The superalgebra $\mathfrak{osp}(1|2)$ can be presented with five generators $x, D, E, |x|^2$ and D^2 with the following defining relations:

$$\begin{aligned}
&\{x, x\} = 2|x|^2, && \{D, D\} = 2D^2, \\
&\{x, D\} = 2E, && [D, E] = D, \\
&[D, |x|^2] = 2x, && [E, x] = x, \\
&[D^2, x] = 2D, && [D^2, E] = 2D^2, \\
&[D^2, |x|^2] = 4E, && [E, |x|^2] = 2|x|^2,
\end{aligned} \tag{6.6}$$

where $[a,b]=ab-ba$ is the commutator and $\{a,b\}=ab+ba$ is the anti-commutator. One can realize four mutually commuting copies of this superalgebra by taking

$$\begin{aligned}
D_i &= \partial_{s_i}-\frac{\mu_i}{s_i}R_i, \quad && D_i^2=D_iD_i,\\
x_i &= s_i, && |x_i|^2=s_i^2,\\
E_i &= s_i\partial_{s_i}+\frac{1}{2},
\end{aligned} \tag{6.7}$$

where $i=1,2,3,4$. Each superalgebra possesses a sCasimir element given by

$$S_i=\frac{1}{2}([D_i,x_i]-1), \tag{6.8}$$

which anticommutes with the odd generators

$$\{S_i,D_i\}=\{S_i,x_i\}=0, \tag{6.9}$$

and thus commutes with the even generators

$$[S_i,E_i]=[S_i,|x_i|^2]=[S_i,D_i^2]=0. \tag{6.10}$$

It is immediate to verify that in the realization (6.7), the reflection R_i verifies the same commutation relations as the sCasimir

$$[R_i,E_i]=[R_i,|x_i|^2]=[R_i,D_i^2]=\{R_i,D_i\}=\{R_i,x_i\}=0. \tag{6.11}$$

This implies that one can construct a Casimir operator of the form

$$Q_i=S_iR_i. \tag{6.12}$$

It is straightforward to verify that Q_i indeed commutes with every generator. These four realizations of $\mathfrak{osp}(1|2)$ can act as building blocks for many other realizations. Let $[n]=\{1,2,\dots,n\}$ and $A\subset[4]$. The operators given by

$$\begin{aligned}
D_A &= \sum_{i\in A}\Big(D_i\prod_{j=i+1}^{\sup A}R_j\Big), \quad && D_A^2=D_AD_A,\\
x_A &= \sum_{i\in A}\Big(s_i\prod_{j=i+1}^{\sup A}R_j\Big), && |x_A|^2=\sum_{i\in A}s_i^2,\\
E_A &= \sum_{i\in A}E_i,
\end{aligned} \tag{6.13}$$

verify the commutation relations (6.6) for any $A\subset[4]$ and thus form new realizations of $\mathfrak{osp}(1|2)$. These result from the repeated application of the coproduct of $\mathfrak{osp}(1|2)$ (see [19]). Moreover, for any A the sCasimir and the Casimir operators are also similarly defined:

$$S_A = \frac{1}{2}([D_A, x_A] - 1), \qquad Q_A = S_A \prod_{i \in A} R_i. \tag{6.14}$$

One can directly check that

$$Q_i = \mu_i, \qquad Q_{jk} = L_{jk}, \qquad Q_B = M_B, \tag{6.15}$$

where Q_{jk} denotes Q_A with $A = \{j, k\}$ and B is any 3-subset of [4]. Another explicit computation gives

$$S_{[4]}^2 - S_{[4]} - \tfrac{3}{4} = \sum_{1 \le i < j \le 4} J_{ij}^2 + (s_1^2 + s_2^2 + s_3^2 + s_4^2) \sum_{i=1}^{4} \frac{\mu_i}{s_i^2}(\mu_i - R_i). \tag{6.16}$$

However, since $|x_{[4]}|^2 = s_1^2 + s_2^2 + s_3^2 + s_4^2$ commutes with $S_{[4]}$ and all the Casimirs, it is central in the algebra generated by the Casimirs and can thus be treated as a constant. Taking $|x_{[4]}|^2 = 1$, it is straightforward by comparing (6.16) and (6.1) that

$$S_{[4]}^2 - S_{[4]} - \tfrac{3}{4} = H. \tag{6.17}$$

Hence, a quadratic combination of the sCasimir of four copies of $\mathfrak{osp}(1|2)$ yields the Hamiltonian of the superintegrable model presented in section 1 and the intermediate Casimirs are its symmetries. Indeed, it can be checked that $[Q_A, H] = 0$ for $A \subset [4]$. The symmetry algebra has the following structure relations

$$\{Q_A, Q_B\} = Q_{(A \cup B) \setminus (A \cap B)} + 2Q_{A \cap B} Q_{A \cup B} + 2Q_{A \setminus (A \cap B)} Q_{B \setminus (A \cap B)}, \tag{6.18}$$

where $A, B \subset [4]$ and $Q_\emptyset = -1/2$ as prescribed by the definitions (6.13) and (6.14). This algebra has already been studied in [20] and is interpreted as a rank 2 Bannai-Ito algebra. To see this, we remark that the Casimirs with $A \subset [3]$ generate the (rank 1) Bannai-Ito algebra. Let $K_1 = Q_{12}, K_2 = Q_{23}$ and $K_3 = Q_{13}$. The recurrence relations (6.18) can then be rewritten as

$$\{K_1, K_2\} = K_3 + \omega_3, \quad \{K_2, K_3\} = K_1 + \omega_1, \quad \{K_3, K_1\} = K_2 + \omega_2, \tag{6.19}$$

where $\omega_1, \omega_2, \omega_3$ are central elements given by

$$\omega_1 = 2Q_3 Q_{123} + 2Q_1 Q_2, \quad \omega_2 = 2Q_1 Q_{123} + 2Q_2 Q_3, \quad \omega_3 = 2Q_2 Q_{123} + 2Q_1 Q_3. \tag{6.20}$$

This corresponds to the Bannai-Ito algebra introduced in [14] which appears in a corresponding superintegrable model with reflections on S^2 as its symmetry algebra [17].

6.4. Wavefunctions

To obtain the solutions to the equation $H\psi = \lambda\psi$ let us first introduce the gauge transformation

$$z \to \tilde{z} \equiv G(\vec{s})^{-1} z G(\vec{s}), \qquad G(\vec{s}) = \prod_{i=1}^{4} |s_i|^{\mu_i}, \tag{6.21}$$

where z is any operator and $\vec{s} \equiv (s_1, s_2, s_3, s_4)$. Under this transformation, the generators of $\mathfrak{osp}(1|2)$ in the realization (6.7) become

$$\begin{aligned}
\tilde{D}_i &= \partial_{s_i} + \frac{\mu_i}{s_i}(1 - R_i), & \tilde{D}_i^2 &= \tilde{D}_i \tilde{D}_i, \\
\tilde{x}_i &= x_i = s_i, & |\tilde{x}_i|^2 &= s_i^2, \\
\tilde{E}_i &= s_i \partial_{s_i} + \gamma_i, & \tilde{R}_i &= R_i, \\
\tilde{S}_i &= -\mu_i R_i, & \tilde{Q}_i &= \mu_i,
\end{aligned} \tag{6.22}$$

where

$$\gamma_A = \sum_{i \in A} (\mu_i + \tfrac{1}{2}). \tag{6.23}$$

These operators also verify (6.6) and correspond to the realization of $\mathfrak{osp}(1|2)$ (or equivalently $sl_{-1}(2)$) arising in the one-dimensional parabose oscillator [5]. Furthermore, the construction (6.13) can be reproduced with this transformed realization to obtain operators of the form $\tilde{D}_A, \tilde{x}_A, \tilde{E}_A, \tilde{S}_A$ and $\tilde{Q}_A$ and is trivially seen to be equivalent to the gauge transformation of the corresponding operators. Hence, we can obtain eigenvalues and eigenfunctions of H by finding eigenfunctions of $\tilde{S}_{[4]}$. Note that since $\tilde{S}_{[4]}$ commutes with $P = R_1 R_2 R_3 R_4$, this is equivalent to finding eigenfunctions of $\tilde{Q}_{[4]}$.

We thus aim to obtain polynomial eigenfunctions of $\tilde{S}_{[4]}$. To do so, let us first introduce $\mathcal{P}_m(\mathbb{R}^n)$, the space of homogeneous polynomials of degree m in the variables $s_1, s_2, \ldots, s_n$. We define $\mathcal{K}_m(\mathbb{R}^n)$ the kernel space of degree m as

$$\mathcal{K}_m(\mathbb{R}^n) = \ker \tilde{D}_{[n]} \cap \mathcal{P}_m(\mathbb{R}^n). \tag{6.24}$$

When $n = 4$, this is an eigenspace of $\tilde{S}_{[4]}$. Indeed, take $\psi_m \in \mathcal{K}_m(\mathbb{R}^4)$ and compute

$$\begin{aligned}
\tilde{S}_{[4]} \tilde{\psi}_m &= \tfrac{1}{2}(\tilde{D}_{[4]} \tilde{x}_{[4]} - \tilde{x}_{[4]} \tilde{D}_{[4]} - 1)\tilde{\psi}_m = \tfrac{1}{2}(\tilde{D}_{[4]} \tilde{x}_{[4]} - 1)\tilde{\psi}_m \\
&= \tfrac{1}{2}(\tilde{D}_{[4]} \tilde{x}_{[4]} + \tilde{x}_{[4]} \tilde{D}_{[4]} - 1)\tilde{\psi}_m = \tfrac{1}{2}(\{\tilde{x}_{[4]}, \tilde{D}_{[4]}\} - 1)\tilde{\psi}_m \\
&= \tfrac{1}{2}(2\tilde{E}_{[4]} - 1)\tilde{\psi}_m = \left[\sum_{i=1}^{4} s_i \partial_{s_i} + \gamma_{[4]} - \tfrac{1}{2}\right] \tilde{\psi}_m,
\end{aligned}$$

where we used the property $\tilde{D}_{[4]}\tilde{\psi}_m = 0$ and the commutation relations (6.6). Since $\tilde{\psi}_m$ is a homogeneous polynomial of degree m, it is an eigenfunction of the Euler operator : $\sum_{i=1}^4 s_i\partial_{s_i}\tilde{\psi}_m = m\tilde{\psi}_m$. This implies

$$\tilde{S}_{[4]}\tilde{\psi}_m = (m + \gamma_{[4]} - \tfrac{1}{2})\tilde{\psi}_m \tag{6.25}$$

and shows that $\mathcal{K}_m(\mathbb{R}^4)$ is an eigenspace of $\tilde{S}_{[4]}$. We use two results in order to construct explicitly the eigenfunctions. First, the space of homogeneous polynomials $\mathcal{P}_m(\mathbb{R}^n)$ admits a decomposition in terms of the kernel spaces. This is called the Fischer decomposition and can be cast as

$$\mathcal{P}_m(\mathbb{R}^n) = \bigoplus_{j=0}^{m} \tilde{x}_{[n]}^j \mathcal{K}_{m-j}(\mathbb{R}^n). \tag{6.26}$$

Second, we use the Cauchy-Kovalevskaia isomorphism (CK-map) between the space of m-homogeneous polynomials in $n-1$ variables and the kernel space of degree m in n variables :

$$\mathbf{CK}_{s_n}^{\mu_n} : \mathcal{P}_m(\mathbb{R}^{n-1}) \to \mathcal{K}_m(\mathbb{R}^n). \tag{6.27}$$

One can compute the CK-map explicitly. To compute $\mathbf{CK}_{s_4}^{\mu_4}$, take $p(s_1, s_2, s_3) \in \mathcal{P}_m(\mathbb{R}^3)$ and let

$$\mathbf{CK}_{s_4}^{\mu_4}[p(s_1, s_2, s_3)] = \sum_{\alpha=0}^{m} s_4^\alpha p_\alpha(s_1, s_2, s_3), \tag{6.28}$$

where $p_\alpha(s_1, s_2, s_3) \in \mathcal{P}_{m-\alpha}(\mathbb{R}^3)$ and $p_0(s_1, s_2, s_3) \equiv p(s_1, s_2, s_3)$. Demand that

$$\tilde{D}_{[4]} \sum_{\alpha=0}^{m} s_4^\alpha p_\alpha(s_1, s_2, s_3) = 0 \tag{6.29}$$

to fix and compute the coefficients $p_\alpha(s_1, s_2, s_3)$. A straightforward calculation yields

$$\mathbf{CK}_{s_4}^{\mu_4} = \sum_{i=0}^{\infty} \frac{(-1)^i (s_4)^{2i}}{i!(\gamma_4)_i (2)^{2i}} \tilde{D}_{[4]}^{2i} + \sum_{i=0}^{\infty} \frac{(-1)^{i+1} (s_4)^{2i+1}}{i!(\gamma_4)_{i+1} (2)^{2i+1}} \tilde{D}_{[4]}^{2i+1}, \tag{6.30}$$

where $(a)_i = a(a+1)\dots(a+n-1)$ denotes the Pochhammer symbol. Similarly, one obtains

$$\mathbf{CK}_{s_n}^{\mu_n} = \sum_{i=0}^{\infty} \frac{(-1)^i (s_n)^{2i}}{i!(\gamma_n)_i (2)^{2i}} \tilde{D}_{[n]}^{2i} + \sum_{i=0}^{\infty} \frac{(-1)^{i+1} (s_n)^{2i+1}}{i!(\gamma_n)_{i+1} (2)^{2i+1}} \tilde{D}_{[n]}^{2i+1}, \tag{6.31}$$

for $n = 2, 3, 4$. Now, iterating the Fischer decomposition (6.26) and the CK-map (6.27), the eigenspace $\mathcal{K}_m(\mathbb{R}^4)$ can be expressed as

$$\mathcal{K}_m(\mathbb{R}^4) \cong \mathbf{CK}_{s_4}^{\mu_4}\Big[\bigoplus_{j_2=0}^{m} \tilde{x}_{[3]}^{m-j_2} \mathbf{CK}_{s_3}^{\mu_3}\Big[\bigoplus_{j_1=0}^{j_2} \tilde{x}_{[2]}^{j_2-j_1} \mathbf{CK}_{s_2}^{\mu_2}\left[\mathcal{P}_{j_1}(\mathbb{R})\right]\Big]\Big]. \tag{6.32}$$

This means that we can explicitly construct a basis of eigenfunctions $\{\tilde{\psi}_{j_1,j_2,j_3}^{(m)}(\vec{s})\}_{j_1+j_2+j_3=m}$ of $\mathcal{K}_m(\mathbb{R}^4)$ with

$$\tilde{\psi}_{j_1,j_2,j_3}^{(m)}(\vec{s}) = \mathbf{CK}_{s_4}^{\mu_4}[\tilde{x}_{[3]}^{j_3} \mathbf{CK}_{s_3}^{\mu_3}[\tilde{x}_{[2]}^{j_2} \mathbf{CK}_{s_2}^{\mu_2}[s_1^{j_1}]]]. \tag{6.33}$$

This calculation can be carried straightforwardly with the help of the identities with $\tilde{\psi}_m \in \mathcal{K}_m(\mathbb{R}^n)$

$$\begin{aligned}
\tilde{D}_{[n]}^{2\alpha}\tilde{x}_{[n]}^{2\beta}\tilde{\psi}_m &= 2^{2\alpha}(-\beta)_\alpha(1-m-\beta-\gamma_{[n]})_\alpha x_{[n]}^{2\beta-2\alpha}\tilde{\psi}_m,\\
\tilde{D}_{[n]}^{2\alpha+1}\tilde{x}_{[n]}^{2\beta}\tilde{\psi}_m &= 2^{2\alpha}\beta(1-\beta)_\alpha(1-m-\beta-\gamma_{[n]})_\alpha x_{[n]}^{2\beta-2\alpha-1}\tilde{\psi}_m,\\
\tilde{D}_{[n]}^{2\alpha}\tilde{x}_{[n]}^{2\beta+1}\tilde{\psi}_m &= 2^{2\alpha}(-\beta)_\alpha(-m-\beta-\gamma_{[n]})_\alpha x_{[n]}^{2\beta+1-2\alpha}\tilde{\psi}_m,\\
\tilde{D}_{[n]}^{2\alpha+1}\tilde{x}_{[n]}^{2\beta+1}\tilde{\psi}_m &= 2^{2\alpha+1}(-\beta)_\alpha(m+\beta+\gamma_{[n]})(1-m-\beta-\gamma_{[n]})_\alpha x_{[n]}^{2\beta-2\alpha}\tilde{\psi},
\end{aligned} \tag{6.34}$$

which follows from (6.6). The result can be presented in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, defined as [2]

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n,\ n+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{1-x}{2}\right], \tag{6.35}$$

with the help of the identity :

$$(x+y)^n P_n^{(\alpha,\beta)}\left(\frac{x-y}{x+y}\right) = \frac{(\alpha+1)_n}{n!}x^n\,{}_2F_1\left[\begin{matrix}-n,\ -n-\beta\\ \alpha+1\end{matrix};-\frac{y}{x}\right]. \tag{6.36}$$

One obtains

$$\tilde{\psi}_{j_1,j_2,j_3}^{(m)}(\vec{s}) = \mathbf{P}_{j_1,j_2,j_3}(\vec{s})\mathbf{Q}_{j_1,j_2}(s_1,s_2,s_3)R_{j_1}(s_1,s_2), \tag{6.37}$$

where

$$\begin{aligned}
\mathbf{P}_{j_1,j_2,j_3}(\vec{s}) &= \frac{c!}{(\gamma_4)_c}(s_1^2+s_2^2+s_3^2+s_4^2)^c\\
&\times\begin{cases} P_c^{(\gamma_4-1,j_1+j_2+\gamma_{[3]}-1)}\left(\frac{s_1^2+s_2^2+s_3^2-s_4^2}{s_1^2+s_2^2+s_3^2+s_4^2}\right) - \frac{s_4\tilde{x}_{[3]}}{s_1^2+s_2^2+s_3^2+s_4^2}P_{c-1}^{(\gamma_4,j_1+j_2+\gamma_{[3]})}\left(\frac{s_1^2+s_2^2+s_3^2-s_4^2}{s_1^2+s_2^2+s_3^2+s_4^2}\right) & \text{if } j_3 = 2c,\\
\tilde{x}_{[3]}P_c^{(\gamma_4-1,j_1+j_2+\gamma_{[3]})}\left(\frac{s_1^2+s_2^2+s_3^2-s_4^2}{s_1^2+s_2^2+s_3^2+s_4^2}\right) - s_3\frac{j_1+j_2+c+\gamma_{[3]}}{c+\gamma_4}P_c^{(\gamma_4,j_1+j_2+\gamma_{[3]}-1)}\left(\frac{s_1^2+s_2^2+s_3^2-s_4^2}{s_1^2+s_2^2+s_3^2+s_4^2}\right) & \text{if } j_3 = 2c+1,\end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathbf{Q}_{j_1,j_2}(s_1,s_2,s_3) &= \frac{b!}{(\gamma_3)_b}(s_1^2+s_2^2+s_3^2)^b\\
&\times\begin{cases} P_b^{(\gamma_3-1,j_1+\gamma_{[2]}-1)}\left(\frac{s_1^2+s_2^2-s_3^2}{s_1^2+s_2^2+s_3^2}\right) - \frac{s_3\tilde{x}_{[2]}}{s_1^2+s_2^2+s_3^2}P_{b-1}^{(\gamma_3,j_1+\gamma_{[2]})}\left(\frac{s_1^2+s_2^2-s_3^2}{s_1^2+s_2^2+s_3^2}\right) & \text{if } j_2 = 2b,\\
\tilde{x}_{[2]}P_b^{(\gamma_3-1,j_1+\gamma_{[2]})}\left(\frac{s_1^2+s_2^2-s_3^2}{s_1^2+s_2^2+s_3^2}\right) - s_2\frac{j_1+b+\gamma_{[2]}}{b+\gamma_3}P_b^{(\gamma_3,j_1+\gamma_{[2]}-1)}\left(\frac{s_1^2+s_2^2-s_3^2}{s_1^2+s_2^2+s_3^2}\right) & \text{if } j_2 = 2b+1,\end{cases}
\end{aligned}$$

$$R_{j_1}(s_1,s_2) = \frac{a!}{(\gamma_2)_a}(s_1^2+s_2^2)^a \times \begin{cases} P_a^{(\gamma_2-1,\gamma_1-1)}\left(\frac{s_1^2-s_2^2}{s_1^2+s_2^2}\right) - \frac{s_1s_2}{s_1^2+s_2^2}P_{a-1}^{(\gamma_2,\gamma_1)}\left(\frac{s_1^2-s_2^2}{s_1^2+s_2^2}\right) & \text{if } j_1 = 2a,\\
s_1P_a^{(\gamma_2-1,\gamma_1)}\left(\frac{s_1^2-s_2^2}{s_1^2+s_2^2}\right) - s_2\frac{a+\gamma_1}{a+\gamma_2}P_a^{(\gamma_2,\gamma_1-1)}\left(\frac{s_1^2-s_2^2}{s_1^2+s_2^2}\right) & \text{if } j_1 = 2a+1.\end{cases}$$

Note that the expressions for $\mathbf{P}_{j_1,j_2,j_3}(s_1,s_2,s_3,s_4)$ and $\mathbf{Q}_{j_1,j_2}(s_1,s_2,s_3)$ contain the operators $\tilde{x}_{[3]}$ and $\tilde{x}_{[2]}$ respectively. Recalling the expressions (6.22) and (6.13), it can be seen that these operators only contain variables s_i and reflection operators R_i. These reflections conveniently account for signs occuring in the solutions without having to give a different expression for every parity combination of the parameters j_1, j_2 and j_3.

By effecting the reverse gauge transformation, we thus obtain a basis for the eigenspace of the operator $S_{[4]}$ given by

$$\psi^{(m)}_{j_1,j_2,j_3}(\vec{s}) = \tilde{\psi}^{(m)}_{j_1,j_2,j_3}(\vec{s})G(\vec{s}), \tag{6.38}$$

where $m = 0, 1, \ldots$ and $j_1 + j_2 + j_3 = m$. With the help of (6.25), they obey the relation

$$S_{[4]}\psi^{(m)}_{j_1,j_2,j_3}(\vec{s}) = (m + \gamma_{[4]} - \tfrac{1}{2})\psi^{(m)}_{j_1,j_2,j_3}(\vec{s}). \tag{6.39}$$

Recalling (6.17), this also implies

$$H\psi^{(m)}_{j_1,j_2,j_3}(\vec{s}) = (m + \gamma_{[4]})(m + \gamma_{[4]} - 2)\psi^{(m)}_{j_1,j_2,j_3}(\vec{s}). \tag{6.40}$$

Finally, we can normalize these eigenfunctions as

$$\Psi^{(m)}_{j_1,j_2,j_3}(\vec{s}) = \frac{\eta_1\eta_2\eta_3}{\sqrt{2}}\psi^{(m)}_{j_1,j_2,j_3}(\vec{s}), \tag{6.41}$$

where

$$\eta_1 = \gamma_2\sqrt{\frac{\Gamma(a+\gamma_{[2]})}{a!\Gamma(a+\gamma_1)\Gamma(a+\gamma_2)}} \times \begin{cases} 1 & \text{if } j_1 = 2a, \\ \sqrt{\frac{a+\gamma_2}{a+\gamma_1}} & \text{if } j_1 = 2a+1, \end{cases} \tag{6.42}$$

$$\eta_2 = \gamma_3\sqrt{\frac{\Gamma(b+j_1+\gamma_{[3]})}{b!\Gamma(b+\gamma_3)\Gamma(b+j_1+\gamma_{[2]})}} \times \begin{cases} 1 & \text{if } j_2 = 2b, \\ \sqrt{\frac{b+\gamma_3}{b+j_1+\gamma_{[2]}}} & \text{if } j_2 = 2b+1, \end{cases} \tag{6.43}$$

$$\eta_3 = \gamma_4\sqrt{\frac{\Gamma(c+j_1+j_2+\gamma_{[4]})}{c!\Gamma(c+\gamma_4)\Gamma(c+j_1+j_2+\gamma_{[3]})}} \times \begin{cases} 1 & \text{if } j_3 = 2c, \\ \sqrt{\frac{c+\gamma_4}{c+j_1+j_2+\gamma_{[3]}}} & \text{if } j_3 = 2c+1, \end{cases} \tag{6.44}$$

so that

$$\int_{S^3} \Psi^{(m)\dagger}_{j_1,j_2,j_3}(\vec{s})\Psi^{(n)}_{k_1,k_2,k_3}(\vec{s})d\vec{s} = \delta_{n,m}\delta_{j_1,k_1}\delta_{j_2,k_2}. \tag{6.45}$$

This can be verified directly from the orthogonality relation of the Jacobi polynomials [2].

6.5. Conclusion

To sum up, we have introduced a new quantum superintegrable model with reflections on the three-sphere. Its symmetries were given explicitly and were shown to realize a rank-two Bannai-Ito algebra. It was observed that the model can be constructed through the combination of four independent realizations of the superalgebra $\mathfrak{osp}(1|2)$. A quadratic expression in the total sCasimir operator was found to coincide with the Hamiltonian while the intermediate Casimir operators were seen to coincide with its symmetries. The exact solutions have been obtained by using a Cauchy-Kovalevskaia extension theorem. We did not find many occurences of this remarkably simple technique in the superintegrability literature and we trust it could find many other applications. Furthermore, an interesting feature of this model is the appearance in a scalar model of the rank 2 Bannai-Ito algebra which arose as a particular case in the analysis of the Dirac-Dunkl equation [20]. One expects that the bivariate Bannai-Ito polynomials will arise as overlaps between wavefunctions of this model separated in different hyperspherical coordinate systems. These polynomials have never been identified so far and we aim to study this question in the near future.

Acknowledgments

The research of HDB is supported by the Fund for Scientific Research-Flanders (FWO-V), project "Construction of algebra realisations using Dirac-operators", grant G.0116.13N. VXG holds a postdoctoral fellowship from the Natural Science and Engineering Research Council of Canada (NSERC). JML holds a scholarship from the Fonds de recherche du Québec – Nature et technologies (FRQNT). The research of LV is supported in part by NSERC.

References

[1] W. Miller, S. Post, P. Winternitz, Classical and quantum superintegrability with applications, J. Phys. A Math. Theor. 46 (42) (2013) 423001.

[2] R. Koekoek, P. A. Lesky, R. F. Swarttouw, Hypergeometric Orthogonal Polynomials, Springer, 2010.

[3] E. G. Kalnins, W. Miller, S. Post, Contractions of 2D 2nd Order Quantum Superintegrable Systems and the Askey Scheme for Hypergeometric Orthogonal Polynomials, SIGMA 9 (2013) 57–84.

[4] E. G. Kalnins, W. MILLER Jr, S. Post, Two-Variable Wilson Polynomials and the Generic Superintegrable System on the 3-Sphere, Methods Appl. SIGMA 7 (2011) 26.

[5] V. X. Genest, M. E. Ismail, L. Vinet, A. Zhedanov, The Dunkl oscillator in the plane: I. Superintegrability, separated wavefunctions and overlap coefficients, J. Phys. A Math. Theor. 46 (14) (2013) 145201.

[6] V. X. Genest, M. E. Ismail, L. Vinet, A. Zhedanov, The Dunkl Oscillator in the Plane II: Representations of the Symmetry Algebra, Commun. Math. Phys. 329 (3) (2014) 999–1029.

[7] V. X. Genest, L. Vinet, A. Zhedanov, The singular and the 2:1 anisotropic Dunkl oscillators in the plane, J. Phys. A Math. Theor. 46 (32) (2013) 325201.

[8] V. X. Genest, L. Vinet, A. Zhedanov, The Dunkl oscillator in three dimensions, J. Phys. Conf. Ser. 512 (13) (2014) 12010.

[9] S. Post, L. Vinet, A. Zhedanov, Supersymmetric quantum mechanics with reflections, J. Phys. A Math. Theor. 44 (43) (2011) 435301.

[10] L. Vinet, A. Zhedanov, A 'missing' family of classical orthogonal polynomials, J. Phys. A Math. Theor. 44 (8) (2011) 085201.

[11] L. Vinet, A. Zhedanov, A limit $q = -1$ for the big q-Jacobi polynomials, Trans. Am. Math. Soc. 364 (10) (2012) 5491–5507.

[12] L. Vinet, A. Zhedanov, A bochner theorem for dunkl polynomials, Symmetry, Integr. Geom. Methods Appl. 7 (2011) 20.

[13] S. Tsujimoto, L. Vinet, A. Zhedanov, Dual -1 Hahn polynomials : "classical" polynomials beyond the Leonard duality, Proc. Am. Math. Soc. 141 (3) (2013) 959–970.

[14] S. Tsujimoto, L. Vinet, A. Zhedanov, Dunkl shift operators and Bannai-Ito polynomials, Adv. Math. (N. Y). 229 (4) (2012) 2123–2158.

[15] V. X. Genest, L. Vinet, A. Zhedanov, Bispectrality of the complementary Bannai-Ito polynomials, Symmetry, Integr. Geom. Methods Appl. 9 (2013) 18.

[16] V. X. Genest, L. Vinet, A. Zhedanov, A continuous limit of the Complementary Bannai-Ito polynomials : Chihara polynomials, Symmetry, Integr. Geom. Methods Appl. 10 (0) (2014) 18–38.

[17] V. X. Genest, L. Vinet, A. Zhedanov, A Laplace-Dunkl Equation on S^2 and the Bannai–Ito Algebra, Commun. Math. Phys. 336 (1) (2015) 243–259.

[18] V. X. Genest, L. Vinet, A. Zhedanov, The Bannai-Ito algebra and a superintegrable system with reflections on the two-sphere, J. Phys. A Math. Theor. 47 (20) (2014) 205202.

[19] V. X. Genest, L. Vinet, A. Zhedanov, Embeddings of the racah algebra into the Bannai-Ito algebra, Symmetry, Integr. Geom. Methods Appl. 11.

[20] H. De Bie, V. X. Genest, L. Vinet, The Z_2^n Dirac-Dunkl operator and a higher rank Bannai-Ito algebra, Adv. Math. (N. Y). 303 (2016) 390–414.

Chapitre 7

A superintegrable model with reflections on S^{n-1} and the higher rank Bannai-Ito algebra

H. de Bie, V.X. Genest, J.M. Lemay et L. Vinet (2017). A superintegrable model with reflections on S^{n-1} and the higher rank Bannai-Ito algebra. *Journal of Physics A: Mathematical and Theoretical 50 (19), 195202*

Abstract. A quantum superintegrable model with reflections on the $(n-1)$-sphere is presented. Its symmetry algebra is identified with the higher rank generalization of the Bannai-Ito algebra. It is shown that the Hamiltonian of the system can be constructed from the tensor product of n representations of the superalgebra $\mathfrak{osp}(1|2)$ and that the superintegrability is naturally understood in that setting. The separated solutions are obtained through the Fischer decomposition and a Cauchy-Kovalevskaia extension theorem.

7.1. Introduction

This paper introduces a superintegrable model with reflections on the $(n-1)$-sphere which has the rank $(n-2)$-Bannai-Ito algebra as its symmetry algebra.

Maximally superintegrable quantum Hamiltonians in n dimensions possess $2n-1$ algebraically independent constants of motion (including H). Often interesting in their own right, these systems form the bedrock for the analysis of symmetries and their study has witnessed significant advances in recent years. Of note is the complete classification of all scalar superintegrable systems in two dimensions whose conserved quantities are at most of order two in momenta [1, 2]. It shows that (for Euclidean signature) all superintegrable systems in that class can be obtained as contractions or special cases of the so-called generic model on the 2-sphere. This developped in parallel with the study of non-linear algebras [3, 4] associated to the bispectrality of the orthogonal polynomials of the Askey tableau [5]. These algebras are usually referred to by the name of the corresponding polynomials.

Separated wavefunctions of the generic model are obtained as joint eigenfunctions of H and one constant of motion. The overlaps between wavefunctions associated to the diagonalization of two different generators are given in terms of Racah polynomials [6]. In view of this, it was somewhat natural to find that the symmetry algebra of the generic scalar system on the 2-sphere is the Racah algebra [7]. This was subsequently put into a cogent framework when it was observed that the description of the generic model could be formulated via the recoupling of three $\mathfrak{sl}(2)$ realizations [8, 9, 10].

Superalgebras somehow subsumes Lie algebras. Indeed Lie algebras can be engendered as even subalgebra of superalgebras from quadratic expressions in the odd generators of the latter. A translation of this has appeared in the realm of superintegrable models. It has indeed been seen [11] that by using reflection operators, a number of generic superintegrable models with different constants can be combined together in a supersymmetric fashion to give a generalized Hamiltonian whose symmetries form the Bannai-Ito algebra [12, 11]. This algebra associated to the Bannai-Ito polynomials [13] has simple defining relations and arises upon considering the tensor product of three $\mathfrak{osp}(1|2)$ superalgebras [14, 15]. It has further been seen that the Racah algebra can be embedded in the Bannai-Ito algebra by using quadratic polynomials in the generators of the latter [16]. This model with reflection operator together with its symmetry algebra could thus be viewed as more basic than the generic scalar one.

In the wake of the classification of two-dimensional systems, the exploration of higher dimensional superintegrable models was undertaken. The generic model on the 3-sphere was shown to be connected to the bivariate Racah polynomials [17]; the model with reflections on S^3 was also constructed and analyzed [18]. Having in mind the recoupling framework for superintegrable models, it becomes clear that the search for the symmetry algebras of higher dimensional version of the generic model amounts to the identification of the Bannai-Ito and Racah algebras of higher ranks. Results in this direction have been obtained.

The higher rank Bannai-Ito algebra was first identified using the Dirac-Dunkl equation as model [19]. It was subsequently constructed in [20] using n-fold products of $\mathfrak{osp}(1|2)$. Similarly, the Racah algebra was extended to arbitrary ranks in [21] by considering multiple tensor products of $\mathfrak{sl}(2)$ realized in terms of Dunkl operators. It was also observed [22] that the generators of this algebra realize the Drinfeld-Kohno relations.

We now bring these advances to bear on superintegrable models by providing here the non-relativistic Hamiltonian with reflections that has for symmetry algebra the higher rank Bannai-Ito algebra recently discovered. We shall also show how these symmetries can be put to use in order to obtain the wavefunctions of this quantum model.

The paper is divided as follows. In section 2, the model with n parameters on S^{n-1} is introduced and its symmetries are given. In section 3, it is shown how it is built out of n copies of the superalgebra $\mathfrak{osp}(1|2)$. This construction will lead to the identification of the symmetry algebra. The wavefunctions are obtained in Section 4 using the Fischer decomposition and a Cauchy-Kovalevskaia extension theorem. (It will be of interest to observe that such a theorem can also be devised in a scalar situation outside the usual Clifford algebra context.) Some concluding remarks are offered in section 5.

7.2. A superintegrable model on S^{n-1}

Let $s_1, s_2, \dots, s_n$ be the Cartesian coordinates of the n-dimensional Euclidean space and consider the embedding of the S^{n-1} sphere given by the constraint: $\sum_{i=1}^{n} s_i^2 = 1$. We shall be interested in the system with n parameters $\mu_1, \mu_2, \dots, \mu_n$ with $\mu_i \geq 0$ for $i = 1, 2, \dots, n$ governed by the Hamiltonian

$$H = \sum_{1 \leq i < j \leq n} J_{ij}^2 + \sum_{i=1}^{n} \frac{\mu_i}{s_i^2}(\mu_i - R_i), \tag{7.1}$$

where

$$J_{ij} = \frac{1}{i}(s_i \partial_{s_j} - s_j \partial_{s_i}), \qquad R_i f(s_i) = f(-s_i), \tag{7.2}$$

are the angular momentum operators and reflection operators, respectively. Denote by $[n]$ the set $\{1, 2, 3, ..., n\}$. The quantities

$$M_A = \left(-\frac{1}{2} + \sum_{i \in A} (\tfrac{1}{2} + \mu_i R_i) + \sum_{\substack{j<k \\ j,k \in A}} \left(-i J_{jk} - s_k \frac{\mu_j}{s_j} R_j + s_j \frac{\mu_k}{s_k} R_k \right) \prod_{l=j}^{k-1} R_l \right) \prod_{m \in A} R_m, \tag{7.3}$$

labelled by subsets $A \subset [n]$ can be seen to commute with H and are thus conserved. Note that when the set A contains only one element, say i, M_i will be taken to be

$$M_i = \mu_i, \qquad i = 1, 2, \dots, n. \tag{7.4}$$

In view of their expression, the number of algebraically independent constant of motion M_A will correspond to the number of independent generators of $\mathfrak{so}(n)$ thus implying that H is superintegrable. Furthermore, a direct computation shows that all reflections are also symmetries of H:

$$[H, R_i] = 0, \qquad i = 1, 2, \dots, n. \tag{7.5}$$

7.3. Algebraic construction from $\mathfrak{osp}(1|2)$

We shall now explain the relation that the Hamiltonian H has with $\mathfrak{osp}(1|2)$. This superalgebra has 5 generators, two odd x and D and three even $E, |x|^2$ and D^2 which satisfy the commutation relations:

$$
\begin{aligned}
&\{x, x\} = 2|x|^2, && \{D, D\} = 2D^2, \\
&\{x, D\} = 2E, && [D, E] = D, \\
&[D, |x|^2] = 2x, && [E, x] = x, \\
&[D^2, x] = 2D, && [D^2, E] = 2D^2, \\
&[D^2, |x|^2] = 4E, && [E, |x|^2] = 2|x|^2,
\end{aligned}
\tag{7.6}
$$

where $[a, b] = ab - ba$ is the commutator and $\{a, b\} = ab + ba$ is the anti-commutator. One can realize mutually commuting copies of this superalgebra by taking

$$
\begin{aligned}
&D_i = \partial_{s_i} - \frac{\mu_i}{s_i} R_i, && D_i^2 = D_i D_i, \\
&x_i = s_i, && |x_i|^2 = s_i^2, \\
&E_i = s_i \partial_{s_i} + \frac{1}{2},
\end{aligned}
\tag{7.7}
$$

where $i = 1, 2, \ldots, n$. Each superalgebra possesses a sCasimir element given by

$$S_i = \frac{1}{2}([D_i, x_i] - 1), \tag{7.8}$$

which anticommutes with the odd generators

$$\{S_i, D_i\} = \{S_i, x_i\} = 0, \tag{7.9}$$

and thus commutes with the even generators

$$[S_i, E_i] = [S_i, |x_i|^2] = [S_i, D_i^2] = 0. \tag{7.10}$$

It is immediate to verify that in the realization (7.7), the reflection R_i obeys the same commutation relations as the sCasimir

$$[R_i, E_i] = [R_i, |x_i|^2] = [R_i, D_i^2] = \{R_i, D_i\} = \{R_i, x_i\} = 0. \tag{7.11}$$

This implies that one can construct a Casimir operator of the form

$$Q_i = S_i R_i. \tag{7.12}$$

It is straightforward to verify that Q_i indeed commutes with every generator. The commuting realizations of $\mathfrak{osp}(1|2)$ can be used as building blocks to construct other realizations. Let $[n] = \{1, 2, \ldots, n\}$ and $A \subset [n]$ as before. The operators given by

$$\begin{aligned} D_A &= \sum_{i\in A}\Big(D_i \prod_{j=1}^{i-1} R_j\Big), & D_A^2 &= D_A D_A, \\ x_A &= \sum_{i\in A}\Big(s_i \prod_{j=1}^{i-1} R_j\Big), & |x_A|^2 &= \sum_{i\in A} s_i^2, \\ E_A &= \sum_{i\in A} E_i, & R_A &= \prod_{i\in A} R_i, \end{aligned} \tag{7.13}$$

verify the commutation relations (7.6) for any $A \subset [n]$ and thus form new realizations of $\mathfrak{osp}(1|2)$. These result from the repeated application of the coproduct of $\mathfrak{osp}(1|2)$ (see [19]). For any A the sCasimir and the Casimir operators are again defined by

$$S_A = \frac{1}{2}([D_A, x_A] - 1), \qquad Q_A = S_A R_A. \tag{7.14}$$

One can directly check that

$$Q_A = M_A, \qquad A \subset [n]. \tag{7.15}$$

Another explicit computation gives

$$S_{[n]}^2 - S_{[n]} - \frac{(n-1)(n-3)}{4} = \sum_{1\leq i<j\leq n} J_{ij}^2 + \left(\sum_{i=1}^n s_i^2\right)\sum_{i=1}^n \frac{\mu_i}{s_i^2}(\mu_i - R_i). \tag{7.16}$$

However, since $|x_{[n]}|^2 = \sum_{i=1}^n s_i^2$ commutes with $S_{[n]}$ and all the Casimirs, it is central and can be treated as a constant. Taking $|x_{[n]}|^2 = 1$, it is straightforward to see from (7.1) and (7.16) that

$$S_{[n]}^2 - S_{[n]} - \frac{(n-1)(n-3)}{4} = H. \tag{7.17}$$

Hence, a quadratic combination of the sCasimir of n copies of $\mathfrak{osp}(1|2)$ yields the Hamiltonian of the superintegrable model presented in section 1 and the $\mathfrak{osp}(1|2)$ Casimirs Q_A will be its symmetries since $[Q_A, H] = 0$ for $A \subset [n]$ in view of (7.17). This makes the Bannai-Ito algebra the symmetry algebra of the model since it is precisely defined as the algebra generated by the $\mathfrak{osp}(1|2)$ intermediate Casimir operators Q_A given by (7.13) and (7.14). The defining relations have been obtained [19] and read:

$$\{Q_A, Q_B\} = Q_{(A\cup B)\setminus(A\cap B)} + 2Q_{A\cap B}Q_{A\cup B} + 2Q_{A\setminus(A\cap B)}Q_{B\setminus(A\cap B)}, \tag{7.18}$$

where $A, B \subset [n]$ and $Q_\emptyset = -1/2$. When $n = 3$, let $K_1 = Q_{\{1,2\}}, K_2 = Q_{\{2,3\}}$ and $K_3 = Q_{\{1,3\}}$. The recurrence relations (7.18) can then be rewritten as

$$\{K_1, K_2\} = K_3 + \omega_3, \quad \{K_2, K_3\} = K_1 + \omega_1, \quad \{K_3, K_1\} = K_2 + \omega_2, \tag{7.19}$$

where $\omega_1, \omega_2, \omega_3$ are central elements given by

$$\omega_1 = 2Q_3Q_{123} + 2Q_1Q_2, \quad \omega_2 = 2Q_1Q_{123} + 2Q_2Q_3, \quad \omega_3 = 2Q_2Q_{123} + 2Q_1Q_3. \tag{7.20}$$

This corresponds to the Bannai-Ito algebra already seen [15] to be the symmetry algebra of the S^2 version of the Hamiltonian H given in (7.1).

7.4. Wavefunctions

We shall indicate in this section how the separated wavefunctions of H can be obtained by exploiting the symmetries that have been exhibited. To that end we shall design in this scalar context an extension map of the Cauchy-Kowalevskaia type that is formulated in terms of Dunkl operators. In order to make these operators appear we shall first perform the gauge transformation

$$z \to \tilde{z} \equiv G(\vec{s})^{-1} z G(\vec{s}), \qquad G(\vec{s}) = \prod_{i=1}^{n} |s_i|^{\mu_i}, \tag{7.21}$$

where z is any operator and $\vec{s} \equiv (s_1, s_2, \ldots, s_n)$. Under this transformation, the $\mathfrak{osp}(1|2)$ generators of the realization (7.7) become

$$\begin{aligned} \tilde{D}_i &= \partial_{s_i} + \frac{\mu_i}{s_i}(1 - R_i), & \tilde{D}_i^2 &= \tilde{D}_i \tilde{D}_i, \\ \tilde{x}_i &= x_i = s_i, & |\tilde{x}_i|^2 &= s_i^2, \\ \tilde{E}_i &= s_i \partial_{s_i} + \gamma_i, & \tilde{R}_i &= R_i, \\ \tilde{S}_i &= -\mu_i R_i, & \tilde{Q}_i &= \mu_i, \end{aligned} \tag{7.22}$$

where

$$\gamma_A = \sum_{i \in A} (\mu_i + \tfrac{1}{2}). \tag{7.23}$$

These operators also verify (7.6) and correspond to the realization of $\mathfrak{osp}(1|2)$ (or equivalently of $sl_{-1}(2)$) associated to the one-dimensional parabose oscillator [23]. The construction (7.13) can be repeated to obtain operators $\tilde{D}_A, \tilde{x}_A, \tilde{E}_A, \tilde{S}_A$ and $\tilde{Q}_A$ that are gauge equivalent to those without tildes. We can hence obtain the eigenvalues and eigenfunctions of H by finding those of $\tilde{S}_{[n]}$. Note that since $\tilde{S}_{[n]}$ commutes with $P = \prod_{i=1}^{n} R_i$, this is equivalent to finding the eigenfunctions of $\tilde{Q}_{[n]}$.

We now wish to obtain the polynomial eigenfunctions of $\tilde{S}_{[n]}$. Denote by $\mathcal{P}_m(\mathbb{R}^n)$ the space of homogeneous polynomials of degree m in the variables $s_1, s_2, \ldots, s_n$ and define $\mathcal{K}_m(\mathbb{R}^n)$ by

$$\mathcal{K}_m(\mathbb{R}^n) = \ker \tilde{D}_{[n]} \cap \mathcal{P}_m(\mathbb{R}^n) \tag{7.24}$$

with $\ker \tilde{D}_{[n]}$ the set of null-eigenfunctions of $\tilde{D}_{[n]}$. $\mathcal{K}_m(\mathbb{R}^n)$ is an eigenspace of $\tilde{S}_{[n]}$. Indeed, take $\psi_m \in \mathcal{K}_m(\mathbb{R}^n)$, one has

$$\begin{aligned}
\tilde{S}_{[n]}\tilde{\psi}_m &= \tfrac{1}{2}(\tilde{D}_{[n]}\tilde{x}_{[n]} - \tilde{x}_{[n]}\tilde{D}_{[n]} - 1)\tilde{\psi}_m = \tfrac{1}{2}(\tilde{D}_{[n]}\tilde{x}_{[n]} - 1)\tilde{\psi}_m \\
&= \tfrac{1}{2}(\tilde{D}_{[n]}\tilde{x}_{[n]} + \tilde{x}_{[n]}\tilde{D}_{[n]} - 1)\tilde{\psi}_m = \tfrac{1}{2}(\{\tilde{x}_{[n]}, \tilde{D}_{[n]}\} - 1)\tilde{\psi}_m \\
&= \tfrac{1}{2}(2\tilde{E}_{[n]} - 1)\tilde{\psi}_m = \left[\sum_{i=1}^{n} s_i\partial_{s_i} + \gamma_{[n]} - \tfrac{1}{2}\right]\tilde{\psi}_m,
\end{aligned}$$

where we used the property $\tilde{D}_{[n]}\tilde{\psi}_m = 0$ and the commutation relations (7.6). Since $\tilde{\psi}_m$ is a homogeneous polynomial of degree m, it is an eigenfunction of the Euler operator : $\sum_{i=1}^{n} s_i\partial_{s_i}\tilde{\psi}_m = m\tilde{\psi}_m$. This implies that

$$\tilde{S}_{[n]}\tilde{\psi}_m = (m + \gamma_{[n]} - \tfrac{1}{2})\tilde{\psi}_m \tag{7.25}$$

and concludes our proof.

Our aim is thus to construct a basis for $\mathcal{K}_m(\mathbb{R}^n)$. This will be done by relying on two constructs. One is a Cauchy-Kovalevskaia (CK)-map between the space of homogeneous polynomials of degree m in $n-1$ variables $\mathcal{P}_m(\mathbb{R}^{n-1})$ and the space $\mathcal{K}_m(\mathbb{R}^n)$ of null-eigenfunctions of $\tilde{D}_{[n]}$ that are homogeneous polynomials of degree m in n variables:

$$\mathbf{CK}_{s_n}^{\mu_n} : \mathcal{P}_m(\mathbb{R}^{n-1}) \to \mathcal{K}_m(\mathbb{R}^n). \tag{7.26}$$

To construct explicitly the map $\mathbf{CK}_{s_n}^{\mu_n}$ take $p(s_1, \ldots, s_{n-1}) \in \mathcal{P}_m(\mathbb{R}^{n-1})$ and let

$$\mathbf{CK}_{s_n}^{\mu_n}[p(s_1, \ldots, s_{n-1})] = \sum_{\alpha=0}^{m} s_n^{\alpha} p_\alpha(s_1, \ldots, s_{n-1}), \tag{7.27}$$

where $p_\alpha(s_1, \ldots, s_{n-1}) \in \mathcal{P}_{m-\alpha}(\mathbb{R}^{n-1})$ and $p_0(s_1, \ldots, s_{n-1}) \equiv p(s_1, \ldots, s_{n-1})$. Demand that

$$\tilde{D}_{[n]} \sum_{\alpha=0}^{m} s_n^{\alpha} p_\alpha(s_1, \ldots, s_{n-1}) = 0 \tag{7.28}$$

and solve for the coefficients $p_\alpha(s_1, \ldots, s_{n-1})$. A straightforward calculation yields

$$\mathbf{CK}_{s_n}^{\mu_n} = \sum_{i=0}^{\infty} \frac{(-1)^i (s_n)^{2i}}{i!(\gamma_n)_i (2)^{2i}} \tilde{D}_{[n-1]}^{2i} + \sum_{i=0}^{\infty} \frac{(-1)^{i+1} (s_n)^{2i+1}}{i!(\gamma_n)_{i+1} (2)^{2i+1}} R_{[n-1]} \tilde{D}_{[n-1]}^{2i+1}, \tag{7.29}$$

where $(a)_i = a(a+1)\ldots(a+i-1)$ denotes the Pochhammer symbol. It can be shown that the resulting map is an isomorphism (the proof follows the one given in [21]).

The second construct is the Fisher decomposition which states that the space of homogeneous polynomials $\mathcal{P}_m(\mathbb{R}^n)$ can be decomposed over spaces $\mathcal{K}_l(\mathbb{R}^n)$ as follows :

$$\mathcal{P}_m(\mathbb{R}^n) = \bigoplus_{j=0}^{m} \tilde{x}_{[n]}^{j} \mathcal{K}_{m-j}(\mathbb{R}^n). \tag{7.30}$$

(This is analogous to the decomposition of $\mathcal{P}_m(\mathbb{R}^n)$ over spaces of spherical harmonics.) Upon using in alternance the Fisher decomposition and the CK map, one shows that the space $\mathcal{K}_m(\mathbb{R}^n)$ can be represented as follows :

$$\mathcal{K}_m(\mathbb{R}^n) \cong \mathbf{CK}_{s_n}^{\mu_n}\Big[\bigoplus_{j_{n-2}=0}^{m} \tilde{x}_{[n-1]}^{m-j_{n-2}} \mathbf{CK}_{s_{n-1}}^{\mu_{n-1}}\Big[\cdots \bigoplus_{j_1=0}^{j_2} \tilde{x}_{[2]}^{j_2-j_1} \mathbf{CK}_{s_2}^{\mu_2}\left[\mathcal{P}_{j_1}(\mathbb{R})\right]\Big]\Big]. \tag{7.31}$$

This implies that a basis for $\mathcal{K}_m(\mathbb{R}^n)$ is provided by the eigenfunctions $\{\tilde{\psi}_{j_1,\ldots,j_{n-1}}^{(m)}(\vec{s})\}_{\sum_{i=1}^{n-1} j_i = m}$ given by

$$\tilde{\psi}_{j_1,\ldots,j_{n-1}}^{(m)}(\vec{s}) = \mathbf{CK}_{s_n}^{\mu_n}\,[\tilde{x}_{[n-1]}^{j_{n-1}} \mathbf{CK}_{s_{n-1}}^{\mu_{n-1}}\,[\ldots \tilde{x}_{[2]}^{j_2} \mathbf{CK}_{s_2}^{\mu_2}[s_1^{j_1}]]]. \tag{7.32}$$

Different bases are obtained by permuting the order in which the CK-extensions are applied. This leads to explicit formulas. The calculation that (7.32) entails can be carried out straightforwardly with the help of the identities with $\tilde{\psi}_m \in \mathcal{K}_m(\mathbb{R}^n)$

$$\begin{aligned}
\tilde{D}_{[n]}^{2\alpha} \tilde{x}_{[n]}^{2\beta} \tilde{\psi}_m &= 2^{2\alpha}(-\beta)_\alpha (1 - m - \beta - \gamma_{[n]})_\alpha x_{[n]}^{2\beta-2\alpha} \tilde{\psi}_m,\\
\tilde{D}_{[n]}^{2\alpha} \tilde{x}_{[n]}^{2\beta+1} \tilde{\psi}_m &= 2^{2\alpha}(-\beta)_\alpha (-m - \beta - \gamma_{[n]})_\alpha x_{[n]}^{2\beta+1-2\alpha} \tilde{\psi}_m,\\
R_{[n]} \tilde{D}_{[n]}^{2\alpha+1} \tilde{x}_{[n]}^{2\beta} \tilde{\psi}_m &= -2^{2\alpha} \beta (1-\beta)_\alpha (1 - m - \beta - \gamma_{[n]})_\alpha x_{[n]}^{2\beta-2\alpha-1} R_{[n]} \tilde{\psi}_m,\\
R_{[n]} \tilde{D}_{[n]}^{2\alpha+1} \tilde{x}_{[n]}^{2\beta+1} \tilde{\psi}_m &= 2^{2\alpha+1}(-\beta)_\alpha (m + \beta + \gamma_{[n]})(1 - m - \beta - \gamma_{[n]})_\alpha x_{[n]}^{2\beta-2\alpha} R_{[n]} \tilde{\psi},
\end{aligned} \tag{7.33}$$

which follows from (7.6). The results can be presented in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ defined as [5]

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left[\begin{matrix} -n,\ n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right] \tag{7.34}$$

and with the help of the identity :

$$(x+y)^n P_n^{(\alpha,\beta)}\left(\frac{x-y}{x+y}\right) = \frac{(\alpha+1)_n}{n!} x^n\, {}_2F_1\left[\begin{matrix} -n,\ -n-\beta \\ \alpha+1 \end{matrix}; -\frac{y}{x}\right]. \tag{7.35}$$

One obtains

$$\tilde{\psi}_{j_1,\ldots,j_{n-1}}^{(m)}(\vec{s}) = \mathbf{P}_n \mathbf{P}_{n-1} \ldots \mathbf{P}_3 Q_{j_1}(s_1, s_2), \tag{7.36}$$

where $\mathbf{P}_k$ is an operator depending on the k variables $s_1, \ldots, s_k$ and on the $k-1$ parameters $j_1, \ldots, j_{k-1}$ which is given by

$$\mathbf{P}_k = \frac{c!}{(\gamma_k)_c} \left(\sum_{i=1}^{k} s_i^2 \right)^c$$
$$\times \begin{cases} P_c^{(\gamma_k-1, j_{[k-2]}+\gamma_{[k-1]}-1)}\left(\frac{s_1^2+\cdots+s_{k-1}^2-s_k^2}{s_1^2+\cdots+s_{k-1}^2+s_k^2}\right) + \frac{s_k}{s_1^2+\cdots+s_k^2} P_{c-1}^{(\gamma_k, j_{[k-2]}+\gamma_{[k-1]})}\left(\frac{s_1^2+\cdots+s_{k-1}^2-s_k^2}{s_1^2+\cdots+s_{k-1}^2+s_k^2}\right)\tilde{x}_{[k-1]}R_{[k-1]} \\ \hfill \text{if } j_{k-1} = 2c, \\ P_c^{(\gamma_k-1, j_{[k-2]}+\gamma_{[k-1]})}\left(\frac{s_1^2+\cdots+s_{k-1}^2-s_k^2}{s_1^2+\cdots+s_{k-1}^2+s_k^2}\right)\tilde{x}_{[k-1]} - \frac{j_{[k-2]}+c+\gamma_{[k-1]}}{c+\gamma_k} P_c^{(\gamma_4, j_{[k-2]}+\gamma_{[k-1]}-1)}\left(\frac{s_1^2+\cdots+s_{k-1}^2-s_k^2}{s_1^2+\cdots+s_{k-1}^2+s_k^2}\right)s_k R_{[k-1]} \\ \hfill \text{if } j_{k-1} = 2c+1 \end{cases}$$

with

$$j_{[m]} = \sum_{i=1}^{m} j_i \tag{7.37}$$

and where $Q_{j_1}(s_1, s_2)$ is the function

$$Q_{j_1}(s_1, s_2) = \frac{a!}{(\gamma_2)_a}(s_1^2+s_2^2)^a \times \begin{cases} P_a^{(\gamma_2-1,\gamma_1-1)}\left(\frac{s_1^2-s_2^2}{s_1^2+s_2^2}\right) + \frac{s_1 s_2}{s_1^2+s_2^2} P_{a-1}^{(\gamma_2,\gamma_1)}\left(\frac{s_1^2-s_2^2}{s_1^2+s_2^2}\right) & \text{if } j_1 = 2a, \\ s_1 P_a^{(\gamma_2-1,\gamma_1)}\left(\frac{s_1^2-s_2^2}{s_1^2+s_2^2}\right) - s_2 \frac{a+\gamma_1}{a+\gamma_2} P_a^{(\gamma_2,\gamma_1-1)}\left(\frac{s_1^2-s_2^2}{s_1^2+s_2^2}\right) & \text{if } j_1 = 2a+1. \end{cases}$$

Note that the expressions for $\mathbf{P}_k$ contain the operators $\tilde{x}_{[k-1]}$ and $R_{[k-1]}$ respectively. Recalling the expressions (7.22) and (7.13), it can be seen that these operators only involve the variables s_i and the reflection operators R_i with $i = 1, \ldots, k-1$. These reflections conveniently account for signs occuring in the solutions and prevent the need to give different expressions for each parity combination of the parameters $j_1, j_2, \ldots, j_{n-1}$.

By effecting the reverse gauge transformation, we thus obtain a basis for the eigenspace of the operator $S_{[n]}$ given by

$$\psi_{j_1,\ldots,j_{n-1}}^{(m)}(\vec{s}) = \tilde{\psi}_{j_1,\ldots,j_{n-1}}^{(m)}(\vec{s})G(\vec{s}), \tag{7.38}$$

where $m = 0, 1, \ldots$ and $j_1 + \cdots + j_{n-1} = m$. Given (7.25), they obey the relation

$$S_{[n]}\psi_{j_1,\ldots,j_{n-1}}^{(m)}(\vec{s}) = (m + \gamma_{[n]} - \tfrac{1}{2})\psi_{j_1,\ldots,j_{n-1}}^{(m)}(\vec{s}). \tag{7.39}$$

Recalling (7.17), this is seen to imply that

$$H\psi_{j_1,\ldots,j_{n-1}}^{(m)}(\vec{s}) = (m + \gamma_{[n]})(m + \gamma_{[n]} - 2)\psi_{j_1,\ldots,j_{n-1}}^{(m)}(\vec{s}). \tag{7.40}$$

The normalized eigenfunctions are given by

$$\Psi^{(m)}_{j_1,\ldots,j_{n-1}}(\vec{s}) = \frac{\nu_1}{\sqrt{2}}\left(\prod_{i=3}^{n}\eta_i\right)\psi^{(m)}_{j_1,\ldots,j_{n-1}}(\vec{s}), \tag{7.41}$$

where

$$\nu_1 = (\gamma_2)_a\sqrt{\frac{\Gamma(a+\gamma_{[2]})}{a!\Gamma(a+\gamma_1)\Gamma(a+\gamma_2)}}\times\begin{cases}1 & \text{if } j_1 = 2a,\\ \sqrt{\frac{a+\gamma_2}{a+\gamma_1}} & \text{if } j_1 = 2a+1,\end{cases} \tag{7.42}$$

$$\eta_k = (\gamma_k)_c\sqrt{\frac{\Gamma(c+j_{[n-2]}+\gamma_{[n]})}{c!\Gamma(c+\gamma_n)\Gamma(c+j_{[n-2]}+\gamma_{[n-1]})}}\times\begin{cases}1 & \text{if } j_{n-1} = 2c,\\ \sqrt{\frac{c+\gamma_n}{c+j_{[n-2]}+\gamma_{[n-1]}}} & \text{if } j_{n-1} = 2c+1,\end{cases} \tag{7.43}$$

so that

$$\int_{S^{n-1}}\Psi^{(m)\dagger}_{j_1,\ldots,j_{n-1}}(\vec{s})\Psi^{(m')}_{k_1,\ldots,k_{n-1}}(\vec{s})d\vec{s} = \delta_{m,m'}\delta_{j_1,k_1}\ldots\delta_{j_{n-1},k_{n-1}}. \tag{7.44}$$

This can be verified directly from the orthogonality relation of the Jacobi polynomials [5].

7.5. Conclusion

To sum up, we have introduced a new quantum superintegrable model with reflections on the $(n-1)$-sphere. Its symmetries were given explicitly and shown to realize the higher rank generalization of the Bannai-Ito algebra. It was observed that the model can be constructed through the combination of n independent realizations of the superalgebra $\mathfrak{osp}(1|2)$. A quadratic expression in the total sCasimir operator was found to coincide with the Hamiltonian while the intermediate Casimir operators were seen to form its symmetries. The exact solutions have been obtained by using a Cauchy-Kovalevskaia extension theorem.

In keeping with the 2-dimensional picture [11] the overlap between wavefunctions associated to different maximal Abelian subalgebras [19] of the Bannai-Ito algebra will be expressed in terms of multivariate Bannai-Ito polynomials that we plan to characterize in the near future.

We have stressed that the Hamiltonian H with reflections on S^{n-1} actually commutes with all reflection operators and that these can hence be diagonalized simultaneously with H. In each of the sectors with definite parity, H reduces to a scalar Hamiltonian that extends to S^{n-1} the generic model on S^2 known to have the Racah algebra as symmetry algebra. It was shown in [21] that these scalar models on S^{n-1} admit the (more involved) higher rank Racah algebra identified in the same article as the algebra generated by intermediate Casimir operators in the n-fold tensor product of realizations of $\mathfrak{sl}(2)$. This indicates that, as in the rank 1 case [16], there is an embedding of the Racah algebra in the Bannai-Ito one for higher ranks also. This rests on the fact that the

$\mathfrak{sl}(2)$ (intermediate) Casimir operators are quadratic expressions that are reflection invariant in the (intermediate) Casimir operators of $\mathfrak{osp}(1|2)$. Details will be given elsewhere.

While in two dimensions, all scalar second order superintegrable models (with Euclidean signature) can be obtained from the generic model, this is obviously not so in higher dimensions. We have here provided a superintegrable multidimensional version with reflections of the master model in two dimensions; There are however other known superintegrable models in arbitrary dimensions that do not derive from the one discussed here. Of particuliar interest are the rational Calogero models which are formulated in terms of Dunkl and reflection operators especially when distinguishable particles are considered. These models are superintegrable and their symmetries have been much studied. (See for instance [**24, 25**] among the many references on this topic.) We intend to revisit the rational Calogero model with the perspective on superintegrable systems brought in this paper.

Acknowledgments

The research of HDB is supported by the Fund for Scientific Research-Flanders (FWO-V), project "Construction of algebra realisations using Dirac-operators", grant G.0116.13N. VXG holds a postdoctoral fellowship from the Natural Science and Engineering Research Council of Canada (NSERC). JML holds an Alexander-Graham-Bell fellowship from NSERC. The research of LV is supported in part by NSERC.

References

[1] E. G. Kalnins, W. Miller, S. Post, Contractions of 2D 2nd Order Quantum Superintegrable Systems and the Askey Scheme for Hypergeometric Orthogonal Polynomials, SIGMA 9 (2013) 57–84.

[2] W. Miller, S. Post, P. Winternitz, Classical and quantum superintegrability with applications, J. Phys. A Math. Theor. 46 (42) (2013) 423001.

[3] A. Zhedanov, "Hidden" symmetry of Askey-Wilson polynomials, Theor. Math. Phys. 89 (2) (1991) 1146–1157.

[4] Y. A. Granovskii, I. Lutzenko, A. Zhedanov, Mutual integrability, quadratic algebras and dynamical symmetry, Ann. Phys. (N. Y). 217 (1) (1992) 1–20.

[5] R. Koekoek, P. A. Lesky, R. F. Swarttouw, Hypergeometric Orthogonal Polynomials, Springer, 2010.

[6] E. G. Kalnins, W. Miller, S. Post, Wilson polynomials and the generic superintegrable system on the 2-sphere, J. Phys. A Math. Theor. 40 (38) (2007) 11525.

[7] Y. A. Granovskii, A. Zhedanov, Nature of the symmetry group of the $6j$ symbol, J. Exp. Theor. Phys. 94 (10) (1988) 1982–1985.

[8] V. X. Genest, L. Vinet, A. Zhedanov, Superintegrability in Two Dimensions and the Racah–Wilson Algebra, Lett. Math. Phys. 104 (8) (2014) 931–952.

[9] V. X. Genest, L. Vinet, A. Zhedanov, The equitable Racah algebra from three $\mathfrak{su}(1,1)$ algebras, J. Phys. A Math. Theor. 47 (2) (2014) 25203.

[10] V. X. Genest, L. Vinet, A. Zhedanov, The Racah algebra and superintegrable models, J. Phys. Conf. Ser. 512 (2014) 12011.

[11] V. X. Genest, L. Vinet, A. Zhedanov, The Bannai-Ito algebra and a superintegrable system with reflections on the two-sphere, J. Phys. A Math. Theor. 47 (20) (2014) 205202.

[12] S. Tsujimoto, L. Vinet, A. Zhedanov, Dunkl shift operators and Bannai-Ito polynomials, Adv. Math. (N. Y). 229 (4) (2012) 2123–2158.

[13] E. Bannai, T. Ito, Algebraic combinatorics I. Association schemes, Benjamin/Cummings Publishing Company, 1984.

[14] V. X. Genest, L. Vinet, A. Zhedanov, The Bannai-Ito polynomials as Racah coefficients of the $sl_{-1}(2)$ algebra, Proc. Am. Math. Soc. 142 (5) (2014) 1545–1560.

[15] V. X. Genest, L. Vinet, A. Zhedanov, A Laplace-Dunkl Equation on S^2 and the Bannai–Ito Algebra, Commun. Math. Phys. 336 (1) (2015) 243–259.

[16] V. X. Genest, L. Vinet, A. Zhedanov, Embeddings of the racah algebra into the Bannai-Ito algebra, Symmetry, Integr. Geom. Methods Appl. 11.

[17] E. G. Kalnins, W. Miller Jr, S. Post, Two-Variable Wilson Polynomials and the Generic Superintegrable System on the 3-Sphere, Methods Appl. SIGMA 7 (2011) 26.

[18] H. De Bie, V. Genest, J.-M. Lemay, L. Vinet, A superintegrable model with reflections on S^3 and the rank two Bannai-Ito algebra, Acta Polytech. 56 (3).

[19] H. De Bie, V. X. Genest, L. Vinet, The Z_2^n Dirac-Dunkl operator and a higher rank Bannai-Ito algebra, Adv. Math. (N. Y). 303 (2016) 390–414.

[20] H. De Bie, V. X. Genest, W. van de Vijver, L. Vinet, Bannai-Ito algebras and the $osp(1|2)$ superalgebra, in: Phys. Math. Asp. Symmetries, Springer International Publishing, 2017, pp. 349–354.

[21] H. De Bie, V. X. Genest, W. van de Vijver, L. Vinet, A higher rank racah algebra and the laplace–dunkl operator, Journal of Physics A: Mathematical and Theoretical 51 (2) (2017) 025203.

[22] P. Iliev, The generic quantum superintegrable system on the sphere and racah operators, Letters in Mathematical Physics 107 (11) (2017) 2029–2045.

[23] V. X. Genest, M. E. Ismail, L. Vinet, A. Zhedanov, The Dunkl oscillator in the plane: I. Superintegrability, separated wavefunctions and overlap coefficients, J. Phys. A Math. Theor. 46 (14) (2013) 145201.

[24] V. B. Kuznetsov, Hidden symmetry of the quantum Calogero-Moser system, Phys. Lett. A 218 (3-6) (1996) 212–222.

[25] M. Feigin, T. Hakobyan, On Dunkl angular momenta algebra, J. High Energy Phys. 2015 (11) (2015) 107.

Chapitre 8

Convolution identities for Dunkl orthogonal polynomials from the $\mathfrak{osp}(1|2)$ Lie superalgebra

E. Koelink, J.M. Lemay et L. Vinet (2019). Convolution identities for Dunkl orthogonal polynomials from the $\mathfrak{osp}(1|2)$ Lie superalgebra. *Journal of Mathematical Physics 60 (9), 091701*

Abstract. New convolution identities for orthogonal polynomials belonging to the $q = -1$ analog of the Askey-scheme are obtained. A specialization of the Chihara polynomials will play a central role as the eigenfunctions of a special element of the Lie superalgebra $\mathfrak{osp}(1|2)$ in the positive discrete series representation. Using the Clebsch-Gordan coefficients, a convolution identity for the Specialized Chihara, the dual −1 Hahn and the Big −1 Jacobi polynomials is found. Using the Racah coefficients, a convolution identity for the Big −1 Jacobi and the Bannai-Ito polynomials is found. Finally, these results are applied to construct a bilinear generating function for the Big −1 Jacobi polynomials.

8.1. Introduction

In [1], Granovskii and Zhedanov proposed an approach to obtain convolution identities for orthogonal polynomials of the Askey-scheme through algebraic methods. The main idea is to study a self-adjoint element of a Lie algebra which corresponds to a recurrence operator diagonalized by orthogonal polynomials in a suitable representation. In the tensor product of representations, the Clebsch-Gordan decomposition and the Racah recoupling can then be used to relate polynomial eigenfunctions in two different bases to arrive at convolution identities. Van der Jeugt [2] expanded on this idea to obtain generalizations of some classical convolution identities for the Laguerre and Hermite polynomials. One of the authors then joined Van der Jeugt [3] to exploit this approach further and obtain convolution identities for the Meixner-Pollaczek, the Hahn and the Jacobi polynomials and their descendants with $\mathfrak{su}(1,1)$ as the underlying Lie algebra and also for the Al-Salam Chihara, q-Racah and Askey-Wilson polynomials using the quantized analog $U_q(\mathfrak{su}(1,1))$. Two subsequent papers [4, 5] extended this work and derived generating functions and Poisson kernels for some involved polynomials by using differential realizations of the discrete series representations of $\mathfrak{su}(1,1)$ and its q-generalization.

The main goal of this paper is to use this construction to obtain convolution identities for orthogonal polynomials belonging to the Bannai-Ito scheme of −1 orthogonal polynomials [6, 7, 8, 9, 10, 11, 12]. These polynomials arise as the $q = -1$ limits of families belonging to the Askey tableau of q-orthogonal polynomials [13]. More precisely, most of its polynomials are defined by $q \to -1$ limits of the Askey-Wilson polynomials and its descendants. The −1 orthogonal polynomials are eigenfunctions of Dunkl operators [14] which involve the reflexion operator R defined by $Rf(x) = f(-x)$ [15, 16]. For this reason, they are also called Dunkl orthogonal polynomials. The first example of such polynomials was introduced by Bannai and Ito as a $q \to -1$ limit of the q-Racah polynomials in the classification of a category of association scheme [6]. They have since been fully characterized [8] and have appeared in various context : superintegrable systems [17, 18, 19] and the transport of quantum information [20, 21, 11] for example. The Lie superalgebra $\mathfrak{osp}(1|2)$, sometimes referred to as $sl_{-1}(2)$, has been found to provide a fruitful algebraic underpinning for a number of −1 polynomials [22, 23, 24]. In particular, the Clebsch-Gordan coefficients of $\mathfrak{osp}(1|2)$ can be expressed in terms of dual −1 Hahn polynomials [25] and the Racah coefficients in terms of Bannai-Ito polynomials [26]. These two results are essential ingredients of the main results of this paper : the convolution identities given in propositions 8.4 and 8.7. The former relates the Specialized Chihara, the dual −1 Hahn and the Big −1 Jacobi polynomials and the latter connects the Big −1 Jacobi and the Bannai-Ito polynomials. These results can also be interpreted in another interesting way. They give connection coefficients for different two-variables polynomials orthogonal with respect to the same measure. This is an interesting feature as the extension to multiple variables of the Bannai-Ito scheme is in its early stages [27, 28, 29]. It is also quite remarkable to have a framework relating so many Dunkl orthogonal polynomials. As additional results, we obtain a generating function for the Specialized Chihara polynomials and follow an approach similar to the one used in [4] to obtain a bilinear generating function for the Big −1 Jacobi polynomials. The discussion in section 5 indicates that the results of [5] can be generalized to Lie superalgebra representations; the complexity of the outcome will however increase considerably.

The paper is structured as follows. The properties of the relevant −1 orthogonal polynomials are surveyed in section 2 and section 3 is dedicated to a review of the superalgebra $\mathfrak{osp}(1|2)$ and its Clebsch-Gordan and Racah coefficients. We proceed in section 4 to the construction of convolution identities. A self-adjoint element of $\mathfrak{osp}(1|2)$ is introduced and its generalized eigenvectors in the positive discrete series representation are obtained. Looking at the tensor product of irreducible representations, the Clebsch-Gordan coefficients are used to construct a first convolution identity. Next, the three-fold tensor product and the Racah coefficients are considered to obtain a second convolution identity. In section 5, we present a first application of these results and derive a bilinear generating function for the Big −1 Jacobi polynomials. Finally, some closing remarks are given in the conclusion.

8.2. Review of Dunkl orthogononal polynomials

The results we present in this paper have the notable feature of connecting various orthogonal polynomials from the Bannai-Ito scheme together. The families involved are the Specialized Chihara polynomials $P_n(\lambda;\mu,\gamma)$, the Big −1 Jacobi polynomials $J_n(x;a,b,c)$, the dual −1 Hahn polynomials $R_n(x;\eta,\xi,N)$ and finally the Bannai-Ito polynomials $B_n(x;\rho_1,\rho_2,r_1,r_2)$. We review in this section some of their properties while etablishing the notation that will be used throughout this paper.

8.2.1. Specialized Chihara polynomials

A one-parameter extension of the generalized Hermite polynomials was introduced in [12] as a special case of the Chihara polynomials which both sit in the $q=-1$ analog of the Askey scheme. We present some of their properties here with a different normalization and a different notation. For simplicity, we name them the Specialized Chihara polynomials and denote them by $P_n(\lambda;\mu,\gamma)=P_n(\lambda)$. They satisfy the 3-term recurrence relation

$$\lambda P_n(\lambda)=[n+1]_\mu^{1/2}P_{n+1}(\lambda)+\gamma(-1)^nP_n(\lambda)+[n]_\mu^{1/2}P_{n-1}(\lambda) \tag{8.1}$$

where $[n]_\mu=n+(1-(-1)^n)\mu$ denotes the μ-number. The Specialized Chihara polynomials can be expressed in terms of Laguerre polynomials in the following way :

$$\begin{aligned}
P_{2n}(\lambda;\mu,\gamma)&=(-1)^n\sqrt{\frac{n!\,\Gamma(\mu+\frac{1}{2})}{\Gamma(n+\mu+\frac{1}{2})}}\;L_n^{(\mu-\frac{1}{2})}\left(\frac{\lambda^2-\gamma^2}{2}\right),\\
P_{2n+1}(\lambda;\mu,\gamma)&=(-1)^n\sqrt{\frac{n!\,\Gamma(\mu+\frac{3}{2})}{\Gamma(n+\mu+\frac{3}{2})}}\left(\frac{\lambda-\gamma}{\sqrt{2\mu+1}}\right)L_n^{(\mu+\frac{1}{2})}\left(\frac{\lambda^2-\gamma^2}{2}\right),
\end{aligned} \tag{8.2}$$

where the Laguerre polynomials are given in terms of the usual hypergeometric function

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_1F_1\left[\begin{matrix}-n\\ \alpha+1\end{matrix};x\right] \tag{8.3}$$

with $(a)_n = a(a+1)\dots(a+n-1)$ denoting the Pochhammer symbol. The Specialized Chihara polynomials satisfy the orthogonality relation

$$\int_F P_n(\lambda)P_m(\lambda)w(\lambda,\mu,\gamma)d\lambda = 2\Gamma(\mu+\tfrac{1}{2})\delta_{n,m} \tag{8.4}$$

where $F = (-\infty, -|\gamma|\ ) \cup (\ |\gamma|, \infty)$ and the weight function is given by

$$w(\lambda,\mu,\gamma) = \mathrm{sign}(\lambda)(\lambda+\gamma)\left(\frac{\lambda^2-\gamma^2}{2}\right)^{\mu-\frac{1}{2}} e^{-\left(\frac{\lambda^2-\gamma^2}{2}\right)}. \tag{8.5}$$

This result can easily be verified from the orthogonality of the Chihara polynomials or from that of the Laguerre polynomials. They satisfy also a specialization of the differential-difference equation obeyed by the Chihara polynomials.

8.2.2. The Big -1 Jacobi polynomials

We now review some of the properties of the Big −1 Jacobi polynomials which shall be needed in the following. Denoted by $J_n(x;a,b,c)$, these polynomials are also part of the $q=-1$ analog of the Askey scheme and were introduced in [9] as a $q \to -1$ limit of the Big q-Jacobi polynomials. They are defined by

$$J_n(x;a,b,c) = \begin{cases} {}_2F_1\left[\begin{matrix}-\frac{n}{2}, \frac{n+a+b+2}{2}\\ \frac{a+1}{2}\end{matrix};\frac{1-x^2}{1-c^2}\right] + \frac{n(1-x)}{(1+c)(a+1)}\ {}_2F_1\left[\begin{matrix}1-\frac{n}{2}, \frac{n+a+b+2}{2}\\ \frac{a+3}{2}\end{matrix};\frac{1-x^2}{1-c^2}\right], & n \text{ even},\\ {}_2F_1\left[\begin{matrix}-\frac{n-1}{2}, \frac{n+a+b+1}{2}\\ \frac{a+1}{2}\end{matrix};\frac{1-x^2}{1-c^2}\right] - \frac{(n+a+b+1)(1-x)}{(1+c)(a+1)}\ {}_2F_1\left[\begin{matrix}-\frac{n-1}{2}, \frac{n+a+b+3}{2}\\ \frac{a+3}{2}\end{matrix};\frac{1-x^2}{1-c^2}\right], & n \text{ odd},\end{cases} \tag{8.6}$$

where ${}_2F_1$ is the standard Gauss hypergeometric function. We shall simply write $J_n(x)$ instead of $J_n(x;a,b,c)$ when the parameters are clear from the context. They satisfy the recurrence relation

$$x\,J_n(x) = A_n\,J_{n+1}(x) + (1-A_n-C_n)\,J_n(x) + C_n\,J_{n-1}(x),$$

with coefficients

$$A_n = \begin{cases}\frac{(n+a+1)(c+1)}{2n+a+b+2}, & n \text{ even},\\ \frac{(1-c)(n+a+b+1)}{2n+a+b+2}, & n \text{ odd},\end{cases} \qquad C_n = \begin{cases}\frac{n(1-c)}{2n+a+b}, & n \text{ even},\\ \frac{(n+b)(1+c)}{2n+a+b}, & n \text{ odd}.\end{cases}$$

It can be seen that for $a, b > -1$ and $|c| \neq 1$ the polynomials $J_n(x)$ are positive-definite. The orthogonality relation of the Big −1 Jacobi polynomials is different for $|c| < 1$ and $|c| > 1$. In what follows, we only need the polynomials for the first case. For $|c|<1$, one has

$$\int_{\mathcal{C}} J_n(x;a,b,c)\, J_m(x;a,b,c)\, \omega(x;a,b,c)\, \mathrm{d}x = \left[\frac{(1-c^2)^{\frac{a+b+2}{2}}}{(1+c)}\right] h_n(a,b)\, \delta_{nm}, \tag{8.7}$$

where the interval is $\mathcal{C}=(-1,-|c|\ )\cup(\ |c|,1)$ and the weight function reads

$$\omega(x;a,b,c) = \mathrm{sign}(x)\,(1+x)\,(x-c)\,(x^2-c^2)^{\frac{b-1}{2}}\,(1-x^2)^{\frac{a-1}{2}}. \tag{8.8}$$

The normalization factor h_n is given by

$$h_n(a,b) = \begin{cases} \frac{2\,\Gamma\left(\frac{n+b+1}{2}\right)\Gamma\left(\frac{n+a+3}{2}\right)\left(\frac{n}{2}\right)!}{(n+a+1)\,\Gamma\left(\frac{n+a+b+2}{2}\right)\left(\frac{a+1}{2}\right)^2_{\frac{n}{2}}}, & n \text{ even},\\ \frac{(n+a+b+1)\,\Gamma\left(\frac{n+b+2}{2}\right)\Gamma\left(\frac{n+a+2}{2}\right)\left(\frac{n-1}{2}\right)!}{2\,\Gamma\left(\frac{n+a+b+3}{2}\right)\left(\frac{a+1}{2}\right)^2_{\frac{n+1}{2}}}, & n \text{ odd}. \end{cases} \tag{8.9}$$

The orthogonality relation for $|c|>1$ and a difference equation can be found in [9].

8.2.3. The dual -1 Hahn polynomials

We now introduce a third family of orthogonal polynomials. The dual −1 Hahn polynomials, denoted by $R_n(x;\eta,\xi,N)$ or $R_n(x)$, depends on two real parameters η,ξ and on an integer parameter N. They have been introduced in [11] as a $q\to-1$ limit of the dual q-Hahn polynomials. They have found applications in the transport of quantum information along spin chains [20] and, of importance here, they have also been shown to arise as the Clebsch-Gordan coefficients of the Lie superalgebra $\mathfrak{osp}(1|2)$ [25]. They satisfy the 3-term recurrence relation

$$xR_n(x) = R_{n+1}(x) + ((-1)^{n+1}(2\xi+(-1)^N 2\eta)-1)R_n(x) + 4[n]_\xi[N-n+1]_\eta R_{n-1}(x). \tag{8.10}$$

They can be expressed as follows in terms of hypergeometric series. For N even, we have

$$R_n(x;\eta,\xi,N) = \begin{cases} 16^{\frac{n}{2}}\left(-\frac{N}{2}\right)_{\frac{n}{2}}\left(\frac{1-2\eta-N}{2}\right)_{\frac{n}{2}}\ {}_3F_2\left[\begin{matrix}-\frac{n}{2},\ \delta+\frac{x+1}{4},\ \delta-\frac{x+1}{4}\\ -\frac{N}{2},\ \frac{1-2\eta-N}{2}\end{matrix};1\right], & n \text{ even},\\ 16^{\frac{n-1}{2}}\left(1-\frac{N}{2}\right)_{\frac{n-1}{2}}\left(\frac{1-2\eta-N}{2}\right)_{\frac{n-1}{2}}(x+2\eta+2\xi+1)\,{}_3F_2\left[\begin{matrix}-\frac{n-1}{2},\ \delta+\frac{x+1}{4},\ \delta-\frac{x+1}{4}\\ 1-\frac{N}{2},\ \frac{1-2\eta-N}{2}\end{matrix};1\right], & n \text{ odd},\end{cases}$$

where $\delta=-\frac{\eta+\xi+N}{2}$ and, for N odd, we have

$$R_n(x;\eta,\xi,N) = \begin{cases} 16^{\frac{n}{2}}\left(\frac{1-N}{2}\right)_{\frac{n}{2}}\left(\frac{2\xi+1}{2}\right)_{\frac{n}{2}}\ {}_3F_2\left[\begin{matrix}-\frac{n}{2},\ \delta+\frac{x+1}{4},\ \delta-\frac{x+1}{4}\\ \frac{1-N}{2},\ \frac{2\xi+1}{2}\end{matrix};1\right], & n \text{ even},\\ 16^{\frac{n-1}{2}}\left(\frac{1-N}{2}\right)_{\frac{n-1}{2}}\left(\frac{2\xi+3}{2}\right)_{\frac{n-1}{2}}(x+2\xi-2\eta+1)\,{}_3F_2\left[\begin{matrix}-\frac{n-1}{2},\ \delta+\frac{x+1}{4},\ \delta-\frac{x+1}{4}\\ \frac{1-N}{2},\ \frac{2\xi+3}{2}\end{matrix};1\right], & n \text{ odd},\end{cases}$$

where $\delta = \frac{\eta+\xi+1}{2}$. The dual -1 Hahn polynomials are orthogonal with respect to the discrete measure

$$\sum_{s=0}^{N} \varpi_s(\eta,\xi,N) R_n(y_s;\eta,\xi,N) R_m(y_s,\eta,\xi,N) = \nu_n(\eta,\xi,N)\delta_{n,m} \tag{8.11}$$

with the following grid points :

$$y_s = \begin{cases} (-1)^s(2s-2\eta-2\xi-2N-1), & N \text{ even}, \\ (-1)^s(2s+2\eta+2\xi+1), & N \text{ odd}. \end{cases} \tag{8.12}$$

For N even, the weights and normalization factors are given by

$$\varpi_s(\eta,\xi,N) = \begin{cases} (-1)^{\frac{s}{2}} \dfrac{\left(-\frac{N}{2}\right)_{\frac{s}{2}} \left(-\frac{N}{2}-\eta+\frac{1}{2}\right)_{\frac{s}{2}} (-N-\eta-\xi)_{\frac{s}{2}}}{\frac{s}{2}! \left(-\frac{N}{2}-\xi+\frac{1}{2}\right)_{\frac{s}{2}} \left(-\frac{N}{2}-\eta-\xi\right)_{\frac{s}{2}}}, & s \text{ even}, \\ (-1)^{\frac{s-1}{2}} \dfrac{\left(-\frac{N}{2}\right)_{\frac{s+1}{2}} \left(-\frac{N}{2}-\eta+\frac{1}{2}\right)_{\frac{s-1}{2}} (-N-\eta-\xi)_{\frac{s-1}{2}}}{\frac{s-1}{2}! \left(-\frac{N}{2}-\xi+\frac{1}{2}\right)_{\frac{s-1}{2}} \left(-\frac{N}{2}-\eta-\xi\right)_{\frac{s+1}{2}}}, & s \text{ odd}, \end{cases} \tag{8.13}$$

$$\nu_n(\eta,\xi,N) = \begin{cases} 16^n \ \frac{n}{2}! \left(-\frac{N}{2}\right)_{\frac{n}{2}} \left(\xi+\frac{1}{2}\right)_{\frac{n}{2}} \left(\frac{-N-2\eta+1}{2}\right)_{\frac{n}{2}} \dfrac{(-N-\eta-\xi)_{\frac{N}{2}}}{\left(\frac{-N-2\xi+1}{2}\right)_{\frac{N}{2}}}, & n \text{ even}, \\ -16^n \ \frac{n-1}{2}! \left(-\frac{N}{2}\right)_{\frac{n+1}{2}} \left(\xi+\frac{1}{2}\right)_{\frac{n+1}{2}} \left(\frac{-N-2\eta+1}{2}\right)_{\frac{n-1}{2}} \dfrac{(-N-\eta-\xi)_{\frac{N}{2}}}{\left(\frac{-N-2\xi+1}{2}\right)_{\frac{N}{2}}}, & n \text{ odd}, \end{cases} \tag{8.14}$$

whileas for N odd, the weights and normalization factors are given by

$$\varpi_s(\eta,\xi,N) = \begin{cases} (-1)^{\frac{s}{2}} \dfrac{\left(\frac{1-N}{2}\right)_{\frac{s}{2}} \left(\xi+\frac{1}{2}\right)_{\frac{s}{2}} (\eta+\xi+1)_{\frac{s}{2}}}{\frac{s}{2}! \left(\eta+\frac{1}{2}\right)_{\frac{s}{2}} \left(\frac{N+3}{2}+\eta+\xi\right)_{\frac{s}{2}}}, & s \text{ even}, \\ (-1)^{\frac{s-1}{2}} \dfrac{\left(\frac{1-N}{2}\right)_{\frac{s-1}{2}} \left(\xi+\frac{1}{2}\right)_{\frac{s+1}{2}} (\eta+\xi+1)_{\frac{s-1}{2}}}{\frac{s-1}{2}! \left(\eta+\frac{1}{2}\right)_{\frac{s+1}{2}} \left(\frac{N+3}{2}+\eta+\xi\right)_{\frac{s-1}{2}}}, & s \text{ odd}, \end{cases} \tag{8.15}$$

$$\nu_n(\eta,\xi,N) = \begin{cases} 16^n \ \frac{n}{2}! \left(\frac{1-N}{2}\right)_{\frac{n}{2}} \left(\xi+\frac{1}{2}\right)_{\frac{n}{2}} \left(-\frac{N}{2}-\eta\right)_{\frac{n}{2}} \dfrac{(\eta+\xi+1)_{\frac{N+1}{2}}}{\left(\eta+\frac{1}{2}\right)_{\frac{N+1}{2}}}, & n \text{ even}, \\ -16^n \ \frac{n-1}{2}! \left(\frac{1-N}{2}\right)_{\frac{n-1}{2}} \left(\xi+\frac{1}{2}\right)_{\frac{n+1}{2}} \left(-\frac{N}{2}-\eta\right)_{\frac{n+1}{2}} \dfrac{(\eta+\xi+1)_{\frac{N+1}{2}}}{\left(\eta+\frac{1}{2}\right)_{\frac{N+1}{2}}}, & n \text{ odd}. \end{cases} \tag{8.16}$$

For future convenience, we introduce

$$z_s = (-1)^{s+N+1}(2s+2\eta+2\xi+1), \qquad \rho_s(\eta,\xi,N) = \begin{cases} \varpi_{N-s}(\eta,\xi,N), & N \text{ even}, \\ \varpi_s(\eta,\xi,N), & N \text{ odd} \end{cases} \tag{8.17}$$

which corresponds to the grid points and weights of the dual −1 Hahn polynomials with the indices reversed when N is even but not when N is odd.

8.2.4. The Bannai-Ito polynomials

We finally present a last family of orthogonal polynomials called the Bannai-Ito polynomials. They were originally discovered by Bannai and Ito [6] in their classification of $P-$ and $Q-$ polynomials association scheme which are in correspondance with orthogonal polynomials satisfying the Leonard duality property. In this original setting, they were observed to be $q \to -1$ limit of the q-Racah polynomials. They have also been shown to correspond to a $q \to -1$ limit of the Askey-Wilson polynomials [8]. The Bannai-Ito polynomials occur in a Bochner-type theorem for first order Dunkl difference operators [8] that has them at the top of the $q = -1$ analog of the q-Askey tableau. Of particular relevance to this paper, they are the Racah coefficients for the Lie superalgebra $\mathfrak{osp}(1|2)$ [26].

The monic Bannai-Ito polynomials $B_n(x; \rho_1, \rho_2, r_1, r_2)$, or $B_n(x)$ for short, depend on 4 parameters ρ_1, ρ_2, r_1, r_2 and the linear combination

$$g = \rho_1 + \rho_2 - r_1 - r_2. \tag{8.18}$$

They are symmetric with respect to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group transformations generated by $\rho_1 \leftrightarrow \rho_2$ and $r_1 \leftrightarrow r_2$. Throughout this section, it will be convenient to write integers as follows

$$n = 2n_e + n_p, \qquad n_p \in \{0, 1\}, \quad n, n_e \in \mathbb{N}. \tag{8.19}$$

The Bannai-Ito polynomials can be defined in terms of two hypergeometric functions

$$\begin{aligned} \frac{1}{\eta_n} B_n(x; \rho_1, \rho_2, r_1, r_2) &= {}_4F_3\left[\begin{matrix} -n_e,\ n_e+g+1,\ x-r_1+\frac{1}{2},\ -x-r_1+\frac{1}{2} \\ 1-r_1-r_2,\ \rho_1-r_1+\frac{1}{2},\ \rho_2-r_1+\frac{1}{2} \end{matrix}; 1\right] \\ &+ \frac{(-1)^n (n_e+n_p+gn_p)\big(x-r_1+\frac{1}{2}\big)}{\big(\rho_1-r_1+\frac{1}{2}\big)\big(\rho_2-r_1+\frac{1}{2}\big)} {}_4F_3\left[\begin{matrix} -n_e-n_p+1,\ n_e+n_p+g+1,\ x-r_1+\frac{3}{2},\ -x-r_1+\frac{1}{2} \\ 1-r_1-r_2,\ \rho_1-r_1+\frac{3}{2},\ \rho_2-r_1+\frac{3}{2} \end{matrix}; 1\right] \end{aligned} \tag{8.20}$$

with the normalization coefficients

$$\eta_n = (-1)^n \frac{(\rho_1 - r_1 + \frac{1}{2})_{n_e+n_p} (\rho_2 - r_1 + \frac{1}{2})_{n_e+n_p} (1 - r_1 - r_2)_{n_e}}{(n_e + g + 1)_{n_e+n_p}}. \tag{8.21}$$

It is also possible to express the Bannai-Ito polynomials as a linear combination of two Wilson polynomials [8].

The $B_n(x)$ satisfy the 3-term recurrence relation

$$xB_n(x) = B_{n+1}(x) + (\rho_1 - A_n - C_n)B_n(x) + A_{n-1}C_n B_{n-1}(x), \tag{8.22}$$

with the initial conditions $B_{-1}(x) = 0$ and $B_0(x) = 1$. The recurrence coefficients A_n and C_n are given by

$$A_n = \begin{cases} \dfrac{(n + 2\rho_1 - 2r_1 + 1)(n + 2\rho_1 - 2r_2 + 1)}{4(n + g + 1)}, & n \text{ even}, \\ \dfrac{(n + 2g + 1)(n + 2\rho_1 + 2\rho_2 + 1)}{4(n + g + 1)}, & n \text{ odd}, \end{cases}$$
$$C_n = \begin{cases} -\dfrac{n(n - 2r_1 - 2r_2)}{4(n + g)}, & n \text{ even}, \\ -\dfrac{(n + 2\rho_2 - 2r_2)(n + 2\rho_2 - 2r_1)}{4(n + g)}, & n \text{ odd}. \end{cases} \tag{8.23}$$

Favard's theorem states that these polynomials will be orthogonal only if they satisfy the positivity conditions $u_n = A_{n-1}C_n > 0$. Since this cannot be achieved for all $n \in \mathbb{N}$, the parameters must verify a truncation condition of the form

$$u_0 = u_{N+1} = 0. \tag{8.24}$$

The integer N is called the truncation parameter.

If these conditions are fulfilled, the Bannai-Ito polynomials $B_n(x)$ satisfy the discrete orthogonality relation

$$\sum_{k=0}^{N} w_k B_n(x_k) B_m(x_k) = h_n \delta_{nm}, \tag{8.25}$$

with respect to a positive set of weights w_k. The orthogonality grid x_k corresponds to the simple roots of the characteristic polynomial $B_{N+1}(x)$. The explicit formulas for the weight function w_k and the grid points x_k depend on the parity of N and more explicitly on the realization of the truncation condition $u_{N+1} = 0$.

If N is even, it follows from (8.23) that the condition $u_{N+1} = 0$ is tantamount to one of the following requirements associated to all possible values of j and ℓ :

$$i)\ r_j - \rho_\ell = \frac{N+1}{2}, \qquad j, \ell \in \{1, 2\}. \tag{8.26}$$

Note that the four possibilities coming from the choices of j and ℓ are equivalent since the polynomials $B_n(x)$ are invariant under the exchanges $\rho_1 \leftrightarrow \rho_2$ and $r_1 \leftrightarrow r_2$.

If N is odd, it follows from (8.23) that the condition $u_{N+1} = 0$ is equivalent to one of the following restrictions:

$$ii)\ \rho_1 + \rho_2 = -\frac{N+1}{2}, \qquad iii)\ r_1 + r_2 = \frac{N+1}{2}, \qquad iv)\ \rho_1 + \rho_2 - r_1 - r_2 = -\frac{N+1}{2}. \tag{8.27}$$

In this paper, we shall only be concerned with the truncation conditions $r_2 - \rho_1 = \frac{N+1}{2}$ when N is even and $\rho_1 + \rho_2 = -\frac{N+1}{2}$ when N is odd. In these cases, the grid points have the expression

$$x_k = (-1)^k(k/2 + \rho_1 + 1/4) - 1/4, \tag{8.28}$$

for $k = 0, \ldots, N$ and using (8.19) the weights take the form

$$w_k = \frac{(-1)^k}{k_e!}\frac{(\rho_1 - r_1 + 1/2)_{k_e+k_p}(\rho_1 - r_2 + 1/2)_{k_e+k_p}(\rho_1 + \rho_2 + 1)_{k_e}(2\rho_1 + 1)_{k_e}}{(\rho_1 + r_1 + 1/2)_{k_e+k_p}(\rho_1 + r_2 + 1/2)_{k_e+k_p}(\rho_1 - \rho_2 + 1)_{k_e}}, \tag{8.29}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol. When N is even, the normalization factors are given by

$$h_n = \frac{n_e!N_e!(1+2\rho_1)_{N_e}(1+\rho_1+\rho_2)_{n_e}(1+n_e+g)_{N_e-n_e}(\frac{1}{2}+\rho_1-r_1)_{n_e+n_p}(\frac{1}{2}+\rho_2-r_1)_{n_e+n_p}}{(N_e-n_e-n_p)!(\frac{1}{2}+\rho_1+r_1)_{N_e-n_e}(\frac{1}{2}+n_e+n_p+\rho_2-r_2)_{N_e-n_e-n_p}(1+n+g)^2_{n_e+n_p}} \tag{8.30}$$

and, for N odd, they are instead given by

$$h_n = \frac{n_e!N_e!(1+2\rho_1)_{N_e+1}(1-r_1-r_2)_{n_e}(1+n_e+g)_{N_e+1-n_e}(\frac{1}{2}+\rho_1-r_1)_{n_e+n_p}(\frac{1}{2}+\rho_1-r_2)_{n_e+n_p}}{(N_e-n_e)!(\frac{1}{2}+\rho_1+r_1)_{N_e+1-n_e-n_p}(\frac{1}{2}+n_e+n_p+\rho_2-r_2)_{N_e+1-n_e-n_p}(1+n+g)^2_{n_e+n_p}}. \tag{8.31}$$

The Bannai-Ito polynomials also verify a difference equation. It was shown in [8] that in fact they diagonalize the most general first order Dunkl difference operator with orthogonal polynomials as eigenfunctions.

8.3. The $\mathfrak{osp}(1|2)$ Lie Superalgebra

This section describes the key entity upon which our study rests namely the Lie superalgebra $\mathfrak{osp}(1|2)$. This superalgebra possesses one even generator J_0 and two odd generators $J_\pm$. The $\mathbb{Z}_2$-grading will be encoded with the help of an involution operator R. The defining relations are

$$[J_0, J_\pm] = \pm J_\pm, \quad \{J_+, J_-\} = 2J_0, \quad [R, J_0] = \{R, J_\pm\} = 0, \quad R^2 = 1 \tag{8.32}$$

where $[A,B] = AB - BA$ and $\{A,B\} = AB + BA$ are the usual commutator and anticommutator. There is a Casimir operator

$$Q = (J_+J_- - J_0 + \tfrac{1}{2})R \tag{8.33}$$

which commutes with all the generators. Moreover, $\mathfrak{osp}(1|2)$ also admits a Hopf algebra structure. In particular, there is an algebra morphism called the coproduct defined on the universal enveloping algebra $\Delta : U(\mathfrak{osp}(1|2)) \to U(\mathfrak{osp}(1|2)) \otimes U(\mathfrak{osp}(1|2))$ which acts as follows

$$\Delta(J_\pm) = J_\pm \otimes R + 1 \otimes J_\pm, \quad \Delta(J_0) = J_0 \otimes 1 + 1 \otimes J_0, \quad \Delta(R) = R \otimes R \tag{8.34}$$

and a $*$-structure given by $J_\pm^* = J_\mp, J_0^* = J_0$ and $R^* = R$. The Hilbert space $\ell^2(\mathbb{Z}_+)$ equipped with the orthonormal basis $e_n^{(\mu,\epsilon)}, n = 0, 1, \dots$ supports irreducible representations (μ, ϵ) of $\mathfrak{osp}(1|2)$ indexed by two parameters $\mu > 0$ and $\epsilon = \pm 1$:

$$
\begin{aligned}
J_0 e_n^{(\mu,\epsilon)} &= (n + \mu + \tfrac{1}{2}) e_n^{(\mu,\epsilon)}, \qquad & R e_n^{(\mu,\epsilon)} &= \epsilon(-1)^n e_n^{(\mu,\epsilon)}, \\
J_+ e_n^{(\mu,\epsilon)} &= [n+1]_\mu^{1/2} e_{n+1}^{(\mu,\epsilon)}, \qquad & J_- e_n^{(\mu,\epsilon)} &= [n]_\mu^{1/2} e_{n-1}^{(\mu,\epsilon)}
\end{aligned}
\tag{8.35}
$$

where $[n]_\mu = n + (1 - (-1)^n)\mu$ denotes again the μ-number. It is called the positive discrete series representation and the decomposition into irreducible components of the tensor product of two such representations is given by

$$
(\mu_1, \epsilon_1) \otimes (\mu_2, \epsilon_2) = \bigoplus_{j=0}^{\infty} \Big(\mu_1 + \mu_2 + \tfrac{1}{2} + j,\ \epsilon_1 \epsilon_2 (-1)^j \Big). \tag{8.36}
$$

This Clebsch-Gordan decomposition series implies that there is a unitary transformation between the direct product and direct sum bases of the representations involved :

$$
e_N^{(\mu_{12},\epsilon_{12})} = \sum_{n_1+n_2=N+j} C_{n_1,n_2}^{N,j} e_{n_1}^{(\mu_1,\epsilon_1)} \otimes e_{n_2}^{(\mu_2,\epsilon_2)} \tag{8.37}
$$

where

$$
\mu_{12} = \mu_1 + \mu_2 + j + \tfrac{1}{2}, \quad \epsilon_{12} = \epsilon_1 \epsilon_2 (-1)^j, \quad j = 0, 1, 2, \dots \tag{8.38}
$$

The Clebsch-Gordan coefficients of $\mathfrak{osp}(1|2)$ are given in terms of the dual -1 Hahn polynomials [25] by

$$
C_{n_1,n_2}^{N,j} = \frac{(-1)^{\phi(n_1,n_2,j)}}{(2\epsilon_2)^{n_1}} \sqrt{\frac{[n_2]_{\mu_2}!\ \rho_j(\mu_2, \mu_1, n_1 + n_2)}{[n_1]_{\mu_1}! [n_1 + n_2]_{\mu_2}!\ \nu_0(\mu_2, \mu_1, n_1 + n_2)}}\ R_{n_1}(z_j; \mu_2, \mu_1, n_1 + n_2) \tag{8.39}
$$

with the μ-factorial defined by $[n]_\mu! = [1]_\mu [2]_\mu \dots [n]_\mu$. Here, we fix the phase factors to be

$$
\phi(n_1, n_2, j) = \frac{n_1(n_1 - 1)}{2} + \frac{j(j+1)}{2} + n_1(n_1 + n_2 + 1). \tag{8.40}
$$

The Clebsch-Gordan decomposition can also be used to recouple the multiple tensor product of irreducible representations in a pairwise fashion. For instance, when considering the three-fold tensor product $(\mu_1, \epsilon_1) \otimes (\mu_2, \epsilon_2) \otimes (\mu_3, \epsilon_3)$ there are two standard ways of doing this. On the one hand, one can decompose the first two spaces into irreducible representations, and then couple the resulting spaces with the third component :

$$
(\mu_1, \epsilon_1) \otimes (\mu_2, \epsilon_2) \otimes (\mu_3, \epsilon_3) = \bigoplus_{j_{12}=0}^{\infty} (\mu_{12}, \epsilon_{12}) \otimes (\mu_3, \epsilon_3) = \bigoplus_{j_{12}=0}^{\infty} \bigoplus_{j_{(12)3}=0}^{\infty} (\mu_{123}, \epsilon_{123}). \tag{8.41}
$$

On the other hand, it is possible to combine the last two spaces first and to bring in the first component subsequently :

$$(\mu_1, \epsilon_1) \otimes (\mu_2, \epsilon_2) \otimes (\mu_3, \epsilon_3) = \bigoplus_{j_{23}=0}^{\infty} (\mu_1, \epsilon_1) \otimes (\mu_{23}, \epsilon_{23}) = \bigoplus_{j_{23}=0}^{\infty} \bigoplus_{j_{1(23)}=0}^{\infty} (\mu_{123}, \epsilon_{123}). \tag{8.42}$$

Focusing on the parameters μ, the following relations stem from the repeated use of (8.38) :

$$\mu_{123} = \mu_1 + \mu_2 + \mu_3 + 1 + j_{123}, \tag{8.43}$$

$$= \mu_{12} + \mu_3 + j_{(12)3}, \tag{8.44}$$

$$= \mu_1 + \mu_{23} + j_{1(23)} \tag{8.45}$$

and

$$\mu_{12} = \mu_1 + \mu_2 + j_{12}, \qquad \mu_{23} = \mu_2 + \mu_3 + j_{23}. \tag{8.46}$$

Analogous relations can be found for the parameters ϵ. These equations imply that the five decomposition integers j are constrained :

$$j_{123} = j_{1(23)} + j_{23} = j_{(12)3} + j_{12}. \tag{8.47}$$

While only three decomposition integers are independent, it will be convenient to use all five to simplify the notation especially when dealing with indices. Now, to each of the two decomposition schemes one can associate a basis. To the scheme (8.41) corresponds

$$f_{n_{123}}^{j_{123},j_{12}} = \sum_{n_{12}+n_3} C_{n_{12},n_3}^{n_{123},j_{(12)3}} e_{n_{12}}^{(\mu_{12},\epsilon_{12})} \otimes e_{n_3}^{(\mu_3,\epsilon_3)} \tag{8.48}$$

$$= \sum_{n_{12}+n_3} \sum_{n_1+n_2} C_{n_{12},n_3}^{n_{123},j_{(12)3}} C_{n_1,n_2}^{n_{12},j_{12}} \; e_{n_1}^{(\mu_1,\epsilon_1)} \otimes e_{n_2}^{(\mu_2,\epsilon_2)} \otimes e_{n_3}^{(\mu_3,\epsilon_3)} \tag{8.49}$$

where the sums run over $n_{123} + j_{123} = n_{12} + n_3$, $n_{12} + j_{12} = n_1 + n_2$ and to (8.42), the basis

$$g_{n_{123}}^{j_{123},j_{23}} = \sum_{n_1+n_{23}} C_{n_1,n_{23}}^{n_{123},j_{1(23)}} e_{n_1}^{(\mu_1,\epsilon_1)} \otimes e_{n_{23}}^{(\mu_{23},\epsilon_{23})} \tag{8.50}$$

$$= \sum_{n_1+n_{23}} \sum_{n_2+n_3} C_{n_1,n_{23}}^{n_{123},j_{1(23)}} C_{n_2,n_3}^{n_{23},j_{23}} \; e_{n_1}^{(\mu_1,\epsilon_1)} \otimes e_{n_2}^{(\mu_2,\epsilon_2)} \otimes e_{n_3}^{(\mu_3,\epsilon_3)} \tag{8.51}$$

where $n_{123} + j_{123} = n_1 + n_{23}$ and $n_{23} + j_{23} = n_2 + n_3$. The connection coefficients for these two bases are called the Racah coefficients. Explicitly,

$$f_{n_{123}}^{j_{123},j_{12}} = \sum_{j_{23}=0}^{j_{123}} \mathcal{R}_{j_{12},j_{23},j_{123}}^{\mu_1,\mu_2,\mu_3} g_{n_{123}}^{j_{123},j_{23}}. \tag{8.52}$$

It has been shown in [26] that the Racah coefficients for the $\mathfrak{osp}(1|2)$ Lie superalgebra can be expressed in terms of Bannai-Ito polynomials given in (8.20). One has

$$\mathcal{R}^{\mu_1,\mu_2,\mu_3}_{j_{12},j_{23},j_{123}} = (-1)^{\varphi}\epsilon_3^{j_{12}}\sqrt{\frac{w_{j_{23}}}{h_{j_{12}}}}\ B_{j_{12}}\left(x_{j_{23}};\tfrac{\mu_2+\mu_3}{2},\tfrac{\mu_1+(-1)^{j_{123}}\mu_{123}}{2},\tfrac{\mu_3-\mu_2}{2},\tfrac{(-1)^{j_{123}}\mu_{123}-\mu_1}{2}\right) \tag{8.53}$$

where the x_k, w_k, h_n are given in equations (8.28–8.31) where the parameters of the Bannai-Ito polynomials $\rho_i, r_i, i = 1,2$ are assumed to be the same as in the polynomial $B_{j_{12}}(x;\rho_1,\rho_2,r_1,r_2)$ above. The choice of phase factor relevant for this paper is

$$\varphi = j_{123}\,\frac{(j_{12}-1)j_{12}}{2} + (j_{123}+1)\left(j_{23}+\frac{(j_{12}+1)j_{12}}{2}\right). \tag{8.54}$$

8.4. Convolution identities

The construction of convolution identities for orthogonal polynomials that we are proposing here will proceed along the following lines : select an appropriate self-adjoint element X from the Lie superalgebra, construct its generalized eigenvectors in a given representation and study their overlaps in the tensor product of representations. In order to obtain orthogonal polynomials, this Lie superalgebra element should act as a three-term recurrence operator in the chosen representation. Furthermore, the chosen element should generate a coideal subalgebra to ensure proper behavior under tensor product of representations. The self-adjoint element we consider here is

$$X_c = J_+ + J_- + cR \tag{8.55}$$

in the $\mathfrak{osp}(1|2)$ Lie superalgebra depending a single parameter $c \in \mathbb{R}$. While this is not the most general self-adjoint element in $\mathfrak{osp}(1|2)$, the addition of a J_0 term would break the coideal property given in equation (8.65). Indeed, this property is non-trivial for algebras with a twisted coproduct such as (8.34). See [3] for example, where this construction is done for $\mathfrak{su}(1,1)$ and $U_q(\mathfrak{su}(1,1))$ where the coproduct is untwisted in the first case and twisted in the second.

With the element X_c at hand, we shall study its discrete series representations and compute its generalized eigenvectors in the next subsection. In the subsequent ones, we will respectively consider the two-fold and three-fold tensor product of irreducible representations and derive two convolution identities.

8.4.1. Action of X_c in the positive discrete series representation

The operator X_c has a tridiagonal structure on the representation space (μ,ϵ) defined in (8.35). One has

$$X_c e_n^{(\mu,\epsilon)} = [n+1]_\mu^{1/2} e_{n+1}^{(\mu,\epsilon)} + c\epsilon(-1)^n e_n^{(\mu,\epsilon)} + [n]_\mu^{1/2} e_{n-1}^{(\mu,\epsilon)}. \tag{8.56}$$

Let v^c_λ denote the eigenvector with eigenvalue λ of X_c. Then, there is an expansion of the form

$$v^c_\lambda = \sum_{n=0}^{\infty} a_n e_n^{(\mu,\epsilon)}, \qquad a_n \in \mathbb{R}. \tag{8.57}$$

Acting on both sides of this equation with the operator X_c gives the following 3-term recurrence relation on the expansion coefficients a_n :

$$\lambda a_n = [n+1]_\mu^{1/2} a_{n+1} + c\epsilon(-1)^n a_n + [n]_\mu^{1/2} a_{n-1}. \tag{8.58}$$

There is a solution of the form $a_n = P_n(\lambda)\cdot a_0$ where the $P_n(\lambda)$ are orthogonal polynomials satisfying the recurrence relation

$$\lambda P_n(\lambda) = [n+1]_\mu^{1/2} P_{n+1}(\lambda) + c\epsilon(-1)^n P_n(\lambda) + [n]_\mu^{1/2} P_{n-1}(\lambda). \tag{8.59}$$

Comparing this equation with the recurrence relation (8.1), one sees directly that the $P_n(\lambda)$ are Specialized Chihara polynomials with the following parameters :

$$P_n(\lambda) = P_n(\lambda; \mu, c\epsilon). \tag{8.60}$$

Taking into account the normalization factor in the orthogonality relation (8.4), the polynomials $P_n(\lambda; \mu, c\epsilon)$ are thus *orthonormal* with respect to the weight function

$$W(\lambda, \mu, c\epsilon) = \frac{w(\lambda, \mu, c\epsilon)}{2\Gamma(\mu + \frac{1}{2})} \tag{8.61}$$

on the interval $F = (-\infty, |c|\ ) \cup (\ |c|, \infty)$ where the $w(\lambda, \mu, c\epsilon)$ are given in (8.5). If one asks that the eigenvectors v^c_λ be orthonormal, it is easy to see that this implies $a_0 = 1$ and that the generalized eigenvectors of X_c are

$$v^c_\lambda = \sum_{n=0}^{\infty} P_n(\lambda; \mu, c\epsilon)\ e_n^{(\mu,\epsilon)}. \tag{8.62}$$

Note that the series in (8.62) does not converge in the representation space and should be considered as a formal eigenvector. We reformulate this result in the following proposition.

Proposition 8.1. *The unitary operator*

$$\begin{aligned} \Lambda: \quad & \ell^2(\mathbb{Z}_+) \to L^2(F, W(\lambda, \mu, c\epsilon)) \\ & e_n^{(\mu,\epsilon)} \quad \mapsto P_n(\lambda, \mu, c\epsilon) \end{aligned} \tag{8.63}$$

is an intertwiner of the operator M_λ which denotes multiplication by λ on $L^2(F, W(\lambda, \mu, c\epsilon))$ and the operator X_c acting in $\ell^2(\mathbb{Z}_+)$:

$$M_\lambda \Lambda = \Lambda X_c. \tag{8.64}$$

Proof. The unitarity of Λ is checked from the fact that it maps an orthonormal basis onto another one. The intertwining relation derives directly from the above computation. $\square$

8.4.2. Action on the tensor product space

We now wish to study the action of the coproduct of X_c on a tensor product of irreducible representations $(\mu_1, \epsilon_1) \otimes (\mu_2, \epsilon_2)$ and obtain its generalized eigenvectors. A straightforward computation yields

$$\Delta(X_c) = X_c \otimes R + 1 \otimes X_0 \tag{8.65}$$

where X_0 denotes the operator X_c with its parameter set to zero. In view of how it acts on the first space in the tensor product, it is natural to study the action of $\Delta(X_c)$ on vectors of the following form :

$$\Delta(X_c) v^c_{\lambda_1} \otimes e^{(\mu_2,\epsilon_2)}_{n_2} = v^c_{\lambda_1} \otimes (\lambda_1 R + X_0) e^{(\mu_2,\epsilon_2)}_{n_2} \tag{8.66}$$

$$= v^c_{\lambda_1} \otimes X_{\lambda_1} e^{(\mu_2,\epsilon_2)}_{n_2} \tag{8.67}$$

where $v^c_{\lambda_1}$ is an eigenvector of X_c given by (8.62) and X_{λ_1} is the operator (8.55) with parameter λ_1. It follows that the generalized eigenvectors of $\Delta(X_c)$ are

$$v^c_{\lambda_1,\lambda_2} = \sum_{n_1,n_2=0}^{\infty} P_{n_1}(\lambda_1; \mu_1, c\epsilon_1) P_{n_2}(\lambda_2; \mu_2, \lambda_1\epsilon_2)\ e^{(\mu_1,\epsilon_1)}_{n_1} \otimes e^{(\mu_2,\epsilon_2)}_{n_2} \tag{8.68}$$

with eigenvalues λ_2. This allows us to establish the following proposition.

Proposition 8.2. *The unitary operator*

$$\begin{aligned} \Upsilon: \quad & \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+) \to L^2(G, W(\lambda_1, \mu_1, c\epsilon_1) W(\lambda_2, \mu_2, \lambda_1\epsilon_2)) \\ & e^{(\mu_1,\epsilon_1)}_{n_1} \otimes e^{(\mu_2,\epsilon_2)}_{n_2} \quad \mapsto P_{n_1}(\lambda_1; \mu_1, c\epsilon_1) P_{n_2}(\lambda_2; \mu_2, \lambda_1\epsilon_2) \end{aligned} \tag{8.69}$$

with

$$G = \{(\lambda_1, \lambda_2) \in \mathbb{R} \mid |\lambda_2| > |\lambda_1| > |c|\}$$

is an intertwiner of the operator M_{λ_2} on $L^2(G, W(\lambda_1, \mu_1, c\epsilon_1) W(\lambda_2, \mu_2, \lambda_1\epsilon_2))$ denoting multiplication by λ_2 and the operator $\Delta(X_c)$ acting in $\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$:

$$M_{\lambda_2} \Upsilon = \Upsilon \Delta(X_c). \tag{8.70}$$

Proof. The proof is similar to that of the previous proposition. Unitarity follows from the mapping of an orthonormal basis onto another one and the intertwining relation is a restatement of the eigenvectors computed above. $\square$

Now, in view of the Clebsch-Gordan decomposition (8.36) of $(\mu_1, \epsilon_1) \otimes (\mu_2, \epsilon_2)$ into irreducible representations, there exists another orthonormal basis $e_N^{(\mu_{12},\epsilon_{12})}$ with $N = 0, 1, \dots$ and

$$\mu_{12} = \mu_1 + \mu_2 + j + \tfrac{1}{2}, \quad \epsilon_{12} = \epsilon_1\epsilon_2(-1)^j, \quad j = 0, 1, \dots \tag{8.71}$$

where j labels the irreducible subspaces of the form $(\mu_{12}, \epsilon_{12})$ in the tensor product space. This basis is often referred to as the *coupled basis*, while the basis $e_{n_1}^{(\mu_1,\epsilon_1)} \otimes e_{n_2}^{(\mu_2,\epsilon_2)}$ is called the *uncoupled basis*. The operator Υ also maps the coupled basis to a set of orthonormal polynomials in $L^2(G, W(\lambda_1, \mu_1, c\epsilon_1)W(\lambda_2, \mu_2, \lambda_1\epsilon_2))$.

Proposition 8.3. *In $L^2(G, W(\lambda_1, \mu_1, c\epsilon_1)W(\lambda_2, \mu_2, \lambda_1\epsilon_2))$, we have*

$$\Upsilon e_N^{(\mu_{12},\epsilon_{12})}(\lambda_1, \lambda_2) = P_N(\lambda_2; \mu_{12}, c\epsilon_{12})\Upsilon e_0^{(\mu_{12},\epsilon_{12})}(\lambda_1, \lambda_2), \tag{8.72}$$

$$\Upsilon e_0^{(\mu_{12},\epsilon_{12})}(\lambda_1, \lambda_2) = K_j(\lambda_2; \mu_2, \mu_1; c)\ J_j\left(\frac{\epsilon_2\lambda_1}{\lambda_2};\ 2\mu_2, 2\mu_1, -\frac{c\epsilon_1\epsilon_2}{\lambda_2}\right) \tag{8.73}$$

with

$$K_j(\lambda_2; \mu_2, \mu_1; c) = \sqrt{\left(\frac{\lambda_2^2 - c^2}{2}\right)^j \left(\frac{\lambda_2 - c\epsilon_1\epsilon_2}{\lambda_2 - (-1)^j c\epsilon_1\epsilon_2}\right) \frac{\Gamma(\mu_1 + \frac{1}{2})\Gamma(\mu_2 + \frac{1}{2})}{\Gamma(\mu_{12} + \frac{1}{2})h_j(2\mu_2, 2\mu_1)}} \tag{8.74}$$

where the notation of section 2 for the Specialized Chihara and the Big −1 Jacobi polynomials is assumed.

Proof. Use the intertwining relation (8.70) to write

$$\lambda_2\Upsilon e_N^{(\mu_{12},\epsilon_{12})} = M_{\lambda_2}\Upsilon e_N^{(\mu_{12},\epsilon_{12})} = \Upsilon\Delta(X_c)e_N^{(\mu_{12},\epsilon_{12})}. \tag{8.75}$$

The action of $\Delta(X_c)$ on the irreducible spaces $(\mu_{12}, \epsilon_{12})$ is given by the relations (8.35). This yields a 3-term recurrence relation on N of the same form as (8.58). Its solution is (8.72) where the initial condition $\Upsilon e_0^{(\mu_{12},\epsilon_{12})}(\lambda_1, \lambda_2)$ remains to be determined. To obtain it, note that the orthonormality of the vectors and the unitarity of Υ implies the following. Using the notation (8.71) and $\tilde{\mu}_{12} = \mu_1 + \mu_2 + \frac{1}{2} + \tilde{j}$, $\tilde{\epsilon}_{12} = \epsilon_1\epsilon_2(-1)^{\tilde{j}}$, one has

$$\begin{aligned}\delta_{N,\tilde{N}}\delta_{j,\tilde{j}} &= \langle e_N^{(\mu_{12},\epsilon_{12})}, e_{\tilde{N}}^{(\tilde{\mu}_{12},\tilde{\epsilon}_{12})}\rangle = \langle \Upsilon e_N^{(\mu_{12},\epsilon_{12})}, \Upsilon e_{\tilde{N}}^{(\tilde{\mu}_{12},\tilde{\epsilon}_{12})}\rangle \\ &= \iint_G P_N(\lambda_2; \mu_{12}, c\epsilon_{12})P_{\tilde{N}}(\lambda_2; \tilde{\mu}_{12}, c\tilde{\epsilon}_{12})\Upsilon e_0^{(\mu_{12},\epsilon_{12})}(\lambda_1, \lambda_2)\Upsilon e_0^{(\tilde{\mu}_{12},\tilde{\epsilon}_{12})}(\lambda_1, \lambda_2) \\ &\qquad \times W(\lambda_1, \mu_1, c\epsilon_1)W(\lambda_2, \mu_2, \lambda_1\epsilon_2)d\lambda_1 d\lambda_2.\end{aligned} \tag{8.76}$$

Integrate first on λ_1 :

$$\begin{aligned}\delta_{N,\tilde{N}}\delta_{j,\tilde{j}} = &\int_{G_2} P_N(\lambda_2;\mu_{12},c\epsilon_{12})P_{\tilde{N}}(\lambda_2;\tilde{\mu}_{12},c\tilde{\epsilon}_{12}) \\ &\times \int_{G_1} \Upsilon e_0^{(\mu_{12},\epsilon_{12})}(\lambda_1,\lambda_2)\Upsilon e_0^{(\tilde{\mu}_{12},\tilde{\epsilon}_{12})}(\lambda_1,\lambda_2)W(\lambda_1,\mu_1,c\epsilon_1)W(\lambda_2,\mu_2,\lambda_1\epsilon_2)d\lambda_1 d\lambda_2\end{aligned} \tag{8.77}$$

where

$$G_1 = (-\lambda_2,-|c|)\cup(|c|,\lambda_2), \qquad G_2 = (-\infty,-|c|)\cup(|c|,\infty). \tag{8.78}$$

If we consider the special case $j=\tilde{j}$, the inner integral on λ_1 must correspond to the orthogonality measure of the Specialized Chihara polynomials $P_N(\lambda_2;\mu_{12},c\epsilon_{12})$ since the corresponding moment problem is determined. This can be checked from the divergence of the series $\sum_{n=1}^{\infty}[n]_\mu^{-1/2}$ which satisfies one of Carleman's conditions for determinacy [30]. Furthermore, applying Υ on both sides of the Clebsch-Gordan decomposition (8.37) with $N=0$, one can deduce that $\Upsilon e_0^{(\mu_{12},\epsilon_{12})}$ is a polynomial of degree j in the variables λ_1 and λ_2. Taking these two observations into account, it is possible to identify $\Upsilon e_0^{(\mu_{12},\epsilon_{12})}$ in terms of Big -1 Jacobi polynomials. Indeed, setting $u=\epsilon_2\lambda_1/\lambda_2$, it is straightforward to identify the resulting weights factors with those of the Big -1 Jacobi polynomials given in (8.8) in the variable u. This gives the result (8.73). The normalization factor (8.74) is obtained by comparing the weight function in the integral on λ_2 with that of the Specialized Chihara polynomials $P_N(\lambda_2;\mu_{12},c\epsilon_{12})$ given in (8.5) and computing the integral. □

Thus, the equations (8.72) and (8.73) give us the action of Υ on the coupled basis. With these results in hand, it is now possible to obtain a convolution identity for the Specialized Chihara, Big −1 Jacobi and dual −1 Hahn polynomials.

Proposition 8.4. *In the notation of section 2 for the Specialized Chihara, Big −1 Jacobi and dual −1 Hahn polynomials, the following convolution identity holds :*

$$\begin{aligned}&K_j(\lambda_2;\mu_2,\ \mu_1;c)P_N(\lambda_2;\mu_{12},c\epsilon_{12})J_j\left(\frac{\epsilon_2\lambda_1}{\lambda_2};\ 2\mu_2,2\mu_1,-\frac{c\epsilon_1\epsilon_2}{\lambda_2}\right) \\ &\quad= \sum_{n_1+n_2=N+j}(-1)^{\phi(n_1,n_2,j)}\left(\frac{\epsilon_2}{2}\right)^{n_1}\sqrt{\frac{[n_2]_{\mu_2}!\ \rho_j(\mu_2,\mu_1,n_1+n_2)}{[n_1]_{\mu_1}![n_1+n_2]_{\mu_2}!\ \nu_0(\mu_2,\mu_1,n_1+n_2)}} \\ &\qquad\qquad \times R_{n_1}(z_j;\mu_2,\mu_1,n_1+n_2)P_{n_1}(\lambda_1;\mu_1,c\epsilon_1)P_{n_2}(\lambda_2;\mu_2,\lambda_1\epsilon_2).\end{aligned} \tag{8.79}$$

Proof. Consider the Clebsch-Gordan decomposition (8.37) and apply the operator Υ on each side of the equation. Using (8.69), (8.72) and (8.73), one obtains the above formula. □

Remark 8.1. *It is also possible to obtain another convolution identity using the inverse basis expansion :*

$$\begin{aligned}&P_{n_1}(\lambda_1;\,\mu_1,c\epsilon_1)P_{n_2}(\lambda_2;\mu_2,\lambda_1\epsilon_2)\\&\qquad=\sum_{N+j=n_1+n_2}(-1)^{\phi(n_1,n_2,j)}\left(\frac{\epsilon_2}{2}\right)^{n_1}\sqrt{\frac{[n_2]_{\mu_2}!\;\rho_j(\mu_2,\mu_1,n_1+n_2)}{[n_1]_{\mu_1}![n_1+n_2]_{\mu_2}!\;\nu_0(\mu_2,\mu_1,n_1+n_2)}}\\&\qquad\times R_{n_1}(z_j;\mu_2,\mu_1,n_1+n_2)K_j(\lambda_2;\mu_2,\mu_1;c)\;P_N(\lambda_2;\mu_{12},c\epsilon_{12})J_j\left(\frac{\epsilon_2\lambda_1}{\lambda_2};\;2\mu_2,2\mu_1,-\frac{c\epsilon_1\epsilon_2}{\lambda_2}\right).\end{aligned}\tag{8.80}$$

Remark 8.2. *Note that the propositions 8.2 and 8.3 both define polynomials in two variables λ_1,λ_2 of the Tratnik-type that are orthogonal with respect to the same measure $W(\lambda_1,\mu_1,c\epsilon_1)W(\lambda_2,\mu_2,\lambda_1\epsilon_2)$ on G. In this picture, the convolution identities (8.79) and (8.80) provide connection coefficients for these two sets of orthogonal polynomials. This is an interesting result especially since the generalization to multiple variables of the Bannai-Ito scheme is still in its early stages. Note that section 4.3 similarly gives orthogonal polynomials in three variables in this class.*

8.4.3. Action on the three-fold tensor product space

We now study how the operator X_c can be extended to the three-fold tensor product space $(\mu_1,\epsilon_1)\otimes(\mu_2,\epsilon_2)\otimes(\mu_3,\epsilon_3)$. This will lead to another convolution identity involving the Racah coefficients of $\mathfrak{osp}(1|2)$. One first computes

$$\begin{aligned}\Delta^2(X_c)\equiv(1\otimes\Delta)\Delta(X_c)&=\Delta(X_c)\otimes R+\Delta(1)\otimes X_0\\&=X_c\otimes R\otimes R+1\otimes X_0\otimes R+1\otimes 1\otimes X_0.\end{aligned}\tag{8.81}$$

The notation Δ^2 is unambiguous because of the coassociativity of the coproduct : $(1\otimes\Delta)\Delta=(\Delta\otimes 1)\Delta$. The eigenvectors of $\Delta^2(X_c)$ can be found by studying its action on vectors of the form $v^c_{\lambda_1,\lambda_2}\otimes e_{n_3}$:

$$\begin{aligned}\Delta^2(X_c)v^c_{\lambda_1,\lambda_2}\otimes e_{n_3}&=(\Delta(X_c)\otimes R+\Delta(1)\otimes X_0)v^c_{\lambda_1,\lambda_2}\otimes e_{n_3}\\&=(1\otimes1\otimes\lambda_2R+1\otimes1\otimes X_0)v^c_{\lambda_1,\lambda_2}\otimes e_{n_3}\\&=(1\otimes1\otimes X_{\lambda_2})v^c_{\lambda_1,\lambda_2}\otimes e_{n_3}\\&=v^c_{\lambda_1,\lambda_2}\otimes X_{\lambda_2}e_{n_3}.\end{aligned}\tag{8.82}$$

It follows naturally that the generalized eigenvectors are

$$v^c_{\lambda_1,\lambda_2,\lambda_3}=\sum_{n_1,n_2,n_3=0}^{\infty}P_{n_1}(\lambda_1;\mu_1,c\epsilon_1)P_{n_2}(\lambda_2;\mu_2,\lambda_1\epsilon_2)P_{n_3}(\lambda_3;\mu_3,\lambda_2\epsilon_3)\;e_{n_1}^{(\mu_1,\epsilon_1)}\otimes e_{n_2}^{(\mu_2,\epsilon_2)}\otimes e_{n_3}^{(\mu_3,\epsilon_3)}$$

with eigenvalues λ_3. This establishes the analog of propositions 8.1 and 8.2.

Proposition 8.5. *The unitary operator*

$$\Theta: \quad \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+) \to L^2(G^{(3)}, W^{(3)}), \tag{8.83}$$

$$e_{n_1}^{(\mu_1,\epsilon_1)} \otimes e_{n_2}^{(\mu_2,\epsilon_2)} \otimes e_{n_3}^{(\mu_3,\epsilon_3)} \mapsto P_{n_1}(\lambda_1;\mu_1,c\epsilon_1)P_{n_2}(\lambda_2;\mu_2,\lambda_1\epsilon_2)P_{n_3}(\lambda_3;\mu_3,\lambda_2\epsilon_3) \tag{8.84}$$

where

$$G^{(3)} = \{(\lambda_1,\lambda_2,\lambda_3) \in \mathbb{R} \mid |\lambda_3| > |\lambda_2| > |\lambda_1| > |c|\}$$

and

$$W^{(3)} = W(\lambda_1,\mu_1,c\epsilon_1)W(\lambda_2,\mu_2,\lambda_1\epsilon_2)W(\lambda_3,\mu_3,\lambda_2\epsilon_3).$$

is an intertwiner of the operator M_{λ_3} on $L^2(G^{(3)}, W^{(3)})$ and the operator $\Delta^2(X_c)$ acting in $\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$:

$$M_{\lambda_3}\Theta = \Theta\Delta^2(X_c). \tag{8.85}$$

Proof. The proof follows those of propositions 8.1 and 8.2. Unitarity follows from mapping an orthonormal basis onto another one. The intertwining relation comes from the computation of the generalized eigenvectors before proposition 8.5. □

The idea is again to act with the operator Θ in different bases in order to obtain a new convolution identity. The bases of interest here are the two that arise when decomposing the representation space $(\mu_1,\epsilon_1) \otimes (\mu_2,\epsilon_2) \otimes (\mu_3,\epsilon_3)$ into irreducible components. As mentioned in section 3, this can be done in two standard ways using the Clebsch-Gordan decomposition. This yields the bases $f_{n_{123}}^{j_{123},j_{12}}$ and $g_{n_{123}}^{j_{123},j_{23}}$ respectively given in (8.48) and (8.50).

Proposition 8.6. *In $L^2(G^{(3)}, W^{(3)})$, we have*

$$\begin{aligned}\Theta f_{n_{123}}^{j_{123},j_{12}} = {} & K_{j_{12}}(\lambda_2;\mu_2,\mu_1;c)\ J_{j_{12}}\left(\frac{\epsilon_2\lambda_1}{\lambda_2};\ 2\mu_2,2\mu_1,-\frac{c\epsilon_1\epsilon_2}{\lambda_2}\right) \\ & \times K_{j_{(12)3}}(\lambda_3;\mu_3,\mu_{12};c)\ J_{j_{(12)3}}\left(\frac{\epsilon_3\lambda_2}{\lambda_3};\ 2\mu_3,2\mu_{12},-\frac{c\epsilon_{12}\epsilon_3}{\lambda_3}\right) P_{n_{123}}(\lambda_3;\mu_{123},c\epsilon_{123}),\end{aligned} \tag{8.86}$$

$$\begin{aligned}\Theta g_{n_{123}}^{j_{123},j_{23}} = {} & K_{j_{23}}(\lambda_3;\mu_3,\mu_2;\lambda_1)\ J_{j_{23}}\left(\frac{\epsilon_3\lambda_2}{\lambda_3};\ 2\mu_3,2\mu_2,-\frac{\lambda_1\epsilon_2\epsilon_3}{\lambda_3}\right) \\ & \times K_{j_{1(23)}}(\lambda_3;\mu_{23},\mu_1;c)\ J_{j_{1(23)}}\left(\frac{\epsilon_{23}\lambda_1}{\lambda_3};\ 2\mu_{23},2\mu_1,-\frac{c\epsilon_1\epsilon_{23}}{\lambda_3}\right) P_{n_{123}}(\lambda_3;\mu_{123},c\epsilon_{123}).\end{aligned} \tag{8.87}$$

Proof. The key to obtain the action of Θ on these two bases is to use the expansions (8.48) and (8.50) in terms of the basis $e_{n_1}^{(\mu_1,\epsilon_1)} \otimes e_{n_2}^{(\mu_2,\epsilon_2)} \otimes e_{n_3}^{(\mu_3,\epsilon_3)}$, act with Θ as per proposition 8.5 and resum the resulting polynomials by using the convolution identity (8.79) twice. Using the notation of proposition 8.3 and of section 2, one obtains the relations above. □

This leads to the following result.

Proposition 8.7. *The convolution identity*

$$K_{j_{12}}(\lambda_2;\mu_2,\mu_1;c)J_{j_{12}}\left(\tfrac{\epsilon_2\lambda_1}{\lambda_2};2\mu_2,2\mu_1,-\tfrac{c\epsilon_1\epsilon_2}{\lambda_2}\right)K_{j_{(12)3}}(\lambda_3;\mu_3,\mu_{12};c)J_{j_{(12)3}}\left(\tfrac{\epsilon_3\lambda_2}{\lambda_3};2\mu_3,2\mu_{12},-\tfrac{c\epsilon_{12}\epsilon_3}{\lambda_3}\right)$$

$$=\sum_{j_{23}=0}^{j_{123}}(-1)^{\varphi}\epsilon_3^{j_{12}}\sqrt{\frac{w_{j_{23}}}{h_{j_{12}}}}\;B_{j_{12}}\left(x_{j_{23}};\tfrac{\mu_2+\mu_3}{2},\tfrac{\mu_1+(-1)^{j_{123}}\mu_{123}}{2},\tfrac{\mu_3-\mu_2}{2},\tfrac{(-1)^{j_{123}}\mu_{123}-\mu_1}{2}\right) \tag{8.88}$$

$$\times K_{j_{23}}(\lambda_3;\mu_3,\mu_2;\lambda_1)J_{j_{23}}\left(\tfrac{\epsilon_3\lambda_2}{\lambda_3};2\mu_3,2\mu_2,-\tfrac{\lambda_1\epsilon_2\epsilon_3}{\lambda_3}\right)K_{j_{1(23)}}(\lambda_3;\mu_{23},\mu_1;c)J_{j_{1(23)}}\left(\tfrac{\epsilon_{23}\lambda_1}{\lambda_3};2\mu_{23},2\mu_1,-\tfrac{c\epsilon_1\epsilon_{23}}{\lambda_3}\right)$$

holds with the relations (8.47) between the $j_{12}, j_{23}, j_{(12)3}, j_{1(23)}, j_{123}$, the notation for the polynomials of section 2 and equation (8.74).

Proof. This formula is obtained by acting with Θ on both sides of the Racah decomposition (8.52). The factors $P_{n_{123}}(\lambda_3;\mu_{123},c\epsilon_{123})$ on the left and on the right cancel out. This is just the manifestation of the well-known Wigner-Eckart theorem in this context. □

Remark 8.3. *It is also possible to obtain a similar convolution identity using the orthogonality of the Racah coefficients :*

$$K_{j_{23}}(\lambda_3;\mu_3,\mu_2;\lambda_1)J_{j_{23}}\left(\tfrac{\epsilon_3\lambda_2}{\lambda_3};2\mu_3,2\mu_2,-\tfrac{\lambda_1\epsilon_2\epsilon_3}{\lambda_3}\right)K_{j_{1(23)}}(\lambda_3;\mu_{23},\mu_1;c)J_{j_{1(23)}}\left(\tfrac{\epsilon_{23}\lambda_1}{\lambda_3};2\mu_{23},2\mu_1,-\tfrac{c\epsilon_1\epsilon_{23}}{\lambda_3}\right)$$

$$=\sum_{j_{12}=0}^{j_{123}}(-1)^{\varphi}\epsilon_3^{j_{12}}\sqrt{\frac{w_{j_{23}}}{h_{j_{12}}}}\;B_{j_{12}}\left(x_{j_{23}};\tfrac{\mu_2+\mu_3}{2},\tfrac{\mu_1+(-1)^{j_{123}}\mu_{123}}{2},\tfrac{\mu_3-\mu_2}{2},\tfrac{(-1)^{j_{123}}\mu_{123}-\mu_1}{2}\right) \tag{8.89}$$

$$\times K_{j_{12}}(\lambda_2;\mu_2,\mu_1;c)J_{j_{12}}\left(\tfrac{\epsilon_2\lambda_1}{\lambda_2};2\mu_2,2\mu_1,-\tfrac{c\epsilon_1\epsilon_2}{\lambda_2}\right)K_{j_{(12)3}}(\lambda_3;\mu_3,\mu_{12};c)J_{j_{(12)3}}\left(\tfrac{\epsilon_3\lambda_2}{\lambda_3};2\mu_3,2\mu_{12},-\tfrac{c\epsilon_{12}\epsilon_3}{\lambda_3}\right)$$

where the relations (8.47) and the same notation are still assumed.

8.5. Bilinear generating function

In this section, we consider a realization of $\mathfrak{osp}(1|2)$ in terms of Dunkl operators. This leads to a generating function for the Specialized Chihara polynomials. Additionally, the convolution identity from proposition 8.4 is used to derive a bilinear generating function for the Big −1 Jacobi polynomials.

We introduce the following realization in terms of Dunkl operators of the $\mathfrak{osp}(1|2)$ Lie superalgebra :

$$J_+=z,\quad J_-=\partial_z+\frac{\mu}{z}(1-R_z),\quad J_0=z\partial_z+\mu+\tfrac{1}{2},\quad R=R_z \tag{8.90}$$

where R_z is just the the reflexion operator acting on the variable z by $R_zf(z)=f(-z)$. These operators verify the relations (8.35) when acting on the orthonormal basis vectors $e_n^{(\mu,\epsilon)}=([n]_\mu!)^{-1/2}\,z^n$.

We shall make use of this model to derive generating functions for the Specialized Chihara polynomials $P_n(\lambda;\mu,c\epsilon)$. First, recall equation (8.62) giving the generalized eigenvectors of the operator X_c and insert the realization (8.90) above to obtain

$$v_\lambda^c(z,\mu,\epsilon)=\sum_{n=0}^{\infty}P_n(\lambda;\mu,c\epsilon)\ \frac{z^n}{[n]_\mu^{1/2}!}. \tag{8.91}$$

If one can obtain an explicit formula for $v_\lambda^c(z,\mu,\epsilon)$ in terms of special functions, this equation will yield the desired generating function. This can be done through the following steps. First, split the sum in the RHS over the even and the odd values of n :

$$v_\lambda^c(z,\mu,\epsilon)=\sum_{k=0}^{\infty}P_{2k}(\lambda;\mu,c\epsilon)\ \frac{z^{2k}}{[2k]_\mu^{1/2}!}+\sum_{k=0}^{\infty}P_{2k+1}(\lambda;\mu,c\epsilon)\ \frac{z^{2k+1}}{[2k+1]_\mu^{1/2}!}. \tag{8.92}$$

Then, rewrite the μ-factorial in terms of Pochhammer symbols with

$$[2k]_\mu!=4^k k!(\mu+\tfrac{1}{2})_k, \qquad [2k+1]_\mu!=2(\mu+\tfrac{1}{2})4^k k!(\mu+\tfrac{3}{2})_k \tag{8.93}$$

and express the polynomials $P_{2k}(\lambda)$ and $P_{2k+1}(\lambda)$ in terms of Laguerre polynomials using equation (8.2) to get

$$v_\lambda^c(z,\mu,\epsilon)=\sum_{k=0}^{\infty}\frac{(-\frac{1}{2}z^2)^k}{(\mu+\frac{1}{2})_k}L_k^{(\mu-\frac{1}{2})}\left(\frac{\lambda^2-c^2}{2}\right)\ +\ \frac{z(\lambda-c\epsilon)}{2\mu+1}\ \sum_{k=0}^{\infty}\frac{(-\frac{1}{2}z^2)^k}{(\mu+\frac{3}{2})_k}L_k^{(\mu+\frac{1}{2})}\left(\frac{\lambda^2-c^2}{2}\right). \tag{8.94}$$

Now, each sum can be reframed in terms of hypergeometric functions with the help of a generating function for the Laguerre polynomials (see equation 1.11.11 of [13]) :

$$\sum_{n=0}^{\infty}\frac{t^n}{(\alpha+1)_n}L_n^{(\alpha)}(x)=e^t\ {}_0F_1\left[{- \atop \alpha+1};-xt\right]. \tag{8.95}$$

Inserting the result into equation (8.91) leads to the following generating function for the Specialized Chihara polynomials.

Proposition 8.8. *The Specialized Chihara polynomials possess the generating function*

$$\sum_{n=0}^{\infty}P_n(\lambda;\mu,c\epsilon)\ \frac{z^n}{[n]_\mu^{1/2}!}=e^{-z^2/2}\left({}_0F_1\left[{- \atop \mu+\frac{1}{2}};\frac{z^2(\lambda^2-c^2)}{4}\right]+\frac{z(\lambda-c\epsilon)}{2\mu+1}{}_0F_1\left[{- \atop \mu+\frac{3}{2}};\frac{z^2(\lambda^2-c^2)}{4}\right]\right).$$

Proof. This result follows from the preceding computation. □

Remark 8.4. *It is also possible to reexpress the hypergeometric functions in terms of the modified Bessel functions of the first type $I_\alpha(x)$ to obtain*

$$\sum_{n=0}^{\infty}P_n(\lambda;\mu,c\epsilon)\ \frac{z^n}{[n]_\mu^{1/2}!}=\frac{e^{-z^2/2}\Gamma(\mu-\frac{1}{2})}{(\frac{z}{2})^{\mu-1/2}(\lambda^2-c^2)^{\frac{\mu}{2}+\frac{1}{4}}}\left[(\lambda^2-c^2)^{\frac{1}{4}}I_{\mu-\frac{1}{2}}(z\sqrt{\lambda^2-c^2})+\tfrac{2\mu-1}{2\mu+1}\ I_{\mu+\frac{1}{2}}(z\sqrt{\lambda^2-c^2})\right].$$

Remark 8.5. *Note that this explicit expression for the generalized eigenvectors $v_\lambda^c(z,\mu,\epsilon)$ in terms of special functions in the realization (8.90) can also be obtained by solving the difference-differential equation $X_c v_\lambda^c(z,\mu,\epsilon) = \lambda v_\lambda^c(z,\mu,\epsilon)$. This also requires to separate the function $v_\lambda^c(z,\mu,\epsilon)$ into an even and odd part and leads to a system of two coupled first order ordinary differential equations.*

Focusing now on obtaining a bilinear generating function, we consider how the generating function given above carries to the tensor product of representations. In fact, the extension of the generalized eigenvectors to the tensor product of two representations in the realization (8.90) is immediate. Explicitly, the generalized eigenvectors of $\Delta^2(X_c)$ are

$$v_{\lambda_1}^c(z_1,\mu_1,\epsilon_1)v_{\lambda_2}^{\lambda_1}(z_2,\mu_2,\epsilon_2) = \sum_{n_1,n_2=0}^{\infty} P_{n_1}(\lambda_1;\mu_1,c\epsilon_1)P_{n_2}(\lambda_2;\mu_2,\lambda_1\epsilon_2)\frac{z_1^{n_1}z_2^{n_2}}{[n_1]_{\mu_1}^{1/2}![n_2]_{\mu_2}^{1/2}!} \tag{8.96}$$

where each $v_\lambda^c(z,\mu,\epsilon)$ admits an expression in terms of special functions as before. The monomials can be cast in the coupled basis by the inverse expansion of (8.37),

$$\frac{z_1^{n_1}z_2^{n_2}}{[n_1]_{\mu_1}^{1/2}![n_2]_{\mu_2}^{1/2}!} = \sum_{N+j=n_1+n_2} C_{n_1,n_2}^{N,j} e_N^{(\mu_{12},\epsilon_{12})}(z_1,z_2), \tag{8.97}$$

to obtain

$$v_{\lambda_1}^c(z_1,\mu_1,\epsilon_1)v_{\lambda_2}^{\lambda_1}(z_2,\mu_2,\epsilon_2) = \sum_{n_1,n_2=0}^{\infty} P_{n_1}(\lambda_1;\mu_1,c\epsilon_1)P_{n_2}(\lambda_2;\mu_2,\lambda_1\epsilon_2)\sum_{N+j=n_1+n_2} C_{n_1,n_2}^{N,j} e_N^{(\mu_{12},\epsilon_{12})}(z_1,z_2).$$

Inverting the order of summation, one gets

$$v_{\lambda_1}^c(z_1,\mu_1,\epsilon_1)v_{\lambda_2}^{\lambda_1}(z_2,\mu_2,\epsilon_2) = \sum_{N,j=0}^{\infty} e_N^{(\mu_{12},\epsilon_{12})}(z_1,z_2)\sum_{n_1+n_2=N+j} C_{n_1,n_2}^{N,j} P_{n_1}(\lambda_1;\mu_1,c\epsilon_1)P_{n_2}(\lambda_2;\mu_2,\lambda_1\epsilon_2)$$

where the second sum now corresponds directly to the first convolution identity (8.79). Using this gives

$$\begin{aligned} v_{\lambda_1}^c(z_1,\mu_1,\epsilon_1)v_{\lambda_2}^{\lambda_1}(z_2,\mu_2,\epsilon_2) &= \sum_{j=0}^{\infty} K_j(\lambda_2;\mu_2,\mu_1;c)J_j\left(\tfrac{\epsilon_2\lambda_1}{\lambda_2};\ 2\mu_2,2\mu_1,-\tfrac{c\epsilon_1\epsilon_2}{\lambda_2}\right) \\ &\quad\times\sum_{N=0}^{\infty} P_N(\lambda_2;\mu_{12},c\epsilon_{12})e_N^{(\mu_{12},\epsilon_{12})}(z_1,z_2). \end{aligned} \tag{8.98}$$

If one obtains an expression for the second sum in terms of special functions, then a generating function for the Big −1 Jacobi polynomials follows immediately. We first look for an explicit realization of the coupled basis vectors $e_N^{(\mu_{12},\epsilon_{12})}(z_1,z_2)$. Consider its expansion in terms of the uncoupled basis given in (8.37) in the realization (8.90) and substitute $n_2 = N+j-n_1$:

$$e_N^{(\mu_{12},\epsilon_{12})}(z_1,z_2) = \sum_{n_1=0}^{N+j} C_{n_1,N+j-n_1}^{N,j}\frac{(\frac{z_1}{z_2})^{n_1}}{[n_1]_{\mu_1}^{1/2}![N+j-n_1]_{\mu_2}^{1/2}!}\ z_2^{N+j}. \tag{8.99}$$

The sum on the RHS corresponds to the generating function for the $\mathfrak{osp}(1|2)$ Clebsch-Gordan coefficients [23]. Taking into account the choice of normalization and phase factor made in section 3, this gives

$$e_N^{(\mu_{12},\epsilon_{12})}(z_1,z_2) = \begin{cases} \dfrac{(z_1^2+z_2^2)^{N/2}}{[N]^{1/2}_{\mu_{12}}!} f_e(j) & \text{if } N \text{ even,} \\[2ex] \dfrac{(z_1^2+z_2^2)^{N/2}}{[N]^{1/2}_{\mu_{12}}!} f_o(j) & \text{if } N \text{ odd} \end{cases} \tag{8.100}$$

where $f_e(j)$ and $f_o(j)$ are functions of j. Let $j = 2j_e + j_p$ with $j_p \in \{0,1\}$ and $j_e \in \mathbb{N}$, then

$$f_e(j) = (-1)^{j_e+j_p} \frac{z_2^j}{[j]^{1/2}_{\mu_2}!} \left[\frac{(\frac{1}{2}+\mu_1)_{j_e+j_p}}{(j_e+j_p+1+\mu_1+\mu_2)_{j_e+j_p}} \right]^{1/2} \tag{8.101}$$
$$\times \left({}_2F_1\left[\begin{matrix} -j_e, \ \frac{1}{2}-j_e-j_p-\mu_2 \\ \frac{1}{2}+\mu_1 \end{matrix} ; -(\tfrac{z_1}{z_2})^2 \right] + \frac{(-1)^{j_p} z_1 (j+2\mu_2 j_p)}{z_2 \epsilon_2 (1+2\mu_1)} {}_2F_1\left[\begin{matrix} 1-j_e-j_p, \ \frac{1}{2}-j_e-\mu_2 \\ \frac{3}{2}+\mu_1 \end{matrix} ; -(\tfrac{z_1}{z_2})^2 \right] \right)$$

and

$$f_o(j) = (-1)^{j_e+j_p} \left(\frac{z_1^2}{z_2^2}+1 \right)^{-1/2} \frac{z_2^j}{[j]^{1/2}_{\mu_2}!} \left[\frac{(\frac{1}{2}+\mu_1)_{j_e+j_p}}{(j_e+j_p+1+\mu_1+\mu_2)_{j_e+j_p}} \right]^{1/2} \tag{8.102}$$
$$\times \left({}_2F_1\left[\begin{matrix} -j_e-j_p, \ -\frac{1}{2}-j_e-\mu_2 \\ \frac{1}{2}+\mu_1 \end{matrix} ; -(\tfrac{z_1}{z_2})^2 \right] + \tfrac{(-1)^{j_p} z_1 (j+1+2\mu_1+2\mu_2 j_p)}{z_2 \epsilon_2 (1+2\mu_1)} {}_2F_1\left[\begin{matrix} -j_e, \ \frac{1}{2}-j_e-j_p-\mu_2 \\ \frac{3}{2}+\mu_1 \end{matrix} ; -(\tfrac{z_1}{z_2})^2 \right] \right).$$

Separating the sum over N according to parities in (8.98) and substituting (8.100) gives

$$\begin{aligned} &v^c_{\lambda_1}(z_1,\mu_1,\epsilon_1) v^{\lambda_1}_{\lambda_2}(z_2,\mu_2,\epsilon_2) \\ &\qquad = \sum_{j=0}^\infty K_j(\lambda_2;\mu_2,\mu_1;c) J_j\left(\tfrac{\epsilon_2\lambda_1}{\lambda_2};\ 2\mu_2, 2\mu_1, -\tfrac{c\epsilon_1\epsilon_2}{\lambda_2}\right) \\ &\qquad \times \left(f_e(j) \sum_{k=0}^\infty P_{2k}(\lambda_2;\mu_{12},c\epsilon_{12}) \frac{(z_1^2+z_2^2)^{\frac{2k}{2}}}{[2k]^{1/2}_{\mu_{12}}!} + f_o(j) \sum_{k=0}^\infty P_{2k+1}(\lambda_2;\mu_{12},c\epsilon_{12}) \frac{(z_1^2+z_2^2)^{\frac{2k+1}{2}}}{[2k+1]^{1/2}_{\mu_{12}}!} \right). \end{aligned}$$

The two sums over k have precisely the form of the sums appearing in equation (8.92). It is thus possible to reexpress both of them in terms of an hypergeometric function by using the generating function of the Laguerre polynomials. This yields

$$\begin{aligned} &v^c_{\lambda_1}(z_1,\mu_1,\epsilon_1) v^{\lambda_1}_{\lambda_2}(z_2,\mu_2,\epsilon_2) \\ &\quad = \exp\left(-\frac{z_1^2+z_2^2}{2} \right) \sum_{j=0}^\infty K_j(\lambda_2;\mu_2,\mu_1;c) J_j\left(\tfrac{\epsilon_2\lambda_1}{\lambda_2};\ 2\mu_2, 2\mu_1, -\tfrac{c\epsilon_1\epsilon_2}{\lambda_2}\right) \\ &\quad \times \left(f_e(j) {}_0F_1\left[\begin{matrix} - \\ \mu_{12}+\frac{1}{2} \end{matrix} ; \tfrac{(z_1^2+z_2^2)(\lambda_2^2-c^2)}{4} \right] + f_o(j) \frac{(z_1^2+z_2^2)^{\frac{1}{2}}(\lambda_2 - c\epsilon_{12})}{2\mu_{12}+1} {}_0F_1\left[\begin{matrix} - \\ \mu_{12}+\frac{3}{2} \end{matrix} ; \tfrac{(z_1^2+z_2^2)(\lambda_2^2-c^2)}{4} \right] \right). \end{aligned}$$

Using proposition 8.8 to express both eigenvectors on the LHS in terms of special functions, the previous equation becomes a generating function for the Big -1 Jacobi polynomials.

Proposition 8.9. *The Big −1 Jacobi polynomials satisfy the bilinear generating function*

$$\left({}_0F_1\left[\begin{matrix}-\\ \mu_1+\frac{1}{2}\end{matrix};\frac{z_1^2(\lambda_1^2-c^2)}{4}\right]+\frac{z_1(\lambda_1-c\epsilon_1)}{2\mu+1}{}_0F_1\left[\begin{matrix}-\\ \mu_1+\frac{3}{2}\end{matrix};\frac{z_1^2(\lambda_1^2-c^2)}{4}\right]\right)$$

$$\times\left({}_0F_1\left[\begin{matrix}-\\ \mu_2+\frac{1}{2}\end{matrix};\frac{z_2^2(\lambda_2^2-\lambda_1^2)}{4}\right]+\frac{z_2(\lambda_2-\lambda_1\epsilon_2)}{2\mu_2+1}{}_0F_1\left[\begin{matrix}-\\ \mu_2+\frac{3}{2}\end{matrix};\frac{z_2^2(\lambda_2^2-\lambda_1^2)}{4}\right]\right)$$

$$=\sum_{j=0}^{\infty}K_j(\lambda_2;\mu_2,\mu_1;c)J_j\left(\tfrac{\epsilon_2\lambda_1}{\lambda_2};\ 2\mu_2,2\mu_1,-\tfrac{c\epsilon_1\epsilon_2}{\lambda_2}\right)$$

$$\times\left(f_e(j){}_0F_1\left[\begin{matrix}-\\ \mu_{12}+\frac{1}{2}\end{matrix};\tfrac{(z_1^2+z_2^2)(\lambda_2^2-c^2)}{4}\right]+f_o(j)\frac{(z_1^2+z_2^2)^{\frac{1}{2}}(\lambda_2-c\epsilon_{12})}{2\mu_{12}+1}{}_0F_1\left[\begin{matrix}-\\ \mu_{12}+\frac{3}{2}\end{matrix};\tfrac{(z_1^2+z_2^2)(\lambda_2^2-c^2)}{4}\right]\right)$$

where $\mu_{12}=\mu_1+\mu_2+\frac{1}{2}+j$ *and* $f_e(j)$*,* $f_o(j)$*,* $K_j(\lambda_2;\mu_1,\mu_2;c)$ *are given by the formulas* (8.101)*,* (8.102)*,* (8.74)*.*

Proof. The result follows from the analysis provided before the statement of this proposition. □

8.6. Conclusion

We considered the discrete series representations of the superalgebra $\mathfrak{osp}(1|2)$ and singled out a self-adjoint element X_c. We constructed the generalized eigenvectors of this special element in the representation spaces and in their two- and three-fold tensor products. Looking at different bases and their overlaps led to our main results : propositions 8.4 and 8.7 which provide convolution identities for −1 orthogonal polynomials and also connection coefficients for two-variable Dunkl polynomials orthogonal with respect to the same measure. This was further used to obtain a bilinear generating function for the Big −1 Jacobi polynomials. This led to interpretations and connections between the Specialized Chihara, the dual −1 Hahn, the Big −1 Jacobi and the Bannai-Ito polynomials.

This study suggests a number of future research questions. A natural extension would be to look at higher dimensional spaces via the n-fold tensor product of representations. This was done for $\mathfrak{su}(1,1)$ in [**31**]. This should lead to new convolution identities and to multivariate Dunkl orthogonal polynomials of Tratnik type. In fact, it is straightforward to extend the unitary operators from propositions 8.1, 8.2 and 8.5 to an arbitrary n-fold tensor product; the main difficulty is to find interesting bases and overlaps. It should be noted however that some investigations on this last point have already been done [**32, 29**]. Another avenue to explore would be how the convolution identities obtained here could be used to derive different generating functions and Poisson kernels. Interesting constructions pointing in this direction have been presented in [4] and [5] for the Lie algebras $\mathfrak{su}(1,1)$ and $U_q(\mathfrak{su}(1,1))$. Finally, a broader project would be to revisit the construction with different representations and realizations. We mention as examples [33] and [34] where similar questions are considered.

References

[1] Y. I. Granovskii, A. S. Zhedanov, New construction of $3nj$-symbols, J. Phys. A Gen. Phys. 26 (17) (1993) 4339–4344.

[2] J. Van Der Jeugt, Coupling coefficients for Lie algebra representations and addition formulas for special functions, J. Math. Phys. 38 (5) (1997) 2728–2740.

[3] H. T. Koelink, J. Van Der Jeugt, Convolutions for orthogonal polynomials from Lie and quantum algebra representations, SIAM J. Math. Anal. 29 (3) (1998) 794–822.

[4] J. Van Der Jeugt, R. Jagannathan, Realizations of $su(1,1)$ and $U_q(su(1,1))$ and generating functions for orthogonal polynomials, J. Math. Phys. 39 (9) (1998) 5062–5078.

[5] H. T. Koelink, J. Van Der Jeugt, Bilinear generating functions for orthogonal polynomials, Constr. Approx. 15 (4) (1999) 481–497.

[6] E. Bannai, T. Ito, Algebraic combinatorics I. Association schemes, Benjamin/Cummings Publishing Company, 1984.

[7] L. Vinet, A. Zhedanov, A 'missing' family of classical orthogonal polynomials, J. Phys. A Math. Theor. 44 (8) (2011) 085201.

[8] S. Tsujimoto, L. Vinet, A. Zhedanov, Dunkl shift operators and Bannai-Ito polynomials, Adv. Math. (N. Y). 229 (4) (2012) 2123–2158.

[9] L. Vinet, A. Zhedanov, A limit $q = -1$ for the big q-Jacobi polynomials, Trans. Am. Math. Soc. 364 (10) (2012) 5491–5507.

[10] V. X. Genest, L. Vinet, A. Zhedanov, Bispectrality of the complementary Bannai-Ito polynomials, Symmetry, Integr. Geom. Methods Appl. 9 (2013) 18.

[11] S. Tsujimoto, L. Vinet, A. Zhedanov, Dual -1 Hahn polynomials : "classical" polynomials beyond the Leonard duality, Proc. Am. Math. Soc. 141 (3) (2013) 959–970.

[12] V. X. Genest, L. Vinet, A. Zhedanov, A continuous limit of the complementary Bannai-Ito polynomials : Chihara polynomials, Symmetry, Integr. Geom. Methods Appl. 10 (0) (2014) 18–38.

[13] R. Koekoek, P. A. Lesky, R. F. Swarttouw, Hypergeometric orthogonal polynomials, Springer, 2010.

[14] L. Vinet, A. Zhedanov, A Bochner theorem for Dunkl polynomials, Symmetry, Integr. Geom. Methods Appl. 7 (2011) 20.

[15] C. F. Dunkl, Reflection groups and orthogonal polynomials on the sphere, Math. Zeitschrift 197 (1) (1988) 33–60.

[16] C. F. Dunkl, Differential-difference operators associated to reflection groups, Trans. Am. Math. Soc. 311 (1) (2006) 167.

[17] V. X. Genest, M. E. Ismail, L. Vinet, A. Zhedanov, The Dunkl oscillator in the plane: I. Superintegrability, separated wavefunctions and overlap coefficients, J. Phys. A Math. Theor. 46 (14) (2013) 145201.

[18] V. X. Genest, M. E. Ismail, L. Vinet, A. Zhedanov, The Dunkl Oscillator in the Plane II: Representations of the Symmetry Algebra, Commun. Math. Phys. 329 (3) (2014) 999–1029.

[19] V. X. Genest, L. Vinet, A. Zhedanov, A Laplace-Dunkl equation on S^2 and the Bannai-Ito algebra, Commun. Math. Phys. 336 (1) (2015) 243–259.

[20] L. Vinet, A. Zhedanov, Dual -1 Hahn polynomials and perfect state transfer, J. Phys. Conf. Ser. 343 (2012) 12125.

[21] L. Vinet, A. Zhedanov, Para-Krawtchouk polynomials on a bi-lattice and a quantum spin chain with perfect state transfer, J. Phys. A Math. Theor. 45 (26) (2012) 265304.

[22] S. Tsujimoto, L. Vinet, A. Zhedanov, From $sl_q(2)$ to a parabosonic Hopf algebra, Symmetry, Integr. Geom. Methods Appl. 7 (2011) 93–105.

[23] G. Bergeron, L. Vinet, Generating functions for the $\mathfrak{osp}(1|2)$ Clebsch-Gordan coefficients, J. Phys. A Math. Theor. 49 (11) (2016) 115202.

[24] P. Baseilhac, V. X. Genest, L. Vinet, A. Zhedanov, An embedding of the Bannai-Ito algebra in $U(\mathfrak{osp}(1|2))$ and -1 polynomials, Letters in Mathematical Physics 108 (7) (2018) 1623–1634.

[25] V. X. Genest, L. Vinet, A. Zhedanov, The algebra of dual -1 Hahn polynomials and the Clebsch-Gordan problem of $sl_{-1}(2)$, J. Math. Phys. 54 (2) (2013) 023506.

[26] V. X. Genest, L. Vinet, A. Zhedanov, The Bannai-Ito polynomials as Racah coefficients of the $sl_{-1}(2)$ algebra, Proc. Am. Math. Soc. 142 (5) (2014) 1545–1560.

[27] V. X. Genest, J. M. Lemay, L. Vinet, A. Zhedanov, Two-variable -1 Jacobi polynomials, Integr. Transform. Spec. Funct. 26 (6) (2015) 411–425.

[28] J.-M. Lemay, L. Vinet, Bivariate Bannai-Ito polynomials, J. Math. Phys. 59 (12) (2018) 121703.

[29] H. De Bie, H. De Clercq, The q-bannai-ito algebra and multivariate $(-q)$-racah and bannai-ito polynomials, arXiv preprint arXiv:1902.07883.

[30] E. Koelink, Spectral theory and special functions, in: Laredo Lect. Orthogonal Polynomials Spec. Funct., Nova Science Publishers, New York, 2004, pp. 45–84.

[31] H. Rosengren, Multivariable orthogonal polynomials and coupling coefficients for discrete series representations, SIAM journal on mathematical analysis 30 (2) (1999) 233–272.

[32] H. De Bie, V. X. Genest, W. van de Vijver, L. Vinet, Bannai-Ito algebras and the $osp(1|2)$ superalgebra, in: Phys. Math. Asp. Symmetries, Springer International Publishing, 2017, pp. 349–354.

[33] W. Groenevelt, E. Koelink, Meixner functions and polynomials related to Lie algebra representations, J. Phys. A. Math. Gen. 35 (1) (2002) 65–85.

[34] W. Groenevelt, Bilinear summation formulas from quantum algebra representations, Ramanujan J. 8 (3) (2004) 383–416.

Conclusion

Pour conclure, voici quelques pistes de recherche découlant directement des travaux présentés dans cette thèse et qui mériteraient sans doute d'être poursuivies.

- Dans l'esprit des deux premiers chapitres, il serait intéressant d'étudier si un processus limite pourrait permettre de résoudre la troncation singulière qui se présente dans les coefficients de récurrence des polynômes de Bannai-Ito. La même situation se présente également pour les polynômes de Bannai-Ito complémentaires. Une version « para » de ces deux familles de polynômes pourrait peut-être être obtenue.

- Un large programme serait d'entreprendre l'exploration plus systématique des polynômes multivariés de type Tratnik du tableau de Bannai-Ito. Bien qu'on commence à avoir une relativement bonne compréhension des polynômes de Bannai-Ito multivariés, le terrain est presqu'entièrement vierge concernant les autres familles. Une piste est l'étude rigoureuse des processus limites $q \to -1$ des extensions multivariés du q-tableau d'Askey.

- Il serait intéressant d'obtenir plus d'interprétations algébriques et physiques des polynômes du tableau de Bannai-Ito. Un chemin possible est l'étude d'autres superalgèbres de la forme $\mathfrak{osp}(n|m)$ ou encore des analogues $q \to -1$ d'algèbres quantiques. Ces structures pourraient potentiellement s'adapter à des constructions algébriques comme celle présentée au chapitre 8.

- Les questions de transport d'information et de calculs quantiques sont une immense source de questions de recherche. Pour rester proche du travail présenté au chapitre 5, mentionnons qu'il pourrait être intéressant de faire la correspondance entre ce modèle et les propriétés de transport sur des graphes en lien avec des schémas d'association.

Contributions de l'auteur

Conformément aux exigences de la FESP, cette annexe vise à détailler la contribution de JML aux articles compris dans cette thèse. La description est sommaire. Chaque article relève d'un effort collectif entre les coauteurs et d'un travail d'équipe au sein de laquelle JML a joué un rôle essentiel. La majorité des calculs et le premier jet pour chaque article sont notament dus à JML. Les contributions sous la forme de commentaires, discussions et suggestions ne sont pas mentionnées.

- Chapitre 1 : Idée originale de LV et AZ. Calculs et rédaction par JML et AZ.
  Rédaction par JML et LV.
- Chapitre 2 : Idée originale de JML, LV et AZ. Calculs par JML.
  Rédaction par JML et LV.
- Chapitre 3 : Idée originale de LV et AZ. Calculs et rédaction par VXG et JML.
- Chapitre 4 : Idée originale et rédaction par JML et LV. Calculs par JML.
- Chapitre 5 : Idée originale LV et AZ. Calculs par JML. Rédaction par JML et LV.
- Chapitre 6 : Idée originale par HDB, VXG et LV. Calculs par VXG et JML.
  Rédaction par HDB, VXG, JML et LV.
- Chapitre 7 : Idée originale et rédaction par HDB, VXG, JML et LV. Calculs par JML.
- Chapitre 8 : Idée originale et calculs par EK, JML et LV. Rédaction par JML.

Bibliographie

[1] W. Rudin, Real and complex analysis, McGraw-hill education, 2006.

[2] G. E. Andrews, R. Askey, R. Roy, Special functions, Vol. 71, Cambridge university press, 2000.

[3] S. Wolfram, The history and future of special functions, Wolfram Technology Conference (2005).

[4] C. Albanese, M. Christandl, N. Datta, A. Ekert, Mirror inversion of quantum states in linear registers, Physical Review Letters 93 (23) (2004) 230502.

[5] R. J. Chapman, M. Santandrea, Z. Huang, G. Corrielli, A. Crespi, M.-H. Yung, R. Osellame, A. Peruzzo, Experimental perfect state transfer of an entangled photonic qubit, Nature communications 7 (2016) 11339.

[6] A. Perez-Leija, R. Keil, A. Kay, H. Moya-Cessa, S. Nolte, L.-C. Kwek, B. M. Rodríguez-Lara, A. Szameit, D. N. Christodoulides, Coherent quantum transport in photonic lattices, Physical Review A 87 (1) (2013) 012309.

[7] L. Vinet, A. Zhedanov, Almost perfect state transfer in quantum spin chains, Phys. Rev. A 86 (2012) 052319.

[8] M. Christandl, L. Vinet, A. Zhedanov, Analytic next-to-nearest-neighbor XX models with perfect state transfer and fractional revival, Physical Review A 96 (3) (2017) 032335.

[9] O. Gal'bert, Y. I. Granovskii, A. Zhedanov, Dynamical symmetry of anisotropic singular oscillator, Physics Letters A 153 (4-5) (1991) 177–180.

[10] E. Jafarov, N. Stoilova, J. Van der Jeugt, Finite oscillator models: the Hahn oscillator, Journal of Physics A: Mathematical and Theoretical 44 (26) (2011) 265203.

[11] N. Mukunda, E. Sudarshan, J. Sharma, C. Mehta, Representations and properties of para-bose oscillator operators. i. energy position and momentum eigenstates, Journal of Mathematical Physics 21 (9) (1980) 2386–2394.

[12] V. X. Genest, M. E. Ismail, L. Vinet, A. Zhedanov, The Dunkl oscillator in the plane: I. superintegrability, separated wavefunctions and overlap coefficients, Journal of Physics A: Mathematical and Theoretical 46 (14) (2013) 145201.

[13] V. X. Genest, L. Vinet, A. Zhedanov, The singular and the 2:1 anisotropic Dunkl oscillators in the plane, Journal of Physics A: Mathematical and Theoretical 46 (32) (2013) 325201.

[14] V. X. Genest, J.-M. Lemay, L. Vinet, A. Zhedanov, The Hahn superalgebra and supersymmetric Dunkl oscillator models, Journal of Physics A: Mathematical and Theoretical 46 (50) (2013) 505204.

[15] T. S. Chihara, An introduction to orthogonal polynomials, Dover publications, 2011.

[16] R. Koekoek, P. A. Lesky, R. F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, Springer, 2010.

[17] V. X. Genest, L. Vinet, A. Zhedanov, A “continuous” limit of the complementary Bannai-Ito polynomials: Chihara polynomials, Symmetry, Integrability and Geometry: Methods and Applications 10 (2014) 38–18.

[18] G. Gasper, M. Rahman, G. George, Basic hypergeometric series, Vol. 96, Cambridge university press, 2004.

[19] E. Bannai, T. Ito, Algebraic combinatorics, Benjamin/Cummings Menlo Park, 1984.

[20] S. Tsujimoto, L. Vinet, A. Zhedanov, Dunkl shift operators and Bannai-Ito polynomials, Advances in Mathematics 229 (4) (2012) 2123–2158.

[21] V. X. Genest, L. Vinet, A. Zhedanov, Bispectrality of the complementary Bannai-Ito polynomials, SIGMA. Symmetry, Integrability and Geometry: Methods and Applications 9 (2013) 018.

[22] S. Tsujimoto, L. Vinet, A. Zhedanov, Dual -1 Hahn polynomials: "classical" polynomials beyond the leonard duality, Proceedings of the American Mathematical Society 141 (3) (2013) 959–970.

[23] L. Vinet, A. Zhedanov, A 'missing' family of classical orthogonal polynomials, Journal of Physics A: Mathematical and Theoretical 44 (8) (2011) 085201.

[24] L. Vinet, A. Zhedanov, A limit $q = -1$ for the Big q-Jacobi polynomials, Transactions of the American Mathematical Society 364 (10) (2012) 5491–5507.

[25] C. F. Dunkl, Y. Xu, Orthogonal polynomials of several variables, Vol. 155, Cambridge University Press, 2014.

[26] I. G. Macdonald, et al., Affine Hecke algebras and orthogonal polynomials, Vol. 157, Cambridge University Press, 2003.

[27] R. Griffiths, Orthogonal polynomials on the multinomial distribution, Australian Journal of Statistics 13 (1) (1971) 27–35.

[28] R. Griffiths, Orthogonal polynomials on the negative multinomial distribution, Journal of Multivariate Analysis 5 (2) (1975) 271–277.

[29] S. Karlin, J. McGregor, Linear growth models with many types and multidimensional Hahn polynomials, in: Theory and application of special functions, Elsevier, 1975, pp. 261–288.

[30] M. Tratnik, Some multivariable orthogonal polynomials of the Askey tableau – discrete families, Journal of mathematical physics 32 (9) (1991) 2337–2342.

[31] M. Tratnik, Some multivariable orthogonal polynomials of the Askey tableau – continuous families, Journal of mathematical physics 32 (8) (1991) 2065–2073.

[32] G. Gasper, M. Rahman, Some systems of multivariable orthogonal Askey-wilson polynomials, in: Theory and applications of special functions, Springer, 2005, pp. 209–219.

[33] G. Gasper, M. Rahman, Some systems of multivariable orthogonal q-Racah polynomials, The Ramanujan Journal 13 (1-3) (2007) 389–405.

[34] T. Koornwinder, Two-variable analogues of the classical orthogonal polynomials, in: Theory and application of special functions, Elsevier, 1975, pp. 435–495.

[35] L. Vinet, A. Zhedanov, How to construct spin chains with perfect state transfer, Physical Review A 85 (1) (2012) 012323.

[36] L. Vinet, A. Zhedanov, Para-Krawtchouk polynomials on a bi-lattice and a quantum spin chain with perfect state transfer, Journal of Physics A: Mathematical and Theoretical 45 (26) (2012) 265304.

[37] W. Miller Jr, S. Post, P. Winternitz, Classical and quantum superintegrability with applications, Journal of Physics A: Mathematical and Theoretical 46 (42) (2013) 423001.

[38] V. X. Genest, L. Vinet, A. Zhedanov, Superintegrability in two dimensions and the Racah–Wilson algebra, Letters in Mathematical Physics 104(8) (2014) 931952.

[39] E. G. Kalnins, W. Miller Jr, S. Post, Contractions of 2d 2nd order quantum superintegrable systems and the askey scheme for hypergeometric orthogonal polynomials, Symmetry, Integrability and Geometry: Methods and Applications 9 (057).

[40] E. G. Kalnins, W. Miller Jr, S. Post, Two-variable Wilson polynomials and the generic superintegrable system on the 3-sphere, Symmetry, Integrability and Geometry: Methods and Applications 7 (2011) 26.

[41] S. Post, Racah Polynomials and Recoupling Schemes of $\mathfrak{su}(1,1)$, Symmetry, Integrability and Geometry: Methods and Applications 11 (2015) 57.

[42] V. X. Genest, L. Vinet, A. Zhedanov, The Bannai-Ito algebra and a superintegrable system with reflections on the two-sphere, Journal of Physics A: Mathematical and Theoretical 47 (20) (2014) 205202.

[43] H. De Bie, H. De Clercq, The q-Bannai-Ito algebra and multivariate $(-q)$-Racah and Bannai-Ito polynomials, arXiv preprint arXiv:1902.07883.

[44] H. T. Koelink, J. Van der Jeugt, Convolutions for orthogonal polynomials from lie and quantum algebra representations, SIAM journal on mathematical analysis 29 (3) (1998) 794–822.

[45] Y. I. Granovskii, A. Zhedanov, New construction of $3nj$-symbols, Journal of Physics A: Mathematical and General 26 (17) (1993) 4339.