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Abstract

Uncertainties as to future supply costs of nonrenewable natural resources, such as oil and gas, raise the issue of the choice of supply sources. In a perfectly deterministic world, an efficient use of multiple sources of supply requires that any given market exhausts the supply it can draw from a low cost source before moving on to a higher cost one; supply sources should be exploited in strict sequence of increasing marginal cost, with a high cost source being left untouched as long as a less costly source is available. We find that this may not be the efficient thing to do in a stochastic world. We show that there exist conditions under which it can be efficient to use a risky supply source in order to conserve a cheaper non risky source. The benefit of doing this comes from the fact that it leaves open the possibility of using it instead of the risky source in the event the latter's future cost conditions suddenly deteriorate. There are also conditions under which it will be efficient to use a more costly non risky source while a less costly risky source is still available. The reason is that this conserves the less costly risky source in order to use it in the event of a possible future drop in its cost.

Keywords: Security of supply; Uncertainty, Nonrenewable resources; Order of use
J.E.L. classification: Q310, D810, D900.

Résumé

L'incertitude sur le coût futur d'une ressource non renouvelable telle le pétrole ou le gaz pose la question du choix des sources d'approvisionnement. Dans un monde déterministe l'utilisation efficace de sources multiples exige que tout marché épuise une source de coût relativement faible avant de passer à une source de coût plus élevé ; l'utilisation des sources d'approvisionnement doit se faire strictement par ordre de coût marginal croissant ; on ne touchera à aucune source de coût élevé tant qu'existe une source moins coûteuse. Nous montrons que cette règle n'est pas forcément efficace en univers stochastique. Il existe des conditions sous lesquelles il peut être efficace de recourir à une source risquée pour conserver une ressource moins coûteuse non risquée. Une telle stratégie permet de se réserver la possibilité d'utiliser cette dernière au cas où les conditions d'accès à la source risquée se détérioreraient subitement. Il existe également des conditions sous lesquelles il sera efficace d'utiliser une source plus coûteuse mais non risquée alors qu'une source risquée mais meilleur marché est toujours disponible. Dans ce cas il s'agit d'économiser la source bon marché dans l'éventualité d'une baisse possible de son coût, qui la rendrait encore plus intéressante.

Mots-clés : Sécurité d'approvisionnement ; Incertitude ; Ordre d'utilisation.
Classification J.E.L. : Q310, D810, D900.

1 Introduction

In a period of high geopolitical risks, the question of the security of supply of nonrenewable natural resources, such as oil and gas, becomes a matter of particular concern for decision makers. The presence of significant uncertainties as to future supply costs raises the issue of the choice of supply sources in order to assure, at a cost, a certain degree of supply independence. One can think, for instance, of imports from a foreign source as being more prone to sudden supply disruptions than is the supply from a domestic source. The decision then becomes whether to use up the risky foreign supply in order conserve the safe domestic one, or whether to leave the risky supply source for future use and use the domestic source now. Evidently this will depend on the relative current costs, but also on what is foreseen for the future.

The issue of supply independence is obviously not relevant in a world of perfect certainty. In a perfectly deterministic world, an efficient use of multiple sources of supply requires that any given market exhaust the supply it can draw from a low cost source before moving on to a higher cost one. Hence, supply sources should be exploited in strict sequence of increasing marginal cost, with a high cost source being left untouched as long as a less costly source is available.[1] Given a positive discount rate, it makes sense to want to delay the high cost as much as possible in such a world.

The question arises as to what extent this principle remains valid in a world where future supply is subject to uncertainty. For instance, faced with the choice between two sources of supply, one of whose supply cost is stochastic, should the currently more expensive source necessarily be avoided as long as the other source is still available, if this more expensive source also happens to be the risky one? What if the risky source is the currently less expensive one? Should it then necessarily be used up before any use is made of the more

[1] This is sometimes referred to as the Herfindahl principle (Herfindahl 1967). For extensions of this principle to account for set-up costs, for general equilibrium issues or for spatial considerations, see Hartwick, Kemp and Long (1986), Kemp and Long (1980), Amigue, Favard, Gaudet and Moreaux (1998), Gaudet, Moreaux and Salant (2001).

costly non risky source? Those are the type of questions we investigate in this paper.

We will assume that the decision maker has at any given time a choice between supplying from a source whose future cost is known with certainty — a domestic source, maybe — and a source whose cost, although currently known, is subject to future disruption — a foreign source, for example. More specifically, we assume that the supply cost from the risky source can be, at any given time, in one of two states: a favorable state, in which it is low, and an unfavorable state, in which it is high. When it is currently low, there is a known probability that it will suddenly jump up in the future to its high level. When it is currently high, there is a known probability that it will suddenly return to its low level in the future. When its cost is low and the alternative source happens to be available at an even lower cost, then, in a deterministic world, you would want to use up the alternative source first. Similarly, when its cost is high and the cost of the alternative source is even higher, then, in a deterministic world, you would want to conserve all of the alternative source for later use. We find that this may not be the efficient thing to do in a stochastic world. In the first case, we show that there exist conditions under which it can be efficient to use some of the risky supply source in order to conserve the cheaper non risky source. The benefit of doing this comes from the fact that it leaves open the possibility of using it instead of the risky source in the event the latter's future cost conditions suddenly deteriorate. In the second case, we show that there are conditions under which it will be efficient to use the more costly non risky source while the risky source is still available. The reason why this might be efficient is that it conserves the less costly risky source in order to use it instead in the event of a possible future drop in its cost.

The next section is devoted to the derivation of conditions that must hold for the consumption path to be efficient. From those efficiency conditions we formulate, in Section 3, general decision criteria for the order of use of the two supply sources. In Section 4 we then derive explicit conditions on the parameters in order for it to be optimal to use the high cost supply source, which may be the risky source or the non risky one, while the low cost one is still available. Section 5 is devoted to a brief look at some special cases. We discuss there the particular case where the non risky source is an artificially constituted reserve as opposed to a natural deposit, the case of a complete embargo on supply, and the restrictions to treating the problem in a certainty equivalent form. We conclude in Section 6.

2 The efficiency conditions

Assume there are at date t two potential sources of supply of the resource, from deposits of size $X_1(t)$ and $X_2(t)$. The two sources of supply are perfect substitutes in consumption. The unit cost of drawing from deposit 1 is known with certainty to be $c_1 > 0$. The unit cost from deposit 2, $c_2(t)$, is stochastic. It can take either the value $c_2 > 0$ or $c_2 + m$, $m > 0$, with:

$$c_2(t+dt) = \begin{cases} \left.\begin{array}{l} c_2 + m \text{ with probability } \lambda dt \\ c_2 \text{ with probability } 1 - \lambda dt \end{array}\right\} \text{when } c_2(t) = c_2 \\ \\ \left.\begin{array}{l} c_2 \text{ with probability } \gamma dt \\ c_2 + m \text{ with probability } 1 - \gamma dt \end{array}\right\} \text{when } c_2(t) = c_2 + m \end{cases} \tag{1}$$

where $0 < \lambda dt, \gamma dt < 1$. Thus, when the current supply cost from deposit 2 is low ($c_2(t) = c_2$), there is a probability λdt that it will jump up to $c_2 + m$ over the interval dt. On the other hand, when the current supply cost from deposit 2 is high ($c_2(t) = c_2 + m$), there is a probability γdt that it will revert to c_2 over the interval dt.

An efficient supply policy $(x_1(t), x_2(t))$ is then one that maximizes:

$$E_t \int_t^{\infty} e^{-rs} \left[U(x_1(s) + x_2(s)) - c_1 x_1(s) - c_2(s) x_2(s) \right] ds \tag{2}$$

subject to (1) and to:

$$\dot{X}_i(s) = -x_i(s), \quad i = 1, 2 \tag{3}$$

$$x_i(s) \geq 0, \quad X_i(0) = X_{i0}, \quad i = 1, 2 \tag{4}$$

where r is the rate of interest, rate at which the instantaneous net benefits are discounted. It is assumed that the gross benefit function satisfies $U' > 0$, $U'' < 0$ and $U'(0) = \infty$. Under those assumptions, some positive amount of the resource will be consumed at any given time unless both deposits are exhausted. It also follows from those assumptions that both deposits will be physically exhausted, possibly in infinite time. If a deposit is exhausted in finite time, the date at which this will occur is stochastic, because $c_2(t)$ is.

Define

$$V(X_1(t), X_2(t), c_2(t)) = \max_{x_1(s), x_2(s)} E_t \int_t^\infty e^{-r(s-t)} \left[U(x_1(s) + x_2(s)) - c_1 x_1(s) - c_2(s) x_2(s)\right] ds, \tag{5}$$

which is the value function expressed in current value at t.[2] Then the Bellman equation associated with this stochastic optimization problem is:[3]

$$\begin{aligned} rV(X_1(t), X_2(t), c_2(t)) \quad = \quad & \max_{x_1(t), x_2(t)} \{U(x_1(t) + x_2(t)) - c_1 x_1(t) - c_2(t) x_2(t) \\ & - V_{X_1}(X_1(t), X_2(t), c_2(t)) x_1(t) - V_{X_2}(X_1(t), X_2(t), c_2(t)) x_2(t) \\ & + E\{\Delta V | c_2(t)\}\} \end{aligned} \tag{6}$$

where

$$E\{\Delta V | c_2(t)\} = \begin{cases} \lambda[V(X_1(t), X_2(t), c_2 + m)) - V(X_1(t), X_2(t), c_2)] & \text{if} \quad c_2(t) = c_2 \\ \gamma[V(X_1(t), X_2(t), c_2)) - V(X_1(t), X_2(t), c_2 + m)] & \text{if} \quad c_2(t) = c_2 + m. \end{cases}$$

The maximization of the right-hand side requires, for $i = 1, 2$:

$$U'(x_1(t)+x_2(t)) - c_i(t) - V_{X_i} \leq 0, \quad [U'(x_1(t)+x_2(t)) - c_i(t) - V_{X_i}] x_i(t) = 0, \quad x_i(t) \geq 0. \tag{7}$$

Thus if a positive supply is being drawn from deposit i at date t, then the marginal benefit derived from making use of this resource flow must be equal to the full marginal cost of producing it, which is the sum of $c_i(t)$, the marginal cost of extracting it, and V_{X_i}, the marginal value foregone by consuming it today rather than keeping it for future consumption.

The two sources being perfect substitutes and the full total cost being linear, this maximization implies that a source will never be used, at any given date, if its full marginal

[2] Note that $V(X_1(t), X_2(t), c_2(t))$ depends only on the current state and not on the current date t, the problem being time autonomous.

[3] This formulation of the Bellman equation makes use of the generalized Itô formula for jump processes. See Brock and Maliaris (1982), pages 122–124.

cost at that date is higher than that of the other source. Therefore, as long as the two full marginal costs differ, we will have, at any given date t, either $x_1(t) > 0$ and $x_2(t) = 0$, or $x_2(t) > 0$ and $x_1(t) = 0$, or $x_1(t) = 0$ and $x_2(t) = 0$ if both deposits are exhausted.[4]

In addition to condition (7), the following intertemporal efficiency condition is also necessary, for any positive interval of time Δt:

$$e^{-r\Delta t}E\{V_{X_i}(X_1(t+\Delta t), X_2(t+\Delta t), c_2(t+\Delta t)\} - V_{X_i}(X_1(t), X_2(t), c_2(t)) = 0. \quad (8)$$

To see this, first consider a small interval of time dt and calculate $e^{-rdt}E\{V_{X_i}(X_1(t+dt), X_2(t+dt), c_2(t+dt)\}$ by expanding around $(dt = 0, X_1(t), X_2(t), c_2(t))$, for both $c_2(t) = c_2 + m$ and $c_2(t) = c_2$, to verify (see Appendix) that, for $i = 1, 2$:

$$e^{-rdt}E\{V_{X_i}\} - V_{X_i} = \begin{cases} \text{when} \quad c_2(t) = c_2 : \\ \{\lambda[V_{X_i}(X_1(t), X_2(t), c_2 + m)) - V_{X_i}(X_1(t), X_2(t), c_2)] \\ -rV_{X_i}(X_1(t), X_2(t), c_2) \\ -V_{X_iX_i}(X_1(t), X_2(t), c_2)x_i(t)\}dt \\ \\ \text{when} \quad c_2(t) = c_2 + m : \\ \{\gamma[V_{X_i}(X_1(t), X_2(t), c_2)) - V_{X_i}(X_1(t), X_2(t), c_2 + m)] \\ -rV_{X_i}(X_1(t), X_2(t), c_2 + m) \\ -V_{X_iX_i}(X_1(t), X_2(t), c_2 + m)x_i(t)\}dt. \end{cases} \quad (9)$$

But differentiating the Bellman equation (6) totally with respect to X_i, we find that the right-hand side of (9) must be zero in both cases (see Appendix). Therefore $e^{-rds}E\{V_{X_i}\} = V_{X_i}$ at any date $s \in [t, t+\Delta]$ and, consequently, the same is true over the entire interval of arbitrary length Δt. It follows that (8) must hold along an optimal path.

The arbitrage condition (8) is Hotelling's rule (Hotelling 1931). It says that the discounted expected marginal valuation of each resource stock must be constant. In other words, the expected marginal valuations must be growing at the rate of interest.

It is well known that in the absence of uncertainty, the efficient use of the two resource deposits would require that they be used in strict sequence, with the lower cost supply source

[4]Under certainty, if $U'(0) = \infty$, as assumed, there will always be at least one of the deposits available, since it would be optimal to take an infinite time to fully exhaust the resource. Whether this remains the case under uncertainty is not the focus of this paper and is of no consequence for its results.

being completely depleted before moving on to the higher cost one (Herfindahl 1967). For instance, if $c_1 < c_2$ and both are known with certainty, then no use will be made of supply source 2 until supply source 1 is exhausted. To see this, first notice that if both supply sources are to be used, then it must be the case that $V_{X_1}(X_1(0), X_2(0)) > V_{X_2}(X_1(0), X_2(0))$, for otherwise the full marginal cost of using source 2 would always be higher than the full marginal cost of using source 1, implying, by condition (7), that source 2 would never be used. But since along an efficient path both V_{X_1} and V_{X_2} are growing at the rate $r > 0$ in order to satisfy the equivalent of condition (8) in the absence of uncertainty, there must come a time, say $t = \tau$, when $c_1 + e^{r\tau}V_{X_1}(X_1(0), X_2(0)) = c_2 + e^{r\tau}V_{X_2}(X_1(0), X_2(0))$. For all $t > \tau$, the full marginal cost of source 2 will be lower than that of source 1 and, by condition (7), only source 2 should be used; for all $t < \tau$, the full marginal cost of source 1 will be lower than that of source 2, and only source 1 should be used. Since both deposits must be physically exhausted, τ must be the date at which source 1 is fully depleted and source 2 takes over.

The question that now arises is the following: If there is uncertainty of the type postulated in (1) about the future cost of supplying from source 2, does efficiency still dictate that the two sources be used in strict sequence? For instance, if $c_1 < Ec_2$, should we refrain from using supply source 2 as long as supply source 1 is still available? It turns out that this is not the case. In what follows, we show that there are conditions on the parameters under which it will be efficient to make use of deposit 2 before deposit 1 is exhausted even if $c_1 < c_2$, so *a fortiori* even if $c_1 < Ec_2$. In fact, it may even turn out to be optimal to exhaust the supply from the higher cost source 2 before source 1 is used up. This does not depend on whether the costlier deposit is the uncertain one or not. Indeed, we also show that even if $c_1 > c_2 + m$, so *a fortiori* even if $c_1 > Ec_2$, it may be efficient under some circumstances to begin using deposit 1 before deposit 2 is used up. Again, it will possibly be optimal to completely exhaust supply source 1 before the less costly supply source 2 is used up.

3 The efficient order of use

Consider first a supply policy such that the non risky source 1 is being used at t and will be exhausted at some time $t+\Delta t$. Suppose furthermore that at $t+\Delta t$ the risky supply source 2 takes over. Therefore the supply policy is characterized by $x_1(t) > 0$ and $x_2(t+\Delta t) > 0$. Notice that Δt may be stochastic, because the policy followed between t and $t+\Delta t$ may depend on $c_2(s)$ for $s > t$. However, whatever the value of Δt, consider the following arbitrage:

1. reduce the quantity supplied from X_1 by one unit at t and increase it by one unit at $t+\Delta t$.

2. reduce the quantity supplied from X_2 by one unit at $t+\Delta t$ and increase it by one unit at t.

This is feasible and leaves the total consumption path of the initial policy unchanged. The expected change in the total cost of that consumption path, conditional on Δt and $c_2(t)$, is:

$$\begin{aligned}
\Delta C &= c_2(t) + V_{X_2}(X_1(t), X_2(t), c_2(t)) \\
&\quad + e^{-r\Delta t}\left[c_1 + E\left\{V_{X_1}(0, X_2(t+\Delta t), c_2(t+\Delta t)) \,|c_2(t), \Delta t\right\}\right] \\
&\quad - \left[c_1 + V_{X_1}(X_1(t), X_2(t), c_2(t))\right. \\
&\quad \left. + e^{-r\Delta t} E\left\{c_2(t+\Delta t) + V_{X_2}(0, X_2(t+\Delta t), c_2(t+\Delta t)) \,|c_2(t), \Delta t\right\}\right].
\end{aligned}$$

Since the intertemporal arbitrage condition (8) must hold for both supply sources along an optimal path, this reduces to:

$$\Delta C = (c_2(t) - c_1) - e^{-r\Delta t}[(c_2(t) - c_1) + E\{\Delta c_2 |\, c_2(t),\ \Delta t\}], \tag{10}$$

where

$$E\{\Delta c_2 |\, c_2(t),\ \Delta t\} \equiv E\{c_2(t+\Delta t) \,|\, c_2(t),\ \Delta t\} - c_2(t)$$

is the expected difference between $c_2(t+\Delta t)$ and $c_2(t)$ conditional on $c_2(t)$ and Δt. The first term on the right-hand side measures the immediate effect on cost at t, while the second term measures the expected effect on cost at $t+\Delta t$, discounted to t.

The arbitrage will be (strictly) profitable if and only if $\Delta C < 0$, which requires:[5]

$$(c_2(t) - c_1)[e^{r\Delta t} - 1] < E\{\Delta c_2 | \; c_2(t), \; \Delta t\}. \tag{11}$$

The left-hand side of the inequality is the total change in cost, valued at $t + \Delta t$, assuming $c_2(t)$ to remain unchanged between t and $t + \Delta t$. The right-hand side is the change in $c_2(t)$ expected to occur between those two dates.

Two cases must be considered, according as to which state prevails at t for the supply cost of the risky source:

Case 1: $c_2(t) = c_2$**.** In this case, $E\{\Delta c_2 | \; c_2(t), \; \Delta t\} > 0$, and:

a. If $c_1 > c_2$, then $\Delta C < 0$ for all Δt, m, λ and γ;

b. If $c_1 < c_2$, then $\Delta C < 0$ if and only if $E\{\Delta c_2 | \; c_2(t), \; \Delta t\}$ is sufficiently large to offset the loss from producing at t at the higher cost c_2 rather than at c_1.

Case 2: $c_2(t) = c_2 + m$**.** In this case, $E\{\Delta c_2 | \; c_2(t), \; \Delta t\} < 0$, and:

a. If $c_1 < c_2 + m$, then $\Delta C > 0$ for all Δt, m, λ and γ;

b. If $c_1 > c_2 + m$, then $\Delta C > 0$ if and only if the negative $E\{\Delta c_2 | \; c_2(t), \; \Delta t\}$ is sufficient to offset the gain from producing at t at the lower cost $c_2 + m$ rather than at c_1.

Case 1a says, not surprisingly, that as long as the risky supply source is currently at its cheapest and cheaper than the non risky source, it always pays to conserve the non risky source.

Case 1b is more interesting. It says that when the risky supply source is currently at its cheapest, but more expensive than the non risky source, then, if the expected cost saving from using it at $t + \Delta t$ more than compensates the known cost increase of using the risky source at t, it will still pay to conserve some of the non risky source, even though its current

[5] Notice that if $c_2(t)$ is known with certainty, so that the right-hand side is identically zero, we immediately recover the result that the more costly source must *never* be used while the less costly source is still available.

cost is less than the lowest possible cost of the risky source, let alone its expected cost. This is clearly a case where, contrary to what efficiency dictates in the absence of uncertainty about the future, it may be efficient to use the source that is currently more expensive even though the less expensive one is still available. The reason why this may be optimal is that the cheap supply thus put aside now, at the cost of having to replace it by the more costly risky source, serves as a reserve that can be used in the unfavorable event that $c_2(t)$ jumps up to $c_2 + m$ in the future. Whether this is optimal or not will depend, for any Δt, on the parameters m, λ and γ, as will be shown explicitly in the next section.

Note that if the suggested arbitrage turns out to be profitable at t, it does not mean that it will be profitable forever, since the cost of the risky source may jump up to $c_2 + m$ at some future date and we then find ourselves in Case 2. On the other hand, it is entirely possible that the cost of the risky source does not jump up to $c_2 + m$ in the finite time until its exhaustion. In that case the more costly source will be exhausted before the less costly one is used up.

Case 2a says that if the cost of the risky source is currently at its highest and it exceeds the cost of the non risky source, the suggested arbitrage is never profitable: it never pays to conserve the less costly non risky source in this situation, for obvious reasons.

The interesting case is now Case 2b. It says that when the cost of the non risky source is greater than even the worst realization of the cost of the risky source (which therefore can only go down), postponing the use of the costlier non risky source in favor of the less costly risky source will be profitable *only if* the expected future gain from a possible drop in the supply cost of the risky source from $c_2 + m$ to c_2 at some date in the future is not sufficient to compensate for the current gain from producing at cost $c_2 + m$ rather than at the higher c_1. Otherwise it becomes efficient to use the more costly non risky source now instead of the less costly alternative, in order to reserve the possibility of using the risky source when its cost is in the more favorable state c_2, instead of $c_2 + m$. This imposes conditions on the parameters which we discuss explicitly in the next section.

Note again that even if the suggested arbitrage fails to be profitable at t, it may not remain unprofitable indefinitely, since the cost of the risky source may later jump down to c_2, in which case we revert to Case 1. But again, if it turns out that the cost never jumps down to c_2 in the finite time during which the non risky source is being used, it is entirely possible that this more costly supply source will be exhausted first.

The same conclusions can be obtained by considering an initial supply policy such that the risky source 2 is being used at t and will be exhausted at $t + \Delta t$, with the non risky source 1 taking over at $t + \Delta t$. If we then transfer one unit of supply from X_2 from t to $t + \Delta t$ and compensate by transferring one unit of supply from X_1 from $t + \Delta t$ to t, we find that the expected change in the total cost, conditional on Δt and $c_2(t)$, is

$$\Delta C = (c_1 - c_2(t)) - e^{-r\Delta t}[(c_1 - c_2(t)) - E\{\Delta c_2 |\ c_2(t),\ \Delta t\}]. \tag{12}$$

This is simply the negative of the expression for the expected cost change in (10). Therefore if the current state is $c_2(t) = c_2$ and $c_1 < c_2$, we get the mirror image of Case 1b: conserving the more costly risky source for later use may be unprofitable. Indeed, conserving the costlier risky source for later then makes sense *only if* $E\{\Delta c_2 |\ c_2(t),\ \Delta t\} > 0$ is not so large as to offset the gain from producing at t at the lower cost c_1 rather than at c_2. Otherwise, efficiency requires that the higher cost risky source be used now, while its cost is favorable, in order to avoid having to use it in the event its cost jumps up to $c_2 + m$.

Similarly, if the current state is $c_2(t) = c_2 + m$ and $c_2 + m < c_1$, we get the mirror image of Case 2b: if $E\{\Delta c_2 |\ c_2(t),\ \Delta t\} < 0$ is sufficiently negative to offset the loss from producing at t at cost c_1 rather than at the lower cost $c_2 + m$, then the arbitrage is profitable. Although the risky source is now cheaper than the non risky source, if the expected drop in cost is sufficiently large, it makes sense to conserve it in the hope of being able to use it in the event its cost falls to c_2.

4 Conditions for conserving the lower cost supply source

We have found two situations where it may be desirable to depart from the principle that a high cost supply source should never be used while a lower cost source is still available. For any time Δt remaining until the exhaustion of the resource under use in the hypothetical initial policy, those situations will occur under particular conditions on the probabilities of regime changes, λ and γ, and the magnitude of the upward or downward change in cost, m. We now consider those conditions. Since in both situations this involves the expected change in $c_2(t)$, we begin by expressing this expected change explicitly in terms of the parameters.

4.1 The expected supply cost of the uncertain source

Let $p(s) \equiv Pr(c_2(t+s) = c_2 + m \mid c_2(t) = c_2 + m)$, the probability that the high cost $c_2 + m$ will prevail at $t + s$ if it prevails at t. Since $c_2(t)$ follows the stochastic process (1), the conditional probability that $c_2(t+s+ds) = c_2 + m$ is $1 - \gamma ds$ if $c_2(t+s) = c_2 + m$ and it is λds if $c_2(t+s) = c_2$. Thus

$$p(s+ds) = p(s)(1-\gamma ds) + (1-p(s))\lambda ds$$

so that

$$\frac{dp(s)}{ds} = -p(s)(\gamma+\lambda) + \lambda. \tag{13}$$

A particular solution to that linear first-order differential equation is

$$p = \frac{\lambda}{\gamma+\lambda}$$

and a solution to the homogenous part is

$$p = ke^{-(\gamma+\lambda)s}.$$

It follows that a general solution is

$$p(s) = \frac{\lambda}{\gamma+\lambda} + \left(\frac{\gamma}{\gamma+\lambda}\right)e^{-(\gamma+\lambda)s} \tag{14}$$

where k was set equal to $\gamma/(\gamma+\lambda)$ using $p(0) = 1$.

Similarly, let $q(s) \equiv Pr(c_2(t+s) = c_2 \mid c_2(t) = c_2)$. Then

$$q(s) = \frac{\gamma}{\gamma+\lambda} + \left(\frac{\lambda}{\gamma+\lambda}\right) e^{-(\gamma+\lambda)s}. \tag{15}$$

Letting $s = \Delta t$, the expected value of $c_2(t)$ after an interval of time Δt, conditional on $c_2(t)$ and Δt, can now be established to be:

$$\begin{aligned}
E\{c_2(t+\Delta t) \mid c_2,\ \Delta t\} &= q(\Delta t)\, c_2 + (1 - q(\Delta t))\,(c_2 + m) \\
&= c_2 + (1 - q(\Delta t))\, m \\
E\{c_2(t+\Delta t) \mid c_2 + m,\ \Delta t\} &= p(\Delta t)\,(c_2 + m) + (1 - p(\Delta t))\, c_2 \\
&= (c_2 + m) - (1 - p(\Delta t))\, m.
\end{aligned}$$

Using (14) and (15), this is rewritten in terms of the parameters as:

$$E\{c_2(t+\Delta t) \mid c_2,\ \Delta t\} = c_2 + m\left(\frac{\lambda}{\gamma+\lambda}\right)\left(1 - e^{-(\gamma+\lambda)\Delta t}\right) \tag{16}$$

$$E\{c_2(t+\Delta t) \mid c_2 + m,\ \Delta t\} = (c_2 + m) - m\left(\frac{\gamma}{\gamma+\lambda}\right)\left(1 - e^{-(\gamma+\lambda)\Delta t}\right) \tag{17}$$

Figure 1 shows these expected costs as a function of Δt. The difference in the expected value of $c_2(t+\Delta t)$ conditional on $c_2(t) = c_2 + m$ and that conditional on $c_2(t) = c_2$ is seen to go to zero when Δt goes to infinity, as both tend to $c_2 + m\lambda/(\gamma+\lambda)$.

From (16) and (17), we get the expected value of the change in $c_2(t)$ conditional on $c_2(t)$ and Δt:

$$E\{\Delta c_2 \mid c_2,\ \Delta t\} = m\left(\frac{\lambda}{\gamma+\lambda}\right)\left(1 - e^{-(\gamma+\lambda)\Delta t}\right) \tag{18}$$

$$E\{\Delta c_2 \mid c_2 + m,\ \Delta t\} = -m\left(\frac{\gamma}{\gamma+\lambda}\right)\left(1 - e^{-(\gamma+\lambda)\Delta t}\right) \tag{19}$$

The behavior of the expected cost change as a function of Δt is illustrated in Figure 2. When $c_2(t) = c_2$, the expected cost change is seen to tend to $m\lambda/(\gamma+\lambda)$ as Δt tends to infinity. When $c_2(t) = c_2 + m$, it tends to $-m\gamma/(\gamma+\lambda)$.

4.2 The conditions on the parameters

We can now express explicitly, in terms of the parameters m, λ, γ, and for any value of Δt, the conditions under which making use of the high cost supply source while the low cost one is still available is efficient, given Δt.

Consider first Case 1b, which is the case where the risky source is currently in the favorable state $c_2(t) = c_2$, but is more expensive that the non risky source ($c_1 < c_2$). Substituting from (18) into (11), we find that it is strictly profitable in that situation to conserve the low cost non risky supply source, in the hope of being able to use it in the event that $c_2(t)$ jumps up to the unfavorable state $c_2 + m$ at $t + \Delta t$, if (and only if):

$$c_2 - c_1 < m \left(\frac{\lambda}{\gamma + \lambda} \right) \left(\frac{1 - e^{-(\gamma+\lambda)\Delta t}}{1 - e^{-r\Delta t}} \right) e^{-r\Delta t}. \tag{20}$$

In Case 2b, the risky source is currently in the unfavorable state $c_2(t) = c_2 + m$ and is less expensive than the non risky source ($c_1 > c_2 + m$). Substituting from (19) into (11), we find that the sufficient (and necessary) condition under which it is strictly profitable to conserve the low cost risky supply in that situation, in the hope of using it in the favorable event that $c_2(t)$ falls to c_2 at $t + \Delta t$, can be written:

$$c_1 - (c_2 + m) < m \left(\frac{\gamma}{\gamma + \lambda} \right) \left(\frac{1 - e^{-(\gamma+\lambda)\Delta t}}{1 - e^{-r\Delta t}} \right) e^{-r\Delta t}. \tag{21}$$

In both inequalities (20) and (21), the left-hand side expresses the cost of the arbitrage at t and the right-hand side represents its expected discounted benefit at $t + \Delta t$. In the first case, the benefit comes from the fact that there is a positive probability that conserving the non risky source will avoid having to use the risky supply source when its cost is high, thus saving m.[6] For a given Δt, the condition is more likely to be satisfied the greater is m, the greater is λ and the lower is γ.

In the second case, the benefit comes from the fact that there is a positive probability that conserving the more expansive non risky source for later will mean giving up the possibility

[6]Notice that the right-hand side of (20) can be rewritten, using (15) as $m(1 - q(\Delta t))e^{-r\Delta t}/(1 - e^{r\Delta t})$, where $(1 - q(\Delta t))$ is the probability of a high cost regime occurring at $t + \Delta t$ when currently in the low cost regime.

of using the risky supply source when its cost is low, thus saving m. This is perhaps more naturally put in terms of the mirror image of Case 2b, generated by the alternative arbitrage described in Section 3, where it is initially hypothesized that the risky supply source is used up to date $t + \Delta t$, at which point the more expensive non risky source takes over. The arbitrage involves exchanging a unit from supply source 2 at t for a unit from supply source 1 at $t + \Delta t$. The benefit then comes from the fact that there is a positive probability that conserving the less costly risky source will permit its use when its cost has fallen to c_2 rather than at its current cost of $c_2 + m$.[7] For a given Δt, the condition is more likely to be satisfied the greater is m, the greater is γ and the lower is λ.

Notice that when Δt tends to zero, meaning that the supply source in use under the initially hypothesized scenario is about to reach exhaustion, then

$$\lim_{\Delta t \to 0} \left(\frac{1 - e^{-(\gamma+\lambda)\Delta t}}{1 - e^{-r\Delta t}} \right) e^{-r\Delta t} = \frac{\gamma + \lambda}{r}.$$

Consequently, conditions (20) and (21) reduce to:

$$c_2 - c_1 < m\frac{\lambda}{r} \tag{22}$$

$$c_1 - (c_2 + m) < m\frac{\gamma}{r} \tag{23}$$

As supply is just about to be exhausted with $c_2(t) = c_2$, the probability that, should $c_2(t)$ jump to $c_2 + m$, it will drop back to c_2 in the future becomes irrelevant in determining whether it pays to conserve the low cost source: time will have run out to profit from this possible drop in costs of the risky source. Similarly, if supply is about to be exhausted with $c_2(t) = c_2 + m$, λ becomes irrelevant.

At the other extreme, when Δt tends to infinity, the right-hand side tends to zero in both conditions (20) and (21). Hence, when Δt is infinite, conditions (20) and (21) are never satisfied. Such would be the case if the low cost supply source were infinitely abundant, either

[7] As in the previous case, the right-hand side of (21) can be rewritten, using (14) as $m(1-p(\Delta t))e^{-r\Delta t}/(1-e^{r\Delta t})$, where $(1 - p(\Delta t))$ is the probability of a low cost regime occurring at $t + \Delta t$ when currently in the high cost regime.

consisting of a resource stock of infinite size or of a technology that allowed an unlimited production flow. Clearly, in such a case, any supply policy that involved using the high cost source at any given time would be dominated by a policy that allowed the same consumption flow from the low cost source.

Since the right-hand side of both conditions (20) and (21) is a continuous function of Δt, it follows that for any given current cost spread between the high cost and low cost supply sources, there is a $\Delta t \geq 0$ beyond which it is not optimal to conserve the high cost supply source while the low cost one is still available. But as long as conditions (20) or (21) are satisfied, there will always be a range of Δt for which it is profitable to do so. Alternatively, for any finite Δt, there are values of the parameters and of the cost spread such that conditions (20) or (21) are satisfied.

5 Some special cases

The preceding analysis lends itself to some interesting specific interpretations. For one, the non risky supply source can be interpreted as an artificially constituted strategic reserve. As a second specific case, a jump in the cost of the risky source can be made to represent a complete embargo. Finally, the formulation of the problem allows us to identify a precise property on the value function that prevents the problem from being given a certainty equivalent interpretation in this particular case, but which could be satisfied in other specific circumstances under similar uncertainty about the future. We now turn briefly to those interpretations.

5.1 An artificially constituted strategic reserve

Suppose that the non risky source 1 is from an artificially constituted stockpile.[8] The difference with the natural deposit is twofold. First, the supply cost c_1 must now be interpreted as net of storage cost. It could therefore be negative, if the unit cost of storing the resource

[8]The management of strategic reserves has been studied, in differing manners, by Teisberg (1981), Hillman and Long (1983), Bergström, Loury and Persson (1985) and Devarajan and Wiener 1989)

exceeds the cost of bringing it to the market. We would then necessarily have $c_1 < c_2(t)$ and would forcibly find ourselves in either Cases 1.b or 2.a of Section 3, with Case 1.b being the interesting one. Of course, if c_1 is positive, the interesting Case 2.b ($c_1 > c_2 + m$) remains theoretically possible, although much less likely than for a natural supply source.

More importantly perhaps, if supply source 1 is an artificially created stock, the flow $x_1(t)$ is not restricted to be nonnegative, since one can draw from supply source 2 in order to add to the stockpile. The conditions just derived can be used to argue that not only can it be efficient to conserve the cheaper non risky source when the supply cost from the risky source is low ($c_2(t) = c_2 > c_1$), but also that it can be efficient to add to it by drawing from the risky supply source more than is consumed.

To see this, let $\alpha > 0$ represent the upper bound to the rate of stockpiling. We must then replace the nonnegativity constraint $x_1(t) \geq 0$ on supply source 1 by $x_1(t) \geq -\alpha$, keeping, of course, the nonnegativity constraint $x_2(t) \geq 0$ on supply source 2. Condition (7) must be modified accordingly, but condition (8) is unchanged. We know from condition (7) that as long as the two full marginal costs differ, one and only one of the constraints will be binding. In other words, unless there are no resources left, at any time t either $x_1(t) = -\alpha$ and $x_2(t) > \alpha$ or $x_1(t) > 0$ and $x_2(t) = 0$.[9] Let the initial supply policy be such that at date t source 2 is being used at a rate $x_2(t) > \alpha$, while $x_1(t) = -\alpha$, so that some stockpiling is going on at rate α. Suppose, furthermore, that starting at some date $t + \Delta t$, when the risky supply source 2 is depleted, consumption needs are met entirely from the non risky supply source 1. Now consider a transfer of one unit of supply from source 2 from t to $t + \Delta t$ and compensate by transferring one unit of supply from source 1 from $t + \Delta t$ to t.[10] This leaves the consumption path unchanged, but changes the expected total cost, conditional on Δt and $c_2(t)$, by an amount given by (12). As already pointed out, this is the negative of (10) and the cases obtained for $c_2(t) = c_2$ and $c_2(t) = c_2 + m$ are simply the mirror images of

[9] Recall that under the assumptions on the utility function, a positive quantity will be consumed at any given time unless both sources are exhausted. Therefore if $x_1(t) = -\alpha$, not only must $x_2(t)$ be strictly positive, it must exceed α in order to have $x_1(t) + x_2(t) > 0$.

[10] This is the alternative arbitrage considered at the end of Section 3.

Case 1 and Case 2 respectively.

It follows that if, at time t, $c_2(t) = c_2 > c_1$, then conserving the more costly risky source for later use is profitable only if the positive expected change in cost $(E\{\Delta c_2 | c_2(t), \Delta t\})$ is not so large that it more than offsets the gain that can be had by producing at t at cost c_1 rather than at the higher cost c_2. Condition (20) on the parameters would then be violated. Otherwise, condition (20) is satisfied and, not only does it become optimal to use the higher cost risky source now, while its cost is c_2, in the hope of avoiding having to use it when its cost is $c_2 + m$, but it is optimal to draw more than is required to meet consumption needs in order to add to the artificial stocks, at the maximal rate α.

If ever the unlikely case $c_2(t) = c_2 + m < c_1$ turned out to pertinent, then, if condition (21) is satisfied, it would make sense to draw down the more costly non risky stockpile in the hope of being able to use the risky source in its more favorable state.

5.2 A complete embargo

A situation of complete temporary embargo can be captured by assuming m to be infinite in (1). Thus, when $c_2(t) = c_2$, there is a probability λdt that it will become infinite in the interval of time dt. When this occurs, supply from source 2 is ruled out for the duration of the embargo, being prohibitively costly. However, when the risky source is in a state of embargo, there is a probability γdt that this embargo will be lifted during the interval dt, as cost drops back to c_2.

It goes without saying that Case 2 is now irrelevant, since $c_2 + m = \infty$. But Case 1 remains, being conditional on $c_2(t) = c_2$. If the state at time t is such that $c_2(t) = c_2 > c_1$ (Case 1b), then, given the positive probability that an embargo will occur over the interval dt, in the form of an infinitely large expected cost change, conserving the cheap supply source 1 is always profitable. This can be seen directly from condition (20), by setting $m = \infty$ in the right-hand side expression. Not only is it always optimal to conserve the low cost non risky supply source when faced with the probability of an embargo, but, if the low cost source in question is an artificial stockpile, it is always profitable to add to it at the maximal rate.

Of course, if the arbitrage is profitable at t, it does not mean that it will be profitable forever, since an embargo may occur in the future during which consumption will have to rely strictly on the non risky supply source.

5.3 Certainty equivalence

Assume $\gamma = 0$. Then the stochastic process (1) describes a situation where there is a probability of a once and for all future jump up in the cost of the risky source, with no chance of it dropping back if the upward jump ever occurred. Since all uncertainty is then lifted if ever $c_2(t) = c_2 + m$, we only have to consider the state $c_2(t) = c_2$, and therefore:

$$E\{\Delta V | c_2(t)\} = \lambda[V(X_1(t), X_2(t), c_2 + m)) - V(X_1(t), X_2(t), c_2)].$$

The Bellman equation (6) can then be written:

$$\begin{aligned}(r+\lambda)V(X_1(t), X_2(t), c_2) \quad = \quad & \max_{x_1(t), x_2(t)} \{U(x_1(t) + x_2(t)) - c_1 x_1(t) - c_2 x_2(t) \\ & -V_{X_1}(X_1(t), X_2(t), c_2) x_1(t) - V_{X_2}(X_1(t), X_2(t), c_2) x_2(t) \\ & +\lambda V(X_1(t), X_2(t), c_2 + m)\} \end{aligned} \tag{24}$$

All the analysis pertaining to Case 1 still goes through, except for having $\gamma = 0$ in condition (20).

Notice that if and only $V(X_1(t), X_2(t), c_2 + m) = 0$, then the problem can be written in a certainty equivalent form, by simply adjusting the discount rate by the factor λ to take account of the uncertainty about the future. In a context where there are two sources of supply, one of them being certain, it would not make sense to assume that because the cost of the risky source jumps up, the value of the remaining stocks becomes zero. This is true even with $m = \infty$.[11] For this reason, certainty equivalence does not hold for the problem at hand. However, in a context where the only source of supply was the risky one, one could imagine conditions on the utility function such that the jump would render the resource

[11] Nor would it make sense if we had $\gamma > 0$, for then not only are there two sources of supply, but there is always the possibility that the cost will jump down in the future after having jumped up.

worthless and certainty equivalence could be used to carry out the analysis.[12]

6 Conclusion

In a world where the future is known with certainty, it is optimal to want to exhaust the cheaper supply source before moving on to a more costly one. This makes sense because, when the future is discounted at a positive rate, postponing the high costs as long as possible minimizes the total cost of a given consumption path. Uncertainty about the future can introduce ambiguity into this conclusion, as we have shown.

We have captured this uncertainty by assuming that when the current cost of a risky supply source is low, there is a positive probability that it will jump up in the future and, when the current cost is high, there is a probability that it will jump down. A feature of such a stochastic process is that the expected change in cost over any positive interval of time is different from zero. We believe this is a good representation of reality, given the important geopolitical risk to which is subject the supply of a number of nonrenewable resources. In such a world, whether the efficient use of multiple supply sources requires conserving the currently cheaper source will depend on the expected future change in costs of the risky source. If the cheaper source currently happens to be the non risky one, then you may want to conserve it for future use if the expected change in cost of the risky source is positive. Even if the cheaper source happens to be the risky source, you may want to conserve it for future use if the expected change in cost is negative. In both cases we have characterized the conditions on the parameters under which it makes sense to conserve the cheap source for future use rather than consuming it while it is known to be cheaper. In fact, if the non risky source is interpreted as an artificially constituted stockpile, by opposition to a natural deposit, it can make sense, for appropriate values of the parameters, to add to it, even though it would currently be cheaper to use it than to use the risky source.

Our analysis includes, as a special limiting case, that of a complete embargo on supply,

[12] An example of this might be, under certain circumstances, the exploitation of a single source of supply under threat of expropriation, a subject dealt with in Long (1975).

which is akin to an infinite upward jump in cost. Even if there is a probability that, once in place, the embargo will be lifted at some subsequent time, it is *always* optimal to conserve a cheap non risky source when faced with the probability of a future embargo. In this extreme case, this is the only way of guaranteeing a supply source in the eventuality of a complete embargo on the risky source. At the other extreme, as the size of the possible jump in cost tends to zero, we recover the deterministic case, under which it is *always* optimal to use the currently cheaper supply source.

We have assumed throughout that one of the two sources is non risky, in the sense that its future cost is known with certainty. This allows for sharper results than if the future cost of both sources were uncertain, but to a different degree. The analysis could easily be extended to treat such a case, but with little gain in insight. Adding uncertainty as to the size of the jump can also be handled without great difficulty, but with little gain.

Appendix

We prove in this Appendix the statement made in (9) and use it to show that it follows from the Bellman equation (6) that

$$e^{-rdt}E\{V_{X_2}(X_1(t+dt), X_2(t+dt), c_2)\} - V_{X_2}(X_1(t), X_2(t), c_2) = 0$$

is a necessary condition for optimality.

Consider the case where $c_2(t) = c_2$ (the adaptation for $c_2(t) = c_2 + m$ is immediate), and the scenario where $x_2(t) > 0$ and $x_1(t) = 0$ (again the adaptation to the alternative case is immediate). Then, for a small interval of time dt, the expected value of $V_{X_2}(X_1(t), X_2(t), c_2(t))$ at $t+ dt$ is

$$\begin{aligned} E\{e^{-rdt}V_{X_2}(X_1(t+dt), X_2(t+dt), c_2(t+dt))\} = \\ & \lambda dt e^{-rdt}V_{X_2}(X_1(t+dt), X_2(t+dt), c_2+m) \\ + & (1-\lambda dt)\, e^{-rdt}V_{X_2}(X_1(t+dt), X_2(t+dt), c_2). \end{aligned}$$

Now calculate $e^{-rdt}V_{X_2}(X_1(t+dt), X_2(t+dt), c_2+m)$ and $e^{-rdt}V_{X_2}(X_1(t+dt), X_2(t+dt), c_2)$ by expanding these functions around $(dt = 0, X_2(t))$, treating in the first instance $X_1(t)$ and c_2+m as parameters since $X_1(t)$ does not change over the interval and c_2+m is constant, and treating $X_1(t)$ and c_2 as parameters in the second instance. This gives, using $dX_2 = -x_2(t)dt$ and with $c_2(t) = c_2$ by assumption:

$$\begin{aligned} E\{e^{-rdt}V_{X_2}(X_1(t+dt), X_2(t+dt), c_2)\} = \\ & \lambda dt\, [V_{X_2}(X_1(t), X_2(t), c_2+m) - rdtV_{X_2}(X_1(t), X_2(t), c_2+m) \\ & - V_{X_2X_2}(X_1(t), X_2(t), c_2+m)x_2(t)dt] \\ & + (1-\lambda dt)\, [V_{X_2}(X_1(t), X_2(t), c_2) - rdtV_{X_2}(X_1(t), X_2(t), c_2) \\ & - V_{X_2X_2}(X_1(t), X_2(t), c_2)x_2(t)dt] \end{aligned}$$

The terms in $(dt)^2$ can be neglected, since they go to zero faster than dt as $dt \to 0$. Hence this becomes:

$$
\begin{aligned}
e^{-rdt}E\{V_{X_2}(X_1(t+dt),X_2(t+dt),c_2)\} &- V_{X_2}(X_1(t),X_2(t),c_2) = \qquad (25)\\
&+ \{\lambda\left[V_{X_2}(X_1(t),X_2(t),c_2+m)-V_{X_2}(X_1(t),X_2(t),c_2)\right]\\
&- rV_{X_2}(X_1(t),X_2(t),c_2)-V_{X_2X_2}(X_1(t),X_2(t),c_2)x_2(t)\}dt,
\end{aligned}
$$

which establishes (9) for $c_2\,(t)=c_2$. The expression that applies when $c_2\,(t)=c_2+m$ is obtained similarly.

The Bellman equation is:

$$
\begin{aligned}
rV(X_1(t),X_2(t),c_2(t)) &= \max_{x_1(t),x_2(t)}\{U(x_1(t)+x_2(t))-c_1x_1(t)-c_2(t)x_2(t)\\
&\quad -V_{X_1}(X_1(t),X_2(t),c_2(t))x_1(t)-V_{X_2}(X_1(t),X_2(t),c_2(t))x_2(t)\\
&\quad +E\{\Delta V|c_2(t)\},\}
\end{aligned}
$$

with $E\{\Delta V|c_2(t)\}$ defined as in Section 2. As noted also in Section 2, since the two sources are perfect substitutes ($\frac{\partial U}{\partial x_1}=\frac{\partial U}{\partial x_2}$), $x_i\,(t)\geq 0$, and the full total cost is linear in $x_i(t)$, the solution to the maximization entails either $x_1(t)>0$ and $x_2(t)=0$, or $x_1(t)=0$ and $x_2(t)>0$, or $x_1(t)=x_2(t)=0$ when the two sources are exhausted.[13] Therefore, if at date t, $x_2(t)>0$ and $x_1(t)=0$ (the adaptation to the alternative case is immediate) and $c_2(t)=c_2$ (the adaptation to the case of $c_2(t)=c_2+m$ is also immediate), the Bellman equation can be written, after replacing $x_2(t)$ by its optimal value $x_2^*(t)$, as:

$$
\begin{aligned}
(r+\lambda)\,V(X_1(t),X_2(t),c_2) &= U(x_2^*\,(t))-c_2(t)x_2^*\,(t)-V_{X_2}(X_1(t),X_2(t),c_2(t))x_2^*\,(t)\\
&\quad +\lambda V(X_1(t),X_2(t),c_2+m)\}
\end{aligned}
$$

Differentiating both sides totally, while applying the envelope theorem and noting that $dX_2\,(t)=-x_2^*\,(t)$ and $dX_1\,(t)=0$ (since $x_1^*(t)=0$ by assumption), we obtain:

$$
(r+\lambda)V_{X_2}(X_1(t),X_2(t),c_2)=\lambda V_{X_2}(X_1(t),X_2(t),c_2+m)-V_{X_2X_2}(X_1(t),X_2(t),c_2(t))x_2^*(t).
$$

[13] It is only at transition dates, when $c_1+V_{X_1}=c_2+V_{X_2}$, that the two full marginal costs do not differ, but that cannot last for any positive interval of time.

Substituting into the right-hand side of (25), we find that it is indeed zero.

This proves the claim that $e^{-rdt}E\{V_{X_2}(X_1(t+dt), X_2(t+dt), c_2)\} - V_{X_2}(X_1(t), X_2(t), c_2) = 0$ is a necessary condition. The proof for the case of $c_2(t) = c_2 + m$ proceeds in the same way, *mutatis mutandis*. As argued in Section 2, it follows that (8) is also necessary.

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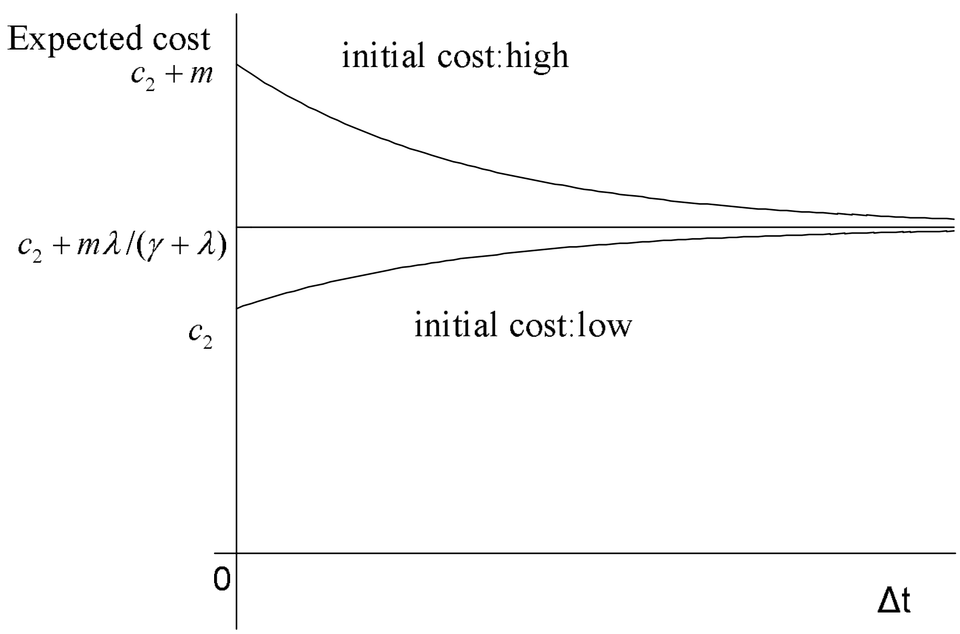

Figure 1: Expected cost after a period Δt has elapsed

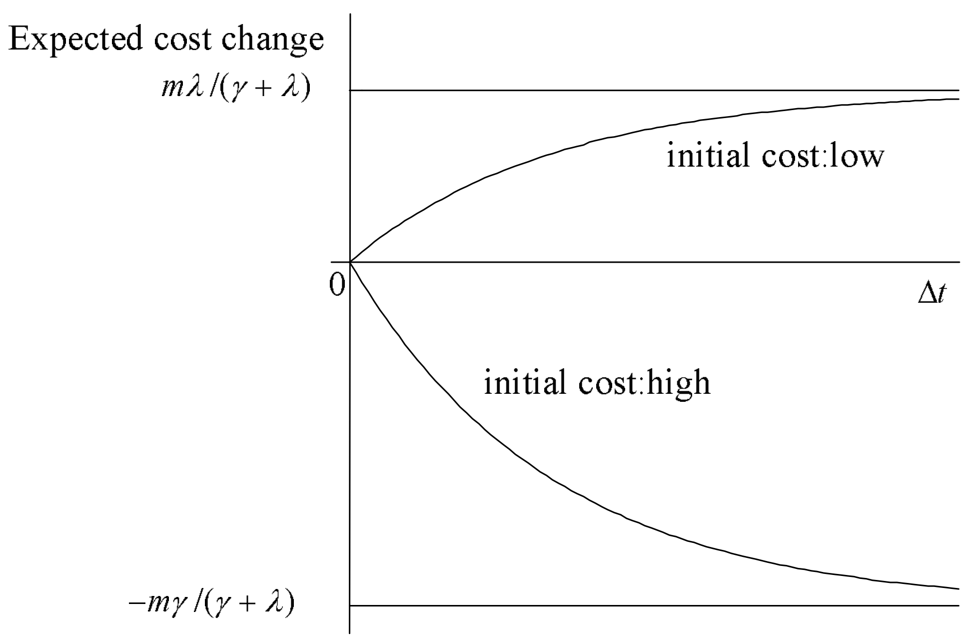

Figure 2: Expected cost change after a period Δt has elapsed