Strategy-proof choice under monotonic additive preferences

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Abstract

We describe the class of strategy-proof mechanisms for choosing sets of objects when preferences are additive and monotonic. *JEL classification numbers: D71.*

1. Introduction

We study strategy-proof mechanisms for choosing a *set* of objects from a given collection of such sets. We assume that the objects are valuable, public in nature, and that their consumption generates no cross-effects. Under these assumptions, a typical preference is naturally represented by a *measure*, i.e., a monotonic additive set function.

Choosing independent public projects or selecting candidates to a given set of unrelated positions are classic illustrations of our model. Collective choice under uncertainty is a perhaps less familiar application. In this case, the objects are the possible *states of nature* and the social decision problem consists in choosing a *bet*, i.e., an uncertain prospect where "collective success" is achieved if a specific event occurs, and failure ensues otherwise. Since a subjective-expected-utility agent ranks bets according to the subjective probability she attaches to the events characterizing them, her preferences are represented by a probability measure.

In all these examples, specifying the class of *feasible* sets of objects is fundamental. Feasibility constraints may take various forms: local public projects must normally be chosen under some aggregate budget constraint, recruitment decisions may involve capacity constraints, lower bounds, diversity requirements, and so on. It is therefore important to study mechanisms whose range may be constrained in many different ways. For any collection of feasible sets, the theorem we shall prove describes the class of strategy-proof mechanisms whose range coincides with that collection.

To illustrate our result, suppose there are three agents and the set of relevant objects is $X = \{a, b, c, d, e\}$. Let agent 1 select which of the sets $\{a, b\}, \{b, c\}, \{a, c\}$ will be

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part of the selected set, and let the agents decide by majority vote which of $\{d\}$ or $\{e\}$ will be included as well. The selected set is the union of these two separate choices. If preferences are additive, it is easy to see that this mechanism is strategy-proof. Note that not all subsets of X can be chosen: the range of the mechanism contains the six sets $\{a,b,d\}$, $\{b,c,d\}$, $\{a,c,d\}$, $\{a,b,e\}$, $\{b,c,e\}$, $\{a,c,e\}$. Observe also that (i) no set in the range is a strict subset of another, (ii) every set in the range is the union of one of the sets $\{a,b\}, \{b,c\}, \{a,c\}$ and one of the sets $\{d\}, \{e\}$, (iii) the choice between $\{a,b\}, \{b,c\}, \{a,c\}$ is dictatorial whereas the choice between $\{d\}$ and $\{e\}$ is majority-based. As we shall see, all strategy-proof mechanisms defined for monotonic additive preferences have a structure similar to the one in this example.

Our work belongs to the literature on strategy-proofness in contexts where the set of social alternatives has a Cartesian product structure and preferences are separable or additive. To see the connection, note that the collection of all subsets of a finite set of objects $\{1, ..., m\}$ may be identified with the Cartesian product $\{0,1\}^m$. Early works in the field, such as Barberà, Sonnenschein and Zhou (1991) and Le Breton and Sen (1999), describe the unconstrained strategy-proof mechanisms –those whose range is the entire set of alternatives.

Subsequent contributions explore the consequences of imposing feasibility constraints on the range of the mechanism. Barberà, Massó and Neme (2005) characterize all the strategy-proof mechanisms for choosing sets of objects (or, equivalently, alternatives in the cube $\{0,1\}^m$) when preferences are either additive or separable. Reffgen and Svensson (2012) generalize the analysis to the case where alternatives form an arbitrary finite product set.

Compared to this strand of work, the specificity of the current paper lies in the assumption that preferences are *monotonic.* We believe that this restriction is worth studying because of its importance in many applications. Although the structure of strategy-proof mechanisms may in general be very sensitive to the restrictions imposed on individual preferences[1], it turns out that this is not the case here: restricting attention to the domain of monotonic additive preferences does not substantially alter the characterization offered in Theorem 1 of Barberà, Massó and Neme (2005).

2. Setup

Let $N = \{1, ..., n\}$, $n \geq 2$, be the set of *agents* and let $X = \{1, ..., m\}$, $m \geq 2$, be a set of *objects*. The set of (social) *alternatives* is $\mathcal{X} := \{A : \varnothing \neq A \subseteq X\}$. We emphasize that each alternative is a *set* of objects. Throughout the paper, $\subseteq$ denotes inclusion and $\subset$ is reserved for strict inclusion.

Agent i's *preference* is a linear order P_i on $\mathcal{X}$. For any nonempty subset of alternatives $\mathcal{A} \subseteq \mathcal{X}$, let $\tau(P_i; \mathcal{A})$ denote the maximal element of P_i in $\mathcal{A}$. We call a preference P_i

[1] Compare Theorem 1 and Theorem 2 in Barberà, Massó and Neme (2005).

additive if there exists a function $u_i : X \to \mathbb{R}$ such that

$$AP_iB \Leftrightarrow \sum_{x\in A} u_i(x) > \sum_{x\in B} u_i(x) \text{ for all } A, B \in \mathcal{X}.$$

We then say that P_i is *represented* by u_i. We call P_i *monotonic additive* if it is represented by a function u_i such that $u_i(x) > 0$ for all $x \in X$. If P_i is monotonic additive, then AP_iB whenever $B \subset A$. Let $\mathcal{P}_a$ denote the set (or *domain*) of additive preferences and $\mathcal{P}_{ma}$ the domain of monotonic additive preferences. We use $\mathcal{P}$ to denote an arbitrary domain of preferences.

For any domain $\mathcal{P}$, a *profile (of preferences in $\mathcal{P}$)* is a list $P = (P_1, ..., P_n) \in \mathcal{P}^N$. If $i \in N$ and $P_i' \in \mathcal{P}$, we denote by $(P_i', P_{-i}) \in \mathcal{P}$ the profile obtained from P by replacing agent i's preference P_i with P_i'. A *social choice function (SCF) on the domain $\mathcal{P}$* is a function $f : \mathcal{P}^N \to \mathcal{X}$. The *range* of f is the set $\mathcal{R}_f = \left\{f(P) : P \in \mathcal{P}^N\right\}$. The SCF f is manipulable by $i \in N$ at $P \in \mathcal{P}^N$ via the report $P_i' \in \mathcal{P}$ if $f(P_i', P_{-i})P_if(P)$. If f is not manipulable by any agent at any profile via any report, it is *strategy-proof*. Our purpose is to describe the class of strategy-proof SCFs on the domain of monotonic additive preferences $\mathcal{P}_{ma}$.

3. Characterization

For any nonempty set of alternatives $\mathcal{A} \subseteq \mathcal{X}$, a *decomposition* of $\mathcal{A}$ is a collection $\{\mathcal{A}_1, ..., \mathcal{A}_L\}$ such that

$$\begin{aligned}
\text{(i) } \mathcal{A}_l &\subseteq \mathcal{X} \text{ for } l = 1, ..., L, \\
\text{(ii) } A_l \cap A_{l'} &= \varnothing \text{ for all } A_l \in \mathcal{A}_l,\ A_{l'} \in \mathcal{A}_{l'},\ l \neq l', \\
\text{(iii) } \mathcal{A} &= \left\{ \bigcup_{l=1}^{L} A_l : (A_1, ..., A_L) \in \mathcal{A}_1 \times ... \times \mathcal{A}_L \right\}.
\end{aligned}$$

A decomposition of $\mathcal{A}$ is *maximal* if there is no other decomposition $\{\mathcal{A}_1', ..., \mathcal{A}_{L'}'\}$ of $\mathcal{A}$ such that $L' > L$. Barberà, Massó and Neme (2005) prove that every set $\mathcal{A} \subseteq \mathcal{X}$ has a unique maximal decomposition; see also Svensson and Torstensson (2008). The sets $\mathcal{A}_1, ..., \mathcal{A}_L$ are the *components* of $\mathcal{A}$.

If $f : \mathcal{P}^N \to \mathcal{X}$ is a SCF on the domain $\mathcal{P}$ and $\{\mathcal{A}_1, ..., \mathcal{A}_L\}$ is the maximal decomposition of its range $\mathcal{R}_f$, then there exist well-defined SCFs $f_1 : \mathcal{P}^N \to \mathcal{A}_1$, ..., $f_L : \mathcal{P}^N \to \mathcal{A}_L$ such that $f(P) = f_1(P) \cup ... \cup f_L(P)$ for all $P \in \mathcal{P}^N$. If $\mathcal{A}_l$ contains a single alternative, say A_l, then obviously $f_l(P) = A_l$ for all $P \in \mathcal{P}^N$. For any $l \in \{1, ..., L\}$, we say that f is *dictatorial on* $\mathcal{A}_l$ if there is an agent $i \in N$ such that $f_l(P) = \tau(P_i; \mathcal{A}_l)$ for all $P \in \mathcal{P}^N$. If $l \in \{1, ..., L\}$ is such that $\mathcal{A}_l$ contains exactly two alternatives, say, A_l and B_l, we say that f is *voting by committees on* $\mathcal{A}_l$ if there exists a nonempty, inclusion-monotonic family $\mathcal{W}_{A_l}$ of nonempty subsets of N such that

$$f_l(P) = \begin{cases} A_l \text{ if } \{i \in N : A_lP_iB_l\} \in \mathcal{W}_{A_l}, \\ B_l \text{ otherwise.} \end{cases}$$

Of course, an equivalent definition obtains by permuting the roles of the alternatives A_l, B_l.[2]

Let us begin by recalling Barberà, Massó and Neme's (2005) characterization of the strategy-proof SCFs on the domain of all additive preferences.

Theorem 1 (Barberà, Massó and Neme, 2005)[3]. *Let $f : \mathcal{P}_a^N \to \mathcal{X}$ be a SCF and let $\{\mathcal{A}_1, ..., \mathcal{A}_L\}$ be the maximal decomposition of $\mathcal{R}_f$. Then f is strategy-proof if and only if (i) f is dictatorial on $\mathcal{A}_l$ for all l such that $|\mathcal{A}_l| \geq 3$ and (ii) f is voting by committees on $\mathcal{A}_l$ for all l such that $|\mathcal{A}_l| = 2$.*

We now turn to our characterization on the domain of monotonic additive preferences. Say that a set of alternatives $\mathcal{A} \subseteq \mathcal{X}$ is a *clutter* if there are no $A, B \in \mathcal{A}$ such that $A \subset B$.

Theorem 2. *Let $f : \mathcal{P}_{ma}^N \to \mathcal{X}$ be a SCF and let $\{\mathcal{A}_1, ..., \mathcal{A}_L\}$ be the maximal decomposition of $\mathcal{R}_f$. Then f is strategy-proof if and only if (i) $\mathcal{R}_f$ is a clutter, (ii) f is dictatorial on $\mathcal{A}_l$ for all l such that $|\mathcal{A}_l| \geq 3$, and (iii) f is voting by committees on $\mathcal{A}_l$ for all l such that $|\mathcal{A}_l| = 2$.*

4. Proof

We derive Theorem 2 from Theorem 1. Let $f : \mathcal{P}_{ma}^N \to \mathcal{X}$ be a SCF and let $\{\mathcal{A}_1, ..., \mathcal{A}_L\}$ be the maximal decomposition of $\mathcal{R}_f$. It is easy to check that f is strategy-proof if conditions (i), (ii), and (iii) hold. Conversely, suppose that f is strategy-proof.

Step 1. The range $\mathcal{R}_f$ is a clutter.

Substep 1.1. The SCF f is unanimous on its range: for all $A \in \mathcal{R}_f$ and all $P \in \mathcal{P}_{ma}^N$, $[\tau(P_i; \mathcal{R}_f) = A \text{ for all } i \in N] \Rightarrow [f(P) = A]$.

The argument is standard. Fix $A \in \mathcal{R}_f$ and $P \in \mathcal{P}_{ma}^N$; and suppose that $\tau(P_i; \mathcal{R}_f) = A$ for all $i \in N$. Since $A \in \mathcal{R}_f$, there exists $\overline{P} \in \mathcal{P}_{ma}^N$ such that $f(\overline{P}) = A$. Consider the sequence of preference profiles $P^0 = \overline{P}$, $P^1 = (P_1, \overline{P}_{-1})$, ..., $P^n = P$ and apply strategy-proofness repeatedly to obtain $A = f(\overline{P}) = f(P^0) = f(P^1) = ... = f(P^n) = f(P)$.

Substep 1.2. The range $\mathcal{R}_f$ is a clutter.

Let $A, B \in \mathcal{R}_f$ and suppose, contrary to the claim, that $A \subset B$. Without loss of generality, assume that

$$\text{there is no } C \in \mathcal{R}_f \text{ such that } B \subset C. \tag{4.1}$$

For each $i \in N$, choose a function $u_i^B : X \to \mathbb{R}_{++}$ such that

$$u_i^B(x) > \sum_{y \in X \setminus A} u_i^B(y) \text{ for all } x \in A, \tag{4.2}$$

$$u_i^B(x) > \sum_{y \in X \setminus B} u_i^B(y) \text{ for all } x \in B, \tag{4.3}$$

[2] Namely: f is voting by committee on $\mathcal{A}_l$ if there exists a nonempty, inclusion-monotonic family $\mathcal{W}_{B_l}$ of nonempty subsets of N such that $f_l(P) = B_l$ if $\{i \in N : B_l P_i A_l\} \in \mathcal{W}_{B_l}$ and $f_l(P) = A_l$ otherwise. To see that the two definitions are equivalent, simply define $\mathcal{W}_{B_l} = \{S : N \setminus S \notin \mathcal{W}_{A_l}\}$.

[3] This is Theorem 1, p. 196. Our reformulation follows that Reffgen and Svensson (2012), p. 671.

and let $P_i^B \in \mathcal{P}_{ma}$ be the monotonic additive preference represented by u_i^B. Because of (4.1) and (4.3), $\tau(P_i^B; \mathcal{R}_f) = B$. Letting $P^B = (P_1^B, ..., P_n^B)$, Substep 1.1 implies $f(P^B) = B$.

Since $A \in \mathcal{R}_f$, there exists $P^A \in \mathcal{P}_{ma}^N$ such that $f(P^A) = A$. Consider the sequence of preference profiles

$$P^0 = P^A,\ P^1 = (P_1^B, P_{-1}^A),\ \dots,\ P^n = P^B.$$

If $f(P^0) \subset f(P^1)$, f is manipulable by agent 1 at P^0 via P_1^B. If $f(P^1) \subset f(P^0)$, f is manipulable by agent 1 at P^1 via P_1^A. It follows that either

$$\text{(i) } f(P^0) \setminus f(P^1) \neq \varnothing \text{ and } f(P^1) \setminus f(P^0) \neq \varnothing,$$

or

$$\text{(ii) } f(P^0) = f(P^1).$$

Suppose first that (i) holds. Because the first statement in (i) means that $f(P^A) \setminus f(P_1^B, P_{-1}^A) = A \setminus f(P_1^B, P_{-1}^A) \neq \varnothing$, (4.2) implies that $f(P^A) P_1^B f(P_1^B, P_{-1}^A)$, contradicting strategy-proofness. Therefore (ii) must hold, and $A = f(P^0) = f(P^1)$. Repeating the argument yields $A = f(P^0) = f(P^1) = ... = f(P^n) = f(P^B)$, contradicting the fact that $f(P^B) = B$.

Step 2. The SCF f is dictatorial on each component $\mathcal{A}_l$ such that $|\mathcal{A}_l| \geq 3$ and f is voting by committees on each component $\mathcal{A}_l$ such that $|\mathcal{A}_l| = 2$.

The proof consists in extending the SCF $f : \mathcal{P}_{ma}^N \to \mathcal{X}$ to a SCF $f^+ : \mathcal{P}_a^N \to \mathcal{X}$ which is strategy-proof and has the same range as f; it then suffices to apply Theorem 1 in Barberà, Massó and Neme (2005) to conclude. The key step in extending f is to show that any linear order *on the range* $\mathcal{R}_f$ which is the restriction to $\mathcal{R}_f$ of an additive preference on $\mathcal{X}$ is also the restriction to $\mathcal{R}_f$ of a *monotonic* additive preference on $\mathcal{X}$:

Substep 2.1. For every $P_i \in \mathcal{P}_a$ there exists $P_i^+ \in \mathcal{P}_{ma}$ such that $AP_iB \Leftrightarrow AP_i^+B$ for all $A, B \in \mathcal{R}_f$.

To prove this claim, fix $P_i \in \mathcal{P}_a$ and let $u_i : X \to \mathbb{R}$ represent P_i. Define $\alpha := \min_{x \in X} u(x)$. If $\alpha > 0$, then $P_i \in \mathcal{P}_{ma}$ and the claim is trivially true. From now on, assume $\alpha \leq 0$. Note that $\alpha < 0$ because P_i is a linear order.

Index the alternatives in the range of f so that $\mathcal{R}_f = \{A_t : t \in T\}$. For every (possibly empty) set $S \subseteq T$, define

$$a_S = \left(\bigcap_{t \in S} A_t\right) \setminus \left(\bigcup_{t \in T \setminus S} A_t\right)$$

with the convention that $\bigcap\limits_{t \in \varnothing} A_t = X$. Observe that for every $x \in X$ there is a unique (possibly empty) set $S(x) \subseteq T$ such that $x \in a_{S(x)}$. Choose $\delta > -\alpha \max_{S \subseteq T} |a_S|$, define

$$u_i^+(x) = u_i(x) + \frac{\delta}{|a_{S(x)}|} \text{ for all } x \in X,$$

and let $P_i^+ \in \mathcal{P}_a$ be the additive preference represented by u_i^+. Since for all $x \in X$

$$u_i^+(x) > u_i(x) - \frac{\alpha \max_{S \subseteq T} |a_S|}{\left|a_{S(x)}\right|} \geq u_i(x) - \alpha \geq 0,$$

we have $P_i^+ \in \mathcal{P}_{ma}$.

It remains to check that the restrictions of P_i and P_i^+ to $\mathcal{R}_f$ coincide. To this end, observe first that for every $S \subseteq T$,

$$\sum_{x \in a_S} u_i^+(x) = \sum_{x \in a_S} u_i(x) + \delta.$$

It follows that for every $t \in T$,

$$\begin{aligned}
\sum_{x \in A_t} u_i^+(x) &= \sum_{S \subseteq T:\, t \in S} \; \sum_{x \in a_S} u_i^+(x) \\
&= \sum_{S \subseteq T:\, t \in S} \left(\sum_{x \in a_S} u_i(x) + \delta \right) \\
&= \sum_{S \subseteq T:\, t \in S} \; \sum_{x \in a_S} u_i(x) + \delta \left|\{S \subseteq T : t \in S\}\right| \\
&= \sum_{x \in A_t} u_i(x) + \delta \left|\{S \subseteq T : t \in S\}\right| \\
&= \sum_{x \in A_t} u_i(x) + 2^{|T|-1} \delta.
\end{aligned}$$

Hence, for any $t, t' \in T$, $\sum_{x \in A_t} u_i^+(x) - \sum_{x \in A_{t'}} u_i^+(x) = \sum_{x \in A_t} u_i(x) - \sum_{x \in A_{t'}} u_i(x)$, which implies that $A_t P^+ A_{t'} \Leftrightarrow A_t P A_{t'}$.

Substep 2.2. Conclusion of the proof.

For every $P \in \mathcal{P}_a^N$, choose a profile $P^+ \in \mathcal{P}_{ma}^N$ such that $A P_i^+ B \Leftrightarrow A P_i B$ for all $A, B \in \mathcal{R}_f$ and all $i \in N$. Define $f^+ : \mathcal{P}_a^N \to \mathcal{X}$ by

$$f^+(P) = f(P^+).$$

By construction, f^+ is a SCF defined on the domain of all additive preferences and it has the same range as f, i.e., $\mathcal{R}_{f^+} = \mathcal{R}_f$. Strategy-proofness of f implies that (i) $f^+(P) = f(P)$ for all $P \in \mathcal{P}_{ma}^N$ (because the alternatives selected by f at two profiles that coincide on the range must be the same) and (ii) f^+ is strategy-proof. Since $\{\mathcal{A}_1, ..., \mathcal{A}_L\}$ is the maximal decomposition of $\mathcal{R}_{f^+}$, it follows from (ii) and Theorem 1 in Barberà, Massó and Neme (2005) that f^+ is dictatorial on each component $\mathcal{A}_l$ such that $|\mathcal{A}_l| \geq 3$ and is voting by committees on each component $\mathcal{A}_l$ such that $|\mathcal{A}_l| = 2$. It now follows from (i) that f itself is dictatorial on each $\mathcal{A}_l$ such that $|\mathcal{A}_l| \geq 3$ and is voting by committees on each $\mathcal{A}_l$ such that $|\mathcal{A}_l| = 2$.■

References

[1] Barberà, S., Massó, J., and Neme, A. (2005), "Voting by committees under constraints," *Journal of Economic Theory*, **122**, 185-205.

[2] Barberà, S., Sonnenschein, H., and Zhou, L. (1991), "Voting by committees," *Econometrica,* **59**, 595–609.

[3] Le Breton, M. and Sen, A. (1999), "Separable preferences, strategyproofness and decomposability," *Econometrica,* **67**, 605–628.

[4] Reffgen, A. and Svensson, L.-G. (2012), "Strategy-proof voting for multiple public goods," *Theoretical Economics*, **7**, 663-688.

[5] Svensson, L.-G. and Torstensson, P. (2008), "Strategy-proof allocation of multiple public goods," *Social Choice and Welfare,* **30**, 181-196.