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Three essays in empirical asset pricing

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RÉSUMÉ

Le thème principal de cette thèse est la rationalisation de nouveaux faits empiriques mettant en évidence les rendements d'actifs financiers et certaines variables macroéconomiques. Les modèles d'évaluation d'actifs financiers supposent que les risques qui affectent les opportunités d'investissement sont liés à un ou plusieurs facteurs macroéconomiques et sont compensés par les rendements. Ces modèles cherchent dès lors à déterminer les principaux facteurs de risque auxquels les investisseurs portent le plus d'attention. D'intéressantes propriétés de la volatilité de la consommation en rapport avec l'évaluation d'actifs financiers ont été mises en exergue dans des études récentes, mais elles ont été principalement reliées à la dimension temporelle des rendements. Une contribution majeure de notre recherche sera de caractériser et de mesurer son impact sur les rendements en coupe transversale.

Le premier chapitre documente des faits empiriques montrant l'existence d'une relation robuste entre l'incertitude macroéconomique et les rendements d'actions. Il met en évidence le fait que les investisseurs de long terme s'intéressent non seulement à la variation des niveaux futur et présent de la consommation (risque de long terme dans le niveau de la consommation), mais aussi et peut-être plus à la variation entre les niveaux futur et présent de l'incertitude entourant cette croissance de la consommation (risque de long terme dans la volatilité de la consommation). Nous montrons que les différences entre les primes de risque de divers portefeuilles d'actions sont également dues à l'hétérogénéité dans leur exposition aux risques liés à la volatilité de la consommation. Les faits empiriques documentés dans ce chapitre suggèrent que les risques liés à la volatilité de la consommation sont fortement corrélés aux primes de risque pour divers horizons d'investissement, plus que les risques liés au niveau de la consommation pour des investissements longs et moins pour les investissements courts. En plus, le risque de long terme lié à la volatilité est valorisé même en présence du risque de long terme lié au niveau. Cette étude est théoriquement motivée par un modèle d'évaluation d'actifs financiers par équilibre dans lequel la consommation suit un processus affine à volatilité stochastique. Un calibrage bien mené de ce modèle d'équilibre général rationaliserait

ces évidences empiriques.

Ensuite, nous analysons les problèmes numériques, analytiques et statistiques qui affectent certaines conclusions tirées de modèles existants d'évaluation d'actifs financiers. Les modèles basés sur la consommation ont regagné de l'intérêt avec de nouveaux liens mis en évidence entre la volatilité du marché et les rendements, l'évaluation des portefeuilles de long terme, ou la prévisibilité des rendements. Des liens sont établis entre les primes de risque et différents types de préférences, où la séparation entre l'aversion au risque et l'élasticité de substitution intertemporelle, et la formation d'habitude sont centrales. Souvent, la solution à ces modèles nécessite une approximation et les quantités d'intérêt sont calculées par simulation.

Le deuxième chapitre propose une modélisation qui permet d'obtenir des formules analytiques pour de nombreuses statistiques habituellement calculées dans le but d'évaluer si un modèle d'évaluation d'actifs financiers est capable de reproduire certains faits empiriques. Le modèle proposé est assez flexible pour capter les diverses dynamiques de la consommation et des dividendes, aussi bien que les divers types de préférences qui ont été adoptées dans les modèles basés sur la consommation. Il permet ainsi la réévaluation dans un cadre commun des différentes méthodes de résolution et approximations usuelles dans les modèles d'équilibre général d'évaluation d'actifs financiers. Ces formules analytiques nous font mieux comprendre les mécanismes économiques qui sous-tendent les résultats empiriques et les limites de validité des approximations usuelles.

Des progrès récents dans la modélisation des rendements d'actions prouvent que les moments d'ordre supérieur en général et l'asymétrie conditionnelle en particulier varient à travers le temps. Finalement, le troisième chapitre développe un modèle affine à facteurs multiples en temps discret et à composantes inobservables dans lequel la variance et l'asymétrie conditionnelles des rendements sont stochastiques. De façon importante et cohérente, nous distinguons la dynamique de la variance conditionnelle de celle de l'asymétrie conditionnelle. Notre approche permet à la distribution des rendements journaliers courants d'être asymétrique conditionnellement aux facteurs courants. Dans notre modèle, l'asymétrie conditionnelle est la résultante, d'une part des effets de levier, et d'autre part de l'asymétrie de la distribution des rendements courants conditionnellement aux facteurs courants. Nous dérivons des formules analytiques pour différentes conditions de moments utiles pour l'inférence par la méthode des moments généralisée. En appliquant notre approche aux rendements journaliers de plusieurs actions et indices boursiers, nous montrons que la distribution des rendements journaliers courants conditionnellement à la volatilité courante est positivement asymétrique et permet reproduire des statistiques échantillonales telles que l'asymétrie inconditionnelle et les corrélations négatives entre rendements courants et carrés des rendements futurs. L'effet de levier est significatif et négatif tandis que l'asymétrie conditionnelle est positive, impliquant que l'asymétrie de la distribution des rendements courants conditionnellement à la volatilité courante domine l'effet de levier dans la détermination de l'asymétrie conditionnelle.

Mots clés: Volatilité de la Consommation, Risque lié à la Volatilité, Rendements en Coupe Transversale, Modèle d'Évaluation d'Actifs Financiers par Équilibre, Prime de Risque des Actions, Énigme du Taux sans Risque, Prévisibilité des Rendements, Modèles Affines, Volatilité Stochastique, Asymétrie Stochastique, Effet de Levier, Méthode des Moments Généralisée.

ABSTRACT

The main theme of this thesis is the rationalization of new stylized facts involving both asset returns and relevant macroeconomic variables. Asset pricing models assume that risks that affect investment opportunities are related to one or several macroeconomic factors, and that these risks are compensated by appropriate returns. These models then aim at determining the risk factors that investors care the most about. Interesting asset pricing properties of consumption volatility have been put forward in earlier studies, but they were mainly related to the time series dimension of asset returns. A major contribution of our research will be to characterize and measure its impact in the cross-sectional dimension.

The first chapter documents empirical facts showing the existence of a strong relationship between macroeconomic uncertainty and stock returns. It provides and supports the evidence that long-term investors care not only about variation between future and present consumption level (long-run consumption level risk), but also and perhaps mostly about variation between future and present macroeconomic uncertainty (long-run consumption volatility risk). We show that differences in risk premia across stocks are also due to the heterogeneity in their exposure to changes in consumption volatility. Empirical facts documented in this chapter suggest that consumption volatility risk is highly correlated to risk premium for various investment horizons, more than consumption level risk for long-period investments and less for short-period investment in stocks. Moreover, long-run volatility risk is priced even in the presence of long-run consumption risk. This study is theoretically motivated by a reduced-form affine general equilibrium model with stochastic volatility. A well-conducted calibration of such a model would rationalize theses empirical findings.

Further, we shed light on numerical, analytical and statistical problems that affect some conclusions of existing asset pricing models. Consumption-based equilibrium asset pricing models have regained some momentum with new insights about the connections between stock market volatility and returns, the pricing of long-run claims, or return predictability. Links are established between risk premiums and different types

of preferences, where separation between the elasticity of intertemporal substitution and risk aversion, and habit formation take center stage. Often, the solution of these models necessitates an approximation and quantities of interest are computed through simulations.

The second chapter proposes a model that delivers closed-form formulas for many of the statistics usually computed to assess the ability of the models to reproduce stylized facts. The proposed model is flexible enough to capture the various dynamics for consumption and dividends as well as the different types of preferences that have been assumed in consumption-based asset pricing models. It then offers a common setting to re-evaluate various methods of resolution and usual approximations in asset pricing general equilibrium models. The availability of closed-form formulas enhances our understanding of the economic mechanisms behind empirical results and of the limits of validity for the usual approximations.

Recent developments in asset return modeling have shown evidence for time-variation in conditional higher moments, especially skewness and leverage effects. Finally, the third chapter develops a discrete time affine multifactor latent variable model of asset returns which allows for both stochastic volatility and stochastic skewness (SVS model). Importantly, we disentangle the dynamics of conditional volatility and conditional skewness in a coherent way. Our approach allows the distribution of current daily returns conditional on current volatility to be asymmetric. In our model, time-varying conditional skewness is driven by the conditional leverage effect and the asymmetry of the distribution of current returns conditional on current volatility.

We derive analytical formulas for various moment conditions that we use for GMM inference. Applying our approach to several equity and index daily returns, we show that the conditional distribution of current daily returns, conditional on current volatility, is positively skewed and helps to match sample return skewness as well as negative cross-correlations between returns and squared returns. The conditional leverage effect is significant and negative. The conditional skewness is positive, implying that the asymmetry of the distribution of current returns conditional on current volatility dominates the leverage effect in determining the conditional skewness.

Keywords: Consumption Volatility, Volatility Risk, Cross-Section of Returns, Equilibrium Asset Pricing, Equity Premium, Risk-free Rate Puzzle, Predictability of Returns, Affine Models, Stochastic Volatility, Stochastic Skewness, Leverage Effect, GMM.

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INTRODUCTION GÉNÉRALE

Cette thèse consiste en trois essais liés l'évaluation empirique des actifs financiers. Cette recherche comporte trois axes principaux. D'abord, elle s'intéresse à la rationalisation des nouveaux faits empiriques mis en évidence sur la relation entre les rendements des portefeuilles d'actions et certaines variables macroéconomiques. Ainsi, nous étudions les implications de l'incertitude macroéconomique, mesurée par la volatilité de la consommation, sur les rendements en coupe transversale. Ensuite, nous développons des solutions analytiques aux problèmes numériques, analytiques et statistiques qui se posent dans le contexte des modèles d'évaluation d'actifs financiers. Enfin, cette thèse s'intéresse au développement, à l'estimation et au diagnostic de nouveaux modèles de séries temporelles des rendements, permettant à la fois d'évaluer analytiquement les actifs financiers et leurs produits dérivés, tout en tenant compte de nouveaux faits stylisés liés à la variation temporelle des caractéristiques de dispersion, d'asymétrie et d'aplatissement de la distribution conditionnelle des rendements d'actions.

Les opportunités d'investissement sont variables dans le temps et les investisseurs font face à de multiples sources de risques financiers et macroéconomiques dont ils doivent se couvrir lorsqu'ils choisissent leurs portefeuilles intertemporellement. Les modèles d'évaluation d'actifs financiers supposent que chacun de ces risques est lié à un facteur financier ou macroéconomique, et cherchent dès lors à donner une réponse à la question centrale suivante : " quels sont les principaux facteurs de risque auxquels les investisseurs portent le plus d'attention ? "

Le premier chapitre de cette thèse, intitulé " Volatilité de la Consommation et Rendements d'Actions en Coupe Transversale ", met en évidence et justifie de façon empirique le fait que les investisseurs s'intéressent non seulement à la variation entre les niveaux futur et présent de la consommation (risque de long terme dans le niveau de la consommation), mais aussi et surtout à la variation entre les niveaux futur et présent de l'incertitude macroéconomique (risque de long terme dans la volatilité de la consommation). Comme dans Bansal et Yaron (2004), incertitude macroéconomique fait référence à la volatilité de la consommation agrégée. Nous répondons à la question suivante : "

les différences entre les primes de risque d'actions sont-elles dues à l'hétérogénéité dans leurs co-mouvements avec la volatilité de la consommation ? ". Nous montrons que les portefeuilles ayant les primes de risque les plus élevées, ont également une plus grande covariance négative avec la variation de long terme dans la volatilité de la consommation. Ce résultat est autant vrai pour les investissements de court terme dans les actions (typiquement une période comme dans la plupart des modèles) que pour les investissements de long terme. Il suggère ainsi que les investisseurs ne préfèrent pas disposer d'actions dont la rentabilité est faible lorsqu'ils font face à une incertitude macroéconomique future élevée relativement au présent. En conséquence, ils demandent une plus grande compensation en termes de primes de risque pour posséder de tels actifs.

La majorité des études supposent que la consommation est homoscédastique. L'hypothèse critique que la volatilité de la consommation est variable dans le temps est cruciale pour la présente étude. Plusieurs articles récents prouvent qu'il existe une relation entre l'incertitude macroéconomique et les opportunités d'investissement et qu'elle est déterminante pour comprendre la formation des prix d'actifs financiers (Bansal et Yaron (2004) et Bansal, Khatchatrian et Yaron (2004)). Kandel et Stambaugh (1990) montrent que la volatilité de la consommation varie en relation avec le cycle des affaires et qu'elle est prévisible par trois variables financières dont le ratio prix/dividende du marché des actions. Les modèles de Markov à changement de régime estimés avec les données de la consommation soutiennent que la variance de la consommation n'est pas la même à travers les différents régimes (Kandel et Stambaugh (1990), Bonomo et Garcia (1993), Lettau, Ludvigson et Wachter (2006)). De même, d'autres articles ayant estimé des modèles autorégressifs généraux à hétéroscédasticité conditionnelle avec ces mêmes données sur la consommation, corroborent l'hypothèse d'une variance conditionnelle variable dans le temps (Bansal, Khatchatrian et Yaron (2004)).

D'importants faits empiriques sur les rendements d'actions en coupe transversale ont été établis par Fama et French (1992). Ces faits stipulent que les firmes dont le ratio valeur comptable/valeur boursière est plus élevé, ont des rendements supérieurs à ceux des firmes dont ce rapport est plus faible. De même, les firmes à faible capitalisation boursière ont des rendements plus élevés que les firmes à grande capitalisation boursière. Le modèle standard d'évaluation d'actifs financiers basé sur la consommation n'a pas permis de justifier ces différences dans les rendements des actions par des différences entre les risques liés à la consommation. De nouveaux arguments ont alors émergé en faveur de facteurs liés aux caractéristiques gouvernant la distribution de la consommation conditionnellement à l'information économique passée. C'est dans ce cadre que d'importantes implications des modèles avec volatilité de la consommation variable ont été mises en exergue dans des études récentes, mais portent principalement sur la dimension temporelle des rendements (Bansal et Yaron (2004), Tauchen (2005), Eraker (2006)). Les implications de cette mesure de l'incertitude macroéconomique pour les rendements en coupe transversale n'ont pas encore été examinées, et c'est dans cette dernière dimension que nous mettons l'accent. Nous utilisons donc, non seulement l'information économique contenue dans le niveau de la consommation courante, mais aussi et surtout celle contenue dans l'incertitude sur le profil de la consommation future, dans le but d'améliorer la performance du modèle standard d'évaluation d'actifs financiers basé sur la consommation, ce dernier n'ayant pas donné les résultats empiriques escomptés.

Alors que les études empiriques s'intéressent principalement aux rendements d'actions pour des investissements courts (typiquement une période dans la plupart des études), Bansal, Dittmar et Kiku (2005) montrent que la relation entre le risque et le rendement varie beaucoup à mesure que l'horizon d'investissement augmente. Nous considérons des investissements pour plusieurs horizons, ayant tous la caractéristique que l'investisseur possède un portefeuille risqué pendant les premières périodes, puis change ensuite pour un portefeuille sans risque qu'il détient jusqu'à l'horizon d'investissement. Nous étudions la sensibilité des rendements de ces investissements par rapport à la variation du niveau de la consommation d'une part, et par rapport à la variation de la volatilité de la consommation d'autre part, sur l'horizon d'investissement. Pour la volatilité de la consommation, nous utilisons une mesure paramétrique provenant de l'estimation d'un modèle autorégressif général à hétéroscédasticité conditionnelle. Nous utilisons la mesure standard du risque d'un portefeuille donnée par la covariance entre les gains et le facteur de risque.

Nous représentons sur un graphique l'évolution du risque du portefeuille en fonction de l'horizon d'investissement, nous trouvons qu'au fur et à mesure que l'horizon d'investissement augmente, il existe des différences significatives entre les risques liés à la volatilité de la consommation, des portefeuilles avec plus faible ratio valeur comptable/valeur boursière aux portefeuilles avec ratio valeur comptable/valeur boursière plus élevé, les derniers possédant des risques plus grands. Plus important encore, alors que les investisseurs font face à la fois au risque lié au niveau de la consommation et au risque lié à la volatilité de la consommation, en utilisant les portefeuilles d'actions classés selon la taille de l'entreprise et le ratio valeur comptable/valeur boursière, nous montrons que la relation entre la volatilité de la consommation et les rendements est surtout une relation de long terme et qu'elle est assez stable. Les investissements longs sont plus sensibles aux risques de long terme dans la volatilité de la consommation que les investissements courts. Nous trouvons par ailleurs que les investissements longs sont moins sensibles aux risques de long terme dans le niveau qu'aux risques de long terme dans la volatilité.

En utilisant la méthode des moments généralisée, nous estimons pour chaque horizon d'investissement et pour chaque durée de détention des actions le modèle linéaire reliant la prime de risque à la covariance entre gains et variation dans le niveau d'une part, et à la covariance entre gains et variation dans la volatilité d'autre part. Nous trouvons que le prix du risque lié à la volatilité de la consommation est négatif et significatif lorsqu'estimé avec les rendements de long terme. Ceci justifierait alors pourquoi les actions des entreprises avec ratio valeur comptable/valeur boursière plus élevé ont des rendements plus élevés : c'est simplement parce que ces actions ont des gains procycliques.

Les deux facteurs macroéconomiques considérés dans cette étude (variation dans le niveau et variation dans la volatilité de la consommation) ont une justification théorique. Dans un modèle d'équilibre général sous forme réduite dont l'agent représentatif possède une fonction d'utilité récursive et dont la consommation suit un modèle à volatilité stochastique (comme dans Tauchen (2005)), nous montrons que l'agent valorise les actifs financiers grâce à un taux marginal de substitution intertemporelle qui dépend à la fois de la variation dans le niveau et de la variation dans la volatilité de la consommation.

Le premier chapitre est ainsi relié à trois littératures toutes récentes. La première littérature est celle qui utilise la volatilité d'une variable d'intérêt de l'investisseur comme facteur de risque dans un modèle d'évaluation d'actifs financiers. Alors que nous nous intéressons à une mesure paramétrique de la volatilité de la consommation agrégée, Ang et al. (2004) utilisent une mesure non paramétrique de la volatilité du marché des actions, tandis qu'Adrian et Rosenberg (2006) utilisent une mesure paramétrique de la volatilité du rendement agrégé du marché des actions.

La seconde littérature est celle qui examine si les actifs financiers sont évalués en fonction de leur exposition aux risques de long terme. Bansal et Yaron (2004), puis Bansal, Dittmar et Kiku (2005) montrent que les risques de long terme sont pris en compte sur les marchés financiers. Parker et Julliard (2005) considèrent les risques de long terme dans le niveau de la consommation pour évaluer les portefeuilles d'actions. Nous examinons en plus le risque de long terme dans la volatilité de la consommation.

La troisième et dernière littérature est celle qui examine les implications des moments d'ordre supérieur de la consommation dans la coupe transversale des rendements. Jacobs et Wang (2004) montrent en utilisant des données microéconomiques, que la variance interindividuelle de la consommation a le potentiel d'expliquer les différences entre les primes de risque d'actions. Cette étude diffère de la leur par le fait que ces auteurs se focalisent sur les risques idiosyncratiques non assurés liés à la consommation individuelle tandis que nous nous intéressons aux risques systématiques liés à la consommation agrégée. De plus, alors que leur facteur de risque peut être vu comme une mesure du degré d'hétérogénéité entre les individus, le nôtre représente plutôt une mesure du degré d'imprécision qui affecte les prévisions des agents sur l'évolution du niveau futur de la consommation totale.

Au cours des deux dernières décennies, les économistes de la finance se sont efforcés de résoudre deux principales énigmes à savoir, l'énigme de la prime de risque d'action et l'énigme du taux sans risque. Le modèle d'évaluation d'actifs financiers basé sur la consommation, introduit par Lucas (1978) et Breeden (1989) a été modifiée par de nouvelles spécifications des préférences des agents, plus à même de justifier la forte prime de risque des actions et le faible taux sans risque. Deux de ces modèles qui se distinguent des autres par leur popularité sont le modèle avec utilité récursive d'Epstein et Zin (1989, 1991) et le modèle de formation d'habitude externe de Campbell et Cochrane (1999). Récemment, ces modèles ont été utilisés pour reproduire de nouveaux faits empiriques sur la relation entre volatilité du marché des actions et les rendements, la valorisation des rentes de long terme, ou la prévisibilité des rendements (voir Bansal et Yaron (2004), Bansal, Gallant et Tauchen (2004), Hansen, Heaton et Li (2004), Lettau, Ludvigson et Wachter (2004)). Les efforts ont été centrés sur le choix de la dynamique de la consommation et des dividendes. De nouveaux modèles pour la dynamique conjointe de la consommation et des dividendes ont été expérimentés, alors qu'originellement l'égalité entre la consommation et les dividendes était souvent supposée. Généralement, la résolution de ces modèles nécessite une ou plusieurs approximations et les quantités d'intérêt sont calculées par simulation.

Dans le deuxième chapitre de cette thèse, intitulé " Un Cadre Analytique d'Évaluation des Modèles de Valorisation d'Actifs Financiers et de Prévisibilité des Rendements ", nous proposons un modèle qui permet d'obtenir des formules analytiques pour de nombreuses statistiques habituellement calculées dans le but d'évaluer si un modèle d'évaluation d'actifs financiers est capable de reproduire certains faits empiriques. Le modèle proposé est assez flexible pour capter les diverses dynamiques de la consommation et des dividendes, aussi bien que les divers types de préférences qui ont été adoptées dans les modèles basés sur la consommation. Il permet ainsi la réévaluation dans un cadre commun des différentes méthodes de résolution et approximations usuelles dans les modèles d'équilibre général d'évaluation d'actifs financiers. Ces formules analytiques nous font mieux comprendre les mécanismes économiques qui sous-tendent les résultats empiriques et les limites de validité des approximations usuelles.

Pour dériver les formules analytiques, nous supposons que les accroissements logarithmiques de la consommation et des dividendes par tête suivent un processus bivarié dont les coefficients du vecteur des moyennes et de la matrice de variances-covariances varient selon une même chaîne de Markov stationnaire et homogène, s_t, qui prend les valeurs $1,\ldots,N$ (si on admet que l'économie est caractérisée par N états de la nature). Plusieurs modèles d'évaluation d'actifs financiers ont été fondés sur des versions simplifiées de ce processus général, mais la principal raison est que ce processus permet d'obtenir des formules analytiques pour de nombreuses statistiques que les chercheurs ont essayé de reproduire : la moyenne et la variance des rendements excédentaires et du taux sans risque, la moyenne et la variance du ratio prix/dividende, le coefficient de détermination de la régression des rendements, des rendements excédentaires, du taux de croissance des dividendes, du taux de croissance de la consommation et de la volatilité de la consommation sur le ratio prix/dividende et le ratio consommation/richesse, l'autocorrélation négative des rendements et des rendements excédentaires à des horizons longs. Nous utilisons aussi ce modèle pour reproduire certains moments du processus de la consommation et des dividendes impliqués par d'autres modèles. Cette dernière approche est utilisée par Mehra et Prescott (1985) dans leur papier qui mît en avant l'énigme de la prime de risque et dès lors, fît école.

Dans les formules que nous développons pour les multiples statistiques, nous supposons que nous avons déjà résolu le modèle donnant le prix de l'actif d'intérêt ou le rapport des gains de cet actif sur son prix. La structure du processus de la consommation et des dividendes implique qu'un tel rapport possède une valeur par régime, ce qui permet d'obtenir des formules analytiques.

Le prix d'un actif quelconque est, bien entend, relié au facteur d'escompte stochastique qui à son tour dépendra du modèle considéré. Nous calculons les prix dans une économie à changement de régime markovien et avec préférences récursives (Epstein et Zin (1989)) ainsi qu'avec formation d'habitude externe (Campbell et Cochrane (1999)). Ces modèles délivrent deux types de rapport gains sur prix : le rapport de la consommation sur le prix du portefeuille de marché et le rapport du dividende sur le prix de l'action. Le premier rapport est inobservable mais Lettau, Ludvigson et Wachter (2001 a,b) ont proposé la contrepartie empirique d'un parent proche, le rapport consommation sur richesse. Une fois que nous distinguons la consommation des dividendes, ces modèles délivrent une mesure de cette importante statistique économique. En plus, dans le cadre de l'utilité récursive, le rapport consommation sur richesse détermine le facteur d'escompte stochastique. Une fois que les équations d'Euler non-linéaires définissant ce rapport dans les différents régimes ont été résolues, tous les autres prix d'actifs peuvent être obtenus analytiquement.

L'importance de dériver des formules analytiques ne doit en aucun cas être pris à la légère. Lettau, Ludvigson et Wachter (2004), qui précisément utilisent un modèle à changement de régime markovien pour la consommation, font la remarque que leur modèle à deux états prend un temps très long pour être résolu et que le modèle à trois états serait à la limite numériquement insolvable. Ces auteurs utilisent un modèle de mise à jour qu'ils doivent résoudre à chaque date étant donné leur nouvelle évaluation des probabilités de transition du processus de Markov. Nos formules peuvent être adaptées à cette approche et allègeront considérablement le processus numérique. Un autre gain en temps d'exécution vient potentiellement des simulations que les chercheurs exécutent pour calculer les régressions en vue des études de prévisibilité. La procédure usuelle est d'essayer de répliquer les statistiques usuelles avec un nombre d'observations identique à celui de l'échantillon aussi bien qu'avec un nombre d'observations beaucoup plus grand pour s'assurer si le modèle est capable de reproduire la prévisibilité en population. On éviterait ce dernier exercice, plus coûteux en temps d'exécution, par l'utilisation de formules analytiques que nous fournissons. Ceci s'applique également aux ratios de variances.

Une autre contribution, non pas la moindre, est d'utiliser nos formules analytiques pour évaluer l'impact des approximations que les chercheurs appliquent à la résolution des modèles. Une approximation omniprésente dans la littérature est la log-linéarisation de Campbell et Shiller (1988). Nous produisons des formules pour plusieurs approximations des rapports gains sur prix dans le modèle d'Epstein et Zin (1989).

Nous appliquons notre cadre analytique à deux articles saillants dus récemment à Lettau, Ludvigson et Wachter (2004) et à Bansal et Yaron (2004). Tous font la promotion du rôle important de l'incertitude macroéconomique lue à travers la volatilité de la consommation, comme facteur de valorisation des actifs financiers. Le premier article modélise le taux de croissance de la consommation comme un processus à changement de régime markovien et utilise les préférences d'Epstein et Zin (1989), et par conséquent correspond directement à notre cadre d'application. Le second article utilise les mêmes préférences mais modélise conjointement la consommation et les dividendes par un processus autorégressif à volatilité stochastique. Pour ce dernier modèle, nous proposons une procédure qui fait correspondre les moments de ce processus avec ceux de notre processus à changement de régime markovien. En inscrivant les deux modèles dans un même cadre, nous sommes en mesure de montrer leurs similitudes et leurs différences en termes d'implications pour l'évaluation d'actifs financiers et la prévisibilité des rendements. Nos formules analytiques nous permettent d'explorer un plus vaste ensemble de paramètres de préférence que dans les articles originels, ce qui permet de mieux comprendre le rôle de ces paramètres dans la détermination des statistiques financières d'intérêt. Nous faisons également correspondre un modèle à changement de régime markovien au processus spécifié par Campbell et Cochrane (1999) pour le rapport du surplus sur la consommation, puis nous fournissons les résultats analytiques pour plusieurs statistiques calculées par simulation dans le papier originel.

Nous étendons considérablement les formules analytiques de Bonomo et Garcia (2004) pour le modèle d'évaluation d'actifs financiers basé sur la consommation de Lucas (1978). Récemment, deux articles ont proposé de développer des formules analytiques pour des modèles d'évaluation d'actifs financiers. Abel (2005) calcule explicitement la prime de risque, la prime de long terme et la prime pour certaines actions dérivées dans un cadre qui inclut les modèles de formation d'habitude et d'externalités sur la consommation. Les formules sont dérivées sous l'hypothèse de log-normalité, d'indépendance et de distribution identique des taux de croissances de la consommation et des dividendes. Nous supposons également la log-normalité, mais conditionnellement à un certain nombre d'états, et notre variable d'état capte la dynamique des taux de croissance. Eraker (2006) produit des formules analytiques pour les prix d'actions et d'obligations dans un modèle d'évaluation d'actifs financiers par équilibre avec des préférences d'Epstein et Zin (1989), sous l'hypothèse que les taux de croissances de la consommation et des dividendes suivent un processus affine. Cependant, il adopte l'approximation de Campbell et Shiller (1988) indispensable pour obtenir ces formules.

Une grande partie des travaux empirique dans le domaine de la finance justifient le fait que, non seulement la variance conditionnelle des rendements change à travers le temps, mais aussi les rendements ont une distribution conditionnelle non-normale dont l'asymétrie varie également en fonction du temps. Ces deux caractéristiques des rendements sont cruciales étant donné que les variations de la variance et de l'asymétrie conditionnelles à travers le temps influencent le comportement des investisseurs et par conséquent leurs choix de portefeuille à travers le temps. Des évidences empiriques montrent l'existence de risques financiers liés à la volatilité et à l'asymétrie des rendements et justifient la prise en compte de ces risques lors de la valorisation des actifs financiers par les investisseurs. En effet, les investisseurs demandent une plus grande prime de risque pour disposer d'actifs dont les gains sont plus volatiles et dont les gains extrêmement faibles sont plus réguliers que les gains extrêmement élevés. La variation temporelle de la variance et de l'asymétrie conditionnelles peuvent alors expliquer les prix d'actifs.

Le troisième et dernier chapitre de cette thèse, intitulé " Modèles Affines à Asymétrie Stochastique ", développe un modèle affine à facteurs multiples en temps discret et à composantes inobservables dans lequel la variance et l'asymétrie conditionnelles des rendements sont stochastiques. Plus important encore, dans le cas du modèle à deux facteurs, le vecteur constitué par rendements, la volatilité et l'asymétrie suit un processus affine. La variation temporelle dans la volatilité des rendements trouve son origine dans les modèles autorégressifs à hétéroscédasticité conditionnelle (ARCH, Engle (1982)) ou ses extensions (GARCH, Bollerslev (1986), et EGRACH, Nelson (1991)). Alors que dans les modèles ARCH et GARCH la volatilité des rendements est complètement déterminée par l'historique des rendements observés, une approche alternative, devenue populaire dans la littérature récente, est le modèle à volatilité stochastique (SV), dans lequel la volatilité des rendements est une composante inobservable qui subit des chocs de source différente de celle générant les chocs sur les rendements. La plupart des applications des modèles GARCH et SV supposent que la distribution conditionnelle des rendements est symétrique. Même si cette hypothèse permet de générer les queues épaisses observées pour la distribution inconditionnelle des rendements, il reste encore à expliquer la variation temporelle et le signe des asymétries conditionnelles (asymétrie et effets de levier) et les queues de la distribution conditionnelle des rendements (voir Hansen (1994)). Les asymétries conditionnelles sont importantes car, pour la valorisation des options par exemple, l'hétéroscédasticité conditionnelle uniquement ne suffit pas à expliquer ce fait empirique important qui dans la littérature est qualifiée de sourire des options.

Au premier plan, nous développons un modèle semi-affine à facteurs multiples, à volatilité stochastique dont les innovations sur les rendements sont asymétriques. Christoffersen, Heston et Jacobs (2006) étudient également un modèle semi-affine des rendements avec asymétrie variable dans le temps. Cependant, l'asymétrie conditionnelle dans leur modèle est liée de façon déterministe à la variance conditionnelle, ce qui est également le cas pour le modèle à un facteur dans notre cas. Cependant, la volatilité et l'asymétrie conditionnelles dans leur modèle subissent les mêmes chocs que les rendements puisqu'il s'agit d'une variante des modèles GARCH. Au contraire, notre modèle à un facteur est une variante des modèles à volatilité stochastique, qui nouvellement peuvent être étudiés dans un cadre affine ne supposant pas la normalité conditionnelle des rendements. Mieux encore, dans notre cas à deux facteurs ou plus, nous brisons le lien déterministe entre la volatilité et l'asymétrie conditionnelles qui se comportent dès lors comme deux facteurs linéairement indépendants caractérisant de manières différentes la dynamique temporelle des rendements et subissant des chocs de sources différentes de celle générant les chocs sur les rendements.

Harvey et Siddique (1999) considèrent également une distribution conditionnelle asymétrique des rendements dont la volatilité et l'asymétrie conditionnelles sont deux facteurs linéairement indépendants avec des dynamiques de type GARCH. Leur asymétrie conditionnelle autorégressive est une façon simple de modéliser l'asymétrie conditionnelle et fournit également une méthodologie d'estimation de l'asymétrie conditionnelle qui est facile à mettre en oeuvre précisément par l'applicabilité du maximum de vraisemblance. Cependant, un défaut d'application, et non pas le moindre, de la modélisation de Harvey et Siddique (1999) est que leur modèle est non-affine et devient couteux en temps d'exécution pour la résolution des modèles d'évaluation d'actifs financiers, précisément à cause de la non-existence de formules analytiques entraînant une résolution numérique ou par simulations. Notre modèle est une alternative convenable au modèle de Harvey et Siddique (1999). Nous modélisons l'asymétrie par une combinaison affine de facteurs stochastiques linéairement indépendants. L'existence de la fonction génératrice des moments offre un cadre de résolution analytique des modèles d'évaluation d'actifs financiers permettant de gagner énormément en temps d'exécution. Nous montrons aussi comment cette fonction génératrice des moments permet d'estimer le modèle par la méthode des moments généralisée en se basant sur des conditions de moments exactes. Dans notre cadre à facteurs stochastiques, nous distinguons l'information de l'agent économique de celui de l'économètre et fournissons explicitement les équivalents GARCH de la volatilité, de l'asymétrie et des effets de levier conditionnels.

L'autre objectif est de développer et d'implémenter un algorithme pour le calcul analytique des moments inconditionnels exacts de la variable observable, dans un modèle semi-affine général en temps discret à facteurs multiples qui englobe notre modèle. Une étude similaire a été conduite par Jiang et Knight (2002) dans le cadre des processus affines en temps continu. Ces auteurs dérivent de manière analytique la fonction caractéristique inconditionnelle conjointe du processus de diffusion vectoriel. Cependant, cette question, bien que d'une importance à ne pas sous-estimer, n'a pas été examinée pour les processus affines en temps discret. Premièrement, les formules analytiques pour les moments inconditionnels permettent d'évaluer l'impact direct des paramètres du modèle sur des moments inconditionnels critiques tels que l'asymétrie, l'aplatissement excédentaire, l'autocorrélation des carrés des rendements et les corrélations croisées entre les rendements et les carrés des rendements. Deuxièmement, les moments inconditionnels en population peuvent être directement comparés à leurs contreparties empiriques. En, cette évaluation s'avère indispensable dans un exercice de calibrage où les paramètres du modèle sont fixés de sorte à reproduire les valeurs échantillonales de certains de ces moments inconditionnels. Plus important encore, cette comparaison entre moments en populations et moments empiriques permet la mise en oeuvre d'une procédure d'estimation du modèle par la méthode des moments généralisée avec l'avantage inqualifiable de se baser sur des conditions de moments exactes. Cette technique d'estimation permet également d'évaluer l'habileté du modèle à répliquer les faits empiriques connus tels que la persistance dans la volatilité des rendements à travers l'autocorrélation des carrés des rendements, l'absence d'autocorrélation des rendements, les effets de levier négatifs à travers les corrélations croisées entre les rendements et les carrés des rendements, l'aplatissement excédentaire positif et l'asymétrie négative. Chacun de ces faits stylisés est pris en compte par une ou plusieurs conditions de moments particulières faisant partie du vecteur des conditions de moments utilisé pour l'estimation du modèle.

Nous appliquons cette nouvelle procédure d'estimation des modèles semi-affines pour notre modèle à un facteur, en utilisant les séries de rendements journaliers de plusieurs portefeuilles d'actions et d'indices boursiers. Pour estimer les facteurs stochastiques, nous appliquons une variante du filtre de Kalman pour les modèles non-linéaires. Les paramètres du modèle sont tous significatifs et les implications du modèle sont frappantes. D'abord, la distribution des rendements journaliers courants conditionnellement à la volatilité courante est positivement asymétrique. De plus, cette asymétrie positive est nécessaire pour reproduire des statistiques échantillonales significatives telles que l'asymétrie inconditionnelle et les corrélations négatives entre rendements courants et carrés des rendements futurs. Ensuite, cette distribution positivement asymétrique engendre également une asymétrie positive de la distribution des rendements courants conditionnellement aux rendements passés. Ce résultat est contraire à certaines conclusions d'une large partie de la littérature existante (Forsberg et Bollerslev (2002)). Finalement, lorsque la distribution des rendements journaliers courants conditionnellement à la volatilité courante est contrainte à la normalité, alors le modèle engendre une asymétrie négative de la distribution des rendements courants conditionnellement aux rendements passés, ce qui corrobore la littérature existante. Cependant, sous cette hypothèse, le modèle ne reproduit plus l'asymétrie et les effets de levier inconditionnels. En plus, les tests de restrictions sur-identifiantes rejettent le modèle contraint aux niveaux conventionnels tandis que ces tests ne rejettent pas le modèle non contraint générant une asymétrie conditionnelle positive de la distribution des rendements courants conditionnellement aux rendements passés.

CHAPTER 1

CONSUMPTION VOLATILITY AND THE CROSS-SECTION OF STOCK RETURNS

Abstract

Interesting asset pricing properties of consumption volatility have been put forward in studies, but they are mainly related to the time series dimension of asset returns. In this chapter we characterize and measure consumption volatility and its impact in the cross-section of asset returns. Motivated by a reduced-form consumption-based general equilibrium model with stochastic volatility, we document a strong relation between macroeconomic uncertainty and stock returns. Our findings suggest that consumption volatility risks are highly correlated with short and long horizon risk premia. Moreover, these risks account for differences in risk premia across size and book-to-market sorted portfolios. We find that long-run consumption volatility risk is economically important even in the presence of long-run consumption level risk. In particular, we find that value stocks pay high average returns because they covary more negatively with long-horizon variation in consumption volatility than what other stocks do. We argue that long-run volatility risk is relevant for interpreting differences in risk compensation across assets.

1.1 Introduction

The question *what do (or should) investors care about?* is central in Asset Pricing and a variety of models continue to provide alternative answers. Investors face time-varying investment opportunities and multiple sources of financial and macroeconomic risks that they should hedge themselves against when constructing financial portfolios. This chapter provides and supports the evidence that investors care not only about variation between future and present consumption levels, but also and perhaps mostly about variation between future and present macroeconomic uncertainty. As in Bansal and Yaron (2004), macroeconomic uncertainty refers to the volatility of aggregate consumption. We answer the following question: *Are differences in risk premia across stocks due to the heterogeneity in their exposure to consumption volatility risk?* We find that portfolios with high risk premia have high negative covariances with long-horizon variation in consumption volatility. This is true for short-period investments as well as for long-period investments. Therefore, this finding suggests that investors dislike assets paying less for higher future macroeconomic uncertainty relative to the present. Consequently, investors will demand a higher risk premium for holding such assets.

The critical consideration that consumption volatility varies over time is central in this study. A recent literature emphasizes that the relationship between macroeconomic uncertainty and investment opportunities is crucial to understand the behavior of asset prices (see, for example, Bansal and Yaron (2004)). Kandel and Stambaugh (1990) find that consumption volatility varies over the business cycle and is predicted by three financial variables.[1] That is, consumption volatility tends to be larger at the end of recessions or immediately after them. Markov-Switching models estimated on consumption data support that consumption growth volatility varies across different regimes (Kandel and Stambaugh (1991), Bonomo and Garcia (1993), Lettau, Ludvigson and Wachter (2006)). Modelling consumption volatility as a GARCH process, Bansal, Khatchatrian and Yaron (2004) find a significant ARCH effect. They also show that this measure of consumption volatility is predicted by the price-dividend ratio.

As choosing a portfolio is equivalent to buying various types of risks, asset pricing models aim at identifying relevant financial and macroeconomic risks that are priced, and at determining if these risks justify the observed pattern across historical asset returns. In other words, they investigate if a relationship between a group of asset returns and the corresponding asset risks is monotonic, has the right sign and is economically significant. Roughly speaking, these models try to explain the size and the value premia. The size premium comes from the fact that stocks of firms with small capitalization (small stocks)

[1] Kandel and Stambaugh (1990) regress consumption volatility at the quarter t on (a) the difference at the end of quarter $t-1$ between Moody's average yield on bonds rated Baa and bonds rated Aaa, (b) the difference at the end of quarter $t-1$ between the Aaa yield and the yield on a U.S. Treasury bill with maturity closest to one month, and (c) the dividend-price ratio at the quarter $t-1$ for the value-weighted portfolio of NYSE stocks. The chi-squared statistic does not reject the hypothesis that consumption volatility does not depend on the predictive variables.

have historically paid higher average returns than those of firms with large capitalization (large stocks). On the other hand, stocks of firms with a high ratio of book value to market value (value stocks) have historically paid higher average returns than those of firms with a low ratio of book value to market value (growth stocks): the difference is known as the value premium.

Since a cross-sectional model with the level of consumption itself has a weak performance in justifying differences across stock returns, a part of the literature continues to deal with consumption by motivating higher moments of consumption as possible priced factors. The volatility of consumption can provide additional information about consumption that should be taken into account in consumption-based cross-sectional models. However, while some useful asset pricing implications of models involving time-varying consumption volatility were put forward in earlier studies (Bansal and Yaron (2004), Tauchen (2005), Eraker (2006)), the implications of this measure of macroeconomic uncertainty have not been investigated for the cross-section of stock returns, and then constitute the focus of this study.

Empirical studies of asset pricing models typically examine the cross-sectional implications of macroeconomic factors for short-period investments by focusing only on one-period returns. However, Bansal, Dittmar and Kiku (2005) show that the risk-return relationship varies extensively as the investment horizon increases. We consider multi-horizon investments where the investor stays in stocks from the beginning, then switches to the safe asset and stays on it until the end of the investment period. We then study how returns on such investments react to the variation in consumption level as well as the variation in consumption volatility between the end and the beginning of the investment period. We use a parametric measure of consumption volatility inferred from a GARCH specification. We examine cross-sectional implications of each of these macroeconomic factors for short and long-period holding returns. We use the standard measure of the asset risk, the covariance between an asset's payoff and the risk factor.

Plotting volatility risks against horizon across multihorizon portfolios sorted on book-to-market, dividend-to-price, earnings-to-price and cash flows-to-price ratios, we find that, for various stock holding periods, there is a significant difference between portfolio risks as the investment horizon increases. Moreover, these risks exhibit a pattern that generally matches that of risk premia across these dimensions. This means that growth stocks have a lower volatility risk than value stocks and the volatility risk of the market portfolio lies between these extreme risks. Moreover, for most of the investment horizons, consumption volatility risk is more correlated with multiperiod returns on the Fama and French size and book-to-market sorted portfolios than consumption level risk. As we show that portfolios with high risk premia covary more negatively with variations in consumption volatility, we further ask whether this explains their higher average returns. We estimate linear models which link risk premia to covariances of returns with factors and find that the price of long-horizon volatility risk is negative and significantly estimated in the cross-section of multiperiod returns.

The two macroeconomic factors that we consider in this chapter are theoretically motivated by a consumption-based model with a representative investor who values its payoffs through a stochastic discount factor that depends log-linearly on both variations in consumption level and variations in consumption volatility. We further show that it is the case in an affine reduced-form general equilibrium model with stochastic volatility as in Bansal and Yaron (2004) and Tauchen (2005). While the logarithm of the standard SDF of the power utility only depends on changes in consumption level, that of the recursive utility depends additionally on changes in consumption volatility through consumption valuation ratios.

This chapter belongs to the recent literature that examines whether stock returns can be priced by their exposure to long-run risks. Long-run risks appear to be a key concern in asset markets (Bansal and Yaron (2004), Bansal, Dittmar and Kiku (2005)). Parker and Julliard (2005) consider long-run consumption risk, showing that ultimate consumption risk of assets, measured by the covariance between returns and the long-horizon consumption growth, can account for the value premium. In addition, we consider long-run volatility risk, measured by the covariance of returns with the long-horizon volatility variation. We show that this risk is priced in financial markets and can also account for the value premium.

This chapter also relates to the growing literature that includes volatility factors in cross-sectional asset pricing models. The volatility of the aggregate stock market return, as well as the volatility of aggregate consumption, provides a measure of macroeconomic uncertainty through the link between financial markets and the real economy. However, it is not directly related to macroeconomic fundamentals. Ang et al. (2006) show that a nonparametric proxy of market volatility is a cross-sectional stock pricing factor.[2] In a parametric approach, Adrian and Rosenberg (2006) model the market return as a GARCH process and decompose its volatility into a short and a long-run component. They find that these volatility components have negative and significant prices of risk in the cross-section of stock returns. Their work can be viewed as using additional information provided by state variables in the market return process to improve the CAPM, while we use additional information provided by state variables in the consumption growth process to improve a cross-sectional consumption-based asset pricing model.

Finally, this chapter builds on work that examines the implications of higher moments of consumption for the cross-section of asset returns. Using household consumption data, Jacobs and Wang (2004) find that the variance of the cross-sectional distribution of consumption growth has some potential to explain asset risk premia. Their result points out that assets with high negative covariance with consumption dispersion also have high returns. We depart from the work of Jacobs and Wang (2004) in that we examine the risk of the volatility of aggregate consumption whereas they focus on the risk of the dispersion in idiosyncratic consumption. Second, as they use microdata to construct risk factors, we use macrodata in this chapter. Third, factors differ in what they refer to. While the variance of the cross-sectional distribution of consumption growth is mostly a degree of heterogeneity across individuals, consumption volatility is mostly the degree of the imprecision that affects agents's expectations about future consumption. Fourth as we mention earlier, our factors incorporate long-run risks in consumption volatility. Finally, their work relates to the literature on market incompleteness, whereas our setup rests upon the complete markets assumption that underlies the representative

[2]To proxy innovations in aggregate market volatility, Ang et al. (2006) use changes in the VIX index from the Chicago Board Options Exchange (CBOE).

agent framework.

The rest of the chapter is organized as follows. Section 1.2 motivates and discusses the cross-sectional risk-return relationship involving different investment horizons and stock holding periods. Section 1.3 presents the data, establishes relevant empirical facts and discusses empirical risk-return relationships through cross-sectional correlations between mean excess returns and volatility risks. Section 1.4 estimates risk prices and provides additional empirical findings and diagnostics. Section 1.5 rationalizes the results from the perspective of existing asset pricing equilibrium models. Section 1.6 concludes.

1.2 Motivating Consumption Volatility Risk in the Cross-Section of Asset Premia

The standard consumption-based asset pricing theory states that an investor cares only about the level of its consumption at each period in time and should then invest in stocks in consequence. However, empirical tests show that the comovement between one-period asset payoffs and one-period consumption growth fails to explain differences in one-period average returns across assets. The literature has grown so far to address the issue of improving the ability of consumption-based models in understanding risk compensations from asset exposures to good and bad news about consumption. These news can be related either to the consumption level or to time-varying consumption moments.

1.2.1 An Underlying Economic Model

To explain the aggregate stock market behavior and asset pricing puzzles, Bansal and Yaron (2004) provide a version of the reduced form general equilibrium model in which investors have concerns about risks from the level of consumption growth, from changes in consumption growth forecasts and from changes in consumption volatility. This induces a time-varying equity risk premium which is associated with conditional covariances of return with innovations in these state variables. In their model, if the representative agent prefers early resolution of uncertainty, has both the coefficient of risk aversion and the elasticity of intertemporal substitution greater than one, then volatility carries a positive risk premium. This adds to the growing set of asset pricing properties of consumption volatility which have so far been mainly established in the time series dimension of asset returns.

We assume a modified version of the Bansal and Yaron (2004) model where agents have concern about consumption level and consumption volatility only. This follows the literature that assumes a long-term investor with recursive preferences (Kreps and Porteus (1978), Epstein and Zin (1989) and Weil (1990)) and specify the dynamics of economic endowments. The current continuation value of investor's utility evolves according to:

$$V_t = \left\{(1-\delta)C_t^{1-\frac{1}{\psi}} + \delta\left[\mathscr{R}_t\left(V_{t+1}\right)\right]^{1-\frac{1}{\psi}}\right\}^{\frac{1}{1-\frac{1}{\psi}}} \quad \text{if } \psi \neq 1 \tag{1.1}$$

$$= \left[C_t\right]^{1-\delta}\left[\mathscr{R}_t\left(V_{t+1}\right)\right]^{\delta} \quad \text{if } \psi = 1, \tag{1.2}$$

where $\mathscr{R}_t\left(V_{t+1}\right) = \left[E\left(V_{t+1}^{1-\gamma}\right) \mid \mathscr{I}_t\right]^{\frac{1}{1-\gamma}}$ and $\mathscr{I}_t$ is the information set of the investor at time t. The parameter of risk aversion is γ, the elasticity of intertemporal substitution (EIS) is ψ, the subjective discount factor is δ and the parameter $\theta \equiv (1-\gamma)\left(1-\psi^{-1}\right)^{-1}$ helps for many interpretations.

Epstein and Zin (1989) show that for such an investor, consumption and portfolio choice induces a restriction on the gross return on any asset i that is given by the Euler equation:

$$E\left[M_{t,t+1}R_{i,t+1} \mid \mathscr{I}_t\right] = 1, \tag{1.3}$$

where $M_{t,t+1}$ is the standard SDF that values consumption as well as any financial payoff one period ahead and is given by:

$$M_{t,t+1} = \left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\right]^{\theta}\left(\frac{1}{R_{w,t+1}}\right)^{1-\theta}. \tag{1.4}$$

$R_{w,t+1}$ is the gross return to the total consumption claim. The logarithm of the Epstein-

Zin SDF is given by:

$$m_{t,t+1} = \theta \ln \delta - \frac{\theta}{\psi} \Delta c_{t+1} - (1-\theta)\, r_{w,t+1},$$

where $r_{w,t+1} = \ln R_{w,t+1}$. The log-linearization of the investor's budget constraint, defined by the Campbell and Shiller (1988)'s approximation of the log-return around a suitable benchmark is:

$$r_{w,t+1} = \rho_0 + \frac{1}{\rho_1} x_t - x_{t+1} + \Delta c_{t+1}, \tag{1.5}$$

where $x_t = \ln C_t - \ln W_t$ is the log consumption-wealth ratio.[3]

The standard SDF of the power utility does not depend on consumption volatility even if consumption growth dynamics contains time-varying volatility. It only depends on the level of consumption growth. On the contrary, an investor with recursive preferences cares about consumption volatility. For such an investor, the intertemporal marginal rate of substitution depends on consumption valuation ratios whose movements can be related to that of consumption volatility (see Bansal, Khatchatrian and Yaron (2004)).

According to (1.4), since consumption growth is observable and the return to total wealth is not, any state variable that is suspected to have a power to price asset returns and consistently with the general equilibrium framework should be linked to the unobservable return. In order to establish this link, researchers assume that equilibrium consumption together with such state variables follow an exogenous model. Here we assume that consumption growth has the following dynamics:

$$\Delta c_{t+1} = \mu_c + \phi_c (h_t - \mu_h) + \sqrt{h_t} u_{t+1} \tag{1.6}$$

$$h_{t+1} = (1-\phi_h)\, \mu_h + \phi_h h_t + \sigma_h \eta_{t+1}, \tag{1.7}$$

[3]The constant ρ_0 is given by:

$$\rho_0 = \ln\left(\frac{1-\rho_1}{\rho_1}\right) - \frac{\ln(1-\rho_1)}{\rho_1}.$$

where $(u_{t+1},\eta_{t+1})^{\top} \sim \mathcal{N}.\mathcal{I}.\mathcal{D}(0,I)$. The gaussian dynamics (1.7) for the volatility of aggregate consumption is also considered by Bansal and Yaron (2004). To the contrary, we do not treat expected consumption growth as a separate state variable in our specification since it is likely to capture similar long-run risks in consumption level that are already captured by consumption growth,[4] and given that we essentially wish to maintain a balance between level and volatility risk sources.

Since shocks to consumption and consequently to total investor's wealth and its marginal rate of substitution are governed by only one state variable which is the consumption volatility, then the log consumption-wealth ratio has the form $x_t = \Phi_0 + \Phi_h h_t$ and the logarithm of the SDF (1.8) becomes,

$$m_{t,t+1} = p_1 - p_c \Delta c_{t+1} - p_h \underbrace{\left(h_{t+1} - \frac{h_t}{\rho_1} \right)}_{\approx \Delta h_{t+1}}, \tag{1.8}$$

where $p_1 = \theta \ln \delta - (1-\theta)\left(\rho_0 + (1-\rho_1)\rho_1^{-1}\Phi_0\right)$ is a constant with no special interest at this stage[5]. The discount coefficient ρ_1 has many asset pricing interpretations, among which those found in Campbell and Shiller (1988), Campbell (1993, 1996) and Campbell and Vuolteenaho (2004). These papers highlight the link of the coefficient ρ_1 to the average consumption-wealth ratio generated by a portfolio strategy of a mutual-fund investor who saves a fraction of his mutual fund every period to finance its consumption.

Since $\rho_1 \approx 1$ as the frequency becomes high, the term $\left(h_{t+1} - \rho_1^{-1} h_t\right)$ will behave as Δh_{t+1} and the logarithm of the SDF will be linear in consumption growth and changes in consumption volatility, where $p_c = \gamma$ is the standard price of level risk measured by the risk aversion parameter, and $p_h = -(1-\theta)\Phi_h$ is the price of volatility risk. The loading of the consumption-wealth ratio on consumption volatility and the price of volatility risk

[4]Note that expected consumption growth is usually empirically proxied by a weighted combination of the lags of consumption growth (see also Bansal, Dittmar and Lundblad (2004)), for example if consumption growth is an ARMA(1,1).

[5]The constant Φ_0 is given by:

$$\Phi_0 = -\frac{\rho_1}{1-\rho_1}\left[\rho_0 + \ln\delta + \left(1 - \frac{1}{\psi}\right)(\mu_c - \phi_c \mu_h) - (1-\phi_h)\mu_h \Phi_h + \frac{1}{2}\theta\sigma_h^2 \Phi_h^2\right].$$

are given by:

$$\Phi_h = -\frac{\rho_1}{(1-\rho_1\phi_h)}\left(1-\frac{1}{\psi}\right)\left[\phi_c+\frac{1}{2}(1-\gamma)\right] \tag{1.9}$$

$$p_h = \frac{\rho_1}{(1-\rho_1\phi_h)}\left(\gamma-\frac{1}{\psi}\right)\left[\phi_c+\frac{1}{2}(1-\gamma)\right]. \tag{1.10}$$

In the asset pricing literature, authors seem to agree that $\gamma > 1$, whereas there is still no consensus on $\psi > 1$ and $\psi^{-1} < \gamma$. Then, the sign of the parameter θ and its position with respect to one are still crucial for asset pricing results. Bansal, Khatchatrian and Yaron (2004) argue that a rise in economic uncertainty leads to a fall in asset prices. In particular the total investor's wealth will fall due to an increase in consumption volatility. To capture a positive relation between consumption volatility and consumption-wealth ratio, the coefficient Φ_h that drives this effect should be positive. On the other hand, only the condition $\psi^{-1} < \gamma$ is required for the volatility risk price to be negative and this can still be the case if $\psi \leq 1$. When the EIS is equal to one, Φ_h is equal to zero and the consumption-wealth ratio is constant. In this case, the Campbell and Shiller's approximation is exact with $\rho_1 = \delta$ and $x_t = \ln(1-\delta)$.[6]

While this reduced-form general equilibrium model suggests that consumption volatility could be a cross-sectional pricing factor, the question of how it affects the cross-section of expected returns have received less attention. Stocks with different sensitivities to the volatility of aggregate consumption should have different expected returns as changes in macroeconomic uncertainty induce changes in investment opportunities. Investors' expectations about future consumption are imprecise, and the degree of that imprecision is measured by consumption volatility. Since consumption is the claim on total investor wealth, the imprecision about expected future consumption also reflects the uncertainty about future wealth. In that sense, movements in consumption volatility

[6]The logarithm of the risk-free rate implied by the model is given by $r_{f,t+1} = q_1 - q_h h_t$ where:

$$q_1 = -p_1 + p_c(\mu_c - \phi_c\mu_h) + (1-\phi_h)\mu_h p_h - \frac{1}{2}\sigma_h^2 p_h^2$$

$$q_h = \left(\frac{1}{\rho_1} - \phi_h\right)p_h + \frac{1}{2}p_c^2.$$

provide additional news about consumption that are likely to influence investment decisions. For this reason, consumption volatility is suspected as empirically relevant for explaining asset returns.

An investor chooses intertemporally its portfolios to face as better as he can bad states of the economy. As well as such an investor dislikes low consumption levels, he also dislikes high uncertainty on future consumption levels. We examine these two concerns of investors by analyzing empirical risk-return relationships involving asset returns and both consumption level and consumption volatility risks.

1.2.2 Valuing Multiple-Horizon Payoffs in the Cross-Section

The failures of the standard consumption-based asset pricing theory in empirical tests brought to researchers to examine relationship between multiperiod returns rather than one-period returns, or long-horizon changes in macroeconomic variables rather than one-horizon changes. Parker and Julliard (2005) show that differences in exposure of one-period stock returns to long-horizon consumption growth account for cross-sectional differences in stock risk premia. They argue that slow adjustment of consumption to return data is a reason of why contemporaneous consumption risk fails to explain expected one-period stock returns. While empirical studies typically deal with one-period returns, Bansal, Dittmar and Kiku (2005) study the relation between consumption risk and stock return when the stock holding period is the same as the investment horizon. They show that this risk-return relationship varies extensively by investment horizon, and that consumption risk almost converges to the long-run relation between dividend and consumption as the horizon increases. We also vary the investment horizon and work directly with returns.

We consider that N stocks denoted $i = 1,..,N$ and the safe asset f are traded in the economy. The one-period return of investing the asset i from $t+j-1$ to $t+j$ is denoted $R_{i,t+j}, j \geq 1$. The total investment horizon is denoted S. An S-period investment in this study starts with a one-period investment in the first period and payoffs are reinvested at the beginning of each of the $(S-1)$ subsequent periods. We are interested in risk-return relationships involving returns on long-horizon investments which consist in investing in the same stock for the first periods and then reinvesting payoffs in the safe asset for the remaining periods. Given an investment horizon S, the gross return on such an investment can be written:

$$R_{it,k,S} = \underbrace{\prod_{j=1}^{k} R_{i,t+j}}_{R_{it,k}} \prod_{j=k+1}^{S} R_{f,t+j} \tag{1.11}$$

and defines the S-period gross return formed by investing from time t in the asset i for the first k periods, and then reinvesting its payoffs from date $t+k$ in the safe asset, for the remaining $(S-k)$ periods.

The excess return with respect to the return on the investment which consists in staying in the safe asset for the whole period is defined by $R^e_{it,k,S} = R_{it,k,S} - R_{ft,S}$. Note that $R_{ft,S}$ is not the return on a bond that bought at time t will deliver a unit consumption at time $t+S$. An investor who buys at time t an S-horizon investment plan consisting to stay in the safe asset for the whole period is now making a risky decision if $S > 1$, since future one-period risk-free rates $R_{f,t+j}$, $j > 1$ are not known at time t and are affected by macroeconomic factors during the investment period. $R_{ft,S}$ is unknown at time t and is not the S-period risk-free rate from t to $t+S$.

The single horizon Euler condition (1.3) implies the multiple horizon Euler condition:

$$E\left[M_{t,t+S} R_{it,k,S} \mid \mathscr{I}_t\right] = 1 \tag{1.12}$$

where $M_{t,t+S} = \prod_{j=1}^{S} M_{t+j-1,t+j}$ is the multiperiod SDF and $R_{it,k,S}$ a compound long-horizon return defined in (1.11). The subscript of $M_{t,t+S}$ indicates that it is pricing S-period holding returns from time t to time $t+S$.

Because what guides investors seems to be the comovement between asset payoffs and risk factors, it is appealing to measure the risk for holding an asset as the covariance between the payoff and the risk factor. The sign of this covariance indicates if the asset and the factor move in the same or opposite direction, whereas its magnitude quantifies the degree of this comovement. One can already observe that the logarithm of $M_{t,t+S}$,

$$m_{t,t+S} = \log M_{t,t+S} \approx Sp_1 - p_c \Delta c_{t,S} - p_h \Delta h_{t,S}, \tag{1.13}$$

features two main horizon-dependent macroeconomic risk factors.

The first factor,

$$\Delta c_{t,S} = c_{t+S} - c_t = \sum_{j=1}^{S} \Delta c_{t+j},$$

is the variation in the level of consumption between the end and the beginning of the investment period and also equals the future S-horizon consumption growth. The covariance of this factor with an asset return measures the S-level risk, or the ultimate consumption risk of the asset if $k = 1$, as termed in Parker and Julliard (2005). These authors argue that ultimate consumption risk is a better asset risk measure than the standard consumption risk in the CCAPM, for example if consumption reacts with lags to stock returns. Alternatively, the link between returns and future long-horizon consumption growth can be simply due to the fact that investors are concerned with long-run risks in consumption. For a given S, the cross-section of S-level risks for k-period holding stocks is defined by the vector:

$$Cov\left(\Delta c_{t,S}, R^e_{\cdot t,k,S}\right) = \left(Cov\left(\Delta c_{t,S}, R^e_{1t,k,S}\right) \quad \cdots \quad Cov\left(\Delta c_{t,S}, R^e_{Nt,k,S}\right)\right)^{\top}. \tag{1.14}$$

An investor would dislike an asset which the excess return has a positive covariance with the variation in the level of consumption. Such an asset pays less in bad states of the economy characterized by low future consumption level relative to the present, and the investor will require a relatively high premium for holding that asset. On the other hand, the investor will dislike the asset i_2 more than the asset i_1 in a situation where both covariances are positive, and the covariance of asset i_1 has the low magnitude. All other things being equal, asset i_2 will have a more higher required level risk premium than asset i_1.

By a similar reasoning, an investor would prefer an asset which the excess return has a negative covariance with the variation in the level of consumption. Such an asset pays more in bad states of the economy characterized by low future consumption level relative to the present, and the investor will be able to give up a relatively high premium for holding that asset. On the other hand, the investor will prefer the asset i_1 more than the asset i_2 in a situation where both covariances are negative, and the covariance of asset i_1 has the high magnitude. All other things being equal, asset i_2 will have a more lower given up level risk premium than asset i_1.

The second factor,

$$\Delta h_{t,S} = h_{t+S} - h_t = \sum_{j=1}^{S} \Delta h_{t+j},$$

is the change in the volatility of consumption between the end and the beginning of the investment period. By similarity with the consumption level case, we define its covariance with an asset return as the S-volatility risk of the asset. The S-horizon volatility variation is relevant if investors have concerns about long-run risks in consumption volatility. Furthermore, if consumption level reacts with lags to returns, to some extent it should also be the case for consumption volatility. In this case, as S increases, the S-volatility risk would provide the better measure of the volatility risk embodied in asset payoffs. The innovation of this chapter is to show that, in addition to long-horizon consumption growth, long-horizon variation in consumption volatility captures the cross-sectional dispersion of stock returns as well, and that long-run volatility risk is economically important even in the presence of long-run consumption risk. For a given S, the cross-section of S-volatility risks for k-period holding stocks is defined by the vector:

$$Cov\left(\Delta h_{t,S}, R^e_{\cdot t,k,S}\right) = \left(Cov\left(\Delta h_{t,S}, R^e_{1t,k,S}\right) \quad \cdots \quad Cov\left(\Delta h_{t,S}, R^e_{Nt,k,S}\right)\right)^{\top}. \quad (1.15)$$

An agent who faces an increase in macroeconomic uncertainty would fear the repercussion on its future wealth and then, he would like to increase precautionary savings. This investor would dislike an asset which the excess return has a negative covariance with the variation in the volatility of consumption. Such an asset pays less in bad states of the economy characterized by high future consumption volatility relative to the present, and the investor will require a relatively high premium for holding that asset. On the other hand, the investor will dislike the asset i_2 more than the asset i_1 in a situation where both covariances are negative, and the covariance of asset i_1 has the low magnitude. All other things being equal, asset i_2 will have a more higher required volatility risk premium that asset i_1.

By a similar reasoning, an investor would prefer an asset which the excess return has a positive covariance with the variation in the volatility of consumption. Such an asset pays more in bad states of the economy characterized by high future consumption volatility relative to the present, and the investor will be able to give up a relatively high premium for holding that asset. On the other hand, the investor will prefer the asset i_1 more than the asset i_2 in a situation where both covariances are positive, and the covariance of asset i_1 has the high magnitude. All other things being equal, asset i_2 will have a more lower given up volatility risk premium that asset i_1.

The risk premium that an investor will require to stay the first k periods in the stock i and the remaining $(S-k)$ periods in the safe asset, instead of staying in the safe asset for the whole period, is defined by the expectation of the corresponding excess return: $E\left[R^e_{it,k,S}\right]$. For a given S, as the risk-free rate part is common for all returns $R_{it,k,S}, i = 1,..,N$, the cross-section of average k-period stock holding returns can be defined by the vector:

$$E\left[R^e_{\cdot t,k,S}\right] = \left(E\left[R^e_{1t,k,S}\right] \quad \cdots \quad E\left[R^e_{Nt,k,S}\right]\right)^\top . \tag{1.16}$$

1.2.3 Level and Volatility Risk-Return Relations

We measure the risk-return relationship at each investment horizon S and for each stock holding period k, through cross-sectional correlations between the vector (1.16) of k-period stock risk premia and the vectors of S-level and S-volatility risks, (1.14) and

(1.15) respectively. These cross-sectional correlations are denoted:

$$\rho_{rc}(S,k) = Corr\left(E\left[R^e_{\cdot t,k,S}\right], Cov\left(\Delta c_{t,S}, R^e_{\cdot t,k,S}\right)\right) \quad (1.17)$$

$$\rho_{rh}(S,k) = Corr\left(E\left[R^e_{\cdot t,k,S}\right], Cov\left(\Delta h_{t,S}, R^e_{\cdot t,k,S}\right)\right) \quad (1.18)$$

According to the theory, the average return of an asset is higher the more positively it covariates with variations in consumption level, and the more negatively it covariates with variations in consumption volatility. Moreover, the more negatively asset payoff covariates with variations in consumption level, and the more positively it covariates with variations in consumption volatility, the lower will be the asset risk premium. Thus, $\rho_{rc}(S,k)'$ and $\rho_{rh}(S,k)$ are expected to be respectively positive and negative, and their magnitudes will assess how important are relationships between k-period holding stock returns and S-horizon variations in consumption level and in consumption volatility respectively. An empirical analysis of these cross-sectional correlations is provided in Section 3.5.

Since the square of the correlation between the explained and the explicative variables measures the R-squared of the projection of the former onto the latter, we note that $[\rho_{rh}(S,k)]^2$ also measures the proportion of variations in risk premium across stocks, which is explained solely by consumption volatility risk. Similarly, $[\rho_{rc}(S,k)]^2$ also measures the proportion of variations in risk premium across stock, which is explained solely by consumption level risk.

Since equation (1.12) is also equivalent to:

$$E\left[R^e_{it,k,S}\right] = Cov\left(-\frac{M_{t,t+S}}{E\left[M_{t,t+S}\right]}, R^e_{it,k,S}\right), \quad (1.19)$$

one can use equation (1.19) to derive the horizon-dependent relationship between risk premium and covariances between returns and the two main macroeconomic factors motivated by our underlying model. It suffices to replace the SDF $M_{t,t+S}$ by one of its log-linear approximations $\widetilde{M}_{t,t+S}$ where:

$$\frac{\widetilde{M}_{t,t+S}}{E\left[M_{t,t+S}\right]} = 1 + \beta_S \left(m_{t,t+S} - E\left[m_{t,t+S}\right]\right). \tag{1.20}$$

The approximated SDF has the same mean as the true SDF and the coefficient β_S would be positive to ensure a positive relationship between the SDF and its approximation.[7]

Substituting (1.20) in (1.19) yields:

$$E\left[R^e_{it,k,S}\right] = p_{c,S} Cov\left(\xi_{\Delta c,t,S}, R^e_{it,k,S}\right) + p_{h,S} Cov\left(\xi_{\Delta h,t,S}, R^e_{it,k,S}\right) \tag{1.23}$$

where $\xi_{\Delta c,t,S} = \Delta c_{t,S} - E\left[\Delta c_{t,S}\right]$ and $\xi_{\Delta h,t,S} = \Delta h_{t,S} - E\left[\Delta h_{t,S}\right]$ are respectively the demeaned S-horizon variations in consumption level and in consumption volatility, and $p_{c,S}$ and $p_{h,S}$ are cross-sectional level and volatility risk prices given by:

$$p_{c,S} = \gamma \beta_S \text{ and } p_{h,S} = p_h \beta_S. \tag{1.24}$$

These prices are respectively positive and negative, and constant across horizons if β_S is a positive constant. From (1.10) it is straightforward that the magnitude of volatility risk price increases for a more risk-averse investor and/or a more persistent volatility process.

Equation (1.23) postulates that investors demand or give up both multihorizon consumption and volatility risk premia to invest in stocks. Each premium is the product of the quantity of the associated risk with a parameter that measures the price (or the compensation) for a unit risk. Since investors require a positive risk premium to hold assets

[7]The special case $\beta_S = 1$ is similar to the SDF approximation of Yogo (2005). Two other special cases are given by:

$$\beta_S = \frac{1}{E\left[M_{t,t+S}\right]} \sqrt{\frac{Var\left[M_{t,t+S}\right]}{Var\left[m_{t,t+S}\right]}} \text{ if } Var\left[\widetilde{M}_{t,t+S}\right] = Var\left[M_{t,t+S}\right] \tag{1.21}$$

$$\beta_S = \frac{1}{E\left[M_{t,t+S}\right]} \frac{Cov\left(M_{t,t+S}, m_{t,t+S}\right)}{Var\left[m_{t,t+S}\right]} \text{ if } \sqrt{E\left[\left(M_{t,t+S} - \widetilde{M}_{t,t+S}\right)^2\right]} \text{ is minimum.} \tag{1.22}$$

It can be shown that the values of β_S in special cases (1.21) and (1.22) are greater than one so that the magnitudes of cross-sectional level and volatility risk prices are respectively greater than the magnitudes of the risk aversion γ and the loading p_h.

that they dislike and are able to require a negative risk premium (give up a positive risk premium) to hold asset that they prefer as discussed earlier in this section, from an economic point of view, the price of the volatility risk should therefore be negative and the price of the level risk positive. Intuitively, the coefficients $p_{c,S}$ and $p_{h,S}$ are expected to be positive and negative respectively. In an empirical study, Section 1.4 provides details for estimating S-level and S-volatility risk prices in the two-factor cross-sectional linear covariance model (1.23). It then analyzes the estimation results and provides some conclusions.

Parker and Julliard (2005) assume an investor whose intertemporal marginal rate of substitution depends solely on the level of consumption and they essentially investigate cross-sectional relations like (1.23), with $k = 1$ and without S-volatility risk. Bansal, Dittmar and Kiku (2005) also deal with similar cross-sectional relations which do not involve S-volatility risk, but in the case $k = S$. However, they decompose the S-level risk into a trend risk and a business cycle risk, which they show are compensated by appropriate multiperiod returns. In addition, since the volatility of aggregate consumption varies in relation with the business cycle, as stated in Kandel and Stambaugh (1990), it could be said that the S-horizon variation in consumption volatility appears to be a business cycle risk factor, as well as the S-horizon variation in consumption level as shown in Parker and Julliard (2005). This issue is also empirically investigated in the next section.

At this stage, S-volatility risks cannot be computed in an empirical study since consumption volatility is unobservable. To measure this risk from the data, we use a parametric measure of consumption volatility provided by a GARCH of Heston and Nandi (2000) with no leverage parameter. That is, we extract consumption volatility from the following dynamics:

$$\Delta c_{t+1} = \mu_c + \phi_c (h_t - \mu_h) + \sqrt{h_t} u_{t+1} \tag{1.25}$$

$$h_{t+1} = (1 - \phi_h) \mu_h + \phi_h h_t + \sigma_h \left(u_{t+1}^2 - 1 \right) \tag{1.26}$$

where $u_{t+1} \sim \mathcal{NID}(0,1)$. We further denote by π the vector $\pi = (\mu_c, \phi_c, \mu_h, \phi_h, \sigma_h)^\top$ and let $\omega_h = (1 - \phi_h) \mu_h - \sigma_h$.

This GARCH specification, although considered for empirical purposes because of its easy estimation and filtering, shares some properties with the stochastic volatility in the underlying equilibrium model that motivates this study. Both volatility dynamics lead to an affine model and are such that the conditional leverage effect is zero and the volatility of volatility is constant.

1.3 Cross-sectional Empirical Facts

1.3.1 Consumption and Return Data

We use quarterly data for consumption of nondurable and services from 1947:1 to 2005:2, taken from the NIPA tables available from the Bureau of Economic Analysis. The associated PCE deflator is further used to convert nominal returns into real returns. We estimate the Heston and Nandi (2000) GARCH(1,1) for consumption growth.[8] Estimation results for this GARCH fit over the entire sample and over the subsample starting in 1963:3 are displayed in Table 1.1. The GARCH and ARCH coefficients of the dynamics are both significant and corroborate the central assumption that consumption volatility is time-varying. Bansal, Khatchatrian and Yaron (2004) estimate a standard GARCH(1,1) for consumption growth and find similar conclusions.

We further use parameter estimates and the extracted consumption volatility to compute estimates of demeaned consumption and volatility factors.[9] The time series of long-horizon changes in consumption volatility is plotted in Figure 1.1 for horizons of four and twelve quarters, corresponding to one-year and three-year changes. The figure shows that during business cycle recessions, long-horizon changes in consumption

[8]We estimate the model with $\phi_c = 0$. The estimation of ϕ_c leads to a negative and insignificant estimate. We notice however that this does not influence the empirical facts documented in this section.

[9]We compute empirical volatility risk factors $\Delta h_{t,S}(\widehat{\pi})$ from the recursion:

$$h_0(\widehat{\pi}) = \widehat{\mu}_h \text{ and } \forall t \geq 0, \tag{1.27}$$

$$h_{t+1}(\widehat{\pi}) = \left(1 - \widehat{\phi}_h\right)\widehat{\mu}_h + \widehat{\phi}_h h_t(\widehat{\pi}) + \widehat{\sigma}_h \left[\frac{\left(\Delta c_{t+1} - \widehat{\mu}_c - \widehat{\phi}_c (h_t(\widehat{\pi}) - \widehat{\mu}_h)\right)^2}{h_t(\widehat{\pi})} - 1 \right] \tag{1.28}$$

where $\widehat{\pi}$ is the consistent maximum likelihood estimator of π.

Table 1.1: **GARCH Fit of Consumption Growth.**
This table presents results for the estimation of model (1.26) over two samples considered in previous studies.

	μ_c	μ_h	ϕ_h	σ_h
(1)	(2)	(3)	(4)	(5)
	A. Sample 1947:2 - 2005:2			
Estimate	**0.00544**	**2.698E-5**	**0.87552**	**4.430E-6**
Std.dev.	0.00031	5.888E-6	0.05799	1.340E-6
	B. Sample 1963:3 - 2005:2			
Estimate	**0.00568**	**1.954E-5**	**0.82368**	**3.502E-6**
Std.dev.	0.00034	4.308E-6	0.09328	1.202E-6

volatility increase as macroeconomic uncertainty becomes more and more higher for the future relatively to the present. Figure 1.2 shows a similar plot for long-horizon changes in the level of aggregate consumption and confirms a permanent fall in the long-run economic growth throughout recessions.

We also use return data constituted with four groups of 5 portfolios sorted on dividend yield, book-to-market, earnings-to-price, and cash flows-to-price ratios, as well as the 25 Fama and French size and book-to-market sorted portfolios. Returns are monthly and span the period 1946:4 to 2005:8. They are aggregated to obtain quarterly returns. The attractiveness of these sets of portfolios in empirical studies is due to the fact that stocks show significant differences in their average excess returns.

Following Parker and Julliard (2005), we stop the sample of single-period returns at 1999:4, so that the horizon S in multiperiod returns and consumption and volatility factors can vary up to five years while maintaining the same sample of returns for the study as we vary S. That is, we use all available consumption and volatility data up to the fourth quarter of 1999 plus S quarters, with $S = 23$ corresponding to the second quarter of 2005. For all portfolios, we compute sample measures of mean returns and level and volatility risks defined in Section 1.2.2.[10] For brevity we present these measures only for representative portfolios and horizons.

Panel A of Table 1.2 shows sample estimates of mean excess returns, covariance

[10]We provide sample estimates of $E[y_t]$ where y_t is $R^e_{t,k,S}$, $\left(R^e_{t,k,S} - E\left[R^e_{t,k,S}\right]\right)(\Delta c_{t,S} - E[\Delta c_{t,S}])$ and

Figure 1.1: **Changes in Consumption Volatility and NBER Recessions.**
This figure displays time series of changes in consumption volatility. The dashed line displays $\Delta h_{t,S}$ for $S > 1$, the solid line displays $\Delta h_{t,1}$. Also displayed are overall trends of these changes during NBER recessions.

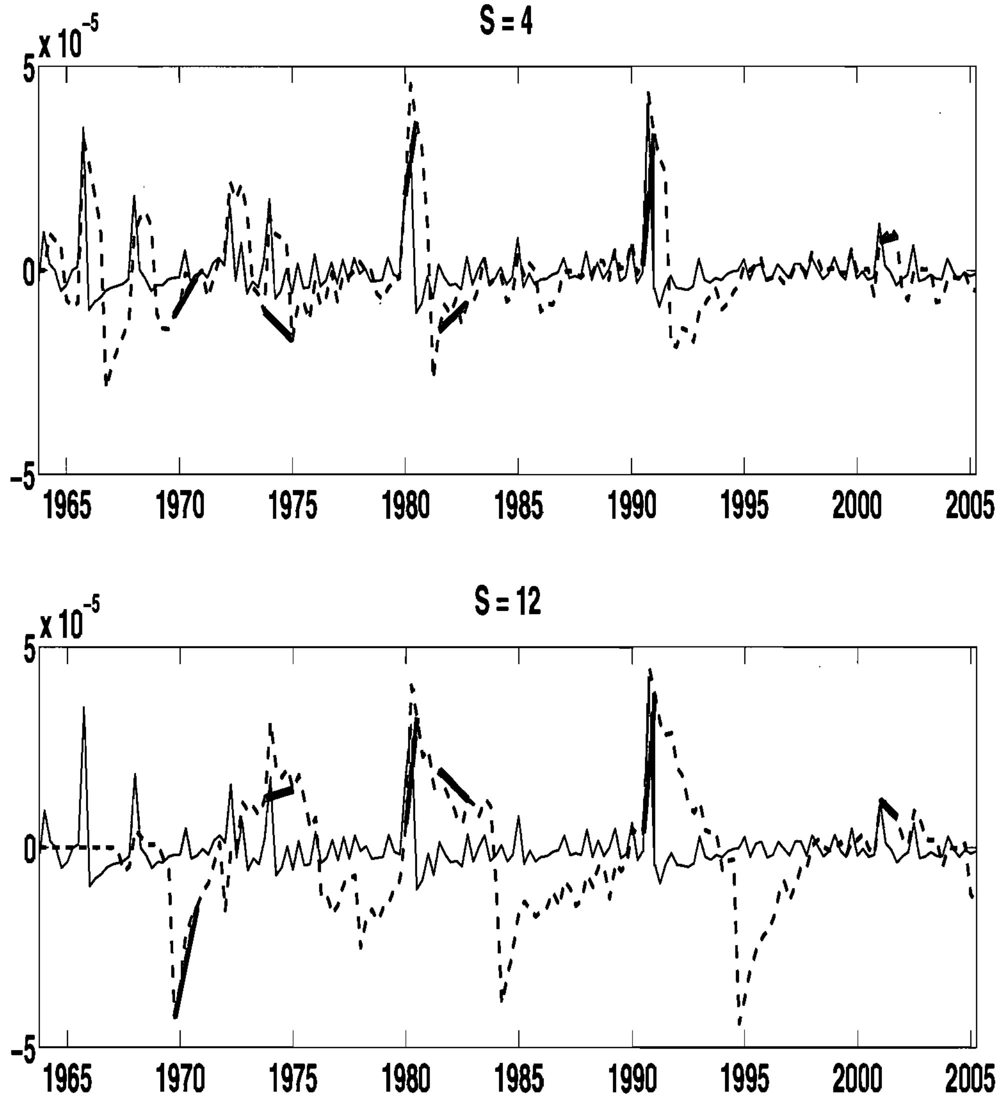
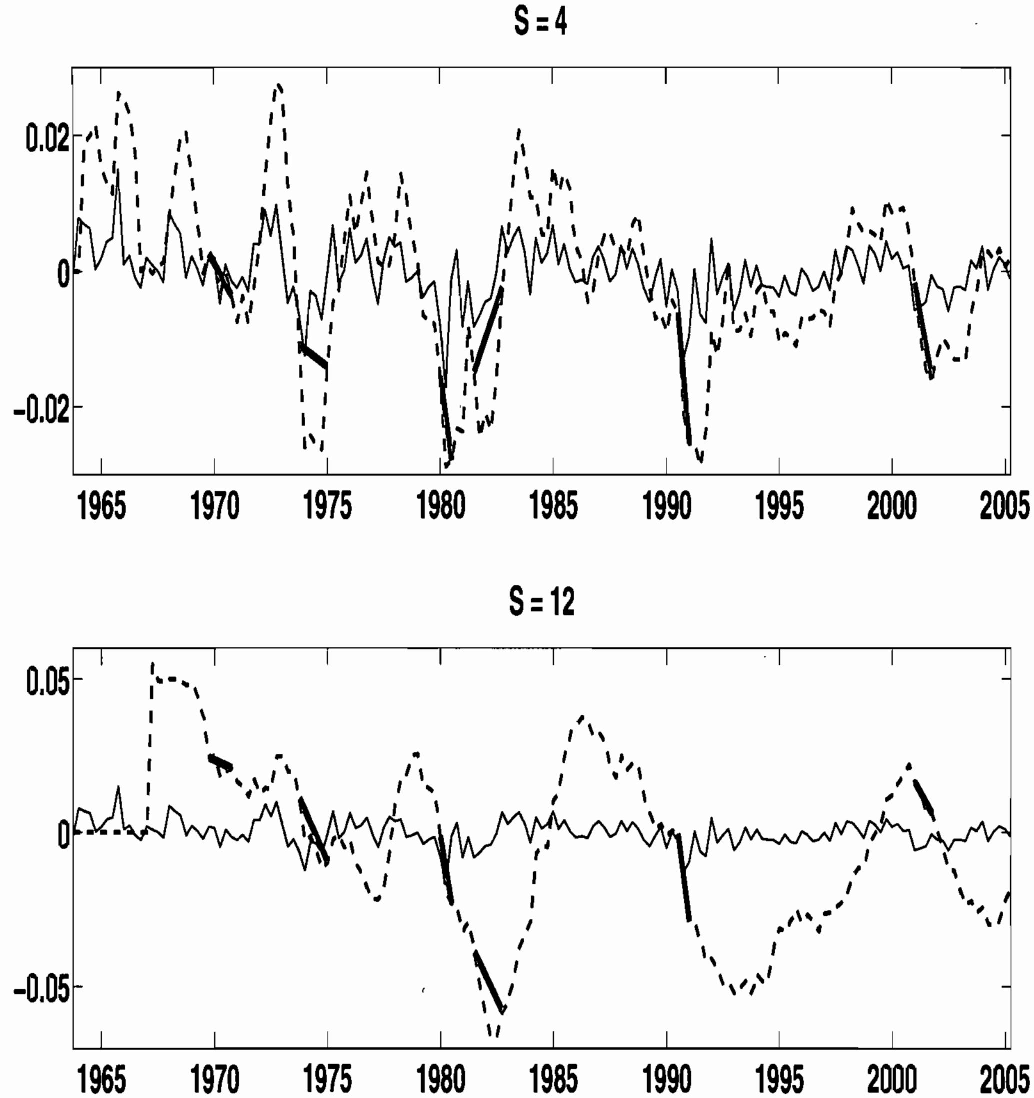


Figure 1.2: **Changes in Consumption Level and NBER Recessions.**
This figure displays time series of changes in consumption level. The dashed line displays $\Delta c_{t,S}$ for $S > 1$, the solid line displays $\Delta c_{t,1}$. Also displayed are overall trends of these changes during NBER recessions.



between excess returns and changes in consumption level and covariance between excess returns and changes in consumption volatility, for single-period investment in stocks and for the growth (labeled **L**), the neutral (labeled **3**) and the value (labeled **H**) stocks sorted across the earnings-to-price dimension. Panel B shows the same estimates for the same portfolios sorted across the cash flows-to-price dimension. For each set of portfolios, estimates of the difference between these statistics for the two extreme portfolios are also displayed (in rows labeled **H-L**).

The data evidence comparable spreads across the two portfolio characteristics; a single-period investment in the highest earnings-to-price firms over height periods pays on average a real quarterly excess returns of 2.65%, whereas in the lowest earnings-to-price firms it pays on average 1.67% per quarter. The highest earnings-to-price firms have more positive covariances of returns with changes in consumption level, and more negative covariances of returns with changes in consumption volatility than the lowest earnings-to-price firms. Except for average excess returns of these portfolios and level risk for the value portfolio, these statistics are at most slightly significant for single-period returns whereas it is the contrary regarding multiperiod returns over the same sample. Table 1.3 shows the same estimates as for Table 1.2 for multiperiod investment in stocks (full-period). In addition to observed positive spreads for average excess returns and level risk on one hand, and negative spread for volatility risk on the other hand between extreme portfolios, estimates of these spreads are significant for excess returns and volatility risk for various horizons considered in the table. This may announce at this stage the importance of explaining differences in stock returns at horizons more than one quarter.

We finally present average excess returns and risk measures for less aggregate port-

$\left(R^e_{t,k,S} - E\left[R^e_{t,k,S}\right]\right)\left(\Delta h_{t,S} - E\left[\Delta h_{t,S}\right]\right)$. Inference is conducted via the central limit theorem:

$$\sqrt{T}\left(\frac{1}{T}\sum_{t=1}^{T} y_t - E\left[y_t\right]\right) \sim \mathcal{N}\left(0, \sum_{j=-\infty}^{+\infty} E\left[\left(y_t - E\left[y_t\right]\right)\left(y_{t-j} - E\left[y_t\right]\right)^{\top}\right]\right)$$

and estimates of asymptotic covariance matrices are calculated using the Newey-West procedure with S lags. While the error in the estimates of the second and the third occurrences of y_t are affected by the error in the mean, we do not account for in the inference. Indeed, this effect is negligible if the mean is well-estimated, meaning that the corresponding error is small enough.

Table 1.2: **Estimates of Excess Returns and Level and Volatility Risks for Single-Period Growth and Value Portfolios.** The entries of the table are sample estimates of $E[y_t]$ where y_t is successively $R^e_{t,1,S}$, $\left(R^e_{t,1,S} - E\left[R^e_{t,1,S}\right]\right)(\Delta c_{t,S} - E[\Delta c_{t,S}])$ and $10^4\left(R^e_{t,1,S} - E\left[R^e_{t,1,S}\right]\right)(\Delta h_{t,S} - E[\Delta h_{t,S}])$. Standard errors are given below the estimates and calculated using the Newey-West procedure with S lags.

	$S=1$			$S=4$			$S=8$			$S=12$			$S=16$		
	R^e	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$	R^e	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$	R^e	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$	R^e	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$	R^e	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$
Panel A. Earnings-to-Price Sorted Portfolios.															
L	**1.54**	**0.0061**	**0.034**	**1.57**	**0.0203**	**-0.028**	**1.62**	**0.0236**	**0.093**	**1.66**	**0.0151**	**0.035**	**1.70**	**0.0049**	**-0.013**
	0.80	0.0037	0.045	0.77	0.0104	0.076	0.69	0.0137	0.103	0.64	0.0118	0.114	0.64	0.0125	0.150
3	**1.61**	**0.0046**	**0.046**	**1.64**	**0.0128**	**-0.052**	**1.70**	**0.0250**	**0.044**	**1.74**	**0.0218**	**-0.019**	**1.79**	**0.0151**	**0.009**
	0.65	0.0031	0.043	0.62	0.0082	0.072	0.58	0.0109	0.075	0.55	0.0112	0.090	0.53	0.0117	0.105
H	**2.60**	**0.0070**	**0.053**	**2.65**	**0.0214**	**-0.170**	**2.71**	**0.0386**	**-0.101**	**2.78**	**0.0406**	**-0.118**	**2.85**	**0.0282**	**-0.115**
	0.76	0.0034	0.047	0.73	0.0098	0.107	0.68	0.0131	0.112	0.65	0.0122	0.118	0.63	0.0135	0.112
H-L	**1.07**	**0.0009**	**0.019**	**1.08**	**0.0011**	**-0.143**	**1.09**	**0.0150**	**-0.194**	**1.12**	**0.0255**	**-0.153**	**1.15**	**0.0232**	**-0.101**
	0.54	0.0027	0.037	0.61	0.0077	0.068	0.63	0.0118	0.067	0.64	0.0100	0.070	0.63	0.0128	0.098
Panel B. Cash Flows-to-Price Sorted Portfolios.															
L	**1.68**	**0.0061**	**0.045**	**1.72**	**0.0210**	**-0.036**	**1.77**	**0.0255**	**0.076**	**1.81**	**0.0191**	**0.035**	**1.85**	**0.0094**	**-0.035**
	0.81	0.0037	0.047	0.78	0.0105	0.081	0.71	0.0138	0.109	0.69	0.0118	0.119	0.69	0.0128	0.151
3	**1.82**	**0.0048**	**0.049**	**1.86**	**0.0149**	**-0.044**	**1.92**	**0.0276**	**0.049**	**1.98**	**0.0248**	**0.001**	**2.03**	**0.0152**	**0.001**
	0.68	0.0031	0.041	0.67	0.0091	0.067	0.62	0.0127	0.079	0.58	0.0132	0.091	0.59	0.0132	0.110
H	**2.66**	**0.0076**	**0.037**	**2.70**	**0.0217**	**-0.120**	**2.77**	**0.0343**	**-0.025**	**2.83**	**0.0329**	**-0.084**	**2.89**	**0.0227**	**-0.085**
	0.67	0.0029	0.040	0.65	0.0084	0.092	0.59	0.0111	0.088	0.54	0.0112	0.098	0.52	0.0128	0.089
H-L	**0.99**	**0.0015**	**-0.008**	**0.99**	**0.0007**	**-0.084**	**0.99**	**0.0089**	**-0.101**	**1.02**	**0.0139**	**-0.118**	**1.04**	**0.0134**	**-0.050**
	0.51	0.0024	0.030	0.57	0.0076	0.067	0.60	0.0116	0.071	0.61	0.0104	0.080	0.61	0.0123	0.100

Table 1.3: **Estimates of Excess Returns and Level and Volatility Risks for Full-Period Growth and Value Portfolios.** The entries of the table are sample estimates of $E[y_t]$ where y_t is successively $R^e_{t,S,S}$, $\left(R^e_{t,S,S} - E\left[R^e_{t,S,S}\right]\right)(\Delta c_{t,S} - E[\Delta c_{t,S}])$ and $10^4\left(R^e_{t,S,S} - E\left[R^e_{t,S,S}\right]\right)(\Delta h_{t,S} - E[\Delta h_{t,S}])$. Standard errors are given below the estimates and are computed using the Newey-West procedure with S lags.

		$S=4$			$S=8$			$S=12$			$S=16$			$S=20$	
	R^e	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$	R^e	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$	R^e	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$	R^e	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$	R^e	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$
Panel A. Earnings-to-Price Sorted Portfolios.															
L	**5.93**	**0.0417**	**0.123**	**10.18**	**0.0558**	**0.458**	**13.93**	**0.0104**	**0.277**	**17.99**	**-0.0100**	**0.018**	**23.38**	**-0.0960**	**-0.331**
	2.43	0.0366	0.165	4.07	0.0755	0.321	6.41	0.1037	0.428	9.50	0.1322	0.634	12.80	0.1949	0.810
3	**6.45**	**0.0335**	**-0.034**	**13.16**	**0.0524**	**-0.074**	**19.84**	**0.0253**	**-0.581**	**27.33**	**0.0361**	**-1.003**	**36.28**	**-0.0291**	**-1.445**
	2.05	0.0276	0.143	3.87	0.0563	0.334	6.15	0.0860	0.425	9.04	0.1495	0.650	11.88	0.2327	0.761
H	**10.63**	**0.0606**	**-0.253**	**22.26**	**0.1493**	**-0.981**	**34.69**	**0.1911**	**-1.927**	**48.40**	**0.2784**	**-2.709**	**64.00**	**0.1906**	**-3.493**
	2.45	0.0322	0.234	4.85	0.0806	0.508	7.49	0.1300	0.587	10.43	0.2133	0.875	13.38	0.2945	1.098
H-L	**4.70**	**0.0189**	**-0.375**	**12.08**	**0.0934**	**-1.439**	**20.76**	**0.1807**	**-2.204**	**30.41**	**0.2884**	**-2.727**	**40.62**	**0.2866**	**-3.162**
	2.16	0.0244	0.227	4.56	0.0820	0.523	6.67	0.1378	0.666	8.30	0.1773	0.882	9.59	0.2150	1.083
Panel B. Cash Flows-to-Price Sorted Portfolios.															
L	**6.60**	**0.0444**	**0.123**	**11.87**	**0.0682**	**0.497**	**16.72**	**0.0367**	**0.351**	**21.92**	**0.0352**	**0.043**	**28.21**	**-0.0493**	**-0.432**
	2.49	0.0366	0.156	4.49	0.0790	0.314	7.35	0.1160	0.499	10.94	0.1480	0.674	14.75	0.2082	0.810
3	**7.38**	**0.0345**	**-0.004**	**14.92**	**0.0407**	**-0.216**	**22.73**	**-0.0169**	**-0.758**	**31.87**	**-0.0505**	**-1.296**	**43.15**	**-0.1770**	**-1.852**
	2.24	0.0311	0.157	4.01	0.0635	0.363	6.55	0.1034	0.500	10.03	0.1804	0.788	13.77	0.2816	1.001
H	**10.86**	**0.0680**	**-0.156**	**22.54**	**0.1752**	**-0.624**	**34.67**	**0.2398**	**-1.500**	**48.18**	**0.3524**	**-2.221**	**63.49**	**0.3349**	**-2.840**
	2.17	0.0276	0.215	4.08	0.0693	0.462	6.18	0.1139	0.456	8.70	0.2017	0.734	10.84	0.2800	0.896
H-L	**4.26**	**0.0236**	**-0.279**	**10.67**	**0.1070**	**-1.121**	**17.94**	**0.2031**	**-1.851**	**26.26**	**0.3172**	**-2.264**	**35.28**	**0.3843**	**-2.408**
	2.02	0.0231	0.188	4.30	0.0780	0.475	6.64	0.1345	0.623	8.89	0.1707	0.804	10.91	0.2141	1.014

Table 1.4: **Estimates of Excess Returns and Level and Volatility Risks for Single-Period Size and Book-to-Market Sorted Portfolios.**
The entries of the table are sample estimates of $E[y_t]$ where y_t is successively $R^e_{t,1,S}$, $\left(R^e_{t,1,S} - E\left[R^e_{t,1,S}\right]\right)(\Delta c_{t,S} - E[\Delta c_{t,S}])$ and $10^4\left(R^e_{t,1,S} - E\left[R^e_{t,1,S}\right]\right)(\Delta h_{t,S} - E[\Delta h_{t,S}])$. Standard errors are given below the estimates and calculated using the Newey-West procedure with S lags.

	$S=1$			$S=8$		$S=12$			$S=16$	
	R^e	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$	R^e	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$
Panel A. Small Portfolios.										
11	**1.31**	**0.0137**	**0.040**	**0.0492**	**-0.074**	**1.39**	**0.0318**	**-0.121**	**0.0102**	**-0.192**
	1.31	0.0060	0.076	0.0279	0.177	1.22	0.0312	0.204	0.0325	0.216
13	**2.64**	**0.0103**	**0.027**	**0.0511**	**-0.137**	**2.84**	**0.0449**	**-0.157**	**0.0265**	**-0.186**
	1.04	0.0049	0.061	0.0200	0.149	0.90	0.0206	0.162	0.0204	0.170
15	**3.61**	**0.0114**	**0.052**	**0.0575**	**-0.176**	**3.84**	**0.0527**	**-0.194**	**0.0361**	**-0.203**
	1.06	0.0053	0.077	0.0204	0.154	0.90	0.0226	0.156	0.0247	0.157
15-11	**2.30**	**-0.0023**	**0.012**	**0.0083**	**-0.102**	**2.45**	**0.0209**	**-0.073**	**0.0259**	**-0.011**
	0.62	0.0028	0.034	0.0168	0.078	0.77	0.0198	0.097	0.0213	0.096
Panel B. Medium-Sized Portfolios.										
31	**1.81**	**0.0069**	**0.062**	**0.0276**	**0.007**	**1.96**	**0.0096**	**-0.042**	**-0.0062**	**-0.099**
	1.07	0.0050	0.067	0.0215	0.137	0.77	0.0202	0.157	0.0175	0.177
33	**2.24**	**0.0066**	**0.051**	**0.0414**	**-0.068**	**2.41**	**0.0405**	**-0.114**	**0.0295**	**-0.131**
	0.80	0.0039	0.059	0.0147	0.110	0.66	0.0151	0.114	0.0161	0.125
35	**3.01**	**0.0070**	**0.058**	**0.0425**	**-0.078**	**3.21**	**0.0376**	**-0.099**	**0.0256**	**-0.134**
	0.85	0.0040	0.055	0.0153	0.112	0.66	0.0165	0.119	0.0178	0.118
35-31	**1.19**	**0.0001**	**-0.005**	**0.0149**	**-0.085**	**1.25**	**0.0280**	**-0.057**	**0.0319**	**-0.035**
	0.65	0.0031	0.043	0.0159	0.092	0.62	0.0144	0.105	0.0141	0.100
Panel C. Large Portfolios.										
51	**1.79**	**0.0056**	**0.039**	**0.0278**	**0.087**	**1.94**	**0.0239**	**0.026**	**0.0170**	**-0.022**
	0.75	0.0033	0.043	0.0130	0.104	0.74	0.0132	0.114	0.0141	0.146
53	**1.60**	**0.0059**	**0.044**	**0.0164**	**0.069**	**1.72**	**0.0149**	**-0.001**	**0.0068**	**0.032**
	0.60	0.0026	0.033	0.0093	0.065	0.56	0.0093	0.081	0.0109	0.086
55	**2.02**	**0.0059**	**0.034**	**0.0320**	**-0.022**	**2.18**	**0.0297**	**-0.055**	**0.0186**	**-0.035**
	0.67	0.0027	0.034	0.0100	0.095	0.57	0.0127	0.114	0.0143	0.103
55-51	**0.23**	**0.0003**	**-0.006**	**0.0043**	**-0.109**	**0.24**	**0.0057**	**-0.081**	**0.0016**	**-0.012**
	0.57	0.0027	0.031	0.0110	0.069	0.62	0.0119	0.080	0.0132	0.085

Table 1.5: **Estimates of Excess Returns and Level and Volatility Risks for Full-Period Size and Book-to-Market Sorted Portfolios.**
The entries of the table are sample estimates of $E[y_t]$ where y_t is successively $R^e_{t,S,S}$, $\left(R^e_{t,S,S} - E\left[R^e_{t,S,S}\right]\right)(\Delta c_{t,S} - E[\Delta c_{t,S}])$ and $10^4\left(R^e_{t,S,S} - E\left[R^e_{t,S,S}\right]\right)(\Delta h_{t,S} - E[\Delta h_{t,S}])$. Standard errors are given below the estimates and calculated using the Newey-West procedure with S lags.

		$S=4$					$S=12$		$S=16$	
				$S=8$						
	R^e	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$	R^e	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$	$R^e\xi_{\Delta c}$	$R^e\xi_{\Delta h}$
Panel A. Small Portfolios.										
11	**5.58**	**0.0844**	**0.076**	**0.2152**	**-0.184**	**10.01**	**0.3560**	**-0.848**	**0.5818**	**-0.322**
	4.57	0.0551	0.409	0.1405	0.942	13.81	0.3079	1.422	0.5942	1.913
13	**11.18**	**0.0740**	**-0.250**	**0.1659**	**-1.203**	**35.92**	**0.2060**	**-2.383**	**0.2956**	**-2.228**
	3.52	0.0426	0.270	0.0946	0.647	10.17	0.1647	1.025	0.3182	1.383
15	**14.79**	**0.0976**	**-0.301**	**0.2418**	**-1.704**	**49.81**	**0.3230**	**-3.607**	**0.4524**	**-4.086**
	3.44	0.0409	0.345	0.1127	0.851	11.47	0.2195	1.192	0.4352	1.584
15-11	**9.21**	**0.0132**	**-0.377**	**0.0266**	**-1.520**	**39.80**	**-0.0331**	**-2.759**	**-0.1295**	**-3.765**
	2.72	0.0323	0.296	0.1024	0.815	8.86	0.2031	1.291	0.3095	1.565
Panel B. Medium-Sized Portfolios.										
31	**6.80**	**0.0399**	**0.162**	**0.0620**	**0.179**	**15.02**	**0.0207**	**-0.472**	**-0.0051**	**-0.633**
	3.02	0.0451	0.250	0.0872	0.568	7.04	0.1376	0.755	0.2312	1.085
33	**9.15**	**0.0579**	**-0.203**	**0.1342**	**-0.832**	**28.68**	**0.1485**	**-1.902**	**0.1839**	**-2.397**
	2.55	0.0320	0.211	0.0734	0.515	7.15	0.1250	0.647	0.2173	0.928
35	**12.44**	**0.0648**	**-0.357**	**0.1403**	**-1.422**	**42.01**	**0.1454**	**-2.696**	**0.2080**	**-3.427**
	2.64	0.0323	0.295	0.0814	0.675	7.89	0.1328	0.865	0.2431	1.200
35-31	**5.64**	**0.0249**	**-0.519**	**0.0783**	**-1.601**	**26.98**	**0.1247**	**-2.224**	**0.2131**	**-2.795**
	2.21	0.0351	0.330	0.0904	0.701	5.85	0.1401	0.873	0.1864	1.136
Panel C. Large Portfolios.										
51	**7.21**	**0.0420**	**0.142**	**0.0710**	**0.548**	**20.26**	**0.0505**	**0.433**	**0.0752**	**-0.066**
	2.51	0.0354	0.155	0.0867	0.363	9.05	0.1421	0.668	0.1835	0.788
53	**6.49**	**0.0434**	**-0.002**	**0.0745**	**-0.104**	**19.88**	**0.0611**	**-0.599**	**0.0643**	**-1.121**
	2.00	0.0285	0.160	0.0653	0.340	6.45	0.1090	0.422	0.1854	0.677
55	**8.27**	**0.0436**	**-0.312**	**0.0989**	**-0.924**	**25.82**	**0.0895**	**-1.960**	**0.1026**	**-2.768**
	2.23	0.0270	0.190	0.0713	0.455	7.14	0.1407	0.600	0.2615	0.956
55-51	**1.06**	**0.0016**	**-0.454**	**0.0278**	**-1.472**	**5.56**	**0.0391**	**-2.393**	**0.0274**	**-2.703**
	2.09	0.0271	0.199	0.0815	0.554	7.10	0.1382	0.771	0.1723	0.817

folios in Tables 1.4 and 1.5. Portfolios are picked among the 25 size and book-to-market sorted portfolios and each two-digit label **xy** in the first column of the tables represents one portfolio. The first digit **x** refers to the size quintiles (1 indicating the smallest firms, 5 the largest), and the second digit **y** refers to book-to-market quintiles (1 indicating the portfolio with the lowest book-to-market ratio, 5 the highest). We present results for the growth (**y=1**), the neutral (**y=3**) and the value (**y=5**) for small (**x=1**), medium-sized (**x=3**) and large (**x=5**) firms. Except for growth stocks, average excess returns and level risk are significantly estimated for all portfolios in single-period investment in stocks over horizons up to twelve quarters as shown in Table 1.4. Evidences for positive spreads in average returns and level risk and for a negative spread in volatility risk between value and growth portfolios are well-related in each size group as the investment horizon increases. The story remains true for multiperiod investments in stocks as shown in Table 1.5 whereas significant volatility risk is more often observed, together with significant average excess returns.

We plot the pattern of S-level and S-volatility risks across book-to-market sorted portfolios, and also across dividend-to-price, earnings-to-price and cash flows-to-price sorted portfolios. The figures are similar to those of Hansen, Heaton and Li (2005) which show the pattern of S-level risk across 5 dividend-to-price sorted portfolios especially when the stock holding period is one quarter. Next, we describe how consumption level and consumption volatility risks are correlated with short and long-period returns, and analyze the pattern of these risks across value and growth portfolios.

1.3.2 Patterns of Level and Volatility Risks

In this subsection, we describe the pattern of consumption volatility risk across stocks. As S varies, we describe how S-volatility risk ranks portfolios from the less to the more riskier, and we compare this ranking to that based on the risk premium. We also compare the ranking by consumption volatility risk to the ranking by consumption level risk. We focus on one-period ($k = 1$) and full period ($k = S$) returns as this makes our findings comparable to results of previous studies (Parker and Julliard (2005), Bansal, Dittmar and Kiku (2005)). We will say that volatility or level risk rank stocks well if the more riskier is a portfolio, the more higher is its volatility or level risk.

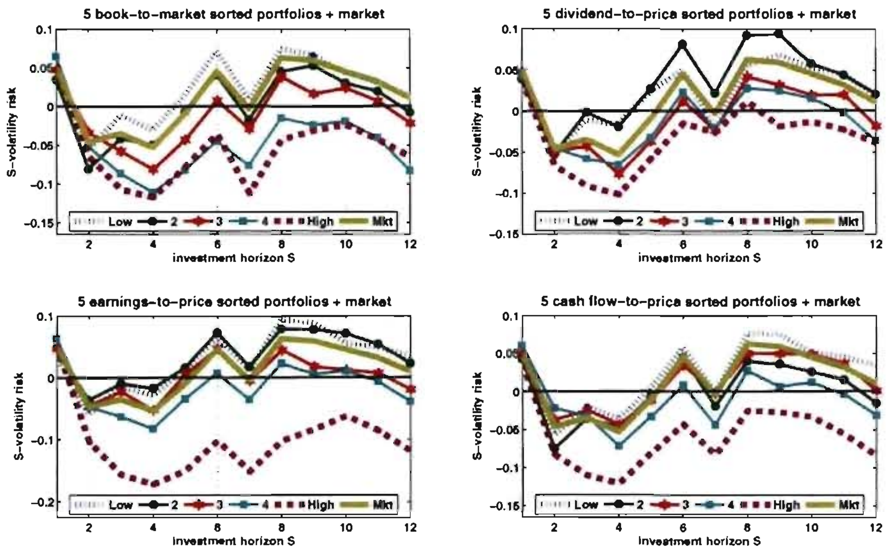
Panel A of Figure 1.3 shows the pattern of consumption volatility risk by investment horizon when stocks are hold for one quarter at the beginning of the investment period. At each investment horizon, the top point represents the less riskier stock and the bottom point the more riskier. It can be observed that difference between volatility risks for the extreme value and the extreme growth portfolios is not apparent for $S = 1$ and $S = 2$. However, for $S > 2$, there is a significant gap between volatility risks of these portfolios, with the value line on the bottom and the growth line on the top, which shows that value assets are more riskier than growth assets when investments are exposed to variations in consumption volatility. Because value stocks covariate highly and negatively with variations in the volatility of aggregate consumption and more so than other stocks, this means that their payoffs are lower than those of other stocks when macroeconomic uncertainty becomes higher in the future relatively to the present. Then value stocks are disliked more than other stocks and investors require a more higher premium to hold them. Not surprisingly in Panel A of Figure 1.3, the market risk (covariance between aggregate stock market return and variations in consumption volatility) lies between extreme risks (value risk and growth risk).

Panel B of Figure 1.3 shows the pattern of consumption level risk by investment horizon when stocks are hold for one quarter at the beginning of the investment period. Contrarily to the pattern of consumption volatility risk across stocks, at each investment horizon, the top point represents the more riskier stock and the bottom point the less riskier. Compared to the pattern of consumption volatility risk, one can observe that for smaller investment horizons where volatility risk sorts stocks as they are ordered according to risk premium, level risk does worst in this sort. Growth assets appear to be more riskier than other assets when exposed to relatively short variations in consumption level, and this clearly appears for $S < 6$ in Panel B of Figure 1.3. However, consistent with a similar pattern plotted in Hansen, Heaton and Li (2005) and with the results of Parker and Julliard (2005), as the investment horizon increases, differences in consumption level risk across stocks become significant, with portfolios ranked as they are sorted according to their risk premium, that is value stocks are more riskier than growth stocks

Figure 1.3: **Volatility and Level Risks for Single-Period Growth and Value Portfolios.**

This figure presents the pattern of S-volatility and S-level Risks across growth and value portfolios when $k = 1$. Risks are computed as covariances of returns with changes in consumption volatility in Panel A, and with changes in consumption level in Panel B.

Panel A. Volatility Risk



Panel B. Level Risk

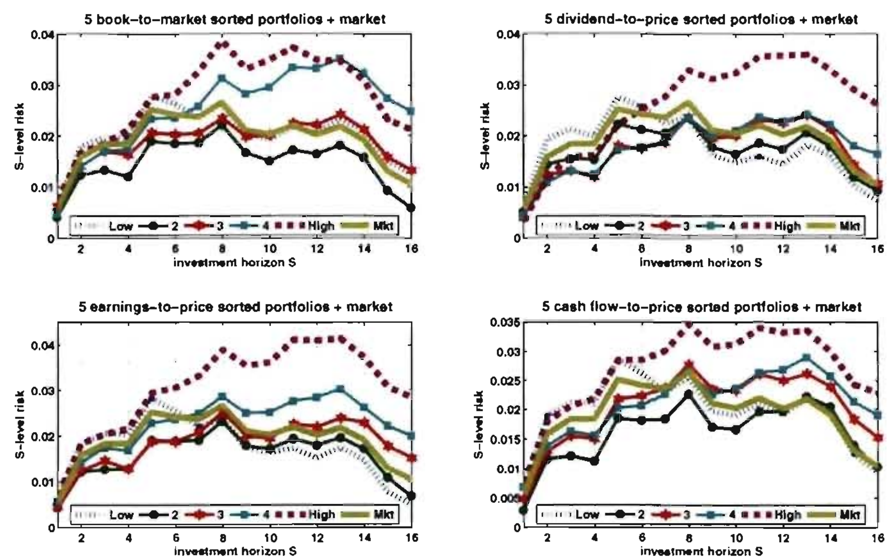
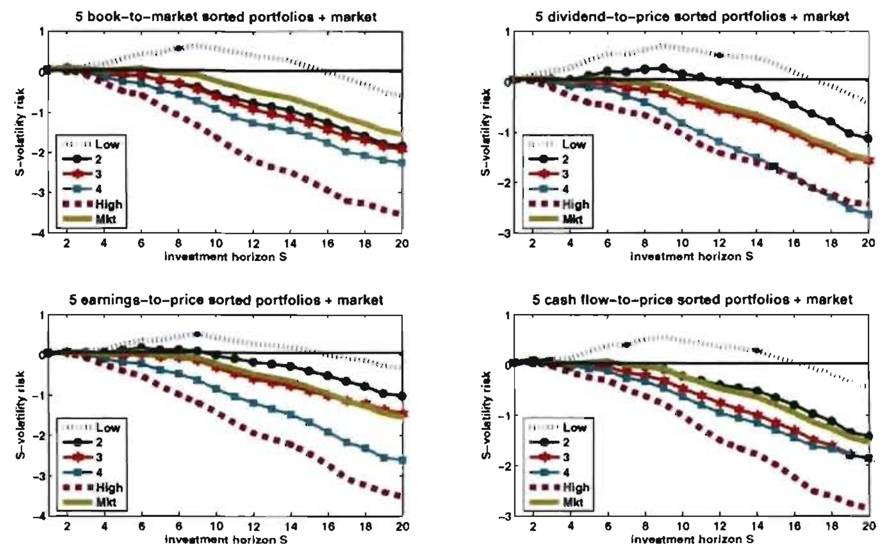
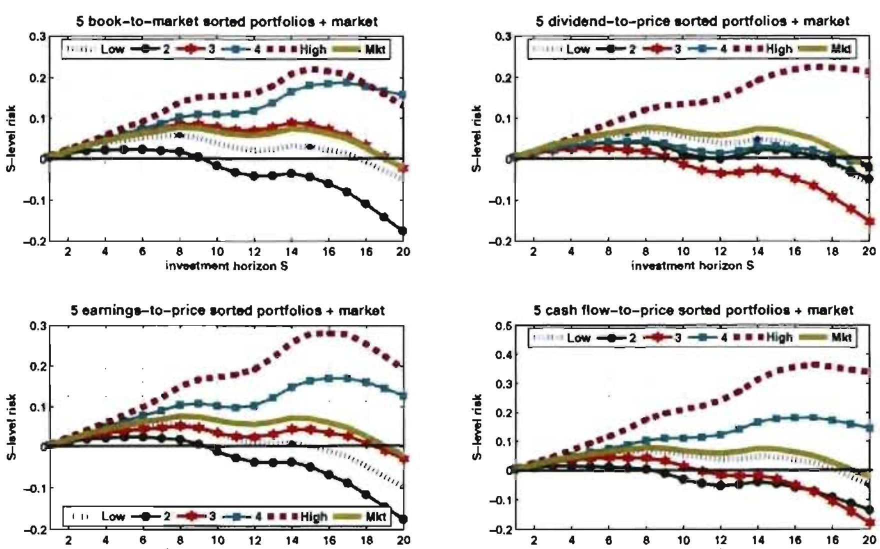


Figure 1.4: **Volatility and Level Risks for Full-Period Growth and Value Portfolios.** This figure presents the pattern of S-volatility and S-level Risks across long-horizon growth and value portfolios ($k = S$). Risks are computed as covariances of returns with changes in consumption volatility in Panel A, and with changes in consumption level in Panel B.

Panel A. Volatility Risk



Panel B. Level Risk



when investment are exposed to long-horizon variations in consumption level.

Panel A of Figure 1.4 shows the pattern of consumption volatility risk by investment horizon when stocks are hold for the full investment period. This pattern clearly shows that the ranking between asset risks is the same between assets as the horizon increases, value stocks having a more pronounced negative covariance with volatility variations than growth stocks. Once again and not surprisingly, the long-horizon market portfolio risk lies between extreme portfolio risks. Since value stocks also have higher mean returns than growth stocks, one can expect that projecting full period stock returns in stock S-volatility risks will give a negative slope coefficient.

Panel B of Figure 1.4 shows the pattern of consumption level risk by investment horizon when stocks are hold for the full investment period. Compared to the similar pattern of volatility risk, one can observe that consumption level risk fails to rank well with the semi-growth portfolio which in all dimensions is riskier than the extreme growth portfolio. In addition, the extreme growth appears to be more riskier than the medium in dividend-to-price and cash flow-to-price dimensions, and even more riskier than the semi-value in the dividend-to-price dimension.

We also plot the pattern of S-volatility and S-level risks for book-to-market sorted portfolios at a less aggregate level, that is when assets are first sorted according to the firm size, and then according to the firm book-to-market in each size group. Corresponding figures are shown in Appendix I. Figures I.2 and I.4 show the pattern of consumption volatility risk for one-period ($k = 1$) and full period ($k = S$) holding stock returns respectively. Figures I.3 and I.5 display similar patterns of consumption level risk. All confirm that the findings at the aggregate level also hold in each size group.

1.3.3 Analyzing the Risk-Return Relationship

While the pattern of volatility and level risks across stocks inform how portfolios are ranked from the less to the more riskier (or from the less to the more preferred), we cannot still assess the strength of the relationship between these risks and the total risk premium that investors require to invest in stocks instead of the safe asset. Even if portfolios are well-ranked by volatility risk at horizons S_1 and S_2, the strength of the

Table 1.6: **Correlations of Returns with Level and Volatility Risks.**
This table presents correlations of the mean excess k-period returns on the 25 Fama and French portfolios with the S-level and S-volatility risks. Risks are computed as covariances of returns with changes in consumption volatility and with changes in consumption level. For each horizon S, the top line represents correlations with S-level risk and the bottom line shows correlations with S-volatility risk. Consumption volatility satisfies the Heston and Nandi (2000) dynamics specified in equation (1.26).

	k						
S	1	2	4	8	12	16	20
1	0.32						
	0.23						
2	0.30	0.54					
	-0.38	**-0.03**					
4	0.45	0.56	0.70				
	-0.77	**-0.74**	**-0.76**				
8	0.70	0.77	0.76	0.68			
	-0.72	**-0.79**	**-0.86**	**-0.88**			
12	0.80	0.84	0.86	0.82	0.59		
	-0.68	**-0.70**	**-0.82**	**-0.87**	**-0.88**		
16	0.76	0.79	0.82	0.86	0.73	0.54	
	-0.63	**-0.70**	**-0.84**	**-0.87**	**-0.87**	**-0.90**	
20	0.77	0.83	0.85	0.86	0.81	0.67	0.45
	-0.50	**-0.53**	**-0.72**	**-0.84**	**-0.84**	**-0.91**	**-0.89**

relationship between risk premium and consumption volatility risk at these horizons can differ widely. In section 1.2.3 we defined this strength through cross-sectional correlations between the two business cycle risks and risk premium. In the following, we compute the sample cross-sectional correlation of each risk with average excess return across the 25 Fama and French size and book-to-market sorted portfolios, in order to assess how strong high excess returns are associated with high level or volatility risk. As discussed previously, the square of this correlation also measures the fraction of the cross-sectional dispersion in mean average excess returns explained by level or volatility risk.

This section examines cross-sectional correlations between risk premium and consumption volatility risk. As S varies for given k, we analyze the strength of the relationship between volatility risk and return at lower investment horizons to the strength at longer horizons. At each horizon, we also compare the relationship between volatility risk and return to the relationship between level risk and return. On the other hand, as k varies for given S, we analyze the strength of the relationship between volatility risk and return, and also oppose volatility risk-return relationship to level risk-return relationship.

Table 1.6 shows correlations between risk premium and consumption level and consumption volatility risks when both the total investment horizon S and the stock holding period k equal one and two quarters, then one, two, three, four and five years. The second column of the table measures how much one-period returns are correlated to variations in consumption volatility, but also in consumption level as in Parker and Julliard (2005). One can observe that one-period stock risk premium is weakly and positively correlated to one-horizon consumption volatility risk and this is not consistent with the theory that, when exposed to variations in consumption volatility, riskier investments should have higher average excess returns. Moreover, while volatility risk-return correlation becomes negative from the horizon of two quarters, it remains weak. However the volatility risk-return correlation grows as the investment horizon increases.

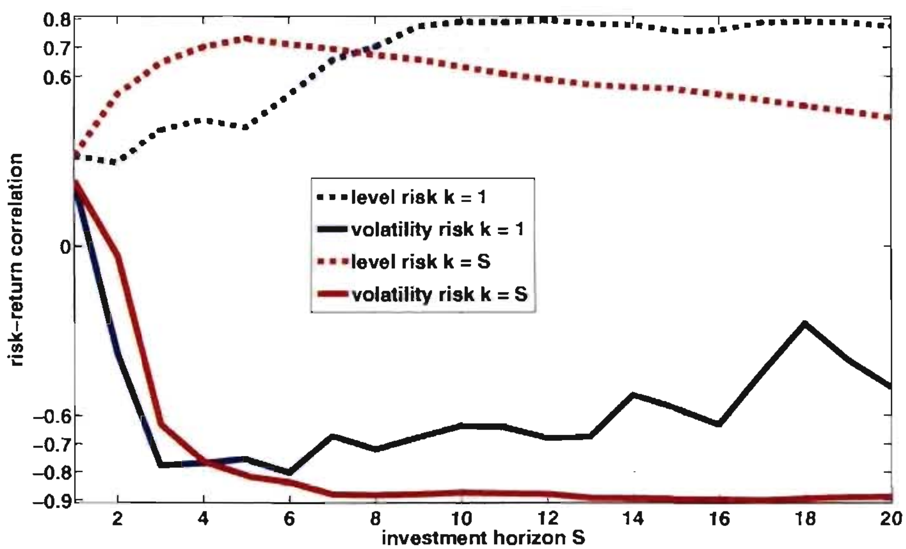
The second column of Table 1.6 also shows the known weak correlation between contemporaneous consumption risk (here the level risk at the horizon of one quarter) and risk premium. This correlation is 0.32 and means that contemporaneous consumption risk explains only about 10% of variations in average stock returns. The level risk-return correlation is still weak at the investment horizon of two quarters, then grows as the horizon increases. If the weak performance of shorter variations in consumption level to explain differences in average stock returns is due to the slow adjustment of consumption to returns as argued by Parker and Julliard (2005), then we argue that the same reason could explain why shorter variations in consumption volatility also performs weakly in explaining differences across average stock returns.

The diagonal line of Table 1.6 measures how much full-period returns are correlated to changes in consumption volatility, and also in consumption level as in Bansal, Dittmar and Kiku (2005). It shows that consumption volatility risk is highly and negatively correlated to full-period stock risk premium, and so more than single-period stock risk

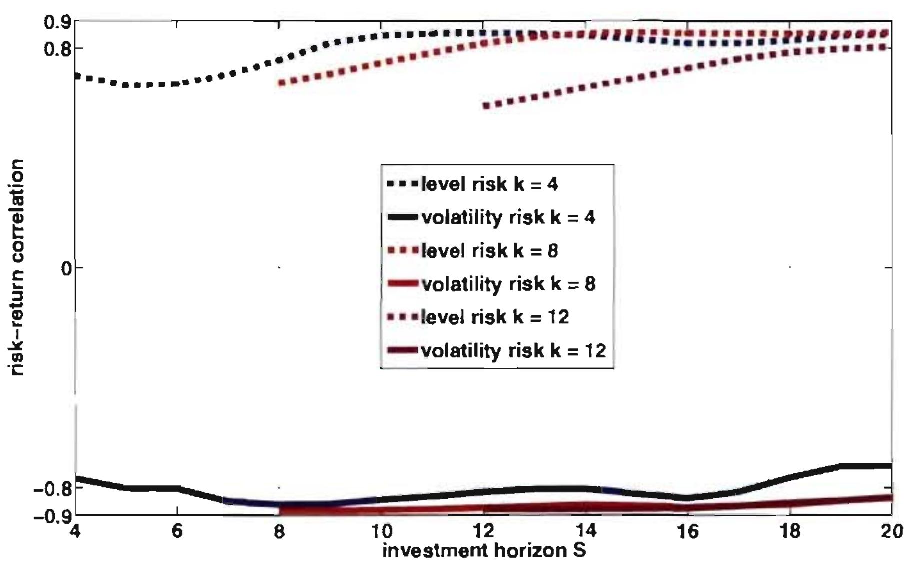
Figure 1.5: **Volatility Risk-Return and Level Risk-Return Relations by Investment Horizon.**

This figure presents the pattern of $\rho_{rc}(S,k)$ (dashed lines) and that of $\rho_{rh}(S,k)$ (solid lines) for $k=1$ and $k=S$ in Panel A, and when k is fixed to 4, 8 and 12 in Panel B, while S varies from 1 to 20.

Panel A. Single-Period and Full-Period Returns



Panel B. MultiPeriod Returns



premium. In contrast consumption level risk is more correlated to one-quarter stock risk premium than to full period stock risk premium. On the other hand, correlation of average excess S-period returns with S-volatility risk dominates that with S-level risk at all horizons $S > 2$. Since the former is quite constant for all investment horizons, it seems that there is a stable long-run relationship between stock returns and variations in consumption volatility. A correlation of -0.88 would also mean that more than 75% of heterogeneity in average long-period stock returns come from the heterogeneity in their exposure to permanent movements in consumption volatility.

The latter facts are well illustrated in Panel A of Figure 1.5 which plots consumption volatility risk-return relationship versus consumption level risk-return relationship for one-quarter and full period holding stocks. The figure shows that average one-quarter returns are more correlated to short-horizon volatility risk than to short-horizon level risk. While this correlation is greater than 0.75 with volatility risk for horizons $2 < S < 7$, it is smaller than 0.55 for level risk for the same horizons. In contrast, average one-period returns are more correlated to long-horizon consumption level risk than to long-horizon consumption volatility risk. With long-horizon consumption volatility risk, this correlation has a downward trend from the horizon $S = 8$, where it is worth -0.72, to the horizon $S = 20$, where it is worth -0.50. On the other hand, with long-horizon consumption level risk, this correlation is close to about 0.80 from the horizon $S = 9$ to the horizon $S = 20$, both of where it is worth 0.77.

Panel A of Figure 1.5 also illustrates that average full-period returns are more correlated to consumption volatility risk than to consumption level risk, with a complete domination of the volatility risk-return relationship in the long run. While the volatility-risk return correlation is close to about -0.90 from the horizon $S = 7$ to the horizon $S = 20$, the level risk-return relationship declines from 0.70 to 0.45 for the same horizons.

Finally, Panel B of Figure 1.5 and Figure I.1 in Appendix I illustrate that changes in consumption volatility are at least as correlated to other multiple-period returns as changes in consumption level, and much more for short investment horizons relatively to the stock holding period.

1.4 Pricing Consumption Volatility Risk in the Cross-Section

The striking pattern of S-volatility risk across stocks and its high correlation with expected excess returns motivates our investigation of how this risk is priced in financial markets, especially when S-level risk is also taken into account. We inquire how much of the cross-sectional differences in stocks is explained by both S-level and S-volatility risks, and this is important since variations in consumption level are uncorrelated to variations in consumption volatility from our GARCH specification. Estimating the volatility risk price in a two-factor model, and evaluating the amount of premium coming from volatility variations will also determine how important are long-run volatility risks in the presence of long-run consumption risks.

1.4.1 Estimation Methodology

Following recent empirical studies of cross-sectional asset pricing (see for example, Cochrane (1996), Jagannathan and Wang (1996), and Jacobs and Wang (2004)), we use the generalized method of moment (GMM, Hansen (1982)) to evaluate the significance of consumption volatility factors. Cochrane (2001, Chapter 15) demonstrates that the GMM approach works well for linear asset pricing models. The cross-sectional model (1.23) satisfies a moment condition of the form:

$$E\left[-\iota b+\left(1-\xi^{\top}(\pi)\,p\right)R\right]=0 \tag{1.29}$$

where $\xi(\pi)$ is the vector of demeaned factors, R is the vector of excess returns, p is the vector of risk prices and b is the constant term introduced to measure by how much the cross-sectional model fails to predict returns. Demeaned factors depend on the parameter vector π that governs the processes (1.25) and (1.26) of consumption growth and consumption volatility. The vector ι is of same length as R and has all its components equal to one. The moment condition (1.29) holds for a given date and a given horizon. We avoid subscripts in variables and parameters to simplify notations in this section. The vectors $\xi(\pi)$ and p have two components each.

Equation (1.29) is also equivalent to:

$$\mu_R = \iota b + \Sigma_{R\xi}(\pi)\, p \tag{1.30}$$

where $\mu_R = E[R]$ and $\Sigma_{R\xi}(\pi) = E\left[R\xi^\top(\pi)\right]$ are respectively the vector of mean excess returns and the covariance matrix of excess returns with factors. The latter depends on the parameter vector π of consumption and volatility processes through $\xi(\pi)$.

Two-Step Estimation With Prespecified Weighting Matrix. If the parameter vector π were known, then the constant b and the factor risk prices p could be consistently estimated by GMM based on the moment condition (1.29), by minimizing the distance between average actual returns $\widehat{\mu}_R$ and average predicted returns $\iota b + \widehat{\Sigma}_{R\xi}(\pi)\, p$ with respect to a positive definite matrix W. $\widehat{\mu}_R$ and $\widehat{\Sigma}_{R\xi}(\pi)$ are the sample counterparts of the mean vector μ_R and the covariance matrix $\Sigma_{R\xi}(\pi)$.

Minimizing the distance:

$$dist(b,p) = \sqrt{\left(\widehat{\mu}_R - \iota b - \widehat{\Sigma}_{R\xi}(\pi)\, p\right)^\top W \left(\widehat{\mu}_R - \iota b - \widehat{\Sigma}_{R\xi}(\pi)\, p\right)} \tag{1.31}$$

with respect to b and p gives:

$$\widehat{b}(\pi) = \left(\iota^\top W \iota\right)^{-1} \iota^\top W \left[\widehat{\mu}_R - \widehat{\Sigma}_{R\xi}(\pi)\,\widehat{p}(\pi)\right] \tag{1.32}$$

$$\widehat{p}(\pi) = \left[\widehat{\Sigma}_{\xi R}(\pi) A \widehat{\Sigma}_{R\xi}(\pi)\right]^{-1} \widehat{\Sigma}_{\xi R}(\pi) A \widehat{\mu}_R \tag{1.33}$$

where $A = W - W\iota\left(\iota^\top W \iota\right)^{-1}\iota^\top W$. For these solutions, the vector of pricing errors and the minimum distance value are given by:

$$\widehat{e}(\pi) = W^{-1}\widehat{B}(\pi)\,\widehat{\mu}_R \tag{1.34}$$

$$\widehat{d}(\pi) = \sqrt{\widehat{e}^\top(\pi)\, W \widehat{e}(\pi)} = \sqrt{\widehat{\mu}_R^\top \widehat{B}(\pi)\,\widehat{\mu}_R} \tag{1.35}$$

where $\widehat{B}(\pi) = A - A\widehat{\Sigma}_{R\xi}(\pi)\left[\widehat{\Sigma}_{\xi R}(\pi) A \widehat{\Sigma}_{R\xi}(\pi)\right]^{-1} \widehat{\Sigma}_{\xi R}(\pi) A$. We then compute the adjusted central R-squared through the formula:

$$R^2(\pi) = 1 - \frac{N-1}{N-K-1} \frac{\widehat{e}^{\top}(\pi) A \widehat{e}(\pi)}{\widehat{\mu}_R^{\top} A \widehat{\mu}_R}, \tag{1.36}$$

where N and K are respectively the number of portfolios and the number of factors. If W is the identity matrix, then the formula (1.36) gives the adjusted central R-squared calculated as if we were doing a linear regression of the average returns on risks measured by covariances between returns and factors. In this case, $\widehat{d}(\pi)/\sqrt{N}$ is the square root of the weighted average of the squared pricing errors and measures how much the expected return based on the fitted model is off for a typical portfolio.

The matrices A and $\widehat{B}(\pi)$ have the property that $AW^{-1}A = A$ and $\widehat{B}(\pi)W^{-1}\widehat{B}(\pi) = \widehat{B}(\pi)$. Let $\widehat{\Sigma}_{bb}(\pi)$ and $\widehat{\Sigma}_{pp}(\pi)$ be the variances of these estimators. In general, $\widehat{b}(\pi)$, $\widehat{p}(\pi)$, $\widehat{\Sigma}_{bb}(\pi)$, $\widehat{\Sigma}_{pp}(\pi)$, $\widehat{e}(\pi)$ and $\widehat{d}(\pi)$ are continuous functions of π. Then, if π is unknown and if $\widehat{\pi}$ is a consistent estimator of π, it will hold that $\widehat{b}(\widehat{\pi})$, $\widehat{p}(\widehat{\pi})$, $\widehat{\Sigma}_{bb}(\widehat{\pi})$ and $\widehat{\Sigma}_{pp}(\widehat{\pi})$ are also consistent estimates of b, p, Σ_{bb} and Σ_{pp}. Even if this method of estimation is consistent, the uncertainty in the estimation of π leads to a larger asymptotic variance than when π is known. We have consistently estimated π by maximum likelihood in Section 1.3. We now use this estimate to compute the estimates $\widehat{b} = \widehat{b}(\widehat{\pi})$, $\widehat{p} = \widehat{p}(\widehat{\pi})$, $\widehat{\Sigma}_{bb} = \widehat{\Sigma}_{bb}(\widehat{\pi})$ and $\widehat{\Sigma}_{pp} = \widehat{\Sigma}_{pp}(\widehat{\pi})$, and also the pricing errors $\widehat{e}(\widehat{\pi})$, the minimum distance $\widehat{d}(\widehat{\pi})$ and the R-squared $R^2(\widehat{\pi})$.

One-Step Estimation With Prespecified Weighting Matrices. Let $f(\pi;\Delta c)$ denotes the density function of u in the model (1.26) satisfied by consumption growth and consumption volatility. In the two-stage estimation procedure, $L(\pi;\Delta c) = \sum \ln f(\pi;\Delta c)$ is first maximized to find an estimator of π that is further plugged into the cross-sectional estimation to obtain estimates of factor risk prices. With the one-step estimation procedure, we estimate the parameter π, simultaneously with the cross-sectional factor risk prices in a full single-stage GMM system. Let $\ell(\pi) = \left(\xi^{\top}(\pi), \frac{\partial \ln f}{\partial \pi^{\top}}\right)^{\top}$. In addition to the moment condition (1.29) we consider the moment condition:

$$E\left[\ell(\pi)\right] = 0 = \mu_\ell(\pi). \tag{1.37}$$

We perform the GMM estimation by placing the weighting matrices W and $\lambda\widehat{\Sigma}_{\ell\ell}^{-1}(\pi)$ respectively on the moments (1.29) and (1.37), and a null matrix on any product of these moments. This one-step estimation can be seen as practically equivalent to the two-step estimation. In the first step, we choose $\widehat{\pi}$ to minimize

$$\widehat{e}^{\top}(\pi)W\widehat{e}(\pi) + \lambda\widehat{\mu}_\ell^{\top}(\pi)\widehat{\Sigma}_{\ell\ell}^{-1}(\pi)\widehat{\mu}_\ell(\pi)$$

where $\widehat{\mu}_\ell(\pi)$ is the sample counterpart of $\mu_\ell(\pi)$, and where $\widehat{e}(\pi)$ is defined as in (1.34). In the second step, we plug $\widehat{\pi}$ into (1.32) and (1.33) to obtain $\widehat{b}$ and $\widehat{p}$. The number λ is large enough to ensure that estimates fit well the consumption growth and volatility processes, match factor conditional or unconditional means, as well as minimize the gap between actual and fitted returns (See also Yogo (2005) and Parker and Julliard (2005)).

Choosing the Prespecified Weighting Matrix. As weighting matrix, we use the second moment matrix of returns $W = \widehat{\Sigma}_{RR}^{-1}$. Hansen and Jagannathan (1997) advocate the use of this matrix instead of the optimal weighting matrix. It has two main economically important features. First, it provides estimates that minimize the distance between a stochastic discount factor that depends in a simple linear way on variations in both consumption level and consumption volatility, and the space of true stochastic discount factors. Second, as well as the optimal weighting matrix, the second moment matrix will make the objective function (1.31) invariant to the initial choice of intertemporal portfolios.[11] The portfolios used for the estimation are formed on economically interesting characteristics (size and book-to-market ratio). The second moment matrix will also form economically interesting combinations of these portfolios instead of unusual ones as the optimal matrix will do, and is more likely to provide small pricing errors

[11]Kandel and Stambaugh (1995) argue that results of several important asset pricing model tests are portfolio-dependent.

(Cochrane 2001, Chap. 11).

The R-squared (1.36) when $W = \widehat{\Sigma}_{RR}^{-1}$ is not interpretable as explanatory power of initial stock risk premia by level and volatility risks. Risk-return correlations in Section 1.3.2 were computed giving each portfolio equal weight. For this reason, only the R-squared (1.36) based on the identity matrix $W = I$ that puts equal weight on initial portfolios, can be used to compare horizon-dependent models since they are all based on equally weighted pricing errors. This R-squared is interpretable in terms of explanatory power of level and volatility risks and is related to squared correlations between risk premium and risks.

1.4.2 Estimation Results

This section will ask whether variations in consumption level and in consumption volatility are statistically significant, as well as if model tests of overindentifying restrictions reject the complete explanation of average stock returns by these factors. However, beyond these econometric issues, we are also and perhaps mostly interested in the economical significance of consumption level and consumption volatility risks for the cross-section of average stock returns. This economical significance contains two major points. Are the prices of the consumption level and consumption volatility risks respectively positive and negative as will be expected from the facts established in Section 1.3 and consistently with the theory? Do these risks explain a sizable percentage of variation in average stock returns?

We perform the estimation of the cross-sectional linear covariance model (1.23) for five values of S, corresponding to investment horizons of one quarter, then one, two, three and four years ($S = 1$, 4, 8, 12 and 16). We rely results based on two-step estimation since one-step estimation results are similar. These results are shown in Tables 1.7 and 1.8. We report the R-squared based on the identity matrix as well as the associated minimum distance between actual and fitted returns.

Table 1.7 shows that, both S-level and S-volatility risk prices are estimated insignificantly at all horizons, in the cross-section of one-period holding stock returns. Estimates of volatility risk price are even positive at horizons of three and four years. How-

Table 1.7: **Estimation of the Price of Volatility Risk in the Cross-Section of Single-Period Stock Returns.**
This table presents results from the two-step estimation described in Section 1.4.1 and based on the weighting matrix $W = \widehat{\Sigma}_{RR}^{-1}$. The entries of the table are the total investment horizon S, the horizon k of the investment in stocks, estimates of the constant term $\widehat{b}_{u,S}$, of the price of the S-level risk $\widehat{p}_{c,S}$ and of the price of the S-volatility risk $\widehat{p}_{h,S}$ (to be multiplied by 10^{-4}), the model J-statistics J_T, the cross-sectional R-squared R^2 and the square root of the weighted average of square pricing errors $\widehat{d}(\pi)$. The two latter statistics are also provided for the identity weighting matrix, namely $R^2(I)$ and $\widehat{d}(I)$. The numbers below the estimates are standard errors and below the J-statistics is the p-value. Covariance matrices are calculated using the Newey-West procedure with S lags.

k	S	$\widehat{b}_{u,S}$	$\widehat{p}_{c,S}$	$\widehat{p}_{h,S}$	J_T	R^2	$\widehat{d}$	$R^2(I)$	$\widehat{d}(I)$
1	1	1.90	13.09	**3.43**	60.52	-0.08	0.69	0.11	0.54
		(0.52)	(37.82)	**(3.51)**	[0.000]				
1	4	1.94	18.51	**-0.48**	109.03	-0.07	0.69	0.56	0.39
		(0.66)	(19.86)	**(2.51)**	[0.000]				
1	8	2.06	9.87	**-1.58**	146.40	-0.06	0.69	0.50	0.42
		(0.65)	(13.54)	**(2.25)**	[0.000]				
1	12	2.06	7.82	**0.58**	225.44	-0.08	0.69	0.60	0.38
		(0.68)	(9.95)	**(2.03)**	[0.000]				
1	16	2.12	6.86	**0.23**	282.95	-0.08	0.70	0.63	0.38
		(0.77)	(10.92)	**(2.08)**	[0.000]				

ever, while consumption level and volatility risks appear not statistically significant, they show some economic significance in explaining the cross-section of average one-period stock risk premiums. Both of these risks explain 60% of variations in average one-period returns at the horizon of three years, and 63% of these variations at the horizon of four years. This percentage is 11% at the horizon of one quarter and reflects the well-documented weakness of contemporaneous consumption risk in explaining differences in stock returns. As discussed in Parker and Julliard (2005), the fact that the cross-sectional model does not perform as this horizon can be related to the low adjustment of consumption to returns. However, the fact that it behaves well for longer horizons, as we can see an increase in the R-squared from the horizon of one quarter, can not only be related to the fact that consumption and consequently volatility have had time to adjust to returns. It also reflects the concerns that investors have about long-run risks both in

Table 1.8: **Estimation of the Price of Volatility Risk in the Cross-Section of Full-Period Stock Returns.**
This table presents results from the two-step estimation described in Section 1.4.1 and based on the weighting matrix $W = \widehat{\Sigma}_{RR}^{-1}$. The entries of the table are the total investment horizon S, the horizon k of the investment in stocks, estimates of the constant term $\widehat{b}_{u,S}$, of the price of the S-level risk $\widehat{p}_{c,S}$ and of the price of the S-volatility risk $\widehat{p}_{h,S}$ (to be multiplied by 10^{-4}), the model J-statistics J_T, the cross-sectional R-squared R^2 and the square root of the weighted average of square pricing errors $\widehat{d}(\pi)$. The two latter statistics are also provided for the identity weighting matrix, namely $R^2(I)$ and $\widehat{d}(I)$. The numbers below the estimates are standard errors and below the J-statistics is the p-value. Covariance matrices are calculated using the Newey-West procedure with S lags.

k	S	$\widehat{b}_{u,S}$	$\widehat{p}_{c,S}$	$\widehat{p}_{h,S}$	J_T	R^2	$\widehat{d}$	$R^2(I)$	$\widehat{d}(I)$
1	1	1.90	13.09	**3.43**	60.52	-0.08	0.69	0.11	0.54
		(0.52)	(37.82)	**(3.51)**	[0.000]				
4	4	3.36	66.28	**-2.51**	67.46	0.05	1.52	0.72	1.29
		(2.74)	(21.26)	**(4.01)**	[0.000]				
8	8	-2.52	37.85	**-5.91**	75.74	0.04	2.51	0.76	2.89
		(3.51)	(17.52)	**(2.29)**	[0.000]				
12	12	-1.16	37.22	**-5.50**	139.85	0.06	3.05	0.75	5.04
		(2.67)	(13.35)	**(1.85)**	[0.000]				
16	16	12.84	28.78	**-5.21**	198.95	0.00	3.96	0.80	6.67
		(5.56)	(10.65)	**(2.13)**	[0.000]				

consumption level and in consumption volatility.

Table 1.8 shows that, both S-level and S-volatility risk prices are estimated significantly at longer horizons, in the cross-section of full-period holding stock returns. The price of the S-volatility risk is everywhere negative but the first horizon. Note from the diagonal of Table 1.6 that a positive rather than a negative correlation between volatility risk and return was reported for this horizon. Note also that the estimated magnitude of the price of volatility risk is almost the same for longer horizons. Consistent with the results of related studies the price of the S-level risk is almost everywhere positive and significantly estimated. Consumption level and consumption volatility risks explain 72% of variations in average full-period holding stock returns at the horizon of one year. This explanatory power increases for longer horizons and reaches 80% at the horizon of four years. Most of this variability may come from S-volatility risk since it is more correlated to long-period risk premia than S-level risk. The RSSE[12], which also measures the distance between the vector of actual returns and the vector of fitted returns, increases from short to long horizons. It shows that the fitted one-period risk premium departs in average from the actual by 0.32% to 0.42% per quarter.

Estimated positive and negative signs for consumption level and consumption volatility risk respectively confirms that these risks are correctly priced, in the sense that portfolios with higher positive covariances of returns with variations in consumption level, and high negative covariances of returns with variations in consumption volatility, will have high average excess returns. Small R^2s from the estimation based on $W = \widehat{\Sigma}_{RR}^{-1}$, mean that with respect to the square root of the second moment matrix, the combination of S-level risks and the combination of S-volatility risks across stocks are not economically important in explaining the combination of average stock returns. This highlights the fact that the cross-sectional R^2 and the corresponding distance between actual and fitted returns are not invariant to portfolio formation (Cochrane (2006), Roll and Ross (1994), Kandel and Stambaugh (1995)) and depend a lot on the estimation method.[13] However, this does not change the fact that S-level risk and S-volatility risks themselves are economically important in explaining the cross-section of average stock returns. The constant term is generally insignificant in all models with $k = S$. The J-statistics for the different estimation exercises vary widely. While it is tempting to interpret these differences, such an interpretation is not possible since the model at a given horizon does not nest that of the previous or the next horizon. We can only conclude that almost all test statistics indicate rejection of the null hypothesis at conventional levels of significance.

Figure 1.6 plots the ratio of fitted to realized expected one-quarter excess returns for the 25 Fama and French size and book-to-market sorted portfolios against the portfolios themselves. Fitted values and pricing errors are generated using the GMM with identity weighting matrix. Figure 1.7 shows similar plots using full-period returns. The statistics for investment horizons $S = 1, 8$ and 16 are displayed both for one-factor models involving level and volatility risks separately and for the two-factor model involving both risks.

[12] Root Sum Squared Errors

[13] R^2 is only well-defined for the estimation with the identity weighting matrix when estimates are equivalent to OLS estimates.

Figure 1.6: **Fitted/Realized Ratio of Portfolio Average Single-Period Excess Returns.** This figure presents the fitted/realized ratio of average one-period excess returns for the 25 Fama and French size and book-to-market sorted portfolios ($k = 1$). Fitted values are based on the model estimates with the identity weighting matrix.

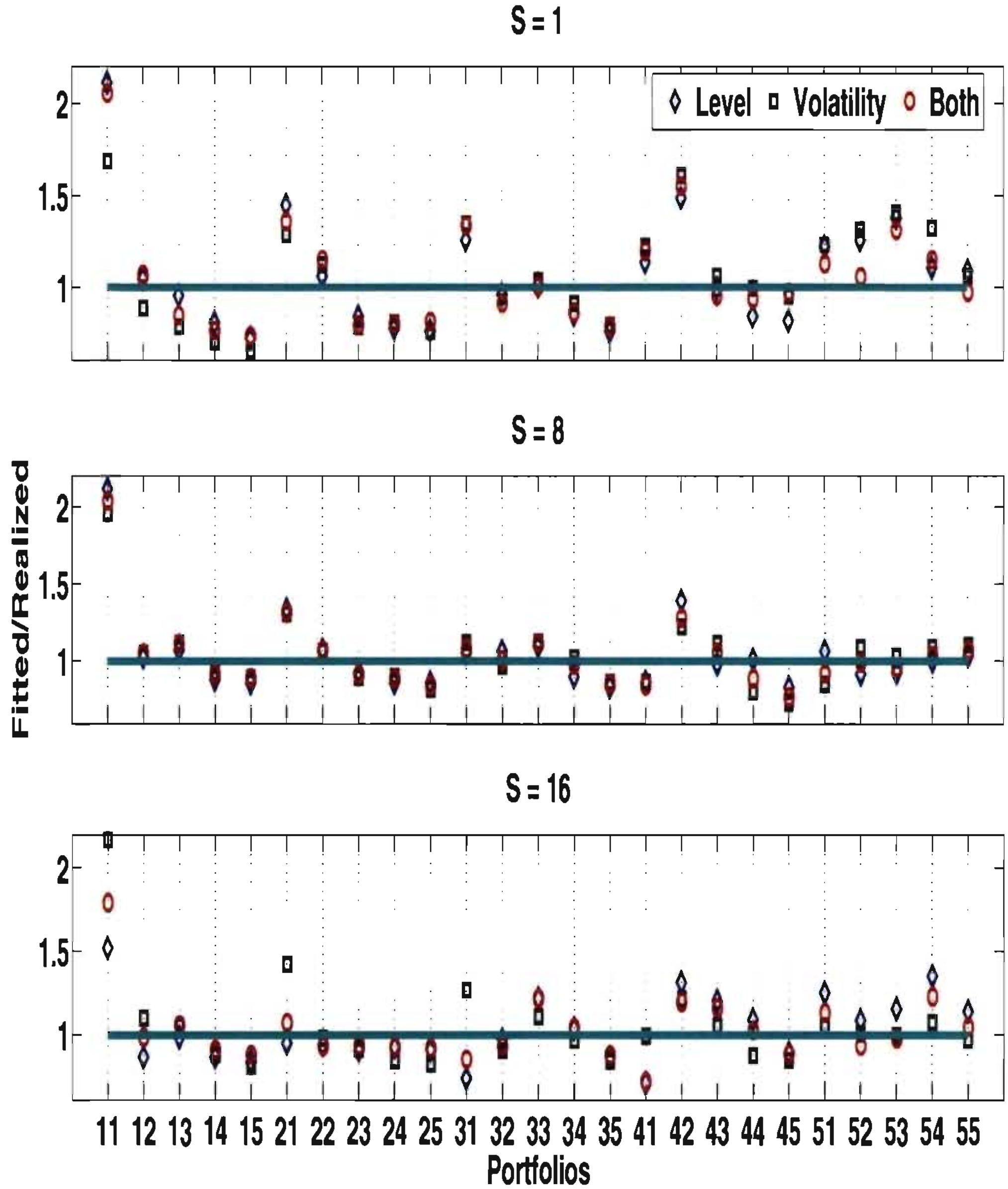


Figure 1.7: **Fitted/Realized Ratio of Portfolio Average Full-Period Excess Returns.** This figure presents the fitted/realized ratio of average multiperiod excess returns for the 25 Fama and French size and book-to-market sorted portfolios ($k = S$). Fitted values are based on the model estimates with the identity weighting matrix.

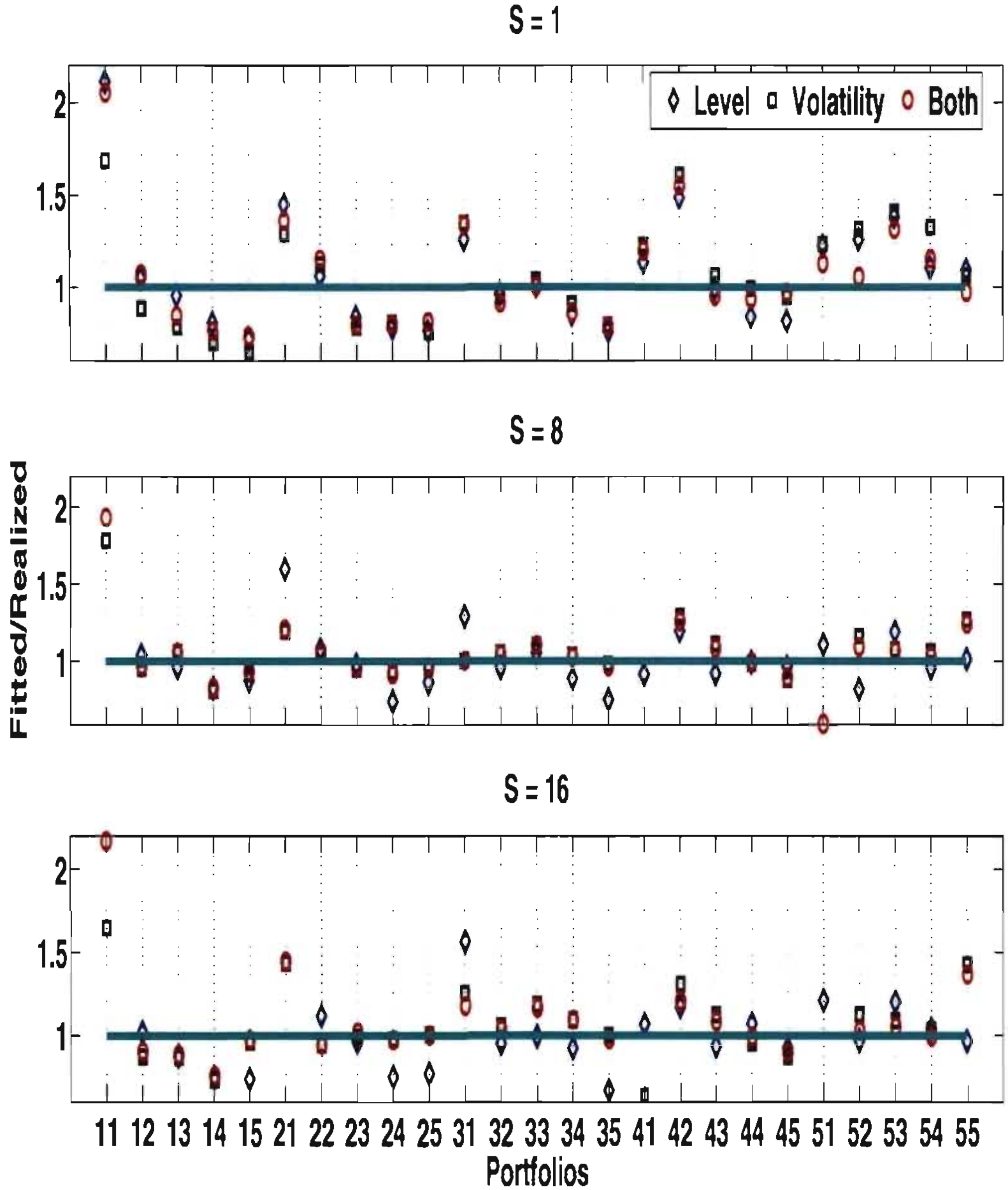
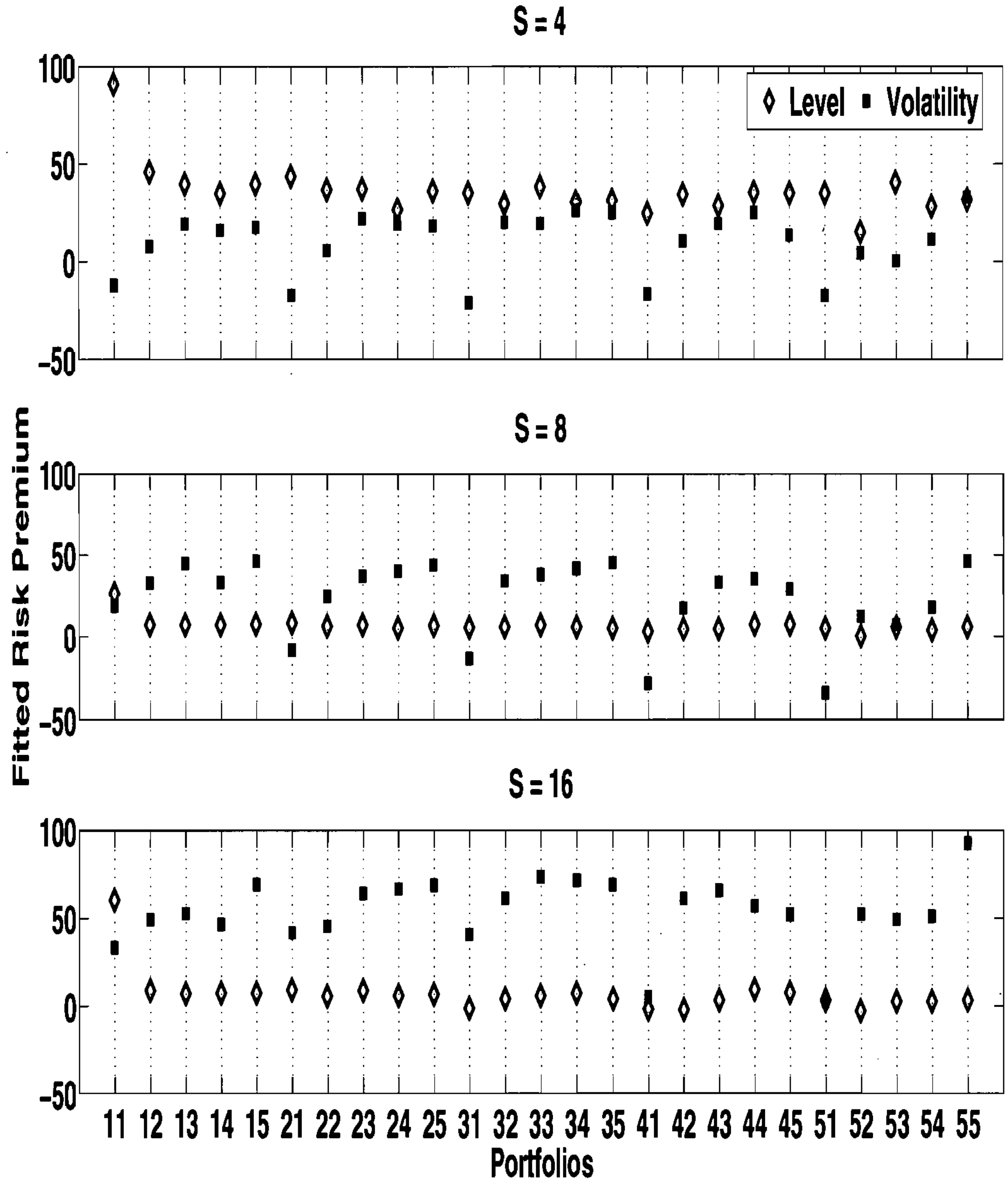


Figure 1.8: **Portfolio Level and Volatility Risk Premia.**

This figure presents the percentage of average excess return that represent level and volatility risk premia, meaning the ratios $\frac{p_{c,S}Cov\left(\xi_{\Delta c,t,S}, R^{e}_{it,k,S}\right)}{E\left[R^{e}_{it,k,S}\right]} \times 100$ and $\frac{p_{h,S}Cov\left(\xi_{\Delta h,t,S}, R^{e}_{it,k,S}\right)}{E\left[R^{e}_{it,k,S}\right]} \times 100$. Fitted values are based on the model estimates with the identity weighting matrix.



If the model fit is perfect, all the points in the corresponding panel of the figure would lie along the horizontal line $Y = 1$. The first panel of the figures shows clearly that few do, both for one-factor and two-factor models, corroborating the failure of the traditional CCAPM. All models have almost comparable fits in the three panels of Figure 1.6. In the first panel ($S = 1$), pricing errors are the highest for portfolios 11 and 42 as fitted values are more than 50% higher than realized values. Portfolio 11 remains poorly priced by all models in the second and the third panels of Figure 1.6 while the pricing error for portfolio 42 reduces for $S = 8, 16$. Except for growth portfolios (11, 21, 31 and 42), volatility risk prices very well one-period returns in each size group and performs better on returns on large firms for long investment horizons. The two last panels of Figure 1.7 confirms that overall, the volatility risk model has a better fit for long-period returns than the level risk model. Finally, Figure 1.8 plots level and volatility risk premiums for multiperiod investments in stocks and for different investment horizons. One can notice that, overall for portfolios with both positive level and volatility risk premiums, level risk premium is more important than volatility risk premium for at short horizons while, in the contrary, the volatility risk premium dominates the level risk premium at longer horizons.

1.5 Rationalizing the Empirical Evidence

This section examines whether evidences documented previously are consistent with the implications of existing parametric general equilibrium models. We adopt the recursive preferences and the consumption dynamics assumed in Section 1.2.1 and we specify the dynamics of dividends. We follow previous studies by choosing reasonable parameter values which calibrate the model such that it reproduces important features of asset markets. Then we further examine its implications for the cross-section of stock returns. We want the model to produce as possible portfolios whose return cross-section mimic that of the observed portfolios. However, since the model does not account for the size dimension, we further concentrate on large portfolios to illustrate the empirical findings and choose parameters to match usual statistics.[14]

1.5.1 Actual Dividend and Share

For each portfolio, quarterly price and dividend series are constructed in the same manner as in Bansal, Dittmar and Lundblad (2005). We observe the monthly nominal return series[15] computed with and without dividend, R^{with}_{t+1} and R^{wout}_{t+1}. Asset price and dividend series are then computed as:

$$P_{t+1} = R^{wout}_{t+1} P_t \tag{1.38}$$

$$D_{t+1} = \left(R^{with}_{t+1} - R^{wout}_{t+1}\right) P_t, \tag{1.39}$$

with $P_0 = 1$. Since the initial price is normalized to 1, these measures represent the actual price and dividend up to a multiplicative constant. Monthly prices are averaged within each quarter to obtain quarterly prices and monthly dividends are summed within each quarter to obtain quarterly dividends. There is no evidence of a seasonal component in quarterly prices. On the contrary, quarterly dividends have strong seasonalities that are removed by taking as measure of dividends in quarter t, an average of the dividends in quarter t and over the previous three quarters $t-3$, $t-2$ and $t-1$. Price and dividend series are then converted into real using the PCE deflator. Dividend-price ratios are then computed. Annualized empirical means and standard deviations for excess returns, dividend growths and dividend-price ratios of the 25 Fama and French size and book-to-market sorted portfolios as well as for the market return, the risk-free rate and the consumption growth are shown in Table 1.9. Log shares are also constructed as the log ratio of dividend to consumption and represent the actual shares up to an additive constant. These log shares are plotted in Figure 1.9.[16]

[14]The model can account for the book-to-market dimension since it does for the dividend-to-price dimension which is similar. As this type of model performs well in explaining the aggregate stock market behavior (Bansal and Yaron (2004), Eraker (2006)), it is also likely to perform well in the class of large portfolios.

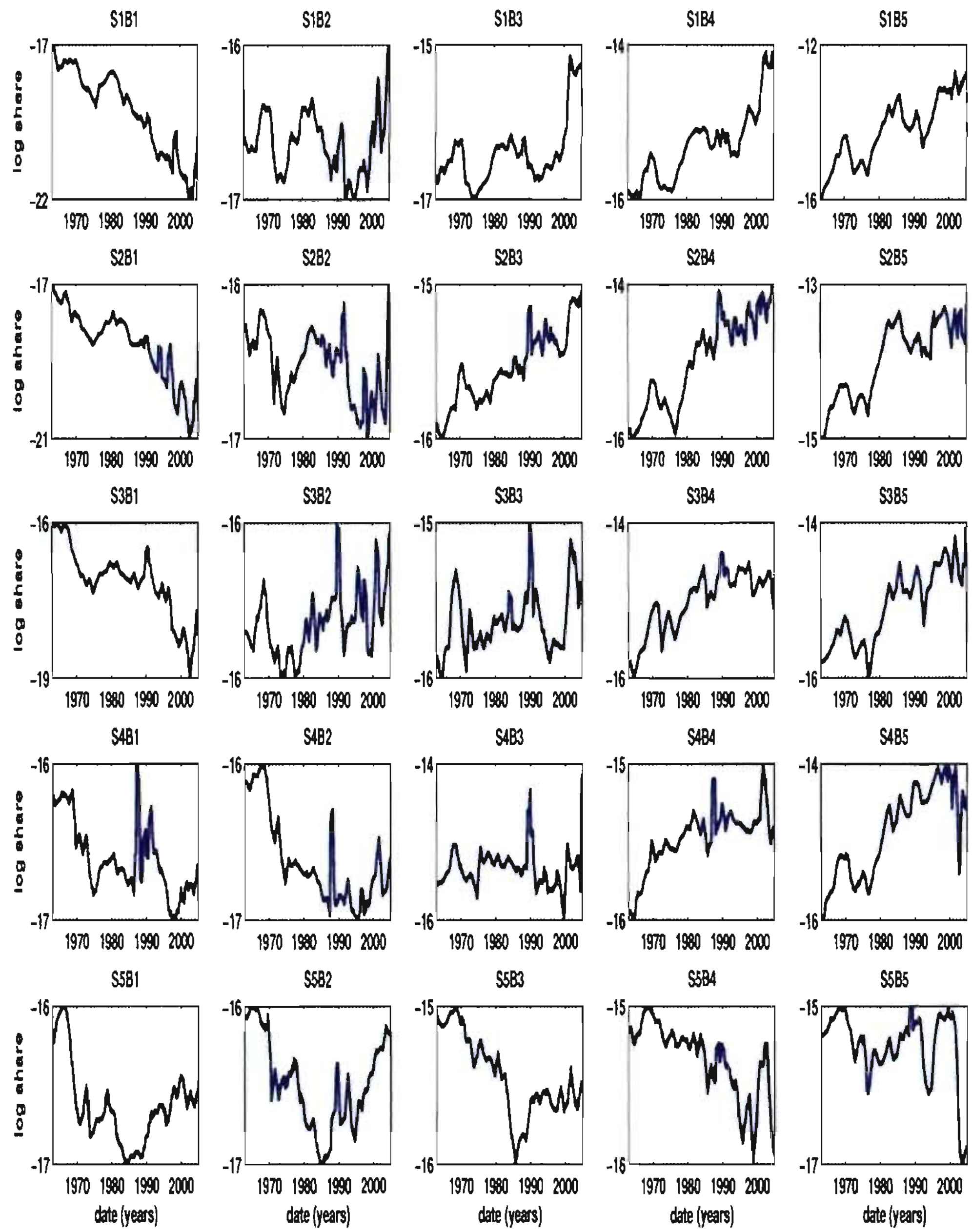
[15]We take return data from: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/

[16]Each label SxBy in the figure represents one portfolio. The first digit **x** refers to the size quintiles (1 indicating the smallest firms, 5 the largest), and the second digit **y** refers to book-to-market quintiles (1

Table 1.9: **Descriptive Statistics for Size and Book-to-Market Sorted Portfolios.** This table presents the annualized descriptive statistics of asset returns from 1963:3 to 2005:2. mean and standard deviation of excess returns and dividend-price ratios are in percentage.

	1963:3-2005:2					
Asset	$E[R^e]$	$\sigma[R^e]$	$E[\Delta d]$	$\sigma[\Delta d]$	$E[\frac{D}{P}]$	$\sigma[\frac{D}{P}]$
(1)	(2)	(3)	(4)	(5)	(6)	(7)
11	5.13	33.75	-5.54	29.26	0.60	0.46
12	11.04	28.10	4.60	25.07	1.38	0.83
13	11.65	24.42	5.65	14.70	2.00	0.96
14	14.18	22.97	7.37	16.16	2.35	1.04
15	15.89	25.42	9.74	21.88	1.95	0.93
21	6.00	26.46	-2.96	38.58	0.99	0.63
22	8.97	21.36	2.58	14.78	2.03	1.02
23	11.79	20.83	5.02	12.61	2.72	1.19
24	12.64	20.63	4.93	13.82	3.33	1.39
25	13.88	21.04	7.22	20.00	2.92	1.29
31	6.02	23.12	-0.99	19.53	1.27	0.75
32	9.57	19.43	3.21	14.35	2.32	1.11
33	9.40	17.40	3.12	14.84	3.22	1.34
34	11.21	19.66	4.99	12.73	3.76	1.50
35	13.48	20.07	6.79	19.78	3.64	1.42
41	7.15	20.38	1.06	26.19	1.62	0.77
42	6.76	17.17	0.85	18.14	2.73	1.13
43	9.40	16.51	4.17	16.58	3.53	1.44
44	11.14	18.31	3.46	13.99	4.20	1.57
45	11.57	18.62	5.16	12.19	3.96	1.38
51	5.61	18.00	2.51	11.64	2.14	0.81
52	6.06	15.47	1.93	11.90	3.34	1.18
53	6.32	14.44	1.29	7.70	4.06	1.52
54	7.22	15.16	0.90	10.50	4.65	1.73
55	7.73	18.12	0.45	17.02	4.89	1.87
MKT	6.10	16.14				
RF	1.79	1.14				
CONS			2.22	1.32		

Figure 1.9: **Log Shares for Size and Book-to-Market Sorted Portfolios.**
This figure presents the pattern of log shares for the 25 Fama and French size and book-to-market sorted portfolios.



1.5.2 Model Consumption Shares, Dividends, Price-Dividend Ratios and Returns

Here we describe how we generate portfolio returns in the economy. Lettau and Wachter (2006) provide a model where benchmark assets are zero-coupon equities paying the aggregate dividend. Here we extend benchmark assets to zero-coupon securities paying dividends on long-lived assets. Let $P^a_{n,t}$ the price at date t of the zero-coupon security paying n periods later from t, the dividend on an arbitrary long-lived asset a. The arbitrary long-lived asset a can be any long-lived primitive asset, any long-lived portfolio or the consumption claim. The Euler equation that requires the no-arbitrage condition for zero coupon securities is given by:

$$P^a_{n,t} = E\left[M_{t,t+1}P^a_{n-1,t+1} \mid \mathscr{I}_t\right], \tag{1.40}$$

with the trivial boundary condition $P^a_{0,t} = D^a_t$. Equation (1.40) can also be written:

$$\frac{P^a_{n,t}}{D^a_t} = E\left[M_{t,t+1}\frac{S^a_{t+1}}{S^a_t}\frac{C_{t+1}}{C_t}\frac{P^a_{n-1,t+1}}{D^a_{t+1}} \mid \mathscr{I}_t\right], \tag{1.41}$$

with the boundary condition:

$$\frac{P^a_{0,t}}{D^a_t} = 1,$$

and where S^a_t denotes the dividend share of total consumption of the asset a up to a multiplicative constant.

Equation (1.41) is the same for all long-lived assets, and its solution depends on the dynamics of the dividend shares. Log dividend shares are usually modeled as stationary processes (see for example Menzly, Santos and Veronesi (2004)). This assumption has two main critical implications in discrete time setting. First, all asset dividends are cointegrated with consumption, with the same normalized cointegration vector $(1,-1)$. Second and more importantly, the stationarity of dividend shares implies that all dividends grow at the same rate as consumption.

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indicating the portfolio with the lowest book-to-market ratio, 5 the highest).

We assume instead that dividends are cointegrated with consumption and an asset specific random walk variable driven by consumption volatility, and that the cointegration vector is also specific to the asset. Furthermore, we assume that the right hand side of cointegration equations are linear combinations of a deterministic trend and a common stationary and persistent variable that helps capturing the predictable part of dividend growth. Formally, we write:

$$d_t^a - (1+\lambda_c^a)\,c_t - v_t^a = \lambda_0^a t + \lambda_z^a z_t \tag{1.42}$$

$$z_{t+1} = \phi_z z_t + \sigma_z \sqrt{h_t}\,\varepsilon_{t+1} \tag{1.43}$$

$$v_{t+1}^a = v_t^a + \lambda_h^a (h_t - \mu_h) + \sigma_v^a \sqrt{h_t}\,u_{t+1}^a, \tag{1.44}$$

where $\left(u_{t+1}, \eta_{t+1}, \varepsilon_{t+1}, u_{t+1}^a\right)^\top \sim \mathcal{N}\left(0, \begin{pmatrix} 1 & 0 & \rho & 0 \\ 0 & 1 & 0 & 0 \\ \rho & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)$ with $\rho > 0$.

Even if the choice of such a process can be justified on various grounds, the first reason why we depart from the common specification of stationary dividend share is an empirical one. Consumption growth and portfolio dividend growth series are very different in terms of mean as well as variance and other moments. Cointegration tests often reject the hypothesis of a cointegration between dividends and consumption (Hansen, Heaton and Li (2005)). However, if the cointegration is strongly assumed, it seems therefore empirically sound to choose a model that does not impose the same cointegration vector between consumption and all dividends as the majority of models do. The pattern of the log shares of the 25 Fama and French size and book-to-market sorted portfolios plotted in Figure 1.9 show the evidence of a trend either in variables or the cointegration equation. Table 1.9 confirms that mean dividend growths are very different across these portfolios.[17]

[17] Equation (1.42) denotes the cointegration equation of asset a up to an additive constant and specifies that the dividend share of the asset is stationary if and only if $\lambda_0^a = 0$, $\lambda_c^a = 0$, $\lambda_h^a = 0$ and $\sigma_h^a = 0$.

The implied dynamics of the asset's dividend growth is given by:

$$\Delta d_t^a = \lambda_0^a + (1+\lambda_c^a)\mu_c - \lambda_z^a(1-\phi_z)z_t + [\lambda_h^a + (1+\lambda_c^a)\phi_c](h_t - \mu_h) + \sqrt{h_t}\left[(1+\lambda_c^a)u_{t+1} + \lambda_z^a\sigma_z\varepsilon_{t+1} + \sigma_v^a u_{t+1}^a\right]. \quad (1.45)$$

Our model can then generate predictable dividend growths with different means, thanks to the cointegration coefficients of dividend and consumption. Moreover, if $\lambda_0^a = 0$, $\lambda_c^a = 0$, $\lambda_h^a = 0$ and $\sigma_v^a = 0$, then the dividend share is identified by z_t up to an additive and a multiplicative constant. In this case, equation (1.45) implies that the dividend share captures the predictable component of the dividend growth. This last point is consistent with the view expressed in Lettau and Ludvigson (2005) that, if consumption follows a random walk like (1.6) and if the consumption-dividend ratio is stationary, then the consumption-dividend ratio captures the predictable component of the dividend growth.[18]

On the other hand,

$$\text{if } \rho = 1 \text{ then } z_t = \sigma_z \sum_{j=0}^{\infty} \phi_z^j\left(\Delta c_{t-j} - \mu_c\right). \quad (1.46)$$

Then, in this particular case of our setting, the process z_t almost plays a similar role as expected consumption growth in the Bansal and Yaron (2004)'s model in predicting the dividend growth using a variable that depends on past consumption levels. The coefficients λ_z and λ_h are negative so that dividends will increase following an increase in expected consumption growth and/or a fall in macroeconomic uncertainty.

Lettau and Wachter (2006) advocate the fact that if primitive assets are long-lived, then it is not easy to model their dividend shares stochastically in a discrete time setting, in a way similar to the continuous time setting of Menzly, Santos and Veronesi (2004), because of the difficulty to keep the shares between zero and one as well as their sum to one. However, equation (1.41) shows that we don't need to model the dividend share

[18]In general, a model that aims at explaining only the aggregate market behavior will not require additional ingredients as for the complete cross-section of asset returns.

itself in order to compute the price-dividend ratio of a long-lived asset or portfolio. It is just sufficient to model the share up to a multiplicative constant. This constant is of no particular interest unless we need to completely characterize asset prices and not only asset valuation ratios (such as price-dividend ratios in our case). The fact is that when the shares are known up to a multiplicative constant, dividends and prices are also known up to the same multiplicative constant and that does not affect the price-dividend ratio since the constant simplifies. For this reason, we drop the constant term in the cointegration equation (1.42) such that s_t^a measures the share up to this constant in our study.[19]

For solving for zero-coupon security valuation ratios in this model, one conjectures that:

$$\frac{P_{n,t}^a}{D_t^a} = \exp\left(A^a(n) + A_z^a(n) z_t + A_h^a(n) h_t\right). \tag{1.48}$$

The solution (1.48) for zero-coupon security valuation ratios then holds with:

$$\begin{aligned} A^a(n) &= p_1 + (\mu_c - \phi_c\mu_h)(1+\lambda_c^a - p_c) + \lambda_0^a - \lambda_h^a\mu_h + (1-\phi_h)\mu_h\left[A_h^a(n-1) - p_h\right] \\ &\quad + \frac{1}{2}\sigma_h^2\left[A_h^a(n-1) - p_h\right]^2 + A^a(n-1) \end{aligned} \tag{1.49}$$

$$A_z^a(n) = -\lambda_z^a(1-\phi_z) + \phi_z A_z^a(n-1) \tag{1.50}$$

$$\begin{aligned} A_h^a(n) &= \lambda_h^a + (1+\lambda_c^a - p_c)\phi_c + \frac{1}{2}(\sigma_v^a)^2 + \frac{1}{2}(1+\lambda_c^a - p_c)^2 + \frac{1}{2}\sigma_z^2\left[\lambda_z^a + A_z^a(n-1)\right]^2 \\ &\quad + \rho(1+\lambda_c^a - p_c)\sigma_z\left[\lambda_z^a + A_z^a(n-1)\right] + \left(\frac{1}{\rho_1} - \phi_h\right)p_h + \phi_h A_h^a(n-1), \end{aligned} \tag{1.51}$$

where $A^a(0) = 0$, $A_z^a(0) = 0$ and $A_h^a(0) = 0$. The $A(\cdot)$ functions of all long-lived assets have the same recursion and differ only through the asset's specific parameter values λ_0^a, λ_c^a, λ_z^a, λ_h^a and σ_v^a.

We further assume that $\phi_c = 0$. The parameters λ_c^a and λ_z^a are constrained by the

[19]Formally, if $\mathscr{A}$ is the set of all primitive long-lived assets, then there are positive constants β^a such that:

$$\sum_{a\in\mathscr{A}} \beta^a S_t^a < 1. \tag{1.47}$$

The complement to one of the sum in (1.47) can then account for the shares of short-lived primitive assets as well as the share of labor income.

equation:

$$(1+\lambda_c^a)+\rho\sigma_z\lambda_z^a=\chi^{a,c}$$

where $\chi^{a,c}$ is the ratio of the covariance between consumption growth and the dividend growth of the asset a to the mean of consumption volatility. This ensures that, increasing λ_z^a will rise the asset premium by reducing the price-dividend ratio, so that value stocks will be assets with high magnitude of λ_z^a.

The price P_t^a of the asset a at date t is the sum of prices of zero coupon securities paying future dividends on asset a. Then, the asset price-dividend ratio is given by the formula:

$$\frac{P_t^a}{D_t^a}=\sum_{n=1}^{\infty}\exp\left(A^a(n)+A_z^a(n)z_t+A_h^a(n)h_t\right), \tag{1.52}$$

where the $A(\cdot)$ functions are defined in (1.49), (1.50) and (1.51). The formula (1.52) is a nice way to compute the price-dividend ratio without a further analytical approximation of the asset return to asset a, similar to the approximation (1.5) of the return on the claim to the aggregate consumption.[20] The gross return on asset a is then given by:

$$R_{t+1}^a=\frac{P_{t+1}^a+D_{t+1}^a}{P_t^a}=\left(\frac{P_{t+1}^a}{D_{t+1}^a}+1\right)\left(\frac{P_t^a}{D_t^a}\right)^{-1}\left(\frac{D_{t+1}^a}{D_t^a}\right), \tag{1.53}$$

where the price-dividend ratio is given by (1.52) and the dividend growth by (1.45).

To understand how consumption volatility risks affect the more complex long-lived asset a, we follow Lettau and Wachter (2006) by concentrating on how these risks influence simple zero-coupon securities paying future dividends on the asset a. Let $R_{n,t+1}^a$ denote the one-period return on the zero-coupon security with the price P_{nt}^a at the date t, that is:

$$R_{n,t+1}^a=\frac{P_{n-1,t+1}^a}{P_{nt}^a}=\frac{P_{n-1,t+1}^a}{D_{t+1}^a}\left(\frac{P_{nt}^a}{D_t^a}\right)^{-1}\frac{D_{t+1}^a}{D_t^a}. \tag{1.54}$$

[20]The coefficients of the Campbell and Shiller (1988)'s approximation depend on preference parameters and empirical studies do not usually address this point. Garcia, Meddahi and Tedongap (2006) show how this approximation can affect some asset pricing statistics and their framework provide closed-form formulas of the Campbell and Shiller's coefficients.

Consumption level and consumption volatility risks of this zero-coupon security at one horizon are given by:

$$Cov\left(r^a_{n,t+1} - r_{f,t+1}, \Delta c_{t+1}\right) = \left[\chi^{a,c} + \rho\sigma_z A^a_z(n-1)\right]\mu_h \tag{1.55}$$

$$Cov\left(r^a_{n,t+1} - r_{f,t+1}, \Delta h_{t+1}\right) = \left[A^a_h(n-1) + A^a_h(n) - \lambda^a_h - q_h\right]\frac{\sigma^2_h}{1+\phi_h}. \tag{1.56}$$

These equations also defined the term structure of one-horizon consumption level and consumption volatility risks of zero-coupon securities. Increasing the magnitude of λ^a_h rises the volatility risk. Increasing the magnitude of λ^a_z will increase both consumption level and consumption volatility risks. Assets with high magnitude of λ^a_z will then have high risk premia. These assets with high risk premia will also be value stocks since increasing the magnitude of λ^a_z also lowers the price-dividend ratio. This is consistent with an empirical result from Bansal, Dittmar and Lundblad (2005) that the coefficient of the projection of the dividend growth into an empirical proxy of expected consumption growth explains differences in risk compensation across assets.

Since we find that consumption volatility is economically relevant as well for the cross-section, the innovation here is that to take macroeconomic uncertainty into account, the dividend growth can be projected into both an empirical proxy of expected consumption growth and that of consumption volatility. In addition to the coefficient λ^a_z, the resulting coefficient λ^a_h then gives the possibility to explain cross-sectional differences in asset returns with further information about consumption which is provided by consumption volatility.

1.5.3 Model Calibration and Implications for Stock Returns

We calibrate the model at the quarterly frequency. Our value of the mean of the consumption growth corresponds to its sample counterpart $\mu_c = 0.00555$.[21] To calibrate the consumption volatility h, we convert the monthly volatility of Bansal and Yaron

[21] Assuming $\phi_c = 0$ makes consumption growth unpredictable. Models with $\phi_c < 0$ and $\phi_c > 0$ may lead to different asset pricing implications and we leave this issue to a further and more elaborated investigation. The estimation of ϕ_c in Section 1.3.1 led to a negative and insignificant estimate.

Table 1.10: **Simulation: Parameter Values and Model Implied Statistics for Large Book-to-Market Sorted Portfolios.**
This table presents portfolio parameters as well as the annualized statistics of asset returns from simulated samples. Mean and standard deviation of excess returns, dividend growths and dividend-price ratios are in percentage.

	Parameters					Statistics					
Asset	λ_0^a	λ_c^a	λ_z^a	λ_h^a	σ_v^a	$E[R^e]$	$\sigma[R^e]$	$E[\Delta d]$	$\sigma[\Delta d]$	$E[\frac{D}{P}]$	$\sigma[\frac{D}{P}]$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
51	-0.006	1.16	-16	-330	4.16	4.93	22.72	2.53	12.39	1.81	0.54
52	-0.013	2.16	-24	-370	4.21	6.16	24.50	1.92	12.88	3.26	1.06
53	-0.015	2.22	-25	-460	2.38	6.69	26.70	1.35	9.73	3.82	1.49
54	-0.018	2.69	-28	-475	3.52	7.17	28.16	0.88	12.19	4.38	1.74
55	-0.023	3.34	-32	-485	6.06	8.07	31.84	0.51	18.19	4.60	1.84
RF						2.00	0.29				
CONS								2.22	2.67		

(2004) into a quarterly volatility, and use the corresponding parameter values.[22] The resulting parameters for the consumption volatility are $\phi_h = 0.962$, $\sigma_h = 1.18 \times 10^{-5}$ and $\mu_h = 1.83 \times 10^{-4}$. We use $\rho = 1$ so that the process z is a weighted combination of past consumption growth levels. Since in the Bansal and Yaron (2004)'s model demeaned expected consumption growth captures the predictable component of dividend growth, we convert it into a quarterly process and use the corresponding parameter values to calibrate the process z that then plays a similar role in our model as we argue earlier. The procedure is similar to what we follow for the consumption volatility. The resulting parameters are $\phi_z = 0.938$ and $\sigma_z = 0.129$.

Our values of preference parameters are $\gamma = 20$ for the risk aversion and $\psi = 1$ for the EIS. These values are also used by Hansen, Heaton and Li (2005). We use $\delta = 0.997$ and this quarterly value of the subjective discount factor corresponds to a monthly value of 0.999 also considered in previous studies. The parameters of the volatility process

[22]To do so, we first represent monthly consumption volatility with a two-state Markov chain as in Garcia, Meddahi and Tedongap (2006). Then, we convert the monthly chain into a quarterly one by multiplying conditional mean and variance by three and compounding three times the transition probability matrix. Finally, we determine the coefficients of the AR(1) process represented by the quarterly Markov chain.

are higher than those estimated in the data. Higher values of the mean and the standard deviation of consumption volatility are necessary to generate actual risk premia as stated in Eraker (2006). The preference parameters considered in this study were not able to generate an annual equity premium larger than 1% using volatility parameters estimated in the data. Table 1.10 displays the complete parameter values used for the calibration assessment and the model implied statistics. The reported statistics are based on 1,000 Monte Carlo experiments, each with 252 quarterly observations. Increasing the size of the Monte Carlo makes little difference in the results.

We start with the analysis on implications for zero-coupon securities. Zero-coupon securities guaranteeing dividends on different assets have the same behavior but with different intensity since this intensity depends on the specific parameters of any asset. We illustrate the implications in the case of zero-coupon securities paying future dividends on the large value portfolio.

Figure 1.10 displays the pattern of the $A(\cdot)$ functions characterizing the price-dividend ratio of a zero-coupon security. The function $A_z^a(n)$ is positive and increasing, and converges to $-\lambda_z^a$. The intuition behind this behavior is that higher levels of z_t correspond to higher expected dividend growth, hence the price of the security that pays the asset dividend in the future will also be higher. The function $A_h^a(n)$ is negative so that a rise in macroeconomic uncertainty induces a fall in asset prices, and decreasing as well as the function $A^a(n)$ so that zero-coupon security prices diminish when the maturity increases. The decreasing and the convergence to $-\infty$ of the function $A^a(n)$ also constitutes a necessary condition for the convergence of the price-dividend ratio (1.52). Since zero-coupon securities with higher maturities have low prices, they are similar to value stocks and should be more riskier.

The term structure of consumption level and consumption volatility risks plotted in Figure 1.11 confirms that risks are higher for longer maturities. Volatility risks are negative and decreasing so that long-maturity securities have higher negative covariances with variations in consumption volatility, as well as higher positive covariances with variations in consumption level than short-maturity securities. The model can then explain the differences in volatility risk premia across short-lived low-price securities and

Figure 1.10: **Plot of the $A(\cdot)$ Functions**
This figure presents the pattern of the $A(\cdot)$ functions for the case of the large value portfolio.

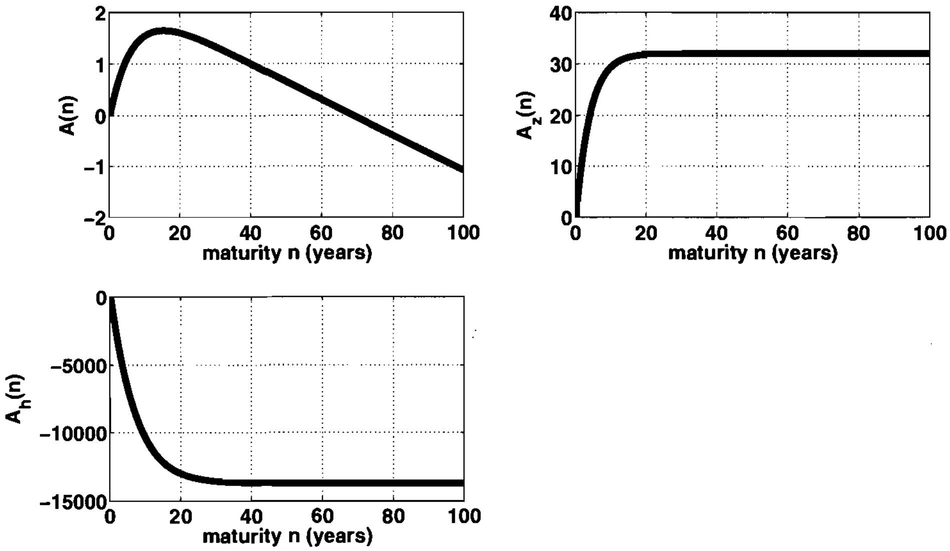


Figure 1.11: **Term Structure of Consumption Level and Consumption Volatility Risks.**
This figure presents the term structure of consumption level and consumption volatility risks for zero-coupon security paying future dividends on the large value portfolio.

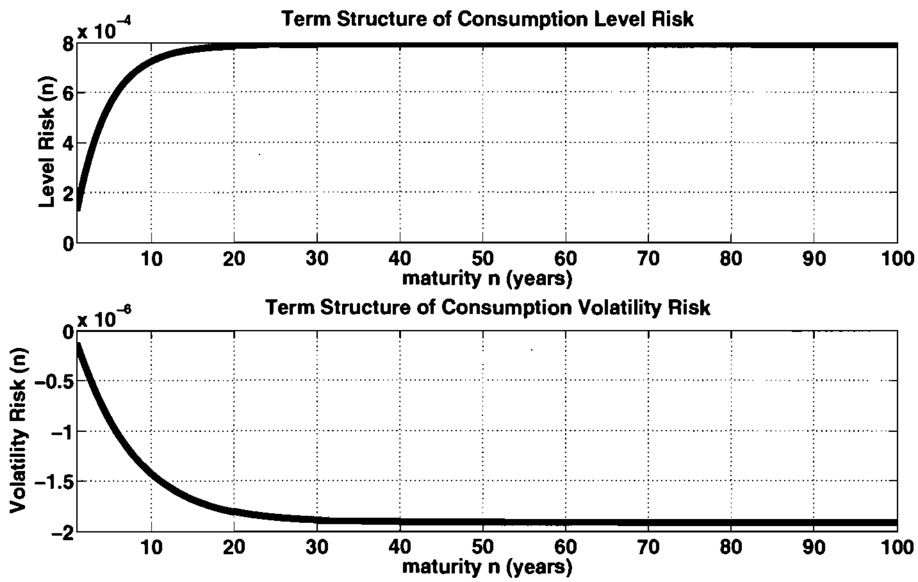


Figure 1.12: **Simulation: Volatility Risk for Large Book-to-Market Sorted Portfolios ($k = 1$).**

This figure presents the pattern of volatility risks across large book-to-market sorted portfolios.

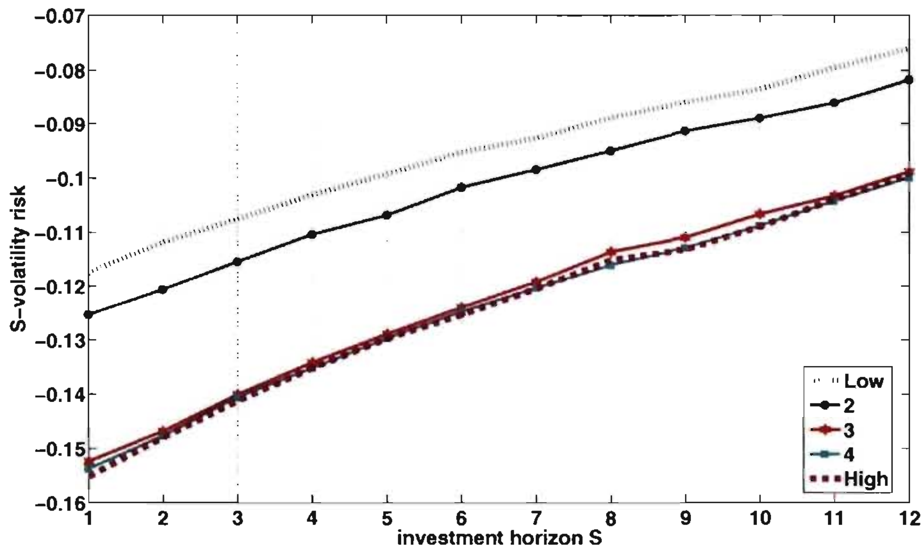
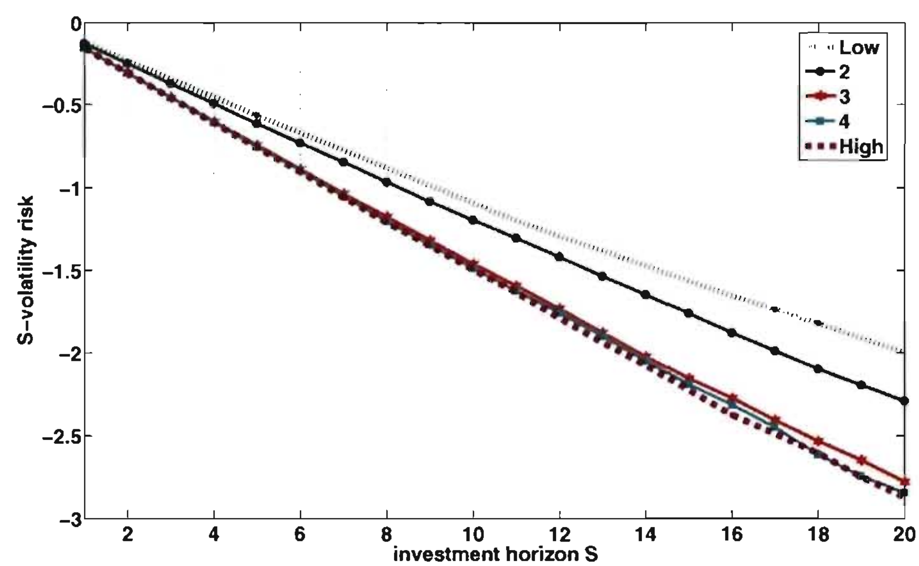


Figure 1.13: **Simulation: Volatility Risk for Large Book-to-Market Sorted Portfolios ($k = S$).**

This figure presents the pattern of volatility risks across large book-to-market sorted portfolios.



high-price securities in the maturity dimension.

We now examine the ability of the model to explain differences in volatility risk premia across long-lived low price-to-dividend stocks and high price-to-dividend stocks. We illustrate the implications in the set of large book-to-market sorted portfolios. While the overall fit of the statistics of these portfolios is reasonable as shown in Table 1.10, the model produces returns that are more volatile than in the data. This arise because the larger is the magnitude of the parameter λ_h^a, the larger is the return and its volatility. It is possible to simplify the model by setting the parameter λ_h^a to zero for all assets. This will lower returns and their volatility and either a more higher parameter of risk aversion or elasticity of intertemporal substitution, or a more higher magnitude of the parameter λ_z^a will be necessary for the model to generate actual returns. In consequence, it will produce low price-dividend ratios than in the data.[23]

Figure 1.12 shows the pattern of volatility risks computed via simulation across large book-to-market sorted portfolios and for one-period holding returns. Figure 1.13 shows the same pattern for full-period returns. S-volatility risks for one-period and full-period holding stocks respectively are negative with a downward trend as the horizon increases, a pattern observed in Figures I.2 and I.4 which plots the similar measure of volatility risk in the data. On the other hand, S-volatility risk for full-period portfolios computed from the model is negative and displays a similar pattern as the same measure computed from the data.

An important point illustrated in Figure 1.13 is the gap between volatility risks for the extreme value and the extreme growth portfolios. The large value is more riskier as in the data. The slightly difference between the data and the model occurs for the semi-growth and the semi-value portfolios. Their volatility risks are more closer to that of the extreme value portfolio than in the data. However, as in the data, there is just a little gap between these risks. The more pronounced trend of all these patterns in the model are explained with the fact that consumption volatility is more persistent in the model than

[23]The results of Bansal and Yaron (2004) also suggest that increase the magnitude of the risk aversion lowers the price-dividend ratio and rises the equity premium. With a risk aversion parameter of 7.5 in their model, they report a price-dividend ratio of 25.02 and an equity premium of 4.01 for the aggregate stock. With a risk aversion parameter of 10, the reported values are respectively 19.98 and 6.84.

in the data. However, the overall message is clear and states that the model replicates the findings in the data that consumption volatility risk accounts for the differences in risk premia across portfolios sorted from growth to value.

1.6 Conclusion

Investors have concerns about consumption volatility because they fear the repercussion of macroeconomic uncertainty on their future wealth. Motivated by an affine general equilibrium model with stochastic volatility, we have documented empirical facts supporting a strong relation between stock returns and changes in consumption volatility. We found that short-period returns are correlated with short-horizon changes in consumption volatility and with long-horizon changes in consumption level, and more so than long-period returns. On the other hand, long-period returns are more correlated with long-run changes in consumption volatility than with long-run changes in consumption level.

The uncertainty on macroeconomic growth as measured by consumption volatility displays a business cycle pattern and has the potential to explain differences in risk premia across the 25 Fama and French size and book-to-market sorted portfolios, even in the presence of long-run consumption risk. The estimation of long-run consumption volatility risk price in the cross-section of long-period returns provides a significant estimate with a negative sign.

A further issue will be to check whether a well-calibrated reduced form consumption-based general equilibrium model, similar to those considered in previous studies for explaining the aggregate stock market behavior, can also rationalize our empirical findings. An attempt to this rationalization leads to promising results.

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CHAPTER 2

AN ANALYTICAL FRAMEWORK FOR ASSESSING ASSET PRICING MODELS AND PREDICTABILITY

Abstract

Consumption-based equilibrium asset pricing models have regained some momentum with new insights about the connections between stock market volatility and returns, the pricing of long-run claims, or return predictability. Links are established between risk premiums and different types of preferences, where separation between the elasticity of intertemporal substitution and risk aversion, and habit formation take center stage. Often, the solution of these models necessitates an approximation and quantities of interest are computed through simulations. We propose a model that delivers closed-form formulas for many of the statistics usually computed to assess the ability of the models to reproduce stylized facts. The proposed model is flexible enough to capture the various dynamics for consumption and dividends as well as the different types of preferences that have been assumed in consumption-based asset pricing models. The availability of closed-form formulas enhances our understanding of the economic mechanisms behind empirical results and of the limits of validity for the usual approximations.

2.1 Introduction

In the last twenty years or so, financial economists have devoted a lot of energy to solving two unyielding puzzles, the equity premium puzzle and the risk-free rate puzzle. The specification of preferences in the basic consumption CAPM model introduced by Lucas (1978) and Breeden (1979) has been modified to accommodate a large equity premium and a rather low risk-free rate. The two most popular models are without a doubt the recursive utility model of Epstein and Zin (1989, 1991) and the external habit model of Campbell and Cochrane (1999). Recently, these models have been used to reproduce new facts about the connections between stock market volatility and returns, the pricing of long-run claims, or return predictability (see in particular Bansal and Yaron, 2004, Bansal, Gallant and Tauchen, 2004, Hansen, Heaton and Li, 2004, Lettau, Ludvigson and Wachter, 2004). The effort then has been centered on the specification of the endowment process. New joint dynamic models have been proposed for consumption and dividend growth, while at the beginning the equality of consumption and dividend was often assumed. Often, the solution of these new full-fledged models necessitates an approximation and quantities of interest are computed through simulations.

In this chapter we propose a model that delivers closed-form formulas for many of the statistics usually computed to assess the ability of the models to reproduce stylized facts. The proposed model is flexible enough to capture the various dynamics for consumption and dividends as well as various types of preferences that have been postulated in consumption-based asset pricing models.

To derive analytical formulas, we assume that the logarithms of real per capita consumption and dividend growth follow a bivariate process where both the means, variances and covariances change according to a Markov variable s_t which takes the values $1,\ldots,N$ (if N states of nature are assumed for the economy), where s_t is a stationary and homogenous Markov chain. Several asset pricing models have been built with constrained versions of this general process, but the main reason of this choice is that it leads to closed-forms formulas for many of the statistics that researchers have attempted to reproduce: the first and second moments of the equity premium and of the risk-free rate, the mean of and the volatility of the price-dividend ratio, the predictability of returns and excess returns by the dividend-price ratio, the predictability of consumption volatility by the dividend-price ratio and the consumption-wealth ratio, and the negative autocorrelation of returns and excess returns at long horizons. We also use this model to match some moments of the consumption and dividend processes implied by other dynamic models. This is the approach taken by Mehra and Prescott (1985) in their seminal paper that puts forward the equity premium puzzle.

In the formulas we will develop for the various statistics we will assume that we have solved the model for the price of the asset of interest or a ratio of the payoff of the asset to its price. As we will see, the structure of the endowment process implies that there will be one such payoff-price ratio per regime and this will help in computing closed-form analytical formulas.

Of course the price of any asset is dependent upon the stochastic discount factor which will be model-dependent. We solve for the prices in a Markov-switching economy with recursive preferences (Epstein and Zin, 1989) and with external habit (Campbell and Cochrane, 1999). These models deliver two fundamental payoff-price ratios: the consumption-market portfolio price ratio and the dividend-equity price ratio. The first ratio is unobservable but Lettau and Ludvigson (2001 a,b) have proposed a close parent with the consumption-wealth ratio. Once we differentiate consumption and dividends, these models deliver a measure of this important economic quantity. Moreover, in the recursive utility framework, the consumption-price ratio enters the stochastic discount factor. Once a solution to the nonlinear Euler set of equations defining this ratio in the various states is found, all other asset prices can be obtained analytically.

The importance of deriving closed-form formulas should not be underestimated. Lettau, Ludvigson and Wachter (2004), who use precisely a Markov-switching model for their endowment, remark that their two-state model takes very long to solve and that a three-state model would be computationally infeasible. They use a learning model that they must solve at each time period given their new assessment of the transition probabilities of the Markov process. Our formulas can be adapted to this approach and will ease considerably the process. Another considerable saving of processing time comes potentially from the simulations researchers run to compute predictability regressions. The usual procedure is to try to replicate the actual statistics with the same number of observations as in the sample as well with a much larger number of observations to see if the model can produce predictability in population. The last exercise, the most costly in computing time, is avoided by using the formulas we provide. The same is true for the variance ratios.

Another useful contribution is to use these formulas to assess the impact of approximations that researchers apply to solve models. One pervasive approximation in asset pricing is the log-linearization of Campbell and Shiller (1988). We provide formulas for several approximations of the payoff-price quantities in the Epstein and Zin (1989) model.

We apply our analytical framework to two prominent recent papers by Lettau, Ludvigson and Wachter (2004) and Bansal and Yaron (2004). Both promote the role of macroeconomic uncertainty measured by the volatility of consumption as a determining factor in the pricing of assets. The first paper models consumption growth as a Markov switching process and uses Epstein and Zin (1989) preferences, and so fits directly our framework. The second paper uses the same preferences but models the consumption-dividend endowment as an autoregressive process with time-varying volatility. For this model, we propose a moment-matching procedure with our Markov-switching process. By putting the two models in the same framework, we are able to point out their similarities and differences for asset pricing implications and predictability. Our analytical formulas allow us to explore a much wider set of preference parameters than in the original papers and thus to better understand their role in determining the financial quantities of interest. We also match the consumption surplus dynamics specified by Campbell and Cochrane (1999) with a Markov switching model and provide analytical results for many of the quantities generated by simulation in the original paper.

This chapter extends considerably the closed-form price-dividend formulas provided in Bonomo and Garcia (1994) for the Lucas (1978) CCAPM model. Recently, two papers have also proposed to develop analytical formulas for asset pricing models. Abel (2005) calculates exact expressions for risk premia, term premia, and the premium on levered equity in a framework that includes habit formation and consumption externalities (keeping up or catching up with the Joneses). The formulas are derived under lognormality and an i.i.d. assumption for the growth rates of consumption and dividends. We also assume lognormality but after conditioning on a number of states and our state variable capture the dynamics of the growth rates. Eraker (2006) produces analytic pricing formulas for stocks and bonds in an equilibrium CCAPM with Epstein-Zin preferences, under the assumption that consumption and dividend growth rates follow affine processes. However, he uses the Campbell and Shiller (1988) approximation to maintain a tractable analytical form of the pricing kernel.

The rest of the chapter is organized as follows. Section 2 describes the Markov-switching model for consumption and dividend growth. Section 3 enumerates several empirical facts and provides analytical formulas for the statistics reproducing these stylized facts. In Section 4, we solve for the price-dividend ratio in asset pricing models. Section 5 provides applications to several asset pricing models for the US post-war economy. Section 6 concludes. Appendix II collects the proofs of main propositions.

2.2 A Markov-Switching Model for Consumption and Dividends

We follow the approach pioneered by Mehra and Prescott (1985) by specifying a stochastic process for the endowment process and solving the model for the prices of the market portfolio, an equity and the risk-free asset in the economy. The goal in this branch of the empirical asset pricing literature is to determine if equilibrium models with reasonable preferences are able to reproduce some stylized facts associated with returns, consumption and dividends.

Contrary to the original model in Lucas (1978)), we make a distinction between consumption and dividends. Consumption is the payoff on the market portfolio while dividends accrue to equity owners. This distinction is nowadays almost always made (see Bansal and Yaron, 2004, Hansen, Heaton and Li (2004) and Lettau, Ludvigson and Wachter (2005) among others), but was introduced originally by Tauchen (1986) and pursued further by Cecchetti, Lam and Mark (1993) and Bonomo and Garcia (1994, 1996). [1]

The main reason for disentangling the consumption and dividend processes is first and foremost an empirical one: the series are very different in terms of mean, variance, and other moments.

[1] Abel (1992) formulates a model with production, but where the labor supply is inelastic and the stock of capital is fixed and does not depreciate, and randomness comes from technology shocks. Then, consumption is equal to the total income of the economy, which is the sum of dividends - the capital income - with labor income. The disentanglement of consumption and dividends appears naturally in an asset pricing model of a production economy. However, usually total income is different from consumption, since there is investment, and although the Euler condition for asset returns still involves discounting the return by the intertemporal marginal rate of substitution in consumption, the latter depends also on leisure (see Brock, 1982, and Danthine and Donaldson, 1995). In Abel's (1992) simple version, labour supply is fixed and there is no investment. Thus, his version of a production economy fits perfectly our empirical framework.

We postulate that the logarithms of consumption and dividends growth follow a bivariate process where both the means, variances and covariances change according to a Markov variable s_t which takes the values $1,\ldots,N$ (if N states of nature are assumed for the economy). The sequence $\{s_t\}$ of Markov variables evolves according to the following transition probability matrix P.

We assume that

$$\zeta_t = \begin{cases} (1,0,0,...,0)^\top & \text{when } s_t = 1 \\ (0,1,0,...,0)^\top & \text{when } s_t = 2 \\ . & . \\ . & . \\ . & . \\ (0,0,0,...,1)^\top & \text{when } s_t = N \end{cases}$$

where s_t is a stationary and homogenous Markov chain. We also assume

$$x_{c,t+1} \equiv \log(C_{t+1}) - \log(C_t) = c_{t+1} - c_t = \mu_c^\top \zeta_t \; + \; (\omega_c^\top \zeta_t)^{1/2} \, \varepsilon_{c,t+1} \tag{2.1}$$

$$x_{d,t+1} \equiv \log(D_{t+1}) - \log(D_t) = d_{t+1} - d_t = \mu_d^\top \zeta_t \; + \; (\omega_d^\top \zeta_t)^{1/2} \, \varepsilon_{d,t+1}, \tag{2.2}$$

where

$$\begin{pmatrix} \varepsilon_{c,t+1} \\ \varepsilon_{d,t+1} \end{pmatrix} \mid \sigma(\varepsilon_{c,\tau}, \varepsilon_{d,\tau}, \tau \leq t; \zeta_m, m \in \mathbf{Z}) \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} 1 & \rho^\top \zeta_t \\ \rho^\top \zeta_t & 1 \end{bmatrix} \right) \tag{2.3}$$

We define the matrix P by

$$P^\top = [p_{ij}]_{1 \leq i,j \leq N}, \;\; p_{ij} = P(s_{t+1} = j \mid s_t = i). \tag{2.4}$$

We assume that the Markov chain is stationary with an ergodic distribution Π, $\Pi \in$

$\mathbb{R}^N$, i.e.,

$$\Pi = E[\zeta_t]. \tag{2.5}$$

We have

$$E[\zeta_t \zeta_t^\top] = Diag(\Pi_1, ..., \Pi_N) \text{ and } Var[\zeta_t] = Diag(\Pi_1, ..., \Pi_N) - \Pi\Pi^\top. \tag{2.6}$$

Bonomo and Garcia (1994, 1996) use the specification (2.1,2.2) with constant correlations for the joint consumption-dividends process to investigate if an equilibrium asset pricing model with different types of preferences can reproduce various features of the real and excess return series. [2] The heteroscedasticity of the endowment process measures economic uncertainty as put forward by Bansal and Yaron (2004).

In the following, we adopt the notation:

$$\forall u \in \mathbb{R}^N, \ A(u) = Diag(\exp(u_1), ..., \exp(u_N))P. \tag{2.7}$$

$$I_t = \sigma(D_\tau, \tau \leq t), \ J_t = \sigma(D_\tau, s_\tau, \tau \leq t) = \sigma(D_\tau, \zeta_\tau, \tau \leq t). \tag{2.8}$$

We also note

$$P^h = [P_{i,j}(h)]_{1 \leq i,j \leq N}.$$

The vector e denotes the $N \times 1$ vector whose all components equal one, while e_i denotes the vector whose i-th component equals zeros and the others equal zero , i.e.,

$$e = (1,,,,1)^\top, \ e_1 = (1,0,...,0)^\top, \ e_2 = (0,1,0,...,0)^\top, ..., \text{ and } e_N = (0,...,0,1)^\top. \tag{2.9}$$

Finally, $\odot$ denotes the element by element multiplication operator, i.e.,

$$X \odot Y = (x_1 y_1, ..., x_N y_N)^\top, \text{ where } X = (x_1, ..., x_N)^\top \text{ and } Y = (y_1, ..., y_N)^\top,$$

[2]Cecchetti, Lam, and Mark (1991) use a two-state homoskedastic specification of (11) for the endowment and similar preferences to try to match the first and second moments of the return series. The authors use two models, one with a leverage economy, another with a pure exchange economy without bonds. In both instances, they are unable to replicate the first and second moments taken together.

and for any real number q, $X^q = (x_1^q, ..., x_N^q)^\top$.

2.3 Analytical Formulas for Statistics Reproducing Stylized Facts

In this section we start by recalling a series of stylized facts that researchers have tried to reproduce with consumption-based equilibrium models. In the formulas we will develop for the various statistics we will assume that we have solved the model for the price of the asset of interest or a ratio of the payoff of the asset to its price. As we will see, the structure of the endowment process implies that there will be one such payoff-price ratio per regime and this will help in computing closed-form analytical formulas. Of course these prices are model-dependent and in the next section we will solve for the prices in a Markov-switching economy with recursive preferences (Epstein and Zin, 1989).

2.3.1 The Stylized Facts

In his survey on consumption-based asset pricing Campbell (2002) enumerates a number of stylized facts about the stock market and its relation to short-term interest rates and consumption growth. We report these stylized facts and others computed with a post-war data set of quarterly consumption, dividends and returns data for the US economy (1947:1 to 2002:4). The empirical predictability results for the quarterly US data from 1947 to 2002 are reported in table 2.1.

1. The average return on stock is high (7.43% per year).

2. The average riskless real interest rate is low (1.20% per year).

3. Real stock returns are volatile (standard deviation of 16.93% per year).

4. The real interest rate is much less volatile (standard deviation of 2.28% per year) and much of the volatility is due to short-run inflation risk. Note however that there might be regimes as shown in Garcia and Perron (1996).

5. Real consumption growth is very smooth (standard deviation of 1.33% per year).

6. Real dividend growth is extremely volatile at short horizons because of seasonality in dividend payments (annualized quarterly standard deviation of 22.50%). At longer horizons it is intermediate between the volatility of stock return and the volatility of consumption growth.

7. Quarterly real consumption growth and real dividend growth have a very weak correlation of 0.15 but the correlation increases at lower frequencies.

8. Real consumption growth and real stock returns have a quarterly correlation of 0.16. The correlation increases at 0.31 at a 1-year horizon and declines at longer horizons.

9. Quarterly real dividend growth and real stock returns have a very weak correlation of 0.11, but correlation increases dramatically at lower frequencies.

10. Real US consumption growth not well forecast by its own history or by the stock market. The first-order autocorrelation of the quarterly growth rate of real non-durables and services consumption is 0.22. The log price-dividend ratio forecasts less than 4.5% of the variation of real consumption growth at horizons of 1 to 4 years.

11. Real US dividend growth has some short-run forecastability arising from the seasonality of dividend payments (autocorrelation of -0.44). But it is not well forecast by the stock market. The log price-dividend ratio forecasts no more than 1.5% of the variation of real dividend growth at horizons of 1 to 4 years.

12. The real interest rate has some positive serial correlation; its first-order autocorrelation is 0.63. However the real interest rate is not well forecast by the stock market.

13. Excess returns of US stock over Treasury bills are highly forecastable. The log price-dividend ratio forecasts 10% of the variance of the excess return at a 1-year horizon, 19% at a 3-year horizon and 26% at a 5-year horizon. Real returns exhibit

a lower predictability, also increasing with the horizon (9% at a 1-year horizon, 15% at a 3-year horizon and 22% at a 5-year horizon).

To reproduce these stylized facts one needs three main types of formulas: formulas for expected returns, formulas for variance ratios of returns, formulas for predictability of returns.

2.3.2 Formulas for Expected Returns

2.3.2.1 Expected Returns on a Dividend-Producing Asset

We define the return process R_{t+1} as

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t}, \tag{2.10}$$

while the aggregated return over h periods is given by

$$R_{t+1:t+h} = \sum_{j=1}^{h} R_{t+j}. \tag{2.11}$$

We define the return process $R_{M,t+1}$ on the (unobservable) market portfolio as

$$R_{M,t+1} = \frac{P_{M,t+1} + C_{t+1}}{P_{M,t}}. \tag{2.12}$$

One important property that we will use in deriving our analytical formulas is the Markov property of the model. We will show that the variables P_t/D_t, $P_{M,t}/C_t$ and $P_{F,t}/1$ (where the $P_{F,t}$ is the price of a bond), are (non-linear) functions of the state variable ζ_t. On the other hand, the state ζ_t takes a finite number of values. Consequently, any real non-linear function $g(\cdot)$ of ζ_t is a linear function of ζ_t. The reason is the following: the function $g(\zeta_t)$ takes the values g_1 in state 1, g_2 in state 2,..., g_N in state N; hence,

$$g(\zeta_t) = \bar{g}^\top \zeta_t \text{ where } \bar{g} = (g_1, g_2, ..., g_N)^\top.$$

This property will allow us to characterize analytically the variables P_t/D_t, $P_{M,t}/C_t$ and

$P_{F,t}/1$ while other data generating processes need either linear approximations or numerical models to solve the model.

In the rest of the chapter, we will adopt the following notation:

$$\frac{P_t}{D_t} = \lambda_1^\top \zeta_t, \tag{2.13}$$

$$\frac{P_{M,t}}{C_t} = \lambda_{1c}^\top \zeta_t, \tag{2.14}$$

$$R_{F,t+1} = \frac{1}{P_{f,t}} = b^\top \zeta_t. \tag{2.15}$$

Observe also that one can write

$$\frac{D_t}{P_t} = \lambda_2^\top \zeta_t \text{ with } \lambda_2 = (\lambda_{11}^{-1}, ..., \lambda_{1N}^{-1})^\top, \text{ where } \lambda_1 = (\lambda_{11}, ..., \lambda_{1N})^\top. \tag{2.16}$$

Likewise,

$$\frac{C_t}{P_{M,t}} = \lambda_{2c}^\top \zeta_t \text{ with } \lambda_2 = (\lambda_{1c1}^{-1}, ..., \lambda_{1cN}^{-1})^\top, \text{ where } \lambda_{1c} = (\lambda_{1c1}, ..., \lambda_{1cN})^\top. \tag{2.17}$$

In Section 4, we will use the asset pricing models to characterize the vectors λ_1, λ_2, λ_{1c}, and b as functions of the parameters of the consumption and dividend growth dynamics and the utility function of the representative agent. In the rest of this section, we will characterize the predictability of the returns and excess returns as well as some other population moments by assuming that λ_1, λ_2, λ_{c1}, and b are known. These formulas depend only on the previous vectors and the dynamics of the dividend growth and the Markov chain.

In order to study the predictability of the returns and excess returns, we need to connect them to the state variable ζ_t and to the dividend growth. We show in the appendix that

$$R_{t+1} = (\lambda_2^\top \zeta_t) \exp(x_{d,t+1})(\lambda_3^\top \zeta_{t+1}) \text{ with } \lambda_3 = \lambda_1 + e, \tag{2.18}$$

where the vectors λ_1 and e are defined in (2.13) and (2.9) respectively. Finally, we denote the excess return by R^e_{t+1}, i.e.,

$$R^e_{t+1} = R_{t+1} - R_{F,t+1}. \tag{2.19}$$

Proposition 2.3.1. ***Characterization of the Expected Values of Returns and Excess Returns.***
We have

$$E[R_{t+1} \mid J_t] = (\lambda_2^\top \zeta_t)\ \exp(\mu_d^\top \zeta_t + \omega_d^\top \zeta_t/2)\ \lambda_3^\top P \zeta_t = \psi^\top \zeta_t, \tag{2.20}$$

where $\psi = (\psi_1, ..., \psi_N)^\top$ *and*

$$\psi_i = \lambda_{2i}\ \exp(\mu_{d,i} + \omega_{d,i}/2)\ \lambda_3^\top P e_i,\ \ i = 1, ..., N. \tag{2.21}$$

Likewise,

$$E[R^e_{t+1} \mid J_t] = (\psi - b)^\top \zeta_t. \tag{2.22}$$

Consequently, $\forall j \geq 2$

$$E[R_{t+j} \mid J_t] = \psi^\top P^{j-1} \zeta_t \text{ and } E[R^e_{t+j} \mid J_t] = (\psi - b)^\top P^{j-1} \zeta_t. \tag{2.23}$$

Finally,

$$E[R_{t+1:t+h} \mid J_t] = \psi_h^\top \zeta_t \ \text{ and } \ E[R^e_{t+1:t+h} \mid J_t] = (\psi_h - b_h)^\top \zeta_t \tag{2.24}$$

where

$$\psi_h = \left(\sum_{j=1}^{h} P^{j-1} \right)^\top \psi \ \text{ and } \ b_h = \left(\sum_{j=1}^{h} P^{j-1} \right)^\top b. \tag{2.25}$$

2.3.2.2 Expected Risk-Free Rate

In the sequel, we will also compute in the application section the frequency with which models produce negative interest rates. The probability that the risk-free rate is negative is given by

$$P(R_{F,t+1} < 1) = E\left[\mathbf{1}_{\{R_{F,t+1}<1\}}\right] = E\left[g^{\top}\zeta_t\right] = g^{\top}\Pi, \tag{2.26}$$

where $g = \left(\mathbf{1}_{\{b_1<1\}}, ..., \mathbf{1}_{\{b_N<1\}}\right)^{\top}$.

2.3.3 Variance Ratios for Returns

In this section we provide variance formulas for the dividend price ratio as well as some covariance formulas between this ratio and the returns at various horizons. We conclude by a formula for the variance ratio that measures the autocorrelation in returns. Cecchetti, Lam and Mark (1990) were the first to reproduce the autocorrelation in returns with a Lucas-type model where the growth rate of the endowment process (represented either by consumption, income or dividends) followed a two-state Markov-switching model in the mean. Bonomo and Garcia (1994) showed that a two-state model with one mean and two variances is closer to the data but cannot reproduce the autocorrelation in returns.

Proposition 2.3.2. ***Some Population Parameters.***

$$Var\left[\frac{D_t}{P_t}\right] = \lambda_2^{\top} Var[\zeta_t]\lambda_2 \text{ and } Var\left[\frac{C_t}{P_{M,t}}\right] = \lambda_{2c}^{\top} Var[\zeta_t]\lambda_{2c}. \tag{2.27}$$

In addition, we have

$$Cov\left(R_{t+1:t+h}, \frac{D_t}{P_t}\right) = \psi_h^\top Var(\zeta_t)\lambda_2, \tag{2.28}$$

$$Cov\left(R_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right) = \psi_h^\top Var(\zeta_t)\lambda_{2c}, \tag{2.29}$$

$$Cov\left(R^e_{t+1:t+h}, \frac{D_t}{P_t}\right) = (\psi_h - b_h)^\top Var(\zeta_t)\lambda_2, \tag{2.30}$$

$$Cov\left(R^e_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right) = (\psi_h - b_h)^\top Var(\zeta_t)\lambda_2. \tag{2.31}$$

We also have

$$\begin{aligned} Var[R_{t+1:t+h}] &= h\theta_2^\top E\left[\zeta_t\zeta_t^\top\right]P^\top\theta_3. \\ &+ h(\theta_1\odot\theta_1)^\top E[\zeta_t\zeta_t^\top]P^\top(\lambda_3\odot\lambda_3) - h^2(\theta_1^\top E[\zeta_t\zeta_t^\top]P^\top\lambda_3)^2 \\ &+ 2\sum_{j=2}^{h}(h-j+1)\theta_1^\top E[\zeta_t\zeta_t^\top]P^\top(\lambda_3\odot((P^{j-2})^\top(\theta_1\odot(P^\top\lambda_3)))), \end{aligned} \tag{2.32}$$

where

$$\theta_1 = \lambda_2\odot(\exp(\mu_{d,1}+\omega_{d,1}/2),\ldots,\exp(\mu_{d,N}+\omega_{d,N}/2))^\top, \tag{2.33}$$

$$\theta_2 = (\theta_1\odot\theta_1\odot(\exp(\omega_{d,1}),\ldots,\exp(\omega_{d,N}))^\top) - (\theta_1\odot\theta_1), \tag{2.34}$$

$$\theta_3 = \lambda_3\odot\lambda_3. \tag{2.35}$$

Likewise,

$$\begin{aligned} Var\left[R^e_{t+1:t+h}\right] &= h\theta_2^\top E\left[\zeta_t\zeta_t^\top\right]P^\top\theta_3 \\ &+ h\left((\theta_1\odot\theta_1)^\top E\left[\zeta_t\zeta_t^\top\right]P^\top(\lambda_3\odot\lambda_3) - 2(\theta_1\odot b)^\top E\left[\zeta_t\zeta_t^\top\right]P^\top\lambda_3\right) \\ &+ h(b\odot b)^\top\Pi - h^2\left(\theta_1^\top E\left[\zeta_t\zeta_t^\top\right]P^\top\lambda_3 - b^\top\Pi\right)^2 \\ &+ 2\sum_{j=2}^{h}(h-j+1)q_j \end{aligned} \tag{2.36}$$

where

$$\begin{aligned} q_j = {} & \theta_1^\top E\left[\zeta_t\zeta_t^\top\right] P^\top \left(\lambda_3 \odot \left(\left(P^{j-2}\right)^\top \left(\theta_1 \odot \left(P^\top \lambda_3\right)\right)\right)\right) \\ & - \theta_1^\top E\left[\zeta_t\zeta_t^\top\right] P^\top \left(\lambda_3 \odot \left(\left(P^{j-2}\right)^\top b\right)\right) \\ & - b^\top E\left[\zeta_t\zeta_t^\top\right] \left(P^{j-1}\right)^\top \left(\theta_1 \odot \left(P^\top \lambda_3\right)\right) + b^\top E\left[\zeta_t\zeta_t^\top\right] \left(P^{j-1}\right)^\top b. \end{aligned} \tag{2.37}$$

Observe that by using (2.32), one gets the variance ratio of aggregate returns which is given by

$$Ratio(h) \equiv \frac{1}{h} \frac{Var[R_{t+1:t+h}]}{Var[R_{t+1:t+1}]}. \tag{2.38}$$

One also gets a similar formula for the excess returns by using (2.36).

2.3.4 Predictability of Returns: An Analytical Evaluation

As mentioned in the previous section on stylized facts there appears to be a strong predictability of returns by the dividend-price ratio, which increases with the horizon. It is important to establish if this predictability measured inevitably in finite samples is reproduced in population by the postulated model. Therefore, we provide below the formulas for the population coefficients of the regressions of aggregated returns over a number of periods on the price-dividend ratio. In the section on applications below we will investigate by simulation to what extent some models produce predictability in finite samples but not in population. Several papers proposed models to reproduce predictability in returns. Bonomo and Garcia (1994) showed by simulation that a model with disappointment averse preferences (a recursive utility model with a Chew-Deckel certainty equivalent, see Epstein and Zin, 1989) and a Markov switching endowment for consumption and dividends was able to reproduce predictability in finite samples. More recently, Bansal and Yaron (2004) also reproduced this predictability with a recursive utility model with a Kreps and Porteus certainty equivalent.

It is common in the asset pricing literature to predict future (excess) returns by the dividend-price ratio. In doing so, one computes the regression of the aggregate returns onto the dividend-price ration and a constant. In the following, we will use the analytical formulas derived above in order to study these predictive ability in population.

When one does the linear regression of a variable, say $y_{t+1:t+h}$, onto by another one, say x_t, and a constant, one gets

$$y_{t+1:t+h} = a_{y,1}(h) + b_{y,1}(h)x_t + \eta_{y,1,t+h}(h)$$

where

$$b_{y,1} = \frac{Cov(y_{t+1:t+h}, x_t)}{Var[x_t]}$$

while the corresponding population coefficient of determination denoted R^2 is given by

$$R^2 = \frac{(Cov(y_{t+1:t+h}, x_t)))^2}{Var[y_{t+1:t+h}]Var[x_t]}.$$

We will use these formulas in the following Proposition in order to characterize the predictive ability of the dividend-price ratio.

Proposition 2.3.3. ***Regression of the Aggregated Returns onto the Dividend-Price Ratio and a Constant.***
Define the population regressions

$$R_{t+1:t+h} = a_1(h) + b_1(h)\frac{D_t}{P_t} + \eta_{1,t+h}(h) \text{ and } R^e_{t+1:t+h} = a^e_1(h) + b^e_1(h)\frac{D_t}{P_t} + \eta^e_{1,t+h}(h). \quad (2.39)$$

Denote the population coefficients of determination by $R^2(h, D/P)$ and $R^2_e(h, D/P)$. Then,

$$b_1(h) = \frac{Cov\left(R_{t+1:t+h}, \frac{D_t}{P_t}\right)}{Var\left[\frac{D_t}{P_t}\right]}, \quad b^e_1(h) = \frac{Cov\left(R^e_{t+1:t+h}, \frac{D_t}{P_t}\right)}{Var\left[\frac{D_t}{P_t}\right]}, \quad (2.40)$$

$$R^2(h, D/P) = \frac{\left(Cov\left(R_{t+1:t+h}, \frac{D_t}{P_t}\right)\right)^2}{Var[R_{t+1:t+h}]Var\left[\frac{D_t}{P_t}\right]}, \ R^2_e(h, D/P) = \frac{\left(Cov\left(R^e_{t+1:t+h}, \frac{D_t}{P_t}\right)\right)^2}{Var[R^e_{t+1:t+h}]Var\left[\frac{D_t}{P_t}\right]}, \quad (2.41)$$

where $Cov\left(R_{t+1:t+h}, \frac{D_t}{P_t}\right)$, $Cov\left(R^e_{t+1:t+h}, \frac{D_t}{P_t}\right)$, $Var\left[\frac{D_t}{P_t}\right]$, $Var[R_{t+1:t+h}]$ and $Var[R^e_{t+1:t+h}]$ are given in (2.28), (2.30), (2.27), (2.32) and (2.36) respectively.

The following Proposition characterizes the predictive ability of the consumption-price ratio:

Proposition 2.3.4. ***Regression of the Aggregated Returns onto the Consumption-Price Ratio and a Constant.***

Define the population regressions

$$R_{t+1:t+h} = a_{1c}(h) + b_{1c}(h)\frac{C_t}{P_{M,t}} + \eta_{1c,t+h}(h),\ R^e_{t+1:t+h} = a^e_{1c}(h) + b^e_{1c}(h)\frac{C_t}{P_{M,t}} + \eta^e_{1c,t+h}(h). \quad (2.42)$$

Denote the population coefficients of determination by $R^2(h,C/P_M)$ and $R^2_e(h,C/P_M)$. Then,

$$b_{1c}(h) = \frac{Cov\left(R_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right)}{Var\left[\frac{C_t}{P_{M,t}}\right]},\quad b^e_{1c}(h) = \frac{Cov\left(R^e_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right)}{Var\left[\frac{C_t}{P_{M,t}}\right]}, \quad (2.43)$$

$$R^2(h,C/P_M) = \frac{\left(Cov\left(R_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right)\right)^2}{Var[R_{t+1:t+h}]Var\left[\frac{C_t}{P_{M,t}}\right]},\ R^2_e(h,C/P_M) = \frac{\left(Cov\left(R^e_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right)\right)^2}{Var[R^e_{t+1:t+h}]Var\left[\frac{C_t}{P_{M,t}}\right]}, \quad (2.44)$$

where $Cov\left(R_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right)$, $Cov\left(R^e_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right)$, $Var\left[\frac{C_t}{P_{M,t}}\right]$, $Var[R_{t+1:t+h}]$ and $Var[R^e_{t+1:t+h}]$ are given in (2.29), (2.31), (2.27), (2.32) and (2.36) respectively.

The two previous propositions characterize the predictive ability of the dividend-price and consumption-price ratios. However, it is common in the literature to use jointly these two variables in the predictive regressions. The following proposition characterizes the joint predictive ability of the dividend-price and consumption-price ratios. However, we do not study in this chapter the empirical counterpart of these joint predictive ability.

Proposition 2.3.5. ***Regression of the Aggregated Returns onto the Dividend-Price and Consumption-Price Ratios and a Constant.***

Define the population regressions

$$\begin{aligned} R_{t+1:t+h} &= \tilde{a}_1(h) + \left(\frac{D_t}{P_t}, \frac{C_t}{P_{M,t}}\right)\tilde{b}_1(h) + \tilde{\eta}_{1,t+h}(h), \\ R^e_{t+1:t+h} &= \tilde{a}^e_1(h) + \left(\frac{D_t}{P_t}, \frac{C_t}{P_{M,t}}\right)\tilde{b}^e_1(h) + \tilde{\eta}^e_{1,t+h}(h). \end{aligned} \quad (2.45)$$

Denote the coefficients of determination by $R^2(h, D/P, C/P_M)$ *and* $R_e^2(h, D/P, C/P_M)$. *Then,*

$$\tilde{b}_1(h) = \Omega^{-1}\left(Cov\left(R_{t+1:t+h}, \frac{D_t}{P_t}\right), Cov\left(R_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right)\right)^\top, \tag{2.46}$$

$$\tilde{b}_1^e(h) = \Omega^{-1}\left(Cov\left(R_{t+1:t+h}^e, \frac{D_t}{P_t}\right), Cov\left(R_{t+1:t+h}^e, \frac{C_t}{P_{M,t}}\right)\right)^\top, \tag{2.47}$$

$$R^2(h, D/P, C/P_M) = \frac{\tilde{b}_1(h)^\top\, \Omega\, \tilde{b}_1(h)}{Var[R_{t+1:t+h}]},\ R_e^2(h, D/P, C/P_M) = \frac{\tilde{b}_1^e(h)^\top\, \Omega\, \tilde{b}_1^e(h)}{Var[R_{t+1:t+h}^e]} \tag{2.48}$$

where $Cov\left(R_{t+1:t+h}, \frac{D_t}{P_t}\right)$, $Cov\left(R_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right)$, $Cov\left(R_{t+1:t+h}^e, \frac{C_t}{P_{M,t}}\right)$, $Var\left[\frac{D_t}{P_t}\right]$, $Var\left[\frac{C_t}{P_{M,t}}\right]$, $Var[R_{t+1:t+h}]$ *and* $Var[R_{t+1:t+h}^e]$ *are given in (2.28), (2.29), (2.30), (2.31), (2.27), (2.27), (2.32) and (2.36) respectively, while the matrix* Ω *is defined by*

$$\Omega = \begin{bmatrix} Var[\frac{D_t}{P_t}] & Cov[\frac{D_t}{P_t}, \frac{C_t}{P_{M,t}}] \\ Cov[\frac{D_t}{P_t}, \frac{C_t}{P_{M,t}}] & Var[\frac{C_t}{P_{M,t}}] \end{bmatrix} \tag{2.49}$$

where

$$Cov\left[\frac{D_t}{P_t}, \frac{C_t}{P_{M,t}}\right] = \lambda_2^\top Var[\zeta_t]\lambda_{2c}. \tag{2.50}$$

2.3.5 Predictability of Consumption Volatility and Growth Rates

Bansal and Yaron (2004) provide empirical evidence for fluctuating consumption volatility. They also provide some evidence that realized consumption volatility predicts and is predicted by the price– dividend ratio.

We start this subsection by characterizing some moments and then we will study the predictability of the aggregate consumption volatility in a subsequent proposition. The consumption variance σ_{ct}^2 defined in (2.1) equals $\omega_c^\top \zeta_t$.

Proposition 2.3.6. *We have*

$$Cov\left(\sigma_{c,t+1:t+h}^2, \frac{D_t}{P_t}\right) = \omega_{ch}^\top Var[\zeta_t]\,\lambda_2 \tag{2.51}$$

$$Cov\left(\sigma_{c,t+1:t+h}^2, \frac{C_t}{P_{M,t}}\right) = \omega_{ch}^\top Var[\zeta_t]\,\lambda_{2c}, \tag{2.52}$$

where

$$\omega_{ch} = \left(\sum_{j=1}^{h} P^j\right)^{\top} \omega_c. \tag{2.53}$$

In addition,

$$Var\left[\sigma^2_{c,t+1:t+h}\right] = \omega_c^{\top} Var\left[\zeta_{t+1:t+h}\right]\omega_c \tag{2.54}$$

where

$$Var\left[\zeta_{t+1:t+h}\right] = \left(hI + 2\sum_{j=2}^{h}(h-j+1)P^{j-1}\right) Var\left[\zeta_t\right]. \tag{2.55}$$

We are now able to study the predictability of the aggregate consumption volatility.

Proposition 2.3.7. ***Regression of Aggregate Consumption Volatility onto Dividend-Price Ratio.***
Define the population regression

$$\sigma^2_{c,t+1:t+h} = a_3(h) + b_3(h)\frac{D_t}{P_t} + \eta_{3,t+h}(h), \tag{2.56}$$

and denote the population coefficient of determination by $R^2\left(h, \sigma_c^2, \frac{D}{P}\right)$. *Then,*

$$b_3(h) = \frac{Cov\left(\sigma^2_{c,t+1:t+h}, \frac{D_t}{P_t}\right)}{Var\left[\frac{D_t}{P_t}\right]}, \tag{2.57}$$

$$R^2\left(h, \sigma_c^2, \frac{D}{P}\right) = \frac{\left(Cov\left(\sigma^2_{c,t+1:t+h}, \frac{D_t}{P_t}\right)\right)^2}{Var\left[\sigma^2_{c,t+1:t+h}\right] Var\left[\frac{D_t}{P_t}\right]}, \tag{2.58}$$

where $Cov\left(\sigma^2_{c,t+1:t+h}, \frac{D_t}{P_t}\right)$, $Var\left[\sigma^2_{c,t+1:t+h}\right]$, *and* $Var[\frac{D_t}{P_t}]$ *are given by (2.51), (2.54), and (2.27) respectively.*

We can also characterize the predictive ability of the consumption-price ratio:

Proposition 2.3.8. ***Regression of Aggregate Consumption Volatility onto Consumption-Price Ratio.***

Define the population regression

$$\sigma^2_{c,t+1:t+h} = a_{3c}(h) + b_{3c}(h)\frac{C_t}{P_{M,t}} + \eta_{3c,t+h}(h), \tag{2.59}$$

and denote the population coefficient of determination by $R^2\left(h, \sigma^2_c, \frac{C}{P_M}\right)$*. Then,*

$$b_{3c}(h) \quad = \quad \frac{Cov\left(\sigma^2_{c,t+1:t+h}, \frac{C_t}{P_{M,t}}\right)}{Var\left[\frac{C_t}{P_{M,t}}\right]}, \tag{2.60}$$

$$R^2\left(h, \sigma^2_c, \frac{C}{P_M}\right) \quad = \quad \frac{\left(Cov\left(\sigma^2_{c,t+1:t+h}, \frac{D_t}{P_t}\right)\right)^2}{Var\left[\sigma^2_{c,t+1:t+h}\right] Var\left[\frac{D_t}{P_t}\right]}, \tag{2.61}$$

where $Cov\left(\sigma^2_{c,t+1:t+h}, \frac{C_t}{P_{M,t}}\right)$, $Var\left[\sigma^2_{c,t+1:t+h}\right]$, *and* $Var[\frac{C_t}{P_{M,t}}]$ *are given by (2.52), (2.54), and (2.27) respectively.*

The two previous propositions characterize the predictive ability of the dividend-price and consumption-price ratios in forecasting aggregate volatility. We now characterize the moments through which we will study the predictability of aggregate consumption and dividend growths in a subsequent proposition.

Aggregate consumption and dividend growth rates over h periods are defined by:

$$\Delta c_{t+1:t+h} = \sum_{j=1}^{h} \Delta c_{t+j} \text{ and } \Delta d_{t+1:t+h} = \sum_{j=1}^{h} \Delta d_{t+j}.$$

The expected values of these multi-period growth rates are given by:

$$E\left[\Delta c_{t+1:t+h} \mid J_t\right] = \mu_{ch}^{\top}\zeta_t \text{ and } E\left[\Delta d_{t+1:t+h} \mid J_t\right] = \mu_{dh}^{\top}\zeta_t$$

where

$$\mu_{ch} = \left(\sum_{j=1}^{h} P^{j-1}\right)^{\top} \mu_c \text{ and } \mu_{dh} = \left(\sum_{j=1}^{h} P^{j-1}\right)^{\top} \mu_d.$$

Proposition 2.3.1. *We have*

$$Cov\left(\Delta c_{t+1:t+h}, \frac{D_t}{P_t}\right) = \mu_{ch}^{\top} Var[\zeta_t]\lambda_2 \text{ and } Cov\left(\Delta c_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right) = \mu_{ch}^{\top} Var[\zeta_t]\lambda_{2c} \quad (2.62)$$

$$Cov\left(\Delta d_{t+1:t+h}, \frac{D_t}{P_t}\right) = \mu_{dh}^{\top} Var[\zeta_t]\lambda_2 \text{ and } Cov\left(\Delta d_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right) = \mu_{dh}^{\top} Var[\zeta_t]\lambda_{2c}. \quad (2.63)$$

In addition,

$$Var[\Delta c_{t+1:t+h}] = \mu_c^{\top} Var[\zeta_{t:t+h-1}]\mu_c + h\omega_c^{\top}\Pi \quad (2.64)$$

$$Var[\Delta d_{t+1:t+h}] = \mu_d^{\top} Var[\zeta_{t:t+h-1}]\mu_d + h\omega_d^{\top}\Pi \quad (2.65)$$

where $Var[\zeta_{t:t+h-1}] = Var[\zeta_{t+1:t+h}]$ *given by (2.55).*

These formulas allow us to study the predictability of growth rates which are characterized in the following propositions.

Proposition 2.3.2. ***Regression of Aggregate Consumption Growth onto Dividend-Price and Consumption-Price Ratios.***
Define the population regressions

$$\Delta c_{t+1:t+h} = a_4(h) + b_4(h)\frac{D_t}{P_t} + \eta_{4,t+h}(h) \quad (2.66)$$

$$\Delta c_{t+1:t+h} = a_{4c}(h) + b_{4c}(h)\frac{C_t}{P_{M,t}} + \eta_{4c,t+h}(h) \quad (2.67)$$

and denote the coefficients of determination by $R^2\left(h, \Delta c, \frac{D}{P}\right)$ *and* $R^2\left(h, \Delta c, \frac{C}{P_M}\right)$ *respec-*

tively. Then,

$$b_4(h) = \frac{Cov\left(\Delta c_{t+1:t+h}, \frac{D_t}{P_t}\right)}{Var\left[\frac{D_t}{P_t}\right]} \text{ and } R^2\left(h, \Delta c, \frac{D}{P}\right) = \frac{\left(Cov\left(\Delta c_{t+1:t+h}, \frac{D_t}{P_t}\right)\right)^2}{Var[\Delta c_{t+1:t+h}]\, Var\left[\frac{D_t}{P_t}\right]} \tag{2.68}$$

$$b_{4c}(h) = \frac{Cov\left(\Delta c_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right)}{Var\left[\frac{C_t}{P_{M,t}}\right]} \text{ and } R^2\left(h, \Delta c, \frac{C}{P_M}\right) = \frac{\left(Cov\left(\Delta c_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right)\right)^2}{Var[\Delta c_{t+1:t+h}]\, Var\left[\frac{C_t}{P_{M,t}}\right]} \tag{2.69}$$

where $Cov\left(\Delta c_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right)$, $Cov\left(\Delta c_{t+1:t+h}, \frac{D_t}{P_t}\right)$ *and* $Var[\Delta c_{t+1:t+h}]$ *are given in (2.62), (2.62) and (2.64) respectively.*

Proposition 2.3.3. ***Regression of Aggregate Dividend Growth onto Dividend-Price and Consumption-Price Ratios.***

Define the population regressions

$$\Delta d_{t+1:t+h} = a_5(h) + b_5(h)\frac{D_t}{P_t} + \eta_{5,t+h}(h) \tag{2.70}$$

$$\Delta d_{t+1:t+h} = a_{5c}(h) + b_{5c}(h)\frac{C_t}{P_{M,t}} + \eta_{5c,t+h}(h) \tag{2.71}$$

and denote the coefficients of determination by $R^2\left(h, \Delta d, \frac{D}{P}\right)$ *and* $R^2\left(h, \Delta d, \frac{C}{P_M}\right)$ *respectively. Then,*

$$b_5(h) = \frac{Cov\left(\Delta d_{t+1:t+h}, \frac{D_t}{P_t}\right)}{Var\left[\frac{D_t}{P_t}\right]} \text{ and } R^2\left(h, \Delta d, \frac{D}{P}\right) = \frac{\left(Cov\left(\Delta d_{t+1:t+h}, \frac{D_t}{P_t}\right)\right)^2}{Var[\Delta d_{t+1:t+h}]\, Var\left[\frac{D_t}{P_t}\right]} \tag{2.72}$$

$$b_{5c}(h) = \frac{Cov\left(\Delta d_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right)}{Var\left[\frac{C_t}{P_{M,t}}\right]} \text{ and } R^2\left(h, \Delta d, \frac{C}{P_M}\right) = \frac{\left(Cov\left(\Delta d_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right)\right)^2}{Var[\Delta d_{t+1:t+h}]\, Var\left[\frac{C_t}{P_{M,t}}\right]} \tag{2.73}$$

where $Cov\left(\Delta d_{t+1:t+h}, \frac{C_t}{P_{M,t}}\right)$, $Cov\left(\Delta d_{t+1:t+h}, \frac{D_t}{P_t}\right)$ *and* $Var[\Delta d_{t+1:t+h}]$ *are given in (2.63), (2.63) and (2.65) respectively.*

2.4 Solving Asset Pricing Models

The benchmark model for equilibrium consumption-based asset pricing is the Lucas (1978) model. We will reserve below the acronym CCAPM for this model. It will appear as a particular case of the so-called Epstein and Zin (1989) model that we will analyze in depth in this chapter. In fact this model is a particular case of the general recursive specification used by Epstein and Zin (1989) in which a representative agent derives his utility by combining current consumption with a certainty equivalent of future utility through an aggregator. Depending on how this certainty equivalent is specified, the recursive utility concept can accommodate several classes of preferences. A class that is used extensively in empirical work is the so-called Kreps-Porteus, where the certainty equivalent conforms with expected utility for ranking timeless gambles, but with a different parameter than the aggregator's parameter. This is what it is usually called the Epstein and Zin (1989) model. We will keep below with this tradition.[3]

Another very influential model is the Campbell and Cochrane (1999) model which extends the basic external habit formation literature. In habit formation models, an investor derives utility not from the absolute level of consumption but from its level relative to a benchmark which is related to past consumption.[4] When this reference level depends on past aggregate per capita consumption, the *catching up* with the Joneses specification of Abel (1990), or on current per capita consumption, the *keeping up* with the Joneses of Abel (1999)[5], it captures the idea that the individual wants to maintain his relative status in the economy. Campbell and Cochrane (1999) specify a slow-moving habit and impose a nonlinear dynamics on the surplus consumption with respect to the habit.

The main goal of this section is to characterize the vectors λ_1, λ_{1c} and b defined in (2.13), (2.14) and (2.15) as function of the parameters describing the dynamics of the

[3]Epstein and Zin (1989) go further by integrating in a temporal setting a large class of atemporal non-expected utility theories, in particular homogeneous members of the class introduced by Chew (1985) and Dekel (1986). The certainty equivalent is then defined implicitly. It includes in particular a disappointment aversion specification, see Bonomo and Garcia (1994).

[4]See among others Abel (1990, 1996), Campbell and Cochrane (1999), Constantinides (1990), Ferson and Constantinides (1991), Heaton (1995), and Sundaresan (1989).

[5]It generalizes Gali's (1994) specification of consumption externalities whereby agents have preferences defined over their own consumption as well as current per capita consumption in the economy.

consumption and dividend growths and the utility function of the representative agent. We provide analytical formulas for these quantities for the three models just described : CCAPM (Lucas, 1978), Epstein and Zin (1989) and Campbell and Cochrane (1999). Solutions for the latter model are provided in appendix III.

2.4.1 The CCAPM

2.4.1.1 Consumption Equals Dividend

We start by assuming that the consumption equals the dividend as in Lucas (1978), which implies

$$\mu_c = \mu_d, \;\; \omega_c = \omega_d, \;\; \rho = (1,1,...,1)^\top. \tag{2.74}$$

Proposition 2.4.1. ***Characterization of the Asset Prices.***
We have

$$\frac{P_t}{D_t} = \delta e^\top \left[Id - \delta A((1-\gamma)\mu_d + (1-\gamma)^2\omega_d/2)\right]^{-1} \exp\left((1-\gamma)\mu_d + (1-\gamma)^2\omega_d/2)^\top \zeta_t\right) \zeta_t, \tag{2.75}$$

where the matrix $A(\cdot)$ is defined in (2.7). Consequently, the i-th component, i=1,...,N, of the vector λ_1 defined in (2.13) are given by

$$\lambda_{1,i} = \delta \exp((1-\gamma)\mu_{d,i} + (1-\gamma)^2\omega_{d,i}/2))e^\top \left[Id - \delta A((1-\gamma)\mu_d + (1-\gamma)^2\omega_d/2)\right]^{-1} e_i. \tag{2.76}$$

In addition, $\lambda_{1c} = \lambda_1$ while the i-th component of the vector b defined in (2.15) is given by

$$b_i = \delta^{-1}\exp(\gamma\mu_{c,i} - \frac{\gamma^2}{2}\omega_{c,i}). \tag{2.77}$$

The formulas in the previous proposition are not new. Cecchetti, Lam and Mark (1990) derived them for homoskedastic models while Bonomo and Garcia (1993) did it for the same model as us. It is also worth noting that the matrix

$$\left[Id - \delta A((1-\gamma)\mu_d + (1-\gamma)^2\omega_d/2)\right]$$

might be singular or leads to negative prices for some parameters (of the consumption growth and utility function). Such cases happen when the maximization problem does not admit a solution. We will see in the results that such examples happen and that one can detect them. Note however that an approximation of the model (e.g., log-linearization) may lead to different results, for instance, provide prices that make sense while the true maximization problem does not admit a solution. The discussion of this issue in more details will follow in papers derived from this chapter.

2.4.1.2 Consumption and Dividend Are Different

Here, we still consider the CCAPM model but we assume that consumption and dividend are different. Henceforth, we use the vectors μ_{cd}, ω_{cd}, μ_{cc} and ω_{cc} defined by

$$\mu_{cd} = -\gamma\mu_c + \mu_d, \;\; \omega_{cd} = \gamma^2\omega_c + \omega_d - 2\gamma\left(\rho \odot (\omega_c)^{1/2} \odot (\omega_d)^{1/2}\right) \tag{2.78}$$

$$\mu_{cc} = (1-\gamma)\,\mu_c, \;\; \omega_{cc} = (1-\gamma)^2\,\omega_c \tag{2.79}$$

Proposition 2.4.2. ***Characterization of the Asset Prices.***
The i-th component, i=1,...,N, of the vector λ_1 defined in (2.13) is given by

$$\lambda_{1,i} = \delta\exp(\mu_{cd,i} + \omega_{cd,i}/2))e^\top\left[Id - \delta A(\mu_{cd} + \omega_{cd}/2)\right]^{-1} e_i, \tag{2.80}$$

where $A(\cdot)$ is defined in (2.7). In addition, the i-th component, i=1,...,N, of the vector λ_{1c} defined in (2.14) is given by

$$\lambda_{1c,i} = \delta\exp\left((1-\gamma)\,\mu_{c,i} + \frac{(1-\gamma)^2}{2}\omega_{c,i}\right)e^\top\left[Id - \delta A\left((1-\gamma)\,\mu_c + \frac{(1-\gamma)^2}{2}\omega_c\right)\right]^{-1} e_i \tag{2.81}$$

Finally, the components of the vector b defined in (2.15) are given by

$$b_i = \delta^{-1}\exp(\gamma\mu_{c,i} - \frac{\gamma^2}{2}\omega_{c,i}). \tag{2.82}$$

2.4.2 The Recursive Utility Model

The representative agent has recursive utility defined over consumption flow C_t as follows:

$$V_t = \left\{(1-\delta)C_t^{1-\frac{1}{\psi}} + \delta\left[\mathscr{R}_t\left(V_{t+1}\right)\right]^{1-\frac{1}{\psi}}\right\}^{\frac{1}{1-\frac{1}{\psi}}} \quad \text{if } \psi \neq 1 \tag{2.83}$$

$$= C_t^{1-\delta}\left[\mathscr{R}_t\left(V_{t+1}\right)\right]^{\delta} \quad \text{if } \psi = 1, \tag{2.84}$$

where V_t is the current continuation value of investor utility, $\mathscr{R}_t\left(V_{t+1}\right) = \left(E\left[V_{t+1}^{1-\gamma} \mid J_t\right]\right)^{\frac{1}{1-\gamma}}$ is the certainty equivalent of the next period continuation value of investor utility, γ is the coefficient of relative risk aversion, ψ is the elasticity of intertemporal substitution, δ is the subjective discount factor and $\theta = (1-\gamma)/(1-1/\psi)$.

When $\psi \neq 1$, it is proved that the stochastic discount factor is given by:

$$M_{t,t+1} = \delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\delta\frac{\frac{P_{M,t+1}}{C_{t+1}}+1}{\frac{P_{M,t}}{C_t}}\right)^{\theta-1}, \tag{2.85}$$

where the market price-consumption ratio is given by:

$$\frac{P_{M,t}}{C_t} = \delta E\left[\left(\frac{P_{M,t+1}}{C_{t+1}}+1\right)^{\theta}\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma} \mid J_t\right]^{\frac{1}{\theta}}. \tag{2.86}$$

If $\theta = 1$, one remarks that (2.85) corresponds to the stochastic discount factor of an investor with time-separable utility and constant relative risk aversion. Also, if the elasticity of intertemporal substitution is different from unity, the ratio of the continuation value to consumption is related to the market price-consumption ratio as follows:

$$\frac{V_t}{C_t} = (1-\delta)^{\frac{1}{1-\frac{1}{\psi}}}\left(\frac{P_{M,t}}{C_t}+1\right)^{\frac{1}{1-\frac{1}{\psi}}}. \tag{2.87}$$

On the other hand, when $\psi = 1$, the benchmark case considered by Hansen, Heaton and Li (2005), the stochastic discount factor is given by:

$$M^1_{t,t+1} = \delta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \left(\frac{\frac{V^1_{t+1}}{C_{t+1}}}{\left(\frac{V^1_t}{C_t}\right)^{\frac{1}{\delta}}}\right)^{1-\gamma}. \tag{2.88}$$

where the utility-consumption ratio is given by:

$$\frac{V^1_t}{C_t} = \left(E\left[\left(\frac{V^1_{t+1}}{C_{t+1}}\right)^{1-\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma} \mid J_t\right]\right)^{\frac{\delta}{1-\gamma}} \tag{2.89}$$

whereas the market price-consumption ratio is constant and equal to $\delta/(1-\delta)$. In the rest of the paper, we will also adopt the notation:

$$\frac{V_t}{C_t} = z^{\top}\zeta_t, \tag{2.90}$$

2.4.2.1 Market Price-Consumption Ratio

We start our analysis by characterizing the vector λ_{1c} defined in (2.14) that characterizes the consumption price ratio. The characterization of this vector is the main difference between Epstein-Zin and CCAPM models. We will show below that when one has the vector λ_{1c}, one gets the vectors dividend price ratio (i.e. the vector λ_1) and the risk-free rate (i.e., the vector b) as for the CCAPM. The following proposition characterizes the vector λ_{1c}.

Proposition 2.4.3. *When the EIS is different from one, the components $\lambda_{1c,i}$, z_i, $i = 1,..,N$, of the vectors λ_{1c} and z are the solution of the following equations:*

$$\lambda_{1c,i} = \delta\left(\sum_{j=1}^{N} p_{ij}\left(\lambda_{1c,j}+1\right)^{\theta}\right)^{\frac{1}{\theta}} \exp\left(\frac{(1-\gamma)}{\theta}\mu_{c,i} + \frac{(1-\gamma)^2}{2\theta}\omega_{c,i}\right). \tag{2.91}$$

$$z_i = (1-\delta)^{\frac{1}{1-\frac{1}{\psi}}}\left(\lambda_{1c,i}+1\right)^{\frac{1}{1-\frac{1}{\psi}}}. \tag{2.92}$$

Instead, if the EIS is equal to one, the components of the vector λ_{1c} and z are given *by:*

$$\lambda_{1c,i} = \frac{\delta}{1-\delta}. \tag{2.93}$$

$$z_i = \exp\left(\delta\mu_{c,i} + \frac{\delta(1-\gamma)}{2}\omega_{c,i}\right)\left(\sum_{j=1}^{N} p_{ij} z_j^{1-\gamma}\right)^{\frac{\delta}{1-\gamma}}. \tag{2.94}$$

Equations (2.91) and (2.94) are highly nonlinear, respectively when $\theta \neq 1$, that is, when Epstein-Zin model is not the CCAPM, and when $\delta \neq 1$, that is when the investor subjectively has a preference for the present relatively to the future. However, it is easy to solve these equations numerically by using numerical algorithms. We did by using the nonlinear equation solver in GAUSS.

As stated earlier, one of the objectives of this paper is to assess the errors due to using some approximations of the price-consumption ratio instead of the formula (2.91), given that these approximations are often considered for models with an EIS different from one. The first simple approximation is to linearize this function around $\lambda^* e$ where λ^* is a positive number. It leads to

$$\left(\sum_{j=1}^{N} p_{ij}\left(\lambda_{1c,j}+1\right)^{\theta}\right)^{\frac{1}{\theta}} \approx \sum_{j=1}^{N} p_{ij}\left(\lambda_{1c,j}+1\right). \tag{2.95}$$

Consequently, one gets for $i = 1,..,N$,

$$\lambda_{1c,i} = \delta\exp\left(\frac{(1-\gamma)}{\theta}\mu_{c,i} + \frac{(1-\gamma)^2}{2\theta}\omega_{c,i}\right)e^{\top}\left[Id - \delta A\left(\frac{(1-\gamma)}{\theta}\mu_c + \frac{(1-\gamma)^2}{2\theta}\omega_c\right)\right]^{-1} e_i. \tag{2.96}$$

The second approximation is the log-linearization of Campbell and Shiller (1988) for the market return, which leads to:

$$r_{M,t+1} = \ln R_{M,t+1} \approx k_0 + k_1 v_{1c}^{\top}\zeta_{t+1} - v_{1c}^{\top}\zeta_t + \Delta c_{t+1}, \tag{2.97}$$

where v_{1c} denotes the logarithm of the price-consumption ratio. Consequently, one gets:

$$\lambda_{1c,i} \approx \exp\left(v_{1c,i}\right), \ i = 1,..,N$$
$$v_{1c,i} = (\ln\delta + k_0) + \frac{(1-\gamma)}{\theta}\mu_{c,i} + \frac{(1-\gamma)^2}{2\theta}\omega_{c,i} + \frac{1}{\theta}\ln\left[\sum_{j=1}^{N} p_{ij}\exp\left(\theta k_1 v_{1c,j}\right)\right]. \quad (2.98)$$

Notice that although the coefficient k_1 is exogenously specified in previous studies, it is an endogenous coefficient which depends on preference parameters as well. The value of coefficients k_1 and k_0 are given by the formulas:

$$k_1 = \frac{1}{1+\exp\left(-\Pi^\top l_{\lambda_c}\right)} \text{ and } k_0 = -\ln k_1 - (1-k_1)\ln\left(\frac{1}{k_1} - 1\right), \quad (2.99)$$

where $l_{\lambda_c} = \left(\ln\lambda_{1c,1},..,\ln\lambda_{1c,N}\right)^\top$ and $\lambda_{1c,i}$, $i = 1,..,N$ are given by (2.91).

Since the coefficients k_1 and k_0 depend on the mean log price-consumption ratio $\bar{v}_{1c}$, then the vector v_{1c} whose components are given by equation (2.98) is also a function of $\bar{v}_{1c}$, that is, $v_{1c} = v_{1c}\left(\bar{v}_{1c}\right)$. In an alternative method to find model-consistent coefficients of the Campbell and Shiller log-linearization, Bansal, Kiku and Yaron (2007) solve for $\bar{v}_{1c}$ through the nonlinear equation which equalizes the mean of the logarithm of the exact model-implied price-consumption ratio with the mean of its log-linear approximation. In our case, this equation is given by:

$$\bar{v}_{1c} = \Pi^\top v_{1c}\left(\bar{v}_{1c}\right), \quad (2.100)$$

and

$$k_1 = \frac{1}{1+\exp\left(-\bar{v}_{1c}\right)} \text{ and } k_0 = -\ln k_1 - (1-k_1)\ln\left(\frac{1}{k_1} - 1\right). \quad (2.101)$$

The third approximation has been recently considered by Hansen, Heaton and Li (2005). Log-linearizing the market return around the endogenous price-consumption ratio and specifying exogenous values for the parameters of this log-linearization has the practical drawback of using, almost surely, wrong parameters to evaluate asset market implications of the asset pricing model. Instead, these authors log-linearize the stochastic discount factor around the unitary elasticity of intertemporal substitution.

We further consider the following notation:

$$v_t = \ln\left(\frac{V_t}{C_t}\right),\ Dv_t^1 = \lim_{\psi\to 1}\frac{\partial v_t}{\partial(1/\psi)} = h^\top \zeta_t \text{ and } Dm_{t,t+1}^1 = \lim_{\psi\to 1}\frac{\partial m_{t,t+1}}{\partial(1/\psi)} = \zeta_{t+1}^\top F\zeta_t, \tag{2.102}$$

where v_t is the logarithm of the utility-consumption ratio (2.87) and Dv_t^1 is its derivative with respect to $1/\psi$ and evaluated at $1/\psi = 1$, $m_{t,t+1}$ is the logarithm of the SDF (2.85) and $Dm_{t,t+1}^1$ is its derivative with respect to $1/\psi$ and evaluated at $1/\psi = 1$.

Hansen, Heaton and Li (2005) establish that the derivative Dv_t^1 is given by the recursion:

$$Dv_t^1 = -\frac{1-\delta}{2\delta}\left(v_t^1\right)^2 + \delta E\left[\frac{\left(V_{t+1}^1\right)^{1-\gamma}}{E\left[\left(V_{t+1}^1\right)^{1-\gamma}\mid J_t\right]}Dv_{t+1}^1 \mid J_t\right], \tag{2.103}$$

from which it follows that the components $h_i,\ i = 1,..,N$ of the vector h characterizing this derivative in our model are given by the equation:

$$h_i = -\frac{1-\delta}{2\delta}\frac{1}{\sum_{j=1}^{N} p_{ij}z_j^{1-\gamma}}\left(\left(\left(l_z^2\right)^\top\right)\odot\left(\left(z^{1-\gamma}\right)^\top P\right)\right)\left[Id - \delta A_{**}(0)\right]^{-1}e_i, \tag{2.104}$$

where $z_i,\ i = 1,..,N$ are given by (2.94), $l_z = (\ln z_1,..,\ln z_N)^\top$ and:

$$A_{**}(u) = Diag\left(\exp(u_1)\left(\frac{z_1^{1-\gamma}}{\sum_{j=1}^{N} p_{1j}z_j^{1-\gamma}}\right),..,\exp(u_N)\left(\frac{z_N^{1-\gamma}}{\sum_{j=1}^{N} p_{Nj}z_j^{1-\gamma}}\right)\right)P,\ \forall u\in\mathbb{R}^N. \tag{2.105}$$

They also establish that the derivative $Dm^1_{t,t+1}$ is given by the equation:

$$\begin{aligned} Dm^1_{t,t+1} &= v^1_{t+1} - \frac{v^1_t}{\delta} + (1-\gamma)\left(Dv^1_{t+1} - E\left[\frac{\left(V^1_{t+1}\right)^{1-\gamma}}{E\left[\left(V^1_{t+1}\right)^{1-\gamma} \mid J_t\right]} Dv^1_{t+1} \mid J_t\right]\right) \\ &= \left(v^1_{t+1} + (1-\gamma) Dv^1_{t+1}\right) - \frac{\left(v^1_t + (1-\gamma) Dv^1_t\right)}{\delta} - \frac{(1-\gamma)(1-\delta)}{2\delta^2}\left(v^1_t\right)^2, \end{aligned} \tag{2.106}$$

from which it follows that the elements $f_{ij},\ 1 \leq i,j \leq N$ of the matrix $F^\top$ characterizing this derivative are given by:

$$f_{ij} = \left(\ln z_j + (1-\gamma) h_j\right) - \frac{(\ln z_i + (1-\gamma) h_i)}{\delta} - \frac{(1-\gamma)(1-\delta)}{2\delta^2}(\ln z_i)^2 \tag{2.107}$$

where $h_i,\ z_i,\ i = 1,..,N$ are given by (2.104) and (2.94) respectively.

Hansen, Heaton and Li (2005) consider the first order Taylor expansion of the SDF (2.85) around the unitary elasticity of intertemporal substitution, that is:

$$\begin{aligned} M_{t,t+1} &= \exp\left(m_{t,t+1}\right) \\ m_{t,t+1} &\approx m^1_{t,t+1} + \left(\frac{1}{\psi} - 1\right) Dm^1_{t,t+1} \end{aligned} \tag{2.108}$$

Let P_ψ denotes the matrix defined by $P_\psi^\top = \left[p_{\psi,ij}\right]_{1 \leq i,j \leq N}$ such that:

$$p_{\psi,ij} = p_{ij} \exp\left(\left(\frac{1}{\psi} - 1\right) f_{ij}\right). \tag{2.109}$$

Given the Hansen, Heaton and Li (2005)'s approximation (2.108), the components $\lambda_{1c,i},\ i = 1,..,N$ of the vector λ_{1c} characterizing the market price-consumption ratio are given by the following formula:

$$\lambda_{1c,i} = \delta\left(\frac{1}{z_i^{1/\delta}}\right)^{1-\gamma} \exp\left(\mu_{cc,i} + \frac{\omega_{cc,i}}{2}\right)\left(\left(z^{1-\gamma}\right)^\top P_\psi\right)\left[Id - \delta A_\psi\left(\mu_{cc} + \frac{\omega_{cc}}{2}\right)\right]^{-1} e_i \tag{2.110}$$

where $z_i,\ i=1,..,N$ are given by (2.94) and

$$A_\psi(u) = Diag\left(z_1^{(1-1/\delta)(1-\gamma)}\exp(u_1),..,z_N^{(1-1/\delta)(1-\gamma)}\exp(u_N)\right)P_\psi,\ \forall u \in \mathbb{R}^N. \tag{2.111}$$

The approximations (2.96), (2.98) and (2.110) can also serve to obtain starting values for a numerical algorithm.

2.4.2.2 Equity Price-Dividend Ratio

Interestingly, when one has the price-consumption ratio, i.e., the vector λ_{1c}, one gets analytically the equity price-dividend ratio.

Proposition 2.4.4. *When the EIS is different from one, the components λ_{1i}, i=1,..,N, of the vector λ_1 characterizing the equity price-dividend ratio are given by the following formula:*

$$\lambda_{1i} = \delta\left(\frac{\delta}{\lambda_{1c,i}}\right)^{\theta-1}\exp\left(\mu_{cd,i}+\frac{\omega_{cd,i}}{2}\right)\left(\left((\lambda_{1c}+e)^{\theta-1}\right)^\top P\right)\left[Id-\delta A_*\left(\mu_{cd}+\frac{\omega_{cd}}{2}\right)\right]^{-1}e_i \tag{2.112}$$

where

$$A_*(u) = Diag\left(\left(\delta\frac{\lambda_{1c,1}+1}{\lambda_{1c,1}}\right)^{\theta-1}\exp(u_1),...,\left(\delta\frac{\lambda_{1c,N}+1}{\lambda_{1c,N}}\right)^{\theta-1}\exp(u_N)\right)P,\ \forall u\in\mathbb{R}^N. \tag{2.113}$$

and $\lambda_{1c,i},\ i=1,..,N$ are given by (2.91).

Instead, if the EIS is equal to one, the components of the vector λ_1 are given by:

$$\lambda_i = \delta\left(\frac{1}{z_i^{1/\delta}}\right)^{1-\gamma}\exp\left(\mu_{cd,i}+\frac{\omega_{cd,i}}{2}\right)\left(\left(z^{1-\gamma}\right)^\top P\right)\left[Id-\delta A_1\left(\mu_{cd}+\frac{\omega_{cd}}{2}\right)\right]^{-1}e_i \tag{2.114}$$

where z_i, $i = 1,..,N$ are given by (2.94) and

$$A_1(u) = Diag\left(z_1^{(1-1/\delta)(1-\gamma)}\exp(u_1),..,z_N^{(1-1/\delta)(1-\gamma)}\exp(u_N)\right)P,\ \forall u \in \mathbb{R}^N,$$

consistently with the notation (2.111).

Likewise, one can also use the log-linearization method to get the price-dividend ratio. The log-linearization of the equity return is given by:

$$r_{t+1} = \ln R_{t+1} \approx k_{m0} + k_{m1} v_1^\top \zeta_{t+1} - v_1^\top \zeta_t + \Delta d_{t+1}. \tag{2.115}$$

Also, although the coefficient k_{m1} has often been exogenously specified in empirical studies, it is an endogenous coefficient which depends on preference parameters as well as k_1. The value of coefficients k_{m1} and k_{m0} are given by the formulas:

$$k_{m1} = \frac{1}{1+\exp\left(-\Pi^\top l_\lambda\right)} \text{ and } k_{m0} = -\ln k_{m1} - (1-k_{m1})\ln\left(\frac{1}{k_{m1}} - 1\right) \tag{2.116}$$

where $l_\lambda = (\ln\lambda_{11},..,\ln\lambda_{1N})^\top$ and $\lambda_{1i},\ i = 1,..,N$ are given by (2.112).

We present below the formulas when one uses the log-linearization for the equity return (simple log-linearization) and one uses the log-linearization for both the market and equity returns (double log-linearization).

In the double log-linearization, one gets:

$$\begin{aligned} \lambda_{1i} &\approx \exp(v_{1i}),\ i = 1,..,N \\ v_{1i} &= \theta\ln\delta + (\theta-1)k_0 + k_{m0} - (\theta-1)v_{1c,i} + \mu_{cd,i} + \frac{1}{2}\omega_{cd,i} \\ &\quad + \ln\left[\sum_{j=1}^{N} p_{ij}\exp\left((\theta-1)k_1 v_{1c,j} + k_{m1}v_{1j}\right)\right]. \end{aligned} \tag{2.117}$$

where $v_{1c,i},\ i = 1,..,N$ are given by (2.98).

In contrast, the simple log-linearization leads to:

$$\lambda_{1i} \approx \exp(\nu_{1i}), \, i = 1,..,N$$
$$\nu_{1i} = \theta \ln \delta + k_{m0} + \mu_{cd,i} + \frac{1}{2}\omega_{cd,i} + \ln \left[\sum_{j=1}^{N} p_{ij} \left(\frac{\lambda_{1c,j}+1}{\lambda_{1c,i}} \right)^{\theta-1} \exp(k_{m1}\nu_{1j}) \right]. \tag{2.118}$$

where $\nu_{1c,i}$, $i = 1,..,N$ are given by (2.98).

The coefficients k_{m1} and k_{m0} also depend on the mean log price-dividend ratio $\bar{\nu}_1$. Then, the vector ν_1 whose components are given by equation (2.117) or (2.118) is also a function of $\bar{\nu}_1$, that is, $\nu_1 = \nu_1(\bar{\nu}_1)$. Bansal, Kiku and Yaron (2007) also solve for $\bar{\nu}_1$ through the nonlinear equation which equalizes the mean of the logarithm of the exact model-implied price-dividend ratio with the mean of its log-linear approximation. In our case, this equation is given by:

$$\bar{\nu}_1 = \Pi^\top \nu_1(\bar{\nu}_1), \tag{2.119}$$

and

$$k_{m1} = \frac{1}{1+\exp(-\bar{\nu}_1)} \quad \text{and} \quad k_{m0} = -\ln k_{m1} - (1-k_{m1})\ln\left(\frac{1}{k_{m1}} - 1\right). \tag{2.120}$$

Alternatively, the approximation (2.108) due to Hansen, Heaton and Li (2006) leads to the following formula for the price-dividend ratio:

$$\lambda_{1i} = \delta \left(\frac{1}{z_i^{1/\delta}} \right)^{1-\gamma} \exp\left(\mu_{cd,i} + \frac{\omega_{cd,i}}{2}\right) \left(\left(z^{1-\gamma}\right)^\top P_\psi \right) \left[Id - \delta A_\psi \left(\mu_{cd} + \frac{\omega_{cd}}{2} \right) \right]^{-1} e_i \tag{2.121}$$

where z_i, $i = 1,..,N$ are given by (2.94) and A_ψ in defined in (2.111).

2.4.2.3 Risk-Free Rate

The following proposition characterizes the components of the vector b in 2.15.

Proposition 2.4.5. *When the EIS is different from one, the components b_i, i=1,..,N, of the vector b characterizing the risk-free rate are given by the following formula:*

$$\frac{1}{b_i} = \delta \exp\left(-\gamma\mu_{c,i} + \frac{1}{2}\gamma^2\omega_{c,i}\right)\sum_{j=1}^{N} p_{ij}\left(\delta\frac{\lambda_{1c,j}+1}{\lambda_{1c,i}}\right)^{\theta-1}, \qquad (2.122)$$

where $\lambda_{1c,i}$, $i = 1,..,N$ are given by (2.91).

Instead, if the EIS is equal to one, the components of the vector b are given by:

$$\frac{1}{b_i} = \delta \exp\left(-\mu_{c,i} - \frac{1}{2}(1-2\gamma)\,\omega_{c,i}\right). \qquad (2.123)$$

Based on approximations, the risk-free rate with the Campbell and Shiller's log-linearization of the market return is given by the formula:

$$\frac{1}{b_i} = \delta^{\theta}\exp\left((\theta-1)(k_0 - v_{1c,i})\right)\exp\left(-\gamma\mu_{c,i} + \frac{1}{2}\gamma^2\omega_{c,i}\right)\sum_{j=1}^{N} p_{ij}\exp\left((\theta-1)k_1 v_{1c,j}\right), \quad (2.124)$$

where $v_{1c,i}$, $i = 1,..,N$ are given by (2.98).

Alternatively, the risk-free rate with the Hansen, Heaton and Li's Taylor expansion of the true SDF is given by the formula:

$$\frac{1}{b_i} = \delta \exp\left(-\mu_{c,i} - \frac{1}{2}(1-2\gamma)\,\omega_{c,i}\right)\frac{\sum_{j=1}^{N} p_{\psi,ij} z_j^{1-\gamma}}{\sum_{j=1}^{N} p_{ij} z_j^{1-\gamma}}, \qquad (2.125)$$

where z_i, $i = 1,..,N$ are given by (2.94) and $p_{\psi,ij}$ $i,j = 1,..,N$ are defined in (2.109).

2.5 Applications to Models of the Post-War US Economy

In this section, we apply the derived formulas in three contexts. First, we estimate a Markov-switching model directly on the quarterly growth rates of real consumption and dividend per capita for the US postwar period. Then we can apply the formulas derived in the two previous sections for the CCAPM and the Epstein-Zin model. In a second application, we analyze the Markov-switching model with Epstein and Zin (1989) preferences proposed by Lettau, Ludvigson and Wachter (2004). In the last application, we calibrate a Markov-switching model in order to match the endowment process used by Bansal and Yaron (2004). The goal is to easily compute population values for several statistics that have been obtained by numerical techniques or by simulation, as well as to produce results for a larger parameter set than the one in the last two papers. This way we will hopefully better understand the economic intuition behind results and assess robustness to changes in the values of preference and endowment parameters.

2.5.1 A Two-State Markov Switching Model with Epstein-Zin Preferences

We start with a simple model, a two-state Markov switching model in both means and variances previously estimated by Bonomo and Garcia (1994, 1996) with annual secular data on consumption and dividends. The estimated parameters are reported in Table 2.2. The first state is a low-mean high-variance state for consumption. Dividend growth is also low in this state while variance is not very different from the variance in the high state. Both states have about the same degree of persistence and consequently the unconditional probabilities are close to 0.5.

In Table 2.3, we report the asset pricing implications of this endowment when the agent has Epstein-Zin preferences. As expected, a high risk aversion is needed to arrive at equity premium values comparable to what is observed in the data. The equity premium increases in both risk and intertemporal substitution, while the risk-free decreases sharply with the elasticity of intertemporal substitution. The reduction of the interest rate may come either from the variance of the market portfolio if θ is negative or from the variance of consumption if θ is positive. To see that, it is easier to look at the Euler condition in a model with jointly lognormal and homoskedastic asset returns and consumption, where the risk-free interest is given by:

$$r_{f,t+1} = -log\delta + \frac{1}{\psi}E_t[\Delta c_{t+1}] + \frac{\theta-1}{2}\sigma_w^2 - \frac{\theta}{2\psi^2}\sigma_c^2 \qquad (2.126)$$

The sign of θ is determined, for a given γ, by the value of ψ. If ψ is less than one, θ is positive, if it is greater than one, θ is negative.

Table 2.1: **Predictability of Returns and Growth Rates: Data.**
This table shows estimates of slope coefficients, and R-squared of regressions $y_{t+1:t+h} = a_y(h) + b_y(h)\frac{D_t}{P_t} + \eta_{y,t+h}(h)$, where the variable y is return, excess return, consumption growth rate or dividend growth rate. Standard errors are Newey and West (1987) corrected using 10 lags. Lines 6 and 11 show variance ratios of aggregate returns and aggregate excess returns respectively. The horizon h is quarterly in regressions and converted into annual in the table. Estimates and standard deviations of slope coefficients are multiplied by 10^{-4} in the table.

h	1	2	3	4	5
			Returns		
Estimate	0.1416	0.2415	0.3027	0.3747	0.5128
Std. Dev.	0.0502	0.0930	0.1166	0.1277	0.1498
R-squared	9.0192	13.5480	15.1060	17.5200	22.3720
Var. Ratio	1.0271	0.9623	0.8660	0.8209	0.9199
			Excess Returns		
Estimate	0.1527	0.2617	0.3247	0.3875	0.5126
Std. Dev.	0.0458	0.0858	0.1066	0.1156	0.1354
R-squared	10.9800	17.1750	19.5180	21.8600	26.2010
Var. Ratio	1.0028	0.9105	0.7880	0.7189	0.8017
			Consumption Growth		
Estimate	-0.0047	-0.0058	-0.0117	-0.0163	-0.0214
Std. Dev.	0.0041	0.0073	0.0098	0.0117	0.0136
R-squared	1.6140	1.0546	2.6146	3.6245	4.6223
			Dividend Growth		
Estimate	0.0045	0.0184	0.0233	0.0370	0.0623
Std. Dev.	0.0176	0.0337	0.0431	0.0503	0.0536
R-squared	0.0488	0.4107	0.4435	0.9214	2.3147

Table 2.2: **Parameters of a Two-State Markov-Switching Model for Quarterly US Data on Consumption and Dividends - 1947:3-2002:4.**

This table shows parameters of a two-state Markov-Switching Model estimated on actual data. μ_c and μ_d are conditional means of consumption and dividend, ω_c and ω_d are conditional variances of consumption and dividend. ρ is the conditional correlation between consumption and dividend shocks. $P^\top$ is the transition matrix across different regimes and Π is the vector of unconditional probabilities of regimes. Means and standard deviations are in percent.

	State 1	State 2
$\mu_c^\top$	1.647	2.798
$\mu_d^\top$	-12.075	13.868
$(\omega_c^\top)^{\frac{1}{2}}$	2.669	1.587
$(\omega_d^\top)^{\frac{1}{2}}$	16.976	19.369
$\rho^\top$	0.003	0.003
	$P^\top$	
State 1	0.687	0.313
State 2	0.301	0.699
$\Pi^\top$	0.490	0.510

Table 2.3: **Asset Pricing Implications of the Two-State Markov Switching Model.** The entries are model population values of asset prices. The price-consumption ratio is given in 2.91 and the price-dividend ratio in 2.112. The input parameters for the model are given in Table 2.2. The expressions $E[R_m - R_f]$ and $E[R_f]$ are respectively the annualized equity premium and mean risk-free rate. The expressions $\sigma(R_m)$, $\sigma(R_f)$, $\sigma\left(\frac{C}{P}\right)$ and $\sigma\left(\frac{D}{P}\right)$ are respectively the annualized volatilities of market return, risk-free rate, consumption-price ratio and dividend-price ratio. The subjective factor of discount δ is set to 0.98.

γ	ψ	$E[R_m - R_f]$	$E[R_f]$	$\sigma(R_m)$	$\sigma(R_f)$	$E\left[\frac{P}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P}\right)$	$\sigma\left(\frac{D}{P}\right)$
10.0	0.5	1.38	6.04	26.90	1.56	24.52	34.79	0.044	0.553
10.0	0.7	1.56	4.80	27.25	1.14	34.38	57.20	0.014	0.350
10.0	1.3	1.79	3.38	27.69	0.66	63.41	210.88	0.004	0.100
10.0	1.5	1.83	3.16	27.77	0.59	72.91	359.03	0.005	0.059
20.0	0.5	3.42	5.37	26.58	1.93	26.67	23.21	0.047	0.790
20.0	0.7	3.73	4.23	26.96	1.43	36.12	29.39	0.015	0.653
20.0	1.3	4.12	2.93	27.44	0.86	60.53	41.98	0.005	0.481
20.0	1.5	4.19	2.73	27.52	0.78	67.57	44.94	0.006	0.453
30.0	0.5	5.50	4.74	26.09	2.31	29.52	17.04	0.048	0.996
30.0	0.7	5.94	3.68	26.45	1.72	38.26	19.35	0.016	0.924
30.0	1.3	6.48	2.47	26.93	1.06	57.63	22.83	0.006	0.829
30.0	1.5	6.57	2.28	27.01	0.96	62.51	23.48	0.008	0.813
40.0	0.5	7.21	4.13	25.49	2.70	33.39	14.02	0.047	1.090
40.0	0.7	7.77	3.14	25.83	2.02	40.88	15.11	0.016	1.074
40.0	1.3	8.44	2.01	26.28	1.25	54.81	16.56	0.007	1.045
40.0	1.5	8.55	1.84	26.35	1.13	57.85	16.80	0.009	1.040

It is interesting to note that the expected price-dividend ratio takes very large values when γ is 10 and ψ is greater than one. These values reflect a lack of convergence. The matrix $[Id - \delta A_* (\mu_{cd} + \omega_{cd}/2)]$ in (2.112) becomes nearly singular and this inflates the value of the price-dividend ratio.

The value of the volatility of the dividend-price ratio appears to be very high for all preference parameter pairs and it increases with the risk aversion.

We compute the R^2 of the regressions of multiperiod future returns on the current dividend-price or consumption-price ratio but we do not find any significant predictability at any horizon for any pair of preference parameters. Neither can this model reproduce the negative autocorrelation observed in both returns and excess returns as reported in Table 2.1.

In the next two sections we will look at two models that have been proposed recently by Bansal and Yaron (2004) and Lettau, Ludvigson and Wachter (2005) to advocate the determining role of economic uncertainty (volatility of consumption) in the formation of asset prices. The latter model uses a Markov-switching endowment process and Epstein-Zin preferences. It is therefore a direct application of our framework and we will be able to compute directly all quantities of interest analytically. In the former model, the endowment follows an autoregressive process, but the preferences are also based on Epstein and Zin (1989). We will see how to set this model in our framework by matching the autoregressive endowment process with a Markov-switching process.

2.5.2 The Lettau, Ludvigson and Wachter (2005) Model

The endowment process in Lettau, Ludvigson and Wachter (2005) is a constrained version of the general process (2.1), (2.2). They assume a consumption process (2.1) where the mean and the variance are governed by two different Markov chains. For the dividend process they simply assume that $D_t = (C_t)^{\lambda}$. Therefore the mean and the standard deviation of dividend growth is simply λ times the mean and the standard deviation of consumption growth, and the correlation parameter is one. We report in table 2.4 the corresponding values of the resulting four-state Markov chain based on the estimates reported in their paper.

Table 2.4: **Parameters of the Four-State Quarterly Markov-Switching Model of Lettau, Ludvigson and Wachter (2006).**

In this table, we report the parameters of the Markov-Switching Model (2.1), (2.2) with $N = 4$, constructed using estimates reported in Lettau, Ludvigson and Wachter (2006). μ_c and μ_d are conditional means of consumption and dividend, ω_c and ω_d are conditional variances of consumption and dividend. ρ is the conditional correlation between consumption and dividend shocks. $P^\top$ is the transition matrix across different regimes and Π is the vector of unconditional probabilities of regimes. Means and standard deviations are in percent.

	State 1	State 2	State 3	State 4
$\mu_c^\top$	0.62	0.62	-0.32	-0.32
$\mu_d^\top$	2.80	2.80	-1.45	-1.45
$(\omega_c^\top)^{\frac{1}{2}}$	0.75	0.40	0.75	0.40
$(\omega_d^\top)^{\frac{1}{2}}$	3.36	1.82	3.36	1.82
		$P^\top$		
State 1	0.960	0.006	0.034	0.000
State 2	0.009	0.957	0.000	0.034
State 3	0.205	0.001	0.789	0.005
State 4	0.002	0.204	0.007	0.787
$\Pi^\top$	0.515	0.343	0.085	0.057

In their model, they assume that investors do not know the state they are in but they know the parameters of the process. Therefore at each period they update their estimate of the probability of being in a state given their current information. In other words they compute filtered probabilities. Based on the latter, they compute numerically the price-consumption and price-dividend ratios that are solutions of the Euler conditions of the equilibrium model. We have seen that given the parameters of the endowment process we could calculate the price-consumption ratio by solving a nonlinear equation and the price-dividend ratio analytically in the Epstein-Zin model. Following this procedure at each point in time by using the filtered probabilities for the Markov chain along the way, we can reproduce easily the full trajectory of the price-dividend ratio. We also intend to carry out this exercise in future research following this chapter. Instead we will assume that investors know the state and compute the various statistics corresponding to the stylized facts we presented earlier.

Since Lettau, Ludvigson and Wachter (2005) focused on the trajectory of the price-dividend ratio and its relationship with consumption volatility, they did not report the values for these statistics and the sensitivity of the various quantities to the values of preference parameters. We include a large set of preference parameters to see how the various economic and financial quantities change as a function of preference parameters.

2.5.2.1 Asset Pricing Implications for the LLW Model

We report in table 2.5 the values of the first two moments of the equity premium and the risk-free rate, as well as the means of the price-dividend and the price-consumption ratios and the standard deviations of the consumption-price and the dividend-price ratios. We have limited the risk aversion parameter γ to this range of values because for values below 15 we obtain negative prices for large ψ values and for values above 30 we start having problems solving the nonlinear system for the price-consumption ratios.

Several comments can be made. While the equity premium can be matched with a risk aversion of 25 to 30, the risk-free rate remains high. Negative prices appear with a γ of 15 and even at 20 convergence problems occur. The expected value of the price-dividend ratio takes very large values. At around a maximum of 11 percent, the volatility of the equity premium is low compared to the data, but the volatility of the risk-free rate matches well the actual value. A comparison with the previous two-state model is instructive. While a higher risk aversion is needed to increase the equity premium it matches better the level of the risk-free rate and the volatility of the equity premium and produces less convergence problems at similar levels of risk aversion. The key parameters to understand these differences are the mean and volatility of dividend growth. Limited at 13.5 percent in the high-volatility state (a direct result of setting λ to 4.5), the volatility is much lower than the 20 percent estimated with the dividend data. Moreover it falls at around 7 percent in the low volatility state. In the two-state model it remained at 16 percent. For the mean, it is the same multiple of the mean of consumption growth in low and high states. This does not seem to be coherent with the data, especially in the high mean state. This state is the most frequently visited with an overall probability of 86 percent. We will come back to these remarks later when we analyze the Bansal and Yaron (2004) model. It will be also a four-state model but the parameters of the dividend

Table 2.5: **Asset Pricing Implications: LLW.**

The entries are model population values of asset prices. The input parameters of the MS model are given in Table 2.4. The expressions $E[R^e]$ and $E[R_f]$ are respectively the annualized equity premium and mean risk-free rate. The expressions $\sigma(R)$, $\sigma(R_f)$, $\sigma\left(\frac{C}{P_M}\right)$ and $\sigma\left(\frac{D}{P}\right)$ are respectively the annualized volatilities of equity return, risk-free rate, consumption-price ratio and dividend-price ratio. The quarterly subjective factor of discount is set to 0.9925.

γ	ψ	$E[R^e]$	$E[R_f]$	$\sigma(R)$	$\sigma(R_f)$	$E\left[\frac{P_M}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P_M}\right)$	$\sigma\left(\frac{D}{P}\right)$
15	0.5	2.74	6.93	8.42	1.27	21.77	243.31	0.036	0.013
15	0.6	3.01	6.22	8.96	1.07	24.59	-1428.28	0.022	0.003
15	0.7	3.21	5.70	9.38	0.92	27.08	-243.94	0.013	0.016
15	1.3	3.83	4.26	10.69	0.53	37.54	-75.01	0.005	0.067
15	1.4	3.89	4.14	10.81	0.50	38.78	-70.99	0.006	0.072
15	1.5	3.94	4.03	10.91	0.47	39.92	-67.84	0.007	0.077
20	0.5	4.11	6.99	8.69	1.26	22.73	55.92	0.038	0.062
20	0.6	4.51	6.23	9.25	1.07	25.40	72.17	0.023	0.054
20	0.7	4.81	5.69	9.69	0.92	27.72	90.73	0.014	0.046
20	1.3	5.70	4.14	11.00	0.53	36.89	296.39	0.006	0.017
20	1.4	5.78	4.01	11.12	0.50	37.93	363.52	0.007	0.014
20	1.5	5.85	3.90	11.22	0.47	38.87	451.93	0.008	0.012
25	0.5	5.57	7.11	8.71	1.28	23.93	30.16	0.038	0.116
25	0.6	6.11	6.29	9.27	1.08	26.40	33.49	0.023	0.116
25	0.7	6.50	5.69	9.69	0.93	28.47	36.30	0.014	0.115
25	1.3	7.68	4.01	10.96	0.53	36.20	46.92	0.006	0.106
25	1.4	7.78	3.87	11.07	0.50	37.02	48.06	0.008	0.105
25	1.5	7.87	3.75	11.17	0.47	37.77	49.10	0.009	0.104
30	0.5	6.92	7.27	8.49	1.32	25.38	20.77	0.038	0.163
30	0.6	7.60	6.37	9.01	1.10	27.56	22.00	0.024	0.170
30	0.7	8.09	5.72	9.40	0.95	29.32	22.95	0.015	0.174
30	1.3	9.54	3.87	10.60	0.53	35.49	25.99	0.007	0.182
30	1.4	9.67	3.72	10.70	0.49	36.11	26.28	0.008	0.182
30	1.5	9.78	3.58	10.80	0.46	36.67	26.53	0.009	0.183

process will be based on the data.

2.5.2.2 Asset Returns and Consumption Volatility Predictability in the LLW Model

We report the R^2 values of the regression of future returns on the current consumption-price ratio in table 2.6 and the same regression on the current dividend-price ratio in table 2.7. Before we compare the results with the data, it is important to emphasize that the statistics we compute in a quarterly model is the predictability of future returns at several horizons (in the table we report 1 to 20 years) based on the current quarterly price-dividend ratio, that is computed with the dividend of the current quarter. In the regressions we carried out in the data and reported in table 2.1, the independent variable was a price-dividend ratio with dividends cumulated over a year. This adds persistence to the regressor and increases the R^2 of the regression. However this difference will not affect our ability to detect the ability of a model to generate predictability. To say it in a few words, the models do not seem to produce predictability at any horizon for any parameter configuration.

It is not the case with excess returns. We also report the R^2 values of the regression of future excess returns on the current consumption-price ratio in table 2.6 and the same regression on the current dividend-price ratio in table 2.7. Even if it is not very high, there is a non-negligible predictability, which increase with risk aversion. The fact that dividends are perfectly correlated with consumption plays certainly a role in the higher predictability for excess returns than for returns.

The other important predictability concerns the volatility of consumption, which plays a key role in explaining asset prices in both Lettau, Ludvigson and Wachter (2005) and Bansal and Yaron (2004). We also report the R^2 values of the regression of future consumption volatilities on the current consumption-price ratio in table 2.6 and the same regression on the current dividend-price ratio in table 2.7. As expected in this model, consumption volatility is highly predictable since both regressors depend only on the consumption states. It is more predictable by the dividend-price ratio since there is more variability in this ratio than in the consumption-price ratio.

Table 2.6: **Predictability by the Consumption-Price Ratio: LLW**

This table shows the R-squared of the regression $y_{t+1:t+h} = a_2(h) + b_2(h)\frac{C_t}{P_{M,t}} + \eta_{2,t+h}(h)$, where y is return, excess return, consumption volatility, consumption growth or dividend growth. The horizon h is quaterly in the regression and converted into annual in the table. The input parameters of the MS model are given in Table 2.4. The quarterly subjective factor of discount is set to 0.9925.

							h									
γ	ψ	Returns			Excess			Volatility			Consumption			Dividend		
		1	3	5	1	3	5	1	3	5	1	3	5	1	3	5
15	0.5	0.97	0.46	0.21	1.06	1.14	1.03	9.90	9.16	8.48	26.13	13.52	7.96	26.13	13.52	7.96
15	0.6	0.28	0.11	0.03	1.22	1.28	1.14	10.58	9.78	9.06	25.93	13.41	7.89	25.93	13.41	7.89
15	0.7	0.05	0.01	0.00	1.32	1.38	1.22	11.11	10.28	9.52	25.76	13.33	7.84	25.76	13.33	7.84
15	1.3	0.30	0.35	0.33	1.61	1.66	1.43	12.85	11.89	11.01	25.22	13.05	7.68	25.22	13.05	7.68
15	1.4	0.37	0.41	0.39	1.63	1.68	1.45	13.02	12.04	11.15	25.17	13.02	7.67	25.17	13.02	7.67
15	1.5	0.43	0.47	0.43	1.65	1.70	1.46	13.17	12.18	11.27	25.13	13.00	7.65	25.13	13.00	7.65
20	0.5	0.40	0.12	0.02	1.76	2.08	2.02	15.89	14.70	13.60	24.28	12.56	7.40	24.28	12.56	7.40
20	0.6	0.04	0.00	0.02	1.95	2.26	2.15	16.70	15.45	14.30	24.03	12.43	7.32	24.03	12.43	7.32
20	0.7	0.01	0.07	0.12	2.07	2.38	2.24	17.33	16.02	14.83	23.84	12.34	7.26	23.84	12.34	7.26
20	1.3	0.70	0.83	0.81	2.37	2.68	2.47	19.27	17.82	16.49	23.24	12.03	7.08	23.24	12.03	7.08
20	1.4	0.79	0.92	0.89	2.40	2.71	2.49	19.44	17.98	16.65	23.19	12.00	7.06	23.19	12.00	7.06
20	1.5	0.88	1.01	0.96	2.42	2.72	2.50	19.60	18.13	16.78	23.14	11.97	7.05	23.14	11.97	7.05
25	0.5	0.24	0.03	0.00	2.20	2.89	3.00	22.11	20.45	18.93	22.36	11.57	6.81	22.36	11.57	6.81
25	0.6	0.00	0.03	0.11	2.35	3.03	3.10	22.88	21.16	19.59	22.13	11.45	6.74	22.13	11.45	6.74
25	0.7	0.05	0.18	0.29	2.45	3.12	3.15	23.45	21.68	20.07	21.95	11.36	6.68	21.95	11.36	6.68
25	1.3	0.87	1.13	1.18	2.69	3.32	3.26	25.14	23.25	21.53	21.43	11.09	6.53	21.43	11.09	6.53
25	1.4	0.97	1.24	1.27	2.70	3.33	3.27	25.29	23.39	21.65	21.38	11.06	6.51	21.38	11.06	6.51
25	1.5	1.06	1.34	1.36	2.72	3.34	3.27	25.42	23.51	21.77	21.34	11.04	6.50	21.34	11.04	6.50
30	0.5	0.29	0.02	0.01	2.23	3.34	3.76	28.21	26.09	24.15	20.48	10.59	6.24	20.48	10.59	6.24
30	0.6	0.02	0.04	0.16	2.32	3.38	3.73	28.77	26.61	24.63	20.31	10.51	6.18	20.31	10.51	6.18
30	0.7	0.03	0.18	0.36	2.37	3.39	3.70	29.18	26.99	24.99	20.18	10.44	6.15	20.18	10.44	6.15
30	1.3	0.74	1.12	1.28	2.48	3.39	3.58	30.36	28.08	25.99	19.82	10.25	6.03	19.82	10.25	6.03
30	1.4	0.83	1.23	1.38	2.49	3.38	3.57	30.46	28.17	26.08	19.79	10.24	6.03	19.79	10.24	6.03
30	1.5	0.92	1.32	1.47	2.49	3.38	3.56	30.54	28.25	26.15	19.76	10.22	6.02	19.76	10.22	6.02

Table 2.7: **Predictability by the Dividend-Price Ratio: LLW.**
This table shows the R-squared of the regression $y_{t+1:t+h} = a_2(h) + b_2(h)\frac{D_t}{P_t} + \eta_{2,t+h}(h)$, where y is return, excess return, consumption volatility, consumption growth or dividend growth. The horizon h is quarterly in the regression and converted into annual in the table. The input parameters of the MS model are given in Table 2.4. The quarterly subjective factor of discount is set to 0.9925.

		h														
γ	ψ	Returns			Excess			Volatility			Consumption			Dividend		
		1	3	5	1	3	5	1	3	5	1	3	5	1	3	5
15	0.5	0.38	0.08	0.00	1.15	1.64	1.80	41.99	38.83	35.95	16.20	8.38	4.93	16.20	8.38	4.93
15	0.6	0.07	0.00	0.03	1.23	1.70	1.83	42.54	39.34	36.42	16.02	8.29	4.88	16.02	8.29	4.88
15	0.7	0.00	0.04	0.11	1.28	1.73	1.84	43.30	40.05	37.07	15.77	8.16	4.80	15.77	8.16	4.80
15	1.3	0.34	0.52	0.61	1.38	1.81	1.87	46.92	43.39	40.17	14.63	7.57	4.45	14.63	7.57	4.45
15	1.4	0.39	0.58	0.66	1.39	1.81	1.87	47.32	43.76	40.51	14.50	7.50	4.42	14.50	7.50	4.42
15	1.5	0.44	0.64	0.71	1.39	1.82	1.87	47.67	44.09	40.82	14.39	7.44	4.38	14.39	7.44	4.38
20	0.5	0.10	0.00	0.05	1.81	2.72	3.08	45.26	41.86	38.75	15.18	7.85	4.62	15.18	7.85	4.62
20	0.6	0.00	0.08	0.20	1.90	2.76	3.04	44.57	41.22	38.16	15.39	7.96	4.69	15.39	7.96	4.69
20	0.7	0.06	0.23	0.38	1.97	2.78	3.02	44.33	41.00	37.96	15.45	7.99	4.71	15.45	7.99	4.71
20	1.3	0.70	1.01	1.12	2.11	2.84	2.97	44.60	41.25	38.19	15.34	7.94	4.67	15.34	7.94	4.67
20	1.4	0.77	1.09	1.20	2.12	2.85	2.97	44.68	41.32	38.25	15.32	7.92	4.66	15.32	7.92	4.66
20	1.5	0.84	1.17	1.27	2.13	2.85	2.96	44.75	41.38	38.31	15.29	7.91	4.66	15.29	7.91	4.66
25	0.5	0.04	0.02	0.15	2.27	3.63	4.23	46.92	43.39	40.17	14.67	7.59	4.47	14.67	7.59	4.47
25	0.6	0.01	0.18	0.37	2.34	3.59	4.08	45.29	41.88	38.77	15.16	7.85	4.62	15.16	7.85	4.62
25	0.7	0.12	0.38	0.59	2.39	3.56	3.98	44.36	41.02	37.98	15.45	7.99	4.70	15.45	7.99	4.70
25	1.3	0.88	1.31	1.48	2.52	3.52	3.76	42.56	39.36	36.43	15.98	8.27	4.87	15.98	8.27	4.87
25	1.4	0.97	1.41	1.57	2.52	3.51	3.74	42.45	39.26	36.34	16.01	8.29	4.88	16.01	8.29	4.88
25	1.5	1.05	1.50	1.65	2.53	3.51	3.73	42.36	39.17	36.26	16.04	8.30	4.88	16.04	8.30	4.88
30	0.5	0.07	0.03	0.20	2.40	4.19	5.13	49.49	45.77	42.37	13.88	7.18	4.23	13.88	7.18	4.23
30	0.6	0.00	0.18	0.42	2.40	4.01	4.79	47.13	43.59	40.35	14.60	7.55	4.45	14.60	7.55	4.45
30	0.7	0.08	0.37	0.65	2.41	3.89	4.57	45.71	42.27	39.13	15.03	7.78	4.58	15.03	7.78	4.58
30	1.3	0.78	1.29	1.55	2.43	3.63	4.06	42.60	39.40	36.47	15.98	8.27	4.87	15.98	8.27	4.87
30	1.4	0.86	1.39	1.64	2.43	3.61	4.02	42.39	39.20	36.29	16.04	8.30	4.88	16.04	8.30	4.88
30	1.5	0.94	1.48	1.73	2.43	3.59	3.99	42.21	39.04	36.14	16.09	8.33	4.90	16.09	8.33	4.90

Table 2.8: **Variance Ratios of Aggregate Returns: LLW**

This table shows the variance ratios $\frac{Var(R_{t+1:t+h}}{hVar(R_{t+1})}$ and $\frac{Var(R^e_{t+1:t+h}}{hVar(R^e_{t+1})}$, where the horizon h is quarterly and converted into annual in the table. The price-consumption ratio is given by (2.91) and the price-dividend ratio by (2.112). The input parameters for the model (2.1)-(2.2) are given in table 2.4. The quarterly subjective factor of discount is set to 0.9925.

		h									
γ	ψ	Returns					Excess				
		1	2	3	4	5	1	2	3	4	5
15	0.5	1.118	1.206	1.255	1.284	1.303	0.937	0.890	0.863	0.847	0.837
15	0.6	1.068	1.118	1.146	1.162	1.173	0.927	0.872	0.842	0.823	0.811
15	0.7	1.035	1.061	1.075	1.083	1.088	0.921	0.861	0.827	0.807	0.793
15	1.3	0.960	0.928	0.911	0.899	0.891	0.905	0.833	0.792	0.767	0.750
15	1.4	0.954	0.919	0.899	0.886	0.878	0.904	0.831	0.789	0.764	0.747
15	1.5	0.950	0.911	0.889	0.875	0.866	0.903	0.829	0.787	0.761	0.744
20	0.5	1.090	1.157	1.195	1.217	1.231	0.923	0.865	0.833	0.814	0.802
20	0.6	1.042	1.073	1.089	1.099	1.105	0.912	0.846	0.810	0.788	0.773
20	0.7	1.010	1.018	1.022	1.023	1.024	0.905	0.834	0.794	0.770	0.754
20	1.3	0.939	0.892	0.865	0.849	0.838	0.889	0.804	0.756	0.727	0.709
20	1.4	0.934	0.883	0.854	0.837	0.825	0.887	0.802	0.754	0.724	0.705
20	1.5	0.929	0.876	0.845	0.826	0.814	0.886	0.800	0.751	0.722	0.702
25	0.5	1.086	1.150	1.186	1.207	1.221	0.921	0.863	0.831	0.813	0.802
25	0.6	1.038	1.066	1.081	1.090	1.096	0.910	0.843	0.807	0.785	0.772
25	0.7	1.007	1.012	1.014	1.015	1.016	0.903	0.831	0.791	0.767	0.752
25	1.3	0.935	0.886	0.858	0.841	0.830	0.886	0.800	0.752	0.723	0.705
25	1.4	0.930	0.877	0.847	0.829	0.817	0.885	0.798	0.749	0.720	0.701
25	1.5	0.926	0.869	0.838	0.818	0.805	0.884	0.796	0.747	0.718	0.698
30	0.5	1.107	1.187	1.232	1.259	1.277	0.933	0.884	0.859	0.845	0.838
30	0.6	1.057	1.100	1.124	1.138	1.148	0.922	0.865	0.835	0.817	0.807
30	0.7	1.025	1.043	1.054	1.060	1.064	0.915	0.853	0.819	0.799	0.787
30	1.3	0.949	0.911	0.889	0.876	0.868	0.899	0.823	0.781	0.756	0.740
30	1.4	0.944	0.901	0.877	0.863	0.853	0.898	0.820	0.778	0.753	0.737
30	1.5	0.939	0.893	0.867	0.852	0.842	0.897	0.819	0.776	0.750	0.734

2.5.2.3 Variance Ratios in the LLW Model

The last point we analyzed is the capacity of the models to produce the negative autocorrelation at long horizons. The variance ratios of returns and excess returns on the stock are reported in table 2.8. When ψ is greater than one, the models are able to produce variance ratios less than one, declining with the horizon, for both returns and excess returns. For excess returns there is negative autocorrelation even for values of ψ less than one, but it is more pronounced above one.

2.5.3 Reproducing the Bansal and Yaron (2004) Model with a Markov-Switching Model

The model of Bansal and Yaron (2004) for the endowment is:

$$x_{t+1} = (1-\rho_x)\mu_x + \rho_x x_t + \varphi_e\sqrt{h_t}e_{t+1} \quad (2.127)$$

$$h_{t+1} = (1-\nu_1)\sigma^2 + \nu_1 h_t + \sigma_w w_{t+1} \quad (2.128)$$

$$x_{c,t+1} = x_t + \sqrt{h_t}\eta_{t+1} \quad (2.129)$$

$$x_{d,t+1} = \mu_{xd} + \phi(x_t - \mu_x) + \varphi_d\sqrt{h_t}u_{t+1} \quad (2.130)$$

with $e_{t+1}, w_{t+1}, \eta_{t+1}, u_{t+1} \sim N.i.i.D.(0,1)$.

Our goal here is to characterize a Markov Switching (MS) model described in Section 2 that has the same features than the endowment model chosen by Bansal and Yaron (2004). The main characteristics of the later endowments are: 1) The expected means of the consumption and dividend growth rates are a linear function of the same autoregressive process of order one denoted x_t. 2) The conditional variances of the consumption and dividend growth rates are a linear function of the same autoregressive process of order one denoted h_t. 3) The variables x_{t+1} and h_{t+1} are independent conditionally to their past. 4) The innovations of the consumption and dividend growth rates are independent given the state variables.

2.5.3.1 Characterizing the Matching Markov-Switching Model

In the MS case, the first characteristic of Bansal and Yaron (2004) Model implies that one has to assume that the expected means of the consumption and dividend growth rates are a linear function of the same Markov chain with two states given that a two-state Markov chain is an AR(1) process. Likewise, the second one implies that the conditional variances of the consumption and dividend growth rates are a linear function of the same two-state Markov chain. The third characteristic implies that the mean and variance Markov chains should be independent. Consequently, we should assume that the Markov chain described in Section 2 has 4 states, two states for the means and two states for the variances and that the transition matrix P is restricted such as the means and variance states are independent; see Table 4. Finally, the last characteristic implies that the vector ρ defined in (2.3) equals zero.

In the rest of this subsection, our goal is to approximate an AR(1) process, say z_t, like x_t or h_t by a two-state Markov chain. Without loss of generality, we assume that the Markov chain y_t takes the values 0 (first state) and 1 (second state) while the transition matrix P_y is given by

$$P_y^{\top} = \begin{pmatrix} p_{y,11} & 1-p_{y,11} \\ 1-p_{y,22} & p_{y,22} \end{pmatrix}.$$

The stationary distribution is

$$\pi_{y,1} = P(y=0) = \frac{1-p_{y,22}}{2-p_{y,11}-p_{y,22}}, \quad \pi_{y,2} = P(y=1) = \frac{1-p_{y,11}}{2-p_{y,11}-p_{y,22}}. \tag{2.131}$$

In addition, we assume that $z_t = a + by_t$. Without loss of generality, we assume that $b > 0$, that is, the second state corresponds to this high value of z_t. Our goal is to characterize the vector $\theta = (p_{y,11}, p_{y,22}, a, b)^{\top}$ that matches the characteristic of the process z_t. The first characteristics that we want to match are the mean, the variance and the first order autocorrelation of the process z_t denoted μ_z, σ_z^2 and ρ_z respectively. Given that the dimension of θ is four, another restriction is needed. For instance, Mehra and Prescott (1985) assumed $p_{y,11} = p_{y,22}$. In contrast, we will focus on matching the kurtosis of the process z_t denoted κ_z. We will show below that matching the mean, variance, kurtosis and first autocorrelation does not fully identify the parameters. However, knowing the sign of the skewness of z_t (denotes sk_z) and the other four characteristics will fully identify the vector θ.

Proposition 2.5.1. Moments of a two-state Markov chain.
We have

$$\begin{aligned}
\mu_z &= a + b\mu_y = a + b\pi_{y,2} \\
\sigma_z^2 &= b^2\sigma_y^2 = b^2\pi_{y,1}\pi_{y,2} \\
sk_z &= sk_y = \frac{1}{\sqrt{\pi_{y,1}\pi_{y,2}}}\left(-\frac{\pi_{y,2}}{\pi_{y,1}} + \frac{\pi_{y,1}}{\pi_{y,2}}\right) \\
\kappa_z &= \kappa_y = \frac{\pi_{y,1}^2}{\pi_{y,2}} + \frac{\pi_{y,2}^2}{\pi_{y,1}} \\
\rho_z &= \rho_y = p_{y,11} + p_{y,22} - 1
\end{aligned} \tag{2.132}$$

The previous proposition, combined with (2.131), characterizes the moments of a Markov chain in terms of the vector θ. As pointed out above, Mehra and Prescott (1985) assumed that $p_{y,11} = p_{y,22}$, which implies $sk_z = 0$ and $\kappa_z = 1$. The empirical evidence reported in Cecchetti, Lam and Mark (1990) suggests that the kurtosis of the expected consumption growth is higher than one and that its skewness is negative.[6]

We will now invert this characterization, that is, we will determine the vector θ in terms of the moments of z_t:

Proposition 2.5.2. Matching an AR(1) process by a two-state Markov chain.
We have

$$\text{if } sk_z \leq 0,\ p_{y,11} = \frac{1+\rho_z}{2} - \frac{1-\rho_z}{2}\sqrt{\frac{k_z-1}{k_z+3}},\ p_{y,22} = \frac{1+\rho_z}{2} + \frac{1-\rho_z}{2}\sqrt{\frac{k_z-1}{k_z+3}} \tag{2.133}$$

$$\text{if } sk_z > 0,\ p_{y,11} = \frac{1+\rho_z}{2} + \frac{1-\rho_z}{2}\sqrt{\frac{k_z-1}{k_z+3}},\ p_{y,22} = \frac{1+\rho_z}{2} - \frac{1-\rho_z}{2}\sqrt{\frac{k_z-1}{k_z+3}} \tag{2.134}$$

[6] Strictly speaking, the process x here is the expected mean of the consumption growth and not the growth. Therefore, the skewness and kurtosis of these two processes are different but connected.

and

$$b = \frac{\sigma_z}{\sqrt{\pi_{y,1}\pi_{y,2}}},\ a = \mu_z - b\pi_{y,2} \tag{2.135}$$

where $\pi_{y,1}$ and $\pi_{y,2}$ are connected to $p_{y,11}$ and $p_{y,22}$ through (2.131).

We will now characterize the moments of the process x_t and h_t of the Bansal and Yaron (2004) model.

Proposition 2.5.3. Moments of the Bansal and Yaron (2004)Model.
The mean μ_x and the first autocorrelation ρ_x of x_t are given in (2.127). The variance, skewness and kurtosis of x_t are given by

$$\sigma_x^2 = \frac{\varphi_e^2\sigma^2}{1-\rho_x^2},\ sk_x = 0,\ \kappa_x = 3\frac{(1-\rho_x^2)^2}{1-\rho_x^4}\left(1 + 2\frac{\rho_x^2}{1-\rho_x^2}\frac{\nu_1}{\sigma^2} + \frac{\sigma_w^2}{\sigma^2(1-\nu_1^2)}\right). \tag{2.136}$$

Likewise,

$$\mu_h = \sigma^2,\ \sigma_h^2 = \frac{\sigma^2}{1-\nu_1^2},\ sk_h = 0,\ \kappa_h = 3,\ \rho_h = \nu_1. \tag{2.137}$$

Observe that the skewness of the expected mean of the growth consumption equals zero in Bansal and Yaron (2004) model as in Mehra and Prescott (1985) model. In contrast, in order to generate a kurtosis higher than one, the Markov switching needs some skewness. Given that the skewness of consumption growth is empirically negative, we will take this identification assumption, that is, we will use (2.133) to identify the transition probabilities $p_{x,11}$ and $p_{x,22}$.

Likewise, the skewness of the variance process is zero in Bansal and Yaron (2004) model which is somewhat unrealistic given that the variance is a positive random variable. A popular variance model is the Heston (1993) model where the stationary distribution of the variance process is a Gamma distribution. Given that the skewness of a Gamma distribution is positive, we make the same assumption on h_t and we therefore use (2.134) to identify the transition probabilities $p_{h,11}$ and $p_{h,22}$.

We do have now the two independent Markov chains that generate the expected mean and variance of consumption growth. Putting together these two processes leads to a four-state Markov chain (low mean and low variance, low mean and high variance, high mean and low variance, high mean and high variance) whose transition probability matrix is given by

$$P^\top = \begin{bmatrix} p_{x,11}p_{h,11} & p_{x,11}p_{h,12} & p_{x,12}p_{h,11} & p_{x,12}p_{h,12} \\ p_{x,11}p_{h,21} & p_{x,11}p_{h,22} & p_{x,12}p_{h,21} & p_{x,12}p_{h,22} \\ p_{x,21}p_{h,11} & p_{x,21}p_{h,12} & p_{x,22}p_{h,11} & p_{x,11}p_{h,12} \\ p_{x,21}p_{h,21} & p_{x,21}p_{h,22} & p_{x,22}p_{h,21} & p_{x,22}p_{h,22} \end{bmatrix} \tag{2.138}$$

where $p_{\cdot,12} = 1 - p_{\cdot,11}$ and $p_{\cdot,21} = 1 - p_{\cdot,2}$, while the vectors μ_c, ω_c, μ_d, and ω_d defined in (2.1) and (2.2) are given by

$$\begin{aligned} \mu_c &= (a_c, a_c, a_c + b_c, a_c + b_c)^\top \\ \omega_c &= (a_h, a_h + b_h, a_h, a_h + b_h)^\top \\ \mu_d &= (\mu_{xd} - \phi\mu_x)e + \phi\mu_c \\ \omega_d &= \varphi_d^2\, \omega_c. \end{aligned} \tag{2.139}$$

2.5.3.2 Reproducing the Stylized Facts

The parameters of the resulting Markov-switching model are given in Table 2.9. We are now able to reproduce some stylized facts that were considered in Bansal and Yaron (2004), that is the first two moments of the equity premium and the risk-free rate and some statistics about the price-dividend ratio, predictability of returns by the price-dividend ratio, predictability of the variance of consumption by the price-dividend ratio and the ratio of variances.

2.5.3.2.1 Asset Pricing Implications The set of statistics reproduced by Bansal and Yaron (2004) is given in Table 2.10. We present an equivalent table generated with the analytical formulas reported in the previous sections and the parameter values of the matching MS process in Table 2.9. We include a larger spectrum of preference parameters than in Bansal and Yaron (2004) to better understand the variation of economic and financial quantities as a function of preference parameters. To gauge the useful-

Table 2.9: **Parameters of the Markov-Switching Model.**
This table shows parameters of the Markov-Switching Model calibrated to match the model of Bansal and Yaron (2004). Calibration is made such that unconditional variance and kurtosis of MS mean and volatility of consumption matched similar moments in the BY model, whereas the implied skewness of MS mean is negative and that of MS volatility is positive. μ_c and μ_d are conditional means of consumption and dividend, ω_c and ω_d are conditional variances of consumption and dividend. $P^{\top}$ is the transition matrix across different regimes and Π is the vector of unconditional probabilities of regimes. Means and standard deviations are in percent. The model is calibrated at the monthly frequency. The correlation vector ρ is set to zero.

	State 1	State 2	State 3	State 4
$\mu_c^{\top}$	-0.181	-0.181	0.236	0.236
$\mu_d^{\top}$	-0.843	-0.843	0.407	0.407
$(\omega_c^{\top})^{\frac{1}{2}}$	0.731	0.941	0.731	0.941
$(\omega_d^{\top})^{\frac{1}{2}}$	3.289	4.233	3.289	4.233
		$P^{\top}$		
State 1	0.981	0.003	0.017	0.000
State 2	0.010	0.973	0.000	0.017
State 3	0.004	0.000	0.993	0.003
State 4	0.000	0.004	0.010	0.985
$\Pi^{\top}$	0.162	0.043	0.627	0.168

Table 2.10: **Asset Pricing Implications: Table IV of Bansal and Yaron (2004)**
The entries are model population values of asset prices. The expressions $E[r_m - r_f]$ and $E[r_f]$ are respectively the annualized equity premium and mean risk-free rate. The expressions $\sigma(r_m)$, $\sigma(r_f)$, and $\sigma(p-d)$ are respectively the annualized volatilities of market return, risk-free rate and price-dividend ratio. The monthly subjective factor of discount is set to 0.998 and the elasticity of intertemporal substitution to $\psi = 1.5$.

	Data		Model	
Variable	Estimate	Std.dev.	$\gamma = 7.5$	$\gamma = 10$
		Returns		
$E[r_m - r_f]$	6.33	(2.15)	4.01	6.84
$E[r_f]$	0.86	(0.42)	1.44	0.93
$\sigma[r_m]$	19.42	(3.07)	17.81	18.65
$\sigma[r_f]$	0.97	(0.28)	0.44	0.57
		Price-Dividend Ratio		
$E[\exp(p-d)]$	26.56	(2.53)	25.02	19.98
$\sigma[p-d]$	0.29	(0.04)	0.18	0.21
$AC1[p-d]$	0.81	(0.09)	0.80	0.82
$AC2[p-d]$	0.64	(0.15)	0.65	0.67

ness of analytical formulas it is essential to remember that in the case of the Bansal and Yaron's model, finding these quantities means either solving the model numerically for each configuration of the preference parameters or computing these quantities by simulation. Numerical solutions take time to achieve a reasonable degree of precision. For simulations, long trajectories are needed to obtain population parameters. Determining which length is appropriate is not a trivial issue, especially when coupled with time considerations.

Table 2.11 is based on the value of 0.998 chosen by Bansal and Yaron (2004) for the time discount parameter. We observe that the values for the first two moments of the equity premium are close to the values found by Bansal and Yaron with their model reported in Table 2.10, but the average risk-free rate is higher. Several interesting observations can be made from this table. First and foremost, the table shows clearly that it is through values greater than 1 for the ψ parameter that the equity premium puzzle is solved. The expected value of the equity return is about equal (around 9%) at $\gamma = 10$ for all values of ψ. However, the risk-free rate drops five points of percentage when ψ goes from 0.5 to 1.5. At a low risk aversion, the magnitude of the drop is less pronounced. In fact, at $\gamma = 10$ the expected value of the price-consumption ratio decreases in a significant way. A second observation concerns the price-dividend ratio. At low values of the risk aversion parameter γ the expectation of the price-dividend ratio increases significantly with the elasticity of intertemporal substitution ψ, while the volatility of the price-dividend does not change much. At low values of the risk aversion parameter γ it is exactly the opposite.

Thanks to analytical formulas it is immediate to reproduce the same table for a slightly larger δ of 0.999. The results are presented in Table 2.12. Again several instructive conclusions can be drawn. Looking only at the moments, one does not see much difference with the previous table, except maybe for the fact that the expected risk-free rate decreases, which is an expected result. However a look at the left side of the table shows that the expected values for the price-consumption ratio and the price-dividend change drastically and take in certain configurations of the preference parameters very large implausible values.

Table 2.11: **Asset Pricing Implications:** $\delta = 0.998$

The entries are model population values of asset prices.The price-consumption ratio is given in 2.91 and the price-dividend ratio in 2.112. The input parameters for the monthly model are given in Table (2.9). The expressions $E[R_m - R_f]$ and $E[R_f]$ are respectively the annualized equity premium and mean risk-free rate. The expressions $\sigma(R_m)$, $\sigma(R_f)$, $\sigma\left(\frac{C}{P}\right)$ and $\sigma\left(\frac{D}{P}\right)$ are respectively the annualized volatilities of market return, risk-free rate, consumption-price ratio and dividend-price ratio. The monthly subjective factor of discount is set to 0.998.

γ	ψ	$E[R_m - R_f]$	$E[R_f]$	$\sigma(R_m)$	$\sigma(R_f)$	$E\left[\frac{P}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P}\right)$	$\sigma\left(\frac{D}{P}\right)$
2.5	0.2	1.10	10.05	15.95	3.17	12.23	12.58	0.544	0.283
2.5	0.5	0.13	5.89	13.32	1.16	25.48	30.11	0.080	0.074
2.5	0.8	0.65	4.55	15.62	0.72	35.91	46.28	0.015	0.093
2.5	1	0.92	4.06	16.72	0.58	41.58	56.09	0.000	0.091
2.5	1.2	1.13	3.73	17.55	0.49	46.46	65.15	0.008	0.087
2.5	1.5	1.37	3.38	18.47	0.40	52.61	77.52	0.015	0.081
5	0.2	0.00	10.58	16.07	2.92	16.24	13.56	0.459	0.268
5	0.5	1.05	6.11	13.68	1.14	29.89	22.77	0.074	0.115
5	0.8	2.36	4.49	16.16	0.72	37.95	27.08	0.016	0.174
5	1	2.92	3.88	17.27	0.58	41.58	28.82	0.000	0.191
5	1.2	3.33	3.46	18.09	0.49	44.37	30.09	0.009	0.201
5	1.5	3.78	3.02	18.96	0.40	47.51	31.44	0.018	0.210
7.5	0.2	-1.66	12.11	16.60	2.89	30.50	14.00	0.268	0.279
7.5	0.5	1.82	6.60	13.26	1.21	37.51	17.51	0.060	0.122
7.5	0.8	3.83	4.49	15.18	0.75	40.46	18.87	0.015	0.210
7.5	1	4.63	3.70	16.05	0.59	41.58	19.39	0.000	0.239
7.5	1.2	5.19	3.15	16.68	0.48	42.37	19.74	0.010	0.257
7.5	1.5	5.79	2.58	17.36	0.36	43.19	20.11	0.019	0.276
10	0.2	-4.85	15.14	18.08	3.24	33063.74	14.88	0.000	0.307
10	0.5	1.86	7.26	12.73	1.34	48.86	15.42	0.045	0.089
10	0.8	4.62	4.51	14.05	0.78	42.94	15.90	0.013	0.187
10	1	5.65	3.52	14.70	0.59	41.58	16.09	0.000	0.222
10	1.2	6.37	2.83	15.19	0.45	40.80	16.22	0.009	0.245
10	1.5	7.12	2.12	15.73	0.32	40.11	16.36	0.019	0.268

Table 2.12: **Asset Pricing Implications:** $\delta = 0.999$

The entries are model population values of asset prices. The price-consumption ratio is given in 2.91 and the price-dividend ratio in 2.112. The input parameters for the monthly model are given in Table 2.9. The expressions $E[R_m - R_f]$ and $E[R_f]$ are respectively the annualized equity premium and mean risk-free rate. The expressions $\sigma(R_m)$, $\sigma(R_f)$, $\sigma\left(\frac{C}{P}\right)$ and $\sigma\left(\frac{D}{P}\right)$ are respectively the annualized volatilities of market return, risk-free rate, consumption-price ratio and dividend-price ratio. The monthly subjective factor of discount is set to 0.999.

γ	ψ	$E[R_m - R_f]$	$E[R_f]$	$\sigma(R_m)$	$\sigma(R_f)$	$E\left[\frac{P}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P}\right)$	$\sigma\left(\frac{D}{P}\right)$
2.5	0.2	1.19	8.75	16.20	3.18	14.35	14.88	0.479	0.248
2.5	0.5	0.15	4.69	13.40	1.16	36.97	46.98	0.058	0.050
2.5	0.8	0.72	3.35	15.95	0.72	63.48	99.86	0.009	0.046
2.5	1	1.02	2.86	17.17	0.58	83.25	156.54	0.000	0.035
2.5	1.2	1.26	2.52	18.10	0.49	104.88	248.74	0.004	0.024
2.5	1.5	1.53	2.16	19.12	0.40	141.35	589.05	0.006	0.011
5	0.2	0.00	9.37	16.34	2.92	20.89	16.29	0.373	0.232
5	0.5	1.17	4.97	13.80	1.14	48.38	29.78	0.048	0.092
5	0.8	2.62	3.31	16.49	0.72	70.92	36.55	0.009	0.136
5	1	3.24	2.68	17.69	0.58	83.25	39.36	0.000	0.147
5	1.2	3.70	2.23	18.56	0.49	93.86	41.43	0.005	0.153
5	1.5	4.19	1.77	19.51	0.40	107.21	43.66	0.008	0.159
7.5	0.2	-1.87	11.18	17.00	2.90	63.01	16.77	0.136	0.244
7.5	0.5	1.92	5.57	13.24	1.22	75.51	20.89	0.031	0.102
7.5	0.8	4.12	3.33	15.22	0.75	81.12	22.55	0.008	0.177
7.5	1	4.98	2.49	16.11	0.58	83.25	23.17	0.000	0.202
7.5	1.2	5.60	1.91	16.77	0.47	84.75	23.61	0.005	0.218
7.5	1.5	6.25	1.30	17.48	0.36	86.31	24.06	0.010	0.234
10	0.2	-4.73	13.38	18.30	3.18	34090.34	19.68	0.000	0.237
10	0.5	1.81	6.32	12.67	1.37	147.73	18.15	0.015	0.071
10	0.8	4.83	3.37	13.99	0.79	92.11	18.59	0.006	0.157
10	1	5.95	2.31	14.65	0.59	83.25	18.78	0.000	0.188
10	1.2	6.73	1.58	15.15	0.45	78.57	18.91	0.005	0.208
10	1.5	7.53	0.82	15.70	0.31	74.61	19.05	0.011	0.229

Table 2.13: **Asset Pricing Implications: Log-linearization $\delta = 0.999$**
The entries are model population values of asset prices. The price-consumption ratio is given by (2.98) and the price-dividend ratio by (2.118). The input parameters for the monthly model are given in Table 2.9. The coefficient k_1 in the log-linearization (2.97) is set to 0.997 and the coefficient k_{m1} in the log-linearization (2.115) is set to 0.996. The expressions $E[R_m - R_f]$ and $E[R_f]$ are respectively the annualized equity premium and mean risk-free rate. The expressions $\sigma(R_m)$, $\sigma(R_f)$, $\sigma\left(\frac{C}{P}\right)$ and $\sigma\left(\frac{D}{P}\right)$ are respectively the annualized volatilities of market return, risk-free rate, consumption-price ratio and dividend-price ratio. The monthly subjective factor of discount is set to 0.999.

γ	ψ	$E[R_m - R_f]$	$E[R_f]$	$\sigma(R_m)$	$\sigma(R_f)$	$E\left[\frac{P}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P}\right)$	$\sigma\left(\frac{D}{P}\right)$
2.5	0.2	1.90	8.52	16.63	3.24	10.32	14.07	0.724	0.277
2.5	0.5	0.76	4.69	13.23	1.16	35.36	36.28	0.058	0.059
2.5	0.8	1.78	3.34	15.14	0.72	48.56	46.31	0.011	0.085
2.5	1.2	2.54	2.53	16.59	0.49	58.00	53.10	0.006	0.095
2.5	1.5	2.88	2.19	17.25	0.40	62.29	56.09	0.012	0.098
2.5	1.8	3.12	1.96	17.71	0.34	65.32	58.19	0.015	0.100
5	0.2	0.29	9.37	16.56	2.92	19.12	15.67	0.416	0.248
5	0.5	1.29	4.88	13.65	1.14	41.19	29.32	0.053	0.089
5	0.8	2.74	3.26	15.96	0.72	50.44	34.56	0.011	0.132
5	1.2	3.74	2.27	17.67	0.49	56.54	37.94	0.007	0.152
5	1.5	4.19	1.85	18.44	0.40	59.21	39.39	0.013	0.159
5	1.8	4.50	1.57	18.97	0.34	61.05	40.37	0.017	0.164
7.5	0.2	-1.54	10.75	17.06	2.88	37.15	17.06	0.214	0.242
7.5	0.5	1.89	5.26	13.40	1.20	48.63	22.59	0.045	0.103
7.5	0.8	3.91	3.24	15.52	0.74	52.58	24.42	0.011	0.173
7.5	1.2	5.26	1.98	17.14	0.48	55.00	25.53	0.007	0.212
7.5	1.5	5.85	1.44	17.87	0.37	56.02	25.99	0.014	0.227
7.5	1.8	6.25	1.08	18.38	0.30	56.71	26.33	0.019	0.237
10	0.2	-4.05	12.53	17.87	3.09	66.74	19.99	0.114	0.225
10	0.5	2.07	5.75	12.87	1.30	56.37	19.32	0.037	0.084
10	0.8	4.84	3.24	14.44	0.77	54.56	19.30	0.010	0.173
10	1.2	6.63	1.67	15.74	0.46	53.66	19.31	0.007	0.228
10	1.5	7.39	1.00	16.35	0.33	53.32	19.33	0.014	0.251
10	1.8	7.92	0.55	16.77	0.24	53.10	19.34	0.019	0.266

Table 2.14: **Coefficients of the Campbell and Shiller (1988)'s log-linearization.** The entries are model implied coefficients of the Campbell and Shiller (1988)'s log-linearization. The price-consumption ratio is given by (2.98) and the price-dividend ratio by (2.118). The input parameters for the monthly model are given in Table 2.9.

γ	ψ	$\delta = 0.998$				$\delta = 0.999$			
		k_1	k_0	k_{m1}	k_{m0}	k_1	k_0	k_{m1}	k_{m0}
2.5	0.2	0.9930	0.0419	0.9934	0.0399	0.9940	0.0368	0.9944	0.0348
2.5	0.5	0.9967	0.0220	0.9972	0.0191	0.9977	0.0160	0.9982	0.0130
2.5	0.8	0.9977	0.0164	0.9982	0.0133	0.9987	0.0100	0.9992	0.0068
2.5	1	0.9980	0.0144	0.9985	0.0113	0.9990	0.0079	0.9995	0.0046
2.5	1.2	0.9982	0.0131	0.9987	0.0099	0.9992	0.0065	0.9997	0.0031
2.5	1.5	0.9984	0.0118	0.9989	0.0085	0.9994	0.0050	0.9999	0.0014
5	0.2	0.9946	0.0334	0.9938	0.0375	0.9958	0.0272	0.9949	0.0322
5	0.5	0.9972	0.0192	0.9963	0.0242	0.9983	0.0127	0.9972	0.0193
5	0.8	0.9978	0.0156	0.9969	0.0210	0.9988	0.0091	0.9977	0.0163
5	1	0.9980	0.0144	0.9971	0.0200	0.9990	0.0079	0.9979	0.0153
5	1.2	0.9981	0.0136	0.9972	0.0193	0.9991	0.0071	0.9980	0.0147
5	1.5	0.9983	0.0129	0.9973	0.0186	0.9992	0.0063	0.9981	0.0141
7.5	0.2	0.9971	0.0198	0.9940	0.0366	0.9986	0.0107	0.9950	0.0315
7.5	0.5	0.9978	0.0158	0.9953	0.0301	0.9989	0.0086	0.9960	0.0260
7.5	0.8	0.9979	0.0148	0.9956	0.0284	0.9990	0.0081	0.9963	0.0245
7.5	1	0.9980	0.0144	0.9957	0.0278	0.9990	0.0079	0.9964	0.0240
7.5	1.2	0.9980	0.0142	0.9958	0.0275	0.9990	0.0078	0.9964	0.0236
7.5	1.5	0.9981	0.0140	0.9958	0.0271	0.9990	0.0077	0.9965	0.0233
10	0.2	1.0000	0.0000	0.9943	0.0349	1.0000	0.0000	0.9957	0.0277
10	0.5	0.9983	0.0126	0.9946	0.0335	0.9994	0.0048	0.9954	0.0292
10	0.8	0.9981	0.0140	0.9948	0.0327	0.9991	0.0072	0.9955	0.0287
10	1	0.9980	0.0144	0.9948	0.0325	0.9990	0.0079	0.9956	0.0285
10	1.2	0.9980	0.0147	0.9949	0.0323	0.9989	0.0083	0.9956	0.0284
10	1.5	0.9979	0.0149	0.9949	0.0321	0.9989	0.0087	0.9956	0.0282

Table 2.15: **Asset Pricing Implications: Log-linearization with $\delta = 0.999$ and Analytical Coefficients**

The entries are model population values of asset prices. The price-consumption ratio is given by (2.98) and the price-dividend ratio by (2.118). The input parameters for the monthly model are given in Table 2.9. For each combination of preference parameters, the coefficients k_1 and k_0 in the log-linearization (2.97), and the coefficients k_{m1} and k_{m0} in the log-linearization (2.115) are given in Table 2.14. The expressions $E[R^e]$ and $E[R_f]$ are respectively the annualized equity premium and mean risk-free rate. The expressions $\sigma(R)$, $\sigma(R_f)$, $\sigma\left(\frac{C}{P_M}\right)$ and $\sigma\left(\frac{D}{P}\right)$ are respectively the annualized volatilities of equity return, risk-free rate, consumption-price ratio and dividend-price ratio. The monthly subjective factor of discount is set to 0.999.

γ	ψ	$E[R^e]$	$E[R_f]$	$\sigma(R)$	$\sigma(R_f)$	$E\left[\frac{P_M}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P_M}\right)$	$\sigma\left(\frac{D}{P}\right)$
2.5	0.2	1.17	8.84	16.04	3.16	13.73	14.65	0.483	0.246
2.5	0.5	0.15	4.69	13.41	1.16	36.83	46.89	0.058	0.050
2.5	0.8	0.73	3.35	15.97	0.72	63.46	98.99	0.009	0.046
2.5	1.2	1.27	2.52	18.12	0.49	104.87	244.58	0.004	0.025
2.5	1.5	1.54	2.16	19.13	0.40	141.27	575.37	0.006	0.012
2.5	1.8	1.73	1.92	19.86	0.34	183.42	5256.19	0.006	0.001
5	0.2	0.06	9.37	16.19	2.92	19.20	16.08	0.392	0.230
5	0.5	1.18	4.96	13.82	1.14	48.00	29.73	0.048	0.093
5	0.8	2.69	3.30	16.60	0.72	70.88	35.90	0.009	0.140
5	1.2	3.81	2.23	18.75	0.49	93.83	40.14	0.005	0.161
5	1.5	4.33	1.77	19.74	0.40	107.09	42.04	0.008	0.169
5	1.8	4.69	1.45	20.43	0.33	117.96	43.37	0.010	0.173
7.5	0.2	-1.73	11.11	16.79	2.90	54.33	16.47	0.156	0.243
7.5	0.5	1.95	5.55	13.26	1.22	74.69	20.83	0.032	0.103
7.5	0.8	4.22	3.33	15.31	0.75	81.06	22.13	0.008	0.184
7.5	1.2	5.77	1.91	16.95	0.47	84.72	22.85	0.005	0.231
7.5	1.5	6.46	1.30	17.69	0.35	86.20	23.14	0.010	0.250
7.5	1.8	6.93	0.88	18.22	0.28	87.19	23.33	0.013	0.262
10	0.2	-4.99	13.67	19.20	3.26	∞	20.15	0.000	0.249
10	0.5	1.83	6.31	12.68	1.37	146.07	18.12	0.015	0.072
10	0.8	4.91	3.37	14.03	0.79	92.04	18.36	0.006	0.162
10	1.2	6.87	1.58	15.24	0.45	78.54	18.47	0.005	0.217
10	1.5	7.71	0.82	15.80	0.30	74.52	18.50	0.011	0.240
10	1.8	8.29	0.31	16.21	0.21	72.15	18.53	0.015	0.256

Table 2.16: **Asset Pricing Implications:** $\delta = 0.998$

The entries are model population values of asset prices.The price-consumption ratio is given in (2.91) and the price-dividend ratio in (2.112). The input parameters for the monthly model are given in Table (2.9). The expressions $E[R^e]$ and $E[R_f]$ are respectively the annualized equity premium and mean risk-free rate. The expressions $\sigma(R)$, $\sigma(R_f)$, $\sigma\left(\frac{C}{P}\right)$ and $\sigma\left(\frac{D}{P}\right)$ are respectively the annualized volatilities of market return, risk-free rate, consumption-price ratio and dividend-price ratio. The monthly subjective factor of discount is set to 0.998.

γ	ψ	$E[R^e]$	$E[R_f]$	$\sigma(R)$	$\sigma(R_f)$	$E\left[\frac{P}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P}\right)$	$\sigma\left(\frac{D}{P}\right)$
5	0.1	8.08	13.81	45.20	7.01	10.97	8.92	1.423	1.238
5	0.125	3.56	13.05	30.69	5.16	12.07	10.02	1.007	0.815
5	0.2	0.00	10.58	16.07	2.92	16.24	13.56	0.459	0.268
5	0.25	-0.26	9.36	13.36	2.29	19.01	15.67	0.308	0.120
5	0.5	1.05	6.11	13.68	1.14	29.89	22.77	0.074	0.115
5	0.75	2.19	4.68	15.82	0.77	36.86	26.54	0.021	0.168
5	1	2.92	3.88	17.27	0.58	41.58	28.82	0.000	0.191
5	1.25	3.42	3.37	18.26	0.47	44.97	30.35	0.011	0.203
5	1.5	3.78	3.02	18.96	0.40	47.51	31.44	0.018	0.210
5	2	4.26	2.56	19.90	0.30	51.07	32.89	0.025	0.219
5	3	4.77	2.08	20.90	0.21	55.12	34.44	0.031	0.226
5	4	5.04	1.83	21.43	0.16	57.37	35.26	0.034	0.230
10	0.1	0.00	17.04	64.86	5.85	19730.89	27.59	0.001	0.523
10	0.125	-4.20	17.23	38.23	4.73	22836.97	19.05	0.001	0.518
10	0.2	-4.85	15.14	18.08	3.24	33063.89	14.88	0.000	0.307
10	0.25	-3.36	13.05	14.51	2.66	120.56	14.78	0.052	0.190
10	0.5	1.86	7.26	12.73	1.34	48.86	15.42	0.045	0.089
10	0.75	4.29	4.83	13.86	0.85	43.45	15.84	0.017	0.176
10	1	5.65	3.52	14.70	0.59	41.58	16.09	0.000	0.222
10	1.25	6.52	2.69	15.30	0.43	40.66	16.25	0.011	0.249
10	1.5	7.12	2.12	15.73	0.32	40.11	16.36	0.019	0.268
10	2	7.89	1.40	16.30	0.18	39.50	16.51	0.029	0.291
10	3	8.68	0.66	16.91	0.05	38.96	16.66	0.040	0.314
10	4	9.09	0.28	17.23	0.05	38.71	16.74	0.046	0.326

Table 2.17: **Asset Pricing Implications: $\delta = 0.999$**

The entries are model population values of asset prices. The price-consumption ratio is given in (2.91) and the price-dividend ratio in (2.112). The input parameters for the monthly model are given in Table 2.9. The expressions $E[R^e]$ and $E[R_f]$ are respectively the annualized equity premium and mean risk-free rate. The expressions $\sigma(R)$, $\sigma(R_f)$, $\sigma\left(\frac{C}{P}\right)$ and $\sigma\left(\frac{D}{P}\right)$ are respectively the annualized volatilities of market return, risk-free rate, consumption-price ratio and dividend-price ratio. The monthly subjective factor of discount is set to 0.999.

γ	ψ	$E[R^e]$	$E[R_f]$	$\sigma(R)$	$\sigma(R_f)$	$E\left[\frac{P}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P}\right)$	$\sigma\left(\frac{D}{P}\right)$
5	0.1	8.78	12.14	47.64	7.10	12.83	10.11	1.277	1.134
5	0.125	3.88	11.60	32.03	5.19	14.43	11.53	0.881	0.734
5	0.2	0.00	9.37	16.34	2.92	20.89	16.29	0.373	0.232
5	0.25	-0.28	8.20	13.43	2.29	25.61	19.26	0.239	0.102
5	0.5	1.17	4.97	13.80	1.14	48.38	29.78	0.048	0.092
5	0.75	2.43	3.51	16.13	0.77	67.52	35.68	0.012	0.132
5	1	3.24	2.68	17.69	0.58	83.25	39.36	0.000	0.147
5	1.25	3.79	2.14	18.75	0.47	96.28	41.86	0.005	0.155
5	1.5	4.19	1.77	19.51	0.40	107.21	43.66	0.008	0.159
5	2	4.73	1.29	20.52	0.30	124.48	46.08	0.011	0.164
5	3	5.30	0.78	21.59	0.20	147.63	48.71	0.012	0.169
5	4	5.60	0.52	22.16	0.15	162.40	50.11	0.013	0.170
10	0.1	0.00	15.82	71.61	5.84	23003.85	58.87	0.001	0.263
10	0.125	-4.35	15.78	40.74	4.71	24500.36	29.25	0.001	0.355
10	0.2	-4.73	13.38	18.30	3.18	34090.76	19.68	0.000	0.237
10	0.25	-3.49	11.81	14.65	2.65	33799.49	18.54	0.000	0.157
10	0.5	1.81	6.32	12.67	1.37	147.73	18.15	0.015	0.071
10	0.75	4.47	3.72	13.79	0.86	95.70	18.54	0.008	0.147
10	1	5.95	2.31	14.65	0.59	83.25	18.78	0.000	0.188
10	1.25	6.89	1.43	15.26	0.42	77.72	18.94	0.006	0.213
10	1.5	7.53	0.82	15.70	0.31	74.61	19.05	0.011	0.229
10	2	8.37	0.05	16.29	0.16	71.25	19.20	0.017	0.250
10	3	9.23	-0.75	16.92	0.03	68.35	19.36	0.024	0.271
10	4	9.67	-1.15	17.24	0.07	67.05	19.44	0.027	0.281

Table 2.18: **Asset Pricing Implications: CS Log-linearization with Exogenous Coefficients, $\delta = 0.999$**

The entries are model population values of asset prices. The price-consumption ratio is given by (2.98) and the price-dividend ratio by (2.117). The input parameters for the monthly model are given in Table 2.9. Coefficients k_1 and k_{m1} in log-linearizations (2.97) and (2.115) are set to 0.997. The expressions $E[R^e]$ and $E[R_f]$ are respectively the annualized equity premium and mean risk-free rate. The expressions $\sigma(R)$, $\sigma(R_f)$, $\sigma\left(\frac{C}{P}\right)$ and $\sigma\left(\frac{D}{P}\right)$ are respectively the annualized volatilities of market return, risk-free rate, consumption-price ratio and dividend-price ratio. The monthly subjective factor of discount is set to 0.999.

γ	ψ	$E[R^e]$	$E[R_f]$	$\sigma(R)$	$\sigma(R_f)$	$E\left[\frac{P}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P}\right)$	$\sigma\left(\frac{D}{P}\right)$
5	0.1	33.93	10.33	225.04	7.52	6.38	99999.00	3.055	0.000
5	0.125	13.68	10.98	38.09	5.29	9.62	6.55	1.467	1.484
5	0.2	1.25	9.37	16.95	2.92	19.12	13.79	0.416	0.294
5	0.25	0.12	8.20	13.54	2.29	24.49	17.97	0.248	0.115
5	0.5	1.11	4.88	13.78	1.14	41.19	31.13	0.053	0.088
5	0.75	2.31	3.45	15.96	0.77	49.31	37.57	0.015	0.122
5	1.25	3.57	2.19	18.32	0.47	57.07	43.79	0.008	0.142
5	1.5	3.93	1.85	18.99	0.40	59.21	45.48	0.013	0.146
5	2	4.40	1.43	19.87	0.31	62.00	47.71	0.019	0.150
5	3	4.90	0.98	20.80	0.21	64.94	50.06	0.025	0.155
5	4	5.15	0.76	21.28	0.16	66.47	51.29	0.027	0.156
10	0.1	0.56	15.82	63.24	5.84	104.15	29.35	0.175	0.483
10	0.125	-3.59	15.46	39.32	4.69	84.92	24.64	0.157	0.409
10	0.2	-4.02	12.53	18.29	3.09	66.74	20.19	0.114	0.232
10	0.25	-2.66	10.78	14.51	2.54	62.62	19.18	0.093	0.147
10	0.5	2.57	5.75	12.92	1.30	56.37	17.65	0.037	0.097
10	0.75	5.16	3.54	14.37	0.83	54.75	17.24	0.013	0.190
10	1.25	7.57	1.54	16.14	0.43	53.59	16.94	0.008	0.277
10	1.5	8.21	1.00	16.67	0.33	53.32	16.90	0.014	0.300
10	2	9.05	0.31	17.36	0.20	52.99	16.82	0.021	0.330
10	3	9.93	-0.40	18.10	0.06	52.67	16.74	0.029	0.361
10	4	10.38	-0.76	18.49	0.03	52.51	16.69	0.033	0.377

Table 2.19: **Endogenous Coefficients of the Campbell and Shiller (1988)'s log-linearization.**
The entries are model implied coefficients of the Campbell and Shiller (1988)'s log-linearization given in (2.99) and (2.116). The input parameters for the monthly model are given in Table 2.9.

γ	ψ	$\delta = 0.998$				$\delta = 0.999$			
		k_1	k_0	k_{m1}	k_{m0}	k_1	k_0	k_{m1}	k_{m0}
5	0.1	0.9905	0.0537	0.9896	0.0580	0.9917	0.0480	0.9907	0.0527
5	0.125	0.9921	0.0462	0.9912	0.0503	0.9933	0.0404	0.9923	0.0450
5	0.2	0.9946	0.0334	0.9938	0.0375	0.9958	0.0272	0.9949	0.0322
5	0.25	0.9955	0.0288	0.9947	0.0331	0.9967	0.0224	0.9957	0.0278
5	0.5	0.9972	0.0192	0.9963	0.0242	0.9983	0.0127	0.9972	0.0193
5	0.75	0.9977	0.0160	0.9968	0.0213	0.9988	0.0095	0.9977	0.0166
5	1	0.9980	0.0144	0.9971	0.0200	0.9990	0.0079	0.9979	0.0153
5	1.25	0.9982	0.0135	0.9972	0.0191	0.9991	0.0070	0.9980	0.0146
5	1.5	0.9983	0.0129	0.9973	0.0186	0.9992	0.0063	0.9981	0.0141
5	2	0.9984	0.0121	0.9974	0.0179	0.9993	0.0056	0.9982	0.0135
5	3	0.9985	0.0113	0.9975	0.0173	0.9994	0.0048	0.9983	0.0129
5	4	0.9986	0.0110	0.9976	0.0170	0.9995	0.0044	0.9983	0.0126
10	0.1	1.0000	0.0001	0.9963	0.0243	1.0000	0.0001	0.9982	0.0130
10	0.125	1.0000	0.0001	0.9952	0.0303	1.0000	0.0001	0.9969	0.0213
10	0.2	1.0000	0.0000	0.9943	0.0349	1.0000	0.0000	0.9957	0.0277
10	0.25	0.9993	0.0059	0.9944	0.0348	1.0000	0.0000	0.9955	0.0288
10	0.5	0.9983	0.0126	0.9946	0.0335	0.9994	0.0048	0.9954	0.0292
10	0.75	0.9981	0.0139	0.9948	0.0328	0.9991	0.0070	0.9955	0.0288
10	1	0.9980	0.0144	0.9948	0.0325	0.9990	0.0079	0.9956	0.0285
10	1.25	0.9980	0.0147	0.9949	0.0322	0.9989	0.0084	0.9956	0.0283
10	1.5	0.9979	0.0149	0.9949	0.0321	0.9989	0.0087	0.9956	0.0282
10	2	0.9979	0.0151	0.9949	0.0319	0.9988	0.0091	0.9956	0.0281
10	3	0.9979	0.0153	0.9950	0.0317	0.9988	0.0094	0.9957	0.0279
10	4	0.9979	0.0154	0.9950	0.0316	0.9988	0.0096	0.9957	0.0278

Table 2.20: **Asset Pricing Implications: CS Log-linearization with Endogenous Coefficients, $\delta = 0.999$**

The entries are model population values of asset prices. The price-consumption ratio is given by (2.98) and the price-dividend ratio by (2.117). The input parameters for the monthly model are given in Table 2.9. Coefficients k_1 and k_{m1} in log-linearizations (2.97) and (2.115) are given in Table 2.19. The expressions $E[R^e]$ and $E[R_f]$ are respectively the annualized equity premium and mean risk-free rate. The expressions $\sigma(R)$, $\sigma(R_f)$, $\sigma\left(\frac{C}{P_M}\right)$ and $\sigma\left(\frac{D}{P}\right)$ are respectively the annualized volatilities of equity return, risk-free rate, consumption-price ratio and dividend-price ratio. The monthly subjective factor of discount is set to 0.999. We report 99999 for values of the price-consumption ratio greater than 10^5.

γ	ψ	$E[R^e]$	$E[R_f]$	$\sigma(R)$	$\sigma(R_f)$	$E\left[\frac{P_M}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P_M}\right)$	$\sigma\left(\frac{D}{P}\right)$
5	0.1	8.18	13.27	42.05	6.85	9.56	8.64	1.510	1.212
5	0.125	3.83	12.02	29.94	5.12	11.81	10.57	0.991	0.756
5	0.2	0.06	9.37	16.19	2.92	19.20	16.08	0.392	0.230
5	0.25	-0.26	8.18	13.41	2.29	24.28	19.26	0.247	0.101
5	0.5	1.18	4.96	13.82	1.14	48.00	29.72	0.048	0.093
5	0.75	2.49	3.51	16.22	0.77	67.46	35.11	0.012	0.136
5	1.25	3.91	2.14	18.95	0.47	96.24	40.57	0.005	0.163
5	1.5	4.32	1.77	19.73	0.40	107.09	42.04	0.008	0.169
5	2	4.88	1.28	20.79	0.30	124.18	44.07	0.011	0.175
5	3	5.47	0.78	21.91	0.20	146.97	46.24	0.012	0.182
5	4	5.79	0.51	22.49	0.15	161.47	47.38	0.013	0.184
10	0.1	0.62	15.82	68.97	5.84	99999.00	39.44	0.000	0.382
10	0.125	2.90	16.85	73.08	6.05	99999.00	23.75	0.000	0.660
10	0.2	5.39	18.41	71.99	6.18	99999.00	13.86	0.000	1.119
10	0.25	6.75	19.47	70.12	6.23	99999.00	10.79	0.000	1.409
10	0.5	1.83	6.31	12.68	1.37	146.07	18.12	0.015	0.072
10	0.75	4.54	3.72	13.82	0.86	95.58	18.33	0.008	0.151
10	1.25	7.04	1.43	15.34	0.42	77.69	18.46	0.006	0.222
10	1.5	7.71	0.82	15.80	0.31	74.52	18.50	0.011	0.240
10	2	8.59	0.04	16.42	0.16	71.04	18.54	0.017	0.264
10	3	9.50	-0.76	17.08	0.03	68.01	18.57	0.024	0.288
10	4	9.96	-1.17	17.43	0.07	66.63	18.58	0.027	0.301

Table 2.21: **Asset Pricing Implications: HHL Taylor Expansion, $\delta = 0.998$**
The entries are model population values of asset prices. The price-consumption ratio is given by (2.110) and the price-dividend ratio by (2.121). The input parameters for the monthly model are given in Table 2.9. The expressions $E[R^e]$ and $E[R_f]$ are respectively the annualized equity premium and mean risk-free rate. The expressions $\sigma(R)$, $\sigma(R_f)$, $\sigma\left(\frac{C}{P_M}\right)$ and $\sigma\left(\frac{D}{P}\right)$ are respectively the annualized volatilities of equity return, risk-free rate, consumption-price ratio and dividend-price ratio. The monthly subjective factor of discount is set to 0.998.

γ	ψ	$E[R^e]$	$E[R_f]$	$\sigma(R)$	$\sigma(R_f)$	$E\left[\frac{P_M}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P_M}\right)$	$\sigma\left(\frac{D}{P}\right)$
5	0.1	14.44	6.04	53.50	8.90	13.09	12.03	1.264	1.036
5	0.125	6.13	9.14	33.58	5.85	13.38	12.27	0.933	0.718
5	0.2	0.32	9.69	16.31	2.99	16.74	14.78	0.448	0.254
5	0.25	-0.17	8.95	13.41	2.31	19.32	16.54	0.304	0.117
5	0.5	1.04	6.08	13.68	1.14	29.93	22.95	0.074	0.114
5	0.75	2.19	4.68	15.82	0.77	36.87	26.56	0.021	0.168
5	1	2.92	3.88	17.27	0.58	41.58	28.82	0.000	0.191
5	1.25	3.42	3.37	18.26	0.47	44.97	30.36	0.011	0.203
5	1.5	3.78	3.02	18.96	0.40	47.52	31.47	0.018	0.210
5	2	4.25	2.56	19.91	0.30	51.09	32.97	0.025	0.218
5	3	4.76	2.08	20.91	0.21	55.17	34.61	0.031	0.225
5	4	5.02	1.84	21.44	0.16	57.42	35.48	0.034	0.229
10	0.1	-3.19	21.23	65.68	6.06	-12.01	23.72	2.414	0.613
10	0.125	-5.97	19.72	38.91	4.92	-22.84	17.47	0.734	0.572
10	0.2	-4.96	15.26	18.07	3.25	-764.53	14.87	0.011	0.307
10	0.25	-3.38	13.05	14.50	2.66	124.62	14.79	0.050	0.189
10	0.5	1.86	7.25	12.73	1.34	48.91	15.42	0.045	0.089
10	0.75	4.29	4.83	13.86	0.85	43.45	15.84	0.017	0.176
10	1	5.65	3.52	14.70	0.59	41.58	16.09	0.000	0.222
10	1.25	6.52	2.69	15.30	0.43	40.66	16.25	0.011	0.249
10	1.5	7.12	2.12	15.73	0.32	40.12	16.36	0.019	0.268
10	2	7.89	1.40	16.30	0.18	39.51	16.51	0.029	0.291
10	3	8.69	0.65	16.91	0.05	38.97	16.66	0.040	0.314
10	4	9.09	0.27	17.23	0.05	38.72	16.74	0.046	0.326

Table 2.22: **Asset Pricing Implications: HHL Taylor Expansion, $\delta = 0.999$**
The entries are model population values of asset prices. The price-consumption ratio is given by (2.110) and the price-dividend ratio by (2.121). The input parameters for the monthly model are given in Table 2.9. The expressions $E[R^e]$ and $E[R_f]$ are respectively the annualized equity premium and mean risk-free rate. The expressions $\sigma(R)$, $\sigma(R_f)$, $\sigma\left(\frac{C}{P_M}\right)$ and $\sigma\left(\frac{D}{P}\right)$ are respectively the annualized volatilities of equity return, risk-free rate, consumption-price ratio and dividend-price ratio. The monthly subjective factor of discount is set to 0.999.

γ	ψ	$E[R^e]$	$E[R_f]$	$\sigma(R)$	$\sigma(R_f)$	$E\left[\frac{P_M}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P_M}\right)$	$\sigma\left(\frac{D}{P}\right)$
5	0.1	16.31	3.10	58.06	9.32	16.40	15.13	1.072	0.873
5	0.125	6.86	7.08	35.58	5.98	16.72	15.22	0.785	0.607
5	0.2	0.36	8.34	16.61	2.99	21.88	18.39	0.358	0.212
5	0.25	-0.19	7.73	13.47	2.31	26.28	20.80	0.233	0.097
5	0.5	1.15	4.94	13.80	1.14	48.53	30.13	0.048	0.091
5	0.75	2.42	3.50	16.13	0.77	67.55	35.74	0.012	0.131
5	1	3.24	2.68	17.69	0.58	83.25	39.36	0.000	0.147
5	1.25	3.79	2.14	18.75	0.47	96.29	41.88	0.005	0.154
5	1.5	4.19	1.77	19.51	0.40	107.26	43.73	0.008	0.159
5	2	4.72	1.29	20.52	0.30	124.63	46.26	0.011	0.164
5	3	5.28	0.78	21.61	0.20	147.98	49.06	0.012	0.167
5	4	5.58	0.52	22.18	0.15	162.92	50.57	0.013	0.169
10	0.1	-4.43	21.32	72.72	6.15	-8.23	43.85	4.203	0.357
10	0.125	-7.19	19.66	41.89	5.02	-13.53	24.54	1.344	0.431
10	0.2	-5.75	14.87	18.78	3.34	-46.18	18.35	0.188	0.264
10	0.25	-3.95	12.51	14.84	2.73	-102.07	17.89	0.064	0.169
10	0.5	1.82	6.31	12.68	1.37	148.34	18.15	0.015	0.071
10	0.75	4.47	3.72	13.79	0.86	95.72	18.54	0.008	0.147
10	1	5.95	2.31	14.65	0.59	83.25	18.78	0.000	0.188
10	1.25	6.89	1.43	15.26	0.42	77.73	18.94	0.006	0.213
10	1.5	7.53	0.82	15.70	0.31	74.63	19.05	0.011	0.229
10	2	8.37	0.05	16.29	0.16	71.27	19.20	0.017	0.250
10	3	9.23	-0.75	16.92	0.03	68.39	19.36	0.024	0.271
10	4	9.68	-1.16	17.25	0.07	67.11	19.44	0.027	0.281

Another interesting issue is the effect of log-linearizing the returns on the market portfolio (equation (2.98)) and on the equity (equation (2.117)). We present two sets of results. First, in table 2.13, we choose reasonable yet arbitrary values for the k parameters in the approximating formulas. Second, in Table 2.15, we set for the k parameters the values implied by the preference parameters. Indeed, in the Campbell and Shiller approximation, the parameter k is a function of the parameters and cannot be set arbitrarily [7]. With arbitrary values, the very large values present in Table 2.12 disappear and one may think that the model is acceptable for all configurations of the preference parameters. However, when we compute the statistics with the k values corresponding to the preference parameters (reported in Table 2.14), the values obtained for all moments are close to the analytical values shown in Table 2.12. This illustrates the fact that log-linearizations must be conducted very carefully. One major obstacle is to compute the k parameters when one does not have analytical formulas for the price-payoff ratios. When the model is solved numerically the values have to be chosen by successive trials. The exact way to proceed and the stopping rule remain unclear to us. Therefore, even if one does not want to model the endowment with a Markov-switching structure, the values obtained with this modeling strategy for these crucial parameters could definitely help for finding the right values of the k parameters.

We also compare the log-linearization of Campbell and Shiller (1988) to the Taylor expansion of Hansen, Heaton and Li (2005). Tables 2.17, 2.20 and 2.22 show that the log-linearization with endogenous coefficients is as well as accurate as the Taylor expansion for values of the EIS around the unity (typically between 0.5 and 1.5). Tables 2.17, 2.18 and 2.22 show that the log-linearization with exogenous coefficients is in general less accurate than the Taylor expansion. When the EIS is low, typically less than 0.25 inclusive, the Taylor expansion can considerably lower the risk-free rate for low risk aversion (then exaggerating the equity premium) and considerably increase the risk-free rate for high risk aversion (then deteriorating the equity premium). The log-linearization with endogenous coefficients seems to be worst when the EIS is low and the risk aver-

[7] The formula for k_0 is given by: $k_0 = -log(k_1) - (1 - k_1)log(1/k_1 - 1)$ where $k_1 = 1/(1 + exp(\overline{(c_t - p_t)})$, with $\overline{(c_t - p_t)})$ the mean log consumption-price ratio. The expressions for k_{m0} and k_{m1} are similar but for the dividend-price ratio instead of the consumption-price ratio

sion is high. In general, for high values of the EIS, the Taylor expansion does well compared to the log-linearization. Finally, results not reported here have also shown that the approach of Bansal, Kiku and Yaron (2007) is comparable to the log-linearization with endogenous coefficients.

An important message of Bansal and Yaron (2004) concerns the role played by time-varying volatility in consumption, a proxy for economic uncertainty. To gauge the sensitivity of the results to time-varying volatility we recompute the same moments by keeping the volatility constant in the Markov-switching model. We now have a two-state model with the parameters reported in Table 2.23. The corresponding asset return moments are given in Table 2.24. We find that they are almost identical to the results we obtained with time-varying volatility reported in Table 2.11. This result is different from the result reported in Bansal and Yaron (2004) and suggests that the action is more in the time-varying mean than in the variance. This point deserves further investigation.

2.5.3.2.2 Predictability of Returns Bansal and Yaron (2004) computed by simulation the R^2 of regressions of the cumulative excess returns from t to $t+h$ on the dividend-price ratio at t. They found that their model with a risk aversion parameter of 10 and an elasticity of intertemporal substitution of 1.5 was able to reproduce some of the predictability observed in the data. The simulation was run with 840 observations as in their data sampling period.

We have derived analytically the R^2 of the same regression in population. In table (2.25) we report the corresponding results with the same configurations of preference parameters that we selected before for asset pricing implications.

The first striking result is the total lack of predictability of excess returns by the dividend-price ratio. This is in contrast with the predictability found in Bansal and Yaron (2004). They report R^2 of 5, 10 and 16 percent at horizons of 1, 3 and 5 years respectively. To identify the source of the difference between these results, we first reproduce by simulation the same statistics both for the original Bansal and Yaron model (2004) and the matching Markov-switching model we have built. Another word of caution is in order before we look at the results. The regression that we run has as a dependent

Table 2.23: **Parameters of the Markov-Switching Model with Constant Volatility.** This table shows parameters of the Markov-Switching Model calibrated to match the model of Bansal and Yaron (2004). Calibration is made such that unconditional variance and kurtosis of MS mean of consumption matched similar moments in the BY model, whereas the implied skewness of MS mean is negative. The MS volatility of consumption is constant as in Case I of Bansal and Yaron (2004). μ_c and μ_d are conditional means of consumption and dividend, ω_c and ω_d are conditional variances of consumption and dividend. $P^{\top}$ is the transition matrix across different regimes and Π is the vector of unconditional probabilities of regimes. Means and standard deviations are in percent. The model is calibrated at the monthly frequency. The correlation vector ρ is set to zero.

	State 1	State 2
$\mu_c^{\top}$	-0.181	0.236
$\mu_d^{\top}$	-0.843	0.407
$(\omega_c^{\top})^{\frac{1}{2}}$	0.780	0.780
$(\omega_d^{\top})^{\frac{1}{2}}$	3.510	3.510
	$P^{\top}$	
State 1	0.983	0.017
State 2	0.004	0.996
$\Pi^{\top}$	0.205	0.795

Table 2.24: **Asset Pricing Implications: Constant Volatility** $\delta = 0.998$
The entries are model population values of asset prices. The price-consumption ratio is given in 2.91 and the price-dividend ratio in 2.112. The input parameters for the monthly model are given in Table 2.23. The expressions $E[R_m - R_f]$ and $E[R_f]$ are respectively the annualized equity premium and mean risk-free rate. The expressions $\sigma(R_m)$, $\sigma(R_f)$, $\sigma\left(\frac{C}{P}\right)$ and $\sigma\left(\frac{D}{P}\right)$ are respectively the annualized volatilities of market return, risk-free rate, consumption-price ratio and dividend-price ratio. The monthly subjective factor of discount is set to 0.998.

γ	ψ	$E[R_m - R_f]$	$E[R_f]$	$\sigma(R_m)$	$\sigma(R_f)$	$E\left[\frac{P}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P}\right)$	$\sigma\left(\frac{D}{P}\right)$
2.5	0.5	0.13	5.89	13.30	1.16	25.48	30.09	0.080	0.073
2.5	1.5	1.37	3.38	18.46	0.40	52.61	77.34	0.015	0.081
5	0.5	1.06	6.11	13.65	1.14	29.88	22.73	0.074	0.114
5	1.5	3.78	3.02	18.95	0.40	47.52	31.38	0.018	0.210
7.5	0.5	1.83	6.60	13.23	1.21	37.49	17.47	0.060	0.119
7.5	1.5	5.79	2.58	17.34	0.36	43.19	20.07	0.019	0.276
10	0.5	1.88	7.25	12.68	1.34	48.78	15.36	0.045	0.082
10	1.5	7.13	2.13	15.70	0.32	40.13	16.31	0.019	0.267

variable the cumulative monthly returns over *yearly* periods (1 to 20) and the *monthly* dividend-price ratio as an independent variable. In Bansal and Yaron (2004) it is a yearly dividend (cumulated monthly dividends over twelve months). Cumulating the dividends will certainly increase the R^2 but would not change the evidence over the actual presence of predictability.

In Table 2.26, we can see that there is strong predictability both in the original Bansal and Yaron (2004) model and the matching Markov switching, so it is not due to a perverse effect of our matching procedure. These results seem to point strongly towards a small sample explanation. Predictability appears in finite sample due to the presence of a very persistent variable on the right hand side[8] but disappears in population regressions. Abel (2005) also finds little or no predictability of excess returns by the dividend-price ratio in a model of preferences with a benchmark level of consumption (habit formation or consumption externalities such as keeping up or catching up with the Joneses) and i.i.d. growth rates of consumption and dividends[9]. However Abel (2005) finds that the return on stock is predictable by the dividend-price ratio.

Table 2.25 reports the analytical R^2 of the regression of returns on equity on the dividend-price ratio for the matching MS model. There seems to be some predictability for values of the elasticity of intertemporal substitution (ψ parameter) below one, but that it disappears for values above one. This is true for all values of the risk aversion parameter γ, the only difference being that predictability increases with γ for all values of ψ. This result about the pivotal value of one for ψ is the opposite of what was found in the previous section for asset pricing implications. The asset return moments were better reproduced for values of ψ greater than one.

To contrast these regression results in population with the finite sample results, we simulate the Markov-switching model over periods of 840 observations, the sample length in Bansal and Yaron (2004), and compute the R^2 of the same regression. The

[8]cite literature on problems in predictability regressions.

[9]Abel (2005) finds that the dividend-price ratio cannot predict the excess rate of return on stock relative to one-period riskless bills, when the excess rate of return is defined as the ratio of the gross rates of return on the two assets. He finds a very small R^2 for plausible values of the preference parameters when the excess rate of return is defined as the arithmetic difference between the rates of return on stocks and one-period riskless bills.

Table 2.25: **Predictability by the Dividend-Price Ratio:** $\delta = 0.998$

This table shows the R-squared of the regression $y_{t+1:t+h} = a_2(h) + b_2(h)\frac{D_t}{P_t} + \eta_{2,t+h}(h)$, where y is return, excess return, consumption volatility, consumption growth or dividend growth. The horizon h is monthly in the regression and converted into annual in the table. The price-consumption ratio is given by (2.91) and the price-dividend ratio by (2.112). The input parameters for the monthly model are given in Table 2.9. The monthly subjective factor of discount is set to 0.998.

							h									
γ	ψ	Returns			Excess			Volatility			Consumption			Dividend		
		1	3	5	1	3	5	1	3	5	1	3	5	1	3	5
2.5	0.2	28.85	46.54	45.96	0.22	0.39	0.42	0.74	0.60	0.50	29.13	35.78	31.14	15.96	23.26	21.78
2.5	0.5	5.85	9.09	8.79	0.00	0.01	0.01	2.08	1.70	1.41	28.69	35.24	30.66	15.72	22.90	21.44
2.5	0.8	1.23	2.10	2.14	0.08	0.16	0.17	0.58	0.48	0.40	29.18	35.84	31.19	15.99	23.30	21.81
2.5	1	0.47	0.83	0.87	0.15	0.28	0.31	0.43	0.35	0.29	29.23	35.91	31.25	16.02	23.34	21.85
2.5	1.2	0.17	0.31	0.33	0.20	0.39	0.44	0.35	0.29	0.24	29.26	35.94	31.27	16.03	23.36	21.87
2.5	1.5	0.02	0.05	0.05	0.27	0.53	0.60	0.29	0.24	0.20	29.28	35.96	31.29	16.04	23.37	21.88
5	0.2	28.34	46.07	45.71	0.00	0.00	0.00	1.68	1.38	1.15	28.82	35.39	30.80	15.79	23.00	21.54
5	0.5	4.33	6.89	6.75	0.11	0.21	0.23	2.24	1.84	1.53	28.63	35.17	30.60	15.69	22.86	21.40
5	0.8	0.56	0.99	1.03	0.38	0.75	0.84	0.63	0.51	0.43	29.17	35.82	31.17	15.98	23.28	21.80
5	1	0.11	0.20	0.22	0.51	1.01	1.14	0.45	0.37	0.30	29.23	35.90	31.24	16.01	23.33	21.85
5	1.2	0.01	0.01	0.01	0.60	1.19	1.36	0.36	0.30	0.25	29.25	35.93	31.27	16.03	23.35	21.87
5	1.5	0.03	0.06	0.07	0.70	1.40	1.60	0.30	0.24	0.20	29.28	35.96	31.29	16.04	23.37	21.88
7.5	0.2	28.05	46.40	46.52	0.04	0.07	0.08	2.72	2.23	1.85	28.47	34.97	30.43	15.60	22.73	21.28
7.5	0.5	5.52	8.61	8.34	0.05	0.09	0.09	4.45	3.65	3.03	27.90	34.27	29.82	15.29	22.28	20.86
7.5	0.8	1.20	2.05	2.10	0.14	0.27	0.30	1.05	0.86	0.72	29.03	35.65	31.02	15.90	23.17	21.70
7.5	1	0.46	0.82	0.86	0.18	0.35	0.39	0.72	0.59	0.49	29.14	35.79	31.14	15.96	23.26	21.78
7.5	1.2	0.17	0.32	0.34	0.21	0.40	0.45	0.57	0.46	0.38	29.19	35.85	31.20	15.99	23.30	21.82
7.5	1.5	0.03	0.05	0.05	0.23	0.45	0.50	0.45	0.37	0.31	29.23	35.90	31.24	16.01	23.33	21.85
10	0.2	27.35	47.26	48.65	0.01	0.02	0.02	3.51	2.88	2.38	28.20	34.64	30.14	15.45	22.51	21.08
10	0.5	7.46	11.22	10.65	0.00	0.00	0.00	14.47	11.87	9.84	24.59	30.21	26.29	13.47	19.63	18.38
10	0.8	2.94	4.80	4.78	0.01	0.03	0.03	2.29	1.88	1.56	28.61	35.15	30.58	15.68	22.84	21.39
10	1	1.69	2.85	2.89	0.02	0.04	0.04	1.44	1.18	0.98	28.90	35.49	30.88	15.83	23.07	21.60
10	1.2	1.05	1.82	1.87	0.02	0.04	0.05	1.09	0.89	0.74	29.02	35.64	31.01	15.90	23.16	21.69
10	1.5	0.57	1.01	1.06	0.03	0.05	0.05	0.83	0.68	0.56	29.10	35.74	31.10	15.94	23.23	21.75

Table 2.26: **Predictability of Returns by the Dividend-Price Ratio: Simulation with $S = 1000$ and $T = 840$.** This table shows the R-squared of the regression $y_{t+1:t+h} = a_1(h) + b_1(h)\frac{D_t}{P_t} + \eta_{1,t+h}(h)$, where y is return or excess return. The horizon h is monthly in the regression and converted into annual in the table. The entries are based on 1000 simulations of consumption and dividend processes, each with 840 monthly observations. The parameter configuration for the MS model is given in Table 2.9. The price-consumption ratio is given by (2.91) and the price-dividend ratio by (2.112). The monthly subjective factor of discount is set to 0.998.

γ	ψ	Excess MS					Excess BY					Returns MS				
		1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
2.5	0.5	1.73	2.96	3.95	4.77	5.39	1.40	2.56	3.50	4.34	5.08	5.79	8.25	9.24	9.37	9.10
2.5	0.8	2.70	4.34	5.53	6.46	7.10	1.49	2.74	3.72	4.57	5.31	2.98	4.61	5.54	5.99	6.18
2.5	1.2	3.45	5.46	6.85	7.89	8.58	1.54	2.82	3.82	4.68	5.42	2.97	4.68	5.73	6.35	6.71
2.5	1.5	3.79	5.96	7.46	8.56	9.28	1.55	2.84	3.86	4.72	5.46	3.21	5.06	6.21	6.91	7.33
5	0.5	2.02	3.40	4.48	5.37	6.03	1.38	2.53	3.47	4.30	5.03	4.69	6.84	7.82	8.06	7.95
5	0.8	3.23	5.17	6.56	7.62	8.35	1.47	2.71	3.69	4.55	5.28	2.74	4.32	5.29	5.83	6.12
5	1.2	4.03	6.38	8.01	9.21	10.00	1.53	2.80	3.81	4.67	5.40	3.12	4.95	6.11	6.82	7.27
5	1.5	4.37	6.89	8.62	9.88	10.71	1.54	2.83	3.85	4.71	5.45	3.48	5.51	6.79	7.60	8.11
7.5	0.5	1.75	3.02	4.07	4.92	5.59	1.36	2.51	3.46	4.30	5.02	5.60	8.03	9.03	9.21	9.00
7.5	0.8	2.61	4.23	5.44	6.39	7.05	1.47	2.71	3.70	4.57	5.30	2.83	4.42	5.36	5.83	6.05
7.5	1.2	3.18	5.06	6.40	7.42	8.11	1.53	2.81	3.83	4.70	5.44	2.72	4.33	5.35	5.96	6.34
7.5	1.5	3.41	5.41	6.81	7.85	8.55	1.55	2.84	3.87	4.75	5.49	2.90	4.62	5.72	6.40	6.83
10	0.5	1.46	2.62	3.63	4.48	5.17	1.35	2.50	3.46	4.31	5.04	7.50	10.43	11.45	11.51	11.13
10	0.8	1.91	3.20	4.20	5.00	5.58	1.47	2.72	3.74	4.62	5.36	3.74	5.62	6.58	6.93	6.98
10	1.2	2.31	3.73	4.76	5.56	6.10	1.53	2.83	3.88	4.77	5.52	2.74	4.31	5.26	5.74	5.98
10	1.5	2.49	3.97	5.02	5.82	6.34	1.56	2.87	3.93	4.83	5.58	2.62	4.16	5.12	5.66	5.96

Table 2.27: **Predictability by the Consumption-Price Ratio:** $\delta = 0.998$

This table shows the R-squared of the regression $y_{t+1:t+h} = a_2(h) + b_2(h)\frac{C_t}{P_{M,t}} + \eta_{2,t+h}(h)$, where y is return, excess return, consumption volatility, consumption growth or dividend growth. The horizon h is monthly in the regression and converted into annual in the table. The price-consumption ratio is given by (2.91) and the price-dividend ratio by (2.112). The input parameters for the monthly model are given in Table 2.9. The monthly subjective factor of discount is set to 0.998.

		h														
γ	ψ	Returns			Excess			Volatility			Consumption			Dividend		
		1	3	5	1	3	5	1	3	5	1	3	5	1	3	5
2.5	0.2	29.05	46.85	46.27	0.22	0.39	0.42	0.01	0.01	0.01	29.38	36.08	31.40	16.09	23.45	21.96
2.5	0.5	6.02	9.35	9.04	0.00	0.01	0.01	0.01	0.01	0.01	29.38	36.08	31.40	16.09	23.45	21.96
2.5	0.8	1.24	2.12	2.17	0.08	0.16	0.18	0.01	0.01	0.01	29.38	36.08	31.40	16.09	23.45	21.96
2.5	1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2.5	1.2	0.17	0.32	0.34	0.20	0.40	0.44	0.01	0.01	0.01	29.38	36.08	31.40	16.09	23.45	21.96
2.5	1.5	0.02	0.05	0.05	0.27	0.53	0.60	0.01	0.01	0.00	29.38	36.08	31.40	16.09	23.45	21.96
5	0.2	28.73	46.68	46.30	0.00	0.00	0.00	0.06	0.05	0.04	29.36	36.06	31.38	16.08	23.43	21.94
5	0.5	4.50	7.16	7.02	0.11	0.21	0.23	0.05	0.04	0.03	29.36	36.06	31.38	16.09	23.44	21.95
5	0.8	0.57	1.01	1.05	0.39	0.75	0.85	0.05	0.04	0.03	29.36	36.06	31.38	16.09	23.44	21.95
5	1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	1.2	0.01	0.01	0.01	0.60	1.20	1.37	0.05	0.04	0.03	29.36	36.07	31.38	16.09	23.44	21.95
5	1.5	0.03	0.06	0.06	0.70	1.40	1.60	0.05	0.04	0.03	29.36	36.07	31.38	16.09	23.44	21.95
7.5	0.2	28.61	47.29	47.38	0.04	0.07	0.08	0.20	0.17	0.14	29.31	36.00	31.32	16.06	23.40	21.91
7.5	0.5	5.96	9.31	9.03	0.05	0.09	0.10	0.14	0.12	0.10	29.33	36.03	31.35	16.07	23.41	21.92
7.5	0.8	1.23	2.12	2.18	0.14	0.28	0.31	0.13	0.11	0.09	29.33	36.03	31.35	16.07	23.42	21.93
7.5	1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
7.5	1.2	0.18	0.33	0.35	0.21	0.40	0.45	0.12	0.10	0.08	29.34	36.03	31.36	16.07	23.42	21.93
7.5	1.5	0.03	0.05	0.06	0.23	0.45	0.51	0.12	0.10	0.08	29.34	36.03	31.36	16.07	23.42	21.93
10	0.2	27.98	48.29	49.66	0.01	0.02	0.02	0.54	0.44	0.37	29.19	35.85	31.20	15.99	23.30	21.82
10	0.5	9.39	14.19	13.53	0.00	0.01	0.01	0.32	0.27	0.22	29.27	35.95	31.29	16.04	23.37	21.88
10	0.8	3.10	5.08	5.06	0.01	0.03	0.03	0.29	0.24	0.20	29.28	35.97	31.30	16.04	23.38	21.89
10	1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	1.2	1.09	1.89	1.95	0.02	0.04	0.05	0.27	0.22	0.19	29.29	35.97	31.30	16.05	23.38	21.89
10	1.5	0.60	1.05	1.10	0.03	0.05	0.05	0.27	0.22	0.18	29.29	35.98	31.31	16.05	23.38	21.89

Table 2.28: **Asset Pricing Implications: MS Matching BY, high γ**

The entries are model population values of asset prices. The price-consumption ratio is given by (2.91) and the price-dividend ratio by (2.112). The input parameters for the model (2.1)-(2.2) are given in Table 2.9. The expressions $E[R_m - R_f]$ and $E[R_f]$ are respectively the annualized equity premium and mean risk-free rate. The expressions $\sigma(R_m)$, $\sigma(R_f)$, $\sigma\left(\frac{C}{P}\right)$ and $\sigma\left(\frac{D}{P}\right)$ are respectively the annualized volatilities of market return, risk-free rate, consumption-price ratio and dividend-price ratio. The monthly subjective factor of discount is set to 0.998.

γ	ψ	$E[R_m - R_f]$	$E[R_f]$	$\sigma(R_m)$	$\sigma(R_f)$	$E\left[\frac{P}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P}\right)$	$\sigma\left(\frac{D}{P}\right)$
5	0.2	0.00	10.58	16.07	2.92	16.24	13.56	0.459	0.268
5	0.5	1.05	6.11	13.68	1.14	29.89	22.77	0.074	0.115
5	0.8	2.36	4.49	16.16	0.72	37.95	27.08	0.016	0.174
5	1	2.92	3.88	17.27	0.58	41.58	28.82	0.000	0.191
5	1.2	3.33	3.46	18.09	0.49	44.37	30.09	0.009	0.201
5	1.5	3.78	3.02	18.96	0.40	47.51	31.44	0.018	0.210
10	0.2	-4.85	15.14	18.08	3.24	33063.74	14.88	0.000	0.307
10	0.5	1.86	7.26	12.73	1.34	48.86	15.42	0.045	0.089
10	0.8	4.62	4.51	14.05	0.78	42.94	15.90	0.013	0.187
10	1	5.65	3.52	14.70	0.59	41.58	16.09	0.000	0.222
10	1.2	6.37	2.83	15.19	0.45	40.80	16.22	0.009	0.245
10	1.5	7.12	2.12	15.73	0.32	40.11	16.36	0.019	0.268
15	0.2	-10.74	17.73	18.90	3.82	36788.63	29.68	0.000	0.169
15	0.5	0.69	8.49	12.47	1.62	86.61	15.18	0.022	0.053
15	0.8	5.09	4.51	13.02	0.85	47.08	14.44	0.010	0.127
15	1	6.61	3.15	13.39	0.59	41.58	14.25	0.000	0.161
15	1.2	7.63	2.23	13.69	0.41	38.74	14.15	0.008	0.185
15	1.5	8.67	1.29	14.03	0.24	36.38	14.05	0.018	0.210
20	0.2	-15.86	19.47	18.74	4.30	34300.58	768.91	0.000	0.007
20	0.5	-0.80	9.37	12.54	1.85	181.33	16.76	0.009	0.060
20	0.8	5.03	4.43	12.73	0.91	50.44	14.64	0.008	0.095
20	1	6.96	2.78	12.95	0.59	41.58	14.13	0.000	0.123
20	1.2	8.25	1.68	13.14	0.38	37.42	13.83	0.008	0.144
20	1.5	9.53	0.58	13.36	0.18	34.14	13.56	0.017	0.167

Table 2.29: **Predictability of Excess Returns by the Dividend-Price Ratio: MS Matching BY, high γ**

This table shows the R-squared of the regression $y_{t+1:t+h} = a_2(h) + b_2(h)\frac{D_t}{P_t} + \eta_{2,t+h}(h)$, where y is return, excess return, consumption volatility, consumption growth or dividend growth. The horizon h is monthly in the regression and converted into annual in the table. The price-consumption ratio is given by (2.91) and the price-dividend ratio by (2.112). The input parameters for the monthly model are given in Table 2.9. The monthly subjective factor of discount is set to 0.998.

		h														
γ	ψ	Returns			Excess			Volatility			Consumption			Dividend		
		1	3	5	1	3	5	1	3	5	1	3	5	1	3	5
5	0.2	28.34	46.07	45.71	0.00	0.00	0.00	1.68	1.38	1.15	28.82	35.39	30.80	15.79	23.00	21.54
5	0.5	4.33	6.89	6.75	0.11	0.21	0.23	2.24	1.84	1.53	28.63	35.17	30.60	15.69	22.86	21.40
5	0.8	0.56	0.99	1.03	0.38	0.75	0.84	0.63	0.51	0.43	29.17	35.82	31.17	15.98	23.28	21.80
5	1	0.11	0.20	0.22	0.51	1.01	1.14	0.45	0.37	0.30	29.23	35.90	31.24	16.01	23.33	21.85
5	1.2	0.01	0.01	0.01	0.60	1.19	1.36	0.36	0.30	0.25	29.25	35.93	31.27	16.03	23.35	21.87
5	1.5	0.03	0.06	0.07	0.70	1.40	1.60	0.30	0.24	0.20	29.28	35.96	31.29	16.04	23.37	21.88
10	0.2	27.35	47.26	48.65	0.01	0.02	0.02	3.51	2.88	2.38	28.20	34.64	30.14	15.45	22.51	21.08
10	0.5	7.46	11.22	10.65	0.00	0.00	0.00	14.47	11.87	9.84	24.59	30.21	26.29	13.47	19.63	18.38
10	0.8	2.94	4.80	4.78	0.01	0.03	0.03	2.29	1.88	1.56	28.61	35.15	30.58	15.68	22.84	21.39
10	1	1.69	2.85	2.89	0.02	0.04	0.04	1.44	1.18	0.98	28.90	35.49	30.88	15.83	23.07	21.60
10	1.2	1.05	1.82	1.87	0.02	0.04	0.05	1.09	0.89	0.74	29.02	35.64	31.01	15.90	23.16	21.69
10	1.5	0.57	1.01	1.06	0.03	0.05	0.05	0.83	0.68	0.56	29.10	35.74	31.10	15.94	23.23	21.75
15	0.2	23.82	43.17	45.89	1.16	1.95	1.98	8.57	7.03	5.83	26.48	32.53	28.30	14.51	21.14	19.79
15	0.5	0.93	1.31	1.18	0.00	0.00	0.00	79.65	65.34	54.19	3.08	3.78	3.29	1.69	2.46	2.30
15	0.8	6.02	9.25	8.88	0.63	1.13	1.20	10.05	8.24	6.84	26.05	32.00	27.84	14.27	20.80	19.47
15	1	4.66	7.35	7.16	1.02	1.80	1.88	5.48	4.49	3.73	27.56	33.85	29.46	15.10	22.00	20.60
15	1.2	3.72	5.97	5.88	1.29	2.25	2.32	3.79	3.11	2.58	28.12	34.54	30.06	15.41	22.45	21.02
15	1.5	2.83	4.64	4.62	1.57	2.68	2.75	2.68	2.20	1.82	28.49	34.99	30.45	15.61	22.74	21.29
20	0.2	19.67	36.45	39.41	3.64	5.51	5.26	19.25	15.79	13.10	22.87	28.09	24.44	12.53	18.26	17.09
20	0.5	0.77	1.21	1.22	0.00	0.01	0.01	87.69	71.93	59.65	0.43	0.52	0.46	0.23	0.34	0.32
20	0.8	5.87	8.75	8.25	0.91	1.61	1.69	27.95	22.92	19.01	20.14	24.74	21.52	11.03	16.08	15.05
20	1	5.90	8.99	8.58	1.95	3.36	3.44	14.91	12.23	10.15	24.44	30.02	26.12	13.39	19.51	18.27
20	1.2	5.36	8.29	7.99	2.75	4.60	4.65	9.99	8.20	6.80	26.07	32.02	27.86	14.28	20.81	19.48
20	1.5	4.62	7.28	7.08	3.57	5.83	5.79	6.81	5.59	4.63	27.12	33.31	28.98	14.86	21.65	20.27

Table 2.30: **Variance Ratios of Aggregate Returns: $\delta = 0.998$**

This table shows the variance ratios $\frac{Var(R_{t+1:t+h})}{hVar(R_{t+1})}$, where the horizon h is monthly and converted into annual in the table. The price-consumption ratio is given by (2.91) and price-dividend ratio by (2.112). The input parameters for the monthly model are given in table 2.9. The monthly subjective factor of discount is set to 0.998.

		h									
γ	ψ	Returns					Excess				
		1	2	3	4	5	1	2	3	4	5
2.5	0.2	1.11	1.21	1.29	1.36	1.43	1.02	1.04	1.06	1.07	1.09
2.5	0.5	1.14	1.27	1.38	1.48	1.56	1.00	1.00	1.00	0.99	0.99
2.5	0.8	1.06	1.12	1.17	1.21	1.24	0.99	0.98	0.97	0.96	0.96
2.5	1	1.04	1.07	1.10	1.13	1.15	0.98	0.97	0.96	0.95	0.94
2.5	1.2	1.02	1.04	1.06	1.07	1.09	0.98	0.96	0.95	0.93	0.92
2.5	1.5	1.01	1.02	1.02	1.03	1.03	0.98	0.96	0.94	0.92	0.91
5	0.2	1.10	1.20	1.28	1.34	1.40	1.00	1.00	1.00	1.00	1.00
5	0.5	1.12	1.23	1.32	1.40	1.47	0.99	0.98	0.98	0.97	0.97
5	0.8	1.04	1.08	1.11	1.13	1.16	0.98	0.96	0.94	0.92	0.91
5	1	1.02	1.03	1.05	1.06	1.07	0.97	0.95	0.92	0.91	0.89
5	1.2	1.00	1.01	1.01	1.01	1.01	0.97	0.94	0.92	0.89	0.88
5	1.5	0.99	0.98	0.98	0.97	0.97	0.97	0.93	0.91	0.88	0.86
7.5	0.2	1.09	1.17	1.24	1.30	1.35	0.99	0.98	0.98	0.97	0.97
7.5	0.5	1.13	1.26	1.37	1.46	1.54	1.00	0.99	0.99	0.98	0.98
7.5	0.8	1.06	1.11	1.16	1.20	1.23	0.99	0.97	0.96	0.95	0.95
7.5	1	1.04	1.07	1.10	1.12	1.14	0.98	0.97	0.96	0.94	0.94
7.5	1.2	1.02	1.04	1.06	1.07	1.08	0.98	0.96	0.95	0.94	0.93
7.5	1.5	1.01	1.02	1.02	1.03	1.03	0.98	0.96	0.95	0.93	0.92
10	0.2	1.06	1.11	1.15	1.19	1.22	1.01	1.01	1.01	1.02	1.02
10	0.5	1.17	1.32	1.45	1.57	1.66	1.00	1.00	1.00	1.00	1.00
10	0.8	1.09	1.18	1.26	1.32	1.38	1.00	1.01	1.01	1.01	1.02
10	1	1.07	1.13	1.19	1.24	1.28	1.01	1.01	1.01	1.02	1.02
10	1.2	1.05	1.10	1.15	1.18	1.21	1.01	1.01	1.02	1.02	1.02
10	1.5	1.04	1.08	1.11	1.13	1.15	1.01	1.01	1.02	1.02	1.03

Table 2.31: **Variance Ratios of Aggregate Returns: Simulation with $S = 1000$ and $T = 840$**

γ	ψ	h											
		1	2	3	4	5	6	7	8	9	10	15	20
2.5	0.5	1.11	1.19	1.27	1.33	1.37	1.41	1.43	1.45	1.45	1.46	1.40	1.26
2.5	0.8	1.04	1.07	1.09	1.11	1.12	1.13	1.13	1.13	1.13	1.12	1.05	0.93
2.5	1.2	1.01	1.01	1.01	1.00	1.00	1.00	0.99	0.98	0.97	0.96	0.88	0.77
2.5	1.5	1.00	0.99	0.98	0.97	0.96	0.95	0.94	0.92	0.91	0.90	0.81	0.71
5.0	0.5	1.09	1.16	1.22	1.27	1.30	1.33	1.35	1.36	1.36	1.36	1.30	1.17
5.0	0.8	1.02	1.03	1.04	1.05	1.05	1.05	1.05	1.04	1.04	1.03	0.95	0.84
5.0	1.2	0.99	0.98	0.97	0.95	0.94	0.93	0.92	0.90	0.89	0.88	0.79	0.69
5.0	1.5	0.98	0.96	0.94	0.92	0.91	0.89	0.87	0.86	0.84	0.83	0.74	0.64
7.5	0.5	1.10	1.19	1.26	1.31	1.36	1.39	1.41	1.42	1.43	1.44	1.38	1.24
7.5	0.8	1.04	1.06	1.08	1.10	1.11	1.12	1.12	1.12	1.11	1.10	1.03	0.92
7.5	1.2	1.01	1.00	1.00	1.00	1.00	0.99	0.98	0.97	0.96	0.95	0.87	0.76
7.5	1.5	1.00	0.98	0.97	0.96	0.96	0.95	0.93	0.92	0.91	0.89	0.81	0.71
10.0	0.5	1.13	1.24	1.33	1.41	1.46	1.51	1.54	1.56	1.57	1.58	1.53	1.38
10.0	0.8	1.07	1.12	1.16	1.20	1.22	1.24	1.25	1.26	1.26	1.26	1.19	1.07
10.0	1.2	1.03	1.05	1.07	1.09	1.10	1.10	1.10	1.10	1.09	1.09	1.01	0.90
10.0	1.5	1.02	1.03	1.04	1.05	1.05	1.05	1.05	1.04	1.03	1.02	0.95	0.84

results are reported in Table 2.26. The R^2 of the finite sample regressions are not too different from the analytical R^2 for $\psi = 0.5$. As the value of the elasticity of intertemporal substitution increases the gap between the population and finite sample statistics increases[10]. Therefore, for some values of the preference parameters, predictability appears to be a finite sample phenomenon while for some others it seems to be a feature of the model.

Lettau and Ludvigson (2001a,b) have put forward that a measure of consumption over wealth has a greater predicting power than the dividend-price ratio. We present in Table 2.27 results of the regression of cumulative returns on the consumption-price ratio in the Epstein-Zin economy. Indeed, we find higher predictability for all preference parameter pairs. In particular, for $\gamma = 10$ and $\psi = 0.5$ the R^2 for the consumption-price ratio is equal to 9.39, 14.19 and 13.53 for 1, 3 and 5 years, as opposed to 7.46, 11.22 and 10.65 for the dividend-price ratio. The remarks made above about the finite sample results for the dividend-price ratio apply equally to the consumption-price ratio. In particular, there is no predictability of excess returns by the price-consumption ratio.

Some predictability of excess returns appears if risk aversion increases for values of ψ greater than one. It is interesting to note that for higher risk aversion the BY model behaves more like the LLW model. The volatility of the stock decreases as well as the level of the price-dividend ratio. These results are illustrated in tables 2.28 and 2.29.

2.5.3.2.3 Predictability of Volatility Another important message found in Bansal and Yaron (2004) is the predictability of consumption volatility by the dividend-price ratio. In Table 2.25, we report the R^2 of the regression of cumulative future consumption volatility over several horizons on the current price-dividend ratio. Results are similar to those obtained for future returns predictability. Not all preference configurations are able to produce predictable volatility. Again only low values of the elasticity of intertemporal substitution are able to generate predictability ($\psi = 0.5$). There is no predictability at all for values of ψ above one. Predictability is the strongest at a one-year horizon.

[10]We have checked that results similar to the analytical are obtained when we simulate with a sample of 2,000 observations

2.5.3.2.4 Variance Ratios There is negative autocorrelation at long horizons in returns. Evidence is provided by the variance ratios computed at several horizons in Table 2.1. The variance ratios are less than one and decrease from year 2 up to year 4.

The corresponding analytical quantities are reported in Table 2.30. Most of the preference parameter combinations produce strong positive autocorrelations increasing with the horizons. Only one set, $\gamma = 5$ and $\psi = 1.5$ produce slight negative autocorrelation. The same results would have been visible in a simulated finite sample setting with 840 observations (see Table 2.31). However, predictability would have appeared stronger for the above-mentioned particular set of parameters and other candidate sets would have appeared.

2.6 Conclusion

Equilibrium asset pricing models have become harder to solve. To reproduce resilient stylized facts, researchers have assumed that the representative investor is endowed with more sophisticated preferences. The fundamentals in the economy, consumption and dividends, have also been modeled with richer dynamics. Often the time required to solve the model numerically or to simulate it to compute the statistics of interest is prohibitive. Therefore, researchers lean towards simpler models, making simplifying assumptions as a compromise between reality and feasibility.

In this chapter, we have provided analytical formulas that should be of great help to assess the ability of these models to reproduce the stylized facts. We have chosen a flexible model for the endowment that can be applied directly to the data, as already done by several researchers, or used to match other processes that are contemplated. In terms of preferences, we have chosen the recursive framework of Epstein and Zin (1989), widely used in the asset pricing literature. We have limited our analysis to the Kreps and Porteus (1978) certainty equivalent. In future research we intend to try to find analytical formulas for other certainty equivalents in the recursive framework and other types of preferences.

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CHAPTER 3

AFFINE STOCHASTIC SKEWNESS MODELS

Abstract

Recent developments in asset return modeling have shown evidence for time-variation not only in conditional variance, but also especially in conditional skewness and leverage effects. We develop a discrete time affine multifactor latent variable model of asset returns which allows for both stochastic volatility and stochastic skewness (SVS model). Importantly, we disentangle the dynamics of conditional volatility and conditional skewness in a coherent way. Our approach allows the distribution of current daily returns conditional on current volatility to be asymmetric. In our model, time-varying conditional skewness is driven by the conditional leverage effect and the asymmetry of the distribution of current returns conditional on current volatility. We derive analytical formulas for various moment conditions that we use for GMM inference. Applying our approach to several equity and index daily returns, we show that the conditional distribution of current daily returns, conditional on current volatility, is positively skewed and helps to match sample return skewness as well as negative cross-correlations between returns and squared returns. The conditional leverage effect is significant and negative. The conditional skewness is positive, implying that the asymmetry of the distribution of current returns conditional on current volatility dominates the leverage effect in determining the conditional skewness.

3.1 Introduction

Most empirical research in Finance put forward evidences that, not only return volatility changes over time, but also returns are conditionally non-normal with time-varying conditional skewness. These two important features of asset returns are critically important as changes in volatility and skewness modify intertemporal opportunities for portfolio choice. Evidence show that volatility and skewness risks are priced in financial markets as people require more premium for holding assets with more volatile and more negatively skewed payoffs. Path dependence in the second and the third moments are then able to explain security prices. This chapter develops an affine multifactor latent variable model of asset returns where both conditional volatility and conditional skewness are time-varying and unobservable factors. Most importantly, in the two-factor case, the vector of returns, volatility and skewness is affine.

Path dependence in return volatility has originally been captured by an autoregressive conditional heteroskedasticity model (ARCH, Engle (1982)) or its extensions (GARCH, Bollerslev (1986) and EGARCH, Nelson (1991)). While return volatility is completely determined as a function of past observed returns in ARCH and GARCH models, an alternative approach, which has become more popular recently, is the stochastic volatility model (SV), where return volatility is an unobserved component which undergoes shocks from a different source other than return shocks. Most empirical applications of stochastic volatility and GARCH models are based on the assumption that the conditional distribution of returns is symmetric. Even if these models help explaining the observed unconditional fat-tailedness of actual returns, there is still a lot to do in explaining unconditional asymmetries (skewness and leverage effects) as well as conditional higher return moments (skewness and kurtosis especially) [see e.g. Hansen (1994)]. Conditionally nonsymmetric return innovations are critically important as in option pricing for example, heteroscedasticity alone does not suffice to explain the option smirk.

The primary goal of this chapter is to develop a semi-affine multifactor stochastic volatility model with skewed return innovations. Christoffersen, Heston and Jacobs (2006) also study a semi-affine model of returns with time-varying volatility and conditional skewness. However, skewness in their model is deterministically related to volatility and both undergo return shocks–since they work in a GARCH setting, whereas in the new SV setting, volatility and skewness evolve as two separate factors with linearly independent transformations, capturing different features of the return dynamics and undergoing shocks from different sources than return shocks.

Harvey and Siddique (1999) also consider a nonsymmetric conditional distribution of return with volatility and skewness as two separate factors which follow GARCH-type processes. Their autoregressive conditional skewness is a simple way to model conditional asymmetry and provides an easy methodology to estimate time-varying conditional skewness because of the availability of the likelihood function. However, the non-affinity of their model is a practical drawback for solving asset pricing and derivative models. For example, in a general equilibrium model with autoregressive conditional skewness of endowment growths, as well as in an option pricing model with autoregressive conditional skewness of returns, asset prices do not exist in closed-form. Then, solving such models involves numerical methods or simulation techniques which take a lot of time to perform and for which it is difficult to assess the errors. Instead, we propose a convenient alternative to autoregressive conditional skewness where skewness—as well as volatility—is viewed as an affine combination of stochastic components. The availability of the moment generating function in our setting leads to a GMM estimation based on exact moment conditions. It also provides an analytical tool for solving asset and derivative pricing models. We distinguish agent and econometrician information sets in our SV setting and provide explicit GARCH counteparts of volatility and conditional skewness and leverage effects.

Another contribution of this chapter is to develop and implement an algorithm for computing exact analytical unconditional moments of observable in a more general discrete time semi-affine multifactor latent variable model that nests our SVS model. A similar study is conducted by Jiang and Knight (2002) in the case of continuous time affine processes. They derive the unconditional joint characteristic function of the diffusion vector process in closed form. However, this issue has not been addressed so far in the literature for discrete time affine models although of critical importance. First, these analytical formulas help in assessing the direct impact of model parameters on critical unconditional return moments such as skewness, excess kurtosis, autocorrelation of squared returns and coskewness. More generally, this is very helpful for calibration exercises where model parameters are set to match important features of the data. Second, the unconditional moments of observable implied by the model can directly be compared to their sample counterparts. This allows for a GMM estimation based on exact moment conditions. Moreover, this estimation technique permits a direct evaluation of the performance of the model in replicating well-known stylized facts like the persistence of volatility through the autocorrelation of squared returns, the absence of autocorrelation of returns, the negative leverage effect via coskewness, the unconditional fat-tailedness and the negative asymmetry of returns. All these well-known empirical facts are driven by particular unconditional moments which are considered in the vector of moment conditions when performing the model estimation.

In this chapter, we apply the new GMM procedure for discrete time affine latent variable models to the estimation of the one-factor SVS model using several equity and index daily returns. We further apply the Unscented Kalman Filter to estimate cumulants of stochastic factors conditional on observable returns, as they are necessary to evaluate the GARCH counterparts of volatility and conditional skewness. Model parameters are significantly estimated and model implications are striking. The distribution of current daily returns conditional on current daily volatility is positively skewed and appears sufficient to match unconditional asymmetry and leverage effects all significant in daily return data. Second, this positively skewed distribution of current daily returns conditional on current daily volatility leads to a positive skewness of current returns conditional on past returns and this result departs from most of the existing literature (e.g. Forsberg and Bollerslev (2002)). Third, when the distribution of current daily returns conditional on current daily volatility is constrained to be normal, then a negative skewness of current returns conditional on past returns comes up to corroborate most of existing findings. However, the model doesn't match unconditional skewness and leverage effects. Moreover, the GMM test for overidentifying restrictions rejects the constrained model at conventional level of significance whereas it does not reject the unconstrained model which leads to a significant positive skewness of current returns conditional on current volatility.

The rest of the chapter is organized as follows. Section 3.2 presents the general semi-affine multifactor latent variable model of asset returns, discusses the nested SVS model and derives GARCH counterparts of volatility and skewness. Section 3.4 presents the procedure to estimate cumulants of the stochastic components of volatility and skewness, conditional on observable returns. Section 3.5 presents arbitrage-free and risk-neutral valuation of assets based on the SVS model. Section 3.6 derives analytical formulas of return moments and presents the GMM estimation based on exact moment conditions. Section 3.7 estimates the univariate SVS model using several equity and index daily returns and provides some diagnostics. It also derives GARCH estimates of volatility and skewness and discusses their model implications. Section 3.8 concludes.

3.2 Affine Models of Returns: An Overview

3.2.1 Definition and General Structure

A discrete time parametric semi-affine multifactor latent variable model of returns with time-varying conditional moments can be characterized by its conditional cumulant-generating function:

$$\begin{aligned}\Psi_t(x,y;\theta) &= \ln E_t\left[\exp\left(xr_{t+1}+y^\top l_{t+1}\right)\right] = \ln E_t\left[\exp\left(xr_{t+1}+\sum_{i=1}^{k} y_i l_{i,t+1}\right)\right] \\ &= A(x,y;\theta)+B(x,y;\theta)^\top l_t = A(x,y;\theta)+\sum_{i=1}^{k} B_i(x,y;\theta)\, l_{it}, \qquad (3.1)\end{aligned}$$

where $E_t[\cdot] \equiv E[\cdot \mid I_t]$ denotes the expectation conditional to a well-specified information set I_t, $l_t = (l_{1t},..,l_{kt})^\top$ is the vector of latent factors and θ is the vector of parameters.[1] In all what follows, parameter θ is withdrawn from functions A and B for expository purposes.

In practice, models are specified through the joint dynamics of observable returns r_{t+1} and latent factors $l_t = (l_{1t},..,l_{kt})^\top$. In general, all conditional moments of returns are affine functions of the latent factors. In particular, a latent factor l_{it} itself can be a specific conditional return moment, equivalent to the fact that derivatives of the functions $A(x,y)$ and $B_i(x,y)$ also satisfy specific conditions. Proposition 3.2.1 below gives necessary and sufficient conditions under which a latent factor is the conditional variance or the

[1] Darolles, Gourieroux and Jasiak (2006) study in details conditions for the stationarity in distribution of vector affine processes. The vector process $\left(r_{t+1}, l_t^\top\right)^\top$ is stationary in distribution if the conditional moment-generating function $E_t\left[\exp\left(xr_{t+h}+y^\top l_{t+h}\right)\right]$ converges to the unconditional moment-generating function $E\left[\exp\left(xr_t+y^\top l_t\right)\right]$ as h approaches infinity.

conditional asymmetry.

Proposition 3.2.1. *The factor l_{it} is the conditional variance of returns if and only if*

$$\left.\frac{\partial^2 A(x,y)}{\partial x^2}\right|_{x=0,y=0} = 0 \text{ and } \left.\frac{\partial^2 B_j(x,y)}{\partial x^2}\right|_{x=0,y=0} = \mathbf{1}_{\{j=i\}}. \tag{3.2}$$

The factor l_{it} is the central conditional third moment of returns if and only if

$$\left.\frac{\partial^3 A(x,y)}{\partial x^3}\right|_{x=0,y=0} = 0 \text{ and } \left.\frac{\partial^3 B_j(x,y)}{\partial x^3}\right|_{x=0,y=0} = \mathbf{1}_{\{j=i\}}. \tag{3.3}$$

Especially, affine models of the form (3.1) with a single latent factor corresponding to the conditional variance have been widely studied in the literature as GARCH and Stochastic Volatility models. An extensive review of this literature is given in Shephard (2005). Example 1 below lists most common cases with normal return shocks conditional to the latent volatility.

Example 1. ***Stochastic Volatility.***
Discrete time parametric semi-affine latent variable models of returns with only one factor which is a conditional return moment, are the following stochastic volatility models which have been considered in many empirical studies. Return dynamics is given by:

$$r_{t+1} = \mu_r - \beta_h\mu_h + \beta_h h_t + \sqrt{h_t}u_{t+1} \tag{3.4}$$

where the volatility process satisfies one of the followings:

$$h_{t+1} = (1-\phi_h)\mu_h - \alpha_h + \left(\phi_h - \alpha_h\lambda_h^2\right)h_t + \alpha_h\left(\varepsilon_{t+1} - \lambda_h\sqrt{h_t}\right)^2, \tag{3.5}$$

$$h_{t+1} = (1-\phi_h)\mu_h + \phi_h h_t + \sigma_h\varepsilon_{t+1}, \tag{3.6}$$

$$h_{t+1} = (1-\phi_h)\mu_h + \phi_h h_t + \sigma_h\sqrt{h_t}\varepsilon_{t+1}, \tag{3.7}$$

and where u_{t+1} and ε_{t+1} are two i.i.d standard normal shocks. The parameter vector p is $(\mu_r, \beta_h, \mu_h, \phi_h, \alpha_h, \lambda_h, \rho_{rh})^\top$ with the volatility dynamics (3.5) whereas it is $(\mu_r, \beta_h, \mu_h, \phi_h, \sigma_h)^\top$ with the autoregressive gaussian volatility (3.6) and $(\mu_r, \beta_h, \mu_h, \phi_h, \sigma_h, \rho_{rh})^\top$ with the *square-root volatility (3.7), where ρ_{rh} denotes the conditional correlation between the shocks u_{t+1} and ε_{t+1}. The particular case $\rho_{rh} = 1$ in the volatility dynamics (3.5) corresponds to the Heston and Nandi (2000)'s GARCH(1,1). For this reason we refer to the dynamics (3.5) as HN-S volatility.*

The A and B functions characterizing the cumulant-generating functions for these models are given by:

$$A(x,y) = (\mu_r - \beta_h \mu_h)x + ((1-\phi_h)\mu_h - \alpha_h)y - \frac{1}{2}\ln(1-2\alpha_h y) \tag{3.8}$$

$$B(x,y) = \beta_h x + \left(\phi_h - \alpha_h \lambda_h^2\right)y + \frac{1}{2}x^2 + \frac{\alpha_h y}{1-2\alpha_h y}(\lambda_h - \rho_{rh}x)^2 \tag{3.9}$$

for the HN-S specification,

$$A(x,y) = (\mu_r - \beta_h \mu_h)x + (1-\phi_h)\mu_h y + \frac{1}{2}\sigma_h^2 y^2 \tag{3.10}$$

$$B(x,y) = \beta_h x + \phi_h y + \frac{1}{2}x^2 \tag{3.11}$$

for the autoregressive gaussian specification and finally

$$A(x,y) = (\mu_r - \beta_h \mu_h)x + (1-\phi_h)\mu_h y \tag{3.12}$$

$$B(x,y) = \beta_h x + \phi_h y + \frac{1}{2}\left(x^2 + 2\rho_{rh}\sigma_h xy + \sigma_h^2 y^2\right) \tag{3.13}$$

for the square-root specification.

One should notice that the volatility processes (3.6) and (3.7) are not well defined since h_t can take negative values for example in simulations of the process.[2] This can also arise with the process (3.5) unless parameters satisfy a couple of constraints. Note also that if the volatility shock ε_{t+1} in (3.6) is allowed to be correlated to the return shock u in (3.4), then the model becomes non-affine.

A known case of a well-defined affine stochastic volatility model assumes that h_t

[2]Because of this limitation, autoregressive gaussian and squared-root stochastic volatility models have been mainly explored in continuous time. To avoid negative values of h_t in simulations for examples, on can find the dynamics of $\ln h_t$ using the Itô lemma and work through the logarithmic model.

follows an autoregressive gamma process (Gourieroux and Jasiak (2001)). However, when combined with the return process (3.4), the model presumes that within a period, return and volatility shocks are mutually independent, what appears to be a counterfactual assumption against the well-documented conditional leverage effect (Black(1976) and Christie (1982)). This counterfactual assumption is not required for classical log-normal stochastic volatility and GARCH models. However, these latter models are less tractable in empirical studies because of their non-affinity. Then, there has always been a trade-off between tractable affine models with counterfactual assumptions and non-tractable non-affine models that do not require these assumptions. In this chapter, we aim at combining both the tractability of our affine model and its ability to take into account important features of the data (fat-tailedness, asymmetry and leverage effect) in a coherent way.

3.2.2 Conditional Leverage Effect and Skewness.

While return models of Example 1 are such that the vector $(r_{t+1}, h_{t+1})^{\top}$ of returns and volatility is affine, the conditional skewness of returns in these models is zero. The literature on asset return models has evolved so far and empirical evidence upon path dependence of conditional skewness as well as its contribution to risk management and asset pricing rose in recent studies. The necessity to model return skewness has become of first order importance.

Existing affine stochastic volatility models basically lead to a couple of equations of the form:

$$r_{t+1} = e(h_t) + \sqrt{h_t}u_{t+1} \tag{3.14}$$

$$h_{t+1} = m(h_t) + \sqrt{v(h_t)}\varepsilon_{t+1} \tag{3.15}$$

where u_{t+1} and ε_{t+1} are two errors with mean zero and unit variance. Written in this form, the conditional skewness of returns is zero unless u_{t+1} is conditionally asymmetric. Also, these models do not allow for the leverage effect unless the shocks u_{t+1} and ε_{t+1} are correlated. However, it is generally assumed that u_{t+1} is gaussian and unfeasible to assume a conditional correlation when at least one of the shocks is non-gaussian. This is a potential limitation that typically arises when u_{t+1} is gaussian and equation (3.15) is such that h_t is an autoregressive gamma process.

Since the leverage effect is the nonzero conditional covariance between returns and volatility, this means that projecting r_{t+1} onto h_{t+1} should lead to a nonzero slope coefficient. Then, another technique to account for skewness and leverage effect in asset returns modeling would be to project returns r_{t+1} onto volatility h_{t+1} and characterize the projection error. This will basically lead to a return equation of the form:

$$r_{t+1} = g(h_t) + \lambda h_{t+1} + \sqrt{h_t - \lambda^2 v(h_t)} u_{t+1} \tag{3.16}$$

where u_{t+1} is an error with mean zero and unit variance. One can still endow u_{t+1} with a suitable distribution conditional on $\langle h_t, h_{t+1} \rangle$ such that combining (3.15) with (3.16) leads to an affine stochastic volatility model of asset returns. The model will now account for the leverage effect through λ. The conditional skewness will also depend on λ as well as the conditional asymmetry of the shock u_{t+1}, if any. We further use a similar technique in our return modeling.

This chapter aims first at developing a semi-affine multifactor latent variable model of returns such that both conditional variance h_t and conditional skewness s_t are stochastic. Moreover, the vector $\left(r_{t+1}, h_{t+1}, s_{t+1} h_{t+1}^{3/2}\right)^{\top}$ is affine in the case of two linearly independent latent factors. It is more easy to think at a semi-affine one-factor model with stochastic volatility as in Example 1, that is such that the equation for volatility dynamics is directly specified, precisely because of tractable properties of the standard normal distribution that governs return and volatility dynamics. It is more challenging to think at a semi-affine two-factor model with stochastic skewness as additional factor, such that both equations for volatility and skewness dynamics are directly specified. The reason is that, while conditional asymmetry of returns appears to be a necessary and sufficient condition to generate time-variation in conditional skewness, asymmetric distributions are not as tractable as the normal distribution. A strategy to get equations which explicitly characterize the joint dynamics of returns, volatility and skewness would be to first specify a semi-affine two-factor model with arbitrary linearly independent latent factors, more easier to think at, and:

- find volatility and conditional skewness in terms of the two arbitrary factors,
- then, invert the previous relationship to determine the two arbitrary factors in terms of volatility and skewness,
- and finally, replace the arbitrary factors in the initial return model to get the joint dynamics of returns, volatility and skewness.

3.3 Return Models with Stochastic Skewness

3.3.1 General Setup

The dynamics of returns in our model is built upon shocks drawn from a standardized inverse gaussian distribution. The cumulant-generating function of a discrete random variable which follows a standardized inverse gaussian distribution of parameter s, denoted $SIG(s)$, is given by:

$$\psi(u;s) = \ln E\left[\exp(uX)\right] = -3s^{-1}u + 9s^{-2}\left(1 - \sqrt{1 - \frac{2}{3}su}\right). \tag{3.17}$$

For such a random variable, one has $E[X] = 0$, $E\left[X^2\right] = 1$ and $E\left[X^3\right] = s$, meaning that s is the skewness of X. In addition to the fact that the SIG distribution is directly parameterized by its skewness, the limiting distribution when the skewness s tends to zero is the standard normal distribution, that is $SIG(0) \equiv \mathcal{N}(0,1)$. This particularity makes the SIG an ideal building block for studying departures from normality.

For each variable in all what follows, the time subscript denotes the date from which the value of the variable is known. We assume that returns follow the dynamics:

$$\begin{aligned} r_{t+1} = \ln\frac{S_{t+1}}{S_t} &= \mu_0 + \sum_{i=1}^{k}\beta_i\left(\sigma_{it}^2 - \mu_i\right) + \sum_{i=1}^{k}\lambda_i\left(\sigma_{i,t+1}^2 - \mu_i\right) + \sum_{i=1}^{k}\sigma_{i,t+1}u_{i,t+1} \\ &= \delta_t + \lambda^\top\sigma_{t+1}^2 + \sigma_{t+1}^\top u_{t+1} \end{aligned} \tag{3.18}$$

where S_t is the price process, $\delta_t = \mu_0 - (\beta+\lambda)^\top \mu + \beta^\top \sigma_t^2$ and $u_{i,t+1} \mid \langle \sigma_{t+1}^2, I_t \rangle \sim SIG\left(\eta_i \sigma_{i,t+1}^{-1}\right)$. If $\eta_i = 0$, then $u_{i,t+1}$ is a standard normal shock. The k return shocks $u_{i,t+1}$ are mutually independent conditionally on $\langle \sigma_{t+1}^2, I_t \rangle$. The vector μ is the unconditional mean of the stationary process σ_t^2. In consequence μ_0 is the unconditional expected return. The time t information set I_t contains past returns $\underline{r_t} = \{r_t, r_{t-1}, ...\}$ and past latent factors $\underline{\sigma_t^2} = \{\sigma_t^2, \sigma_{t-1}^2, ...\}$.

The process σ_t^2 is assumed to be affine with the conditional cumulant generating function

$$\psi_t^\sigma(y) = \ln E\left[\exp\left(y^\top \sigma_{t+1}^2\right) \mid I_t\right] = a(y) + b(y)^\top \sigma_t^2. \tag{3.19}$$

In this case, the vector $\left(r_{t+1}, (\sigma_{t+1}^2)^\top\right)^\top$ is semi-affine in the sense of Bates (2006). Its conditional cumulant generating function is given by:

$$\Psi_t(x,y) = \ln E\left[\exp\left(x r_{t+1} + y^\top \sigma_{t+1}^2\right) \mid I_t\right] = A(x,y) + B(x,y)^\top \sigma_t^2,$$

with

$$A(x,y) = \left(\mu_0 - (\beta+\lambda)^\top \mu\right) x + a(f(x,y)) \tag{3.20}$$

$$B(x,y) = \beta x + b(f(x,y)) \tag{3.21}$$

where $f(x,y) = (f_1(x,y_1), .., f_k(x,y_k))^\top$ with $f_i(x,y_i) = y_i + \lambda_i x + \psi(x;\eta_i)$.

Since the factors σ_{it}^2 are nonnegative, we assume that the vector σ_t^2 follows a multivariate autoregressive gamma process. This process also represents the discrete-time counterpart to many of the multivariate affine diffusions that have previously been examined in the literature. It follows that the log conditional Laplace transform of the vector σ_t^2 has the exponential affine form (3.19) with:

$$a(y) = -\sum_{i=1}^{k} \nu_i \ln(1-\alpha_i y_i) \text{ and } b_i(y) = \sum_{j=1}^{k} \frac{\phi_{ij} y_j}{1-\alpha_j y_j}.$$

The $k \times k$ matrix $\Phi = \left[\phi_{ij}\right]$ represents the persistence matrix of the vector σ_t^2 and the autoregressive gamma processes σ_{it}^2 are mutually correlated if the off-diagonal elements of Φ are nonzero. More specifically for the one-factor model that we focus on in this chapter, the unique latent state variable σ_{1t}^2 has the following conditional cumulant generating function:

$$\psi_t^{\sigma}(y_1) = \ln E\left[\exp\left(y_1 \sigma_{1,t+1}^2\right) \mid I_t\right] = a(y_1) + b_1(y_1)\sigma_{1t}^2$$

where

$$a(y_1) = -\nu_1 \ln(1 - \alpha_1 y_1) \text{ and } b_1(y_1) = \frac{\phi_1 y_1}{1 - \alpha_1 y_1}.$$

The parameter ϕ_1 is the persistence of the factor and the parameters ν_1 and α_1 are related to persistence and unconditional mean μ_1 and variance ω_1 as follows:

$$\nu_1 = \frac{\mu_1^2}{\omega_1} \text{ and } \alpha_1 = \frac{(1-\phi_1)\,\omega_1}{\mu_1}.$$

Proposition 3.3.1. *Conditional on I_t, the mean μ_t^r, the variance h_t and the skewness s_t of returns are given by:*

$$\mu_t^r = \mu_0 - (\beta + \lambda)^\top \mu + \beta^\top \sigma_t^2 + \lambda^\top m_t^{\sigma} = c_{0\mu} + \sum_{i=1}^{k} c_{i\mu}\sigma_{it}^2 = c_{0\mu} + c_{\mu}^\top \sigma_t^2, \quad (3.22)$$

$$h_t = \lambda^\top V_t^{\sigma} \lambda + e^\top m_t^{\sigma} = c_{0h} + \sum_{i=1}^{k} c_{ih}\sigma_{it}^2 = c_{0h} + c_h^\top \sigma_t^2, \quad (3.23)$$

$$s_t h_t^{3/2} = (\lambda \otimes \lambda)^\top S_t^{\sigma} \lambda + 3e^\top V_t^{\sigma} \lambda + \eta^\top m_t^{\sigma} = c_{0s} + \sum_{i=1}^{k} c_{is}\sigma_{it}^2 = c_{0s} + c_s^\top \sigma_t^2, \quad (3.24)$$

where the coefficients c_{nl} depend on model parameters,

$$m_t^{\sigma} = E\left[\sigma_{t+1}^2 \mid I_t\right], \; V_t^{\sigma} = E\left[\left(\sigma_{t+1}^2 - m_t^{\sigma}\right)\left(\sigma_{t+1}^2 - m_t^{\sigma}\right)^\top \mid I_t\right]$$

and

$$S_t^{\sigma} = E\left[\left(\left(\sigma_{t+1}^2 - m_t^{\sigma}\right) \otimes \left(\sigma_{t+1}^2 - m_t^{\sigma}\right)\right)\left(\sigma_{t+1}^2 - m_t^{\sigma}\right)^\top \mid I_t\right].$$

The vector e denotes the $k \times 1$ vector of ones.

The linearity of conditional volatility and conditional third moment of returns in terms of latent state variables comes from the fact that the elements of the vector m_t^{σ} and of the matrices V_t^{σ} and S_t^{σ} are also linear in these variables. This is a consequence of the affine structure of the process σ_t^2. Also, note that the bivariate vector $\left(h_t, s_t h_t^{3/2}\right)^{\top}$ is not deterministically related to contemporaneous and past returns as for GARCH-type processes as in Harvey and Siddique (2000). This is the reason why we label our model stochastic volatility and skewness (SVS model).

Proposition 3.3.2. *Conditional on I_t, the covariance between returns and volatility (leverage effect) and the covariance between returns and skewness are given by:*

$$Cov(r_{t+1}, h_{t+1} \mid I_t) = c_h^{\top} V_t^{\sigma} \lambda = c_{0,rh} + \sum_{i=1}^{k} c_{i,rh} \sigma_{it}^2 = c_{0,rh} + c_{rh}^{\top} \sigma_t^2, \tag{3.25}$$

$$Cov\left(r_{t+1}, s_{t+1} h_{t+1}^{3/2} \mid I_t\right) = c_s^{\top} V_t^{\sigma} \lambda = c_{0,rs} + \sum_{i=1}^{k} c_{i,rs} \sigma_{it}^2 = c_{0,rs} + c_{rs}^{\top} \sigma_t^2, \tag{3.26}$$

where the c's depend on parameters.

It should be noted that, in our SVS model, although the parameter η dictates the contemporaneous conditional asymmetry of returns—that is, the asymmetry of returns conditional on factors of the same date—it is not the only parameter that characterizes the conditional skewness of returns as defined in equation (3.24). The parameter λ plays a central role in generating conditional asymmetry in returns, even if returns are normally distributed conditional upon contemporaneous factors, that is when $\eta = 0$.

It is also not surprising that the vector λ governs the conditional leverage effect since it represents the slope vector of the linear projection of returns on factors of the same date. For a negative correlation between spot returns and variance, and consistently with the postulate of Black (1976) and the leverage effect documented by Christie (1982) and others, the parameter λ may be expected to be negative. If $\lambda = 0$, there is no leverage effect. There is also no skewness unless $\eta \neq 0$. Then, the contemporaneous conditional asymmetry in this model reinforces the effects of the leverage parameter λ.

While $\sigma_{1t}^2, .., \sigma_{kt}^2$ are the primitive predictive variables in our SVS model, predictability when $k \geq 2$ can also be directly related to conditional variance and skewness which are more economically interpretable. For example, empirical facts support that an increase in return variance leads to an increase in expected returns. This comes from the fact that agents require more risk premium when the stock payoff become more volatile, meaning that it becomes more riskier to invest in the stock. As well as agents dislike high return volatility, they prefer positive return skewness since it implies that higher and even extreme positive values of return are more likely to realize. Then, agents are ready to deliver some premium in exchange of a positive skewness, or to require some premium to compensate a negative skewness.

When $k \geq 2$ and if $c_{1h}c_{2s} \neq c_{1s}c_{2h}$ without loss of generality, one can invert relations (3.23) and (3.24) to obtain σ_{1t}^2 and σ_{2t}^2 in terms of h_t and $s_t h_t^{3/2}$. Using inverted relations in (3.22) gives expected returns in terms of volatility and skewness:

$$\mu_t = c_{0\mu}^* + c_{1\mu}^* h_t + c_{2\mu}^* s_t h_t^{3/2} + \sum_{i=3}^{k} c_{i\mu}^* \sigma_{it}^2 \tag{3.27}$$

where

$$c_{0\mu}^* = c_{0\mu} + c_{1\mu}\frac{c_{0s}c_{2h} - c_{0h}c_{2s}}{c_{1h}c_{2s} - c_{1s}c_{2h}} + c_{2\mu}\frac{c_{0h}c_{1s} - c_{0s}c_{1h}}{c_{1h}c_{2s} - c_{1s}c_{2h}}$$

$$c_{1\mu}^* = \frac{c_{1\mu}c_{2s} - c_{1s}c_{2\mu}}{c_{1h}c_{2s} - c_{1s}c_{2h}} \text{ and } c_{2\mu}^* = \frac{c_{1h}c_{2\mu} - c_{1\mu}c_{2h}}{c_{1h}c_{2s} - c_{1s}c_{2h}}$$

$$c_{i\mu}^* = c_{1\mu}\frac{c_{is}c_{2h} - c_{ih}c_{2s}}{c_{1h}c_{2s} - c_{1s}c_{2h}} + c_{2\mu}\frac{c_{ih}c_{1s} - c_{is}c_{1h}}{c_{1h}c_{2s} - c_{1s}c_{2h}}$$

Moreover if $k = 2$, it turns out that the vector $\left(r_{t+1}, h_{t+1}, s_{t+1}h_{t+1}^{3/2}\right)^\top$ is semi-affine with the conditional characteristic function:

$$\begin{aligned}\Psi_{*t}(x, y_1, y_2) &= \ln E\left[\exp\left(x r_{t+1} + y_1 h_{t+1} + y_2 s_{t+1} h_{t+1}^{3/2}\right) \mid I_t\right] \\ &= A_*(x, y_1, y_2) + B_h(x, y_1, y_2)\, h_t + B_s(x, y_1, y_2)\, s_t h_t^{3/2},\end{aligned} \tag{3.28}$$

with

$$A_*(x,y_1,y_2) = c_{0h}y_1 + c_{0s}y_2 + \frac{c_{0s}c_{2h} - c_{0h}c_{2s}}{c_{1h}c_{2s} - c_{1s}c_{2h}} B_1(x, c_{1h}y_1 + c_{1s}y_2, c_{2h}y_1 + c_{2s}y_2)$$
$$+ \frac{c_{0h}c_{1s} - c_{0s}c_{1h}}{c_{1h}c_{2s} - c_{1s}c_{2h}} B_2(x, c_{1h}y_1 + c_{1s}y_2, c_{2h}y_1 + c_{2s}y_2), \quad (3.29)$$

$$B_h(x,y_1,y_2) = \frac{c_{2s}}{c_{1h}c_{2s} - c_{1s}c_{2h}} B_1(x, c_{1h}y_1 + c_{1s}y_2, c_{2h}y_1 + c_{2s}y_2)$$
$$- \frac{c_{1s}}{c_{1h}c_{2s} - c_{1s}c_{2h}} B_2(x, c_{1h}y_1 + c_{1s}y_2, c_{2h}y_1 + c_{2s}y_2), \quad (3.30)$$

and

$$B_s(x,y_1,y_2) = \frac{c_{1h}}{c_{1h}c_{2s} - c_{1s}c_{2h}} B_2(x, c_{1h}y_1 + c_{1s}y_2, c_{2h}y_1 + c_{2s}y_2)$$
$$- \frac{c_{2h}}{c_{1h}c_{2s} - c_{1s}c_{2h}} B_1(x, c_{1h}y_1 + c_{1s}y_2, c_{2h}y_1 + c_{2s}y_2), \quad (3.31)$$

where the functions A and $B = (B_1, B_2)^\top$ are defined in (3.20) and (3.21). In this case, the advantage of the SVS model is that unobserved variables are directly interpretable as conditional variance and skewness instead of arbitrary factors.

While the IG-GARCH model of Christoffersen, Heston and Jacobs (2006) implies a strong relationship between conditional variance and skewness, in our two-factor case, we disentangle these two moments while maintaining a semi-affine structure of the model. This separation between the volatility and the conditional skewness comes from the decomposition of return shocks into two linearly independent components whose individual variances have specific dynamics.

3.3.2 Continuous-Time Limits

We are interested in continuous-time versions of our SVS models. In appendix IV, we derive the continuous-time versions of our one-factor SVS model. We show that the two-factor SVS model has two interesting continuous-time limits. Writing the SVS model for a small time interval, we consider letting the time interval shrink to zero.

Compound autoregressive processes as σ_t^2 in our case have been widely discussed by Gourieroux and Jasiak (2006) as well as Lamberton and Lapeyre (1992). They show that the continous time limit of a univariate autoregressive gamma process is a square-root process. It follows that the dynamics of σ_{1t}^2 converges to the square-root diffusion:

$$d\sigma_{1t}^2 = \kappa_1\left(\varpi_1 - \sigma_{1t}^2\right)dt + e_1\sigma_{1t}dw_{1t}$$

where w_{1t} is a Wiener process and κ_1, ϖ_1 and e_1 are related to the initial parameters as follows:

$$\kappa_1 = -\ln\phi_1, \quad \varpi_1 = \frac{\nu_1\alpha_1}{1-\phi_1}, \quad \text{and} \quad e_1^2 = \frac{-2\ln\phi_1}{1-\phi_1}\alpha_1. \tag{3.32}$$

The two continuous-time limits of the one-factor SVS model differ from their return processes. We consider that δ_t is constant. The reason is that in continuous time in the return equation (3.18) one cannot identify β_i and λ_i separately. To avoid this identification problem, we set $\beta_i \doteq 0$ in the continuous time limit. If both η_1 approaches zero, then the return process converges to:

$$d\ln S_t = \left[\mu_0 + \lambda_1\left(\sigma_{1t}^2 - \mu_1\right)\right]dt + \sigma_{1t}dz_{1t} \tag{3.33}$$

where z_{1t} is a Wiener process. Instead, if η_1 is constant, then the return process converges to:

$$d\ln S_t = \left[\mu_0 + \lambda_1\left(\sigma_{1t}^2 - \mu_1\right) - \frac{3\sigma_{1t}^2}{\eta_1}\right]dt + \frac{\eta_1}{3}dy_{1t} \tag{3.34}$$

where y_{1t} is a pure-jump inverse gaussian process with degree of freedom $9\sigma_{1t}^2/\eta_1^2$ on interval $[t, t+dt]$. The stock price in this case converges to a pure-jump process with stochastic intensity.[3]

[3]The inverse Gaussian process has been investigated by Barndorff-Neilsen and Levendorskii (2000), Jensen and Lunde (2001), and Bollerslev and Forsberg (2002). See also the excellent overview of related processes in Barndorff-Nielsen and Shephard (2001).

3.3.3 GARCH vs. SVS

In GARCH models, the information set I_t is exactly $\underline{r_t}$ so that both the economic agent and the econometrician view the same information set. This is an implicit strong assumption in GARCH models. In the SVS model, the econometrician doesn't observe $\underline{\sigma_t^2}$, only known by the economic agent. While the moments in Proposition 3.3.1 are conditional on $I_t = \underline{r_t} \bigcup \underline{\sigma_t^2}$, one can also derived their GARCH counterparts, meaning same return moments now conditional on econometrician's information, $\underline{r_t}$ only. Let $\mu_t^{r,G}$, h_t^G and s_t^G respectively denote the mean, the variance and the skewness of r_{t+1} conditional on $\underline{r_t}$. One has:

$$\mu_t^{r,G} = c_{0\mu} + c_\mu^\top G_{\mu t}, \tag{3.35}$$

$$h_t^G = c_{0h} + c_h^\top G_{\mu t} + c_\mu^\top G_{ht} c_\mu, \tag{3.36}$$

$$s_t^G \left(h_t^G\right)^{3/2} = c_{0s} + c_s^\top G_{\mu t} + c_\mu^\top G_{ht} c_h + \left(c_\mu \otimes c_\mu\right)^\top G_{st} c_\mu \tag{3.37}$$

where

$$G_{\mu t} = E\left[\sigma_t^2 \mid \underline{r_t}\right], \tag{3.38}$$

$$G_{ht} = E\left[\sigma_t^2 \left(\sigma_t^2\right)^\top \mid \underline{r_t}\right] - E\left[\sigma_t^2 \mid \underline{r_t}\right] E\left[\sigma_t^2 \mid \underline{r_t}\right]^\top, \tag{3.39}$$

$$\begin{aligned} G_{st} = {} & E\left[\left(\sigma_t^2 \otimes \sigma_t^2\right)\left(\sigma_t^2\right)^\top \mid \underline{r_t}\right] - 3E\left[\left(\sigma_t^2 \otimes \sigma_t^2\right) \mid \underline{r_t}\right] E\left[\sigma_t^2 \mid \underline{r_t}\right]^\top \\ & + 2\left(E\left[\sigma_t^2 \mid \underline{r_t}\right] \otimes E\left[\sigma_t^2 \mid \underline{r_t}\right]\right) E\left[\sigma_t^2 \mid \underline{r_t}\right]^\top. \end{aligned} \tag{3.40}$$

are mean, variance and third central moment of the latent vector σ_t^2 conditional upon observed returns $\underline{r_t}$.

Disentangling agent and econometrician information sets in return modeling can be crucial. In our SVS model with only one latent factor, return conditional variance and central third moment are perfectly correlated to the agent, whereas it is the contrary to the econometrician unless returns are unpredictable ($c_\mu = 0$). Under return predictability, our one latent variable SVS model generates, conditional to observable returns, an asymmetry that is not perfectly correlated to the variance. This is the contrary in the

IG-GARCH model of Christoffersen, Heston and Jacobs (2006) where these two conditional moments are perfectly correlated. Also, While these authors restrict the conditional skewness of returns to be negative, Feunou (2006) provides an empirical evidence that conditional skewness, although centered around a negative value, can be positive at some dates. The autoregressive conditional skewness of Harvey and Siddique (2000) can also attain positive values. This can arise in our SVS model as we don't impose any restriction on parameters.

The GARCH counterparts of the leverage effect and of the conditional covariance between returns and skewness are defined by:

$$Cov\left(r_{t+1}, h_{t+1}^{G} \mid \underline{r_t}\right) \text{ and } Cov\left(r_{t+1}, s_{t+1}^{G}\left(h_{t+1}^{G}\right)^{3/2} \mid \underline{r_t}\right).$$

These two quantities are difficult to express in terms of the moments of the latent vector σ_t^2 conditional on observed returns $\underline{r_t}$ and instead we consider the following two quantities which are more easier:

$$Cov\left(r_{t+1}, h_{t+1} \mid \underline{r_t}\right) = c_{0,rh} + c_{rh}^{\top} G_{\mu t} + c_{\mu}^{\top} G_{ht} \Phi^{\top} c_h \tag{3.41}$$

$$Cov\left(r_{t+1}, s_{t+1} h_{t+1}^{3/2} \mid \underline{r_t}\right) = c_{0,rs} + c_{rs}^{\top} G_{\mu t} + c_{\mu}^{\top} G_{ht} \Phi^{\top} c_s, \tag{3.42}$$

where Φ represents the persistence matrix of the latent vector.

3.4 Filtering

Various strategies to deal with non-linear state-space systems have been proposed in the filtering literature: the Extended Kalman Filter, the Particle Filter and more recently the Unscented Kalman Filter that we apply in this chapter.[4] Since our SVS model has the standard state space representation, on can use Kalman Filter-based techniques to compute $G_{\mu t}$, G_{ht} and G_{st}. As these methods will not guarantee that $E\left[\sigma_{it}^2 \mid \underline{r_t}\right]$ is positive, it would be more convenient to filter $\omega_{it} = \ln \sigma_{it}^2$. Let $\omega_t = \left(\omega_{1t}, .., \omega_{kt}\right)^{\top}$.

[4]See Leippold and Wu (2003) and Bakshi, Carr and Wu (2005) for application in finance, Julier et al. (1995) and Jullier and Uhlmann (1996) for details and Wan and van der Merwe (2001) for textbook treatment.

The basic framework of Kalman filter techniques involves estimation of the state of a discrete-time nonlinear dynamic system of the form:

$$r_{t+1} = H\left(\omega_{t+1}, u_{t+1}^{*}\right) \tag{3.43}$$

$$\omega_{t+1} = F\left(\omega_{t}, \varepsilon_{t+1}^{*}\right), \tag{3.44}$$

where u_{t+1}^{*} and ε_{t+1}^{*} are not necessarily but conventionally two gaussian noises. For this reason, we log-normally approximate our model, which in the one-factor case leads to:

$$H\left(\omega_{1,t+1}, u_{1,t+1}^{*}\right) = \mu_0 + \lambda_1 \exp\left(\omega_{1,t+1}\right) + \exp\left(\frac{\omega_{1,t+1}}{2}\right)\left[\exp\left(\ln\left(\frac{9}{s\left(\omega_{1,t+1}\right)\sqrt{s\left(\omega_{1,t+1}\right)^2+9}}\right) + \sqrt{\ln\left(\frac{s\left(\omega_{1,t+1}\right)^2+9}{9}\right)}\,u_{1,t+1}^{*}\right) - \frac{3}{s\left(\omega_{1,t+1}\right)}\right] \tag{3.45}$$

and

$$F\left(\omega_{1t}, \varepsilon_{1,t+1}^{*}\right) = \ln\left(\frac{m\left(\omega_{1t}\right)^2}{\sqrt{m\left(\omega_{1t}\right)^2 + v\left(\omega_{1t}\right)}}\right) + \sqrt{\ln\left(\frac{m\left(\omega_{1t}\right)^2 + v\left(\omega_{1t}\right)}{m\left(\omega_{1t}\right)^2}\right)}\,\varepsilon_{1,t+1}^{*}. \tag{3.46}$$

where

$$\begin{aligned} s\left(\omega_{1,t+1}\right) &= \eta_1 \exp\left(-\frac{\omega_{1,t+1}}{2}\right) \\ m\left(\omega_{1t}\right) &= \left(1-\phi_1\right)\mu_1 + \phi_1 \exp\left(\omega_{1t}\right) \\ v\left(\omega_{1t}\right) &= \left(1-\phi_1\right)^2 \sigma_1^2 + \frac{2\left(1-\phi_1\right)\phi_1\sigma_1^2}{\mu_1}\exp\left(\omega_{1t}\right). \end{aligned}$$

Details on this log-normal approximation for one-factor as well as two-factor models are provided in appendix VI.

Let $\omega_{t|\tau}$ be the estimate of ω_t using returns up to and including time τ, $\underline{r_{\tau}}$, and let $P_{t|\tau}^{\omega\omega}$ be its covariance. Given the join distribution of $\left(\omega_t^{\top}, u_{t+1}^{*\top}, \varepsilon_{t+1}^{*\top}\right)^{\top}$ conditionally to

$\underline{r_t}$, the filter predicts what future state and returns will be using process models. Optimal predictions and associated mean squared errors are given by:

$$\omega_{t+1|t} = E\left[\omega_{t+1} \mid \underline{r_t}\right] = E\left[F\left(\omega_t, \varepsilon^*_{t+1}\right) \mid \underline{r_t}\right] \tag{3.47}$$

$$r_{t+1|t} = E\left[r_{t+1} \mid \underline{r_t}\right] = E\left[H\left(\omega_{t+1}, \varepsilon^*_{t+1}\right) \mid \underline{r_t}\right] \tag{3.48}$$

$$P^{\omega\omega}_{t+1|t} = E\left[\left(\omega_{t+1} - \omega_{t+1|t}\right)\left(\omega_{t+1} - \omega_{t+1|t}\right)^\top \mid \underline{r_t}\right] \tag{3.49}$$

$$P^{rr}_{t+1|t} = E\left[\left(r_{t+1} - r_{t+1|t}\right)\left(r_{t+1} - r_{t+1|t}\right)^\top \mid \underline{r_t}\right] \tag{3.50}$$

$$P^{\omega r}_{t+1|t} = E\left[\left(\omega_{t+1} - \omega_{t+1|t}\right)\left(r_{t+1} - r_{t+1|t}\right)^\top \mid \underline{r_t}\right]. \tag{3.51}$$

The join distribution of $\left(\omega_t^\top, u^{*\top}_{t+1}, \varepsilon^{*\top}_{t+1}\right)^\top$ conditionally to $\underline{r_t}$ is conventionally assumed gaussian. To the contrary of the standard Kalman filter where the functions H and F are linear, the precise values of the conditional moments (3.47) to (3.51) can not be determined analytically in our model because the functions H and F are strongly nonlinear. Alternative methods produce approximations of these conditional moments.

The Extended Kalman Filter linearizes the functionals H and F in the state-space system to determine the conditional moments analytically. While this simple linearization maintains a first-order accuracy, it can introduce large errors in the true posterior mean and covariance of the transformed random variable which may lead to sub-optimal performance and sometimes to divergence of the filter. The Particle Filter uses Monte-Carlo simulations of the relevant distributions to get estimates of moments. In contrast, the Uncented Kalman Filter adresses the approximation issues of the Extended Kalman filter and the computational issues of the Particle Filter. It represents the distribution of $\left(\omega_t^\top, u^{*\top}_{t+1}, \varepsilon^{*\top}_{t+1}\right)^\top$ conditional on $\underline{r_t}$ by a minimal set of carefully chosen points. This reduces the computational burden but maintain second-order accuracy. Details on the Unscented Kalman Filter are provided in appendix VIII.

The next step is to use current returns to update estimate (3.47) of the state. In the Kalman filter, a linear update rule is specified, where the weights are chosen to minimize the mean squared error of the estimate. This rule is given by:

$$\omega_{t+1|t+1} = \omega_{t+1|t} + K_{t+1}\left(r_{t+1} - r_{t+1|t}\right) \tag{3.52}$$

$$P^{\omega\omega}_{t+1|t+1} = P^{\omega\omega}_{t+1|t} - K_{t+1} P^{rr}_{t+1|t} K^{\top}_{t+1} \tag{3.53}$$

$$K_{t+1} = P^{\omega r}_{t+1|t}\left(P^{rr}_{t+1|t}\right)^{-1}. \tag{3.54}$$

Once the Kalman recursion outlined above delivers the estimates $\omega_{t|t}$ and $P^{\omega\omega}_{t|t}$ for the whole sample, the statistics $G_{\mu t}$, G_{ht} and G_{st} can be computed using approximations of moments of a nonlinear function of a gaussian random variable. Without loss of generality, appendix VII derives corresponding formulas in the univariate case.

3.5 Arbitrage-Free and Risk-Neutral Pricing

In the context of asset and derivative pricing, one would like to find a probability measure under which the expected gross return on a security equals the gross return on the safe security. To define such a probability measure, it is sufficient to define a Radon-Nikodym derivative which changes the historical measure into the risk-neutral measure (see Christoffersen et al. (2006)). This is also equivalent to define a stochastic discount factor $M_{t,t+1}$ (as in Gourieroux and Monfort (2006)) from which investors value financial payoffs. The stochastic discount factor $M_{t,t+1}$ satisfies the following conditions:

$$E\left[M_{t,t+1} \mid I_t\right] = \exp\left(-r_{f,t+1}\right) \quad \text{and} \quad E\left[M_{t,t+1}\exp\left(r_{t+1}\right) \mid I_t\right] = 1. \tag{3.55}$$

where $r_{f,t+1}$ refers to the risk-free rate from date t to date $t+1$, known at date t since the final payoff of a safe security is known in advance.

The conditions (3.55) are the familiar Euler conditions for the safe and the risky securities. Recent asset pricing general equilibrium models decompose log returns into exogenous consumption growth with specified dynamics and an endogenous part that depends on the price-consumption ratio solved through Euler conditions (See as examples Bansal and Yaron (2004) and Tauchen (2005)). They follow the economic definition of returns as the ratio of future payoffs to current price. In the alternative approach used in this chapter, we follow the statistical definition of log returns as a sum of endogenous expected returns which depend on a variable like δ_t (see also Duan, Ritchken and Sun (2005)) and exogenous return innovation with specified dynamics. Solving for δ_t through the conditions (3.55) necessitates the knowledge of the exact form of the pricing kernel or equivalently of the change of measure.

From the affinity of our SVS models we conjecture that the stochastic discount factor has the form:

$$M_{t,t+1} = \exp\left(\varsigma_t + \kappa r_{t+1} + \pi^\top \sigma_{t+1}^2\right) = \exp\left(\varsigma_t + \kappa r_{t+1} + \sum_{i=1}^{k} \pi_i \sigma_{i,t+1}^2\right). \tag{3.56}$$

This form of the change of measure is different from that considered in previous studies in option pricing. Heston and Nandi (2000) and Christoffersen, Heston and Jacobs (2006) conjecture that the change of measure is log-linear in returns only. Including latent variables governing the return dynamics as we do in this chapter is more familiar with the context of general equilibrium models. For example, in a affine general equilibrium model with stochastic volatility as in Bansal and Yaron (2004) and Tauchen (2005), the change of measure of a representative investor with recursive preferences of Epstein and Zin (1989), depend log-linearly on both the return on aggregate wealth and the volatility of aggregate consumption.

From conditions (3.55) one has that:

$$\varsigma_t = -A(1+\kappa,\pi) - B(1+\kappa,\pi)^\top \sigma_t^2 \tag{3.57}$$

$$r_{f,t+1} = [A(1+\kappa,\pi) - A(\kappa,\pi)] + [B(1+\kappa,\pi) - B(\kappa,\pi)]^\top \sigma_t^2. \tag{3.58}$$

While the pricing kernel is completely determined in many asset pricing models with endogenous risk-free rate and equity premium, particularly in equilibrium models cited in this chapter, an alternative literature considers that the risk-free rate is constant (Heston and Nandi (2000) and Christoffersen, Heston and Jacobs (2006)), then transmits the indeterminacy to the change of measure through ς_t. If the risk-free rate is constant and equal to r_f in our models, then the endogenous risk premium is given by:

$$\mu_0 - r_f = (\beta + \lambda)^\top \mu + a(f(\kappa, \pi)) - a(f(1+\kappa, \pi)) \tag{3.59}$$

where

$$\beta = b(f(\kappa, \pi)) - b(f(1+\kappa, \pi)). \tag{3.60}$$

Equation (3.59) gives the risk-premium as function of agent preferences characterized by the parameters κ and π. As shown in appendix V, in a general equilibrium model with the recursive utility of Epstein and Zin (1989) and unitary elasticity of intertemporal substitution, the parameter κ is the opposite of the risk aversion parameter while the parameter π is a function of the risk-aversion and the subjective discount factor.

The joint dynamics of returns and latent variables under the risk-neutral distribution is characterized by the following cumulant generating function:

$$\Psi_t^*(x,y) = \ln E^*\left[\exp\left(xr_{t+1} + y^\top \sigma_{t+1}^2\right) \mid I_t\right] = A^*(x,y) + B^*(x,y)^\top \sigma_t^2$$

where $E^*[\cdot \mid I_t]$ denotes the expectation associated with the density $M_{t,t+1}\exp\left(r_{f,t+1}\right)$ and

$$A^*(x,y) = A(x+\kappa, y+\pi) - A(\kappa,\pi) \text{ and } B^*(x,y) = B(x+\kappa, y+\pi) - B(\kappa,\pi).$$

Let $\psi_{t,t+h}^{*r}(x)$ denotes the conditional log-moment generating function of aggregate returns $\sum_{i=1}^{h} r_{t+i}$. One has

$$E^*\left[\exp\left(x\sum_{i=1}^{h} r_{t+i}\right) \mid I_t\right] = \exp\left(\psi_{t,t+h}^{*r}(x)\right) = \exp\left(A_r^*(x;h) + B_r^*(x;h)^\top \sigma_t^2\right),$$

where the sequence of functions $A_r^*(x;h)$ and $B_r^*(x;h)$ satisfy the following recursion:

$$A_r^*(x;h) = A_r^*(x;h-1) + A^*(x, B_r^*(x;h-1)) \text{ and } B_r^*(x;h) = B^*(x, B_r^*(x;h-1)),$$

with $A_r^*(x;1) = A^*(x,0)$ and $B_r^*(x;1) = B^*(x,0)$.

3.6 Unconditional Moments and GMM Estimation of Semi-Affine Latent Variable Models.

In this section we show a simple procedure to compute analytically unconditional moments of observable in a semi-affine multifactor latent variable model. We further confront these analytical moments to their sample counterparts in a single step optimal GMM estimation. The estimation of latent variable models and in particular of discrete time stochastic volatility models like (3.6) and (3.7) have become a challenging issue in financial econometrics literature. From an econometric viewpoint a practical drawback of stochastic volatility models is the intractability of the likelihood function. Because volatility is an unobserved component and the model is non-gaussian, the likelihood function is only available in the form of a multiple integral. Also, in the case of the univariate lognormal stochastic autoregressive volatility model, Quasi Maximum Likelihood (QML) and Method of Moments estimators are not very reliable (see Jacquier, Polson, and Rossi, 1994; Andersen and Sørensen, 1996). Exact likelihood-oriented methods require simulations and are thus computer intensive (see Danielsson, 1994; Jacquier, Polson, and Rossi, 1994).

In the case of semi-affine models whose the cumulant-generating function takes the form (3.1), Bates (2006) provides an algorithm to perform the estimation via Approximated Maximum Likelihood (AML). In this chapter we show that in such models, relevant unconditional moments of observable (here the returns) can be derived analytically. Examples of such moments are mean, variance, skewness, kurtosis and autocorrelations of squared returns. This allows for a GMM-based estimation of the vector of parameters θ that is more easier to perform as it is done very quickly and is not computationally intensive. Moreover, the existence of closed-form formulas helps analyzing the impact of several model parameters on critical return moments (for example, skewness, kurtosis and autocorrelation of squared returns). This also enhances our understanding of mechanisms behind analytical results and of the limits of validity of methods based on approximations. We present the model estimation in the more general setting of semi-affine multifactor latent variable model of returns presented in Section 3.2.1. It is worthwhile to notice that the procedure can be extended to a setting where r_{t+1} is a vector.

3.6.1 Analytical Expressions of Unconditional Moments

Given the joint cumulant-generating function (3.1), the conditional moment-generating function of the vector of latent variables l_t is given by:

$$E_t\left[\exp\left(y^\top l_{t+1}\right)\right] = \exp\left(A_l(y) + B_l(y)^\top l_t\right) \tag{3.61}$$

where $A_l(y) \equiv A(0,y)$ and $B_l(y) \equiv B(0,y)$. The unconditional moment-generating function of the latent vector is then given by:

$$E\left[\exp\left(y^\top l_{t+1}\right)\right] = E\left[E_t\left[\exp\left(y^\top l_{t+1}\right)\right]\right] = E\left[\exp\left(A_l(y) + B_l(y)^\top l_t\right)\right], \tag{3.62}$$

from which we deduce that the cumulant-generating function $\Psi_l(y) = \ln E\left[\exp\left(y^\top l_t\right)\right]$ satisfies:

$$\Psi_l(y) = A_l(y) + \Psi_l(B_l(y)). \tag{3.63}$$

This function can be found analytically as for affine (jump-)diffusion processes as in Jiang and knight (2002). Since the unconditional cumulant generating function can be expressed as an infinite polynomial whose coefficients are unconditional cumulants, we notice that it is sufficient in a discrete time setting to find the derivatives of $\Psi_l(y)$ at $y = 0$, and this can be done through equation (3.63), since $B_l(0) = 0$.

Similarly, the conditional moment-generating function of observable returns r_t given the joint cumulant-generating function (3.1) can be written:

$$E_t\left[\exp\left(x r_{t+1}\right)\right] = \exp\left(A_r(x) + B_r(x)^\top l_t\right) \tag{3.64}$$

where $A_r(x) \equiv A(x,0)$ and $B_r(x) \equiv B(x,0)$. The unconditional moment-generating function of observable returns is then given by:

$$E\left[\exp\left(xr_{t+1}\right)\right]=E\left[E_t\left[\exp\left(xr_{t+1}\right)\right]\right]=E\left[\exp\left(A_r(x)+B_r(x)^\top l_t\right)\right], \tag{3.65}$$

from which we deduce that the cumulant-generating function $\Psi_r(x)=\ln E\left[\exp\left(xr_t\right)\right]$ satisfies:

$$\Psi_r(x)=A_r(x)+\Psi_l\left(B_r(x)\right). \tag{3.66}$$

Proposition 3.6.1. *The n-th unconditional cumulant of the observable returns r_t is the number $\kappa_r(n)$ given by:*

$$\kappa_r(n)=\frac{\partial^n \Psi_r}{\partial x^n}(0)=\frac{\partial^n A_r}{\partial x^n}(0)+\left.\frac{\partial^n}{\partial x^n}\left(\Psi_l\left(B_r(x)\right)\right)\right|_{x=0}. \tag{3.67}$$

As we mentioned earlier, cross-moments of returns can also be computed analytically and this can be performed through cross-cumulants of couples $\left(r_{t+1},r_{t+1+j}\right)^\top, j>0$. The unconditional moment-generating function of such couples is easily obtained in case of affine models (See Darolles et al. 2006). One has:

$$E\left[\exp\left(xr_{t+1}+zr_{t+1+j}\right)\right]=E\left[\exp\left(A_{r,j}(z)+A\left(x,B_{r,j}(z)\right)+B\left(x,B_{r,j}(z)\right)^\top l_t\right)\right]. \tag{3.68}$$

The functions $A_{r,j}$ and $B_{r,j}$ satisfy the forward recursions:

$$A_{r,j}(z)=A_{r,j-1}(z)+A_l\left(B_{r,j-1}(z)\right) \tag{3.69}$$

$$B_{r,j}(z)=B_l\left(B_{r,j-1}(z)\right), \tag{3.70}$$

with the initial conditions $A_{r,1}(z)=A_r(z)$ and $B_{r,1}(z)=B_r(z)$. It comes from the equation (3.68) that the unconditional cumulant-generating function

$$\Psi_{r,j}(x,z)=\ln E\left[\exp\left(xr_t+zr_{t+j}\right)\right]$$

is given by:

$$\Psi_{r,j}(x,z) = A_{r,j}(z) + A\left(x, B_{r,j}(z)\right) + \Psi_l\left(B\left(x, B_{r,j}(z)\right)\right). \tag{3.71}$$

Proposition 3.6.2. *Given $n > 0$ and $m > 0$, the unconditional cross-cumulant of order (n,m) of the observable returns r_t is the number $\kappa_{r,j}(n,m)$ given by:*

$$\begin{aligned}\kappa_{r,j}(n,m) &= \frac{\partial^{n+m}\Psi_{r,j}}{\partial x^n \partial z^m}(0,0) \\ &= \frac{\partial^{n+m}}{\partial x^n \partial z^m}\left(A\left(x, B_{r,j}(z)\right)\right)\Big|_{x=0,z=0} + \frac{\partial^{n+m}}{\partial x^n \partial z^m}\left(\Psi_l\left(B\left(x, B_{r,j}(z)\right)\right)\right)\Big|_{x=0,z=0}.\end{aligned} \tag{3.72}$$

Note that one should have $\kappa_{r,j}(n,0) = \kappa_{r,j}(0,n) = \kappa_r(n)$ for any $j > 0$ because of the stationarity of the return process. Since $B_r(0) = 0$, the formulas (3.67) and (3.72) show that cumulants of the latent vector l_t are essential to compute cumulants and cross-cumulants of returns. As pointed out earlier, these derivatives of the function $\Psi_l(y)$ at $y = 0$ can be solved analytically through equation (3.63), since $B_l(0) = 0$. Let the operator $\mathscr{D}$ defines the Jacobian matrix of a real matrix function of a matrix of real variables, as defined in Magnus and Neudecker (1988) (Ch. 9, Sec. 4, Page 173).

Proposition 3.6.3. *The n-th unconditional cumulant of the latent vector l_t is the $k^{n-1} \times k$ matrix $\kappa_l(n)$ given by:*

$$\kappa_l(n) = \mathscr{D}^n \Psi_l(0), \tag{3.73}$$

where $\mathscr{D}^n \Psi_l(0)$ is found through the equation

$$\mathscr{D}^n \Psi_l(0) = \mathscr{D}^n A_l(0) + \mathscr{D}^n \left(\Psi_l\left(B_l(y)\right)\right)\big|_{y=0}, \tag{3.74}$$

and depends on $\mathscr{D}\Psi_l(0)$, $\mathscr{D}^2\Psi_l(0)$,...,$\mathscr{D}^{n-1}\Psi_l(0)$, $\mathscr{D}B_l(0)$, $\mathscr{D}^2 B_l(0)$,...,$\mathscr{D}^n B_l(0)$.

Note that while the matrix $\kappa_l(n)$ of all cumulants of order n has k^n elements, only $\binom{n}{n+k-1}$ of these elements are distinct due to the equality of some partial derivatives of the function $\Psi_l(y)$. The higher order derivatives of composite functions in (3.67), (3.72) and (3.74) are evaluated through the chain rule given by the Faà di Bruno's formula which the multivariate version is detailed in Constantine and Savits (1996). In the case of a univariate latent variable ($k = 1$), it is very easy to find higher order cumulants of the latent variable. This task is more cumbersome and tedious for $k > 1$. In this latter case, when $n = 1$, the solution to the equation (3.74) is given by:

$$\mathscr{D}\Psi_l(0) = \mathscr{D}A_l(0)\left[Id_k - \mathscr{D}B_l(0)\right]^{-1}. \tag{3.75}$$

Note that $\mathscr{D}B_l(0)$ represents the persistence matrix of the latent vector l_t. However when $n > 1$, it can be shown that the matrix $\mathscr{D}^n\Psi_l(0)$ satisfies:

$$\mathscr{D}^n\Psi_l(0) - \left((\mathscr{D}B_l(0))^{\otimes(n-1)}\right)^\top \mathscr{D}^n\Psi_l(0)\,\mathscr{D}B_l(0) = \mathscr{D}^n A_l(0) + C_n \tag{3.76}$$

where the matrix C_n depends on the matrices $\left\{\mathscr{D}^j B_l(0)\right\}_{1\leq j\leq n-1}$ and $\left\{\mathscr{D}^j\Psi_l(0)\right\}_{2\leq j\leq n}$ through the multivariate Faà di Bruno's formula. As example, the second unconditional cumulant of the latent vector is given by:

$$\mathscr{D}^2\Psi_l(0) - \mathscr{D}B_l(0)^\top \mathscr{D}^2\Psi_l(0)\,\mathscr{D}B_l(0) = \mathscr{D}^2 A_l(0) + \left(Id_k \otimes \mathscr{D}\Psi_l(0)\right)\mathscr{D}^2 B_l(0). \tag{3.77}$$

It turns out from (3.76) that $\mathscr{D}^n\Psi_l(0)$ is solution to a matrix equation that can be written $X - \Delta X \Gamma = \Lambda$. Jameson (1968) and Jiang and Wei (2005) study this matrix equation in the general case and derive the explicit solution by means of characteristic polynomials. Using the *vec* operator, the solution to the matrix equation $X - \Delta X\Gamma = \Lambda$ is given by:

$$vec(X) = \left[Id - \left(\Gamma^\top \otimes \Delta\right)\right]^{-1} vec(\Lambda).$$

In the particular case where the matrices Δ and Γ are diagonal, solving this equation is more easier and elements of the solution matrix $X = \left[x_{ij}\right]$ are given by $x_{ij} = \lambda_{ij}/(1 - \delta_i\gamma_j)$, where $\Delta = Diag(\delta_1, \delta_2, \ldots)$, $\Gamma = Diag(\gamma_1, \gamma_2, \ldots)$ and $\Lambda = \left[\lambda_{ij}\right]$. This is the case in our general multivariate latent variable model when the components of the multivariate function $B_l(y) = \left(B_{l,1}(y), .., B_{l,k}(y)\right)^\top$ satisfy $B_{l,j}(y_1, .., y_k) = B_{l,j}(y_j)$. In this case, the persistence matrix $\mathscr{D}B_l(0)$ is diagonal and its diagonal elements represent individual persistence of latent factors l_{jt}. It is sufficient to have $B_i(x, y_1, .., y_k) = B_i(x, y_i)$ in (3.1) and (3.21).

We have just provided analytical formulas for computing return cumulants and cross-cumulants $\kappa_{r,j}(n,m)$, $j > 0$, $n \geq 0$, $m > 0$. This also allows us to compute analytically the corresponding return moments and cross-moments $\mu_{r,j}(n,m) = E\left[r_t^n r_{t+j}^m\right]$ through the relationship between multivariate moments and cumulants, derived and proved in Constantine and Savits (1996), as an application of the multivariate Faà di Bruno's formula. For example:

$$\mu_{10} = \kappa_{10}$$
$$\mu_{20} = \kappa_{10}^2 + \kappa_{20}$$
$$\mu_{11} = \kappa_{10}\kappa_{01} + \kappa_{11}$$
$$\mu_{30} = \kappa_{10}^3 + 3\kappa_{10}\kappa_{20} + \kappa_{30}$$
$$\mu_{21} = \kappa_{10}^2\kappa_{01} + 2\kappa_{10}\kappa_{11} + \kappa_{01}\kappa_{20} + \kappa_{21}$$
$$\mu_{40} = \kappa_{10}^4 + 6\kappa_{10}^2\kappa_{20} + 4\kappa_{10}\kappa_{30} + 3\kappa_{20}^2 + \kappa_{40}$$
$$\mu_{31} = \kappa_{10}^3\kappa_{01} + 3\kappa_{10}^2\kappa_{11} + 3\kappa_{10}\kappa_{01}\kappa_{20} + 3\kappa_{10}\kappa_{21} + \kappa_{01}\kappa_{30} + 3\kappa_{20}\kappa_{11} + \kappa_{31}$$
$$\mu_{22} = \kappa_{10}^2\kappa_{01}^2 + \kappa_{10}^2\kappa_{02} + 4\kappa_{10}\kappa_{01}\kappa_{11} + \kappa_{01}^2\kappa_{20} + 2\kappa_{10}\kappa_{12} + 2\kappa_{01}\kappa_{21} + \kappa_{20}\kappa_{02} + 2\kappa_{11}^2 + \kappa_{22}$$

where μ_{nm} and κ_{nm} respectively denote $\mu_{r,j}(n,m)$ and $\kappa_{r,j}(n,m)$ for simplification.

3.6.2 GMM Procedure

Notice that all these moments are functions of the parameter vector θ that governs both the dynamics of returns and that of the latent factors. We can then choose N pertinent moments to perform the GMM estimation of the return model. In this chapter, we choose N pertinent ones among all the moments $\mu_{r,j}(n,m) = E\left[r_t^n r_{t+j}^m\right]$ such that $1 \leq j \leq J$, $0 \leq n \leq Q$ and $0 < m \leq Q - n$, meaning N among $Q + JQ(Q-1)/2$ moments of order less than or equal to Q. Since the moments of observed returns implied by a given model can directly be compared to their sample equivalent, our estimation setup is more likely to evaluate the performance of a given model in replicating well-known stylized facts like autocorrelation of squared returns, absence of autocorrelation of returns, leverage effect which can be captured via coskewness, unconditional fat-taildness and asymmetry of returns. All these well-known empirical facts can be considered as relevant part of our moment conditions by choosing corresponding moments.

Let $g_t(\theta) = \left[r_t^{n_i} r_{t+j_i}^{m_i} - \mu_{r,j_i}(n_i, m_i)\right]_{1 \leq i \leq N}$ denotes the $N \times 1$ vector from the retained moments. We have $E\left[g_t(\theta)\right] = 0$ and we define the sample counterpart of this moment condition as follows:

$$\widehat{g}(\theta) = \begin{pmatrix} \widehat{E}\left[r_t^{n_1} r_{t+j_1}^{m_1}\right] - \mu_{r,j_1}(n_1, m_1) \\ \cdot \\ \cdot \\ \widehat{E}\left[r_t^{n_N} r_{t+j_N}^{m_N}\right] - \mu_{r,j_N}(n_N, m_N) \end{pmatrix}. \tag{3.78}$$

Given the $N \times N$ matrix $\widehat{W}$ used to weight the moments, the GMM estimator $\widehat{\theta}$ of the parameter vector is given by:

$$\widehat{\theta} = \arg\min_{\theta} \widehat{g}(\theta)^{\top} \widehat{W} \widehat{g}(\theta). \tag{3.79}$$

Interestingly, the variance-covariance matrix of $g_t(\theta)$ does not depend on the vector of parameter θ. This is a huge advantage since with a nonparametric empirical variance-covariance matrix of moment conditions, the optimal GMM procedure can be implemented in one step. Also and most importantly, two different models can be estimated via same moment conditions and weighting matrix. In this case, the minimum value of the GMM objective function itself is a criterion for comparison of alternative models.

In some cases, this GMM procedure also has a huge numerical advantage compared to the maximum likelihood estimation even when the likelihood function can be derived. Maximum likelihood estimation becomes difficult to perform numerically especially when the support of the likelihood function is parameter-dependent. This is the case in the IG-GARCH model of Christoffersen, Heston and Jacobs (2006) that can also be estimated through this GMM method.

On the other hand, the maximum likelihood estimation of semi-affine latent variable models of Bates (2006) and the quasi-maximum likelihood estimation based on the Kalman recursion have the downside that critical unconditional higher moments (skewness and kurtosis) of returns can be poorly estimated due to the second order approximation of the distribution of the latent variable conditional on observable returns. Moreover, in single-stage estimation and filtering methods like the Unscented Kalman Filter and the Bates (2006)'s algortihm, one can argue that approximations affect both parameter and state estimations.

Instead, our GMM procedure matches critical higher moments exactly and requires no approximation for parameter estimation. Provided with the GMM estimates of model parameters, Bates (2006)'s procedure or any other filtering procedure like the Unscented Kalman Filter can be followed for the state estimation. In this sense, approximations required by these techniques will only affect state estimation. In future research following this chapter, Monte Carlo experiments are performed to assess this two-stage estimation and filtering.

3.7 Estimation of SVS Models Using Daily Equity Returns.

3.7.1 Parameter Estimation

We estimate the SVS models using daily returns on S&P500 and CRSP indexes as well as daily returns on six Fama and French portfolios. As explained in Fama and French (1993), the six portfolios are the outcome of the intersection of two independent sorts. Stocks are sorted into two size groups— **S** (small; that is, market capitalization below the NYSE median) and **B** (big; that is, market capitalization above the NYSE median)—and into three book-to-market groups—**G** (growth; that is, in the bottom 30 percent of the NYSE book-to-market), **N** (neutral; that is, in the middle 40 percent of the NYSE book-to-market) and **V** (value; that is, in the top 30 percent of the NYSE book-to-market). The six portfolios are commonly labelled SG, SN, SV, BG, BN and BV.

Table 3.1 summarizes basic descriptive statistics of these returns. It shows the well-documented facts that asset returns are negatively skewed and fat-tailed. Small stocks are generally more negatively skewed than big stocks and a growth portfolio has lower average returns and higher negative skewness compared to a value portfolio of the same size.

Table 3.1: **Summary Statistics of Stocks Returns for the Period 1990-2005.**

$r_t \times 100$	Mean	Median	Std.	Skew.	Kurt.	Max.	Min.
SG	0.024	0.110	1.205	-0.463	6.613	7.102	-8.992
SN	0.055	0.110	0.847	-0.462	6.363	4.555	-5.668
SV	0.061	0.120	0.788	-0.608	6.993	3.990	-5.869
BG	0.040	0.060	1.086	-0.059	6.783	6.269	-8.034
BN	0.045	0.060	0.904	-0.191	7.043	5.647	-6.699
BV	0.043	0.070	0.888	-0.300	6.882	5.136	-6.486
CRSP	0.040	0.071	0.979	-0.208	7.211	5.180	-6.856
S&P500	0.036	0.041	1.013	-0.015	6.694	5.731	-6.867

To perform the GMM procedure for each series, we need to decide which moments to choose. To achieve this task, we refer to the relative importance of return moments. We consider the moments

$$\left\{E\left[r_t^j\right]\right\}_{j=1}^4$$

in order to match the critical first moments of asset returns. Figure 3.2 displays autocorrelations of square returns which as shown are significant up to the twentieth lag. As the positive and significant autocorrelation of square returns appears to be a critical empirical fact, we consider the moments

$$\left\{E\left[r_t^2 r_{t+j}^2\right]\right\}_{j=1}^5$$

in order to match these autocorrelations. The negative and significant cross-correlation between returns and square returns for various leads as shown in Panel A of Figure 3.3 is an empirical fact characterizing the well-known leverage effect. Panel B of Figure 3.3 shows similar cross-correlations for various lags. The cross-correlation between returns and cube returns is also shown to be positive and significant at least for the first three

Figure 3.1: **Return Series.**

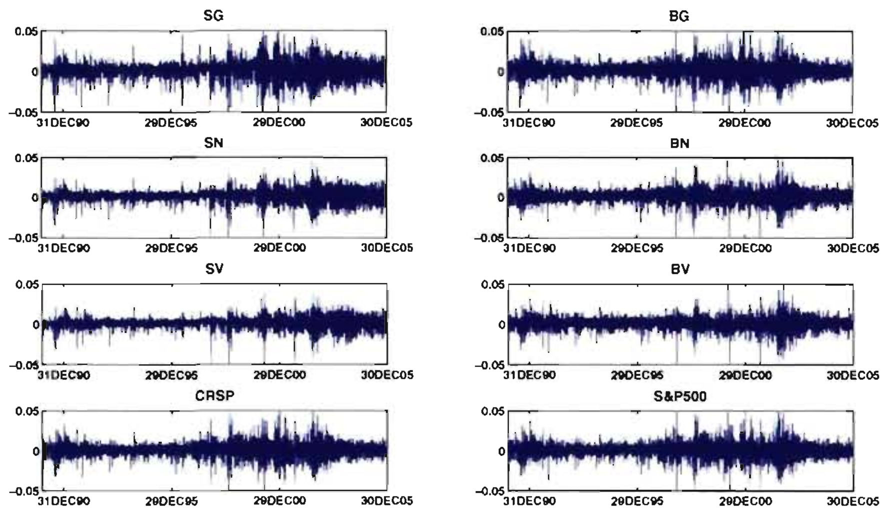


Figure 3.2: **Autocorrelation of Squared Returns.**

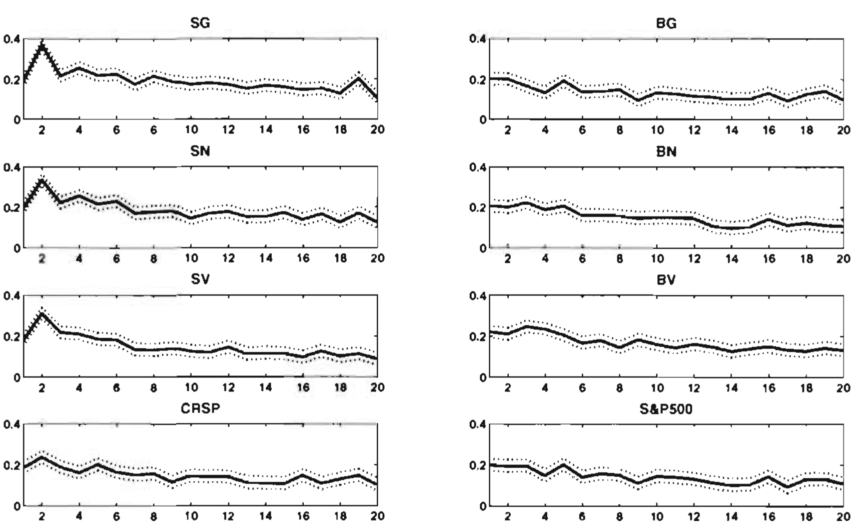
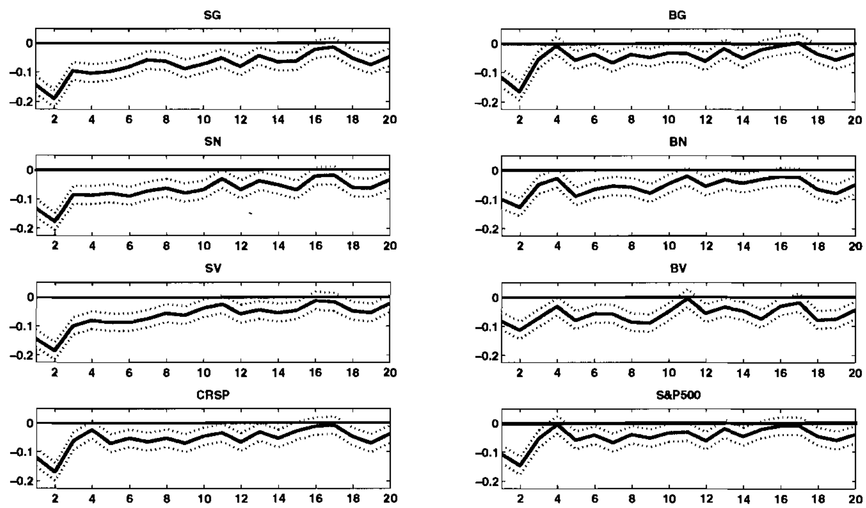


Figure 3.3: **Cross-Correlations Between Returns and Squared Returns.**

Panel A: $Corr\left(r_t, r_{t+j}^2\right)$



Panel B: $Corr\left(r_t^2, r_{t+j}\right)$

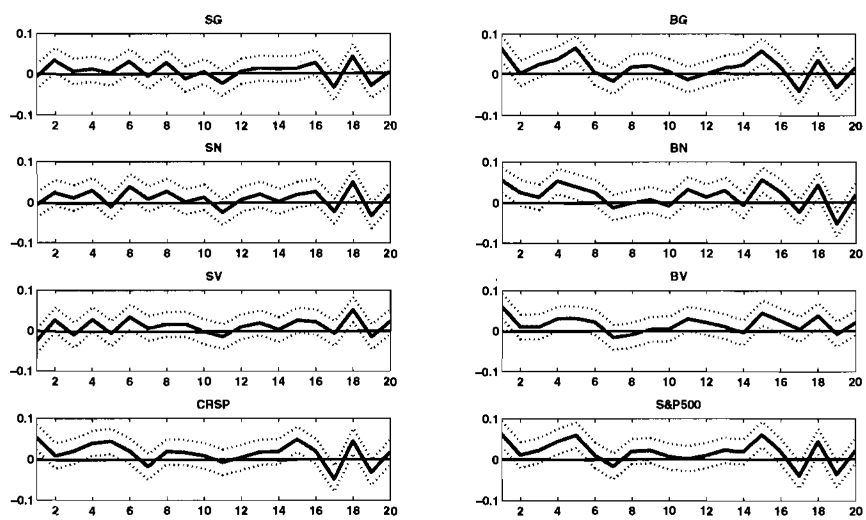
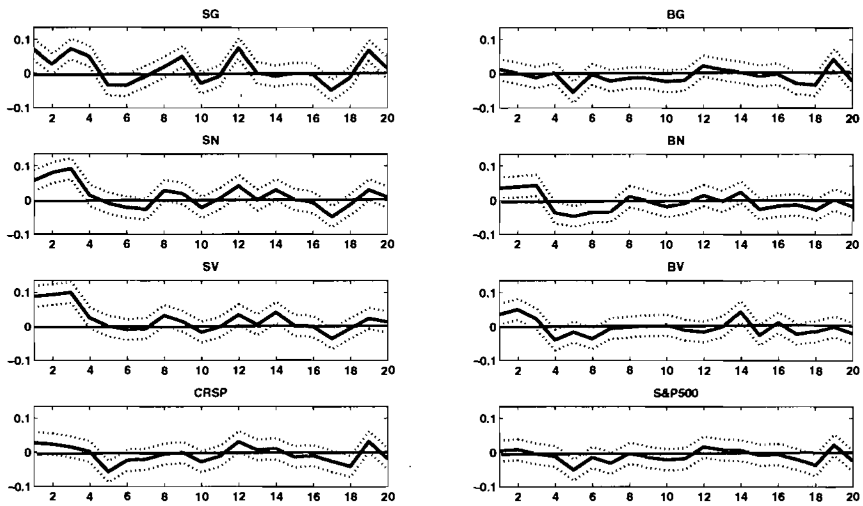
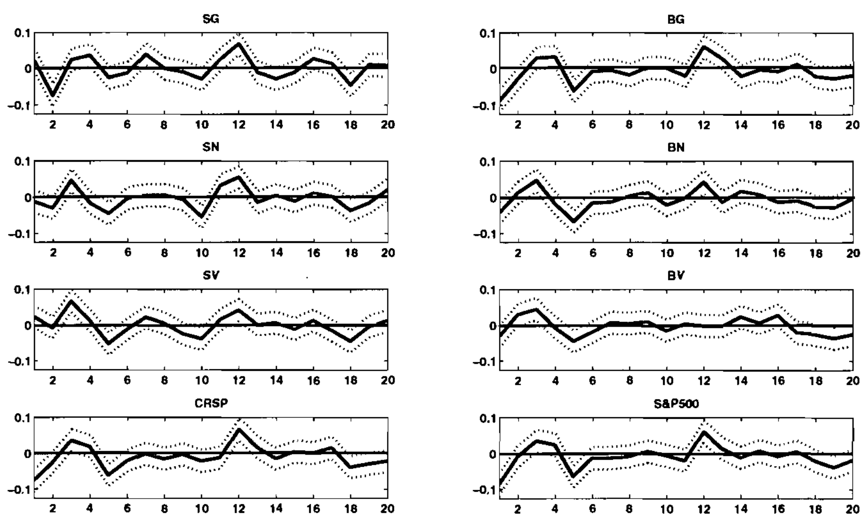


Figure 3.4: **Cross-Correlations Between Returns and Cubed Returns.**

Panel A: $Corr\left(r_t, r_{t+j}^3\right)$



Panel B: $Corr\left(r_t^3, r_{t+j}\right)$



leads as shown in Panel A of Figure 3.4, especially for small stocks. Panel B of Figure 3.4 shows similar cross-correlations for various lags. To assess the ability of our SVS models to match these important features of return data we add the set of moments

$$\left\{E\left[r_t r_{t+j}^2\right], E\left[r_t r_{t+j}^3\right]\right\}_{j=1}^3.$$

We weight the 15 moments with the diagonal of the inverse of the covariance matrix of moments:

$$\widehat{W} = Diag\left\{\left(\widehat{Var}\left[g_t\right]\right)^{-1}\right\}.$$

This matrix is nonparametric and puts more weight on moments with low magnitude. Estimation results for one-factor SVS models are shown in Table 3.2. Indeed, we use 14 moment conditions in our GMM procedure since we don't estimate the unconditional mean of returns μ_0, set to its sample counterpart. Also, the parameter β_1 is not estimated. The reason is that, due to the high persistence of the factor, it becomes difficult in the return equation (3.18) to identify β_1 and λ_1 separately. To avoid this identification problem, we set $\beta_1 = 0$.

Panel A of Table 3.2 shows the estimation results in the case of the contemporaneous conditional asymmetry of returns, that is, when $\eta_1 \neq 0$ is estimated. Starting with the measure equation (3.18), estimation output confirm that projecting returns onto the latent factor results in a significant negative coefficient λ_1 and corroborates the story that an increase in contemporaneous volatility lowers asset payoffs. Most importantly is the significance and the positivity of the coefficient η_1 in the return equation. This is a new evidence in asset return dynamics which says that the distribution of the daily returns conditional upon their contemporaneous volatilities is asymmetric. This result differs from the findings of Forsberg and Bollerslev (2002) that the distribution of daily returns conditional upon their realized volatilities is normal.

Coming to the state dynamics, estimation results show that the latent variable governing the daily return dynamics is highly persistent, with significant estimates of the coefficient of persistence of 0.967 and 0.956 for the S&P500 and the CRSP indexes respectively. This also means that daily return volatility as perceived by agents is highly

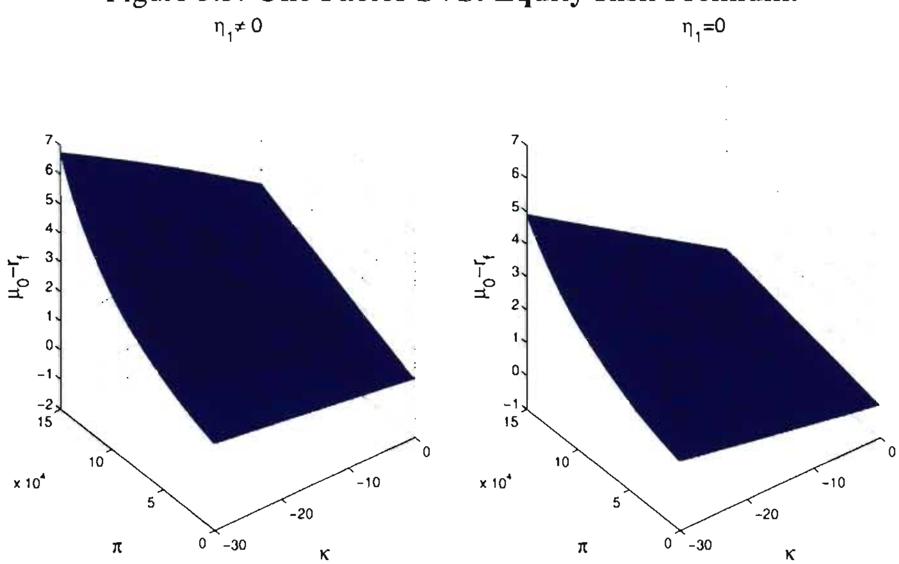
Table 3.2: **One-Factor SVS: GMM Estimation Results.**

Param.	S&P500	SG	SN	SV	BG	BN	BV	CRSP
Panel A: Contemporaneous Conditional Asymmetry: $\eta_1 \neq 0$								
λ_1	-22.39	-26.37	-42.68	-55.52	-22.87	-24.95	-21.23	-27.50
	4.01	3.96	7.12	8.57	3.39	5.26	6.17	4.60
η_1	7.44E-3	5.74E-3	4.06E-3	4.11E-3	8.47E-3	5.31E-3	3.21E-3	7.44E-3
	1.37E-3	1.17E-3	1.12E-3	1.03E-3	1.49E-3	1.26E-3	1.29E-3	1.50E-3
μ_1	9.67E-5	1.33E-4	6.55E-5	5.49E-5	1.10E-4	7.77E-5	7.59E-5	8.80E-5
	9.62E-6	1.42E-5	6.68E-6	5.08E-6	1.04E-5	8.23E-6	8.16E-6	9.06E-6
ϕ_1	0.967	0.996	1.008	0.968	0.954	0.970	1.007	0.956
	0.037	0.028	0.020	0.025	0.032	0.043	0.028	0.039
$\sqrt{\omega_1}$	1.02E-4	1.37E-4	6.36E-5	5.38E-5	1.17E-4	8.50E-5	8.38E-5	9.66E-5
	1.43E-5	1.81E-5	7.16E-6	5.38E-6	1.58E-5	1.37E-5	1.60E-5	1.24E-5
J-Stat	9.36	11.52	12.19	20.12	11.54	7.47	5.14	8.92
p-value	0.40	0.24	0.20	0.02	0.24	0.59	0.82	0.44
Panel B: Contemporaneous Conditional Normality: $\eta_1 = 0$								
λ_1	-12.01	-22.41	-37.03	-50.62	-13.14	-16.30	-16.10	-19.13
	2.95	3.55	6.61	8.02	2.61	4.60	5.72	3.63
μ_1	1.01E-4	1.37E-4	6.93E-5	5.85E-5	1.15E-4	8.02E-5	7.74E-5	9.23E-5
	1.01E-5	1.49E-5	7.36E-6	5.50E-6	1.13E-5	8.57E-6	8.55E-6	9.73E-6
ϕ_1	0.974	1.034	1.048	1.016	0.971	1.000	1.024	1.001
	0.035	0.028	0.023	0.028	0.031	0.042	0.027	0.040
$\sqrt{\omega_1}$	1.09E-4	1.37E-4	6.25E-5	5.22E-5	1.25E-4	8.66E-5	8.42E-5	9.97E-5
	1.56E-5	1.87E-5	7.29E-6	5.62E-6	1.72E-5	1.45E-5	1.65E-5	1.35E-5
J-Stat	18.66	18.85	16.30	23.80	20.06	13.65	7.56	15.65
p-value	0.03	0.03	0.06	0.00	0.02	0.14	0.58	0.07

persistent as well, since it is a linear function of the latent factor. All the estimates of Panel A are significant and overall, the J-test of over-identifying restrictions does not reject the models, with a minimum p-value of 0.20 except for the small value portfolio rejected with a p-value of 0.02.

We now assess how important is the contemporaneous conditional non-normality in asset return modeling. Panel B of Table 3.2 shows the estimation results in the case of the contemporaneous conditional normality of returns, that is, with the constraint $\eta_1 = 0$. As in the first panel, all the parameters are significantly estimated. Compared to results of Panel A however, there is a decrease in the magnitude of the leverage parameter and an increase in the persistence of the factors— estimates of the persistence become greater or equal to 1 for four of the six assets. Moreover, and most importantly, models are or tend to be rejected in the data. There is a sharp decrease in the p-values compared to Panel A. For the S&P500 and the CRSP indexes, the p-values decrease from 0.40 and 0.44 to 0.03 and 0.06 respectively.

Figure 3.5: **One Factor SVS: Equity Risk Premium.**



Most importantly, the contemporaneous conditional asymmetry is important in determining the equity premium. Using the GMM estimates for the S&P500, we evaluate the formula (3.59) for a range of preference parameters both when $\eta_1 \neq 0$ and when

Table 3.3: **One-Factor SVS: c's Coefficients.**

Coeff.	S&P500	SG	SN	SV	BG	BN	BV	CRSP
			Contemporaneous Conditional Asymmetry: $\eta_1 \neq 0$					
$c_{0\mu}$	2.46E-3	3.72E-3	3.37E-3	3.56E-3	2.79E-3	2.33E-3	2.05E-3	2.72E-3
$c_{1\mu}$	-21.659	-26.261	-43.026	-53.736	-21.822	-24.196	-21.372	-26.299
c_{0h}	3.15E-6	5.23E-7	-5.25E-7	1.77E-6	5.03E-6	2.35E-6	-5.14E-7	3.85E-6
c_{1h}	0.97081	0.99683	1.0062	0.97798	0.95987	0.97321	1.0062	0.96316
c_{0s}	2.27E-8	2.98E-9	-2.17E-9	6.75E-9	4.05E-8	1.20E-8	-1.67E-9	2.70E-8
c_{1s}	6.75E-3	5.63E-3	4.22E-3	3.43E-3	7.34E-3	4.74E-3	3.31E-3	6.38E-3
c_{0r}	-2.45E-10	-7.69E-12	-1.11E-11	-1.66E-10	-6.53E-10	-1.64E-10	-6.83E-12	-4.86E-10
c_{1r}	-1.51E-4	-2.93E-5	4.26E-5	-1.82E-4	-2.48E-4	-1.36E-4	2.67E-5	-2.43E-4
			Contemporaneous Conditional Normality: $\eta_1 = 0$					
$c_{0\mu}$	1.54E-3	3.43E-3	3.24E-3	3.62E-3	1.86E-3	1.76E-3	1.71E-3	2.17E-3
$c_{1\mu}$	-11.691	-23.169	-38.812	-51.4	-12.751	-16.304	-16.485	-19.142
c_{0h}	2.65E-6	-4.68E-6	-3.33E-6	-9.05E-7	3.37E-6	-7.61E-9	-1.86E-6	-6.45E-8
c_{1h}	0.975	1.029	1.040	1.012	0.972	1.000	1.023	1.001
c_{0s}	-2.98E-10	-1.46E-9	-1.01E-9	-9.90E-11	-5.29E-10	-3.31E-15	-1.97E-10	-2.79E-13
c_{1s}	-2.19E-4	6.42E-4	6.31E-4	2.22E-4	-3.05E-4	8.70E-7	2.17E-4	8.65E-6
c_{0r}	-9.94E-11	-4.86E-10	-3.36E-10	-3.31E-11	-1.76E-10	-1.10E-15	-6.57E-11	-9.30E-14
c_{1r}	-7.30E-5	2.14E-4	2.11E-4	7.40E-5	-1.02E-4	2.90E-7	7.25E-5	2.88E-6

Table 3.4: **One-Factor SVS: Moment Matching for the S&P500 Index and Small Portfolios.**

		S&P500			SG			SN			SV		
		Sample	$\eta_1 \neq 0$	$\eta_1 = 0$	Sample	$\eta_1 \neq 0$	$\eta_1 = 0$	Sample	$\eta_1 \neq 0$	$\eta_1 = 0$	Sample	$\eta_1 \neq 0$	$\eta_1 = 0$
$E[r_t]$	0	3.68E-4	3.64E-4	3.64E-4	2.47E-4	2.43E-4	2.43E-4	5.53E-4	5.49E-4	5.49E-4	6.16E-4	6.13E-4	6.13E-4
$E[r_t^2]$	1	1.03E-4	1.02E-4	1.03E-4	1.45E-4	1.46E-4	1.47E-4	7.21E-5	7.32E-5	7.50E-5	6.24E-5	6.42E-5	6.59E-5
$E[r_t^3]$	1	9.72E-8	1.13E-7	-3.22E-7	-7.03E-7	-7.16E-7	-1.20E-6	-1.62E-7	-1.71E-7	-3.33E-7	-1.83E-7	-1.91E-7	-3.25E-7
$E[r_t^4]$	1	7.07E-8	7.11E-8	6.91E-8	1.39E-7	1.38E-7	1.35E-7	3.23E-8	3.23E-8	3.13E-8	2.63E-8	2.53E-8	2.42E-8
$E[r_t r_{t+5}^2]$	0	-1.11E-7	-1.77E-7	-9.11E-8	-3.69E-7	-5.40E-7	-5.20E-7	-7.03E-8	-1.72E-7	-1.63E-7	-6.36E-8	-1.32E-7	-1.34E-7
$E[r_t^3 r_{t+5}]$	0	-7.37E-9	2.65E-9	1.29E-9	-6.27E-9	1.42E-8	1.38E-8	-2.51E-9	3.67E-9	3.47E-9	-2.56E-9	3.23E-9	3.30E-9
$E[r_t^2 r_{t+5}^2]$	1	2.27E-8	2.09E-8	2.15E-8	4.70E-8	4.77E-8	5.00E-8	1.10E-8	1.14E-8	1.19E-8	8.10E-9	8.22E-9	8.61E-9
$E[r_t r_{t+5}^3]$	0	-6.07E-9	2.65E-9	1.29E-9	-8.54E-9	1.42E-8	1.38E-8	-6.42E-10	3.67E-9	3.47E-9	-1.01E-10	3.23E-9	3.30E-9
$E[r_t r_{t+4}^2]$	0	2.42E-8	-1.84E-7	-9.46E-8	-3.91E-7	-5.42E-7	-5.02E-7	-7.97E-8	-1.70E-7	-1.53E-7	-5.58E-8	-1.38E-7	-1.32E-7
$E[r_t^3 r_{t+4}]$	0	3.09E-9	2.74E-9	1.33E-9	8.43E-9	1.42E-8	1.33E-8	-9.24E-10	3.64E-9	3.30E-9	5.12E-10	3.34E-9	3.24E-9
$E[r_t^2 r_{t+4}^2]$	1	1.95E-8	2.13E-8	2.18E-8	5.13E-8	4.78E-8	4.90E-8	1.22E-8	1.14E-8	1.16E-8	8.66E-9	8.37E-9	8.54E-9
$E[r_t r_{t+4}^3]$	0	-1.35E-9	2.74E-9	1.33E-9	1.13E-8	1.42E-8	1.33E-8	5.93E-10	3.64E-9	3.30E-9	1.04E-9	3.34E-9	3.24E-9
$E[r_t^2 r_{t+3}]$	0	9.35E-8	-1.91E-7	-9.81E-8	6.22E-8	-5.44E-7	-4.84E-7	5.57E-8	-1.68E-7	-1.44E-7	2.61E-8	-1.44E-7	-1.29E-7
$E[r_t r_{t+3}^2]$	1	-9.87E-8	-1.91E-7	-9.81E-8	-3.58E-7	-5.44E-7	-4.84E-7	-7.75E-8	-1.68E-7	-1.44E-7	-7.83E-8	-1.44E-7	-1.29E-7
$E[r_t^3 r_{t+3}]$	0	4.28E-9	2.83E-9	1.37E-9	5.53E-9	1.43E-8	1.29E-8	2.41E-9	3.61E-9	3.14E-9	3.03E-9	3.46E-9	3.19E-9
$E[r_t^2 r_{t+3}^2]$	1	2.24E-8	2.16E-8	2.21E-8	4.66E-8	4.79E-8	4.81E-8	1.12E-8	1.13E-8	1.13E-8	8.85E-9	8.52E-9	8.47E-9
$E[r_t r_{t+3}^3]$	1	-4.84E-10	2.83E-9	1.37E-9	1.67E-8	1.43E-8	1.29E-8	4.80E-9	3.61E-9	3.14E-9	4.55E-9	3.46E-9	3.19E-9
$E[r_t^2 r_{t+2}]$	0	6.64E-8	-1.99E-7	-1.02E-7	1.72E-7	-5.47E-7	-4.67E-7	7.26E-8	-1.67E-7	-1.36E-7	6.91E-8	-1.50E-7	-1.27E-7
$E[r_t r_{t+2}^2]$	1	-3.28E-7	-1.99E-7	-1.02E-7	-7.56E-7	-5.47E-7	-4.67E-7	-2.08E-7	-1.67E-7	-1.36E-7	-1.82E-7	-1.50E-7	-1.27E-7
$E[r_t^3 r_{t+2}]$	0	-8.86E-10	2.92E-9	1.41E-9	-1.79E-8	1.43E-8	1.25E-8	-1.66E-9	3.58E-9	2.98E-9	-4.46E-10	3.57E-9	3.14E-9
$E[r_t^2 r_{t+2}^2]$	1	2.23E-8	2.20E-8	2.24E-8	6.46E-8	4.81E-8	4.72E-8	1.43E-8	1.13E-8	1.11E-8	1.09E-8	8.67E-9	8.40E-9
$E[r_t r_{t+2}^3]$	1	9.50E-10	2.92E-9	1.41E-9	6.01E-9	1.43E-8	1.25E-8	4.18E-9	3.58E-9	2.98E-9	4.27E-9	3.57E-9	3.14E-9
$E[r_t^2 r_{t+1}]$	0	1.94E-7	-2.07E-7	-1.06E-7	-5.28E-10	-5.49E-7	-4.51E-7	3.09E-8	-1.65E-7	-1.28E-7	7.71E-9	-1.56E-7	-1.24E-7
$E[r_t r_{t+1}^2]$	1	-2.30E-7	-2.07E-7	-1.06E-7	-5.55E-7	-5.49E-7	-4.51E-7	-1.44E-7	-1.65E-7	-1.28E-7	-1.32E-7	-1.56E-7	-1.24E-7
$E[r_t^3 r_{t+1}]$	0	-1.00E-8	3.02E-9	1.45E-9	5.37E-9	1.44E-8	1.20E-8	-6.90E-10	3.55E-9	2.84E-9	1.04E-9	3.70E-9	3.09E-9
$E[r_t^2 r_{t+1}^2]$	1	2.26E-8	2.24E-8	2.27E-8	4.36E-8	4.82E-8	4.64E-8	1.05E-8	1.12E-8	1.08E-8	7.94E-9	8.83E-9	8.34E-9
$E[r_t r_{t+1}^3]$	1	5.32E-10	3.02E-9	1.45E-9	1.65E-8	1.44E-8	1.20E-8	2.94E-9	3.55E-9	2.84E-9	4.04E-9	3.70E-9	3.09E-9

Table 3.5: **One-Factor SVS: Moment Matching for Large Portfolios and the CRSP Index.**

		BG			BN			BV			CRSP		
		Sample	$\eta_1 \neq 0$	$\eta_1 = 0$	Sample	$\eta_1 \neq 0$	$\eta_1 = 0$	Sample	$\eta_1 \neq 0$	$\eta_1 = 0$	Sample	$\eta_1 \neq 0$	$\eta_1 = 0$
$E[r_t]$	0	4.03E-4	3.99E-4	3.99E-4	4.52E-4	4.48E-4	4.48E-4	4.34E-4	4.32E-4	4.32E-4	4.05E-4	4.01E-4	4.01E-4
$E[r_t^2]$	1	1.18E-4	1.17E-4	1.18E-4	8.19E-5	8.24E-5	8.24E-5	7.91E-5	7.93E-5	7.95E-5	9.60E-5	9.53E-5	9.61E-5
$E[r_t^3]$	1	6.58E-8	9.33E-8	-4.84E-7	-3.08E-8	-3.82E-8	-2.63E-7	-1.08E-7	-1.13E-7	-2.45E-7	-7.92E-8	-4.24E-8	-4.70E-7
$E[r_t^4]$	1	9.43E-8	9.52E-8	9.17E-8	4.68E-8	4.67E-8	4.45E-8	4.25E-8	4.25E-8	4.17E-8	6.60E-8	6.48E-8	6.14E-8
$E[r_t r_{t+5}^2]$	0	-1.35E-7	-2.27E-7	-1.36E-7	-1.25E-7	-1.32E-7	-8.98E-8	-1.00E-7	-1.30E-7	-9.86E-8	-1.27E-7	-1.96E-7	-1.64E-7
$E[r_t^3 r_{t+5}]$	0	-9.89E-9	4.06E-9	2.34E-9	-5.40E-9	2.16E-9	1.35E-9	-3.28E-9	1.91E-9	1.38E-9	-6.86E-9	3.67E-9	3.07E-9
$E[r_t^2 r_{t+5}^2]$	1	2.95E-8	2.71E-8	2.83E-8	1.50E-8	1.43E-8	1.49E-8	1.37E-8	1.46E-8	1.49E-8	2.07E-8	1.89E-8	2.06E-8
$E[r_t r_{t+5}^3]$	0	-8.97E-9	4.06E-9	2.34E-9	-3.87E-9	2.16E-9	1.35E-9	-1.24E-9	1.91E-9	1.38E-9	-6.59E-9	3.67E-9	3.07E-9
$E[r_t r_{t+4}^2]$	0	1.94E-8	-2.40E-7	-1.42E-7	-1.61E-8	-1.37E-7	-8.98E-8	-1.96E-8	-1.29E-7	-9.55E-8	-1.64E-8	-2.06E-7	-1.64E-7
$E[r_t^3 r_{t+4}]$	0	5.29E-9	4.26E-9	2.42E-9	-1.41E-9	2.23E-9	1.35E-9	-5.10E-10	1.89E-9	1.34E-9	2.27E-9	3.84E-9	3.06E-9
$E[r_t^2 r_{t+4}^2]$	1	2.47E-8	2.77E-8	2.87E-8	1.43E-8	1.45E-8	1.48E-8	1.48E-8	1.45E-8	1.47E-8	1.85E-8	1.94E-8	2.06E-8
$E[r_t r_{t+4}^3]$	0	1.15E-10	4.26E-9	2.42E-9	-3.00E-9	2.23E-9	1.35E-9	-2.96E-9	1.89E-9	1.34E-9	9.84E-11	3.84E-9	3.06E-9
$E[r_t^2 r_{t+3}]$	0	1.18E-7	-2.54E-7	-1.47E-7	6.08E-8	-1.43E-7	-8.98E-8	5.15E-8	-1.28E-7	-9.25E-8	8.61E-8	-2.18E-7	-1.64E-7
$E[r_t r_{t+3}^2]$	1	-1.32E-7	-2.54E-7	-1.47E-7	-5.36E-8	-1.43E-7	-8.98E-8	-8.85E-8	-1.28E-7	-9.25E-8	-1.07E-7	-2.18E-7	-1.64E-7
$E[r_t^3 r_{t+3}]$	0	4.83E-9	4.46E-9	2.50E-9	3.78E-9	2.29E-9	1.35E-9	3.26E-9	1.88E-9	1.31E-9	4.13E-9	4.01E-9	3.06E-9
$E[r_t^2 r_{t+3}^2]$	1	2.74E-8	2.84E-8	2.92E-8	1.57E-8	1.47E-8	1.48E-8	1.52E-8	1.45E-8	1.45E-8	2.01E-8	1.98E-8	2.05E-8
$E[r_t r_{t+3}^3]$	1	-2.04E-9	4.46E-9	2.50E-9	3.45E-9	2.29E-9	1.35E-9	1.61E-9	1.88E-9	1.31E-9	1.65E-9	4.01E-9	3.06E-9
$E[r_t^2 r_{t+2}]$	0	5.17E-8	-2.69E-7	-1.53E-7	8.04E-8	-1.48E-7	-8.98E-8	5.10E-8	-1.27E-7	-8.95E-8	5.92E-8	-2.29E-7	-1.64E-7
$E[r_t r_{t+2}^2]$	1	-4.61E-7	-2.69E-7	-1.53E-7	-1.93E-7	-1.48E-7	-8.98E-8	-1.59E-7	-1.27E-7	-8.95E-8	-3.57E-7	-2.29E-7	-1.64E-7
$E[r_t^3 r_{t+2}]$	0	-4.43E-9	4.67E-9	2.58E-9	1.09E-9	2.37E-9	1.35E-9	2.16E-9	1.87E-9	1.28E-9	-2.88E-9	4.20E-9	3.06E-9
$E[r_t^2 r_{t+2}^2]$	1	3.02E-8	2.91E-8	2.96E-8	1.48E-8	1.50E-8	1.48E-8	1.40E-8	1.44E-8	1.43E-8	2.28E-8	2.03E-8	2.05E-8
$E[r_t r_{t+2}^3]$	1	5.01E-11	4.67E-9	2.58E-9	3.13E-9	2.37E-9	1.35E-9	3.65E-9	1.87E-9	1.28E-9	2.65E-9	4.20E-9	3.06E-9
$E[r_t^2 r_{t+1}]$	0	2.44E-7	-2.84E-7	-1.59E-7	1.35E-7	-1.54E-7	-8.98E-8	1.37E-7	-1.25E-7	-8.66E-8	1.62E-7	-2.41E-7	-1.64E-7
$E[r_t r_{t+1}^2]$	1	-3.11E-7	-2.84E-7	-1.59E-7	-1.44E-7	-1.54E-7	-8.98E-8	-1.04E-7	-1.25E-7	-8.66E-8	-2.43E-7	-2.41E-7	-1.64E-7
$E[r_t^3 r_{t+1}]$	0	-1.43E-8	4.89E-9	2.66E-9	-3.46E-9	2.44E-9	1.35E-9	-2.34E-9	1.85E-9	1.24E-9	-8.56E-9	4.39E-9	3.06E-9
$E[r_t^2 r_{t+1}^2]$	1	3.03E-8	2.99E-8	3.01E-8	1.50E-8	1.52E-8	1.48E-8	1.43E-8	1.43E-8	1.41E-8	1.98E-8	2.09E-8	2.05E-8
$E[r_t r_{t+1}^3]$	1	1.81E-9	4.89E-9	2.66E-9	2.82E-9	2.44E-9	1.35E-9	2.57E-9	1.85E-9	1.24E-9	3.12E-9	4.39E-9	3.06E-9

$\eta_1 = 0$. Figure 3.5 plots the annualized equity premium in terms of preferences for our one-factor SVS model. One observes that the maximum premium generated in this range of preference parameters when $\eta_1 \neq 0$ is almost 2% more than the same premium generated with $\eta_1 = 0$.

For the S&P500 index and the small book-to-market sorted stocks, Table 3.4 compares unconditional moments of returns computed from the parameter estimates through the analytical formulas, to their sample counterparts. A straightforward remark is how accurate the model with $\eta_1 \neq 0$ matches the selected moments better than the model with $\eta_1 = 0$. Especially, the third row of the table shows that the unconditional skewness in not matched with $\eta_1 = 0$ and this is also true for the unconditional leverage effect as shown in rows 18 and 24 of the table. Over all, this rises the role of the parameter η_1 in capturing third order return moments. Table 3.5 shows similar comparisons for the large book-to-market sorted portfolios and the CSRP index and the same observations hold.

Finally, as we mentioned previously, the choice of the moments to be used in the GMM procedure is crucial when intended to reproduce important empirical facts. While the cross-correlation between returns and cube returns is in general not significant for big stocks and market indexes, one can observe that, although used for the estimation procedure, this moment is not matched by the GMM estimates, except for the first lead where it appears significant for some of these stocks. However, for small stocks, this moment is significant empirically as shown in Panel A of Figure 3.4 for the three first leads, and Table 3.4 shows that the GMM estimates reproduce the moment as well. Next, we filter the latent factors using the GMM estimates of parameters.

3.7.2 State Estimation.

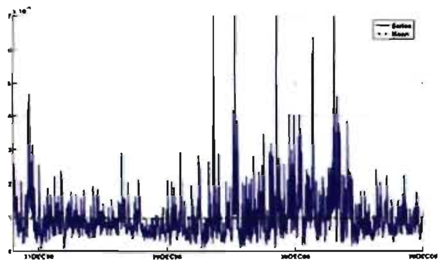
We use the Unscented Kalman Filter algorithm with our GMM estimates to filter the latent factor σ_{1t}^2 that we use to compute the GARCH counterparts of conditional volatility and skewness, i.e G_{ht} and G_{st}. We do this exercise for all estimations in Table 3.2 such that the estimate of the factor persistence ϕ_1 is well below the unity. This is the case for the big growth stock and the S&P500 index in both panels of the table, and for the small growth, the small value, the big neutral stocks and the CRSP index in Panel A of the table.

Figure 3.6 displays the time series of the GARCH counterparts of volatility and skewness for the big growth stock and the S&P500 index both for the contemporaneous conditional asymmetry ($\eta_1 \neq 0$) as well as for the contemporaneous conditional normality ($\eta_1 = 0$). Asset returns in our sample as plotted in Figure 3.1, are characterized by moderately high volatility at the beginning of the sample (1990-1992), followed by low volatility (1993-1996), then high volatility (1997-2003) and low or moderately high volatility at the end of the sample (2004-2005). This volatility pattern is well-matched by the volatility time series plotted in the first and the second rows of Figure 3.6. Also notice the slightly difference between volatility time series in different columns of the figure, due to the effect of the positive parameter η_1. The volatility pattern in left panels of Figure 3.6 is more tightened.

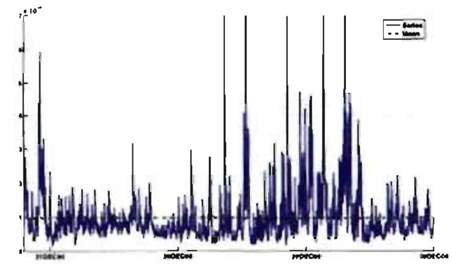
The third and the fourth rows of the figure show the pattern of the GARCH counterpart of conditional skewness. Overall results are striking. Conditional skewness is negative when $\eta_1 = 0$ as displayed in Figure 3.6, and this is consistent with the IG-GARCH model of Christoffersen, Heston and Jacobs (2006). It should also be noticed that these authors constrain their IG-GARCH model to display negative conditional skewness. On the other hand, as we mentioned earlier, maximum likelihood methods can poorly match higher order unconditional return moments and this may also affect conditional higher order moments. Compared to our model, we also found that critical unconditional third order moments of returns, skewness and leverage effects, are not matched by our GMM estimation procedure when we assumed contemporaneous conditional normality. In contrast, if contemporaneous conditional asymmetry is allowed, we found that our GMM procedure matches unconditional skewness and leverage effects very well and, in this case, the pattern of conditional skewness displayed in Figure 3.6 shows that conditional skewness is positive and even with a mean with large magnitude compared to the contemporaneous conditional normality case. Figure 3.7 confirms that these results hold for other portfolios as well.

Figure 3.6: **Portfolios Volatility and Skewness: S&P500 and Others**

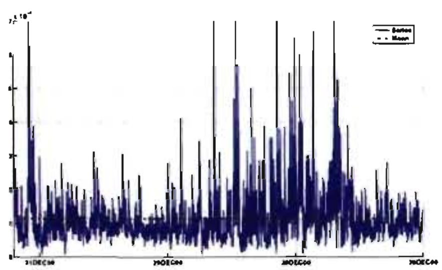
S&P500 Volatility $\eta_1 \neq 0$



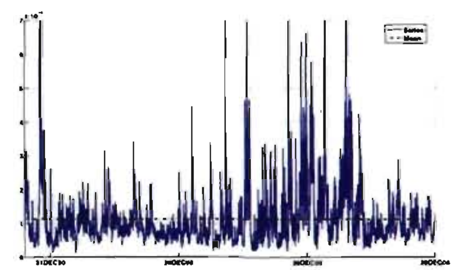
S&P500 Volatility $\eta_1 = 0$



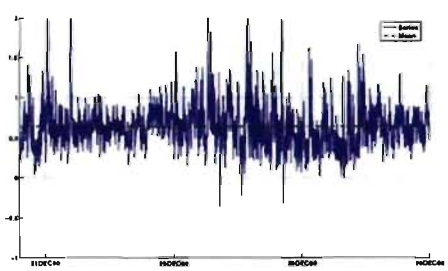
BG Volatility $\eta_1 \neq 0$



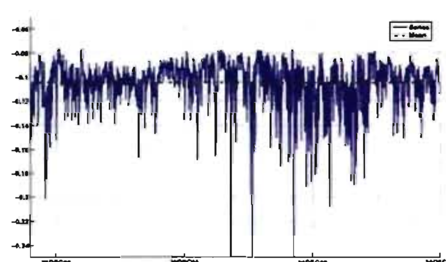
BG Volatility $\eta_1 = 0$



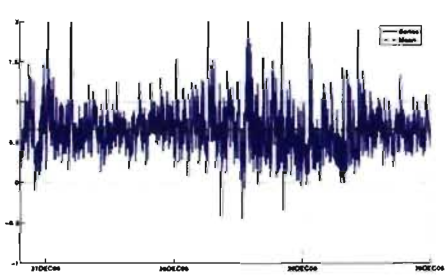
S&P500 Skewness $\eta_1 \neq 0$



S&P500 Skewness $\eta_1 = 0$



BG Skewness $\eta_1 \neq 0$



BG Skewness $\eta_1 = 0$

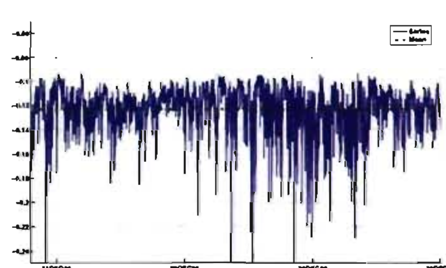
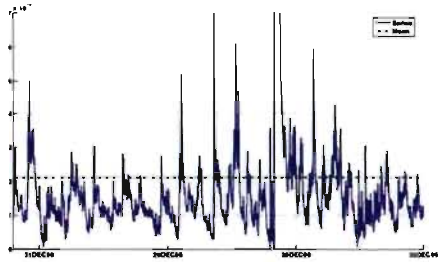
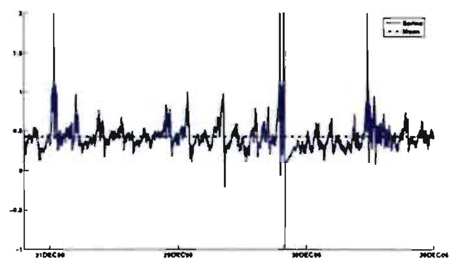


Figure 3.7: **Portfolios Volatility and Skewness: Others**

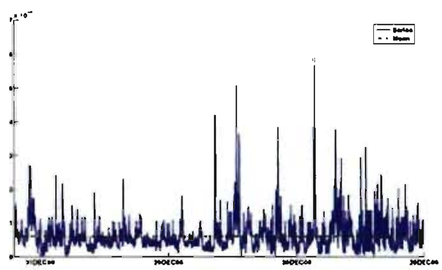
SG Volatility $\eta_1 \neq 0$



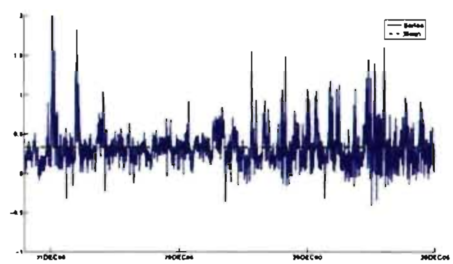
SG Skewness $\eta_1 \neq 0$



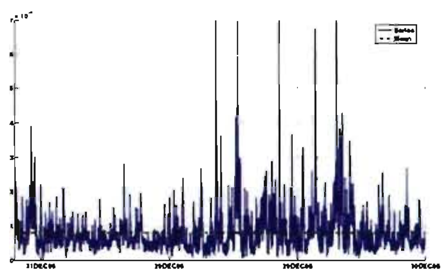
SV Volatility $\eta_1 \neq 0$



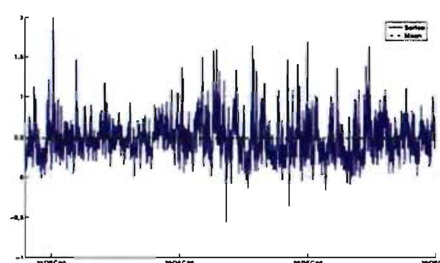
SV Skewness $\eta_1 \neq 0$



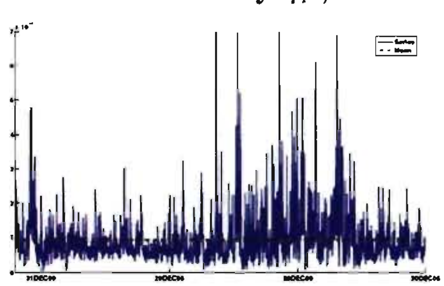
BN Volatility $\eta_1 \neq 0$



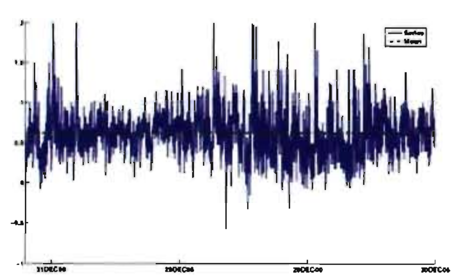
BN Skewness $\eta_1 \neq 0$



CRSP Volatility $\eta_1 \neq 0$



CRSP Skewness $\eta_1 \neq 0$



3.8 Conclusion

In this chapter, we provide a new affine multivariate latent variable model for asset returns in which conditional volatility and skewness are stochastic. We characterize these critical conditional return moments as well as their GARCH counterparts. The model allows for closed-form asset and option pricing formulas and can be well-utilized in term structure as well.

We also develop a GMM procedure for the estimation of a more general affine multivariate latent variable model that nests our SVS specification. This procedure has a huge computational advantage compared to maximum likelihood-based techniques and perfectly matches critical higher order return moments while other methods generally fail to. We apply this procedure to our univariate SVS model and use the GMM estimates and the Unscented Kalman Filter to derive the GARCH counterparts of volatility and skewness.

Results point out that stochastic skewness appears to be relevant in asset pricing. Moreover and more striking, a positively-skewed distribution of returns conditional on contemporaneous latent volatility fits the unconditional return skewness and leverage effects, whereas a normal distribution doesn't. Most importantly, this positively-skewed distribution generates positive conditional skewness, in contrast to negative conditional skewness more consistent with previous studies.

These striking results open a room for a new relevant issue which deserves further investigation: can a return model with well-fitted unconditional third order moments produce negative conditional skewness? This constitutes an ongoing research together with the estimation of the two factor SVS model and the application of SVS models in asset and derivative pricing as well as term structure of interest rates.

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CONCLUSION GÉNÉRALE

Les investisseurs s'intéressent à la volatilité de la consommation parce qu'ils craignent les répercussions de l'incertitude macroéconomique sur leur richesse future. Motivés par un modèle d'équilibre général sous forme réduite dans lequel la consommation suit un processus affine à volatilité stochastique, le premier chapitre de cette thèse a documenté des faits empiriques importants reliant les rendements des actions à la volatilité de la consommation.

Les investisseurs peuvent choisir pour un horizon d'investissement donné, de détenir les actions pour une courte période au début de l'intervalle d'investissement, disons pendant la première période seulement, puis des obligations sans risque pour le restant de l'intervalle d'investissement. Adoptant une telle stratégie pour des horizons d'investissement courts, nous avons trouvé que le risque lié à la variation dans le niveau de la consommation est moins corrélé au rendement escompté que le risque lié à la variation dans la volatilité de la consommation pendant l'intervalle d'investissement. L'inverse se produit pour des horizons d'investissement longs.

Par ailleurs, les investisseurs peuvent choisir pour un horizon d'investissement donné, de détenir les actions pour une plus longue période, voire tout le long de l'intervalle d'investissement. Pour une telle stratégie et pour des horizons d'investissement courts, nous avons trouvé que le risque lié à la variation dans le niveau de la consommation est plus corrélé au rendement escompté que le risque lié à la variation dans la volatilité de la consommation pendant l'intervalle d'investissement. L'inverse se produit pour des horizons d'investissement longs.

Ainsi, les rendements d'actions possédées pour une courte période sont plus sensibles aux variations de court terme dans la volatilité de la consommation tandis qu'il existe une relation stable de long terme entre la volatilité de la consommation et les rendements d'actions. L'estimation des prix des risques en utilisant les portefeuilles d'actions des firmes classées selon la capitalisation boursière et le rapport valeur comptable/valeur boursière, a montré que la volatilité de la consommation reste un facteur significatif dans la valorisation des actions, même lorsque les investisseurs tiennent déjà compte de la valorisation basée sur le niveau de la consommation.

La volatilité de la consommation varie avec le cycle des affaires et apparaît comme un facteur important permettant d'expliquer les différences entre les primes de risque d'actions par les différences entre les sensibilités de leurs gains vis-à-vis de l'incertitude macroéconomique. Une recherche future s'intéressera à la rationalisation des faits empiriques établis à l'aide d'un modèle d'équilibre général similaire à celui ayant motivé cette étude.

Les modèles d'évaluation d'actifs financiers par équilibre sont devenus difficile à résoudre. Pour reproduire les faits empiriques robustes, les chercheurs ont supposé que l'agent représentatif est doté de préférences plus sophistiquées. Les agrégats fondamentaux de l'économie, à savoir la consommation et les dividendes, ont également bénéficié de dynamiques plus riches. Souvent, le temps requis pour résoudre numériquement le modèle ou pour le simuler afin de calculer les statistiques d'intérêt est abusif. Par conséquent, les chercheurs se sont tournés vers des modèles plus simples, faisant des hypothèses simplificatrices comme compromis entre la réalité et la faisabilité.

Dans le deuxième chapitre de cette thèse, nous avons fourni des formules analytiques qui devraient énormément aider à évaluer l'habilité de ces modèles à reproduire les faits empiriques. Nous avons choisi un modèle flexible pour la consommation et les dividendes, pouvant être directement appliqué aux données comme l'ont précédemment fait plusieurs chercheurs, ou encore pouvant être utilisé pour reproduire d'autres types de processus qui ont été examinés. En termes de préférences, nous avons choisi le cadre de l'utilité récursive d'Epstein et Zin (1989), largement utilisé dans la littérature sur l'évaluation d'actifs financiers. Nous avons limité notre analyse à l'équivalent certain de Kreps et Porteus (1978). Dans la recherche future, nous avons l'intention d'essayer d'obtenir des formules analytiques pour d'autres équivalents certains dans le cadre de l'utilité récursive, mais aussi pour d'autres types de préférences.

Dans le troisième chapitre de cette thèse, nous avons fourni un nouveau modèle affine multivarié à variables latentes pour les rendements journaliers. Dans ce modèle, la variance et l'asymétrie conditionnelles sont des combinaisons linéaires de facteurs stochastiques. Nous avons caractérisé ces moments conditionnels critiques tels que perçus par l'agent économique, ainsi que leurs contreparties telles que vues par l'économètre. Le modèle permet d'obtenir des formules analytiques aussi bien pour les moments en population des rendements que pour les prix d'actifs financiers. Nous développons ensuite une procédure d'estimation par la méthode des moments généralisée. Nous argumentons que cette procédure présente un énorme avantage par rapport à l'estimation par maximum de vraisemblance. En outre elle permet de reproduire parfaitement des moments critiques des rendements tels que l'asymétrie et l'aplatissement tandis que la plupart des méthodes y échouent.

Nous avons appliqué cette nouvelle procédure d'estimation au cas univarié de notre modèle et avons estimé le facteur latent grâce à une variante du filtre de Kalman non-linéaire. Les résultats ont montré que l'asymétrie inconditionnelle est déterminante pour l'évaluation d'actifs financiers. Plus frappant encore, une asymétrie positive de la distribution des rendements courants conditionnellement à la volatilité courante est nécessaire et suffisante pour reproduire l'asymétrie et les effets de levier inconditionnels, mais engendre une asymétrie positive de la distribution des rendements courants conditionnellement aux rendements passés, ce qui est contraire aux résultats empiriques connus.

Ce résultat étonnant et robuste demande d'examiner plus rigoureusement la question de savoir si un modèle reproduisant parfaitement les asymétries inconditionnelles génèrerait une asymétrie conditionnelle négative. Cette dernière question constitue une recherche en cours, ainsi que l'estimation du modèle bivarié et ses implications pour la valorisation des produits dérivés et la structure à terme des taux d'intérêt.

Appendix I

Additional Figures: Consumption Level and Volatility Risks at a Less Aggregate Level

Figure I.1: **Cross-Sectional Correlations, by Stock Holding Period, Between Risk Premium and Consumption Level and Consumption Volatility Risks.**
This figure presents the patterns of $\rho_{rc}(S,k)$ and $\rho_{rh}(S,k)$ when S is fixed to 8, 12, 16 and 20, while k varies from 1 to S.

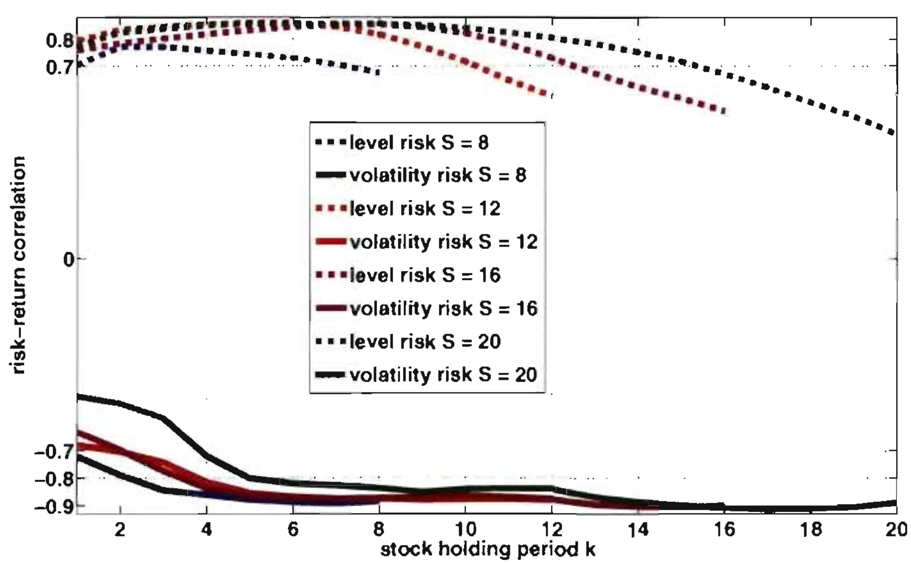


Figure I.2: **Volatility Risk for One-Period Book-to-Market Sorted Portfolios in Size Dimension.**
This figure presents the pattern of S-volatility risk across one-period book-to-market sorted portfolios in size dimension ($k = 1$). Risks are computed as covariances of returns with variations in consumption volatility.

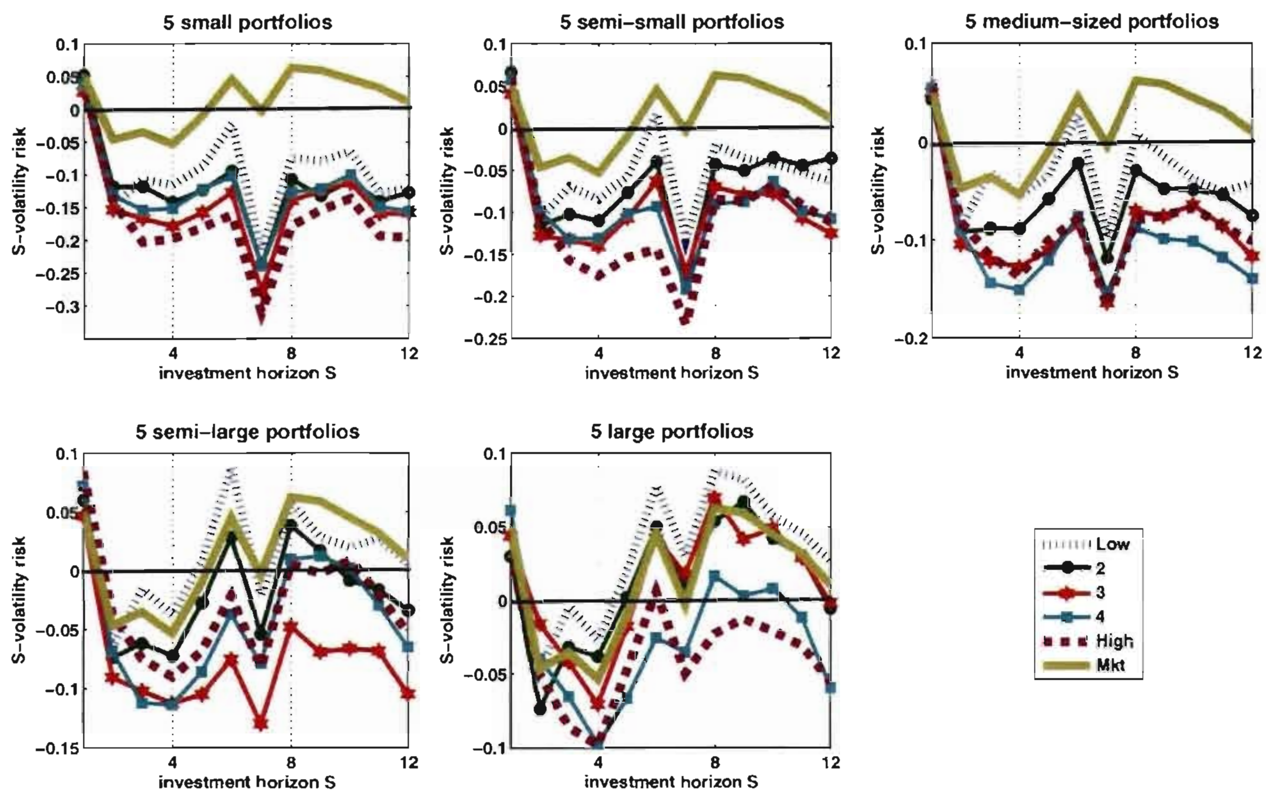


Figure I.3: **Level Risk for One-Period Book-to-Market Sorted Portfolios in Size Dimension.**
This figure presents the pattern of S-level risk across one-period book-to-market sorted portfolios in size dimension ($k = 1$). Risks are computed as covariances of returns with variations in consumption level.

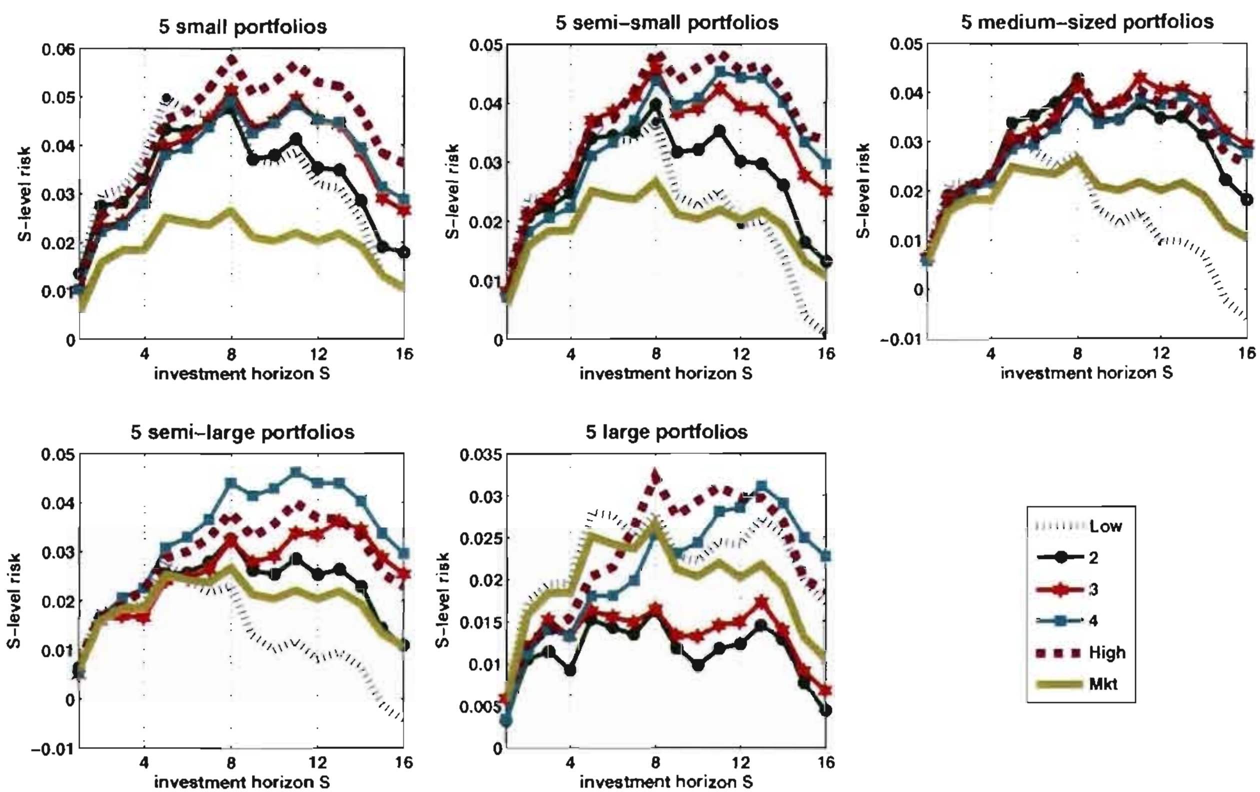


Figure I.4: **Volatility Risk for Multiperiod Book-to-Market Sorted Portfolios in Size Dimension.**
This figure presents the pattern of S-volatility risk across long-horizon book-to-market sorted portfolios in size dimension ($k = S$). Risks are computed as covariances of returns with variations in consumption volatility.

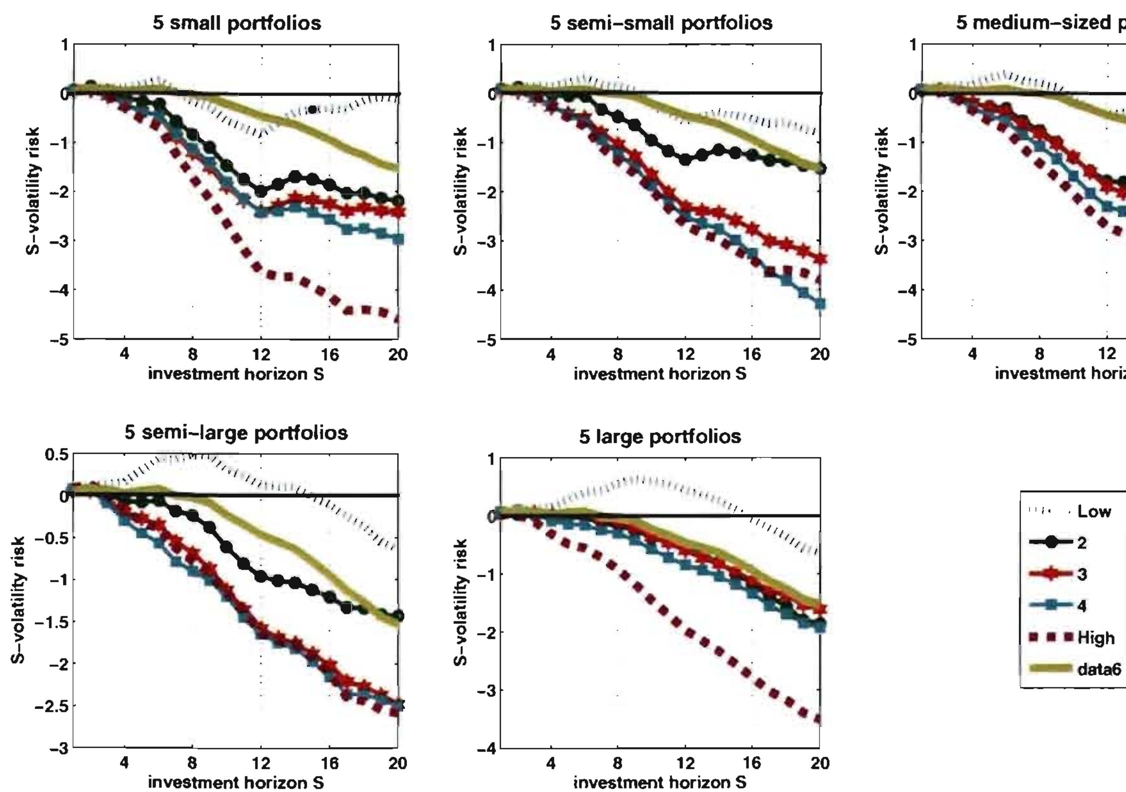
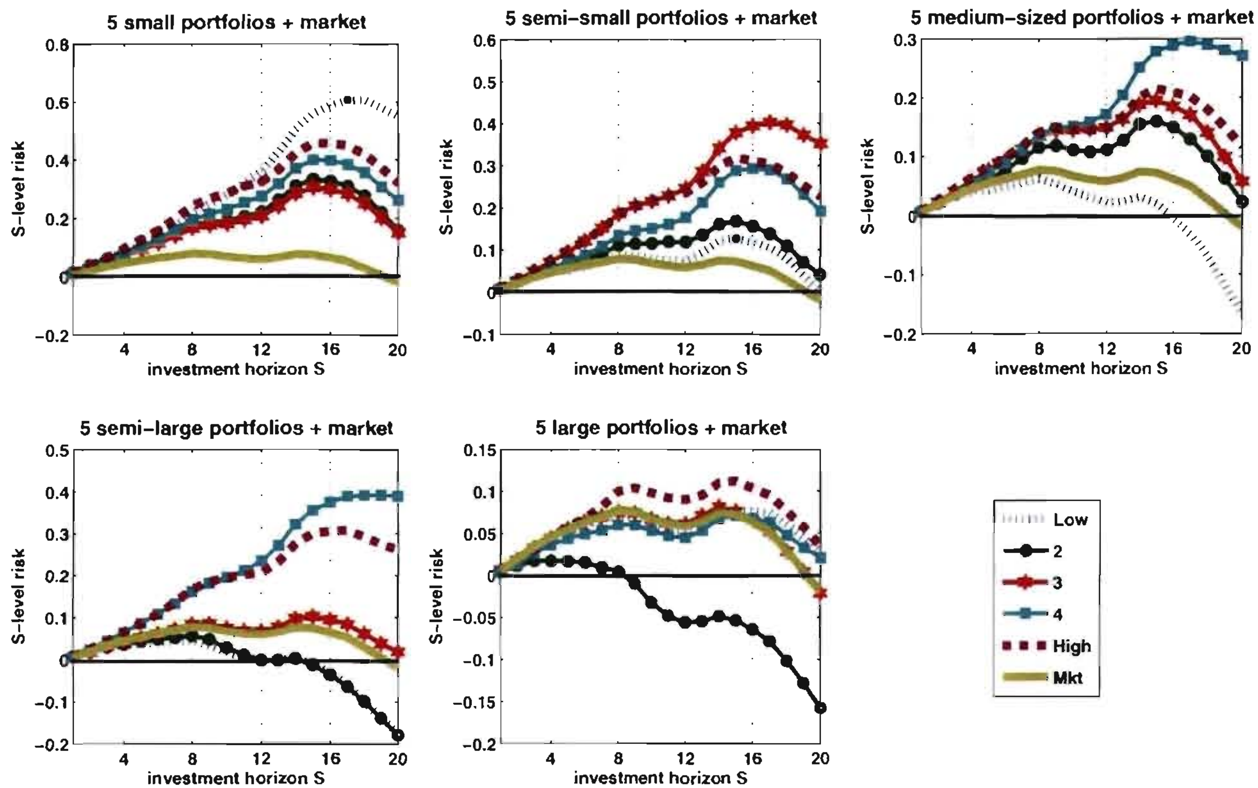


Figure I.5: **Level Risk for Multiperiod Book-to-Market Sorted Portfolios in Size Dimension.**
This figure presents the pattern of S-level risk across long-horizon book-to-market sorted portfolios in size dimension ($k = S$). Risks are computed as covariances of returns with variations in consumption level.



Appendix II

Technical Appendix of Chapter 2

Before showing the propositions, we provide some results on Markov switching models. It is well known that (see, e.g., Hamilton (1994), page 679)

$$E[\zeta_{t+h} \mid J_t] = P^h \zeta_t, \tag{A.1}$$

In addition,

$$\forall h,\ P^h \Pi = \Pi. \tag{A.2}$$

For any vectors $A, B, \in \Re^N$, we have

$$(A^\top \zeta_t)(B^\top \zeta_t) = (A \odot B)^\top \zeta_t. \tag{A.3}$$

Meddahi and Taamouti (2004) showed that: $\forall h \geq 2, \forall u_i \in \Re, i = 1, ..., h,$ we have

$$E\left[\exp\left(\sum_{i=1}^{h} u_i x_{t+i}\right) \mid J_t\right] = e^\top \prod_{i=2}^{h} A(\mu u_{h+1-i} + \omega u_{h+1-i}^2/2)\ \ \exp((u_1 \mu + \omega u_1^2/2)^\top \zeta_t)\zeta_t; \tag{A.4}$$

$\forall u \in \Re, \forall h \geq 2,$ we have

$$E\left[\exp\left(u \sum_{i=1}^{h} x_{t+i}\right) \mid J_t\right] = e^\top \left(A(\mu u + \omega u^2/2)\right)^{h-1} \exp((u\mu + (u^2/2)\omega)^\top \zeta_t)\zeta_t; \tag{A.5}$$

finally,

$$\forall u \in \Re^N, \forall h \geq 1,\ E[\exp(u^\top \zeta_{t+h}) \mid J_t] = e^\top A(u) P^{h-1} \zeta_t, \tag{A.6}$$

$$E[\exp(u^\top \zeta_{t+1})\zeta_{t+1} \mid J_t] = A(u)\zeta_t. \tag{A.7}$$

Lemma 1: *For any vectors $A,\ B \in \Re^N$ and for any integer h, $h > 0$, we have*

$$Var\left[\sum_{j=1}^{h}(A^\top \zeta_{t+j-1})(B^\top \zeta_{t+j})\right] = h(A \odot A)^\top E[\zeta_t \zeta_t^\top]P^\top(B \odot B) - h^2(A^\top E[\zeta_t \zeta_t^\top]P^\top B)^2$$
$$+2\sum_{j=2}^{h}(h-j+1)A^\top E[\zeta_t \zeta_t^\top]P^\top(B \odot ((P^{j-2})^\top(A \odot (P^\top B)))). \tag{A.8}$$

Proof of Lemma 1. Define the random variable u_t as $u_t = (A^\top \zeta_{t-1})(B^\top \zeta_t)$. We have

$$\begin{aligned} Var\left[\sum_{j=1}^{h}(A^\top \zeta_{t+j-1})(B^\top \zeta_{t+j})\right] &= Var\left[\sum_{j=1}^{h} u_{t+j}\right] \\ &= hVar[u_t] + 2\sum_{j=2}^{h}(h-j+1)Cov(u_{t+1}, u_{t+j}). \end{aligned} \tag{A.9}$$

We first compute $Var[u_t]$. We have,

$$E[u_t] = A^\top E[\zeta_t \zeta_{t+1}^\top]B = A^\top E[\zeta_t E[\zeta_{t+1}^\top \mid \zeta_t]]B = A^\top E[\zeta_t \zeta_t^\top P^\top]B = A^\top E[\zeta_t \zeta_t^\top]P^\top B. \tag{A.10}$$

In addition,

$$u_t^2 = (A^\top \zeta_t)^2(B^\top \zeta_{t+1})^2 = ((A \odot A)^\top \zeta_t)((B \odot B)^\top \zeta_{t+1}).$$

Hence, the same calculations done in the proof of (A.10) yield to

$$E[u_t^2] = (A \odot A)^\top E[\zeta_t \zeta_t^\top]P^\top(B \odot B). \tag{A.11}$$

By combining (A.10) and (A.11), one gets

$$Var[u_t] = (A \odot A)^\top E[\zeta_t \zeta_t^\top]P^\top(B \odot B) - (A^\top E[\zeta_t \zeta_t^\top]P^\top B)^2. \tag{A.12}$$

We now compute $Cov(u_{t+1}, u_{t+j})$. For $j \geq 2$, we have

$$\begin{aligned}
E[u_{t+1}u_{t+j}] &= E[(A^\top \zeta_t)(B^\top \zeta_{t+1})(A^\top \zeta_{t+j-1})(B^\top \zeta_{t+j})] \\
&= E[(A^\top \zeta_t)(B^\top \zeta_{t+1})(A^\top \zeta_{t+j-1})(B^\top E[\zeta_{t+j} \mid \zeta_{t+j-1}])] \\
&= E[(A^\top \zeta_t)(B^\top \zeta_{t+1})(A^\top \zeta_{t+j-1})(B^\top P\zeta_{t+j-1})] \\
&= E[(A^\top \zeta_t)(B^\top \zeta_{t+1})((A \odot (P^\top B))^\top \zeta_{t+j-1})],
\end{aligned}$$

where the last equality follows from (A.3). Hence,

$$\begin{aligned}
E[u_{t+1}u_{t+j}] &= E[(A^\top \zeta_t)(B^\top \zeta_{t+1})((A \odot (P^\top B))^\top E[\zeta_{t+j-1} \mid \zeta_{t+1}])] \\
&= E[(A^\top \zeta_t)(B^\top \zeta_{t+1})((A \odot (P^\top B))^\top P^{j-2} \zeta_{t+1})] \\
&= E[(A^\top \zeta_t)(B \odot ((P^{j-2})^\top (A \odot (P^\top B)))^\top \zeta_{t+1})],
\end{aligned}$$

where again the last equality follows from (A.3). Therefore,

$$\begin{aligned}
E[u_{t+1}u_{t+j}] &= A^\top E[\zeta_t \zeta_{t+1}^\top](B \odot ((P^{j-2})^\top (A \odot (P^\top B)))) \\
&= A^\top E[\zeta_t \zeta_t^\top] P^\top (B \odot ((P^{j-2})^\top (A \odot (P^\top B)))).
\end{aligned} \tag{A.13}$$

By combining (A.10) and (A.13), one gets

$$Cov[u_{t+1}, u_{t+j}] = A^\top E[\zeta_t \zeta_t^\top] P^\top (B \odot ((P^{j-2})^\top (A \odot (P^\top B)))) - (A^\top E[\zeta_t \zeta_t^\top] P^\top B)^2. \tag{A.14}$$

By plugging (A.12) and (A.14) into (A.9), one gets (A.8).

Proof of Proposition 2.4.1. See Bonomo and Garcia (1994).

Proof of (2.18): We have

$$\begin{aligned}
R_{t+1} &= \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{D_t}{P_t} \frac{D_{t+1}}{D_t} \left(\frac{P_{t+1}}{D_{t+1}} + 1 \right) = (\lambda_2^\top \zeta_t) \exp(x_{t+1})(\lambda^\top \zeta_{t+1} + 1) \\
&= (\lambda_2^\top \zeta_t) \exp(x_{t+1})(\lambda_3^\top \zeta_{t+1}),
\end{aligned}$$

where the last equality holds given that $e^\top \zeta_{t+1} = 1$.

Proof of Proposition 2.3.1. Given the information J_t, the processes ζ_{t+1} and x_{t+1} are independent. Therefore,

$$\begin{aligned}
E[R_{t+1} \mid J_t] &= E[(\lambda_2^\top \zeta_t)\ \exp(x_{t+1})\ (\lambda_3^\top \zeta_{t+1}) \mid J_t] \\
&= (\lambda_2^\top \zeta_t)\ E[\exp(x_{t+1}) \mid J_t]\ E[(\lambda_3^\top \zeta_{t+1}) \mid J_t] \\
&= (\lambda_2^\top \zeta_t)\ \exp(\mu^\top \zeta_t + \omega^\top \zeta_t/2)\ \lambda_3^\top E[\zeta_{t+1} \mid J_t] \\
&= (\lambda_2^\top \zeta_t)\ \exp(\mu^\top \zeta_t + \omega^\top \zeta_t/2)\ \lambda_3^\top P \zeta_t,
\end{aligned}$$

i.e., (2.20). Consequently, $\forall j \geq 2$

$$E[R_{t+j} \mid J_t] = \psi^\top E[\zeta_{t+j-1} \mid J_t] = \psi^\top P^{j-1} \zeta_t,$$

i.e., (2.23). Finally,

$$E[R_{t+1:t+h} \mid J_t] = E\left[\sum_{j=1}^{h} R_{t+j} \mid J_t\right] = \psi^\top \left(\sum_{j=1}^{h} P^{j-1}\right) \zeta_t,$$

i.e., (2.24).

Proof of Proposition 2.3.2. By using (2.16), one gets

$$Var\left[\frac{D_t}{P_t}\right] = Var[\lambda_2^\top \zeta_t] = \lambda_2^\top Var[\zeta_t] \lambda_2,$$

i.e. (2.27). We also have

$$\begin{aligned}
Cov\left(R_{t+1:t+h}, \frac{D_t}{P_t}\right) &= Cov(E[R_{t+1:t+h} \mid J_t], \lambda_2^\top \zeta_t) = Cov(\psi_h^\top \zeta_t, \lambda_2^\top \zeta_t) = \psi_h^\top Cov(\zeta_t, \zeta_t^\top \lambda_2) \\
&= \psi_h^\top Var(\zeta_t) \lambda_2,
\end{aligned}$$

i.e., (2.28).

Observe that conditional on the information set $\{\zeta_\tau, \tau \in \mathbf{N}\}$, the variables R_{t+j}, $j = 1, ..., h$, are independent. Therefore,

$$\begin{aligned}
Var[R_{t+1:t+h}] &= Var[E[R_{t+1:t+h} \mid \{\zeta_\tau, \tau \in \mathbf{N}\}]] + E[Var[R_{t+1:t+h} \mid \{\zeta_\tau, \tau \in \mathbf{N}\}]] \\
&= Var\left[\sum_{j=1}^{h} E[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbf{N}\}]\right] + E\left[\sum_{j=1}^{h} Var[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbf{N}\}]\right].
\end{aligned} \quad (A.15)$$

Given that $R_{t+j} = (\lambda_2^\top \zeta_{t+j-1})\ (\lambda_3^\top \zeta_{t+j})\ \exp(x_{t+j})$, we have

$$\begin{aligned} E[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbf{N}\}] &= (\lambda_2^\top \zeta_{t+j-1})\ (\lambda_3^\top \zeta_{t+j}) E[\exp(x_{t+j}) \mid \{\zeta_\tau, \tau \in \mathbf{N}\}] \\ &= (\lambda_2^\top \zeta_{t+j-1})\ (\lambda_3^\top \zeta_{t+j}) \exp(\mu^\top \zeta_{t+j-1} + \omega^\top \zeta_{t+j-1}/2) \\ &= (\theta_1^\top \zeta_{t+j-1})\ (\lambda_3^\top \zeta_{t+j}), \end{aligned} \tag{A.16}$$

and

$$\begin{aligned} Var[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbf{N}\}] &= (\lambda_2^\top \zeta_{t+j-1})^2\ (\lambda_3^\top \zeta_{t+j})^2 Var[\exp(x_{t+j}) \mid \{\zeta_\tau, \tau \in \mathbf{N}\}] \\ &= ((\lambda_2 \odot \lambda_2)^\top \zeta_{t+j-1})\ ((\lambda_3 \odot \lambda_3)^\top \zeta_{t+j}) \\ &\quad (\exp(2\mu^\top \zeta_{t+j-1} + 2\omega^\top \zeta_{t+j-1}) - \exp(2\mu^\top \zeta_{t+j-1} + \omega^\top \zeta_{t+j-1})) \\ &= (\theta_2^\top \zeta_{t+j-1})\ (\theta_3^\top \zeta_{t+j}), \end{aligned} \tag{A.17}$$

where θ_1, θ_2 and θ_3 are defined in (2.33), (2.34), and (2.35) respectively. Consequently,

$$\begin{aligned} E\left[\sum_{j=1}^{h} Var[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbf{N}\}]\right] &= E\left[\sum_{j=1}^{h} (\theta_2^\top \zeta_{t+j-1})\ (\theta_3^\top \zeta_{t+j})\right] \\ &= \theta_2^\top \sum_{j=1}^{h} E\left[\zeta_{t+j-1} \zeta_{t+j}^\top\right] \theta_3 \\ &= \theta_2^\top \sum_{j=1}^{h} E\left[\zeta_{t+j-1} E[\zeta_{t+j}^\top \mid J_{t+j-1}]\right] \theta_3 \\ &= \theta_2^\top \sum_{j=1}^{h} E\left[\zeta_{t+j-1} \zeta_{t+j-1}^\top P^\top\right] \theta_3, \end{aligned}$$

i.e.,

$$E\left[\sum_{j=1}^{h} Var[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbf{N}\}]\right] = h\theta_2^\top E\left[\zeta_t \zeta_t^\top\right] P^\top \theta_3. \tag{A.18}$$

In addition, we have

$$Var\left[\sum_{j=1}^{h} E[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbf{N}\}]\right] = Var\left[\sum_{j=1}^{h} (\theta_1^\top \zeta_{t+j-1})\ (\lambda_3^\top \zeta_{t+j})\right].$$

Therefore, by using (A.8), one gets

$$Var\left[\sum_{j=1}^{h} E[R_{t+j} \mid \{\zeta_\tau, \tau \in \mathbf{N}\}]\right] = h(\theta_1 \odot \theta_1)^\top E[\zeta_t \zeta_t^\top] P^\top (\lambda_3 \odot \lambda_3) - h^2(\theta_1^\top E[\zeta_t \zeta_t^\top] P^\top \lambda_3)^2$$
$$+ 2\sum_{j=2}^{h} (h-j+1)\theta_1^\top E[\zeta_t \zeta_t^\top] P^\top (\lambda_3 \odot ((P^{j-2})^\top (\theta_1 \odot (P^\top \lambda_3)))). \quad \text{(A.19)}$$

Finally, by combining (A.15) with (A.18) and (A.19), one gets (2.32).

Appendix III

The Campbell and Cochrane (1999) Model

The Euler equation in Campbell and Cochrane (1999) is given by

$$1 = E\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{S_{t+1}}{S_t}\right)^{-\gamma} R_{t+1} \mid J_t\right]. \tag{A.1}$$

If we assume that,

$$x_{s,t+1} \equiv \log(S_{t+1}) - \log(S_t) = \mu_s^\top \zeta_t + (\omega_s^\top \zeta_t)^{1/2} \varepsilon_{c,t+1} \tag{A.2}$$

where

$$\begin{pmatrix} \varepsilon_{c,t+1} \\ \varepsilon_{d,t+1} \\ \varepsilon_{s,t+1} \end{pmatrix} \mid \sigma(\varepsilon_{c,\tau}, \varepsilon_{d,\tau}, \varepsilon_{s,\tau}, \tau \leq t; \zeta_m, m \in \mathbf{Z})$$

$$\sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{bmatrix} 1 & \rho^\top \zeta_t & \rho_{cs}^\top \zeta_t \\ \rho^\top \zeta_t & 1 & \rho_{ds}^\top \zeta_t \\ \rho_{cs}^\top \zeta_t & \rho_{ds}^\top \zeta_t & 1 \end{bmatrix}\right). \tag{A.3}$$

Observe that Campbell and Cochrane assumed

$$\rho_{cs} = e = (1, 1, \ldots, 1)^\top \text{ and } \rho_{ds} = \rho. \tag{A.4}$$

Under (A.3), (A.1) becomes

$$1 = E\left[\delta \exp(-\gamma x_{cs,t+1})\, R_{t+1} \mid J_t\right]. \tag{A.5}$$

with

$$x_{cs,t+1} \equiv x_{c,t+1} + x_{s,t+1} = \mu_{cs}^{\top}\zeta_t \; + \; (\omega_{cs}^{\top}\zeta_t)^{1/2} \, \varepsilon_{cs,t+1} \tag{A.6}$$

where

$$\mu_{cs} = \mu_c + \mu_s, \; \omega_{cs} = \omega_c + \omega_d + 2\rho_{cs} \odot (\omega_c)^{1/2} \odot (\omega_s)^{1/2}, \tag{A.7}$$

$$\varepsilon_{cs,t+1} \mid \sigma(\varepsilon_{c,\tau}, \varepsilon_{d,\tau}, \varepsilon_{x,\tau}, \tau \leq t; \zeta_m, m \in \mathbf{Z}) \;\sim\; \mathcal{N}(0,1).$$

Consequently, their model is like a CCAPM model where one has x_{cs} in the SDF instead of having x_c. In order to compute the price-payoff ratios with these preferences, it is important to derive the joint dynamics of $(\varepsilon_{cs,t+1}, \varepsilon_{d,t+1}, \varepsilon_{c,t+1})^{\top}$. It will then be sufficient to plug these formulas in the CCAPM model to derive the vectors λ.

We have

$$\begin{aligned} &\begin{pmatrix} \varepsilon_{cs,t+1} \\ \varepsilon_{d,t+1} \\ \varepsilon_{c,t+1} \end{pmatrix} \mid \sigma(\varepsilon_{c,\tau}, \varepsilon_{d,\tau}, \varepsilon_{s,\tau}, \tau \leq t; \zeta_m, m \in \mathbf{Z}) \\ &\sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{bmatrix} 1 & \rho_{csd}^{\top}\zeta_t & \rho_{csc}^{\top}\zeta_t \\ \rho_{csd}^{\top}\zeta_t & 1 & \rho^{\top}\zeta_t \\ \rho_{csc}^{\top}\zeta_t & \rho^{\top}\zeta_t & 1 \end{bmatrix} \right), \end{aligned} \tag{A.8}$$

where

$$\rho_{csc} = (w_c + \rho_{cs} \odot w_c^{1/2} \odot w_s^{1/2}) \odot w_c^{-1/2} \odot \omega_{cs}^{-1/2}, \tag{A.9}$$

$$\rho_{csd} = (\rho_{cd} \odot w_c^{1/2} \odot w_d^{1/2} + \rho_{ds} \odot w_d^{1/2} \odot w_s^{1/2}) \odot w_d^{-1/2} \odot \omega_{cs}^{-1/2}. \tag{A.10}$$

Define

$$\mu_{csd} = -\gamma\mu_{cs} + \mu_d, \; \omega_{csd} = \gamma^2 \omega_{cs} + \omega_d - 2\gamma(\rho_{csd} \odot \omega_{cs}^{1/2} \odot \omega_d^{1/2}) \tag{A.11}$$

$$\mu_{csc} = -\gamma\mu_{cs} + \mu_c =, \ \omega_{csc} = \gamma^2\omega_{cs} + \omega_c - 2\gamma(\rho_{csc} \odot \omega_{cs}^{1/2} \odot \omega_c^{1/2}) \tag{A.12}$$

Proposition III.0.1. ***Characterization of the Asset Prices.***
The i-th component, i=1,...,N, of the vector λ_1 defined in (2.13) is given by

$$\lambda_{1,i} = \delta \exp(\mu_{csd,i} + \omega_{csd,i}/2))e^\top \left[Id - \delta A(\mu_{csd} + \omega_{csd}/2)\right]^{-1} e_i, \tag{A.13}$$

where $A(\cdot)$ is defined in (3.20). In addition, the i-th component, i=1,...,N, of the vector λ_{1c} defined in (2.14) is given by

$$\lambda_{1c,i} = \delta \exp(\mu_{csc,i} + \omega_{csc,i}/2))e^\top \left[Id - \delta A(\mu_{csc} + \omega_{csc}/2)\right]^{-1} e_i, \tag{A.14}$$

Finally, the components of the vector b defined in (2.15) are given by

$$b_i = \delta^{-1} \exp(\gamma\mu_{cs,i} - \frac{\gamma^2}{2}\omega_{cs,i}). \tag{A.15}$$

Appendix IV

Continuous Time Limits of SVS Models

Here we derive the continuous time limit of our one-factor SVS model. For simplicity we consider that δ_t is constant. Our discrete time process is built in two steps. We first specify the distribution of log returns conditional upon the latent factor:

$$r_{t+1} = \mu_0 + \lambda_1 \left(\sigma_{1,t+1}^2 - \mu_1\right) + \sigma_{1,t+1} u_{1,t+1}$$

and next, we specify the dynamics of the latent factor, namely a univariate autoregressive gamma process, ARG(1).

$$E\left[\exp\left(y_1 \sigma_{1,t+1}^2\right) \mid I_t\right] = \exp\left(a\left(y_1\right) + b_1\left(y_1\right) \sigma_{1,t}^2\right)$$

where

$$b_1\left(y_1\right) = \frac{\phi_1 y_1}{1 - \alpha_1 y_1} \text{ and } a\left(y_1\right) = -\nu_1 \ln\left(1 - \alpha_1 y_1\right).$$

Compound autoregressive processes has been widely discussed by Gourieroux and Jasiak (2006) as well as Lamberton and Lapeyre (1992) who established a more general result. They show that the continous time limit of a univariate autoregressive gamma process, ARG(1), is a square-root process, CIR. Thus, it follows that:

$$d\sigma_{1t}^2 = \kappa_1 \left(\varpi_1 - \sigma_{1t}^2\right) dt + e_1 \sigma_{1t} dw_{1t}$$

where w_{1t} is a Wiener process and κ_1, ϖ_1 and e_1 are related to the initial parameters as follows:

$$\kappa_1 = -\ln \phi_1, \quad \varpi_1 = \frac{\nu_1 \alpha_1}{1 - \phi_1} \text{ and } e_1^2 = \frac{-2\ln \phi_1}{1 - \phi_1} \alpha_1.$$

The return process has two interesting continuous time limits, depending on how we parameterize the skewness parameter η_1. It is useful to write down the model for a time interval Δ:

$$\ln \frac{S_{t+\Delta}}{S_t} = \left[\mu_0 + \lambda_1 \left(\sigma_{1,t+\Delta}^2 - \mu_1\right)\right] \Delta + \sqrt{\Delta}\sigma_{1,t+\Delta} u_{1,t+\Delta}$$

where S_t denotes the price.

Case 1: η_1 shrinks to zero as $\Delta \longrightarrow 0$. In this case $u_{1,t+\Delta}$ converges to standard normal distribution, which implies that

$$d\left(\ln\left(S_t\right)\right) = \left[\mu_0 + \lambda_1 \left(\sigma_{1t}^2 - \mu_1\right)\right] dt + \sigma_{1t} dz_{1t}$$

where z_{1t} is a Wiener process.

Case 2: $\eta_1 = constant$. We can always write:

$$u_{1,t+\Delta} = \frac{\eta_1}{3\sqrt{\Delta}\sigma_{1,t+\Delta}} y_{1,t+\Delta} - \frac{3\sqrt{\Delta}\sigma_{1,t+\Delta}}{\eta_1}$$

where $y_{1,t+\Delta}$ follows a standard inverse gaussian of parameter

$$\frac{9\Delta\sigma_{1,t+\Delta}^2}{\eta_1^2}.$$

Thus

$$\sqrt{\Delta}\sigma_{1,t+\Delta} u_{1,t+\Delta} = \frac{\eta_1}{3} y_{1,t+\Delta} - \frac{3\Delta\sigma_{1,t+\Delta}^2}{\eta_1}.$$

This implies that

$$\begin{aligned} \ln \frac{S_{t+\Delta}}{S_t} &= \left[\mu_0 + \lambda_1 \left(\sigma_{1,t+\Delta}^2 - \mu_1\right)\right] \Delta + \frac{\eta_1}{3} y_{1,t+\Delta} - \frac{3\Delta\sigma_{1,t+\Delta}^2}{\eta_1} \\ &= \left[\mu_0 + \lambda_1 \left(\sigma_{1,t+\Delta}^2 - \mu_1\right) - \frac{3\sigma_{1,t+\Delta}^2}{\eta_1}\right] \Delta + \frac{\eta_1}{3} y_{1,t+\Delta}. \end{aligned}$$

The continous time limit is then given by:

$$d\ln S_t = \left[\mu_0 + \lambda_1\left(\sigma_{1t}^2 - \mu_1\right) - \frac{3\sigma_{1t}^2}{\eta_1}\right]dt + \frac{\eta_1}{3}dy_{1t}$$

where y_{1t} is a pure-jump inverse gaussian process with degree of freedom $9\sigma_{1t}^2/\eta_1^2$ on interval $[t, t+dt]$.

Appendix V

SVS in General Equilibrium

Consider an investor whose lifetime utility V_t depends recursively on its consumption flow C_t as follows:

$$V_t = C_t^{1-\delta} \left[\mathscr{R}_t \left(V_{t+1}\right)\right]^{\delta}, \tag{A.1}$$

where $\mathscr{R}_t \left(V_{t+1}\right) = \left(E\left[V_{t+1}^{1-\gamma} \mid J_t\right]\right)^{\frac{1}{1-\gamma}}$ is the certainty equivalent of the next period lifetime utility, δ is the subjective discount factor and γ is the coefficient of risk aversion. For this investor, the elasticity of intertemporal substitution is unitary.

Hansen, Heaton and Li (2005) use the shadow valuation of consumption and lifetime utility to show that investor values consumption claim through a stochastic discount factor whose the logarithm is given by:

$$m_{t,t+1} = \ln \delta - \gamma \Delta c_{t+1} + (1-\gamma)\left(z_{v,t+1} - \frac{z_{v,t}}{\delta}\right), \tag{A.2}$$

where $\Delta c_{t+1} = c_{t+1} - c_t = \ln C_{t+1} - \ln C_t$ and $z_{v,t} = v_t - c_t = \ln V_t - \ln C_t$ are respectively the consumption growth rate and the log utility-consumption ratio. It can also be shown that the return on a claim to total consumption for this investor is given by:

$$r_{t+1} = -\ln \delta + \Delta c_{t+1}. \tag{A.3}$$

In this case, the logarithm of the SDF becomes:

$$m_{t,t+1} = (1-\gamma)\ln \delta - \gamma r_{t+1} + (1-\gamma)\left(z_{v,t+1} - \frac{z_{v,t}}{\delta}\right), \tag{A.4}$$

From the recursion (A.1) and the equation (A.3) on has:

$$z_{v,t} = \frac{\delta}{1-\gamma} \ln E\left[\exp\left((1-\gamma)\left(\ln \delta + z_{v,t+1} + r_{t+1}\right)\right)\right]. \tag{A.5}$$

If the dynamics of returns is given by (3.18) and we conjecture that $z_{v,t} = \Upsilon_0 + \Upsilon^\top \sigma_t^2$, then it follows that Υ solves the equation:

$$\Upsilon = \frac{\delta}{1-\gamma} B\left((1-\gamma), (1-\gamma)\Upsilon\right), \tag{A.6}$$

and

$$\Upsilon_0 = \frac{\delta}{1-\delta}\left[\ln\delta + \frac{A\left((1-\gamma), (1-\gamma)\Upsilon\right)}{1-\gamma}\right]. \tag{A.7}$$

If σ_t^2 is a multivariate autoregressive gamma process as defined in Section 3.3, then the element Υ_i solves the equation:

$$\begin{aligned}\alpha_i(1-\gamma)^2\Upsilon_i^2 - (1-\gamma)\left\{\left[1-\delta\left(\phi_i - \alpha_i\beta_i(1-\gamma)\right)\right] - \alpha_i\left[\lambda_i(1-\gamma) + \psi(1-\gamma;\eta_i)\right]\right\}\Upsilon_i \\ + \delta\beta_i(1-\gamma) + \delta\left(\phi_i - \alpha_i\beta_i(1-\gamma)\right)\left[\lambda_i(1-\gamma) + \psi(1-\gamma;\eta_i)\right] = 0.\end{aligned} \tag{A.8}$$

Equation (A.8) has two solutions:

$$\Upsilon_i^- = \frac{b_\Upsilon - \sqrt{\Delta_\Upsilon}}{2\alpha_i(1-\gamma)} \quad \text{and} \quad \Upsilon_i^+ = \frac{b_\Upsilon + \sqrt{\Delta_\Upsilon}}{2\alpha_i(1-\gamma)} \tag{A.9}$$

where

$$\begin{aligned}b_\Upsilon &= \left[1-\delta\left(\phi_i - \alpha_i\beta_i(1-\gamma)\right)\right] - \alpha_i\left[\lambda_i(1-\gamma) + \psi(1-\gamma;\eta_i)\right] \\ \Delta_\Upsilon &= \left\{\left[1+\delta\left(\phi_i - \alpha_i\beta_i(1-\gamma)\right)\right] - \alpha_i\left[\lambda_i(1-\gamma) + \psi(1-\gamma;\eta_i)\right]\right\}^2 - 4\delta\phi_i.\end{aligned}$$

Finally, the loading Υ_i of the log utility-consumption ratio on the factor σ_{it}^2 is then $\Upsilon_i = \Upsilon_i^-$ since the root Υ_i^+ has the unappealing property that

$$\lim_{\alpha_i \to 0} \alpha_i \Upsilon_i^+ \neq 0$$

which would mean the impact of the factor would grow without bound as it becomes unimportant as pointed out by Tauchen (2005).

Comparing (3.56) and (A.4) shows that:

$$\varsigma_t = (1-\gamma)\left[\ln\delta - \frac{1-\delta}{\delta}\Upsilon_0 - \left(\frac{\Upsilon}{\delta}\right)^{\top}\sigma_t^2\right] \tag{A.10}$$

$$\kappa = -\gamma \tag{A.11}$$

$$\pi = (1-\gamma)\,\Upsilon. \tag{A.12}$$

Appendix VI

Second Order Lognormal Approximation of Positive Random Variables

The second order lognormal approximation of a positive random variable X with mean μ_x and variance σ_x^2 is given by:

$$X \approx \exp\left(\ln\left(\frac{\mu_x^2}{\sqrt{\mu_x^2+\sigma_x^2}}\right)+\sqrt{\ln\left(\frac{\mu_x^2+\sigma_x^2}{\mu_x^2}\right)}\varepsilon_X\right) \tag{A.1}$$

where ε_X is a standard normal random variable.

Given (A.1), the second order lognormal approximation of a standardized inverse gaussian random variable u with positive skewness s is given by:

$$u \approx \exp\left(\ln\left(\frac{9}{s\sqrt{s^2+9}}\right)+\sqrt{\ln\left(\frac{s^2+9}{9}\right)}\varepsilon\right)-\frac{3}{s} \tag{A.2}$$

where ε is a standard normal random variable.

Given (A.1), the second order lognormal approximation for the dynamics of a stationary univariate autoregressive gamma process X_{t+1} with mean μ_x, variance σ_x^2 and persistence ϕ_x is given by:

$$X_{t+1} \approx \exp\left(\ln\left(\frac{m(X_t)^2}{\sqrt{m(X_t)^2+v(X_t)}}\right)+\sqrt{\ln\left(\frac{m(X_t)^2+v(X_t)}{m(X_t)^2}\right)}\varepsilon_{X,t+1}\right) \tag{A.3}$$

where

$$m(X_t) = (1-\phi_x)\mu_x+\phi_x X_t \tag{A.4}$$

$$v(X_t) = (1-\phi_x)^2\sigma_x^2+\frac{2(1-\phi_x)\phi_x\sigma_x^2}{\mu_x}X_t \tag{A.5}$$

and $\varepsilon_{X,t+1}$ is a i.i.d. standard normal shock.

The second order lognormal approximation of a couple (X,Y) of positive random variables with means μ_x and μ_y, variances σ_x^2 and σ_y^2 and correlation ρ_{xy} is given by:

$$X \approx \exp\left(\ln\left(\frac{\mu_x^2}{\sqrt{\mu_x^2+\sigma_x^2}}\right)+\sqrt{\ln\left(\frac{\mu_x^2+\sigma_x^2}{\mu_x^2}\right)}\varepsilon_X\right) \tag{A.6}$$

$$Y \approx \exp\left(\ln\left(\frac{\mu_y^2}{\sqrt{\mu_y^2+\sigma_y^2}}\right)+\sqrt{\ln\left(\frac{\mu_y^2+\sigma_y^2}{\mu_y^2}\right)}\left(c_{xy}\varepsilon_X+\sqrt{1-c_{xy}^2}\varepsilon_Y\right)\right) \tag{A.7}$$

where ε_X and ε_Y are uncorrelated standard normal random variables and:

$$c_{xy} = \frac{\ln\left(1+\rho_{xy}\frac{\sigma_x\sigma_y}{\mu_x\mu_y}\right)}{\sqrt{\ln\left(1+\frac{\sigma_x^2}{\mu_x^2}\right)\ln\left(1+\frac{\sigma_y^2}{\mu_y^2}\right)}}. \tag{A.8}$$

Given (A.6) and (A.7), the second order lognormal approximation for the dynamics of a stationary bivariate autoregressive gamma process (X_{t+1}, Y_{t+1}) with means μ_x and μ_y, variances σ_x^2 and σ_y^2, persistences ϕ_x and ϕ_y and correlation ρ_{xy} is given by:

$$\ln X_{t+1} \approx \ln\left(\frac{m_1(X_t)^2}{\sqrt{m_1(X_t)^2+v_1(X_t)}}\right)+\sqrt{\ln\left(\frac{m_1(X_t)^2+v_1(X_t)}{m_1(X_t)^2}\right)}\varepsilon_{X,t+1} \tag{A.9}$$

$$\begin{aligned}\ln Y_{t+1} \approx{}& \ln\left(\frac{m_2(Y_t)^2}{\sqrt{m_2(Y_t)^2+v_2(Y_t)}}\right)\\ &+\sqrt{\ln\left(\frac{m_2(Y_t)^2+v_2(Y_t)}{m_2(Y_t)^2}\right)}\left(c(X_t,Y_t)\,\varepsilon_{X,t+1}+\sqrt{1-c(X_t,Y_t)^2}\varepsilon_{Y,t+1}\right)\end{aligned} \tag{A.10}$$

where

$$m_1(X_t) = (1-\phi_x)\mu_x + \phi_x X_t \tag{A.11}$$

$$v_1(X_t) = (1-\phi_x)^2\sigma_x^2 + \frac{2(1-\phi_x)\phi_x\sigma_x^2}{\mu_x}X_t \tag{A.12}$$

$$m_2(Y_t) = (1-\phi_y)\mu_y + \phi_y Y_t \tag{A.13}$$

$$v_2(Y_t) = (1-\phi_y)^2\sigma_y^2 + \frac{2(1-\phi_y)\phi_y\sigma_y^2}{\mu_y}Y_t \tag{A.14}$$

$$\tag{A.15}$$

and $\varepsilon_{X,t+1}$ and $\varepsilon_{Y,t+1}$ are uncorrelated i.i.d. standard normal shocks and

$$c(X_t,Y_t) = \frac{\ln\left(1+\rho(X_t,Y_t)\frac{\sqrt{v_1(X_t)v_2(Y_t)}}{m_1(X_t)m_2(Y_t)}\right)}{\sqrt{\ln\left(1+\frac{v_1(X_t)}{m_1(X_t)^2}\right)\ln\left(1+\frac{v_2(Y_t)}{m_2(Y_t)^2}\right)}} \tag{A.16}$$

where

$$\rho(X_t,Y_t) = \frac{\rho_{xy}(1-\phi_x\phi_y)\sigma_x\sigma_y}{\sqrt{v_1(X_t)v_2(Y_t)}}. \tag{A.17}$$

Appendix VII

The Unscented Kalman Filter

The Unscented Kalman Filter is essentially an approximation of a nonlinear transformation of probability distribution coupled with the Kalman Filter. It has been introduced in the engineering literature by Julier et al. (1995) and Jullier and Uhlmann (1996). (See also Wan and van der Merwe (2001) for general introduction) and, to our knowledge, was first imported in Finance by Leippold and Wu (2003).

The Unscented Filter selects a set of sigma points in the distribution of $\left(\omega_t^\top, u_{t+1}^{*\top}, \varepsilon_{t+1}^{*\top}\right)^\top$ conditional on $\underline{r_t}$. This distribution is assumed gaussian with mean

$$\bar{\chi} = \left(\omega_{t|t}^\top, \bar{u}^\top, \bar{\varepsilon}^\top\right)^\top$$

and variance

$$P^{\chi\chi} = \begin{pmatrix} P_{t|t}^{\omega\omega} & P^{\omega u} & P^{\omega\varepsilon} \\ P^{u\omega} & P^{uu} & P^{u\varepsilon} \\ P^{\varepsilon\omega} & P^{\varepsilon u} & P^{\varepsilon\varepsilon} \end{pmatrix}.$$

Following Julier et al. (1995) we consider the $2n+1$ sigma points $\chi_i = \left(\omega_{i,t|t}^\top, u_i^\top, \varepsilon_i^\top\right)^\top$ with associated weights W_i defined by:

$$\begin{aligned} \chi_0 &= \bar{\chi}, & W_0 &= \kappa/(n+\kappa) \\ \chi_i &= \bar{\chi} + \left(\sqrt{(n+\kappa)P^{\chi\chi}}\right)_i, & W_i &= 1/2(n+\kappa) \\ \chi_{i+n} &= \bar{\chi} - \left(\sqrt{(n+\kappa)P^{\chi\chi}}\right)_i, & W_i &= 1/2(n+\kappa), \end{aligned} \tag{A.1}$$

where n is the dimension of the vector $\left(\omega_t^\top, u_{t+1}^{*\top}, \varepsilon_{t+1}^{*\top}\right)^\top$, κ is an appropriately chosen real number and $\left(\sqrt{(n+\kappa)P^{\chi\chi}}\right)_i$ is the ith column of the matrix $(n+\kappa)P^{\chi\chi}$.

These sigma points are transformed through state and observation functions to obtain:

$$\omega_{i,t+1|t} = F\left(\omega_{i,t|t}, u_i\right) \text{ and } r_{i,t+1|t} = H\left(\omega_{i,t+1|t}, \varepsilon_i\right)$$

from which approximations of predicted means and covariances are computed as:

$$\widehat{\omega_{t+1|t}} = \sum_{i=0}^{2n} W_i \omega_{i,t+1|t} \text{ and } \widehat{r_{t+1|t}} = \sum_{i=0}^{2n} W_i r_{i,t+1|t} \tag{A.2}$$

$$\widehat{P^{\omega\omega}_{t+1|t}} = \sum_{i=0}^{2n} W_i \left(\omega_{i,t+1|t} - \widehat{\omega_{t+1|t}}\right)\left(\omega_{i,t+1|t} - \widehat{\omega_{t+1|t}}\right)^\top \tag{A.3}$$

$$\widehat{P^{rr}_{t+1|t}} = \sum_{i=0}^{2n} W_i \left(r_{i,t+1|t} - \widehat{r_{t+1|t}}\right)\left(r_{i,t+1|t} - \widehat{r_{t+1|t}}\right)^\top \tag{A.4}$$

$$\widehat{P^{\omega r}_{t+1|t}} = \sum_{i=0}^{2n} W_i \left(\omega_{i,t+1|t} - \widehat{\omega_{t+1|t}}\right)\left(r_{i,t+1|t} - \widehat{r_{t+1|t}}\right)^\top . \tag{A.5}$$

Appendix VIII

Approximated Moments of a Function of a Normal Random Variable

Consider a normal random variable X with mean μ_x and variance σ_x^2. Let $Y = f(X)$, where f is a twice differentiable real function. The variable Y admits the second order Taylor approximation

$$Y = f(\mu_x) + f'(\mu_x)(X-\mu_x) + \frac{1}{2}f''(\mu_x)(X-\mu_x)^2 \tag{A.1}$$

which implies that the mean of Y can be approximated by:

$$\mu_y = E[Y] = f(\mu_x) + \frac{1}{2}f''(\mu_x)\sigma_x^2 \tag{A.2}$$

It follows that:

$$Y-\mu_y = f'(\mu_x)(X-\mu_x) + \frac{1}{2}f''(\mu_x)\left[(X-\mu_x)^2-\sigma_x^2\right] \tag{A.3}$$

$$\begin{aligned}(Y-\mu_y)^2 &= f'(\mu_x)^2(X-\mu_x)^2 + f'(\mu_x)f''(\mu_x)\left[(X-\mu_x)^3-\sigma_x^2(X-\mu_x)\right]\\ &+\frac{1}{4}f''(\mu_x)^2\left[(X-\mu_x)^4-2\sigma_x^2(X-\mu_x)^2+\sigma_x^4\right]\end{aligned} \tag{A.4}$$

$$\begin{aligned}(Y-\mu_y)^3 &= f'(\mu_x)^3(X-\mu_x)^3 + \frac{3}{2}f'(\mu_x)^2f''(\mu_x)\left[(X-\mu_x)^4-\sigma_x^2(X-\mu_x)^2\right]\\ &+\frac{3}{4}f'(\mu_x)f''(\mu_x)^2\left[(X-\mu_x)^5-2\sigma_x^2(X-\mu_x)^3+\sigma_x^4(X-\mu_x)\right]\\ &+\frac{1}{8}f''(\mu_x)^3\left[(X-\mu_x)^6-3\sigma_x^2(X-\mu_x)^4+3\sigma_x^4(X-\mu_x)^2-\sigma_x^6\right]\end{aligned} \tag{A.5}$$

The third and fifth central moments of X are zero whereas the fourth and sixth central moments of X are respectively $3\sigma_x^4$ and $15\sigma_x^6$. Based on that, taking expectations of (A.4) and (A.5) gives the following approximations for the variance and the third moment of

Y:

$$\sigma_y^2 = Var[Y] = f'(\mu_x)^2 \sigma_x^2 + \frac{1}{2} f''(\mu_x)^2 \sigma_x^4 \tag{A.6}$$

$$E\left[(Y-\mu_y)^3\right] = 3f'(\mu_x)^2 f''(\mu_x) \sigma_x^4 + f''(\mu_x)^3 \sigma_x^6. \tag{A.7}$$