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On the Manipulability of Proportional Representation

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Abstract. This paper presents a new model of voter behaviour under methods of proportional representation (PR). We abstract away from rounding, and assume that a party securing k percent of the vote wins exactly k percent of the available seats. Under this assumption PR is not manipulable by any voter aiming at maximisation of the number of seats in the parliament of her most preferred party. However in this paper we assume that voters are concerned, first and foremost, with the distribution of power in the post-election parliament. We show that, irrespective of which positional scoring rule is adopted, there will always exist circumstances where a voter would have an incentive to vote insincerely. We demonstrate that a voter's attitude toward uncertainty can influence her incentives to make an insincere vote. Finally, we show that the introduction of a threshold - a rule that a party must secure at least a certain percentage of the vote in order to reach parliament - creates new opportunities for strategic voting. We use the model to explain voter behaviour at the most recent New Zealand general election.

1 Introduction

This paper investigates opportunities for strategic voting under proportional representation (PR). The classical Gibbard-Sattethwaite theorem [19, 20] is not applicable in this framework and, if we assume that a party securing k percent of the vote wins exactly k percent of the available seats, then any voter who is concerned with maximising the number of seats in the parliament of her favourite party, will not have any incentives to vote insincerely. In practical implementations of PR a technique of rounding off will need to be applied whenever the number of candidates to be elected exceeds the number of ballots cast (see e.g., [26], chapter 4). Cox and Shugart [13] demonstrated that this need to "round off" can render PR manipulable. If a party is in a position where receiving a few more or a few less votes will not alter the number of seats it will take, then some of that party's supporters may peel off, and attempt to influence the distribution of the remaining seats. However the observed insincere behaviour of voters, in particular at the New Zealand general election held September 17th, 2005, cannot be explained by rounding off.

 This paper presents a new model of voter behaviour under PR. The purpose of the model is to demonstrate that under each and every method of PR there can arise circumstances where a voter has an incentive to vote insincerely. The model revolves around the following observations. A parliamentary election held under a method of PR will determine how parliamentary seats are to be allocated. Once the election is over, and all the results are in, a government formation process begins. We will model that process with a simple game, where the voting weight of each parliamentary party is given by the number of seats it has won. The solution of the game will be a government, a set of ministers, a set of policies to be enacted, and so on. Under PR, a small change in the way ballots are cast will only ever result in a small change in the voting weights of the parties in the post-election government formation game. But there do exist circumstances where a small change in the way ballots are cast can effect a significant change in certain other facets of the government formation game, in-particular in the voting powers of the parties and in the set of feasible solutions. If a voter is concerned about these latter matters, and finds themself in one of these circumstances, then we will see that they have an incentive to vote insincerely.

 Rounding is not the cause of these effects. The cause is the parliamentary quota, and the circumstance is when the total number of seats won by some subset of the parties is expected to lie "close" to the quota. To illustrate, suppose three parties contest an election held under a method of PR, and let the vector $\mathbf{x}$ denote the resultant parliament (where the ith coordinate gives the number of seats won by the ith party). Compare the parliaments $\mathbf{x}^{(1)} = (49, 49, 3)$, $\mathbf{x}^{(2)} = (50, 49, 2)$, and $\mathbf{x}^{(3)} = (51, 49, 1)$. Suppose that in the parliament a strict majority is sufficient to pass any motion. We contend that there is no reason to a priori believe that either the government-formation game or its solution would be significantly different were the post-election parliament to be $\mathbf{x}^{(1)}$ rather than $\mathbf{x}^{(2)}$. We contend that a comparison of $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ does not lead to the same conclusion: if $\mathbf{x}^{(2)}$ is the election result, then any two of the three parties could form a coalition government (as could all three); if $\mathbf{x}^{(3)}$ is the election result, the first party would likely form a government alone.

 In our model we will assume that voters are aware that a post-election government formation game will take place, but that they cannot solve it. We will assume that voters first and foremost concern with the distribution of power in the post-election parliament. We abstract away from rounding, and assume that a party securing k percent of the vote wins exactly k percent of the

available seats. We show that, irrespective of which positional scoring rule is adopted, there will always exist circumstances where a voter would have an incentive to vote insincerely. We demonstrate that a voter's attitude toward uncertainty can influence his or her incentives to make an insincere vote. We show that the introduction of a threshold - a rule that a party must secure at least a certain percentage of the vote in order to reach parliament - creates new opportunities for strategic voting. We also show that in some circumstances any would-be manipulators are in danger of undershooting and overshooting and need to carefully coordinate their efforts.

This paper was initially motivated by a desire to explain the behaviour of voters at the New Zealand general election held September 17th, 2005. The New Zealand electoral system is mixed member proportional (MMP), similar to the system run in Germany. Anecdotal evidence has suggested that at the election some voters voted insincerely even though their doing so could have cost their most-preferred-party seats. We shall show that the model presented can account for such behaviour. We also investigate how the aforementioned incentives to manipulate interrelate to the rounding. We show that the rounding may actually be considered as deterrent to manipulation since it requires a greater degree of coordination from manipulators in order to avoid under and overshooting.

To date, research on this topic has been rather sparse. Austin-Smith and Banks [6], Baron and Diermeier [8], and De Sinopoli and Iannantuoni [14, 15, 16] have constructed multi-stage spatial models of political systems that incorporate proportional representation. In these models voters (i) have preferences over the set of policies that governments might pursue but (ii) do not necessarily vote for the party to which they are ideologically closest. Voters might support parties exposing views more extreme than their own in a bid to counteract votes from other voters whose opinions lie on the opposite side of the policy spectrum. In all the aforementioned papers it is assumed that, at the ballot box, voters can indicate a preference for just one party (i.e. the positional scoring rule is plurality). Karp et al [22] modelled split voting in NZ, but, unlike us, they assume voters use their party vote sincerely and their electorate vote strategically.

Below, Sections 2 (Parliament choosing rules), 3 (Indices of voting power), and 4 (Voters) describe our model. Theoretical results are presented in Section 5. Section 6 uses the model to explain voter behaviour at the most recent New Zealand general election, and Section 7 concludes.

2 Parliament Choosing Rules

We model a parliamentary election. We assume that a parliamentary body is to be elected, that the body contains a fixed number k of seats, and that m political parties are competing for those seats. We assume n voters are eligible to vote, and all do.

Voters have preferences on the set of political parties A. We will denote the parties by $a_1, \ldots, a_m$. Every voter has a favourite party, a second favourite, and so on. No voter is indifferent between any two parties. Every voter's preferences can then be represented as a linear order on A. Let $\mathcal{L}(A)$ be the set of all possible linear orders. The Cartesian product $\mathcal{L}(A)^n$ will then represent preferences of the whole society. Elements of this Cartesian product are called *profiles*. The collection of all ballot papers will also be a profile. At the ballot box, voters do not necessarily rank the parties in the order of their sincere preference.

We assume each voter forms an expectation of what will transpire at the election. We follow Cox and Shugart and assume that these expectations "are publicly generated - by, for example,

polls and newspapers' analysis" of the parties' prospects - "so that diversity of opinion in the electorate is minimised" ([13], page 303).

The result of the election will be a parliament. Any parliament can be represented by a point in the simplex

$$S^{m-1} = \left\{ (x_1, \ldots, x_m) \mid \sum_{i=1}^{m} x_i = 1 \right\},$$

where x_i is the fraction of the seats the ith party wins at the election. In this paper we will ignore rounding and assume that a party can win any portion of the m seats. Rounding (or apportionment) is an important issue, but its necessity and its consequences have been analysed elsewhere ([13, 26]). We exclude consideration of rounding in order to focus more directly on other causes of manipulative behaviour. We presume that every party decides on a party list before the election, i.e. ranks its candidates in a certain order with no ties. After the fractions of the seats each party has won is known, the composition of the parliament is decided on the basis of those party lists. If a party is allowed to have k MPs then the first k candidates from the party list become MPs.

A *parliament choosing rule* is employed to calculate the distribution of seats in the parliament. A parliament choosing rule is a composition of a *score function* and a *seat allocation rule*.

Given a profile $R = (R_1, \ldots, R_n)$ and a set of alternatives A, a score function assigns to each $a_i \in A$ a real number. The greater this number, the better a_i is supposed to have done. There are a wide variety of score functions ([23] has a comprehensive list of them). In this paper we will work with *normalised positional score functions*.

Let $w_1 \geq w_2 \geq \ldots \geq w_m = 0$ be m real numbers which we shall refer to as *weights*, and let $\mathbf{w} = (w_1, \ldots, w_m)$. Let $\mathbf{v} = (i_1, \ldots, i_m)$, where i_k indicates the number of voters that stated that they rank alternative a kth best. Then, given a profile $R = (R_1, \ldots, R_n)$, the positional score of alternative a is given by:

$$sc_{\mathbf{w}}(a) = \mathbf{w} \cdot \mathbf{v} = w_1 i_1 + \ldots + w_m i_m.$$

Well known vectors of weights and their respective scores include:

- the Plurality score $sc_{\mathbf{p}}(a)$, where $\mathbf{p} = (1, 0, \ldots, 0)$,

- the Borda score $sc_{\mathbf{b}}(a)$, where $\mathbf{b} = (m - 1, m - 2, \ldots, 1, 0)$,

- the Antiplurality score $sc_{\mathbf{a}}(a)$, where $\mathbf{a} = (1, \ldots, 1, 0)$.

The *vector of normalised positional scores* is given by

$$\mathbf{sc_w} = \frac{1}{\sum_{i=1}^{m} sc_{\mathbf{w}}(a_i)} \left(sc_{\mathbf{w}}(a_1), sc_{\mathbf{w}}(a_2), \ldots, sc_{\mathbf{w}}(a_m) \right).$$

Clearly, $\mathbf{sc_w} \in S^{m-1}$.

Definition 1. *A* normalised positional score function *is a mapping*

$$F_s \colon \mathcal{L}(A)^n \to S^{m-1},$$

which assigns to every profile its vector of normalised positional scores for some fixed vector of weights $\mathbf{w}$.

Given a vector of scores $\mathbf{sc} \in S^{m-1}$, a seat allocation rule determines the distribution of seats in parliament $(x_1, \ldots, x_m)$.

Definition 2. *A* seat allocation rule *is any mapping*

$$F_a \colon S^{m-1} \to S^{m-1}.$$

There are two main examples of such rules.

Example 1 (Identity seat allocating rule). *F_a is the identity function, i.e., $F_a(\mathbf{x}) = \mathbf{x}$.*

For the next example, we fix a *threshold*, which is a positive real number ϵ such that $0 < \epsilon \leq 1/m$. We define a *threshold function* $\delta_\epsilon \colon [0, 1] \to [0, 1]$ so that

$$\delta_\epsilon(x) = \begin{cases} 0 & \text{if } x < \epsilon, \\ x & \text{if } x \geq \epsilon. \end{cases}$$

Example 2 (Threshold seat allocating rule). *Let ϵ be a positive real number such that $0 < \epsilon \leq 1/m$. Suppose $\mathbf{x} \in S^{m-1}$. Then we define $y_i = \delta_\epsilon(x_i)$ and $z_i = y_i / \sum_{i=1}^{m} y_i$. We now set $F_a(\mathbf{x}) = \mathbf{z}$, where $\mathbf{z} = (z_1, \ldots, z_m)$. The restriction $\epsilon \leq 1/m$ guarantees F_a is always defined.*

Definition 3. *A* parliament choosing rule *is a composition $F = F_a \circ F_s$ of a score function and a seat allocation rule:*

$$F_a \circ F_s \colon \mathcal{L}(A)^n \to S^{m-1}.$$

If the identity seat allocating rule is employed, we shall refer to the parliament choosing rule as pure proportional representation. If a threshold seat allocating rule is employed, we shall refer to the parliament choosing rule as proportional representation with a threshold.

In practice only the plurality score has been used in systems of proportional representation. Nevertheless we do not want to restrict our generality here as other scores may be considered in the future (Brams and Potthoff [12], for example, suggested combining PR and approval voting scores). Note that there is a significant difference between parliament choosing rules and choose-k rules (see [11] and the references therein). A choose-k rule picks a k-element subset of the set of alternatives, which is clearly inappropriate in our context when the parties and not the candidates are the alternatives. A parliament choosing rule reveals not only which parties win parliamentary seats, but also how many seats each of them gets.

3   Indices of Voting Power

Choosing a parliament is effectively a fair division problem. It might be thought desirable to allocate each political party a quantity of seats in direct proportion to its support in society. Suppose we do desire this, and suppose we accept that the support for a party can be measured by the score it is assigned, by a score function, at an election: then PR is an obvious choice for a parliament choosing rule.

But does PR provide a satisfactory solution to the fair division problem? For sure, each party gets a (roughly) "fair" share of parliamentary representation. However, once the election is over a government has to be formed and a coalition arrangement may need to be negotiated. The political power of each player in the government formation game may not be proportional

to either its score or its parliamentary representation. PR can divide seats up "fairly" but it is unlikely to divide power up "fairly."

We will assume that the distribution of power in a parliament can be computed by a (normalised) *voting power index*. Given a parliament $(x_1, \ldots, x_m)$, a voting power index P computes a vector of voting powers $\mathbf{p} = (p_1, \ldots, p_m)$, where p_i denotes the proportion of power held by party a_i.

Before formally defining a power index we need the following, standard, definitions. A *weighted voting game* is a simple m-person game characterised by a non-negative real vector $(w_1, \ldots, w_m)$, where w_i represents the ith player's voting weight, and a quota q. The quota gives the minimum number of votes necessary to establish a winning coalition. A coalition C is winning if $\sum_{i \in C} w_i > q$. Given a parliament, the formation of the government is a weighted voting game with weights $x_1, \ldots, x_m$ and quota $\frac{1}{2}$ (we will assume throughout that any strict majority of votes is sufficient to pass any motion in parliament), i.e. the players are the parties and their weights are the proportion of parliamentary seats that they hold.

Let $N = \{1, 2, \ldots, n\}$ and let $v = (N, W)$ be a simple n-person game with $W \subseteq 2^N$ being the set of all winning coalitions. A coalition C is called a *minimal winning coalition* if $C \in W$ and $C \setminus \{i\} \notin W$ for all $i \in C$. A party is called a *dummy* if it does not belong to any minimal winning coalition.

Definition 4. *Any mapping $P: S^{m-1} \to S^{m-1}$ is called a voting power index if the following conditions hold. Suppose $\mathbf{p} = P(\mathbf{x})$, then*

PI1. If the ith party is a dummy, then $p_i = 0$,

PI2. If the set of minimal winning coalitions of parliament $\mathbf{x}$ is the same as the set of minimal winning coalitions of the parliament $\mathbf{y}$, then $P(\mathbf{x}) = P(\mathbf{y})$.

This definition follows Holler and Packel's definition of a power index for games [21]. Allingham [5] requires also a monotonicity condition. However the Deegan-Packel index [17] and the Public Good Index [21] do not satisfy the monotonicity requirement and we do not include it. Non-monotonic indices have their justification in Riker's "size principle" [24], which says that "... participants create coalitions just as large as they believe will ensure winning and no larger" (p. 47).

Perhaps the best known voting power indices are the Banzhaf (Bz) and Shapley-Shubik (S-S) indices (see [7, 10, 27]). These indices count, in different ways, how many times a player is critical for some winning coalition. According to Felsenthal and Machover, these two indices "have, by and large, been accepted as valid measures of a priori voting power. Some authors have a preference for one or another of these two indices; many regard them as equally valid. Although other indices have been proposed — ... — none has achieved anything like general recognition as a valid index." [18], page 9.

It is worth pointing out that more seats do not necessarily translate into more power. For instance, compare the parliaments $(x_1, x_2, x_3) = (98/100, 1/100, 1/100)$ and $(x_1, x_2, x_3) = (51/100, 48/100, 1/100)$; party a_2 has no more power in the second than in the first.

4 Voters

We do not assume that the n voters participating in the parliamentary election are (directly) "policy-motivated", nor that they are (directly) concerned with the distribution of seats in

the post-election parliament. Instead, we assume that voters are primarily concerned with the amount of power each of the different parties gain.

We assume that each voter has in mind one particular power index (let us say the ith voter has in mind P_i). We assume that each voter is able to rank all possible vectors of power that the index they have in mind could produce, and that this ranking is consistent with their preferences over the set of political parties. In short, the ith voter has an order $\succeq_i$ on S^{m-1} consistent with his or her preference order on A. The two preference orders (on A and on S^{m-1}) belonging to the ith voter are encapsulated in her utility vector $\mathbf{u}_i$. More precisely, we assume that the ith voter has a vector of utilities $\mathbf{u}_i = (u_1^{(i)}, \ldots, u_m^{(i)})$, normalised so that $\min_j u_j^{(i)} = 0$, such that:

- the ith voter prefers party a_j to party a_k iff $u_j^{(i)} > u_k^{(i)}$;

- given any two vectors of power indices $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_m)$ we have $\mathbf{p} \succeq_i \mathbf{q}$ iff $\mathbf{p} \cdot \mathbf{u}_i \geq \mathbf{q} \cdot \mathbf{u}_i$, where $\cdot$ is the dot product in $\mathbb{R}^m$.

Two voters i and j will be said to be of the same *type* iff their power indices P_i and P_j are the same and $\succeq_i = \succeq_j$ on the range of $P_i = P_j$. Thus a voter's type can be viewed as a pair $(P, \succ)$, where P is a power index and $\succ$ is an order on the range of P.

Example 3. *If we denote the strict preference component of $\succeq_i$ as $\succ_i$, and the ith voter prefers a_1 to a_2 to a_3, etc., then we must have*

$$\mathbf{e}_1 \succ_i \mathbf{e}_2 \succ_i \ldots \succ_i \mathbf{e}_m, \tag{1}$$

where $\mathbf{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the vector whose only nonzero coordinate is 1 in the jth place.

Definition 5. *Let L be a linear order on the set A of political parties:*

$$a_{i_1} > a_{i_2} > \ldots > a_{i_m}.$$

We say that the voter's type $(P, \succ)$ is consistent with L if

$$\mathbf{e}_{i_1} \succ \mathbf{e}_{i_2} \succ \ldots \succ \mathbf{e}_{i_m},$$

which means that this voter prefers the parliament where all the power belongs to a_{i_1} to the parliament where all the power belongs to a_{i_2}, etc.

Fix a voter i. Set $U_i(j)$ equal to this voter's jth largest utility. We will say that the ith voter is *uncertainty averse* if the function $j \mapsto U_i(j)$ is concave down and *uncertainty seeking* if the function $j \mapsto U_i(j)$ is concave up.

In the case of $m = 3$ for ease of exposition we rename parties a_1, a_2, a_3 as A, B, C, respectively. If a voter prefers a_1 to a_2 to a_3, we will denote this as $A > B > C$.

Example 4. *Consider the case where $m = 3$ and a voter prefers A to B to C. Suppose this voter is comparing the vectors of power $\mathbf{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\mathbf{q} = (0, 1, 0)$. The vector $\mathbf{p}$ corresponds to a post-election situation where none of the three parties has an outright majority, and a coalition government will need to be formed. If a voter anticipates, prior to the election, that $\mathbf{p}$ will be the outcome, then she may be uncertain about the composition of the next government. The vector $\mathbf{q}$ corresponds to a post-election situation where party a_2 has total power, and can form a government by itself. A voter of the opinion that $\mathbf{q}$ will be the outcome of the election will have no doubt as to the composition of the next government. This voter will rank $\mathbf{p}$ over $\mathbf{q}$ if she is uncertainty seeking, or $\mathbf{q}$ over $\mathbf{p}$ if she is uncertainty averse.*

Let L be a linear order on the set of alternatives. We will now define a relation $\gg_L$ on parliaments associated with L. For any two parliaments $\mathbf{x}$ and $\mathbf{y}$ we write $\mathbf{x} \gg_L \mathbf{y}$ if $P(\mathbf{x}) \succ P(\mathbf{y})$ for every voter whose type $(P, \succ)$ is consistent with the linear order L. This relation is not complete.

Consider the case of $m = 3$. Suppose that pure PR is used. In this case the identity seat allocation rule is used and the vector of normalised scores for the three political parties coincides with the parliament elected. Every parliament $\mathbf{x} = (x_1, x_2, x_3)$ can be then represented by a point of the triangle S^2, whose barycentric co-ordinates are x_1, x_2 and x_3. For every voter i, associated with every possible parliament $\mathbf{x} = (x_1, x_2, x_3)$ there will be a vector of voting power indices $P_i(\mathbf{x}) \in S^2$. Regardless of her index P_i, whenever the parliament $\mathbf{x}$ falls strictly inside one of the triangles AKM, MKL, or KLC, then $P_i(\mathbf{x})$, will be $(1, 0, 0)$, $(0, 1, 0)$, or $(0, 0, 1)$, respectively, since two parties in this parliament will be dummies (axiom PI1). Should the parliament $\mathbf{x}$ fall inside the inner triangle, then all $P_i(\mathbf{x})$, will equal $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ (axiom PI2).
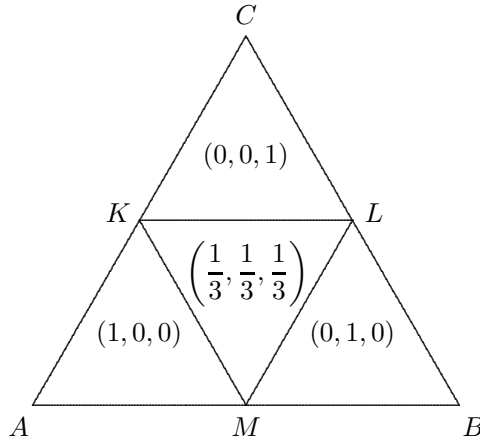
Figure 1: Pure PR. Regions where normalised scores give equal vectors of power indices.

Should the parliament fall on the perimeter of the inner triangle (excluding points M, K, and L) the vector of power indices will depend on the index of voting power used by the voters. For example, the vector $P_i(\mathbf{x})$ of a voter who uses $P_i = $ Bz power index will be either $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$ or some permutation thereof, and the vector $P_i(\mathbf{x})$ of a voter who uses $P_i = $ S-S power index will be $(\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$ or, again, some permutation of. Finally, if the parliament coincides with one of the vertices of the inner triangle M, K, and L, the vector of indices, regardless of the index, will be $(\frac{1}{2}, \frac{1}{2}, 0)$ or a permutation thereof.

Example 5. *Consider again the case where $m = 3$ and a voter prefers A to B to C (denote this linear order by L). Then for the four parliaments $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{m}$, located inside the triangles AKM, MKL, KLC, KLM, respectively, we will have*

$$\mathbf{x} \gg_L \mathbf{y} \gg_L \mathbf{z}, \qquad \mathbf{x} \gg_L \mathbf{m} \gg_L \mathbf{z}.$$

This is because, for every voter whose type $(P, \succ)$ is compatible with L

$$(1, 0, 0) \succ (0, 1, 0) \succ (0, 0, 1), \qquad (1, 0, 0) \succ (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \succ (0, 0, 1).$$

5   The manipulability of proportional representation

In this section we will show that under any type of proportional representation there will exist circumstances where a voter has an incentive to vote insincerely. For the pure PR these incentives depend on the attitude of a particular voter to uncertainty, in which case we say that the rule is weakly manipulable. We will then show that the introduction of a threshold creates additional opportunities to vote strategically and that some incentives to vote strategically are no longer related to attitude to uncertainty. We call the latter phenomenon strong manipulability.

Definition 6. *Let R be a profile such that $R_{i_1} = \ldots = R_{i_k} = L$ for some group of indices $I = \{i_1, \ldots, i_k\}$ and a linear order L on A. A parliament choosing rule F is said to be:*

- *weakly manipulable at R if there exists a linear order L' on A and a certain type of voters $(P, \succ)$ consistent with L such that for the profile R', which results when $R_{i_1}, \ldots, R_{i_k}$ in R are replaced with L',*
$$P(F(R')) \succ P(F(R));$$

- *strongly manipulable at R if there exists a linear order L' such that for the profile R' constructed as above*
$$F(R') \gg_L F(R).$$

Let us emphasise the difference between the two grades of manipulability. A profile is weakly manipulable if voters of just one particular type are in position to vote insincerely and get an advantage through this misrepresentation. It may happen that some voters whose views are represented by a linear order L are able to get an advantage and some are not. For example, uncertainty seeking voters with views L may be better off by misrepresenting their preferences by declaring them to be L' and at the same time uncertainty averse voters with the same views might be worse off as a result. A profile is strongly manipulable if all voters whose type is consistent with L will be better off.

In this paper we consider only susceptibility to *micro manipulation*, when a small percentage of the voters try to coordinate their efforts. This term was coined by Donald Saari and we refer the reader to [25] for more justification of the concept. Roughly speaking, F is micro (weakly or strongly) manipulable if, as $n \to \infty$, the manipulating group may consist of an arbitrary small fraction of the society.

Our results are obtained for the case $m = 3$. In fact, this is the main case. It is clear that if we are able to demonstrate manipulability of a parliament choosing rule for $m = 3$ parties, it will be manipulable for any $m \geq 3$. Austin-Smith and Banks [6], and Baron and Diemeier [8] also assume $m = 3$.

Theorem 1. *Let the parliament choosing rule be pure PR. Then the rule is always weakly manipulable but never strongly manipulable. Moreover,*

1. *If $\mathbf{w} = \mathbf{a}$, i.e. for the antiplurality score, the rule is not manipulable by uncertainty averse voters.*

2. *If* **w** = **p**, *i.e. for the plurality score, the rule is not manipulable by uncertainty seeking voters.*

Proof. Without loss of generality, let us consider a voter with preference $A > B > C$ who believes that if she votes sincerely, the outcome — in terms of scores — will correspond to the point X shown on Figure 2. Irrespective of the positional scoring rule, by voting insincerely she cannot improve the score of A, nor worsen the score of C. If she votes insincerely, she will expect the vector of scores to fall in the shaded area. By insincerely reporting her preferences to be $B > A > C$, she will move the vector of scores she expects horizontally east. This she can do so long as the score function is not antiplurality. By insincerely reporting $A > C > B$, she moves the vector of scores she expects north west, parallel to BC, and this misrepresentation is possible except in the event the score function is plurality.
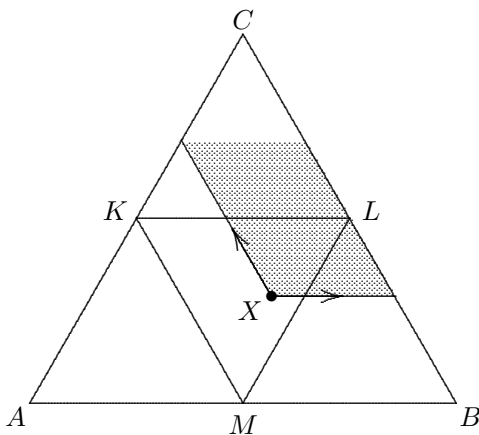

Figure 2: Pure PR. Possible directions of change under a manipulation attempt

A small group of voters all of whom have preference $A > B > C$ cannot escape from the region inside KLC. They would not wish to escape into KLC, nor out of AKM. But if they were uncertainty averse, they would seek, by voting strategically, to move the expected vector of scores from inside MKL (or from on segment ML) to inside MBL. If they were uncertainty seeking, they would be keen to move the expected vector of scores the other way. In either case, if the vector of scores they expect to transpire if they vote sincerely is "close" enough to ML, and if the score function permits, an incentive to manipulate exists. It is not true, however, that all voters, whose type is consistent with $A > B > C$, will have an incentive to manipulate in the same fashion, hence the manipulative opportunities are only weak. It is interesting to note that if a group of voters with preference $A > B > C$ expect that if they all vote sincerely the vector of scores will lie "in the vicinity of ML.", the uncertainty averse and uncertainty seeking members of this group would then attempt to manipulative against each other, even though they have identical preferences on the set of parties. □

It is interesting to note that manipulation can be more difficult for uncertainty seeking voters than for uncertainty averse ones. The former might be in danger of overshooting and need some degree of coordination. In Figure 3 we see how a micro manipulation attempt can

have a disastrous consequences if too many uncertainty seeking voters try to escape the region MLB into the region KLM. They might end up in the region KLC and be much worse off compared to the sincere voting outcome.
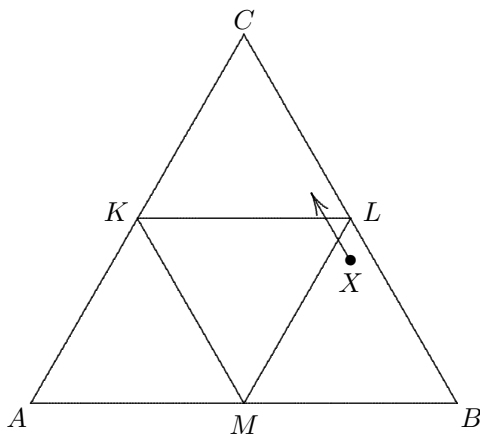

Figure 3: Pure PR. Possibility of overshooting for uncertainty seeking voters

We now show that the introduction of a threshold creates opportunities for strong manipulation but removes the danger of overshooting in any micro manipulation attempt.

Theorem 2. *Let the parliament choosing rule be proportional representation rule with a threshold. Then the rule is strongly manipulable iff* $\mathbf{w} \neq \mathbf{a}$.

Proof. Since the threshold seat allocation rule F_a is now used, normalised positional scores and parliaments are different and S^2 represents only the vector of scores $\mathbf{sc}$. The respective parliament then will be $\mathbf{x} = F_a(\mathbf{sc})$. The partition on Figure 4 shows the power indices that parties get for each vector of normalised scores.
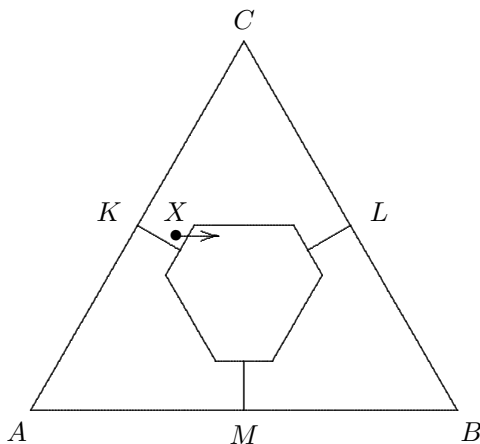
Figure 4: PR with a threshold. Possibility of strong manipulation

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The introduction of a threshold changes the shape of the regions in which the associated vector of power indices is constant. The central region, in which $P_i(\mathbf{x}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, becomes a hexagon. In it the identity $\mathbf{x} = \mathbf{sc}$ still holds. The three regions in which $P_i(\mathbf{x})$ is respectively equal to $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are not convex anymore.

Suppose that a small group of voter with preference $A > B > C$ believe that if they vote sincerely the resulting normalised score will correspond to the point X. At this point, B does not score highly enough to overcome the threshold. If at the election this group insincerely state their preferences to be $B > A > C$, they may be able (so long as the score function is not antiplurality) to push B over the threshold, and move the expected vector of scores inside the hexagon. When this group votes truthfully, the vector of voting power is anticipated to be $(0, 0, 1)$. Untruthful voting could bring about the vector of voting power $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. This is an unambiguously better prospect for all voters with preference $A > B > C$, regardless of their power index and their vector of utilities: hence the introduction of a threshold can create opportunities for strong manipulation. $\square$

Since the shape of the central region is now different and all three acute corners are cut off there is no danger of overshooting during any micro manipulation attempt. When $m \geq 4$, both strategic overshooting and undershooting are possible regardless of whether or not a threshold is present. The following example illustrates that when $m=4$, and a threshold is in operation, strategic overshooting is possible. That strategic voters could undershoot will be illustrated in Section 6.

Assume now that four parties $(A, B, C,$ and $D)$ are contesting an election. Assume that the scoring rule is plurality, and that the parliament-choosing rule is proportional representation with a five percent threshold. Suppose that the ith voter has a preference order $A > B > C > D$, uses the S-S power index, and has utility vector $\mathbf{u}^{(i)} = (10, 9, 6, 0)$.

Suppose that in the event all voters report truthfully the outcome of the election would be:

vector of scores	6	44	5	45
parliament	6	44	5	45
S-S powers	3/12	3/12	1/12	5/12

Suppose that 1% of voters had insincerely reported their first preference to be B rather than A, ceteris paribus. The outcome of the election would then have been:

vector of scores	5	45	5	45
parliament	5	45	5	45
S-S powers	1/6	1/3	1/6	1/3

Suppose that 2% of voters had insincerely reported their first preference to be B rather than A, ceteris paribus. The outcome of the election would then have been:

vector of scores	4	46	5	45
parliament	0	47.92	5.21	46.88
S-S powers	0	1/3	1/3	1/3

(The figures for the last parliament do not sum due to rounding.)

In the first scenario, where all voters report truthfully, we have $\mathbf{u}^{(i)} \cdot \mathbf{p} = 5.25$, where $\mathbf{p}$ is the corresponding vector of power indices. In the second scenario, where 1% of the electorate

vote strategically, we have $\mathbf{u}^{(i)} \cdot \mathbf{p} = 5.67$ (rounded). In the third scenario, where 2% of voters are insincere, we have $\mathbf{u}^{(i)} \cdot \mathbf{p} = 5.00$.

Thus the ith voter prefers the second scenario to the first, but the first to the third. When considering a strategic vote, such a voter would likely be concerned about the possibility of overshooting. Perhaps the ith voters attitude to uncertainty would influence his or her desire to chance an untruth.

We note that this example shows that the strategic overshooting can occur during a micro manipulation attempt as it is clear that in our example the numbers 1% and 2% can be made arbitrary small.

6   The 2005 New Zealand General Election

The NZ electoral system is mixed member proportional, with a 5% threshold. Voters have two votes - an electoral (district) vote, and a party vote. A first-past-the-post election is run in 69 electorates, with the winner of each electorate becoming an MP. Party votes are tallied nationally. The Saint-Lague formula is then applied to the party votes to determine how many seats in total each party is entitled to (if a party neither wins an electorate nor more than than 5% of the party vote then it is excluded from consideration). If a party has fewer electorate wins than places in the highest 120 Saint-Lague quotients then its parliamentary representation is topped-up accordingly from the party list.

The most recent New Zealand general election took place on September 17th 2005. At the election 28.71% of voters gave their electorate vote and their party vote to different parties ([1]) (down from 39.04% in 2002). Perhaps most of these voters split because their first choice either did not stand a candidate in the relevant electorate or because their first choice had no hope of winning the relevant electorate. But the 28.71% figure is high enough to suggest a reasonable amount of insincere voting went on. In particular, anecdotal evidence (reports to the authors) has suggested that some voters with preferences

$$\text{Labour} > \text{Greens} > ...$$

may have cast their party vote for the Greens. We use our model to suggest why.

The two opinion polls closest to the election gave the following results:

Poll	Date	Labour	National	NZ First	Greens
TVNZ Colmar Brunton	15 September	38%	41%	5.5%	5.1%
Herald Digipoll	16 September	44.6%	37.4%	4.5%	4.6%

Table 1: Results of the last two opinion polls

Results of previous polls are available on [4]. The Green Party were not expected to win an electorate seat, and NZ First were expected to win at most one. As it turned out, neither party won an electorate seat. Table 2, below, shows the actual election result. Also shown is what would have transpired had 0.4% of the electorate not given their party vote to the Greens and given them Labour, ceteris paribus.

	Hypothesised			Actual		
Party	Party Vote	Seats	SS	Party Vote	Seats	SS
Labour	41.10	50	0.324	41.50	54	0.414
National	39.10	48	0.262	39.10	50	0.214
NZ First	5.72	7	0.143	5.72	7	0.214
Green Party	5.30	6	0.110	4.90	0	0.0
Maori Party	2.12	4	0.076	2.12	4	0.081
United Future	2.67	3	0.043	2.67	3	0.048
ACT	1.51	2	0.029	1.51	2	0.014
Progressive	1.16	1	0.014	1.16	1	0.014

Table 2: Results of the NZ 2005 general election

Election results were obtained from [1]. Alternative election scenarios can be investigated at [2]. Voting power indices were calculated at [3].

Now suppose that a group of voters at the 2005 general election behave as do voters in our model. Suppose that all members of this group were solely concerned with how Shapley-Shubik power will be distributed in the post-election parliament. Suppose that all members of this group rank the Labour party first, the Green Party second, and attribute zero or negligibly small utility to the powers of all the other parties contesting the election. Further suppose that each group member believes that the election outcome laid out in the right-hand-side of Table 2 is a distinct possibility. Such a supposition is not unreasonable, given the pre-election polls. Then members of this group may have an incentive to party vote Green. The existence and strength of such incentives will depend on each individual voter's utilities.

Define

$$\alpha_i = \frac{\mathbf{u}_i(\text{Greens})}{\mathbf{u}_i(\text{Labour})}$$

to be the ratio of utilities of the Greens and Labour calculated for the ith voter. To construct Figure 5 below we first fix the party votes obtained by all parties other than Labour and the Greens. We then allow the Greens' party vote to vary from 4.9% to 8.9% (and, necessarily, Labour's party vote to vary from 41.5% to 37.5%). For each possible Green party vote we show, on the vertical axis, the minimum value of α_i the ith voter must have in order to prefer the outcome arising from this Green party vote to the hypothesised outcome arising when the Greens secure 4.9%. For example, suppose the ith voter is comparing the outcome arising when the Greens win 5.3% of the party vote to the outcome arising when the Greens win 4.9% (i.e. he or she is comparing the parliament on the left-hand-side of Table 2 to that on the right-hand-side of Table 2). This voter prefers the former to the latter provided his or her utilities comply with $\alpha_i > 0.826$.

The shape of the graph reflects the working of the Saint-Lague formula. Consider the situation, for example, when the Greens have 5.3% of the party vote (the actual election result). At this point, the 119th largest Saint-Lague quotient belongs to Labour, and the 120th largest to National. As the Greens' party vote increases (to the detriment of Labour's) past 5.4%, the Green party capture the 120th largest quotient from National. The Greens then win a 7th parliamentary seat, and National lose their 48th. As the Greens' party vote rises further, their 7th largest quotient eventually exceeds Labour's 50th largest. No seat changes hands, but

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the Greens then have the 119th largest quotient, and Labour the 120th largest. As the Greens' party vote increases past 5.7%, Labour's party vote decreases to the point where its' 50th largest quotient falls below National's 48th largest. National's 48th seat is then restored at the expense of Labour's 50th. The cycle then repeats itself as the Greens' party vote continues to increase.

The ith voter prefers a parliament with the Greens on between 6 and 11 seats, and with National on 47, to a parliament without the Greens, and with National on 50, provided they have $\alpha_i > 0.676$. This voter prefers a parliament with the Greens on between 6 and 11 seats, and with National on 48, to a parliament without the Greens, and with National on 50, provided that $\alpha_i > 0.897$.

The ith voter unreservedly prefers a parliament with a small number of Greens to a parliament without the Greens if she has $\alpha_i > 0.897$. Only those voters who value Green power nearly as highly as Labour power would meet this criteria. Such a voter would have a clear incentive to give their party vote to the Greens, despite their first preference being for Labour. If this voter party votes Green, she increases the likelihood that the Greens will reach parliament. If sufficiently many other group members feel and act the same way, then the Greens *will* enter parliament.

The ith voter with $0.676 < \alpha_i < 0.897$ prefers some parliaments where the Greens are present to those where the Greens are not, but not all. Such group members would not have an unambigious incentive to party vote Green unless they knew precisely how many other group members were also going to use their vote strategically. By voting strategically they would be at risk of both overshooting and undershooting.
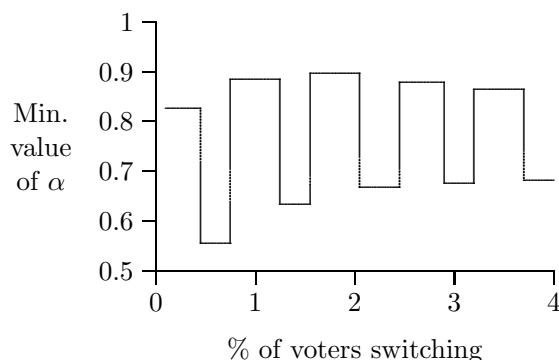
Figure 5 Graph of minimal utility ratio for which the manipulation is successful

We conjecture, then, that at the 2005 NZ general election certain voters with preference Labour > Green > ... felt they preferred the power configuration of a parliament with a small Green presence to that of a Green-less parliament, thought that polling data showed the Greens might not cross the threshold, and so party voted Green in order to increase the likelihood that the Greens would enter parliament. We do not suggest that these voters coordinated, nor that they had knowledge of the preferences or intentions of others beyond what was available from widely disseminated polling data. We conjecture that the Greens were polling so close to the threshold that these voters were not overly concerned about damaging Labour's prospects without improving the Greens showing (strategically undershooting). We conjecture that these voters were not worried about overshooting because they felt the proportion of the

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electorate that was (i) concerned about configurations of parliamentary power, (ii) had preference Labour > Green > . . ., and (iii) had $0.897 < \alpha$, was relatively small (in particular, less than 4.0%).

Also, It may be the case that, in practice, there are many kinds of voters - some voters may be concerned with the post-election distribution of voting power, others with the post-election distribution of seats, others with the policies to be pursued by the next government, etc. It would be quite possible, but unnecessarily complicated, to incorporate all these different kinds of voter in our model.

7 Conclusion

This paper has presented a new model of voter behaviour under methods of proportional representation. We showed that if voters are mindful of how voting power will be distributed in the post-election parliament, then incentives to vote insincerely will exist under any method of PR. We showed that attitudes to uncertainty may influence their incentives to vote insincerely. We demonstrated that introducing a threshold could encourage greater numbers of voters to vote strategically in the same manner. Studying an example of the most recent New Zealand general election we observe that, with two major minor parties having approximately 5% support in the society, the existing 5% threshold may be too high resulting in a high level of insincere voting. We illustrate that rounding can to a certain degree deter voters from manipulation since it may cause both undershooting and overshooting.

Questions this paper raises that future research could address include: How do incentives to vote strategically vary with the choice of positional scoring rule? What if the scoring rule is not positional? Finally, the undershooting/overshooting phenomenon as deterrent of manipulation deserves a thorough investigation.

References

[1] http://www.electionresults.govt.nz/

[2] http://www.elections.org.nz/mmp.html

[3] http://www.warwick.ac.uk/ ecaae/

[4] New Zealand general election, 2005. Wikipedia. The Free Encyclopedia. http://www.wikipedia.org/

[5] Allingham, M.G. (1975) Economic Power and Value of Games. Zeitschrift für Nationalökonomie **35**: 293–299.

[6] Austin-Smith, D. and Banks, J. (1988) Elections, Coalitions, and Legislative Outcomes. American Political Science Review **82**: 405-422

[7] Banzhaf, J.F. (1965) Weighted Voting does not work: A mathematical analysis. Rutgers Law Review **35**: 317-343

[8] Baron, D.P. and Diermeier D. (2001) Elections, Governments, and Parliaments in Proportional Representation Systems. The Quarterly Journal of Economics **116**: 933–967

[9] Brams, S.J, Jones, M.A., and Kilgour, D.M. (2002) Single-Peakedness, Coalition Formation, and Disconnected Coalitions. Journal of Theoretical Politics **14**: 185–202

[10] Brams, S.J. (1975) Game Theory and Politics, New York.

[11] Brams, S.J. and Fishburn, P.C. (2002) Voting Procedures. In: Arrow, Sen and Suzumura (eds.) Handbook of Social Choice and Welfare, Volume 1, North-Holland.

[12] Brams, S. and Potthoff, R. (1998) Proportional Representation: Broadening the Options. Journal of Theoretical Politics **10**: 147-178.

[13] Cox, G.W. and Shugart, M.W. (1996) Strategic Voting Under Proportional Representation. The Journal of Law, Economics, and Organisation **12**: 299-324.

[14] De Sinopoli, F. and Iannantuoni, G. (2001) A Spatial Voting Model where Proportional Rule Leads to Two-Party Equilibria. Research Paper Series 31, Tor Vergata University, CEIS.

[15] De Sinopoli, F. and Iannantuoni, G. (2002) Some Results on Strategic Voting and Proportional Representation with Multidimensional Policy Space. Universersidad Carlos III De Madrid. Departmento de Economia. Working Paper 02-57. Economics Series 21.

[16] De Sinopoli, F. and Iannantuoni, G. (2005) Extreme Voting under Proportional Representation: The Multidimensional Case. University of Cambridge. Faculty of Economics. Working Paper 0531

[17] Deegan, J., Jr. and Packel, E.W. (1979) A New Index of Power for Simple n-Person Games. International Journal of Game Theory **7**:113–123.

[18] Felsenthal, D.S. and Machover, M. (1998) The Measurement of Voting Power. Edward Elgar Publishing.

[19] Gibbard A (1973) Manipulation of Voting Schemes: A General Result: Econometrica **41**:587–601.

[20] Satterthwaite MA (1975) Strategy-proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions: Journal of Economic Theory **10**:187–217.

[21] Holler, M.J. and Packel, E.W. (1983) Power, Luck and the Right Index. Zeitschrift für Nationalökonomie **43**: 21–29.

[22] Karp, J.A., Vowles, J., Banducci, S.A., Donovan, T. (2002) Strategic Voting, Party Activity, and Candidate Effects: Testing Explanations for Split Voting in New Zealand's New Mixed System. Electoral Studies **21**: 1-22

[23] McCabe-Dansted, J.C., Slinko, A.M. (2006) Exploratory Analysis of Similarities between Social Choice Rules, Group Decision and Negotiation, **15**: 77–107.

[24] Riker, W.H. (1962) The Theory of Political Coalitions, New Heaven, CT: Yale University Press.

[25] Saari, D. (1990) Susceptibility to manipulation. Public Choice **64**: 21–41

[26] Saari, D. (1994) Geometry of Voting. Springer-Verlag

[27] Shapley, L.S. and Shubik, M. (1954) A method for evaluating the distribution of power in a committee system. American Political Science Review **48**: 787–792

[28] Taylor, A.D. (1995) Mathematics and politics : strategy, voting, power and proof. Springer-Verlag