Threshold Luce Rules

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Abstract.

In the late 1950's, Luce proposed two different theories of imperfect utility discrimination that have had a lasting impact on economics. One model (1956) gave rise to the literature on *just noticeable differences* while the other (1959) laid the foundations for the literature on *discrete choice*. In this paper, I present a unified model of imperfect utility discrimination that generalizes Luce's two models; and addresses the main limitations of each. Surprisingly, choice behavior consistent with this model is characterized by two conditions from Luce's monograph (1959).

Keywords: just noticeable difference, semi-order, Luce Choice axiom, strict utility model; imperfect utility discrimination. **JEL:** D01, D03.

During the late 50's, Luce proposed two theories of decision-making that have had a profound impact on economics: *semi-orders* (1956), which gave rise to the literature on "just noticeable differences" and incomplete preferences; and the *strict utility* model[2] (1959), which laid the foundations for the multinomial logit model (McFadden, 1974) and the subsequent literature on discrete choice.

While the literatures that developed from Luce's two models have little in common, the models themselves share the same basic goal: to capture imperfect utility discrimination. In the semi-order model, the imperfection relates to the decision-maker's inability to distinguish between alternatives whose utilities are close. In this sense, it is a theory of *limited* discrimination. In contrast, the strict utility model is a theory of *error-prone* discrimination. Since the decision-maker selects each alternative with odds proportional to its utility, she becomes less likely to "mistakenly" choose an inferior alternative as its utility decreases.

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[2]Luce (1959) does not use this name for his model. It is due to Block and Marschak (1960, pp. 179-180).

Despite their impact, Luce's two models are not without limitations. For one, the semi-order model is agnostic about behavior on the *small scale*. When the decision-maker cannot distinguish between the utilities of two alternatives, she is "indifferent" between the two and the model is silent about how she chooses. In turn, the strict utility model is not so convincing on the *large scale*. Even when the utility of one alternative is very small relative to every other alternative, it continues to be chosen with positive probability.

In this paper, I propose a two-stage model of imperfect utility discrimination, called the *threshold Luce model*, that not only unifies Luce's two models, but also addresses the main limitations of each. When faced with a set of alternatives, the decision-maker first eliminates every feasible alternative that she clearly discriminates as inferior. Then, the decision-maker selects each of the remaining alternatives (among which she discriminates imperfectly) with probability proportional to its utility. On the small scale, the model collapses to the strict utility model; and, on the large scale, to the semi-order model. What is more, it unifies Luce's two theories in a natural way: the semi-order that the decision-maker uses to eliminate alternatives in the first stage is based on the utility function that determines her choice probabilities in the second stage.

After presenting the model more formally in Section 1, I state my main result (Theorem 1) in Section 2. Surprisingly, it shows that choice behavior consistent with the threshold model is characterized by two conditions from Luce's monograph (1959): the original version of his well-known *Choice axiom*, which includes an "often ignored" (Luce, 2008) part pertaining to alternatives that are chosen with zero probability; and a transitivity requirement that weakens Block and Marschak's (1960) *Strong stochastic transitivity*.

The result makes the threshold Luce model and the two conditions that characterize it mutually reinforcing. In one direction, the fact that axioms from Luce's monograph characterize the threshold model suggest that it is the "right" way to unify his theories of imperfect utility discrimination. In the other, the threshold model breathes new life into Luce's "old" conditions. When his monograph was first published, critics suggested that Luce dealt with zero probabilities in a "distinctly artificial manner" (Luce and Suppes, p. 336). To the contrary, Theorem 1 shows that his original Choice axiom is essential for a natural model of imperfect utility discrimination. In the same vein, Theorem 1 provides a fresh perspective on stochastic transitivity. It suggests that such conditions may play an important role in decision-making procedures that combine a "coarse" process of elimination with a more "refined" or deliberative process of choice.[3]

In Section 3, I present two additional results (Theorems 2 and 3) that are used to prove my main result; and, in Section 4, I show how these preliminary results are relevant for extensions of the model that vary the structure of the first stage and the level of connection to the second stage. Not only does this serve to highlight the flexibility of my approach, but it clarifies the connection between the original version of Luce's Choice axiom and a growing body of work that extends the strict utility model to accommodate zero-probability choice (e.g., McCausland, 2009; Lindberg, 2012; Dogan and Yildiz, 2016; Ahumada and Ulku, 2017; Echenique and Saito, 2017; and Cerreia-Vioglio et al., 2018). Formally, the "gap" between the Choice axiom and the *extended Luce models* studied in these papers is spanned by a minor variation on Luce and Suppes' (1965) *Product rule.*

1. The model

Let X denote a countable set of *alternatives* and $\mathcal{F} = \{S \subseteq X : |S| \in \mathbb{N}\}$ the collection of (finite) *menus* on X. A *random choice function* on $\mathcal{F}$ is a mapping $p : X \times \mathcal{F} \to [0,1]$ such that $\sum_{x \in S} p(x, S) = 1$ for every menu $S \in \mathcal{F}$. In other words, $p(x, S)$ is the probability of choosing the feasible alternative $x \in S$; and, by extension, $p(R, S) := \sum_{x \in R} p(x, S)$ the probability of choosing some alternative in the subset $R \subseteq S$.

My focus is a model of random choice that combines Luce's models of imperfect utility discrimination into a cohesive two-stage procedure. To formalize, first let $\mathbb{R}_{++}$ denote the strictly positive real numbers and $\mathcal{I}_{++}$ the set of closed intervals on $\mathbb{R}_{++}$. Then, consider

[3] The only previous work to hint at this connection is Roberts (1971). Following Luce (1959, pp. 34-37), he used probability thresholds to define a family of binary relations from binary stochastic choice data (i.e., for a given threshold, let x be preferred to y if the probability of choosing x over y exceeds the threshold). Roberts then showed that a generalized version of Strong stochastic transitivity is necessary for a family of semi-orders to be induced by the strict utility model in this way.

the transitive and irreflexive (equivalently, transitive and asymmetric) binary relations[4] $\gg$ and $>$ on $\mathcal{I}_{++}$ such that, for distinct intervals $[a,b]$ and $[c,d]$:[5]

$[a,b] \gg [c,d]$ if $a > d$; and

$[a,b] > [c,d]$ if $a \geq c$ and $b \geq d$.

In words, $[a,b] \gg [c,d]$ if the interval $[a,b]$ lies *entirely* above the interval $[c,d]$; and $[a,b] > [c,d]$ if each endpoint of $[a,b]$ lies (weakly) above the corresponding endpoint of $[c,d]$. Finally, define a *utility function* (on X) to be a mapping $v : X \to \mathbb{R}_{++}$ that assigns a strictly positive value $v(x)$ to each alternative $x \in X$; and a *utility correspondence* (on X) to be a mapping $B : X \to \mathcal{I}_{++}$ that assigns an interval (or *range*) of utilities $B(x)$ to each $x \in X$.

In the first stage of the procedure, the decision-maker assigns a range of utilities $B(x)$ to each alternative x in the feasible set S; and eliminates any alternative whose range lies entirely below that of another feasible alternative. The set of remaining alternatives defines a *choice correspondence*[6] Γ_B on $\mathcal{F}$ such that

$$\Gamma_B(S) := \{x \in S : B(z) \gg B(x) \text{ for no } z \in S\} \text{ for each menu } S \in \mathcal{F}.$$

In the second stage, the decision-maker assigns a specific utility value $v(x)$ to each of the remaining alternatives; and uses these to determine the relative choice probabilities, selecting $x \in \Gamma_B(S)$ with probability

$$\frac{v(x)}{\sum_{y \in \Gamma_B(S)} v(y)}.$$

What makes the procedure cohesive is the close connection between the utilities assigned in the two stages. For each alternative, (i) the second-stage utility must be drawn from the range of first-stage utilities; and, moreover, (ii) the two utilities must be *co-monotonic*.

Definition. *A* ***threshold Luce rule*** *is a pair (B, v) consisting of a utility correspondence B on X and a utility function v on X such that, for all $x, y \in X$: (i) $v(x) \in B(x)$; and (ii) $B(x) > B(y) \iff v(x) > v(y)$. The rule (B, v) represents a random choice function p if,*

[4]For the benefit of unfamiliar reader, I define these properties in Appendix A.

[5]Since $[a, \infty) := \{x \in \mathbb{R}_{++} : x \geq a\} \in \mathcal{I}_{++}$, I allow $b, d \in \mathbb{R}_{++} \cup \{\infty\}$ and define $[a, \infty] := [a, \infty)$.

[6]A *choice correspondence* C on $\mathcal{F}$ selects a subset of alternatives $\emptyset \subset C(S) \subseteq S$ for each menu $S \in \mathcal{F}$.

for each menu $S \in \mathcal{F}$ and every alternative $x \in S$:

$$p(x,S) = \begin{cases} \frac{v(x)}{\sum_{y \in \Gamma_B(S)} v(y)} & \text{if } x \in \Gamma_B(S) \\ 0 & \text{otherwise.} \end{cases}$$

The remarks below show how the model relates to Luce's theories of utility discrimination.

Remark 1A. Luce (1956) defined a class of transitive and irreflexive binary relations that accommodate intransitive indifference. Formally, $\succ$ is a *semi-order* if it is irreflexive and, for all $x, y, z, w \in X$, satisfies:

Ferrers property. $[x \succ w$ and $y \succ z] \implies [x \succ z$ or $y \succ w]$; and,
Semi-transitivity. $[x \succ y$ and $y \succ z] \implies [x \succ w$ or $w \succ z]$.[7]

While both of these properties require $\succ$ to be transitive, neither requires the indifference relation $\sim$ to be transitive (where, as usual, one defines $x \sim y$ if $x \not\succ y$ and $y \not\succ x$). Nonetheless, Luce showed that every semi-order $\succ$ can be *represented by a utility correspondence* B in the sense that for all $x, y \in X$,

$$x \succ y \iff B(x) \gg B(y).$$

Later work (e.g., Theorem 7 of Fishburn, 1985) tightened Luce's result, showing that a binary relation $\succ$ is a semi-order if and only if it has a *non-nested* representation B in the sense that for all $x, y \in X$,

$$B(x) \setminus B(y) \neq \emptyset \implies B(y) \setminus B(x) \neq \emptyset.$$

Due to the close connection between the two stages of a threshold Luce rule (B, v), the first-stage utility correspondence B is non-nested.[8] So, the binary relation $\succ_B$ defined by

$$x \succ_B y \text{ if } B(x) \gg B(y)$$

[7] Luce (1956) gives a more complicated axiomatization. The one here is due to Scott and Suppes (1958).

[8] In fact, requirements (i)-(ii) of the definition are equivalent to the requirements in Beja and Gilboa's (1992) *generalized utility representation* of semi-orders.

is a semi-order. What is more, $\succ_B$ *rationalizes* the first-stage choice correspondence Γ_B.[9]

Remark 1B. Luce's strict utility model (1959, p. 23) uses a utility function v to represent a random choice function p. In particular, for each $S \in \mathcal{F}$ and every $x \in S$, the model specifies

$$p(x,S) := \frac{v(x)}{\sum_{y \in S} v(y)}.$$

This is a special case of the threshold Luce model where no alternative is eliminated in the first stage (i.e., $\Gamma_B(S) = S$ for all $S \in \mathcal{F}$). One (B, v) parametrization with this feature defines $B(x) := [v(x), \infty)$ for all $x \in X$ (so that $B(x)$ is bounded from below by $v(x)$ and unbounded from above).

2. Main result

In this section, I axiomatize the model and provide identification results for its parameters.

(a) Axiomatization

My characterization relies on two conditions from Luce's monograph (1959). The first is his main axiom (p. 6).

Luce Choice axiom (LCA). *For all menus $R, S, T \in \mathcal{F}$ such that $R \subset S \subseteq T$:*

$$p(R,T) = \begin{cases} p(R,S) \times p(S,T) & \text{if } p(x,\{x,y\}) > 0 \text{ for all } \{x,y\} \subseteq T\text{; and} \\ p(R \setminus \{x\}, T \setminus \{x\}) & \text{if } p(x,\{x,y\}) = 0 \text{ for some } \{x,y\} \subseteq T. \end{cases}$$

Conventionally, this axiom is stated in conjunction with the assumption that the random choice function p is *positive* (i.e., $p(x,T) > 0$ for each $T \in \mathcal{F}$ and every $x \in T$). In that case, it reduces to the requirement that, for all menus $R, S, T \in \mathcal{F}$ such that $R \subset S \subseteq T$,

$$p(R,T) = p(R,S) \times p(S,T). \qquad (\text{LCA}_+)$$

[9] A binary relation $\succ$ *rationalizes* a choice correspondence C if $C(S) = \{x \in S : z \succ x \text{ for no } z \in S\}$ for all $S \in \mathcal{F}$.

This formula equates the probability of choosing in the menu R with the probability conditional on any intermediate menu $R \subset S \subseteq T$. For a positive random choice function p, LCA$_+$ is equivalent[10] to the well-known *Independence of irrelevant alternatives* (IIA), which specifies that the odds of choosing x over y do not depend on the other alternatives available.[11] Formally, for all menus $T \in \mathcal{F}$ and alternatives $x, y \in T$,

$$\frac{p(x,T)}{p(y,T)} = \frac{p(x,\{x,y\})}{p(y,\{x,y\})}.$$

Luce's well-known Theorem 3 (1959, p. 23) shows that a positive random choice function p has a strict utility representation if and only if p satisfies LCA$_+$ (or, equivalently, IIA).

The original version of the Luce Choice axiom imposes the conditional probability formula in LCA$_+$ if binary choice probabilities discriminate "imperfectly" among the alternatives in T. Otherwise, some feasible alternative $x \in T$ is "perfectly" discriminated as inferior and the second line applies, which stipulates that x can be eliminated without affecting the probability of choosing within $R \subset T$.

The second axiom is Luce's stochastic transitivity requirement (see pp. 27 and 143):

Luce transitivity (LT).[12] *For all alternatives* $x, y, z \in X$ *such that* $p(x,\{x,y\}), p(y,\{y,z\}) \geq 1/2$*:*

$$\max\{p(x,\{x,y\}), p(y,\{y,z\})\} = 1 \implies p(x,\{x,z\}) = 1.$$

This condition is less demanding than Block and Marschak's (1960, p. 190) *Strong stochastic transitivity*, which requires

$$p(x,\{x,z\}) \geq \max\{p(x,\{x,y\}), p(y,\{y,z\})\}$$

even when $\max\{p(x,\{x,y\}), p(y,\{y,z\})\} < 1$. However, the two conditions are equivalent when p satisfies the Luce Choice axiom (by Lemma 6 of Appendix E). In my setting, this means that Luce transitivity is effectively more demanding than either of the *Moderate* or

[10]See Lemma 5 of Appendix E, which generalizes Lemma 3 of Luce (1959, p. 9).

[11]For an important critique of IIA, see Debreu (1960); or McFadden (1974), who adapts Debreu's example.

[12]This axiom later appeared as Axiom 8 (*Strong dominance transitivity*) in Echenique and Saito.

Weak stochastic transitivity conditions from the literature.[13] Under the assumption that $p(x,\{x,y\}), p(y,\{y,z\}) \geq 1/2$, the stronger of these conditions requires

$$p(x,\{x,z\}) \geq \min\{p(x,\{x,y\}), p(y,\{y,z\})\}.$$

Theorem 1A. *A random choice function p can be represented by a threshold Luce rule (B, v) if and only if it satisfies the Luce Choice axiom and Luce transitivity.*

As I show in Section 3, the two axioms play conceptually distinct roles in the representation: the Luce Choice axiom ensures that p can be represented by a general two-stage procedure while Luce transitivity ensures that the first stage of the procedure exhibits the required structure and connection to the second stage. As I show in Section 4, the modularity of the characterization makes it easier to generalize the result.

(b) Identification

The notions of "perfect" and "imperfect" discrimination (that I used to provide intuition for the Luce Choice axiom) are related to parameter identification in the threshold Luce model. To formalize these notions:

Definition. *Alternatives x and y are* ***imperfectly discriminated****, which I denote $x \frown y$, if both are chosen with positive probability from $\{x,y\}$ so that $p(x\{x,y\}), p(y\{x,y\}) > 0$. In turn, x and y are* ***linked*** *by imperfect discrimination if there is some some sequence of alternatives $z_1, ..., z_n \in X$ such that $x{=}z_1 \frown ... \frown z_n{=}y$. Since linking defines an equivalence relation, it partitions the domain X into* ***linked components****.*

The first-stage semi-order $\succ_B$ (see Remark 1A) is identified by perfect discrimination:

$$x \succ_B y \ \text{ if } \ p(y,\{x,y\}) = 0. \qquad (\succ_B\text{-Id})$$

This is clear from the representation: to be chosen with zero probability, y must be eliminated by x in the first stage. While the semi-order $\succ_B$ may then be used to put bounds on the

[13] For the unfamiliar reader, I restate all three of these transitivity conditions in Appendix A.

size of the utility interval $B(x)$ associated with each alternative $x \in X$, there is no way to identify these intervals exactly.[14]

In turn, the second-stage utility v is identified by imperfect discrimination. As in the strict utility model, the relative utility $v(x)/v(y)$ of imperfectly discriminated alternatives $x, y \in X$ is uniquely determined by the relative choice probability $p(x, \{x, y\})/p(y, \{x, y\})$. By extension, the relative utility of alternatives linked by a sequence $x{=}z_1 \frown ... \frown z_n{=}y$ of imperfect discriminations is uniquely determined by the ratio

$$\frac{v(x)}{v(y)} = \prod_{i=1}^{n-1} \frac{p(z_i, \{z_i, z_{i+1}\})}{p(z_{i+1}, \{z_i, z_{i+1}\})}. \qquad (v\text{-Id})$$

However, there is no way to determine the relative utility of alternatives from different linked components. To obtain relative identification of v across the entire domain, the following condition is required:

Linked domain (LD).[15] *Every pair of alternatives $x, y \in X$ is linked.*

Luce (1959, p. 25) suggests that an even stronger regularity requirement (called *Finitely connected domain* in Appendix A) is reasonable when the alternatives are distinguished by "minor" differences.[16] Whatever the merits of his argument, it is worth emphasizing that Finitely connected domain and Linked domain both impose strong restrictions on the first stage. In particular, they require $\succ_B$ to satisfy the following property:

Sorites property. For all $x, y \in X$:

$$x \succ y \implies [x = z_0 \sim ... \sim z_{n+1} = y \text{ for some } z_1, ..., z_n \in X].^{17}$$

This property formalizes the classical paradox of the same name: any pair of alternatives

[14] Where v is the (relative) utility determined by (v-Id), the lower end of $B(x)$ must be between $\sup\{v(y) : y \in X \text{ and } x \succ_B y\}$ and $\inf\{v(y) : y \in X \text{ and } x \not\succ_B y\}$; and, the upper end of $B(x)$ between $\sup\{v(y) : y \in X \text{ and } y \not\succ_B x\}$ and $\inf\{v(y) : y \in X \text{ and } y \succ_B x\}$. See Definition 2 of Luce (1956) for a similar approach.

[15] This axiom previously appeared as Condition 4 in Krantz et al. (1989, p. 417) and Axiom 6 (*Richness*) in Echenique and Saito (2018). Given the Luce Choice axiom, it is equivalent to a weaker condition that allows for linking on non-binary menus.

[16] For a random choice function that satisfies the Luce Choice axiom and Luce transitivity, it turns out that Linked domain is actually equivalent to this condition. See Lemma 7 of Appendix E.

[17] Recall that $z \sim z'$ if $z \not\succ z'$ and $z' \not\succ z$ (see Remark 1A).

that are perfectly discriminated (such as a "heap of sand" and a "non-heap of sand") can nonetheless be linked by imperfect discrimination. This property is quite limiting since it precludes the possibility that $\sim_B$ is transitive.

The next result summarizes the foregoing observations about identification.

Theorem 1B. *For a threshold Luce representation* (B, v) *of a random choice function* p*:*

(i) *the semi-order* $\succ_B$ *is uniquely identified; and,*

(ii) *the utility function* v *is identified, within each linked component, up to a constant factor.*

Accordingly, v *is unique up to a constant factor if and only if the random choice function* p *satisfies Linked domain. In that case, the semi-order* $\succ_B$ *satisfies the Sorites property.*

In combination with Theorem 1A, this clarifies the connection to a result from Luce's monograph (p. 25) that is often overlooked.

Remark 2. Luce's Theorem 4 (which is restated in Appendix A) establishes a strict utility representation for the domain of *undiscriminated menus* $\mathcal{F}_+ := \{S \in \mathcal{F} : x \frown y \text{ for all } x, y \in S\}$ where all binary choice probabilities are non-zero. Specifically, if p satisfies the axioms of Theorem 1A *as well as* the Finitely connected domain condition discussed above, then there is a utility function v (unique up to a constant factor) that represents p on $\mathcal{F}_+$. Luce's result follows from Theorems 1A-B by noting that, in the threshold model, no alternative in an undiscriminated menu is eliminated in the first stage (i.e., $\Gamma_B(S) = S$ for all $S \in \mathcal{F}_+$).

3. Preliminary results

In this section, I present two results that I use to establish the sufficiency of the axioms in Theorem 1A. These results are interesting in their own right; and central to the extensions discussed in Section 4 below.

(a) Extended Luce rules

To establish sufficiency in Theorem 1A, the first step is to show that the Luce Choice axiom is "almost" sufficient to obtain the following general two-stage representation of behavior:

Definition. *An **extended Luce rule** (Γ, v) is a pair consisting of: a choice correspondence Γ on $\mathcal{F}$; and a utility function v on X. The rule (Γ, v) represents a random choice function p if, for each menu $S \in \mathcal{F}$ and every alternative $x \in S$:*

$$p(x, S) = \begin{cases} \frac{v(x)}{\sum_{y \in \Gamma(S)} v(y)} & \textit{if} \quad x \in \Gamma(S) \\ 0 & \textit{otherwise.} \end{cases}$$

To simplify, I write $(\succ, v)$ instead of (Γ, v) when Γ is rationalized by a binary relation $\succ$.

As noted in the introduction, this kind of representation has been studied in a number of recent papers. Compared with the threshold model, it imposes less structure on the first stage and does not require any kind of connection between the two stages of the representation. As it turns out, the gap between the Luce Choice axiom and an extended Luce representation is spanned by the following condition:

Quadruple Product rule (4-PR). *For all distinct alternatives $x, y, z, w \in X$ such that $x \frown y \frown z \frown w \frown x$:*

$$\frac{p(x, \{x, y\}) \times p(y, \{y, z\}) \times p(z, \{z, w\}) \times p(w, \{w, x\})}{p(y, \{x, y\}) \times p(z, \{y, z\}) \times p(w, \{z, w\}) \times p(x, \{w, x\})} = 1.$$

To interpret this condition, suppose the decision-maker chooses exactly *once* from each of the four menus in the formula above. Then, one can use her behavior to reveal preference in the conventional way (i.e., aPb if a is chosen from $\{a, b\}$). Assuming that her choices are independent across menus, the numerator of the formula reflects the probability of the revealed preference cycle $xPyPzPwPx$. In turn, the denominator reflects the probability of the "opposite" cycle $xPwPzPyPx$. In other words, the Quadruple Product rule specifies that four-cycles of revealed preference are equally likely to arise in *either* direction.

This interpretation dates back to Luce (1959, p. 17), who showed that his Choice axiom implies a similar condition for *three* alternatives.[18] This condition, later called the *Product*

[18]For the proof, see Step 4 in the proof of Theorem 2 from Appendix C. To see that the Luce Choice axiom does not imply the Quadruple product rule, see Example 2 in Appendix E.

rule by Luce and Suppes (1965, p. 341), stipulates that, for every three-cycle of imperfect discriminations $x \frown y \frown z \frown x$,

$$\frac{p(x,\{x,y\}) \times p(y,\{y,z\}) \times p(z,\{z,x\})}{p(y,\{x,y\}) \times p(z,\{y,z\}) \times p(x,\{z,x\})} = 1.$$

It turns out that both product rules are necessary to identify utilities from choice probabilities. To see this, suppose that p violates the Quadruple Product rule on the four-cycle of imperfect discriminations $x \frown y \frown z \frown w \frown x$. Using equation ($v$-Id), one can then derive the contradiction that

$$\frac{v(x)}{v(z)} = \frac{p(x,\{x,y\}) \times p(y,\{y,z\})}{p(y,\{x,y\}) \times p(z,\{y,z\})} \neq \frac{p(x,\{x,w\}) \times p(w,\{w,z\})}{p(w,\{x,w\}) \times p(z,\{w,z\})} = \frac{v(x)}{v(z)}.$$

To see that the Product rule is also necessary, substitute $w = x$ in the preceding argument.

More broadly, the same kind of argument shows that a "general" product rule covering cycles of *arbitrary* length is necessary for an extended Luce representation. The next result follows from the fact that this general condition (called the *Strong Product rule* in Appendix A) is implied by the combination of the Luce Choice axiom and the Quadruple Product rule.

Theorem 2. *For a random choice function p, the following are equivalent:*

(i) *p satisfies the Luce Choice axiom and the Quadruple Product rule;*
(ii) *p can be represented by an extended Luce rule $(\succ, v)$ where $\succ$ is transitive and irreflexive;*
(iii) *p satisfies the Luce Choice axiom and can be represented by an extended Luce rule (Γ, v).*

For (ii), the identification of the binary relation $\succ$ and the utility function v are the same as in Theorem 1B. For (iii), the choice correspondence Γ is uniquely identified, for all $S \in \mathcal{F}$, by $\Gamma(S) := \{x \in S : p(x,S) > 0\}$.

The equivalence $(i) \Leftrightarrow (iii)$ shows that the Quadruple Product rule is necessary *and* sufficient for an extended Luce representation when p satisfies the Luce Choice axiom. In turn, $(i) \Rightarrow (ii)$ shows that the "rationality" of the first-stage comes for free: when p satisfies the Luce Choice axiom, the Quadruple Product rule ensures that Γ is rationalized by a transitive and irreflexive binary relation. Finally, $(ii) \Rightarrow (iii)$ shows that the Luce Choice axiom is necessary to ensure the rationality of the first stage.

(b) Utility-based semi-orders

Since the Luce Choice axiom and Luce transitivity imply the Quadruple Product rule (by Lemma 2(a) of Appendix D), Theorem 2 ensures that these axioms are sufficient for an extended Luce representation.

To complete the sufficiency portion of Theorem 1A, it remains to show that the first and second stages have the required structure. Given what can be identified from behavior (in Theorem 1B), the relevant question is what conditions are necessary and sufficient for the first-stage binary relation $\succ_B$ and the second-stage utility function v to be related as in the threshold Luce model.

Definition. *A binary relation $\succ$ is **strongly based on** a utility function v if there is a utility correspondence B such that, for all $x, y \in X$: (i) $v(x) \in B(x)$; and (ii) $B(x) > B(y) \iff v(x) > v(y)$.*[19]

The answer turns out to be the following "mixed" transitivity requirement.

v-***transitivity.*** *For all $x, y, z \in X$:*

(i) $[x \succ y$ and $v(y) \geq v(z)] \implies x \succ z$; (ii) $[v(x) \geq v(y)$ and $y \succ z] \implies x \succ z$.

Besides the recent work on *necessary and possible (NaP) preferences*[20] (see Giarlotta, 2018), this kind of requirement has received little attention. To my knowledge, mixed transitivity conditions have only figured prominently in work by Fishburn (1970, p. 321; 1985, p. 130), Roberts (1971, p. 256), and Bordes (1979, p. 191).

With this in mind, it is worth considering v-transitivity in more detail. To do so, first let $Int(a, b) := \{\alpha a + (1 - \alpha)b : \alpha \in (0, 1)\}$ denote the open interval with endpoints $a, b \in \mathbb{R}$ (i.e., (a, b) or (b, a) depending on the relative size of a and b); and consider the following

[19] As discussed in Remark 1A, conditions (i) and (ii) ensure that the binary relation $\succ$ is a semi-order.

[20] I thank Paola Manzini and Marco Mariotti for pointing out the connection. Formally, an *NaP preference* $(\succsim_N, \succsim_P)$ is a pair of binary relations on X such that: (i) $\succsim_N$ is transitive and reflexive; (ii) $\succsim_N$ refines $\succsim_P$; (iii) for all $x, y \in X$, $x \succsim_N y$ or $y \succsim_P x$; and (iv) for all $x, y, z \in X$, $[x \succsim_P y \succsim_N z$ or $x \succsim_N y \succsim_P z] \implies x \succsim_P z$. To see the relationship to Theorem 3 below, let $\geq_v$ denote the binary relation induced by a utility function v; and $\succsim$ the binary relation defined from $\succ$ by $x \succsim y$ if $y \not\succ x$. Then, whenever $\succ$ is irreflexive and satisfies v-transitivity, the pair $(\geq_v, \succsim)$ defines an NaP preference.

properties of a binary relation:

Definition. *For a given utility function v, the following properties of a binary relation $\succ$ apply to all $x, y, z \in X$:*

> ***v-consistency.*** $x \succ y \implies v(x) > v(y)$.
>
> ***v-transitive indifference.*** $[x \sim y \sim z$ *and* $v(y) \notin Int(v(x), v(z))] \implies x \sim z$.

The first property requires $\succ$ to be consistent with (the weak order defined by) the utility function v. In turn, the second property requires the indifference $x \sim y \sim z$ to be transitive when the utility of the "middle" alternative y is *not* between the utilities of the "extreme" alternatives x and z. Intuitively, this ensures that the indifference relation $\sim$ must exhibit some degree of transitivity.

Next, consider the following refinements of a binary relation:

Definition. *Given an irreflexive binary relation $\succ$, consider the binary relations defined by:*

$$x \succ^* y \quad \text{if} \quad x \succ z \sim y \text{ or } x \sim z \succ y \text{ for some alternative } z \in X; \text{ and}$$
$$x \succ^{**} y \quad \text{if} \quad x \succ^* y \text{ and } y \not\succ^* x.$$

The binary relation $\succ^*$ *refines* the original binary relation $\succ$ (i.e., for all $x, y \in X$, $x \succ^* y \implies x \succ y$) by adding "indirect" preferences that arise via intermediate alternatives. It also refines $\succ^{**}$, which removes the "ambiguous" comparisons from $\succ^*$. Finally, it is well-known that $\succ^{**}$ refines the original relation $\succ$ when the latter is transitive; and $\succ^{**}$ coincides with $\succ^*$ when $\succ$ is a semi-order (see Theorem 2 of Fishburn, 1985).

Theorem 3. *For an irreflexive binary relation $\succ$, the following statements are equivalent:*

(i) $\succ$ *is (a semi-order that is) strongly based on the utility function v;*

(ii) $\succ$ *satisfies v-transitivity;*

(iii) $\succ$ *satisfies transitivity, v-consistency and v-transitive indifference; and*

(iv) $\succ^*$ *satisfies v-consistency.*

The equivalence (i) $\Leftrightarrow$ (ii) is the result anticipated above: v-transitivity is necessary and sufficient for $\succ$ to be strongly based on v. In turn, statements (iii) and (iv) recast v-transitivity in more familiar terms.[21] First, (iii) shows that v-transitivity not only requires $\succ$ to be transitive and consistent with v, but also requires some degree of transitivity for the indifference relation $\sim$. In turn, (iv) shows that v-transitivity is equivalent to a stronger consistency requirement, namely that $\succ^*$ (a refinement of $\succ$) is consistent with v.

Using definitions ($\succ_B$-Id) and (v-Id) to "translate" from the *deterministic* setting of preference and utility into the *stochastic* setting of random choice shows that Luce transitivity is necessary for the v-transitivity of $\succ_B$. In fact, Luce transitivity is also sufficient for this purpose (by the argument given in Lemma 4(a) of Appendix D). Combined with Theorems 2 and 3, this establishes the sufficiency portion of Theorem 1A.

4. Generalizations

Luce transitivity jointly determines (a) the structure of the first stage and (b) its connection to the second. In this section, I relax this requirement and consider some natural generalizations of the threshold Luce model along these two dimensions. As with the original model, Theorem 2 makes it straightforward to characterize these generalizations by translating the desired deterministic properties into the stochastic setting.[22]

(a) Structure of the first stage

For the first stage to be rationalized by a semi-order, one must impose stochastic versions of the Ferrers property and semi-transitivity (defined in Remark 1B). In particular:

Stochastic Ferrers property. For all $x, y, z, w \in X$:

[21] For an irreflexive binary relation $\succ$, v-transitivity is also equivalent to v-consistency and the following:

Roberts' property. *For all* $x, y, z \in X$, $[v(x) \geq v(y) \geq v(z)$ *and* $x \sim z] \implies x \sim y \sim z$.

Roberts' (1971, p. 250) notion of *compatibility* between a binary relation $\succ$ and a utility function v requires this same property but weakens v-consistency to the requirement that, for all $x, y \in X$, $x \succ y \implies v(x) \geq v(y)$. It follows that a binary relation $\succ$ strongly based on v is compatible with v. To see that the converse is not true, fix a linear order on X and let v denote the constant utility function (such that $v(x) = v(y)$ for all $x, y \in X$). Then, $\succ$ is compatible with v but $\succ$ violates v-transitivity.

[22] Fishburn (1978) exploited the same idea of translation, though his focus was a different model.

$p(w, \{x, w\}), p(z, \{y, z\}) = 0 \implies \min\{p(z, \{x, z\}), p(w, \{y, w\})\} = 0.$

Stochastic semi-transitivity. For all $x, y, z, w \in X$:

$p(y, \{x, y\}), p(z, \{y, z\}) = 0 \implies \min\{p(w, \{x, w\}), p(z, \{z, w\})\} = 0.$

When combined with the Luce Choice axiom, these stochastic rationality conditions are sufficient for an extended Luce representation $(\succ, v)$ where $\succ$ is a semi-order. This follows from Theorem 2 and the observation that the Stochastic Ferrers property strengthens the Quadruple Product rule when p satisfies the Luce Choice axiom. To see this, suppose $x \frown y \frown w \frown z \frown x$. If both *diagonal pairs* are perfectly discriminated ($x \not\frown w$ and $y \not\frown z$), then the Stochastic Ferrers property also requires one of the *adjacent pairs* to be perfectly discriminated ($x \not\frown y$, $y \not\frown w$, $w \not\frown z$, or $z \not\frown x$). So, suppose $x \frown w$. Since the Luce Choice axiom implies the Product rule,

$$\frac{p(x, \{x, y\}) \times p(y, \{y, w\})}{p(y, \{x, y\}) \times p(w, \{y, w\})} = \frac{p(x, \{x, w\})}{p(w, \{x, w\})} = \frac{p(x, \{x, z\}) \times p(z, \{z, w\})}{p(z, \{x, z\}) \times p(w, \{z, w\})}.$$

By re-arranging this identity, one obtains the formula from the Quadruple Product rule.

The same approach may be used to characterize extended Luce models that impose more (or, in some cases, less) structure on the first stage.[23] It suffices to translate the relevant rationality conditions into the stochastic setting. For parsimony, I relegate the details to Theorem 2* of the Online Appendix.

(b) Relationship to the second stage

The threshold Luce model imposes a tight connection between the two stages of the representation. The next definition suggests some natural ways to weaken this connection.

Definition. *A binary relation $\succ$ is **based on** a utility function v if there is a utility correspondence B such that: (i) for all $x \in X$, $v(x) \in B(x)$; and, (ii') for all $x, y \in X$, $B(x) > B(y) \implies v(x) > v(y)$. In turn, $\succ$ is **weakly based on** v if there is a utility correspondence B such that condition (i) holds.*

[23]As above, one can dispense with the Quadruple Product rule when $\succ$ satisfies the Ferrers property.

A binary relation that is based on a utility function v may have less structure than one that is *strongly* based on v. While the latter requires $\succ$ to be a semi-order (see Remark 1A), this is no longer true when co-monotonicity is weakened to condition (ii').[24] In turn, a binary relation that is *weakly* based on a utility function v may be less closely related to v than one that is based on v. The next example serves to illustrate:

Example 1. *Consider the random choice function p on $X := \{x_1, x_2, x_3\}$ represented by the pair $(\succ, v)$ where: $x_3 \sim x_2 \succ x_1 \sim x_3$; and $v(x_i) = i$ for all $x_i \in X$. Then, it is straightforward to check that p satisfies the Stochastic Ferrers property and Stochastic semi-transitivity. However, it violates Luce transitivity since*

$$p(x_3, \{x_2, x_3\}) = {}^3\!/_5 \text{ and } p(x_2, \{x_1, x_2\}) = 1 \text{ but } p(x_3, \{x_1, x_3\}) = {}^3\!/_4 < 1.$$

In this example, the issue is that the semi-order $\succ$ puts alternative x_2 "indirectly" above alternative x_3 while the utility function v does the opposite. In other words, the binary relation $\succ^*$ (defined in Section 3(b) above) violates v-consistency. From Lemma 1, it follows that $\succ$ cannot be based on v. The utility correspondence B defined below shows that $\succ$ is nonetheless *weakly* based on v:

$$B(x_2) := [{}^3\!/_2, 4],\ B(x_3) := [1, {}^7\!/_2],\ \text{and } B(x_1) := [0, {}^5\!/_4].$$

The next condition is the stochastic analog of the requirement that $\succ$ satisfies v-consistency:

Weak consistency (WC). *For every sequence of imperfect discriminations $z_1 \frown ... \frown z_n$:*

$$p(z_n, \{z_1, z_n\}) = 0 \implies \prod_{i=1}^{n-1} \frac{p(z_i, \{z_i, z_{i+1}\})}{p(z_{i+1}, \{z_i, z_{i+1}\})} > 1.$$

This condition states that, on a sequence linking perfectly discriminated alternatives, the "net" odds cannot favor the inferior alternative. It is straightforward to show that, when the Luce Choice axiom holds, Weak consistency strengthens Moderate stochastic transitivity (see Lemma 3(a) of Appendix D).

[24] In this case, $\succ$ is only required to be an *interval order* (see Appendix A for the definition).

The choice function in Example 1 satisfies Weak consistency since $p(x_1, \{x_1, x_2\}) = 0$ and

$$\frac{p(x_3, \{x_1, x_3\})}{p(x_1, \{x_1, x_3\})} \times \frac{p(x_2, \{x_2, x_3\})}{p(x_3, \{x_2, x_3\})} = \frac{v(x_3)}{v(x_1)} \times \frac{v(x_2)}{v(x_3)} = 2 > 1.$$

More generally, Weak consistency is necessary and sufficient for the first stage to be *weakly* based on the second. (The proof extends the arguments used in Theorems 1A and 3.)

Given the discussion in Section 4(a), one can then impose the desired structure on the first stage by translating the relevant rationality conditions into the stochastic setting. For extended Luce rules $(\succ, v)$ where the first-stage binary relation $\succ$ is a semi-order that is *weakly* based on the second-stage utility function v, for instance, it suffices to combine Weak consistency with the Luce Choice axiom and the stochastic versions of the Ferrers and Semi-transitivity properties.

Translating the requirement that v refines the binary relation $\succ^{**}$ from Section 3(b) imposes a tighter connection between the two stages of the representation. In fact, this *Moderate consistency* axiom is necessary and sufficient for the first stage to be based on the second. As above, this result is a simple extension of Theorems 1A and 3.[25] For parsimony, I relegate the details to Theorem 1* of the Online Appendix.

5. Conclusion

To conclude, I briefly discuss some implications of my results for related work.

(a) Extended Luce rules

Theorems 1 and 2 are formally related to the recent literature on extended Luce rules.

The applications of Theorem 2 discussed in Section 4(a) cover many of the results from this literature (see e.g., Theorems B-D in Appendix A). However, Theorem 2 cannot help to axiomatize the entire class of extended Luce rules—since this result requires the first stage to be rationalized by a (transitive and irreflexive) binary relation. Two recent papers (Ahumada

[25] An even stronger condition obtains by translating the requirement that v refines the binary relation $\succ^*$ defined in Section 3(b). From Theorem 3, this *Strong consistency* axiom is equivalent to Luce transitivity when the Luce Choice axiom holds.

and Ulku; Echenique and Saito) characterize the entire class of extended Luce rules in terms of the Strong Product rule mentioned in Section 3(a) (see Theorem A in Appendix A). In part, the value of Theorem 2 is to show that one can replace this requirement with the more intuitive Luce Choice axiom in applications where the first stage is rational.

In turn, Theorem 1 generalizes the main result of Echenique and Saito (see Theorem E in Appendix A), who characterize a special class of threshold Luce rules where the interval around $v(x)$ takes the form $B(x) := [v(x)/\sqrt{1+\alpha}, v(x) \times \sqrt{1+\alpha}]$ for some constant $\alpha \in \mathbb{R}_+$.[26] Their result relies on the Strong Product rule and a "calibration" condition (*Path monotonicity* in Appendix A) that strengthens Weak consistency.

(b) Zero-frequency choice

Discrete choice data almost always includes options that are never chosen. The phenomenon has only become more prevalent with the advent of big data. Besides the work on extended Luce rules, the threshold model is also related to some other recent work that address "zeroes" in discrete choice.

Matejka and McKay (2015) propose a theoretical model that accommodates zero-probability choice. In their model, the decision-maker has a prior over the utility of alternatives, which she can update at a cost proportional to the reduction in entropy between her prior and posterior beliefs. A different notion of attention costs motivates the threshold Luce model. Instead of incurring *information costs* of refining her beliefs, the decision-maker incurs a *carrying cost* for each alternative she considers. Based on her prior, she ignores alternatives that are sufficiently unlikely to have the highest realized utility.

Gandhi et al. (2017) address some econometric issues associated with zero-frequency choice. As they explain, the popular estimation procedure proposed by Berry et al. (1995) does not work when the data contains zeroes. Typically, the literature has "ignored" the problem by lumping zero-frequency choices with the outside option; or "corrected" it by replacing zeroes with small choice probabilities. Both tricks introduce bias into the estimation. Gandhi et al. suggest an approach that avoids bias by using the data to estimate

[26] Clearly, a binary relation $\succ$ with such a representation is *strongly* based on the utility function v.

upper and lower bounds for the true choice probabilities. Conceptually, this is similar to identification in the threshold Luce model, which relies on choice data to determine ranges of "true" utilities for each alternative.

(c) Luce's conjecture

Luce (1959, pp. 27 and 112) conjectured that Luce transitivity is necessary to represent a random choice function satisfying the Luce Choice axiom by a *one-dimensional utility scale*. While he never explained the phrase in italics, the current paper suggests a number of natural interpretations.

One such interpretation might require the first stage to be *strongly* based on the second while another might only require the first stage to be *weakly* based on the second. Although the conjecture is false under the second interpretation (by Example 1), it is true under the first (by Theorem 1A). Considering that extended Luce representations were not known at the time, this is remarkable. Even more remarkable, the first stage of the required representation specifies a type of binary relation which Luce (1956) himself proposed only a few years before making the conjecture.

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Appendix A – Background

(a) Representation of binary relations

Definition. *The following properties of a binary relation $\succ$ on X apply to all alternatives $x, y, z, w \in X$:*

P1 (*irreflexivity*). $x \sim x$.
P2 (*transitivity*). $[x \succ y$ *and* $y \succ z] \implies x \succ z$.
P3 (*Ferrers property*). $[x \succ w$ *and* $y \succ z] \implies [x \succ z$ *or* $y \succ w]$.
P4 (*semi-transitivity*). $[x \succ y$ *and* $y \succ z] \implies [x \succ w$ *or* $w \succ z]$.
P5 (*negative transitivity*). $x \succ z \implies [x \succ y$ *or* $y \succ z]$.

*A binary relation $\succ$ is: a **pre-order** if it satisfies P1 and P2; an **interval order** if it satisfies P1 and P3; a **semi-order** if it satisfies P1, P3 and P4; and a **weak order** if it satisfies P1 and P5.*

Each of P3-P5 implies P2; and P5 implies both P3 and P4. So, a weak order is a semi-order which, in turn, is an interval order. What is more, $\succ$ is a weak order if and only if the indifference relation $\sim$ is transitive.

The following conditions impose some additional structure on the intervals used to represent a binary relation. As in the main text, let B denote a utility correspondence and v a utility function.

Definition. *The following regularity conditions for a pair (B, v) apply to all alternatives $x, y, z \in X$:*

R1 (*weak co-monotonicity*). $B(x) > B(y) \implies v(x) > v(y)$.
R2 (*non-nestedness*). $B(x) \setminus B(y) \neq \emptyset \implies B(y) \setminus B(x) \neq \emptyset$.
R3 (*clustering*). $[B(x) \cap B(z) \neq \emptyset$ *and* $B(y) \cap B(z) \neq \emptyset] \implies B(x) \cap B(y) \neq \emptyset$.

A pair (B, v) *represents* a binary relation $\succ$ if: (i) B represents $\succ$; and (i) $v(x) \in B(X)$ for all $x \in X$. The next remark (which adapts the results of Fishburn, 1985) ensures such a representation for the pre-order classes defined above.

Remark 3. *Suppose $\succ$ is an irreflexive relation on X. Then:*

(a) *$\succ$ is an interval order if and only if it can be represented by a pair (B, v).*
(b) *$\succ$ is a semi-order if and only if it can be represented by a pair (B, v) that satisfies R2.*
(c) *$\succ$ is a weak order if and only if it can be represented by a pair (B, v) that satisfies R2 and R3.*

In each of the statements (a)-(c), the pair (B, v) can be strengthened to a pair (B, v) that satisfies R1.

Proof. (**a**) Fishburn's Theorems 6 and 8 establish that $\succ$ is an interval order if and only if it can be represented by a utility correspondence B. To complete the result, define $v(x) := [\inf B(x) + \sup B(x)]/2$. Then, it is clear that $v(x) \in B(x)$ for all $x \in X$. What is more, (B, v) satisfies R1. ❙

(**b**) Similarly, Fishburn's Theorems 7 and 8 show that $\succ$ is a semi-order if and only if it can be represented by a utility correspondence B that satisfies R2. To complete the result, simply define v as in (a). ❙

(**c**) Finally, part (b) establishes that $\succ$ is a weak order only if it can be represented by a pair (B, v) that satisfies R1 and R2. Clearly, R3 is also necessary for such a representation. Conversely, if the representation (B, v) satisfies R2 and R3, it follows directly that $\succ$ is a weak order. ■

(b) Extended Luce rules

For the reader's convenience, I state the main results from the related literature on extended Luce rules.[27] Before doing so, I first state some additional axioms mentioned in the text.

[27] To simplify the statement of these results, I omit the (obvious) uniqueness properties of the representation.

(i) Axioms

Strong stochastic transitivity (SST). *For all distinct alternatives $x, y, z \in X$:*

$$p(x, \{x, y\}), p(y, \{y, z\}) \geq 1/2 \implies p(x, \{x, z\}) \geq \max\{p(x, \{x, y\}), p(y, \{y, z\})\}.$$

Moderate stochastic transitivity (MST). *For all distinct alternatives $x, y, z \in X$:*

$$p(x, \{x, y\}), p(y, \{y, z\}) \geq 1/2 \implies p(x, \{x, z\}) \geq \min\{p(x, \{x, y\}), p(y, \{y, z\})\}.$$

Weak stochastic transitivity (WST). *For all distinct alternatives $x, y, z \in X$:*

$$p(x, \{x, y\}), p(y, \{y, z\}) \geq 1/2 \implies p(x, \{x, z\}) \geq 1/2.$$

Product rule (PR).[28] *For all distinct alternatives $x, y, z \in X$ such that $x \frown y \frown z \frown x$:*

$$\frac{p(x, \{x, y\})}{p(y, \{x, y\})} \times \frac{p(y, \{y, z\})}{p(z, \{y, z\})} \times \frac{p(z, \{x, z\})}{p(x, \{x, z\})} = 1.$$

Finitely connected domain (FCD).[29] *For every pair of alternatives $x, y \in X$ such that $p(y, \{x, y\}) = 0$, there is a finite sequence of imperfect discriminations $y = z_1 \frown ... \frown z_n = x$ such that $\max\{p(z_i, \{z_i, z_{i+1}\})\}_{i=1}^{n} \leq 1/2$.*

For every menu $S \in \mathcal{F}$, let $\overset{S}{\frown}$ denote the binary relation defined by $x \overset{S}{\frown} y$ if $p(x, S), p(y, S) > 0$. Then, a (generalized) *sequence of imperfect discriminations* is a pair $(z_i, S_i)_{i=1}^{n}$ consisting of: (i) a sequence $z_1, ..., z_n \in X$ and (ii) a sequence $S_1, ..., S_{n-1} \in \mathcal{F}$ such that $z_1 \overset{S_1}{\frown} z_2 ... z_{n-1} \overset{S_{n-1}}{\frown} z_n$. The pair $(z_i, S_i)_{i=1}^{n}$ is a *cycle of imperfect discriminations* if $z_1 = z_n$.

Strong Product rule (SPR).[30] *For every cycle of imperfect discriminations $(z_i, S_i)_{i=1}^{n}$:*

$$\prod_{i=1}^{n-1} \frac{p(z_i, S_i)}{p(z_{i+1}, S_i)} = 1. \qquad (*)$$

(ii) Results

Luce's Theorem 4. *Suppose p satisfies LCA for $\{T \in \mathcal{F} : |T| \leq 3\}$,*[31] *as well as LT and FCD. Then, there is a strictly positive utility function v such that for every menu $S \in \mathcal{F}$ and alternative $x \in S$*

$$p(x, S) = \frac{v(x)}{\sum_{y \in S} v(y)}$$

provided that (i) $p(y, \{y, z\}) \neq 0, 1$ for all $y, z \in S$ and (ii) p satisfies LCA for the menu S.

Theorem A (Ahumada and Ulku; Echenique and Saito). *Suppose X is finite. Then, p satisfies SPR if and only if it can be represented by an extended Luce rule (Γ, v).*

Maximization (Max).[32] *For all $x \in S$: $p(x, S) > 0 \iff p(x, \{x, y\}) > 0$ for all $y \in S$.*
Stochastic attention (SA).[33] *For all $x \in S$: $p(x, S) = 0 \implies p(y, S) = p(y, S \setminus \{x\})$ for all $y \in S$.*

Theorem B (Ahumada and Ulku; Echenique and Saito).[34] *Suppose X is finite. Then, p satisfies SPR, Max*

[28] Luce and Suppes (1965, p. 341) credit Luce (1959, Theorem 2) as the original source of this rule.

[29] In fact, Luce's condition (Definition 1 on p. 25) imposes the stronger hypothesis that $p(y, \{x, y\}) < 1/2$.

[30] This axiom is called *Extended Cyclical Independence* by Ahumada and Ulku; and *Cyclical Independence* by Echenique and Saito. The name used here instead evokes the connection to the Product rule.

[31] A random choice function p satisfies LCA for a collection $\mathcal{G} \subseteq \mathcal{F}$ if it satisfies the axiom for each menu $T \in \mathcal{G}$.

[32] This axiom (P1 in Fishburn, 1978) combines Echenique and Saito's Axioms 2 (*Weak Regularity*) and 4 (*Probabilistic β*).

[33] This is the stochastic analog of Masatlioglu et al.'s (2012) attention filter property (or Axiom 2 in Fishburn, 1975).

[34] Ahumada and Ulku replace Maximization with their Conditions 1-2.

and SA if and only if it can be represented by an extended Luce rule $(\succ, v)$ where $\succ$ is a pre-order.

Theorem C (Lindberg; Cerreia-Vioglio et al.; Dogan and Yildiz).[35] *Suppose X is finite. Then, p satisfies LCA_+ if and only if it can be represented by an extended Luce rule $(\succ, v)$ where $\succ$ is a weak order.*

McCausland considers the case where $X \subseteq \mathbb{R}^n_+$. In that setting, the *(vector) dominance* order $\gneq$ is defined by $x \gneq y$ if $x_i \geq y_i$ for all $i \in \{1, ..., n\}$ and $x \neq y$. Vector dominance suggests a natural concept of monotonicity:

Dominance monotonicity (DM). *For all $x, y \in X$: $x \gneq y \iff p(y, \{x, y\}) = 0$.*

Theorem D (McCausland).[36,37] *Suppose $X \subseteq \mathbb{R}^n_+$. Then, p satisfies LCA, MST and DM if and only if it can be represented by an extended Luce rule $(\geq, v)$ such that, for all $x, y \in X$, $x \geq y \implies v(x) \geq v(y)$.*

A binary relation $\succ$ is a *multiplicative semi-order* based on a utility function v if it can be represented by utility correspondence B where, for all $x \in X$, $B(x) := [v(x)/\sqrt{1+\alpha}, v(x) \times \sqrt{1+\alpha}]$ for some constant $\alpha \in \mathbb{R}_+$.

Path monotonicity (PM). *For all sequences of imperfect discriminations $z_1 \frown ... \frown z_n$ and $z'_1 \frown ... \frown z'_m$:*

$$[p(z_n, \{z_1, z_n\}) = 0 \text{ and } p(z'_m, \{z'_1, z'_m\}) \neq 0, 1] \implies \prod_{i=1}^{n-1} \frac{p(z_i, \{z_i, z_{i+1}\})}{p(z_{i+1}, \{z_i, z_{i+1}\})} > \prod_{i=1}^{m-1} \frac{p(z'_i, \{z'_i, z'_{i+1}\})}{p(z'_{i+1}, \{z'_i, z'_{i+1}\})}.$$

Theorem E (Echenique and Saito). *Suppose X is finite. Then, p satisfies SPR, Max, SA and PM if and only if it can be represented by an extended Luce rule $(\succ, v)$ where $\succ$ is a multiplicative semi-order based on v. In the representation, v is unique up to a non-negative scalar if and only if p satisfies Linked domain.*

Appendix B – Proof of Theorem 3

Note: For the reader's convenience, I have underlined the key assumptions used to prove each step of Lemma 1. This makes it easier to see how the arguments extend directly to establish Theorem 3^* of the Online Appendix.

Lemma 1. *If $\succ$ is irreflexive and $\succ^*$ satisfies v-consistency, then $\succ$ is a semi-order based on v.*

Proof. Suppose $\succ$ is irreflexive and $\succ^*$ satisfies v-consistency.

Step 1. $\succ$ is a semi-order on X.

First, note that $\succ$ is transitive. If $x \succ y \succ z$, then $v(x) > v(y) > v(z)$ and, consequently, $x \succsim z$ by v-consistency. If $x \sim z$, then $y \succ z \sim x \succ y$. So, $y \succ^* z \succ^* y$. Since $\succ^*$ satisfies v-consistency, $v(y) > v(z) > v(y)$.

For P3, let $x \succ y$ and $z \succ w$. By way of contradiction, suppose $w \succsim x$ and $y \succsim z$. Since $\succ$ is transitive, $y \sim z \succ w \sim x \succ y$. So, $y \succ^* w \succ^* y$. Since $\succ^*$ satisfies v-consistency, $v(y) > v(w) > v(y)$.

For P4, let $x \succ y \succ z$. By way of contradiction, suppose $z \succsim w \succsim x$. Since $\succ$ is transitive, $y \succ z \sim w \sim x \succ y$. So, $y \succ^* w \succ^* y$. Since $\succ^*$ satisfies v-consistency, $v(y) > v(w) > v(y)$. ❙

Following Fishburn (1985, pp. 21-23), first define the relations $\succ^-$ and $\succ^+$ on X by

$$x \succ^- y \quad \text{if} \quad x \succ z \sim y \text{ for some } z \in X; \text{ and}$$
$$x \succ^+ y \quad \text{if} \quad x \sim z \succ y \text{ for some } z \in X.$$

[35] Dogan and Yildiz replace LCA_+ with an equivalent condition called *rejection supermodularity*.
[36] McCausland decomposes Dominance monotonicity into two separate conditions (his Assumptions 2 and 4).
[37] McCausland also imposes an additional condition (his Assumption 3) which ensures that v is log-concave.

Step 2. $\succ^-$ and $\succ^+$ are weak orders on X.

Since $\succ$ is an interval order on X (by Step 1), this follows from Fishburn's Theorem 2 (1985, p. 22). ▮

Let $X^- := \{x^- : x \in X\}$ and $X^+ := \{x^+ : x \in X\}$ and consider the relation $\succ_\pm$ on $X^\pm := X^- \cup X^+$ defined by

$$\begin{aligned} x^- \succ_\pm y^- \quad & \text{if} \quad x \succ^- y \text{ or } [x \sim^- y \text{ and } x \succ^+ y]; \\ x^+ \succ_\pm y^+ \quad & \text{if} \quad x \succ^+ y \text{ or } [x \sim^+ y \text{ and } x \succ^- y]; \\ x^- \succ_\pm y^+ \quad & \text{if} \quad x \succ y; \text{ and} \\ x^+ \succ_\pm y^- \quad & \text{if} \quad x \succsim y. \end{aligned}$$

Step 3.[38] *$\succ_\pm$ is a weak order on $X^\pm$. What is more, $x^\pm \sim_\pm y^\pm \Leftrightarrow [x \sim^- y,\ x \sim^+ y$ and $\{x^\pm, y^\pm\} \subseteq X^-, X^+]$.*

To see that $\succ_\pm$ is irreflexive, suppose otherwise. In particular, let $x^- \succ_\pm x^-$. (The case where $x^+ \succ_\pm x^+$ is similar.) Since $x^- \succ_\pm x^-$, $x \succ^- x$ or $x \succ^+ x$. Both cases contradict Step 1 (i.e., $\succ$ is an interval order on X).

To see that $\succ_\pm$ is negatively transitive, suppose $x^\pm \succ_\pm z^\pm$ and consider some alternative $y^\pm \in X$. By way of contradiction, suppose $y^\pm \succsim_\pm x^\pm$ and $z^\pm \succsim_\pm y^\pm$. There are eight different cases to consider:

1. $(x^\pm, y^\pm, z^\pm) = (x^+, y^+, z^+)$. By definition, $x^+ \succ_\pm z^+$, $y^+ \succsim_\pm x^+$ and $z^+ \succsim_\pm y^+$. So, $x \succsim^+ z \succsim^+ y \succsim^+ x$. Since $\succ^+$ is a weak order by Step 2, $x \sim^+ z \sim^+ y \sim^+ x$. By definition, $x^+ \succ_\pm z^+$ and $x \sim^+ z$ imply $x \succ^- z$. So, $x \succ w \sim z$ for some $w \in X$. If $w \succsim y$, then $x \succ w \succsim y$. Since $\succ$ is transitive, $x \succ^- y$. Since $y \sim^+ x$, this contradicts $y^+ \succsim_\pm x^+$. Otherwise, $y \succ w$. Then, $y \succ w \sim z$. So, $y \succ^- z$ by definition. Since $z \sim^+ y$, this contradicts $z^+ \succsim_\pm y^+$.

2. $(x^\pm, y^\pm, z^\pm) = (x^-, y^-, z^-)$. By definition, $x^- \succ_\pm z^-$, $y^- \succsim_\pm x^-$ and $z^- \succsim_\pm y^-$. This case is similar to case 1.

3. $(x^\pm, y^\pm, z^\pm) = (x^-, y^-, z^+)$. By definition, $x \succ z$, $y^- \succsim_\pm x^-$ and $z \succsim y$. Since $\succ$ is transitive, $x \succ z \succsim y$ implies $x \succ^- y$. In turn, $x \succ^- y$ implies $x^- \succ_\pm y^-$. But, this contradicts $y^- \succsim_\pm x^-$.

4. $(x^\pm, y^\pm, z^\pm) = (x^-, y^+, z^-)$. By definition, $x^- \succ_\pm z^-$, $y \succsim x$ and $z \succ y$. Since $\succ$ is transitive, $z \succ y \succsim x$ implies $z \succ^- x$. In turn, $x^- \succ_\pm z^-$ implies $x \succsim^- z$. But, this contradicts $z \succ^- x$ by Step 2.

5. $(x^\pm, y^\pm, z^\pm) = (x^+, y^-, z^-)$. By definition, $x \succsim z$, $y \succ x$ and $z^- \succsim_\pm y^-$. This case is similar to case 3.

6. $(x^\pm, y^\pm, z^\pm) = (x^-, y^+, z^+)$. By definition, $x \succ z$, $y \succsim x$ and $z^+ \succsim_\pm y^+$. Since $\succ$ is transitive, $y \succsim x \succ z$ implies $y \succ^+ z$. In turn, $y \succ^+ z$ implies $y^+ \succ_\pm z^+$. But, this contradicts $z^+ \succsim_\pm y^+$.

7. $(x^\pm, y^\pm, z^\pm) = (x^+, y^-, z^+)$. By definition, $x^+ \succ_\pm z^+$, $y \succ x$ and $z \succsim y$. Since $\succ$ is transitive, $z \succsim y \succ x$ implies $z \succ^+ x$. In turn, $x^+ \succ_\pm z^+$ implies $x \succsim^+ z$. But, this contradicts $z \succ^+ x$ by Step 2.

8. $(x^\pm, y^\pm, z^\pm) = (x^+, y^+, z^-)$. By definition, $x \succsim z$, $y^+ \succsim_\pm x^+$, and $z \succ y$. This case is similar to case 6.

By definition of $\succ_\pm$, $x^\pm \sim_\pm y^\pm$ if and only if $[x \sim^- y,\ x \sim^+ y$ and $\{x^\pm, y^\pm\} \subseteq X^-, X^+]$. ▮

Next, let $\succ^v$ denote the binary relation on $X^* := X^\pm \cup X$ induced by v:

$$\begin{aligned} x^* \succ^v y^* \quad & \text{if} \quad v(x) > v(y) \text{ and } [\{x^*, y^*\} \subseteq X^+, X \text{ or } X^-]; \\ x^\pm \succ^v y \quad & \text{if} \quad x^\pm = y^+; \text{ and} \\ x \succ^v y^\pm \quad & \text{if} \quad y^\pm = x^-. \end{aligned}$$

In turn, extend $\succ_\pm$ from $X^\pm$ to $X^* := X^\pm \cup X$ by letting $x^* \sim_\pm y^*$ if $\{x^*, y^*\} \cap X \neq \emptyset$.

Finally, let $\rhd^v_\pm$ denote the binary relation on X^* defined by the lexicographic composition of $\succ_\pm$ with $\succ^v$:

$$x^\pm \rhd^v_\pm y^\pm \quad \text{if} \quad x^* \succ_\pm y^* \text{ or } [x^* \sim_\pm y^* \text{ and } x^* \succ^v y^*].$$

Step 4. $\rhd^v_\pm$ is acyclic.

[38] The proof follows the same kind of reasoning as the proof of Fishburn's Theorem 3 (pp. 23-24).

By way of contradiction, suppose $\rhd^v_\pm$ contains a cycle C on $Y \subseteq X$. Since $\succ_\pm$ is a weak order on $X^\pm$ and $\succ^v$ is a weak order on X, C contains a sub-cycle C' on $Y' \subseteq Y$ that "alternates" two-by-two between $Y^\pm$ and Y as follows:

$$...x^+ \rhd^v_\pm x \succ^v y \rhd^v_\pm y^- \succ_\pm z^+ \rhd^v_\pm z...$$

Since $\rhd^v_\pm$ is a weak order on $Y^\pm$, $r^+ \succ^v_\pm s^-$ cannot hold for all r^+ and s^- in C' such that $r^+ \rhd^v_\pm r \succ^v s \rhd^v_\pm s^-$. Without loss of generality, suppose $y^- \succsim^v_\pm x^+$. Since $y^- \not\sim_\pm x^+$ by Step 3, $y^- \succ_\pm x^+$. So, $y \succ x$ by definition. Since $\succ$ satisfies v-consistency, $v(y) > v(x)$. Then, $y \succ^v x$ by definition. But, this contradicts $x \succ^v y$. ▎

By Step 4, the transitive closure $tc(\rhd^v_\pm)$ of $\rhd^v_\pm$ is a pre-order. So, $tc(\rhd^v_\pm)$ admits a weak order extension $\unrhd^v_\pm$ by the Szpilrajn Theorem. Since X is countable, $\unrhd^v_\pm$ also admits a utility representation $u : X^\pm \cup X \to \mathbb{R}$ such that $u(x) = v(x)$ for all $x \in X$. Finally, define $B : X \to \mathcal{I}_{++}$ such that $B(x) := [u(x^-), u(x^+)]$ for all $x \in X$.

Step 5. The pair (B, v) represents $\succ$.

For all $x \in X$, $v(x) \in B(x)$. For all $x, y \in X$: $x \succ y \iff x^- \succ^v_\pm y^+ \iff u(x^-) > u(y^+) \iff B(x) \gg B(y)$. ▎

Step 6. The pair (B, v) satisfies R1.

Let $B(x) > B(y)$. By way of contradiction, suppose $v(y) \geq v(x)$. By definition of $B(x) > B(y)$, $u(x^+) \geq u(y^+)$ and $u(x^-) \geq u(y^-)$ (with at least one of the equalities strict). Then, since $v(y) \geq v(x)$, $x^+ \succsim_\pm y^+$ and $x^- \succsim_\pm y^-$ (with at least one of the preferences strict). So, $x^+ \succ_\pm y^+$ and $x^- \succ_\pm y^-$ by Step 3. Since (the asymmetric part of) $\succ^*$ satisfies v-consistency, $v(x) > v(y)$. But, this contradicts $v(y) \geq v(x)$. ▎

Step 7. The pair (B, v) satisfies R2.

Suppose $u(x^+) > u(y^+)$. (The reasoning for $u(x^-) > u(y^-) \Rightarrow u(x^+) > u(y^+)$ is similar.) There are two cases:

1. If $x^+ \succ_\pm y^+$, then: (i) $x \succ^+ y$; or (ii) $x \sim^+ y$ and $x \succ^- y$. In sub-case (ii), $x^- \succ^v_\pm y^-$ holds by definition. In sub-case (i), suppose $y \succ^- x$. Then, $x \sim z \succ y \succ w \sim x$ for some $z, w \in X$. But, this contradicts the fact that $\succ$ is a semi-order. So, $x \succsim^- y$. Since $x \succ^+ y$ as well, $x^- \succ^v_\pm y^-$.

2. If $x^+ \sim_\pm y^+$ and $v(x) > v(y)$, then $x^- \sim_\pm y^-$ by Step 3. Since $v(x) > v(y)$ as well, $x^- \succ^v_\pm y^-$.

So, $x^- \succ^v_\pm y^-$ in both cases. Then, $u(x^-) > u(y^-)$ as required. ■

Proof of Theorem 3. Suppose $\succ$ is irreflexive.

(**i**) ⇒ (**ii**) Suppose $\succ$ is a semi-order based on the utility function v. Given Remark 3 in Appendix A, there exists a (B, v) representation of $\succ$ that satisfies R1 and R2. Let $x \succ y$ and $v(y) \geq v(z)$. (The case where $v(x) \geq v(y)$ and $y \succ z$ is similar.) Then, since (B, v) satisfies R2, $B(y) > B(z)$, $B(z) > B(y)$ or $B(y) = B(z)$. Since $v(y) \geq v(z)$, R1 then implies $B(y) > B(z)$ or $B(y) = B(z)$. Since (B, v) represents $\succ$, $x \succ y$ implies $B(x) \gg B(y)$. So, $B(x) \gg B(y) > B(z)$ or $B(x) \gg B(y) = B(z)$. In either case, $B(x) \gg B(z)$ so that $x \succ z$. ▎

(**ii**) ⇒ (**iii**) Suppose $\succ$ satisfies v-transitivity. To see that $\succ$ satisfies v-consistency, let $x \succ y$. By way of contradiction, suppose $v(y) \geq v(x)$. Then, $x \succ x$ by v-transitivity, which is a contradiction. So, $v(x) > v(y)$.

To see that $\succ$ satisfies v-transitive indifference, let $x \sim y \sim z$ and $v(y) \geq \max\{v(x), v(z)\}$. (The case where $v(y) \leq \min\{v(x), v(z)\}$ is similar.) Towards a contradiction, suppose $x \not\sim z$. If $z \succ x$, then $y \succ x$ by v-transitivity, which contradicts $x \sim y$. Otherwise, $x \succ z$. By the same kind of reasoning, this leads to the contradiction $y \succ z$.

Finally, to see that $\succ$ is transitive, let $x \succ y \succ z$. Then, $v(y) > v(z)$ by v-consistency. So, $x \succ z$ by v-transitivity. ▎

(**iii**) ⇒ (**iv**) Suppose $\succ$ (is a pre-order that) satisfies v-consistency and v-transitive indifference. To see that $\succ^*$ satisfies v-consistency, let $x \succ z \sim y$ for some $z \in X$. (The case where $x \sim z \succ y$ for some $z \in X$ is similar.) By way of contradiction, suppose $v(y) \geq v(x)$. By v-consistency, $x \succ z$ implies $v(x) > v(z)$ so that $v(y) \geq v(x) > v(z)$. Since $x \sim y \sim z$, v-transitive indifference then implies $x \sim z$. But, this contradicts $x \succ z$. ▎

(**iv**) ⇒ (**i**) See Lemma 1. ■

Appendix C – Proof of Theorem 2

Note: Given a sequence of imperfect discriminations $(z_i, S_i)_{i=1}^n$, let $\mathbb{P}(z_i, S_i)_{i=1}^n := \prod_{i=1}^{n-1} \frac{p(z_i,S_i)}{p(z_{i+1},S_i)}$.

(**i**) $\Rightarrow$ (**ii**) Suppose p satisfies LCA and 4-PR.

Note: The proofs of Steps 1 to 4 below do not rely on 4-PR.

Step 1.[39] *$\succ_B$ is a pre-order where $\succ_B$ is defined by $x \succ_B y$ if $p(x, \{x, y\}) = 0$ (as in ($\succ_B$-Id) from the text).*

To show that $\succ_B$ is transitive, suppose $x \succ_B y \succ_B z$. By definition, $p(y, \{x, y\}), p(z, \{y, z\}) = 0$. So, $p(x, \{x, y, z\}) = p(x, \{x, z\}), p(x, \{x, y\})$ by LCA(ii). Thus, $1 = p(x, \{x, y\}) = p(x, \{x, y, z\}) = p(x, \{x, z\})$. So, $x \succ_B z$. Since $\succ_B$ is irreflexive by definition, the binary relation $\succ_B$ is a pre-order. ∎

Step 2.[40] *For all $S \in \mathcal{F}$, $\Gamma(S) = \max_{\succ_B} S$ (where Γ is defined as in Theorem 2).*

If $|S| = 1$, then there is nothing to prove. So, suppose $|S| > 1$. Starting from $S_0 := S$, recursively define $S_i := S_{i-1} \setminus \{x\}$ where $x \in S_{i-1}$ is such that $y \succ_B x$ for some $y \in S_{i-1}$.

To see that this iterative elimination process must stop at $\widehat{S} := \max_{\succ_B} S$ regardless of the order of elimination, suppose that it stops at $T \supset \widehat{S}$. Then, there is some $x \in T$ and $y_1 \in S \setminus T$ such that $y_1 \succ_B x$ and $z \not\succ_B x$ for all $z \in T$. So, y_1 must have been eliminated at a prior stage by some $y_2 \in S$ such that $y_2 \succ_B y_1$. Since $y_1 \succ_B x$, the transitivity of $\succ_B$ (see Step 1) implies $y_2 \succ_B x$. Since the process stops at T, $y_2 \in S \setminus T$; and one can then repeat the same argument with y_2 in place of y_1. Since S is finite, this line of reasoning implies $T = \{x\}$ and $y \succ_B x$ for all $y \in S \setminus \{x\}$. This is a contradiction: since $|S| > 1$ and $T = \{x\}$, there must be some stage where x eliminates an alternative $y \in S \setminus \{x\}$.

Since the elimination process stops at $\widehat{S}$, (the second line of) LCA implies $p(x, \widehat{S}) = p(x, S)$ for all $x \in \widehat{S}$. To complete the proof, suppose $p(x, \widehat{S}) = 0$ for some $x \in \widehat{S}$. Since $p(\widehat{S}, \widehat{S}) = 1$, $\widehat{S} \neq \{x\}$. Then, by (the first line of) LCA, $p(x, \widehat{S}) = p(x, \{x, y\}) \times p(\{x, y\}, \widehat{S})$ for all $y \in \widehat{S} \setminus \{x\}$. Since $p(x, \widehat{S}) = 0$ and $p(x, \{x, y\}) > 0$, $p(\{x, y\}, \widehat{S}) = 0$. So $1 = p(\widehat{S}, \widehat{S}) \leq \sum_{y \in \widehat{S} \setminus \{x\}} p(\{x, y\}, \widehat{S}) = 0$, which is a contradiction. ∎

Step 3.[41] *If $x, y \in \Gamma(S)$, then $p(x, \{x, y\}) \times p(y, S) = p(y, \{x, y\}) \times p(x, S)$.*

From Step 2, $p(z, S) = p(z, \Gamma(S)) > 0$ for $z \in \{x, y\}$. By (the first line of) LCA, $p(z, S) = p(z, \{x, y\}) \times p(\{x, y\}, \Gamma(S))$ for $z \in \{x, y\}$. Since $p(\{x, y\}, \Gamma(S)) > 0$, cross-multiplying the identities for $p(x, S)$ and $p(y, S)$ gives the result. ∎

Step 4.[42] *p satisfies the Product rule (PR) (which is defined in Appendix A above).*

Let $LHS := \frac{p(x,\{x,y\})}{p(y,\{x,y\})} \times \frac{p(y,\{y,z\})}{p(z,\{y,z\})} \times \frac{p(z,\{x,z\})}{p(x,\{x,z\})}$ and $\Pi := p(x, \{x, y, z\}) \times p(y, \{x, y, z\}) \times p(z, \{x, y, z\})$. Then:

$$LHS \times \Pi = \frac{p(x, \{x, y\}) \times p(y, \{x, y, z\})}{p(y, \{x, y\})} \times \frac{p(y, \{y, z\}) \times p(z, \{x, y, z\})}{p(z, \{y, z\})} \times \frac{p(z, \{x, z\}) \times p(x, \{x, y, z\})}{p(x, \{x, z\})} = \Pi.$$

by repeated application of the identity in Step 3. Since $\Pi \neq 0$ by Step 2, $LHS = 1$. ∎

Step 5.[43] *p satisfies the Strong Product rule (SPR) (which is defined in Appendix A above).*

Fix a cycle of imperfect discriminations $(z_i, S_i)_{i=1}^n$. By strong induction on n, I show that

$$\mathbb{P}(z_i, S_i)_{i=1}^n = 1. \qquad (**)$$

The base cases $n = 3, 4, 5$ are straightforward:

[39] The argument establishing transitivity is equivalent to the argument in Luce's Lemma 4 (p. 10).

[40] The last paragraph of the proof follows the argument in Luce's Lemma 1 (p. 6).

[41] This result generalizes Luce's Lemma 3 (p. 7); but the proof relies on the same type of argument.

[42] This result is Luce's Theorem 2 (p. 16).

[43] This generalizes the argument for the same result in the strict utility model (see fn. 9 of Luce and Suppes, 1965).

- $n = 3$: $\mathbb{P}(z_i, S_i)_{i=1}^3 = \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^3 = 1$ where the first equality holds by Step 3; and the second equality by the fact that the factors of $\mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^3$ are reciprocals.

- $n = 4$: $\mathbb{P}(z_i, S_i)_{i=1}^4 = \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^4 = 1$ where the equalities hold by Steps 3 and 4, respectively.

- $n = 5$: $\mathbb{P}(z_i, S_i)_{i=1}^5 = \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^5 = 1$ where the equalities hold by Step 3 and 4-PR, respectively.

For the induction step, suppose $(**)$ holds for $n \le m$ and let $n = m + 1$.

By Step 3, $(z_i, \{z_i, z_{i+1}\})_{i=1}^{m+1}$ is a cycle of imperfect discriminations. If $p(z_j, \{z_j, z_k\}) \in (0, 1)$ for some $j \in \{1, ..., , m-2\}$ and $k \in \{j+2, ..., m\}$, then $z_j \frown z_k$. One can use this link to divide $(z_i, \{z_i, z_{i+1}\})_{i=1}^{m+1}$ into two new cycles of imperfect discriminations, namely:

$$z_j \frown ... \frown z_k \frown z_j \quad \text{and} \quad z_k \frown ... \frown z_m \frown z_1 \frown ... \frown z_j \frown z_k$$

From the induction hypothesis,

$$\mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=j}^{k-1} \times \frac{p(z_k, \{z_j, z_k\})}{p(z_j, \{z_j, z_k\})} = 1 \quad \text{and} \quad \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=k}^{m+1} \times \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^{j-1} \times \frac{p(z_j, \{z_j, z_k\})}{p(z_k, \{z_j, z_k\})} = 1.$$

Using Step 3 and these two identities,

$$\mathbb{P}(z_i, S_i)_{i=1}^{m+1} = \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^{m+1} = \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=j}^{k-1} \times \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=k}^{m+1} \times \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^{j-1} = 1.$$

To complete the proof, I show that $p(z_j, \{z_j, z_k\}) \in (0, 1)$ for some $j \in \{1, ..., , m-2\}$ and $k \in \{j+2, ..., m\}$. By way of contradiction, suppose $p(z_j, \{z_j, z_k\}) \in \{0, 1\}$ for all $j \in \{1, ..., , m-2\}$ and $k \in \{j+2, ..., m\}$. (Since $m + 1 > 5$, the alternatives z_1, z_2, z_3, z_4 and z_m are all distinct.)

Suppose $p(z_1, \{z_1, z_3\}) = 1$. (The case where $p(z_1, \{z_1, z_3\}) = 0$ is similar.) Then, $z_1 \succ_B z_3$. If $p(z_1, \{z_1, z_4\}) = 0$, then $z_4 \succ_B z_1$ so that $z_4 \succ_B z_3$ by transitivity, which contradicts the assumption that $z_3 \sim z_4$ (i.e., $z_3 \frown z_4$).[44] So, $p(z_1, \{z_1, z_4\}) = 1$ (and hence $z_1 \succ_B z_4$). By the same reasoning, $z_1 \succ_B z_4$ implies $p(z_2, \{z_2, z_4\}) = 1$ (and hence $z_2 \succ_B z_4$). By pursuing this line of reasoning, one obtains the following:

$$p(z_2, \{z_2, z_4\}) = 1 \Rightarrow p(z_2, \{z_2, z_m\}) = 1 \Rightarrow p(z_3, \{z_3, z_m\}) = 1 \Rightarrow p(z_3, \{z_1, z_3\}) = 1.$$

Since $p(z_1, \{z_1, z_3\}) = 1$ by assumption, the last implication above gives the desired contradiction. ∎

Let $\overset{*}{\frown}$ denote the (symmetric) relation on X where $x \overset{*}{\frown} y$ if x and y are linked. As noted in the text, $\overset{*}{\frown}$ defines an equivalence relation on X. Thus, $(X/\overset{*}{\frown}) := \{X_j\}_{j \in J}$ partitions the domain X into linked components that are indexed by some (potentially infinite) set $J = \{1, 2, ...\} \subseteq \mathbb{N}$.

First, define a strictly increasing function $f : J \to [1, +\infty)$ such that $f(1) = 1$. For each component $X_j \in (X/\overset{*}{\frown})$, fix some $\hat{x}^j \in X_j$ and set $v(\hat{x}^j) := f(j)$. For each $y^j \in X_j \setminus \{\hat{x}^j\}$, fix a sequence of imperfect discriminations $(z_i, S_i)_{i=1}^n$ from $z_1 = y^j$ to $z_n = \hat{x}^j$ and set $v(y^j) := \mathbb{P}(z_i, S_i)_{i=1}^n \times f(j)$. Since $\mathbb{P}(z_i, S_i)_{i=1}^n$ is finite, $v(y_j) \in \mathbb{R}_{++}$. So, by construction, v is a mapping such that $v : X \to \mathbb{R}_{++}$.

Step 6. p is represented by the extended Luce rule (Γ, v).

Fix some $S \in \mathcal{F}$ and $x \in S$. By Step 2, $\Gamma(S) = \max_{\succ_B} S$. If $x \notin \Gamma(S)$, then $p(x, S) = 0$ as required. Otherwise, $x \in \Gamma(S)$. By definition, $\Gamma(S) \subseteq X_j$ for some component $X_j \in (X/\overset{*}{\frown})$. By SPR,

$$p(x, S) \times v(y) = \left[p(x, S) \times \frac{v(y)}{f(j)}\right] \times f(j) = \left[p(y, S) \times \frac{v(x)}{f(j)}\right] \times f(j) = p(y, S) \times v(x)$$

[44] Given the definition of $\succ_B$, it turns out that $\sim$ coincides with $\frown$. Since the conceptual basis for each relation is somewhat different, I continue to use both symbols (selecting the one that is more appropriate in the specific context).

for all $y \in \Gamma(S)$. By adding up these equations for all $y \in \Gamma(S)$, one obtains:

$$p(x,S) \times \left[\sum_{y\in\Gamma(S)} v(y)\right] = \sum_{y\in\Gamma(S)} [p(x,S) \times v(y)] = \sum_{y\in\Gamma(S)} [p(y,S) \times v(x)] = \left[\sum_{y\in\Gamma(S)} p(y,S)\right] \times v(x) = 1 \times v(x).$$

By re-arranging the preceding expression, one obtains $p(x,S) = v(x)/\left[\sum_{y\in\Gamma(S)} v(y)\right]$ as required. ❙

To conclude the proof, note that the uniqueness of $\succ_B$ follows from Steps 1-2 above. In turn, SPR implies that v is unique up to scalar multiplication within each linked component. ❙

(**ii**) ⇒ (**iii**) Suppose p is represented by an extended Luce rule (Γ, v) such that Γ rationalized by a pre-order $\succ$. From the uniqueness of the representation (shown above), $p(y, \{x,y\}) = 0$ if $x \succ y$; and $\Gamma(S) = \max_{\succ} S$ for all $S \in \mathcal{F}$. Applying these definitions to the representation, it is easy to check that p satisfies LCA. ❙

(**iii**) ⇒ (**i**) Suppose p is represented by an extended Luce rule (Γ, v) (whether or not p happens to satisfy LCA). Using the representation, it is easy to check that p satisfies 4-PR. ■

Appendix D – Proof of Theorem 1

Lemma 2. *For a random choice function p that satisfies the Luce Choice axiom:*

(a) *p satisfies the Quadruple Product rule if it satisfies Luce transitivity; but* **(b)** *the converse need not hold.*

Proof. Fix a random choice function p that satisfies LCA.

(a) Suppose $x \frown y \frown z \frown w \frown x$. I first show $p(x, \{x,z\})$ or $p(y, \{y,w\}) \in (0,1)$. Towards a contradiction, suppose $p(x, \{x,z\}) = p(y, \{y,w\}) = 1$. (The case where $p(x, \{x,z\}) = p(y, \{y,w\}) = 0$ is similar.) If $p(z, \{z,w\}) \in [1/2, 1]$, then $p(x, \{x,z\}) = 1$ implies $p(x, \{x,w\}) = 1$ by LT, which contradicts $w \frown x$. So, $p(z, \{z,w\}) \in [0, 1/2)$. Similarly, $p(y, \{y,w\}) = 1$ and $y \frown z$ imply $p(w, \{z,w\}) \in [0, 1/2)$. So, $p(z, \{z,w\}) + p(w, \{z,w\}) < 1$, a contradiction.

By Step 4 in the proof of [(**i**) ⇒ (**ii**) from] Theorem 2, LCA implies the Quadruple Product rule. Since $p(x, \{x,z\})$ or $p(y, \{y,w\}) \in (0,1)$, the argument in Section 4(a) then implies the required identity. ❙

(b) Suppose p on $X := \{x,y,z,w\}$ is represented by $(\succ, v)$ where: $x \succ y$ and $z \succ w$ (but $a \sim b$ for all other $a, b \in X$); and $v(a) = 1$ for all $a \in X$. To see that p violates LT, notice that $p(x, \{x,y\}) = 1$ but $p(x, \{x,z\}), p(y, \{y,z\}) = 1/2$. ■

Lemma 3. *For a random choice function p that satisfies the Luce Choice axiom:*

(a) *p satisfies Moderate stochastic transitivity if it satisfies Weak consistency; but* **(b)** *the converse need not hold.*

Proof. Fix a random choice function p that satisfies LCA.

(a) First observe that LCA implies that p satisfies S2 and PR (by Steps 1 and 4 in the proof of Theorem 2). Next, suppose $\min\{p(x, \{x,y\}), p(y, \{y,z\})\} \geq 1/2$ for distinct $x, y, z \in X$. There are four cases:

1. Let $p(x, \{x,y\}) = p(y, \{y,z\}) = 1$. Then, by S2, $p(x, \{x,z\}) = 1 = \min\{p(x, \{x,y\}), p(y, \{y,z\})\}$.

2. Let $p(x, \{x,y\}) = 1$ and $p(y, \{y,z\}) < 1$. If $p(x, \{x,z\}) = 0$, then $p(z, \{y,z\}) = 1$ by S2, which contradicts $p(y, \{y,z\}) \geq 1/2$. If $p(x, \{x,z\}) = 1$, then $p(x, \{x,z\}) > p(y, \{y,z\}) = \min\{p(x, \{x,y\}), p(y, \{y,z\})\}$ as required. Otherwise, $p(x, \{x,z\}) \in (0,1)$. In that case,

$$\frac{p(y, \{y,z\})}{p(z, \{y,z\})} \times \frac{p(z, \{x,z\})}{p(x, \{x,z\})} < 1$$

by WC. Since $p(y, \{y,z\}) \geq 1/2$, it again follows that $p(x, \{x,z\}) > p(y, \{y,z\}) = \min\{p(x, \{x,y\}), p(y, \{y,z\})\}$.

3. Let $p(x, \{x,y\}) < 1$ and $p(y, \{y,z\})\} = 1$. This case is similar to case 2.

4. Let $\max\{p(x,\{x,y\}),p(y,\{y,z\})\} < 1$. If $p(x,\{x,z\}) = 0$, then $p(z,\{y,z\}) > p(x,\{x,y\}) \geq 1/2$ by (the reasoning in) case 2. But, this contradicts $p(y,\{y,z\}) \geq 1/2$. If $p(x,\{x,z\}) = 1$, then $p(x,\{x,z\}) \geq \min\{p(x,\{x,y\}),p(y,\{y,z\})\}$ as required. Otherwise, $p(x,\{x,z\}) \in (0,1)$. In that case,

$$\frac{p(x,\{x,z\})}{p(z,\{x,z\})} = \frac{p(x,\{x,y\})}{p(y,\{x,y\})} \times \frac{p(y,\{y,z\})}{p(z,\{y,z\})}$$

by PR. Then, since $\min\{p(x,\{x,y\}),p(y,\{y,z\})\} \geq 1/2$, $p(x,\{x,z\}) \geq \min\{p(x,\{x,y\}),p(y,\{y,z\})\}$. ❙

(b) The random choice function p from the proof of Lemma 2(b) satisfies MST and LCA. However, it violates WC. To see this, notice that $p(y,\{x,y\}) = 0$ and $y\frown z\frown x$ but $p(y,\{y,z\}) \times p(z,\{x,z\}) = p(z,\{y,z\}) \times p(x,\{x,z\})$. ■

Lemma 4. *For a random choice function p that satisfies the Luce Choice axiom:*

(a) *p satisfies Weak consistency if it satisfies Luce transitivity; but* **(b)** *the converse need not hold.*

Proof. Fix a random choice function p that satisfies LCA.

(a) Since p satisfies LCA and LT, p satisfies 4-PR by Lemma 2(a). So, p satisfies SPR by Step 5 of Theorem 2. Now, suppose y is linked to x by a sequence of imperfect discriminations $(z_i,\{z_i,z_{i+1}\})_{i=1}^n$ and $p(y,\{x,y\}) = 0$. The proof that $\mathbb{P}(z_i,\{z_i,z_{i+1}\})_{i=1}^n < 1$ holds is by strong induction on n.

For the base case $n = 3$, $y = z_1\frown z_2\frown z_3 = x$. If $p(y,\{y,z_2\}) \in [1/2,1)$, then $p(x,\{x,z_2\}) = 1$ by LT. So, $p(y,\{y,z_2\}) < 1/2$. Similarly, $p(z_2,\{x,z_2\}) < 1/2$. Combining these two inequalities gives $\mathbb{P}(z_i,\{z_i,z_{i+1}\})_{i=1}^3 < 1$.

For the induction step $n = m+1$, suppose the desired identity holds for every sequence of imperfect discriminations $(z_i,\{z_i,z_{i+1}\})_{i=1}^n$ such that $n \leq m$. If $p(z_j,\{z_j,z_k\}) \in (0,1)$ for any $1 \leq j,k \leq m+1$ such that $k > j+1$, then $y = z_1\frown\ldots\frown z_j\frown z_k\frown\ldots\frown z_{m+1} = x$ so that

$$\mathbb{P}(z_i,\{z_i,z_{i+1}\})_{i=1}^j \times \frac{p(z_j,\{z_j,z_k\})}{p(z_k,\{z_j,z_k\})} \times \mathbb{P}(z_i,\{z_i,z_{i+1}\})_{i=k}^{m+1} < 1$$

by the induction hypothesis. Since p satisfies SPR, $p(z_j,\{z_j,z_k\}) = \mathbb{P}(z_i,\{z_i,z_{i+1}\})_{i=j}^k \times p(z_k,\{z_j,z_k\})$. By combining this with the previous inequality,

$$\mathbb{P}(z_i,\{z_i,z_{i+1}\})_{i=1}^{m+1} = \mathbb{P}(z_i,\{z_i,z_{i+1}\})_{i=1}^j \times \mathbb{P}(z_i,\{z_i,z_{i+1}\})_{i=j}^k \times \mathbb{P}(z_i,\{z_i,z_{i+1}\})_{i=k}^{m+1} < 1.$$

So, suppose $p(z_j,\{z_j,z_k\}) \in \{0,1\}$ for all $1 \leq j,k \leq m+1$ such that $k > j+1$.

In case $p(z_1,\{z_1,z_3\}) = 0$, the base case $n = 3$ implies $p(z_1,\{z_1,z_2\}),p(z_2,\{z_2,z_3\}) < 1/2$. What is more, $p(z_2,\{z_2,z_4\}) = 0$. Otherwise, $p(z_2,\{z_2,z_4\}) = 1$ which implies $p(z_2,\{z_2,z_3\}) > 1/2$ by the base case $n = 3$. By extending this type of reasoning, $p(z_1,\{z_1,z_3\}) = 0$ implies $p(z_i,\{z_i,z_{i+1}\}) < 1/2$ for all $1 \leq i \leq m$. Similarly, $p(z_1,\{z_1,z_3\}) = 1$ implies $p(z_i,\{z_i,z_{i+1}\}) > 1/2$ for all $1 \leq i \leq m$.

Now, consider $p(z_1,\{z_1,z_m\}) \in \{0,1\}$. If $p(y,\{y,z_m\}) = 1$, then $p(x,\{x,y\}) = 1$ and LT imply $p(x,\{x,z_m\}) = 1$, which contradicts $p(x,\{x,z_m\}) \in (0,1)$. So, $p(y,\{y,z_m\}) = 0$. Similarly, $p(z_2,\{z_2,x\}) = 0$. Since $y = z_1\frown\ldots\frown z_m$ and $z_2\frown\ldots\frown z_{m+1} = x$, the induction hypothesis then implies $\mathbb{P}(z_i,\{z_i,z_{i+1}\})_{i=1}^m < 1$ and $\mathbb{P}(z_i,\{z_i,z_{i+1}\})_{i=2}^{m+1} < 1$.

Since $p(z_1,\{z_1,z_3\}) \in \{0,1\}$, the observations from the two previous paragraphs together imply $p(z_i,\{z_i,z_{i+1}\}) < 1/2$ for all $1 \leq i \leq m$. Combining these m inequalities gives $\mathbb{P}(z_i,\{z_i,z_{i+1}\})_{i=1}^{m+1} < 1$. ❙

(b) The random choice function p from the example in Section 4(b) satisfies WC and LCA but violates LT. ■

Note: For the reader's convenience, I have underlined the key assumptions used to prove each step of Theorem 1. This makes it easier to see how the arguments extend directly to establish Theorem 1* of the Online Appendix.

Proof of Theorem 1. **(A)** ($\Leftarrow$) Suppose p satisfies LCA and LT.

Since it satisfies LCA and LT, p satisfies Weak consistency (WC) and Weak stochastic transitivity (WST) by Lemmas 3(a) and 4(a). Let $\succ_p$ denote the binary relation on X defined by $x \succ_p y$ if $p(y, \{x, y\}) < 1/2$. By WST, $\succ_p$ is a weak order. (It is irreflexive by definition. To that it is negatively transitive, let $x \succ_p y$. By way of contradiction, suppose $y \succsim_p z \succsim_p x$. Then, $y \succsim_p x$ by WST, which contradicts $x \succ_p y$.)

First, consider the equivalence relation $\overset{*}{\frown}$ defined after Step 5 of Theorem 2. By definition, the partition $(X/\overset{*}{\frown})$ coarsens $(X/{\sim_p})$. Number the components of $(X/\overset{*}{\frown})$ in increasing order of $\succ_p$ (i.e., recursively define $X^1 := X$, $X^{j+1} := X \setminus (\cup_{k=1}^{j} X_k)$ and $X_j \in (X/\overset{*}{\frown})$ so that $X_j \supseteq \arg\min_{\succ_p} X^j$ contains the $\succ_p$-minimal alternatives in X^j).

Next, define the strictly increasing function $f : J \to [1, +\infty)$ from the proof of Theorem 2 as follows:

$$f(j+1) := f(j) \times \mathbb{P}(z_i^j, \{z_i^j, z_{i+1}^j\})_{i=1}^k + 1$$

where $(z_i^j, \{z_i^j, z_{i+1}^j\})_{i=1}^k$ is a sequence of imperfect discriminations linking $\overline{x}_j \in \arg\max_{\succ_p} X_j$ to $\underline{x}_j \in \arg\min_{\succ_p} X_j$. Define $v : X \to \mathbb{R}_{++}$ as in the proof of Theorem 2 by picking $\hat{x}^j \in \arg\min_{\succ_p} X_j$ for each $X_j \in (X/\overset{*}{\frown})$.

Step 1. $(\succ_B, v)$ represents p where $\succ_B$ is defined as in ($\succ_B$-Id).

By Lemma 2(a), LCA and LT imply 4-PR. The result then follows from the proof of Theorem 2. ❙

Step 2. $\succ_B$ is a semi-order that is based on v.

Since p satisfies LCA and 4-PR, p satisfies SPR by Step 5 in the proof of Theorem 2.

By Theorem 3, it suffices to show that $\succ_B^*$ satisfies v-consistency. To see this, let $x \succ_B z \sim_B y$ for some $z \in X$. (The case where $x \sim_B z \succ_B y$ for some $z \in X$ is similar.) There are two possibilities.

If $x \in X_j$ and $y, z \in X_k$ for $j \neq k$, then $k < j$ since $x \succ_B z$. By definition of v,

$$v(x) \geq f(j) \geq v(\overline{z}) + 1 > v(y)$$

where $\overline{z} \in \arg\max_{\succ_p} X_k$. So, $v(x) > v(y)$ as required.

Otherwise, $x, y, z \in X_j$. First observe $p(y, \{x, y\}) \in [0, 1/2)$. Otherwise, $p(y, \{x, y\}) \in [1/2, 1]$. Then, since $p(x, \{x, z\}) = 1$, LT implies $p(y, \{y, z\}) = 1$. But, this contradicts $y \sim_B z$. This leaves two possibilities:

1. Suppose $p(y, \{x, y\}) = 0$. By definition, $v(x) := \mathbb{P}(w_i, S_i)_{i=1}^n \times f(j)$ for some sequence of imperfect discriminations $x = w_1 \overset{S_1}{\frown} w_2 \overset{S_2}{\frown} \ldots \overset{S_{n-2}}{\frown} w_{n-1} \overset{S_{n-1}}{\frown} w_n = \hat{x}^j$; and $v(y) := \mathbb{P}(w_i', S_i')_{i=1}^m \times f(j)$ for some sequence of imperfect discriminations $y = w_1' \overset{S_1'}{\frown} w_2' \overset{S_2'}{\frown} \ldots \overset{S_{n-2}'}{\frown} w_{n-1}' \overset{S_{n-1}'}{\frown} w_n' = \hat{x}^j$. Then, since p satisfies WC and SPR,

$$1 > \frac{\mathbb{P}(w_i', \{w_i', w_{i+1}'\})_{i=1}^m \times f(j)}{\mathbb{P}(w_i, \{w_i, w_{i+1}\})_{i=1}^n \times f(j)} = \frac{\mathbb{P}(w_i', S_i')_{i=1}^m \times f(j)}{\mathbb{P}(w_i, S_i)_{i=1}^n \times f(j)} = \frac{v(y)}{v(x)}.$$

2. Suppose $p(y, \{x, y\}) \in (0, 1/2)$. Then, SPR implies

$$1 > \frac{p(y, \{x, y\})}{p(x, \{x, y\})} = \frac{\mathbb{P}(w_i', S_i')_{i=1}^m \times f(j)}{\mathbb{P}(w_i, S_i)_{i=1}^n \times f(j)} = \frac{v(y)}{v(x)}.$$

In either case, $v(x) > v(y)$ as required. ❙

($\Rightarrow$) Suppose p is represented by a threshold Luce rule (B, v). Given Theorem 2, I show that p satisfies LT. Fix $x, y, z \in X$ such that $p(x, \{x, y\}), p(y, \{y, z\}) \in [1/2, 1]$. Suppose $p(x, \{x, y\}) = 1$. (The case where $p(y, \{y, z\}) = 1$ is similar.) If $p(y, \{y, z\}) \in [1/2, 1)$, then $v(y) > v(z)$. Then, since $\succ_B$ satisfies v-transitivity (by Theorem 3), $p(x, \{x, z\}) = 1$. Otherwise, $p(y, \{y, z\}) = 1$. In that case, $p(x, \{x, z\}) = 1$ follows from the transitivity of $\succ_B$. ❙

(B) By construction, $\succ_B$ is unique. Since p satisfies SPR, LD is necessary and sufficient for the uniqueness of v; and it implies that $\succ_B$ satisfies the Sorites property. ∎

Appendix E – Additional Results

Lemma 5. *A positive random choice function p satisfies Strong stochastic transitivity if and only if it satisfies IIA.*

Proof. Fix a positive random choice function p. Given Luce's Lemma 3 (which specializes the argument given in Step 3 of Theorem 2 above), it suffices to show that IIA implies SST. Given $R \subset S \subseteq T$ and $y \in S$, IIA implies

$$p(y,T) \times p(R,S) = p(y,T)\sum_{x\in R} p(x,S) = p(y,T) \times p(y,S)\sum_{x\in R}\frac{p(x,\{x,y\})}{p(y,\{x,y\})} = p(y,S)\sum_{x\in R} p(x,T) = p(y,S) \times p(R,T).$$

Summing this identity over all $y \in S$ gives $p(S,T) \times p(R,S) = p(S,S) \times p(R,T)$. So, $p(R,T) = p(R,S) \times p(S,T)$. ■

Lemma 6. *For a random choice function p that satisfies the Luce Choice axiom, p satisfies Strong stochastic transitivity if and only if it satisfies Luce transitivity.*

Proof. Fix a random choice function p that satisfies LCA and LT. It suffices to show that it satisfies SST.

Since p satisfies LCA and LT, it satisfies 4-PR by Lemma 2(a). So, p satisfies SPR by Step 5 in Theorem 2. Now, suppose $\min\{p(x,\{x,y\}), p(y,\{y,z\})\} \in [1/2, 1]$ and $p(x,\{x,z\}) < 1$. (If $p(x,\{x,z\}) = 1$, then there is nothing to prove.) If $\max\{p(x,\{x,y\}), p(y,\{y,z\})\} = 1$, then $p(x,\{x,z\})\} = 1$ by LT, which is a contradiction. So, suppose $\max\{p(x,\{x,y\}), p(y,\{y,z\})\} < 1$. If $p(x,\{x,z\}) = 0$, then $p(z,\{y,z\})\} = 1$ by LT, which is a contradiction. So, $p(x,\{x,z\}) \in (0,1)$. By SPR,

$$\frac{p(x,\{x,z\})}{p(z,\{x,z\})} = \frac{p(x,\{x,y\})}{p(y,\{x,y\})} \times \frac{p(y,\{y,z\})}{p(z,\{y,z\})}.$$

Without loss of generality, suppose $p(x,\{x,y\}) \geq p(y,\{y,z\}) \geq 1/2$. From the identity above,

$$\frac{p(x,\{x,z\})}{p(z,\{x,z\})} = \frac{p(x,\{x,y\})}{p(y,\{x,y\})} \times \frac{p(y,\{y,z\})}{p(z,\{y,z\})} \geq \frac{p(x,\{x,y\})}{p(y,\{x,y\})}.$$

In turn, this implies $p(x,\{x,z\}) \geq \max\{p(x,\{x,y\}), p(y,\{y,z\})\}$. ■

Lemma 7. *For a random choice function p that satisfies the Luce Choice axiom and Luce transitivity, p satisfies Finitely connected domain if and only if it satisfies Linked domain.*

Proof. Fix a random choice function p that satisfies LCA, LT, and LD. It suffices to show that it satisfies FCD.

Since p satisfies LCA and LT, it satisfies WC by Lemma 4(a). Fix $x, y \in X$ such that $p(y,\{x,y\}) = 0$. Since p satisfies LD, $x = z_1 \frown ... \frown z_n = y$ for some sequence $z_1, ..., z_n \in X$. If $p(z_i,\{z_i,z_{i+1}\}) \geq 1/2 \geq p(z_{i+1},\{z_{i+1},z_{i+2}\})$ for some $1 \leq i \leq n-2$, then LT implies $z_i \frown z_{i+2}$. By deleting z_{i+1}, one obtains a shorter sequence linking x to y. (The same reasoning applies if $p(z_i,\{z_i,z_{i+1}\}) \leq 1/2 \leq p(z_{i+1},\{z_{i+1},z_{i+2}\})$ for some $1 \leq i \leq n-2$.) Ultimately, these deletions leads to a sequence $z'_1, ..., z'_m \in X$ such that:

- $p(z'_i,\{z'_i,z'_{i+1}\}) \leq 1/2$ for all $i = 1, ..., m-1$; or
- $p(z'_i,\{z'_i,z'_{i+1}\}) \geq 1/2$ for all $i = 1, ..., m-1$ and $p(z'_j,\{z'_j,z'_{j+1}\}) > 1/2$ for some $1 \leq j \leq m-1$.

Since $p(y,\{x,y\}) = 0$, the first case contradicts WC. So, the second case obtains. ■

Example 2. *Consider the random choice function p on $X := \{x,y,z,w\}$ with binary choice probabilities given by*

$$p(x,\{x,y\}) = p(y,\{y,z\}) = p(z,\{z,w\}) = 1/2,\ p(x,\{x,z\}) = p(y,\{y,w\}) = 1 \text{ and } p(x,\{x,w\}) = 1/3.$$

These binary choice probabilities are consistent with the Luce Choice axiom. In fact, assuming that p satisfies the Luce Choice axiom, they pin down p completely. However, p violates the Quadruple Product rule since $x \frown y \frown z \frown w \frown x$ but

$$\frac{p(x,\{x,y\}) \times p(y,\{y,z\}) \times p(z,\{z,w\}) \times p(w,\{w,x\})}{p(y,\{x,y\}) \times p(z,\{y,z\}) \times p(w,\{z,w\}) \times p(x,\{w,x\})} = 2 \neq 1.$$

Online Appendix

(a) Extension of Theorem 3

Theorem 3*. *For a utility function $v : X \to \mathbb{R}$ and an irreflexive relation $\succ$ on X:*

(1.1) *$\succ$ is an interval order that is weakly based on v if and only if $\succ$ satisfies P3 and v-consistency.*

(1.2) *$\succ$ is an interval order that is based on v if and only if $\succ$ satisfies P3 and $\succ^{**}$ satisfies v-consistency.*

(2.1) *$\succ$ is a semi-order that is weakly based on v if and only if $\succ$ satisfies P3, P4 and v-consistency.*

(3.1) *$\succ$ is a weak order that is based on v if and only if $\succ$ satisfies P5 and v-consistency.*

Proof. In each case, necessity is obvious. For sufficiency, define B as in the proof of Lemma 1. Then:

(**1**.**1**) Since it satisfies P3, $\succ$ is an interval order. By Steps 2-5 in the proof of Lemma 1, (B, v) represents $\succ$. ▎

(**1**.**2**) Since $\succ$ satisfies P3, it is transitive and coarsens $\succ^{**}$. Then, by the argument in part (1.1), $\succ$ is an interval order represented by (B, v). Since $\succ^{**}$ satisfies v-consistency, (B, v) satisfies R1 by Step 6 (in the proof of Lemma 1). ▎

(**2**.**1**) Since it satisfies P3 and P4, $\succ$ is a semi-order. By the argument in part (1.1), (B, v) represents $\succ$. Since $\succ$ is a semi-order, (B, v) satisfies R2 by Step 7 (in the proof of Lemma 1). ▎

(**3**.**1**) Since it satisfies P5, $\succ$ is a weak order. By the argument in part (2.1), (B, v) represents $\succ$ and satisfies R2. Then, since $\succ$ is a semi-order, (B, v) also satisfies R3 (which is tantamount to the transitivity of $\sim$). Finally, since $\succ$ is a weak order, $\succ$ coincides with $\succ^{**}$. Since $\succ$ satisfies v-consistency, $\succ^{**}$ satisfies the same property. So, satisfies R1 by Step 6 (in the proof of Lemma 1). ■

(b) Extension of Theorem 2

By translating P2-P5 (from Appendix A), one obtains the following conditions, which apply to all $x, y, z, w \in X$:

S2. $p(x, \{x, y\}), p(y, \{y, z\}) = 1 \implies p(x, \{x, z\}) = 1$.[45]
S3 (*Stochastic Ferrers*). $p(w, \{x, w\}), p(z, \{y, z\}) = 0 \implies \min\{p(z, \{x, z\}), p(w, \{y, w\})\} = 0$.
S4 (*Stochastic semi-transitivity*). $p(y, \{x, y\}), p(z, \{y, z\}) = 0 \implies \min\{p(w, \{x, w\}), p(z, \{z, w\})\} = 0$.
S5. $p(x, \{x, y\}), p(y, \{y, z\}) > 0 \implies p(x, \{x, z\}) > 0$.[46]

Theorem 2*.[47] *A random choice function p satisfies the Luce Choice axiom and:*

(1) *S3 if and only if it can be represented by an extended Luce rule $(\succ, v)$ where $\succ$ an interval order.*
(2) *S3-S4 if and only if it can be represented by an extended Luce rule $(\succ, v)$ where $\succ$ is a semi-order.*
(3) *S5 if and only if it can be represented by an extended Luce rule $(\succ, v)$ where $\succ$ is a weak order.*

Proof. If $\succ_B$ is defined as in the proof of Theorem 2, then S3-S5 translate P3-P5. By the argument in Section 4(a) of the text, S3 strengthens 4-PR when p satisfies LCA. So, Theorem 2 implies (1)-(3). ■

Remark 4. Parts (2) and (3) provide alternatives to Theorems B and C from Appendix A.

[45] This property also appears as Axiom 7 (*Dominance transitivity*) in Echenique and Saito.
[46] This property also appears as Axiom P2 in Fishburn (1978).
[47] To simplify, I omit the uniqueness properties of the representation (which are the same as Theorem 2).

(c) Extension of Theorem 1

Define the binary relation $\succ_B$ as in ($\succ_B$-Id). Then, the appropriate consistency conditions for the binary relations $\succ_B^*$ and $\succ_B^{**}$ (defined in Section 3(b) of the text) may be stated as follows:

Strong consistency (SC). *For every sequence of imperfect discriminations $z_1 \frown ... \frown z_n$:*

$$z_1 \succ_B^* z_n \implies \prod_{i=1}^{n-1} \frac{p(z_i, \{z_i, z_{i+1}\})}{p(z_{i+1}, \{z_i, z_{i+1}\})} > 1.$$

Moderate consistency (MC). *For every sequence of imperfect discriminations $z_1 \frown ... \frown z_n$:*

$$z_1 \succ_B^{**} z_n \implies \prod_{i=1}^{n-1} \frac{p(z_i, \{z_i, z_{i+1}\})}{p(z_{i+1}, \{z_i, z_{i+1}\})} > 1.$$

Lemma 8. *For a random choice function p that satisfies the Luce Choice axiom, the following are equivalent:*

(i) *p satisfies S3-S4 and Moderate consistency; and*

(ii) *p satisfies Luce transitivity; and*

(iii) *p satisfies Strong consistency.*

Proof. Fix a random choice function p that satisfies LCA.

(**i**) $\Rightarrow$ (**ii**) First observe the following. Since p satisfies LCA and S3, p satisfies 4-PR by the argument in Section 4(a) of the text. So, p satisfies SPR by Step 5 in the proof of Theorem 2. Next, suppose $p(x, \{x, y\}) = 1$ and $p(y, \{y, z\}) \in [1/2, 1]$. (The case where $p(x, \{x, y\}) \in [1/2, 1]$ and $p(y, \{y, z\}) = 1$ is similar.) By S2, $p(x, \{x, z\}) > 0$. By way of contradiction, suppose $p(x, \{x, z\}) \in (0, 1)$. Then, $z \sim_B x \succ_B y$ so that $z \succ_B^* y$.

In fact, $z \succ_B^{**} y$. If $y \succ_B z$, then $x \succ_B z$ by S2, which is a contradiction. If $y \sim_B w \succ_B z$ for some $w \in X$, then $x \succ_B z$ or $w \succ_B y$ by S3, which is a contradiction. If $y \succ_B w \sim_B z$ for some $w \in X$, then $x \succ_B z$ or $z \succ_B y$ by S4, which is a contradiction. Since $z \succ_B^{**} y$, $p(z, \{y, z\}) > p(y, \{y, z\})$ by MC, which is a contradiction. ❙

(**ii**) $\Rightarrow$ (**iii**) As in the proof of Lemma 4, p satisfies 4-PR and SPR. Suppose y is linked to x by a sequence of imperfect discriminations $z_1 \frown ... \frown z_n$ and $x \succ_B^* y$. If $x \succ y$, then $\mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^n < 1$ by Lemma 4(a). Otherwise, $p(y, \{x, y\}) \in (0, 1)$. Since $x \succ_B^* y$, suppose there exists some $w \in X$ such that $y \sim_B x \succ_B w \sim_B y$. (The case where $x \sim_B w \succ_B y \sim_B x$ is similar.) Then, $p(x, \{x, y\}) \in (1/2, 1)$. Otherwise, $p(y, \{x, y\}) \in [1/2, 1)$ and $p(x, \{x, w\}) = 1$ imply $p(y, \{y, w\}) = 1$ by LT, which contradicts $p(y, \{y, w\}) \in (0, 1)$. Since $p(x, \{x, y\}) \in (1/2, 1)$, SPR then implies the desired inequality $\mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^n < 1$. ❙

(**iii**) $\Rightarrow$ (**i**) To show S3, let $p(y, \{x, y\}), p(w, \{z, w\}) = 0$. By way of contradiction, suppose $p(x, \{x, w\}), p(y, \{y, z\}) \in [0, 1)$. Since p satisfies S2 (by the proof of Theorem 2), $p(x, \{x, w\}), p(y, \{y, z\}) \in (0, 1)$ and $p(x, \{x, z\}) \in (0, 1)$. So, $x \succ_B^* z \succ_B^* x$ by definition. Then, $p(x, \{x, z\}) > p(z, \{x, z\}) > p(x, \{x, z\})$ by SC, which is a contradiction.

To show S4, let $p(y, \{x, y\}), p(z, \{y, z\}) = 0$. By way of contradiction, suppose $p(w, \{z, w\}), p(x, \{x, w\}) \in [0, 1)$. Since p satisfies S2, $p(w, \{z, w\}), p(x, \{x, w\}) \in (0, 1)$. So, $y \succ_B^* w \succ_B^* y$ by definition. Then, $p(y, \{y, w\}) > p(w, \{y, w\}) > p(y, \{y, w\})$ by SC, which is a contradiction. ■

Lemma 9. *For a random choice function p that satisfies the Luce Choice axiom:*

(a) *p satisfies Luce transitivity if it satisfies S5; but* **(b)** *the converse need not hold.*

Proof. Fix a random choice function p that satisfies LCA.

(a) Since p satisfies LCA and S5, p satisfies 4-PR by the argument in Section 4(a) of the text. So, p satisfies PR by Step 4 of Theorem 2. Now, suppose $p(x, \{x, y\}) = 1$ and $p(y, \{y, z\}) \in [1/2, 1]$. (The case where $p(x, \{x, y\}) \in [1/2, 1]$

and $p(y,\{y,z\})=1$ is similar.) Then, by S5, $p(x,\{x,z\})>0$. By way of contradiction, suppose $p(x,\{x,z\})\in(0,1)$. Then, $p(y,\{x,y\})>0$ by S5, which is a contradiction. ❙

(b) Suppose p on $X:=\{x,y,z\}$ is represented by $(\succ,v)$ such that: $y\sim z\succ x\sim y$; and $\langle v(x),v(y),v(z)\rangle=\langle 1,2,3\rangle$. Then, p satisfies LT. However, it violates S5 since $p(x,\{x,y\})={}^1\!/_3$ and $p(y,\{y,z\})={}^2\!/_5$ but $p(x,\{x,z\})=0$. ∎

Theorem 1*.[48] *A random choice function p satisfies the Luce Choice axiom as well as*

(0.1) *the Quadruple Product rule and Weak consistency if and only if it can be represented by an extended Luce rule $(\succ,v)$ where $\succ$ is a pre-order that satisfies v-consistency.*

(1.1) *S3 and Weak consistency iff it can be represented by $(\succ,v)$ where $\succ$ is an interval order weakly based on v.*

(1.2) *S3 and Moderate consistency iff it can be represented by $(\succ,v)$ where $\succ$ is an interval order based on v.*

(2.1) *S3-S4 and Weak consistency iff it can be represented by $(\succ,v)$ where $\succ$ is a semi-order weakly based on v.*

(2.2) *S3-S4 and Moderate consistency iff it can be represented by $(\succ,v)$ where $\succ$ is a semi-order based on v.*

(2.3) *Strong consistency iff it can be represented $(\succ,v)$ where $\succ$ is a semi-order based on v.*

(3.1) *S5 iff it can be represented by $(\succ,v)$ where $\succ$ is a weak order based on v.*

Proof. Fix a random choice function p. In each case, necessity is obvious.

For sufficiency, define $(\succ_B,v)$ as in the proof of Theorem 1. Then:

(0.1) Since p satisfies WC, it satisfies WST by Lemma 3(a). Then, by Step 1 in the proof of Theorem 1, $(\succ_B,v)$ represents p. To see that $\succ_B$ satisfies v-consistency, suppose $x\succ_B y$. Then, by the same kind of reasoning as Step 2 in the proof of Theorem 1 (when $x\in X_j$ and $y\in X_k$ for $j\neq k$), $v(x)>v(y)$. ❙

(1.1) Since p satisfies LCA and S3, it satisfies 4-PR by the argument in Section 4(a); and $\succ_B$ satisfies P3. Since p satisfies WC, it satisfies WST by Lemma 3(a). By the argument in part (0.1), $(\succ_B,v)$ represents p. By Theorem 3*(1.1), it suffices to show that $\succ_B$ satisfies v-consistency. To see this, suppose $x\succ_B y$:

- If $x\in X_j$ and $y\in X_k$ for $j\neq k$, then $v(x)\geq f(j)\geq v(\overline{z})+1>v(y)$ as in Step 2 (from Theorem 1).

- Otherwise, $x,y\in X_j$. By definition, $u(x):=\mathbb{P}(w_i,S_i)_{i=1}^n\times f(j)$ for some sequence of imperfect discriminations $(w_i,S_i)_{i=1}^n$ linking x to $\hat{x}^j$; and $v(y):=\mathbb{P}(w'_i,S'_i)_{i=1}^m\times f(j)$ for some sequence $(w'_i,S'_i)_{i=1}^m$ linking y to $\hat{x}^j$. Since $p(y,\{x,y\})=0$, WC implies $v(x)=\mathbb{P}(w_i,S_i)_{i=1}^n\times f(j)>\mathbb{P}(w'_i,S'_i)_{i=1}^m\times f(j)=v(y)$ as in Step 2 (from Theorem 1). ❙

(1.2) By the argument in part (1.1), $(\succ_B,v)$ represents p; and $\succ_B$ satisfies P3. If $\succ_B^{**}$ satisfies v-consistency, then Theorem 3*(1.2) implies that $\succ_B$ is an interval order based on v. To see that $\succ_B^{**}$ satisfies v-consistency, fix some alternatives $x\succ_B^{**} y$ and suppose $x\succ_B z\sim_B y$ for some $z\in X$. (The case where $x\sim_B z\succ_B y$ is similar.) Then, $v(x)>v(y)$ by the same type of reasoning as Step 2 (in the proof of Theorem 1). The main difference is that $p(y,\{x,y\})\in[0,{}^1\!/_2)$ follows from MC rather than LT. (If $p(y,\{x,y\})=1$, then $p(y,\{y,z\})=1$ by Step 1 of Theorem 2, contradicting $z\sim_B y$. Since $x\succ_B^{**} y$, MC implies $p(y,\{x,y\})\in[{}^1\!/_2,1)$. So, $p(y,\{x,y\})\in[0,{}^1\!/_2)$ as required.) ❙

(2.1) By the argument in part (1.1), $(\succ_B,v)$ represents p; and $\succ_B$ satisfies P3 and v-consistency. Since p satisfies S4, $\succ_B$ satisfies P4. Then, by Theorem 3*(2.1), $\succ_B$ is a semi-order weakly based on v. ❙

(2.2-2.3) These statements follow directly from Lemma 8 and Theorem 1. ❙

(3.1) Since p satisfies LCA and S5, it satisfies WC by Lemmas 4(a) and 9(a). Then, by the argument in part (0.1), $(\succ_B,v)$ represents p; and $\succ$ satisfies v-consistency. Since p satisfies S5, $\succ_B$ satisfies P5. Then, by Theorem 3*(3.1), $\succ_B$ is a weak order based on v. ∎

Remark 5. Parts (2.2) and (2.3) provide alternatives to Theorem 1A.

[48] To simplify, I omit the uniqueness properties of the representation (which are the same as Theorem 1).