PERIODIC AUTOREGRESSIVE CONDITIONAL HETEROSKEDASTICITY

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RÉSUMÉ

Dans cette étude, nous proposons une classe de processus ARCH périodiques. Cette structure est semblable à celle des processus linéaires périodiques. Les processus P-ARCH partagent beaucoup de similarités avec les processus périodiques linéaires mais ont aussi, à cause des non linéarités, des caractéristiques spécifiques. Nous étudions de façon analytique les pertes d'efficacité en terme de prévisions dues à des erreurs de spécifications lorsque les données suivent un processus P-ARCH et qu'un modèle ARCH (saisonnier) est estimé. Le papier inclut également une étude de Monte Carlo qui complémentaire les résultats théoriques et une application au taux de change DM - livre Sterling. Plusieurs extensions, telles que P-EGARCH et P-IGARCH, sont aussi proposées.

Mots clés : persistance dans la volatilité, structures périodiques, taux de change.

ABSTRACT

High frequency asset returns generally exhibit time dependent and seasonal clustering of volatility. This paper proposes a new class of models featuring periodicity in conditional heteroskedasticity explicitly designed to capture the repetitive seasonal time variation in the second order moments. The structures of this new class of Periodic ARCH, or P-ARCH, models share many properties with the periodic ARMA processes for the mean. The implicit relation between P-GARCH structures and time-invariant seasonal weak GARCH processes documents how neglected autoregressive conditional heteroskedastic periodicity may give rise to a loss in efficiency. The importance and magnitude of this informational loss are quantified for a variety loss functions through the use of Monte Carlo simulation methods. An empirical example for the daily bilateral Deutschmark – British Pound spot exchange rate highlights the practical relevance of the new P-GARCH class of models. Extensions to other periodic ARCH structures, including P-IGARCH and P-EGARCH processes along with possible discrete time periodic representations of stochastic volatility models subject to time deformation, are also discussed, along with issues related to multivariate representations and the possibility of common persistence in the seasonal volatility across multiple time series.

Key words: volatility clustering, seasonality, periodic structures, ARCH, GARCH, P-GARCH, exchange rates.
1. Introduction

Numerous previous studies have documented seasonal effects in the means and standard deviations of financial asset returns and payoffs. For instance, Keim (1983) finds that the return on small company stocks tend to be higher in the month of January. Possible explanations for this phenomenon include the year end tax-loss selling theories by Dyl (1977) and Reiganum (1983), or the presence of seasonality in the risk-return tradeoff relationship as studied by Connor and Korajczyk (1988), Korajczyk and Viallet (1989), Ng, Engle and Rothschild (1992), and Demos, Sentana and Shah (1993). In a similar vein Gallant, Rossi and Tauchen (1992) report, that the variance of the U.S. Standard and Poor composite price index in October is almost a tenfold of the variance for March, while Shiller (1994) observe that ten out of the twenty-five stock market crashes that occurred in the U.S. since 1928 were concentrated in the month of October; see also Schwert (1990a). In a related context, Bollerslev and Hodorick (1994) find further corroborating evidence regarding seasonality in the conditional heteroskedasticity of aggregate stock market dividend yields. The presence of seasonality in asset returns is even more pronounced at higher sampling frequencies, however. In particular, Lakonishok and Smidt (1988) and Schwert (1990b) argue that returns on Mondays are systematically lower than any other days of the week, while French and Roll (1986), French, Schwert and Stambaugh (1987) and Nelson (1991) demonstrate that daily stock return volatility tend to be higher following non-trading days, although proportional less than the time period of the market closure. Significant high frequency intraday patterns in the means and variances of stock returns have also been documented by Wood, McInish and Ord (1985) and Harris (1986) among others. Similar effects in the volatility of daily exchange rates have been found by Baillie and Bollerslev (1989), while Baillie and Bollerslev (1991) and Dacorogna, Müller, Nagler, Olsen, and Pictet (1993) report distinct intraday volatility patterns in the foreign exchange market.

The fact that asset returns exhibit volatility clustering within and across different days and throughout the year has interesting implications both from a theoretical and an empirical point of view. For instance, seasonal habit persistence in preferences and its effect on asset pricing, as studied by Hansen and Sargent (1990), Ferson and Harvey (1992) and Braun, Constantinides and Ferson (1993), may explain part of the observed phenomena. Along these lines, the fairly regular and institutionalized rhythm of releasing information to the general public, like annual corporate reports and dividend announcements, or the calendar of releases of economic data

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1 A crash is here defined as a drop in the stock market prices exceeding six percent between two successive trading days.
by government agencies, may contribute to the volatility being structured with seasonal-specific patterns. Dynamic patterns may also emerge as a result of systematic market closures and agents rational reaction in an asymmetric informational environment; see e.g. the market-microstructure theories by Kyle (1985), Admati and Pfleiderer (1988) and Andersen (1993a). Besides these theoretical issues, important questions related to the judiciously choice of a parametric structure to best capture the observed dynamics of the seasonal conditional heteroskedasticity also arises. This is the theme of the present paper.

In order to motivate the new developments, it is helpful to recall some commonly used time-series models to forecast seasonality in the mean. The framework generally adopted in that context is that of seasonal ARIMA models, possibly involving an unobserved component structure, as discussed by Nerlove, Grether and Carvalho (1979), Bell and Hillmer (1984), Hylleberg (1986), and Ghysels (1994a), among others. The basic idea of a linear time-invariant autoregressive structure involving seasonal lags can easily be adopted as a possible parameterization for the conditional variance. Such would lead to a seasonal ARCH model, as used, for instance, by Bollerslev and Hodrick (1994). An alternative approach to analyzing the mean behavior of seasonal time series is to employ ARIMA models whose parameters change seasonally. Initially proposed by Gladyshev (1961), such models have gained considerable interest in recent years. These models, referred to as periodic models because of the seasonal parameter variation, are now well documented both with respect to their theoretical properties as well as their empirical relevance. In particular, Tiao and Grupe (1980) establish the formal link between the former class of models, namely, seasonal ARIMA models, and periodic ARIMA models, while empirical evidence supporting periodic linear structures for a wide variety of macroeconomic time series may be found in Osborn (1988), Osborn and Smith (1989) and Ghysels and Hall (1992), among others.

The same periodic parameter variation originally proposed to capture the repetitive seasonal behavior in the conditional means of economic time series may be used to formulate conditional heteroskedasticity analogues of periodic ARIMA models. In its simplest form, it is natural to consider a periodic ARCH, or P-ARCH, model, in which the autoregressive conditional heteroskedasticity is characterized by seasonally varying autoregressive coefficients. Then by analogy to the results for periodic ARMA structures for the mean obtained by Tiao and

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2 This paper proposes a class of models featuring periodicity in conditional heteroskedasticity. An alternative approach, not pursued here, is that of stochastic switching-regime models with a different Markov switching scheme for each season; see Ghysels (1993, 1994b).
Grupe (1980), it is possible to establish an implicit relationship between the P-ARCH class of models and weak ARCH type models with seasonal lags. By this relationship a seasonal ARCH representation entails an informational loss in efficiency relative to the true P-ARCH model and by the same token, the P-ARCH model outperform the seasonal ARCH representation in terms of volatility predictability. It is important to recognize, that this analogue between ARCH models and linear ARMA structures in the mean only goes through when considering the linear projections figuring in weak ARCH models, as defined by Drost and Nijman (1993). A strong ARCH structure, which is often implicitly imposed in Maximum Likelihood estimation, does not yield a direct correspondence between a representation with seasonality in the laws and one with seasonality in the lags. Consequently, the informational loss alluded to above may be even more severe for ARCH models than with linear ARMA structures.

The plan of the rest of the paper is as follows. Section 2 is devoted to the definition of the new class of P-GARCH models, including a discussion of their relationship with more conventional seasonal GARCH models. In section 3 the practical estimation of P-GARCH models and the finite sample loss in efficiency from not modelling the periodic autoregressive structure is assessed through a small scale Monte Carlo experiment. The estimation results in section 4 for a daily time series of Deutschmark - British Pound exchange rates illustrate the empirical relevance of periodic ARCH models in characterizing the seasonality in the volatility of asset returns. Section 5 concludes with a discussion of possible extensions of periodicity in conditional heteroskedasticity to the case of P-IGARCH and P-EGARCH models, along with possible discrete time periodic representations of stochastic volatility models subject to time deformation, as well as issues related to multivariate ARCH models and the possibility of periodicity in common persistence.

2. Periodic GARCH Models

Following the seminal paper by Engle (1982), the ARCH class of models for time varying conditional heteroskedasticity have become very widely applied. To illustrate the ideas, consider the ARCH(q) model for the discrete time real valued stochastic process, \( \{e_t\} \).

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3 Bollerslev, Chou and Kroner (1992) provide a survey of empirical applications in finance, while the important theoretical developments have been surveyed recently by Bera and Higgins (1993) and Bollerslev, Engle and Nelson (1994).
\[ E[\varepsilon_t \mid I_{t-1}^{\varepsilon}] = 0, \]  
(1)

\[ E[\varepsilon_t^2 \mid I_{t-1}^{\varepsilon}] = \sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2, \]  
(2)

where \( I_{t-1}^{\varepsilon} \) denotes the usual Borel \( \sigma \)-field filtration based on the realization of the \( \{\varepsilon_t\} \) process up to time \( t-1 \). Now, instead of having a fixed parameter structure for the conditional variance equation, it is possible to draw on the similarity of the AR(q) model and periodic AR processes, and consider a time-varying coefficient model for conditional heteroskedasticity according to which,

\[ E[\varepsilon_t^2 \mid I_{t-1}^{\varepsilon}, s] = \sigma_{st}^2 = \sum_{s_t=1}^{S} d_{st} \sigma_s + \sum_{i=1}^{q} \sum_{s_t=1}^{S} d_{st} \alpha_i \varepsilon_{t-i}^2, \]  
(3)

where \( d_{st} = 1 \) if \( s \) is the stage of the periodic cycle at time \( t \), and \( d_{st} = 0 \) otherwise. Note that \( s \) appears in the conditioning set in equation (3), so that the coefficients of the model are allowed to vary periodically with the observable stage of a cycle with length \( S \).\(^4\) The most straightforward case is where the periodic cycle is purely repetitive; i.e., \( d_{st} = 1 \) if \( s = t \mod S \), and zero otherwise. In most empirically relevant cases, however, \( s \) may be governed by a variable deterministic cycle with upper bound \( S \). For example, with daily data non-trading days usually take place after every fifth trading day, but some weeks have holidays which interrupt the regular weekly pattern. In such a case, \( S = 5 \), but not all trading day cycles actually attain five consecutive trading days.

The GARCH(p,q) model, introduced by Bollerslev (1986) often provides a more parsimonious representation than the ARCH(q) model for \( \sigma_t^2 \) defined in equation (2). Specifically, this model postulates that,

\(^4\) The lag length \( q \) in equation (3) is assumed to be independent of \( S \). This entails no loss in generality, as \( q \) may be set equal to the maximal order of lags across all periods.
\[
\sigma_i^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{i-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{j-i}^2,
\]

(4)

or equivalently,

\[
\varepsilon_i^2 = \omega + \sum_{i=1}^{\text{Max}(p,q)} (\alpha_i + \beta_i) \varepsilon_{i-i}^2 + \nu_i - \sum_{j=1}^{p} \beta_j \nu_{i-j},
\]

(5)

where \( \nu_i \equiv \varepsilon_i^2 - \sigma_i^2 \), and \( \alpha_i \equiv 0, \beta_j \equiv 0 \) for \( i > q \) and \( j > p \), respectively. Note, by definition, the \( \{\nu_i\} \) process is serially uncorrelated with mean zero. Hence, the representation of the GARCH\((p,q)\) process in equation (5) may be interpreted as an ARMA\((\text{Max}(p,q),p)\) process for \( \{\varepsilon_i^2\} \). Suitable regularity conditions, as discussed by Bollerslev (1986), ensure that the \( \{\varepsilon_i^2\} \) process is covariance stationary and hence has a Wold representation and a spectral decomposition. Now, defining \( v_{st} \equiv \varepsilon_i^2 - E[\varepsilon_i^2 | \varepsilon_{s-1}, s] \equiv \varepsilon_i^2 - \sigma_{st}^2 \), the analogue of a P-GARCH process becomes quite apparent,

\[
\varepsilon_i^2 = \sum_{s=t}^{S} \omega_s + \sum_{i=1}^{\text{Max}(p,q)} \sum_{s=t}^{S} d_{st} (\alpha_s + \beta_s) \varepsilon_{i-s}^2 - \sum_{j=1}^{p} \beta_s \nu_{i-j} + \nu_{st}.
\]

(6)

Of course, for the P-GARCH\((p,q)\) model to be well defined, the conditional variance, \( \sigma_{st}^2 \), must be positive almost surely. Necessary and sufficient conditions on the \( \omega_s, \alpha_s \) and \( \beta_s \) parameters for this to hold true may easily be verified on a case-by-case basis following the approach in Nelson and Cao (1992). Analogously to the periodic ARMA processes discussed above, it is possible to interpret the P-GARCH model defined in equation (6) as an ARMA process for \( \{\varepsilon_i^2\} \) with a time-varying but periodic correlation structure; see Bollerslev (1988) for a discussion of the autocorrelation structure of the time-invariant GARCH\((p,q)\) model.

The similarities between periodic ARMA and periodic GARCH processes do not carry through straightforwardly, however. In particular, because of the non-linearities, the above class of GARCH processes defined in terms of conditional expectations is not closed under temporal and cross-sectional aggregation; see Drost and Nijman (1993) and Nijman and Sentana (1993).
It is therefore not possible without further qualifications to straightforwardly apply the formula derived by Tiao and Grupe (1980) to characterize the relationship between P-GARCH and seasonal fixed parameter GARCH processes. However, the complications that arise from the non-linearities in the GARCH formulation may be circumvented by considering the wider class of weak GARCH processes. To facilitate this discussion, it is convenient to rewrite the model in equation (6) as,

$$\sigma^2_{st} = \omega_s + \alpha_s(L)\epsilon^2_t + \beta_s(L)\sigma^2_{st},$$  \hspace{1cm} (7)$$

where $\alpha_s(L) = \sum_{i=1,q} \alpha_{is} L^i$ and $\beta_s(L) = \sum_{j=1,p} \beta_{js} L^j$. Following Drost and Nijman (1993), $\{\epsilon_t\}$ is then defined to follow a weak P-GARCH process when $\sigma^2_{st}$ in (7) corresponds to the best linear projection of $\epsilon^2_t$ on the space spanned by $\{1, \epsilon_{t-1}, \epsilon_{t-2}, \ldots, \epsilon^2_{t-1}, \epsilon^2_{t-2}, \ldots\}$ given period $s$,

$$E[\epsilon^2_t - \sigma^2_{st} \mid s] = E[(\epsilon^2_t - \sigma^2_{st}) \epsilon_{t-s} \mid s] = E[\epsilon^2_t - \sigma^2_{st} \mid s] = 0 \hspace{1cm} (8)$$

for $i = 1, 2, \ldots$. This also gives rise to the alternative representation,

$$\epsilon^2_t = \omega_s + [\alpha_s(L) + \beta_s(L)]\epsilon^2_t + [1 - \beta_s(L)]\eta_{st},$$  \hspace{1cm} (9)$$

where $\eta_{st} = \epsilon^2_t - P(\epsilon^2_t \mid \epsilon^2_{t-1}, \epsilon^2_{t-2}, \ldots, \epsilon_{t-1}, \epsilon_{t-2}, \ldots, s)$, and $P(\cdot)$ denotes the corresponding linear projection.\(^5\) Note, the projections in (8) still involve seasonal conditioning and therefore produce a periodic autocorrelation structure. The specification of a weak P-GARCH model obviously entails an informational loss when compared to the conventional P-GARCH formulation in equation (6), which defines $\sigma^2_{st}$ as the conditional expectation of $\epsilon^2_t$ based on the full information set implied by the Borel $\sigma$-field filtration of $\{\epsilon_t\}$ augmented with seasonal conditioning.

However, by considering the wider class of weak GARCH processes defined above, it is possible to carry out the mechanics of the Tiao and Grupe (1980) formula. This formula essentially amounts to the removal of the seasonal conditioning in (8) by averaging out the autocorrelation structure across all seasons, thus resulting in a time invariant seasonal weak\(^5\)

\(^5\) It is worth noting at this point, that in many practical applications one may restrict the $\beta_s(L)$ polynomial to be independent of $s$, resulting in a P-GARCH process with periodic patterns only in the autoregressive part of the model.
GARCH processes. Formally this operation is most easily accomplished by constructing a skip-sampled vector representation of the squared residuals collecting all observations over a single periodic cycle. Since there are S such squared residuals, define $\varepsilon_t = (\varepsilon_{S_t}^2, \varepsilon_{S(t-1)+S-1}^2, \ldots, \varepsilon_{S(t-1)+2}^2, \varepsilon_{S(t-1)+1}^2)$, where $t$ refers to the time index. Likewise, define $\Pi_t$ as the $S \times 1$ vector of innovations that appear in the weak $p$-GARCH($p,q$) model. Since the vectors obtained this way cover an entire periodic cycle, they encompass all possible parameter variations. To illustrate, consider the simple case where $S = 2$ with alternating periods, each of which obey a GARCH(1,1) structure. Then using equation (9), the following bivariate vector system becomes apparent,

$$
\begin{pmatrix}
1 & -\alpha_1^2 \beta_{12} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{2t}^2 \\
\varepsilon_{2(t-1)+1}^2
\end{pmatrix}
= 
\begin{pmatrix}
\omega_2 \\
\omega_1
\end{pmatrix}
+ 
\begin{pmatrix}
0 & 0 \\
\beta_1 & 0
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{2(t-1)}^2 \\
\varepsilon_{2(t-2)+1}^2
\end{pmatrix}
+ 
\begin{pmatrix}
\beta_{12} & 0 \\
0 & \beta_{11}
\end{pmatrix}
\eta_{22t}
$$

(10)

or more compactly,

$$
\varepsilon_t^2 = 
\begin{pmatrix}
\omega_2 + (\alpha_1^2 + \beta_{12})\omega_1 \\
\omega_1
\end{pmatrix}
+ 
\begin{pmatrix}
(\alpha_1 + \beta_{11})(\alpha_1^2 + \beta_{12}) \\
\alpha_1 + \beta_{11}
\end{pmatrix}
\varepsilon_{t-1}^2 + 
\begin{pmatrix}
\beta_{11} & \beta_{11}(\alpha_1^2 + \beta_{12}) \\
0 & \beta_{11}
\end{pmatrix}
\eta_t.
$$

(11)

Note, equation (11) defines a time invariant representation for the $\{\varepsilon_t^2\}$ process. Thus, given that $(\alpha_1 + \beta_{11})(\alpha_1^2 + \beta_{12}) < 1$, so that the eigenvalues of the first-order autoregressive lag matrix are greater than one, the $\{\varepsilon_t^2\}$ process has a Wold decomposition and a spectral

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6 Osborn (1991) notes the similarities between the operation of averaging out correlations and that of cross-sectorial aggregation.

7 For instance, in applications of periodic models to quarterly or monthly data $t$ could be an annual time index. In the application of periodic ARCH processes to daily or intraday sampling frequencies, $t$ may correspond to a weekly time scale with a vector representation of daily series, or to the daily sampling of a vector of hourly processes.

8 This example also forms the basis for the numerical simulations reported in section 3 below.

9 Note, unlike usual multivariate ARCH processes as surveyed by Bollerslev, Engle and Nelson (1994), the vector ARCH process in (10) or (11) does not involve any conditional cross-covariances, as each element of the $\{\varepsilon_t^2\}$ vector process represent the same underlying univariate process sampled at a different points in time.
representation. Of course, these arguments are not constrained to the P-GARCH(1,1) model with $S = 2$. Under suitable regularity conditions ensuring that the roots of the characteristic autoregressive polynomial for the corresponding vector system have modulus greater than one, any weak P-GARCH(p,q) process may be expressed in terms its fundamental time invariant MA representation,

\[ \varepsilon_t = \omega + \sum_{j=0}^\infty A_j \eta_{t-j} = \omega + A(L)\eta_t, \]  

(12)

where the elements of the $\omega$ vector and $A_j$ coefficient matrices are determined by the polynomials for the underlying model defined in equation (9). From equation (12), the multivariate covariance generating function and spectral representation may therefore be written as,

\[ F(e^{i2}) = A(e^{i2}) Q A(e^{i2})^*, \]  

(13)

where $-\pi \leq z \leq \pi$, and $Q$ denotes the unconditional covariance matrix for the $S$ dimensional innovation process, $\{\eta_t\}$.

Now, following Tiao and Grupe (1980), it is possible to show that there exist parameters $\sigma^2_\eta, \alpha_1, ..., \alpha_P$ and $\beta_1, ..., \beta_Q$ such that,

\[ \sigma^2_\eta \left[ \sum_{j=1}^{\text{Max}(P,Q)} (\alpha_j + \beta_j) e^{-i2} \right]^2 \right] \]  

(14)

where $R(e^{-i2}) = S^{\frac{1}{2}}(1, e^{-i2}, ..., e^{-(S-1)i2})$. Equation (14) establishes a relationship between the parameters for the P-GARCH(p,q) model and the corresponding weak GARCH(P,Q) process with parameters $\alpha_i$ and $\beta_j$ for $i = 1, ..., P$ and $j = 1, ..., Q$. Unfortunately, this relationship is only implicit, and does lend itself to an analytical characterization of the weak GARCH(P,Q) parameters as an explicit function of the $S$ parameter vectors for the underlying P-GARCH(p,q)
model\textsuperscript{10}, see Osborn (1991) for further discussion along these lines within the context of periodic ARMA processes. In the numerical results reported on below, the practical implications of this implicit correspondence is therefore explored through the actual estimation of seasonal GARCH models for a P-GARCH data generating processes.\textsuperscript{11}

3. Estimation and Inference

A variety of estimation and testing procedures have been suggested for conducting inference in ARCH type models; for a more extensive discussion of these procedures we refer to the survey articles by Bera and Higgins (1993), and Bollerslev, Engle, and Nelson (1994). The scope of this section is not to contribute to the basic theory of estimation and hypothesis testing in ARCH models as such. Instead, our aim is merely to comment on some of the specific issues that arise in the estimation and testing of periodic ARCH structures.

To illustrate, let $\theta$ denote the vector of unknown parameters for all $S$ seasons; i.e., $\theta' = (\theta_1', \theta_2', ..., \theta_S')$ where for the P-GARCH$(p,q)$ model $\theta_s = (\omega_s, \alpha_{1s}, ..., \alpha_{qs}, \beta_{1s}, ..., \beta_{ps})$. The log likelihood function for a strong P-ARCH model may then be conveniently written as the sum of the corresponding conditional log likelihoods for each of the $S$ seasonal cycles,

$$L_T(\theta | l_t^S) = \sum_{t=1}^{T} \log f_t(\epsilon_t; \theta).$$

\textsuperscript{10} It is fairly easy to show, however, that the autoregressive order of the time invariant weak GARCH process necessarily exceeds the order of the periodic cycle; i.e., $P \geq S$ whenever $p \neq 0$.

\textsuperscript{11} It is difficult to evaluate the loss in efficiency or prediction accuracy foregone from ignoring seasonal conditioning as the error distributions are time-varying; see the discussion in Baillie and Bollerslev (1992) and Bollerslev, Engle and Nelson (1994) pertaining to prediction and model evaluation within a general ARCH context. However, the unconditional MSE for $\epsilon_t^2$ from the weak P-GARCH$(p,q)$ model may be expressed as the sum of the MSE's for each of the $S$ seasons,

$$\text{MSE}_{\text{WPG}} = \sum_{s=1}^{S} \exp \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log 2\pi F_s(e^{-iz}) \, dz \right],$$

where $F_s(\cdot)$ denotes the $s$'th diagonal element of the spectral density matrix in (13); see Kolmogorov (1941) and Janacek (1975). Similarly, the MSE for the corresponding weak GARCH$(P,Q)$ model with spectral representation in equation (14) may be written as,

$$\text{MSE}_{\text{WG}} = \exp \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log 2\pi R(e^{-iz}) \, F(e^{i\xi}) \, R(e^{-iz'}) \, dz \right].$$

It is important to recognize the limitations of the simple MSE criterion in this context, however, as it assigns equal weights to the errors in a heteroskedastic environment. Also note, that the informational loss associated with the relaxation from strong to weak P-GARCH is not being assessed by these expressions.
Furthermore, assuming that the one-step ahead prediction errors are conditional normally distributed,

\[ l_{ai}(c_i; \theta) = -0.5 \{ \ln(2\pi) + \ln(\sigma_{ai}^2(\theta)) + \hat{e}_{ai}^2 / \sigma_{ai}^2(\theta) \} . \]  

Under appropriate regularity conditions the Maximum Likelihood Estimates (MLE) for the true parameters \( \theta_0 \), say \( \hat{\theta}_T \), obtained by maximizing equation (15) then satisfies,

\[ T^{1/2} \left( \hat{\theta}_T - \theta_0 \right) \xrightarrow{d} N(0, A(\theta_0)^{-1}) , \]  

where \( A(\theta_0) \) denotes the Hessian evaluated at \( \theta_0 \). In many applications with high frequency data the assumption of conditional normality underlying equation (16) may be violated, however. Fortunately, \( \hat{\theta}_T \) remains consistent under quite general conditions, and may be given a Quasi MLE (QMLE) interpretation. Since the outer product of the gradients and the inverse of the Hessian doesn’t cancel out in this situation, the asymptotic covariance matrix for the QMLE takes the form \( A(\theta_0)^{-1}B(\theta_0)A(\theta_0)^{-1} \), where \( B(\theta_0) \) is equal to the outer product of the gradients; for further discussion along these lines and a formal proof for the GARCH(1,1) case see Weiss (1986), Bollerslev and Wooldridge (1992), Lumsdaine (1992), and Lee and Hansen (1993). Interestingly, even though no formal analytical results are available for the weak ARCH case, the simulations in Drost and Nijman (1992) suggest that for large sample sizes the QMLE procedure is generally very reliable for the estimation of weak GARCH structures also. The simulations reported on below further corroborate this evidence.

Testing within the context of P-GARCH models poses no new conceptual difficulties. In particular, if estimates are available both under the null and alternative hypotheses, a conventional Wald statistic using an estimate of the robust covariance matrix outlined above may be employed. However, suppose that the null hypothesis of interest dictates the lack of a periodic structure; i.e., \( \theta_1 = \theta_2 = \ldots = \theta_s \). In this situation estimates may only be available under the null hypothesis, and a score test may be more convenient. Specifically, let \( \nabla_{\hat{\theta}} \theta_s(e_t; \theta) \) denote the score function for the \( t \)’th observation which belongs to season \( s \). Then, under appropriate regularity conditions, the scores for the full parameter vector evaluated under the null hypothesis will satisfy a martingale central limit theorem,
\[ T^{-1/2} \sum_{t=1}^{T} d_{st}^t \nabla_{\theta} \ell_{st}(c_t; \theta_0) \rightarrow N(0, A(\theta_0)) , \] (18)

which forms the basis for the standard asymptotic chi-square LM test calculated as quadratic form in \( \sum_{t=1}^{T} d_{st}^t \nabla_{\theta} \ell_{st}(c_t; \theta_0) \). This test is particularly easy to implement for the hypothesis that \( \theta_1 = \theta_2 = \ldots = \theta_s \), as it only requires estimates for the non-periodic specification, and then checks whether the score function evaluated with data from each of the \( S \) seasons separately is close to zero. The score test may also be robustified to fit within the QMLE context in order to accommodate the possibility of a misspecified conditional normal likelihood function; see Wooldridge (1990) and Bollerslev and Wooldridge (1992) for details. We shall not pursue this testing strategy any further here, however.

Instead, we now turn to the results from a small scale simulation study designed to gauge an idea about the reliability of the QMLE procedure applied to the estimation of P-GARCH structures, and to better assess the potential importance of allowing for seasonal conditioning in ARCH modeling. The estimations are based on a sample size of \( T=2000 \), which is comparable to the number of observations in the empirical example presented in the next section. The true Data Generation Process (DGP) is a P-GARCH(1,1) model with a periodic cycle of two periods.\(^{12}\) The model parameters are indicated at the top of table 1. Note, that even though \( \alpha_1 + \beta_1 > 1 \), the model is covariance stationarity as \( (\alpha_1 + \beta_1)(\alpha_{12} + \beta_1) < 1 \), with the other model parameters defined so that the unconditional variance equals unity, whereas the unconditional seasonal variances equal 1.1 and 0.9, respectively.\(^{13}\) All of the reported results are based on \( N=1000 \) replications.\(^{14}\)

It is immediately clear from panel A of the table, that the MLE procedure does a remarkable good job of uncovering the true unknown parameters for the correctly specified P-

\(^{12}\) Additional simulation results for other P-GARCH formulations and non-periodic GARCH models with seasonal dummies in the conditional variance equation are available upon request.

\(^{13}\) The unconditional seasonal variances equal \( \sigma_1^2 = [\omega_2(\alpha_1+\beta_1) + \omega_2][1 - (\alpha_1+\beta_1)(\alpha_{12}+\beta_1)]^{-1} \) and \( \sigma_2^2 = [\omega_2(\alpha_{12}+\beta_1) + \omega_2][1 - (\alpha_1+\beta_1)(\alpha_{12}+\beta_1)]^{-1} \), respectively.

\(^{14}\) To avoid start-up problems 2000 initial observations were discarded for each of the replications. The normal random variables were generated by the RNDNS subroutine in the GAUSS computer language.
GARCH(1,1) models in columns six and seven. The mean of the parameter estimates, $\hat{\theta}_T = \sum_{i=1}^{N} \hat{\theta}_{T,i}$ where $\hat{\theta}_{T,i}$ denote the estimate from the $i$th replication, are all very close to the underlying true values, $\theta_0$. It is also worth noting, that the sample standard errors for the estimates across the 1000 Monte Carlo replications reported in parentheses are in close accordance with the average standard error estimates based on the robustified covariance matrix estimator reported in square brackets. Conventional inference procedures for the P-GARCH model based on the asymptotic distribution in (17) with $A(\hat{\theta}_T)^{-1} B(\hat{\theta}_T) A(\hat{\theta}_T)^{-1}$ in the place of $A(\theta_0)^{-1}$ should therefore work well in practice. It is particularly noteworthy, that this finding carries over to the estimated weak GARCH models. The Monte Carlo standard errors and the average QMLE standard error estimates are similarly close for these models. Interestingly, the average estimates for $\alpha_1$ are approximately equal to $0.270$ for all three weak GARCH models. Also, the implied unconditional variances based on the average parameter estimates for the three weak GARCH models equal 0.966, 0.982 and 0.988 respectively, whereas the unconditional seasonal variances for the GARCH model with $\omega_1 \neq \omega_2$ equal 0.979 and 0.936, respectively.

The first two sets of rows in panel B report the average values of the Akaike (1973) and Schwarz (1978) information criteria; i.e., $\text{AIC} = N^{-1} \sum_{i=1}^{N} 2 - \ln(\hat{\theta}_{T,i}^T | \hat{\theta}_{0,i}^T) - 2k$ and $\text{SIC} = N^{-1} \sum_{i=1}^{N} 2 - \ln(\hat{\theta}_{T,i}^T | \hat{\theta}_{0,i}^T) - \ln(T) k$, where $k = \dim(\hat{\theta}_{T,i})$ refer to the number of estimated model parameters. The numbers reported in parentheses give the proportion of times that a particular model was favored by one of the two information criteria. It is evident that both selection criteria are very effective in discriminating between the true P-GARCH models and the corresponding non-seasonal weak GARCH models. The AIC criteria favors the P-GARCH model with $\omega_1 \neq \omega_2$ 16.7 percent of the times compared to the more parsimonious SIC criteria which correctly identifies that $\omega_1 = \omega_2$ in 99.3 percent of the replications.

The second group of numbers in panel B report the average simulated values for various loss functions designed to measure the difference between the true and the estimated conditional variances. The Mean Square Error criteria $\text{MSE}_A = N^{-1} T^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} [\sigma^2_{st}(\hat{\theta}_0) - \sigma^2_{st}(\hat{\theta}_{T,i})]^2$ is always minimized for one of the two P-GARCH formulations, with a slight advantage for the true DGP that restricts $\omega_1 = \omega_2$. The increase in the value of the $\text{MSE}_A$

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15 To start up the recursions for $\sigma^2_{st}(\theta)$, $\epsilon^2_{st}(\theta)$ and $\phi^2_{st}(\theta)$ were fixed at their unconditional seasonal sample analogous.

16 By a Central Limit Theorem argument, the Monte Carlo standard error for $\bar{\theta}_T$ may be consistently estimated by $N^{-1/2}$ times the sample standard error for the N estimates reported in parentheses.
criteria for the three weak GARCH models is quite dramatic. While the MSE loss criteria may be a natural choice in evaluating competing estimates for the mean, it is less obvious in a heteroskedastic environment. A natural alternative would be to consider the Heteroskedasticity adjusted MSE, or the relative squared error loss, defined by $\text{HMSE}_A = N^{-1} T^{-1} \sum_{i=1,N} \sum_{t=1,T} \left[ \sigma_{i,t}^2(\theta_0) \sigma_{i,t}^2(\hat{\theta}_T) - 1 \right]^2$. This criteria favors the P-GARCH model with $\omega_1 = \omega_2$ for 93.0 percent of the simulations. Interestingly, the average simulated value for $\text{HMSE}_A$ from this model equals only 0.013, compared to 0.096 for the non-periodic GARCH model with a seasonal dummy in the conditional variance equation. Ignoring the conditional heteroskedasticity result in a substantially larger loss of $\text{HMSE}_A = 5.561$. Similar findings are available for the Logarithmic Loss function, 

$$LL_A = N^{-1} T^{-1} \sum_{i=1,N} \sum_{t=1,T} \ln[\sigma_{i,t}^2(\theta_0) \sigma_{i,t}^2(\hat{\theta}_T)]^2,$$

which assigns proportionally higher weights to wrong predictions in low variance regimes.

Of course, in practical applications the true conditional variance is unknown, so that the loss function will have to be based on the difference between the estimated conditional variance and the realized squared residuals. One such criteria often used in practice is the MSE analog of $\text{MSE}_B = N^{-1} T^{-1} \sum_{i=1,N} \sum_{t=1,T} [\varepsilon_{i,t}(\theta_0) - \sigma_{i,t}(\hat{\theta}_T)]^2$. Interestingly, judged by this criteria the three non-periodic weak GARCH models and the true P-GARCH models all result in fairly similar average squared error losses, even though the two P-GARCH models yield the smallest loss among the different specifications for 37.9 and 40.0 percent of the replications, respectively. However, as noted above, the use of an MSE type loss function in a heteroskedastic environment is somewhat problematic. In particular, by the $\text{MSE}_B$ criteria a non-linear least squares regression of the squared residuals on all the variables in the time t-1 information will always produce the lowest possible in-sample loss. With this in mind, it is interesting to note that the the Heteroskedasticity adjusted MSE, $\text{HMSE}_B = N^{-1} T^{-1} \sum_{i=1,N} \sum_{t=1,T} \left[ \varepsilon_{i,t}(\theta_0) \sigma_{i,t}^2(\hat{\theta}_T) - 1 \right]^2$, almost unambiguously favor the P-GARCH specifications. The $\text{HMSE}_B$'s for the estimated P-GARCH models are just slightly below the implied true value of two, whereas the smallest

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17 As discussed in Bollerslev, Engle and Nelson (1994), the most natural loss function may ultimately depend upon the goals of the particular application; see also West, Edison and Cho (1993).

18 For instance, West and Cho (1994) use an MSE criteria in the evaluation of alternative models for the conditional variance of U.S. dollar exchange rates, while Pagan and Schwert (1990) employ the MSE and the LL loss function defined below in their analysis of competing specifications for U.S. stock market volatility.
average value for the three weak GARCH models is 2.162. The findings for the Logarithmic Loss function, \( \text{LL}_B = N^{-1} \frac{T-1}{N} \sum_{t=1}^T \ln(\Sigma e^2_{t,j}(\theta_0)\sigma^2_{t,j}(\theta^*_T))^{-1} \), are comparable.

Summing up, the simulation evidence for the P-GARCH(1,1) DGP reported in table 1, and similar results for other DGP’s available upon request, confirm the reliability of the QMLE based inference procedures in the P-GARCH context, including the estimation of implied weak GARCH models. The simulations also illustrate the potential importance and informational loss associated with neglected periodic heteroskedasticity. In the next section we shall therefore rely on the tools analyzed here in the specification and estimation of a simple P-GARCH model for a time series of daily exchange rates.

4. Modeling Periodicity in Daily Deutschmark - British Pound Exchange Rate Volatility

Several recent studies have found that the volatility of daily U.S. dollar exchange rates tend to be highly persistent and well approximated by an integrated or long-memory type GARCH process; see e.g. Engle and Bollerslev (1986), McCurdy and Morgan (1988), Baillie and Bollerslev (1989), Hsieh (1989), Baillie, Bollerslev and Mikkelsen (1993), and Taylor (1994). At the same time, Bollerslev and Engle (1993) argue that, even though the volatility processes for the daily Deutschmark - U.S. dollar and the British Pound - U.S. dollar exchange rates are both highly persistent, it appears that the non-stationarity is common across the two rates, so that the volatility of the corresponding bilateral Deutschmark - British Pound rate shows less persistence. Thus, to circumvent issues related to the appropriate modeling of the long-run volatility persistence, we shall here concentrate on the dynamics of the Deutschmark - British Pound rate. The data covers the period from January 3, 1984 through December 31, 1991, for a total of 1974 observations.\(^{19}\)

The logarithmic first differences of high frequency nominal spot exchange rates are widely accepted as being approximately uncorrelated through time; see op. cit. Following standard practice, we shall therefore concentrate on modeling the conditional variance of the mean adjusted percentage returns,

\[
y_t = 100 \{\ln(s_t) - \ln(s_{t-1})\} = \mu + \varepsilon_t,
\]

\(^{19}\) The Deutschmark - British Pound exchange rate is constructed from the corresponding daily U.S. dollar rates recorded by the IMF in International Financial Statistics.
for $t=1,2,...,1974$. The insignificant Ljung and Box (1978) portmanteau test for up to twentieth order serial correlation in $\varepsilon_t(\hat{\Theta}_T)$ reported in the first column of table 2, $Q(20)$, confirm that $\{y_t\}$ is an approximate martingale process.\textsuperscript{20} However, in line with previous evidence in the literature for other financial markets, the second column in the table indicate that the volatility in the Deutschmark - British Pound foreign exchange market is systematically higher on days following market closures. The estimate for the periodic non-trading day dummy, $d_{st}$, in the variance for $\varepsilon_t$ is highly significantly, and suggests an average increase in the volatility of about fifty percent.\textsuperscript{21} At the same time, the very large value of the $Q^2(20)$ portmanteau test for the squared residuals, $\varepsilon_t^2(\hat{Theta})$, indicate that the inclusion of this dummy variable in the variance does not explain any of the pronounced volatility clustering in the exchange rate changes. The last three rows of the table therefore report the QMLE for a simple GARCH(1,1) formulation, a GARCH(1,1) model that includes the seasonal non-trading day dummy in the conditional variance equation, and a P-GARCH(1,1) model. Following the analysis in Baillie and Bollerslev (1989), the $d_{st}$ dummy variable is entered in the conditional variance equation to allow for an impulse effect of the market closure,

$$\sigma_{st}^2 = \omega + d_{st}\omega_s - (\omega + d_{st-1}\omega_s)(\alpha_1 + \beta_1 + d_{st}\alpha_{1s}) + \alpha_1 \varepsilon_{t-1}^2 + d_{st}\alpha_{1s} \varepsilon_{t-1}^2 + \beta_1 \sigma_{st-1}^2. \quad (20)$$

The estimates and summary statistics from these three models clearly illustrates, that even though the GARCH(1,1) model parsimoniously captures the own temporal dependence in the second order moments of the returns, the non-trading day effects remain very significant. Both the AIC and the SIC information criteria strongly favor the inclusion of the seasonal dummy variable in the conditional variance equation of the GARCH(1,1) model. However, the same information criteria as well as the robust t-statistic for the $\alpha_{1s}$ parameter estimate from the P-GARCH(1,1) model suggest the importance of allowing for a dynamic periodic structure in better characterizing the non-trading day effect. Interestingly, the sum of the estimated autoregressive coefficients for the P-GARCH model when $d_{st} = 0$ equals $\hat{\alpha}_1 + \hat{\beta}_1 = 1.000$, whereas on days

\textsuperscript{20} Note that the nominal significance levels for the standard portmanteau test for serial correlation in the mean tend to be very conservative in the presence of ARCH effects; see e.g. the simulation evidence in Diebold (1988) and Bollerslev and Mikkelsen (1993).

\textsuperscript{21} The $d_{st}$ dummy is defined to be equal to one on Mondays and other days following closures of the Deutschmark or the British Pound - U.S. dollar market during regular European trading hours. The sample consist of 456 such non-trading periods, corresponding to 23.1 percent of the observations.
following market closures $\hat{a}_1 + \hat{a}_{1s} + \hat{\beta}_1 = 0.889$. Thus, shocks to the conditional variance that occur when the market is closed appear to be less informative about the future volatility and tend to die out at a faster rate than shocks that occur during regular trading days. As noted above, the gain obtain by allowing for the richer P-GARCH structure will ultimately depend upon the particular application of the model. It is noteworthy, however, that the Heteroskedasticity adjusted MSE criteria advocated in the previous section is also minimized for the P-GARCH specification.

5. Conclusion and Extensions to Other Periodic Heteroskedastic Models

Of course, the idea of using periodic structures in formulating time series models for conditional heteroskedasticity is not limited to the GARCH class of models, but may be exploited for other parametric formulations, both univariate and multivariate. In concluding, we shall discuss several such possible extensions of the basic P-GARCH structure developed above, in the hope that they may lead to a better understanding and/or prediction of the observed volatility clustering phenomenon. It would be easy to simply take the different parametric models that have been suggested in the literature, and then define a periodic version for each; hence, EGARCH would lead to P-EGARCH, IGARCH to P-IGARCH, ARCH-M to P-ARCH-M, TARCH to P-TARCH, STARCH to P-STARCH and so on. Obviously, such an unguided generalization would not be very useful. Instead, we shall here focus on a few specific cases which we believe would be especially interesting to pursue in future work.

As noted above, in many applications of the conventional GARCH(p,q) model to high frequency data, it is often found that the parameter estimates for the autoregressive polynomial sum to approximately one. This observation was the primary motivation behind the definition of the so-called Integrated GARCH(p,q), or IGARCH(p,q), class of models proposed by Engle and Bollerslev (1986). In the IGARCH(p,q) model $\alpha(1) + \beta(1) = 1$. For periodic heteroskedasticity, it may therefore be useful to extend the IGARCH class of models to the P-IGARCH(p,q) process defined in equation (6), with the restriction that, $\alpha_s(1) + \beta_s(1) = 1$ for all $s = 1, ..., S$. Such a restriction provides a direct ARCH analogue of the I(1) restriction in linear periodic ARMA processes studied by Ghysels and Hall (1993).

Multivariate ARCH models are easily over-parameterized, and direct periodic multivariate extensions of existing parameterizations surely will not make it any easier. It is possible, however, that by appropriately restricting the parameterization of the long-run implications of the heteroskedastic periodicity feasible and particularly useful multivariate extensions may be
obtained. To illustrate, consider a multivariate ARCH model for the $N \times 1$ vector process, $\{c_t\}$, with $N \times N$ time-varying covariance matrix, $\Omega_t$. Bollerslev and Engle (1993) define the $\{c_t\}$ process to be co-persistent in variance if at least one element of $E[\Omega_{t+h} | \Omega_t] = \sigma^2 < \infty$ for increasing forecast horizon, $h$, yet there exists a non-trivial linear combination, $\{\gamma_t\}$, with finite unconditional variance; i.e., $\lim_{h \to \infty} E[\gamma_{t+h} | \Omega_t] = \sigma^2 < \infty$ for all $t$. A natural extension of this idea would be to allow for periodic co-persistence. Such a process would have to satisfy the condition that, $\lim_{h \to \infty} E[\gamma_t, \Omega_{s+t} \gamma_t, \gamma_t, \gamma_t, s] = \sigma^2 < \infty$ for all seasons, $s = 1, 2, ..., S$, and time periods, $t$. This definition of periodic co-persistence amounts to the restriction that the corresponding stacked multivariate process, $\{c_{st}\}$, containing all $S$ periods in analogy to the univariate stacked process defined in section 2, has a non-trivial linear combination, $(\gamma_{s1}, ..., \gamma_{sS})$, which is co-persistent in the sense of Bollerslev and Engle (1993). Such a condition might naturally help to ensure a more parsimonious multivariate seasonal representation.

It is generally agreed that the GARCH(p,q) model in equation (4) captures most of the predictable variation in the volatility of high frequency asset returns. Yet, the simple GARCH(p,q) model doesn’t allow for any asymmetric responses in the conditional variance function, sometimes referred to as leverage effects in the context of stock market volatility. To circumvent this shortcoming Nelson (1991) proposed the Exponential GARCH, or EGARCH, model. In the EGARCH model $(\ln(\sigma^2_t))$ follows an ARMA process, in which the innovation sequence is constructed in such a way that positive and negative shocks have a distinct effect on the future conditional variances. Extending this formulation to allow for seasonal conditioning, the corresponding unconstrained P-EGARCH(p,q) model may be defined by,

$$
\ln(\sigma^2_{st}) = \omega_s + \alpha_s(L) \beta_s(L)^{-1} (\theta_s z_{t-1} + \gamma_s [ | z_{t-1} | - E[| z_{t-1} |]) \tag{21}
$$

where $z_t = \varepsilon_t \sigma^{2}_{st}$ has conditional mean zero and unit variance. Obviously, the process in (21) is easily over-parameterized. For instance, suppose we restrict attention to the periodicity due to non-trading day effects as analyzed in the previous section; i.e., $S = 2$, but with a variable though perfectly predictable periodic pattern. Then, even the simple P-EGARCH(1,1) model involves a total of ten parameters. For higher order models and more complicated periodic cycle, the number of parameters increases rapidly at the rate $S((p+q) + 3)$, so that some simplifying assumptions will have to be imposed. The case where $\alpha_s(L) = \alpha(L)$ and
$\beta_s(L) = \beta(L)$ for all $s$ corresponds to a process with periodic asymmetries. Hence, a negative shock after, say a non-trading day, may have a different impact than on any other day. Conversely, with $\theta_s = \theta$ and $\gamma_s = \gamma$ for all $s$, but periodic ARMA polynomials, the dynamic responses are similar to the P-GARCH($p,q$) model analyzed above.

In the analysis of theoretical pricing models in finance, it is often more convenient to work with continuous time stochastic differential equations, as opposed to discrete time stochastic ARCH type difference equations. To that end, Nelson (1990) has shown, that when the process is sampled at increasingly higher frequencies, the non-periodic EGARCH(1,1) model defined in (21) consistently approximates the continuous time stochastic volatility model,

$$d[\ln(y_t)] = \theta \sigma_t^2 \, dt + \sigma_t \, dW_{1t},$$  \hspace{1cm} (22)

$$d[\ln(\sigma_t^2)] = -\alpha [\ln(\sigma_t^2) - \beta] \, dt + dW_{2t},$$  \hspace{1cm} (23)

where $W_{1t}$ and $W_{2t}$ denote independent standard Wiener processes. In an extension of this formulation, Ghysels and Jasiak (1994) have recently proposed a new class of stochastic volatility models with time deformation. According to this formulation the Ornstein-Uhlenbeck process in (23) evolves in operational time, $\tau = g(t)$, as opposed to calendar time, $t$,

$$d[\ln(\sigma_t^2)] = -\alpha [\ln(\sigma_t^2) - \beta] \, d\tau + dW_{2\tau},$$  \hspace{1cm} (24)

Assuming that the operational time scale is measurable with respect to the usual calendar time filtration, Ghysels and Jasiak (1994) demonstrate, that a natural discrete time representation for the stochastic volatility model defined by equations (22) and (24) is given by,

$$\ln[y_t] - \ln(y_{t-1})^2 = \kappa + \ln(\sigma_t^2) + \xi_t,$$  \hspace{1cm} (25)

$$\ln(\sigma_t^2) = \exp[\lambda (g(t) - g(t-1))] \ln(\sigma_{t-1}^2) + \xi_t.$$  \hspace{1cm} (26)

\footnote{See Andersen (1993b) for a discussion of ARCH and stochastic volatility models.}
where \( \kappa \) and \( \lambda \) denote scalar constants, and the two innovation processes, \( \{ \zeta_t \} \) and \( \{ \xi_t \} \), are serially uncorrelated with mean zero.\(^{23}\) Candidate variables for explaining the pace of the operational clock in high frequency financial applications include past trading volume and absolute price changes. If however, the time deformation is taken to be purely deterministic and only dependent on the season, it follows that the discrete time process for the conditional variance may be written as,

\[
\ln(\sigma_{st}^2) = \sum_{s=1}^{S} d_{st} \phi_s \ln(\sigma_{s,t-1}^2) + \xi_{st}. \tag{27}
\]

Note, both the AR coefficients and the innovation variances take on different values according to the season. The stochastic volatility model with seasonal time deformation therefore provides a rational for the estimation of discrete time periodic ARCH type models. We leave further theoretical and empirical work on this particular model and the other heteroskedastic periodic structures discussed above for future work.

\(^{23}\) See Ghysels and Jasiak (1994) and Harvey, Ruiz and Shephard (1994) for further details.
Table 1, Panel A

Finite Sample Distributions for P-GARCH(1,1) Data Generating Process

\[ y_t = \mu + \varepsilon_t \]
\[ \varepsilon_t | \varepsilon_{t-1} \sim N(0, \sigma^2_t) \]
\[ s = 1, 2 \]
\[ T = 1, 2, \ldots, 2000 \]
\[ \sigma^2_t = \omega + d_1 \omega_j + \alpha_1 \varepsilon_{t-1}^2 + \ldots + \alpha_4 \varepsilon_{t-4}^2 + d_1 \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma^2_{t-1} \]
\[ \mu = 0.0 \]
\[ \omega = 0.05 \]
\[ \omega_j = \omega_2 = 0.0 \]
\[ \alpha_1 = \ldots = \alpha_4 = 0.0 \]
\[ \alpha_{1f} = 0.4666 \]
\[ \alpha_{1z} = 0.0727 \]
\[ \beta_1 = 0.7 \]

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Table 1, Panel B

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Notes: Panel A of the table reports the mean of the Quasi Maximum Likelihood Estimates (QMLE) for the different models across the N=500 Monte Carlo replications under the P-GARCH(1,1) Data Generating Process (DGP) indicated at the top of the table. The sample standard deviations for the 500 estimates are reported in parentheses, with the corresponding mean of the QMLE based standard error estimates in square brackets. Panel B of the table gives the average value of the Akaike (AIC) and Schwartz Information Criterion (SIC) for each of the different model specifications. MSE_A and MSE_B denote the average Mean Square Error for the true conditional variance and squared innovations; i.e., N^{-1} T^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} (\sigma_t^2(\theta_0) - \sigma_t^2(\hat{\theta}_t))^2 and N^{-1} T^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} (\epsilon_t^2(\theta_0) - \epsilon_t^2(\hat{\theta}_t))^2 respectively, where i indicate the order of the replication. HMSE_A and HMSE_B refer to corresponding Heteroskedasticity adjusted MSE; i.e., N^{-1} T^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} (\hat{\sigma}_t^2(\theta_0)\sigma_t^2(\hat{\theta}_t) - 1^2 and N^{-1} T^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} (\epsilon_t^2(\theta_0)\epsilon_t^2(\hat{\theta}_t) - 1^2. The Logarithmic Loss functions LL_A and LL_B are calculated as N^{-1} T^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \ln(\sigma_t^2(\theta_0)\sigma_t^2(\hat{\theta}_t) - 1^2 and N^{-1} T^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \ln(\epsilon_t^2(\theta_0)\epsilon_t^2(\hat{\theta}_t) - 1^2. The proportion of times that a specification was favored by one of the information criteria or a particular loss function are reported in parentheses.
Table 2
GARCH Models for Daily Deutschemark - British Pound Exchange Rate

\[ y_t = 100\{\ln(s_t) - \ln(s_{t-1})\} = \mu + \varepsilon_t \]

\[ \varepsilon_{t+1} \mid \varepsilon_{t+1}^2 \sim N(0, \sigma_{\varepsilon_{t+1}}^2) \quad s=1,2 \quad T=1,2,...,1974 \]

\[ \sigma_{\varepsilon_t}^2 = \omega + \alpha_t \varepsilon_{t-1}^2 + \beta_t \sigma_{\varepsilon_{t-1}}^2 \]

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<td>( \mu )</td>
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<td>(0.009)</td>
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<tr>
<td>( \omega )</td>
<td>0.221</td>
<td>0.200</td>
<td>0.264</td>
<td>0.259</td>
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<td>(0.012)</td>
<td>(0.011)</td>
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<td>( \omega_2 )</td>
<td>-</td>
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<td>( \alpha_1 )</td>
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<td>-</td>
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<td>0.137</td>
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<td>( \alpha_{1s} )</td>
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<tr>
<td>( \beta_1 )</td>
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AIC: -2626.2, -2601.1, -2221.2, -2191.0, -2179.3
SIC: -2652.5, -2617.9, -2243.5, -2219.0, -2212.9

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<td>( b_4 )</td>
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<td>Q(20)</td>
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<td>Q^2(20)</td>
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<td>LL</td>
<td>9.67</td>
<td>9.52</td>
<td>8.79</td>
<td>8.21</td>
<td>8.66</td>
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Notes: The table reports the Quasi Maximum Likelihood Estimates (QMLE) for the daily percentage returns on the Deutschemark - British Pound exchange rate from January 3, 1984 through December 31, 1991, for a total of 1,974 observations. Robust standard errors are reported in parentheses. The value of the corresponding Akaike and Schwartz Information Criteria are given in the AIC and SIC rows. The sample skewness and kurtosis for the standardized residuals, \( \xi_t(\beta_t)\sigma_{\varepsilon_t}^2(\beta_t)^{-1/2} \), are denoted by \( b_3 \) and \( b_4 \), respectively. Q(20) and Q^2(20) refer to the Ljung-Box portmanteau tests for up to twentieth order serial correlation in the standardized and the squared standardized residuals, respectively. The three different loss criteria are calculated as MSE = \( T^{-1} \sum_{t=1,T} [\xi_t^2(\beta_t) - \sigma_{\varepsilon_t}^2(\beta_t)^{-1} - 1]^2 \), HMSE = \( T^{-1} \sum_{t=1,T} [\xi_t^2(\beta_t)\sigma_{\varepsilon_t}^2(\beta_t)^{-1}]^2 \), and LL = \( T^{-1} \sum_{t=1,T} \ln[\xi_t^2(\beta_t)\sigma_{\varepsilon_t}^2(\beta_t)^{-1}]^2 \).
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