Université de Montréal
Faculté des arts et des sciences
Département de sciences économiques

CAHIER 9329

STRATEGYPROOF COLLECTIVE CHOICE IN ECONOMIC AND POLITICAL ENVIRONMENTS

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September 1993

Financial support from the FCAR is gratefully acknowledged.

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Ce cahier a également été publié au Centre de recherche et développement en économique (C.R.D.E.) (publication no 1993).

Dépôt légal - 1993
Bibliothèque nationale du Québec
Bibliothèque nationale du Canada

ISSN 0709-9231

RÉSUMÉ

Cet article synthétise quelques résultats récents sur la non-manipulabilité des règles de décision collective dans les modèles de choix économiques et politiques.

Mots clés : non-manipulabilité, domaines restreints.

ABSTRACT

The present paper surveys some recent results characterizing strategyproof collective choice rules when preferences satisfy conditions that are meaningful in economic or political environments.

Key words : strategyproofness, restricted domains.

1. INTRODUCTION

Individual decision theory postulates that agents act according to their preferences. In collective choice theory, the relationship between preferences and choice is less straightforward. Presumably, a group of agents that must make a decision affecting all of its members will somehow rely on their preferences. As these might conflict, some sort of procedure is needed to arbitrate among them and find a compromise. To be sure, procedures may be highly complex. For our purpose, however, describing the precise institutional environment in which the decision process takes place is irrelevant. The assumption is that developed societies have reached a point where the numerous institutional details surrounding the choice have been codified and can be assumed to be fixed, at least over the time period relevant to the choice problem under consideration. A procedure may thus be modelled as a *choice rule* mapping profiles of individual preferences into social alternatives. This choice rule is assumed to be known to all agents.

Agents, however, act according to their preferences. In a collective choice process summarized by a choice rule, an agent's action merely consists in reporting her preferences over the social alternatives. There is no reason to *assume* that reporting will be faithful : if an agent can secure a better alternative by announcing preferences that differ from the truth, she may do so. Clearly, collective decisions based on such misreports may turn out to be quite unsatisfactory. It is therefore important to know whether choice rules immune to manipulations can be devised.

Strategyproofness is the strongest possible form of robustness against manipulations. A choice rule is called strategyproof if telling the truth is a dominant strategy for every agent. This means that, no matter what the others report, a lie never pays off. Such a requirement is attractive but hard to meet. A fundamental result, established by Gibbard (1973) and Satterthwaite (1975), states that a strategyproof rule which is flexible enough to allow the possible election of at least three alternatives, must be dictatorial : there is an agent whose preference dictates the final outcome irrespective of the other agents' reports.

Most of the analysis below is developed in a unified framework; it is therefore convenient to formally describe it at this point. The collective choice problem consists in picking an element from a set A of social *alternatives*. Elements of A are denoted by a, b and so on. The *agents* form a finite set N and are indexed by subscripts running from 1 to n. Agent i's *preference* over the social alternatives is represented by a complete and transitive binary relation R_i over A. The notation a P_i b means that a is strictly preferred to b : a R_i b and not b R_i a. Indifference is denoted by I_i : a I_i b if and only if a R_i b and b R_i a. A strict preference is one that forbids indifference between distinct alternatives : a I_i b implies a = b. A preference profile is a list R = $(R_1, ..., R_n)$. The notation R_{-i} stands for the sublist $(R_1, ..., R_{i-1}, R_{i+1}, ..., R_n)$. Denoting by $D_i(A)$ the domain of admissible preferences for agent i, a *choice rule* is a mapping $f : \prod_{i \in N} D_i(A) \to A$. When the preference domain is common to all agents, it is denoted by D(A). The range of f will be denoted by A_f. The choice rule f is manipulable by agent i at profile R via the misreport R_i' if $f(R_i', R_{-i})\ P_i\ f(R)$. It is strategyproof if no agent can manipulate it at any profile via any misreport : $f(R)\ R_i\ f(R_i', R_{-i})$ for all R, i, and R_i'. The formal expression of the Gibbard-Satterthwaite result is the following :

Theorem 1. *Let D(A) contain all strict preferences over A. Let $f : D(A)^N \to A$ be a strategyproof choice rule whose range A_f contains at least three alternatives. Then f is dictatorial : there exists i ∈ N such that*

$$\forall R \in D(A)^N,\ \forall a \in A_f,\ f(R)\ R_i\ a.$$

This disappointing result holds under the so-called universal domain assumption : every strict preference over the social alternatives is conceivable and may thus be reported.[1] While this may be a natural assumption when the set of alternatives has no particular structure, it is unreasonably strong when that set arises from some specific economic or political problem. As an example, consider the problem of choosing the levels of public expenditures on education and defense. Due to technological constraints, the set of alternatives will typically be a compact subset of $\mathbb{R}_+^2$. In such a framework, it is natural to assume that preferences are continuous, monotonic and maybe convex. Under these restrictions, however, the Gibbard-Satterthwaite theorem simply does not apply. Whether nontrivial strategyproof rules exist is an issue that must be addressed anew. Of course, additional assumptions on the preferences may be appropriate in other contexts. The recent years have seen the emergence of a literature that systematically investigates the structure of strategyproof collective choice rules in

1 The theorem also holds if D(A) is taken to be the domain of all preferences. In fact, one may allow indifference over some pairs of alternatives and forbid it over others. Beja (1991) shows that the theorem is valid whenever D(A) contains (but may be larger than) the domain of all strict preferences.

economic and political environments. The purpose of this essay is to give the reader an introduction to that field.

The picture that emerges from the papers that will be surveyed is the following. In environments where preferences satisfy relatively general requirements, a strategyproof choice rule must be either very rigid (i.e., its range is severely restricted) or grossly unjust (namely, it displays some form of dictatorship) : the tradeoff expressed by the Gibbard-Satterthwaite theorem thus persists. By contrast, imposing a strong a priori structure on the preferences allows for strategyproof rules that are reasonably flexible and equitable. Even though this general conclusion is not surprising, it is by no means obvious. Indeed, consider the proposition obtained by replacing the domain D(A) in the statement of Theorem 1 with some strictly smaller domain. This new proposition is not implied by Theorem 1 because strategyproofness now becomes a less demanding requirement. But neither does it imply Theorem 1 because dictatorship also becomes a weaker conclusion.

The paper is organized as follows. Restrictions on preferences that allow for nontrivial strategyproof choice rules will be discussed first. Two such restrictions, single-peakedness and separability, have been discussed in some detail in the literature. Section 2 is devoted to the popular assumption of single-peaked preferences. With such preferences, the well-known median voter choice rule is strategyproof. Conversely, virtually every strategyproof rule can be viewed as a generalized median rule. The various characterizations proposed in the literature will be reviewed and multidimensional extensions will be discussed. Section 3 assumes separable preferences. This means that the agents have well-defined "marginal" preferences over each component of the social alternatives. Under some conditions, strategyproof rules must then be decomposable. That is to say, each component of the social outcome is chosen on the basis of the corresponding marginal preferences only. Using this result, several characterization results will be derived for some important subdomains of separable preferences. Section 4 concerns domain restrictions that are more standard in economics, such as continuity and convexity. When preferences are continuous, the negative conclusion of the Gibbard-Satterthwaite theorem will be proven to carry over. If convexity is also required, a little more - but still not much - flexibility is allowed : the range of a nondictatorial strategyproof rule must be of dimension one.

Two final comments are in order about the scope of the present essay. With the exception of a few early contributions that have turned out to be very influential, most of the papers reviewed here are very recent. I have tried to minimize the overlaps with

Muller and Satterthwaite's (1985) survey, to which the reader is referred for complementary information, especially about older papers.

Secondly, the present survey is exclusively concerned with environments where social alternatives are public in nature. Until very recently, private-commodity environments had received little attention : the only references that I am aware of are Hurwicz (1972), Dasgupta, Hammond and Maskin (1979), and Satterthwaite and Sonnenschein (1981). In the last few years, however, several papers have started sorting out the implications of strategyproofness in a variety of models involving private commodities. Among them are Alcalde and Barberà (1992), Barberà and Jackson (1993), Ching (1992b, c), Hurwicz and Walker (1990), Moulin-Shenker (1992), Moreno (1992), Moreno and Walker (1991), Olson (1991), Shenker (1992a, b), Sprumont (1992, 1993), Yi (1992) and Zhou (1991b). These papers will not be discussed here because of lack of space, but also because it seems a little early to do so. Even though some nice results have been proved, substantial difficulties persist. In particular, and contrary to the pure public-alternative case, complicated strategyproof rules that cannot be defined for two-agent problems become available when there are at least three agents. Consequently, most characterization results either focus on the two-agent case or combine strategyproofness with other properties.

2. SINGLE-PEAKED PREFERENCES

The present section explores the nature of strategyproof choice rules when preferences are single-peaked. If the set of alternatives is one-dimensional, this assumption means that each agent has some ideal alternative, or peak, and that moving away from this peak always hurts. In the first subsection, I set up the basic one-dimensional model and explain in some detail why the well-known median peak rule is strategyproof. I then describe Moulin's (1980) characterization of the "peak-only" strategyproof rules. A rule is peak-only if the decision is based solely on the peaks of the reported preferences. Subsection 3 shows that there is very little loss of generality in making this seemingly severe restriction : a strategyproof rule whose range is an interval must be peak-only. The next subsection is devoted to two alternative characterizations due to Ching (1992a) and Barberà, Gul and Stachetti (1992). The last subsection discusses some characterization results obtained by Barberà, Gul and Stachetti (1992), Barberà, Massó and Neme (1992), and Peters, van der Stel and Storcken (1991a) under a multidimensional version of the single-peakedness assumption.

2.1 The basic model and the median rule

Let us assume that there is an odd number of agents, say $n = 2p + 1$. One alternative, or outcome, must be chosen from some interval $A = [\underline{a}, \bar{a}] \subset \mathbb{R}$. Each agent i's preference R_i over A is single-peaked. This means that there exists an alternative $a^*(R_i) \in A$ such that for all a, b $\in$ A for which $b < a \leq a^*(R_i)$ or $a^*(R_i) \leq a < b$, we have $a \, P_i \, b$. In the literature, $a^*(R_i)$ is interchangeably called the ideal point or the peak of the preference R_i. The latter term is somewhat misleading as it fails to indicate that $a^*(R_i)$ belongs to the set of alternatives. It is nevertheless more commonly used and I will therefore stick to it. It should be emphasized that two preferences that have the same peak need not be identical : they may disagree when comparing alternatives lying on opposite sides of this peak. The following auxiliary concept will prove convenient. For a single-peaked preference R_i and an alternative a, $r_i(a)$ will denote the alternative that lies on the other side of the peak of R_i and is just as good as a. More precisely, the mapping $r_i : A \to A$ is defined as follows : for $a \in [\underline{a}, a^*(R_i)]$, $r_i(a) = b$ if there exists $b \in [a^*(R_i), \bar{a}]$ such that $a \, I_i \, b$, and $r_i(a) = \bar{a}$ otherwise; for $a \in [a^*(R_i), \bar{a}]$, $r_i(a) = b$ if there exists $b \in [\underline{a}, a^*(R_i)]$ such that $a \, I_i \, b$, and $R_i(a) = \underline{a}$ otherwise. The set of continuous single-peaked preferences over A will be denoted by SP(A). A choice rule is a mapping from $SP(A)^N$ into A. A slightly more general version of this model, which allows for "plateaus" rather than peaks, is analyzed in Moulin (1984).

Single-peaked preferences make sense in a broad variety of political and economic models. Moulin (1988) records the following examples. If the alternatives are candidates running for an election, and if they can be ordered along some one-dimensional scale such as the traditional left-right scale, it is reasonable to assume that a voter will prefer a candidate whose position is, say, slightly too leftist (compared to his ideal position) over any other that is even farther left. In one-dimensional location models, where consumers' preferences are determined by transportation costs, the single-peakedness assumption is again quite natural. A third example is provided in public economics by the standard one public good - one private good model. Under a linear cost function and a uniform cost sharing rule, the set of alternatives is just a line segment. Single-peakedness follows here from strict convexity of the preferences defined over the public good - private good space.

What makes the assumption of single-peaked preferences attractive is that it allows us to construct nontrivial strategyproof choice rules. The simplest one, that was first studied by Black (1948) and drew continued attention ever since, is the so-called median (peak) choice rule. If $B = \{a_1, ..., a_m\}$ is a subset of A containing an odd number $m = q + 1$ of alternatives, the median of B, denoted med B, is the number $a \in B$ such that $\#\{a_i \in B \mid a_i \leq a\} \geq q + 1$ and $\#\{a_i \in B \mid a \leq a_i\} \geq q + 1$.

Definition 2.1. *The median (peak) rule f^m chooses for each preference profile the median of the peaks of the preferences :*

$$\forall R \in SP(A)^N, f^m(R) = med\{a^*(R_1), ..., a^*(R_n)\} .$$

It is well known and easy to check that this rule is strategyproof. Nevertheless, decomposing the argument into elementary steps will highlight some basic properties that will play an important role in the analysis of all strategyproof rules.

First of all, note that the median rule only takes into account the peaks of the preferences. Any other piece of information is merely ignored. Any choice rule that possesses this property will be called *peak-only*.

Definition 2.2. *The choice rule $f : SP(A)^N \to A$ is peak-only if*

$$\forall R, R' \in SP(A)^N, f(R) = f(R') \text{ whenever } a^*(R_i) = a^*(R'_i) \text{ for all } i \in N .$$

Of course, the peak-only property drastically restricts the scope for manipulations : no agent can hope to affect the choice without modifying his reported peak.

A second key feature of the median rule is that its sensitivity to the peaks of the preferences is, in fact, quite limited. If an agent's peak is smaller than the chosen alternative, any change in this peak that does not make it greater than the chosen alternative will not affect the decision. Likewise, the only way that an agent whose peak is greater than the choice could possibly affect this choice is by reporting a peak smaller than the choice. This property was called *uncompromisingness* by Border and Jordan (1983).

Definition 2.3. *The choice rule $f : SP(A)^N \to A$ is uncompromising if*

$$\forall R \in SP(A)^N, \forall i \in N, \forall R'_i \in SP(A) ,$$

$$\text{if } a^*(R_i) < f(R) \text{ and } a^*(R'_i) \leq f(R), \text{ then } f(R) = f(R'_i, R_{-i}) ;$$

$$\text{if } f(R) < a^*(R_i) \text{ and } f(R) \leq a^*(R'_i), \text{ then } f(R) = f(R'_i, R_{-i}) .$$

Finally, if an agent whose peak is smaller (greater) than the median peak reported a peak greater (smaller) than the median, she could only push the median farther away from her peak. This is not profitable because preferences are single-peaked. The median choice rule is therefore strategyproof.

2.2 A first characterization : minmax rules

The foregoing discussion suggests a natural question : apart from the median rule, which rules, if any, are strategyproof? The first systematic treatment of that issue is found in Moulin (1980). In Moulin's model, the set of alternatives is the real line, but the arguments are easily transposed to our framework. The paper fully characterizes strategyproof rules under the *assumption* that they are peak-only. It also describes subclasses of peak-only strategyproof rules that are of particular interest from a normative viewpoint. The analysis starts with the following observation. Rather than choosing the median peak, suppose we picked the highest one. This procedure would still be strategyproof for exactly the same reasons as the median rule is. We could as well select the smallest peak, or the second smallest, and so on. Now, we can describe such biased procedures by assuming the existence of a number of "phantom voters", whose votes are fixed, and choosing the median of the agents' peaks and these fixed ballots. It turns out that every peak-only strategyproof rule that is anonymous, i.e., symmetric in the peaks, can be described in that way.

Theorem 2.1. *The choice rule $f : SP(A)^N \to A$ is peak-only, strategyproof and anonymous if and only if there exist numbers $a_1, \ldots, a_{n+1} \in A$ such that*

$$\forall R \in SP(A)^N, f(R) = med\{a^*(R_1), \ldots, a^*(R_n), a_1, \ldots, a_{n+1}\} .$$

These rules will be called *extended median rules (with n + 1 parameters)*. Moulin's proof of Theorem 2.1, which will not be presented here, works by establishing the result for one-agent choice rules and using an induction argument.

Theorem 2.1 has an interesting corollary. If we let $a_1 = a_2 = \ldots = a_{n+1}$, the corresponding extended median rule becomes totally insensitive to the preferences. We might want to exclude such undesirable rules and, in particular, require (Pareto) efficiency. When preferences are single-peaked, this simply means that the choice should fall between the smallest and the largest reported peaks. With only n - 1 fixed ballots at our disposal, the median of those ballots and the peaks will obviously be efficient. Conversely, n - 1 parameters are enough to describe any efficient rule that is peak-only, strategyproof and anonymous. Indeed, for such a rule f and for any $a^* \in A$, we know from Theorem 2.1 that if $a^*(R_i) = a^*$ for all $i \in N$, then $f(R) = \text{med}\{a^*, ..., a^*, a_1, ..., a_{n+1}\} = a^*$. But this requires that $a_i = \underline{a}$ and $a_j = \bar{a}$ for at least one i and one j, say, $i = n$ and $j = n + 1$. Since for every R, $\text{med}\{a^*(R_1), ..., a^*(R_n), a_1, ..., a_{n-1}, \underline{a}, \bar{a}\} = \text{med}\{a^*(R_1), ..., a^*(R_n), a_1, ..., a_{n-1}\}$, we are done. We have proved :

Theorem 2.2. *The choice rule $f : SP(A)^N \to A$ is peak-only, strategyproof, anonymous and efficient if and only if it is an extended median rule with $n - 1$ parameters, i.e., there exist $a_1, ..., a_{n-1} \in A$ such that*

$$\forall R \in SP(A)^N, f(R) = med\{a^*(R_1), ..., a^*(R_n), a_1, ..., a_{n-1}\} .$$

This is the content of Moulin's Proposition 2. The reader may have noticed that the full force of the efficiency condition was not used in deriving the result. Efficiency can be replaced by the weaker property of *unanimity* : whenever all peaks coincide, this common peak should be the choice. As a matter of fact, even the unanimity condition is stronger than needed. Indeed, under strategyproofness, it is implied by the weaker condition of *surjectivity*. To see why, fix a profile R at which all peaks are equal, say, to a^*. By surjectivity there exists R' such that $a^* = f(R')$. By strategyproofness, $f(R') = f(R_1, R'_{-1})$. Repeating this argument shows that $a^* = f(R') = f(R)$, as desired. Theorem 2.2 may thus be modified as follows :

Theorem 2.2'. *The choice rule $f : SP(A)^N \to A$ is peak-only, strategyproof, anonymous and surjective if and only if it is an extended median rule with $n - 1$ parameters.*

The last result in Moulin (1980) does not rely on any anonymity or efficiency condition. Let me call *coalition* any (possibly empty) subset S of N (not necessarily distinct from N). The set of all coalitions will be denoted by 2^N. Moulin shows :

Theorem 2.3. *The choice rule $f : SP(A)^N \to A$ is peak-only and strategyproof if and only if for each coalition S there exists a number $a_S \in A$ such that*

$$\forall R \in SP(A)^N, f(R) = \min_{S \subset N} \left\{ \max_{i \in S} \{a^*(R_i), a_S\} \right\} .$$

I will refer to such rules as *minmax rules*. Minmax rules generalize the extended median rules. In the two-agent case, for instance, they take the form

$$f(R_1, R_2) = \min\Big\{a_\emptyset, \max\{a^*(R_1), a_{\{1\}}\}, \max\{a^*(R_2), a_{\{2\}}\},$$
$$\max\{a^*(R_1), a^*(R_2), a_{\{1,2\}}\}\Big\}. \qquad (2.1)$$

Observe that there is no loss of generality in assuming

$$a_{\{1,2\}} \le a_{\{1\}}, a_{\{2\}} \le a_\emptyset . \qquad (2.2)$$

(Indeed, if $a_\emptyset \le a_{\{i\}}$, then $a_\emptyset \le \max\{a^*(R_i), a_{\{i\}}\}$, so that the right-hand side of the latter inequality plays no role in (2.1). Therefore, $a_{\{i\}}$ may as well be set equal to $a_\emptyset$. Likewise, if $a_{\{i\}} \le a_{\{1,2\}}$, then $\max\{a^*(R_i), a_{\{i\}}\} \le \max\{a^*(R_1), a^*(R_2), a_{\{1,2\}}\}$, and since the latter term plays no role in (2.1), $a_{\{1,2\}}$ can be replaced by $a^*(R_i)$.) Now, assuming (2.2), it appears that $a_{\{1,2\}} \le f(R) \le a_\emptyset$ for each R : the range of f is thus constrained by the parameters $a_{\{1,2\}}$ and $a_\emptyset$. In the extreme case where $a_{\{1,2\}} = a_\emptyset$, the outcome is independent of the preferences. On the other hand, the parameters $a_{\{1\}}$ and $a_{\{2\}}$ determine the relative influence of each agent. For instance, if $\underline{a} = a_{\{1,2\}} = a_{\{2\}}$ and $a_{\{1\}} = a_\emptyset = \bar{a}$, then $f(R) = a^*(R_2)$ for every R : agent two dictates the choice. If $a_{\{1\}}$ and $a_{\{2\}}$ both equal some common value a and if $\underline{a} = a_{\{1,2\}}$ and $a_\emptyset = \bar{a}$, then $f(R) = \text{med}\{a^*(R_1), a^*(R_2), a\}$, an extended median rule.

2.3 A peak-only lemma

The main shortcoming of Moulin's work is the peak-only assumption. On the face of it, this is a severe restriction. A lot of information is neglected by restricting attention from the preference relations to just their peaks. One may suspect, however, that the restriction is related to the strategyproofness condition itself. After all, the manipulability of a choice rule ought to be correlated with its level of sophistication : the more sensitive a rule is to the fine details of the preferences, the more vulnerable it should be to misreports. How simple must a choice rule be in order to resist manipulations? When preferences are single-peaked, it turns out that every strategyproof rule whose range is an interval must be peak-only. A variant of this result was proved by Barberà-Gul-Stachetti (1992) in a (multidimensional) discrete model with strict preferences and a weaker version of it was established by Ching (1992a). The proof that will be presented here is adapted from Ching's. It is worth mentioning that in a variety of other contexts as well, strategyproof rules that are not "too bizarre" rely exclusively on the agents' most preferred alternatives. Proving this fact constitutes a key step in many papers in the literature. Unfortunately, the proofs remain model-specific and are often quite complicated.

Lemma 2.1. *A strategyproof rule $f : SP(A)^N \to A$ whose range is an interval (possibly degenerated to a single point) is peak-only.*

Proof. Let f be a strategyproof rule whose range A_f is an interval. Throughout the proof, we use the convention that $[a, b] = \emptyset$ if $b < a$.

Fact 1. *f is unanimous on its range* : $\forall R \in SP(A)^N$, $\forall a \in A_f$ $f(R) = a$ *whenever* $a^*(R_i) = a$ *for all* $i \in N$.

Fix $a \in A_f$ and suppose $a^*(R_i) = a$ for all i. Let R' be such that $f(R') = a$. By strategyproofness, $f(R_1, R'_{-1}) = f(R') = a$. Repeated application of this argument leads to $f(R) = a$.

Fact 2. (i) *For each R and i such that* $f(R) < a^*(R_i)$, *for each* R'_i *such that* $f(R) \leq a^*(R'_i)$, *either* $f(R'_i, R_{-i}) = f(R)$ *or* $f(R'_i, R_{-i}) \in [r_i(f(R)), r'_i(f(R))]$.

(ii) *For each R and i such that* $a^*(R_i) < f(R)$, *for each* R'_i *such that* $a^*(R'_i) \leq f(R)$, *either* $f(R'_i, R_{-i}) = f(R)$ *or* $f(R'_i, R_{-i}) \in [r'_i(f(R)), r_i(f(R))]$.

Let us prove (i). If $f(R) < f(R'_i, R_{-i}) < r_i(f(R))$, then $f(R_i, R'_{-i})\ P_i\ f(R)$ and i manipulates at R via R'_i. If $f(R'_i, R_{-i}) < f(R)$ or $r'_i(f(R)) < f(R'_i, R_{-i})$, then $f(R)\ P'_i\ f(R'_i, R_{-i})$ and i manipulates at (R'_i, R_{-i}) via R_i. Statement (i) follows. The proof of (ii) is similar.

Fact 3. *f is "own-peak-only" : for each i, R, and* R'_i *such that* $a^*(R_i) = a^*(R'_i)$, *we have* $f(R) = f(R'_i, R_{-i})$.

Fix i, say i = n, and fix R and R'_n such that $a^*(R_n) = a^*(R'_n)$. We must show that

$$f(R) = f(R'_n, R_{-n}) \; . \qquad (2.3)$$

If $f(R) = a^*(R_n)$, strategyproofness immediately yields (2.3). Suppose next, without loss of generality, that $f(R) < a^*(R_n)$. If $r'_n(f(R)) < r_n(f(R))$, (2.3) follows directly from Fact 2(i). Next, consider the case where $r_n(f(R)) \leq r'_n(f(R))$. Let $S = \{j \in N \mid a^*(R_n) < a^*(R_j)\}$. We may suppose that $S = \{1, ..., s\}$ (with the convention that $S = \emptyset$ if $s = 0$) and let $a^*(R_1) \leq ... \leq a^*(R_s)$. For every $j \in S$, construct R'_j such that $a^*(R'_j) = a^*(R_n)$ and $a^*(R_j) < r'_j(f(R)) < r_j(f(R))$. By repeated use of Fact 2(i),

$$f(R'_1, ..., R'_s, R_{s+1}, ..., R_n) = f(R) . \tag{2.4}$$

Consider now the profile $(R'_1, ..., R'_s, R_{s+1}, ..., R_{n-1}, R'_n)$. By Facts (2.i) and (2.4), there are only two possible cases.

Case 1. $f(R'_1, ..., R'_s, R_{s+1}, ..., R_{n-1}, R'_n) = f(R)$.

In order to establish (2.3), we need only show that

$$f(R'_1, ..., R'_s, R_{s+1}, ..., R_{n-1}, R'_n) = f(R'_n, R_{-n}) . \tag{2.5}$$

I first claim that

$$f(R'_1, ..., R'_s, R_{s+1}, ..., R_{n-1}, R'_n) = f(R_1, R'_2, ..., R'_s, R_{s+1}, ..., R_{n-1}, R'_n) . \tag{2.6}$$

Suppose that this is not the case. By Fact 2(i), $f(R_1, R'_2, ..., R'_s, R_{s+1}, ..., R_{n-1}, R'_n) \in [r'_1(f(R'_1, ..., R'_s, R_{s+1}, ..., R_{n-1}, R'_n)), r_1(f(R'_1, ..., R'_s, R_{s+1}, ..., R_{n-1}, R'_n))]$. This means, in particular, that

$$f(R) < a^*(R_1) < f(R_1, R'_2, ..., R'_s, R_{s+1}, ..., R_{n-1}, R'_n) .$$

Since the range of f is an interval,

$$f(R_1, R'_2, ..., R'_s, R_{s+1}, ..., R_{n-1}, R'_n) \neq a^*(R_1) \in A_f . \tag{2.7}$$

For $j = 2, ..., s$ and $j = n$, construct R''_j such that $a^*(R''_j) = a^*(R_1)$ and $r'_j(f(R_1, R'_2, ..., R'_s, R_{s+1}, ..., R_{n-1}, R'_n)) < r''_j(f(R_1, R'_2, ..., R'_s, R_{s+1}, ..., R_{n-1}, R'_n))$. For $j = s + 1, ..., n - 1$, construct R''_j such that $a^*(R''_j) = a^*(R_1)$ and $r_j(f(R_1, R'_2, ..., R'_s, R_{s+1}, ..., R_{n-1}, R'_n)) < r''_j(f(R_1, R'_2, ..., R'_s, R_{s+1}, ..., R_{n-1}, R'_n))$. By repeated application of Fact 2(ii), $f(R_1, R''_{-1}) = f(R_1, R'_2, ..., R'_s, R_{s+1}, ..., R_{n-1}, R'_n)$, which together with (2.7) contradicts Fact 1 and proves (2.6). Repeating the argument yields (2.5).

Case 2. $f(R_1', \dots, R_s', R_{s+1}, \dots, R_n) \in [r_n(f(R)), r_n'(f(R))]$.

An argument similar to the one used in Case 1 yields a contradiction to Fact 1. This proves Fact 3.

To complete the proof of the lemma, observe that an own-peak-only rule must be peak-only. ▪

The range condition in Lemma 2.1 cannot be dropped. Suppose that the range of f consists of two alternatives, and that the decision is made by majority voting over them. This is a perfectly strategyproof procedure that nonetheless uses nonpeak information. The condition can, however, be weakened. All that really matters is that every single-peaked preference on A be single-peaked on the range of f as well. Barberà and Jackson (1992) prove the peak-only property under that weaker condition.

Using Lemma 2.1, we may now strengthen Moulin's results. In Theorems 2.1 and 2.3, the peak-only condition can be replaced by the assumption that the range of f is an interval. In Theorem 2.2, the peak-only condition can be dropped altogether, since efficiency implies that the range of f is A itself. As strategyproofness and surjectivity imply `unanimity, the peak-only condition is also redundant in Theorem 2.2'. Summing up, we obtain :

Corollary to Theorem 2.1. *The following statements are equivalent :*

(i) $f : SP(A)^N \to A$ *is strategyproof, anonymous, and its range is an interval.*

(ii) $f : SP(A)^N \to A$ *is an extended median rule with* $n + 1$ *parameters.*

Corollary to Theorems 2.2 and 2.2'. *The following statements are equivalent :*

(i) $f : SP(A)^N \to A$ *is strategyproof, anonymous and efficient.*

(ii) $f : SP(A)^N \to A$ *is strategyproof, anonymous and unanimous.*

(iii) $f : SP(A)^N \to A$ *is strategyproof, anonymous and surjective.*

(iv) $f : SP(A)^N \to A$ *is an extended median rule with* $n - 1$ *parameters.*

Corollary to Theorem 2.3. *The following statements are equivalent :*

(i) $f : SP(A)^N \to A$ *is strategyproof and its range is an interval.*

(ii) $f : SP(A)^N \to A$ *is a minmax rule.*

2.4 Alternative characterizations : augmented median rules and committee rules

In a recent paper, Ching (1992a) offers a characterization of peak-only strategyproof rules that constitutes an interesting alternative to Moulin's. The class that he considers can be described as follows. The outcome is the median of the agents' peaks and n + 1 parameters. But these parameters are selected from some larger pool of 2^n parameters according to the reported peaks.

Definition 2.4. *For any* $R \in SP(A)^N$, *let* R^* *be any permutation of* N *such that* $a^*(R_{R^*(1)}) \leq ... \leq a^*(R_{R^*(n)})$. *The choice rule* $f : SP(A)^N \to A$ *is an augmented median rule if for each coalition S there exists a number* $a_S \in A$ *such that*

$$\forall R \in SP(A)^N, f(R) = med\{a^*(R_1), ..., a^*(R_n), a_{\emptyset}, a_{\{R^*(1)\}}, a_{\{R^*(1),R^*(2)\}}, ..., a_N\} .$$

Thus, if the reported peaks are $a^*(R_1) < a^*(R_2) < ... < a^*(R_n)$, the n + 1 parameters effectively used to compute the choice are $a_{\emptyset}, a_{\{1\}}, a_{\{1,2\}}, a_{\{1,2,3\}}, ..., a_N$. If the peaks of agents 1 and 2 were permuted, the parameters would be $a_{\emptyset}, a_{\{2\}}, a_{\{1,2\}}, a_{\{1,2,3\}}, ...,$ a_N. Using Lemma 2.1, Ching's result yields the following characterization :

Theorem 2.4. *The following statements are equivalent :*

(i) $f : SP(A)^N \to A$ *is strategyproof and its range is an interval.*

(ii) $f : SP(A)^N \to A$ *is an augmented median rule.*

The fact that an augmented median rule is indeed strategyproof is not obvious. Checking it is actually quite tedious, even when there are as few as two agents. Ching's approach consists in proving that every augmented median rule can be rewritten as a minmax rule. By contrast, the proof that every strategyproof rule whose range is an interval must be an augmented median rule is quite direct. The first two steps establish properties of peak-only strategyproof rules that are parallel to the ones we already pointed out when discussing the simple median rule :

(i) Every peak-only strategyproof rule is uncompromising.

(ii) Under a peak-only strategyproof rule, manipulating by reporting a peak that lies on the right (left) of the chosen alternative while one's true peak lies on the left (right) must leave the collective decision between the originally chosen alternative and one's reported peak.

Using these two facts, one can then show that every peak-only strategyproof rule is an augmented median rule. Applying Lemma 2.1 completes the proof.

Barberà, Gul and Stachetti (1992) investigate the structure of strategyproof rules when preferences satisfy a multidimensional extension of the single-peakedness condition. In the one-dimensional case, their results imply one more characterization of the strategyproof rules whose ranges are intervals. The model they use differs from the one presented in Subsection 2.1 in two respects : the set of alternatives is finite and preferences are strict. Although the arguments used by Barberà, Gul and Stachetti do rely on these assumptions, it turns out that their characterization can be translated into our framework.

To understand their approach, let us consider again the simple median rule. For any alternative a, any coalition of at least p + 1 agents can guarantee that the selected outcome will not exceed a by reporting peaks that are all smaller than or equal to a. No coalition containing fewer than p + 1 agents has that power. Let us therefore agree to call a coalition winning at a if and only if it contains at least p + 1 agents. Observe that at any preference profile, the median rule selects the smallest alternative a such that the agents whose peaks do not exceed a form a winning coalition at a. This rule possesses two particular features. First, any coalition that is winning at some alternative is winning at every alternative. This is a neutrality property. Secondly, whether a coalition is winning only depends on its size, not on the identity of its members. This is an anonymity property. One suspects that both properties, albeit attractive from a normative viewpoint, are orthogonal to the issue of manipulability : every choice rule that is based on some (possibly nonneutral or nonanonymous) system of winning coalitions should be strategyproof. Of course, a system of winning coalitions must satisfy some conditions :

Definition 2.5. *A left (winning) coalition system W (defined on A) associates with each alternative $a \in A$ a (possibly empty) set of (possibly empty) coalitions $W(a)$ such that*

(i) *if $S \in W(a)$ and $S \subset T \subset N$, then $T \in W(a)$,*

(ii) *if $S \in W(a)$ and $a < b \leq \bar{a}$, then $S \in W(b)$,*

(iii) $W(\bar{a}) = 2^N$,

(iv) *$\{a \mid S \in W(a)\}$ is closed for each $S \subset N$.*

Every coalition belonging to W(a) is called (left) winning at a.

Definition 2.6. *The choice rule f is a committee rule if there exists a left winning coalition system W such that*

$$\forall R \in SP(A)^N, f(R) = min\{a \mid \{i \in N \mid a^*(R_i) \leq a\} \in W(a)\}.$$

Committee rules are called generalized median in Barberà, Gul and Stachetti (1992). The committee terminology is motivated by the fact that W(a) is usually called a committee in cooperative game theory; it also avoids confusion with Moulin's extended median rules and Ching's augmented median rules. A committee rule picks the smallest alternative a such that the agents whose peaks are not greater than a constitute a left winning coalition at a. (Of course, a symmetric definition could be given in terms of right winning coalitions.)

It is easy to verify that committee rules are strategyproof (the key observation is once again that they are uncompromising). Under our range condition, the converse is also true :

Theorem 2.5. *The following statements are equivalent :*

(i) *$f : SP(A)^N \to A$ is strategyproof and its range is an interval,*

(ii) *$f : SP(A)^N \to A$ is a committee rule.*

An indirect but fairly easy proof consists in showing that the class of committee rules coincides with that of minmax rules. Theorem 2.5 then follows from the corollary to Theorem 2.3.

2.5 Multidimensional extensions

Assuming that the set of alternatives is one-dimensional is evidently very restrictive. In recent years, several papers have explored the problem of characterizing strategyproof choice rules when alternatives are multidimensional and preferences satisfy some generalized single-peakedness condition. A first class of preferences over (a convex subset of) $\mathbb{R}^m$ that generalize the single-peaked preferences consists of the strictly convex preferences possessing a unique ideal point. More generally, one may consider the so-called star-shaped preferences. The main results for these two domains are due to Border and Jordan (1983), Bordes, Laffond and Le Breton (1990), and Zhou (1991a). As they are essentially negative, they will be reviewed in Section 4. A third class consists of those star-shaped preferences that are separable. Positive results for that case were obtained by Border and Jordan (1983); they will be discussed in Section 3, which is devoted to the separability restriction. A fourth generalization of the single-peakedness assumption was introduced by Barberà, Gul and Stachetti (1992) and further studied by Barberà, Massó and Neme (1993), and Peters, van der Stel and Storcken (1991a). The purpose of this subsection is to give a brief account of those papers.

Barberà, Gul and Stachetti concentrate on collective decision problems where alternatives are m-dimensional vectors whose components can take a finite number of integer values. The set A is thus the Cartesian product of m finite sets, say $A = \prod_{h=1}^{m} A^h = \prod_{h=1}^{m} \{a^h, a^h + 1, .., \bar{a}^h\}$. Here and in the sequel, superscripts will always refer to the components of the social alternatives. To illustrate the flexibility of the approach, let me just mention the following application. Suppose that some organization (such as a firm or a club) has to select a set of individuals (workers or members) from a given pool of m applicants. Let us identify each individual with one coordinate and assign value one to coordinate h if individual h is selected and zero if that individual is turned down. Our original problem is equivalent to choosing an alternative in the set $A = \{0, 1\}^m$. The reader will have no trouble finding other examples that fit the formalism.

Let me now describe the restrictions on the preferences. Call two alternatives contiguous if they differ by one unit in one component and are identical in the others. Any two distinct alternatives a and b can be linked through a sequence of contiguous alternatives. The smallest number needed to complete this link is the distance between a and b; it is denoted by d(a, b). The mapping d is just the usual "city block" metric : $d(a, b) = \sum_{h=1}^{m} |a^h - b^h|$. Two conditions are imposed on the preferences. First, preferences are strict : no agent is allowed to be indifferent between two distinct alternatives. Since the number of alternatives is finite, this is perhaps not a severe limitation. Secondly, each preference is single-peaked in the following sense :

Definition 2.7. *A (strict) preference R_i on A is (multidimensional) single-peaked if there exists $a^*(R_i) \in A$ such that $a^*(R_i)R_i\, a$ for all $a \in A$ and $a\, R_i\, b$ for all $a, b \in A$ such that $d(a^*(R_i), b) = d(a^*(R_i), a) + d(a, b)$.*

The set of all multidimensional single-peaked (strict) preferences on A will be denoted by MSP(A). A few comments are in order.

(1) The restriction of a single-peaked preference to any Cartesian product B contained in A is also single-peaked. Moreover, the peak of the restriction is the projection on B of the peak of the original preference.

(2) Graphically, single-peakedness means that any alternative a contained in the "box" between b and $a^*(R_i)$ is preferred to b. This is also the only restriction. If a does not belong to the box between b and a*, and if b does not belong to the box between a and a*, there exist two single-peaked preferences with peak at a* that rank a and b differently. In this sense, the domain of single-peaked preferences is relatively rich.

(3) Single-peakedness does not imply separability (see Definition 3.1 in the next section). Suppose, for instance, that $A^1 = \{1, 2, 3\}$, $A^2 = \{1, 2\}$, and $A = A^1 \times A^2$. Consider the preference P_i represented by the utility function $u_i(1, 1) = 0$, $u_i(2, 1) = 2$, $u_i(3, 1) = 1$, $u_i(1, 2) = -2$, $u_i(2, 2) = -1$, and $u_i(3, 2) = -3$. This preference is single-peaked but not separable.

For any choice rule f on $MSP(A)^N$, let f^h denote the mapping that associates with each preference profile R the h^{th} coordinate of f(R). Consider now the following class of rules. For each coordinate h and every profile R, determine $f^h(R)$ according to some left winning coalition system W^h defined on A^h, namely let

$$f^h(R) = min\{a^h \in A^h \mid \{i \in N \mid a^*(R_i)^h \le a^h\} \in W^h(a^h)\} . \tag{2.8}$$

Call such a rule a "multidimensional committee rule". Notice that by changing the h^{th} coordinate of her reported peak, an agent can only affect the h^{th} coordinate of the outcome. In view of comment (1) above, this manipulation can only push the outcome away from her peak, which is not profitable. Multidimensional committee rules are therefore strategyproof. Barberà, Gul and Stachetti show that no other choice rule whose range is a Cartesian product can be strategyproof :

Theorem 2.6. *Suppose that the range of the choice rule f : $MSP(A)^N \to A$ is a Cartesian product. Then f is strategyproof if and only if it is a multidimensional committee rule.*

The assumption that the range of the choice rule is a Cartesian product limits the domain of application of Theorem 2.6. In many economic and political situations, the set of feasible alternatives is typically not a Cartesian product. In the example mentioned earlier, where a firm has to hire workers from a given pool of applicants, it is natural to assume a maximal number of vacancies. The purpose of Barberà, Massó and Neme (1993) is to take such restrictions into account. To do so, they distinguish between the set of *conceivable* alternatives, B, which is assumed to be an m-fold Cartesian product, and over which preferences are defined; and the set of *feasible* alternatives, A, which is included in B. Their results hold for feasible sets that have what they call the "Orthant Property". The restriction is the following. Suppose that the alternative a is unfeasible. If we can hit a feasible alternative by decreasing (increasing) the h^{th} coordinate of a while keeping the other ones fixed, then we should not be able to find a feasible alternative by increasing (decreasing) the h^{th} coordinate and maintaining the other ones constant. Moreover, we should not find a feasible alternative by simultaneously decreasing coordinates along which an increase would make a feasible and increasing coordinates along which a decrease would permit feasibility. This is a fairly mild restriction. Its purpose is to rule out some very arbitrary sets like those where a single alternative in the interior of a box is unfeasible.

Using multidimensional committee rules for choosing from a feasible set A which is not a Cartesian product poses two difficulties. First of all, we should focus on the agents' best alternatives among the feasible ones rather than on their unconstrained peaks. Let me call f a *multidimensional committee rule on A* if and only if the following holds :

$$\forall R \in MSP(B)^N,\ \forall h \in \{1, ..., m\},$$

$$f^h(R) = \min\{a^h \in B^h \mid \{i \in N \mid a^*(R_i \mid A)^h \leq a^h\} \in W^h(a^h)\}, \tag{2.9}$$

where each W^h is a left coalition system defined on B^h and $a^*(R_i \mid A)$ is agent i's best alternative in A. Formula (2.9), unfortunately, could still produce unfeasible results. As an illustration, suppose that N = {1, 2, 3, 4}, B = $\{0, 1\}^2$ and A = {(0, 0), (0, 1), (1, 0)} (i.e., at most one candidate can be selected from a pool of two). For h = 1, 2 and $a^h \in \{0, 1\}$, let $S \in W^h(a^h)$ if and only if $\#S \geq 3$ (i.e., coalitions of at least three agents can block the election of a candidate). Suppose that the agents' best alternatives in A are respectively (0, 1), (0, 1), (1, 0) and (1, 0). The outcome would then be (1, 1), which is not feasible. Assuming that A has the Orthant Property, Barberà, Massó and Neme identify a condition, which they call the "Intersection Property", that is necessary and sufficient for a multidimensional committee rule on A to always select a feasible alternative. This condition, which is too complicated to be described in detail here, essentially requires that those families of winning coalitions whose power could lead to unfeasible outcomes have some elements in common, so that divergent recommendations cannot occur.

Having restricted our attention to those multidimensional committee rules on A that have the Intersection Property, we face a second difficulty. Such rules may fail to be strategyproof. Suppose N = {1, 2, 3}, B = $\{0,1\}^2$, A = {(0, 0), (0, 1), (1, 0)} and, for each a^h, $S \in W^h(a^h)$ if and only if $\#S \geq 3$. It is easy to check that the choice rule f determined by the left coalition systems W^h always produces feasible outcomes. But it is not strategyproof. Suppose that agent one's peak on B is (0, 0) while agent two's peak is (0, 1). Let P_3 and P_3' be preferences such that (1, 1) P_3 (1, 0) P_3 (0, 1) P_3 (0, 0) and (0, 1) P_3' (0, 0) P_3' (1, 1) P_3' (1, 0). Since (0, 1) = $f(P_1, P_2, P_3')\ P_3\ f(P_1, P_2, P_3)$ = (0, 0), agent three can manipulate f. The problem arises from the fact that assuming single-peakedness on the set of conceivable alternatives B need not imply many restrictions on the preferences over the set of feasible alternatives A. It all depends on the shape of A : it may even be that every preference on A is the restriction of some single-peaked preference on B. Further restrictions are therefore necessary for positive results. The authors consider the set MSP(A) of those single-peaked preferences on B *whose unconstrained peak happens to lie in A*. Assuming that A satisfies the Orthant Property and that f is a choice rule from $MSP(A)^N$ onto A, they show that f is strategyproof if and only if it is a multidimensional committee rule on A satisfying the Intersection Property.

Closely related to the work of Barberà, Massó and Neme is the paper by Peters, van der Stel and Storcken (1991a). These authors consider feasible sets that are arbitrary nonempty subsets of Euclidean spaces. Preferences are binary relations on the feasible set A; they must have at least one peak (i.e., a maximal element in A) and any alternative b in the box between a peak and some other alternative a must be at least as good as a. These are roughly the same restrictions as in Barberà, Massó and Neme, except that completeness and transitivity are not assumed and that indifferences are allowed. The authors focus exclusively on those choice rules that operate on the basis of a list of reported peaks. They show that if such a rule is strategyproof, each coordinate of the outcome must be determined on the basis of the corresponding coordinates of the reported peaks by using a minmax formula à la Moulin (1980). The authors do not investigate under which conditions the resulting outcome is feasible.

3. SEPARABLE PREFERENCES

A major breakthrough achieved in the seventies was the discovery that the negative conclusions of the Gibbard-Satterthwaite theorem could be overcome in the following important particular case. Suppose that in addition to a set of feasible public decisions there exists a perfectly divisible good, usually called money, that is available in unlimited supply to perform compensatory payments across agents. A social alternative now consists of a feasible public decision along with a vector of net money transfers to the agents. In order to be feasible, these transfers must, of course, sum up to a nonpositive number. Suppose next that each agent's preference is representable by a utility function that is separable and linear in the transfer he receives. Under this assumption, often referred to as quasilinearity, Clark and Groves were able to construct nontrivial strategyproof choice rules that always select an efficient public decision (i.e., one that maximizes the sum of the agents' utilities). Green and Laffont (1979) later showed that these properties are characteristics of the Clark-Groves rules.[2]

Clearly, quasilinearity is just one very specific separability assumption. In recent years, several authors have come to realize that interesting insights could be gained in a

2 The Clark-Groves rules may waste money at some profiles of preferences and are therefore not (fully) efficient. Walker (1980) showed that this inefficiency occurs at "virtually all" profiles. Hurwicz and Walker (1990) extended this result to a larger class of choice problems that includes pure exchange economies.

variety of collective choice models by imposing other separability assumptions. Most of them are particular expressions of the same general restriction. This restriction is twofold. First, as in Subsection 2.5, social alternatives are supposed to comprise well-defined components or dimensions. This is standard modelling practice. In traditional economic models, bundles of goods are represented by vectors. A similar framework is often used in applied consumer analysis, where commodities are viewed as bundles of characteristics. In political theory, the program of a candidate is commonly described as a point in some Euclidean space, each coordinate representing the candidate's stand on a specific issue : in a two-dimensional model, for instance, the first coordinate might represent the proposed level of public expenditure on education, while the spendings on defense might be measured along the second coordinate. In social choice, finally, the conflict between efficiency and libertarism was formalized by Sen in a model where each different component of a social alternative describes the "personal issue" of a different member of society.

Having postulated a multidimensional structure over the set of alternatives, the second part of the restriction concerns the preferences. Consider two social alternatives a and b differing in only one component, say the first one. Suppose that an agent prefers a over b. Keeping the first component of a and b fixed, perform a common change in any of the remaining components of a and b to obtain a' and b'. Separability requires that a' be preferred to b'. In the two-dimensional example previously described, this would mean that preferences over public spending on defense are independent of the level of expenditures on education, and conversely. In other words, separable preferences induce well-defined "marginal" preferences over each dimension of the collective choice problem under consideration.

The next subsection formalizes the setup just described and presents an important and general result : if preferences are rich enough, (onto) strategyproof rules must be decomposable, that is to say, each component of the social outcome is chosen on the basis of the corresponding marginal preferences only. Using this result, Subsection 2 establishes characterization results for some important subdomains of separable preferences. All these results hold under the assumption that the number of alternatives is finite and that preferences are strict. Subsection 3 studies the case where indifferences are allowed.

3.1 A framework, and a decomposability lemma

The set of alternatives, A, is supposed to be the Cartesian product of m sets, $A^1, ..., A^m$, which are finite but otherwise arbitrary. The set of components is $M = \{1, ..., m\}$. For any $h \in M$, the notation a^{-h} denotes the sublist $(a^1, ..., a^{h-1}, a^{h+1}, ..., a^m)$ and $A^{-h} = \prod_{k \in M \setminus \{h\}} A^k$. Agent i's preference over A belongs to some domain of admissible preferences denoted by $D_i(A)$. A choice rule is a mapping $f : \prod_{i \in N} D_i(A) \to A$. Various restrictions will be considered but, in any event, each $D_i(A)$ contains only strict preferences. Any $R_i \in D_i(A)$ thus has a unique maximal element in A, or peak, which is denoted by $a^*(R_i)$. Admissible preferences must be separable :

Definition 3.1. *The preference $R_i \in D_i(A)$ is separable if the following condition holds :*

$$\forall h \in M, \forall a^h, b^h \in A^h, \forall x^{-h}, y^{-h} \in A^{-h},$$
$$(a^h, x^{-h})\, R_i\, (b^h, x^{-h}) \Leftrightarrow (a^h, y^{-h})\, R_i\, (b^h, y^{-h}) ;$$

The domain of all (strict) separable preferences over A is denoted by S(A). The separability assumption implies the existence of a well-defined "marginal preference" over each component set A^h :

Definition 3.2. *The marginal preference over A^h induced by the preference $R_i \in S(A)$ is the relation R_i^h defined as follows :*

$$\forall a^h, b^h \in A^h, \quad a^h R_i^h b^h \Leftrightarrow (a^h, x^{-h})\, R_i\, (b^h, x^{-h}) \text{ for all } x^{-h} \in A^{-h} .$$

The notation R^h will stand for the list of marginal preferences $(R_1^h, ..., R_n^h)$. Any admissible domain $D_i(A) \subset S(A)$ induces for each component h a *marginal domain* $D_i(A^h)$ containing every preference over A^h that is the marginal preference induced by some preference in $D_i(A)$.

This setup is a simplified version of Le Breton and Sen's (1992) model. The first difference is that they do not assume that the sets $A^1, ..., A^m$ are finite. But forbidding indifference between alternatives only makes sense if finiteness is assumed anyway. Secondly, Le Breton and Sen allow for partially separable preferences (for which marginals may not be defined on every component but perhaps only on groups of components).

The separability assumption naturally suggests to decompose the collective choice procedure. Since marginal preferences over component sets are well-defined, it makes sense to choose each component of the final outcome on the basis of the corresponding marginal preferences only :

Definition 3.3. *The choice rule* $f : \prod_{i \in N} D_i(A) \to A$ *is decomposable (into marginal choice rules) if for each* $h \in M$ *there exists a mapping* $\phi^h : \prod_{i \in N} D_i(A^h) \to A^h$ *such that*

$$\forall R \in \prod_{i \in N} D_i(A), f^h(R) = \phi^h(R^h).$$

If each *marginal choice rule* ϕ^h is strategyproof, it is obvious that the separability of the preferences guarantees that f cannot be manipulated. Thus, on any domain of strict separable preferences, a rule that decomposes into strategyproof marginal rules is itself strategyproof. A simple example suffices to show that the converse statement is false. Suppose that n = 2 and A^1 and $A^2 = \{0, 1\}$. Let $D^1(A) = D^2(A)$ contain the (unique) preference R_0 in S(A) that has its peak at (0, 1) and ranks (1, 1) above (0, 0) and the (unique) preference R_0' in S(A) that has its peak at (0, 0) and ranks (0, 1) above (1, 0). Consider the following rule. The outcome is agent two's (reported) preferred alternative among agent one's (reported) best and second best alternatives : $f(R_0, R_0) = f(R_0, R_0') = f(R_0', R_0) = (0, 1)$ and $f(R_0', R_0') = (0, 0)$. This rule is strategyproof but clearly not decomposable since $f^2(R_0, R_0') \neq f^2(R_0', R_0')$.

It appears, however, that examples of this sort cannot be constructed when i) the range of f covers the entire set of alternatives and ii) the domains of preferences are rich enough. This result was proved by Le Breton and Sen (1992). I will present a simplified version of their main theorem, which is easier to interpret and conveys the essence of their message. The key concept is given in the following definition :

Definition 3.4. *The domain* $D_i(A) \subset S(A)$ *is rich if and only if it satisfies the following two conditions :*

(i) *for any component h and any $a^h \in A^h$, there exists a preference R_i^h in $D_i(A^h)$ whose maximal element in A^h is a^h;*

(ii) *if R_0^1, ..., R_0^m belong, respectively, to $D_i(A^1)$, ..., $D_i(A^m)$, then every preference R_i whose marginal R_i^h equals R_0^h for every component h belongs to $D_i(A)$.*

According to the first part of this definition, we may choose the marginal domains almost freely *for each component.* We could, for instance, let each $D_i(A^h)$ contain all strict preferences over A^h, but we could also require that these preferences be single-peaked (provided that the structure of each A^h makes this assumption meaningful). On the other hand, the second part of the definition forbids any restriction *across components.*

Lemma 3.1. *For each $i \in N$, let $D_i(A)$ be a rich domain of strict separable preferences, and let f be a choice rule from $\prod_{i \in N} D_i(A)$ onto A. Then f is strategyproof if and only if it is decomposable into strategyproof marginal rules.*

The proof of sufficiency is straightforward. Necessity is much harder to establish. The first part of the argument consists in showing that the choice rule is constant over those profiles which induce the same marginal preference over a given component set and which are unanimous over the remaining ones : if for some $h \in M$ and all $i \in N$, $R_i^h = \overline{R}_i^h$ and $a*(R_i)^{-h} = a*(\overline{R}_i)^{-h} = a^{-h}$, then $f(R) = f(\overline{R})$. Now, fix an alternative a, a component h, and consider two preferences whose maximal elements coincide with a on all components different from h. For any profile R of such preferences, define $f^{a,h}(R^h) = f^h(R)$. The function $f^{a,h}$ is well defined because of our previous claim and because each $D_i(A)$ is rich. The next step consists in demonstrating that for each component h, $f^{a,h}$ is strategyproof and independent of a. When this is done, the rest follows easily. Fix a component h and two profiles R, $\overline{R}$ such that $R^h = \overline{R}^h$. Letting $f(R) = a$ and $f(\overline{R}) = b$, we must show that $a^h = b^h$. Since $f^{a,h}$ and $f^{b,h}$ are strategyproof, $f^{a,h}(R^h) = a^h$ and $f^{b,h}(\overline{R}^h) = b^h$. But since the choice of a and b is irrelevant, $f^{a,h}(R^h) = f^{b,h}(\overline{R}^h)$ and we are done.

3.2 Some consequences

According to Lemma 3.1, the structure of the (surjective) strategyproof rules defined over (the product of) rich individual domains of separable preferences is completely determined by the structures of the strategyproof rules defined over (the products of) the marginal domains. The strength of this result is that it relies only on the richness condition and not on the particular structure of the marginal preferences. This means that, under the richness condition, interesting multicomponent strategyproof rules will exist if and only if the domains of marginal preferences are restricted in a way that guarantees the existence of nontrivial one-component strategyproof rules. The following two theorems illustrate this point.

Suppose that for each component h and each agent i, the domain of marginal preferences consists of all strict preferences over A^h. If each A^h contains at least three alternatives, it follows from Lemma 3.1 and the Gibbard-Satterthwaite theorem that a surjective strategyproof rule must be componentwise dictatorial :

Theorem 3.1. *Let $\#A^h \geq 3$ for every $h \in M$, let $D_i(A) = S(A)$ for every $i \in N$, and let f be a surjective choice rule over $\prod_{i \in N} D_i(A)$. Then f is strategyproof if and only if for every $h \in M$ there exists some $i(h) \in N$ such that*

$$\forall R \in \prod_{i \in N} D_i(A),\ \forall a^h \in A^h,\ f^h(R)\ R^h_{i(h)}\ a^h .$$

A variant of this result is obtained by replacing S(A) with AS(A), the set of additively separable (strict) preferences. A preference R_i is *additively separable* if for each component h there exists a real-valued function u_i^h over A^h such that for any two distinct alternatives a and b, a R_i b if and only if $\sum_{h \in M} u_i^h(a^h) \geq \sum_{h \in M} u_i^h(b^h)$. Additively separable preferences form a strict subset of the class of all separable preferences; they do, however, constitute a rich domain.

One can obtain a second corollary to Lemma 3.1 by keeping the assumption that agents may have any separable preference but restricting every component set A^h to only two elements. It is well known that binary strategyproof choice rules are based on "committees". A m-component surjective strategyproof rule will therefore decompose into m one-component committee rules. Formally, we have :

Definition 3.5. *A committee W is a nonempty collection of nonempty subsets of N satisfying the property that $T \in W$ whenever $S \in W$ and $S \subset T \subset N$. If $S \in W$, S is called winning (under W); it is otherwise losing.*

Theorem 3.2. *Assume that $A^h = \{a^h, b^h\}$ for all $h \in M$ and that $D_i(A) = S(A)$ for all $i \in N$. Let f be a surjective choice rule over $\prod_{i \in N} D_i(A)$. Then f is strategyproof if and only if for every $h \in M$ there exists a committee W^h such that*

$$\forall R \in \prod_{i \in N} D_i(A), f^h(R) = a^h \Leftrightarrow \{i \in N \mid a^h R_i^h b^h\} \in W^h .$$

This theorem is formally equivalent to the main result of Barberà, Sonnenschein and Zhou (1991), although the latter is formulated in a different setting. Up to notational modifications, Barberà, Sonnenschein and Zhou use the following model. There is a set N = {1, ..., n} of agents, or "voters", and a set M = {1, ..., m} of candidates or "issues". The problem consists in choosing a *subset* of candidates. Each voter $i \in N$ is endowed with a (strict) separable preference $\succsim_i$ over the set 2^M of all (possibly empty) subsets of M. Separability means that a candidate is worth being added to any list of candidates if and only if she is a "good" candidate : for each $H \subset M$ and $h \in M \backslash H$, $H \cup \{h\} \succsim_i H$ if and only if $\{h\} \succsim_i \emptyset$. This assumption rules out externality effects across candidates. The set of candidates that is most preferred by a voter thus consists of all the good ones : $h \in \mathrm{Argmax}(\succsim_i; 2^M)$ if and only if $\{h\} \succsim_i \emptyset$. Barberà, Sonnenschein and Zhou call a choice rule f "voting by committees" if for each candidate h there exists a committee W^h such that for all preference profiles $(\succsim_1, ..., \succsim_n)$, $h \in f(\succsim_1, ..., \succsim_n)$ if and only if $\{i \in N \mid h \in \mathrm{Argmax}(\succsim_i); 2^M)\} \in W^h$. In other words : Each voter submits his most preferred subset of candidates; candidate h is then elected if and only if her name appears on all the lists submitted by the members of a coalition that is winning under W^h. Note that if there exists a unique committee W for all h and if any two coalitions of equal size must be either both winning or both losing, the procedure reduces to a quota system : any candidate whose name appears on at least, say, q ballots, is elected. Barberà, Sonnenschein and Zhou's main result is that *a surjective choice rule is strategyproof if and only if it is voting by committee*. If, in addition, anonymity and neutrality are required from the choice rule, we are left with the quota systems.

To see how this result is related to Theorem 3.2, let $A^h = \{0, 1\}$ for all $h \in M$. There is an obvious one-to-one correspondence between the set $A = \prod_{h \in M} A^h$ and 2^M: the set $H \in 2^M$ corresponds to the alternative $a \in A$ for which $a^h = 1$ if $h \in M$ and $a^h = 0$ otherwise. Likewise, there is a one-to-one correspondence between the separable preferences on A and those on 2^M. The marginal preference R_i^h over A^h that is induced by the preference R_i corresponding to $\succsim_i$ is given by the condition that $1 \; R_i^h \; 0$ if and only if $\{h\} \succsim_i \emptyset$. This establishes the equivalence between Barberà, Sonnenschein and Zhou's main result and Theorem 3.2.

3.3 Allowing for indifferences

In classical economic models, the component sets A^h are usually not finite. The assumption that preferences are strict then becomes highly unsatisfactory. (It is, in particular, incompatible with continuity.) If indifferences are allowed, the method of proof used for Lemma 3.1 no longer works and, in fact, no decomposability result comparable to Lemma 3.1 exists yet in the literature.[3] Several characterizations have nevertheless been obtained for various subclasses of separable preferences. Quadratic and, more generally, star-shaped preferences were analyzed by Border and Jordan (1981, 1983) and Chichilnisky and Heal (1981), Euclidean preferences were studied by Laffond (1980), Kim and Roush (1984), Peters, van der Stel and Storcken (1992, 1993) and van der Stel (1993); continuous preferences are the subject of Le Breton and Weymark (1993).

In Border and Jordan's (1983) very influential paper, the set of alternatives A is equal to $\mathbb{R}^m$. Each alternative may be interpreted as a bundle of public commodities (I avoid the term public good because it implicitly implies increasing preferences). A first domain considered by Border and Jordan consists of the so-called star-shaped preferences. A preference is star-shaped if it possesses a unique ideal point and if moving away from it along an arbitrary ray is always harmful. Formally,

Definition 3.6. *The preference R_i is star-shaped if there exists a (unique) point $a^*(R_i) \in \mathbb{R}^m$ such that $a^*(R_i) \; P_i (\lambda a + (1 - \lambda) a^*(R_i)) \; P_i \; a$ for all $a \in \mathbb{R}^m \setminus \{a^*(R_i)\}$ and $\lambda \in (0, 1)$.*

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3 Weymark (1993) proposes a domain richness condition under which a strategyproof rule *that only depends on the peaks of the preferences* decomposes into a product of marginal strategyproof rules. More work is needed to dispense with the peak-only restriction.

The class of star-shaped preferences (over $\mathbb{R}^m$) will be denoted by $SS(\mathbb{R}^m)$. A smaller class consists of the quadratic preferences :

Definition 3.7. *The preference R_i is quadratic if it admits a numerical representation of the form* $u_i(a) = -\sum_{h,k \in M} \alpha^{hk}(a^h - a^*(R_i)^h)(a^k - a^*(R_i)^k)$, *where (α^{hk}) is a symmetric positive definite matrix.*

The class of quadratic preferences (over $\mathbb{R}^m$) will be denoted $Q(\mathbb{R}^m)$. The positive results in Border and Jordan (1983) hold for quadratic or star-shaped preferences that are, in addition, separable. The notion of separability is just the same as for strict preferences (see Definition 3.1). The notation $SSS(\mathbb{R}^m)$ will stand for the class of separable star-shaped preferences and $SQ(\mathbb{R}^m)$ for the class of separable quadratic preferences. Observe that any member R_i of $SQ(\mathbb{R}^m)$ admits a numerical representation of the form $u_i(a) = -\sum_{h \in M} \alpha^h(a^h - a^*(R_i)^h)^2$, where each α^h is positive. Border and Jordan's main result is the following :

Theorem 3.3. *The choice rule $f : SQ(\mathbb{R}^m)^N \to \mathbb{R}^m$ is strategyproof and surjective if and only if for each $h \in M$ there exists a strategyproof and surjective choice rule ϕ^h : $Q(\mathbb{R})^N \to \mathbb{R}$ such that*

$$\forall R \in SQ(\mathbb{R}^m)^N, f^h(R) = \phi^h(R^h).$$

In other words, f is decomposable into a product of one-dimensional choice rules, each of which is strategyproof and surjective.

Border and Jordan complement their decomposability result with a characterization of the surjective strategyproof rules defined on $Q(\mathbb{R})^N$. This is useful because one-dimensional quadratic preferences form a *strict* subset of the single-peaked preferences. Indeed, a quadratic preference R_i over $\mathbb{R}$ is entirely determined by its peak : a R_i b if and only if $|a - a^*(R_i)| \leq |b - a^*(R_i)|$. Therefore, the characterization theorems of Section 2 need not apply. Yet, Border and Jordan show that a surjective choice rule over $Q(\mathbb{R})^N$ is strategyproof if and only if it is a minmax rule à la Moulin (1980). This result was also proved by Kim and Roush (1981).

Theorem 3.3 can be adapted to separable star-shaped preferences. Notice that the h^{th} marginal domain induced by the domain of m-dimensional separable star-shaped preferences consists of all single-peaked preferences over the h^{th} coordinate of $\mathbb{R}^m$. Border and Jordan (1981, 1983) prove the following result :

Theorem 3.4. *The choice rule* $f : SSS(\mathbb{R}^m)^N \to \mathbb{R}^m$ *is strategyproof and surjective if and only if for each* $h \in M$ *there exists a peak-only, uncompromising and surjective choice rule* $\phi^h : SP(\mathbb{R})^N \to \mathbb{R}$ *such that*

$$\forall R \in SSS(\mathbb{R}^m)^N,\ f^h(R) = \phi^h(R^h).$$

An interesting subclass of the separable quadratic preferences is obtained by assuming that the coefficients α^1, ..., α^m in their utility representations are all equal. These preferences are usually called Euclidean or spatial : alternatives that are closer to one's peak are necessarily better. Spatial preferences are popular in location theory and in models of political competition. Border and Jordan (1983) provide a simple example that shows that for such preferences, surjective strategyproof rules need not be decomposable. Let $E(\mathbb{R}^m)$ denote the set of Euclidean preferences on $\mathbb{R}^m$. Suppose $m = n = 2$. For any $a \in \mathbb{R}^2$, let $K(a)$ denote the set $\{x \in \mathbb{R}^2 \mid x^1 \leq a^1,\ x^2 \leq a^2$, and $x^2 - x^1 \leq a^2 - a^1\}$. Define $f : E(\mathbb{R}^2)^{\{1,2\}} \to \mathbb{R}^2$ as follows : for each profile R, f(R) is the maximal element of R_2 in $K(a^*(R_1))$. Obviously, agent 2 cannot manipulate this (surjective) choice rule and it is easily seen that neither can agent 1. Yet the rule is not decomposable. If $a^*(R_1) = (1, 1)$, $a^*(R_2) = (0, 1)$, and $a^*(\overline{R}_2) = (1, 1)$, then $f(R_1, R_2) = (1/2, 1/2)$ while $f(R_1, \overline{R}_2) = (1, 1)$: the second coordinate of the choice is modified even though the profile of marginal preferences over that coordinate has not changed. In fact, it is easy to verify that no relabelling of the coordinate axes would permit a decomposition.

Laffond (1980) provides a detailed analysis of m-dimensional, two-agent problems with Euclidean preferences. His results include a complete characterization of those strategyproof rules that are surjective, anonymous and continuous with respect to the preferences. Notice that since an Euclidean preference is fully determined by its peak, the appropriate notion of continuity is obvious. Laffond's characterization relies on the notion of self-polar cone. A cone K in $\mathbb{R}^m$ is self-polar if its polar $\{y \in \mathbb{R}^m \mid \forall x \in K,\ x \cdot y \leq 0\}$ is equal to $-K$.

Theorem 3.5. *The choice rule* $f : E(\mathbb{R}^m)^{\{1,2\}} \to \mathbb{R}^m$ *is strategyproof, anonymous, surjective and continuous if and only if there exists a closed, convex, self-polar cone K with nonempty interior such that*

$$\forall R \in E(\mathbb{R}^m)^{\{1,2\}},\ f(R) \text{ is the maximal element of } R_1 \text{ in } a^*(R_2) + K .$$

Further results on Euclidean preferences can be found in Kim and Roush (1984) and Peters, van der Stel and Storcken (1992). For the case where $m = 2$ and $n \geq 2$, Kim and Roush characterize all solutions that are strategyproof, anonymous and continuous. They show that any such solution can be constructed as follows. Choose a pair of orthogonal axes in $\mathbb{R}^2$. Each point x in the extended real plane can be represented by taking the projections on these axes as new coordinates. Let $\tilde{x}$ denote that representation. Choose $n + 1$ fixed points $a_1, ..., a_{n+1}$ in the extended real plane. To each profile R of Euclidean preferences, assign the point f(R) given by

$$\tilde{f}(R)^h = med(\tilde{a^*}(R_1)^h, ..., \tilde{a^*}(R_n)^h, \tilde{a}_1^h, ..., \tilde{a}_n^h) \quad \text{for } h = 1, 2 .$$

Kim and Roush further show that if efficiency is added to their first three requirements, the outcome must be determined coordinatewise according to the simple formulas

$$\tilde{f}(R)^h = med(\tilde{a^*}(R_1)^h, ..., \tilde{a^*}(R_n)^h) \qquad \text{for } h = 1, 2 .$$

There are no longer any fixed points, which implies that n must be odd. Peters, van der Stel and Storcken (1992) strengthen this result by showing that an efficient strategyproof rule defined on $E(\mathbb{R}^m)^N$ must be continuous. The continuity requirement in Kim and Roush's characterization is therefore redundant. Peters, van der Stel and Storcken also demonstrate that if $m > 2$, or if $m \geq 2$ and n is even, efficiency, strategyproofness, and anonymity are incompatible. Additional results on Euclidean preferences and, more generally, on preferences generated by arbitrary strictly convex norms, are contained in van der Stel (1993) and Peters, van der Stel and Storcken (1991b, 1993).

Let me close this subsection on separable weak preferences with a brief account of the recent paper by Le Breton and Weymark (1993). These authors investigate the case where preferences are separable, continuous, but otherwise unrestricted. Let $SC(\mathbb{R}^m)$

denote this domain.[4] Le Breton and Weymark prove that a surjective choice rule defined on $SC(\mathbb{R}^m)^N$ must be coordinatewise dictatorial :

Theorem 3.6. *The choice rule $f : SC(\mathbb{R}^m)^N \to \mathbb{R}^m$ is strategyproof and surjective if and only if for every coordinate h there exists some agent i(h) such that*

$$\forall R \in SC(\mathbb{R}^m)^N,\ \forall a^h \in \mathbb{R},\ f^h(R)\ R^h_{i(h)}\ a^h .$$

If, in addition, the choice rule is required to be efficient, then it must be dictatorial : i(h) is the same agent for every coordinate h.

The proof of Theorem 3.6 is very long and quite involved. It is patterned on the proof of Barberà and Peleg's (1990) characterization result for the case where *all* continuous preferences are allowed (see Section 4). Le Breton and Weymark first establish their characterization for those profiles of continuous separable preferences which have a unique maximal element and then extend it to the domain of all profiles of continuous and separable preferences. As in Barberà and Peleg, the characterization on the subdomain is achieved by studying the properties of the "option sets" generated by a strategyproof choice rule. (Given a rule and the preferences of a group of agents, the option set of the remaining agents consists of those alternatives that they can enforce by reporting admissible preferences.) The fact, however, that nonseparable preferences cannot be used in analyzing option sets greatly complicates the argument.

4. BROAD DOMAINS

The results described so far in this survey hold under very stringent restrictions on the preferences. Traditional economic theory, however, does not impose such restrictions. It takes the view that while preferences within a given economic environment may naturally have considerable structure, such as continuity, strict convexity or strict monotonicity, they are not limited to any particular parametric form. If a preference is admissible, any other that is sufficiently close to it should be admissible as well. In Satterthwaite and Sonnenschein's (1981) terminology, the domain of preferences should be "broad". The present section seeks to understand the nature of strategyproof procedures defined on broad domains.

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4 LeBreton and Weymark's original framework is, in fact, more general. They assume that the set of alternatives is a product of Tychonoff spaces satisfying the first axiom of countability, with each component set containing at least three alternatives. The m-dimensional Euclidean space $\mathbb{R}^m$ meets these requirements.

The set of alternatives will be (some subset of) $\mathbb{R}^m$. Each point in that set should be interpreted as a feasible bundle of public commodities. I emphasize that the public character of the alternatives matters a lot. It means that each agent cares about all dimensions of $\mathbb{R}^m$. In private good allocation problems, by contrast, an agent cares only about the dimensions along which his own consumption is measured. This restriction, which *does* affect the possibility of constructing strategyproof rules, is not considered in the results that will be reviewed.

After a brief account in Subsection 4.1 of the early results of Satterthwaite and Sonnenschein (1981) and Border and Jordan (1981, 1983), Subsection 4.2 will describe in some detail the paper by Barberà and Peleg (1990). These authors established a version of the Gibbard-Satterthwaite theorem that is valid for continuous preferences. This required a completely novel technique of proof that was later applied by Zhou (1991a) to characterize strategyproof rules defined on profiles of continuous and convex preferences. A review of the recent paper by Barberà and Jackson (1992), which ties Zhou's characterization with Moulin's (1980) one-dimensional results, will conclude the survey.

4.1 Early works

In a seminal article on strategyproof choice in economic environments, Satterthwaite and Sonnenschein (1981) established a result reminiscent of the Gibbard-Satterthwaite theorem. I should mention that their paper does contain results about private good allocation rules; they will not be discussed here. The model is the following. The set of alternatives A is a compact, convex subset of $\mathbb{R}^m$ whose interior is nonempty. Each preference R_i is represented by a utility function u_i belonging to a convex subset U of a linear function space. This set must be open in the C^2 topology. The latter requirement formalizes the idea that the preference domain should be broad. For instance, the set of all concave twice-differentiable utility functions is not open : an arbitrarily small perturbation of a linear (hence, concave) utility function may destroy its concavity. The set of strictly concave twice-differentiable utility functions, however, is open.

Loosely speaking, Satterthwaite and Sonnenschein's main result characterizes the local behavior of a strategyproof choice rule at those preference profiles where it is smooth. Some notations are needed in order to state their result formally. Let u_{-i}

denote the subprofile $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n)$. Given a choice rule f, the *choice set* of agent i at u_{-i} consists of those alternatives that i can enforce given that the others report the preferences u_{-i} :

$$A_i(u_{-i}; f) = \{a \in A \mid a = f(u_i, u_{-i}) \text{ for some } u_i \in U\} .$$

This set will also be written as $A_i(u_{-i})$ whenever the reference to the choice rule need not be stressed. A *regular profile* is a preference profile where each agent faces a smooth choice set that varies smoothly with the others' preferences. Formally, $u \in U^N$ is regular if i) f is continuously differentiable at u, ii) for all i, $A_i(u_{-i})$ is continuously differentiable in u_{-i} and is a smooth manifold of dimension at most $m - 1$ in a neighborhood of f(u), and iii) for all i, f(u) is the unique and regular maximizer of u_i on $A_i(u_{-i})$.[5]

Theorem 4.1. *Let $f : U^N \to A$ be a strategyproof choice rule. Then for every regular preference profile $u^0 \in U^N$, there exists an agent $i \in N$ and a neighborhood $W(u^0_{-i})$ around u^0_{-i} such that*

$$\forall u_i \in U, \forall u_{-i} \in W(u^0_{-i}), f(u) \text{ maximizes } u_i \text{ over } A_i(u^0_{-i}; f) .$$

The point is that agent i's choice set is not affected by small changes in the reported preferences of the other agents around u^0_{-i}. Locally, the chosen alternative therefore depends only on agent i's preferences. Theorem 4.1 is thus a "local dictatorship" result. As recently stressed by Zhou (1991a) and Moulin and Shenker (1992), local dictatorship by no means implies global dictatorship. In the single-peaked model, for instance, the median-peak agent is a local dictator at every regular profile. Yet, the median rule is regarded as a reasonable procedure. Theorem 4.1 should therefore not be interpreted as a definitely negative result.

The first global negative result in an economic setting was derived by Border and Jordan (1981, 1983). Recall that a surjective strategyproof choice rule defined over the profiles of separable quadratic preferences on $\mathbb{R}^m$ factorizes into m one-dimensional strategyproof and surjective rules (Theorem 3.3). It turns out that as soon as the

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5 By strategyproofness, f(u) is always a maximizer of u_i on $A_i(u_{-i})$. It is regular if neither the gradient of u_i at f(u), nor its bordered Hessian, vanishes.

separability assumption is even slightly relaxed, an impossibility result prevails. To make this statement precise, let $Q_\varepsilon(\mathbb{R}^m)$ denote the set of preferences R_i that admit a numerical representation of the form $u_i(a) = -\sum_{h,k=1}^{m} \alpha^{hk}(a^h - a^*(R_i)^h)(a^k - a^*(R_i)^k)$ with $|\alpha^{hk}| < \varepsilon |\alpha^{hh}|$ whenever $h \neq k$.

Loosely speaking, these preferences are ε-close to being separable. Border and Jordan proved the following theorem :

Theorem 4.2. *Let $\varepsilon > 0$ and $m \geq 2$. Let $f : Q_\varepsilon(\mathbb{R}^m)^N \to \mathbb{R}^m$ be a surjective strategyproof rule. Then f is dictatorial : there exists $i \in N$ such that*

$$\forall R \in Q_\varepsilon(\mathbb{R}^m)^N, f(R) = a^*(R_i).$$

The main weakness of this result lies in the surjectivity assumption. Recall that the Gibbard-Satterthwaite theorem holds for any choice rule whose range contains as few as three alternatives. By contrast, the condition that f be onto $\mathbb{R}^m$ implicitly requires a much greater degree of flexibility from the choice rule. The difficult but fundamental issue is precisely to determine how much flexibility is compatible with strategyproofness.

4.2 A methodological breakthrough

In most economic models, preferences defined over subsets of Euclidean spaces are assumed to be continuous. This simple restriction contradicts the universal domain assumption of the Gibbard-Satterthwaite theorem. Moreover, the standard proofs of the theorem are not easily modified to handle the continuity restriction. Indeed, they typically rely on monotonicity arguments that involve modifying the relative position of one alternative in a given preference profile. Such perturbations may obviously destroy the continuity of the preferences. Barberà and Peleg (1990) developed a method of proof that avoids these difficulties and established a version of the Gibbard-Satterthwaite theorem which is valid on the domain of all continuous preferences. In as much as this particular restriction might not be the most relevant one from an economic viewpoint (one may also wish to impose convexity and monotonicity), their contribution is essentially methodological. But their proof technique has been successfully applied to other domains of preferences over public alternatives, e.g., by Barberà, Sonnenschein and Zhou (1991) (see Subsection 3.2 above), Le Breton and Weymark (1993) (see Subsection 3.3), Bordes, Laffond and Le Breton (1990) and, most notably, Zhou (1991a) (see Subsection 4.3 below). Moreover, the bulk of the recent literature on strategyproofness in private-commodity environments also follows their approach.

Let A be a connected subset of $\mathbb{R}^m$ and let C(A) denote the set of all continuous preferences over A. Barberà and Peleg's result can be expressed as follows :

Theorem 4.3. *A strategyproof choice rule $f : C(A)^N \to A$ whose range A_f contains at least three alternatives is dictatorial : there exists $i \in N$ such that*

$$\forall R \in C(A)^N,\ \forall a \in A_f,\ f(R) R_i\, a .$$

Because of its considerable impact on the literature, the proof of Theorem 4.3 will be presented below. Before going into the details, however, an informal overview may be helpful. The key concept is that of an *option set*. As already mentioned, given a choice rule f and agent i's preference R_i, the option set for the remaining agents consists of those alternatives that they can enforce by reporting subprofiles of admissible preferences : $A_{-i}(R_i) = \{a \in A \mid \exists\, R_{-i} \in C(A)^{N\setminus\{i\}} : a = f(R_i, R_{-i})\}$. Observe that agent i is a dictator if and only if the option set faced by the remaining agents always consists of (a subset of) i's most preferred alternatives in the range of f. The basic idea is to demonstrate the existence of such an agent by deriving increasingly stronger properties of the option sets generated by a strategyproof rule.

The proof is roughly divided into three parts. Consider a strategyproof rule f defined on the profiles of continuous preferences $C(A)^N$. The first part establishes the following preliminary results : (i) the rule must respect unanimity over its range : when all agents have the same preference, the choice must be one of their most preferred alternatives in the range; (ii) the range of the rule must be closed; and (iii) if the rule is dictatorial over the subdomain $C^*(A)^N$ of profiles of continuous preferences having a unique maximal element over the range, then it is dictatorial over the whole domain $C(A)^N$. In view of the latter fact, the second part of the proof focuses on the restriction of the choice rule f to $C^*(A)^N$. Notice that this restricted rule f* is itself strategyproof and that its range coincides with the range of f because of the preliminary result (i).

Using results (i) and (ii), the authors study the properties of the option set generated by f* for agents 2, ..., n. They reach the conclusion that either this option set is a singleton for every preference of agent 1 or it coincides with the full range of f for every preference of this agent. In the former case, agent 1 is a dictator for f*, hence also for f. The latter case is taken care of in the third part of the proof by using an induction argument on the number of agents.

Proof of Theorem 4.3.

Let $f : C(A)^N \to A$ be a strategyproof rule with $\#A_f \geq 3$. For any $B \subset A$ and $R \in C(A)$, Argmax(R; B) will denote the set of maximal elements of R over B. If this set is a singleton, its unique element will be written argmax(R; B).

Fact 1. f is unanimous over its range : $\forall R^0 \in C(A), f(R^0, ..., R^0) \in Argmax(R^0; A_f)$.

Let $a \in A_f$. Pick $R \in C(A)^N$ such that $f(R) = a$. By strategyproofness, $f(R^0) \; R^0 \; f(R_1, R^0, ..., R^0) \; R^0 \; ... \; R^0 \; f(R_1, ..., R_{n-1}, R^0) \; R^0 \; f(R) = a$.

Fact 2. A_f is closed.[6]

Let $\overline{A_f}$ denote the closure of A_f. Suppose $a \in \overline{A_f} \backslash A_f$. Let $\{a^k\}$ be a sequence in A_f converging to a. Pick $R^0 \in C(A)$ such that $argmax(R^0; \overline{A_f}) = a$. Since $a \notin A_f$, $f(R^0, ..., R^0) = b \neq a$. Since $a \; P^0 b$ and since R^0 is continuous, $a^k \; P^0 b$ for every large enough k. Select such a k and choose R^1 such that argmax $(R^1; \overline{A_f}) = a^k$. By Fact 1, $f(R^1, ..., R^1) = a^k$. By strategyproofness, $f(R^0, ..., R^0) \; R^0 \; f(R^1, R^0, ..., R^0) \; R^0 \; ... \; R^0 \; f(R^1, ..., R^1)$. Hence, $b \; R^0 \; a^k$, a contradiction.

Now, let C*(A) denote the set of preferences in C(A) that have a unique maximal element in A_f and let f* denote the restriction of f to $C^*(A)^N$. Notice that f* is strategyproof and that its range A_{f*} equals A_f because of Fact 1. Furthermore,

6 The proof presented here, which is more direct than Barberà and Peleg's, is due to Le Breton and Weymark (1993).

Fact 3. If f^* is dictatorial, then f is dictatorial. More precisely : $\forall i \in N$, $\{\forall R \in C^*(A)^N$, $f^*(R) \in Argmax(R_i; A_{f^*})\} \Rightarrow \{\forall R \in C(A)^N, f(R) \in Argmax(R_i; A_f)\}$.

Choose $i \in N$, say, $i = 1$. Suppose, by way of contradiction, that there exist $R \in C(A)^N$ and $b \in A_f$ such that $b\ P_1\ f(R)$. Construct $\hat{R}_1$ such that $\text{argmax}(\hat{R}_1; A_f) = b$ and $f(R)\ \hat{P}_1$ a for each $a \in A_f \setminus \{f(R)\}$ such that $f(R)\ R_1\ a$. Observe that

$$f(\hat{R}_1, R_{-1}) = f(R) .$$

Indeed, if $(\hat{R}_1, R_{-1}) \neq f(R)$, either $f(\hat{R}_1, R_{-1})\ \hat{R}_1\ f(R)$, in which case $f(\hat{R}_1, R_{-1})\ P_1\ f(R)$ and agent one manipulates at R via $\hat{R}_1$; or $f(R)\ \hat{P}_1\ f(\hat{R}_1, R_{-1})$ and he manipulates at $(\hat{R}_1, R_{-1})$ via R_1. For every $i \neq 1$, construct now $\hat{R}_i$ such that $\text{argmax}(\hat{R}_i; A_f) = f(\hat{R}_1, R_{-1})$. By repeated application of strategyproofness,

$$f(\hat{R}) = f(\hat{R}_1, R_{-1}) .$$

Therefore, $f(\hat{R}) = f^*(\hat{R}) = f(R)$. But $b\ \hat{P}_1\ f^*(\hat{R})$ and $b \in A_{f^*}$, contradicting the assumption that $f^*(R) \in \text{Argmax}(R_1; A_{f^*})$ for each $R \in C^*(A)^N$.

For each $i \in N$ and $R_i \in C^*(A)$, define

$$A^*_{-i}(R_i) = \{a \in A \mid \exists\, R_{-i} \in C^*(A)^{N\setminus\{i\}} : a = f^*(R_i, R_{-i})\} .$$

Because $f^*(R_i, \cdot) : C^*(A)^{N\setminus\{i\}} \to A$ is a strategyproof choice rule whose range is precisely $A^*_{-i}(R_i)$, we have :

Corollary to Fact 1. $\forall R \in C^*(A)^N$ *such that* $R_j = R^0$ *for all* $j \neq i$, $f(R) = f^*(R) = argmax(R^0; A^*_{-i}(R_i))$.

Corollary to Fact 2. $\forall R_i \in C^*(A)$, $A^*_{-i}(R_i)$ *is closed.*

The next fact is a "peak-only" property of option sets :

Fact 4. $\forall \overline{R}_i, \bar{\bar{R}}_i \in C^*(A)$, $\text{argmax}(\overline{R}_i; A_f) = \text{argmax}(\bar{\bar{R}}_i; A_f) = a^* \Rightarrow A^*_{-i}(\overline{R}_i) = A^*_{-i}(\bar{\bar{R}}_i)$.

Assume, by way of contradiction, that there exists, say, $b \in A^*_{-i}(\bar{\bar{R}}_i) \setminus A^*_{-i}(\overline{R}_i)$. By the corollary to Fact 1, $a^* \in A^*_{-i}(\overline{R}_i)$. Hence, $b \neq a^*$. Let us denote by d the Euclidean metric and let $B(a, \partial)$ denote the open ball of center a and radius ∂. Choose $\partial > 0$ such that $B(b, 2\partial) \cap A^*_{-i}(\overline{R}_i) = \emptyset$. For each $a \in A$ and each $j \neq i$, define

$$\hat{u}_j(a) = \frac{d(a,\ A\backslash B(b,\ \partial))}{d(a,\ b)\ +\ d(a,\ A\backslash B(b,\ \partial))},$$

$$\hat{\hat{u}}_j(a) = \frac{d(a,\ A\backslash B(a^*,\ \partial))}{2(d(a,\ a^*)\ +\ d(a,\ A\backslash B(a^*,\ \partial))},$$

$$u^*_j(a) = \hat{u}_j(a) + \hat{\hat{u}}_j(a)\ ,$$

and let R^*_j be the preference represented by u^*_j. Alternative b is this preference's maximal element in A, while a^* is a local peak. Now, by the corollary to Fact 1, $f(\bar{\bar{R}}_i, R^*_{-i}) = \text{argmax}(R^*_j; A^*_{-i}(\bar{\bar{R}}_i)) = b$, since $b \in A^*_{-i}(\bar{\bar{R}}_i)$ and $b\ P^*_j\ a$ for all $a \in A^*_{-i}(\bar{\bar{R}}_i) \setminus \{b\}$. Likewise, $f(\overline{R}_i, R^*_{-i}) = a^*$. But then f is manipulable by i at $(\bar{\bar{R}}_i, R^*_{-i})$ via $\overline{R}_i$, a contradiction.

Fact 5. One of the following statements holds true :

(i) $A^*_{-1}(R_1)$ *is a singleton for each* $R_1 \in C^*(A)$,

(ii) $A^*_{-1}(R_1) = A_f$ *for each* $R_1 \in C^*(A)$.

Suppose not. Then there exists $\overline{R}_1$ and three distinct alternatives a, b, $c \in A_f$ such that $a, b \in A^*_{-1}(\overline{R}_1)$ and $c \notin A^*_{-1}(\overline{R}_1)$. Without loss of generality, assume that $\text{argmax}(\overline{R}_1; A_f) = a$ and (by Fact 4) that $c\ \overline{P}_1\ b$. Choose $\hat{R}_2$ such that $\text{argmax}(\hat{R}_2; A_f) = c$ and $\text{argmax}(\hat{R}_2; A^*_{-1}(\overline{R}_1)) = b$ (such a preference exists by the corollary to Fact 2). By the corollary to Fact 1, $f(\overline{R}_1, \hat{R}_2, ..., \hat{R}_2) = b$. Since, by Fact 1, $f(\hat{R}_2, ..., \hat{R}_2) = c$, agent one manipulates at $(\overline{R}_1, \hat{R}_2, ..., \hat{R}_2)$ via $\hat{R}_2$, a contradiction.

The rest of the proof proceeds now by induction on the number of agents. The case n = 1 is trivial. Suppose the theorem holds for all k = 1, ..., n − 1 and let f be defined for n agents.

If (i) of Fact 5 holds, then agent one is a dictator for f* since the singleton $A^*_{-1}(R_1)$ must consist of the maximal element of R_1 in A_f. By Fact 3, f is therefore dictatorial.

If (ii) holds, $f(R_1, \cdot)$ is an (n−1)-agent strategyproof rule for every R_1 and thus dictatorial by the induction hypothesis. If the dictator is the same for all R_1, the proof is complete. Suppose on the contrary that there exist $\overline{R}_1$, $\hat{R}_1$, and two distinct agents i, j ∈ N \ {1} such that i is a dictator for $f(\overline{R}_1, \cdot)$ and j is a dictator for $f(\hat{R}_1, \cdot)$. Without loss of generality, we may assume that $\overline{R}_1 \in C^*(A)$. Therefore, there exist a, b ∈ A_f such that a $\overline{P}_1$ b. Consider a profile $(\overline{R}_1, \tilde{R}_{-1})$ such that $\text{argmax}(\tilde{u}_i; A_f) = b$ and $\text{argmax}(\tilde{u}_j; A_f) = a$. We have $f(\overline{R}_1, \tilde{R}_{-1}) = b$ since b is i's most preferred alternative in the range and i is a dictator for $f(\overline{R}_1, \cdot)$. Likewise, $f(\hat{R}_1, \tilde{R}_{-1}) = a$. Thus, agent one manipulates at $(\overline{R}_1, \tilde{R}_{-1})$ via $\hat{R}_1$, a contradiction. The proof is now complete. ▪

4.3 Recent results

In a recent article, Zhou (1991a) pursues the line of research initiated by Barberà and Peleg. Like these authors, Zhou is concerned with choice problems involving pure public alternatives only. The interest of his paper is that it concentrates on domain restrictions that are as close as possible to the standard ones in economic theory.

The set of alternatives, A, is assumed to be a (nonempty) convex subset of $\mathbb{R}^m$. In the first part of the paper, preferences over A are assumed to be continuous and convex. Zhou establishes for this case a theorem which is an exact analogue of the Gibbard-Satterthwaite theorem for the universal domain case. In essence, both theorems state that a strategyproof rule is either extremely rigid or dictatorial. For Gibbard and Satterthwaite, an extremely rigid rule is one whose range contains less than three alternatives. For Zhou, a rule is extremely rigid if the dimension of its range is smaller than two.[7] Letting CC(A) denote the set of continuous and convex preferences over A, we have :

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7 The dimension of a subset B of $\mathbb{R}^m$ is the dimension of the smallest affine superset of B. Notice that this is *not* the definition used in the study of manifolds.

Theorem 4.4. *A strategyproof rule $f : CC(A)^N \to A$ whose range A_f is at least of dimension two is dictatorial : there exists $i \in N$ such that*

$$\forall R \in CC(A)^N,\ \forall a \in A_f,\ f(R) R_i\, a\ .$$

The dimension condition is related to the cardinality condition of the Gibbard-Satterthwaite theorem. If the dimension of A_f is at least two, then A_f contains at least three alternatives : Zhou's condition implies Gibbard and Satterthwaite's. The converse is not true. However, if A_f contains three alternatives *that do not lie on the same line*, then the dimension of A_f is at least two. Note also that neither condition can be dispensed with. When A_f contains only two alternatives, simple majority voting and, more generally, any form of voting by committee are strategyproof. In Zhou's model, if A_f is a one-dimensional (connected) set, the restriction to A_f of any preference in CC(A) is a single-plateau preference. The median rule discussed in Section 2 is easily adapted to provide a nondictatorial strategyproof rule in this framework (see Moulin, 1984).

Theorem 4.4 substantially improves upon the early results discussed in Subsection 4.1. In contrast with Satterthwaite and Sonnenschein's result (Theorem 4.1), Zhou's theorem expresses a global impossibility. It is also free of any technical assumption of smoothness on the choice rule. Contrary to the Border and Jordan result (Theorem 4.2), Theorem 4.4 takes feasibility constraints into account by assuming that the set of alternatives is a convex subset of an Euclidean space rather than an entire space. More importantly yet, the surjectivity assumption is dropped and the trade-off between flexibility and equity is very clearly delineated.

Zhou proves Theorem 4.4 by establishing a stronger result that holds for any sufficiently large domain of preferences. Say that the domain D(A) is *abundant* on the set $B \subset A$ if the restriction to B of every quadratic preference over A is the restriction to B of some preference in D(A). Zhou shows that *if D(A) is abundant on a (nonempty) convex subset B of A, every strategyproof rule f on $D(A)^N$ is dictatorial if its range is included in B and is at least of dimension two*. The proof follows Barberà and Peleg's (1990) approach. In its first part, it focuses on the subdomain D*(A) of those preferences in D(A) which are strictly convex and have a unique maximal element in the range of f. Properties of the option sets generated by the restriction f* of f to $D^*(A)^N$ are derived. The requirement that preferences be convex seriously complicates the argument because it forbids using the double-peaked preferences utilized by Barberà and Peleg (see Fact 5 of the proof in Subsection 4.2). The main steps are listed below, where $A^*_{-i}(R_i)$ is the option set faced by the agents other than i under the rule f* when i's preference is R_i, and where $\mathrm{argmax}(R_i; A^*_f)$ denotes the maximal element of R_i in the range A_{f*} of f* :

1) $\forall i \in N, \forall R_i \in D^*(A), A^*_{-i}(R_i)$ is closed;

2) $\forall i \in N, \forall R_i \in D^*(A), \mathrm{argmax}(R_i; A_{f*}) \in A^*_{-i}(R_i)$;

3) $\forall i \in N, \forall R_i \in D^*(A), A^*_{-i}(R_i)$ is star-shaped (in A_{f*}) with respect to $\mathrm{argmax}(R_i; A_{f*})$;

4) $\forall i \in N, \forall R_i, R_i' \in D^*(A), A^*_{-i}(R_i) = A^*_{-i}(R_i')$ whenever $\mathrm{argmax}(R_i; A_{f*}) = \mathrm{argmax}(R_i'; A_{f*})$;

5) $\forall i \in N$, one of the following statements holds :

   i) $\forall R_i \in D^*(A), A^*_{-i}(R_i) = \{\mathrm{argmax}(R_i; A_{f*})\}$,
   ii) $\forall R_i \in D^*(A), A^*_{-i}(R_i) = A_{f*}$;

6) $\exists i \in N : \forall R_i \in D^*(A), A^*_{-i}(R_i) = \{\mathrm{argmax}(R_i; A_{f*})\}$.

Having established the existence of a dictator on $D^*(A)^N$, Zhou then shows that he must be a dictator on the domain $D(A)^N$ as well.

In many economic problems, preferences are not only continuous and convex but also increasing. Because the domain of quadratic preferences contains preferences that are not increasing, neither Theorem 4.4 nor the stronger result from which it is derived can be applied directly. In the second part of his paper, Zhou presents an example that shows how his results can nevertheless be used in an indirect fashion. Consider the following fairly standard public choice problem. The set of alternatives takes the form

$$A = \{a \in \mathbb{R}^m_+ \mid \sum_{h=1}^{m} p^h a^h \leq 1\} ,$$

where $p^1, \ldots, p^m$ and I are positive real numbers representing, respectively, the prices of the public goods 1, ..., m and the total budget to be spent on these goods. Assume further that $m \geq 3$ and let $CSCSI(\mathbb{R}^m_+)$ denote the set of continuous, strictly convex and strictly increasing preferences over $\mathbb{R}^m_+$. A choice rule is a mapping f : $CSCSI(\mathbb{R}^m_+)^N \rightarrow A$.

Let ∂A denote the upper boundary of the feasible set, i.e., $\partial A = \{a \in \mathbb{R}^m_+ \mid \sum_{h=1}^{m} p^h a^h = I\}$. It is easy to check that the restriction to ∂A of every quadratic preference coincides with the restriction of some continuous, strictly convex and strictly increasing preference : the domain CSCSI $(\mathbb{R}^m_+)$ is thus abundant on ∂A. Zhou's main result therefore implies that a strategyproof rule will be dictatorial if its range is a subset of ∂A whose dimension is at least two. As may be expected, the latter condition is fulfilled by any strategyproof rule f which is *unanimous* (i.e., for which $f(R^0, \ldots, R^0) = \text{argmax}(R^0; A)$ for every $R^0 \in CSCSI(\mathbb{R}^m_+)$); in fact, the range of a unanimous rule coincides with ∂A. This leads us to the following conclusion :

Theorem 4.5. *If $m \geq 3$, if $p^1, \ldots, p^m, I > 0$, and if $A = \{a \in \mathbb{R}^m_+ \mid \sum_{h=1}^{m} p^h a^h \leq I\}$, then every unanimous strategyproof choice rule $f : CSCSI(\mathbb{R}^m_+) \rightarrow A$ is dictatorial.*

Informative as it may be, the above theorem remains a partial result. More work is needed to understand the implications of the sole strategyproofness requirement in contexts where preferences are increasing.

It may be appropriate to close this section by mentioning the recent paper by Barberà and Jackson (1992). As already pointed out, the restriction of a continuous and convex preference to some connected one-dimensional subset of $\mathbb{R}^m$ is a single-plateau preference. If the original preference is strictly convex, its restriction will be single-peaked. This observation prompted Barberà and Jackson to combine Moulin's (1980) and Zhou's (1991a) results to provide a full characterization of strategyproof rules when preferences defined over some convex subset A of $\mathbb{R}^m$ are continuous, strictly convex and have a unique maximal element in A. If the range of the rule is at least of dimension two, we know from Zhou's work that the rule is dictatorial. If its range is one-dimensional, we expect the rule to be of the minmax variety as in Moulin (1980). Some care must be taken, however, because the restriction of an admissible preference to the range need not be single-peaked if the latter is not connected. The restricted preferences will nevertheless have at most two "twin peaks". Barberà and Jackson prove that the choice will still be based on a minmax formula; in case of a double-peaked preference, which peak enters the formula is determined through some tie-breaking device. The authors provide a characterization of those devices that preserve strategyproofness.

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