Organizing Time Banks: Lessons from Matching Markets[*]

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Abstract

A time bank is a group of people that set up a common platform to trade services among themselves. There are several well-known problems associated with this type of time banking, e.g., high overhead costs and difficulties to identify feasible trades. This paper constructs a non-manipulable mechanism that selects an individually rational and time-balanced allocation which maximizes exchanges among the members of the time bank (and those allocations are efficient). The mechanism works on a domain of preferences where agents classify services as unacceptable and acceptable (and for those services agents have specific upper quotas representing their maximum needs).

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JEL Classification: D82; D47.

1 Introduction

Time banks have now been established in at least 34 countries. In the United Kingdom, for example, there are more than 300 time banks, and in the United States time banks are operating in at least 40 states.[1] A time bank is a group of individuals and/or organizations in a local community that set up a common platform to trade services among themselves. Services supplied by members of a time bank typically include legal assistance, gardening services, medical services, child care and language lessons. Members of a time bank earn time credit for each time unit they supply to members of the bank and the earned credit can be spent to receive services from other members of the bank.[2] For

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[1]"Time Banking: An Idea Whose Time Has Come?". Yes Magazine.

[2]Some time banks are not based on a "one-for-one" time system, meaning that members of the time bank need not get one unit of time back for each unit of time they supply (Croall, 1997).

example, a gardener who supplies two hours of time may, for example, get a haircut and one hour of child care in return for his gardening services. Even if time banks traditionally have had a very simple organization, most of the nowadays existing time banks take advantage of computer databases for record keeping, and a physical coordinator keeps track of transactions and match requests for services with those who can provide them.

A critical factor for a time bank to function smoothly is the coordination device matching requests for services with those who can provide them. Our basic observation is that this type of service exchange shares many features with some classical markets previously considered in the matching literature, including, e.g., housing markets (Scarf and Shapley, 1974; Abdulkadiroğlu and Sönmez, 1999; Aziz, 2016b), organ markets (Roth et al., 2004; Biró et al., 2009; Ergin et al., 2017), marriage markets (Gale and Shapley, 1962), and markets for school seats (Abdulkadiroğlu and Sönmez, 2003; Kesten and Ünver, 2015). In particular, if a time bank is organized as a matching market, the time bank will have a structure of what in the matching literature is known as a many-to-many matching market. This follows since any member of a time bank can trade services with any other member of the very same time bank and there are no obstacles that prevent a member of a time bank to supply and receive multiple services from members of the very same time bank. Such matching markets have previously been considered by, e.g., Echenique and Oviedo (2006), Konishi and Ünver (2006), and Hatfield and Kominers (2016).

The above mentioned matching markets are centralized as the agents in the system (e.g., tenants, patients, or students) report their preferences over the items to be allocated (e.g., houses, organs, or school seats) to a clearing house and a mechanical procedure determines the final allocation based on the reported preferences and a set of predetermined axioms. Even if time banks often take advantage of computer databases, there is no mechanical procedure that determines the trade of services among the members in the bank based on reported preferences, and it is exactly in this respect that time banks can learn from classical matching markets.

By organizing a time bank as a matching market, it will be possible to solve a number of problems which have been associated with time banks across the world. For example, time banks typically encounter long run organizational sustainability since they experience high overhead costs, e.g., as staff is needed to keep the organization running and, in particular, to help out in the coordination process (Seyfang, 2004). Moreover, it may be challenging for a physical coordinator to identify and coordinate longer trading cycles, and members of time banks often experience that time credits are comparatively easy to earn but harder to spend.[3]

A computer-based clearing house (e.g., an internet-based interface for reporting needs and requests together with an algorithm for matching needs and requests), on the other hand, can help in reducing costs related to coordination and it can identify and coordinate longer cycles in order to maximize trade in the time bank. In addition, problems related to participation and maximality can be solved by designing the matching algorithm in such a way that it always recommends maximal trades where an agent can never lose by joining a time bank (individual rationality and maximality) and by requiring that all members of the bank should receive exactly as much time back as they supply to the bank (time-balance). Furthermore, those allocations turn out to be efficient.

Given the interest in allocations that are individually rational, maximal, and time-balanced, a first

[3] This is the reverse situation compared to conventional credits which generally are hard to earn, but easy to spend.

observation is that such allocations always exist on the general preference domain.[4] However, even if an allocation satisfying these specific properties can be identified, two new problems arise. First, it is often natural to require that the algorithm should be designed in such fashion that it is in the best interest for all agents to report their preferences truthfully (non-manipulability). This property is incompatible with individual rationality, efficiency and time-balance on a general preference domain (Sönmez, 1999, Corollary 1).[5] Second, because members of a time bank can exchange multiple time units, it is not clear that it is easy for members to generally rank any two "consumption bundles". For example, is two hours of hairdressing, two hours of gardening and one hour of babysitting strictly better, equally good, or less preferred to one hour of hairdressing, one hour of gardening and three hours of housekeeping? Hence, it may be an obstacle for members to report their preferences if multiple time units are on stake and if multiple agents are allowed to be involved in a cyclical trade.

As we demonstrate, the above two problems can be solved simultaneously by considering a restricted preference domain. This restricted domain is an extension of the dichotomous domain popularized by Bogomolnaia and Moulin (2004).[6] In the considered domain, individual preferences are completely described by (i) partitioning the members of the bank (or, equivalently, the services that the members provide) into two disjoint subsets containing acceptable and unacceptable members, and by (ii) specifying a member specific upper time bound for each acceptable member. The former condition reflects that an agent is not necessarily interested in all services provided in the bank (an agent's "horizontal" preference) whereas the latter condition captures the idea that an agent may, for example, be interested in at most one haircut but can accept up to 10 hours of babysitting (an agent's "vertical" preference). One advantage of adopting this preference domain is that it facilitates for agents to report their preferences as not all possible bundles have to be ranked strictly.[7] Agents then strictly prefer receiving more time units from acceptable services to receiving fewer time units from acceptable services (without exceeding upper bounds and receiving unacceptable services). In this sense, an agent may have many different indifference classes and preferences are not dichotomous but rather polychotomous.

We define and apply a priority mechanism to solve the problem of exchanging time units between members in a time bank. It is demonstrated that the priority mechanism can be formulated as a min-cost flow problem (Proposition 1). Consequently, it is not only possible to identify time-balanced trades, it is also computationally feasible. The definition of the priority mechanism is flexible as it can be adopted on the restricted preference domain or the general domain. Our main result shows that the priority mechanism is non-manipulable on the restricted preference domain and it always makes a selection from the set of individually rational, maximal, and time-balanced allocations (Theorem 1). To prove this result, a number of novel graph theoretical techniques are needed. In particular, Appendix

[4]This follows since the allocation in which all agents receive their initial endowments is individually rational and satisfy time-balance. The conclusion then follows directly from the observation that the number of individually rational allocations that satisfy time-balance is finite and, consequently, that there exists an allocation among those which maximizes trade in the time bank.

[5]This impossibility should come as no surprise given the results in, e.g., Hurwicz (1972), Green and Laffont (1979), Roth (1982), Alcalde and Barberà (1994), Barberà and Jackson (1995), or Schummer (1999).

[6]In fact, Bogomolnaia and Moulin (2004) and a series of subsequent papers, argue that it is natural to consider a dichotomous domain in problems involving "time sharing".

[7]The strict preference domain is often considered in the matching literature. However, the dichotomous domain is much smaller in size than the strict preference domain, but is is not a subset of the strict domain since indifference relations are allowed in the former but not in the latter domain.

B demonstrates an equivalence result between the min-cost flow problem and a circulation-based maximization problem.[8]

Due to the above mentioned impossibility, a priority mechanism where non-manipulability is abandoned is considered on the general domain. In this case, the priority mechanism is demonstrated to be at least be partly non-manipulable in the sense that any agent that regards the selection of the priority mechanism as most preferred from the set of individually rational, efficient and time-balanced allocations will be unable to manipulate the outcome of the mechanism in his advantage (Theorem 2).

A variety of real-life problems have previously been considered in the matching literature including the above mentioned house allocation problem, kidney exchange problem and school choice problem. There are, however, several differences between these problems and the time banking problem. For example, in the time banking problem, an agent may receive and supply multiple time units. In the school choice problem and the kidney exchange problem, on the other hand, students are allocated at most one school seat and a patient is involved in at most one kidney exchange, respectively. Furthermore, in many matching problems including, e.g., the school choice problem and the house allocation problem, preferences are typically strict and indifference relations are consequently not allowed (the kidney exchange problem is often defined on a dichotomous domain). Generalizations to allow for a weak preference structure have recently been proposed by Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012). However, both these papers only allow agents to trade at most one object. The papers closest to the model investigated here are Athanassoglou and Sethuraman (2011), Aziz (2016a), Biró et al. (2017) and Manjunath and Westkamp (2017), which we describe below.

Athanassoglou and Sethuraman (2011) and Aziz (2016a) consider a housing market where initial endowments as well as allocations are described by a vector of fractions of the houses in the economy. The fractional setting makes it possible to analyze, e.g., efficiency based on (first-order) stochastic dominance, and it is demonstrated that the efficiency and fairness notions of interest conflict with non-manipulability. Even if a similar impossibility is present in the model considered in this paper, the fractional setting is analyzed using different axioms and mechanisms. In addition, Athanassoglou and Sethuraman (2011) and Aziz (2016a) are unable to find any positive results related to non-manipulability in their, respectively, considered reduced preference domains.

Biró et al. (2017) consider, as this paper, a model where agents are endowed with multiple units of an indivisible and agent-specific good, and search for balanced allocations. In their reduced preference domain, agents have responsive preferences over consumption bundles. On this reduced domain, they demonstrate that, for general capacity configurations, no mechanism satisfies individual rationality, efficiency, and non-manipulability. Given this negative finding, they characterize the capacity configurations for which individual rationality, efficiency and non-manipulability are compatible. They also demonstrate that for these capacity configurations, their defined Circulation Top Trading Cycle Mechanism is the unique mechanism that satisfies all three properties of interest. Hence, the main difference between this paper and Biró et al. (2017) is that they consider a different preference domain and, consequently, need a different mechanism to escape the impossibility result.

Finally, Manjunath and Westkamp (2017) have independently considered a model closely related

[8]The min-cost flow problem is considered in the main part of the paper since it is more intuitive and, moreover, can be introduced using minimal notation.

to the one considered here.[9] In their model, an agent can supply distinct services and in our model each agent supplies multiple copies of one service. They also require time-balance and consider a preference domain classifying services as unacceptable and acceptable (no need to specify upper bounds on services since each service is available in one unit). Given this, Manjunath and Westkamp (2017) define a priority mechanism over the set of individually rational and efficient allocations. The main differences between their work and ours is that (i) they allow agents having distinct services whereas each agent in our model has a specific service that comes in multiple copies (ii) their priority mechanism chooses from the set of individually rational and efficient allocations whereas ours chooses from the set of individually rational and maximal allocations (and as we show, any priority mechanism may choose different allocations in their setting and in ours), and (iii) for the non-manipulability result they use a bipartite graph approach whereby capacities for unacceptable services are reduced one-by-one (following the priority order) whereas we use a direct circulation based graph with upper capacities on edges (where the min-cost flow corresponds to the allocation chosen by the priority mechanism). In summary, Manjunath and Westkamp (2017) have a more general model and this paper has a more demanding objective, since we find not only efficient but also maximal allocations.

The remaining part of the paper is outlined as follows. Section 2 introduces the theoretical framework and some basic definitions. The priority mechanism is presented in Section 3. The main results are presented in Section 4. Section 5 discusses our results and concludes. All proofs are relegated to the Appendix.

2 The Model and Basic Definitions

This section introduces the time banking problem together with some definitions and axioms.

2.1 Agents, Bundles, and Allocations

Let $N = \{1, \dots, n\}$ denote the finite set of agents. Each agent $i \in N$ is endowed with $t_i \in \mathbb{N}$ units of time that can be used to exchange services with the agents in N. Let $t = (t_1, \dots, t_n)$ denote the vector of time endowments. Because the exact nature of the services is of secondary interest, the problem will be described in terms of the time that an agent receives from and provides to other agents in N. Let x_{ij} denote the time that agent $i \in N$ receives from agent $j \in N$, or, equivalently, the time that agent j provides to agent i. Here, x_{ii} represents the time that agent $i \in N$ receives from or, equivalently, spends with himself. It is assumed that x_{ij} belongs to the set $\mathbb{N}_0$ of non-negative integers (including 0) representing standardized time units (e.g., 0 minutes for zero units, 30 minutes for one unit, 60 minutes for two units, etc.)

The time that agent $i \in N$ receives from the agents in N can be described by the bundle (or vector) $x_i = (x_{i1}, \dots, x_{in})$. The bundle where agent $i \in N$ spends all time with himself is denoted by ω_i (where $\omega_{ii} = t_i$ and $\omega_{ij} = 0$ for $j \neq i$). An allocation $x = (x_1, \dots, x_n)$ is a collection of n

[9] As of July 19, 2018 we have only seen a conference presentation of Manjunath and Westkamp (2017) and a preliminary draft sent to us, no working paper is available on the webpages of the authors.

bundles (one for each agent in N). An allocation is *feasible* if

$$\sum_{j=1}^{n} x_{ij} = t_i \text{ for all } i \in N, \tag{1}$$

$$\sum_{j=1}^{n} x_{ji} = t_i \text{ for all } i \in N. \tag{2}$$

This means any agent i receives the same amount of time from other agents that the agent supplies to other agents (recall that an agent can receive time from and spend time with himself). In this sense, any feasible allocation satisfies the time-balance conditions (1) and (2). In the remaining part of the paper, it is understood that any allocation is feasible.

2.2 Preferences and Preference Domains

A preference relation for agent $i \in N$ is a complete and transitive binary relation R_i over feasible bundles such that $x_i R_i x_i'$ whenever agent i finds bundle x_i at least as good as bundle x_i'. Let P_i and I_i denote the strict and the indifference part of R_i, respectively. Let $\mathcal{R}_i$ denote the set of all preference relations of agent $i \in N$. A (preference) profile R is a list of individual preferences $R = (R_1, \ldots, R_n)$. The general domain of profiles is denoted by $\mathcal{R} = \mathcal{R}_1 \times \cdots \times \mathcal{R}_n$. A profile $R \in \mathcal{R}$ may also be written as (R_i, R_{-i}) when the preference relation R_i of agent $i \in N$ is of particular importance.

A restricted preference domain $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}_1 \times \cdots \times \tilde{\mathcal{R}}_n \subset \mathcal{R}$ will be considered for our main results. As explained in the Introduction, this restricted domain is based on the idea that any preference relation $R_i \in \tilde{\mathcal{R}}_i$ (1) partitions the set of agents $N \backslash \{i\}$ into two disjoint sets containing acceptable and unacceptable agents, denoted by $A_i(R_i) \subseteq N \setminus \{i\}$ and $U_i(R_i) = N \setminus (A_i(R_i) \cup \{i\})$, respectively, and (2) associates with each acceptable agent $j \in A_i(R_i)$ an upper bound $\bar{t}_{ij} \in \mathbb{N}_0$ on how much time agent i at most would like to receive from agent j. Here one may may interpret (1) as agent i's "horizontal preference" over acceptable and unacceptable services and (2) as agent i's "vertical preference" of how much agent i needs at most of each service. Then, for agent $i \in N$, the preference relation R_i belongs to $\tilde{\mathcal{R}}_i$ if for any allocations x and y:

(i) $\omega_i P_i x_i$ if $x_{ik} > 0$ for some $k \in U_i(R_i)$ or $x_{ij} > \bar{t}_{ij}$ for some $j \in A_i(R_i)$,

(ii) $x_i I_i y_i$ if both $\omega_i P_i x_i$ and $\omega_i P_i y_i$,

(iii) $y_i P_i x_i$ if $y_i R_i \omega_i$, $x_i R_i \omega_i$ and $\sum_{j \in A_i(R_i)} y_{ij} > \sum_{j \in A_i(R_i)} x_{ij}$, or

(iv) $y_i I_i x_i$ if $y_i R_i \omega_i$, $x_i R_i \omega_i$ and $\sum_{j \in A_i(R_i)} y_{ij} = \sum_{j \in A_i(R_i)} x_{ij}$.

The first condition states that an agent strictly prefers not to be involved in any trade rather than receiving time from an unacceptable agent or exceeding his upper bound from an acceptable agent. The second condition means that an agent is indifferent between any two bundles containing an unacceptable agent or exceeding his upper bound from an acceptable agent. The last two conditions reflect a monotonicity property and state that an agent weakly prefers bundles with weakly more acceptable agents whenever bundles do not contain any unacceptable agents and as long as the time bounds $\bar{t}_{ij}$ are not exceeded for acceptable agents.

Remark 1. For the restricted domain $\tilde{\mathcal{R}}$, a report R_i for agent $i \in N$ is given by a set of acceptable agents $A_i(R_i)$ together with an upper time bound $\bar{t}_{ij}$ for each $j \in A_i(R_i)$. An equivalent formulation of the reported preference for agent $i \in N$ is a vector $\bar{t}_i = (\bar{t}_{i1}, \ldots, \bar{t}_{in}) \in \mathbb{N}_0^n$ where $\bar{t}_{ii} = t_i$. Then $\bar{t}_{ij} = 0$ stands for $j \in U_i(R_i)$, i.e., agent i is willing to accept at most zero time units from agent j. Whether the first or the second formulation is used is just a matter of choice.

Remark 2. For any agent $i \in N$ and $R_i \in \tilde{\mathcal{R}}_i$, the preference R_i is polychotomous in the following way: for any $h = 0, 1, \ldots, \min\{t_i, \sum_{j \in A_i(R_i)} \bar{t}_{ij}\} = m$, all allocations x and y such that for all $j \in A_i(R_i)$ $x_{ij} \leq \bar{t}_{ij}$ and $y_{ij} \leq \bar{t}_{ij}$ for all $j \in A_i(R_i)$, $x_{ik} = 0 = y_{ik}$ for all $k \in U_i(R_i)$ and $\sum_{j \in A_i(R_i)} y_{ij} = h = \sum_{j \in A_i(R_i)} x_{ij}$ are ranked indifferent by R_i. Let $\mathcal{I}(h)$ denote this indifference class. Then under R_i all allocations in $\mathcal{I}(m)$ are strictly preferred to all allocations in $\mathcal{I}(m-1)$, and in general, for $h = 1, \ldots, m$, under R_i all allocations in $\mathcal{I}(h)$ are strictly preferred to all allocations in $\mathcal{I}(h-1)$. Thus, R_i contains $m+2$ indifference classes (where $\mathcal{I}(0) = \{\omega_i\}$ and ω_i is strictly preferred to all allocations which are positive for some unacceptable service or exceeds the time bound for an acceptable service). In this sense, preferences belonging to $\tilde{\mathcal{R}}_i$ are polychotomous.

2.3 Axioms and Mechanisms

Let $\mathcal{F}(R)$ denote the set of all feasible allocations at profile $R \in \tilde{\mathcal{R}}$. Allocation $x \in \mathcal{F}(R)$ is *individually rational* if, for all $i \in N$, $x_i R_i \omega_i$. Allocation $x \in \mathcal{F}(R)$ *Pareto dominates* allocation $x' \in \mathcal{F}(R)$ if $x_i R_i x'_i$ for all $i \in N$ and $x_j P_j x'_j$ for some $j \in N$. An allocation is *efficient* if it is not Pareto dominated by any feasible allocation. An allocation x is *maximal* at R if $\sum_{i \in N} \sum_{j \in A_i(R_i)} x_{ij} \geq \sum_{i \in N} \sum_{j \in A_i(R_i)} x'_{ij}$ for all individually rational allocations x'. All individually rational and maximal allocations at profile $R \in \tilde{\mathcal{R}}$ are gathered in the set $\mathcal{X}(R) \subset \mathcal{F}(R)$. Note that $\mathcal{X}(R) \neq \emptyset$ for all $R \in \tilde{\mathcal{R}}$ and that any $x \in \mathcal{X}(R)$ is efficient.[10]

A mechanism φ with domain $\tilde{\mathcal{R}}$ chooses for any profile $R \in \tilde{\mathcal{R}}$ a feasible allocation $\varphi(R) \in \mathcal{F}(R)$. Mechanism φ is manipulable at profile $R \in \tilde{\mathcal{R}}$ by an agent $i \in N$ if there exists R'_i such that $R' = (R'_i, R_{-i}) \in \tilde{\mathcal{R}}$, and for $x = \varphi(R)$ and $x' = \varphi(R')$ we have $x'_i P_i x_i$. If mechanism φ is not manipulable by any agent $i \in N$ at any profile $R \in \tilde{\mathcal{R}}$, then φ is non-manipulable (on the domain $\tilde{\mathcal{R}}$).

3 Priority Mechanisms

Often in real life the chosen allocation is based on a priority mechanism: any such mechanism uses a priority-ordering, which may be deduced from a lottery or from a schematic update based on previous allocation rounds. Let $\pi : N \mapsto N$ be an exogenously given priority-ordering where the highest ranked agent is $i \in N$ with $\pi(i) = 1$, the second highest ranked agent is $i' \in N$ with $\pi(i') = 2$, and so on.

Given $R \in \tilde{\mathcal{R}}$, $i \in N$ and $\mathcal{Z}^* \subseteq \mathcal{X}(R)$, allocation $x \in \mathcal{Z}^*$ belongs to the set $\mathcal{X}^{i,\mathcal{Z}^*}(R)$ if $x_i R_i x'_i$ for all $x' \in \mathcal{Z}^*$, i.e., if allocation x is weakly preferred to any allocation in the set $\mathcal{Z}^*$ under preference R_i. In the special case where the set $\mathcal{Z}^*$ is based on the choice made by some agent $i' \neq i$ for some profile $R \in \tilde{\mathcal{R}}$, i.e., where $\mathcal{Z}^* = \mathcal{X}^{i',\mathcal{Z}^{**}}(R)$ for some $\mathcal{Z}^{**} \subseteq \mathcal{X}(R)$, the set $\mathcal{X}^{i,\mathcal{Z}^*}(R)$ is denoted by $\mathcal{X}^{i,i'}(R)$.

[10] If x is not efficient, then there exists an individually rational allocation x' such that $x'_i R_i x_i$ for all $i \in N$ and $x'_j P_j x_j$ for some $j \in N$. But then $\sum_{i \in N} \sum_{j \in A_i(R_i)} x_{ij} < \sum_{i \in N} \sum_{j \in A_i(R_i)} x'_{ij}$ meaning that x is not maximal, a contradiction.

Definition 1. An allocation $x \in \mathcal{X}(R)$ is agent-i-optimal at profile $R \in \tilde{\mathcal{R}}$ if $x \in \mathcal{X}^{i,\mathcal{X}(R)}(R)$.

Note the difference between the sets $\mathcal{X}^{i,\mathcal{X}(R)}(R)$ and $\mathcal{X}^{i,\mathcal{Z}^*}(R)$. The former set contains all agent *i*'s most preferred allocations in the set $\mathcal{X}(R)$ whereas the latter set contains all agent *i*'s most preferred allocations in a subset $\mathcal{Z}^*$ of $\mathcal{X}(R)$.

Definition 2. Let π be a priority ordering and $N = \{i_1, \dots, i_n\}$ be such that $\pi(i_k) = k$ for all $k = 1, \dots, n$. Then $x \in \mathcal{X}(R)$ is a π-priority allocation at profile $R \in \tilde{\mathcal{R}}$ if:

(i) x belongs to $\mathcal{X}^{i_1,\mathcal{X}(R)}(R)$,

(ii) x belongs to $\mathcal{X}^{i_k,i_{k-1}}(R)$ for all $k = 2, \dots, n$.

One way to think about the set of priority allocations is the following. First, the highest ranked agent identifies all his most preferred allocations in the set $\mathcal{X}(R)$. Then the agent with the second highest priority identifies all his most preferred allocations in the set identified by the highest ranked agent, then the agent with the third highest priority identifies all his most preferred allocations in the set identified by the second highest ranked agent, and so on. Formally, this means that if x is a π-priority allocation, then:

$$x \in \mathcal{X}^{i_n,i_{n-1}}(R) \subseteq \mathcal{X}^{i_{n-1},i_{n-2}}(R) \subseteq \ldots \subseteq \mathcal{X}^{i_2,i_1}(R) \subseteq \mathcal{X}^{i_1,\mathcal{X}(R)}(R) \subseteq \mathcal{X}(R). \tag{3}$$

Note that a priority allocation is agent-i-optimal for the agent $i \in N$ with $\pi(i) = 1$. Moreover, all agents in N are, by construction, indifferent between all allocations in the set $\mathcal{X}^{i_n,i_{n-1}}(R)$.

Definition 3. A mechanism φ is a priority mechanism if there exists a priority ordering π such that for all profiles $R \in \tilde{\mathcal{R}}$ the mechanism φ selects a π-priority allocation from the set $\mathcal{X}(R)$.

Since a priority mechanism always makes a selection from the set $\mathcal{X}(R)$, it chooses an individually rational, maximal, and time-balanced allocation (which is efficient).

4 Results

As we show in Section 5, it is impossible to construct an individually rational, efficient, and non-manipulable mechanism on the general domain $\mathcal{R}$. Our first main result demonstrates that this impossibility can be avoided on the restricted domain $\tilde{\mathcal{R}}$ if trades are based on a priority mechanism.

Theorem 1. Any priority mechanism with domain $\tilde{\mathcal{R}}$ is non-manipulable.

Below we demonstrate that a priority mechanism can be formulated as a min-cost flow problem (Proposition 1). To formulate this problem, a bipartite graph needs to be defined and specific values must be attached to the vertices and the edges in the graph.

Definition 4. For any profile $R \in \tilde{\mathcal{R}}$, the bipartite graph $g = (N, M, E, u)$ is defined by two disjoint sets of vertices, N and M, a set of edges, E, and a profile of upper bounds $u = (u(i,l))_{(i,l) \in E}$ on the flow between any two edges, defined by:

(i) $N = \{1, \dots, n\}$,

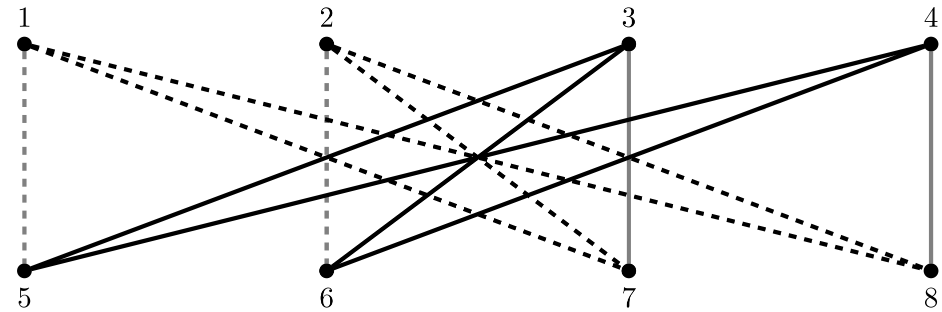

Figure 1: Edge capacity 1 is color-coded by gray, while capacity 2 is denoted by black edges. The edges connecting two copies of the same agent are marked by dashed lines.

(ii) $M = \{n+1, n+2, \ldots, n+n\}$,

(iii) $E = \{(i, n+j) \in N \times M : j \in A_i(R_i)$ or $j = i\}$, and

(iv) for all $i \in N$ and each edge $(i, n+j) \in E$ where $j \in A_i(R_i)$ we set $u(i, n+j) = \bar{t}_{ij}$ and $u(i, n+i) = t_i$.

Example 1. Let $N = \{1, 2, 3, 4\}$, $t_1 = t_2 = 1$ and $t_3 = t_4 = 2$. Let $R \in \tilde{\mathcal{R}}$ be such that $A_1(R_1) = A_2(R_2) = \{3, 4\}$ (with $\bar{t}_{13} = \bar{t}_{14} = \bar{t}_{23} = \bar{t}_{24} = 1$) and $A_3(R_3) = A_4(R_4) = \{1, 2\}$ (with $\bar{t}_{31} = \bar{t}_{32} = \bar{t}_{41} = \bar{t}_{42} = 2$). The constructed graph g is depicted in Figure 1.

The interpretation of the graph g is that the agents in M should be regarded as copies of the agents in N and in particular, agent $n + i \in M$ is the copy of agent $i \in N$. Furthermore, agents $i \in N$ and $n + j \in M$ are connected by an edge if agent j is acceptable for agent i or if $j = i$. Because an allocation will be defined by the flows between the agents in N and M, the above construction guarantees that $n+j \in M$ can only provide time for an agent $i \in N$ if agent i finds agent j acceptable or if agent j is his own copy. Finally the upper bound on flow from $n + j$ to i where $j \in A_i(R_i)$ is equal to the upper bound of how much time agent i wants from agent j. A flow x specifies for each $(i, l) \in E$ a non-negative integer $x_{il} \in \mathbb{N}_0$.[11] Any flow x is equivalent to an allocation in the usual sense: $x_{ii} = x_{i(n+i)}$, $x_{ij} = x_{i(n+j)}$ for all $j \in A_i(R_i)$, and $x_{ij} = 0$ for all $j \in U_i(R_i)$.

Recall that the time-balance conditions (1) and (2) must hold for any allocation. In the language of min-cost flow problems, this means that the required flow (between the vertices in the bipartite graph g) is dictated by conditions (1) and (2) which must be reformulated for the bipartite setting as follows:

$$\sum_{j \in A_i(R_i) \cup \{i\}} x_{i(n+j)} = t_i \text{ for all } i \in N, \tag{1'}$$

$$\sum_{i \in A_j(R_j) \cup \{i\}} x_{j(n+i)} = t_i \text{ for all } i \in N. \tag{2'}$$

A natural interpretation of the bipartite graph is therefore that agents in M supply time to the demanding agents in N. To obtain a maximal outcome, it is important to prevent flows between agents in N

[11] In general, flows may assign real numbers to edges, but for our purpose we restrict flows to assign integers.

and their own copies in M whenever there are other feasible flows or, equivalently, to prevent agents to supply time to their own copies whenever it is feasible to supply time to other distinct agents (by the time-balance conditions, any agent supplying time to other agents also receives in return more time from acceptable agents). This can be achieved by introducing an artificial cost whenever agents supply time to themselves. Let, for this purpose, c_{il} denote the cost associated when $l \in M$ is supplying time to agent i, and let, in particular, for each $(i, l) \in E$:

$$c_{il} = \begin{cases} -1 & \text{if } l = n + i \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

For a given profile $R \in \tilde{\mathcal{R}}$, a given graph $g = (N, M, E, u)$ and given costs $c = (c_{il})_{(i,l) \in E}$, the (artificial) cost is minimized at any allocation $x \in \mathcal{F}(R)$ that solves the following maximization problem:[12]

$$\max \sum_{(i,l) \in E} c_{il} x_{il} \text{ s.t. conditions (1'), (2'), } x_{il} \in \mathbb{N}_0 \text{ and } x_{il} \leq u(i, l) \text{ for all } (i, l) \in E. \tag{5}$$

An allocation $x \in \mathcal{F}(R)$ is a *maximizer* if it is a solution of the maximization problem (5). Let $\mathcal{V}(R, c) \subseteq \mathcal{F}(R)$ denote the set of all maximizers at profile $R \in \tilde{\mathcal{R}}$ for given costs $c = (c_{il})_{(i,l) \in E}$. For notational convenience, the value of an allocation x at cost c is given by $V(x, c) = \sum_{(i,l) \in E} c_{il} x_{il}$.

Lemma 1. If allocation x belongs to $\mathcal{V}(R, c)$ at profile $R \in \tilde{\mathcal{R}}$, then $x \in \mathcal{X}(R)$.

The set of maximizers $\mathcal{V}(R, c)$ is non-empty for any profile $R \in \tilde{\mathcal{R}}$ since $\mathcal{V}(R, c) \subseteq \mathcal{X}(R)$ and $\mathcal{X}(R)$ is non-empty and finite for all $R \in \tilde{\mathcal{R}}$. However, as stated above, agents need not be indifferent between all allocations in the set $\mathcal{V}(R, c)$ since $\mathcal{V}(R, c) \subseteq \mathcal{X}(R)$. Hence, in order to define a priority mechanism based on a solution to maximization problem (5), a refined selection from the set $\mathcal{V}(R, c)$ is necessary which will be based on the priority-ordering π.

To modify the costs c in order to take the priority-ordering π into account, let $\varepsilon_0 \in (0, 1)$ and $\varepsilon_{i-1} = (1 + t_i)\varepsilon_i$ for each $i \in \{1, \ldots, n\}$. By construction of ε_i, it follows that:[13]

$$1 > \varepsilon_0 \geq \varepsilon_i > \sum_{k=i+1}^{n} t_k \varepsilon_k > 0 \text{ for all } i \in \{0, \ldots n - 1\}. \tag{6}$$

To guarantee a larger flow to agents with higher priorities, the value associated with a flow will be monotonically increasing with higher priorities. More specifically, let for each $(i, l) \in E$:

$$\tilde{c}_{il} = \begin{cases} -1 & \text{if } l = n + i \\ \varepsilon_{\pi(i)} & \text{otherwise.} \end{cases}$$

The above construction means that the agent with the highest priority (i.e., the agent with $\pi(i) = 1$)

[12]Note that costs of edges are non-positive and the max-cost flow problem is equivalent to the usual min-cost flow problem.

[13]To see this, note that $\varepsilon_{n-1} = (1 + t_n)\varepsilon_n > t_n \varepsilon_n$ since $\varepsilon_n > 0$ and, consequently, $\varepsilon_{n-2} = (1 + t_{n-1})\varepsilon_{n-1} = \varepsilon_{n-1} + t_{n-1}\varepsilon_{n-1} > t_n \varepsilon_n + t_{n-1}\varepsilon_{n-1}$. Condition (6) then follows by repeating these arguments.

will receive the highest edge weight (for edges $(i,l) \in E\backslash\{(i,n+i)\}$), the agent with the second highest priority (i.e., the agent with $\pi(i)=2$) will receive the second highest edge weight, and so on.

Our second main result demonstrates that a mechanism that selects an allocation from the set of maximizers for each profile in $\tilde{\mathcal{R}}$ and any given priority-ordering is a priority mechanism. From Theorem 1, it is already known that such a mechanism is non-manipulable on the domain $\tilde{\mathcal{R}}$.

Proposition 1. For a given priority-ordering π, a mechanism φ selecting for each profile $R \in \tilde{\mathcal{R}}$ an allocation from $\mathcal{V}(R,\tilde{c})$ is a priority mechanism based on π.

5 Discussion

5.1 Singleton Cores

Theorem 1 establishes that in our time-banking problem there exist mechanisms which are individually rational, efficient, and non-manipulable on the domain $\tilde{\mathcal{R}}$. This is surprising as previously a number of impossibility results for the combination of these axioms have been established by applying a singleton cores result by Sönmez (1999). Below we connect his result to time banking.

Let $\tilde{\mathcal{R}}^1$ denote the set of all profiles $R \in \tilde{\mathcal{R}}$ such that for all $i \in N$ and all $j \in A_i(R_i)$ we have $\bar{t}_{ij}=1$ and $t_i=1$ (i.e., any agent demands at most one time unit of any acceptable service and any agent provides at most one unit of time). This corresponds to the classical dichotomous domain by Bogomolnaia and Moulin (2004). Then it is easy to check that the domain $\tilde{\mathcal{R}}^1$ satisfies Assumption A and B of Sönmez (1999).[14] Hence, his main result applies, which shows the following: if there exists an individually rational, efficient, and non-manipulable mechanism, then for any profile where the core is non-empty we have (i) the core is single-valued and (ii) the mechanism chooses a core allocation. However, here for any $R \in \tilde{\mathcal{R}}^1$, if the core of R is non-empty, then the set of individually rational and efficient allocations is a singleton (and the core is a singleton).[15] But then any priority mechanism chooses this allocation for the profile R.

Once non-unitary endowments are allowed (as it is the case for time banks), the domain $\tilde{\mathcal{R}}$ does not satisfy Assumption B by Sönmez (1999). We show this in the example below.

Example 2. We use the instance introduced in Example 1. Recall that, $N=\{1,2,3,4\}$, $t_1=t_2=1$ and $t_3=t_4=2$, and $R \in \tilde{\mathcal{R}}$ is such that $A_1(R_1)=A_2(R_2)=\{3,4\}$ (with $\bar{t}_{13}=\bar{t}_{14}=\bar{t}_{23}=\bar{t}_{24}=1$) and $A_3(R_3)=A_4(R_4)=\{1,2\}$ (with $\bar{t}_{31}=\bar{t}_{32}=\bar{t}_{41}=\bar{t}_{42}=2$). If 3 comes before 4 in the priority order π, then $(3,3,12,0)$ is the unique π-priority allocation (where this stands for 1 receiving one time unit from 3, 2 one unit from 3 and 3 receiving one unit from each 1 and 2). If 4 comes before 3 in the priority order π, then $(4,4,0,12)$ is the unique π-priority allocation. Note that $(3,3,12,0)P_3(3,4,1,2)P_3\omega_3$ but there exists no R_3' such that $(3,3,12,0)P_3'\omega_3P_3'(3,4,1,2)$ (as

[14] In our framework (without externalities) Assumption A says that for any allocation x we have $x_iI_i\omega_i$ if and only if $x_i=\omega_i$ and Assumption B says that whenever for two allocations x and y with $x_iP_iy_i$ and $x_iR_i\omega_i$, there exists a preference relation R_i' such that $x_iR_i'\omega_iR_i'y_i$.

[15] Note that for any $R \in \tilde{\mathcal{R}}^1$, if the set of individually rational and efficient allocations is not a singleton, then any two individually rational and efficient allocations dominate (via some coalition) each other and the core must be empty: more formally, for $R \in \tilde{\mathcal{R}}$ and any two distinct individually rational and efficient allocations x and y, we have for $S=\{i \in N : x_{ii}=0\}$ we have for all $i \in S$, $x_iR_iy_i$, and for some $j \in S$, $x_jP_jy_j$, i.e., x dominates y with the coalition S (and the same argument applies for y in the role of x and x in the role of y). Thus, the core (which consists of all undominated allocations) is empty.

$(3, 3, 12, 0)P_3'\omega_3$ implies $1 \in A_3(R_3')$ and $\bar{t}_{31}' \geq 1$, and thus $(3, 4, 1, 2)P_3'\omega_3$), i.e., Assumption B is violated for the domain $\tilde{\mathcal{R}}$.

The above example also shows that in general we do not have dichotomous preferences in the domain $\tilde{\mathcal{R}}$. We may have many distinct indifference classes for preferences in the domain $\tilde{\mathcal{R}}$ and yet by Theorem 1, there exists an individually rational, efficient, and non-manipulable mechanism.

Finally, we show that a priority mechanism with the same order may select different allocations when choosing from the set of individually rational and efficient allocations (as in Manjunath and Westkamp (2017)).

Example 3. Let $N = \{1, 2, 3, 4\}$ and $t_1 = t_2 = t_3 = t_4 = 1$. Let $R \in \tilde{\mathcal{R}}$ be such that $A_1(R_1) = \{2\}$, $A_2(R_2) = \{3\}$, $A_3(R_3) = \{1, 4\}$, and $A_4(R_4) = \{3\}$ (with $\bar{t}_{12} = \bar{t}_{23} = \bar{t}_{31} = \bar{t}_{34} = \bar{t}_{43} = 1$). Then $\mathcal{X}(R) = \{(2, 3, 1, 4)\}$, i.e. there is a unique individually rational and maximal allocation which is chosen by any priority mechanism. However, the allocation $(1, 2, 4, 3)$ is individually rational and efficient which is selected by any priority mechanism which chooses from the whole set of individually rational and efficient allocations and where agent 4 occupies the first position in the priority order.

5.2 General Domain

On the general domain, there does not exist any mechanism satisfying individual rationality, efficiency, and non-manipulability. This is a simple consequence of (Sönmez, 1999, Corollary 1): The general domain contains as subdomain marriage markets where N is partitioned by men M and women W where for any "marriage market" R we have $t_i = 1$ and R_i is strict for all $i \in N$, and both (i) $A_i(R_i) = W$ for all $i \in M$ and $\bar{t}_{ij} = 1$ for all $j \in W$ and (ii) $A_i(R_i) = M$ for all $i \in W$ and $\bar{t}_{ij} = 1$ for all $j \in W$. For such marriage markets, the core is non-empty and not a singleton, i.e., by (Sönmez, 1999, Corollary 1) there does not exist any individually rational, efficient, and non-manipulable mechanism.

Our final result demonstrates that this impossibility can, at least partly, be escaped. For this, with slight abuse of notation, let for any $R \in \mathcal{R}$ the set $\mathcal{X}(R)$ stand for the set of all individually rational and efficient allocations under R. Then one can adapt the definition of a priority mechanism as in Section 3. We show that a priority mechanism is partly non-manipulable on the general domain $\mathcal{R}$ in the sense that any agent $i \in N$ who finds the selection of the priority mechanism to be agent-i-optimal at a given profile in $\mathcal{R}$ will be unable to manipulate the mechanism at that specific profile.

Theorem 2. For any profile $R \in \mathcal{R}$ and any given priority-ordering π, a priority mechanism is non-manipulable by any agent $i \in N$ that finds the selection of the mechanism agent-i-optimal at profile R. In particular, the agent $i \in N$ with $\pi(i) = 1$ cannot manipulate a priority mechanism at any profile $R \in \mathcal{R}$.

5.3 Concluding Remarks

This paper has modeled a time bank as a matching market. On a restricted but yet natural preference domain, it has been demonstrated that a priority mechanism can be formulated as a min-cost flow problem and, furthermore, that such mechanism is non-manipulable and always makes a selection from the set of individually rational, efficient, and time-balanced allocations. No mechanism with these properties exists on the general preference domain (Sönmez, 1999, Corollary 1).

Given that non-manipulability must be relaxed to obtain individual rationality, efficiency and time-balance on the general preference domain, this paper has demonstrated that non-manipulability need not be completely abandoned as manipulation possibilities can be prevented for some agents even on the general domain. Results with a similar flavor has previously been obtained in the literature. For example, on the marriage market (Gale and Shapley, 1962), it is well-known that there exists no mechanism that prevents both men and women from manipulating but no man (or woman) can successfully manipulate a mechanism that always selects the men-optimal (women-optimal) stable matching (Dubins and Freedman, 1981; Roth, 1982). Another example is the assignment market (Shapley and Shubik, 1972) where it is well-known that either the buyers or the sellers can manipulate any individually rational and stable mechanism on the general domain but where it is possible to construct mechanisms that prevent at least one of these groups from manipulating (Demange and Gale, 1985). A final example is from Andersson et al. (2014) where it is shown that it is impossible for an agent to successfully manipulate an envy-free and budget-balanced mechanism if it selects the agent's most preferred envy-free and budget-balanced outcome for each preference profile on a general preference domain (this rule is also minimally manipulable in the sense of Andersson et al., 2014).

Even if the considered priority mechanism has been demonstrated to satisfy all properties of interest on a restricted preference domain (and even partly on the general domain), the mechanism can be criticized from a fairness perspective as it discriminates low priority agents. For this reason, it is important to characterize the entire class of mechanisms that satisfies the axioms of interest to see if such discrimination can be avoided or not (or alternatively, one might randomize over priority orderings). Moreover, even if the considered domain restriction is natural for the time banking problem, it may also be of importance to find a maximal domain result where the above mentioned impossibility can be escaped as this will give important information about how much more detailed preferences may be reported to a time bank. Both open problems are left for future research.

Appendix A: Proofs

Appendix A contains the proofs of all results except Theorem 1, which is in Appendix B.

Proof of Lemma 1. Suppose that allocation x belongs to $\mathcal{V}(R, c)$. The fact that x is feasible and individually rational follows directly from the construction of the graph $g = (N, M, E, u)$ and by definition of the maximization problem (5), i.e., $n + j \in M$ is only connected to an agent $i \in N$ if agent $j \in A_i(R_i) \cup \{i\}$, all flows are between connected agents and the flow never exceeds the upper bounds $\bar{t}_{ij}$ on any edge $(i, n + j) \in E$.

To show that allocation x is maximal, it will be demonstrated that x minimizes the total flow between agents $i \in N$ and their respective clones $i + n \in M$. Because $x \in \mathcal{V}(R, c)$ is a maximizer, it follows that:

$$\sum_{(i,l)\in E} c_{il} x_{il} \geq \sum_{(i,l)\in E} c_{il} x'_{il} \text{ for any feasible allocation } x' \text{ in program (5).} \tag{7}$$

Given the construction of the costs in condition (4), it now follows from condition (7) that:

$$\sum_{i=1}^{n} c_{i(n+i)} x_{i(n+i)} \geq \sum_{i=1}^{n} c_{i(n+i)} x'_{i(n+i)}.$$

Because $c_{i(i+n)} = -1$ for all $i \in N$, by condition (4), the above inequality can be rewritten as:

$$\sum_{i=1}^{n} x'_{i(n+i)} \geq \sum_{i=1}^{n} x_{i(n+i)}.$$

But this condition means that allocation x minimizes the total flow between agents $i \in N$ and their respective clones $i + n \in M$ among all feasible allocations, which is the desired conclusion. $\square$

Proof of Proposition 1. It is first demonstrated that $\mathcal{V}(R, \tilde{c}) \subseteq \mathcal{V}(R, c)$ for each profile $R \in \tilde{\mathcal{R}}$. Suppose now that $x \in \mathcal{V}(R, c)$ but $x' \notin \mathcal{V}(R, c)$ for some x' that is feasible in the optimization program defined in (5). To reach the conclusion, it is sufficient to show $x' \notin \mathcal{V}(R, \tilde{c})$.

Note that $x \in \mathcal{V}(R, c)$ and $x' \notin \mathcal{V}(R, c)$ imply $V(x, c) > V(x', c)$. This conclusion together with $c_{il} \in \{-1, 0\}$ and $x_{il} \in \mathbb{N}_0$ for all $(i, l) \in E$ and $\varepsilon_0 < 1$ gives $V(x, c) > V(x', c) + \varepsilon_0$. Because $\tilde{c}_{il} \geq c_{il}$ for all $(i, l) \in E$ by construction, it holds that $V(x, \tilde{c}) \geq V(x, c)$. This together with the above inequalities imply $V(x, \tilde{c}) > V(x', c) + \varepsilon_0$. To complete this part of the proof, we show that $V(x', c) + \varepsilon_0 \geq V(x', \tilde{c})$, since this condition together with the above conclusions then show $V(x, \tilde{c}) > V(x', \tilde{c})$, i.e., that $x' \notin \mathcal{V}(R, \tilde{c})$.

To demonstrate $V(x', c) + \varepsilon_0 \geq V(x', \tilde{c})$, we partition E into two disjoint sets, E^1 and E^2, where the former set contains all edges (i, l) in E where $l \neq i + n$ and the latter contains all edges (i, l) in E where $l = i + n$. Consequently, $c_{il} = 0 < \tilde{c}_{il} = \varepsilon_i$ for all $(i, l) \in E^1$ and $c_{il} = \tilde{c}_{il} = -1$ for all $(i, l) \in E^2$. Hence, the inequality $V(x', c) + \varepsilon_0 \geq V(x', \tilde{c})$ can be rewritten as:

$$\begin{aligned}
V(x', c) + \varepsilon_0 &= \sum_{(i,l) \in E} c_{il} x'_{il} + \varepsilon_0, \\
&= \sum_{(i,l) \in E^1} c_{il} x'_{il} + \sum_{(i,l) \in E^2} c_{il} x'_{il} + \varepsilon_0, \\
&= \sum_{(i,l) \in E^2} \tilde{c}_{il} x'_{il} + \varepsilon_0, \\
&\geq \sum_{(i,l) \in E} \tilde{c}_{il} x'_{il} \\
&= \sum_{(i,l) \in E^1} \tilde{c}_{il} x'_{il} + \sum_{(i,l) \in E^2} \tilde{c}_{il} x'_{il}, \\
&= \sum_{(i,l) \in E^1} \varepsilon_i x'_{il} + \sum_{(i,l) \in E^2} \tilde{c}_{il} x'_{il}, \\
&= V(x', \tilde{c}).
\end{aligned}$$

or, equivalently, as:

$$\varepsilon_0 \geq \sum_{(i,l)\in E^1} \varepsilon_i x'_{il}. \tag{8}$$

Conditions (6) and (1') together with the fact that $\varepsilon_i x_{il} \geq 0$ for all $(i,l) \in N \times M$ now give:

$$\varepsilon_0 > \sum_{i\in N} \varepsilon_i t_i \geq \sum_{(i,l)\in E^1} \varepsilon_i x'_{il}.$$

But then condition (8) must hold. Hence, $\mathcal{V}(R,\tilde{c}) \subseteq \mathcal{V}(R,c)$. This conclusion and Lemma 1 imply that for a given priority ordering π, any mechanism φ choosing for each profile $R \in \tilde{\mathcal{R}}$ an allocation from $\mathcal{V}(R,\tilde{c})$, selects a π-priority allocation from $\mathcal{X}(R)$.

To conclude the proof, it needs only to be demonstrated that φ is a priority mechanism. But this follows directly from the construction of the weights ε_i. To see this, recall from condition (6) that $\varepsilon_i > \sum_{k=i+1}^{n} t_k\varepsilon_k$ for all $i \in \{1,\ldots,n-1\}$. Hence, assigning *one* additional time unit to agent i in maximization problem (5) is strictly preferred to assigning t_j time units to each agent $j \in N$ with $\pi(i) < \pi(j)$. Thus, $\mathcal{V}(R,\tilde{c})$ is a selection from $\mathcal{V}(R,c) \subseteq \mathcal{X}(R)$ that first maximizes the number of time units that agent $i_1 \in N$ with $\pi(i_1) = 1$ exchanges with acceptable agents (i.e., a selection from the set $\mathcal{Z}^{i_1,\mathcal{V}(R,c)}(R)$), and then maximizes the number of time units that agent $i_2 \in N$ with $\pi(i_1) = 2$ exchanges with acceptable agents (i.e., a selection from the set $\mathcal{Z}^{i_2,i_1}(R)$), and so on. This is the definition of a priority mechanism. □

Proof of Theorem 2. To obtain a contradiction, suppose that the priority mechanism φ is agent-i-optimal at profile $R \in \mathcal{R}$ but that agent $i \in N$ can manipulate the mechanism at profile R. This means that there are two profiles $R \in \mathcal{R}$ and $R' = (R'_i, R_{-i}) \in \mathcal{R}$ such that $x = \varphi(R)$, $x' = \varphi(R')$ and $x'_i P_i x_i$. It will be demonstrated that $x' \in \mathcal{X}(R)$ because if this is the case, then the mechanism φ cannot be agent-i-optimal since $x = \varphi(R)$ and $x'_i P_i x_i$. Hence, to obtain the desired contradiction, it needs to be established that x' is individually rational *and* efficient at profile R, i.e., that $x' \in \mathcal{X}(R)$.

It is first proved that x' is individually rational at profile R, i.e., that $x'_j R_j \omega_j$ for all $j \in N$. The relation $x'_j R_j \omega_j$ for $j \neq i$ follows directly as $R_j = R'_j$ and $x' = \varphi(R') \in \mathcal{X}(R')$. Relation $x'_i R_i \omega_i$ follows by the assumption $x'_i P_i x_i$ and the fact that $x_i R_i \omega_i$ (as $x = \varphi(R) \in \mathcal{X}(R)$). Hence, $x'_j R_j \omega_j$ for all $j \in N$.

It is next proved that x' is efficient at profile R, i.e., that there is no allocation x'' that Pareto dominates x' at profile R. To obtain a contradiction, suppose that there is an allocation x'' that Pareto dominates x' at profile R (without loss of generality, it can be assumed that x'' is efficient). This means that $x''_j R_j x'_j$ for all $j \in N$ and $x''_j P_j x'_j$ for some $j \in N$ and, in particular, that $x''_i R_i x'_i P_i x_i$. As x' is individually rational at profile R, by the above conclusion, it follows that x'' is individually rational at profile R. But then because x'' is individually rational and efficient at profile R, it follows that $x'' \in \mathcal{X}(R)$. Then the mechanism φ cannot be agent-i-optimal since $x''_i P_i x_i$. Hence, x' is efficient at profile R.

Hence, $x' \in \mathcal{X}(R)$ and it then follows that a priority mechanism is non-manipulable by any agent $i \in N$ that finds the selection of the mechanism agent-i-optimal at profile $R \in \mathcal{R}$.

The fact that agent $i \in N$ with $\pi(i) = 1$ cannot manipulate a priority mechanism at any profile

$R \in \mathcal{R}$ follows directly from the above conclusion and the fact that a priority mechanism, by definition, always selects an agent-i-optimal allocation for each profile $R \in \mathcal{R}$ for the agent $i \in N$ with $\pi(i) = 1$. □

Appendix B: Proof of Theorem 1

This Appendix first introduces a graph theoretical tool, referred to as the circulation-based model (Appendix B.1). It will then be demonstrated that the circulation-based model, without loss of generality, can replace the min-cost flow problem when analysing the priority mechanism (Appendix B.2). These insights enable us to prove Theorem 1 (Appendix B.3).

Appendix B.1: The Circulation-Based Model

Let $\mathbb{Z}$ denote the set containing all integers. For any profile $R \in \tilde{\mathcal{R}}$, construct a weighted directed graph $D_R = (V, A)$ with capacities $c : A \mapsto \mathbb{N}_0$ and weights $w : A \mapsto \mathbb{Z}$ on its arcs. For ease of notation, we write D instead of D_R whenever the profile R is unambiguous. Each agent $i \in N$ is represented by two vertices, denoted by i^{in} and i^{out}. These $2n$ vertices build the vertex set V of the graph D. We draw a directed arc between each pair of type (i^{in}, i^{out}), pointing to i^{out} and refer to this arc as the *inner arc* of agent $i \in N$. The inner arc has capacity $c(i^{in}, i^{out}) = t_i$. If agent i finds agent j acceptable, then (j^{out}, i^{in}) belongs to the (directed) arc set A of the graph D. Any such arc is called *regular* and has capacity $c(j^{out}, i^{in}) = \bar{t}_{ij}$, i.e., the upper time bound on how much time agent i wants from agent j. Note also that the vertices of type i^{in} have incoming regular arcs and a single outgoing inner arc, while vertices of type i^{out} have outgoing regular arcs and a single incoming inner arc. We define in Appendix B.2 the weights $w : A \mapsto \mathbb{Z}$ using a priority order. An instance of the model is illustrated in Figure 2 (the figure contains some concepts which are explained later in the Appendix).

Definition 5. A *circulation* is a function $C : A \mapsto \mathbb{N}_0$ where:

(i) $C(u, v) \leq c(u, v)$ for every $(u, v) \in A$,

(ii) $\sum_{(u,v)\in A} C(u, v) = \sum_{(v,w)\in A} C(v, w)$ for every vertex $v \in V$.

Condition (i) is a capacity constraint which ensures that agents do not exchange services beyond their time endowment $t_i = c(i^{in}, i^{out})$, and that the upper time bound $\bar{t}_{ij}$ on how much time agent i wants from agent j is not exceeded. Condition (ii) is the classical flow conservation rule, stating that the total flow of the incoming arcs of a vertex equals the total flow of the outgoing arcs, i.e., that an agent provides and receives the same amount of time. The latter condition can also be formulated as:

$$C(i^{in}, i^{out}) = \sum_{(j^{out}, i^{in})\in A} C(j^{out}, i^{in}) = \sum_{(i^{out}, k^{in})\in A} C(i^{out}, k^{in}) \text{ for every agent } i \in N.$$

We call $C(i^{in}, i^{out})$ the *flow value at agent* i. Circulations in a graph D are in one-to-one correspondence with allocations in the time banking problem, e.g., for an allocation x the corresponding flow value of the inner arc at agent i is $C(i^{in}, i^{out}) = t_i - x_{ii}$ and the flow value of any regular arc at agent i is $C(j^{out}, i^{in}) = x_{ij}$ for all $j \in N$. The *allocation value* for agent i is defined as $t_i - x_{ii}$. Another

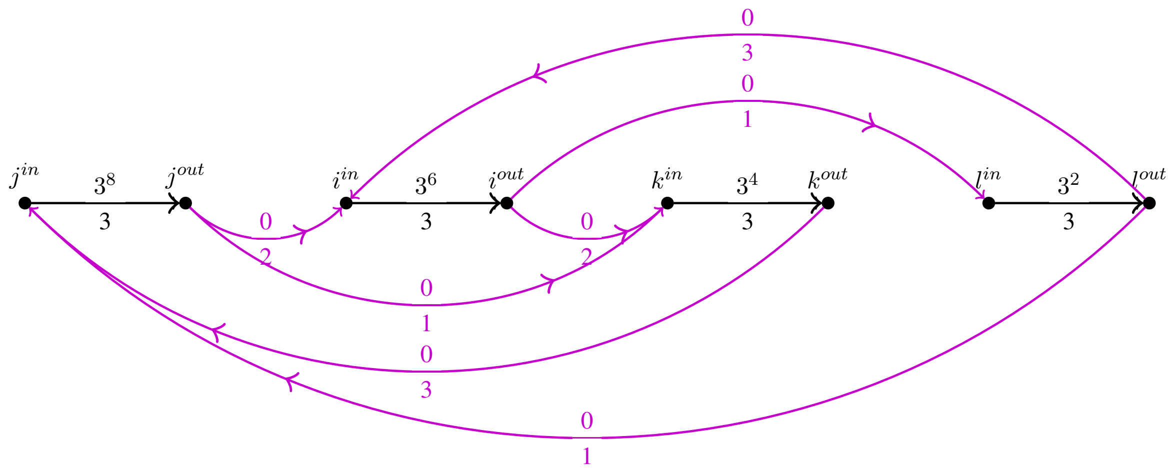

Figure 2: Agents are denoted by i, j, k and l. Inner arcs are marked by horizontal lines, while regular arcs are bent. Arc weights and capacities are written above and below each arc, respectively. The arc weights of agents i, j, k and l on the inner arcs are given by 3^6, 3^8, 3^4 and 3^2, respectively. All arc weights on regular arcs are set to zero. Each agent has an endowment of 3. The max weight circulation saturates all regular edges except (l^{out}, j^{in}) which is left empty, and (l^{out}, i^{out}) which carries one unit of flow. Hence, agent i sends to 2 time units to agent k and 1 time unit to agent l, agent j sends 2 time units to agent i and 1 time unit to agent k, agent k sends 3 time units to agent j, and agent l sends 1 time unit to agent i.

way of expressing this is that the allocation value $t_i - x_{ii}$ of agent i in the time banking problem equals the flow value $C(i^{in}, i^{out})$ at agent i in the circulation model.

Appendix B.2: Replacement Result

This section demonstrates that by placing appropriate weights on the arcs in the graph D, the maximum weight circulations correspond to the outcome of the min-cost flow problem used in Section 4 to identify the outcome of the priority mechanism (Proposition 2). This result implies that the circulation-based model can be adopted in the proof of Theorem 1.

Let π be a priority ordering. Let t_{max} be the largest time endowment of any agent in N, and define the weight $w(u, v)$ on each arc (u, v) in the directed graph $D = (V, A)$ by:

$$w(u,v) = \begin{cases} t_{max}^{2(n+1-\pi(i))} & \text{if } (u,v) = (i^{in}, i^{out}), \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

In Figure 2 this means that agents j and l have the highest and the lowest priorities, respectively (note also that $t_{max} = 3$ since all agents, by assumption, have capacity 3). Let $w(C)$ denote the *weighted sum of flow values* of the agents in N at circulation C, i.e., $w(C) = \sum_{i \in N} C(i^{in}, i^{out}) \cdot w(i^{in}, i^{out})$.

Proposition 2. For any given profile $R \in \tilde{\mathcal{R}}$, let C be a maximum weight circulation where the weights are defined by condition (9). Let C' be the circulation corresponding to an allocation x' selected for R by a priority mechanism φ based on π. Then $C'(i^{in}, i^{out}) = C(i^{in}, i^{out})$ for each $i \in N$.

Proof. As in the statement of the proposition, let C be a maximum weight circulation and let C' be the circulation corresponding to an allocation x' selected by a priority mechanism. Suppose, to obtain a contradiction, that $C'(j^{in}, j^{out}) \neq C(j^{in}, j^{out})$ for some $j \in N$. Let agent i be the agent with the highest priority in π where this holds. Suppose also, without loss of generality, that $\pi(k) = k$ for all $k \in N$. To reach the desired contradiction, we consider two cases.

Case (i): $C'(i^{in}, i^{out}) < C(i^{in}, i^{out})$. In this case, the maximum weight circulation C assigns a higher allocation value to agent i than the priority mechanism. We show by induction that this contradicts the rules of the priority mechanism. Suppose first that agent i is the highest ranked agent according to the priority order π and recall that the priority mechanism, by construction, restricts the set of maximal allocations to those that maximize the allocation value of i (see condition (3) in Section 3). Thus there is no allocation that assigns agent i a higher allocation value than the allocations in this chosen set, and, consequently, no circulation that assigns agent i a higher value. Hence, agent i cannot be the agent with the highest priority. Suppose now that agent i is the second highest ranked agent according to the priority order π. Again, by condition (3) this agent restricts the set of allocations further. And so, the maximum weight circulation C is still in the chosen set when agent i restricts the set of allocations further, and it can, consequently, not have a higher allocation value for agent i than C'. This argument can be repeated inductively to reach the conclusion that it cannot be the case that $C'(i^{in}, i^{out}) < C(i^{in}, i^{out})$.

Case (ii): $C'(i^{in}, i^{out}) > C(i^{in}, i^{out})$. Note first that both C and C' are feasible circulations at profile R. Because agent i is the agent with the highest priority in π where $C'(i^{in}, i^{out}) \neq C(i^{in}, i^{out})$, by assumption, it follows that $C'(k^{in}, k^{out}) = C(k^{in}, k^{out})$ for all agents $k = 1, \ldots, i-1$. It will be demonstrated that agents $i+1, \ldots, n$ cannot make up for the loss C suffered on arc (i^{in}, i^{out}) and thus C cannot be of maximum weight since C' is a feasible circulations at profile R. Recall first that the set $\mathbb{N}_0$ contains only positive integers, so the difference between $C'(i^{in}, i^{out})$ and $C(i^{in}, i^{out})$ is at least 1. By construction of the weights on the inner arcs, defined by condition (9), it then follows that:

$$[C'(i^{in}, i^{out}) - C(i^{in}, i^{out})] \cdot t_{max}^{2(n-i+1)} \geq t_{max}^{2(n-i+1)}. \tag{10}$$

Note next that, in the the extreme case, all agents with lower priorities than agent i have flow value zero in C' and a flow value of t_{max} in C. This means that the weighted sum of the flow values at agents $i+1, \ldots, n$ at circulation C is at most:

$$t_{max} \cdot \sum_{j=i+1}^{n} t_{max}^{2(n-j+1)}. \tag{11}$$

Now, the value of the sum (11) is strictly lower than the right hand side of inequality (10). Consequently, even in the the extreme case when all agents with lower priorities than agent i have flow value zero in C' and a flow value of t_{max} in C, it holds that $w(C') > w(C)$. However, this contradicts that C is a maximum weight circulation since C' is a feasible circulation at graph D_R. $\square$

Appendix B.3: The Proof

Let φ be the priority mechanism based on π where $\pi(i) = i$ for all $i \in N$. To obtain a contradiction, suppose that φ can be manipulated by some agent $i \in N$ at a profile $R \in \tilde{\mathcal{R}}$. This means that there are two profiles $R \in \tilde{\mathcal{R}}$ and $R' = (R'_i, R_{-i}) \in \tilde{\mathcal{R}}$ such that for $x = \varphi(R)$ and $x' = \varphi(R')$ we have $x'_i P_i x_i$. Note that $R'_i \neq R_i$. Let C^1 and C^2 be the maximum weight circulations for the graphs D_R and $D_{R'}$ induced by the profiles R and $R' = (R'_i, R_{-i})$, respectively.

The next lemma shows that we may suppose that the set of acceptable agents reported by agent i at preference relation R'_i is a proper subset of the set of acceptable agents reported by agent i at preference relation R_i.

Lemma 2. *Without loss of generality, we may suppose $A_i(R'_i) \subseteq A_i(R_i)$.*

Proof. We first show $U_i(R_i) \subseteq U_i(R'_i)$. To see this, suppose $j \in U_i(R_i)$ but $j \notin U_i(R'_i)$, i.e., that agent j is unacceptable under R_i but acceptable under R'_i. Since $x'_i P_i x_i$, it must then hold that $x'_{ij} = 0$ by definition of the preferences in $\tilde{\mathcal{R}}_i$. Hence, any regular arc of type (j^{out}, i^{in}) where $j \notin U_i(R'_i)$ in the graph $D_{R'}$ but $j \in U_i(R_i)$ in the graph D_R will not be active in the solution C^1 at profile R'. Hence, $U_i(R_i) \subseteq U_i(R'_i) \cup \{j \in A_i(R'_i) : x'_{i(n+j)} = 0\}$. But then we may choose R''_i such that $A(R''_i) = A_i(R'_i) \backslash \{j \in A_i(R'_i) : x'_{i(n+j)} = 0\}$ and $\bar{t}''_{ik} = \bar{t}'_{ik}$ for all $k \in A(R''_i)$, and C^1 remains a solution for $R'' = (R''_i, R_{-i}) \in \tilde{\mathcal{R}}$. But for $x'' = \varphi(R'')$ this implies $x''_i I_i x'_i$ and $x''_i P_i x_i$. Hence, $A_i(R''_i) \subseteq A_i(R_i)$ and $x''_i P_i x_i$. □

Recall now that, for any profile in $R \in \tilde{\mathcal{R}}$, each agent $k \in N$ reports a set of acceptable agents $A_k(R_k)$ together with an upper bound on how much time $\bar{t}_{kj}$ agent $k \in N$ at most would like to receive from each acceptable agent $j \in A_k(R_k)$. By Remark 1, the report R_k is equivalent to the vector $\bar{t}_k$ where $\bar{t}_{kk} = t_k$ and $\bar{t}_{kj} = 0$ for all $j \in U_k(R_k)$. This together with the conclusion in Lemma 2 imply that there exists at least one agent j that is acceptable for agent i under R_i where agent i reports a strictly lower or higher time bound $\bar{t}'_{ij}$ at profile R' than under profile R (i.e., $\bar{t}'_{ij} < \bar{t}_{ij}$ or $\bar{t}'_{ij} > \bar{t}_{ij}$). In general, a manipulation R'_i by agent i can consist of both underreporting and overreporting $\bar{t}_{ij}$'s for acceptable agents. There are two possible cases for manipulations: one with overreporting and the other with only underreporting time bounds.

First, consider the case where there is overreporting. If there exists $j \in N\backslash\{i\}$ such that $x'_{ij} > \bar{t}_{ij}$, then by definition of $\tilde{\mathcal{R}}_i$, $\omega_i P_i x'_i$ and since x is individually rational under R, we have $x_i P_i x'_i$, a contradiction. Otherwise $x'_{ij} \leq \bar{t}_{ij}$ for all $j \in N\backslash\{i\}$ and we can just replace $\bar{t}'_i$ with $\bar{t}''_i$ such that $\bar{t}''_{ij} = \min\{\bar{t}_{ij}, \bar{t}'_{ij}\}$ for all $j \in N\backslash\{i\}$. Let R''_i denote i's preference associated with $\bar{t}''_i$. Then x' is still a maximizer for the profile (R''_i, R_{-i}) and therefore the manipulation only consists of underreporting upperbounds which are below $\bar{t}_i$.

Second, it remains to establish that agent i cannot manipulate by underreporting time bounds for acceptable agents, i.e., $\bar{t}'_{ij} \leq \bar{t}_{ij}$ for all $j \in N\backslash\{i\}$. Below we are going to show that agent i cannot gain by underreporting one time bound for an acceptable agent. This is enough to establish that agent i never can gain by reporting a lower bound for several agents at the same time. Because any such misreport can be decomposed into a sequence of manipulations in which at each step only one upper bound $\bar{t}_{ij}$ is changed at the time and agent i is never made better off at any step. Formally, let $k \in A_i(R_i)$ for which $\bar{t}'_{ik} < \bar{t}_{ik}$ and consider the misreport $\bar{t}^{(1)}_i$ where $\bar{t}^{(1)}_{ij} = \bar{t}_{ij}$ for all $j \neq k$ and $\bar{t}^{(1)}_{ik} = \bar{t}'_{ik}$. Let $x^{(1)}$ be the allocation chosen by the priority mechanism when i reports $t^{(1)}$. Below we show that agent i cannot gain by reporting $\bar{t}^{(1)}_i$ instead of $\bar{t}_i$. In particular, $\sum_{j \in A_i(R_i)} x_{ij} \geq \sum_{j \in A_i(R_i)} x^{(1)}_{ij}$. Thus, $x_i R_i x^{(1)}_i$. If there is another agent $\ell \neq k$ such that $\bar{t}^{(1)}_{i\ell} \neq \bar{t}'_{i\ell}$ then consider $\bar{t}^{(2)}$

where $\bar{t}_{ij}^{(2)} = \bar{t}_{ij}^{(1)}$ for all $j \neq \ell$ and $\bar{t}_{i\ell}^{(2)} = \bar{t}'_{i\ell}$. Suppose again that agent i cannot gain by reporting $\bar{t}_i^{(2)}$ instead of $\bar{t}_i^{(1)}$. This means again that $\sum_{j \in A_i(R_i)} x_{ij}^{(1)} \geq \sum_{j \in A_i(R_i)} x_{ij}^{(2)}$. Thus, by transitivity $x_i R_i x^{(2)}$. This argument can be repeated inductively until the point that $\bar{t}_i^{(p)} = \bar{t}'_i$, and if in each step agent i never gains by reporting $\bar{t}_i^{(j)}$ instead of $\bar{t}^{(j-1)}$ we have shown that agent i cannot gain by reporting $\bar{t}'_i$ instead of $\bar{t}_i$. Hence, to complete the proof of Theorem 1, it is enough to show that agent i cannot gain by misreporting $\bar{t}'_{ij}$ for one agent $j \in A_i(R_i)$.

It only remains to rule out that agent i cannot gain by reporting a strictly lower time bound $\bar{t}_{ij}$. Translating this into the terminology of the circulation-based model, this can equivalently be expressed as the flow value $C(i^{in}, i^{out})$ at agent i in a maximum weight circulation cannot be increased by reducing the capacity on a regular arc (j^{out}, i^{in}). Given this insight, a large part of the remaining proof will focus on a regular arc (j^{out}, i^{in}).

Recall now that C^1 denotes the maximum weight circulations for the true preferences R induced by the graph D_R, and that C^2 denotes the maximum weight solution for the misrepresented preferences R' induced by the graph $D_{R'}$. Furthermore, by the assumption that agent i can manipulate the priority mechanism, it follows that C^2 has a larger flow value at agent i than C^1 does, i.e., $C^2(i^{in}, i^{out}) > C^1(i^{in}, i^{out})$. By construction of the weights in condition (9), the circulation value of C^2 cannot be the same as the circulation value of C^1 if the flow value differs for at least one agent. Thus, the circulation value of C^2 must be strictly smaller than the circulation value of C^1, i.e., $w(C^2) < w(C^1)$. Note also that the circulation C^2 is a feasible circulation in D_R since the flows remain below the capacity on each edge and it preserves flow conservation. However, the circulation C^2 is not optimal in the graph D_R since the circulation value of C^2 is strictly smaller than the circulation value of C^1 and the circulation C^1 is optimal in D_R.

Consider next the function defined by the circulation $C^1 - C^2$ where $C^1(u, v) - C^2(u, v) \in \mathbb{Z}$ for each arc (u, v) in the graph D_R. This function assigns a negative value to the arc (u, v) if the flow through the arc is larger at circulation C^2 than at circulation C^1. For convenience, one can think of these "negative" arcs as arcs turned backwards, with the usual positive flow value on them. Since both C^1 and C^2 are circulations in the graph D_R, their difference also obeys flow conservation and as such, it can be decomposed into cycles.

Note first that a cycle decomposition of $C^1 - C^2$ need not be unique for the profiles R and R'. To obtain one such decomposition, we use a simple inductive algorithm that produces a cycle decomposition of $C^1 - C^2$ in a finite number of iterations. This algorithm uses the flow value of $C^1 - C^2$ on each arc (u, v) in the graph D_R but will not use any information about the arc weights (arc weights are considered below). First, identify a cycle, say $\mathcal{C}$, based on the circulation $C^1 - C^2$ and take its forward or backward arc with a lowest absolute flow value on it. Suppose that the lowest absolute flow value at some agent in the cycle $\mathcal{C}$ is q, then q feasible cycles of type $\mathcal{C}$ can be identified. These cycles represent the first q cycles in the decomposition of $C^1 - C^2$. Then, reduce the flow value on each arc included in the cycle $\mathcal{C}$ by q. This will give an "updated" circulation-based on the "original" circulation $C^1 - C^2$. Now, the arc with the lowest flow value in the updated circulation is guaranteed to null its flow value. Hence, the updated circulation has one less arc and, consequently, one less cycle than the original circulation. Note, however, that the remaining cycles in the updated circulation still obeys flow conservation. We proceed in this manner until the whole circulation $C^1 - C^2$ is decomposed into cycles. Note also that since $\mathbb{N}_0$ is restricted to a set of positive bounded integers, this procedure ends in a finite number of iterations. Moreover, the absolute flow value on an arc monotonically (but not strictly monotonically) decreases in each inductive step.

Note that the cycles in the decomposition are not necessarily arc-disjoint from each other (i.e., several distinct cycles in the decomposition can pass through the same arc), but due to the inductive argument above, each arc in the cycle decomposition is either a forward arc or a backward arc, depending on the sign of $C^1(u,v)-C^2(u,v)$. More precisely, forward arcs are positive, while backward arcs are negative. Thus, it cannot be the case that one cycle in the decomposition uses an arc with positive value, while another cycle uses the same arc with negative value.

Consider now a cycle decomposition of the circulation $C^1 - C^2$ and add the arc weights to the arcs in all cycles included in the decomposition. Based on the sign of the total weight of a cycle in the decomposition, we distinguish positive, zero and negative weight cycles. A positive weight cycle is called an *augmenting cycle*. Note that all augmenting cycles pass through (j^{out}, i^{in}), because any augmenting cycle which does not pass through (j^{out}, i^{in}) would increase the circulation value of C^2 in $D_{R'}$, which is impossible since C^2 is optimal in the graph $D_{R'}$.

Lemma 3. Suppose that $C^1 - C^2$ is decomposed into cycles using the inductive decomposition algorithm from the above. Then:

(i) there exists an augmenting cycle,

(ii) a cycle of weight zero consists exclusively of arcs of weight zero,

(iii) there are no negative weight cycles.

Proof. The proof of Part (i) follows directly since $w(C^1) > w(C^2)$ and $w(C^1)$ equals $w(C^2)$ plus the weight of each cycle in the cycle decomposition of $C^1 - C^2$. Part (ii) follows by construction of the weights, i.e., a cycle of weight zero consists exclusively of arcs of weight zero (obviously, no combination of the weights on inner arcs with coefficients in the open interval between 0 and t_{max} can add up to zero).

Part (iii) is proved by contradiction. Suppose that there is a cycle $\mathcal{C}$ of negative total weight in the cycle decomposition of $C^1 - C^2$. Let the reverse of $\mathcal{C}$ be denoted by $\overleftarrow{\mathcal{C}}$. The reverse $\overleftarrow{\mathcal{C}}$ has positive total weight and preserves the sign of $C^2 - C^1$ on each of its arcs by construction of the inductive decomposition algorithm. Moreover, we will show that, $\overleftarrow{\mathcal{C}}$ can be added to C^1 without violating flow conservation or any capacity constraint in D_R. Thus, $C^1 + \overleftarrow{\mathcal{C}}$ is a circulation of larger weight than C^1. Let now (u,v) be an arbitrary arc in the reverse cycle $\overleftarrow{\mathcal{C}}$. It will be demonstrated that:

$$0 \leq C^1(u,v) + \overleftarrow{\mathcal{C}}(u,v) \leq c(u,v). \tag{12}$$

Condition (12) implies that C^1 cannot be a maximum weight circulation in the graph D_R which contradicts our assumption. We need to consider two cases. Suppose first that $\overleftarrow{\mathcal{C}}(u,v) \geq 0$. Then:

$$C^1(u,v) + \overleftarrow{\mathcal{C}}(u,v) \leq C^1(u,v) + [C^2(u,v) - C^1(u,v)] = C^2(u,v) \leq c(u,v).$$

Note also that because $C^1(u,v)$ and $\overleftarrow{\mathcal{C}}(u,v)$ are non-negative at the arc (u,v), it follows directly that $C^1(u,v) + \overleftarrow{\mathcal{C}}(u,v) \geq 0$. Hence, condition (12) holds when $\overleftarrow{\mathcal{C}}(u,v) \geq 0$. Suppose next that

$\overleftarrow{\mathcal{C}}(u,v) < 0$. In this case:

$$C^1(u,v) + \overleftarrow{\mathcal{C}}(u,v) < C^1(u,v) \leq c(u,v).$$

Furthermore:

$$C^1(u,v) + \overleftarrow{\mathcal{C}}(u,v) \geq C^1(u,v) + [C^2(u,v) - C^1(u,v)] = C^2(u,v) \geq 0.$$

Hence, condition (12) also holds when $\overleftarrow{\mathcal{C}}(u,v) < 0$. $\square$

Lemma 3 thus demonstrated that all cycles in the cycle decomposition of $C^1 - C^2$, which pass through an inner arc, are augmenting cycles. However, we do not know whether these cycles use the arc (j^{out}, i^{in}) as a forward arc or as a backward arc. The following lemma sheds light on this.

Lemma 4. Suppose that $C^1 - C^2$ is decomposed into cycles using the inductive decomposition algorithm from the above, and let (j^{out}, i^{in}) be an arbitrary arc in some cycle in the cycle decomposition of $C^1 - C^2$. Then (j^{out}, i^{in}) is a forward arc.

Proof. Note first that $C^2(j^{out}, i^{in})$ is bounded from above by the decreased capacity of (j^{out}, i^{in}) in $D_{R'}$. If $C^1(j^{out}, i^{in}) \leq C^2(j^{out}, i^{in})$, then C^1 is feasible in the graph $D_{R'}$ and has a larger weight than C^2, which contradicts the optimality of C^2 in the graph $D_{R'}$. Thus, $C^1(j^{out}, i^{in}) - C^2(j^{out}, i^{in}) > 0$, which implies that (j^{out}, i^{in}) is a forward arc in all cycles in the decomposition of $C^1 - C^2$. $\square$

Finally, consider the flow value $C^1(i^{in}, i^{out}) - C^2(i^{in}, i^{out})$. To prove Theorem 1, we only need to establish that $C^1(i^{in}, i^{out}) - C^2(i^{in}, i^{out}) \geq 0$ because this contradicts the assumption that $x_i' P_i x_i$. For this condition to be false, the arc (i^{in}, i^{out}) must be a backward arc in at least one cycle in the cycle decomposition of $C^1 - C^2$. However, as concluded in the above, being a backward arc in one cycle also implies being a backward arc in all cycles. From Lemma 3 we know that all cycles that passes through (i^{in}, i^{out}) are augmenting cycles. Lemma 4 then states that the augmenting cycles use (j^{out}, i^{in}) as a forward arc, and they must, consequently, leave i^{in} either as a forward arc, along the only outgoing arc (i^{in}, i^{out}), or as a backward arc, along any of the regular arcs running to i^{in}. Neither of these two cases allows (i^{in}, i^{out}) to be a backward arc. This concludes the proof and shows that agent i cannot manipulate the priority mechanism φ at any profile $R \in \tilde{\mathcal{R}}$.

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