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A NOTE ON ORDINALLY EQUIVALENT PARETO SURFACES

Yves SPRUMONT[1]

1 Département de sciences économiques and Centre de recherche et développement en économique (C.R.D.E.), Université de Montréal

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C.P. 6128, succursale Centre-ville
Montréal (Québec) H3C 3J7

Télécopieur (FAX) : (514) 343-7221
Courrier électronique (E-mail) : econo@ere.umontreal.ca

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RÉSUMÉ

Deux surfaces de Pareto sont *ordinalement équivalentes* si elles sont reliées par une transformation ordinale, c'est-à-dire une liste de transformations monotones des utilités individuelles. Sinon, elles sont *ordinalement distinctes*. Pour au moins trois individus et sous certaines conditions de régularité, nous construisons une "base ordinale" S^* de l'ensemble S de toutes les surfaces de Pareto : chaque surface dans S est ordinalement équivalente à une surface dans S^* et toutes les surfaces dans S^* sont ordinalement distinctes. De plus, deux surfaces ordinalement équivalentes sont reliées par une transformation ordinale unique (et son inverse).

Mots clés : équivalence ordinale, surface de Pareto, marchandage

ABSTRACT

Two Pareto surfaces are *ordinally equivalent* if they can be mapped onto each other through an ordinal transformation, i.e., a list of monotonic transformations of the individuals' utility levels. Otherwise, they are *ordinally distinct*. Assuming at least three individuals and some regularity conditions, we construct a set S^* of Pareto surfaces which is an "ordinal basis" of the set S of all surfaces : every surface in S is ordinally equivalent to some surface in S^* and all surfaces in S^* are ordinally distinct. We also show that any two ordinally equivalent surfaces are related through a unique ordinal transformation (and its inverse).

Key words : ordinal equivalence, Pareto surface, bargaining

1. Introduction

Let us call two Pareto surfaces (in some utility space) ordinally equivalent if they can be mapped onto each other through an ordinal transformation, i.e., a list of monotonic transformations of the individuals' utility levels. Let us call them ordinally distinct otherwise.

In order to build a normative ordinally welfarist theory of collective choice, an answer to the following questions is desirable: i) how rich is the set of ordinally distinct Pareto surfaces? and ii) what are the ordinal transformations mapping two given ordinally equivalent surfaces onto each other?

We answer these questions under some regularity conditions. Since the two-individual case was analyzed in Shapley (1969), we assume that there are at least three individuals. Regarding the first question, we construct a set $\mathcal{S}^*$ of Pareto surfaces which is an "ordinal basis" of the set $\mathcal{S}$ of all Pareto surfaces: any surface in $\mathcal{S}$ is ordinally equivalent to some surface in $\mathcal{S}^*$ and all surfaces in $\mathcal{S}^*$ are ordinally distinct. As it turns out, $\mathcal{S}^*$ contains infinitely many elements. Regarding the second question, we show (constructively) that any two ordinally equivalent surfaces are linked through a unique ordinal transformation (and its inverse). These results are in sharp contrast with the two-individual case where all regular Pareto surfaces are ordinally equivalent to each other and any two such surfaces can be mapped onto each other through infinitely many ordinal transformations. We briefly discuss the implications of our results for the ordinal approach to distribution problems (Shapley, 1969, Shubik, 1984).

2. Definitions

The set of individuals is denoted by I. It is finite and contains at least three elements. If $i \in I$, $e(i)$ stands fo the vector in $\mathbb{R}^I$ whose jth coordinate is 1 if $j = i$ and 0 otherwise. Vector inequalities are written $\leq, <, \ll$. By a *(Pareto) surface (in $\mathbb{R}^I$)* we mean a subset S of $\mathbb{R}^I$ satisfying the following conditions:

$$\text{for any } x, y \in S, x \leq y \Rightarrow x = y, \tag{1}$$

$$S \subset \mathbb{R}_+^I \text{ and } e(i) \in S \text{ for each } i \in I, \tag{2}$$

$$\text{for each } x \in \mathbb{R}_+^I \setminus \{0\}, \text{there exists some } \lambda \in \mathbb{R} \text{ such that } \lambda x \in S. \tag{3}$$

Points in S are intepreted as utility vectors. Condition (1) is the essential restriction: it states that no point in S Pareto dominates a point in S. Conditions (2) and (3) are normalization and regularity restrictions; they definitely entail some loss of generality. We denote by $\mathcal{S}(I)$ the set of Pareto surfaces in $\mathbb{R}^I$. A typical three-individual Pareto surface is depicted on Figure 1.

Let $\mathcal{F}$ denote the set of increasing bijections from $[0,1]$ onto itself. If $f_i \in \mathcal{F}$ for each $i \in I$, define $f = (f_i)_{i\in I} : [0,1]^I \to [0,1]^I$ by $f(x) = (f_i(x_i))_{i\in I}$ for each $x \in [0,1]^I$. Call such a mapping an *ordinal transformation*. It is important that each f_i has x_i as sole argument; in particular, not every increasing bijection from $[0,1]^I$ onto itself is an ordinal transformation. Denote by $\mathcal{F}^I$ the set of all ordinal transformations. Two Pareto surfaces S, S' are *ordinally equivalent* if there exists $f \in \mathcal{F}^I$ such that $S' = f(S)$, in which case the inverse mapping f^{-1} is well-defined, belongs to $\mathcal{F}^{\,I}$, and ensures that $S = f^{-1}(S')$: ordinal equivalence *is* an equivalence relation on $\mathcal{S}(I)$.

3. THREE INDIVIDUALS

Throughout this section, $I = \{1,2,3\}$. Before entering the formal analysis, it is worthwile showing by means of a simple example that not all Pareto surfaces are ordinally equivalent to each other. Figure 2 depicts a Pareto surface S which is not ordinally equivalent to the simplex Δ of $\mathbb{R}^3$. (We have not drawn the part of S that lies in the interior of $\mathbb{R}^3_+$ because it is irrelevant to the argument.) Suppose, by way of contradiction, that $S = f(\Delta)$ for some ordinal transformation f. Observe first that $f_i(\frac{1}{2}) = \frac{1}{2}$ for each i: indeed if, say, $f_1(\frac{1}{2}) < \frac{1}{2}$, then $f_j(\frac{1}{2}) > \frac{1}{2}$ for $j = 2,3$,which is impossible. Since $f(\frac{1}{4},\frac{3}{4},0) \in S$ and $f_3(0) = 0$, it follows that $0 < f_1(\frac{1}{4}) < \frac{1}{2}$. Assume that $f_1(\frac{1}{4}) \geq \frac{1}{4}$. This inequality implies the following string of inequalities: $f_3(\frac{3}{4}) \leq \frac{3}{4}$, $f_2(\frac{1}{4}) \geq \frac{1}{4}$, $f_1(\frac{3}{4}) \leq \frac{3}{4}$, $f_3(\frac{1}{4}) \geq \frac{1}{4}$, and, finally, $f_2(\frac{3}{4}) \leq \frac{3}{4}$. Thus, $f(\frac{1}{4},\frac{3}{4},0)$ belongs to the line segment $[a,b)$. But then $| f_2(\frac{3}{4}) - \frac{3}{4} | < | f_1(\frac{1}{4}) - \frac{1}{4} | = | f_3(\frac{3}{4}) - \frac{3}{4} | = | f_2(\frac{1}{4}) - \frac{1}{4} | = | f_1(\frac{3}{4}) - \frac{3}{4} | = | f_3(\frac{1}{4}) - \frac{1}{4} | = | f_2(\frac{3}{4}) - \frac{3}{4} |$, a contradiction. Assuming $f_1(\frac{1}{4}) \leq \frac{1}{4}$ leads to a similar contradiction. In fact, the above argument shows that S is ordinally distinct from every surface that coincides with the simplex in the planes $x_i = 0$, $i \in I$.

Thus, the set of surfaces having the latter property is not rich enough to constitute an ordinal basis of $\mathcal{S}(\{1,2,3\})$. On the other hand, it is also redundant: some of its elements, though not identical, are ordinally equivalent. To check the latter assertion, let f_0 be an increasing bijection from $[0,1]$ onto itself such that $f_0(\alpha) + f_0(\beta) = 1$ whenever $\alpha + \beta = 1$, and $f_0(\frac{1}{3}) \neq \frac{1}{3}$. Then $(f_0, f_0, f_0)(\Delta)$ coincides with Δ in the planes $x_i = 0$, $i \in I$, but is not equal to Δ. The reader may check that the set of surfaces containing the three line segments $[e(i), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})]$ is also redundant.

We now define a class of Pareto surfaces that will be "just right". Figure 3 illustrates the definition. Let $\mathbb{N} = \{1,2,...\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{N}_{0-1} = \mathbb{N} \cup \{-1,0\}$. For each $n \in \mathbb{N}$, define

$$\alpha^n = \frac{1}{2}(\alpha^{n-1} + \alpha^{n-2}),$$

with the convention that $\alpha^{-1} = 1$ and $\alpha^0 = 0$. Thus, $\alpha^1 = \frac{1}{2}$, $\alpha^2 = \frac{1}{4}$, $\alpha^3 = \frac{3}{8}$, and generally,

$$\alpha^n = \left\{ \begin{array}{l} \frac{1}{3}\frac{(2^n-1)}{2^n} \text{ if } n \text{ is even,} \\ \frac{1}{3}\frac{(2^n+1)}{2^n} \text{ if } n \text{ is odd.} \end{array} \right\} \tag{4}$$

Let $a^{0,1} := (\alpha^{-1}, \alpha^0, \alpha^0)$, $a^{0,2} := (\alpha^0, \alpha^{-1}, \alpha^0)$, $a^{0,3} := (\alpha^0, \alpha^0, \alpha^{-1})$. For each $i \in I$ and each $n \in \mathbb{N}$, define $a^{n,i} = (a_1^{n,i}, a_2^{n,i}, a_3^{n,i})$ recursively by

$$a_j^{n,i} = \frac{1}{2}(a_j^{n-1,i} + a_{j+1}^{n-1,i}) \text{ for each } j \in I,$$

with the convention that $3 + 1 = 1$. Thus, for instance, $a^{1,1} = (\alpha^1, \alpha^0, \alpha^1)$, $a^{2,1} = (\alpha^2, \alpha^2, \alpha^1)$, $a^{3,1} = (\alpha^2, \alpha^3, \alpha^3)$, and so on. Finally, letting $[x, y]$ denote the line segment linking x, y in $\mathbb{R}^I$, let us define, for each $n \in \mathbb{N}_0$ and every $i \in I$,

$$A^{n,i} = [a^{n,i}, a^{n+1,i}].$$

Let $A^n = \cup_{i \in I} A^{n,i}$, $A^i = \cup_{n \in \mathbb{N}_0} A^{n,i}$, and $A = \cup_{n \in \mathbb{N}_0} \cup_{i \in I} A^{n,i}$. Call a Pareto surface S *normal* if $A \subset S$.

Theorem. *Let S be a Pareto surface in $\mathbb{R}^{\{1,2,3\}}$. There exist a unique normal surface S^* to which S is ordinally equivalent and a unique ordinal transfomation f ensuring that $f(S^*) = S$.*

Proof. Let S be a Pareto surface. We claim that there exists a unique ordinal transformation $f = (f_1, f_2, f_3)$ such that

$$f(A) \subset S. \tag{5}$$

To prove this claim we will construct a mapping $f : [0,1]^I \to [0,1]^I$ satisfying (5) and verify that it is an ordinal transformation. We will then check that no other ordinal transformation satisfies (5). We will use the following terminology and notation. A permutation $\pi : I \to I$ is *proper* if either $\pi(i) = i$ for all i or $\pi(i) \neq i$ for all i. If $x \in [0,1]^I$ and π is a permutation on I, x^π is defined by $(x^\pi)_i = x_{\pi(i)}$ for each $i \in I$. A real-valued mapping g defined on a subset of $[0,1]$ is increasing on a subset E of its domain if $\alpha < \beta \Rightarrow g(\alpha) < g(\beta)$ for all $\alpha, \beta \in E$.

Together with the convention $s^{-1} = (1,1,1)$, the conditions

$$(s_1^n, s_2^n, s_3^{n-1}), (s_1^n, s_2^{n-1}, s_3^n), (s_1^{n-1}, s_2^n, s_3^n) \in S, \tag{6}$$

$n \in \mathbb{N}_0$, recursively determine a unique sequence $\{s^n\}^{n \in \mathbb{N}_{0-1}}$, where $s^n = (s_1^n, s_2^n, s_3^n)$. This is explained in Shubik (1984) and is illustrated in Figure 4; we omit the formal proof. For each $n \in \mathbb{N}_{0-1}$, define

$$f_i(\alpha^n) = s_i^n \text{ for each } i \in I. \tag{7}$$

Taking $n = 0$ and $n = -1$,(7) implies in particular that $f_i(0) = 0$ and $f_i(1) = 1$ for each $i \in I$.

We now define f on the rest of $[0,1]^I$. The definition is recursive. Let $N>0$ and suppose we have defined $f_i(\frac{m}{2^n})$ for $m=0,1,...,2^n$, $n=0,...,N-1$,and $i\in I$. We define $f_i(\frac{m}{2^N})$ for $m=1,3,...,2^{N-1}$ and $i\in I$ as follows.

Assume N is even. Define $m^*=\frac{2^N-1}{3}$: this is an odd integer number. From (4), $\alpha^N=\frac{m^*}{2^N}$. Since (7) holds for $n=N$, $f_i(\frac{m^*}{2^N})=s_i^N$ for each $i\in I$.

Observe next that $\alpha^{N-2}=\frac{m^*-1}{2^N}$. By definition of A^{N-2},

$$(\frac{m^*}{2^N},\frac{m^*+2}{2^N},\frac{m^*-1}{2^N})^\pi\in A^{N-2} \text{ for each proper } \pi. \tag{8}$$

Since $f_i(\frac{m^*-1}{2^N})=s_i^{N-2}$ and $f_i(\frac{m^*}{2^N})=s_i^N$, conditions (5) and (8) determine $f_i(\frac{m^*+2}{2^N})$ for each i. Observe next that $\alpha^{N-3}=\frac{m^*+3}{2^N}$. By definition of A^{N-3},

$$(\frac{m^*}{2^N},\frac{m^*-2}{2^N},\frac{m^*+3}{2^N})^\pi\in A^{N-3} \text{ for every proper } \pi, \tag{9}$$

$$(\frac{m^*+2}{2^N},\frac{m^*-4}{2^N},\frac{m^*+3}{2^N})^\pi\in A^{N-3} \text{ for every proper } \pi. \tag{10}$$

Since $f_i(\frac{m^*}{2^N})$, $f_i(\frac{m^*+2}{2^N})$, and $f_i(\frac{m^*+3}{2^N})$ are determined for all i, conditions (5), (9), (10) determine $f_i(\frac{m^*-2}{2^N})$ and $f_i(\frac{m^*-4}{2^N})$ for all i. We continue in this way until $f_i(\frac{m}{2^N})$ is defined for each $m=1,3,...,2^{N-1}$ and $i\in I$. A similar construction is performed when N is odd.

We have now defined each f_i on the rational numbers in $[0,1]$. By construction, $f(a)\in S$ for any a in A whose coordinates are rational. Before extending the mapping f_1,f_2,f_3 to the whole of $[0,1]$, we verify that they are increasing.

Check first that for each $i\in I$ and $n\in\mathbb{N}$, $s_i^{n-2}<s_i^n<s_i^{n-1}$ if n is even while the opposite strict inequalities hold if n is odd. Likewise, $\alpha^{n-2}<\alpha^n<\alpha^{n-1}$ if n is even and the opposite inequalities hold when n is odd. Therefore (7) implies that each f_i is increasing on $\{\alpha^n \mid n\in\mathbb{N}_{0-1}\}$, i.e,

$$\text{for any } m,n\in\mathbb{N}_{0-1} \text{ and } i\in I,\ \alpha^m<\alpha^n\Rightarrow f_i(\alpha^m)<f_i(\alpha^n). \tag{11}$$

Next, we claim that for every $n\in\mathbb{N}$,

$$\text{for each } i\in I \text{ and } m=1,2,...,2^n,\ f_i(\frac{m-1}{2^n})<f_i(\frac{m}{2^n}). \tag{12}$$

Assume that n is even. Let $m^*=\frac{2^n-1}{3}$. Note that $\alpha^{n-2}=\frac{m^*-1}{2^n},\alpha^n=\frac{m^*}{2^n}$, and $\alpha^{n-1}=\frac{m^*+1}{2^n}$. It follows from (11) that

$$f_i(\frac{m^*-1}{2^n})<f_i(\frac{m^*}{2^n})<f_i(\frac{m^*+1}{2^n}) \text{ for each } i\in I. \tag{13}$$

For each $t=1,...,n$, let $m(t)=\frac{2^n-2^t}{3}$ if t is even and $m(t)=\frac{2^n-2^{t+1}}{3}$ if t is odd. For each t, let

$$E(t)=\{\frac{m(t)}{2^n},\frac{m(t)+1}{2^n},...,\frac{m(t)+2^t}{2^n}\}.$$

Thus, $E(1) = \{\frac{m^*-1}{2^n}, \frac{m^*}{2^n}, \frac{m^*+1}{2^n}\}$, $E(2) = \{\frac{m^*-1}{2^n}, ..., \frac{m^*+3}{2^n}\}$, $E(3) = \{\frac{m^*-5}{2^n}, ..., \frac{m^*+3}{2^n}\}$, ..., and $E(n) = \{0, \frac{1}{2^n}, ..., \frac{2^n-1}{2^n}, 1\}$. To prove that each f_i is increasing on $E(n)$, we proceed by induction. Inequalities (13) tell us that each f_i is increasing on $E(1)$. Fix now $t \geq 1$ and suppose that each f_i is increasing on $E(t)$. We claim that each f_i is increasing on $E(t+1)$. To fix ideas, let us assume that t is even; the argument is only slightly different when t is odd. By the induction hypothesis,

$$f_i(\frac{m(t)}{2^n}) < f_i(\frac{m(t)+1}{2^n}) < ... < f_i(\frac{m(t)+2^t}{2^n}) \text{ for each } i \in I. \tag{14}$$

If $t = n$, we are done. If $t < n$, notice that for each proper permutation π, $(\frac{m(t)}{2^n}, \frac{m(t)}{2^n}, \frac{m(t)+2^t}{2^n})^\pi$, $(\frac{m(t)+1}{2^n}, \frac{m(t)-1}{2^n}, \frac{m(t)+2^t}{2^n})^\pi$, ... $(\frac{m(t)+2^t}{2^n}, \frac{m(t)-2^t}{2^n}, \frac{m(t)+2^t}{2^n})^\pi$ belong to A. By definition of f, we therefore know that for each proper π,

$$\begin{aligned} f((\frac{m(t)}{2^n}, \frac{m(t)}{2^n}, \frac{m(t)+2^t}{2^n})^\pi) &\in S, \\ f((\frac{m(t)+1}{2^n}, \frac{m(t)-1}{2^n}, \frac{m(t)+2^t}{2^n})^\pi) &\in S, \\ &\cdots \\ f((\frac{m(t)+2^t}{2^n}, \frac{m(t)-2^t}{2^n}, \frac{m(t)+2^t}{2^n})^\pi) &\in S. \end{aligned} \tag{15}$$

Since S is a Pareto surface -hence, in particular, satisfies (1)-, inequalities (14) and conditions (15) imply that

$$f_i(\frac{m(t)}{2^n}) > f_i(\frac{m(t)-1}{2^n}) > ... > f_i(\frac{m(t)-2^t}{2^n}) \text{ for each } i \in I. \tag{16}$$

Since $m(t+1) = m(t) - 2^t$, (14) and (16) mean that each f_i is increasing on $E(t+1)$, as was to be proved. A similar argument works when n is odd.

Since the mappings f_1, f_2, f_3 are defined on all rational numbers in $[0, 1]$ and are increasing, they each admit a unique increasing extension to the whole of $[0, 1]$. This completes our construction of an ordinal transformation f satisfying condition (5).

It remains to be proved that f is indeed the unique ordinal tranformation satisfying that condition. To see why this is true, let g be an ordinal transformation meeting (5). It is enough to prove that, for each $n \in \mathbb{N}_{0-1}$,

$$g_i(\alpha^n) = f_i(\alpha^n) \text{ for each } i \in I, \tag{17}$$

for (5) then forces g to coincide with f everywhere. The argument is as follows. Since g is an ordinal transformation, $g_i(\alpha^0) = g_i(0) = 0$ and $g_i(\alpha^{-1}) = g_i(1) = 1$ for each $i \in I$. Next, fix $N > 0$ and assume that (17) holds for $n = 0, ..., N-1$. We establish that it holds for $n = N$. Recall (7) and assume, by way of contradiction, that, say, $g_1(\alpha^N) > s_1^N$. Consider the three points $(\alpha^N, \alpha^N, \alpha^{N-1})$, $(\alpha^N, \alpha^{N-1}, \alpha^N)$, $(\alpha^{N-1}, \alpha^N, \alpha^N) \in A$. Since by the induction hypothesis $g_i(\alpha^{N-1}) = s_i^{N-1}$ for each i, condition (5) implies that $(g_1(\alpha^N), g_2(\alpha^N), s_3^{N-1})$, $(g_1(\alpha^N), s_2^{N-1}, g_3(\alpha^N))$, $(s_1^{N-1}, g_2(\alpha^N), g_3(\alpha^N)) \in S$. By definition of s^N, however, (6) holds for $n = N$. Therefore $g_1(\alpha^N) > s_1^N \Rightarrow g_2(\alpha^N) < s_2^N \Rightarrow g_3(\alpha^N) > s_3^N \Rightarrow g_1(\alpha^N) < s_1^N$, a contradiction.

We have proved that there exists a unique ordinal transformation f satisfying (5). To complete the proof of the theorem, define $S^* = f^{-1}(S)$. By (5), $A \subset S^*$. Thus S^* is a normal surface to which S is ordinally equivalent. To check uniqueness, suppose S is ordinally equivalent to a normal surface $S^{**} \neq S^*$.There must then exist an ordinal transformation g different from the identity such that $S^* = g(S^{**})$. This implies that $S = f \circ g(S^{**})$, hence $f \circ g(A) \subset S$. This is impossible since f is the only ordinal transformation such that $f(A) \subset S$. ■

Since each normal surface is ordinally equivalent to itself, it follows from our theorem that all such surfaces are ordinally distinct. Intuitively speaking, this means that a lot of ordinal information is contained in a Pareto surface. Moreover, this information may be exploited very easily because ordinally equivalent surfaces are related by a unique ordinal transformation (and its inverse).

To be more precise, consider the problem of selecting an equitable utility vector from each Pareto surface. Define a (*distribution*) *rule* to be a mapping ξ which chooses from each Pareto surface S a utility vector $\xi(S)$ in S. Call such a rule *ordinal* if $\xi(f(S)) = f(\xi(S))$ for every surface S and every ordinal transformation f.

According to the theorem, a rule ξ is ordinal if and only if it is constructed as follows. Define ξ *arbitrarily* on the set S^* of normal surfaces. For each surface S, let S^* be the unique normal surface ordinally equivalent to S and let f be the unique ordinal transformation such that $f(S^*) = S$. Define $\xi(S) = f(\xi(S^*))$. An example is the rule described by Shubik (1984): it is the only ordinal rule which is egalitarian on S^*. More generally, an interesting class of ordinal rules are those which are egalitarian on some subset of Pareto surfaces rich enough to ensure that every surface is ordinally equivalent to one of its members. Although it seems natural to choose this subset equal to S^*, this choice should be defended on axiomatic grounds.

4. At least three individuals

We return to our general framework: the number of individuals is finite, at least three, but otherwise arbitrary. Several ordinal bases of the set of Pareto surfaces can be constructed by using the theorem of Section 3. We will define only one. If $x \in \mathbb{R}^I$ and $J \subset I$, let $(x_J, 0)$ denote the vector in $\mathbb{R}^I$ whose ith coordinate is x_i if $i \in J$ and 0 otherwise. If S is a Pareto surface in $\mathbb{R}^I$, let $S_J = \{x_J \in \mathbb{R}^J \mid (x_J, 0) \in S\}$. Clearly, S_J is a Pareto surface in $\mathbb{R}^J$. We call S *normal* if it meets the following two conditions:

$$S_{\{1,2,3\}} \text{ is a normal surface in } \mathbb{R}^{\{1,2,3\}} \text{ in the sense of Section 3,} \tag{18}$$

$$[e(1), e(i)] \subset S \text{ for each } i \in I \setminus \{1,2,3\}. \tag{19}$$

We postpone the discussion of this definition until we have proved the following corollary to our theorem.

Corollary. *Let I contain at least three individuals and let S be a Pareto surface in $\mathbb{R}^I$. There exist a unique normal surface S^* to which S is ordinally equivalent and a unique ordinal transformation f ensuring that $f(S^*) = S$.*

Proof. Let S be a Pareto surface in $\mathbb{R}^I$. By the theorem of Section 3, there exist a unique normal problem $S^*_{\{1,2,3\}}$ in $\mathbb{R}^{\{1,2,3\}}$ to which $S_{\{1,2,3\}}$ is ordinally equivalent and a unique $(f_1, f_2, f_3) \in \mathcal{F}^{\{1,2,3\}}$ such that $(f_1, f_2, f_3)(S^*_{\{1,2,3\}}) = S_{\{1,2,3\}}$. For each $i \in I \setminus \{1,2,3\}$, define the increasing bijection f_i from $[0,1]$ onto itself by the condition that $(f_1(\alpha), 0, ..., 0, f_i(1-\alpha), 0, ..., 0) \in S$ for every $\alpha \in [0,1]$. Then, letting $f = (f_i)_{i\in I}$, it is clear that $S^* := f^{-1}(S)$ is the unique normal surface in $\mathbb{R}^I$ to which S is ordinally equivalent and that f is the only ordinal transformation ensuring that $f(S^*) = S$. ■

As in the three-individual case, it follows from the above corollary that all normal surfaces in $\mathbb{R}^I$ are ordinally distinct and thus form an ordinal basis of $\mathcal{S}(I)$. Furthermore, any two ordinally equivalent surfaces are related by a unique ordinal transformation (and its inverse). This implies in particular that ordinal distribution rules exist whenever there are at least three individuals (even if we demand that the utility vector selected from every surface be strictly positive.)

To be sure, the normalization defined by (18) and (19) is highly asymmetric when there are more than three individuals and may therefore be of little use to define attractive distribution rules. It should be stressed, however, that all normalization procedures will display a substantial degree of asymmetry. In the four-individual case, for instance, the normalization condition that $(\frac{1}{2}, \frac{1}{2}, 0, 0), (0, \frac{1}{2}, \frac{1}{2}, 0), (0, 0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, 0, \frac{1}{2})$ belong to the surface —a very weak form of symmetry (we do not even require that $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $(0, \frac{1}{2}, 0, \frac{1}{2})$ belong to the surface)—,already entails some loss of generality. Indeed, let β be a strictly concave decreasing bijection from $[0,1]$ onto itself and consider a surface S containing $(\alpha, 1-\alpha, 0, 0), (0, \alpha, 1-\alpha, 0), (0, 0, \alpha, 1-\alpha)$ and $(\beta(\alpha), 0, 0, \alpha)$ for each $\alpha \in [0,1]$. Such a surface is not ordinally equivalent to any surface satisfying the weak symmetry property just mentioned because there is no $a \gg 0$ such that $(a_1, a_2, 0, 0), (0, a_2, a_3, 0), (0, 0, a_3, a_4), (a_1, 0, 0, a_4) \in S$.

5. References

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FIGURE 1

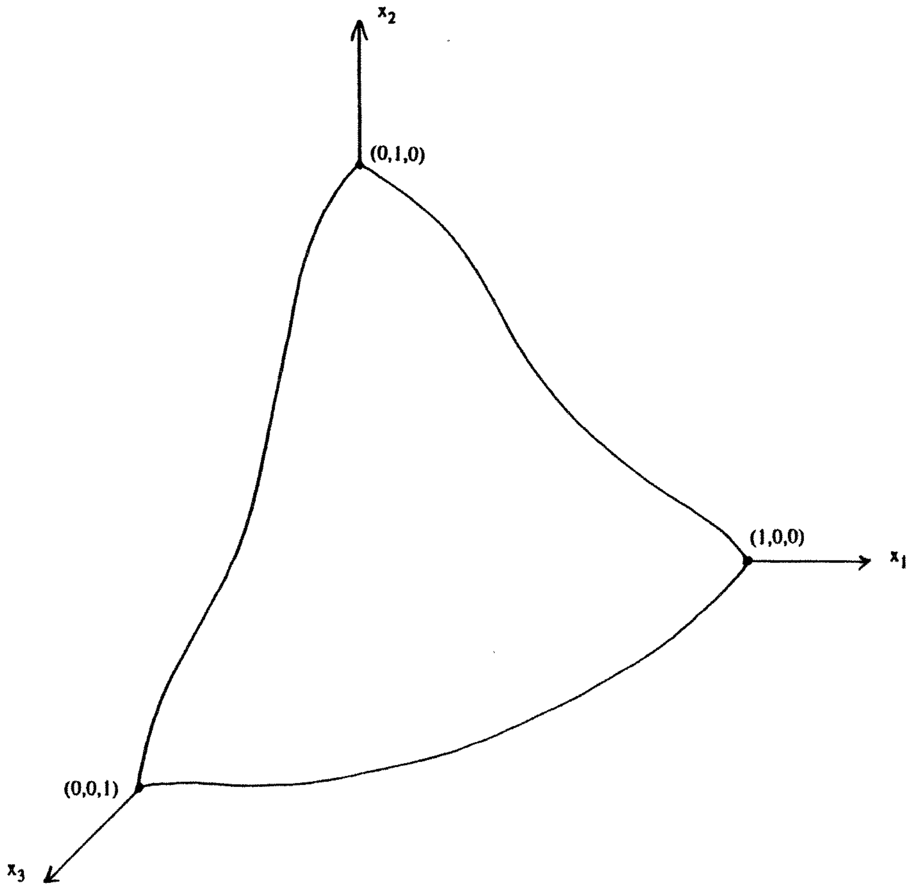


FIGURE 2

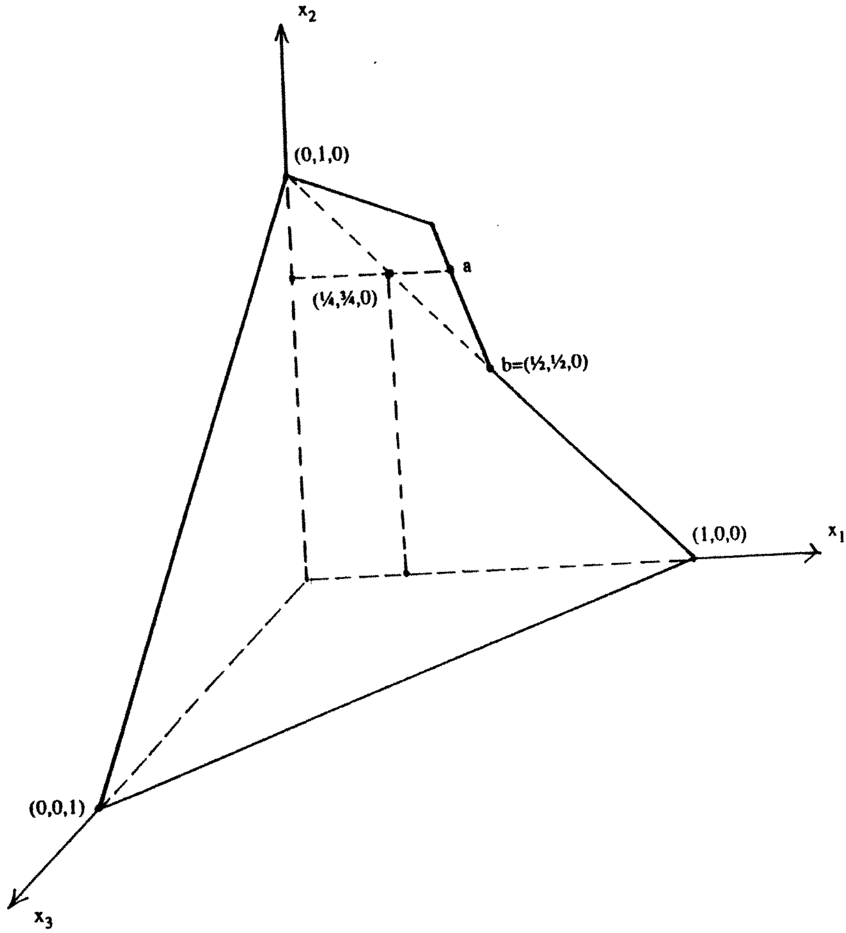
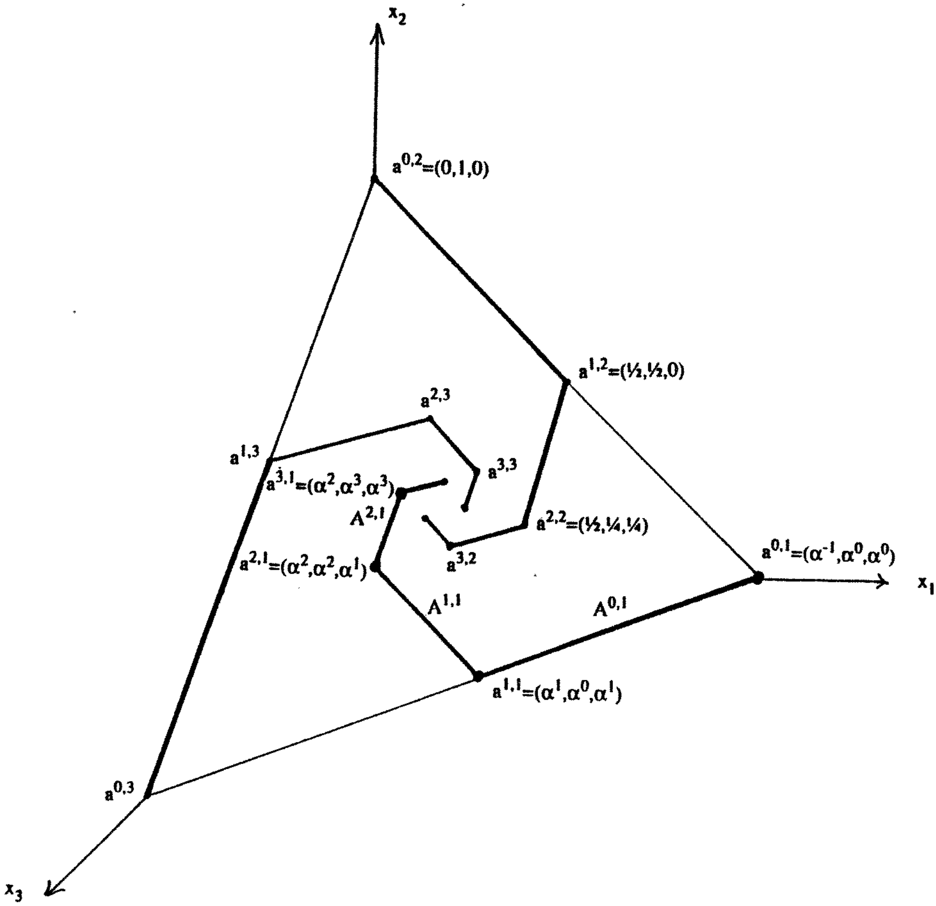
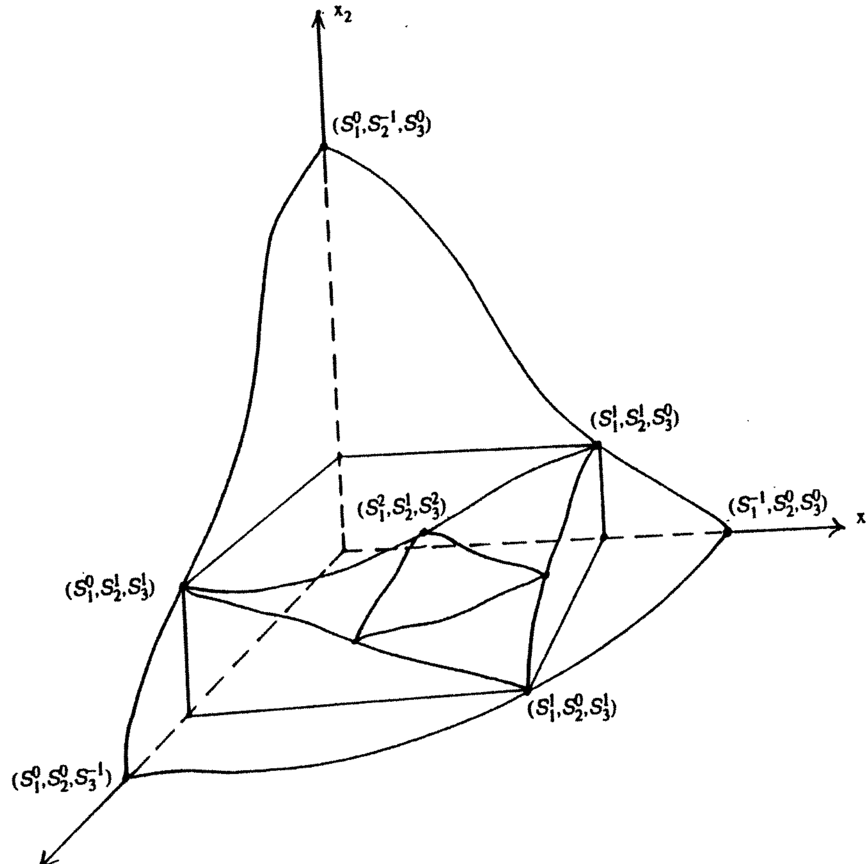


FIGURE 3



x2
a0,2=(0,1,0)
a1,2=(½,½,0)
a2,3
a1,3
a3,3
a3,1=(α2,α3,α3)
A2,1
a2,2=(½,¼,¼)
a3,2
a2,1=(α2,α2,α1)
a0,1=(α-1,α0,α0)
x1
A1,1
A0,1
a1,1=(α1,α0,α1)
a0,3
x3

FIGURE 4



x_2
(S_1^0, S_2^{-1}, S_3^0)
(S_1^1, S_2^1, S_3^0)
(S_1^2, S_2^1, S_3^2)
(S_1^{-1}, S_2^0, S_3^0)
x_1
(S_1^0, S_2^1, S_3^1)
(S_1^1, S_2^0, S_3^1)
(S_1^0, S_2^0, S_3^{-1})
x_3

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