Université de Montréal

Essays on matching and preference aggregation

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à la mémoire de mon père Paounsomda Bonkoungou

à ma mère Sompidian Zabre

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Résumé

Cette thèse est une collection de trois articles dont deux portent sur le problème d'appariement et un sur le problème d'agrégation des préférences. Les deux premiers chapitres portent sur le problème d'affectation des élèves ou étudiants dans des écoles ou universités. Dans ce problème, le mécanisme d'acceptation différée de Gale et Shapley dans sa version où les étudiants proposent et le mécanisme connu sous le nom de mécanisme de Boston sont beaucoup utilisés dans plusieurs circonscriptions éducatives aux Etats-Unis et partout dans le monde. Le mécanisme de Boston est sujet à des manipulations. Le mécanisme d'acceptation différée pour sa part n'est pas manipulable mais il n'est pas efficace au sens de Pareto. L'objectif des deux premiers chapitres est de trouver des mécanismes pouvant améliorer le bien-être des étudiants par rapport au mécanisme d'acceptation différée ou réduire le dégré de vulnérabilité à la manipulation par rapport au mécanisme de Boston.

Dans le Chapitre 1, nous étudions un jeux inspiré du système d'admission précoce aux Etats-Unis. C'est un système d'admission dans les collèges par lequel un étudiant peut recevoir une décision d'admission avant la phase générale. Mais il y a des exigences. Chaque étudiant est requis de soumettre son application à un seul collège et de s'engager à s'inscrire s'il était admis. Nous étudions un jeu séquentiel dans lequel chaque étudiant soumet une application et à la suite les collèges décident de leurs admissions dont les étudiants acceptent. Nous avons montré que selon une notion appropriée d'équilibre parfait en sous-jeux, les résultats de ce mécanisme sont plus efficaces que celui du mécanisme d'acceptation différée.

Dans le Chapitre 2, nous étudions un mécanisme centralisé d'admissions dans les universités françaises que le gouvernement a mis en place en 2009 pour mieux orienter les étudiants dans les établissements universitaires. Pour faire face aux écoles dont les places sont insuffisantes par rapport à la demande, le système défini des priorités qui repartissent les étudiants en grandes classes d'équivalence. Mais le système repose sur les préférences exprimées pour départager les ex-aequos. Nous avons prouvé que l'application du mécanisme d'acceptation différée avec étudiant proposant aprés avoir briser les ex-aequos est raisonable. Nous appelons ce mécanisme mécanisme français. Nous avons montré que le mécanisme français réduit la vulnérabilité à la manipulation par rapport au mécanisme de Boston et améliore le bien-être des étudiants par rapport au mécanisme standard d'acceptation différée où les ex-aequos sont brisés de façon aléatoire.

Dans le Chapitre 3, nous introduisons une classe de règles pour combiner les préférences individuelles en un ordre collectif. Le problème d'agrégation des préférences survient lorsque les membres d'une faculté cherchent une stratégie pour offrir une position sans savoir quel candidat va accepter l'offre. Il est courant de classer les candidats puis donner l'offre suivant cet ordre. Nous avons introuduit une classe de règles appélée règles de dictature sérielle augmentée dont chacune est paramétrée par une liste d'agents (avec répétition) et une règle de vote par comité. Pour chaque profile de préférences, le premier choix de l'agent en tête de la liste devient le premier choix collectif. Le choix du deuxème agent sur la liste, parmi les candidats restants, devient le deuxième choix collectif. Et ainsi de suite jusqu'à ce qu'il reste deux candidats auquel cas le comité vote pour classer ces derniers. Ces règles sont succinctement caractérisées par la non-manipulabilité et la neutralité sous l'extension lexicographique des préférences. Nous avons montré aussi que ces règles sont non-manipulables sous une variété d'extensions raisonable des préférences.

Mots-clés : Appariement, mécanisme d'acceptation différée, mécanisme de Boston, mécanisme français, agrégation des préférences, règle non-manipulable, règle de dictature sérielle augmentée.

Abstract

This thesis is a collection of two papers on matching and one paper on preference aggregation.

The first two chapters are concerned with the problem of assigning students to schools. For this problem, the student proposing version of Gale and Shapley's deferred acceptance mechanism and a mechanism known as Boston mechanism are widely used in many school districts in U.S and around the world. The Boston mechanism is prone to manipulation. The deferred acceptance mechanism is not manipulable ; however, it is not Pareto efficient. The first two chapters of this thesis deal with the problem of either improving the welfare of students over deferred acceptance or reducing the degree of manipulation under Boston.

In Chapter 1, we study a decentralized matching game inspired from the early decision system in the U.S : It is a college admission system in which students can receive admission decisions before the general application period. But there are two requirements. First, each student is required to apply to one college. Second, each student commits to attend the college upon admitted. We propose a game in which students sequentially make one application each and colleges ultimately make admission decisions to which students commit to accept. We show that up to a relevant refinement of subgame perfect equilibrium notion, the expected outcomes of this mechanism are more efficient than that of deferred acceptance mechanism.

In Chapter 2, we study a centralized university admission mechanism that the French government has implemented in 2009 to better match students to

university schools. To deal with oversubscribed schools, the system defined priorities that partition students into very coarse equivalence classes but relies on student reported preferences to further resolve ties. We show that applying student-proposing deferred acceptance mechanism after breaking ties is a reasonable procedure. We refer to this mechanism as French mechanism. We show that this mechanism is less manipulable than the Boston mechanism and more efficient than the standard deferred acceptance in which ties are broken randomly.

In Chapter 3, we introduce a class of rules called augmented serial rules for combining individual preferences into a collective ordering. The aggregation problem appears when faculty members want to devise a strategy for offering an open position without knowing whether any given applicant will ultimately accept an offer. It is a commonplace to order the applicants and make offers accordingly. Each of these augmented serial rules is parametrized by a list of agents (with possible repetition) and a committee voting rule. For a given preference profile, the collective ordering is determined as follows : The first agent's most preferred alternative becomes the top-ranked alternative in the collective ordering, the second agent's most preferred alternative (among those remaining) becomes the second-ranked alternative and so on until two alternatives remain, which are ranked by the committee voting rule. The main result establishes that these rules are succinctly characterized by neutrality and strategy-proofness under the lexicographic extension. Additional results show that these rules are strategy-proof under a variety of other reasonable preference extensions.

Keywords : Matching, deferred acceptance, Boston mechanism, French mechanism, preference aggregation, strategy-proofness, augmented serial rule.

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Chapitre 1

Pareto Dominance over Deferred Acceptance through Decentralization

1.1 Introduction

Since the successful design of the medical labor market clearinghouse in the United States, two-sided matching theory has mainly focused on centralized matching markets. Many matching markets, including markets for physicians, osteopaths, gastroenterologists, lawyers, have moved to centralized matching markets ; or at least have experimented with the possibility. At the same time, numerous matching markets remain decentralized ; and some have even returned to a decentralized mechanism after a period of operating through a clearinghouse. The market for gastroenterologists for instance, introduced a clearinghouse in 1986 ; returned to decentralized market in 1990 ; and, reintroduced a clearinghouse in 2006 (Niederle and Roth, 2009). And yet, little is known about decentralized markets and how they compare to centralized systems. To broaden the scope of market design, we need to understand decentralized markets and their potential.

Empirical studies have shown that centralized markets which succeeded all share a stability property (Roth, 1984b). A matching is **stable** if it matches all agents to acceptable partners and no unmatched pair prefers one another to their proposed partners or one of their proposed partners. A stable matching exists for any market. Also, when preferences are strict, there is a stable matching that each agent on the same side finds at least as desirable as any stable matching. This matching is obtained by applying the agent-proposing **deferred acceptance** (DA) algorithm to this market (Gale and Shapley, 1962). Currently, DA is used in many cities in the United States (like New York and Boston) to assign students to public schools. While DA has the advantage of being stable, it has been criticized for its lack of Pareto efficiency (Kesten, 2010). In general, efficiency is not compatible with stability (Roth, 1982; Ergin, 2002; Abdulkadiroğlu et al., 2009). At the same time, no dominant strategy incentive compatible rule (strategy-proof rule) Pareto dominates DA (Kesten, 2010; Abdulkadiroğlu et al., 2009; Alva and Manjunath, 2017). A natural question is whether one can consider weaker incentive requirements for which students find the expected outcomes more desirable than the outcome produced by DA.

In this paper, we model and analyse a decentralized market and compare the outcomes produced to the outcome produced by DA. We consider a perfect information game that captures the possibility of coordinated actions. In the United States, many colleges have early admissions programs, where they can admit students before the general application period. In an important program, **early decision**, colleges offer binding early admissions. Each applicant is required to (1) apply to exactly one college and (2) commit to attend that college if admitted. Our matching game captures this feature : first, students make applications sequentially. Each student applies to one college, if any. Then, colleges sequentially decide on which applicants to admit. Finally, as reflected in the binding commitment, each student attends the college to which he is admitted (if any).

While the classic equilibrium concept for a perfect information game is subgame perfect Nash equilibrium (SPE), there are undesirable SPEs in the game just described. They involve students playing strategies that are weakly dominated in some subgames. Since we do not expect such equilibria in practice, we consider a refinement of SPE that induces **undominated Nash equilibria in every subgame (SPUE)**. We show that the outcomes according to this solution concept weakly Pareto dominate the outcome produced by DA. This result is part of a new trend of literature that reconsiders the possibility of Pareto dominance over DA (Abdulkadiroğlu et al., 2011; Troyan, 2012; Dur and Morrill, 2016).

An early literature (Roth, 1984a; Crawford, 1991) established that when some students depart from a matching problem, the remaining students are better-off in the new student-proposing DA outcome. This result has been recently generalized (Kojima and Manea, 2010) to show that when some students remove from their acceptable set some colleges they do not obtain, then the other students are better-off in the DA outcome of the resulting problem. The game suggested in this paper can be viewed as a decentralized version of this procedure. However, for the game we consider, this involves showing that the procedure is actually an SPE.

It is very well-known that in a simultaneous game where participants cannot observe each other decision and coordinate their actions, the SPE outcomes are equivalent to the set of stable matchings (Alcalde and Romero-Medina, 2000; Sotomayor, 2004; Echenique and Oviedo, 2004). However, there are decentralized markets where perfect information could be a good approximation, the economics job market for new economists being a case in a point. Indeed, this market features a platform, the Economics Job Market Rumors, where many of the offers and acceptances are published. [1]

The purpose of this paper is not to describe outcomes in markets as they are currently organized ; but rather, to offer insights for alternative design. Because of technology advances, markets can now be organized by designing platforms that help participants to coordinate their actions. The results of this paper show that there might be gains from moving toward

1. https ://www.econjobrumors.com/wiki.php.

such institutions.

The paper most closely related to ours is (Suh and Wen, 2008), which studies a similar game in a marriage market. The key difference is that they propose a strong assumption on preferences which ensures that the student optimal stable matching is the unique SPE outcome. In a recent independent contribution, Dur et al. (2017) studied a game in which students sequentially submit preference relations and the Boston mechanism is applied afterwards. By introducing a refinement, they also show that the outcomes produced by the game Pareto dominate that of DA. Next, (Haeringer and Wooders, 2011) study a decentralized job matching market where workers apply many times. They show that the worker optimal stable matching is the unique SPE outcome. More broadly, the paper is related to a number of contributions in decentralized matching such as Diamantoudi et al. (2015), Pais (2008), Wu (2015) and Romero-Medina and Triossi (2014).

The rest of the paper is organized as follows : in section 2, we formally introduce the model, then we derive the results in Section 3. We collect all proofs in the appendices.

1.2 Model

1.2.1 Many-to-one matching market

Let $S = \{s_1, ..., s_n\}$ denote a finite set of students and $C = \{c_1, ..., c_m\}$ a finite set of colleges. Let $S \cup C$ denote the set of agents with a generic agent denoted by v. Remaining unmatched is denoted by $\emptyset$.

Each student s has a strict preference relation P_s over $C \cup \{\emptyset\}$. Given $s \in S$ and P_s, let R_s denote the weak preference relation associated with P_s : for each $\{v, v'\} \subset C \cup \{\emptyset\}$, $v \; R_s \; v'$ if $v \; P_s \; v'$ or $v = v'$. Let $\mathcal{P}$ denote the set of preference relations and $\mathcal{P}^S$ the set of preference profiles $P = (P_s)_{s \in S}$. For $S' \subset S$, we often write $(P_{S'}, P_{-S'})$ instead of P.

Each college c has a maximum number $q_c \in \mathbb{N}$ of students it can admit, its

capacity. Let $q := (q_c)_{c\in C}$ denote the profile of capacities. Each college c has a strict preference relation $\succ_c$ over the set 2^S of all subsets of S. Given $c \in C$, and c's preference relation $\succ_c$, $\succeq_c$ is the weak preference relation associated with $\succ_c$ and $Ch_{\succ_c} : 2^S \to 2^S$ is c's choice function induced by $\succ_c$ as follows : for each $S' \in 2^S$, $Ch_{\succ_c}(S') := \max_{\succ_c} 2^{S'}$. Note that $Ch_{\succ_c}$ is well-defined because $\succ_c$ is strict. It is convenient to work with choice functions instead of colleges' preference relations. We assume that each preference satisfies the following properties :

Susbtitutable. *For each pair S', S'' of sets such that $S' \subseteq S'' \subseteq S$, we have $Ch_{\succ_c}(S'') \cap S' \subseteq Ch_{\succ_c}(S')$.*

Acceptant. *For each $S' \subseteq S$, $|Ch_{\succ_c}(S')|= min\{q_c, |S'|\}$.*

In the sequel, we write Ch_c without referring to c's underlying preference relation $\succ_c$. Let $Ch = (Ch_c)_{c\in C}$ denote a profile of substitutable and acceptant choice functions.

A **market** is a tuple (S, C, P, Ch, q) that consists of a set S of students, a set C of schools, a preference profile P, a profile of choice functions Ch and a capacity vector q. Since S, C and q mostly remain fixed throughout the paper, we suppress them and instead denote market (S, C, P, Ch, q) by $M = (P, Ch)$.

A **matching** is a function $\mu : S \cup C \to 2^{S\cup C}$ such that

(1) for each $s \in S$, $|\mu(s)|\leq 1$ and $\mu(s) \subset C \cup \{\emptyset\}$,

(2) for each $c \in C$, $|\mu(c)|\leq q_c$ and $\mu(c) \subseteq S$ and

(3) for each $s \in S$ and each $c \in C$, $\mu(s) = \{c\}$ if and only if $s \in \mu(c)$.

By convention, for each $v \in S \cup C$, we write μ_v instead of $\mu(v)$ and for each $s \in S$ and $c \in C$, $\mu_s = c$ instead of $\mu_s = \{c\}$. Let $\mathcal{M}$ denote the set of matchings. We extend agents' preferences to the set $\mathcal{M}$ in the natural way : for each agent v and $\mu, \mu' \in \mathcal{M}$, v prefers μ to μ' if and only if he or it prefers μ_v to μ'_v. Similarly, v finds μ at least as desirable as μ' if and only if he or it finds μ_v at least as desirable as μ'_v. We write $\mu \; R \; \mu'$ if for each $s \in S$, $\mu_s \; R_s \; \mu'_s$.

If $\mu_v = \emptyset$, then agent v is **unmatched** under μ. Let $M = (P, Ch)$. A matching μ is **individually rational** (IR) at M if for each $s \in S$, $\mu_s \; R_s \; \emptyset$ and for each $c \in C$, $Ch_c(\mu_c) = \mu_c$. Next, we say that μ is **blocked** by the pair $(s, c) \in S \times C$ at M if $c \; P_s \; \mu_s$ and $s \in Ch_c(\mu_c \cup \{s\})$. Finally, we say that μ is **stable** at M if it is IR at M and it is not blocked by any pair at M.

A (matching) rule $\varphi : \mathcal{P}^S \to \mathcal{M}$ selects a matching $\varphi(P) \in \mathcal{M}$ for each $P \in \mathcal{P}^S$. Given $P \in \mathcal{P}^S$ and $\mu, \mu' \in \mathcal{M}$, we say that μ **Pareto dominates** μ' (for students) at P if $\mu_s \; R_s \; \mu'_s$ for each $s \in S$ and for some $s \in S$, $\mu_s \; P_s \; \mu'_s$. We say that μ **weakly Pareto dominates** μ' at P if μ Pareto dominates μ' at P or $\mu = \mu'$. A rule φ Pareto dominates a rule φ' if for each $P \in \mathcal{P}^S$, $\varphi(P)$ weakly Pareto dominates $\varphi'(P)$ at P and for some $P \in \mathcal{P}^S$, $\varphi(P)$ Pareto dominates $\varphi'(P)$. Finally, rule φ weakly Pareto dominates rule φ' if φ Pareto dominates φ or $\varphi = \varphi'$.

1.2.2 Matching game

Let $\pi : \{1, 2, ..., n\} \to S$ be a bijection and $\mathcal{O}$ the set of all such bijections. Each bijection describes an order according to which students apply. Given $\pi \in \mathcal{O}$, we index the students in such a way that for each $t = 1, 2, ..., n$, $s_{\pi(t)} = \pi(t)$. Given $\pi \in \mathcal{O}$, the game form induced by it is as follows :

Applications phase : student $s_{\pi(1)}$ either applies to some college $c \in C$ or chooses $\emptyset$ to remain unmatched. Let $a_1 \in C \cup \{\emptyset\}$ denote his decision. For each $t = 2, ..., n$, student $s_{\pi(t)}$ observes all application decisions $a_1, ..., a_{t-1}$ and takes an action $a_t \in C \cup \{\emptyset\}$.

Admissions phase : colleges offer admissions, sequentially in the order of their index numbers. Given any application decisions $a_1, ..., a_n$, college c_1 offers admissions to a subset $o(c_1) \subseteq \{s_{\pi(t)} \in S | a_t = c_1\}$ of its applicants. For each $j = 2, ..., m$, college c_j observes all decisions $a_1, ..., a_n, o(c_1), ..., o(c_{j-1})$; and offers admissions to a subset $o(c_j) \subseteq \{s_{\pi(t)} \in S | a_t = c_j\}$ of its applicants.

Each student's action consists of applying to a college or choosing to remain unmatched. Each college's action consists of admitting some students from among its applicants. A **history** is an ordered collection of actions. Let h^0 be the history where no student has yet applied. Given $\pi \in \mathcal{O}$ and $t = 0, ..., n-1$, the ordered collection $h^t := (h^0, a_1, ..., a_t)$ of actions is a history after which $s_{\pi(t+1)}$ has to play. Let $\mathcal{H}^t_\pi$ denote the set of such histories. Any history after which all decisions are made is a **terminal history**. The outcome attached to the terminal history $(h^0, a_1, ..., a_n, o(c_1), ..., o(c_m))$ is the matching μ defined as follows : for each $t = 1, ..., n$, (i) if $a_t = \emptyset$, then $\mu_{s_{\pi(t)}} = \emptyset$, (ii) if $a_t = c$ and $s_{\pi(t)} \in o(c)$, then $\mu_{s_{\pi(t)}} = c$ and (iii) if $a_t = c$ and $s_{\pi(t)} \notin o(c)$, then $\mu_{s_{\pi(t)}} = \emptyset$. The game form just described is a well-defined **finite extensive-form game of perfect-information**. Given an order π and a market M, let $G[\pi, M]$ denote the game induced by π and M.

Let $\pi \in \mathcal{O}$ and $t = 1, ..., n$. A **strategy** $\sigma_{s_{\pi(t)}}$ of student $s_{\pi(t)}$ is a function $\sigma_{s_{\pi(t)}} : \mathcal{H}^{t-1}_\pi \to C \cup \{\emptyset\}$ specifying an application decision $\sigma_{s_{\pi(t)}}(h^{t-1}) \in C \cup \{\emptyset\}$ for each $h^{t-1} \in \mathcal{H}^{t-1}_\pi$. Let Δ^t_π denote the set of such strategies and $\Delta_\pi = \times^n_{t=1} \Delta^t_\pi$ the set of strategy profiles for students in the game form with order π.

It is straightforward to see that for each history $h^n = (h^0, a_1, ..., a_n)$ and each $c \in C$, the optimal admission of c is $o(c) = Ch_c(\{s_{\pi(t)} \in S | a_t = c\})$. This is because c does not get any further applications and therefore can choose only from $\{s_{\pi(t)} \in S | a_t = c\}$. Henceforth, we ignore colleges' strategies and, abusing language, only speak of a students' strategy profile as equilibrium of $G[\pi, M]$.

In any game $G[\pi, M]$, one can use Kuhn (1953)'s backwards-induction algorithm to find a strategy profile that induces a Nash equilibrium in every subgame of $G[\pi, M]$. Such a strategy profile is a **subgame perfect Nash equilibrium (SPE)** of $G[\pi, M]$.

1.3 Results

We first describe the student-proposing deferred acceptance (DA) algorithm (Gale and Shapley, 1962). Given a market $M = (P, Ch)$,

Step 1. Every student proposes to his most preferred acceptable college under P (if any). Let $\hat{S}_c^1$ be the set of students proposing to college c. College c tentatively accepts the students in $S_c^1 = Ch_c(\hat{S}_c^1)$ and rejects the students in $\hat{S}_c^1 \setminus S_c^1$.

Step t. $(t \geq 2)$. Every student who was rejected at Step $(t-1)$ proposes to his next most preferred acceptable college under P (if any). Let $\hat{S}_c^t$ be the set of students proposing to college c. College c tentatively accepts the students in $S_c^t = Ch_c(S_c^{t-1} \cup \hat{S}_c^t)$ and rejects the students in $(S_c^{t-1} \cup \hat{S}_c^t) \setminus S_c^t$.

The algorithm terminates when each student is either accepted or rejected by all of his acceptable colleges. The tentative acceptances of the later stage become final. Given a choice profile Ch, let $\boldsymbol{DA^{Ch}}$ denote the rule that selects for each P, the DA outcome for $M = (P, Ch)$. We simply call it DA.

We now give an example where a DA outcome Pareto dominates an SPE outcome.

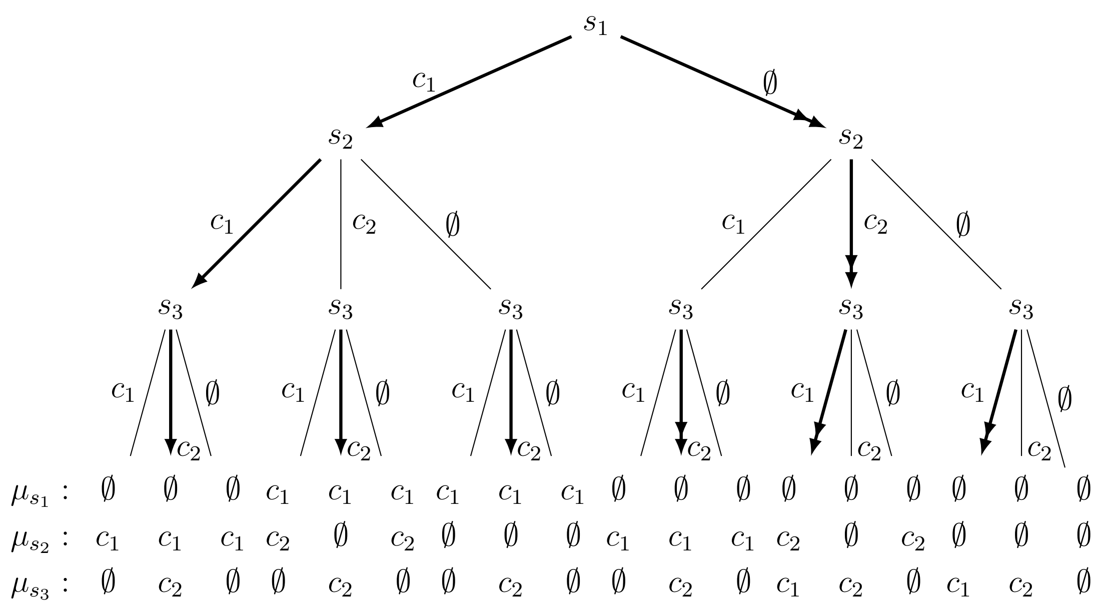
Example 1. *Let* $S = \{s_1, s_2, s_3\}$, $C = \{c_1, c_2\}$ *with capacities* $q_{c_1} = q_{c_2} = 1$, *and preferences are as follows :*

P_{s_1}	P_{s_2}	P_{s_3}	$\succ_{c_1}$	$\succ_{c_2}$
$\emptyset$	c_2	c_1	s_2	s_1
	c_1	c_2	s_1	s_3
			s_3	s_2

Students move under the order specifies index numbers. We represent only the relevant part of the tree. [2]

2. Specifically, we omit colleges' moves and the part where s_1 applies to c_2 and the subsequent subgame since such application will be accepted and c_2 is not acceptable to s_1. We also represent the outcome in the terminal histories of the tree obtained when colleges choose optimally from among their applicants.

FIGURE 1.1 – SPE outcome Pareto dominated by DA



By Kuhn's algorithm, there are two SPEs. They are represented by the sequence of connected arrows and the sequence of connected double arrows represented. In both SPE outcomes, s_1 is unmatched. Clearly, the outcome of the first SPE, $\begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & c_1 & c_2 \end{pmatrix}$, is Pareto dominated by the DA outcome, $\begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & c_2 & c_1 \end{pmatrix}$, the second SPE outcome.

As the example suggests, the underlying feature of the first SPE is that some students apply to colleges that they deem unacceptable ; but they remain unmatched in the equilibrium. While such behavior do not affect the outcome of the applicants in question, it may induce a subsequent student to apply to a college that is less preferred for him than the one to which he would have otherwise applied.

Note that s_1's strategy of applying to the unacceptable college c_1 is weakly dominated. Indeed, his strategy of choosing to remain unmatched weakly dominates his strategy of applying to c_1 : under c_1's strategy of admitting s_1, he prefers the outcome of the first strategy to the outcome of the later. The instance of an SPE containing a weakly dominated strategy is representative of a more general issue : an SPE may have strategies that are weakly dominated in some subgames, that is, their restrictions to some subgames may contain weakly dominated strategies. We deal with these non-intuitive strategies next.

1.3.1 Subgame perfect undominated Nash equilibrium (SPUE)

Given an order π, a market M and a non-terminal history $h \in \bigcup_{t=1}^{n-1} \mathcal{H}_\pi^t$, let $G[\pi, M|_h]$ denote the subgame of $G[\pi, M]$ that starts at h. Moreover, given $t = 1, ..., n$ and $\sigma_{s_{\pi(t)}} \in \Delta_\pi^t$, let $\sigma_{s_{\pi(t)}}|_h$ denote the restriction of $\sigma_{s_{\pi(t)}}$ to $G[\pi, M|_h]$. Let $\sigma|_h = (\sigma_s|_h)_{s \in S}$. The matching attached to the terminal history reached when σ is executed starting from h (and colleges choose optimally) is denoted by $\mu(\sigma|_h)$.

Given $\pi \in \mathcal{O}$, a market $M = (P, Ch)$ and $t = 1, ..., n$, a strategy $\sigma_{s_{\pi(t)}}$ **weakly dominates** a strategy $\sigma'_{s_{\pi(t)}}$ in a subgame $G[\pi, M|_{h^{t-1}}]$, if for each strategy profile $\sigma_{-s_{\pi(t)}}$, student $s_{\pi(t)}$ finds $\mu(\sigma|_{h^{t-1}})$ at least as desirable as $\mu(\sigma'_{s_{\pi(t)}}, \sigma_{-s_{\pi(t)}}|_{h^{t-1}})$ with at least one preference. A strategy $\sigma_{s_{\pi(t)}}$ is **undominated** in a subgame $G[\pi, M|_{h^{t-1}}]$ if it is not weakly dominated in $G[\pi, M|_{h^{t-1}}]$ by any strategy. A strategy profile $\sigma \in \Delta_\pi$ is undominated (in $G[\pi, M]$) if for each $t = 1, ..., n$, and each $h^{t-1} \in \mathcal{H}_\pi^{t-1}$, $\sigma_{s_{\pi(t)}}$ is undominated in $G[\pi, M|_{h^{t-1}}]$.

Remark. *Any strategy that involves an application to an unacceptable college c is weakly dominated in the subgame starting at the history in which it occurs. Either c admits the applicant in question, in which case he prefers choosing to remain unmatched to applying to c. Or c does not admit him, which is the same outcome as choosing to remain unmatched.*

We consider SPEs in which no strategy is dominated in a subgame. Hence, we obtain the following refinement of SPEs.

Subgame perfect undominated Nash equilibrium (SPUE). *Given* $\pi \in \mathcal{O}$ *and a market* M*, the strategy* $\sigma \in \Delta_\pi$ *induces an undominated Nash equilibrium in every subgame of* $G[\pi, M]$*.*

Given $\pi \in \mathcal{O}$ and a market M, let $\mathcal{M}^{SPUE}(G[\pi, M])$ denote the SPUE outcomes of $G[\pi, M]$. This mild refinement of SPE produces outcomes that weakly Pareto dominate the DA outcome.

1.3.2 Equilibrium characterization

Applications to unacceptable colleges are wasted and therefore ruled out in any SPUE. However, these are not the only « wasted » applications. Any application to an acceptable college that is rejected is also wasted. The applicant is unmatched in the corresponding SPUE outcome, yet such application may affect other students.

Example 2. *Consider a variation of Example* 1 *where student* s_1*'s preference relation becomes the one in which* c_1 *is the unique college that is acceptable to him. Applying Kuhn's backwards-induction algorithm again yields the same SPEs identified in the tree of this example. The SPE outcomes are now the DA outcome* $\begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & c_1 & c_2 \end{pmatrix}$ *and the matching* $\begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & c_2 & c_1 \end{pmatrix}$*. Student* s_1 *applied to* c_1 *in an SPE but is not admitted. The other students prefer the outcome of the SPE where he chose to remain unmatched to the one where he applied to* c_1*.*

Given $s \in S$ and $c \in C$, let $P_s^c \in \mathcal{P}$ denote student s's preference relation in which c is the unique college that is acceptable to him and $P_s^\emptyset \in \mathcal{P}$ a preference relation in which he finds no college acceptable.

Let $\pi \in \mathcal{O}$ and $M = (P, Ch)$. We define the following profiles :

(i) $\boldsymbol{P}(h^0) := P$.

(ii) For each $t = 2, ..., n$ and each history $h^{t-1} = (h^0, a_1, ..., a_{t-1})$,

$$\boldsymbol{P}(h^{t-1}) := (P_{s_{\pi(1)}}^{a_1}, ..., P_{s_{\pi(t-1)}}^{a_{t-1}}, P_{s_{\pi(t)}}, ..., P_{s_{\pi(n)}}).$$

To elaborate, consider a history $h^{t-1} = (h^0, a_1, ..., a_{t-1})$. First, $\boldsymbol{P}(h^0)$ is the initial profile P according to M. Next, $\boldsymbol{P}(h^{t-1})$ is the profile obtained from P as follows : (i) for each $t' < t$, we replace the preference relation $P_{s_{\pi(t')}}$ of $s_{\pi(t')}$ with $P^{a_{t'}}_{s_{\pi(t')}}$; and (ii) for each $t' \geq t$, we let $P_{s_{\pi(t')}}$ unchanged, that is, $\boldsymbol{P}_{s_{\pi(t')}}(h^{t-1}) = P_{s_{\pi(t')}}$.

Consider now an SPE that involves a wasted application to an acceptable college. The student who played such a strategy has an alternative strategy where he chooses to remain unmatched at the history where he made such application. It turns out that such a strategy may be part of another SPE. Yet, the subsequent students finds the outcome of the later one as desirable as the outcome of the first one.

In naming it, we relate this form of strategic behavior to a social choice theory concept of similar property.[3] Let $M = (P, Ch)$, $\pi \in \mathcal{O}$ and $t = 1, ..., n$. Let $\sigma_{s_{\pi(t)}} \in \Delta^t_\pi$. Then,

Bossy strategy. *For each history $h^{t-1} \in \mathcal{H}^{t-1}_\pi$, letting $DA^{Ch}_{s_{\pi(t)}}(\boldsymbol{P}(h^{t-1})) := v$,*

(i) if $v = c$ for some $c \in C$, then $\sigma_{s_{\pi(t)}}(h^{t-1}) = v$ and

(ii) if $v = \emptyset$, then $\sigma_{s_{\pi(t)}}(h^{t-1}) = v'$ with $v' \, R_{s_{\pi(t)}} \, \emptyset$ and $v' \neq \emptyset$ for at least one history.

In a bossy strategy $\sigma_{s_{\pi(t)}}$, (i) at each history h^{t-1} with $DA^{Ch}_{s_{\pi(t)}}(\boldsymbol{P}(h^{t-1})) = c$, $s_{\pi(t)}$ applies to c and (ii) at each history h^{t-1} with $DA^{Ch}_{s_{\pi(t)}}(\boldsymbol{P}(h^{t-1})) = \emptyset$, $s_{\pi(t)}$ either chooses to remain unmatched or applies to an acceptable college ; and, in at least one such history he applies to an acceptable college.

At some history h^{t-1} such that $DA^{Ch}_{s_{\pi(t)}}(\boldsymbol{P}(h^{t-1})) = \emptyset$; a bossy strategy may recommend an application to an acceptable college. It turns out that the student $s_{\pi(t)}$ is not admitted by this college in the SPUE of the subgame

3. We thank William Thomson for making a parallel between **non-bossiness**, a social choice theory concept, and the property embodied in the strategy mentioned ; which led to the choice of bossy strategy. This is an analogy to non-bossiness introduced for social choice functions (Satterthwaite and Sonnenschein, 1981) : A social choice function is bossy when some agent can change the outcome for others without changing his own.

that follows. Then, $s_{\pi(t)}$ gets the same thing whether or not he makes the corresponding application. However, choosing to remain unmatched would have helped the remaining students.

A strategy without the application mentioned has a flavour of some solidary. Let $M = (P, Ch)$, $\pi \in \mathcal{O}$, $t = 1, ..., n$ and $\sigma_{s_{\pi(t)}} \in \Delta_{\pi}^{t}$.

Solidary strategy. *For each history $h^{t-1} \in \mathcal{H}_{\pi}^{t-1}$,*

$$\sigma_{s_{\pi(t)}}(h^{t-1}) = DA^{Ch}_{s_{\pi(t)}}(\boldsymbol{P}(h^{t-1})).$$

Theorem 1. *Given $\pi \in \mathcal{O}$ and a market M, the subgame perfect undominated Nash equilibria of $G[\pi, M]$ correspond to the strategy profiles $\sigma = (\sigma_s)_{s \in S}$ such that for each student $s \in S$, σ_s is either a bossy strategy or a solidary strategy.*

Appendix 1 contains the proof of Theorem 1.

1.3.3 Pareto dominance over deferred acceptance

Our main result is the following :

Theorem 2. *The outcome of every SPUE weakly Pareto dominates DA.*

Appendix 2 contains the proof of Theorem 2. The following corollary follows from the optimality of the outcome of DA.

Corollary 1. *The outcome of DA is the unique stable matching that can arise in an SPUE.*

Since the order of applicants can affect the SPUE outcome, it is important to understand the markets for which the outcome is order independent (see Moldovanu and Winter (1995) for a related concept). Before doing so we first show that some key features of the SPUE outcome are order independent for any market. A student who is matched at some order is also matched at any other order. The same thing holds for an unmatched student. This result resembles a feature of the set of stable matchings (Roth, 1984b, 1986),[4] known as rural hospital theorem. A student who is matched at some stable matching is also matched at any other stable matching and the same thing holds for an unmatched student. Furthermore, a college with unfilled seats at some stable matching is matched to the same set of students at each stable matching.

Rural hospital properties. *Given a non-empty subset* $\mathcal{M}' \subseteq \mathcal{M}$ *and* $\{\mu, \mu'\} \subset \mathcal{M}'$,

(i) for each $v \in C \cup S$, $|\mu_v| = |\mu'_v|$ *and*

(ii) for each $c \in C$, $|\mu_c| < q_c$ *implies* $\mu_c = \mu'_c$.

Proposition 1. *Given a market* $M = (P, Ch)$*, the set of SPUE outcomes associated with the set of all possible orderings* $\mathcal{O}$ *satisfies the rural hospital properties.*

Appendix 3 contains the proof of Proposition 1. We note that Proposition 1 is not a negative result as in the medical labor market. It is positive in the sense that the order does not change a matched student to unmatched or an unmatched to matched.

Order independent $\mathcal{G}$-Outcome. *for each* $\{\pi, \pi'\} \subset \mathcal{O}$, $\mathcal{M}^{SPUE}(G[\pi, M]) = \mathcal{M}^{SPUE}(G[\pi', M])$.

Corollary 2. *If a market* $M = (P, Ch)$ *induces an order independent* $\mathcal{G}$*-outcome, then for each* $\pi \in \mathcal{O}$, $\mathcal{M}^{SPUE}(G[\pi, M]) = DA^{Ch}(P)$.

Appendix 4 contains the proof of Corollary 2. Next, we characterize markets that induce order independent $\mathcal{G}$- outcome. We first need a new concept. Given a rule φ and a market $M = (P, Ch)$, let $P^{\varphi} \in \mathcal{P}^S$ denote a profile defined in such a way that for each student s, $\varphi_s(P)$ is his unique acceptable

4. Hatfield and Milgrom (2005) partially obtained the rural hospital theorem on the set of stable matchings, when colleges have substitutable preferences coupled with another condition.

college, if any. Let $P^{\varphi}_{S'}$ denote the restriction of P^{φ} to S'. Let φ be a rule and S' a group of students. Suppose that we replace $P_{S'}$ by $P^{\varphi}_{S'}$. Then, we require that the rule selects the same matching. [5]

Claims consistency. *For each $S' \subseteq S$,*

$$\varphi(P^{\varphi}_{S'}, P_{-S'}) = \varphi(P).$$

Theorem 3. *A market $M = (P, Ch)$ induces an order independent $\mathcal{G}$-outcome if and only if DA^{Ch} is claims consistent at P.*

Appendix 5 contains the proof of Theorem 3. It is worth connecting claims consistency to the popular version. Let Ch be a choice profile. Let (S, P, Ch, q) be a market, $S' \subseteq S$ a group of students and q' a profile of capacities such that for each $c \in C$ $q'_c \leq q_c$. We call the market $(S', P_{S'}, Ch|_{2^{S'}}, q')$ a sub-market of (S, P, Ch, q) where $Ch|_{2^{S'}}$ is the restriction of Ch to $2^{S'}$. Next, the extended DA rule $\widetilde{DA}^{Ch}$ is defined as follows : for each $P \in \mathcal{P}^S$ and each $S' \subset S$, $\widetilde{DA}^{Ch}$ selects the DA outcome for $(S', P_{S'}, Ch|_{2^{S'}}, q')$.

Let $\mu \in \mathcal{M}$ and $S' \subset S$. We define the sub-market $(S', P_{S'}, Ch|_{2^{S'}}, q^{\mu})$ with respect to S' and μ as the one that results from a departure of the students in $S \setminus S'$ with their outcomes at μ ; that is, for each $c \in C$, $q^{\mu}_c = q_c - |\mu_c \setminus S'|$. Then, given a choice profile Ch and $\mathcal{P}'^S \subset \mathcal{P}^S$, DA^{Ch} is consistent on $\mathcal{P}'^S$ if for each $P \in \mathcal{P}'^S$ and each $S' \subseteq S$, letting $\mu = \widetilde{DA}^{Ch}(S, P, Ch, q)$, we have

$$\mu|_{S'} = \widetilde{DA}^{Ch}(S', P_{S'}, Ch|_{2^{S'}}, q^{\mu}).$$

The matching literature has focused on $\mathcal{P}^S$ and searched for choice profiles Ch for which DA^{Ch} is consistent on $\mathcal{P}^S$. The following condition for responsive preferences (Ergin, 2002) and for acceptant substitutable preferences (Kumano, 2009) is the answer. [6] A reversal in a choice profile Ch consists of three distinct students i, j, k and two distinct colleges c, c' such that there exist

5. Variants of consistency have been explored in the literature, ranging from consistency, bilateral consistency to average consistency (Thomson, 2011).

6. Let $\succ_c$ be c's preference relation over $S \cup \{\emptyset\}$. A preference relation $\bar{\succ}_c$ over 2^S is **responsive** to $\succ_c$, if for any subset S' of students with $|S'| < q_c$, $S' \cup \{s\} \bar{\succ}_c S' \cup \{s'\}$ if and only if $s \succ_c s'$ and $S' \cup \{s\} \bar{\succ}_c S'$ if and only if $s \succ_c \emptyset$.

two disjoint subsets $S_c, S_{c'} \subset S \setminus \{i,j,k\}$ verifying (C) $j \notin Ch_c(S_c \cup \{i,j\})$, $k \notin Ch_c(S_c \cup \{j,k\})$ and $i \notin Ch_{c'}(S_{c'} \cup \{i,k\})$ and (S) $|S_c|= q_c - 1$ and $|S_{c'}|= q_{c'} - 1$. A choice profile has **no reversal** if there is no reversal in it. Given a choice profile Ch, DA^{Ch} is consistent on $\mathcal{P}^S$ if and only if Ch has no reversal (Klijn, 2011).

Given a choice profile Ch that has no reversal and each $P \in \mathcal{P}^S$, DA^{Ch} is claims consistent at P.[7] However, claims consistency additionally characterizes markets (P, Ch) for which Ch may have a reversal.

Theorem 3 also characterizes markets for which the DA outcome is Pareto efficient. Together with Corollary 1 and Theorem 2, we can conclude that it is only on those markets that SPUE outcomes do not improve over the DA outcome. Otherwise, there are some SPUE outcomes that Pareto dominate the DA outcome. Furthermore, there is some SPUE outcome that is Pareto efficient.

Proposition 2. *For each market $M = (P, Ch)$, there is an order $\pi \in \mathcal{O}$ such that some $\mu \in \mathcal{M}^{SPE}(G[\pi, M])$ is Pareto efficient.*

Appendix 6 contains the proof of Proposition 2.

Given $\mu \in \mathcal{M}$, let $\mu_C = \{s \in S| \ \mu_s \neq \emptyset\}$ denote the set of students matched under μ. We combine Theorem 1 and Proposition 1 to derive the following corollary.

Corollary 3. *Let $M = (P, Ch)$.*

(i) For each $\pi \in \mathcal{O}$, each $\mu \in \mathcal{M}^{SPUE}(G[\pi, M])$ and each $v \in S \cup C$, we have $|\mu_v|= |DA_v^{Ch}(P)|$.

(ii) If $DA_C^{Ch}(P) = S$, then for each $\pi \in \mathcal{O}$, $\mathcal{M}^{SPUE}(G[\pi, M])$ is a singleton.

Appendix 7 contains the proof of Corollary 3. According to Corollary 3 (ii), for each market $M = (P, Ch)$ such that $DA_C^{Ch}(P) = S$ and each order π,

7. Since Ch has no reversal, for each $P \in \mathcal{P}^S$, $DA^{Ch}(P)$ is Pareto efficient (Kumano, 2009). Let $S' \subseteq S$ and $P' := (P_{S'}^{DA}, P_{-S'})$. It is proven (Kojima and Manea, 2010a) that $DA^{Ch}(P') \, R' \, DA^{Ch}(P)$. Thus, for each $s \in S'$, $DA_s^{Ch}(P') = DA_s^{Ch}(P)$. Therefore, if $DA^{Ch}(P') \neq DA^{Ch}(P)$ then, $DA^{Ch}(P')$ Pareto dominates $DA^{Ch}(P)$ at P.

$\mathcal{M}^{SPUE}(G[\pi, M])$ is a singleton. Each student can unambiguously compare his components in the outcomes of various orders. Then, we can focus on markets satisfying the hypothesis of (ii) and ask the following question : what are students' preferences regarding their orders ? We ask whether students prefer to play earlier or later.

Let $S' \subset S$ and $\{\pi, \pi'\} \subset \mathcal{O}$. Then, S' has the same relative ordering in π and π', which we denote $\pi|_{S'} = \pi'|_{S'}$, if for each $\{s, s'\} \subset S'$, s is ordered before s' in π if and only if s is ordered before s' in π'. Formally, for each $\{s, s'\} \subset S'$, $\pi^{-1}(s) < \pi^{-1}(s')$ if and only if $\pi'^{-1}(s) < \pi'^{-1}(s')$.

Proposition 3. *Let $M = (P, Ch)$ be such that $DA_C^{Ch}(P) = S$. Let $s \in S$ and $\{\pi, \pi'\} \subset \mathcal{O}$ be such that (i) $\pi|_{S \setminus \{s\}} = \pi'|_{S \setminus \{s\}}$ and (ii) $\pi'^{-1}(s) < \pi^{-1}(s)$. Then, s is at least as good in the SPUE of $G[\pi, M]$ as in the SPUE of $G[\pi', M]$.*

Appendix 8 contains the proof of Proposition 3.

1.4 Conclusion

Although matching markets are decentralized, the literature has privileged the study of centralized matching markets. However, empirical studies show that stability is a constraint for a successful design of centralized markets. This has the consequence that deferred acceptance (DA) is used in almost all well-known centralized markets. At the same time DA fails to achieve efficiency. We showed that decentralization can result in outcomes that Pareto improve over DA. This is, in our knowledge, the first contribution to obtain such a result in decentralized setting.

Chapitre 2

University Admissions Reform in France : An Analysis of a New Mechanism

2.1 Introduction

University admissions policies in France have been one of the most hotly debated issues over the last two years. At the centre of the debate is a nation-wide centralized admissions system that the Ministry of Higher Education (MHE) implemented in 2009 to simplify the process for students and to better match students to university school programs. This system replaced a decentralized admissions system that was producing ‘poor matches’. Under the previous system, universities did not use a uniform application process and did not coordinate their admission decisions. Since students had to prepare different applications for each school, most applied to a small number of schools. Moreover, students could potentially receive multiple offers which prompted many to hold on to current offers while waiting for better ones. Consequently, a considerable number of seats remained vacant even though some students never received an offer of admission.

As a solution, a task force of academic leaders commissioned the design of a centralized clearinghouse to coordinate admissions. It was implemented nationwide in 2009 and since then, nearly 800,000 students participate each year for the allocation of seats in nearly 12,000 programs.[1] In its June 1, 2016's memorandum to the student association « Droits des Lycéens », the MHE explained how the system prioritizes applicants at oversubscribed schools.[2] We model the French university admissions system as a school choice problem in which students are strategic agents and school seats are objects that come with priorities.

In this paper, we introduce a new mechanism for school choice motivated by the French university admissions system. We systematically compare it to current school choice mechanisms with respect to how it balances trade-offs among design goals : The new mechanism is a compromise between existing mechanisms. We also introduce a new notion of degree of manipulability to rank mechanisms.

Since school choice was established as a field of market design fifteen years ago, two mechanisms with opposite properties have dominated the debate. First, there is the old Boston (or immediate acceptance) mechanism which has received considerable attention because it is widely used in real-life. Broadly, this mechanism considers the first choices of students first ; and, up to capacity and following the priorities, immediately makes assignments. Then it looks at the second choices of students who didn't get their first choices and proceeds as before immediately filling the remaining vacancies and so on. Unfortunately, the immediate acceptance feature of this mechanism provides it with poor incentive properties. Clearly, it is in a student's best interest to avoid top-ranking a popular school in which he has low priority. A student who otherwise reports such preferences could miss out on his second choice simply because other students with lower priority ranked it first. If the student had ranked his second choice first, he would have been

1. All public universities in France but l'Univresité Paris Dauphine have joined the clearinghouse.

2. The document can be assessed at :
`http://www.youscribe.com/BookReader/Index/2734749/?documentId=2913196`.

more likely to be assigned a seat at that school. Evidently, the mechanism has a justified envy problem ; a student might prefer one school to his assignment and at the same time some of that school' seats may be assigned to students who have lower priority. An (individually rational and non-wasteful) matching that eliminates justified envy is said to be **stable**. [3]

The second mechanism that has been influential in school choice is Gale and Shapley (1962)'s famous Deferred Acceptance (DA) algorithm. It produces stable matchings which are student-optimal stable matchings. [4] In addition, this mechanism has a good incentive properties : No student has an interest in misrepresenting his preferences (Dubins and Freedman, 1981; Roth, 1982). A mechanism with this property is said to be **strategy-proof**. Not only does a strategy-proof mechanism elicit truthful preferences but it also simplifies the strategic aspect of reporting. Finally, a strategy-proof mechanism levels the playing field by preventing sophisticated students from taking advantage of strategically less sophisticated students (Pathak and Sönmez, 2008). Because of these properties, economists have argued for the use of DA over the Boston mechanism. [5]

However, one salient feature of real-world school choice lessens the theoretical appeal of DA. School priorities often exhibit indifferences (or ties) among students. The French system also has this feature. Indeed, there are selective schools which use academic files to strictly order students ; and non-selective schools which give priorities according to educational district but leave students within the same district tied. With coarse priorities, DA

3. A matching is wasteful if the capacity of some school is not fully assigned and some student prefers it to his assignment.

4. The outcome of DA is student-optimal when priorities are strict. Otherwise, the outcome of DA (with exogenous tie breaking) may be Pareto dominated by another stable matching.

5. A third mechanism that has been proposed is an adaptation of the top trading cycles algorithm to school choice. Students on the top of school priorities are granted such priorities which they can trade in cycles. This mechanism is strategy-proof but not stable. However, across many school districts in which this mechanism has been suggested, the public has rejected the idea that priorities, which are often given according to residence or sibling attendance, be used for trade.

performs poorly on the basis of welfare. Indeed, it is a common practice to randomly break ties and apply DA with the corresponding strict priorities. One problem is that the outcome of DA in this case may not be a student-optimal stable matching (Ergin and Erdil, 2008). What is more, this practice further entails significant welfare loss (Abdulkadiroğlu et al., 2009). This issue has motivated the literature to reconsider manipulable mechanisms (like the Boston mechanism). As a series of recent papers (Abdulkadiroğlu et al., 2011; Miralles, 2009; Featherstone and Niederle, 2016; Troyan, 2012) clearly point out, there are economic environments where each of the equilibrium outcomes of the Boston mechanism could Pareto dominate DA. The first of the cited papers, for instance, considers an environment where students have correlated preferences and schools have no priorities and shows that the equilibrium outcomes of the Boston mechanism Pareto improve upon the dominant strategy outcome of DA.[6] This raises the question of whether there is a third mechanism that could balance the incentive and welfare properties better than DA or the Boston mechanism.

The French system departs from random tie breaking by relying on students' reported preferences to break ties. The simplest rule consists of breaking ties so that students who rank a school higher receive priority at that school over students who don't.

In this paper, we show that by using an endogenous tie breaking rule one can improve the welfare of students over DA with random tie breaking. At the same time this new rule reduces the incentives for manipulation relative to the Boston mechanism. In particular, we study the set of stable matching mechanisms in which ties in school priorities are broken according to the French tie breaking rule. We refer to this stability concept as French priority stability or **FP-stability** for short. The drawback, as we show, is that FP-stable matching mechanisms are generally manipulable. We use a notion

6. This conclusion is in contrast with an earlier one from Ergin and Sönmez (2006) according to which, in a complete information environment, the dominant strategy outcome of DA weakly Pareto dominates each of the Nash equilibrium outcomes of the Boston mechanism.

introduced by Pathak and Sönmez (2013) to compare mechanisms according to their vulnerability to manipulation. A mechanism is **less manipulable** than another if any preference profile that is prone to manipulation under the first is also prone to manipulation under the second and the reverse does not hold.

In its October, 2017's report, the « Cour des comptes », which is the French national court of auditors, revealed that the clearinghouse has been implementing a school-optimal FP-stable matching mechanism. [7] In this paper, we study all FP-stable matching mechanisms and show that the student-optimal FP-stable matching mechanism is the most efficient and least manipulable mechanism in the FP-stable set. For this reason, we refer to it as the **French mechanism**. It is a unified mechanism from which a variety of school choice mechanisms – including the Boston mechanism, DA and two other mechanisms discussed in the literature – emerge as special cases.

The shortcoming of the French mechanism is its manipulability. In one forum, the following was posted to a thread discussing a student who was not assigned to a university spot :

« *(...) She should have placed the faculty of her district as her first choice.* »

One media source explains this issue in the following terms :

> « *Candidates must balance between strategy and preferences in ordering their choices. There is a chance of being admitted in a program if it is placed on top of the list.* »

We study the performance of the French mechanism from two different perspectives. From an ex-post perspective, students know all information about school priorities. In this environment, we show that the French mechanism is

7. The report is published after we accomplished the first version of this paper. It can be accessed in the following address :
`https://www.ccomptes.fr/sites/default/files/2017-10/20171019-rapport-admission-post-bac_0.pdf`.

less manipulable than the Boston mechanism. From ex-ante perspective, students do not know all information about priorities. In that setting, we consider an environment of correlated economies that generalizes the framework of Abdulkadiroğlu et al. (2011). We show that the Baysian Nash equilibrium outcomes of the French mechanism Pareto improve upon the dominant strategy outcome of DA. In contrast, the equilibrium outcomes of the Boston mechanism do not Pareto dominate DA nor do they systematically Pareto dominate the French mechanism. Correlated economies provide a good approximation of several school choice environments. For one, students often express similar preferences about schools based on criteria as job opportunities, academic reputation and safety. What is more, in real-world school choice, schools often use a general exam score and set thresholds and ranges for which students are perceived to be equivalent. This causes priorities to be coarse and correlated.

The French university admissions system has an additional concern related to selective schools. These are schools whose objective is to admit the most qualifying student body possible. These schools use various criteria including grades, letter of recommendations and student's class rankings to arrive at a strict ordering of students. Since their seats are assigned through a manipulable mechanism, it would be desirable if the strategic behavior did not jeopardize this goal. Ideally, admission to these schools should reflect student's qualifications instead of their strategic sophistication. Ideally, we would like to have a mechanism for which the set of students assigned to each selective school is insensitive to manipulation. Under the French tie breaking rule this requirement is not compatible with other design goals most noticeable FP-stability. Instead, we require that no student be able to receive a seat at a selective school through manipulation. We say that a school is **not strategically accessible** via a mechanism if no student can manipulate it and obtain a seat at that school. We prove that no selective school is strategically accessible via the French mechanism.

A natural question, that we have addressed, is how the presence of ties affects the incentive property of the French mechanism. We first show that by transforming some ties into strict relations does not entail less manipulable mechanisms : We cannot draw a conclusion from Pathak and Sönmez (2013)'s notion. We propose the following notion of relative accessibility. A mechanism is **less strategically accessible** than another if any school which is strategically accessible by some student via the first mechanism is also strategically accessible by the same student via the second and the reverse does not hold. We show that some transformations of ties into strict relations induce a French mechanism that is less strategically accessible than the French mechanism induced by the original priorities.

Related Literature

We are not aware of any existing paper that has studied the French university admissions system. However, a variety of other papers are similarly motivated by real-world school admissions reforms. In one paper, Chen and Kesten (2017) study school choice reforms in China and proved that in some provinces these reforms have led to less manipulable and more stable mechanisms (which they called parallel mechanisms). Broadly, they characterized a parametric family of mechanisms in which DA and the Boston mechanism emerge as extreme cases. In another paper, Doğan and Yenmez (2016) study the Chicago Board of Education decision to implement a new admissions procedure. They show that an alternative but unified enrolment system improves upon on student's welfare. Third, Westkamp (2013) study the sequential university admissions system used in Germany. He showed that a student-optimal procedural stable matching mechanism Pareto improve upon the German mechanism.

On the methodological front, the current paper is built on a notion introduced by Pathak and Sönmez (2013) and a model developed by Abdulkadiroğlu et al. (2011). The paper has a methodological contribution. Indeed, the idea of comparing mechanisms by their strategic assessibility is new in the matching literature. The paper is also related to Abdulkadiroğlu et al.

(2015) who augmented student's choices by adding a « targeting » variable that they used to break ties. They proved that expanding choice improves the welfare of students over DA with random tie breaking.

Finally, the current paper is part of a vast research program, school choice, dedicated to improving the practice of school admissions. To this end, one part of the literature discusses practical considerations regarding the design of school choice mechanisms (Abdulkadiroğlu et al., 2005, 2006; Pathak, 2016). A second part of the literature addresses concerns about the welfare loss associated with DA suggesting more efficient but nonetheless manipulable alternatives (Kesten, 2010; Ergin and Erdil, 2008). The third part of the literature developes an axiomatic approach to school choice mechanisms including DA (Kojima and Manea, 2010a; Morrill, 2013a; Ehlers and Klaus, 2014) and the Boston mechanism (Kojima and Ünver, 2014; Afacan, 2013; Doğan and Klaus, 2016).[8] Roth and Sotomayor (1990) provide a good treatment of matching markets more generally. Though not directed related to the current paper, there has also been a lot of recent work on ties in matching (Ehlers, 2014; Jaramillo and Manjunath, 2012; Bogomolnaia et al., 2005; Ehlers and Erdil, 2010).

It is very well-understood that not all school admissions design goals are compatible. Some goals should be traded off against others. A significant part of the literature is dedicated to clarifying the nature of these trade-offs. This paper is part of that tradition ; it does not, as it might appear, endorse the use of the French mechanism over DA or the Boston mechanism.

The remainder of the paper is organized as follows. In Section 2, we present the model and the three mechanisms of interest. In Section 3, we present the results. In Section 4, we conclude. We defer all proofs to the Appendix.

8. The top trading cycles mechanism has also been the object of axiomatic study (Abdulkadiroğlu et al., 2017; Morrill, 2013b).

2.2 School choice model and three mechanisms

We consider the **school choice** model introduced by Abdulkadiroğlu and Sönmez (2003).[9] There are a finite set of students and a finite set of university schools (schools for short). Each student demands one seat and each school has a maximum capacity. Each student has an outside option that we denote by $\emptyset$.

Formally, a school choice problem consists of :

1. a finite set $I = \{1, 2, \ldots, n\}$ of $n > 2$ students,
2. a finite set $S = \{a, b, c, \ldots\}$ of $m > 2$ schools,
3. a capacity vector $q = (q_a, q_b, q_c, \ldots)$,
4. a list of preference relations $R = (R_1, \ldots, R_n)$,
5. a list of basic priorities $\succeq = (\succeq_a, \succeq_b, \succeq_c, \ldots)$ and
6. a list of tiebreakers $\pi = (\pi_a, \pi_b, \pi_c, \ldots)$.

For each student i, R_i is a preference relation over $S \cup \{\emptyset\}$ which we assume to be strict.[10] For each student i and each preference relation R_i, let P_i denote the asymmetric part of R_i. Then, for each $i \in I$ and $a, b \in S$, $a\ R_i\ b$ if and only if $a\ P_i\ b$ or $a = b$. We say that school a is an **acceptable choice** for student i under R_i if $a\ P_i\ \emptyset$. Let $\mathcal{R}$ denote the set of preference relations and $\mathcal{R}^I$ the set of preference profiles $R = (R_i)_{i \in I}$. For each school a, $\succeq_a$ is a weak priority order over I.[11, 12] If two students i and j are in tie under $\succeq_a$, we write $i \sim_a j$. If student i has higher priority than student j under $\succeq_a$, we write $i \succ_a j$.[13] We say that school a has a **complete indifference priority** under $\succeq_a$ if for each $i, j \in I$, $i \sim_a j$. In that case, we denote the basic priority as $\succeq_a^{CI}$. Given a school a, a basic priority $\succeq_a$ and a $i \in I$, let

9. Balinski and Sönmez (1999) figure among the earlier literature that initiated the application of matching to school admissions problem.

10. A preference relation is strict if it is complete, transitive and antisymmetric.

11. A priority order is weak if it is complete and transitive.

12. We assume that each student is acceptable to each school.

13. Alternatively, the relation $\succeq_a$ can be viewed as a subset of $I \times I$ such that $(i, j) \in \succeq_a$ if and only if $i \succeq_a j$. Therefore, $i \sim_a j$ if and only if $(i, j) \in \succeq_a$ and $(j, i) \in \succeq_a$. And $i \succ_a j$ if and only if $(i, j) \in \succeq_a$ and $(j, i) \notin \succeq_a$.

$E^i_{\succeq_a} = \{j \in I | j \sim_a i\}$ be the indifference class that contains student i. Given $\succeq_a$ and a subset I' of students, let $\succeq_a|_{I'}$ denote the restriction of $\succeq_a$ to I'. If the basic priority $\succeq_a$ is strict, we say that school a is a **selective school**. Otherwise, we say that school a is **non-selective**. For each school a, π_a is a strict order over I. We call a pair $\tau = (\succeq, \pi)$ consisting of a basic priority profile $\succeq$ and a tiebreaker profile π a priority profile. Given a priority profile $(\succeq, \pi)$ and a school a, let $\succeq_{\pi_a}$ denote the strict priority order that is obtained from $\succeq_a$ by breaking all ties in $\succeq_a$ according to π_a. In the remainder of the paper, we refer to a pair $(R, \succeq)$ of a preference profile and a basic priority profile as an **economy**. We say that an economy $(R, \succeq)$ is **correlated** if students have the same ordinal preferences and schools have the same basic priorities, that is, for each $i, j \in I$, $R_i = R_j$ and for each $a, b \in S$, $\succeq_a = \succeq_b$. We say that students have **correlated preferences** under the preference profile R if for each $i, j \in I$, $R_i = R_j$.

A **matching** is a function $\mu : I \to S \cup \{\emptyset\}$ such that for each school a, $|\mu^{-1}(a)| \leq q_a$. Let $\mathcal{M}$ denote the set of matchings. A **mechanism** is a function $\varphi : \mathcal{R}^I \to \mathcal{M}$ that assigns to each preference profile R a matching $\varphi(R)$. We denote by $\varphi_i(R)$ the component for student i. We now require desirable behaviors of a mechanism.

For a mechanism to attract much participants, it should promote voluntary participation. We say that a matching μ is **individually rational** under R if for each student i, we have $\mu(i)\ R_i\ \emptyset$. A mechanism φ is **individually rational** if for each preference profile R, $\varphi(R)$ is individually rational. Given an economie $(R, \succeq)$ in which $\succeq$ is a strict basic priority profile, we say that a student-school pair (i, a) is a blocking pair for μ under $(R, \succeq)$ if $a\ P_i\ \mu(i)$ and either $|\mu^{-1}(a)| < q_a$ or for some student $j \in \mu^{-1}(a)$, $i \succ_a j$. We say that a matching μ is **stable** under $(R, \succeq)$ if μ is individually rational and there is no blocking pair for it under $(R, \succeq)$. In school choice, school seats are merely objects to be assigned to students. Therefore, only student preferences need to be considered for welfare evaluations. A matching μ is (Pareto) **efficient** under R if there is no other matching μ' such that for each student i, $\mu'(i)\ R_i\ \mu(i)$ and for some student j, $\mu'(j)\ P_j\ \mu(j)$. Otherwise, we say that the matching μ' **Pareto dominates** the matching μ. A mechanism φ is **efficient** if for each preference profile R, $\varphi(R)$ is efficient. A mechanism φ is **strategy-proof** if for each preference profile R and each student i, there is no preference relation R'_i such that $\varphi_i(R'_i, R_{-i})\ P_i\ \varphi_i(R)$. Many mechanisms used in real-world are not strategy-proof. However, some of them are less prone to manipulation than others. Pathak and Sönmez (2013) introduced a notion to compare mechanisms by their vulnerability to manipulation. We say that a preference profile R is **vulnerable** under mechanism φ if there is a student i and an alternative preference relation R'_i such that $\varphi_i(R'_i, R_{-i})\ P_i\ \varphi_i(R)$. In that case, we say that student i **manipulates** φ at R via R'_i and that mechanism φ is **manipulable** under R. A mechanism ϕ is as manipulable as mechanism φ if whenever a preference profile R is vulnerable under φ, then it is also vulnerable under ϕ. A mechanism φ is **less manipulable** than a mechanism ϕ if ϕ is as manipulable as φ and there is at least one preference profile which is vulnerable under ϕ but not under φ. Two mechanisms are **equally manipulable** if the same set of preference profiles are vulnerable under both mechanisms. A mechanism φ is **strongly less manipulable** than a mechanism ϕ if φ is less manipulable than ϕ and for each preference profile R, any student who can manipulate φ under R can also manipulate ϕ under R.[14]

In the previous incentive notions, there is no indication to the set of schools towards which the strategic behavior is oriented nor the set of students who have those incentives. It might be desirable that some schools be spared from the strategic behavior. In our application, selective schools are such schools. We say that school a is strategically accessible by student i via mechanism φ if student i can manipulate φ and obtain a seat at school a,

14. If a mechanism φ is strongly less manipulable than a mechanism ϕ then, for sure φ is less manipulable than ϕ and in addition, at any preference profile R where a given student i can manipulate φ, student i can also manipulate ϕ at R.

that is, there is a preference profile R and a preference relation R_i' such that $\varphi_i(R_i', R_{-i}) = a$ and $a \; P_i \; \varphi_i(R)$. We say that school a is **not strategically accessible** via mechanism φ if that school is not strategically accessible by any student via φ. As stated in the following lemma an individually rational mechanism is strategy-proof if no school is strategically accessible.

Lemma 1. *Let φ be an individually rational mechanism. Then, if no school is strategically accessible via φ then φ is strategy-proof.*[15]

For each school a, let I_a^{φ} be the set of students for whom school a is not strategically accessible via φ. We consider a mechanism ϕ to be as strategically accessible as mechanism φ if any school which is not strategically accessible via ϕ by some student is also not strategically accessible via φ by this student. That is, for each school a, $I_a^{\varphi} \supseteq I_a^{\phi}$. We consider a mechanism φ to be **less strategically accessible** than mechanism ϕ if ϕ is as strategically accessible as φ and there is a school a which is not strategically accessible via φ by some student but strategically accessible via ϕ by this student. That is, for each school a, $I_a^{\varphi} \supseteq I_a^{\phi}$ and for some school b, $I_b^{\varphi} \supsetneq I_b^{\phi}$. We say that school a is strategically accessible by student i via mechanism φ under R if there is a preference relation R_i' such that $a = \varphi_i(R_i', R_{-i}) \; P_i \; \varphi_i(R)$. A mechanism φ is **strongly less strategically accessible** than a mechanism ϕ if for each preference profile R each school which is strategically accessible by a student via φ under R is also strategically accessible by the same student via ϕ under R and there is a preference profile under which some school is not strategically accessible by some student via φ but is strategically accessible by this student. Two mechanisms are **equally strategically accessible** if each school is strategically accessible by the same set of students via both mechanisms.

It is important to note the following distinction in our definition of strategic accessibility. Suppose that school a is strategically accessible via me-

15. For the proof, let φ be an individually rational mechanism and suppose that φ is manipulable by student i. Then, there is R and R_i' such that $\varphi_i(R_i', R_{-i}) \; P_i \; \varphi_i(R)$. Because φ is individually rational, then for some school a, $\varphi_i(R_i', R_{-i}) = a$. Therefore, school a is strategically accessible.

chanism φ by student i. We do not require that in at least one instance in which he can manipulate and obtain a seat at school a, school a be the best school that student i can obtain via a manipulation although it is plausible to suppose that when such strategic opportunity is available, he will instead manipulate and obtain a seat at the school that he finds best. However, we will show that under the French mechanism, if a school is strategically accessible by some student, then it is strategically accessible in an instance in which that school is the best that the student in question can manipulate and obtain a seat at.

2.2.1 Deferred Acceptance mechanism

The Deferred Acceptance (DA) mechanism is suggested by Gale and Shapley (1962) and adapted for school choice by Abdulkadiroğlu and Sönmez (2003). In the current context of weak priorities, it consists of exogenously breaking ties according to the tiebreakers and simulating rounds of applications and rejections of its standard algorithm. Let $\tau = (\succeq, \pi)$ be a priority profile and R a preference profile. Recall that $\succeq_\pi$ is a strict priority profile obtained from $\succeq$ by breaking ties according to π. The steps of the algorithm are as follows :

Step 1 : Each student applies to his first acceptable choice (if any). Each school a rejects the lowest ranked applicants under $\succeq_{\pi_a}$ who are in excess of its capacity and provisionally keeps its remaining applicants.

In general, at

Step ℓ, $\ell > 1$: Each student who is rejected at Step $\ell - 1$ applies to his next acceptable choice (if any). Each school a considers its new applicants together with its applicants who were held at Step $\ell - 1$ and rejects the lowest ranked under $\succeq_{\pi_a}$ who are beyond its capacity and provisionally keeps the remaining applicants.

The algorithm terminates whenever each student is either held or has applied to all his acceptable choices. The outcome associated with this process is the matching in which each student who is accepted at the terminal step is matched to the corresponding school and the others remain unmatched. Let $DA^{\succeq_\pi}(R)$ denote this matching. Given the priority profile $\tau = (\succeq, \pi)$, let DA^τ denote the mechanism that assigns to each preference profile R the matching $DA^{\succeq_\pi}(R)$. Let $\succeq$ and $\succeq'$ be two strict basic priority profiles and R a preference profile. Let r_1 and r_2 be the number of steps that the deferred acceptance producing $DA^{\succeq}(R)$ and $DA^{\succeq'}(R)$ terminate, respectively. Let $r \leq min\{r_1, r_2\}$. We say that the algorithms that lead to $DA^{\succeq}(R)$ and $DA^{\succeq'}(R)$ have the same steps until r if at each Step $\ell \leq r$ of both algorithms, the same applications and the same rejections are made.

2.2.2 Boston mechanism

Prior to the 2005-2006 school year, the Boston public schools were using a mechanism that simulates many rounds of immediate acceptances applications-rejections mechanism. In fact, it is a popular school choice mechanism today. We next describe it formally. Given a priority profile $\tau = (\succeq, \pi)$ and a preference profile R, it is executed as follows :

Step 1 : Each student applies to his first acceptable choice (if any). Each school a rejects the lowest ranked applicants under $\succeq_{\pi_a}$ who are in excess of its capacity and immediately accepts the remaining applicants. Each school a reduces its capacity by the number of students it accepted. Let q_a^1 be the number of seats that remain after the current step.

In general, at

Step ℓ, $\ell > 1$: Each student who is rejected at Step $\ell-1$ applies to his ℓ's acceptable choice (if any). Each school a consider its new applicants, rejects the lowest ranked ones under $\succeq_{\pi_a}$ who are in excess of its capacity that remains after Step $\ell - 1$ and immediately accepts the remaining applicants. Each school a reduces its capacity that remains after Step $\ell - 1$ by the number of students it accepted. Let q_a^ℓ be the number of seats that remain after the current step.

This procedure terminates when each student is either accepted or has applied to all of his acceptable choices. In the later case he remains unassigned. Let BM^{τ} denote the mechanism that assigns to each preference profile R the outcome obtained at the end of this procedure when the priority profile is τ. We refer to this mechanism as **Boston mechanism**.[16]

Note that at each step of the Boston mechanism, acceptances are immediate. Once a school accepts an applicantion, it no longer considers the possibility of rejecting him in later steps. There are other variants of the Boston mechanism.

(a) First Preference First mechanism

More than 50 school districts in England had been using a school admissions mechanism known as **First Preference First** (FPF) until 2007 (Pathak and Sönmez, 2013). The set of schools is partitioned into two sets. One set is referred to as **equal preference schools** and the other as **first preference first schools**. Given a priority profile $\tau = (\succeq, \pi)$, the outcome of the FPF mechanism indexed by τ, which we denote by FPF^{τ}, for the preference profile R is the application of DA with R and the following strict priorities :

- Each equal preference school a uses the strict priority $\succeq_{\pi_a}$,
- Each first preference first school a uses the following strict priority : Any student who ranks school a at a certain position has higher priority than any student who ranks that school lower than that position. Among students who rank school a the same, the priority is determined according to $\succeq_{\pi_a}$.

Remark. *The Boston mechanism is a special case of the first preference first mechanism when each school is a first preference first school ; and, DA a special case when each school is an equal preference school.*

16. This mechanism is also called immediate acceptance mechanism to stress that, in contrast to DA, acceptances at each step are immediate.

(b) Secure Boston mechanism

In studying the Boston mechanism, Dur et al. (2016) were concerned with students loosing priority at schools where they have guaranteed seats. Student i has a guaranteed seat at school a with basic priority order $\succeq_a$ if there are at most q_a students (including student i) who either have higher priority than student i or are in tie with student i under $\succeq_a$.[17] Let $G_{\succeq_a}$ denote the set of students who have guaranteed seats at school a under $\succeq_a$. Note that this set may be empty. They propose a mechanism in which no student looses his priority at a guaranteed school. They refer to this mechanism as **secure Boston** mechanism that we describe. Let $\tau = (\succeq, \pi)$ be a priority profile. For each preference profile R, the outcome that the mechanism selects for it is the application of DA for R and the following modified priority profile $\widehat{\succeq}$: For each school a and each $i, j \in I$:

— if $i \in G_{\succeq_a}$ and $i \succ_{\pi_a} j$, then $i \mathrel{\widehat{\succ}_a} j$,

— if $i, j \notin G_{\succeq_a}$, then

- if student i has ranked school a higher than student j under R, then $i \mathrel{\widehat{\succ}_a} j$,
- if student i and j have ranked school a the same and $i \succ_{\pi_a} j$, then $i \mathrel{\widehat{\succ}_a} j$.

Let sBM^{τ} denote the secure Boston mechanism induced by $\tau = (\succeq, \pi)$. Then, for each preference profile R, $sBM^{\tau}(R) = DA^{\widehat{\succeq}}(R)$.

Remark. ***Top trading cycles mechanism***

A third mechanism which is interesting mentioning is the so-called top trading cycles mechanism. It is implemented via the following algorithm : Let a preference profile R and a priority profile $\tau = (\succeq, \pi)$ be given.

Step 1 : Each student who finds no school acceptable is removed from consideration. Each of the remaining students points to his most preferred school. Each school a points to its highest ordered student under $\succeq_{\pi_a}$. There is at least one cycle because the set of students is finite as

17. A student i has guaranteed seat at school a under $\succeq_a$ if $|\{j \in I | j \succeq_a i\}| \leq q_a$.

well as the set of schools. For each of these cycles, each student is assigned to his first choice. The students in each cycle are removed from consideration and the capacity of each school in that cycle is reduced by one.

In general, at

Step ℓ, $\ell > 1$: Each school which has no remaining seat is removed from consideration. Each student who finds no remaining school acceptable is removed from consideration. Each of the remaining students points to his most preferred remaining school. Each remaining school a points to its highest ordered remaining student under $\succeq_{\pi_a}$. Because the set of remaining students is finite as well as the set of remaining schools, there is at least one cycle. For each of these cycles, each student is assigned to the school that he has pointed to. The students in each cycle are removed from consideration and the capacity of each school in that cycle is reduced by one.

The algorithm terminates when either all schools have been removed from consideration or all students have been removed from consideration.

Observe that DA exogenously breaks ties according to the tiebreaker profile and simulates the rounds of applications-rejections. Another alternative is to break ties endogenously, that is, based on student preferences. The mechanism that we describe next adopts this approach.

2.2.3 French mechanism

In the French university admissions system, there are selective schools which strictly order students based on their academic performances and non-selective schools which define priority based on student's educational district. France is divided into 30 educational districts. For each non-selective school, each student who belongs to the same division as the school has higher priority over each student who is outside that district. But students in the district remain in tie. Similarly, students outside the district remain in tie. We describe next how the French system breaks ties.

A. French tie breaking rule

To begin, the set S of schools is partitioned into T categories. Let $\{\mathcal{S}_1, \ldots, \mathcal{S}_T\}$ be such a partition. Given a preference profile R, we say that school a is student i's ℓ'th **absolute** choice under R_i if there are exactly ℓ schools that he finds at least as good as school a under R_i.[18] Given a school a in category t, we say that school a is student i's ℓ'th **relative** choice under R_i if there are exactly ℓ schools in $\mathcal{S}_t$ that he finds at least as good as school a under R_i, that is, there are exactly ℓ schools that he finds at least as good as school a under the restriction $R_i|_{\mathcal{S}_t}$.[19]

We first sketch a literal description of the French tie breaking.

— **First priority** : For any tie, the student who relatively ranks the school at a certain position has higher priority over the student who relatively ranks it lower than that position.

— **Second priority** : Among students in ties who relatively rank the school the same, the student who absolutely ranks the school at a certain position has higher priority over the student who absolutely ranks it lower than that position.

— **Third priority** : Among students in ties who rank the school relatively and absolutely the same, the student who is ordered higher in the tiebreaker of the school has higher priority over the student who is ordered lower.

We propose a simple way of formalizing this tie breaking procedure. Given a preference profile and a tiebreaker profile, we construct a strict ordering of all students which is served to break ties.

Let a preference profile R and a tiebreaker profile π be given. Given a school $a \in \mathcal{S}_t$, we construct two weak (possibly strict) orderings of students, one based on how students relatively rank school a under R and the other based on how they absolutely rank school a under R.

18. That is, $|\{b \in S | b\ R_i\ a\}| = \ell$.
19. That is, if $a \in \mathcal{S}_t$, then we have $|\{b \in S |\ b\ R_i|_{\mathcal{S}_t}\ a\}| = \ell$.

Ordering according to relative rankings : Let $\succeq_a^B$ be a weak (possibly strict) ordering of students such that for each $i, j \in I$,

— if student i relatively ranked school a higher than did student j under R, then $i \succ_a^B j$ and

— if student i and student j have relatively ranked school a the same under R, then $i \sim_a^B j$.

In the following representation, we identify each position, in the preference profile, where school a is relatively ranked by at least one student and collect all students who ranked it at that position in the same indifference class.

$$\begin{array}{ccccccc}
 & R_1|_{\mathcal{S}_t} & \dots & R_n|_{\mathcal{S}_t} & \text{school } a \text{ ranked} & \underline{\quad \succeq_a^B \quad} \\
1 & . & \dots & . & & \\
2 & a & \dots & . & \longleftarrow \square \longrightarrow & \{1, \dots, \} \\
\vdots & \vdots & \dots & \vdots & & \\
 & . & \dots & . & & \vdots \\
|\mathcal{S}_t| & . & \dots & a & \longleftarrow \square \longrightarrow & \{, \dots, n\}
\end{array}$$

Ordering according to absolute rankings : Let $\succeq_a^A$ be a weak (possibly strict) ordering of students such that for each $i, j \in I$,

— if student i absolutely ranked school a higher than did student j under R, then $i \succ_a^A j$ and

— if student i and student j have absolutely ranked school a the same under R, then $i \sim_a^A j$.

We construct a similar representation as before.

$$\begin{array}{ccccccc}
 & R_1 & \dots & R_n & \text{school } a \text{ ranked} & \underline{\quad \succeq_a^A \quad} \\
1 & . & \dots & . & & \\
2 & a & \dots & . & \longleftarrow \square \longrightarrow & \{1, \dots, \} \\
\vdots & \vdots & \dots & \vdots & & \\
 & . & \dots & . & & \vdots \\
m & . & \dots & a & \longleftarrow \square \longrightarrow & \{, \dots, n\}
\end{array}$$

Let $\widehat{\pi}_a$ be a strict ordering of students constructed as follows :

— first, the ties in $\succeq_a^A$ are broken according to π_a. Let $\succeq_{\pi_a}^A$ be the strict ordering obtained and

— $\widehat{\pi}_a$ is the strict ordering obtained when ties in $\succeq_a^B$ are broken according to $\succeq_{\pi_a}^A$.

The ordering $\widehat{\pi}_a$ is used to break ties in any basic priority of school a. For each basic priority $\succeq_a$ of school a, let $\succeq_{\widehat{\pi}_a}$ be the strict priority obtained from $\succeq_a$ by breaking ties in $\succeq_a$ according to $\widehat{\pi}_a$. Let $\succeq_{\widehat{\pi}}= (\succeq_{\widehat{\pi}_a})_{a\in S}$.

We define a function f from the set of problems to the set of strict basic priority profiles such that for each problem $(R, \succeq, \pi)$, $f(R, \succeq, \pi) = \overline{\succeq}$ in which $\overline{\succeq} =\succeq_{\widehat{\pi}}$ and we have (i) for each school a, $\widehat{\pi}_a$ is the strict ordering obtained as above with the preference profile R and the tiebreaker π_a and (ii) $\succeq_{\widehat{\pi}}$ is obtained from $\succeq$ by breaking ties according to $\widehat{\pi}$.

Remark.

(i) In constructing the French tie breaking for a problem $(R, \succeq, \pi)$, we consider the ranking of unacceptable schools under R as if they were acceptable. This is required to obtain well-defined priority orders because we define a priority order of a school as a weak order of students in which all students are acceptable. This is without loss of generality because the mechanisms that we consider are all individually rational.

(ii) Each student has a first relative choice for each category. Therefore, in contrast to absolute choices, a student has more than one (exactly T) first relative choices. He may also have, more than one second relative choices. For each of the categories that contain more than one school, he has a second relative choice.

Observe that when there is only one school per category or there is only one category, the French tie breaking rule takes the following simplified version. Let a priority profile $(\succeq, \pi)$ and a preference profile R be given. For each school a and each indifference class $E_{\succeq_a}$ and two students $i, j \in E_{\succeq_a}$, student i has higher priority than student j under the French priority for school a if and only if either student i has ranked school a higher than student j or both students have ranked it the same and student i is ordered higher than student j under π_a.

B. French priority stability

After breaking ties according to the French rule, it is natural to study mechanisms that do not violate the strict priorities obtained.

Let $\tau = (\succeq, \pi)$ be a priority profile and R a preference profile. Let $\overline{\succeq} = f(R, \succeq, \pi)$ be the strict priority profile obtained by breaking ties in $\succeq$ according to the French rule. We say that a matching μ is French priority stable or **FP-stable** for short under (R, τ) if μ is stable under $(R, \overline{\succeq})$. We say that a mechanism φ is **τ-FP-stable** if for each preference profile R, $\varphi(R)$ is FP-stable under (R, τ).

Proposition 4. *(Gale and Shapley, 1962) For each priority profile τ and each preference profile R, there is an FP-stable matching under (R, τ) that each student finds at least as good as any other FP-stable matching under (R, τ).*

We refer to the matching identified in this proposition student-optimal FP-stable matching and the mechanism that selects it for each preference profile, student-optimal FP-stable matching mechanism.

Remark. *An alternative procedure would be to apply the top trading cycles algorithm after the French tie breaking. Obviously this procedure will violate FP-stability.*

C. French mechanism : Simplified and generalized

We refer to the student-optimal FP-stable matching mechanism as the **French mechanism**.[20] However, we consider two versions. Let $\tau = (\succeq, \pi)$ be a priority profile.

Simplified French mechanism : Suppose that there is one school per category or there is only one category. In that case, for each priority profile τ, we refer to the French mechanism as **simplified French mechanism** that we denote by FM^{τ}.

Generalized French mechanism : Suppose that there is at least two categories and that each category contains more than one school. In that case, for each priority profile τ, we refer to the French mechanism as **generalized French mechanism** that we denote by gFM^{τ}.

For the rest of the paper, except Section 3.3, we focus on simplified French mechanisms for tractability consideration. But many results hold for the generalized case.

D. The French mechanism unifies school choice mechanisms

We say that a mechanism φ is a French mechanism if there is a priority profile τ such that $\varphi = FM^{\tau}$. We show that the mechanisms that we have previously described are French mechanisms.

DA is a French mechanism : Let a priority profile $\tau = (\succeq, \pi)$ be given. The DA mechanism DA^{τ} is the French mechanism where the basic priority profile is $\succeq_{\pi}$ and the tiebreaker profile is π. Since there is no tie in $\succeq_{\pi}$, we have $FM^{(\succeq_{\pi}, \pi)} = DA^{(\succeq_{\pi}, \pi)}$. Therefore, for any priority profile $\tau = (\succeq, \pi)$, we

20. This mechanism has superior properties over any other FP-stable matching mechanism according to many respects. A second rational for singling out this mechanism is that it has been a reference in school choice.

have

$$DA^{\tau} = FM^{(\succeq_{\pi},\pi)}.$$

Boston is a French mechanism : Let a priority profile $\tau = (\succeq, \pi)$ be given. The Boston mechanism BM^{τ} is the French mechanism where the basic priorities are the complete indifference priority profile $\succeq^{CI}$ but the tiebreaker profile is $\succeq_{\pi}$. First, the following lemma shows that $FM^{(\succeq^{CI},\succeq_{\pi})} = BM^{(\succeq^{CI},\succeq_{\pi})}$.

Lemma 2. *When schools have complete indifference priorities, the French mechanism is equivalent to the Boston mechanism.*

The proof appears in the Appendix 10. Now, because $BM^{(\succeq,\pi)} = BM^{(\succeq^{CI},\succeq_{\pi})}$, for each priority profile $\tau = (\succeq, \pi)$, we have :

$$BM^{\tau} = FM^{(\succeq^{CI},\succeq_{\pi})}.$$

First Preference First is a French mechanism : Let a priority profile $\tau = (\succeq, \pi)$ be given. The first preference first mechanism FPF^{τ} is a French mechanism where each equal preference school a has the basic priority $\succeq_{\pi_a}$ and each first preference first school a has a complete indifference basic priority $\succeq_a^{CI}$. Let $\widehat{\succeq}$ denote such a basic priority profile and $\overline{\succeq} = f(R, \widehat{\succeq}, \pi)$. In that case, the priorities constructed for each school a in the first preference first mechanism coincides with $\overline{\succeq}_a$. We have $FPF^{(\succeq,\pi)}(R) = DA^{\overline{\succeq}}(R)$. Therefore, for each priority profile $\tau = (\succeq, \pi)$,

$$FPF^{\tau} = FM^{(\widehat{\succeq},\pi)}.$$

Secure Boston is a French mechanism : The secure Boston mechanism suggested by Dur et al. (2016) also turns out to be a French mechanism. Let a priority profile $\tau = (\succeq, \pi)$ be given. For each school a, let q_a^g be the number of students who have guaranteed seats at that school. Let $\succeq^{IatB}$ be the basic priority profile in which for each school a, ties occur only at the bottom and the strict part concerns the q_a^g students who have guaranteed seat at school a and ordered according to $\succeq_{\pi_a}$.[21]

Example 3. ***Illustrating IatB basic priority***

Let $(\succeq, \pi)$ *be priority profile such that students in* $\{1, \ldots, q_a^g\}$ *have guaranteed seat at school* a *and* $\succeq_{\pi_a}: 1 \ldots q_a^g \ldots$. *Then, we have :*

$\succeq_a^{IatB}$
1
$\vdots$
q_a^g
$q_a^g + 1, \ldots, m.$

Then, for each priority profile $\tau = (\succeq, \pi)$, we have the following equivalence :

$$sBM^\tau = FM^{(\succeq^{IatB}, \pi)}.$$

The following table summarizes these equivalences.

21. Ehlers and Westkamp (2011) studied a more general but similar class of priority orders and refer to them as indifference at the bottom priority structures. The only difference is that the strict part can be any subset of students.

Table 2.1 – French mechanism unifying school choice mechanisms

French mechanism/ version	Equivalent mechanisms	Priority*
simplified	DA	A
simplified	Boston	B
simplified	First Preference First	A or B
simplified	Secure Boston	C

(*) Indicates the modified priority of each school under FM :
A : strict priority
B : complete indifference priority
C : IatB priority.

2.3 Results

We study the French mechanism according to two different perspectives. In ex-post perspective students know the exogenous tiebreakers. In ex-ante perspective, they do not know this information.

2.3.1 Ex-post perspective

The classic modeling choice in school choice is the complete information environment. We maintain the same modeling choice and study all FP-stable matching mechanisms.

We know from the previous section that under strict basic priorities the French mechanism and DA are equivalent. It is well-known that in this case, DA may not be efficient. Ergin (2002) characterizes the class of priorities that induce efficient DA. Therefore, the French mechanism may not be efficient for some strict priorities. It is not also difficult to construct examples of weak basic priorities for which any FP-stable matching mechanism is not efficient under the reported preferences.

For strict basic priorities the French mechanism, equivalent to DA, is strategy-proof while any other FP-stable matching mechanism is not. This result places the French mechanism at the frontier of least manipulable mechanisms in the FP-stable matching mechanisms. However, if there are some ties in the basic priorities, the French mechanism may not be strategy-proof.

Example 4. ***Manipulability of the French mechanism***

Consider an example of four students and three schools a, b *and* c *where each school has a capacity of one but school* c *has a capacity of two. Let a priority profile* $\tau = (\succsim, \pi)$ *such that* $\pi_a = \pi_b = \pi_c$ *and a preference profile* R *be specified as below. Let* $\overline{\succsim} = f(R, \succsim, \pi)$.

$\succsim_a$	$\succsim_b$	$\succsim_c$	π_a
$1,2$	$1,2$	$3,4$	1
$3,4$	$3,4$	$1,2$	2
			3
			4

R_1	R_2	R_3	R_4
c	a	c	c
a	b	a	b
b	c	b	a

$\overline{\succsim}_a$	$\overline{\succsim}_b$	$\overline{\succsim}_c$
2	2	3
1	1	4
3	4	1
4	3	2

We have

$$FM^{\tau}(R) = DA^{\overline{\succsim}}(R) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ b & a & c & c \end{pmatrix}.$$

Student 1 *is assigned to his third choice school. Suppose now that he reports the preference relation* R_1^a *in which he ranks school* a *first. Let* $R^a = (R_1^a, R_{-1})$ *and* $\overline{\succsim}' = f(R^a, \succsim, \pi)$. *Because* $\overline{\succsim}'_a : 1\,2\,3\,4$, *we have* $FM_1^{\tau}(R^a) = a$. *Since student* 1 *prefers school* a *to school* b, *he manipulates* FM^{τ} *at* R *via* R_1^a.

The fact that the French mechanism is the least manipulable mechanism among FP-stable matching mechanisms for strict basic priorities, can be extended, with no surprise, to any weak basic priority.

Proposition 5. *For any priority profile* τ,

(i) the French mechanism FM^{τ} *is equally or less manipulable than any* τ*-FP-stable matching mechanism and*

(ii) the French mechanism FM^τ is equally or strongly less manipulable than the school optimal τ-FP-stable matching mechanism.

The proof of part (i) appears in Appendix 11 and part (ii) in Appendix 13. Because the French mechanism is manipulable for some priorities, this proposition has the implication that it is not possible to design FP-stable matching mechanisms which are strategy-proof.

Corollary 4. *There is no class of mechanisms $(\varphi^\tau)_\tau$ indexed by priority profiles such that for each priority profile τ, φ^τ is strategy-proof and τ-FP-stable.*

We continue by establishing that any agent can replicate his truthful outcome via a simple strategy.

Lemma 3. *Let τ be a priority profile, R a preference profile and R_i^a a preference relation in which student i ranks school a first. If $FM_i^\tau(R) = a$, then $FM_i^\tau(R_i^a, R_{-i}) = a$.*

The proof appears in Appendix 12. While the French mechanism is manipulable, there is a limit to successfully manipulating it. In any instance, it is potentially manipulable by only students who obtained less than their second choice schools.

Lemma 4. *No student can manipulate the French mechanism or the Boston mechanism to obtain a seat at his true first choice.*

This lemma is actually a corollary of a forthcoming proposition (Proposition 9). We defer its proof to the corollary in question. Turning to strategic accessibility, the French mechanism also remains the least strategically accessible mechanism among FP-stable matching mechanisms.

Proposition 6. *For any priority profile τ,*

(i) the French mechanism FM^τ is equally or less strategically accessible than any τ-FP-stable matching mechanism and

(ii) the French mechanism FM^τ is equally or strongly less strategically accessible than the school-optimal τ-FP-stable matching mechanism.

The proof appears in Appendix 13. We next provide an example in which the French mechanism is strongly less manipulable and less strategically accessible than the school optimal FP-stable matching mechanism.

Example 5. *Consider an example of three students and three schools and let a priority profile $\tau = (\succeq, \pi)$ such that $\pi_a = \pi_b = \pi_c : 1\,2\,3$ and a preference profile R be specified as below.*

$\succeq_a$	$\succeq_b$	$\succeq_c$
3	1	2
1, 2	2, 3	1, 3

R_1	R_2	R_3
a	b	c
b	a	b
c	c	a

Then the outcomes for R of the French FM^τ and the school optimal τ-FP-stable matching mechanism are respectively

$$\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 \\ b & c & a \end{pmatrix}.$$

Therefore, the French mechanism is not manipulable under R while every student can manipulate the school-optimal version. No school is strategically accessible under R via the French mechanism. But every school is strategically accessible by at least one student via the school-optimal τ-FP-stable matching mechanism.

We next compare the French mechanism to known school choice mechanisms. First, DA is evidently less manipulable than the French mechanism (or both mechansims are equally manipulable). But compared to other manipulable mechanisms like the Boston mechanism, the French mechanism performs better. For some priority profiles, the Boston mechanism is strategy-proof. In those cases, DA and Boston are equivalent (Kumano, 2013).

Theorem 4. *Let τ be a priority profile in which the basic priority of each equal preference school is strict. Then the French mechanism FM^τ is equally or less manipulable than the first preference first mechanism FPF^τ.*

The proof of the theorem appears in Appendix 14. Since the Boston mechanism is a first preference first mechanism in which every school is a first preference first school, we have the following corollary.

Corollary 5. *For each priority profile τ, the French mechanism FM^τ is equally or less manipulable than the Boston mechanism BM^τ.*

In the following example, the Boston mechanism is manipulable but the French mechanism is not. Let $\tau = (\succeq, \pi)$ be such that $\pi_a = \pi_b = \pi_c : 1\,2\,3$ and the basic priority profile is specified as follows :

$\succeq_a$	$\succeq_b$	$\succeq_c$
1, 2	3	2, 3
3	1, 2	1

Let a preference profile R be such that for each $i = 1, 2, 3$, $R_i : a\,b\,c$. Then $FM^\tau_1(R) = a$, $FM^\tau_2(R) = c$ and $FM^\tau_3(R) = b$. Both student 1 obtained his first choice and student 3 his first choice without object a. Therefore, they cannot manipulate FM^τ at R. In addition, student 2 cannot obtain neither a nor b with any strategy even if it involves top ranking them. Therefore, R is not vulnerable under FM^τ. Now, $BM^\tau(R) = FM^\tau(R)$ and if student 2 ranks school b first then he is assigned to it. Therefore, R is vulnerable under BM^τ.

While the ranking of mechanisms in Corollary 5 is based on inclusion of vulnerable preference profiles, the result cannot be strengthened by further considering the inclusion of manipulating agents.

Example 6. ***French is not strongly less manipulable than Boston***

Consider an example of a set $I = \{1, 2, 3\}$ of three students and a set $S = \{a, b, c\}$ of three schools each with capacity one. A priority profile $\tau = (\succeq, \pi)$ such that $\pi_a = \pi_b = \pi_c$ and a preference profile R are specified in the tables below :

$\succeq_a$	$\succeq_b$	$\succeq_c$	π_a
1, 2	3	2	1
3	1	3	2
	2	1	3

R_1	R_2	R_3
b	a	a
a	b	b
c	c	c

Then, we have

$$FM^\tau(R) = \begin{pmatrix} 1 & 2 & 3 \\ c & a & b \end{pmatrix} \quad \textit{and} \quad BM^\tau(R) = \begin{pmatrix} 1 & 2 & 3 \\ b & a & c \end{pmatrix}.$$

Suppose that student 1 *reports the preference relation* $R'_1 : a\ b\ c$ *under* FM^τ. *Then we have* $FM^\tau_1(R'_1, R_{-1}) = a$. *Therefore, student* 1 *manipulates* FM^τ *at* R *via* R'_1 *but cannot manipulate* BM^τ *at* R *because he obtained his first choice. In addition, student* 3 *can manipulate the Boston mechanism* BM^τ *at* R *but cannot manipulate the French mechanism* FM^τ *at* R.

Because the French mechanism is less manipulable than the Boston mechanism, whenever the latter is strategy-proof, so is the first.

Corollary 6. *The set of priority profiles for which the Boston mechanism is strategy-proof is a proper subset of the ones for which the French mechanism is strategy-proof.*

This corollary further shows how the French mechanism is less manipulable than the Boston mechanism. Finally, we compare the French mechanism to the Boston mechanism from the viewpoint of strategic accessibility.

Proposition 7. *For each priority profile* τ, *the French mechanism* FM^τ *is equally or less strategically accessible than the Boston mechanism* BM^τ.

This proposition is actually the corollary of a forthcoming proposition. For this reason, we defer the proof to the corollary in question. Comparing mechanisms by their vulnerability to manipulation does not point to the set of schools towards which manipulations are centred around. We first ask whether there are FP-stable matching mechanisms in which selective schools can achieve the goal of recruiting a qualified student body. The idea of basing admissions to selective schools on academic files justifies the requirement that access to this kind of schools be prevented from strategic misrepresentation. First, we show as a lemma that strategic accessibility as we define it is effective via the French mechanism.

Lemma 5. *If a school is strategically accessible by a student via the French mechanism, then it is also strategically accessible when it is the best school that he can obtain via manipulation.*

The proof of the lemma appears in Appendix 18. Fortunately, via the French mechanism, selective schools are spared from a possible bias of the mechanism due to its manipulability. A manipulable mechanism can bias the selection goal of selective schools by assigning non qualified students to these schools simply because these students are much strategically sophisticated. Of course, the strategic behavior of students towards non-selective schools can affect the student body at selective schools.

Theorem 5. *For each priority profile τ,*

(i) no selective school is strategically accessible via the French mechanism FM^{τ} and

(ii) no equal preference school is strategically accessible via the first preference first mechanism FPF^{τ}.

The proof appears in Appendix 15 in which it is a corollary of the following lemma and the fact that the first preference first mechanism is a French mechanism.

Lemma 6. *No student can manipulate the French mechanism to obtain a seat at a school where he is not in tie with any other student.*

The proof appears in Appendix 15. This lemma is precise about the set of students who can engage in manipulation and towards which schools under the French mechanism. Only students in tie at a given school can manipulate the French mechanism to obtain a seat at that school. Motivated by this observation we ask the natural question of how the size of indifference classes affects the incentive properties of the French mechanism. From the lemma, it might be tempting to conclude that when we transform some ties into strict relations, the induced French mechanism will be less manipulation than the initial one. However, a careful look reveals that this conclusion can be misleading.

Example 7. *Inconsistent splits can be detrimental*
Consider an example of three students $I = \{1, 2, 3\}$ *and three schools* $S = \{a, b, c\}$ *each with a capacity of one. Let* $\tau = (\succeq, \pi)$ *be such that* $\pi_a = \pi_b = \pi_c : 1\,2\,3$ *and another basic priority* $\succeq'_b$ *of school* b *are specified as below along with a preference profile* R*. Let* $\tau' = (\succeq', \pi)$ *where* $\succeq_b$ *is replaced by* $\succeq'_b$ *in* $\succeq$*.*

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq'_b$
$1, 2$	3	$1, 3$	3
3	$1, 2$	2	2
			1

R_1	R_2	R_3
b	b	c
c	c	b
a	a	a

Then,

$$FM^{\tau}(R) = \begin{pmatrix} 1 & 2 & 3 \\ b & a & c \end{pmatrix} \text{ and } FM^{\tau'}(R) = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}.$$

Students 1 *and* 3 *obtained their first choices under* $FM^{\tau}(R)$ *and therefore cannot manipulate* FM^{τ} *under* R*. Furthermore, by ranking school* c *first, student* 2 *cannot get a seat there because* $3 \succ_c 2$*. Therefore, student* 2 *cannot manipulate* FM^{τ} *under* R*.*

However, by ranking school c *first student* 1 *is assigned a seat there because* $1 \succ_{\pi_c} 3$*. Student* 1 *can manipulate* $FM^{\tau'}$ *under* R*.*

In this example, we split the indifference class in $\succeq_b$ inconsistently with π_b, that is, student 1 has higher priority than student 2 under π_b but $2 \succ'_b 1$. However, even if we split the indifference classes consistently with π the problem remains.

Example 8. ***Consistent split does not entail comparing mechanisms by vulnerability to manipulation***

Consider a set $I = \{1,2,3,4\}$ *of students and a set* $S = \{a,b,c,d\}$ *of schools each with a capacity of one. A preference profile* R *and a priority profile* $\tau = (\succeq, \pi)$ *such that* $\pi_a = \pi_b = \pi_c : 1\,2\,3\,4$ *are specified below :*

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
1, 2	3	4	1, 4
3, 4	1, 2, 4	2	3
		1	2
		3	

R_1	R_2	R_3	R_4
c	c	c	b
b	b	a	c
d	a	b	d
a	d	d	a

Then we have

$$FM^{\tau}(R) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ d & c & a & b \end{pmatrix}.$$

Consider the following basic priority $\succeq'_b$ *for school* b *in which we split the indifference class consistently with* π_b *and let* $\succeq' = (\succeq_{-b}, \succeq'_b)$ *and* $\tau' = (\succeq', \pi)$.

$\succeq'_b$
3
1, 2
4

Then we have

$$FM^{\tau'}(R) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ d & a & b & c \end{pmatrix}.$$

Let R_2^b *be a preference relation n which student 2 ranks school* b *first. Then, we have* $FM_2^{\tau'}(R_2^b, R_{-2}) = b$. *Therefore, student 2 manipulates* $FM^{\tau'}$ *under* R.

Next, each of students 2, 3 and 4 obtains at least their second choices under $FM^{\tau}(R)$ *and, by Lemma 4, cannot manipulate* FM^{τ} *under* R. *Furthermore, student 1 cannot manipulate* FM^{τ} *under* R. *Indeed, student 1 can manipulate* FM^{τ} *only to obtain a seat at school* b. *Suppose that he reports the* *preference relation R_1^b. Let $\mu = FM^\tau(R_1^b, R_{-1})$ and suppose that $\mu(1) = b$. Then, $\mu(4) = c$. Because $2 \succ_a 3$, $\mu(2) = a$. Therefore, because $3 \succ_b 1$, we have $\mu(3) = b$. Contradicting the assumption that $\mu(1) = b$. Therefore, the preference profile R is not vulnerable under FM^τ.*

This means that we cannot compare French mechanisms induced by a priority profile and its consistent split via the « less manipulability » measure. A split consists of transforming some ties into strict relations without reversing the existing strict relations in the basic priorities. A basic priority order $\succeq'_a$ is a **split** of $\succeq_a$ if for each $i, j \in I$, $i \succ_a j$ implies $i \succ'_a j$.[22] A basic priority order $\succeq'_a$ is a **π_a-consistent split** of the priority order $\succeq_a$ if $\succeq'_{\pi_a} = \succeq_{\pi_a}$.[23] A priority order $\succeq'_a$ is a **minimal π_a-consistent isolation** of student i in the priority order $\succeq_a$ if

(1) $\succeq'_a$ is a π_a-consistent split of $\succeq_a$,

(2) student i is in a tie with another student in $\succeq_a$ but not in $\succeq'_a$ and

(3) we have $\succeq'_a|_{I_1} = \succeq_a|_{I_1}$ and $\succeq'_a|_{I_2} = \succeq_a|_{I_2}$,
where $I_1 = \{j \in I | j \succ_{\pi_a} i\}$ and $I_2 = \{j \in I | i \succ_{\pi_a} j\}$.

We say that $(\succeq', \pi)$ consists of **one minimal consistent isolation** in $(\succeq, \pi)$ if there is a school a such that $\succeq'_a$ is a minimal π_a-consistent isolation of some student in $\succeq_a$ and for each school $b \neq a$, $\succeq'_b = \succeq_b$.

Example 9. *Illustrating minimal consistent isolation*

Consider an example of six students and a tiebreaker π_a : 1 2 3 4 5 6 for school a.

$\succeq_a$		$\succeq'_a$
1	$\Rightarrow$	1
2, 3, 4, 5, 6		2, 3
		4
		5, 6

22. A priority order $\succeq'_a$ is a split of $\succeq_a$ if $\succeq'_a \subset \succeq_a$.
23. The priority order $\succeq'_a$ is a π_a-consistent split of the priority order $\succeq_a$ if $\succeq_{\pi_a} \subset \succeq'_a \subset \succeq_a$.

The basic priority order $\succeq'_a$ *is a* π_a*-consistent isolation of student* 4 *because in* $\succeq_a$ *student* 4 *is in a tie but not in* $\succeq'_a$ *and the broken ties do not reverse the order* π_a.

While consistent splits do not induce less manipulable mechanisms in the set of all preference profiles, we would like to know whether we could recover the comparison in real-world school choice where preferences might be correlated. We consider the domain of correlated preferences and obtain a positive result.

Proposition 8. *Let* τ' *be a consistent split of* τ. *Then, in the domain of correlated preferences, the French mechanism* $FM^{\tau'}$ *is equally or strongly less manipulable than the French mechanism* FM^{τ}.

The proof appears in Appendix 16. We continue with a definition and a lemma.

Definition 1. *Given a priority profile* $(\succeq, \pi)$ *and two distinct students* i *and* j, *a tie* $i \sim_b j$ *with student* i *is quasi-cyclic if there is a third (distinct) student* k, *a second distinct school* a, *a third school* c *distinct from* b *and two subsets* I_a *and* I_b *of students such that*

(1) $k \succ_{\pi_a} i$ *and either*

(1-a) student i *has higher priority than student* j *under* π_b *or*

(1-b) there is a subset $I'_c \subset I \setminus \{i, j\}$ *such that* $|I_c| = q_c$, I'_c *and* I_b *are disjoint,* I'_c *and* $I_a \cup \{k\}$ *are disjoint if* $c \neq a$ *and* $I'_c = I_a \cup \{k\}$ *if* $c = a$ *and for each* $\ell \in I'_c$, $\ell \succ_{\pi_c} j$.

and,

(2) the subsets $I_a, I_b \subset I \setminus \{i, j, k\}$ *are disjoint and such that* $|I_a| = q_a - 1$, $|I_b| = q_b - 1$, *for each* $\ell \in I_a$, $\ell \succ_{\pi_a} i$ *and for each* $\ell \in I_b$, $\ell \succeq_b j$.

We say that a tie with a student in a priority structure is acyclic if it has no quasi-cycle.

Then, the lemma in question is stated below.

Lemma 7. *Given a priority structure $\tau = (\succeq, \pi)$, the following are equivalent :*

(1) the French mechanism FM^{τ} is strategy-proof

(2) every tie with any student in $\succeq$ is acyclic.

The proof of the lemma appears in the Appendix 17. When the basic priority profile is a complete indifference priority profile, the French mechanism is equivalent to the Boston mechanism. In that case, the conditions (1), (1-a), (1-b) and (2) of Definition 1 concern the tiebreaker profile. If (1-a) holds, then we have $k\ \pi_a\ i$ and $i\ \pi_b\ j$. If (1-a) does not hold, then (1-b) holds and $j\ \pi_b\ i$ and there is a student ℓ such that $\ell\ \pi_c\ j$. In any case, there are three students i, j and k and two schools a and b such that $k\ \pi_a\ j$ and $j\ \pi_b\ i$. This is precisely the quasi-cycle condition identified by Kumano (2013) that characterizes manipulable Boston mechanisms.[24] Therefore, the lemma above generalizes the main characterization of Kumano (2013).

We show in an example that inconsistent splits can also be detrimental to comparing mechanisms by the « strategic accessibility » measure.

Example 10. ***Inconsistent split can be detrimental to comparing mechanisms***

Consider a set $I = \{1, 2, 3\}$ of students, a set $S = \{a, b, c\}$ of schools each with a capacity of one. A preference profile R, priority profile $(\succeq, \pi)$ and a split $\succeq'_a$ of $\succeq_a$ that is not consistent with π_a are specified below :

$\succeq_a$	$\succeq_b$	$\succeq_c$
1, 3	1, 2	2
2	3	1
		3

$\succeq'_a$	π_a
3	1
1	3
2	2

R_1	R_2	R_3
a	b	a
b	a	b
c	c	c

where $\pi_b = \pi_c = \pi_a$. The split of the indifference class at $\succeq_a$ is not consistent with π_a.

24. Chen (2014) shows that this condition is equivalent to the one where each pair of two schools have join capacity that exceeds the number of students.

Consider the tie $1 \sim_b 2$*. Because* $1 \succ_{\pi_a} 3$ *and* $1 \succ_c 3$*, there is no student other than student* 1 *and* 2 *such that condition (1) of Definition 1 is satisfied with student* 1*. School* b *is not strategically accessible via* FM^{τ}*. Let* R_1^b *be a preference relation in which student* 1 *ranks school* b *first and* $R^b = (R_1^b, R_{-1})$*. Now we have* $FM_1^{\tau'}(R) = c$ *and* $FM_1^{\tau'}(R^b) = b$*, that is, school* b *is strategically accessible by student* 1 *via* $FM^{\tau'}$ *but not via* FM^{τ}*.*

We are able to rank mechanisms induced by priority profiles and consistent splits.

Theorem 6. *Let* τ' *be a consistent split of* τ*. Then, the French mechanism* $FM^{\tau'}$ *is equally or less strategically accessible than the French mechanism* FM^{τ}*.*

The proof of the theorem appears in Appendix 19. The following result is a corollary of the above theorem which is also nothing but Proposition 7.

Corollary 7. *(Proposition 7) Let* τ *be a priority profile. Then, the French mechanism* FM^{τ} *is less strategically accessible than the Boston mechanism* BM^{τ} *or both mechanisms are equally strategically accessible.*

The proof appears in Appendix 20. By splitting all indifference classes, we evidently obtain a strategy-proof mechanism which is trivially less strategically accessible. However, we can derive nested ranked mechanisms by isolating one agent at each step until we reach the strategy-proof mechanism induced by the strict priority.

Theorem 7. *Let* $\tau = (\succeq, \pi)$ *be a priority profile. If the French mechanism* FM^{τ} *is strategically accessible, then there is a sequence* $\{\tau^t\}_{t=0}^{T}$ *of priority profiles such that*

(i) $\tau^0 = \tau$*,*

(ii) FM^{τ^T} *is not strategically accessible and*

(iii) for each $t = 0, \ldots, T-1$*,* τ^{t+1} *consists of one minimal consistent isolation in* τ^t *and the French mechanism* $FM^{\tau^{t+1}}$ *is less strategically accessible than the French mechanism* FM^{τ^t}*.*

The proof appears in Appendix 21. These results concern ex-post perspective. We consider an ex-ante perspective in which we compare the French mechanism to DA.

2.3.2 Ex-ante perspective

We consider now an ex-ante perspective in which no student knows the tiebreakers. In practice, they are randomly determined. We maintain the general model in which each school has its own tiebreaker. In this framework, a mechanism selects a lottery over matchings. We next introduce relevant notions.

We assume that there is a null object denoted $\emptyset$ whose capacity is $q_\emptyset = n$ and that an unmatched student is matched to the null object and vice versa. A probabilistic assignment matrix is a $n \times (m+1)$ matrix $p = (p_{ia})_{i \in I, a \in S'}$ in which $S' = S \cup \{\emptyset\}$ and p_{ia} represents the probability that student i either is assigned to school a if $a \in S$ or is unassigned if $a = \emptyset$. A probabilistic assignment is feasible if for each $i \in I$ and each $a \in S'$, we have (i) $p_{ia} \in [0,1]$, (ii) $\sum_{i \in I} p_{ia} \leq q_a$ and (iii) $\sum_{b \in S'} p_{ib} = 1$. According to a generalization of Birkhoff (1946) and Von Neumann (1953) theorem (Kojima and Manea, 2010b), any feasible probabilistic assignment is a convex combination of deterministic assignments. Therefore, a probabilistic assignment is a random matching defined as probability distribution over the set $\mathcal{M}$ of matchings. A random mechanism is a function that assigns to each preference profile a random matching.

We redefine the random counterpart of our mechanisms. Let Ω denote the set of all tiebreaker profiles.

Random deferred acceptance :

Given a weak basic priority profile $\succeq$, each school a draws a tiebreaker π_a with the uniform probability $\frac{1}{n!}$. For each preference profile R, the deterministic matching via DA is $DA^{(\succeq,\pi)}(R)$. Therefore, the outcome of the random version of DA which we denote by $rDA^{\succeq}$ for a preference profile R is

$$rDA^{\succeq}(R) = \sum_{\pi \in \Omega} \frac{1}{(n!)^m} DA^{(\succeq,\pi)}(R).$$

Random Boston mechanism :

Given a weak basic priority profile $\succeq$, each school a draws a tiebreaker π_a with the uniform probability $\frac{1}{n!}$. For each preference profile R, the deterministic matching via the Boston mechanism is $BM^{(\succeq,\pi)}(R)$. Therefore, the outcome of the random version of the Boston mechanism which de denote by $rBM^{\succeq}$ for the preference profile R is

$$rBM^{\succeq}(R) = \sum_{\pi \in \Omega} \frac{1}{(n!)^m} BM^{(\succeq,\pi)}(R).$$

Random French mechanism :

Given a weak basic priority profile $\succeq$, each school a draws a tiebreaker π_a with the uniform probability $\frac{1}{n!}$. For each preference profile R, the deterministic matching via the French mechanism is $FM^{(\succeq,\pi)}(R)$. Therefore, the outcome of the random version of the French mechanism which we denote by $rFM^{\succeq}$ for a preference profile R is

$$rFM^{\succeq}(R) = \sum_{\pi \in \Omega} \frac{1}{(n!)^m} FM^{(\succeq,\pi)}(R).$$

We first derive a property shared by the Boston mechanism and the French mechanism. No student can increase his chances of obtaining his first choices by simply reshuffling his bottom choices.

Proposition 9. *Let R be a preference profile and $\succeq$ a basic priority profile. Let R'_i be student i's strategy that differs from R_i only on the ranking of the last $m-k$ choices. Let $p = rFM^{\succeq}(R)$ and $p' = rFM^{\succeq}(R'_i, R_{-i})$. Then, for each school a ranked in the first k choices under R_i, $p_{ia} = p'_{ia}$.*

The proof appears in Appendix 22. Since the Boston mechanism is a

French mechanism, the above proposition also holds for the Boston mechanism. As a corollary, no student can manipulate to obtain his first true choice.

Corollary 8. *(Lemma 4) In a complete information and when the tiebreaker profile is known to students, no student can manipulate the French mechanism or the Boston mechanism to obtain a seat at his first true choice.*

We now develop an environment to compare the French mechanism to DA. The following model by Abdulkadiroğlu et al. (2011), ACY for short, serves as a basis. They consider an environment in which students have correlated ordinal preferences but differentiated cardinal preferences and schools have complete indifference basic priorities. Correlated ordinal preferences reflects real-world school choice. First, students form their preferences about schools based on criteria that range from job opportunities, safety to academic reputation. While perfect correlation of preferences may be a very demanding assumption, it is a good approximation and a necessary assumption to generate a clear result. Tractability consideration is also a secondary argument. If correlated ordinal preferences is motivated from real-world school choice, complete indifferences of school priorities seems less common. In addition, when there are no priorities, the very nature of DA, which is respecting priorities, is not being put to test. Troyan (2012) shows that the main result of ACY is not robust to introducing priorities.

In this paper, we broaden the environment of ACY by considering correlated priorities, that is, schools have the same priorities. Priorities derived form general exam scores is a case in a point. The environment of ACY, in which schools have complete indifference priorities, emerges as a special case.

Let $(R, \succeq)$ be a correlated economy. Suppose that each school a has $K \geq 1$ indifference classes under $\succeq_a$ and let I_ℓ be the set of students in the ℓ's indifference class and $n_\ell = |I_\ell|$. We assume that $\sum_{a \in A} q_a \geq n$. Let $R_* : a_1 \ldots a_m$ denote the common ordinal preference (each school is acceptable). On the other hand, students may have different cardinal utilities. Each student has a vNM utility values $v = (v_1, \ldots, v_m)$ about schools drawn from a finite set $\mathcal{V} := \{(v_1, \ldots, v_m) \in [0,1]^m | v_1 > \ldots > v_m\}$. We further suppose that no student know the cardinal utilities of the other students. For each vNM utility values v of student i and a random assignment p, his expected utility is $\sum_{a \in S'} v_a p_{ia}$. The equilibrium notions is a Bayesian Nash equilibrium in undominated strategies for the French mechanism and weakly dominant strategies for DA. Because students are fully identified by their types (indifference classes and utility values), we are interested in symmetric equilibria in which each type of student plays the same strategy. Let $f_\ell(v)$ denote the probability that a student in I_ℓ draws utility values v from $\mathcal{V}$.

Let $\Delta(\mathcal{R})$ denote the set of probability distributions over the set $\mathcal{R}$ of preference relations. A strategy for a student in I_ℓ is a function $\sigma_\ell : \mathcal{V} \rightarrow \Delta(\mathcal{R})$.

Theorem 8. *In any correlated economy, any type of student finds the outcome of any symmetric equilibrium of the French mechanism at least as good as the outcome of DA with symmetric tie breaking.*

The proof appears in Appendix 24. The proof of the theorem crucially relies on a lemma stated without proof in ACY for the case where schools have no basic priorities. It turns out that the proof is not trivial. Given two lotteries p and p', we say that p stochastically dominates p' under R_i if for each school a we have

$$\sum_{b:bR_ia} p_{ib} \geq \sum_{b:bR_ia} p'_{ib}.$$

In that case, we write that $p \; R_i^{sd} \; p'$. It is well-known that $p \; R_i^{sd} \; p'$ if and only if for each vNM utility values v, $\sum_a v_a p_{ia} \geq \sum_a v_a p'_{ia}$. For each $\ell \in \{1, \ldots, K\}$, let

$$t_\ell := min\{t | \sum_{k=1}^{t} q_{a_k} \geq n_1 + \ldots + n_\ell\}$$

be the index of the marginal school for students in I_ℓ. If students report their preferences truthfully, no student in I_ℓ is assigned to a school that is ranked lower that school with positive probability. The lemma in question is the following.

Lemma 8. *Under the French mechanism* $rFM^{\succeq}$*,*

(a) any strategy in which a student in I_ℓ *ranks any school in* $\{a_{t_\ell+1}, \ldots, a_m\}$ *in the first* t_ℓ *rankings is weakly dominated and*

(b) any strategy in which a student in I_ℓ *ranks any school in* $\{a_{t_\ell}, \ldots, a_m\}$ *in the first* $t_\ell - 1$ *rankings is weakly dominated.*

The proof appears in Appendix 23. By Lemma 2, with complete indifference priorities, the French mechanism reduces to the Boston mechanism and Theorem 8 is nothing but ACY's main result stated as a corollary here :

Corollary 9 (ACY)**.** *In any correlated economy in which each school has a complete indifference basic priority, any type of student finds the outcome of any symmetric equilibrium of the Boston mechanism at least as good as the outcome of DA with symmetric tie breaking.*

When school have priorities, the French mechanism makes a difference in that the Boston mechanism no longer systematically Pareto dominates DA nor does the Boston mechanism systematically Pareto dominate the French mechanism. [25]

Example 11. *Consider an example of three students* 1*,* 2 *and* 3 *and three schools* a*,* b *and* c *each with a capacity of one. The common basic priority is*

$\succeq_a$
$1, 2$
3

The vectors of cardinal utilities are given as follows :

	v^1	v^2	v^3
a	1	1	1
b	0.3	0.3	0.5
c	0	0	0

25. See Troyan (2012) for a more general result.

We suppose that there is complete information.[26] *Under the French mechanism and the Boston mechanism, it is a weakly dominant strategy for student* 1 *and* 2 *to report their preferences truthfully. Therefore, under the Boston mechanism it is a best response for student* 3 *to report a preference in which he ranks b first. Consequently, the assignments under the French mechanism, the Boston mechanism and DA are as follows*

	FM or DA			*BM*		
	a	*b*	*c*	*a*	*b*	*c*
1	1/2	1/2	0	1/2	0	1/2
2	1/2	1/2	0	1/2	0	1/2
3	0	0	1	0	1	0

At the equilibrium of the French mechanism or DA, both student 1 *and* 2 *have expected utilities* $EU_i^j = 1/2(1) + 1/2(0.4) + 0(0) = 0.7$, $j = FM, DA$ *and* $i = 1, 2$. *At the equilibrium of the Boston mechanism, both student* 1 *and* 2 *have expected utilities* $EU_i^{BM} = 1/2(1) + 0(0.4) + 1/2(0) = 0.5$, $i = 1, 2$. *Therefore,* $EU_i^{BM} < EU_i^{DA}$, $i = 1, 2$ *and the equilibrium outcome of the Boston mechanism does not Pareto dominate the dominant strategy outcome of DA.*

In addition, the expected utilities of student 3 *are* $EU_3^{FM} = 0$ *and* $EU_3^{BM} = 0.5$. *While student* 3 *prefers the equilibrium outcome of the Boston mechanism to the equilibrium outcome of French mechanism, student* 1 *and* 2 *prefer the equilibrium outcome of French mechanism to the equilibrium outcome of the Boston mechanism. Therefore, the French mechanism and the Boston mechanism cannot be Pareto ranked at these equilibria.*

2.3.3 Generalized French mechanism

We consider now the generalized French mechanism for which we generalize some important results. The proof of each of these results is almost verbatim the proof of the result corresponding to the simplified case.

26. This is a particular case where $f_1(v^1) = 1$ and $f_2(v^3) = 1$.

Theorem (Theorem 4 bis and Corollary 5 bis)**.** *Let τ be a priority profile such that the basic priority of each equal priority school is strict. Then the generalized French mechanism gFM^{τ} is equally or less manipulable than the first preference first mechanism FPF^{τ}.*

Theorem (Theorem 5 (i) bis)**.** *No selective school is strategically accessible via the generalized French mechanism.*

Proposition (Proposition 8 bis)**.** *Let τ' be a consistent split of τ. In the domain of correlated preferences the generalized French mechanism $gFM^{\tau'}$ is equally or strongly less manipulable than the generalized French mechanism gFM^{τ}.*

2.4 Conclusion

School admissions mechanisms remain one of the education policies that the public pays much attention. The DA and the Boston mechanism have dominated the literature (on school choice). In particular, in a model with ties in school priorities, no other mechanism that deals with these ties other than breaking them randomly has been explored. Motivated by the French university admissions system, this paper presents a novel way that relies on student preferences to break ties. Clearly, this procedure destroys the incentive to truthfully reveal preferences. But the incentive is reduced compared to the Boston mechanism.

Because it elicits cardinal preferences, by allowing students to influence how ties are broken, in an environment of correlated economies its equilibrium outcomes Pareto improve upon the equilibrium outcome of DA. Because selective schools rely on a manipulable mechanism to recruit their student body, the later could be made up of sophisticated students, probably with lower academic files. However, we show that under the French mechanism, no student could end up at such a school via manipulation. We further investigate how the size of indifference classes affects the incentive property of the mechanism. By transforming some ties into strict relations, the corresponding mechanism becomes less strategically accessible. Although this reduction operation does not induce less manipulable mechanisms in general, in the particular domain of correlated preferences it does.

Our study is the first, to our knowledge, that studies the French university admissions system ; in particular, the comparison of its mechanism to known school choice mechanisms. Our study enriches the school choice literature by adding a mechanism that balances trade-offs better than do DA or the Boston mechanism : It is less manipulable than the Boston mechanism in ex-post perspective and more efficient than DA in ex-ante perspective and correlated economies. However, future work is needed to evaluate these advantages via experiments where students face real incentives.

Chapitre 3

Strategy-proof preference Aggregation : Augmented Serial Rules

3.1 Introduction

In the academic job market, departments often face the difficult situation in which they must devise a strategy for offering an open position without knowing ex-ante whether any given applicant would accept it. The usual practice is to order the applicants in order of collective preferences and make offers accordingly. In this paper, we study rules for combining (aggregating) the individual preferences of the faculty members into a collective ordering.

In the example described, a faculty member may find that it is preferable to misrepresent his preferences and arrive at a collective ordering that he finds superior to the one that would have resulted from sincere report. To avoid this incentive issue, we are focused on strategy-proof aggregation.

We are also interested in fair aggregation rules. Two notions of fairness are probably appealing. From the agents' perspective, **anonymity** is a fairness requirement that a rule be invariant up to renaming the agents. Second, when the alternatives are considered for a fairness judgement, **neutrality** requires that a rule be invariant to how they are labelled. Neutrality promotes non-discrimination among applicants based on their names. In practice, such a discrimination is prohibited by law (as the employment discrimination law in the United States).

Anonymity is a compelling property when the agent's opinions have equal weight. However, it is not uncommon to encounter situations that involve agents with different decision powers as in the U.N security council. In the job hiring, for example, the opinion of some faculty members may be weighed higher, the members who are specialized in the open position being a case in a point. Nevertheless, some minimal fairness with respect to the agents is still to require. For example, it is arguably not desirable to completely exclude an agent from the decision process. For this reason, we formulate a **no dummy** requirement, that every agent's opinion matters (every agent has an impact on the collective choice in at least one preference profile).

We introduce aggregation rules called augmented serial rules and study their robustness to preference misrepresentations. Each of these rules is parametrized by a **list** of agents (with possible repetition) and a **committee voting rule**. For a given preference profile, the collective ordering is determined as follows : The first agent's most preferred alternative becomes the top-ranked alternative in the collective ordering, the second agent's most preferred alternative (among those remaining) becomes the second-ranked alternative and so on until two alternatives remain — which are then ranked by the committee voting rule.

Strategy-proof preference aggregation involves a technical difficulty about how to model individual preferences over collective orderings. Suppose that a lie induces a collective ordering that differs from the one induced by truth-telling. To formalize the robustness of an aggregation rule to preference misrepresentations, we need to extend the agent's preferences over the induced collective orderings. We formulate a refinement over extensions and look for strategy-proof rules under this refinement. Many of the extensions studied in the literature are refinements. For example, Bossert and Storcken (1992) first propose a formalisation (extension) based on Kemeny distance. Unfortunately, this type of extension makes it very difficult to find appealing strategy-proof rules. Bossert and Sprumont (2014) consider the minimal refinement. They show that this extension yields a rich class of possibilities. Unfortunately, this extension is incomplete. A third extension, the lexicographic extension, which is also well studied in the literature is also a refinement.

We first show that no aggregation rules, other than dictatorial rules, are neutral and strategy-proof under every refinement. However, interesting aggregation rules emerge under the lexicographic extension. More precisely, augmented serial rules are succinctly characterized by strategy-proofness and neutrality. An important subfamily of these rules further satisfy the no dummy requirement. A small class of these rules are further strategy-proof under Kemeny extension.

Our results are related to object allocation problem, the following allocation rules called serial dictatorships have received a considerable attention because they are the simplest strategy-proof and efficient rules « par excellence » (Hylland and Zeckhauser, 1979; Svensson, 1999) : Agents move in turn, according to a given ordering, to pick their favorite objects from among those that remain. The augmented serial rules are the preference aggregation's natural counterparts. While in serial dictatorship rules, agents who move earlier get their best objects, this feature is not present in a broad class of augmented serial rules. Suppose that the faculty members arrived at a given collective ordering of the applicants. The appointee is the first applicant (in the ordering) who is available for an offer. Because there is uncertainty regarding applicants who could be available, the appointee is not necessary the first ranked applicant. Therefore, there is no certain advantage for a faculty member to suggest the top candidate, unless this member has suggested the whole collective ordering.

In other related work, Harless (2016) extends the analysis of Bossert and Sprumont (2014) to solidarity properties ; and Athanasoglou (2016) finds the betweenness extension weak and argues for the Kemeny extension. In a recent paper, Athanasoglou (2017) introduces a rule inspired from an important rule studied by Bossert and Sprumont (2014) and formulates interesting solidarity properties.

The remainder of the paper is organized as follows. We present the formal model in Section 2 and the results in Section 3. We conclude in Section 4. We defer long proofs to the Appendix.

3.2 Model for preference aggregation and properties

There is a finite set A of $m \geq 2$ alternatives and a finite set N of $n \geq 2$ agents. Let $\mathcal{R}$ denote the set of strict orderings over A. Each agent i has a strict preference relation $R_i \in \mathcal{R}$ over A. We interchangeably write $a\ R_i\ b$ and $(a, b) \in R_i$ to denote that agent i finds alternative a at least as good as alternative b. Let P_i denote the asymmetric part of R_i. An element $R_N \in \mathcal{R}^N$ is a preference profile. We write R instead of R_N and R_{-i} instead of $R_{N\setminus i}$. For each $i \in N$, each $R_i \in \mathcal{R}$ and each non empty subset $B \subset A$, let $max_B R_i$ denote the alternative ranked first by R_i in B. We often denote an ordering by listing the alternatives such that from left to right they are listed in decreasing order. By convention, a preference relation will be indicated by subscript and a social ordering by a superscript. A(n aggregation) **rule** is a function $f : \mathcal{R}^N \to \mathcal{R}$.

We next formulate properties. First, neutrality is a fairness property that is concerned with alternatives. It requires that a rule be invariant to how alternatives are labelled. Let $\pi : A \to A$ be a permutation of A. For each $R^0 \in \mathcal{R}$, let $\pi R^0 \in \mathcal{R}$ be the ordering defined as follows : for each $\{a, b\} \subset A$, $\pi(a)\ \pi R^0\ \pi(b)$ if and only if $a\ R^0\ b$. For each $R \in \mathcal{R}^N$, let $\pi R := (\pi R_1, \ldots, \pi R_n)$.

Neutrality. *For each $R \in \mathcal{R}^N$ and each permutation π of A,*

$$f(\pi R) = \pi f(R).$$

With regard to agents, the minimal requirement is that none of them be ignored in the decision process. The following requirement is a fairness property that every agent's opinion (preference) matters.

No dummy. *For each agent $i \in N$, there is $R \in \mathcal{R}^N$ and $R'_i \in \mathcal{R}$ such that*

$$f(R'_i, R_{-i}) \neq f(R).$$

There is a classical requirement in social choice that no collective ordering be excluded from being chosen. This is often referred to as agents' sovereignty.

Ontoness. $f(\mathcal{R}^N) = \mathcal{R}$.

The last property is concerned with dominant-strategy incentive compatibility. Its formal definition requires that we extend agents' preferences over alternatives to preferences over collective orderings. An **anonymous and complete extension** is a function $\boldsymbol{R}$ that maps $\mathcal{R}$ to the set of strict order relations over $\mathcal{R}$. For simplicity, for each agent $i \in N$ and each $R_i \in \mathcal{R}$, let $\boldsymbol{R}(R_i) = \boldsymbol{R}_i$. We will focus on extensions which satisfy the following requirement that we refer to as refinement.

Refinement (of extensions). *For each agent $i \in N$, each preference relation $R_i \in \mathcal{R}$ and two collective orderings $R^0, R^1 \in \mathcal{R}$,*

$$R_i \cap R^0 \subsetneq R^1 \Rightarrow R^1 \; \boldsymbol{P}_i \; R^0.$$

Remark. *Bossert and Sprumont (2014) proved that any refinement has the following appealing feature. Given an agent i with preference relation R_i, if $R_i \cap R^0 \subsetneq R^1$ then for each $B \subset A$, the alternative recommended by R^1 in B is at least as good as the one recommended by R^0 in B according to R_i ; and there is an instance in which he prefers. An agent i with preference relation R_i has an unambiguous preference for R^1 over R^0.*

The following property prevents agents from misrepresenting their preferences.

Strategy-proofness under *R*. *For each* $R \in \mathcal{R}^N$ *and each* $i \in N$*, there is no* $R'_i \in \mathcal{R}$ *such that,*

$$f(R'_i, R_{-i}) \; \boldsymbol{P}_i \; f(R).$$

We address the question whether there is a rule which is strategy-proof under every refinement. Unfortunately, when coupled with neutrality, no appealing rule is available.

Theorem 9. *Suppose that there are at least three alternatives. Then, the only rule that is strategy-proof under every refinement and neutral is a dictatorial rule which is a rule that always selects the preference relation of a given agent, for every preference profile.*

The proof appears in Appendix 25. This result shows that « strategy-proofness under every refinement » is a very demanding requirement. The relaxation that we explore consists of describing possibilities on particular refinements.

3.3 Results

We describe rules which are strategy-proof under every refinement and onto when there are two alternatives. Let $A = \{a, b\}$. The family of strategy-proof rules can be described by simple games or winning coalitions. A coalition $T \subset N$ is winning with respect to a rule f if for each preference profile R such that for each $i \in T$, $R_i : a\, b$ and for each $i \in T^c$, $R_i : b\, a$ then $f(R) : a\, b$. On the other hand, let ω be a collection of non-empty coalitions and define a rule S^a_ω as follows :

$$\text{for each } R, \; S^a_\omega(R) = a\, b \text{ if and only if } \{i \in N | a \; P_i \; b\} \in \omega.$$

Simple game *S*. *There is a collection* ω *of non-empty coalitions such that :*

i. for each $T \in \omega$ *and* $T' \subset N$*, if* $T' \supset T$ *then* $T' \in \omega$ *and*

ii. $S = S^x_\omega$ *for some* $x \in A$.

Let $\widehat{\omega}$ denote the collection of minimum coalitions associated with ω :

$$\widehat{\omega} = \{T \in \omega | \text{there is no } T' \in \omega, T' \subsetneq T\}.$$

Note that if ω is not empty, $\widehat{\omega}$ is not empty as well. Given $\widehat{\omega}$, we define a rule $\widehat{S}^a_{\widehat{\omega}}$ as follows :

$$\widehat{S}^a_{\widehat{\omega}}(R) = a\, b \text{ if and only if for some } T \in \widehat{\omega},\ \ T \subset \{i \in N | a\ P_i\ b\}.$$

If $\widehat{\omega}$ satisfies (i), then $S^a_\omega = \widehat{S}^a_{\widehat{\omega}}$. To see this, let R be a profile and suppose that $S^a_\omega(R) = a\, b$. Then $\{i \in N | a\ P_i\ b\} \in \omega$. Therefore, there is $T \in \omega$ such that $T \subset \{i \in N | a\ P_i\ b\}$ and for no $T' \in \omega$, $T' \subsetneq T$. Consequently, $T \in \widehat{\omega}$ and thus $\widehat{S}^a_{\widehat{\omega}}(R) = a\, b$. Conversely, let R be a profile such that $\widehat{S}^a_{\widehat{\omega}}(R) = a\, b$. Then there is $T \in \widehat{\omega}$ such that $T \subset \{i \in N | a\ P_i\ b\}$. Because $\widehat{\omega} \subset \omega$, $T \in \omega$. By (i), $\{i \in N | a\ P_i\ b\} \in \omega$ thus $S^a_\omega(R) = a\, b$.

We focus on the rules based on minimum coalitions. Let $\mathcal{C}$ denote the set of collections of non-empty coalitions that satisfy (i) above and $\widehat{\mathcal{C}} = \{\widehat{\omega} | \omega \in \mathcal{C}\}$.

Strong simple game S. *There is a collection ω of non-empty coalitions such that*

i. for each $T \in \omega$ and $T' \subset N$, if $T' \supset T$ then $T' \in \omega$,

ii. $T \in \omega$ if and only if $T^c \notin \omega$ and

ii. $S = \widehat{S}^x_{\widehat{\omega}}$ for some $x \in A$.

Lemma 9. *(Moulin, 1983, P. 64) Suppose that there are two alternatives.*

i. A rule is strategy-proof under every refinement and onto if and only if it is simple game.

ii. A rule is strategy-proof under every refinement and neutral if and only if it is a strong simple game.

Since a strong simple game is neutral, we simply write $\widehat{S}_{\widehat{\omega}}$ instead of $\widehat{S}^a_{\widehat{\omega}}$, without mention of an alternative as parameter.

3.3.1 Lexicographic extension

In the light of Theorem 9, we need to focus on particular refinements in order to obtain appealing rules. The following extension yields interesting results. An agent $i \in N$ with preference relation $R_i \in \mathcal{R}$ compares two collective orderings $R^0 : a_1 \dots a_m$ and $R^1 : b_1 \dots b_m$ by first comparing the alternatives a_1 and b_1 under R_i. If $b_1\ P_i\ a_1$, then agent i prefers R^1 to R^0. If $a_1 = b_1$, then he compares the next ranked alternatives, that is, a_2 and b_2; and so on. Formally,

Lexicographic extension, $\boldsymbol{R}^{lex}$. *For each $i \in N$, each $R_i \in \mathcal{R}$ and two collective orderings $R^0 : a_1 \dots a_m$ and $R^1 : b_1 \dots b_m$,*

i. $R^0\ \boldsymbol{R}_i^{lex}\ R^0$ and

ii. $R^1\ \boldsymbol{P}_i^{lex}\ R^0$ if and only if either $b_1\ P_i\ a_1$ or there is $t > 1$ such that $a_1 = b_1, \dots, a_{t-1} = b_{t-1}$ and $b_t\ P_i\ a_t$.

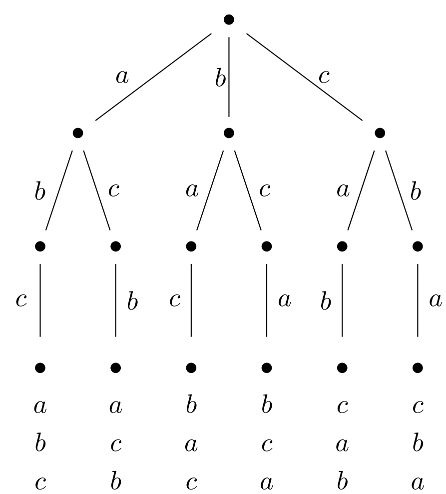
This extension is a refinement. To see this, let $i \in N$ and $R_i, R^0, R^1 \in \mathcal{R}$ such that $R_i \cap R^0 \subsetneq R^1$. Let $R^0 : a_1 \dots a_m$ and $R^1 : b_1 \dots b_m$. Suppose that $a_1 \neq b_1$. If $a_1\ P_i\ b_1$, then because $a_1\ P^0\ b_1$, we have $(a_1, b_1) \in R_i \cap R^0$. Because $R_i \cap R^0 \subsetneq R^1$, we have $(a_1, b_1) \in R^1$ which contradicts the fact that $b_1\ P^1\ a_1$. Therefore, $b_1\ P_i\ a_1$ and thus $R^1\ \boldsymbol{P}_i^{lex}\ R^0$. Let $t > 1$ and suppose that $a_1 = b_1 \dots a_{t-1} = b_{t-1}$ and $a_t \neq b_t$. Then R^0 and R^1 can be presented as follows :

R^0	R^1
a_1	a_1
$\vdots$	$\vdots$
a_{t-1}	a_{t-1}
a_t	b_t
$\vdots$	$\vdots$
a_m	b_1

Obviously, $a_t\ P^0\ b_t$ and $b_t\ P^1\ a_t$. If $a_t\ P_i\ b_t$, then because $a_t\ P^0\ b_t$, we have $(a_t, b_t) \in R_i \cap R^0$. Because $R_i \cap R^0 \subsetneq R^1$, $(a_t, b_t) \in R^1$ which contradicts the fact that $b_t\ P^1\ a_t$. Therefore, $b_t\ P_i\ a_t$ and thus $R^1\ \boldsymbol{P}_i^{lex}\ R^0$.

We describe rules which have interesting properties under this extension. Consider a sequential selection (without repetition) from A until all alternatives have been considered. Consider the tree consisting of all such selections.

At the end of each sequence of selections we attach the collective ordering consistent with the order in which the alternatives have been selected. As an example, consider the case of three alternatives and let $A = \{a, b, c\}$. We have the following tree.



Let h^0 be the initial history in which no alternative is yet selected. Let $\mathcal{H}^0 = \{h^0\}$. For each $k = 0, \ldots, m$, let $\mathcal{H}^k$ denote the set of histories where exactly k alternatives are selected. For each $h \in \bigcup_{t=0}^{m-1} \mathcal{H}^t$, let $A(h)$ denote the set of alternatives not yet selected at h. Let $\Gamma : \bigcup_{k=0}^{m-2} \mathcal{H}^k \to N \cup (A \times \widehat{\mathcal{C}})$ be a function such that

i. for each $h \notin \mathcal{H}^{m-2}$, $\Gamma(h) \in N$,

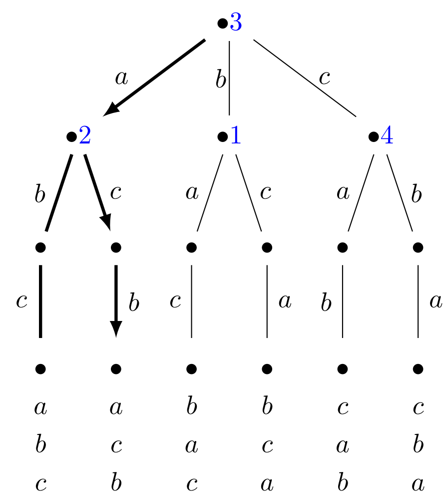
ii. for each $h \in \mathcal{H}^{m-2}$, $\Gamma(h) \in (A(h) \times \widehat{\mathcal{C}})$.

It is a function from $\bigcup_{t=0}^{m-2} \mathcal{H}^t$ to $N \cup (A \times \widehat{\mathcal{C}})$ which assigns to each history an agent except histories in $\mathcal{H}^{m-2}$ where an history is assigned to an element in $(A \times \widehat{\mathcal{C}})$ in such a way that the couple selected contains one of the two remaining alternatives. We define a rule that takes Γ as a parameter.

We start by an informal definition of the rule. For simplicity let $A = \{a, b, c\}$. Given a function Γ and a preference profile R, let R^Γ denote the collective ordering obtained as follows : the most preferred alternative of the agent $\Gamma(h^0)$ is ranked first in the collective ordering R^Γ. In the tree, the selection leads to an history h. Then, the most preferred alternative of the agent $\Gamma(h)$ among those remaining is ranked next, and so on.

Example 12. *Augmented sequential rules f^Γ. Let preference profile R and a parameter Γ be as specified below.*

R_1	R_2	R_3
b	c	a
c	a	c
a	b	b



Then, $f^\Gamma(R) : a\,c\,b$.

For a formal definition, let Γ be a parameter. For each preference profile R, there is a sequence $(h^k)_{k=0}^m$ of consecutive histories starting from the initial history h^0 to the terminal history h^m such that

1. the history h^1 is reached following the selection of the most preferred alternative of agent $\Gamma(h^0)$ under R in $A(h^0)$,
2. if $m > 3$, then for each $k = 1, \ldots, m-3$, the history h^{k+1} is reached following the selection of the most preferred alternative of agent $\Gamma(h^k)$ under R in $A(h^k)$,
3. the history h^{m-1} is reached following the selection of the alternative ranked first by $\widehat{S}^x_{\widehat{\omega}}(R|_{A(h^{m-2})})$ where $(x, \widehat{\omega}) = \Gamma(h^{m-2})$.

Let R^Γ be the collective ordering attached to the terminal history h^m that follows h^{m-1}. We define a rule which selects this collective ordering for each preference profile.

Augmented sequential rule $\boldsymbol{f^\Gamma}$. *For each preference profile $R \in \mathcal{R}^N$,*

$$f^\Gamma(R) = R^\Gamma.$$

Theorem 10. *A rule is strategy-proof under lexicographic extension and onto if and only if it is an augmented sequential rule.*

The proof appears in Appendix 25. In augmented sequential rules the choice of the next agent who should prescribe the next ranked alternative is contingent on the history. Therefore, an important class of these rules are not neutral. However, there is an important subclass of neutral rules. They are rules where the choice of the next agent who should choose the next ranked alternative depends on the number of alternatives chosen so far. For each $k \in \{0, \ldots, m-2\}$ and each $h, h' \in \mathcal{H}^k$, $\Gamma(h) = \Gamma(h')$. In that case the rule can be described as follows : there is a list $\sigma : \{1, \ldots, m-2\} \to N$ of agents (with possible repetition) and a strong simple game $\widehat{S}_{\widehat{\omega}}$. For each preference profile R, the most preferred alternative of $\sigma(1)$ listed first in A, that is $a_1 = max_A R_{\sigma(1)}$, is the alternative ranked first in the collective ordering. The most preferred alternative of the agent $\sigma(2)$ listed second, that is $a_2 = max_{A\setminus\{a_1\}} R_{\sigma(2)}$, is ranked second in the collective ordering and so on until two alternatives, say a and b, remain. The alternative ranked first in $\widehat{S}_{\widehat{\omega}}(R|_{\{a,b\}})$, is ranked next and the alternative that remains is ranked last. Let $R^{(\sigma,\widehat{\omega})}$ be the collective ordering obtained. We refer to the agent who chooses the tth ranked alternative from those remaining, dictator for position t.

Augmented serial rule $\boldsymbol{f^{(\sigma,\widehat{\omega})}}$. *For each $R \in \mathcal{R}^N$,*

$$f^{(\sigma,\widehat{\omega})}(R) = R^{(\sigma,\widehat{\omega})}.$$

Theorem 11. *A rule is strategy-proof under lexicographic extension and neutral if and only if it is an augmented serial rule.*

The proof appears in Appendix 26. In addition, no dummy further shrinks augmented serial rules to a class in which each agent is either a dictator for at least one position or is a member of an element of the set of coalitions that define the strong simple game.

Proposition 10. *A rule f is strategy-proof under lexicographic extension, neutral and no dummy if and only if it is an augmented serial rule $f^{(\sigma,\widehat{\omega})}$ such that for each agent $i \in N$ either there is a position t such that $\sigma(t) = i$ or there is $T \in \widehat{\omega}$ such that $i \in T$.*

The proof appears in Appendix 27. While lexicographic extension yields possibilities, there are other refinements.

3.3.2 Inverse lexicographic and Kemeny extension

In this section, we consider two other refinements. First, it is straightforward to define an extension in the same spirit as lexicographic extension but comparing collective orderings from the bottom up (ie in the opposite order to the way presented by the lexicographic extension).

Inverse lexicographic extension, $\boldsymbol{R}^{ilex}$. *For each agent $i \in N$, each preference relation $R_i \in \mathcal{R}$ and two collective orderings $R^0 : a_1 \dots a_m$ and $R^1 : b_1 \dots b_m$,*

i. R^0 $\boldsymbol{R}_i^{ilex}$ R^0 *and*

ii. R^1 $\boldsymbol{P}_i^{ilex}$ R^0 *if and only if either $a_m\ P_i\ b_m$ or there is $t < m$ such that $a_m = b_m, \dots, a_{t+1} = b_{t+1}$ and $a_t\ P_i\ b_t$.*

This extension is also a refinement. We define a rule called **inverse augmented sequential rule**. For each preference profile R, there is a sequence $(h^k)_{k=0}^m$ of consecutive histories starting from the initial history h^0 to the terminal history h^m such that

1. the history h^1 is reached following the selection of the least preferred alternative of agent $\Gamma(h^0)$ under R in $A(h^0)$,

2. if $m > 3$, then for each $k = 1, \ldots, m-3$, the history h^{k+1} is reached following the selection of the least preferred alternative of agent $\Gamma(h^k)$ under R in $A(h^k)$,
3. the history h^{m-1} is reached following the selection of the alternative ranked last by $\widehat{S}^x_{\widehat{\omega}}(R|_{A(h^{m-2})})$ where $(x, \widehat{\omega}) = \Gamma(h^{m-2})$.

We define a rule which selects for each preference profile R the collective ordering attached to the terminal history h^m and refer to it as inverse augmented sequential rule. We can straightforwardly adapt the proof of Theorem 10 to obtain the following result :

Theorem 12. *A rule is strategy-proof under inverse lexicographic extension and onto if and only if it is an inverse augmented sequential rule.*

The last extension pertains to the so-called Kemeny extension. This is an extension in which each agent compares two collective orderings by comparing the number of pairs over which they differ from his preference relation. Let $\delta : \mathcal{R}^2 \to \mathbb{N} \cup \{0\}$ be a measure defined as follows : for each $(R^0, R^1) \in \mathcal{R}^2$,

$$\delta(R^0, R^1) = |\{(a, b) \in A \times A | (a, b) \notin R^0 \cap R^1\}|.$$

Kemeny extension $\boldsymbol{R}^{kem}$. *For each $i \in N$ and each $R_i, R^0, R^1 \in \mathcal{R}$,*

i. $R^0 \; \boldsymbol{R}_i^{kem} \; R^0$ *and*

ii. $\delta(R_i, R^1) \; < \; \delta(R_i, R^0) \; \Rightarrow R^1 \; \boldsymbol{P}_i^{kem} \; R^0$.

Note that it is open how an agent $i \in N$ with a preference relation R_i compares two collective orderings R^0 and R^1 such that $\delta(R_i, R^1) = \delta(R_i, R^0)$. It was open whether beside dictatorial rules there are rules which are strategy-proof and onto under Kemeny extension. The following proposition provides a partition of augmented serial rules between those which are strategy-proof under Kemeny extension and those which are not.

Proposition 11. *Let $f^{(\sigma, \widehat{\omega})}$ be an augmented serial rule. Then, $f^{(\sigma, \widehat{\omega})}$ is strategy-proof under Kemeny extension if and only if there is $i \in N$ such that for each $t \in \{1, \ldots, m-2\}$,*

$$\sigma(t) = i.$$

The proof appears in Appendix 28. When there are only three alternatives, every augmented serial rule is such that one agent is trivially a dictator for all positions except probably the next to last one. Therefore, we have the following corollary.

Corollary 10. *Suppose that* $m = 3$*. Then every augmented serial rule is strategy-proof under Kemeny extension.*

3.4 Conclusion

We introduced rules called augmented serial rules for aggregating preferences. These rules are simple to describe and furthermore they are strategy-proof across reasonable extensions. They are succinctly characterized by strategy-proofness under lexicographic extension and neutrality. A subfamily of these rules are strategy-proof under the Kemeny extension. We also introduced a no dummy requirement that no agent be excluded from the decision process. An appealing class of augmented serial rules satisfy this requirement.

While the paper exhibits rules which are strategy-proof under Kemeny extension and onto (beside dictatorial rules), future work is required to characterize them all.

Annexes

Proofs from Chapter 1

The proofs use known results that we collect first. Let φ be a rule and Ch a profile of choice functions.

DA possesses some form of monotonicity (Kojima and Manea, 2010a). Let $s \in S$, $v \in C \cup \{\emptyset\}$ and $\{P_s, P'_s\} \subset \mathcal{P}$. We say that P'_s is an (individually rational) IR- monotonic transformation of P_s at v,[1] in notation $\boldsymbol{P'_s\, i.r.m.t\, P_s}$ at v, if any college that is ranked above both v and $\emptyset$ under P'_s is also ranked above v under P_s, that is

$$\text{for each } c \in C,\ \ c\ P'_s\ v \ \text{ and } \ c\ P'_s\ \emptyset \ \Rightarrow \ c\ P_s\ v.$$

Of course, $P_s\, i.r.m.t\, P_s$ at $v \in C \cup \{\emptyset\}$. The following instance deserves a separate illustration as it will be the main form of IR-monotonic transformation we will be using. Given $c \in C$ and $s \in S$, let P^c_s be student s's preference relation where c is his unique acceptable college and $P^{\emptyset}_s$ his preference relation where no college is acceptable.

Remark. *Let $s \in S$, $P_s \in \mathcal{P}$, $v \in C \cup \{\emptyset\}$ and $\mu \in \mathcal{M}$ be such that $v\ \ R_s\ \mu_s\ R_s\ \emptyset$. Then, $P^v_s\ i.r.m.t\ \ P_s$ at μ_s.*

We say that P' is an IR-monotonic transformation of P at matching μ, in notation $P'\ \ i.r.m.t\ \ P$ at μ, if for each $s \in S$, $P'_s\ \ i.r.m.t\ \ P_s$ at μ_s.

1. See (Kojima and Manea, 2010a) for further discussions.

IR-monotonicity. *For each* $\{P, P'\} \subset \mathcal{P}^S$, P' *i.r.m.t* P *at* $\varphi(P) \Rightarrow \varphi(P')\ R'\ \varphi(P)$.

Next, we will use the following incentive compatibility property a rule might possess.

Strategy-proofness. *For each* $P \in \mathcal{P}^S$, *each* $s \in S$ *and each* $P'_s \in \mathcal{P}, \varphi_s(P)\ R_s\ \varphi_s(P'_s, P_{-s})$.

For each choice profile Ch, DA^{Ch} is strategy-proof (Lemma 1). Beside this result, the choice functions themselves have others properties. We collect the relevant ones.

Path-independence. *For each* $c \in C$, *each* $S' \subset S$ *and each* $s \in S, Ch_c(Ch_c(S') \cup \{s\}) = Ch_c(S' \cup \{s\})$.

Irrelevance of rejected students. *For each* $c \in C$, *each* $S' \subsetneq S$ *and each* $s \notin S'$, $s \notin Ch_c(S' \cup \{s\}) \Rightarrow\ Ch_c(S' \cup \{s\}) = Ch_c(S')$.

We now gather in a lemma the useful results.

Lemma 10. *Let* Ch *be a choice profile. Then*

(1) DA^{Ch} *is IR-monotonic (Kojima and Manea (2010a); Theorem 1).*

(2) DA^{Ch} *is strategy-proof (Hatfield and Milgrom, 2005).*

(3) For each $c \in C$, Ch_c *is path-independent (Ehlers and Klaus (2016), Lemma 1).*

(4) For each $c \in C$, Ch_c *satisfies irrelevance of rejected students (Aygün et al., 2012).*

Point 2 of Lemma 1 was first established when each college's preference is substitutable and satisfies the law of aggregate demand, that is, for each $c \in C$ with choice function Ch_c, each $S' \subseteq S$ and each $S'' \subseteq S$, $S' \subseteq S''$ implies $|Ch_c(S')| \leq |Ch_c(S'')|$ (Hatfield and Milgrom, 2005). However, an acceptant preference satisfies this law (Ehlers and Klaus, 2016).

Our proofs also use the following result (Abdulkadiroğlu et al., 2009) when each college's preference over subsets of students is responsive to its preference over individual students. The result is in fact obtained in a more general model encompassing substitutable and acceptant preferences (Alva and Manjunath, 2017).

Lemma 11 (Abdulkadiroğlu et al. (2009), claim in Theorem 1)**.** *Let* $M = (P, Ch)$ *and* $\mu \in \mathcal{M}$*. If* μ *Pareto dominates* $DA^{Ch}(P)$ *at* P*; then, the same set of students are matched in both* μ *and* $DA^{Ch}(P)$*, in notation,* $\mu_C = DA^{Ch}_C(P)$*.*

Finally, by convention we simplify the notation in such a way that given an order π and $t = 1, ...n$, $s_t := s_{\pi(t)}$. Since each statement will involve a unique order, this convention should not create a confusion.

Appendix 1. *Proof of Theorem 1*

The proof has two parts. In Part 1, we show that each strategy profile in which each student plays either a bossy strategy or a solidary strategy is an SPUE. Any SPUE is actually of this kind (Part 2).

Part 1 : Let $\pi \in \mathcal{O}$ and $M = (P, Ch)$. Let $\sigma \in \Delta_\pi$ be a strategy profile such that for each $s \in S$, σ_s is either a bossy strategy or a solidary strategy. Then, σ is an SPUE of $G[\pi, M]$. The remaining part consists of proving that σ is a backwards-induction strategy.

By convention, whenever we consider a history $h^t = (h^0, a_1..., a_t)$, $a_{t'}$ is the application that $s_{t'}$ makes. To simplify the notation, we suppress the reference to the choice profile Ch in DA^{Ch} and just write $DA := DA^{Ch}$. We continue by establishing two important results as lemmas. Lemma 3 will serve as an induction base of an induction argument we use for the general proof and Lemma 4 will be the main part of the induction step. For each $v \in C \cup \{\emptyset\}$, let $S_v(h^0) = \emptyset$ and for each $t = 2, ..., n$ and each history

$h^{t-1} = (h^0, a_1, ..., a_{t-1})$, let $S_v(h^{t-1}) := \{s_{t'} | t' \leq t-1, a_{t'} = v\}$. If $v = c$ for some $c \in C$, then $S_c(h^{t-1})$ is the applications that c received up to h^{t-1}. If $v = \emptyset$, then $S_\emptyset(h^{t-1})$ is the set of students who chose to remain unmatched up to h^{t-1}.

Lemma 12. *Let $h^{n-1} \in \mathcal{H}_\pi^{n-1}$ and $\mu^{n-1} := DA(\boldsymbol{P}(h^{n-1}))$.*

(i) Assume that s_t makes an application at h^{n-1} what solidary strategy would have recommended at h^{n-1}. Let $v' \in C \cup \{\emptyset\}$ be such a decision and $h^n = (h^{n-1}, v')$ be the history following such application. Then, for each $c \in C$, $\mu_c^{n-1} = Ch_c(S_c(h^n))$.

(ii) Assume that $\mu_{s_n}^{n-1} = \emptyset$ and s_n applies to an acceptable college c' under P_{s_n}. Let $h'^n = (h^{n-1}, c')$ be the history following such application. Then, for each $c \in C$, $\mu_c^{n-1} = Ch_c(S_c(h'^n))$.

Proof of Lemma 12. (i). Since we only have two histories to consider, we further simplify the notation in such a way that $S_c := S_c(h^{n-1})$ and $S'_c := S_c(h^n)$.

Now consider the DA algorithm that produces μ^{n-1} for $(\boldsymbol{P}(h^{n-1}), Ch)$. Recall that $\boldsymbol{P}_{s_n}(h^{n-1}) = P_{s_n}$ and for each $t \neq n$, $\boldsymbol{P}_{s_t}(h^{n-1}) = P_{s_t}^v$ for some $v \in C \cup \{\emptyset\}$. Therefore, each student in $S \setminus \{s_n\}$ makes a proposal (if any) no further than the first step of the algorithm. By assumption, $v' = \mu_{s_n}^{n-1}$ and

$$s_n \in S'_{v'}. \tag{1}$$

Let $c \in C$ and consider the set of students proposing to c in the DA algorithm. There are two cases regarding s_n's proposals.

Case 1 : s_n did not propose to c.

Then, (in the first step) c received only proposals from students in S_c.[2] Therefore, $\mu_c^{n-1} = Ch_c(S_c)$. Because s_n did not propose to c in the DA process, $s_n \notin \mu_c^{n-1}$. Now by (1) $s_n \notin S'_c$ and we conclude that $S'_c = S_c$. By substitution, $\mu_c^{n-1} = Ch_c(S'_c) = Ch_c(S_c(h^n))$.

2. S_c is the set of students from whom c receives applications before s_n's turn in the game $G[\pi, M]$. Thus, for each $s \in S_c$, $\boldsymbol{P}_s(h^{t-1}) = P_s^c$.

Case 2 : s_n proposed to c.

If s_n proposed to c in Step 1 of the algorithm, then $\mu_c^{n-1} = Ch_c(S_c \cup \{s_n\})$. If s_n proposed to c in a step later than Step 1, then $\mu_c^{n-1} = Ch_c(Ch_c(S_c) \cup \{s_n\})$. Since Ch_c is path-independent, $Ch_c(Ch_c(S_c) \cup \{s_n\}) = Ch_c(S_c \cup \{s_n\})$. In any case,

$$\mu_c^{n-1} = Ch_c(S_c \cup \{s_n\}). \tag{2}$$

First, assume that $s_n \in \mu_c^{n-1}$. Then $c = v'$ and by (1), $s_n \in S'_c$. Hence $S'_c = S_c \cup \{s_n\}$. Then combining this and (2) we obtain $\mu_c^{n-1} = Ch_c(S'_c)$. Second, assume that $s_n \notin \mu_c^{n-1}$. Then from (2), $s_n \notin Ch_c(S_c \cup \{s_n\})$. Since Ch_c satisfies irrelevance of rejected students, $Ch_c(S_c \cup \{s_n\}) = Ch_c(S_c)$. Again $c \neq v'$ and by (1), $s_n \notin S'_c$. Thus, $S'_c = S_c$. Consequently, $Ch_c(S_c \cup \{s_n\}) = Ch_c(S_c) = Ch_c(S'_c)$. Finally, combining the later result with (2) we obtain $\mu_c^{n-1} = Ch_c(S'_c) = Ch_c(S_c(h^n))$.

(ii) For this case, we simplify the notation in such a way that for each $v \in C \cup \{\emptyset\}$, $S^*_v := S_v(h'^n)$. With regard to the proof of (i), we only need to consider college c' to which s_n made an application. Because s_n applied to c', $S^*_{c'} = S_{c'} \cup \{s_n\}$. Consider the DA algorithm that produces μ^{n-1} for $(\boldsymbol{P}(h^{n-1}), Ch)$. Since $\mu_{s_n}^{n-1} = \emptyset$ and c' is acceptable to him, s_n has proposed to that college in some step of the algorithm. Then using (2) we obtain $\mu_{c'}^{n-1} = Ch_{c'}(S_{c'} \cup \{s_n\})$. Combining this with $S^*_{c'} := S_{c'} \cup \{s_n\}$ we obtain $\mu_{c'}^{n-1} = Ch_{c'}(S^*_{c'}) = Ch_{c'}(S_{c'}(h'^n))$. □

Lemma 13. *Let $t = 1, ..., n-1$, $h^{t-1} \in \mathcal{H}_\pi^{t-1}$ and $\mu^{t-1} := DA(\boldsymbol{P}(h^{t-1}))$.*

(i) Assume that student s_t makes an application at h^{t-1} what solidary strategy would have recommended at h^{t-1}. Let $v \in C \cup \{\emptyset\}$ be such decision, $h^t = (h^{t-1}, v)$ the history that follows and $\mu^t := DA(\boldsymbol{P}(h^t))$. Then, the same set of students are matched in both μ^{t-1} and μ^t, that is, $\mu_C^{t-1} = \mu_C^t$.

(ii) Assume that $\mu_{s_t}^{t-1} = \emptyset$ and s_t applies to an acceptable college c' under P_{s_t} and let $h'^t = (h^{t-1}, c')$ and $\mu'^t := DA(\boldsymbol{P}(h'^t))$. Then, $\mu_C^{t-1} = \mu_C'^t$.

Proof of Lemma 13. (i) First, because $\boldsymbol{P}_{s_t}(h^{t-1}) = P_{s_t}$, we have $\boldsymbol{P}(h^{t-1}) = (P_{s_t}, \boldsymbol{P}_{-s_t}(h^{t-1}))$ and $\boldsymbol{P}(h^t) = (P_{s_t}^v, \boldsymbol{P}_{-s_t}(h^{t-1}))$. Next, because $P_{s_t}^v$ *i.r.m.t* P_{s_t}

at v, we have $\boldsymbol{P}(h^t)$ $i.r.m.t$ $\boldsymbol{P}(h^{t-1})$ at $DA(\boldsymbol{P}(h^{t-1}))$. Because DA is IR-monotonic, $DA(\boldsymbol{P}(h^t))$ $\boldsymbol{R}(h^t)$ $DA(\boldsymbol{P}(h^{t-1}))$ or equivalently,

$$\mu^t \ \boldsymbol{R}(h^t) \ \mu^{t-1}.$$

For s_t, $\mu^t_{s_t} = v$ because $\boldsymbol{P}_{s_t}(h^t) = P^v_{s_t}$ and $\mu^{t-1}_{s_t} = v$. For each student $s \neq s_t$, $\boldsymbol{P}_s(h^t) = \boldsymbol{P}_s(h^{t-1})$. Student s_t is matched to v in both μ^{t-1} and μ^t and the preference relation of each $s \neq s_t$ is the same in both $\boldsymbol{P}(h^{t-1})$ and $\boldsymbol{P}(h^t)$. Therefore,

$$\mu^t \ \boldsymbol{R}(h^{t-1}) \ \mu^{t-1}. \tag{3}$$

Now (3) and Lemma 2 give $\mu^{t-1}_C = \mu^t_C$.

(ii) First, by definition $\boldsymbol{P}_{s_t}(h^{t-1}) = P_{s_t}$ and $\boldsymbol{P}_{s_t}(h'^t) = (P^c_{s_t}, \boldsymbol{P}_{-s_t}(h^{t-1}))$. Since $DA_{s_t}(\boldsymbol{P}(h^{t-1})) = \emptyset$, DA is strategy-proof and $DA_{s_t}(\boldsymbol{P}(h'^t))$ is IR at $(\boldsymbol{P}_{s_t}(h'^t), Ch)$ we have $DA_{s_t}(\boldsymbol{P}(h'^t)) = \emptyset$. Because $\boldsymbol{P}(h'^t)$ $i.r.m.t$ $\boldsymbol{P}(h^{t-1})$ at $DA(\boldsymbol{P}(h^{t-1}))$ and DA is IR-monotonic, we have $DA(\boldsymbol{P}(h'^t))$ $\boldsymbol{R}(h'^t)$ $DA(\boldsymbol{P}(h^{t-1}))$ or equivalently

$$\mu'^t \ \boldsymbol{R}(h'^t) \ \mu^{t-1}.$$

Now with $\boldsymbol{P}_{-s_t}(h'^t) = \boldsymbol{P}_{-s_t}(h^{t-1})$ and $\mu^{t-1}_{s_t} = \emptyset = \mu'^t_{s_t}$, we conclude that

$$\mu'^t \ \boldsymbol{R}(h^{t-1}) \ \mu^{t-1}. \tag{4}$$

Finally, with (4) and Lemma 2 we obtain $\mu'^t_C = \mu^{t-1}_C$. □

We now turn to the proof of Theorem 1. We show that the strategy profile σ is a backwards-induction strategy of $G[\pi, M]$, by induction on $t = 1, ..., n$, starting from n.

Induction base $(t = n)$ **:** We verify that at any history h^{n-1}, s_n is at least as good at following σ_{s_n} at h^{n-1} as any other decision. Assume first that $DA_{s_n}(\boldsymbol{P}(h^{n-1})) = c'$ for some $c' \in C$. We show that if s_n applies to c', then c' will admit him. This is in fact the conclusion of Lemma 12. By this result, if c' is s_n's most preferred college according to P_{s_t}, then we are done.

Otherwise, let $c \in C$ be such that

$$c \ P_{s_n} \ c' \tag{5}$$

and assume that s_n applies to c at h^{n-1}. We show that c will not admit him. Let $\mu^{n-1} = DA(\boldsymbol{P}(h^{n-1}))$ and for the purpose of this proof let $S_c^* = S_c(h^{n-1})$. We next show that $\mu_c^{n-1} \subset S_c^*$. Let $s_t \in \mu_c^{n-1}$ with $t \neq n$. Then s_t applied to c and therefore $s_t \in S_c^*$. Therefore, $\mu_c^{n-1} \subseteq S_c^*$ and $\mu_c^{n-1} \cup \{s_n\} \subseteq S_c^* \cup \{s_n\}$. Since c's preference is substitutable, if $s_n \in Ch_c(S_c^* \cup \{s_n\})$, then

$$s_n \in Ch_c(S_c^* \cup \{s_n\}) \cap (\mu_c^{n-1} \cup \{s_n\}) \subseteq Ch_c(\mu_c^{n-1} \cup \{s_n\}). \tag{6}$$

From (5), (6) and the fact that $\mu_{s_n}^{n-1} = c'$, the pair (s_n, c) blocks μ^{n-1} at $(\boldsymbol{P}(h^{n-1}), Ch)$, contradicting the stability of μ^{n-1} at $(\boldsymbol{P}(h^{n-1}), Ch)$. In conclusion, $s_n \notin Ch_c(S_c^* \cup \{s_n\})$ and c will not admit s_n. Assume now that $DA_{s_n}(\boldsymbol{P}(h^{n-1})) = \emptyset$. Then, the last conclusion says that if s_n applies to an acceptable college, then he will not be admitted. In conclusion, σ_{s_n} is a component of a backwards-induction strategy profile.

Induction hypothesis : Let t be such that $t < n$ and assume that for each t' with $t < t' \leq n$ and each $h^{t'-1} \in \mathcal{H}_\pi^{t'-1}$, student $s_{t'}$ follows $\sigma_{s_{t'}}$ at $h^{t'-1}$.

Induction step : Let $h^{t-1} \in \mathcal{H}_\pi^{t-1}$. We show that s_t is at least as good at following σ_{s_t} at h^{t-1} as any other decision. We distinguish two cases :

Case 1 : $\sigma_{s_t}(h^{t-1}) = c'$ for some $c' \in C$.

We consider the case where s_t applies to c' (Case 1.1) and a case where s_t applies to a college c with $c \, P_{s_t} \, c'$ (Case 1.2), if any.

Case 1.1 : s_t applies to c'. Then c' will admit him.

Let $h^t, ..., h^n$ be the histories reached after s_t's application and each of the remaining students follows his strategy in σ. For each $t' = t-1, ..., n$, let $\mu^{t'} := DA(\boldsymbol{P}(h^{t'}))$. We know that $s_t \in \mu_{c'}^{t-1}$ since $DA_{s_t}(\boldsymbol{P}(h^{t-1})) := \sigma_{s_t}^*(h^{t-1}) = c'$ and $\mu^{t-1} := DA(\boldsymbol{P}(h^{t-1}))$. Now apply Lemma 13 to obtain $s_t \in \mu_C^t$,..., $s_t \in \mu_C^{n-1}$. Next because $\boldsymbol{P}_{s_t}(h^{n-1}) = P_{s_t}^{c'}$ and μ^{n-1} is IR at $(\boldsymbol{P}(h^{n-1}), Ch)$, $s_t \in \mu_C^{n-1}$ implies that $\mu_{s_t}^{n-1} = c'$. Finally, apply Lemma 12 to obtain that $s_t \in \mu_{c'}^{n-1} = Ch_{c'}(S_{c'}(h^n))$. Therefore, c' will admit s_t.

By Case 1.1, if c' is s_t's most preferred college, then we are done. Otherwise, let c be such that

$$c\ P_{s_t}\ c' = \sigma_{s_t}(h^{t-1}). \tag{7}$$

Case 1.2 : s_t applies to c. Then c will not admit him.
Let $h'^t, ..., h'^n$ be the histories reached after this application and each of the remaining students follows his strategy in σ. For each $t' = t-1, ..., n-1$, let $\mu'^t := DA(\boldsymbol{P}(h'^t))$. By definition, $\boldsymbol{P}(h^{t-1}) = (P_{s_t}, \boldsymbol{P}_{-s_t}(h^{t-1}))$ and $\boldsymbol{P}(h'^t) = (P^c_{s_t}, \boldsymbol{P}_{-s_t}(h^{t-1}))$. Because DA is strategy-proof, $DA_{s_t}(P_{s_t}, \boldsymbol{P}_{-s_t}(h^{t-1}))\ \ R_{s_t}$ $DA_{s_t}(P^c_{s_t}, \boldsymbol{P}_{-s_t}(h'^t))$ or equivalently

$$\mu^{t-1}_{s_t}\ \ R_{s_t}\ \ \mu'^t_{s_t}. \tag{8}$$

By (7) and (8) and because R_{s_t} is strict, $c\ P_{s_t}\ \mu'^t_{s_t}$ and thus $c \neq \mu'^t_{s_t}$. Now because $\boldsymbol{P}_{s_t}(h'^t) = P^c_{s_t}$ and $\mu'^t_{s_t}$ is IR at $(\boldsymbol{P}(h'^t), Ch)$, we have $\mu'^t_{s_t} = \emptyset$. Next, apply Lemma 13 to obtain that $s_t \notin \mu'^t_C, ..., s_t \notin \mu'^{n-1}_C$. Thus, $s_t \notin \mu'^{n-1}_c$. Finally apply Lemma 12 to obtain that $s_t \notin Ch_c(S_c(h'^n)) = \mu'^{n-1}_c$. Hence, c will not admit s_t.

Case 2 : $DA_{s_t}(\boldsymbol{P}(h^{t-1})) = \emptyset$.

By an argument similar to Case 1.2, s_t cannot be admitted by an acceptable college. Since applying to an unacceptable college is weakly dominated at the following subgame, s_t is at least as good as following σ_{s_t}.

In conclusion, σ is a backwards-induction strategy of $G[\pi, M]$.

Part 2 : Let $\sigma \in \Delta_\pi$ be an SPUE of $G[\pi, M]$. Then, for each $s \in S$, σ_s is either a bossy strategy or a solidary strategy.

Assume by contradiction that there exists some t, such that σ_{s_t} is neither a bossy strategy nor a solidary strategy of $G[\pi, M]$. Since the restriction of σ_{s_t} to every subgame is not weakly dominated, s_t does not apply to an unacceptable college under P_{s_t} at each history. Thus, there exists a history $h^{t-1} \in \mathcal{H}^{t-1}_\pi$ and a college c such that $DA_{s_t}(\boldsymbol{P}(h^{t-1})) = c$ and $\sigma_{s_t}(h^{t-1}) \neq c$. First, note that s_n is at least as good as following either a bossy strategy or a solidary strategy at each history he has to play. [3] Then, $t \neq n$. Inductively,

3. Note however that it is not a dominant strategy because it depends on firms strate-

let $t' < n$ and assume that for each $t'' = t' + 1, ..., n$, $t \neq t''$. We show that $t \neq t'$. By simplicity, let $\sigma_{s_n}(h^{t'-1}) = c'$ for some $c' \in C$. By Case 1.2 above, if $c' P_{s_{t'}} c$, then c' will not admit $s_{t'}$. However, by Case 1.1, c would have admit $s_{t'}$, had he applied to it. This contradicts the optimality of $\sigma_{s_{t'}}(h^{t'-1})$. In conclusion, $t \neq t'$ and t does not exist. This contradiction finishes the proof.

Appendix 2. *Proof of Theorem 2*

Let $\pi \in \mathcal{O}$ and $M = (P, Ch)$. It is enough if we prove that for each $\mu \in \mathcal{M}^{SPUE}(G[\pi, M])$ and each $s \in S$, $\mu_s \; R_s \; DA_s^{Ch}(P)$. Let σ be an SPUE of $G[\pi, M]$. By Theorem 1, σ is such that for each $s \in S$, σ_s is either a bossy strategy or a solidary strategy. Let $h^0, ..., h^n$ be the histories in the execution path of σ. For each $t' = 0, ..., n$, let $\mu^{t'} := DA(\boldsymbol{P}(h^{t'}))$. Let $t = 1, ..., n$ and recall that, by definition, for each $t' \leq t$, $\boldsymbol{P}_{s_t}(h^{t'}) = P_{s_t}$. Therefore, by (3) we have $\mu_{s_t}^{t-1} \; R_{s_t} ... \; R_{s_t} \; \mu_{s_t}^0$. First, $\mu_{s_t}^{t-1}$ is the outcome for s_t under σ. Second, $\mu^0 := DA^{Ch}(P)$. Therefore, it follows that

$$\mu_{s_t}^{t-1} \;\; R_{s_t} \;\; \mu_{s_t}^0.$$

Appendix 3. *Proof of Proposition 1*

Let $\pi \in \mathcal{O}$ and $M = (P, Ch)$. We prove that $\bigcup_{\pi \in \mathcal{O}} \mathcal{M}^{SPUE}(G[\pi, M])$ satisfies the rural hospital properties. Let $\{\mu, \mu'\} \subset \bigcup_{\pi \in \mathcal{O}} \mathcal{M}^{SPUE}(G[\pi, M])$ and $\mu^* := DA^{Ch}(P)$. By Theorem 2, $\mu \; R \; \mu^*$ and $\mu' \; R \; \mu^*$. It is sufficient if we prove that μ and μ^* satisfies part (i) and (ii) of the definition of the rural hospital properties.

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gies.

Part (i) follows from Lemma 2. For part (ii), assume that for some $c \in C$, $|\mu_c| < q_c$. By part (i), $|\mu^*| = |\mu_c| < q_c$. This part is complete if we show that $\mu_c \subseteq \mu_c^*$. Let $s \in \mu_c$. If $\mu_s^* \neq c$, then $c\, P_s\, \mu_s^*$, in contradiction with the stability of μ^* at M because $|\mu_c^*| < q_c$ and c's preference is acceptant. Thus, $\mu_c \subseteq \mu_c^*$; proving that $\mu_c = \mu_c^*$. Alva and Manjunath (2017) independently proved a similar result in a more general model.

Appendix 4. *Proof of Corollary 2*

Let $M = (P, Ch)$ and assume that it induces an order independent $\mathcal{G}$-outcome. Let $\mu \in \bigcup_{\pi \in \mathcal{O}} \mathcal{M}^{SPUE}(G[\pi, M])$. We show that $\mu = DA^{Ch}(P)$. Let $s \in S$ and $\pi \in \mathcal{O}$ be such that $\pi(1) = s$. Now by Theorem 1, $\mu_s = DA_s^{Ch}(P)$.

Appendix 5. *Proof of Theorem 3*

" $\Rightarrow$ ". Let $M = (P, Ch)$ and assume that it induces an order independent $\mathcal{G}$-outcome and let μ be this unique outcome. Consider DA^{Ch}. In the remainder we drop the reference to Ch in DA^{Ch}. By Corollary 1,

$$\text{for each } s \in S, \mu_s = DA_s(P). \tag{9}$$

We now show that DA is claims consistent at P. Let $S' \subseteq S$. We must show that $DA(P_{S'}^{DA}, P_{-S'}) = DA(P)$. Now let $s \in S$. We distinguish two cases :

Case 1 : $s \in S'$. Because $(P_{S'}^{DA}, P_{-S'})$ *i.r.m.t* P at $DA(P)$, and DA is IR-monotonic, it is easily established that $DA_s(P_{S'}^{DA}, P_{-S'}) = DA_s(P)$.

Case 2 : $s \notin S'$. Let $\pi \in \mathcal{O}$ be such that all students in S' are ordered first, and $\pi(|S'|+1) = s$. Let $h^{|S'|} = (h^o, a_1, ..., a_{|S'|})$ be a history in the execution path histories of the solidary strategy profile of $G[\pi, M]$. By (9), we have

$a_1 = DA_{s_1}(P)$,...,$a_{|S'|} = DA_{s_{|S'|}}(P)$ and thus $\boldsymbol{P}(h^{|S'|}) = (P^{DA}_{S'}, P_{-S'})$. Since $s = s_{|S'|+1}$, according to solidary strategy,

$$\mu_s = DA_s(P^{DA}_{S'}, P_{-S'}). \tag{10}$$

Combining (9) and (10) yield $DA_s(P^{DA}_{S'}, P_{-S'}) = DA_s(P)$.

Case 1 and Case 2 establish that DA is claims consistent at P.

"$\Leftarrow$". Let $M = (Ch, P)$ and assume that DA is claims consistent at P. We prove that M induces an order independent $\mathcal{G}$-outcome. Given $\pi \in \mathcal{O}$, let $\mu^\pi := \mathcal{M}^{SPUE}(G[\pi, M])$. We now show by induction on $t = 1, ..., n$ that for each $\pi \in \mathcal{O}$ and each $s \in S$, $\mu^\pi_s = DA_s(P)$. By Theorem 1, for each $\pi \in \mathcal{O}$ and each $s \in S$,

$$\pi(1) = s \Rightarrow \mu^\pi_s = DA_s(P). \tag{11}$$

Relation (11) is the induction base ($t = 1$). As an induction hypothesis, let $t > 1$ and assume that for each $t' < t$, each $\pi \in \mathcal{O}$ and each $s \in S$, $\pi(t') = s$ implies $\mu^\pi_s = DA_s(P)$. We now prove the induction step. Let $\pi \in \mathcal{O}$ and let $S' \equiv \{s_{t'} | t' < t\}$. Let $h^{t-1} = (h^o, a_1, ..., a_{t-1})$ be a history on the execution path of a solidary strategy. Then, by the induction assumption, for each $t' < t$, $a_{t'} = DA_{s_{t'}}(P)$. Thus, $\boldsymbol{P}(h^{t-1}) = (P^{DA}_{S'}, P_{-S'})$ and by Theorem 1,

$$\mu^\pi_{s_t} = DA_{s_t}(P^{DA}_{S'}, P_{-S'}). \tag{12}$$

Since DA is claims consistent at P, $DA_{s_t}(P^{DA}_{S'}, P_{-S'}) = DA_{s_t}(P)$; and together with (12) we obtain $\mu^\pi_{s_t} = DA_{s_t}(P)$.

Appendix 6. *Proof of Proposition 2*

Let $M = (P, Ch)$ and $\mu = DA^{Ch}(P)$. Let $\pi \in \mathcal{O}$ be constructed as follows : there is $s \in S$ such that for each $s' \in S$, $\mu_{s'}\ R_{s'}\ \mu_s$. Let $\pi(1) = s$. Let $\mu' = DA^{Ch}(P^{DA}_s, P_{-s})$. Then, there is $s' \in S \setminus \{s\}$ such that for each $s'' \in S \setminus \{s\}$, $\mu'_{s''}\ R_{s''}\ \mu'_{s'}$. Let $\pi(2) = s''$. We proceed so until a complete ordering. Then, μ is the solidary strategy outcome of $G[\pi, M]$. However, μ is Pareto efficient (Bando, 2014).

Appendix 7. *Proof of Corollary 3*

Part (i) follows from Theorem 2 and Lemma 2. For (ii) let $M = (P, Ch)$ be a market such that $DA_C(P) = S$ where $DA := DA^{Ch}$. Let $\pi \in \mathcal{O}$ and $\{\sigma^1, \sigma^2\} \subset \Delta_\pi$ two SPUEs of $G[\pi, M]$. We show that they have the same execution path. For each $i = 1, 2$ and each $s \in S$, σ^i_s is either a bossy strategy or a solidary strategy. By definition, $\boldsymbol{P}(h^0) = P$. Thus, since $DA_{s_1}(\boldsymbol{P}(h^0)) \in C$, $\sigma^1_{s_1}(h^0) = DA_{s_1}(\boldsymbol{P}(h^0)) = \sigma^2_{s_1}(h^0) = c_1$. Let $h^1 := (h^0, c_1)$. By Lemma 4 (i) and the fact that $DA_{s_2}(\boldsymbol{P}(h^0)) \in C$, we have $DA_{s_2}(\boldsymbol{P}(h^1)) \in C$. Because $\sigma^i_{s_2}$ is either a bossy strategy or a solidary strategy, $\sigma^1_{s_2}(h^1) = DA_{s_2}(\boldsymbol{P}(h^1)) = \sigma^2_{s_2}(h^1) = c_2$. Let $h^2 = (h^1, c_2)$. Now by induction, we can easily establish that σ^1 and σ^2 have the same execution path. Thus, the same matching is attached to the common path.

Appendix 8. *Proof of Proposition 3*

Let $M = (P, Ch)$ be such that $DA_C(P) = S$ where $DA := DA^{Ch}$. Let $s \in S$ and $\{\pi, \pi'\} \subset \mathcal{O}$ be such that $\pi|_{S\setminus\{s\}} = \pi'|_{S\setminus\{s\}}$ and $\pi'^{-1}(s) < \pi^{-1}(s)$. Without loss of generality, assume that π and π' are adjacent as represented below ; the difference between π and π' occurs only on the elements in boxes :

$$\pi' : s_1 ... s_{t-1} \boxed{s\ s'} ... s_n$$
$$\pi : s_1 ... s_{t-1} \boxed{s'\ s} ... s_n.$$

By Corollary 2, $\mathcal{M}^{SPUE}(G[\pi, M])$ and $\mathcal{M}^{SPUE}(G[\pi', M])$ are each a singleton. Let $\mu^\pi := \mathcal{M}^{SPUE}(G[\pi, M])$ and $\mu^{\pi'} := \mathcal{M}^{SPUE}(G[\pi', M])$. Let $\sigma \in \Delta_\pi$

be an SPUE of $G[\pi, M]$ and $\sigma' \in \Delta_{\pi'}$ an SPUE of $G[\pi', M]$. Consider the histories in their execution paths. Since the ordering of the first $t-1$ students and the market are the same in both games, the first t histories in these paths are the same. We represent them as follows :

$$\text{Execution path of } \sigma' : \quad h^0, ..., h^{t-1}, h'^t, ..., h'^n.$$
$$\text{Execution path of } \sigma : \quad h^0, ..., h^{t-1}, h^t, ..., h^n.$$

We show that $\mu_s^{\pi} \ R_s \ \mu_s^{\pi'}$. Since s decides after history h^{t-1} in $G[\pi', M]$ and after h^t in $G[\pi, M]$, we have

$$\mu_s^{\pi'} = DA_s(\boldsymbol{P}(h^{t-1})) \tag{13}$$

and

$$\mu_s^{\pi} = DA_s(\boldsymbol{P}(h^t)). \tag{14}$$

Now because $\boldsymbol{P}(h^t)$ *i.r.m.t* $\boldsymbol{P}(h^{t-1})$ at $DA(\boldsymbol{P}(h^{t-1}))$, $DA(\boldsymbol{P}(h^t)) \quad \boldsymbol{R}(h^t)$ $DA(\boldsymbol{P}(h^{t-1}))$ as DA is IR-monotonic. Because $\boldsymbol{R}_s(h^t) = R_s$, with (13) and (14) we obtain $\mu_s^{\pi} \ R_s \ \mu_s^{\pi'}$.

Proofs from Chapter 2

Appendix 9. *A useful lemma*

Lemma 14. *Let $\succeq$ be a strict priority profile, R a preference profile and i a student and suppose that $a = DA_i^{\succeq}(R)$.*

(a) Let b be a school and $\succeq'$ a priority profile such that $\succeq'_b|_{-i} = \succeq_b|_{-i}$ and for each school a' distinct from school b, $\succeq'_{a'} = \succeq_{a'}$. If $a \ P_i \ b$, then $DA_i^{\succeq'}(R) = a$.

(b) Let $\succeq'$ be a priority profile such that for each school b, $\succeq'_b |_{-i} = \succeq_b |_{-i}$ and $\succeq'_a = \succeq_a$. If school a is the first choice of student i, then $DA_i^{\succeq'}(R) = a$.

(c) Let $\succeq'$ be a priority profile such that, $\succeq'_{-a} = \succeq_{-a}$, $\succeq'_a |_{-i} = \succeq_a |_{-i}$ and the order of student i is higher under $\succeq'_a$ than under $\succeq_a$. Then, $DA_i^{\succeq'}(R) = a$.

(d) Let b be a school distinct from school a and R_i' a preference relation obtained from R_i by moving school b to the last position. Then $DA_i^{\succeq}(R_i', R_{-i}) = a$.

Proof.

(a) This follows directly from the DA algorithm. More specifically, if the relative order of a student in a school that he did not apply during the DA process changes, then the outcome remains the same for the new DA process.

(b) This is a repeated application of part (a).

(c) First, we know that DA respects improvement of the priority order (Balinski and Sönmez, 1999). So $DA_i^{\succeq'}(R)\ R_i\ DA_i^{\succeq}(R) = a$. Suppose now that $DA_i^{\succeq'}(R)\ P_i\ a$. Because $DA^{\succeq'}$ is individually rational, we have $b = DA_i^{\succeq'}(R)$ for some school b. Let R_i^b be a preference relation for student i in which school b is his first choice and $R^b = (R^b, R_{-i})$. Then, we have $DA_i^{\succeq'}(R^b) = b$ (Roth, 1982). Because $\succeq_b' = \succeq_b$ and $\succeq_{-b}' |_{-i} = \succeq_{-b} |_{-i}$, part (b) above implies that $DA_i^{\succeq}(R^b) = b$. Finally, $b = DA^{\succeq}(R^b)\ P_i\ DA_i^{\succeq}(R) = a$ implies that student i manipulates $DA^{\succeq}$ at R via R_i^b, contradicting the fact that $DA^{\succeq}$ is strategy-proof.

(d) If $a\ P_i\ b$, then the conclusion follows directed from the DA algorithm. If $b\ P_i\ a$, then $DA^{\succeq}(R_i', R_{-i})\ R_i\ a$ (Kojima and Manea, 2010a) – in Theorem 2.[4] If $DA^{\succeq}(R_i', R_{-i})\ P_i\ a$, then student i manipulates $DA^{\succeq}$ at R via R_i', contradicting the fact that $DA^{\succeq}$ is strategy-proof. □

Appendix 10. *Proof of Lemma 2*

Let $\tau = (\succeq^{CI}, \pi)$ be a priority profile, R a preference profile and $\overline{\succeq} = f(R, \succeq^{CI}, \pi)$. In studying the Boston mechanism, Pathak and Sönmez (2008) propose a model where some students are strategic while others are not and

4. This is because DA is weak Maskin monotonic.

report their preferences truthfully. They constructed adjusted economies that allow them to characterize the Nash equilibria of the game. Most importantly, when all students are not strategic, the adjusted economy for $(R, \succeq_\pi^{CI})$ coincides with $(R, \overline{\succeq})$. By Proposition 1 and Proposition 2 therein, there is a unique stable matching under $(R, \overline{\succeq})$ which also coincides with the outcome of the Boston mechanism $BM^{(\succeq^{CI}, \succeq_\pi)}$ for R. Therefore, $BM^\tau(R) = FM^\tau(R)$.

Appendix 11. *Proof of Proposition 5*

Let $\tau = (\succeq, \pi)$ be given and φ a τ-FP-stable matching mechanism. Let R be a preference profile that is vulnerable under FM^τ. There are two cases to consider :

Case 1 : $\varphi(R) = FM^\tau(R)$.

Since R is vulnerable under FM^τ, there is a student i and a preference relation R_i' such that $FM_i^\tau(R_i', R_{-i})\ P_i\ FM_i^\tau(R)$. Since FM^τ is individually rational, $FM_i^\tau(R_i', R_{-i}) = a$ for some school a. Because $\varphi(R) = FM^\tau(R)$, we have

$$a\ P_i\ \varphi_i(R). \tag{15}$$

Let R_i^a be a preference relation for student i in which school a is his only acceptable school and $R^a = (R_i^a, R_{-i})$. By Lemma 3, $FM_i^\tau(R^a) = a$. Let $\overline{\succeq}' = f(R^a, \succeq, \pi)$. Because $FM_i^\tau(R^a) = a$ and $FM^\tau(R^a)$ is stable under $(R^a, \overline{\succeq}')$, then by the the fact that the set of students who are assigned is the same for all stable matchings and $\varphi(R^a)$ is stable under $(R^a, \overline{\succeq}')$, student i is assigned at $\varphi(R^a)$. Therefore, we have $\varphi_i(R^a) = a$. Equation (15) implies $\varphi_i(R^a)\ P_i\ \varphi_i(R)$, which shows that R is vulnerable under φ.

Case 2 : $\varphi(R) \neq FM^\tau(R)$.

Let $\overline{\succeq} = f(R, \succeq, \pi)$. By optimality of $FM^\tau(R)$, for each student i, $FM_i^\tau(R)\ R_i\ \varphi_i(R)$ and a strict relation for some student j. Because φ is individually rational, $FM^{\tau}_j(R) = a$ for some school a. Therefore, we have

$$a\ P_j\ \varphi_j(R). \tag{16}$$

Let R^a_j be a preference relation for student j in which school a is his only acceptable school, $R^a = (R^a_j, R_{-j})$ and $\overline{\succeq}' = f(R^a, \succeq, \pi)$. By Lemma 3, $FM^{\tau}_j(R^a) = a$. By arguments similar to **Case 1**, $\varphi_j(R^a) = a$. Therefore, Equation (16) implies that R is vulnerable under φ.

Appendix 12. *Proof of Lemma 3*

Let $\tau = (\succeq, \pi)$ and $\overline{\succeq} = f(R, \succeq, \pi)$. By definition, $FM^{\tau}(R) = DA^{\overline{\succeq}}(R)$. Suppose that $FM^{\tau}_i(R) = a$. First, we use the following property of DA (Roth, 1982).

$$DA^{\overline{\succeq}}_i(R^a) = a. \tag{17}$$

Let $\overline{\succeq}' = f(R^a, \succeq, \pi)$. Because $R^a_{-i} = R_{-i}$, we have

$$\overline{\succeq}'|_{-i} = \overline{\succeq}|_{-i}. \tag{18}$$

Because school a is the first choice of student i under R^a_i, Equation (17) implies that under the DA process that leads $DA^{\overline{\succeq}}(R^a)$, student i did not propose to a school other than a. By Equation (18) and Lemma 14 (b), we have

$$DA^{\overline{\succeq}}(R^a) = DA^{(\overline{\succeq}_a, \overline{\succeq}'_{-a})}(R^a). \tag{19}$$

From $\overline{\succeq}_a$ to $\overline{\succeq}'_a$ the priority order of student i has weakly improved because he has ranked school a first under R^a_i. By Lemma 14 (c), we have

$$DA^{\overline{\succeq}'}_i(R^a) = DA^{(\overline{\succeq}_a, \overline{\succeq}'_{-a})}_i(R^a). \tag{20}$$

Equations (17), (19) and (20) imply that $DA^{\overline{\succeq}'}_i(R^a) = a$. Therefore, $FM^{\tau}_i(R^a) = a$.

Appendix 13. *Proof of Proposition 6 and Proposition 5 (ii)*

Let $\tau = (\succeq, \pi)$ be a priority profile and R a preference profile. Suppose that student i can manipulate FM^τ under R to obtain a seat at school a. That is, school a is strategically accessible by student i via FM^τ. Let φ be a τ-FP-stable matching mechanism. Then, there is a preference relation R_i' for student i such that $FM_i^\tau(R_i', R_{-i}) = a$ and $a\ P_i\ FM_i^\tau(R)$. By optimality of $FM^\tau(R)$, we have $FM_i^\tau\ R_i\ \varphi_i(R)$. Therefore, $a\ P_i\ \varphi_i(R)$. Let R_i^a be a preference relation for student i in which school a is the unique acceptable school and $R^a = (R_i^a, R_{-i})$. By Lemma 3, $FM_i^\tau(R^a) = a$. Let $\overline{\succeq}' = f(R^a, \succeq, \pi)$. By the fact that the set of students who are matched is the same in all stable matchings (Roth, 1982) and the fact that $\varphi(R^a)$ is stable under $(R, \overline{\succeq}')$, we have $\varphi_i(R^a) = a$. Therefore, $\varphi_i(R^a) = a\ P_i\ \varphi_i(R)$. This shows that the mechanism φ is manipulable by student i under R, proving Proposition 5 (ii) when φ is the school optimal τ-FP-stable matching mechanism. The relation also shows that school a is strategically accessible to student i via φ, also proving Proposition 6 (i) and Proposition 6 (ii) when φ is the school optimal τ-FP-stable matching mechanism.

Appendix 14. *Proof of Theorem 4*

Let $\tau = (\succeq, \pi)$ be a priority profile and R a preference profile. Let $\overline{\succeq} = f(R, \succeq, \pi)$. Suppose that R is not vulnerable under the first preference first mechanism FPF^τ.

Step 1 : *If any student i prefers a first preference first school a to $FPF_i^\tau(R)$, then all the capacity of school a are filled in the first step with students who* *have higher priority than student i under $\succeq_{\pi_a}$.*

Suppose, to the opposite, that a seat of school a is either (1) assigned in a step later than the first step or is assigned in the first step to a student j such that $i \succ_{\pi_a} j$. Let R_i^a be student i's preference relation in which he ranks school a first and $R^a = (R_i^a, R_{-i})$. Let $\succeq'$ be the adjusted priority order for R and τ. Then, $FPF^\tau(R^a) = DA^{\succ'}(R^a)$.

Case (1) : There are less students who rank school a first than its capacity. Therefore, student i is among the the top q_a ranked students under $\succeq_{\pi_a}$ who rank school a first under (R_i^a, R_{-i}). Therefore, student i is among the top q_a ranked students under the adjusted priority order $\succeq'_a$. Because $FPF^\tau(R^a)$ is stable under $(R^a, \succeq')$, student i is assigned to a seat at school a under $FPF^\tau(R^a)$. Because $a \; P_i \; FPF_i^\tau(R)$, FPF^τ is manipulable under R. This contradicts the assumption that R is not vulnerable under FPF^τ.

Case (2) : There are less than q_a students who ranked a first and have higher priority than student i under $\succeq_{\pi_a}$. Therefore, student i is among the top q_a ranked students under $\succeq'_a$. By the same reason, $FPF_i^\tau(R^a) = a$, and the same contradiction follows.

Step 2 : *R is not vulnerable under the French mechanism FM^τ.*

We next show that

$$FM^\tau(R) = FPF^\tau(R).$$

Let $\mu = FPF(R)$. Let $\succeq'$ be the priority profile obtained by adjusting $\succeq$ as in the first preference first mechanism. First, we claim that μ is FP-stable under (R, τ). First, by assumption, the basic priority of each equal preference school is strict. Therefore, $\overline{\succeq}_a = \succeq_a$ and μ has no blocking pair involving such a school under $(R, \overline{\succeq})$. Suppose that there is a student i and a first preference school a such that $a \; P_i \; \mu(i)$. By **Step 1**, any student $j \in \mu^{-1}(a)$ has ranked school a first and $j \succ_{\pi_a} i$. Therefore, school a is the first relative rank and the first absolute rank of student j. Then $j \; \overline{\succ}_a \; i$. By optimality of $FM^\tau(R)$, for each student i,

$$FM_i^\tau(R)\ R_i\ \mu(i). \tag{21}$$

For each student j who is matched to his first choice under μ, $FM_j^\tau(R) = \mu(j)$. Suppose that for some student j, $FM_j^\tau(R)\ P_j\ \mu(j)$. Since μ is the student-optimal stable matching under $(R, \succeq')$, Equation (32) implies that $FM^\tau(R)$ is not stable under $(R, \succeq')$. Because μ is stable under $(R, \succeq')$, Equation (32) implies also that there is a blocking pair (i, a) for $FM^\tau(R)$ under $(R, \succeq')$. School a has no seat available, otherwise, μ wouldn't be stable under $(R, \succeq')$. If school a is an equal preference school, then by the assumption that its basic priority is strict, $\succeq'_a = \overline{\succeq}_a$. Therefore, (i, a) is a blocking pair for $FM^\tau(R)$ under $(R, \overline{\succeq})$, which contradicts that $FM^\tau(R)$ is FP-Stable under (R, τ). Therefore, school a is a first preference first school. By $a\ P_i\ \mu(i)$ and **Step 1**, each student in $\mu^{-1}(a)$ ranked school a first. Therefore, school a is assigned to one more student under $FM^\tau(R)$. Therefore, school a has a vacant seat under μ. This contradicts $a\ P_i\ \mu(i)$ and FP-stability of μ under (R, τ). Therefore, $FM^\tau(R) = FPF^\tau(R)$.

We complete now the proof of the theorem. Let $\mu = FM^\tau(R)$ and suppose that for some school a, $a\ P_i\ \mu(i)$. We consider two cases :

Case (1) : School a is an equal preference school. Since school a has a strict basic priority order by assumption, Theorem 5 (i) implies that school a is not strategically accessible via FM^τ.

Case (2) : School a is a first preference first school. Let $I' = \mu^{-1}(a)$. Since $a\ P_i\ FM_i^\tau(R) = \mu(i)$, by stability of μ under $(R, \overline{\succeq})$, we have $|I'| = q_a$. By **Step 1**, each student in I' ranked school a first and for each $j \in I'$, $j \succ_{\pi_a} i$. Let $\mu' = \varphi(R^a)$ and suppose that $\mu'(i) = a$. Then, for some $j \in I'$, $\mu'(j) \neq a$. Therefore, $a\ P_j\ \mu'(j)$.

Let $\overline{\succeq}' = f(R^a, \succeq, \pi)$. Since student i and student j both ranked school a first and $j \succ_{\pi_a} i$, we have $j\ \overline{\succ}'_a\ i$. Because μ' is stable under $(R^a, \overline{\succeq}')$, this contradicts the facts that $\mu'(i) = a$ and $a\ P_j\ \mu'(j)$. Therefore, by Lemma 3, student i could not manipulate FM^τ and obtain a seat at school a under R. That is, R is not vulnerable under FM^τ.

Appendix 15. *Proof of Lemma 6 and Theorem 5*

Let $\tau = (\succeq, \pi)$ be given. Let i be a student, R a preference profile, R'_i an alternative preference relation of student i, $FM^\tau(R) = \mu$ and $FM^\tau(R'_i, R_{-i}) = \nu$ and suppose that

$$a = \nu(i) \; P_i \; \mu(i), \tag{22}$$

in which student i does not have an equal priority with any other student under $\succeq_a$. Let R^a_i be his preference relation in which he ranked school a first and $R^a = (R^a_i, R_{-i})$. Let $\overline{\succeq} = f(R, \succeq, \pi)$ and $\overline{\succeq}' = f(R^a, \succeq, \pi)$. By Lemma 3, $FM^\tau_i(R^a) = DA^{\overline{\succeq}'}_i(R^a) = a$. From argument similar to the one that established Equation (19), we have

$$DA^{(\overline{\succeq}'_a, \overline{\succeq}_{-a})}_i(R^a) = a. \tag{23}$$

Because, student i does not have equal priority with any other student at $\succeq_a$, we have

$$\overline{\succeq}_a = \overline{\succeq}'_a. \tag{24}$$

Therefore, by replacing Equation (24) in (23), we obtain the following

$$DA^{\overline{\succeq}}_i(R^a) = a. \tag{25}$$

Because $DA^{\overline{\succeq}}(R) = \mu$, Equations (22) and (25), imply that student i manipulates $DA^{\overline{\succeq}}$ under R via R^a_i. This contradicts the fact that $DA^{\overline{\succeq}}$ is strategy-proof.

Because the basic priority of each selective school is strict, Theorem 5 follows.

Appendix 16. *Proof of Proposition 8*

Let R be a profile of correlated preferences, $\tau = (\succeq, \pi)$ a priority profile and $\tau' = (\succeq', \pi)$ a π-consistent split of τ. In accordance with the definition of the French tie breaking, let π' be the strict tie breaking rule constructed under π and R. Because preferences are correlated under R, that is, students have the same preference relation under R, for each school a, $\pi'_a = \pi_a$. Hence,

$$\pi' = \pi. \tag{26}$$

Because τ' is a consistent split of τ, we also have $\succeq'_\pi = \succeq_\pi$. Therefore, by Equation (26)

$$\succeq'_{\pi'} = \succeq_{\pi'} . \tag{27}$$

Consequently, the following equally holds. [5]

$$FM^{\tau}(R) = FM^{\tau'}(R).$$

Suppose that student i can manipulate $FM^{\tau'}$ under R and obtain a seat at school b. We show that he can also manipulate FM^{τ} under R and obtain a seat at school b. Let R_i^b be a preference relation in which student i ranks school b first and $R' = (R_i^b, R_{-i})$. By Lemma 3, $FM_i^{\tau'}(R') = b$. By Lemma 4, school b is not the first choice of student i under R_i. Hence, school b is not the first choice in the common preference relation. Let π''_b be the French tie breaking rule constructed under π the preference profile R'. Because no student but student i ranked school b first, student i is ordered first under π''_b. Next, because $R'_{-i} = R_{-i}$, we have $\pi''|_{-i} = \pi'|_{-i}$. By replacing this equality in Equation (27), we have :

$$\succeq_{\pi''}|_{-i} = \succeq'_{\pi''}|_{-i}. \tag{28}$$

That is, the ordering of students other than student i is the same under $\succeq_{\pi''_b}$ and $\succeq'_{\pi''_b}$. Then we can compare the ordering of student i under $\succeq_{\pi''_b}$ and $\succeq'_{\pi''_b}$.

Step 1 : The position of student i under $\succeq_{\pi'_b}$ did not decrease compared to $\succeq'_{\pi'_b}$.

5. Recall that $FM^{\tau}(R) = DA^{\succeq_{\pi'}}(R)$ and $FM^{\tau'}(R) = DA^{\succeq'_{\pi'}}(R)$.

Let $E^i_{\succeq_b}$ be the indifference class containing student i under $\succeq_b$. Because π''_b simply breaks ties in $\succeq_b$, the ordering of student i relative to any student not in $E^i_{\succeq_b}$ does not change. Because student i is ordered first under π''_b, by breaking the indifference class $E^i_{\succeq_b}$, student i is ordered higher than any student in $E^i_{\succeq}$ under $\succeq_{\pi''_b}$. Observe that because $\succeq'_b$ is a split of $\succeq_b$, the ordering of student i relative to any student not in $E^i_{\succeq_b}$ is the same under both $\succeq_{\pi''_b}$ and $\succeq'_{\pi''_b}$. Therefore, the ordering of student i is higher under $\succeq_{\pi''_b}$ compared to $\succeq'_{\pi''_b}$ or his ordering remains the same under both.

Step 2 : Student i is assigned to a seat at school b under $FM^{\tau}(R')$.

First, recall that $FM^{\tau}(R') = DA^{\succeq_{\pi''}}(R') := \mu$. Since $FM_i^{\tau'}(R') = b$ and $FM^{\tau'}(R') = DA^{\succeq'_{\pi''}}(R')$, we have $DA_i^{\succeq'_{\pi''}}(R') = b$. By Equation (28) and Lemma 14 (b), the later equality implies that

$$DA_i^{(\succeq'_{\pi''_b}, \succeq_{\pi''_{-b}})}(R') = b.$$

This equation, Equation (28) and Lemma 14 (c) imply that $DA_i^{\succeq_{\pi''}}(R') = b$. Therefore, $FM_i^{\tau}(R') = b$. Because $b \; P_i \; FM_i^{\tau'}(R)$ and $FM^{\tau}(R) = FM^{\tau'}(R)$, we have $b \; P_i \; FM_i^{\tau}(R)$. Then $FM_i^{\tau}(R') = b$ implies student i can manipulate FM^{τ} under R and obtain a seat at school b.

Towards providing an example in which the French mechanism induced by a priority profile $\tau = (\succeq, \pi)$ is manipulable but not the mechanism induced by a split of τ, let i be a student who can manipulate FM^{τ} under R. Let $\tau' = (\succeq', \pi)$ be a priority profile such for each school a, $\succeq'_a$ is a minimal π_a-consistent isolation of i. By Lemma 6, student i cannot manipulate $FM^{\tau'}$ under R.

Appendix 17. *Proof of Lemma 7*

(1) ⇒ (2). We prove this par contrapositive. Suppose that for some priority structure $\tau = (\succeq, \pi)$, a tie $i \sim_b j$ with student i has a quasi-cycle constituted of three distinct students i, j and k, three schools a, b and c and three subsets I_a, I_b and I_c of students satisfying (1), (1-a) or (1-b) and (2) of Definition 1.

Claim 1. *Student i can manipulate FM^τ to obtain a seat at school b.*

Consider the following components of some preference profiles R and R' :

R_i	$R_{I_a\cup\{k\}}$	$R_{I_b\cup\{j\}}$
a	a	b
b	b	a
$\emptyset$	$\emptyset$	$\emptyset$

R'_i	$R'_{I_a\cup\{k\}}$	R'_{I_b}	$R'_{I_c\cup\{j\}}$
a	a	b	c
b	b	a	b
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

If (1-a) holds consider the preference profile R in which the components of students in $I_a \cup I_b \cup \{i, j, k\}$ are specified as above and the component of each student not in that set is a preference relation in which no school is acceptable. Then, $FM_i^\tau(R) = \emptyset$. Let R_i^b be a preference relation in which school b is the first choice of student i. Because $i \; \pi_b \; j$, we have $FM_i^\tau(R_i^b, R_{-i}) = b$. Thus, student i manipulates FM^τ to obtain a seat at school b. If (1-a) does not hold, consider a preference profile R' in which the components of students in $I_a \cup I_b \cup I_c \cup \{i, j, k\}$ are specified as above and the component of each student not in that set is a preference relation in which no school is acceptable. Since $j \; \pi_b \; i$, we have $FM_i^\tau(R') = \emptyset$. However, $FM_i^\tau(R_i^b, R'_{-i}) = b$. Thus, student i manipulates FM^τ to obtain a seat at school b.

(2) ⇒ (1). We prove this by the contrapositive. Let $\tau = (\succeq, \pi)$ be a priority structure and suppose that student i manipulates FM^τ at R via R'_i to obtain a seat at school b. That is, $b = FM_i^\tau(R'_i, R_{-i})$ and $b \; P_i \; FM^\tau(R)$.

Claim 2. *There is a tie $i \sim_b j$ with student i that has a quasi-cycle.*

Let school a be the first choice of student i under R_i. By Lemma 4, $b \neq a$. Let R_i^b be a preference relation of student i in which school b is his first choice and $R^b = (R_i^b, R_{-i})$. By Lemma 3, $FM_i^\tau(R^b) = b$. Let $\overline{\succeq} = f(R, \succeq, \pi)$ and $\overline{\succeq}' = f(R^b, \succeq, \pi)$. Consider the mechanism $DA^{(\overline{\succeq}'_b, \overline{\succeq}_{-b})}$.

Claim 3. *$DA_i^{(\overline{\succeq}'_b, \overline{\succeq}_{-b})}(R) = b$.*

First, because $FM^\tau(R^b) = b \; P_i \; FM^\tau(R)$, we have $b \; P_i \; DA_i^{\overline{\succeq}}(R)$. Second, because $\overline{\succeq}'_b|_{-i} = \overline{\succeq}_b|_{-i}$, the algorithms to $DA^{\overline{\succeq}}(R)$ and $DA^{(\overline{\succeq}'_b, \overline{\succeq}_{-b})}(R)$ have the same steps before the step in which student i applied to school b. Therefore, $b \; R_i \; DA_i^{(\overline{\succeq}'_b, \overline{\succeq}_{-b})}(R)$. Suppose now that $b \; P_i \; DA_i^{(\overline{\succeq}'_b, \overline{\succeq}_{-b})}(R)$. By Equation (19), in which school b plays the role of school a, we have $DA_i^{(\overline{\succeq}'_b, \overline{\succeq}_{-b})}(R^b) = DA_i^{\overline{\succeq}'}(R^b) = b$. Hence, student i manipulates $DA^{(\overline{\succeq}'_b, \overline{\succeq}_{-b})}$ at R via R^b, contrary to $DA^{(\overline{\succeq}'_b, \overline{\succeq}_{-b})}$ being strategy-proof. Hence, $DA_i^{(\overline{\succeq}'_b, \overline{\succeq}_{-b})}(R) = b$.

Let $r > 1$ be the first step of the DA algorithm that leads to $\mu = DA^{(\overline{\succeq}'_b, \overline{\succeq}_{-b})}(R)$ in which student i proposed to school b. Since student i can manipulate FM^τ to obtain a seat at school b, by Lemma 6, student i has a tie with another student under $\succeq_b$. Thus, there is at least one student in $E^i_{\succeq_b}$ distinct from student i.

Consider the DA processes for $(R, \overline{\succeq})$ and $(R, (\overline{\succeq}'_b, \overline{\succeq}_{-b}))$ and note that they have the same steps until $r - 1$. We continue by establishing two steps.

Step 1 : In at least one step in $\{r, \ldots, \overline{r}\}$ of the DA process for $(R, (\overline{\succeq}'_b, \overline{\succeq}_{-b}))$, school b either held a student from $E^i_{\succeq_b}$ distinct from student i or received an application from such a student.

Suppose, to the contrary, that this is not the case. Then either any application to school b from a student in $E^i_{\succeq_b}$ distinct from student i has been received and rejected before Step r or no such a student ever applied to school b. In either case, school b received the application of student i without any other student from $E^i_{\succeq_b}$ to be ever considered. Because the DA processes under $(R, \overline{\succeq})$ and $(R, (\overline{\succeq}'_b, \overline{\succeq}_{-b}))$ have the same steps until $r - 1$, then at Step r, school b is holding on the same set of students and received the same applications in both processes. Let $I' = E^i_{\succeq_b} \setminus \{i\}$, and note that $\overline{\succeq}'_b|_{I\setminus I'} = \overline{\succeq}_b|_{I\setminus I'}$. Because school b does not hold and does not receive an application from a student in I', both processes have the same steps until r. It is now easy to see that, by induction, they have the same steps until $\overline{r}$. Because $DA_i^{(\overline{\succeq}'_b,\overline{\succeq}_{-b})}(R) = b$, we have $DA_i^{\overline{\succeq}}(R) = b$. This contradicts the fact that $b \; P_i \; DA_i^{\overline{\succeq}}(R) = FM^{\tau}(R)$.

Step 2 : In at least one step in $\{r, \dots, \overline{r}\}$, at least one student from $E^i_{\succeq_b}$ distinct from student i is rejected by school b in the DA process for $(R, (\overline{\succeq}'_b, \overline{\succeq}_{-b}))$.

Suppose, to the contrary, that this is not the case. Because in at least one step in $\{r, \dots, \overline{r}\}$ school b held or has received an application from at least one student from $E^i_{\succeq_b}$, distinct from student i, then one such student is assigned a seat at school b under $DA^{(\overline{\succeq}'_b,\overline{\succeq}_{-b})}(R)$. Therefore, the set of students who could be rejected by school b after Step $r-1$, is a subset of the set $\{j' | i \succ_b j'\}$ of student who have less priority than student i under $\succeq_b$. We claim that

$$DA^{\overline{\succeq}}(R) = DA^{(\overline{\succeq}'_b,\overline{\succeq}_{-b})}(R).$$

First, recall that the DA processes under $(R, \overline{\succeq})$ and $(R, (\overline{\succeq}'_b, \overline{\succeq}_{-b}))$ have the same steps until $r-1$. Therefore, at Step r of both processes each school a' has received the same new applications and thus has to choose from the same set $\overline{I}^r_{a'}$ of students. Let k be a student who is rejected by school b and j a student who is tentatively accepted by school b both at Step r of the DA process for $(R, (\overline{\succeq}'_b, \overline{\succeq}_{-b}))$. Then $i \succ_b j$ and $k \succeq_b i$. Therefore, $k \succ_b j$. Then, by definition of $\overline{\succeq}_b$ we have $j \; \overline{\succ}_b \; k$. Hence, the set of students in $\overline{I}^r_b$ who are in excess of capacity under $\overline{\succeq}'_b$ remained lower ranked under $\overline{\succeq}_b$. Therefore, school b holds on to the same set of students at Step r of both processes. Both processes have the same steps until r. By induction, both processes have the same steps and thus $DA^{\overline{\succeq}}(R) = DA^{(\overline{\succeq}'_b,\overline{\succeq}_{-b})}(R)$. Because $DA_i^{(\overline{\succeq}'_b,\overline{\succeq}_{-b})}(R) = b$ by Claim 3, we have $DA_i^{\overline{\succeq}}(R) = b$. This contradicts the fact that $b\ P_i\ DA_i^{\overline{\succeq}}(R) = FM^{\tau}(R)$.

Let r' be the first step in which a student from $E^i_{\succeq_b}$, distinct from student i, is rejected after Step r. Let j be one such student. We now construct a quasi-cycle of the tie $i \sim_b j$ with student i. For each school a', let $I^{r'}_{a'}$ be the set of students that school a' held at step r' of the DA process under $(R, (\overline{\succeq}'_b, \overline{\succeq}_{-b}))$. Let $k \in I^{r'}_a$ and $I_a = I^{r'}_a \setminus \{k\}$. Let $I_b = I^{r'}_b \setminus \{i\}$. Since school a is the first choice of student i, for each $\ell \in I_a \cup \{k\}$, $\ell \succ_{\pi_a} i$. Next, because FM^{τ} does not violate the basic priorities, for each $\ell \in I_b$, $\ell \succeq_b j$. Furthermore, I_a and I_b are disjoint because no student is held by two schools at each step. Therefore, i, j, k and I_a and I_b satisfy (1) and (2) of Definition 1. It remains to verify that (1-a) or (1-b) of Definition 1 is satisfied. Because student j is rejected by school b, he has at least one acceptable school under R_j. Let c be his first choice under R_j. We consider two cases :

Case 1 : $c = b$. Because student j is rejected by school b and $DA_i^{(\overline{\succeq}'_b, \overline{\succeq}_{-b})}(R) = b$, we have $i \mathrel{\overline{\succ}'_b} j$. Since $\overline{\succeq}' = f(R^b, \succeq, \pi)$, both student i and j have ranked school b first under R^b. Hence, $i\ \pi_b\ j$.

Case 2 : $c \neq b$. Let $I'_c = I^{r'}_c$. Because student j is rejected by school b, he is also rejected by school c at a step less than r'. Hence, at Step r', the capacity of school c is full, that is, $|I'_c| = q_c$. Because school c is the first choice of student j under R_j, for each $\ell \in I'_c$, $\ell \mathrel{\overline{\succ}_c} j$ and, by definition, for each $\ell \in I'_c$, $\ell \succ_{\pi_c} j$. Because no student is held by two distinct schools at the same step, I_c and I_b are disjoint, I'_c and $I_a \cup \{k\}$ are disjoint if $c \neq a$ and if $c = a$, $I'_c = I^{r'}_a = I_a \cup \{k\}$.

Case 1 and Case 2 show that (1-a) or (1-b) of the Definition 1 is satisfied.

Appendix 18. *Proof of Lemma 5*

Let $\tau = (\succeq, \pi)$ be a priority profile and suppose that school a is strategically accessible to student i via FM^{τ}. By Claim 2 there is a tie $i \sim_a j$ with student i which has a quasi-cycle. By Claim 1 and the example constructed wherein student i can manipulate FM^{τ} and obtain a seat at school a under a preference profile R in which school a is his second choice. By Lemma 4, school a is the best school that student i can manipulate and obtain a seat at under R.

Appendix 19. *Proof of Theorem 6*

Let $\tau' = (\succeq', \pi)$ be a consistent split of $\tau = (\succeq, \pi)$ and suppose that school b is strategically accessible to student i via mechanism $FM^{\tau'}$. Then, for some R and R_i', $FM_i^{\tau'}(R_i', R_{-i}) = b$ and $b \; P_i \; FM_i^{\tau'}(R)$. By Claim 2, there is a tie $i \sim_b' j$ with student i which has a quasi-cycle. By the assumption of consistent split, $i \sim_b j$ and $\succeq_\pi' = \succeq_\pi$. Hence, the quasi-cycle of the tie $i \sim_b' j$ with student i is also a quasi-cycle of the tie $i \sim_b j$ with student i. By Claim 1, student i can manipulate FM^{τ} to obtain a seat at school b, showing that school b is also strategically accessible by student i via FM^{τ}.

Appendix 20. *Proof of Corollary 7*

Let $\tau = (\succeq, \pi)$ be given and $\tau' = (\succeq^{CI}, \succeq_\pi)$. Then $BM^{\tau} = FM^{\tau'}$. First, for each school a and each student i, we have $\succeq_{\pi_a}|_{E^i_{\succeq_a}} = \pi_a|_{E^i_{\succeq_a}}$, that is, the restriction of the tiebreaker $\succeq_{\pi_a}$ and π_a to the indifference class containing student i coincide. Therefore, $FM^{(\succeq, \succeq_\pi)} = FM^{\tau}$. Second, $(\succeq, \succeq_\pi)$ is a consistent split of $(\succeq^{CI}, \succeq_\pi)$. Therefore, by Theorem 6, $FM^{\tau'} = BM^{\tau}$ is as strategically accessible as FM^{τ}. For any strict basic priority, the French mechanism is strategy-proof and thus not strategically accessible whereas for some strict basic priorities, the Boston mechanism is strategically accessible.

Appendix 21. *Proof of Theorem 7*

Let $\tau^0 = (\succeq, \pi)$ be a priority profile and suppose that school a is strategically accessible by student i via FM^{τ^0}. Let $\succeq'_a$ be a π_a-consistent minimal isolation of student i in $\succeq_a$. Let $\succeq' = (\succeq'_a, \succeq_{-a})$ and $\tau^1 = (\succeq', \pi)$. By Lemma 6, school a is not strategically accessible by student i via FM^{τ^1}. Together with Proposition 7, FM^{τ^1} is less strategically accessible than FM^{τ^0}. If FM^{τ^1} is not strategically accessible, let $T = 1$ and $\tau^1 = \succeq_\pi$. Otherwise, choose a school which is strategically accessible and proceed as before. It is easy now to follow this procedure and construct the desired sequence.

Appendix 22. *Proof of Proposition 9*

Let R and R'_i be as specified in the proposition. We say that R'_i is a reshuffling of the last $m-k$ choices of R_i. Let a be a school ranked in the first k choices under R_i. Let π be (arbitrary) given and $\tau = (\succeq, \pi)$. Let $\overline{\succeq} = f(R, \succeq, \pi)$ and $\overline{\succeq}' = f(R', \succeq, \pi)$. By definition, $FM^\tau(R) = DA^{\overline{\succeq}}(R)$ and $FM^\tau(R') = DA^{\overline{\succeq}'}(R')$. To prove the proposition, it is enough to show that, $DA_i^{\overline{\succeq}}(R) = a$ if and only if $DA_i^{\overline{\succeq}'}(R'_i, R_{-i}) = a$.

Case 1 : $DA_i^{\overline{\succeq}}(R) = a$.

First, observe that for each school b ranked in the first k choices of R_i, we have $\overline{\succeq}_b = \overline{\succeq}'_b$ and for each school b ranked in the last $m-k$ choices of R_i, we have $\overline{\succeq}_b|_{-i} = \overline{\succeq}'_b|_{-i}$. Second, because R_i and R'_i agree in the first k ranking, the steps of the algorithms that leads to $DA^{\overline{\succeq}}(R)$ and $DA^{\overline{\succeq}'}(R')$ are the same. This is because schools that have different priorities are ranked below

a under both R_i and R'_i and the relative orderings of students other than i at those schools are the same. Therefore, $DA^{\overline{\succsim}}(R) = DA^{\overline{\succsim}'}(R')$.

Case 2 : $DA_i^{\overline{\succsim}}(R) \neq a$.

If $DA_i^{\overline{\succsim}}(R) = b$ for some school b ranked in the first k choices of R_i, then $DA_i^{\overline{\succsim}'}(R') = b \neq a$. If $DA_i^{\overline{\succsim}}(R)$ is some school b ranked in the last $m-k$ choices of R_i or $\emptyset$, then $DA_i^{\overline{\succsim}'}(R') \neq a$. Otherwise, Case 1 and $DA_i^{\overline{\succsim}'}(R') = a$ would imply that $DA_i^{\overline{\succsim}}(R) = a$ because R_i is also a reshuffling of the last $m-k$ choices and a is ranked in the first k choices of R'_i.

Appendix 23. *Proof of Lemma 8*

Let $k = t_\ell$.

Proof of (a). Let R_i be such that some school in $\{a_{k+1}, \ldots, a_m\}$ is ranked above some school in $\{a_1, \ldots, a_k\}$ under R_i. We now construct a strategy for student i which stochastically dominates R_i with respect to R_*.

Construction of the strategy.

Step 1 :

(a). Starting from the bottom of R_i and moving to the top, let b be the first school in $\{a_{k+1}, \ldots, a_m\}$ which is ranked above a school in $\{a_1, \ldots, a_k\}$. Let a be the most preferred school under R_* in the set of schools that are ranked below b under R_i, that is, $a = max_{R_*}\{a' | b \ P_i \ a'\}$.

(b). Let $R_i^{a\leftrightarrow b}$ be the preference relation obtained from R_i by swapping (the positions of) school a and b. Let $R_i^{(1)} := R_i^{a\leftrightarrow b}$.

More generally, for $t \geq 2$,

Step k :

(a). If under $R_i^{(t-1)}$ some school in $\{a_{k+1}, \ldots, a_m\}$ is ranked above some school in $\{a_1, \ldots, a_k\}$, then starting from the bottom of $R_i^{(k-1)}$ and moving to the top, let b be the first school in $\{a_{k+1}, \ldots, a_m\}$ that is ranked above a school in $\{a_1, \ldots, a_k\}$. Let a be the most preferred school under R_* in the set of schools that are ranked below b under $R_i^{(t-1)}$.

(b). Let $R_i^{(t-1)a \leftrightarrow b}$ be the preference relation obtained from $R_i^{(t-1)}$ by swapping (the positions of) school a and b. Let $R_i^{(t)} := R_i^{(t-1)a \leftrightarrow b}$.

At each step of this procedure, some school in $\{a_{k+1}, \ldots, a_m\}$ is switched to a lower position. More precisely, if school b is such a school, then b will be switched to a lower position at subsequent steps provided that it is still ranked above some school in $\{a_1, \ldots, a_k\}$. Then, school b is eventually ranked below those schools. Thus, the procedure terminates at a finite number of steps. Let $\widehat{R}_i$ be the preference relation obtained at the end of this procedure. We provide an example of this construction.

Example 13. ***Construction of the strategy***

Let $S = \{a, b, c, d, e, f\}$ and suppose that student i belongs to the first indifference class and that the marginal school for this indifference class is school c. In the left hand table below, R_* is the common ranking and R_i is a preference relation reported by student i.

R_*	R_i
a	c
b	e
c	f
d	b
e	a
f	d

R_i		R_i^1		R_i^2		$R_i^3 = \widehat{R}_i$
c		c		c		c
e		$\boldsymbol{e}$		a		a
$\boldsymbol{f}$	$\rightarrow$	$\boldsymbol{a}$	$\rightarrow$	$\boldsymbol{e}$	$\rightarrow$	b
b		b		$\boldsymbol{b}$		e
$\boldsymbol{a}$		f		f		f
d		d		d		d

The schools in bold face are the ones whose positions have been swapped to obtain the next preference ordering.

Then, we have :

Claim 4. *The strategy* $\widehat{R}_i$ *weakly dominates the strategy* R_i.

Proof. Let a and b be the two schools identified in Step 1 (a) as denoted there. Let S_1 be the set of schools in $\{a_1, \ldots, a_k\}$ different than a that are ranked below school b under R_i. Let S_2 be the set of schools that are ranked above b under R_i and S_3 the set of schools in $\{a_{k+1}, \ldots, a_m\}$ that are ranked below b. Note that, by construction, there is no school in $\{a_{k+1}, \ldots, a_m\}$ which is ranked in between two schools in S_1 under R_i otherwise, b would not be the first school in $\{a_{k+1}, \ldots, a_m\}$ which is ranked above a school in $\{a_1, \ldots, a_k\}$. In the following tables, we clarify these notations.

Example 14. *Consider an example in which school* a_4 *is the marginal school for the first indifference class and student* i *belongs to that class. Suppose that student* i *reports the preference relation* R_i. *Then the sets* S_1, S_2 *and* S_3 *and schools* a *and* b *are represented in the right hand table.*

R_*		R_i
$\boldsymbol{a_1}$	S_2	a_7
$\boldsymbol{a_2}$	S_2	a_1
$\boldsymbol{a_3}$		$b = a_5$
$\boldsymbol{a_4}$	S_1	a_4
a_5	S_1	a_3
a_6		$a = a_2$
a_7	S_3	a_6

Let $R^{a\leftrightarrow b} = (R_i^{a\leftrightarrow b}, R_{-i})$, $p' = rFM^{\succeq}(R^{a\leftrightarrow b})$ and $p = rFM^{\succeq}(R)$. We show that $p' \; R_*^{sd} \; p$.

By swapping a and b in R_i there is (possibly not) a reallocation of probability shares between schools in the lotteries assigned to student i. We need to track the flow of these shares. We use the following representation of the allocation for student i under $rFM^{\succeq}(R)$ and $rFM^{\succeq}(R^{a\leftrightarrow b})$.

	R_i						$R_i^{a\leftrightarrow b}$					
	a_1	a_2	a_3	$\dots$	a_m	$\emptyset$	a_1	a_2	a_3	$\dots$	a_m	$\emptyset$
π_1		$\frac{1}{(n!)^m}$							$\frac{1}{(n!)^m}$			
π_2	$\frac{1}{(n!)^m}$										$\frac{1}{(n!)^m}$	
$\vdots$												
$\pi_{\|\Omega\|}$					$\frac{1}{(n!)^m}$		$\frac{1}{(n!)^m}$					

We say that there is a flow of probability shares from school a' to a school b' if there is a profile of tiebreakers π such that $FM_i^{(\succeq,\pi)}(R) = a'$ and $FM_i^{(\succeq,\pi)}(R^{a\leftrightarrow b}) = b'$. In the above representation, there is a flow of probability shares from school a_2 to school a_3, from school a_1 to school a_m and from school a_m to school a_1.

The idea of the proof is to show that the only flows that are possible are the ones in which shares of probability are shifted from school b to schools in S_1, which is an improvement for student i, and the ones in which shares of probability are shifted from schools in S_1 to school a, which is also an improvement for student i. The rest of the proof consists of showing that any other flow is not possible. In any of the following steps, we let $\succeq = f(R, \succeq, \pi)$ and $\succeq' = f(R^{a\leftrightarrow b}, \succeq, \pi)$ (if there is no confusion).

Step 1 : There is no flow of probability shares from a school in S_2 to a school in S_2 nor from a school in $S \setminus S_2$ to a school in S_2.
This follows from Proposition 9 in which for each $a' \in S_2$, $p_{ia'} = p'_{ia'}$.

Step 2 : There is no flow of probability shares from a school in S_1 to a distinct school in S_1. Suppose, to the contrary, that there are two distinct schools $c, d \in S_1$ and a tiebreaker profile π such that $FM_i^{(\succeq,\pi)}(R) = DA_i^{\succeq}(R) = c$ and $FM_i^{(\succeq,\pi)}(R^{a\leftrightarrow b}) = DA_i^{\succeq'}(R^{a\leftrightarrow b}) = d$. From R_i to $R_i^{a\leftrightarrow b}$ only the positions of a and b changed. Therefore, for each school a' other than a and b, we have

$\succeq_{a'}=\succeq'_{a'}$ and

$$\succeq_a|_{-i}=\succeq'_a|_{-i} \quad \text{and} \quad \succeq_b|_{-i}=\succeq'_b|_{-i}. \tag{29}$$

Let R'_i be a preference relation for student i obtained from R_i by moving school a and b to the last and next to last positions. Since i is not matched to a or to b under $DA^{\succeq}(R)$, a successive application of Lemma 14 (d) implies that $DA_i^{\succeq}(R'_i, R_{-i}) = c$. The same argument leads to the conclusion that $DA_i^{\succeq'}(R'_i, R_{-i}) = d$. Because school a and b are ranked below school c under R'_i, by Equation (29) and Lemma 14 (a), we have $DA_i^{(\succeq'_{\{a,b\}},\succeq_{-\{a,b\}})}(R'_i, R_{-i}) = c$. Note now that $(\succeq'_{\{a,b\}}, \succeq_{-\{a,b\}}) =\succeq'$. Therefore, we have $DA_i^{\succeq'}(R'_i, R_{-i}) = c$, which contradicts the conclusion that $DA_i^{\succeq'}(R'_i, R_{-i}) = d$.

Step 3 : There is no flow of probability shares from a school in S_1 to school b. Suppose, to the contrary, that for some $c \in S_1$, a tiebreaker profile π, we have $FM_i^{(\succeq,\pi)}(R) = DA_i^{\succeq}(R) = c$ and $FM_i^{(\succeq,\pi)}(R^{a\leftrightarrow b}) = DA_i^{\succeq'}(R^{a\leftrightarrow b}) = b$. Let R'_i be a preference relation for student i obtained from R_i by moving school a to the last position. By Equation (29) and Lemma 14 (d), we have $DA_i^{\succeq}(R'_i, R_{-i}) = c$ and $DA_i^{\succeq'}(R'_i, R_{-i}) = b$. By Equation (29) and Lemma 14 (a), we have $DA_i^{(\succeq_a,\succeq'_{-a})}(R'_i, R_{-i}) = b$. Next, we have $\succeq_b|_{-i}=\succeq'_b|_{-i}$ and student i is ordered higher at $\succeq_b$ than $\succeq'_b$. Therefore, by Lemma 14 (c), $DA_i^{(\succeq_a,\succeq'_{-a})}(R'_i, R_{-i}) = b$ implies that $DA_i^{(\succeq_{\{a,b\}},\succeq'_{-\{a,b\}})}(R'_i, R_{-i}) = b$. Finally, observe that $(\succeq_{\{a,b\}}, \succeq'_{-\{a,b\}}) = \succeq$. Hence, $DA_i^{\succeq}(R'_i, R_{-i}) = b$, which contradicts the conclusion that $DA_i^{\succeq}(R'_i, R_{-i}) = c$.

Step 4 : There is no flow of probability shares from school a to a school in S_1. This follows from an argument similar to that of Step 3.

Step 5 : There is no flow of probability shares from any school in S_3 to any school in $S \setminus S_3$ nor from any school in S_3 to a distinct school in S_3. More precisely, for each $a' \in S_3$, $p_{ia'} = p'_{ia'} = 0$. Because school b is the first school which is ranked above a school in $\{a_1, \ldots, a_k\}$ under R_i, any school in S_3 is ranked below any school in $\{a_1, \ldots, a_k\}$ under R_i. The conclusion follows from the fact that $\sum_{j=1}^{k} q_{a_j} \geq n$.

We now show that $p' \; R_*^{sd} \; p$, by showing that for each $\ell \leq m$,

$$p'_{ia_1} + \ldots + p'_{ia_\ell} \geq p_{ia_1} + \ldots + p_{ia_\ell}. \tag{30}$$

According to Step 1 and Step 5, only for each school $a' \in S_2 \cup S_3 = S \setminus (\{a,b\} \cup S_1)$, $p'_{ia'} = p_{ia'}$. Therefore, we only need to show that Equation (30) holds at the positions ℓ of the schools in $\{a,b\} \cup S_1$. Let t be the position of school a in R_*, that is, $a = a_t$. Note that school a has the lower index and school b has the greatest index. Further, all schools with an index less than t are ranked above b under R_i; that is, if $t \geq 2$, then $\{a_1 \ldots, a_{t-1}\} \subset S_2$. If $t \geq 2$, then the total probability shares of schools in $\{a_1 \ldots, a_{t-1}\}$ under p and p' are equal. Let α be this common probability shares.

Case 1 : Position $\ell = t$.

Because there is no flow of probability share from a to any other school, we have $p'_{ia} \geq p_{ia}$. Therefore, we have

$$\alpha + p'_{ia_t} \geq \alpha + p_{ia_t},$$

which shows that Equation (30) holds for position ℓ.

Case 2 : Position ℓ of some school in S_1.

Let $S_1^+(\ell)$ be the set of schools a' in S_1 whose positions are less than or equal to ℓ and such that $p'_{ia'} - p_{ia'} \geq 0$. Let $S_1^-(\ell)$ be the set of schools a' in S_1 whose positions are less than or equal to ℓ and such that $p'_{ia'} - p_{ia'} < 0$. By Step 2 through Step 4, for each $a' \in S_1^-(\ell)$, the gap $p_{ia'} - p'_{ia'}$ of probability shares flowed to school a. Therefore, we have

$$p'_{ia} - p_{ia} \geq \sum_{a' \in S_1^-(\ell)} (p_{ia'} - p'_{ia'}).$$

A rearrangement of this relation leads to

$$p'_{ia} + \sum_{a' \in S_1^-(\ell)} p'_{ia'} \geq p_{ia} + \sum_{a' \in S_1^-(\ell)} p_{ia'}.$$

Because for each $a' \in S_1^+(\ell)$, $p'_{ia'} \geq p_{ia'}$, we have

$$\alpha + p'_{ia} + \sum_{a' \in S_1^-(\ell)} p'_{ia'} + \sum_{a' \in S_1^+(\ell)} p'_{ia'} \geq p_{ia} + \sum_{a' \in S_1^-(\ell)} p_{ia'} + \sum_{a' \in S_1^+(\ell)} p_{ia'} + \alpha,$$

which shows that Equation (30) holds for position ℓ.

Case 3 : Position ℓ of school b.

By Step 1 and Step 5, probability shares of school b (possibly) flowed to schools in $\{a\} \cup S_1$ and no probability shares of a school in $\{a\} \cup S_1$ flowed to a school not in $\{a\} \cup S_1$. Therefore, $p_{ib} \geq p'_{ib}$ and

$$p_{ib} - p'_{ib} = \sum_{a' \in S_1 \cup \{a\}} p'_{ia'} - \sum_{a' \in S_1 \cup \{a\}} p_{ia'}.$$

By simple rearrangement, we obtain the following relation

$$\alpha + p'_{ib} + \sum_{a' \in S_1 \cup \{a\}} p'_{ia'} = p_{ib} + \sum_{a' \in S_1 \cup \{a\}} p_{ia'} + \alpha.$$

which shows that Equation (30) holds for position ℓ.

All cases being considered, we have $p' \; R^{sd}_* \; p$. Since the stochastic dominance relation is transitive (Bogomolnaia and Moulin, 2001), $rFM^{\succeq}(\widehat{R}_i, R_{-i}) \; R^{sd}_* \; rFM^{\succeq}(R)$.

We finally give and example in which $rFM^{\succeq}(\widehat{R}_i, R_{-i}) \; P^{sd}_* \; rFM^{\succeq}(R)$. To this end, let $p' = rFM^{\succeq}(\widehat{R}_i, R_{-i})$ and $p = rFM^{\succeq}(R)$. Let a be the school in $\{a_{k+1}, \ldots, a_m\}$ which is ranked higher under R_i and S' the set of schools which are ranked above a under R_i. Note that $S' \subset \{a_1, \ldots, a_k\}$. Let R' be a preference profile in which $R'_i = R_i$, each student $j \neq i$ ranks schools in S' first as in R_i and all schools in $\{a_1, \ldots, a_k\}$ are ranked first under R_j. Because all students rank schools in S' first and the same and $S' \subset \{a_1, \ldots, a_k\}$, we have $\sum_{a' \in S'} p_{ia'} < 1$. Because i ranks school a higher than any other student, we have $p_{ia} = 1 - \sum_{a' \in S'}$. By successive application of Step 1, for each $a' \in S'$, $p_{ia'} = p'_{ia'}$.[6] Because each student ranks all schools in $\{a_1, \ldots, a_k\}$ in the first k positions under $(\widehat{R}_i, R_{-i})$ and $\sum_{j=1} q_{a_j} \geq n$, we have $\sum_{j=1}^{k} p'_{ia_j} = 1$. Thus, all probability shares of school a flowed to schools in $\{a_1, \ldots, a_k\}$, which proves that $p' \; P^{sd}_* \; p$. □

6. This is because the set of schools in S' are ranked above school a and remain so at each step of the construction of $\widehat{R}_i$. Thus, they are ranked above the first school in $\{a_{k+1}, \ldots, a_m\}$ which is ranked above a school in $\{a_1, \ldots, a_k\}$ under $R_i^{(t)}$ from the bottom up.

Proof of (b). Regarding part (a) of the lemma, it is enough to show that the strategy $\widehat{R}_i$ would be dominated if a_k is not ranked kth. Suppose that a_k is not ranked kth under $\widehat{R}_i$. Then, letting $R_i^{(1)} = \widehat{R}_i$ and at each step t swapping a_k and the most preferred school which is ranked below a_k under $R_i^{(t)}$, we can follow the proof argument of part (a) and show that the strategy obtained weakly dominates $\widehat{R}_i$.

Appendix 24. *Proof of Theorem 8*

Let $(R, \succeq)$ be a correlated economy. The proof involves computing DA's assignment first and second proving that at any symmetric equilibrium of the French mechanism $rFM^{\succeq}$, each agent has a deviation that secures him an expected utility at least as greater than that of DA's assignment. We first compute DA's assignment in the following.

Assignment under DA : It is a weakly dominant strategy for each student to report his preferences truthfully. We are interested in the assignment of DA under truth telling. Note that students have a common ordinal preference $a_1 \ldots a_m$ and schools have the same priorities. For each $\ell \in \{1, \ldots, T\}$, let

$$t_\ell := min\{t | \sum_{k=1}^{t} q_{a_k} \geq n_1 + \ldots + n_\ell\}$$

be the index of the marginal school for students in I_ℓ. Under DA's assignment, no student in I_ℓ is assigned a seat of a school with an index less than t_ℓ.

Assignment of students in I_1 : The probability that each student in I_1 is assigned to the school with index t is

$$P_{1a_t}^{DA} = \begin{cases} \dfrac{min\{q_{a_t}, n_1 - \sum_{t'=1}^{t_1-1} q_{a_{t'}}\}}{n_1} & \text{if } t \leq t_1 \\ 0 & \text{if } t > t_1. \end{cases}$$

Assignment of students in I_ℓ with $\ell > 1$:

(a) if $t_\ell = t_{\ell-1}$, then the probability that each student in I_ℓ is assigned to the school with index t is

$$P^{DA}_{\ell a_t} = \begin{cases} 1 & \text{if } t = t_\ell \\ 0 & \text{otherwise.} \end{cases}$$

(b) if $t_\ell > t_{\ell-1}$, then the probability that a student in I_ℓ is assigned to a school with index t is

$$P^{DA}_{\ell a_t} = \frac{\widehat{q}_{a_t}}{n_\ell},$$

where

$$\widehat{q}_{a_t} = \begin{cases} \sum_{t'=1}^{t_{\ell-1}} q_{a_{t'}} - (n_1 + \ldots + n_{\ell-1}) & \text{if } t = t_{\ell-1} \\ q_{a_t} & \text{if } t_{\ell-1} < t < t_\ell \\ n_1 + \ldots + n_\ell - \sum_{t'=1}^{t_\ell - 1} q_{a_{t'}} & \text{if } t = t_\ell \\ 0 & \text{otherwise.} \end{cases}$$

Given DA's assignment for R, we compare the distribution of school seats of students in indifference classes under both DA and any undominated strategy of the French mechanism.

Claim 5. *The same number of seats of each school is allocated to each indifference class under bother $rDA^{\succeq}$ and any undominated strategy of $rFM^{\succeq}$.*

Proof. Let R' be an undominated strategy under $rFM^{\succeq}$, $\ell \geq 1$ and I_ℓ an indifference class. Because schools have the same priorities and they are respect under the French mechanism, the outcome students in $I' = \bigcup_{k=1}^{\ell} I_k$ is simply given by

$$rFM^{\succeq}_{I'}(R') = rFM^{\succeq|_{I'}}(R'_{I'}),$$

where $rFM^{\succeq|_{I'}}$ is the extension of $rFM^{\succeq}$ to problems where the set of students is I'.[7]

Indifference class I_1. By Lemma 8, each student in I_1 ranks any school in $\{a_1, \ldots, a_{t_1}\}$ in his top t_1 rankings under R and school a_{t_1} at t_1'th ranking. Therefore, all seats of each school of index in $\{1, \ldots, t_1 - 1\}$ are allocated to only students in I_1 and the following number of seats of the school of index t_1 is allocated to students in I_1

$$n_1 - \sum_{t'=1}^{t_1-1} q_{a_{t'}}.$$

This proves the claim for the first indifference class.

Let $\ell > 1$ and suppose the claim holds up to the indifference class $I_{\ell-1}$.

Indifference class I_ℓ. By the induction assumption, all seats of schools of index $t < t_{\ell-1}$ (if any) are allocated to indifference classes 1 through $\ell - 1$ as under DA. Therefore, only seats of school of index t_1 or lower remain. Further, the following number of seats of the school of index $t_{\ell-1}$ remains

$$\sum_{t'=1}^{t_{\ell-1}} q_{a_{t'}} - (n_1 + \ldots + n_{\ell-1}). \tag{31}$$

(a) $t_\ell = t_{\ell-1}$. Then, the number in Equation (31) is greater than n_ℓ ; hence, n_ℓ number of seats of the school of index t_ℓ is allocated to the indifference class I_ℓ.

(b) $t_{\ell-1} < t_\ell$. Then the number in Equation (31) is less than n_ℓ. Hence, this number of the seats of school of index $t_{\ell-1}$ is allocated to the indifference class I_ℓ. In addition, By Lemma 8, each student in I_ℓ ranks schools in $\{a_1, \ldots, a_{t_\ell}\}$ in the top t_ℓ under R and the school of index t_ℓ at t_ℓ'th ranking. Therefore, all seats of each school of index t with $t_{\ell-1} < t < t_\ell$ are allocated to the

7. The extension is defined as follows : let $\mathcal{M}'$ be the set of matchings $\mu : I' \to I' \cup S$. Then, for each profile π of orderings of I', defining $\tau := (\succeq|_{I'}, \pi)$, $FM^\tau : \mathcal{R}^{I'} \to \mathcal{M}'$ such that for each $R \in \mathcal{R}^{I'}$, $FM^\tau(R)$ is the assignment selected by FM^τ for R. Then, the associated random analogue $rFM^{\succeq|_{I'}}$ is also defined.

indifference class I_ℓ. Accordingly, a total of

$$\sum_{t'=1}^{t_\ell-1} q_{a_{t'}} - (n_1 + \ldots + n_{\ell-1})$$

seats of schools of indexes t with $t_{\ell-1} \leq t < t_\ell$ are allocated to I_ℓ. Therefore, the following number (which is the gap needed to cover the indifference class I_ℓ) of seats of the school of index t_ℓ are allocated to the indifference class I_ℓ :

$$n_1 + \ldots + n_\ell - \sum_{t'=1}^{t_\ell-1} q_{a_{t'}}.$$

This proves that the claim holds for the indifference class I_ℓ. Therefore, the claim holds for each indifference class. □

We are now ready to finish the proof of Theorem 8. Let $(\sigma^*_\ell)_{\ell \in T}$ be a symmetric equilibrium of the French mechanism $rFM^{\succeq}$. Let $P^{FM}_{\ell a}(\sigma)$ be the probability that a student in the indifference class ℓ be assigned to school a when he plays according to σ_ℓ while the other students play according to σ^*. The probability that a given student in I_ℓ be assigned to a school a in equilibrium is $\sum_{v \in \mathcal{V}} P_{\ell a}(\sigma^*(v)) f_\ell(v)$. Therefore, the total seats assigned to students in I_ℓ in equilibrium of $rFM^{\succeq}$ is

$$n_\ell \sum_{v \in \mathcal{V}} P_{\ell a}(\sigma^*(v)) f_\ell(v).$$

By Claim 5, the same number of seats of each school is allocated to the indifference class I_ℓ under both $rDA^{\succeq}(R)$ and any symmetric equilibrium of $rFM^{\succeq}$. Therefore, we have

$$n_\ell \sum_{v \in \mathcal{V}} P_{\ell a}(\sigma^*(v)) f_\ell(v) = n_\ell P^{DA}_{\ell a}$$

from which we derive that

$$\sum_{v \in \mathcal{V}} P_{\ell a}(\sigma^*(v)) f_\ell(v) = P^{DA}_{\ell a}.$$

Consider the following strategy $\widehat{\sigma}_\ell$ of a student in the indifference class I_ℓ. For each type $\widehat{v}$, play $\sigma^*(v)$ with probability $f_\ell(v)$. Therefore, the probability that such student be assigned to school a is

$$P_{\ell a}^{FM}(\widehat{\sigma}) := \sum_{v \in \mathcal{V}} P_{\ell a}(\sigma^*(v)) f_\ell(v) = P_{\ell a}^{DA}.$$

This student is able to replicate the assignment of $rDA^{\succeq}$ for R via the strategy $\widehat{\sigma}_\ell$. The equilibrium strategy must be associated with an expected utility for this student at least as greater than that of $rDA^{\succeq}$.

Proofs from Chapter 3

Appendix 25. ***Proof of Theorem 9 and Theorem 10***

A choice rule is a function $\varphi : \mathcal{R}^N \to A$. A choice rule φ is strategy-proof if for each $R \in \mathcal{R}^N$, each $i \in N$ and each $R'_i \in \mathcal{R}$, $\varphi(R)\ R_i\ \varphi(R'_i, R_{-i})$. A choice rule φ is onto if $\varphi(\mathcal{R}^N) = A$. In our proofs we need the following result.

Result. *(Gibbard, 1973; Satterthwaite, 1975) Suppose that there are at least three alternatives. A choice rule is strategy-proof and onto if and only if it is a dictatorial rule.*

Part 1 : Proof of Theorem 10

« **If part** ». It can be easily verified that every augmented sequential rule is strategy-proof under lexicographic extension and onto.

« **Only if part** ». Suppose that a rule f is strategy-proof under lexicographic extension and onto. We show that f is an augmented sequential rule. If $m = 2$, then by Lemma 9 f is a simple game and is trivially an augmented sequential rule. Therefore, we assume that $m \geq 3$ for the rest of the proof. For each $R \in \mathcal{R}^N$ and each $t \leq m$, let $f_t(R)$ denote the alternative ranked at the tth position of $f(R)$. We consider two cases :

Step 1 : Consider the following conjecture : Let $k \in \{1, \dots, m-2\}$. The ranking over the first k positions of f are determined as follows : there is a function $\Gamma : \bigcup_{t=0}^{k-1} \mathcal{H}^t \to N$ such that for each $R \in \mathcal{R}^N$, there is a sequence of consecutive histories $(h^t)_{t=0}^{k-1}$ starting from $h^0 \in \mathcal{H}^0$ such that $f_1(R)$ is the alternative ranked first by $R_{\Gamma(h^0)}$ in $A(h^0)$. The history h^1 is reached following the selection of $f_1(R)$. Iteratively, for each $t \leq k-1$, $f_{t+1}(R)$ is the alternative ranked first by $\Gamma(h^t)$ in $A(h^t)$. The history h^{t+1} is reached following the selection of the alternative $f_{t+1}(R)$.

We prove the above conjecture by induction. Let $\varphi : \mathcal{R}^N \to A$ be a choice rule defined as follows : for each $R \in \mathcal{R}^N$, $\varphi(R) = f_1(R)$.

Lemma 15. *φ is strategy-proof and onto.*

First, suppose to the contrary that there is $i \in N$, R and R'_i such that

$$\varphi(R'_i, R_{-i}) \; P_i \; \varphi(R). \tag{32}$$

By definition, $\varphi(R'_i, R_{-i})$ and $\varphi(R)$ are the alternatives ranked first in $f(R'_i, R_{-i})$ and $f(R)$ respectively. By lexicographic extension, Equation 32 implies that $f(R'_i, R_{-i}) \; \boldsymbol{P}_i^{lex} \; f(R)$ which contradicts the assumption that f is strategy-proof under lexicographic extension. Second, let $a \in A$. Let R^0 be a collective ordering in which a is ranked first. Because f is onto, there is $R \in \mathcal{R}^N$ such that $f(R) = R^0$. Therefore, $\varphi(R) = f_1(R) = a$. Because $m \geq 3$, we draw the following result :

Result (Gibbard-Satterthwaite theorem)**.** *φ is a dictatorial rule.*

Let i be the dictator for φ. Let $\Gamma^0 : \mathcal{H}^0 \to N$ such that $\Gamma(h^0) = i$. This is the induction base.

Induction base : For each $R \in \mathcal{R}^N$, $f_1(R)$ is the alternative ranked first by $R_{\Gamma(h^0)}$ in $A(h^0)$. By this induction base, if $m = 3$ then the conjecture holds trivially.

Induction hypothesis : Suppose that $m > 4$ and that the conjecture holds for some $k \in \{1, \dots, m-3\}$. Let $\Gamma^k : \bigcup_{t=0}^{k-1} \mathcal{H}^t \to N$ be the function identified in the conjecture.

Induction step : Let $\Gamma^{k+1} : \bigcup_{t=0}^{k} \mathcal{H}^t \to N$ be a function such that for each $h \in \bigcup_{t=0}^{k-1} \mathcal{H}^t$, $\Gamma^{k+1}(h) = \Gamma^k(h)$. Let $h \in \mathcal{H}^k$ and $a_1 \dots a_k$ the sequence of selection up to the history h. Let $\mathcal{R}(A(h))$ be the set of orderings of $A(h)$. For each i and each $R_i \in \mathcal{R}(A(h))$, let $\widehat{R}_i \in \mathcal{R}$ be an ordering of A that coincides with R_i in $A(h)$ and which coincides with $a_1 \dots a_k$ in the first k positions. By the induction hypothesis, for each $R \in \mathcal{R}(A(h))^N$, $f(\widehat{R}) : a_1 \dots a_k \dots$. Therefore, $f_{k+1}(\widehat{R}) \in A(h)$. Let $\varphi : \mathcal{R}(A(h))^N \to A(h)$ be a choice rule such that for each $R \in \mathcal{R}(A(h))^N$, $\varphi(R) = f_{k+1}(\widehat{R})$.

Lemma 16. *The choice rule φ is strategy-proof and onto.*

Démonstration. The proof follows a similar argument as that of Lemma 15.

□

Because $k \leq m-3$, $|A(h)| \geq 3$ and thus, by Result 3.4 there is a dictator i for φ. Let $\Gamma^{k+1}(h) = i$. Therefore, the conjecture holds for $k+1$ with the function Γ^{k+1}.

Step 2 : There is a function $\Gamma : \bigcup_{t=0}^{m-2} \mathcal{H}^t \to N \cup (A \times \widehat{\mathcal{C}})$ such that for each $h \notin \mathcal{H}^{m-2}$, $\Gamma(h) \in N$ and such that for each $h \in \mathcal{H}^{m-2}$, $\Gamma(h) \in (A(h) \times \widehat{\mathcal{C}})$ and $f = f^\Gamma$. Let $\Gamma : \bigcup_{t=0}^{m-2} \mathcal{H}^t \to N \cup (A \times \widehat{\mathcal{C}})$ be a function such that for each $h \notin \mathcal{H}^{m-2}$, $\Gamma(h) = \Gamma^{m-3}(h)$. Let $h \in \mathcal{H}^{m-2}$, $\{a, b\} = A(h)$ and $a_1 \dots a_{m-2}$ the sequence of selection up to the history h. Let $\mathcal{R}(\{a, b\})$ be the set of orderings of $\{a, b\}$. For each $i \in N$ and each $R_i \in \mathcal{R}(\{a, b\})$, let $\widehat{R}_i \in \mathcal{R}$ be an ordering of A in which the alternatives ranked in the first $m-2$ positions are ranked as $a_1 \dots a_{m-2}$. By Step 1, for each $R \in \mathcal{R}(\{a, b\})^N$, $f(\widehat{R}) : a_1 \dots a_{m-2} \dots$. Therefore, $f_{m-1}(\widehat{R}) \in \{a, b\}$. Let $\varphi : \mathcal{R}(\{a, b\})^N \to \mathcal{R}(\{a, b\})$ be a rule such that for each $R \in \mathcal{R}(\{a, b\})^N$, $\varphi(R) : a\ b$ where $f_{m-1}(\widehat{R}) = a$ and $f_m(\widehat{R}) = b$.

Lemma 17. *The choice rule φ is strategy-proof and onto.*

Démonstration. The proof follows a similar argument as that of Lemma 15.

□

Because $|A(h)| = 2$, by Lemma 9 there is a simple game $\widehat{S}^x_{\widehat{\omega}}$ with $x \in \{a, b\}$ such that $\widehat{S}^x_{\widehat{\omega}} = \varphi$. Let $\Gamma(h) = (x, \widehat{\omega})$. Having constructed the function $\Gamma : \bigcup_{t=0}^{k-2} \mathcal{H}^t \to N \cup (A \times \widehat{\mathcal{C}})$ then, it is now clear that :

Conclusion : $f = f^{\Gamma}$.

Part 2 : Proof of Theorem 9

Suppose that $m \geq 3$ and let f be a rule that is strategy-proof under every refinement and neutral. By Theorem 11, f is an augmented serial rule.

A dictatorial rule is clearly strategy-proof under every refinement and neutral. We verify that any other augmented serial rule is not strategy-proof under inverse lexicographic extension. Let $f^{(\sigma,\widehat{\omega})}$ be an augmented serial rule and suppose that it is not a dictatorial rule. Let $\widehat{N} = \bigcup_{T \in \widehat{\omega}} T$. We consider two cases :

Case 1 : $|\widehat{N}| > 1$. Let $i := \sigma(m-2)$. Let $R \in \mathcal{R}^N$ be such that R_i : $d_1 \dots d_{m-3}\, a\, b\, c$ and for each $j \in N \setminus \{i\}$, R_j : $d_1 \dots d_{m-3}\, a\, c\, b$. Let R'_i : $d_1 \dots d_{m-3}\, b\, c\, a$. There is $T \in \widehat{\omega}$ such that $T \subset N \setminus \{i\}$; otherwise, $\widehat{\omega} = \{\{i\}\}$.[8] This equality contradicts the assumption that $|\widehat{N}| > 1$. Consequently, $f(R) : d_1 \dots d_{m-3}\, a\, c\, b$ and $f(R'_i, R_{-i}) : d_1 \dots d_{m-3}\, b\, a\, c$. Because $b\ P_i\ c$, then $f^{\Gamma}(R'_i, R_{-i})\ \boldsymbol{P}_i^{ilex}\ f^{\Gamma}(R)$.

Case 2 : $|\widehat{N}| = 1$. Let $j := \sigma(m-1)$. Let t be the first position, from $m-2$ to the top where j is no longer a dictator for that position. Such position exists because f^{Γ} is not a dictatorial rule. Let $i := \sigma(t)$. Let $R \in \mathcal{R}^N$ be such that $R_i : b_1 \dots b_t \dots b_m$ and for each $i' \in N \setminus \{i\}$, $R_{i'} : b_1 \dots b_t\, b_{t+2} \dots b_m\, b_{t+1}$. If $t > 1$ let $R'_i : b_1 \dots b_{t-1}\, b_{t+1}\, b_t\, b_{t+2} \dots b_m$.[9] Then, $f^{\Gamma}(R) = R_j$ and $f^{(\sigma,\widehat{\omega})}(R'_i, R_{-i}) = R'_i$. Because $b_{t+1}\ P_i\ b_m$, we have $f^{\Gamma}(R'_i, R_{-i})\ \boldsymbol{P}_i^{ilex}\ f^{\Gamma}(R)$.

Appendix 26. *Proof of Theorem 11*

8. This result is the implication of the following result (Moulin, 1983, P. 64) : if for some $T \subset N$ for each $T' \in \widehat{\omega}$, $T \cap T' \neq \emptyset$, then there is $T'' \in \widehat{\omega}$ such that $T'' \subset T$. If instead $i \in T$, for each $T \in \widehat{\omega}$, then $\{i\} \in \widehat{\omega}$. Since $\widehat{\omega}$ consists of minimal coalitions, $\widehat{\omega} = \{\{i\}\}$.

9. If $t = 1$, let $R'_i : b_2\, b_1\, b_3 \dots b_m$.

« **If part** ». It is easily verified that every augmented serial rule is strategy-proof under lexicographic extension and neutral.
« **Only if part** ». Let f be a rule that is strategy-proof under lexicographic extension and neutral. It is a matter of verification to prove that a neutral rule is onto. Therefore, by Theorem 10 f is an augmented sequential rule, that is, there is a parameter Γ such that $f = f^{\Gamma}$. We show that Γ satisfies the following points :

i. for each $k \in \{0, \ldots, m-2\}$ and each $h, h' \in \mathcal{H}^k$, $\Gamma(h) = \Gamma(h')$ and
ii. for each $h \in \mathcal{H}^{m-2}$, if $(a, \widehat{\omega}) = \Gamma(h)$ for some $a \in A(h)$, then $\widehat{S}^a_{\widehat{\omega}}$ is a strong simple game.

Proof of (ii). Let $h \in \mathcal{H}^{m-2}$, $\{a, b\} = A(h)$ and $(a, \widehat{\omega}) = \Gamma(h)$. By Lemma 9, $\widehat{S}^a_{\widehat{\omega}}$ is a strong simple game.

Proof of (i). Suppose to the contrary that for some $k \in \{1, \ldots, m-2\}$ and some $h, h' \in \mathcal{H}^k$, $\Gamma(h) \neq \Gamma(h')$. Without loss of generality, suppose that h and h' are close, that is, there is an history $h'' \in \mathcal{H}^{k-1}$ from which one can reach h by the selection of an alternative a and from which one can reach h' by the selection of an alternative b. By definition, $\Gamma(h'') \in N$. Let $j = \Gamma(h'')$. Let $i = \Gamma(h)$ and $\ell = \Gamma(h')$. Let $a_1, \ldots, a_{k-1}$ be the sequence of selections up to history h''. Let π be a permutation such that $\pi(a) = b$ and for each $x \notin \{a, b\}$, $\pi(x) = x$. Let R and πR be defined as follows :

R_j	R_i	R_ℓ	πR_j	πR_i	πR_ℓ
a_1	a_1	a_1	a_1	a_1	a_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{k-1}	a_{k-1}	a_{k-1}	a_{k-1}	a_{k-1}	a_{k-1}
a	a	c	b	b	c
b	b	b	a	a	a
c	c	a	c	c	b
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Then, $f^{\Gamma}(R) = a_1, \ldots, a_{k-1}\ a\ b\ c \ldots$ and $f^{\Gamma}(\pi R) = a_1, \ldots, a_{k-1}\ b\ c\ a \ldots$. Therefore, $f^{\Gamma}(\pi R) \neq \pi f^{\Gamma}(R)$. This relation contradicts the fact that $f = f^{\Gamma}$ and that f is neutral.

Suppose now that for some $h, h' \in \mathcal{H}^{m-2}$, $\Gamma(h) \neq \Gamma(h')$. Without loss of generality, suppose that h and h' are close. Let h'' be the history that precedes h and h'. That is, the history h is reached from h'' by the selection of an alternative a and h' is reached from h'' by the selection of an alternative b. Let c be the third alternative not yet selected at history h''. Let $j = \Gamma(h'')$, $(\widehat{\omega}, b) = \Gamma(h)$ and $(\widehat{\omega'}, a) = \Gamma(h')$. Because $\widehat{\omega} \neq \widehat{\omega'}$, either $\widehat{\omega} \setminus \widehat{\omega'} \neq \emptyset$ or $\widehat{\omega'} \setminus \widehat{\omega} \neq \emptyset$. Without loss of generality let $T \in \widehat{\omega} \setminus \widehat{\omega'}$. Let π be a permutation of A such that $\pi(a) = b$ and for each $x \notin \{a, b\}$, $\pi(x) = x$. Let R and πR be as follows.

$R_{1'}$	$R_{2'}$	$R_{3'}$
$\vdots$	$\vdots$	$\vdots$
a	a	c
b	c	b
c	b	a

$\pi R_{1'}$	$\pi R_{2'}$	$\pi R_{3'}$
$\vdots$	$\vdots$	$\vdots$
b	b	c
a	c	a
c	a	b

There are two cases to consider.

Case 1 : $j \in T$. Let R be a preference profile such that for each $i \in T$, $R_i = R_{1'}$ and for each $i \notin T$, $R_i = R_{3'}$. Because $T \in \widehat{\omega}$, $f^{\Gamma}(R) : \dots a\, b\, c$. Because $T \notin \widehat{\omega'}$ and the fact that $\widehat{\omega'}$ is a strong simple game, $T^c \in \widehat{\omega'}$. Thus $f^{\Gamma}(\pi R) : \dots b\, c\, a$. Next, $\pi f^{\Gamma}(R) : \dots b\, a\, c \neq f^{\Gamma}(\pi R)$. This relation violates the assumption that f^{Γ} is neutral.

Case 2 : $j \notin T$. Let R be a preference profile such that for each $i \in T$, $R_i = R_{1'}$, $R_j = R_{2'}$ and for each $i \notin T \cup \{j\}$, $R_i = R_{3'}$. Then $f^{\Gamma}(R) : \dots a\, b\, c$ and $f^{\Gamma}(\pi R) : \dots b\, c\, a$. Thus, $\pi f^{\Gamma}(R) : \dots b\, a\, c \neq f^{\Gamma}(\pi R)$. This relation again violates the assumption that f^{Γ} is neutral.

Appendix 27. *Proof of Proposition 10*

« **If part** » : Let $f^{(\sigma,\widehat{\omega})}$ be an augmented serial rule. If an agent is a dictator for some position, then clearly he is not dummy for $f^{(\sigma,\widehat{\omega})}$. Let $i \in N$ and suppose that for some $T \in \widehat{\omega}$, $i \in T$. Let $\{a,b\} \subset A$ and $R \in \mathcal{R}^N$ be such that for each $j \in T$, $R_j : c_1 \ldots c_{m-2}\, a\, b$ and for each $j \in N \setminus T$, $R_j : c_1 \ldots c_{m-2}\, b\, a$. Let $R'_i \in \mathcal{R}$ be such that $R'_i : c_1 \ldots c_{m-2}\, b\, a$ and $R' = (R'_i, R_{-i})$. Because $T \in \widehat{\omega}$, $T \setminus \{i\} \notin \widehat{\omega}$. Because $\{j | a\ P'_j\ b\} = T \setminus \{i\}$, we have $\{j | a\ P'_j\ b\} \notin \widehat{\omega}$. Therefore, $f^{(\sigma,\widehat{\omega})}(R) : c_1 \ldots c_{m-2}\, a\, b$ and $f^{(\sigma,\widehat{\omega})}(R') : c_1 \ldots c_{m-2}\, b\, a$. Agent i is not dummy for $f^{(\sigma,\widehat{\omega})}$.

« **Only if part** ». Let f be a rule and suppose that it is strategy-proof under lexicographic extension, neutral and no dummy. By Theorem 10, $f = f^{(\sigma,\widehat{\omega})}$. Suppose that an agent $i \in N$ is not a dictator for a position and $i \notin \bigcup_{T \in \widehat{\omega}} T$. Let $R \in \mathcal{R}^N$ and $R'_i \in \mathcal{R}$. Then $f^{(\sigma,\widehat{\omega})}(R'_i, R_{-i}) = f^{(\sigma,\widehat{\omega})}(R)$.

Finally, we show that there are augmented serial rules which are no dummy. If n is odd let ω be the majority voting on N defined as follows : $T \in \omega$ if and only if $|T| > |T^c|$. If n is even, then let ω be the majority voting on $N \setminus \{\sigma(1)\}$.

Appendix 28. *Proof of Proposition 11*

Let $f^{(\sigma,\widehat{\omega})}$ be an augmented serial rule.

« **If part** ». Suppose that there is $i_d \in N$ such that for each $t \in \{1, \ldots, m-2\}$, $\sigma(t) = i$. We claim that f^{Γ} is strategy-proof under Kemeny extension. Let $R \in \mathcal{R}^N$ and $R_{i_d} : a_1 \ldots a_{m-2}\, a\, b$. Then, either $f(R) : a_1 \ldots a_{m-2}\, a\, b$ or $f(R) : a_1 \ldots a_{m-2}\, b\, a$. Therefore, $\delta(R_{i_d}, f(R)) \leq 1$. Let $i \in N$ and $R'_i \in \mathcal{R}$ be such that $\delta(R_i, f(R'_i, R_{-i})) < \delta(R_i, f(R))$. Without loss of generality, suppose that $f(R'_i, R_{-i}) : a_1 \ldots a_{m-2}\, a\, b$ and $f(R) : a_1 \ldots a_{m-2}\, b\, a$. Therefore, $a\ P_i\ b$. Thus, $\widehat{S}_{\widehat{\omega}}((R'_i, R_{-i})|_{\{a,b\}})\ P_i|_{\{a,b\}}\ \widehat{S}_{\widehat{\omega}}(R|_{\{a,b\}})$. This contradicts the fact that $\widehat{S}_{\widehat{\omega}}$ is strategy-proof under every refinement. If $m = 3$, then every augmented serial rule trivially satisfies the condition that for each $t \in \{1, \ldots, m-2\}$, $\sigma(t) = \sigma(1)$.

« **Only if part** ». Suppose that $m > 3$. Suppose that $|\sigma(\{1, \ldots, m-2\})| > 1$. Let $j := \sigma(m-2)$. Then $T = \{t' | t' < m-2, \sigma(t') \neq j\}$ is not empty. Let

$t := max\, T$ and $i := \sigma(t)$. Then, i is the dictator for position t and j is the dictator for positions $t+1$ through $m-2$. Let $R \in \mathcal{R}^N$ be a profile such that $R_i : a_1 \dots a_t \dots a_m$ and for each $i' \in N \setminus \{i\}$, $R_{i'} : a_1 \dots a_t\, a_m\, a_{m-1} \dots a_{t+1}$. If $t > 1$, let $R'_i : a_1 \dots a_{t-1}\, a_{t+1}\, a_t\, a_{t+2} \dots a_m$.[10] We consider two cases :

Case 1 : Agent i is not a dictator for position $m-1$. Then, $f(R) : a_1 \dots a_t\, a_m a_{m-1} \dots a_{t+1}$, and $f(R'_i, R_{-i}) : a_1 \dots a_{t-1} a_{t+1}\, a_t\, a_m \dots a_{t+2}$. By routine computation, $\delta(R_i, f(R)) = \frac{(m-t-1)(m-t)}{2}$ and $\delta(R_i, f(R'_i, R_{-i}) = \frac{(m-t-1)(m-t-2)}{2} + 1$.[11] Then, $\delta(R_i, f(R'_i, R_{-i})) - \delta(R_i, f(R)) = t-(m-2)$. Recall that $t < m-2$. Therefore, we have $\delta(R_i, f(R'_i, R_{-i})) < \delta(R_i, f(R))$. This shows that f^Γ is not strategy-proof under Kemeny extension.

Case 2 : Agent i is a dictator for position $m-1$. Then, $f(R) : a_1 \dots a_t\, a_m \dots a_{t+3}\, a_{t+1}\, a_{t+2}$ and if $t > m-3$, then we have $f(R'_i, R_{-i}) : a_1 \dots a_{t-1}\, a_{t+1}\, a_t\, a_m \dots a_{t+4}\, a_{t+2}\, a_{t+3}$.[12] Then, $\delta(R_i, f(R)) = \frac{(m-t-1)(m-t)}{2} - 1$ and $\delta(R_i, f(R'_i, R_{-i})) = \frac{(m-t-1)(m-t-2)}{2}$. Because $\delta(R_i, f(R'_i, R_{-i})) - \delta(R_i, f(R)) = t - (m-2)$ and $t < m-2$, we have $\delta(R_i, f(R'_i, R_{-i})) < \delta(R_i, f(R))$. This shows that f^Γ is not strategy-proof under Kemeny extension.

10. If $t = 1$, let $R'_i : a_2\, a_1\, a_3 \dots a_m$.

11. To see this, note that $f(R)$ is obtain from R_i by reversing the rank $a_{t+1} \dots a_m$ on the button of R_i. Therefore, $\delta(R_i, f(R)) = \delta(a_{t+1} \dots a_m, a_m \dots a_{t+1}) = \frac{(m-t-1)(m-t)}{2}$ (the Kemeny distance between two reverse orderings of q elements is $1 + 2 \dots + (q-1) = \frac{(q-1)q}{2}$). Next, let $R^0 : a_1 \dots a_t\, a_{t+1}\, a_m \dots a_{t+2}$. Then, $\delta(R_i, f(R'_i, R_{-i})) = \delta(R_i, R^0) + \delta(R^0, f(R'_i, R_{-i})) = \frac{(m-t-1)(m-t-2)}{2} + 1$ because $\delta(R_i, R^0) = 1$ and $\delta(R^0, f(R'_i, R_{-i}) = \dfrac{(m-t-1)(m-t-2)}{2}$.

12. It $t = m-3$, $f(R'_i, R_{-i}) : a_1 \dots a_{t-1}\, a_{t+1}\, a_t\, a_{m-1}\, a_m$.

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