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ADDITIONAL TESTS FOR A UNIT ROOT ALLOWING FOR A BREAK IN THE TREND FUNCTION AT AN UNKNOWN TIME

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RÉSUMÉ

Cet article s'inscrit dans la littérature sur les tests de racine unitaire tenant compte d'un changement dans la fonction de tendance de séries chronologiques lorsque la position du point de rupture est inconnue. Les études précédentes se sont attardées principalement à l'approche "innovational outlier" (IO) qui modélise une rupture se produisant lentement à travers le temps, contrairement à l'approche "additive outlier" (AO) où la rupture est supposée soudaine. Les distributions limites sont dérivées pour plusieurs modèles issus de l'approche AO et les valeurs critiques asymptotiques sont tabulées. Une attention particulière est accordée au choix de la date de rupture. Contrairement à certaines études précédentes, où la rupture dans la fonction de tendance est considérée uniquement sous l'hypothèse alternative de stationnarité, les comportements limites des statistiques dans les cas IO et AO sont aussi dérivés lorsque le changement dans la fonction de tendance est permis sous l'hypothèse nulle d'une racine unitaire. Il est démontré que les distributions limites sont invariantes asymptotiquement à un déplacement de la moyenne, mais ne le sont pas à un changement de pente. Enfin, on expose les résultats de simulations approfondies. Elles sont utilisées pour déterminer les propriétés de niveau et de puissance des statistiques dans le cas d'échantillons finis en s'attardant à l'effet des choix de la date de rupture et du nombre de retards dans l'autorégression.

Mots-clés : rupture de tendance, hypothèse de racine unitaire, tendance stochastique, tendance déterministe, changement structurel.

ABSTRACT

This paper adds to the current literature on unit root tests that allow for a shift in the trend function of a time series where the location of the break point is unknown. Previous results in the literature focused mainly on the "innovational outlier" (IO) approach which models the break as occurring slowly over time as opposed to the "additive outlier" (AO) approach which models the change as sudden. Limiting distributions are derived for several models in the AO framework and asymptotic critical values are tabulated. Particular attention is given to the ways in which the break date is chosen. Unlike some previous studies where the break in the trend function is considered only under the alternative hypothesis of trend stationarity, the limiting behavior of statistics in both the IO and AO frameworks are also derived in the case where a shift in the trend function is permitted under the null hypothesis of a unit root. It is shown that the limiting distributions are asymptotically invariant to a mean shift but are not asymptotically invariant to a slope change. Finally, the results of extensive Monte Carlo simulations are reported. These simulations are used to assess the size and power properties of the statistics in finite samples with attention focused on the effects due to the choices of the break date and the autoregressive lag length.

Key words: breaking trend, unit root hypothesis, stochastic trend, deterministic trend, structural change.

1. INTRODUCTION

During the past decade there has been an on–going debate as to whether shocks to macroeconomic times series should be regarded as permanent or temporary. The debate was sparked by the important findings of Nelson and Plosser (1982). By using the statistical techniques of Dickey and Fuller (1979) they concluded that most macroeconomic time series are best characterized by unit root processes implying that shocks to these series are permanent. This view was challenged in Perron (1989,1990) where it was shown that a rejection of the unit root hypothesis is possible for many macroeconomic times series once allowance is made for a one–time shift in the trend function. Thus, many macroeconomic time series may be better characterized as having temporary shocks fluctuating around a broken deterministic trend function. In this framework, the majority of shocks to the economy are temporary while the breaks in the intercept or slope can be viewed as permanent shocks which are rare or occur infrequently. Two methods of modeling the changes were proposed in Perron (1989,1990). The first, called the "additive outlier" (AO), views the break as occurring suddenly while the second, called the "innovational outlier" (IO), views the break as evolving more slowly over time. The choice of AO versus IO depends on the view one is taking as to the dynamics of the transition path following a break.

A key assumption of the framework proposed in Perron (1989,1990) is that the date of the break in the trend function is fixed (exogenous) and chosen independently of the data. This assumption has drawn much criticism in the recent literature based on the argument that break dates are often chosen after looking at the data leaving room for "data mining". This criticism was first pointed out by Christiano (1992). Several recent studies have proposed procedures which address the choice of break date issue. These include Banerjee, Lumsdaine and Stock (1992) (henceforth BLS (1992)), Perron (1994), Perron and Vogelsang (1992), and Zivot and Andrews (1992).

The strategy used in all four studies was to endogenize the choice of the break date by making it data dependent (i.e. totally correlated with the data). Two approaches in endogenizing the choice of the break point have been considered. Both approaches require estimation of a Dickey–Fuller type regression at all possible (allowable) break dates. The first procedure is to choose the break date which minimizes the Dickey–Fuller t–statistic across all possible regressions. The second procedure is to choose the break date which maximizes (or minimizes, depending on the context) a statistic which tests the significance of one or more of the coefficients on the trend break dummy variables.

Within the four studies, asymptotic results are available for many combinations of trend breaks, choice of break year, and choice of AO or IO models. For non–trending data results are provided in Perron and Vogelsang (1992) for both the AO and IO models where the break date is chosen both by minimizing the D–F t–statistics and by the significance of the coefficient on a mean–break dummy variable. For trending data, Zivot and Andrews (1992) provide results for the IO models where the break date is chosen by minimizing the D–F t–statistic. These include a model with a mean–break only (crash model), a model with a mean–break and a slope change (crash/changing growth model) and a model with a smooth change in slope only (changing growth model). BLS (1992) give results in the IO framework for the crash model and the changing growth model where the break date is chosen both by the D–F t–statistics and by the significance of a trend break dummy parameter. BLS (1992) do not provide results for the crash/changing growth model.

With the exception of Perron and Vogelsang (1992), in all of the above studies the assumption is made that no break has occurred under the null hypothesis of unit root when deriving the asymptotic results. It is shown in Perron and Vogelsang (1992) that for non–trending data, the unit root statistics will be asymptotically invariant to a mean shift under the null hypothesis, but this invariance does not hold in finite samples. Mean and slope changes are nuisance parameters as far as a unit root test is concerned. Clearly it is important to have results determining how mean and slope changes affect the limiting distributions in the models that allow trending data.

This paper has several goals. First, asymptotic distributions are derived for statistics in the crash model and the crash/changing growth model parameterized in the AO framework. Statistics are considered whereby the break date is chosen by minimizing the D–F t–statistic and using the significance of trend–break dummy parameters. Asymptotic results are also provided for the crash/changing growth model parameterized in the IO framework with the break date being chosen via the significance of trend–break dummy parameters. These results are obtained under the assumption that no break has occurred under the null hypothesis. Second, asymptotic results are derived under the assumption that a break has occurred under the null hypothesis of a unit root. Results are given for all three of the models that permit trending data. It is shown that the limiting distributions of the statistics are invariant to a mean change, but are not invariant to a change in slope. When a change in slope is present, asymptotic results are obtained which differ from the no–slope–change asymptotic results. It is shown, however, that for slope changes typically encountered in practice, the slope–change asymptotics are a poor approximation to the finite sample distributions. It is only for very large slope changes that the slope–change asymptotics adequately approximate the finite sample distributions. Finally, results of extensive finite sample simulations are reported to illustrate, i) how well the asymptotic distributions approximate the finite sample distributions, ii) how finite sample size is affected by shifts in the mean and slope, and iii) how finite sample size and power vary between the AO and IO frameworks including how size and power depend on the choice of the break year. In particular, it is found that mean breaks and slope changes can cause size distortions in finite samples. However, the magnitudes of mean breaks and slope changes necessary to cause size distortion are much larger than those frequently seen in actual economic data.

The rest of the paper is organized as follows. In the next section the models and statistics are defined and described. In Section 3 limiting null distributions are derived for both the case where no break has occurred as well as the case where a break has occurred. In Section 4 the finite sample distributions obtained by simulation are presented. Section 5 contains the results of finite sample size and power simulations. Section 6 has concluding remarks, and proofs are relegated to a mathematical appendix.

2. THE MODELS AND STATISTICS

This section lays out the models and statistical procedures that can be used to test for a unit root in a trending series when allowance is made for a break in the trend function at an unknown date. The discussion is brief, and the reader is referred to Perron (1989, 1994), Zivot and Andrews (1992), and BLS (1992) for further details. To be consistent with the existing literature, it is assumed that at most one break has occurred in the trend function. The date of the break (should it occur) is denoted by T_b^c with $1 < T_b^c < T$, where T is the sample size. (The superscript "c" is used to denote the "correct" or true break date). Since the break date is assumed to be unknown, it will be necessary to run regressions allowing break dates that are different from the true break date. Therefore, T_b is used to denote the break date used in a particular regression. The models are labeled as follows. The first, labeled Model 1, allows a shift in the intercept of a trending series. The second, labeled Model 2, allows both a change in intercept and slope. The third, labeled Model 3, allows a "smooth" change in the slope by requiring the end points of the two segments of the broken trend to be joined.

2.1 The Additive Outlier Model

The AO model applies to cases where the break is assumed to occur instantly and is not affected by the dynamics of the series. This terminology is taken from the literature on time series with outliers (e.g. Tiao (1985)). The AO models can be parameterized, respectively, for ($t = 1, ..., T$) as:

$$(1.1) \quad y_t = \mu + \beta t + \theta DU_t^c + z_t, \qquad \text{(Model 1)}$$

$$(1.2) \quad y_t = \mu + \beta t + \theta DU_t^c + \gamma DT_t^{*c} + z_t, \qquad \text{(Model 2)}$$

$$(1.3) \quad y_t = \mu + \beta t + \gamma DT_t^{*c} + z_t, \qquad \text{(Model 3)}$$

where $DU_t^c = 1(t > T_b^c)$, $DT_t^{*c} = 1(t > T_b^c)(t - T_b^c)$ and $1(\cdot)$ is the indicator function. The error z_t is specified to be an ARMA(p+1,q) process defined as $A(L)z_t = B(L)e_t$, where e_t is i.i.d. $(0,\sigma^2)$ with finite fourth moment. A(L) and B(L) are polynomials in L of order $p+1$ and q, respectively, where A(L) can be factored as $A(L) = (1 - \alpha L)A^*(L)$, and $A^*(L)$ is a p^{th} order polynomial in L. It is assumed that $A^*(L)$ and B(L) have all roots outside the unit circle. The initial condition, y_0, is assumed to be a fixed constant. Under the null hypothesis of a unit root $\alpha = 1$, $\mu = y_0$, and z_t can be expressed as $z_t = z_{t-1} + \psi^*(L)e_t$, where $\psi^*(L) \equiv A^*(L)^{-1}B(L)$. Under the alternative hypothesis $|\alpha| < 1$, and z_t is a stationary process that can be expressed as $z_t = \psi(L)e_t$, where $\psi(L) \equiv A(L)^{-1}B(L)$. Note that $\psi^*(L) = (1 - \alpha L)\psi(L)$ and that $\psi(0) = \psi^*(0) = 1$. The parameters θ and γ measure the magnitudes of the possible trend break. For example, when there is a unit root, the intercept of $\{y_t\}$ is y_0 up to time T_b^c and $y_0 + \theta$ afterwards. When the series is stationary, the intercept is μ up to time T_b^c and $\mu + \theta$ afterwards.

Testing for a unit root in the AO framework consists of a two-step procedure. The first step involves detrending the series using the following regressions estimated by OLS, ($t = 1, ..., T$):

$$(2.1) \quad y_t = \mu + \beta t + \theta DU_t + \tilde{y}_t^1, \qquad \text{(Model 1)}$$

$$(2.2) \quad y_t = \mu + \beta t + \theta DU_t + \gamma DT_t^* + \tilde{y}_t^2, \qquad \text{(Model 2)}$$

$$(2.3) \quad y_t = \mu + \beta t + \gamma DT_t^* + \tilde{y}_t^3, \qquad \text{(Model 3)}$$

where $DU_t = 1(t > T_b)$ and $DT_t^* = 1(t > T_b)(t - T_b)$. Note that regressions (2.1)–(2.3) are estimated using the break date T_b which may or may not be the same as the true break date (should it occur). The second step then tests the unit root hypothesis using the t–statistic for testing $\alpha = 1$ in the regressions ($t = k + 1, ..., T$):

(3.1) $\tilde{y}^j_t = \Sigma^k_{i=0}\omega_i D(T_b)_{t-i} + \alpha\tilde{y}^j_{t-1} + \Sigma^k_{i=1}c_i\Delta\tilde{y}^j_{t-i} + u_t,$ (j = 1,2)

(3.2) $\tilde{y}^3_t = \alpha\tilde{y}^3_{t-1} + \Sigma^k_{i=1}c_i\Delta\tilde{y}^3_{t-i} + u_t,$

where $\tilde{y}^j_t$ are the residuals from regression (2.j) (j = 1,2,3) and $D(T_b)_t = 1(t=T_b+1)$. The inclusion of the k+1 dummy variables $D(T_b)_{t-i}$ (i = 0, ..., k) in (3.1) is necessary to ensure that the limiting distributions of the t–statistics from (3.1) are invariant to the correlation structure of the errors. See Perron and Vogelsang (1993a) for details. Additionally, the inclusion of these dummy variables ensures that the limiting distributions of the t–statistics from (3.1) are the same as in the IO model below. The one–time dummies are not needed in Model 3 as the t–statistic on α is invariant to the correlation structure of the data with the appropriate choice of k. However, the limiting distribution of the t–statistic will be different from the IO model as argued by Perron and Vogelsang (1993a). The t–statistics for testing α = 1 using regressions (3.1) and (3.2) are denoted by $t_{\hat{\alpha}}(j,AO,T_b,k)$, where j indicates the model (j=1,2,3), T_b indicates the break date used and k indicates the lag length of the autoregression. Since T^c_b is assumed unknown, and the appropriate order of the autoregression is usually unknown, procedures which select T_b and k are necessary. These issues are discussed after outlining the IO approach.

2.2 The Innovational Outlier Model

The IO model is applicable to cases where it is more reasonable to view the break in the trend as occurring more slowly over time. In practice, the dynamic path of adjustment of the shift could take any form. However, a natural and convenient way to model the dynamics is to assume that the series reacts to shocks to the trend function in the same way that it responds to shocks to the innovation process. This assumption can be captured using the following specification. Under the null hypothesis of a unit root $\{y_t\}$ is determined by (t = 1, ..., T):

(4.1a) $y_t = y_{t-1} + \beta + \psi^*(L)(\theta D(T^c_b)_t + e_t)$ (Model 1),

(4.2a) $y_t = y_{t-1} + \beta + \psi^*(L)(\theta D(T^c_b)_t + \gamma DU^c_t + e_t)$ (Model 2),

(4.3a) $y_t = y_{t-1} + \beta + \psi^*(L)(\gamma DU^c_t + e_t)$ (Model 3).

Note that the immediate impact of a change in slope, say, for Model 3 is given by γ, while the long–run impact is given by $\psi^*(1)\gamma$. Under the alternative hypothesis of stationary errors, the models are (for $t = 1, ..., T$):

$$y_t = \mu + \beta t + \psi(L)(\theta DU_t^c + e_t), \quad (4.1b) \quad \text{(Model 1)}$$

$$y_t = \mu + \beta t + \psi(L)(\theta DU_t^c + \gamma DT_t^{*c} + e_t), \quad (4.2b) \quad \text{(Model 2)}$$

$$y_t = \mu + \beta t + \psi(L)(\gamma DT_t^{*c} + e_t). \quad (4.3b) \quad \text{(Model 3)}$$

Here, like in the unit root case, the immediate impact of a slope change in Model 3 is given by γ, however, the long–run impact is now given by $\psi(1)\gamma$. Models (4.ja) and (4.jb) (j=1,2) can be nested, and the unit root hypothesis tested in the following Dickey–Fuller (1979) type regressions ($t = k + 1, ..., T$) estimated by OLS:

$$y_t = \mu + \beta t + dD(T_b)_t + \theta DU_t + \alpha y_{t-1} + \Sigma_{i=1}^{k} c_i \Delta y_{t-i} + u_t, \quad (5.1)$$

$$y_t = \mu + \beta t + dD(T_b)_t + \theta DU_t + \gamma DT_t^* + \alpha y_{t-1} + \Sigma_{i=1}^{k} c_i \Delta y_{t-i} + u_t. \quad (5.2)$$

For Model 3, a nesting of (4.3a) and (4.3b) is not considered. Instead, the regression considered by Zivot and Andrews (1992) and BLS (1992) is used,

$$y_t = \mu + \beta t + \gamma DT_t^* + \alpha y_{t-1} + \Sigma_{i=1}^{k} c_i \Delta y_{t-i} + u_t. \quad (5.3)$$

Under the null hypothesis of a unit root, $\alpha = 1$ and $\theta = 0$ in (5.1) and (5.2) and $\gamma = 0$ in (5.2) and (5.3). Note that (5.3) does not permit a break in the trend under the null hypothesis. The null hypothesis can be tested using the t–statistics for testing $\alpha = 1$ in regressions (5.1)–(5.3). These t–statistics, for given values of T_b and k, are denoted by $t_{\hat{\alpha}}(j,IO,T_b,k)$ (j=1,2,3).

2.3 Procedures for Selecting T_b and k

To implement the above tests, some choice of T_b and k must be made. Two data dependent methods for choosing T_b are considered in this paper. The first, used by Zivot and Andrews (1992), Perron (1994), Perron and Vogelsang (1992) and BLS (1992) involves choosing T_b such that $t_{\hat{\alpha}}(j,m,T_b,k)$ (j=1,2,3 and m = AO,IO) is minimized[1]. This choice of T_b corresponds to the break date which is most likely to reject the unit root hypothesis. Let $T_b(t_{\hat{\alpha}})$ denote the value of T_b chosen by this procedure and let $t_{\hat{\alpha}}(j,m,T_b(t_{\hat{\alpha}}),k)$ =

[1]This approach is in the spirit of the supremum statistic used widely in the literature on tests for structural change with an unknown break point (e.g. Andrews (1993)).

$\inf_{T_b \in \Omega} t_{\hat{\alpha}}(j,m,T_b,k)$ denote the resulting t–statistic. Ω denotes the set of possible break points. Following Zivot and Andrews (1992), Perron (1994) and Perron and Vogelsang (1992) let $\Omega = (k + 2, ..., T-2)$ be the set of all possible breaks. Other authors, including BLS (1992) suggest trimming (ignoring) 15% of the breaks at each end point, but 15% trimming is not considered here. Some trimming is necessary for the asymptotic results to go through based on arguments in Zivot and Andrews (1992). However, using results from Perron (1994), limiting distributions can be obtained without requiring any trimming.

The second method of choosing T_b corresponds to the methods used by BLS (1992), Christiano (1992), Perron (1994) and Perron and Vogelsang (1992). This approach involves choosing T_b to maximize some statistic (usually a t or F statistic) which tests the significance of one or more of the break parameters (θ,γ). The following statistics are used. For Model 1, T_b is chosen using the maxima of $t_{\hat{\theta}}$ and $|t_{\hat{\theta}}|$ from regressions (2.1) and (5.1). The resulting value of T_b from these procedures are denoted by $T_b(t_{\hat{\theta}})$ and $T_b(|t_{\hat{\theta}}|)$. The maximum of $|t_{\hat{\theta}}|$ is used when the direction of the break is unknown while the maximum of $t_{\hat{\theta}}$ is used when the direction of the break is known *a priori* to be positive. If the direction of the break is known to be negative, then the minimum of $t_{\hat{\theta}}$ is used. This mild assumption that the direction of the break is known can lead to more powerful tests (see Perron (1994) and Perron and Vogelsang (1992)). For Model 2, three methods of choosing T_b are considered. They are the maxima of $t_{\hat{\gamma}}$, $|t_{\hat{\gamma}}|$ and $F_{\hat{\theta},\hat{\gamma}}$ (the minimum of $t_{\hat{\gamma}}$ if the break is negative) from regressions (2.2) and (5.2). $F_{\hat{\theta},\hat{\gamma}}$ is the F–statistic for testing the joint hypothesis that $\theta = \gamma = 0$. Again, the resulting values of T_b are denoted by $T_b(\cdot)$ using the appropriate argument. For Model 3, T_b is chosen using the maxima of $t_{\hat{\gamma}}$ and $|t_{\hat{\gamma}}|$ (the minimum of $t_{\hat{\gamma}}$ if the break is negative) from regressions (2.3) and (5.3). $T_b(t_{\hat{\gamma}})$ and $T_b(|t_{\hat{\gamma}}|)$ denote the respective values of T_b resulting from these maximizations. Finally, the value of $t_{\hat{\alpha}}$ corresponding to one of these choices of T_b is denoted by $t_{\hat{\alpha}}(j,m,T_b(\cdot),k)$. Note that some amount of trimming is required to obtain limiting distributions in these cases. (The results of Perron (1994) only apply to the statistics whereby T_b is chosen to minimize $t_{\hat{\alpha}}$).

Two procedures for choosing the truncation lag parameter k are considered. The first procedure is to simply choose a fixed value for $k = \bar{k}$. The resulting t–statistic is denoted by $t_{\hat{\alpha}}(j,m,l,\bar{k})$ (j=1,2,3; m=AO,IO; l=$T_b(\cdot)$). The second procedure follows Perron (1989,1990,1994) and Perron and Vogelsang (1992) by choosing k using a data dependent method. For a given value of T_b, k is chosen in the following way. Given a maximal value of k, say *kmax*, the significance of the coefficient on the last included lagged first difference is tested using a t–statistic at some pre–specified significance level. If the coefficient is insignificant, the model is estimated using *kmax*−1 lags. Again, the significance of the coefficient on the last included lagged first difference is tested. The procedure continues until significance is found for the coefficient of the last included lagged first difference. Significance is tested using asymptotic normality of the t–statistics on the coefficients of the lagged first differences. In the pure AR(p) case, this procedure selects a value for k which is greater than or equal to p with probability that approaches one asymptotically as long as *kmax* is chosen greater than p. This guarantees that the limiting distributions are the same as in the case where the true value of p is known (see Hall (1990)). In the more general case where the errors possibly have an MA component, results in Ng and Perron (1994) show that limiting distributions will also be the same as in the case where k is some function of T as long as *kmax* is increased at an appropriate rate as the sample size grows. The resulting statistics from this procedure are denoted by $t_{\hat{\alpha}}(j,m,l,k(t\text{–sig}))$.

3. THE LIMITING DISTRIBUTIONS OF THE STATISTICS

The discussion of the asymptotic distributions is divided into two cases. The first case is where no break has occurred ($\theta = \gamma = 0$). For completeness, asymptotic results are stated for all of the models outlined in Section 2. New results pertain to AO Models 1, 2 and 3 for all choices of T_b and for IO Model 2 where T_b is chosen using the t–statistic and F–statistic associated with the trend break parameters. The second case is where a break has occurred under the null hypothesis of a unit root ($\theta \neq 0$ and/or $\gamma \neq 0$). This case has not been considered elsewhere in the literature. Results are given for all the models outlined in Section 2. Where needed, asymptotic critical values are provided via simulation methods. For simplicity, the results are presented and proven for the case where the the errors are uncorrelated and no lagged first differences are included in regressions (3.1), (3.2) and (5.1)–(5.3). Thus $\psi(L)$ is set equal to 1 and k is set equal to 0. It will be argued in Section 3.2 that the results obtained in this special case remain valid under more general error structures and when k is chosen using the data dependent procedure described above. Since all the statistics are invariant to the values of y_0 and β under the null hypothesis, they are set equal to zero.

3.1 Limiting Distributions When No Break Has Occurred

The results in this section are valid for $\theta = \gamma = 0$. Imposing $y_0 = \beta = 0$ and using the simplification that $\psi(L) = 1$, the DGP is the same in all models and can be written as:

(6) $y_t = y_{t-1} + e_t$.

Consider first the limiting behavior of $t_{\hat{\alpha}}(j,m,T_b(t_{\hat{\alpha}}),k=0)$ (j=1,2,3; m=AO,IO). Let $W_j(r,\lambda)$ denote the residuals from the projection of a standard Wiener process $W(r)$ onto the subspace generated by the functions $\{1,r,du(r,\lambda)\}$ (j=1), $\{1,r,du(r,\lambda),dt^*(r,\lambda)\}$ (j=2) and $\{1,r,dt^*(r,\lambda)\}$ (j=3), where $du(r,\lambda) = 1(r > \lambda)$, $dt^*(r,\lambda) = 1(r > \lambda)(r - \lambda)$ on (0,1) and $\lambda = T_b/T$. Under the null hypothesis (6), Zivot and Andrews (1992) prove that:

$$(7) \quad t_{\hat{\alpha}}(j,IO,T_b(t_{\hat{\alpha}}),k=0) \Rightarrow \inf_{\lambda\in\Lambda}\{R_j(\lambda)\} \qquad (j=1,2,3)$$

where $R_j(\lambda) = \int_0^1 W_j(r,\lambda)dW(r)\left[\int_0^1 W_j(r,\lambda)^2 dr\right]^{-1/2}$, Λ is a closed subset of the interval (0,1), and $\Rightarrow$ denotes weak convergence in distribution. Perron (1994) has shown that this result also holds for Λ = [0,1]. Results for the AO models are obtained by combining results in Perron and Vogelsang (1993b) and Zivot and Andrews (1992). Under the null hypothesis (6),

$$(8) \quad t_{\hat{\alpha}}(j,AO,T_b(t_{\hat{\alpha}}),k=0) \Rightarrow \inf_{\lambda\in\Lambda}\{R_j(\lambda)\} \qquad (j = 1, 2),$$

$$(9) \quad t_{\hat{\alpha}}(3,AO,T_b(t_{\hat{\alpha}}),k=0) \Rightarrow \inf_{\lambda\in\Lambda}\{R_{3,AO}(\lambda)\},$$

where
$$R_{3,AO}(\lambda) = \left[\int_0^1 W_3(r,\lambda)^2 dr\right]^{-1/2} \times \left[\int_0^1 W_3(r,\lambda)dW(r) - a_{22}^{-1}\int_\lambda^1 (r - \lambda)W^*(r)dr \int_\lambda^1 W_3(r,\lambda)dr\right],$$

with $a_{22} = \lambda^3(1 - \lambda)^3/3$ and $W^*(r)$ the residuals from a projection of $W(r)$ onto the subspace generated by $\{1,r\}$. Perron (1994) has shown that (8) and (9) also hold for Λ = [0,1] so that trimming is not required in practice.

Notice that the limiting distributions are identical in the AO and IO cases for Models 1 and 2 (the crash models). Without the one–time dummies, $D(T_b)_{t-i}$ (i=0, ..., k), in regression (3.1) this equivalence will no longer hold. See Perron and Vogelsang (1993a) for details. For Model 3 (the changing growth model) the limiting distribution is different in the AO case from that in the IO case. Critical values for (7) (and (8)) can be found in Zivot and Andrews (1992) and are reproduced for Models 1 and 2 in Tables 1.A and 2.A on the lines marked T = ∞. Critical values for (9) can be found in Perron and Vogelsang

(1993a).

Now consider the limiting distributions of $t_{\hat{\alpha}}(j,m,T_b,k=0)$ ($j=1,2,3$, $m=$AO,IO) where T_b is chosen using the maximal statistics based on the coefficients of the trend break dummy variables. Note that in the AO models, $t_{\hat{\theta}}$, $t_{\hat{\gamma}}$, and $F_{\hat{\theta},\hat{\gamma}}$ will diverge as $T \to \infty$. Non-degenerate limiting distributions are obtained by considering $T^{-1/2}t_{\hat{\theta}}$, $T^{-1/2}t_{\hat{\gamma}}$ and $T^{-1}F_{\hat{\theta},\hat{\gamma}}$. Thus, choosing T_b to maximize these rescaled statistics will yield valid asymptotic testing procedures. Of course, in finite samples it makes no difference whether one maximizes the scaled or unscaled statistics. Limiting distributions for Model 1 are as follows. It is shown in BLS (1992) that, under (6),

(10) $t_{\hat{\alpha}}(1,IO,T_b(t_{\hat{\theta}}),k=0) \Rightarrow R_1(\lambda_1^*)$,

(11) $t_{\hat{\alpha}}(1,IO,T_b(|t_{\hat{\theta}}|),k=0) \Rightarrow R_1(\lambda_2^*)$,

where $\lambda_1^* = \underset{\lambda\in\Lambda}{\text{argmax}}\, Q_{1,IO}(\lambda)$, $\lambda_2^* = \underset{\lambda\in\Lambda}{\text{argmax}}\, |Q_{1,IO}(\lambda)|$ and,

$$Q_{1,IO}(\lambda) = H_1(\lambda)K_1(\lambda)^{-1/2}, \quad \text{with}$$

$$H_1(\lambda) = B_1(\lambda) - \int_\lambda^1 W^*(r)dr \int_0^1 W^*(r)dW(r) / \int_0^1 W^*(r)^2 dr,$$

$$K_1(\lambda) = a_{11} - \left(\int_\lambda^1 W^*(r)dr\right)^2 / \int_0^1 W^*(r)^2 dr, \text{ and}$$

$$B_1(\lambda) = \lambda(4-3\lambda)W(1) - W(\lambda) - 6\lambda(1-\lambda)\int_0^1 r\,dW(r),$$

$$a_{11} = \lambda(1-\lambda)(3\lambda^2 - 3\lambda + 1).$$

Critical values for (10) and (11) can be found in Perron (1994). In the case of the AO model, it is shown in the appendix that under (6),

(12) $t_{\hat{\alpha}}(1,AO,T_b(t_{\hat{\theta}}),k=0) \Rightarrow R_1(\lambda_1^*)$,

(13) $t_{\hat{\alpha}}(1,AO,T_b(|t_{\hat{\theta}}|),k=0) \Rightarrow R_1(\lambda_2^*)$,

where $\lambda_1^* = \underset{\lambda\in\Lambda}{\text{argmax}}\, Q_{1,AO}(\lambda)$, $\lambda_2^* = \underset{\lambda\in\Lambda}{\text{argmax}}\, |Q_{1,AO}(\lambda)|$ and

$$Q_{1,AO}(\lambda) = [a_{11}L_1(\lambda)]^{-1/2} \int_\lambda^1 W^*(r)dr, \quad \text{with}$$

$$L_1(\lambda) = \int_0^1 W^*(r)^2 dr - [\int_\lambda^1 W^*(r)dr]^2/a_{11}.$$

Critical values for (12) and (13) were calculated via simulation methods using i.i.d. N(0,1) errors with 1,000 steps to approximate the Wiener processes using 10,000 replications. They are given in Tables (1.B) and (1.C) on the lines $T = \infty$.

Limiting distributions for Model 2 are as follows. All proofs of these results are given in the appendix. For the IO model under (6),

(14) $t_{\hat{\alpha}}(2,IO,T_b(t_{\hat{\gamma}}),k=0) \Rightarrow R_2(\lambda_1^*)$,

(15) $t_{\hat{\alpha}}(2,IO,T_b(|t_{\hat{\gamma}}|),k=0) \Rightarrow R_2(\lambda_2^*)$,

(16) $t_{\hat{\alpha}}(2,IO,T_b(F_{\hat{\theta},\hat{\gamma}}),k=0) \Rightarrow R_2(\lambda_3^*)$,

where $\lambda_1^* = \underset{\lambda \in \Lambda}{\text{argmax}}\ Q^A_{2,IO}(\lambda)$, $\lambda_2^* = \underset{\lambda \in \Lambda}{\text{argmax}}\ |Q^A_{2,IO}(\lambda)|$,

$\lambda_3^* = \underset{\lambda \in \Lambda}{\text{argmax}}\ Q^B_{2,IO}(\lambda)$, and

$$Q^A_{2,IO}(\lambda) = [H_2(\lambda) - K_1(\lambda)^{-1}K_3(\lambda)H_1(\lambda)] * [K_2(\lambda) - K_1(\lambda)^{-1}K_3(\lambda)^2]^{-1/2},$$

$$Q^B_{2,IO}(\lambda) = [H_1(\lambda)^2K_2(\lambda) - 2H_1(\lambda)K_3(\lambda)H_2(\lambda) + H_2(\lambda)^2K_1(\lambda)] * [K_1(\lambda)K_2(\lambda) - K_3(\lambda)^2]^{-1}, \quad \text{with,}$$

$$H_2(\lambda) = B_2(\lambda) - \int_\lambda^1 (r-\lambda)W^*(r)dr \int_0^1 W^*(r)dW(r) / \int_0^1 W^*(r)^2 dr,$$

$$K_2(\lambda) = a_{22} - [\int_\lambda^1 (r-\lambda)W^*(r)dr]^2 / \int_0^1 W^*(r)^2 dr,$$

$$K_3(\lambda) = a_{12} - \int_\lambda^1 W^*(r)dr \int_\lambda^1 (r-\lambda)W^*(r)dr / \int_0^1 W^*(r)^2 dr \quad \text{and}$$

$$B_2(\lambda) = \lambda(1-\lambda)^2 W(1) + \int_\lambda^1 (r-\lambda)dW(r) - (1-\lambda)^2(1+2\lambda)\int_0^1 r\,dW(r),$$

$$a_{12} = \lambda^2(1-\lambda)^2(2\lambda-1)/2.$$

Critical values for (14) – (16) were calculated via simulations. Critical values for (14) and (15) are tabulated in Table 2F. Critical values for (16) are tabulated in Table 2E on the line $T = \infty$. In the AO model, under (6), it is shown in the appendix that,

(17) $\quad t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\gamma}}),k{=}0) \;\Rightarrow\; R_2(\lambda_1^*)$,

(18) $\quad t_{\hat{\alpha}}(2,AO,T_b(|t_{\hat{\gamma}}|),k{=}0) \;\Rightarrow\; R_2(\lambda_2^*)$,

(19) $\quad t_{\hat{\alpha}}(2,AO,T_b(F_{\hat{\theta},\hat{\gamma}}),k{=}0) \;\Rightarrow\; R_2(\lambda_3^*)$,

where $\lambda_1^* = \underset{\lambda\in\Lambda}{\operatorname{argmax}}\; Q^A_{2,AO}(\lambda)$, $\lambda_2^* = \underset{\lambda\in\Lambda}{\operatorname{argmax}}\; |Q^A_{2,AO}(\lambda)|$,

$\lambda_3^* = \underset{\lambda\in\Lambda}{\operatorname{argmax}}\; Q^B_{2,AO}(\lambda)$, and

$$Q^A_{2,AO}(\lambda) = [(a_{22} - a_{12}^2/a_{11})L_1(\lambda) - L_3(\lambda)^2]^{-1/2}L_3(\lambda)$$

$$Q^B_{2,AO}(\lambda) = [L_1(\lambda)(a_{11}a_{22} - a_{12}^2) - a_{11}L_3(\lambda)^2]^{-1}$$

$$\times \left[a_{22}\left(\int_\lambda^1 W^*(r)dr\right)^2 - 2a_{12}\int_\lambda^1 W^*(r)dr\int_\lambda^1 (r-\lambda)W^*(r)dr + a_{11}\left(\int_\lambda^1 (r-\lambda)W^*(r)dr\right)^2\right],$$

with $L_3(\lambda) = \int_\lambda^1 (r-\lambda)W^*(r)dr - (a_{12}/a_{11})\int_\lambda^1 W^*(r)dr$.

Critical values for (17) – (19) were calculated via simulations and are tabulated in Tables (2B) – (2D) on the lines $T = \infty$.

Limiting distributions for Model 3 are as follows. Results for Model (3,IO) are proven in BLS (1992). Under (6),

(20) $\quad t_{\hat{\alpha}}(3,IO,T_b(t_{\hat{\gamma}}),k{=}0) \;\Rightarrow\; R_3(\lambda_1^*)$,

(21) $\quad t_{\hat{\alpha}}(3,IO,T_b(|t_{\hat{\gamma}}|),k{=}0) \;\Rightarrow\; R_3(\lambda_2^*)$,

where $\lambda_1^* = \underset{\lambda\in\Lambda}{\operatorname{argmax}}\; Q_{3,IO}(\lambda)$, $\lambda_2^* = \underset{\lambda\in\Lambda}{\operatorname{argmax}}\; |Q_{3,IO}(\lambda)|$, and

$$Q_{3,IO}(\lambda) = K_2(\lambda)^{-1/2}H_2(\lambda).$$

Critical values for (21) can be found in BLS (1992). Finally, it is shown in the appendix for Model (3,AO) that under (6),

(22) $\quad t_{\hat{\alpha}}(3,AO,T_b(t_{\hat{\gamma}}),k{=}0) \;\Rightarrow\; R_{3,AO}(\lambda_1^*)$,

$$(23) \quad t_{\hat{\alpha}}(3,AO,T_b(|t_{\hat{\gamma}}|),k=0) \Rightarrow R_{3,AO}(\lambda_2^*),$$

where $\lambda_1^* = \underset{\lambda \in \Lambda}{\operatorname{argmax}}\, Q_{3,AO}(\lambda)$, $\lambda_2^* = \underset{\lambda \in \Lambda}{\operatorname{argmax}}\, |Q_{3,AO}(\lambda)|$ and,

$$Q_{3,AO}(\lambda) = [a_{22}L_2(\lambda)]^{-1/2}\int_\lambda^1 (r-\lambda)W^*(r)dr.$$

with $L_2(\lambda) = \int_0^1 W^*(r)^2 dr - a_{22}^{-1}[\int_\lambda^1 (r-\lambda)W^*(r)dr]^2$.

Critical values for (22) and (23) were calculated via simulations and are tabulated in Table 2F.

3.2 Limiting Distributions Under More General Error Processes

The previous asymptotic results were obtained assuming i.i.d. errors and using $k=0$ in the various regressions. The results of Section 3.1 remain valid for more general error processes. In the case where the errors follow an AR(p) process, it is straightforward to extend the results of Dickey and Fuller (1979) to show that the above distributions remain valid provided $k \geq p$. This is formally proved for Model (3,AO) when the break date is known by Vogelsang (1993). Similarly, the distributions remain valid when the errors follow an ARMA(p,q) process provided that k increases at an appropriate rate as the sample size grows as in Said and Dickey (1984). For the case where k is chosen using the significance of the coefficient on the last included lag (the t–sig procedure), the limiting distributions remain valid in the AR(p) case provided $kmax \geq p$. This follows from results in Hall (1990). Likewise, in the ARMA(p,q) case, the limiting distributions remain valid as long as $kmax$ increases at an appropriate rate as the sample size grows. This result is proven by Ng and Perron (1994).

3.3 Limiting Distributions When a Break Occurs

This sub–section describes the limiting behavior of $t_{\hat{\alpha}}(j,m,T_b,k=0)$ (j=1,2,3 m=AO,IO) when the values of θ and γ are different from zero. Throughout this section, "true break date" and "incorrect break date" refer to λ_c and λ ($\lambda \neq \lambda_c$) respectively and note that λ_c and λ remained fixed as T increases. In finite samples, we adopt the conventions that $T_b^c = [\lambda_c T]$ and $T_b = [\lambda T]$ where $[x]$ denotes the integer part of x. When regressions (2.1) – (2.3), (5.1) and (5.2) are estimated using the true break date , $t_{\hat{\alpha}}$ will be exactly invariant to the value of θ under the null hypothesis. Similarly, when regressions (2.2), (2.3) and (5.2) are estimated using the true break date, $t_{\hat{\alpha}}$ will be exactly invariant to the value of γ under the null hypothesis. On the other hand, when regression (5.3) is estimated at the true break date, $t_{\hat{\alpha}}$ will not be invariant to γ. Moreover, when any of the above regressions are estimated at the incorrect break date, $t_{\hat{\alpha}}$ will not be invariant to θ,γ in finite samples. In fact, $t_{\hat{\alpha}}$ will no longer be invariant to γ asymptotically. Because the testing procedures involve estimation at incorrect break dates, the limiting results in Section 3.1 will no longer be valid when $\gamma \neq 0$.

Asymptotically, the dependence upon θ and γ can be made precise. It can be shown that all of the statistics are asymptotically invariant to the magnitude of the intercept change, θ. Perron and Vogelsang (1992) prove asymptotic invariance with respect to θ in models that do not permit trending data. Proving invariance with respect to θ of the statistics considered in this paper involves simple extensions to the results in Perron and Vogelsang (1992) and a formal proof is omitted. When $\gamma \neq 0$, results are given for Models 2 and 3 as these are the models that permit a change in slope. As before, results are derived assuming i.i.d. errors so that $\psi(L) = 1$. The values of y_0 and β are again set equal to zero as the statistics remain invariant to y_0 and β under the null hypothesis. Finally, since the statistics are asymptotically invariant to θ, the value of θ is set equal to zero for simplicity.

The DGP can be written as,

(24) $$y_t = \gamma DT_t^{*c} + z_t, \qquad z_t = z_{t-1} + e_t.$$

The following limiting results are proven in the appendix. Results are given both when the model is estimated at the true break date as well as when the model is estimated at the incorrect break date. Consider estimating the model using the true break date ($\lambda = \lambda_c$). It is shown in Perron (1989) and Perron and Vogelsang (1993b) that under (24),

(25) $$t_{\hat{\alpha}}(2,m,[\lambda_c T],k=0) \Rightarrow R_2(\lambda_c) \quad (m=IO,AO),$$

(26) $$t_{\hat{\alpha}}(3,AO,[\lambda_c T],k=0) \Rightarrow R_{3,AO}(\lambda_c).$$

Things are different for Model (3,IO). It is shown in the appendix that under (24),

(27) $$T^{-1/2} t_{\hat{\alpha}}(3,IO,[\lambda_c T],k=0) = O_p(1).$$

It is easy to establish that $T^{-1/2} t_{\hat{\alpha}}(3,IO,[\lambda_c T],k=0)$ converges to a random variable which has support over the entire real line. Thus, $t_{\hat{\alpha}}(3,IO,[\lambda_c T],k=0)$ will diverge to minus infinity with non–zero probability. Therefore, even when the date of the break is known, the asymptotic size of a unit root test using $t_{\hat{\alpha}}(3,IO,T_b^c,k=0)$ will be distorted if $\gamma \neq 0$.

When Models (2,IO), (2,AO), (3,IO) and (3,AO) are estimated at the incorrect break date ($\lambda \neq \lambda_c$), it is shown in the appendix that under (24),

$$(28) \qquad T^{-1/2}t_{\hat{\alpha}}(j,m,[\lambda T],k=0) = O_p(1) \quad (j=2,3;\ m=AO,IO).$$

It can be shown that the limits of $T^{-1/2}t_{\hat{\alpha}}(j,m,[\lambda T],k=0)$ ($j=2,3$; $m=AO,IO$) are nonrandom functions of γ, λ and λ_c, and that the sign of these limits will depend on λ and λ_c. It is tedious but straightforward to show that given λ_c, there exists λ such that the limit of $T^{-1/2}t_{\hat{\alpha}}(j,m,[\lambda T],k=0)$ ($j=2,3$; $m=AO,IO$) is negative. Thus, $t_{\hat{\alpha}}(j,m,[\lambda T],k=0)$ ($j=2,3$; $m=AO,IO$) diverges to $-\infty$ for some values of λ. The implication of this result is that when $\gamma \neq 0$:

$$(29) \qquad t_{\hat{\alpha}}(j,m,T_b(t_{\hat{\alpha}}),k=0) \Rightarrow -\infty \quad (j=2,3;\ m=AO,IO).$$

The consequences of (29) is that the size of unit root tests using $t_{\hat{\alpha}}(j,m,T_b(t_{\hat{\alpha}}),k=0)$ ($j=2,3$; $m=AO,IO$) will approach one as the sample size grows when $\gamma \neq 0$. What is not evident from (29) is how large T must be, given γ, before size distortions become problematic. The simulation results in Section 5 will provide some insight to the size distortions caused by γ in finite samples.

Another way of explaining the distortions caused by $\gamma \neq 0$ is to examine the values of λ that are chosen by minimizing $t_{\hat{\alpha}}$. If λ_c is chosen for λ asymptotically, then size distortions would disappear as T grows. However, by looking at the behavior of $\lim_{T\to\infty} T^{-1/2}t_{\hat{\alpha}}(j,m,[\lambda T],k=0)$ ($j=2,3$; $m=AO,IO$) it is easy to establish that $\operatorname{argmin}\{t_{\hat{\alpha}}(j,m,[\lambda T],k=0)\} \neq \lambda_c$. As an illustration, $\lim_{T\to\infty} T^{-1/2}t_{\hat{\alpha}}(2,AO,[\lambda T],k=0)$ was computed for $\lambda \in (0,1)$ using $\lambda_c = 0.2$, $\gamma = 5$ and $\sigma^2 = 1$ and is plotted in Figure 1. As can be seen in the figure, $\lim_{T\to\infty} T^{-1/2}t_{\hat{\alpha}}(j,m,[\lambda T],k=0)$ attains a global minimum at $\lambda \approx 0.1$, not at $\lambda = 0.2$. Since $t_{\hat{\alpha}}(j,m,[\lambda T],k=0)$ diverges to $-\infty$ at $\lambda = \operatorname{argmin}\{t_{\hat{\alpha}}(j,m,[\lambda T],k=0)\}$, the asymptotic size of the test will be one. In order to give some sense of how the break dates are chosen in finite samples, the break dates that were chosen in the simulations for Table 3 were recorded. For example, when $\gamma = 5$ and $\sigma^2 = 1$, $\lim_{T\to\infty} T^{-1/2}t_{\hat{\alpha}}(j,m,[\lambda T],k=0)$ attains minima at $\lambda \approx 0.1$, 0.43 and 0.73 for $\lambda_c = 0.2$, 0.5 and 0.8 respectively. With T = 100 the mode of the chosen break dates were $\lambda = 0.12$, 0.44 and 0.75 respectively, which are quite close to the asymptotic values. With T = 250 the mode of the chosen break dates were 0.13, 0.45 and 0.75 respectively, and with T = 500 the modes were 0.13, 0.45 and 0.76, which are again close to the asymptotic values. So, one can view the size distortions of these statistics resulting from the wrong break date being chosen.

Clearly then, given γ, if T is large enough, size distortions from using the $t_{\hat{\alpha}}(j,m,T_b(t_{\hat{\alpha}}),k=0)$ ($j=2,3$; $m=$AO,IO) statistics could lead to false inference. One way around this problem is to find procedures which tend to pick the true break date as the sample size grows. Such a procedure will lead to asymptotically valid tests in Models (2,AO), (2,IO) and (3,AO) as $t_{\hat{\alpha}}$ is invariant to γ at the true break date under the null hypothesis. In Model (3,IO) however, choosing the true break date will not yield a valid test as $t_{\hat{\alpha}}$ is not invariant to γ even at the true break date.

In the AO models, the true break date will be chosen as the sample size grows using the procedures whereby T_b is chosen using $t_{\hat{\gamma}}$ and $F_{\hat{\theta},\hat{\gamma}}$. This can be established by examining the behavior of $t_{\hat{\gamma}}$ and $F_{\hat{\theta},\hat{\gamma}}$ in the presence of a slope change in Models (2,AO) and (3,AO). Under (24) the following results are proven in the appendix. For Model (2,AO), if $\lambda \neq \lambda_c$,

$$(30) \quad T^{-1/2}t_{\hat{\gamma}} = O_p(1) \quad T^{-1}F_{\hat{\theta},\hat{\gamma}} = O_p(1),$$

if $\lambda = \lambda_c$,

$$(31) \quad T^{-1/2}t_{\hat{\gamma}} \Rightarrow \begin{cases} -\infty & \gamma < 0 \\ \infty & \gamma > 0 \end{cases},$$

$$(32) \quad T^{-1}F_{\hat{\theta},\hat{\gamma}} \Rightarrow \infty.$$

For Model (3,AO), if $\lambda \neq \lambda_c$,

$$(33) \quad T^{-1/2}t_{\hat{\gamma}} = O_p(1),$$

if $\lambda = \lambda_c$,

$$(34) \quad T^{-1/2}t_{\hat{\gamma}} \Rightarrow \begin{cases} -\infty & \gamma < 0 \\ \infty & \gamma > 0 \end{cases}.$$

These results have the following implications. In Model (2,AO) because $T^{-1/2}t_{\hat{\gamma}}$ and $T^{-1}F_{\hat{\theta},\hat{\gamma}}$ diverge at the true break date but are $O_p(1)$ at the incorrect break date, as the sample size grows the argmax of $T^{-1/2}t_{\hat{\gamma}}$ and $T^{-1}F_{\hat{\theta},\hat{\gamma}}$ will be λ_c, provided $\lambda_c \in \Lambda$. Similarly in Model (3,AO), $T^{-1/2}t_{\hat{\gamma}}$ diverges at the true break date but is $O_p(1)$ at incorrect break dates. Thus, as the sample size grows, the argmax of $T^{-1/2}t_{\hat{\gamma}}$ will be λ_c, provided $\lambda_c \in \Lambda$.

For Model (2,IO) the behavior of $t_{\hat{\gamma}}$ and $F_{\hat{\theta},\hat{\gamma}}$ are much different. Under (24) it is shown in the appendix that,

if $\lambda = \lambda_c$,

(35) $\quad t_{\hat{\gamma}} = O_p(1) \qquad F_{\hat{\theta},\hat{\gamma}} = O_p(T),$

if $\lambda \neq \lambda_c$,

(36) $\quad t_{\hat{\gamma}} = O_p(T^{1/2}) \qquad F_{\hat{\theta},\hat{\gamma}} = O_p(T),$

Notice that $t_{\hat{\gamma}}$ is bounded at the true break date but diverges at wrong break dates. On the other hand, $F_{\hat{\theta},\hat{\gamma}}$ diverges at the same rate regardless of the break date. Clearly, the true break date will not be chosen using $t_{\hat{\gamma}}$. It is possible that the true break date might be chosen using $F_{\hat{\theta},\hat{\gamma}}$. For this to happen, $T^{-1}F_{\hat{\theta},\hat{\gamma}}$ would have to attain a global maximum at λ_c. Such a proof is not available at this time. Moreover, finite sample results given in Section 5 cast doubt as to whether such a conjecture holds.

Using (30) through (34), the following limiting results are easily obtained ($\gamma \neq 0$):

(37) $\quad t_{\hat{\alpha}}(2,\text{AO},m,k=0) \Rightarrow R_2(\lambda_c) \qquad\qquad (m = T_b(t_{\hat{\gamma}}), T_b(|t_{\hat{\gamma}}|), T_b(F_{\hat{\theta},\hat{\gamma}})),$

(38) $\quad t_{\hat{\alpha}}(3,\text{AO},m,k=0) \Rightarrow R_{3,\text{AO}}(\lambda_c) \qquad\qquad (m = T_b(t_{\hat{\gamma}}), T_b(|t_{\hat{\gamma}}|)).$

The limiting distributions given by (37) and (38) are equivalent to the case where the break date is known as in Perron (1989). Critical values for (37) can be found in Perron (1989), and critical values for (38) can be found in Perron and Vogelsang (1993b). An obvious point to notice is that the limiting distributions given by (37) and (38) are different from the case when $\gamma = 0$ and have larger (smaller in absolute value) critical values. For example, the 5% critical value for $t_{\hat{\alpha}}(2,\text{AO},T_b(F_{\hat{\theta},\hat{\gamma}}),k=0)$ is -4.61 when $\gamma = 0$ and is -4.24 when $\gamma \neq 0$. In practice, one could always just use the critical values for the $\gamma = 0$ case as all the tests will have an asymptotic size no larger than the given nominal size. A drawback, though, is a loss of power should γ be different from zero. A simple way to avoid some of the potential power loss would be to first perform a pretest for a change in slope that is valid for both unit root and stationary errors. Perron (1991) and Vogelsang

(1992) provide such tests. Of course, any time a pretest is conducted, the sixe of the ultimate test may, in principle, be distorted in finite samples. An investigation of the properties of a such a two step testing procedure is left as a future research.

4. SIMULATION OF THE FINITE SAMPLE CRITICAL VALUES

In this section, finite sample critical values are presented for the statistics $t_{\hat{\alpha}}(1,AO,m,k)$ ($m = T_b(t_{\hat{\alpha}}), T_b(t_{\hat{\theta}}), T_b(|t_{\hat{\theta}}|)$), $t_{\hat{\alpha}}(2,AO,m,k)$ ($m = T_b(t_{\hat{\alpha}}), T_b(t_{\hat{\gamma}}), T_b(|t_{\hat{\gamma}}|), T_b(F_{\hat{\theta},\hat{\gamma}})$), and $t_{\hat{\alpha}}(2,IO,T_b(F_{\hat{\theta},\hat{\gamma}}),k)$ under the null hypothesis of a unit root using two procedures for selecting the truncation lag parameter k. The purpose is to assess the adequacy of the asymptotic approximations and to provide a range of finite sample critical values. The latter are reported for the models that have not been explored elsewhere in the literature. Finite sample critical values are available for the other statistics in Zivot and Andrews (1992), Perron (1994), and BLS (1992).

The following data–generating process was used for all of the simulations,

$$(39) \qquad y_t = y_{t-1} + \gamma DU_t^c + e_t, \qquad y_0 = 0.$$

The initial condition, y_0, and β are set equal to zero since the statistics are invariant to their values under the null hypothesis. The magnitude of the mean break, θ, is also set equal to zero since the statistics are asymptotically invariant to the value of θ. For most of the simulations, γ was set equal to zero, however, for the $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\alpha}}),k)$ statistic, simulations are also reported for several sample sizes and several values of γ. The error process, e_t, was specified to be i.i.d. N(0,1). Even though the finite sample distributions of the statistics depend on the correlation structure of the errors, they are asymptotically invariant to the correlation in the errors under the regularity conditions described in Section 3.2 with the appropriate addition of lagged first differences of the data. By using i.i.d. errors for the simulations, it is possible to clearly determine the effects of the choice of k on the finite sample distributions. The use of i.i.d. errors will also help to delineate the affects that γ has on the finite sample distributions. The effects of more general error processes are given in Section 5.

The critical values were simulated for three sample sizes, T = 50, 100 and 150, except for Model(2,AO), $\gamma \neq 0$, where T = 100, 250 and 500 were used. In the case where k is fixed, results are given for $k = 0$, 2, and 5. The results for $k = 2$, 5 measure the effects of overparameterization on the finite sample critical values. In the case where k is chosen using the significance of the t–statistic of the coefficient on the last included lagged first difference, the size of the t–test was set at 10%. In addition, *kmax* was set equal to 5. Larger values of *kmax* were not considered due to the excessive computational time. Each simulation involved 2,000 replications, and the N(0,1) random deviates were obtained from the routine RAN1 of Press, *et al* (1986) written in the C language. To minimize sampling variability across simulations, the same set of generated data was used across simulations using identical sample sizes, T. The initial seed used for the random number generator was –T in all simulations.

Results for Model (1,AO) are given in Tables 1A – 1C. Table 1A presents the finite–sample critical values for the statistic $t_{\hat{\alpha}}(1,AO,T_b(t_{\hat{\alpha}}),k)$, Table 1B for $t_{\hat{\alpha}}(1,AO,T_b(t_{\hat{\theta}}),k)$ and Table 1C for $t_{\hat{\alpha}}(1,AO,T_b(|t_{\hat{\theta}}|),k)$. Looking at the tables, the first thing to notice is that for all three statistics, the asymptotic distributions are good approximations to the finite sample distributions when $k = 0$. Second, as k becomes larger, the critical values become larger (smaller in absolute value) at a given sample size. Thus overparameterizing the truncation lag will lead to undersized tests in practice. Third, when T_b is chosen using the t–statistic on the coefficient of the last included lagged first difference (the k(t–sig) lines), the critical values are much smaller than the asymptotic and fixed k critical values. This is due to correlation, in finite samples, between the t–statistic on the coefficient of the last included lag and the t–statistic for testing $\alpha = 1$. As noted earlier, this correlation disappears asymptotically. The simulations show that it vanishes slowly. Finally, note that the critical values are larger for $t_{\hat{\alpha}}(1,AO,T_b(t_{\hat{\theta}}),k)$ as compared to $t_{\hat{\alpha}}(1,AO,T_b(|t_{\hat{\theta}}|),k)$. For example, the 5% asymptotic critical value for $t_{\hat{\alpha}}(1,AO,T_b(t_{\hat{\theta}}),k)$ is –4.01 while the 5% critical value for $t_{\hat{\alpha}}(1,AO,T_b(|t_{\hat{\theta}}|),k)$ is –4.17. If both procedures tend to pick the same break date, then using $t_{\hat{\alpha}}(1,AO,T_b(t_{\hat{\theta}}),k)$ may result in higher power. Thus, imposing the mild assumption that the direction of the break is known *a priori* can lead to a more powerful test. Further evidence on power is given in the next section.

Tables 2A through 2D present finite–sample critical values for Model (2,AO). Similar observations can be made as in Model 1. In all cases, the asymptotic distribution is a good approximation to the exact distributions for T = 150. In all cases, when k is chosen using the k(t–sig) procedure, critical values are substantially smaller compared to the fixed k and asymptotic distributions. For the statistics where T_b is chosen using $t_{\hat{\gamma}}$ and $|t_{\hat{\gamma}}|$, the critical values are larger when the direction of the break is assumed known. For example the 5% critical values for $t_{\hat{\alpha}}(2,AO,T_b(|t_{\hat{\gamma}}|),k)$ and $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\gamma}}),k)$ are –4.50 and –4.28 respectively. In the case where T_b is chosen using $F_{\hat{\theta},\hat{\gamma}}$, the critical values are smaller than when T_b is chosen using $t_{\hat{\gamma}}$.

In Table 2E results are reported for Model (2,IO) where T_b is chosen using $F_{\hat{\theta},\hat{\gamma}}$.

The asymptotic distribution of $t_{\hat{\alpha}}(2,IO,T_b(F_{\hat{\theta},\hat{\gamma}}),k)$ is not as good an approximation to the finite–sample distribution as compared to the same statistic in the AO model. Finite sample critical values for the other Model (2,IO) statistics can be found in Perron (1994).

Up to this point, all the finite sample critical values were reported for the case where no break had occurred under the unit root null hypothesis. The results in Section 3.3 showed that the asymptotic distributions are not invariant to the value of the slope change parameter γ. To determine the effects of γ on the finite sample distributions in Model (2,AO), simulations were run for $\gamma = 0.5$, 1.0, 2.0 and 5.0. Three sample sizes were used, T = 100, 250, 500 and three breaks dates were considered corresponding to $\lambda_c = 0.2$, 0.5, 0.8. The value of k was set equal to zero in all simulations to avoid any effects caused by the choice of the truncation lag parameter. The results are given in Table 3.

Panel (a) reports the critical values for T = 100, Panel (b) for T = 250 and Panel (c) for T = 500. What is evident from the table is that for a given γ, the critical values become more and more negative as the T increases although the divergence is slow for small γ. For example, with $\lambda_c = 0.5$ and $\gamma = 2.0$ the 5% critical values range from –6.12 at T = 100 to –9.19 at T = 500. To give some intuition behind these results, $\lim_{T\to\infty} T^{-1/2} t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\alpha}}),k=0)$ was computed for $\gamma = 0.5$, 1.0, 2.0, 5.0 with $\lambda_c = 0.5$ and $\sigma^2 = 1$ yielding –0.0832 (γ=0.5), –0.165 (γ=1.0), –0.321 (γ=2.0) and –0.689 (γ=5.0). Consider for illustration $\gamma = 0.5$. For a given T, the asymptotics imply that $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\alpha}}),k=0) \approx T^{1/2}\times(-0.0832)$. When T = 500, $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\alpha}}),k=0) \approx -1.86$ according to the asymptotics. Compare this to –4.23, the median of the finite sample distribution (see Table 3 Panel (c)). Even with a relatively large sample of 500 observations, $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\alpha}}),k=0)$ is little affected by $\gamma = 0.5$. Suppose T = 5,000; the asymptotics imply that $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\alpha}}),k=0) \approx -5.88$. So, for small γ, $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\alpha}}),k=0)$ diverges quite slowly. Now suppose that $\gamma = 5.0$. With T = 500 the asymptotics imply that $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\alpha}}),k=0) \approx -15.41$. This is close to the median value of –14.13 from Table 3 Panel (c). For T = 250, $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\alpha}}),k=0) \approx -10.89$ compared to the median value of –10.43 from Table 3 Panel (b). So, for large γ, $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\alpha}}),k)$ diverges quite fast. In practice, if a very large slope change is suspected, the $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\alpha}}),k)$ statistic should not be used. On the other hand, since the statistics that pick T_b based on the trend break parameters tend to pick the true break date as T grows, they will suffer less from distortion due to large γ and should be more reliable tests in practice.

5. FINITE–SAMPLE SIZE AND POWER SIMULATIONS

This section presents finite–sample size and power simulation results. The purpose of these simulations is to determine the following, a) how size and power are affected by the choice of k in the presence of more general error processes, b) how size and power are affected by θ and γ, and c) how power varies across procedures for choosing T_b. The focus of the simulations is placed on Model (1,AO), and Models (2,AO) and (2,IO). Similar simulation results can be found in Perron (1994) for Models (1,IO), (3,AO) and (3,IO).

The DGP used for all simulations is of the form,

$$(40) \qquad y_t = \theta DU_t + \gamma DT_t^{*c} + z_t, \qquad z_t = \alpha z_{t-1} + \Sigma_{i=1}^4 \varphi(i)\Delta z_{t-i} + e_t + \psi e_{t-1},$$

where $e_t \sim$ i.i.d. N(0,1). Note that (40) has an ARMA(5,1) error specification and is the same DGP used in Perron (1994). This will allow easy comparison of results in Perron (1994) and provide similar finite–sample size/power results for all of the models outlined in Section 2. For the size simulations, α was set equal to 1. For the power simulations α was set equal to 0.8. The sample size for all simulations was T = 100, with 1,000 replications. Regressions were run for fixed $k = 0, 1, ..., 5$ and for k(t–sig) with $kmax = 5$. For the simulations where $\theta,\gamma \neq 0$, the true break date was $T_b^c = 50$ ($\lambda_c = 0.5$). Several values of θ and γ were considered. In Model (1,AO) the values of θ were 0, 2, 5, and 10. In Models (2,AO) and (2,IO) the values were $\theta = 0, 5, 10$ and $\gamma = 0, 1, 2$. Seven different error specifications were used. They include,

(1) $\varphi(i) = 0$ (i = 1,2,3,4) and $\psi = 0$,
(2) $\varphi(1) = 0.6$, $\varphi(i) = 0$ (i = 2,3,4) and $\psi = 0$,
(3) $\varphi(1) = -0.6$, $\varphi(i) = 0$ (i = 2,3,4) and $\psi = 0$,
(4) $\varphi(1) = 0.4$, $\varphi(2) = 0.2$ and $\varphi(3) = \varphi(4) = \psi = 0$,
(5) $\varphi(1) = 0.3$, $\varphi(2) = 0.3$, $\varphi(3) = 0.24$, $\varphi(4) = 0.14$ and $\psi = 0$,
(6) $\varphi(i) = 0$ (i = 1,2,3,4) and $\psi = 0.5$
(7) $\varphi(i) = 0$ (i = 1,2,3,4) and $\psi = -0.5$.

Experiment (1) has i.i.d. errors. This specification will be used to isolate the effects of choosing k too large. Experiment (2) has positive correlation in the errors and is quite common in empirical data. Experiment (3) has negative correlation in the errors. Experiments (4) and (5) have higher order correlation in the errors. These experiments will be useful in isolating the effects on size of picking k too small. Finally, experiments (6) and (7) have MA(1) errors to determine how well the k(t–sig) procedure performs in the presence of MA errors. For fixed k, the 5% asymptotic critical values were used, and for k(t–sig) the appropriate 5% finite sample critical values for T = 100 were used.

The simulation results are presented in Tables 4 and 5. These tables report results for Models (2,AO) and (2,IO) only and for k chosen using the k(t–sig) procedure. Results are not reported for Model 1 as they are qualitatively the same as Model 2. Results for fixed k are not reported as they are very similar to the results reported by Perron and Vogelsang (1992) and Perron (1994). Tables of additional results are available from the authors.

We begin by summarizing the results pertaining to the choice of k. When k is chosen less than the true order of the process substantial size distortion often occurs. In most cases the exact size will be much greater than the nominal size. If k is chosen at least as big as the true order of the process, the exact size is rarely greater than the nominal size. However, power will be lost if the lag structure is over parameterized. When the k(t–sig) procedure is used to pick k, the exact size is close to the nominal size in all cases except when there is negative MA component as in experiment (7). In this case the exact size is substantially inflated above the nominal size. Power using k(t–sig) is generally quite good. It is greater than when k is larger than the true order of the process and is nearly as high as when k is set equal to p in the case of an autoregressive process. Overall, the k(t–sig) procedure has good size and power properties and clearly dominates using a fixed k.

Consider now how a change in mean or slope affects the exact size of the tests. For the procedures where T_b is chosen to minimize $t_{\hat{\alpha}}$, the tests become oversized as θ and γ grow. For example, consider $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\alpha}}),k(t\text{–sig}))$ in Table 4. In experiment (1) where the true process is a random walk, the exact size is 0.040 when $\theta = 0$. The exact size is 0.108 for $\theta = 5.0$ and 0.597 for $\theta = 10$. When $\gamma = 2.0$ the exact size is 0.076. Similar results occur for Models (1,IO), (1,AO) and (2,IO). In particular, for $t_{\hat{\alpha}}(2,IO,T_b(t_{\hat{\alpha}}),k(t\text{–sig}))$ the exact size is 0.065 and 0.314 for $\gamma = 1.0$ and 2.0 respectively as shown in Table 5. Perron (1994) also found the same size distortions in Models (3,IO) and (3,AO). In general when T_b is chosen by minimizing $t_{\hat{\alpha}}$, the exact size will tend to grow as the magnitude of either θ or γ grows, but the size distortions do depend on the parameters of the noise process. For example, when there is positive correlation in the first differences of the errors as in experiment (2), the tests have exact sizes that are no larger than the nominal sizes. If the first differences of the errors are negatively correlated as in experiment (3), the exact size is much greater than the nominal size. Thus, size distortions depend on the magnitudes of θ and γ as well as the correlation structure of the errors.

The effects of θ and γ on exact size when T_b is chosen using the trend break parameters in the IO models are very similar to the procedures where T_b is chosen to minimize $t_{\hat{\alpha}}$. Again, as θ or γ grow, the exact size increases for a given sample size. For example, in Table 5, the exact size for $t_{\hat{\alpha}}(2,IO,T_b(F_{\hat{\theta},\hat{\gamma}}),k(t\text{-}sig))$ in experiment (1) is 0.189 and 0.707 for $\theta = 5.0$ and 10.0 respectively and is 0.106 and 0.233 for $\gamma = 1.0$ and 2.0 respectively. Similar results for Models (1,IO) and (3,IO) are reported in Perron (1994). These results occur because the statistics on the dummy variables are not choosing the true break date enough of the time.

Things are much different in the AO models as the $t_{\hat{\gamma}}$ and $F_{\hat{\theta},\hat{\gamma}}$ statistics do a much better job picking the true break date. Results in Tables 4 and 5 clearly show that the exact size does not tend to grow as θ or γ increase in Model (2,AO). Similar results hold for Model (1,AO). Perron (1994) also shows this to be the case for Model (3,AO). In fact, as the magnitude of the break grows, the exact size tends to be below the nominal size. This occurs because as the magnitude of the break grows, the correct break date is chosen more often, and the finite sample distributions will be better approximated by the limiting distributions in the known break date case. In practice, if a very large break is suspected in the data, tests carried out using the AO model where T_b is chosen using the significance of the break dummy parameters will have the best overall size properties.

It is important to note here, though, that the magnitudes of θ and γ where size distortion becomes a problem are on the order of 5 to 10 times the standard deviation of the innovation errors for θ and at least 1 to 2 times the standard deviation of the innovation errors for γ unless the first differences of the errors are positively correlated in which case there are little size distortions. For many macroeconomic time series, intercept shifts are often less than 5 standard deviations, slope changes are often less than 0.5 standard deviations and the first differences of the errors are often positively correlated. Therefore, size distortions are not necessarily a problem in practice, but care must be used if a series is suspected of having large trend shifts or negative correlation in the first differences of the errors. To illustrate these issues, θ and γ were estimated (imposing a unit root) for the long historical GNP series considered by Perron (1992) who applied the statistics from Model (2,IO). Without going into the details, it was found that $\hat{\theta}$ and $\hat{\gamma}$ were generally no larger (in magnitude) than -8 and 0.7 respectively relative to the standard deviation of the errors. In addition, most of the series exhibited positive correlation in their first differences. Thus, experiment (2) most closely mimics the data considered by Perron (1992). From Table 5 one can see that size distortions are minimal for $\theta = 10$ and $\gamma = 1.0$. Therefore, the size of the tests used by Perron (1992) are not adversely affected by θ and γ.

Next, consider the effects that θ and γ have on the power of the tests. Meaningful conclusions are most easily drawn for Model (2,AO) as size distortions are minimal. Consider the power of $t_{\hat{\alpha}}(2,AO,m,k(t\text{-sig}))$ (m = $T_b(|t_{\hat{\gamma}}|)$, $T_b(t_{\hat{\gamma}})$, $T_b(F_{\hat{\theta},\hat{\gamma}})$). As θ or γ rise, power tends to fall. For example, the power of $t_{\hat{\alpha}}(2,AO,T_b(F_{\hat{\theta},\hat{\gamma}}),k(t\text{-sig}))$ in experiment (1) is 0.339 when $\theta = \gamma = 0.0$ but falls to 0.194, 0.163 for θ = 5, 10 and 0.193, 0.184 for γ = 1.0, 2.0. Part of the reason for this fall in power is that the true break date is chosen more often as θ and γ rise, and the critical values for the known break date case would be more appropriate. Hence, the critical values from the unknown break date case are too negative for a given nominal size and power suffers.

Finally, when making the mild *a priori* assumption that the direction of the break is known, power is higher. For example, a comparison of the power of $t_{\hat{\alpha}}(2,AO,T_b(|t_{\hat{\gamma}}|),k(t\text{-sig}))$ and $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\gamma}}),k(t\text{-sig}))$ from Table 4 reveals the following. Regardless of the true error process, power is generally higher using $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\gamma}}),k(t\text{-sig}))$ as compared to $t_{\hat{\alpha}}(2,AO,T_b(|t_{\hat{\gamma}}|),k(t\text{-sig}))$. This makes sense for the following reason. Since both procedures tend to pick the same break date, and if the direction of the break is known, the values of the statistics will be equal. Then, since $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\gamma}}),k(t\text{-sig}))$ has larger (smaller in absolute value) critical values than $t_{\hat{\alpha}}(2,AO,T_b(|t_{\hat{\gamma}}|),k(t\text{-sig}))$, the former will be a more powerful statistic. This general result holds across all three models in both the AO and IO frameworks.

The practical implications of the size and power simulations can be summarized as follows. The k(t–sig) procedure should be used to choose k as this procedure has good size and power properties. Choosing a fixed k is not recommended as substantial size distortions and/or low power could result. If a very large break is suspected, even under the unit root hypothesis, the AO model with T_b chosen using the significance of the trend break dummy parameters provides tests that have good finite sample size properties. If the magnitudes of the possible shift in the trend function are not too large then all of the procedures will have good finite sample size. Finally, higher power will result if the mild *a priori* assumption is made that the direction of the break is known.

6. CONCLUSIONS

This paper adds to the current literature on unit root tests that allow for a shift in the trend function when the location of the break point is unknown. Previous results in the literature focused mainly on the IO approach where the break is modeled as occurring slowly over time as opposed to the AO approach where the break is sudden. Limiting distributions were derived for several models in the AO framework and asymptotic critical values tabulated. The limiting distributions of the statistics in both the IO and AO frameworks were also derived in the case where the shift in the trend function occurred under the null hypothesis of a unit root. It was shown that the limiting distributions are asymptotically invariant to a shift in intercept but are not asymptotically invariant to a shift in slope. In fact, many of the tests currently proposed in the literature are no longer asymptotically valid testing procedures when a slope shift occurs under the null hypothesis. However, these asymptotic results were shown to provide poor approximations to finite sample distributions for trend breaks of the magnitudes typically encountered in practice. Indeed, for typical changes in slope under the null hypothesis, the usual no–break asymptotic results remain a good approximation.

Several methods for choosing the break date, T_b, and the truncation lag parameter, k, were investigated. A simulation to assess the finite sample size and power of the tests suggested the following guidelines. First, the choice of k should be made using a data dependent method rather than using a fixed k due to better size and power properties. Second, the mild *a priori* assumption that the direction of the break is known will result in more powerful tests. Finally, except for the procedures in the AO framework where T_b is chosen using the significance of the trend break parameters, size tends to grow as the magnitude of the shifts grow. Therefore, if the magnitude of the shift is suspected to be very large, the AO framework is preferred since it permits tests with less size distortion. In practice, though, the magnitudes of trend shifts that are commonly seen in macroeconomic data are not large enough to cause substantial size distortion. Hence, the choice between the AO and IO models should be based on the dynamics of the shift.

MATHEMATICAL APPENDIX

The proofs of the limiting distributions stated in the text are given in this appendix. Weak convergence for a fixed λ is established for the various statistics. The results of Zivot and Andrews (1992) can then be applied to give the limiting results for the minimal and maximal statistics via the continuous mapping theorem. Recall that Zivot and Andrews (1992) require $\lambda \in \Lambda$ where Λ is a closed subset of (0,1). In the case of the statistics whereby the break is chosen to minimize $t_{\hat{\alpha}}$, results from Perron (1994) permit this condition to be relaxed so that $\Lambda = [0,1]$.

The following notation is used throughout the appendix. The symbol $\Rightarrow$ is used to denote weak convergence in distribution and $W^*(r)$ denotes the residuals from a projection of W(r), a standard Wiener process, onto the space spanned by $\{1,r\}$ defined on (0,1). A '$^-$' is used to denote the residuals from a projection of a variable onto the space spanned by $\{1, t\}$. A '$^{\sim}$' is used to denote the residuals from a projection of a variable onto the space spanned by $\{1, t, DU_t\}$. Let Σ be short for $\Sigma_{t=1}^{T}$, and let Σ' be short for $\Sigma_{t=T_b+1}^{T}$. Finally, given a matrix X, M_X is used to denote the matrix $I - X(X'X)^{-1}X'$.

The data-generating process (DGP) can be written in matrix form as,

$$\text{(A.1a)} \qquad Y = D_2^c\gamma + Z, \qquad Z = Z_{-1} + e,$$

where $Y \equiv \{y_t\}$, $Z \equiv \{z_t\}$, $Z_{-1} \equiv \{z_{t-1}\}$, $e \equiv \{e_t\}$, $D_2^c \equiv \{DT_t^{*c}\}$. The small 'c' superscript on the 'D' variables means that these dummy variables are defined using the true break date T_b^c. A 'D' variable without a superscript 'c' (e.g. D_1, D_2) is used to denote dummy variables defined using the break date used in the estimation (T_b). In addition, define the following matrices: $Y_{-1} \equiv \{y_{t-1}\}$, $D_1^c \equiv \{DU_t^c\}$, $D \equiv [D_1\ D_2]$, $D^c \equiv [D_1^c\ D_2^c]$, $N \equiv \{1, t\}$, $\tau_1 = \text{diag}(T^{-1/2}, T^{-3/2})$, $\tau_2 = \text{diag}(T^{-1/2}, T^{-1})$. Note that θ is set equal to zero as all the statistics are asymptotically invariant to θ under the null hypothesis. It is useful to rewrite (A.1a) in several ways:

$$\text{(A.1b)} \qquad Y = Y_{-1} + D_1^c\gamma + e,$$

$$\text{(A.1c)} \qquad Y_{-1} = (D_2^c - D_1^c)\gamma + Z_{-1}$$

$$\text{(A.1d)} \qquad D_2^c = (Y_{-1} - Z_{-1})/\gamma + D_1^c.$$

Detrending both sides of (A.1a) – (A.1d) by $\{1, t\}$ yields, (A.2a) $\tilde{Y} = \tilde{D}_2^c\gamma + \tilde{Z}, \qquad \tilde{Z} = \tilde{Z}_{-1} + \tilde{e}$

(A.2b) $\tilde{Y} = \tilde{Y}_{-1} + \tilde{D}_1^c\gamma + \tilde{e},$

(A.2c) $\tilde{Y}_{-1} = (\tilde{D}_2^c - \tilde{D}_1^c)\gamma + \tilde{Z}_{-1}$

(A.2d) $\tilde{D}_2^c = (\tilde{Y}_{-1} - \tilde{Z}_{-1})/\gamma + \tilde{D}_1^c.$

Likewise, detrending both sides of (A.1a) – (A.1d) by $\{1, t, DU_t\}$ yields,

(A.3a) $\hat{Y} = \hat{D}_2^c\gamma + \hat{Z}, \qquad \hat{Z} = \hat{Z}_{-1} + \hat{e}$

(A.3b) $\hat{Y} = \hat{Y}_{-1} + \hat{D}_1^c\gamma + \hat{e},$

(A.3c) $\hat{Y}_{-1} = (\hat{D}_2^c - \hat{D}_1^c)\gamma + \hat{Z}_{-1}$

(A.3d) $\hat{D}_2^c = (\hat{Y}_{-1} - \hat{Z}_{-1})/\gamma + \hat{D}_1^c.$

The next three lemmas establish the convergence of various sample moments that are used in the proofs.

Lemma A1. Set $\gamma = 0$. Under (A.1a) these limiting results hold:

i) $\tau_1\tilde{D}'\tilde{D}\tau_1 = \begin{bmatrix} T^{-1}\tilde{D}_1'\tilde{D}_1 & T^{-2}\tilde{D}_1'\tilde{D}_2 \\ T^{-2}\tilde{D}_2'\tilde{D}_1 & T^{-3}\tilde{D}_2'\tilde{D}_2 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix},$

ii) $T^{-1}\tilde{D}_1'M_{\tilde{Y}_{-1}}\tilde{e} \Rightarrow \sigma H_1(\lambda),$

iii) $T^{-3/2}\tilde{D}_2'M_{\tilde{Y}_{-1}}\tilde{e} \Rightarrow \sigma H_2(\lambda),$

iv) $T^{-1}\tilde{D}_1'M_{\tilde{Y}_{-1}}\tilde{D}_1 \Rightarrow K_1(\lambda),$

v) $T^{-3}\tilde{D}_2'M_{\tilde{Y}_{-1}}\tilde{D}_2 \Rightarrow K_2(\lambda),$

vi) $T^{-2}\tilde{D}_2'M_{\tilde{Y}_{-1}}\tilde{D}_1 \Rightarrow K_3(\lambda),$

vii) $T^{-2}\tilde{Y}'M_{\tilde{D}_1}\tilde{Y} \Rightarrow \sigma^2 L_1(\lambda),$

viii) $T^{-2}\tilde{Y}'M_{\tilde{D}_2}\tilde{Y} \Rightarrow \sigma^2 L_2(\lambda),$

ix) $T^{-5/2}\tilde{D}_2 M_{\tilde{D}_1}\tilde{Y} \Rightarrow \sigma L_3(\lambda),$

x) $T^{-3}\tilde{D}_2'M_{\tilde{D}_1}\tilde{D}_2 \Rightarrow a_{22} - a_{12}^2/a_{11},$

<u>Proof:</u> Since $\gamma = 0$, $Y = Z$. Convergence of sample moments follow from arguments in

Ouliaris, Park and Phillips (1989) and Perron (1989). As a preliminary, consider the limits of $T^{-1/2}\tilde{D}_1'\tilde{e}$ and $T^{-3/2}\tilde{D}_2'\tilde{e}$:

$$T^{-1/2}\tilde{D}_1'\tilde{e} = T^{-1/2}D_1'e - T^{-1/2}D_1'N\tau_1(\tau_1 N'N\tau_1)^{-1}\tau_1 N'e$$

$$= T^{-1/2}\Sigma' e_t + [\ 1-\lambda,\ T^{-2}\Sigma' t\ ] * \begin{bmatrix} 1 & T^{-2}\Sigma t \\ T^{-2}\Sigma t & T^{-3}\Sigma t^2 \end{bmatrix}^{-1} * \begin{bmatrix} T^{-1/2}\Sigma e_t \\ T^{-3/2}\Sigma t e_t \end{bmatrix}$$

$$\Rightarrow \sigma\{\ W(1) - W(\lambda) + [\ (1-\lambda),\ (1-\lambda^2)/2\ ] * \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}^{-1} * \begin{bmatrix} W(1) \\ \int_0^1 r dW(r) \end{bmatrix}\}$$

$$= \sigma[\ \lambda(4-3\lambda)W(1) - W(\lambda) - 6\lambda(1-\lambda)\int_0^1 r dW(r)] \equiv \sigma B_1(\lambda).$$

$$T^{-3/2}\tilde{D}_2'\tilde{e} = T^{-3/2}D_2'e - T^{-3/2}D_2'N\tau_1(\tau_1 N'N\tau_1)^{-1}\tau_1 N'e$$

$$= T^{-3/2}\Sigma'(t-T_b)e_t$$

$$+ [\ T^{-2}\Sigma'(t-T_b),\ T^{-3}\Sigma' t(t-T_b)\ ] * \begin{bmatrix} 1 & T^{-2}\Sigma t \\ T^{-2}\Sigma t & T^{-3}\Sigma t^2 \end{bmatrix} * \begin{bmatrix} T^{-1/2}\Sigma e_t \\ T^{-3/2}\Sigma t e_t \end{bmatrix}$$

$$\Rightarrow \sigma\{\ W(1) - W(\lambda)$$

$$+ [\ (1-\lambda)^2/2,\ (1-\lambda)^2(2+\lambda)/6\ ] * \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}^{-1} * \begin{bmatrix} W(1) \\ \int_0^1 r dW(r) \end{bmatrix}\}$$

$$= \sigma[\int_\lambda^1 (r-\lambda)dW(r) + \lambda(1-\lambda)^2 W(1) - (1-\lambda)^2(2+\lambda)\int_0^1 r dW(r)] \equiv \sigma B_2(\lambda).$$

i) $$\tau_1\tilde{D}'\tilde{D}\tau_1 = \tau_1 D'D\tau_1 - \tau_1 D'N\tau_1(\tau_1 N'N\tau_1)^{-1}\tau_1 N'D\tau_1 = \begin{bmatrix} (1-\lambda) & T^{-2}\Sigma'(t-T_b) \\ T^{-2}\Sigma'(t-T_b) & T^{-3}\Sigma' t(t-T_b) \end{bmatrix}$$

$$- \begin{bmatrix} (1-\lambda) & T^{-2}\Sigma' t \\ T^{-2}\Sigma'(t-T_b) & T^{-3}\Sigma' t(t-T_b) \end{bmatrix} * \begin{bmatrix} 1 & T^{-2}\Sigma t \\ T^{-2}\Sigma t & T^{-3}\Sigma t^2 \end{bmatrix}^{-1} * \begin{bmatrix} (1-\lambda) & T^{-2}\Sigma'(t-T_b) \\ T^{-2}\Sigma' t & T^{-3}\Sigma' t(t-T_b) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (1-\lambda) & (1-\lambda)^2/2 \\ (1-\lambda)^2/2 & (1-\lambda)^3/3 \end{bmatrix}$$

$$- \begin{bmatrix} (1-\lambda) & (1-\lambda^2)/2 \\ (1-\lambda)^2/2 & (1-\lambda)^2(2+\lambda)/6 \end{bmatrix} * \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}^{-1} * \begin{bmatrix} (1-\lambda) & (1-\lambda)^2/2 \\ (1-\lambda^2)/2 & (1-\lambda)^2(2+\lambda)/6 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda(1-\lambda)(3\lambda^2-3\lambda+1) & \lambda^2(1-\lambda)^2(2\lambda-1)/2 \\ \lambda^2(1-\lambda)^2(2\lambda-1)/2 & \lambda^3(1-\lambda)^3/3 \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}.$$

ii) $T^{-1/2}\tilde{D}_1' M_{\tilde{Y}_{-1}} \tilde{e} = T^{-1/2}\tilde{D}_1'\tilde{e} - T^{-3/2}\tilde{D}_1'\tilde{Y}_{-1}(T^{-2}\tilde{Y}_{-1}'\tilde{Y}_{-1})^{-1}T^{-1}\tilde{Y}_{-1}'\tilde{e}$

$$\Rightarrow \sigma[B_1(\lambda) - \int_\lambda^1 W*(r)dr \int_0^1 W*(r)dW(r)/\int_0^1 W*(r)^2dr] \equiv \sigma H_1(\lambda)$$

iii) $T^{-3/2}\tilde{D}_2' M_{\tilde{Y}_{-1}} \tilde{e} = T^{-3/2}\tilde{D}_2'\tilde{e} - T^{-5/2}\tilde{D}_2'\tilde{Y}_{-1}(T^{-2}\tilde{Y}_{-1}'\tilde{Y}_{-1})^{-1}T^{-1}\tilde{Y}_{-1}'\tilde{e}$

$$\Rightarrow \sigma[B_2(\lambda) - \int_\lambda^1 (r-\lambda)W*(r)dr \int_0^1 W*(r)dW(r)/\int_0^1 W*(r)^2dr] = \sigma H_2(\lambda).$$

iv) $T^{-1}\tilde{D}_1' M_{\tilde{Y}_{-1}} \tilde{D}_1 = T^{-1}\tilde{D}_1'\tilde{D}_1 - (T^{-3/2}\tilde{D}_1'\tilde{Y}_{-1})^2(T^{-2}\tilde{Y}_{-1}'\tilde{Y}_{-1})^{-1}$

$$\Rightarrow a_{11} - \{\int_\lambda^1 W*(r)dr\}^2/\int_0^1 W*(r)^2dr = K_1(\lambda).$$

v) $T^{-3}\tilde{D}_2' M_{\tilde{Y}_{-1}} \tilde{D}_2 = T^{-3}\tilde{D}_2'\tilde{D}_2 - (T^{-5/2}\tilde{D}_2'\tilde{Y}_{-1})^2(T^{-2}\tilde{Y}_{-1}'\tilde{Y}_{-1})^{-1}$

$$\Rightarrow a_{22} - \{\int_\lambda^1 (r-\lambda)W*(r)dr\}^2/\int_0^1 W*(r)^2dr = K_2(\lambda).$$

vi) $T^{-2}\tilde{D}_2' M_{\tilde{Y}_{-1}} \tilde{D}_1 = T^{-2}\tilde{D}_2'\tilde{D}_1 - T^{-5/2}\tilde{D}_2'\tilde{Y}_{-1}(T^{-2}\tilde{Y}_{-1}'\tilde{Y}_{-1})^{-1}T^{-3/2}\tilde{Y}_{-1}'\tilde{D}_1$

$$\Rightarrow a_{12} - \int_\lambda^1 W*(r)dr \int_\lambda^1 (r-\lambda)W*(r)dr/\int_0^1 W*(r)^2dr = K_3(\lambda).$$

vii) $T^{-2}\tilde{Y}' M_{\tilde{D}_1} \tilde{Y} = T^{-2}\tilde{Y}'\tilde{Y} - (T^{-3/2}\tilde{Y}'\tilde{D}_1)^2(T^{-1}\tilde{D}_1'\tilde{D}_1)^{-1}$

$$\Rightarrow \sigma^2[\int_0^1 W*(r)^2dr - \{\int_\lambda^1 W*(r)dr\}^2/a_{11}] = \sigma^2 L_1(\lambda).$$

viii) $T^{-2}\tilde{Y}' M_{\tilde{D}_2} \tilde{Y} = T^{-2}\tilde{Y}'\tilde{Y} - (T^{-5/2}\tilde{Y}'\tilde{D}_2)^2(T^{-3}\tilde{D}_2'\tilde{D}_2)^{-1}$

$$\Rightarrow \sigma^2[\int_0^1 W*(r)^2dr - \{\int_\lambda^1 (r-\lambda)W*(r)dr\}^2/a_{22}] = \sigma^2 L_2(\lambda).$$

ix) $T^{-5/2}\tilde{D}_2 M_{\tilde{D}_1} \tilde{Y} = T^{-5/2}\tilde{D}_2'\tilde{Y} - T^{-2}\tilde{D}_2'\tilde{D}_1(T^{-1}\tilde{D}_1'\tilde{D}_1)^{-1}T^{-3/2}\tilde{D}_1'\tilde{Y}$

$$\Rightarrow \sigma[\int_\lambda^1 (r-\lambda)W*(r)dr - (a_{12}/a_{11})\int_\lambda^1 W*(r)dr] = \sigma L_3(\lambda).$$

x) $T^{-3}\tilde{D}_2 M_{\tilde{D}_1} \tilde{D}_2 = T^{-3}\tilde{D}_2'\tilde{D}_2 - (T^{-2}\tilde{D}_2'\tilde{D}_1)^2(T^{-1}\tilde{D}_1'\tilde{D}_1)^{-1} \Rightarrow a_{22} - a_{12}^2/a_{11}.$

Lemma A2. Let $\gamma \neq 0$ and suppose that $\lambda = \lambda_c$. Under (A.1a) these limiting results hold:

i) $T^{-1/2}\tilde{Y}'M_{\tilde{Y}_{-1}}\tilde{D}\tau_2 = O_p(1)$,

ii) $\tau_2\tilde{D}'M_{\tilde{Y}_{-1}}\tilde{D}\tau_2 = O_p(1)$,

iii) $T^{-1}\hat{D}_2'M_{\hat{Y}_{-1}}\hat{Y} = O_p(1)$,

iv) $T^{-2}\hat{D}_2'M_{\hat{Y}_{-1}}\hat{D}_2 = O_p(1)$,

v) $T^{-3/2}\tilde{Y}_{-1}M_{\tilde{D}_2}(\tilde{Y} - \tilde{Y}_{-1}) = O_p(1)$,

vi) $T^{-2}\tilde{Y}_{-1}M_{\tilde{D}_2}\tilde{Y}_{-1} = O_p(1)$.

<u>Proof:</u> Note that $\tilde{D}_1 = \tilde{D}_1^c$ and $\tilde{D}_2 = \tilde{D}_2^c$ since $T_b = T_b^c$.

i) Using (A.2b) and (A.2d) write (noting that $M_{\tilde{Y}_{-1}}\tilde{Y}_{-1} = 0$),

$$T^{-1/2}\tilde{Y}'M_{\tilde{Y}_{-1}}\tilde{D}\tau_2 = \begin{bmatrix} T^{-1}\tilde{Y}'M_{\tilde{Y}_{-1}}\tilde{D}_1 \\ T^{-3/2}\tilde{Y}'M_{\tilde{Y}_{-1}}\tilde{D}_2 \end{bmatrix} = \begin{bmatrix} T^{-1}(\tilde{D}_1\gamma+\tilde{e})'M_{\tilde{Y}_{-1}}\tilde{D}_1 \\ T^{-3/2}(\tilde{D}_1\gamma+\tilde{e})'M_{\tilde{Y}_{-1}}(\tilde{D}_1 - \tilde{Z}_{-1}/\gamma) \end{bmatrix}.$$

Using (A.2c) write,

$$\begin{aligned} T^{-1}(\tilde{D}_1\gamma+\tilde{e})'M_{\tilde{Y}_{-1}}\tilde{D}_1 &= T^{-1}(\tilde{D}_1\gamma+\tilde{e})'\tilde{D}_1 - T^{-2}(\tilde{D}_1\gamma+\tilde{e})'[(\tilde{D}_2-\tilde{D}_1)\gamma + \tilde{Z}_{-1}] \\ &\quad \times \{T^{-3}[(\tilde{D}_2-\tilde{D}_1)\gamma + \tilde{Z}_{-1}]'[(\tilde{D}_2-\tilde{D}_1)\gamma + \tilde{Z}_{-1}]\}^{-1}T^{-2}[(\tilde{D}_2-\tilde{D}_1)\gamma + \tilde{Z}_{-1}]'\tilde{D}_1 \\ &= T^{-1}\tilde{D}_1'\tilde{D}_1\gamma - [T^{-2}\tilde{D}_1'\tilde{D}_2\gamma^2 + o_p(1)] \\ &\quad \times [T^{-3}\tilde{D}_2'\tilde{D}_2\gamma^2 + o_p(1)]^{-1}[T^{-2}\tilde{D}_2'\tilde{D}_1\gamma + o_p(1)] + o_p(1) = O_p(1). \end{aligned}$$

$$\begin{aligned} T^{-3/2}(\tilde{D}_1\gamma+\tilde{e})'M_{\tilde{Y}_{-1}}(\tilde{D}_1 - \tilde{Z}_{-1}/\gamma) &= T^{-3/2}(\tilde{D}_1\gamma+\tilde{e})'(\tilde{D}_1 - \tilde{Z}_{-1}/\gamma) \\ &\quad - T^{-2}(\tilde{D}_1\gamma+\tilde{e})'[(\tilde{D}_2-\tilde{D}_1)\gamma + \tilde{Z}_{-1}]\{T^{-3}[(\tilde{D}_2-\tilde{D}_1)\gamma + \tilde{Z}_{-1}]'[(\tilde{D}_2-\tilde{D}_1)\gamma + \tilde{Z}_{-1}]\}^{-1} \\ &\quad \times T^{-5/2}[(\tilde{D}_2-\tilde{D}_1)\gamma + \tilde{Z}_{-1}]'(\tilde{D}_1 - \tilde{Z}_{-1}/\gamma) \\ &= -T^{-3/2}\tilde{D}_1'\tilde{Z}_{-1} + [T^{-2}\tilde{D}_1'\tilde{D}_2\gamma^2 + o_p(1)] \\ &\quad \times [T^{-3}\tilde{D}_2'\tilde{D}_2\gamma^2 + o_p(1)]^{-1}[T^{-5/2}\tilde{D}_2'\tilde{Z}_{-1} + o_p(1)] + o_p(1) = O_p(1). \end{aligned}$$

ii) $\tau_2\tilde{D}'M_{\tilde{Y}_{-1}}\tilde{D}\tau_2 = \begin{bmatrix} T^{-1}\tilde{D}_1'M_{\tilde{Y}_{-1}}\tilde{D}_1 & T^{-3/2}\tilde{D}_1'M_{\tilde{Y}_{-1}}\tilde{D}_2 \\ T^{-3/2}\tilde{D}_1'M_{\tilde{Y}_{-1}}\tilde{D}_2 & T^{-2}\tilde{D}_2'M_{\tilde{Y}_{-1}}\tilde{D}_2 \end{bmatrix}.$

Using (A.2c) write,

$$T^{-1}\tilde{D}_1'M_{\tilde{Y}_{-1}}\tilde{D}_1 = T^{-1}\tilde{D}_1'\tilde{D}_1 - \{T^{-2}\tilde{D}_1'[(\tilde{D}_2-\tilde{D}_1)\gamma + \tilde{Z}_{-1}]\}^2$$
$$\times \{T^{-3}[(\tilde{D}_2-\tilde{D}_1)\gamma + \tilde{Z}_{-1}]'[(\tilde{D}_2-\tilde{D}_1)\gamma + \tilde{Z}_{-1}]\}^{-1}$$
$$= T^{-1}\tilde{D}_1'\tilde{D}_1 - [T^{-2}\tilde{D}_1'\tilde{D}_2\gamma + o_p(1)]^2[T^{-3}\tilde{D}_2'\tilde{D}_2\gamma^2 + o_p(1)]^{-1} = O_p(1).$$

Using (A.2c) and (A.2d) write,

$$T^{-3/2}\tilde{D}_1'M_{\tilde{Y}_{-1}}\tilde{D}_2 = T^{-3/2}\tilde{D}_1'M_{\tilde{Y}_{-1}}(\tilde{D}_1 - \tilde{Z}_{-1}/\gamma)$$
$$= -T^{-3/2}\tilde{D}_1'\tilde{Z}_{-1}/\gamma + [T^{-2}\tilde{D}_1'\tilde{D}_2\gamma + o_p(1)]$$
$$\times [T^{-3}\tilde{D}_2'\tilde{D}_2\gamma^2 + o_p(1)]^{-1}[T^{-5/2}\tilde{D}_2'\tilde{Z}_{-1}/\gamma + o_p(1)] + o_p(1) = O_p(1).$$

$$T^{-2}\tilde{D}_2'M_{\tilde{Y}_{-1}}\tilde{D}_2 = T^{-2}(\tilde{D}_1 - \tilde{Z}_{-1}/\gamma)'M_{\tilde{Y}_{-1}}(\tilde{D}_1 - \tilde{Z}_{-1}/\gamma) = T^{-2}\tilde{Z}_{-1}'M_{\tilde{Y}_{-1}}\tilde{Z}_{-1}/\gamma^2 + o_p(1)$$
$$= \left[T^{-2}\tilde{Z}_{-1}'\tilde{Z}_{-1} - [T^{-5/2}\tilde{D}_2'\tilde{Z}_{-1}/\gamma + o_p(1)]^2[T^{-3}\tilde{D}_2'\tilde{D}_2\gamma^2 + o_p(1)]^{-1}\right]/\gamma^2 + o_p(1) = O_p(1).$$

iii) Using (A.3b), (A.3c) and (A.3d) write

$$T^{-1}\hat{D}_2'M_{\hat{Y}_{-1}}\hat{Y} = T^{-1}[\hat{D}_1 - \hat{Z}_{-1}/\gamma]'M_{\hat{Y}_{-1}}[\hat{D}_1\gamma + \hat{e}]$$
$$= -T^{-1}\hat{Z}_{-1}'M_{\hat{Y}_{-1}}\hat{e}/\gamma \text{ (since } \hat{D}_1 = 0 \text{ when } T_b = T_b^c)$$
$$=\{T^{-1}\hat{Z}_{-1}'\hat{e} - T^{-5/2}\hat{Z}_{-1}'(\hat{D}_2\gamma + \hat{Z}_{-1})[T^{-3}(\hat{D}_2\gamma + \hat{Z}_{-1})'(\hat{D}_2\gamma + \hat{Z}_{-1})]^{-1}$$
$$\times T^{-3/2}(\hat{D}_2\gamma + \hat{Z}_{-1})'\hat{e}\}/\gamma = O_p(1).$$

iv) Using (A.3c) and (A.3d) write,

$$T^{-2}\hat{D}_2'M_{\hat{Y}_{-1}}\hat{D}_2 = T^{-2}\hat{Z}_{-1}'M_{\hat{Y}_{-1}}\hat{Z}_{-1}/\gamma^2$$
$$= \left[T^{-2}\hat{Z}_{-1}'\hat{Z}_{-1} - [T^{-5/2}\hat{D}_2'\hat{Z}_{-1}/\gamma + o_p(1)]^2[T^{-3}\hat{D}_2'\hat{D}_2\gamma^2 + o_p(1)]^{-1}\right]/\gamma^2 + o_p(1) = O_p(1).$$

v) Using (A.1b) and (A.1c) write,

$$T^{-3/2}\tilde{Y}_{-1}M_{\tilde{D}_2}(\tilde{Y}-\tilde{Y}_{-1}) = T^{-3/2}[(\tilde{D}_2-\tilde{D}_1)\gamma+\tilde{Z}_{-1}]'M_{\tilde{D}_2}(\tilde{D}_1\gamma+\tilde{e})$$
$$= T^{-3/2}(\tilde{Z}_{-1}-\tilde{D}_1\gamma)'M_{\tilde{D}_2}(\tilde{D}_1\gamma+\tilde{e}) \quad (\text{by } M_{\tilde{D}_2}\tilde{D}_2=0)$$
$$= T^{-3/2}\tilde{Z}_{-1}'M_{\tilde{D}_2}\tilde{D}_1\gamma + o_p(1) = O_p(1).$$

vi) Using (A.2c) write,

$$T^{-2}\tilde{Y}_{-1}M_{\tilde{D}_2}\tilde{Y}_{-1} = T^{-2}[(\tilde{D}_2-\tilde{D}_1)\gamma+\tilde{Z}_{-1}]'M_{\tilde{D}_2}[(\tilde{D}_2-\tilde{D}_1)\gamma+\tilde{Z}_{-1}]$$
$$= T^{-2}(\tilde{Z}_{-1}-\tilde{D}_1\gamma)'M_{\tilde{D}_2}(\tilde{Z}_{-1}-\tilde{D}_1\gamma) = T^{-2}\tilde{Z}_{-1}'M_{\tilde{D}_2}\tilde{Z}_{-1} + o_p(1) = O_p(1),$$

Lemma A3. Set $\gamma \neq 0$ and suppose that $\lambda \neq \lambda_c$. Under (A.1a) these limiting results hold:

i) $T^{-1/2}\tilde{Y}'M_{\tilde{Y}_{-1}}\tilde{D}\tau_1 = O_p(1),$

ii) $\tau_1\tilde{D}'M_{\tilde{Y}_{-1}}\tilde{D}\tau_1 = O_p(1),$

iii) $T^{-2}\hat{D}_2'M_{\hat{Y}_{-1}}\hat{Y} = O_p(1),$

iv) $T^{-3}\hat{D}_2'M_{\hat{Y}_{-1}}\hat{D}_2 = O_p(1),$

v) $T^{-2}\tilde{Y}_{-1}M_{\tilde{D}_2}(\tilde{Y}-\tilde{Y}_{-1}) = O_p(1),$

vi) $T^{-3}\tilde{Y}_{-1}M_{\tilde{D}_2}\tilde{Y}_{-1} = O_p(1),$

vii) $T^{-2}\tilde{Y}_{-1}M_{\tilde{D}}(\tilde{Y}-\tilde{Y}_{-1}) = O_p(1),$

viii) $T^{-3}\tilde{Y}_{-1}M_{\tilde{D}}\tilde{Y}_{-1} = O_p(1).$

Proof:

i) Using (A.2b) write,

$$T^{-1/2}\tilde{Y}'M_{\tilde{Y}_{-1}}\tilde{D}\tau_1 = \begin{bmatrix} T^{-1}\tilde{Y}'M_{\tilde{Y}_{-1}}\tilde{D}_1 \\ T^{-2}\tilde{Y}'M_{\tilde{Y}_{-1}}\tilde{D}_2 \end{bmatrix} = \begin{bmatrix} T^{-1}(\tilde{D}_1^c\gamma+\tilde{e})'M_{\tilde{Y}_{-1}}\tilde{D}_1 \\ T^{-2}(\tilde{D}_1^c\gamma+\tilde{e})'M_{\tilde{Y}_{-1}}\tilde{D}_2 \end{bmatrix}.$$

Using (A.2c) write,

$$T^{-1}(\tilde{D}_1^c\gamma+\tilde{e})'M_{\tilde{Y}_{-1}}\tilde{D}_1 = T^{-1}(\tilde{D}_1^c\gamma+\tilde{e})'\tilde{D}_1 + T^{-2}(\tilde{D}_1^c\gamma+\tilde{e})'[(\tilde{D}_2^c-\tilde{D}_1^c)\gamma+\tilde{Z}_{-1}]$$
$$\times\{T^{-3}[(\tilde{D}_2^c-\tilde{D}_1^c)\gamma+\tilde{Z}_{-1}]'[(\tilde{D}_2^c-\tilde{D}_1^c)\gamma+\tilde{Z}_{-1}]\}^{-1}T^{-2}[(\tilde{D}_2^c-\tilde{D}_1^c)\gamma+\tilde{Z}_{-1}]'\tilde{D}_1$$

$$= T^{-1}\tilde{D}_1^{c\prime}\tilde{D}_1\gamma + o_p(1)$$

$$- [T^{-2}\tilde{D}_1^{c\prime}\tilde{D}_2^c\gamma^2 + o_p(1)][T^{-3}\tilde{D}_2^{c\prime}\tilde{D}_2^c\gamma^2 + o_p(1)]^{-1}[T^{-2}\tilde{D}_2^{c\prime}\tilde{D}_1\gamma + o_p(1)] = O_p(1)$$

$$T^{-2}(\tilde{D}_1^c\gamma + \tilde{e})' M_{\tilde{Y}_{-1}}\tilde{D}_2 = T^{-2}(\tilde{D}_1^c\gamma + \tilde{e})'\tilde{D}_2 + T^{-2}(\tilde{D}_1^c\gamma + \tilde{e})'[(\tilde{D}_2^c - \tilde{D}_1^c)\gamma + \tilde{Z}_{-1}]$$

$$\times \{T^{-3}[(\tilde{D}_2^c - \tilde{D}_1^c)\gamma + \tilde{Z}_{-1}]'[(\tilde{D}_2^c - \tilde{D}_1^c)\gamma + \tilde{Z}_{-1}]\}^{-1}T^{-3}[(\tilde{D}_2^c - \tilde{D}_1^c)\gamma + \tilde{Z}_{-1}]'\tilde{D}_2$$

$$= T^{-2}\tilde{D}_1^{c\prime}\tilde{D}_2\gamma + o_p(1)$$

$$- [T^{-2}\tilde{D}_1^{c\prime}\tilde{D}_2^c\gamma^2 + o_p(1)][T^{-3}\tilde{D}_2^{c\prime}\tilde{D}_2^c\gamma^2 + o_p(1)]^{-1}[T^{-3}\tilde{D}_2^{c\prime}\tilde{D}_2\gamma + o_p(1)] = O_p(1)$$

ii) $$\tau_1\tilde{D}' M_{\tilde{Y}_{-1}}\tilde{D}\tau_1 = \begin{bmatrix} T^{-1}\tilde{D}_1' M_{\tilde{Y}_{-1}}\tilde{D}_1 & T^{-2}\tilde{D}_1' M_{\tilde{Y}_{-1}}\tilde{D}_2 \\ T^{-2}\tilde{D}_1' M_{\tilde{Y}_{-1}}\tilde{D}_2 & T^{-3}\tilde{D}_2' M_{\tilde{Y}_{-1}}\tilde{D}_2 \end{bmatrix} = O_p(1),$$

using arguments similar to (*i*).

iii) Using (A3.b) write,

$$T^{-2}\hat{D}_2' M_{\hat{Y}_{-1}}\hat{Y} = T^{-2}\hat{D}_2' M_{\hat{Y}_{-1}}(\hat{D}_1^c\gamma + \hat{e})$$

$$= T^{-2}\hat{D}_2' M_{\hat{Y}_{-1}}\hat{D}_1^c\gamma + o_p(1) = O_p(1), \quad \text{using arguments similar to } (i).$$

iv) Follows from arguments similar to (*i*).

v) Using (A.2b) and (A.2c) write,

$$T^{-2}\tilde{Y}_{-1}M_{\tilde{D}_2}(\tilde{Y} - \tilde{Y}_{-1}) = T^{-2}[(\tilde{D}_2^c - \tilde{D}_1^c)\gamma + \tilde{Z}_{-1}]' M_{\tilde{D}_2}(\tilde{D}_1^c\gamma + \tilde{e})$$

$$= T^{-2}\tilde{D}_2^{c\prime} M_{\tilde{D}_2}\tilde{D}_1^c\gamma + o_p(1) = O_p(1).$$

vi) Using (A.2c) write,

$$T^{-3}\tilde{Y}_{-1}M_{\tilde{D}_2}\tilde{Y}_{-1} = T^{-3}[(\tilde{D}_2^c - \tilde{D}_1^c)\gamma + \tilde{Z}_{-1}]' M_{\tilde{D}_2}[(\tilde{D}_2^c - \tilde{D}_1^c)\gamma + \tilde{Z}_{-1}]$$

$$= T^{-3}\tilde{D}_2^c M_{\tilde{D}_2}\tilde{D}_2^c\gamma^2 + o_p(1) = O_p(1).$$

vii) Using (A.2b) and (A.2c) write,

$$T^{-2}\tilde{Y}_{-1}M_{\tilde{D}}(\tilde{Y} - \tilde{Y}_{-1}) = T^{-2}[(\tilde{D}_2^c - \tilde{D}_1^c)\gamma + \tilde{Z}_{-1}]' M_{\tilde{D}}(\tilde{D}_1^c\gamma + \tilde{e})$$

$$= T^{-2}\tilde{D}_2^{c\prime} M_{\tilde{D}} \tilde{D}_1^c \gamma + o_p(1) = O_p(1).$$

viii) Using (A.2c) write

$$T^{-3}\tilde{Y}_{-1} M_{\tilde{D}} \tilde{Y}_{-1} = T^{-3}[(\tilde{D}_2^c - \tilde{D}_1^c)\gamma + \tilde{Z}_{-1}]' M_{\tilde{D}}[(\tilde{D}_2^c - \tilde{D}_1^c)\gamma + \tilde{Z}_{-1}]$$

$$= T^{-3}\tilde{D}_2^c M_{\tilde{D}} \tilde{D}_2^c \gamma^2 + o_p(1) = O_p(1).$$

Using the results from the Lemmas, the proofs of the limiting distributions will follow easily. Proofs are not given in the order as presented in the text but are given by model so as to minimize the details.

Proofs for Model (1,AO):

To prove (12) and (13) all that needs to be shown is that the limiting distribution of $T^{-1/2} t_{\hat{\theta}}$ is $Q_{1,AO}(\lambda)$. Using regression (2.1) and (A.2a) with $\gamma = 0$, write

$$T^{-1/2} t_{\hat{\theta}} = T^{-3/2}\tilde{Z}'\tilde{D}_1[T^{-1}s^2 T^{-1}\tilde{D}_1'\tilde{D}_1]^{-1/2} = T^{-3/2}\tilde{Z}'\tilde{D}_1[T^{-2}\tilde{Z}' M_{\tilde{D}_1} \tilde{Z} T^{-1}\tilde{D}_1'\tilde{D}_1]^{-1/2}$$

$$\Rightarrow \int_\lambda^1 W^*(r)dr/(a_{11}L_1(\lambda))^{1/2} = Q_{1,AO}(\lambda),$$

by Lemma A1 (*i*) and (*vii*).

Proofs for Model (2,IO):

Proofs are first given for the case where $\gamma = 0$ followed by proofs when $\gamma \neq 0$. Since the limiting results do not depend on the $D(T_b)_t$ variable in regression (5.2), it is ignored when writing down the statistics. To prove (14) through (16) all that needs to be shown is that $t_{\hat{\gamma}}$ and $F_{\hat{\theta},\hat{\gamma}}$ converge to $Q^A_{2,IO}(\lambda)$ and $Q^B_{2,IO}(\lambda)$ respectively. Using regression (5.2) with $k = 0$ (omitting the $D(T_b)_t$ term) and (A.2b) and (A.2c) with $\gamma = 0$, $t_{\hat{\gamma}}$ can be written as,

$$t_{\hat{\gamma}} = \left[T^{-3/2}\tilde{D}_2' M_{\tilde{Z}_{-1}} \tilde{e} - T^{-2}\tilde{D}_2' M_{\tilde{Z}_{-1}} \tilde{D}_1 (T^{-1}\tilde{D}_1' M_{\tilde{Z}_{-1}} \tilde{D}_1)^{-1} T^{-1/2}\tilde{D}_1' M_{\tilde{Z}_{-1}} \tilde{e}\right]$$

$$* \left[s^2\{T^{-3}\tilde{D}_2' M_{\tilde{Z}_{-1}} \tilde{D}_2 - T^{-4}(\tilde{D}_2' M_{\tilde{Z}_{-1}} \tilde{D}_1)^2 (T^{-1}\tilde{D}_1' M_{\tilde{Z}_{-1}} \tilde{D}_1)^{-1}\}\right]^{-1/2}$$

$$\Rightarrow [H_2(\lambda) - K_3(\lambda)H_1(\lambda)/K_1(\lambda)]/[K_2(\lambda) - K_3(\lambda)^2/K_1(\lambda)]^{-1/2} = Q^A_{2,IO}(\lambda),$$

by Lemma A1 (*ii*) – (*vi*) and the result $s^2 \to \sigma^2$ from Perron (1989).

Similarly, $F_{\hat{\theta},\hat{\gamma}}$ can be written as,

$$F_{\hat{\theta},\hat{\gamma}} = \tilde{e}' M_{\tilde{Y}_{-1}} \tilde{D}\tau_1(\tau_1\tilde{D}' M_{\tilde{Y}_{-1}} \tilde{D}\tau_1)^{-1}\tau_1\tilde{D}M_{\tilde{Y}_{-1}}\tilde{e}/s^2$$

$$\Rightarrow [H_1(\lambda)^2K_2(\lambda) - 2H_1(\lambda)K_3(\lambda)H_2(\lambda) + H_2(\lambda)^2K_1(\lambda)]$$

$$\times [K_1(\lambda)K_2(\lambda) - K_3(\lambda)^2]^{-1} = Q^B_{2,IO}(\lambda),$$

by Lemma A1 (*ii*) – (*vi*). Suppose that $\gamma \neq 0$. Results for $t_{\hat{\alpha}}$ are established first. When $\lambda = \lambda_c$, $t_{\hat{\alpha}}$ is invariant to γ under the null hypothesis, and its limiting distribution is given by (25). Suppose that $\lambda \neq \lambda_c$. Three steps are used to establish (28) (j=2, m=IO).

<u>Step 1:</u> Using (5.2) (ignoring $D(T_b)_t$) write

$$T(\hat{\alpha} - 1) = T^{-2}\tilde{Y}_{-1}' M_{\tilde{D}}(\tilde{Y} - \tilde{Y}_{-1})/[T^{-3}\tilde{Y}_{-1}' M_{\tilde{D}}\tilde{Y}_{-1}] = O_p(1),$$

by Lemma A3 (*vii*) and (*viii*).

<u>Step 2:</u>

$$s^2 = T^{-1}[(\tilde{Y} - \tilde{Y}_{-1} - (\hat{\alpha} - 1)\tilde{Y}_{-1}]' M_{\tilde{D}}[(\tilde{Y} - \tilde{Y}_{-1} - (\hat{\alpha} - 1)\tilde{Y}_{-1}]$$

$$= T^{-1}(\tilde{Y} - \tilde{Y}_{-1})' M_{\tilde{D}}(\tilde{Y} - \tilde{Y}_{-1}) - 2T(\hat{\alpha} - 1)T^{-2}\tilde{Y}_{-1}' M_{\tilde{D}}(\tilde{Y} - \tilde{Y}_{-1})$$

$$+ T^2(\hat{\alpha} - 1)^2T^{-3}\tilde{Y}_{-1}' M_{\tilde{D}}\tilde{Y}_{-1}$$

$$= T^{-1}(\tilde{D}^c_1\gamma + \tilde{e})' M_{\tilde{D}}(\tilde{D}^c_1\gamma + \tilde{e}) + O_p(1) \text{ (using Step 1 and Lemma A3)}$$

$$= O_p(1).$$

<u>Step 3:</u> Again using (5.2) write,

$$T^{-1/2}t_{\hat{\alpha}}(2,IO,[\lambda T],k{=}0) = T^{-2}\tilde{Y}_{-1}' M_{\tilde{D}}(\tilde{Y} - \tilde{Y}_{-1})/[s^2T^{-3}\tilde{Y}_{-1}' M_{\tilde{D}}\tilde{Y}_{-1}]^{1/2} = O_p(1),$$

which follows from Steps 1, 2. Consider $t_{\hat{\gamma}}$ and $F_{\hat{\theta},\hat{\gamma}}$ when $\gamma \neq 0$. Suppose that $\lambda = \lambda_c$.

(A4) $\quad t_{\hat{\gamma}} = T^{-1}\hat{D}_2' M_{\hat{Y}_{-1}}\hat{Y}/(s^2T^{-2}\hat{D}_2' M_{\hat{Y}_{-1}}\hat{D}_2)^{1/2} = O_p(1),$

which follows from Lemma A2 (*iii*) and (*iv*) and $s^2 \to \sigma^2$.

(A5) $\quad T^{-1}F_{\hat{\theta},\hat{\gamma}} = T^{-1/2}\tilde{Y}' M_{\tilde{Y}_{-1}}\tilde{D}\tau_2(\tau_2\tilde{D}' M_{\tilde{Y}_{-1}}\tilde{D}\tau_2)^{-1}\tau_2\tilde{D}M_{\tilde{Y}_{-1}}\tilde{Y}T^{-1/2}/s^2 = O_p(1),$

which follows from Lemma A2 (*i*) & (*ii*). (A4) and (A5) establish (35). Suppose $\lambda \neq \lambda_c$, (A6) $T^{-1/2}t_{\hat{\gamma}} = T^{-2}\hat{D}_2'M_{\hat{Y}_{-1}}\hat{Y}/(s^2T^{-3}\hat{D}_2'M_{\hat{Y}_{-1}}\hat{D}_2)^{1/2} = O_p(1)$,

which follows from Lemma A3 (*iii*) and (*iv*) and $s^2 = O_p(1)$.

(A7) $T^{-1}F_{\hat{\theta},\hat{\gamma}} = T^{-1/2}\tilde{Y}'M_{\tilde{Y}_{-1}}\tilde{D}\tau_1(\tau_1\tilde{D}'M_{\tilde{Y}_{-1}}\tilde{D}\tau_1)^{-1}\tau_1\tilde{D}M_{\tilde{Y}_{-1}}\tilde{Y}T^{-1/2}/s^2 = O_p(1)$,

which follows from Lemma A3 (*i*) and (*ii*). (A6) and (A7) establish (36).

Proofs for Model (2,AO):

As presented for Model (2,IO), proofs are first given for the case where $\gamma = 0$ followed by proofs when $\gamma \neq 0$. Consider results (17) – (19). All that needs to be established are the limiting distributions of $T^{-1/2}t_{\hat{\gamma}}$ and $T^{-1}F_{\hat{\theta},\hat{\gamma}}$. Using regression (2.2) the statistics can be written as,

$$T^{-1/2}t_{\hat{\gamma}} = (T^{-5/2}\tilde{D}_2'M_{\tilde{D}_1}\tilde{Y})(T^{-1}s^2T^{-3}\tilde{D}_2'M_{\tilde{D}_1}\tilde{D}_2)^{-1/2},$$

$$T^{-1}F_{\hat{\theta},\hat{\gamma}} = T^{-1}\tilde{Y}'\tilde{D}\tau_1(\tau_1\tilde{D}'\tilde{D}\tau_1)^{-1}\tau_1\tilde{D}'\tilde{Y}T^{-1}/(T^{-1}s^2),$$

where

(A8) $$T^{-1}s^2 = T^{-2}\tilde{Y}'M_{\tilde{D}_1}\tilde{Y} - T^{-5/2}\tilde{Y}'M_{\tilde{D}_1}\tilde{D}_2(T^{-3}\tilde{D}_2'M_{\tilde{D}_1}\tilde{D}_2)^{-1}T^{-5/2}\tilde{D}_2'M_{\tilde{D}_1}\tilde{Y}$$

$$\Rightarrow \sigma^2[L_1(\lambda) - L_3(\lambda)^2/(a_{22} - a_{12}^2/a_{11})]$$ by Lemma A1 (*vii*), (*ix*) and (*x*).

Using (A8) and Lemma A1 it directly follows that,

$$T^{-1/2}t_{\hat{\gamma}} \Rightarrow L_3(\lambda)[(a_{22} - a_{12}^2/a_{11})L_1(\lambda) - L_3(\lambda)^2]^{-1/2} = Q^A_{2,AO}(\lambda),$$

$$T^{-1}F_{\hat{\theta},\hat{\gamma}} \Rightarrow \left[(a_{11}a_{22} - a_{12}^2)L_1(\lambda) - a_{11}L_3(\lambda)^2\right]^{-1}$$

$$\times \left[a_{22}(\int_\lambda^1 W^*(r)dr)^2 - 2a_{12}\int_\lambda^1 W^*(r)dr\int_\lambda^1(r-\lambda)W^*(r)dr + a_{11}(\int_\lambda^1(r-\lambda)W^*(r)dr)^2\right]$$

$$= Q^B_{2,AO}(\lambda),$$ as required.

Now consider the case where $\gamma \neq 0$. Using (A2.a) and regression (2.2), the statistics can be written as,

(A.9) $T^{-1/2}t_{\hat{\gamma}} = T^{-1/2}\tilde{D}_2'M_{\tilde{D}_1}\{\tilde{D}_2^c\gamma + \gamma\tilde{Z}\}(s^2\tilde{D}_2'M_{\tilde{D}_1}\tilde{D}_2)^{-1/2}$,

$$\text{(A.10)}\quad T^{-1}F_{\hat{\theta},\hat{\gamma}} = T^{-1}\{\tilde{D}_2^c\gamma + \tilde{Z}\}'\tilde{D}(\tilde{D}'\tilde{D})^{-1}\tilde{D}'\{\tilde{D}_2^c\gamma + \tilde{Z}\}/s^2.$$

Consider the limiting behavior of (A.9) and (A.10) when (2.2) is estimated at the true break date ($\lambda = \lambda_c$). In this case s^2 is invariant to γ and $T^{-1}s^2 = O_p(1)$. Using this fact:

$$\text{(A.11)}\quad T^{-1/2}t_{\hat{\gamma}} = T^{1/2}\left[T^{-3}\tilde{D}_2^{c\prime}M_{\tilde{D}_1^c}(\tilde{D}_2^c\gamma + \tilde{Z})(T^{-1}s^2T^{-3}\tilde{D}_2^{c\prime}M_{\tilde{D}_1^c}\tilde{D}_2^c)^{-1/2}\right]$$

$$= \gamma T^{1/2}O_p(1),$$

$$\text{(A.12)}\quad T^{-1}F_{\hat{\theta},\hat{\gamma}} = T\left[T^{-3/2}(\tilde{D}_2^c\gamma + \tilde{Z})'\tilde{D}\tau_1(\tau_1\tilde{D}'\tilde{D}\tau_1)^{-1}\tau_1\tilde{D}'(\tilde{D}_2^c\gamma + \tilde{Z})T^{-3/2}\right]/(T^{-1}s^2)$$

$$= \gamma^2 T O_p(1).$$

Results (31) and (32) immediately follow from (A.11) and (A.12). Next consider the behavior of the statistics when the wrong break date is used in regression (2.2) ($\lambda \neq \lambda_c$). It is no longer true that s^2 is invariant to γ as,

$$\text{(A.13)}\quad T^{-2}s^2 = T^{-3}(\tilde{D}_2^c\gamma + \tilde{Z})'M_{\tilde{D}}(\tilde{D}_2^c\gamma + \tilde{Z}) = \gamma^2T^{-3}\tilde{D}_2^{c\prime}M_{\tilde{D}}\tilde{D}_2^c + o_p(1) = \gamma^2 O_p(1).$$

Now, rewrite (A.11) and (A.12) as,

$$\text{(A.14)}\quad T^{-1/2}t_{\hat{\gamma}} = T^{-3}\tilde{D}_2'M_{\tilde{D}_1}\{\tilde{D}_2^c + \gamma\tilde{Z}\}(T^{-2}s^2T^{-3}\tilde{D}_2'M_{\tilde{D}_1}\tilde{D}_2)^{-1/2} = O_p(1),$$

$$\text{(A.15)}\quad T^{-1}F_{\hat{\theta},\hat{\gamma}} = T^{-3/2}(\tilde{D}_2^c\gamma + \tilde{Z})'\tilde{D}\tau_1(\tau_1\tilde{D}'\tilde{D}\tau_1)^{-1}\tau_1\tilde{D}'(\tilde{D}_2^c\gamma + \tilde{Z})T^{-3/2}(T^{-2}s^2)^{-1} = O_p(1),$$

This proves (30). All that remains to show for Model (2,AO) is (28) ($j = 2$, $m =$ AO) (that $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\alpha}}),k=0)$ diverges when $\gamma \neq 0$ and $\lambda \neq \lambda_c$). Using results in Perron and Vogelsang (1993a), the proof is analogous to the proof for Model (2,IO) and is omitted.

Proofs for Model (3,IO):

The only results that need to be shown are (27) and (28) ($j = 3$, $m =$ IO) in which case $\gamma \neq 0$. Suppose that $\lambda = \lambda_c$. Three steps are used to establish (27).

Step 1: Using (5.3) write,

$$T^{1/2}(\hat{\alpha} - 1) = T^{-3/2}\tilde{Y}_{-1}'M_{\tilde{D}_2}(\tilde{Y} - \tilde{Y}_{-1})/[T^{-2}\tilde{Y}_{-1}'M_{\tilde{D}_2}\tilde{Y}_{-1}] = O_p(1),$$

by Lemma A2 (v) and (vi).

Step 2:

$$s^2 = T^{-1}(\tilde{Y} - \tilde{Y}_{-1})'M_{\tilde{D}_2}(\tilde{Y} - \tilde{Y}_{-1}) - 2T^{1/2}(\hat{\alpha} - 1)T^{-3/2}\tilde{Y}_{-1}'M_{\tilde{D}_2}(\tilde{Y} - \tilde{Y}_{-1})$$
$$+ T(\hat{\alpha} - 1)^2T^{-2}\tilde{Y}_{-1}'M_{\tilde{D}_2}\tilde{Y}_{-1}$$
$$= T^{-1}(\tilde{D}_1^c\gamma + \tilde{e})'M_{\tilde{D}_2}(\tilde{D}_1^c\gamma + \tilde{e}) + O_p(1) \quad \text{(by Step 1 and Lemma A2)}$$
$$= O_p(1).$$

Step 3: Again using (5.3) write,

$$T^{-1/2}t_{\hat{\alpha}}(3,IO,T_b,k=0) = T^{-3/2}\tilde{Y}_{-1}'M_{\tilde{D}_2}(\tilde{Y} - \tilde{Y}_{-1})/[s^2T^{-2}\tilde{Y}_{-1}'M_{\tilde{D}_2}\tilde{Y}_{-1}]^{1/2} = O_p(1)$$

which follows from Steps 1 and 2. This proves (27). Suppose now that $\lambda \neq \lambda_c$. To establish (28) (j=3, m=IO) three steps are again needed.

Step 1: Using (5.3) write,

$$T(\hat{\alpha} - 1) = T^{-2}\tilde{Y}_{-1}'M_{\tilde{D}_2}(\tilde{Y} - \tilde{Y}_{-1})/[T^{-3}\tilde{Y}_{-1}'M_{\tilde{D}_2}\tilde{Y}_{-1}] = O_p(1),$$

by Lemma A3 (v) and (vi).

Step 2:

$$s^2 = T^{-1}(\tilde{Y} - \tilde{Y}_{-1})'M_{\tilde{D}_2}(\tilde{Y} - \tilde{Y}_{-1}) - 2T(\hat{\alpha} - 1)T^{-2}\tilde{Y}_{-1}'M_{\tilde{D}_2}(\tilde{Y} - \tilde{Y}_{-1})$$
$$+ T^2(\hat{\alpha} - 1)^2T^{-3}\tilde{Y}_{-1}'M_{\tilde{D}_2}\tilde{Y}_{-1}$$
$$= T^{-1}(\tilde{D}_1^c\gamma + \tilde{e})'M_{\tilde{D}_2}(\tilde{D}_1^c\gamma + \tilde{e}) + O_p(1) \quad \text{(using Step 1 and Lemma A3)}$$
$$= O_p(1).$$

Step 3: Again using (5.3) write,

$$T^{-1/2}t_{\hat{\alpha}}(3,IO,T_b,k=0) = T^{-2}\tilde{Y}_{-1}'M_{\tilde{D}_2}(\tilde{Y} - \tilde{Y}_{-1})/[s^2T^{-3}\tilde{Y}_{-1}'M_{\tilde{D}_2}\tilde{Y}_{-1}]^{1/2} = O_p(1)$$

which follows from Steps 1 and 2. This proves (28) (j=3, m=IO).

Proofs for Model (3,AO):

As before, proofs are first given for the case where $\gamma = 0$ followed by proofs when $\gamma \neq 0$. Consider results (22) and (23). All that needs to be established is the limiting distribution of $T^{-1/2}t_{\hat{\gamma}}$. Using regression (2.3) the t–statistic can be written as,

$$T^{-1/2}t_{\hat{\gamma}} = (T^{-5/2}\tilde{D}_2'\tilde{Y})(T^{-1}s^2T^{-3}\tilde{D}_2'\tilde{D}_2)^{-1/2} = (T^{-5/2}\tilde{D}_2'\tilde{Y})(T^{-2}\tilde{Y}M_{\tilde{D}_2}\tilde{Y}T^{-3}\tilde{D}_2'\tilde{D}_2)^{-1/2}$$

$$\Rightarrow \int_{\lambda}^{1}(r-\lambda)W^*(r)dr/(a_{22}L_2(\lambda))^{1/2} = Q_{3,AO}(\lambda), \text{ by Lemma A1 } (i) \text{ and } (viii).$$

Now suppose that $\gamma \neq 0$. To prove (33) and (34) use (2.3) and (A.2a) to write,

$$T^{-1/2}t_{\hat{\gamma}} = T^{-1/2}\{\tilde{D}_2'(\tilde{D}_2^c\gamma + \tilde{Z})\}(s^2\tilde{D}_2'\tilde{D}_2)^{-1/2},$$

where $s^2 = T^{-1}(\tilde{D}_2^c\gamma + \tilde{Z})'M_{\tilde{D}_2}(\tilde{D}_2^c\gamma + \tilde{Z})$. When (2.3) is estimated at the true break date ($\lambda = \lambda_c$), $\tilde{D}_2^c = \tilde{D}_2$ and $M_{\tilde{D}_2}\tilde{D}_2^c = 0$ giving $T^{-1}s^2 = T^{-2}\tilde{Z}'M_{\tilde{D}_2}\tilde{Z} = O_p(1)$ by Lemma A1 (*viii*). It immediately follows that,

$$\text{(A.16)} \qquad T^{-1/2}t_{\hat{\gamma}} = T^{1/2}\{T^{-3}\tilde{D}_2^{c\prime}(\tilde{D}_2^c\gamma + \tilde{Z})\}(T^{-1}s^2T^{-3}\tilde{D}_2^{c\prime}\tilde{D}_2^c)^{-1/2} = \gamma T^{1/2}O_p(1).$$

Result (34) is a direct consequence of (A.16). When (2.3) is estimated at the wrong break date ($\lambda \neq \lambda_c$), $M_{\tilde{D}_2}\tilde{D}_2^c \neq 0$ and it follows that,

$$\text{(A17)} \qquad T^{-2}s^2 = T^{-3}(\tilde{D}_2^c\gamma + \tilde{Z})'M_{\tilde{D}_2}(\tilde{D}_2^c\gamma + \tilde{Z}) = T^{-3}\tilde{D}_2^{c\prime}M_{\tilde{D}_2}\tilde{D}_2^c\gamma^2 + o_p(1) = O_p(1).$$

Using (A17) it is easily established that,

$$T^{-1/2}t_{\hat{\gamma}} = \{T^{-3}\tilde{D}_2'(\tilde{D}_2^c\gamma + \tilde{Z})\}(T^{-2}s^2T^{-3}\tilde{D}_2'\tilde{D}_2)^{-1/2} = \gamma O_p(1).$$

This proves (33). All that is left to prove for Model (3,AO) is (28) ($j = 3$, $m = AO$). This result can be proved analogously to the proof in Model (3,IO) using results from Perron and Vogelsang (1993a) and is therefore omitted.

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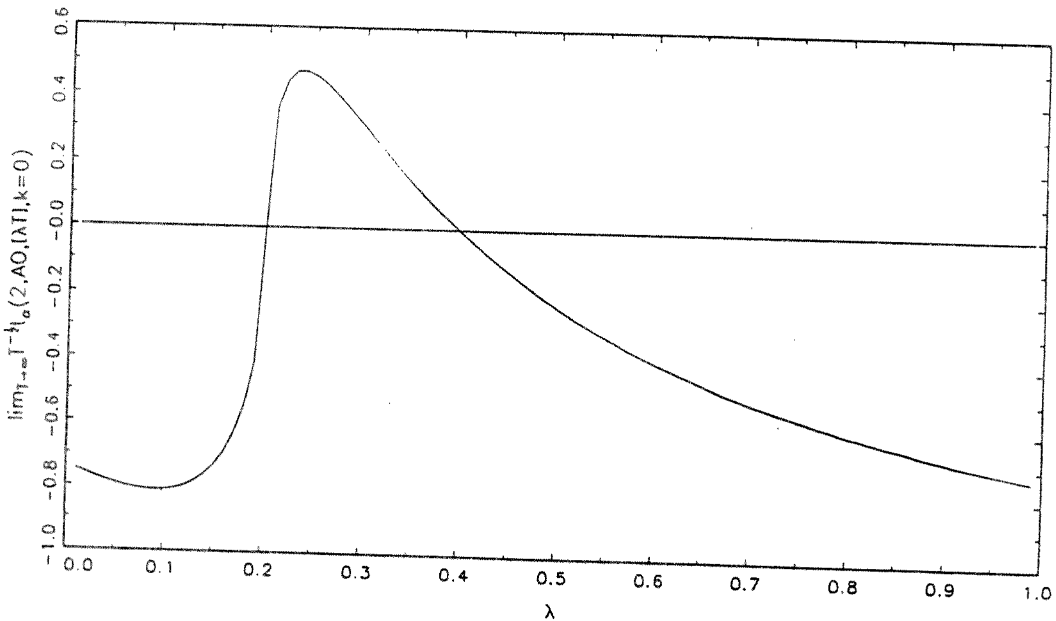

Figure 1: $\lim_{T\to\infty} T^{-\frac{1}{2}} t_\alpha(2, AO, [\lambda T], k=0)$, $\gamma=5.0$, $\lambda_c=0.2$, $\sigma^2=1.0$.

Table 1A : Model (1,AO); Distribution of $t_{\hat{\alpha}}(1,AO,T_b(t_{\hat{\alpha}}))$; Choosing T_b Minimizing $t_{\hat{\alpha}}$.

		1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
T=50	k=0	-5.49	-5.20	-4.91	-4.60	-3.59	-2.77	-2.55	-2.33	-2.02
	k=2	-5.41	-4.98	-4.66	-4.31	-3.37	-2.58	-2.39	-2.18	-1.93
	k=5	-5.22	-4.72	-4.48	-4.11	-3.16	-2.41	-2.21	-2.03	-1.88
	k=k(t-sig)	-5.71	-5.44	-5.16	-4.86	-3.87	-3.04	-2.78	-2.62	-2.38
T=100	k=0	-5.50	-5.19	-4.95	-4.64	-3.72	-2.97	-2.71	-2.53	-2.32
	k=2	-5.30	-5.00	-4.69	-4.43	-3.54	-2.80	-2.56	-2.38	-2.17
	k=5	-4.99	-4.73	-4.49	-4.23	-3.38	-2.65	-2.43	-2.27	-2.03
	k=k(t-sig)	-5.68	-5.36	-5.06	-4.79	-3.86	-3.06	-2.86	-2.64	-2.43
T=150	k=0	-5.37	-5.11	-4.84	-4.56	-3.71	-2.93	-2.67	-2.52	-2.23
	k=2	-5.27	-5.03	-4.71	-4.44	-3.62	-2.82	-2.59	-2.38	-2.08
	k=5	-5.13	-4.83	-4.61	-4.31	-3.47	-2.72	-2.47	-2.24	-1.97
	k=k(t-sig)	-5.61	-5.27	-5.03	-4.73	-3.81	-3.04	-2.77	-2.54	-2.28
T = ∞		-5.41	-5.02	-4.80	-4.58	-3.75	-2.99	-2.77	-2.56	-2.32

Table 1B : Model (1,AO); Distribution of $t_{\hat{\alpha}}(1,AO,T_b(t_{\hat{\theta}}))$; Choosing T_b Maximizing $t_{\hat{\theta}}$.

		1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
T=50	k=0	-4.80	-4.54	-4.25	-3.93	-2.83	-1.84	-1.57	-1.34	-1.11
	k=2	-4.69	-4.34	-4.00	-3.70	-2.62	-1.72	-1.44	-1.21	-0.89
	k=5	-4.43	-4.06	-3.71	-3.39	-2.42	-1.58	-1.38	-1.20	-0.95
	k=k(t-sig)	-5.15	-4.78	-4.50	-4.14	-3.02	-1.99	-1.71	-1.47	-1.18
T=100	k=0	-4.95	-4.56	-4.24	-3.90	-2.87	-2.00	-1.70	-1.45	-1.22
	k=2	-4.70	-4.39	-4.05	-3.78	-2.73	-1.82	-1.55	-1.33	-1.11
	k=5	-4.51	-4.14	-3.87	-3.53	-2.60	-1.74	-1.46	-1.28	-1.00
	k=k(t-sig)	-5.08	-4.66	-4.41	-4.08	-2.93	-2.01	-1.76	-1.52	-1.31
T=150	k=0	-4.67	-4.37	-4.09	-3.82	-2.85	-1.88	-1.63	-1.36	-1.11
	k=2	-4.59	-4.30	-4.00	-3.70	-2.74	-1.81	-1.53	-1.26	-0.93
	k=5	-4.50	-4.23	-3.87	-3.63	-2.62	-1.74	-1.47	-1.26	-0.86
	k=k(t-sig)	-4.87	-4.56	-4.31	-3.95	-2.90	-1.94	-1.68	-1.36	-1.11
T = ∞		-4.57	-4.24	-4.01	-3.74	-2.77	-1.85	-1.57	-1.34	-1.02

Table 1C : Model (1,AO); Distribution of $t_{\hat{\alpha}}(1,AO,T_b(|t_{\hat{\theta}}|))$;
Choosing T_b Maximizing $|t_{\hat{\theta}}|$.

		1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
T=50	k=0	-5.04	-4.70	-4.41	-4.14	-3.09	-2.14	-1.84	-1.64	-1.36
	k=2	-4.96	-4.49	-4.18	-3.86	-2.87	-1.95	-1.69	-1.45	-1.13
	k=5	-4.52	-4.15	-3.88	-3.56	-2.57	-1.72	-1.48	-1.27	-1.03
	k=k(t-sig)	-5.38	-4.91	-4.64	-4.29	-3.25	-2.24	-1.95	-1.72	-1.36
T=100	k=0	-5.02	-4.70	-4.40	-4.11	-3.08	-2.17	-1.96	-1.70	-1.48
	k=2	-4.78	-4.47	-4.24	-3.92	-2.93	-1.99	-1.77	-1.52	-1.28
	k=5	-4.65	-4.35	-4.05	-3.71	-2.81	-1.87	-1.59	-1.41	-1.15
	k=k(t-sig)	-5.21	-4.84	-4.55	-4.26	-3.18	-2.23	-1.96	-1.71	-1.48
T=150	k=0	-4.79	-4.50	-4.32	-4.03	-3.04	-2.11	-1.83	-1.62	-1.30
	k=2	-4.70	-4.41	-4.18	-3.89	-2.94	-2.04	-1.74	-1.52	-1.28
	k=5	-4.65	-4.33	-4.13	-3.78	-2.82	-1.92	-1.69	-1.45	-1.16
	k=k(t-sig)	-5.00	-4.73	-4.45	-4.19	-3.11	-2.15	-1.88	-1.65	-1.36
$T = \infty$		-4.70	-4.40	-4.17	-3.90	-2.94	-2.05	-1.78	-1.52	-1.22

Table 2A : Model (2,AO); Distribution of $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\alpha}}))$;
Choosing T_b Minimizing $t_{\hat{\alpha}}$.

		1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
T=50	k=0	-6.04	-5.63	-5.29	-5.01	-3.97	-3.12	-2.92	-2.76	-2.58
	k=2	-5.62	-5.24	-4.98	-4.65	-3.66	-2.87	-2.72	-2.57	-2.42
	k=5	-5.55	-5.15	-4.71	-4.38	-3.40	-2.63	-2.46	-2.33	-2.22
	k=k(t-sig)	-6.16	-5.85	-5.49	-5.18	-4.26	-3.46	-3.25	-3.06	-2.84
T=100	k=0	-5.82	-5.53	-5.24	-4.93	-4.00	-3.19	-3.00	-2.79	-2.62
	k=2	-5.56	-5.23	-4.97	-4.69	-3.80	-3.04	-2.82	-2.62	-2.42
	k=5	-5.30	-4.97	-4.73	-4.47	-3.60	-2.87	-2.69	-2.54	-2.34
	k=k(t-sig)	-5.96	-5.63	-5.40	-5.11	-4.16	-3.37	-3.15	-2.92	-2.77
T=150	k=0	-5.76	-5.40	-5.13	-4.87	-3.98	-3.21	-3.03	-2.82	-2.60
	k=2	-5.53	-5.26	-5.00	-4.72	-3.86	-3.06	-2.87	-2.70	-2.50
	k=5	-5.38	-5.06	-4.81	-4.57	-3.71	-2.96	-2.75	-2.61	-2.40
	k=k(t-sig)	-5.89	-5.60	-5.35	-5.03	-4.12	-3.28	-3.06	-2.87	-2.66
$T=\infty$		-5.57	-5.30	-5.08	-4.82	-3.98	-3.25	-3.06	-2.91	-2.72

Table 2B : Model (2,AO); Distribution of $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\gamma}}))$;
Choosing T_b Maximizing $t_{\hat{\gamma}}$.

		1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
T=50	k=0	-5.17	-4.81	-4.49	-4.16	-2.92	-1.72	-1.39	-1.12	-0.78
	k=2	-4.85	-4.67	-4.22	-3.83	-2.69	-1.61	-1.29	-1.08	-0.82
	k=5	-4.54	-4.07	-3.75	-3.43	-2.42	-1.47	-1.24	-1.04	-0.80
	k=k(t-sig)	-5.40	-4.95	-4.66	-4.32	-3.07	-1.86	-1.48	-1.23	-0.92
T=100	k=0	-5.09	-4.64	-4.40	-4.06	-2.91	-1.73	-1.35	-1.07	-0.73
	k=2	-4.85	-4.54	-4.22	-3.87	-2.78	-1.62	-1.25	-1.03	-0.70
	k=5	-4.59	-4.26	-4.01	-3.72	-2.61	-1.54	-1.29	-1.02	-0.74
	k=k(t-sig)	-5.08	-4.77	-4.51	-4.19	-3.01	-1.75	-1.40	-1.13	-0.81
T=150	k=0	-4.87	-4.56	-4.32	-4.00	-2.85	-1.61	-1.28	-0.98	-0.62
	k=2	-4.77	-4.44	-4.22	-3.91	-2.79	-1.56	-1.23	-0.96	-0.61
	k=5	-4.68	-4.35	-4.08	-3.79	-2.66	-1.50	-1.19	-0.91	-0.53
	k=k(t-sig)	-5.13	-4.73	-4.46	-4.17	-2.93	-1.68	-1.28	-1.02	-0.62
$T=\infty$		-4.86	-4.55	-4.28	-3.95	-2.84	-1.58	-1.22	-0.88	-0.51

Table 2C : Model (2,AO); Distribution of $t_{\hat{\alpha}}(2,AO,T_b(|t_{\hat{\gamma}}|))$; Choosing T_b Maximizing $|t_{\hat{\gamma}}|$.

		1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
T=50	k=0	-5.46	-5.06	-4.77	-4.40	-3.27	-2.32	-2.07	-1.88	-1.58
	k=2	-5.05	-4.73	-4.42	-4.13	-3.08	-2.20	-2.00	-1.79	-1.57
	k=5	-4.74	-4.37	-4.04	-3.71	-2.73	-1.99	-1.78	-1.63	-1.39
	k=k(t-sig)	-5.49	-5.15	-4.90	-4.61	-3.50	-2.51	-2.24	-2.05	-1.74
T=100	k=0	-5.26	-4.90	-4.61	-4.30	-3.28	-2.32	-2.02	-1.73	-1.50
	k=2	-5.09	-4.70	-4.43	-4.10	-3.12	-2.21	-1.93	-1.72	-1.41
	k=5	-4.78	-4.46	-4.24	-3.93	-2.98	-2.11	-1.88	-1.64	-1.41
	k=k(t-sig)	-5.31	-5.07	-4.81	-4.48	-3.41	-2.40	-2.11	-1.85	-1.54
T=150	k=0	-5.16	-4.80	-4.52	-4.26	-3.22	-2.20	-1.92	-1.73	-1.41
	k=2	-5.01	-4.74	-4.44	-4.19	-3.14	-2.16	-1.89	-1.59	-1.33
	k=5	-4.91	-4.58	-4.33	-4.03	-3.06	-2.15	-1.88	-1.65	-1.38
	k=k(t-sig)	-5.44	-5.03	-4.76	-4.43	-3.35	-2.31	-2.01	-1.77	-1.41
T = ∞		-5.04	-4.73	-4.50	-4.20	-3.20	-2.20	-1.90	-1.59	-1.32

Table 2D : Model (2,AO); Distribution of $t_{\hat{\alpha}}(2,AO,T_b(F_{\hat{\theta},\hat{\gamma}}))$; Choosing T_b Maximizing $F_{\hat{\theta},\hat{\gamma}}$.

		1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
T=50	k=0	-5.75	-5.32	-4.97	-4.62	-3.52	-2.60	-2.36	-2.18	-2.00
	k=2	-5.40	-4.86	-4.53	-4.26	-3.23	-2.35	-2.15	-1.92	-1.69
	k=5	-4.97	-4.53	-4.20	-3.89	-2.85	-2.06	-1.87	-1.68	-1.44
	k=k(t-sig)	-5.89	-5.45	-5.11	-4.77	-3.70	-2.71	-2.48	-2.25	-1.93
T=100	k=0	-5.41	-5.03	-4.82	-4.48	-3.50	-2.59	-2.33	-2.14	-1.90
	k=2	-5.18	-4.84	-4.56	-4.25	-3.30	-2.36	-2.15	-1.97	-1.69
	k=5	-4.79	-4.56	-4.34	-4.07	-3.12	-2.26	-2.00	-1.83	-1.51
	k=k(t-sig)	-5.53	-5.19	-4.93	-4.64	-3.62	-2.64	-2.37	-2.11	-1.80
T=150	k=0	-5.20	-4.96	-4.68	-4.40	-3.45	-2.54	-2.29	-2.04	-1.79
	k=2	-5.19	-4.82	-4.58	-4.26	-3.36	-2.42	-2.18	-1.93	-1.67
	k=5	-4.98	-4.72	-4.46	-4.18	-3.20	-2.32	-2.09	-1.89	-1.64
	k=k(t-sig)	-5.51	-5.14	-4.93	-4.62	-3.58	-2.60	-2.29	-2.06	-1.79
T = ∞		-5.15	-4.86	-4.61	-4.31	-3.35	-2.46	-2.20	-1.94	-1.63

Table 2E : Model (2,IO); Distribution of $t_{\hat{\alpha}}(2,IO,T_b(F_{\hat{\theta},\hat{\gamma}}))$; Choosing T_b Maximizing $F_{\hat{\theta},\hat{\gamma}}$

		1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
T=50	k=0	-5.85	-5.39	-5.05	-4.71	-3.50	-2.33	-1.97	-1.68	-1.29
	k=2	-5.77	-5.33	-5.03	-4.67	-3.39	-2.07	-1.66	-1.28	-0.56
	k=5	-5.88	-5.44	-5.07	-4.64	-3.26	-1.73	-1.12	-0.61	0.16
	k=k(t-sig)	-6.31	-5.92	-5.62	-5.18	-3.85	-2.15	-1.55	-1.01	-0.09
T=100	k=0	-5.60	-5.26	-5.02	-4.63	-3.52	-2.43	-2.12	-1.84	-1.48
	k=2	-5.56	-5.23	-4.98	-4.58	-3.46	-2.25	-1.82	-1.51	-0.98
	k=5	-5.48	-5.22	-4.94	-4.55	-3.37	-2.14	-1.73	-1.21	-0.73
	k=k(t-sig)	-5.85	-5.55	-5.25	-4.95	-3.67	-2.39	-1.96	-1.53	-0.88
T=150	k=0	-5.48	-5.15	-4.92	-4.59	-3.49	-2.40	-2.08	-1.84	-1.47
	k=2	-5.54	-5.25	-4.93	-4.61	-3.46	-2.31	-1.99	-1.70	-1.33
	k=5	-5.52	-5.14	-4.88	-4.59	-3.44	-2.21	-1.83	-1.46	-0.95
	k=k(t-sig)	-5.84	-5.49	-5.14	-4.86	-3.62	-2.33	-1.98	-1.68	-1.11
$T=\infty$		-5.71	-5.41	-5.16	-4.86	-3.91	-2.79	-2.44	-2.13	-1.67

Table 2F : Additional Asymptotic Distributions

Statistic	1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
$t_{\hat{\alpha}}(2,IO,T_b(t_{\hat{\gamma}}))$	-5.28	-4.95	-4.62	-4.28	-2.94	-1.64	-1.33	-0.98	-0.59
$t_{\hat{\alpha}}(2,IO,T_b(\lvert t_{\hat{\gamma}}\rvert))$	-5.57	-5.20	-4.91	-4.59	-3.47	-2.15	-1.86	-1.59	-1.30
$t_{\hat{\alpha}}(3,AO,T_b(t_{\hat{\gamma}}))$	-4.67	-4.36	-4.08	-3.77	-2.65	-1.57	-1.22	-0.90	-0.49
$t_{\hat{\alpha}}(3,AO,T_b(\lvert t_{\hat{\gamma}}\rvert))$	-4.87	-4.58	-4.34	-4.04	-3.08	-2.14	-1.87	-1.61	-1.30

Table 3 : Model (2,AO); Distribution of $t_{\hat{\alpha}}(2,AO,T_b(t_{\hat{\alpha}}))$;
Choosing T_b Minimizing $t_{\hat{\alpha}}$.

Panel (a) : T = 100, k = 0, $\gamma \neq 0$.

		1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
T_b^0=20	γ=0.5	–5.70	–5.38	–5.09	–4.83	–3.87	–3.09	–2.89	–2.75	–2.54
(λ_c=0.2)	γ=1.0	–5.84	–5.52	–5.22	–4.95	–3.97	–3.21	–3.02	–2.83	–2.61
	γ=2.0	–6.93	–6.64	–6.38	–6.10	–5.12	–4.18	–3.93	–3.72	–3.53
	γ=5.0	–9.82	–9.52	–9.19	–8.91	–7.94	–7.01	–6.76	–6.54	–6.29
T_b^0=50	γ=0.5	–5.68	–5.38	–5.12	–4.87	–3.96	–3.14	–2.92	–2.68	–2.49
(λ_c=0.5)	γ=1.0	–5.83	–5.58	–5.31	–5.03	–4.14	–3.24	–2.99	–2.77	–2.47
	γ=2.0	–6.70	–6.40	–6.12	–5.84	–4.98	–4.08	–3.83	–3.57	–3.27
	γ=5.0	–9.13	–8.81	–8.56	–8.28	–7.33	–6.49	–6.22	–6.02	–5.78
T_b^0=80	γ=0.5	–5.78	–5.30	–5.03	–4.70	–3.69	–2.85	–2.63	–2.43	–2.18
(λ_c=0.8)	γ=1.0	–5.45	–5.17	–4.87	–4.50	–3.50	–2.58	–2.35	–2.18	–1.96
	γ=2.0	–5.85	–5.44	–5.17	–4.83	–3.79	–2.80	–2.48	–2.24	–2.04
	γ=5.0	–7.34	–7.08	–6.72	–6.37	–5.33	–4.27	–3.98	–3.69	–3.48

Panel (b) : T = 250, k = 0, $\gamma \neq 0$.

		1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
T_b^0=50	γ=0.5	–5.70	–5.38	–5.12	–4.81	–3.92	–3.18	–3.00	–2.85	–2.64
(λ_c=0.2)	γ=1.0	–6.51	–6.28	–5.99	–5.72	–4.81	–3.86	–3.58	–3.37	–3.11
	γ=2.0	–8.70	–8.39	–8.16	–7.89	–7.06	–6.16	–5.90	–5.62	–5.28
	γ=5.0	–13.30	–13.03	–12.83	–12.83	–11.67	–10.76	–10.50	–10.27	–9.95
T_b^0=125	γ=0.5	–5.79	–5.37	–5.16	–4.94	–4.02	–3.19	–2.92	–2.70	–2.48
(λ_c=0.5)	γ=1.0	–6.38	–6.03	–5.81	–5.53	–4.67	–3.73	–3.41	–3.22	–2.99
	γ=2.0	–8.04	–7.70	–7.47	–7.21	–6.34	–5.43	–5.16	–4.86	–4.72
	γ=5.0	–12.20	–11.78	–11.54	–11.33	–10.43	–9.58	–9.34	–9.14	–8.91
T_b^0=200	γ=0.5	–5.36	–5.09	–4.81	–4.51	–3.51	–2.70	–2.48	–2.26	–2.08
(λ_c=0.8)	γ=1.0	–5.63	–5.28	–5.00	–4.71	–3.64	–2.65	–2.39	–2.17	–1.84
	γ=2.0	–6.50	–6.13	–5.90	–5.60	–4.54	–3.47	–3.15	–2.84	–2.47
	γ=5.0	–9.42	–9.06	–8.78	–8.47	–7.38	–6.29	–5.97	–5.66	–5.29

Table 3 : (Continued)

Panel (c) : T = 500, $k = 0$, $\gamma \neq 0$.

		1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
T_b^c=100	γ=0.5	−5.93	−5.64	−5.45	−5.18	−4.17	−3.32	−3.09	−2.95	−2.77
(λ_c=0.2)	γ=1.0	−7.60	−7.35	−7.13	−6.84	−5.91	−4.93	−4.60	−4.38	−4.03
	γ=2.0	−10.91	−10.63	−10.38	−10.14	−9.27	−8.40	−8.12	−7.81	−7.57
	γ=5.0	−17.60	−17.30	−17.09	−16.89	−15.98	−15.07	−14.83	−14.64	−14.28
T_b^c=250	γ=0.5	−5.99	−5.60	−5.38	−5.11	−4.23	−3.35	−3.01	−2.76	−2.31
(λ_c=0.5)	γ=1.0	−7.11	−6.78	−6.51	−6.31	−5.42	−4.49	−4.20	−3.91	−3.54
	γ=2.0	−9.69	−9.40	−9.19	−8.95	−8.08	−7.20	−6.90	−6.67	−6.42
	γ=5.0	−15.76	−15.50	−15.24	−15.01	−14.13	−13.24	−13.00	−12.80	−12.55
T_b^c=400	γ=0.5	−5.31	−5.01	−4.81	−4.53	−3.52	−2.60	−2.35	−2.12	−1.94
(λ_c=0.8)	γ=1.0	−5.95	−5.61	−5.40	−5.10	−4.02	−2.97	−2.63	−2.30	−1.90
	γ=2.0	−7.46	−7.22	−6.99	−6.64	−5.56	−4.45	−4.11	−3.82	−3.50
	γ=5.0	−11.98	−11.66	−11.37	−11.09	−9.97	−8.80	−8.49	−8.11	−7.66

Table 4 : Finite Sample Size and Power Simulations; Model (2,AO), $t_{\hat{\alpha}}(2,AO,\cdot,k(t\text{-sig}))$.

DGP: $y_t = \theta DU_t^c + \gamma DT_t^{*c} + z_t$, $z_t = \alpha z_{t-1} + \Sigma_{i=1}^4 \varphi(i)\Delta z_{t-i} + (1 + \psi L)e_t$

e_t ~ i.i.d. N(0,1); T = 100, T_b^c = 50; 1,000 replications; 5% nominal size; kmax = 5.

	Size ($\alpha = 1$)					Power ($\alpha = 0.8$)				
	θ,γ	$\theta\,(\gamma=0)$		$\gamma\,(\theta=0)$		θ,γ	$\theta\,(\gamma=0)$		$\gamma\,(\theta=0)$	
T_b	0.0	5.0	10.0	1.0	2.0	0.0	5.0	10.0	1.0	2.0
(1) $\varphi(i) = 0.0$ (i=1,...,4), $\psi = 0.0$										
$T_b(t_{\hat{\alpha}})$	.040	.108	.507	.044	.076	.295	.435	.861	.236	.386
$T_b(\lvert t_{\hat{\gamma}}\rvert)$	.049	.048	.032	.036	.032	.301	.098	.042	.239	.230
$T_b(t_{\hat{\gamma}})$	.050	.050	.040	.072	.061	.350	.133	.055	.376	.371
$T_b(F_{\hat{\theta},\hat{\gamma}})$	.052	.050	.020	.027	.021	.339	.194	.163	.193	.184
(2) $\varphi(1) = 0.6$, $\psi = \varphi(i) = 0.0$ (i=2,3,4)										
$T_b(t_{\hat{\alpha}})$	.033	.036	.072	.030	.029	.832	.731	.804	.696	.714
$T_b(\lvert t_{\hat{\gamma}}\rvert)$	.036	.036	.023	.032	.031	.738	.473	.118	.726	.425
$T_b(t_{\hat{\gamma}})$	.043	.033	.032	.061	.061	.797	.548	.178	.821	.815
$T_b(F_{\hat{\theta},\hat{\gamma}})$	.038	.037	.030	.035	.024	.740	.657	.646	.650	.659
(3) $\varphi(1) = -0.6$, $\psi = \varphi(i) = 0.0$ (i=2,3,4)										
$T_b(t_{\hat{\alpha}})$	.037	.214	.802	.043	.101	.115	.455	.926	.128	.251
$T_b(\lvert t_{\hat{\gamma}}\rvert)$	.039	.037	.032	.030	.030	.114	.042	.027	.088	.089
$T_b(t_{\hat{\gamma}})$	.045	.046	.037	.057	.053	.138	.064	.004	.165	.164
$T_b(F_{\hat{\theta},\hat{\gamma}})$	.052	.028	.016	.023	.019	.132	.067	.061	.073	.065
(4) $\varphi(1) = 0.4$, $\varphi(2) = 0.2$, $\psi = \varphi(3) = \varphi(4) = 0.0$										
$T_b(t_{\hat{\alpha}})$	.023	.028	.063	.019	.025	.631	.556	.725	.493	.515
$T_b(\lvert t_{\hat{\gamma}}\rvert)$	.024	.028	.019	.021	.022	.584	.328	.081	.555	.532
$T_b(t_{\hat{\gamma}})$	.027	.027	.019	.050	.047	.655	.400	.133	.700	.683
$T_b(F_{\hat{\theta},\hat{\gamma}})$	.019	.028	.018	.021	.016	.592	.497	.466	.469	.447

Table 4 : (Continued)

DGP: $y_t = \theta DU_t^c + \gamma DT_t^{*c} + z_t$, $z_t = \alpha z_{t-1} + \Sigma_{i=1}^4 \varphi(i)\Delta z_{t-i} + (1 + \psi L)e_t$

e_t - i.i.d. N(0,1); T = 100, T_b^c = 50; 1,000 replications; 5% nominal size; kmax = 5.

	Size ($\alpha = 1$)					Power ($\alpha = 0.8$)				
	θ,γ	$\theta\ (\gamma = 0)$		$\gamma\ (\theta = 0)$		θ,γ	$\theta\ (\gamma = 0)$		$\gamma\ (\theta = 0)$	
T_b	0.0	5.0	10.0	1.0	2.0	0.0	5.0	10.0	1.0	2.0
(5) $\varphi(1) = 0.3$, $\varphi(2) = 0.3$, $\varphi(3) = 0.25$, $\varphi(4) = 0.14$, $\psi = 0.0$										
$T_b(t_{\hat{\alpha}})$	.011	.008	.013	.013	.013	.854	.788	.783	.769	.758
$T_b(\lvert t_{\hat{\gamma}}\rvert)$	.010	.003	.005	.010	.009	.734	.556	.240	.806	.795
$T_b(t_{\hat{\gamma}})$	.010	.009	.012	.015	.022	.831	.645	.350	.870	.867
$T_b(F_{\hat{\theta},\hat{\gamma}})$	.007	.003	.001	.005	.006	.708	.626	.621	.677	.684
(6) $\psi = 0.5$, $\varphi(i) = 0.0$ (i=1,...,4)										
$T_b(t_{\hat{\alpha}})$	.053	.068	.193	.039	.047	.304	.302	.612	.211	.253
$T_b(\lvert t_{\hat{\gamma}}\rvert)$	.059	.047	.041	.035	.035	.260	.163	.070	.221	.213
$T_b(t_{\hat{\gamma}})$	.061	.057	.043	.067	.069	.303	.216	.101	.332	.325
$T_b(F_{\hat{\theta},\hat{\gamma}})$	.049	.057	.030	.035	.029	.295	.231	.164	.188	.173
(7) $\psi = -0.5$, $\varphi(i) = 0.0$ (i=1,...,4)										
$T_b(t_{\hat{\alpha}})$	.500	.622	.906	.408	.574	.888	.911	.983	.786	.854
$T_b(\lvert t_{\hat{\gamma}}\rvert)$	.298	.175	.112	.265	.242	.685	.150	.013	.640	.626
$T_b(t_{\hat{\gamma}})$	.230	.143	.072	.320	.302	.701	.148	.010	.715	.709
$T_b(F_{\hat{\theta},\hat{\gamma}})$	.391	.239	.209	.254	.230	.723	.591	.591	.613	.612

Table 5 : Finite Sample Size and Power Simulations; Model (2,IO), $t_{\hat{\alpha}}(2,IO,\cdot,k(t\text{-sig}))$.

DGP: $y_t = \theta DU_t^c + \gamma DT_t^{*c} + z_t$, $z_t = \alpha z_{t-1} + \Sigma_{i=1}^{4}\varphi(i)\Delta z_{t-i} + (1 + \psi L)e_t$

e_t ~ i.i.d. N(0,1); T = 100, T_b^c = 50; 1,000 replications; 5% nominal size; kmax = 5.

	Size ($\alpha = 1$)					Power ($\alpha = 0.8$)				
	θ,γ	$\theta\ (\gamma=0)$		$\gamma\ (\theta=0)$		θ,γ	$\theta\ (\gamma=0)$		$\gamma\ (\theta=0)$	
T_b	0.0	5.0	10.0	1.0	2.0	0.0	5.0	10.0	1.0	2.0
(1) $\varphi(i) = 0.0$ (i=1,...,4), $\psi = 0.0$										
$T_b(t_{\hat{\alpha}})$	.034	.119	.637	.065	.314	.211	.442	.976	.233	.688
$T_b(\lvert t_{\hat{\gamma}}\rvert)$	.047	.119	.471	.126	.494	.214	.158	.339	.418	.851
$T_b(t_{\hat{\gamma}})$	.052	.067	.212	.212	.640	.221	.109	.238	.573	.940
$T_b(F_{\hat{\theta},\hat{\gamma}})$	.067	.189	.707	.106	.233	.309	.564	.989	.345	.447
(2) $\varphi(1) = 0.6$, $\psi = \varphi(i) = 0.0$ (i=2,3,4)										
$T_b(t_{\hat{\alpha}})$	.034	.043	.115	.038	.006	.756	.639	.841	.623	.683
$T_b(\lvert t_{\hat{\gamma}}\rvert)$	.048	.051	.104	.055	.116	.651	.158	.382	.759	.843
$T_b(t_{\hat{\gamma}})$	.049	.048	.061	.122	.193	.677	.109	.238	.874	.925
$T_b(F_{\hat{\theta},\hat{\gamma}})$	.062	.074	.151	.061	.093	.747	.715	.893	.730	.773
(3) $\varphi(1) = -0.6$, $\psi = \varphi(i) = 0.0$ (i=2,3,4)										
$T_b(t_{\hat{\alpha}})$	.030	.328	.958	.159	.755	.097	.619	.998	.323	.938
$T_b(\lvert t_{\hat{\gamma}}\rvert)$	.038	.263	.650	.297	.879	.089	.260	.507	.507	.973
$T_b(t_{\hat{\gamma}})$	.040	.136	.328	.438	.950	.103	.159	.339	.662	.995
$T_b(F_{\hat{\theta},\hat{\gamma}})$	.058	.407	.977	.184	.217	.127	.706	.999	.310	.298
(4) $\varphi(1) = 0.4$, $\varphi(2) = 0.2$, $\psi = \varphi(3) = \varphi(4) = 0.0$										
$T_b(t_{\hat{\alpha}})$	.028	.037	.127	.037	.065	.549	.507	.803	.438	.536
$T_b(\lvert t_{\hat{\gamma}}\rvert)$	.047	.047	.120	.056	.121	.488	.328	.081	.599	.697
$T_b(t_{\hat{\gamma}})$	.045	.041	.060	.117	.211	.525	.284	.140	.729	.818
$T_b(F_{\hat{\theta},\hat{\gamma}})$	.054	.067	.171	.058	.095	.595	.620	.862	.559	.595

Table 5 : (Continued)

DGP: $y_t = \theta DU_t^c + \gamma DT_t^{*c} + z_t$, $z_t = \alpha z_{t-1} + \Sigma_{i=1}^{4}\varphi(i)\Delta z_{t-i} + (1 + \psi L)e_t$

e_t ~ i.i.d. $N(0,1)$; $T = 100$, $T_b^c = 50$; 1,000 replications; 5% nominal size; kmax = 5.

T_b	Size ($\alpha = 1$)					Power ($\alpha = 0.8$)				
	θ,γ	$\theta\ (\gamma = 0)$		$\gamma\ (\theta = 0)$		θ,γ	$\theta\ (\gamma = 0)$		$\gamma\ (\theta = 0)$	
	0.0	5.0	10.0	1.0	2.0	0.0	5.0	10.0	1.0	2.0
(5) $\varphi(1) = 0.3$, $\varphi(2) = 0.3$, $\varphi(3) = 0.25$, $\varphi(4) = 0.14$, $\psi = 0.0$										
$T_b(t_{\hat{\alpha}})$	.067	.083	.144	.073	.098	.823	.758	.754	.718	.702
$T_b(\lvert t_{\hat{\gamma}}\rvert)$	.064	.085	.154	.081	.112	.760	.613	.339	.803	.815
$T_b(t_{\hat{\gamma}})$	.062	.058	.089	.094	.155	.800	.613	.345	.888	.886
$T_b(F_{\hat{\theta},\hat{\gamma}})$	.063	.080	.128	.079	.092	.708	.798	.817	.793	.772
(6) $\psi = 0.5$, $\varphi(i) = 0.0$ (i=1,...,4)										
$T_b(t_{\hat{\alpha}})$	.051	.079	.294	.060	.131	.238	.282	.768	.196	.386
$T_b(\lvert t_{\hat{\gamma}}\rvert)$	.066	.080	.249	.103	.240	.202	.178	.265	.328	.591
$T_b(t_{\hat{\gamma}})$	.072	.067	.116	.172	.361	.219	.145	.157	.459	.740
$T_b(F_{\hat{\theta},\hat{\gamma}})$	.092	.124	.363	.089	.169	.306	.385	.835	.294	.416
(7) $\psi = -0.5$, $\varphi(i) = 0.0$ (i=1,...,4)										
$T_b(t_{\hat{\alpha}})$	.447	.656	.999	.435	.896	.843	.955	1.00	.857	.994
$T_b(\lvert t_{\hat{\gamma}}\rvert)$	.373	.457	.679	.561	.964	.699	.361	.375	.914	.998
$T_b(t_{\hat{\gamma}})$	.235	.258	.343	.662	.988	.646	.262	.317	.963	1.00
$T_b(F_{\hat{\theta},\hat{\gamma}})$	.443	.708	.999	.456	.469	.750	.968	1.00	.849	.833

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