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COOPERATIVE OR NONCOOPERATIVE BEHAVIOR?

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RÉSUMÉ

Dans un modèle abstrait à deux agents, toute fonction de choix compatible avec l'hypothèse que les agents agissent de façon non coopérative est aussi compatible avec l'hypothèse d'un comportement coopératif. L'inverse est faux.

Mots clés : rationalisation, équilibre de Nash, optimalité parétienne

ABSTRACT

In an abstract two-agent model, we show that every deterministic joint choice function compatible with the hypothesis that agents act noncooperatively is also compatible with the hypothesis that they act cooperatively. The converse is false.

Key words : rationalization, Nash equilibrium, Pareto efficiency

1. Introduction

The empirical restrictions imposed on individual behavior by the assumption of preference maximization have been thoroughly analyzed, both in the context of abstract choice (this is the literature on choice functions with key contributions by Chernoff (1954), Arrow (1959), and Sen (1971) to mention only a few) and in the context of consumer choice (this is the theory of revealed preference, a sample of which is contained in the book edited by Chipman, Hurwicz, Richter, and Sonnenschein (1971)).

When several individuals interact, choices are harder to explain. Cooperative theories (based on Pareto efficiency) and noncooperative theories (based on Nash equilibrium) both provide valuable insights. Comparing the empirical restrictions imposed by such competing theories on collective behavior is a complex problem that has been widely overlooked. This paper shows — in an admittedly rudimentary framework — that any deterministic collective behavior compatible with the hypothesis that individuals act noncooperatively is also compatible with the hypothesis that they act cooperatively. The converse is false.

2. A model, and a theorem

Let $A_1 = \{1, ..., m_1\}$, $A_2 = \{1, ..., m_2\}$ be two nonempty finite sets of integers representing the conceivable *actions* of two agents, 1 and 2. We refer to elements of $A := A_1 \times A_2$ as *joint actions*. A *row* (of A) is a set of the form $\{(a_1, a_2) \mid a_2 \in A_2\}$ where a_1 is some fixed action in A_1. *Columns* are similarly defined. Suppose we observe the joint actions taken by our agents as the sets of actions actually available to them vary. These joint choices can be described by a joint choice function.

Definition 1. Let $\mathcal{A} = \{B = B_1 \times B_2 \mid \emptyset \neq B \subset A\}$ be the set of nonempty Cartesian product sets included in A. A *joint choice function* f assigns to each $B \in \mathcal{A}$ a nonempty set $f(B) \subset B$. If $f(B)$ is a singleton for each B, we call f *deterministic.*

We do not know the agents' preferences and wonder whether they behave noncooperatively or cooperatively.

Definition 2. The joint choice function f is *noncooperatively rationalizable* if there exist two complete and transitive preference relations $\succcurlyeq_1, \succcurlyeq_2$ on A such that $f(B)$ coincides with the set of Nash equilibria of the game $(B, \succcurlyeq_1, \succcurlyeq_2)$ for every

$B \in \mathcal{A}$. We then say that $\succcurlyeq_1, \succcurlyeq_2$ rationalize f noncooperatively. The function f is *cooperatively rationalizable* if there exist two complete and transitive preference relations $\succcurlyeq_1, \succcurlyeq_2$ such that $f(B)$ coincides with the set of (Pareto) efficient joint actions in $(B, \succcurlyeq_1, \succcurlyeq_2)$ for every B. In that case, $\succcurlyeq_1, \succcurlyeq_2$ rationalize f cooperatively.

In the above definition, Nash equilibrium and Pareto efficiency are given their usual meaning. A Nash equilibrium of $(B, \succcurlyeq_1, \succcurlyeq_2)$ is a joint action $b^* \in B$ such that $b^* \succcurlyeq_1 (b_1, b_2^*)$ for all $b_1 \in B_1$ and $b^* \succcurlyeq_2 (b_1^*, b_2)$ for all $b_2 \in B_2$. Letting $\sim_i$ denote the indifference relation associated with $\succcurlyeq_i$, $b^* \in B$ is efficient in $(B, \succcurlyeq_1, \succcurlyeq_2)$ if for every $b \in B$, $b \succcurlyeq_1 b^*$ and $b \succcurlyeq_2 b^*$ imply that $b \sim_1 b^*$ and $b \sim_2 b^*$. We are now ready to state our main result.

Theorem 1. *If a deterministic joint choice function is noncooperatively rationalizable, then it is also cooperatively rationalizable.*

The proof is presented in the next section. It should be clear that if two preferences $\succcurlyeq_1, \succcurlyeq_2$ rationalize a *deterministic* joint choice function cooperatively, they cannot be too different. Clearly, they cannot conflict within a given row or column: if, say, $(a_1, a_2) \succ_1 (a_1, b_2)$ and $(a_1, a_2) \prec_2 (a_1, b_2)$, then $(\{a_1\} \times \{a_2, b_2\}, \succcurlyeq_1, \succcurlyeq_2)$ admits two efficient joint actions, contradicting our assumption that f is deterministic. While $\succcurlyeq_1, \succcurlyeq_2$ need not be identical, there is in fact no loss in assuming that they are. To be more precise, call a joint choice function f *team-rationalizable* if there exists a single complete and transitive preference relation $\succcurlyeq$ on A which *team-rationalizes* it in the sense that $f(B)$ coincides with the set of maximal elements of $\succcurlyeq$ in B for every $B \in \mathcal{A}$. The following easy lemma is proved in the next section.

Lemma 1. *A deterministic joint choice function is cooperatively rationalizable if and only if it is team-rationalizable.*

An equivalent expression of Theorem 1 follows at once:

Theorem 1bis. *If a deterministic joint choice function is noncooperatively rationalizable, then it is also team-rationalizable.*

Before turning to the proofs, several comments are in order.

(1) A cooperatively rationalizable joint choice function need not be noncooperatively rationalizable, even if it is deterministic. An archetypal example is as follows: $A_1 = A_2 = \{1, 2\}$, $f(\{a\}) = \{a\}$ for each $a \in A$, $f(\{a_1\} \times \{1, 2\}) = \{(a_1, 2)\}$ for $a_1 = 1, 2$, $f(\{1, 2\} \times \{a_2\}) = \{(2, a_2)\}$ for $a_2 = 1, 2$, and $f(A) = \{(1, 1)\}$.

(2) A noncooperatively rationalizable joint choice function f need not be team-rationalizable if it is not deterministic. In that case, there still exists a complete

preference relation whose set of maximal elements in B coincides with $f(B)$ for each B but its strict component might cycle. Such cycles do not contradict the existence of maximal elements because f is not defined on all subsets of A but only on the Cartesian products. Here is an example. Let $A_1 = A_2 = \{1, 2, 3\}$. Define the complete and transitive preference $\succcurlyeq_1$ by

$$(1,3) \sim_1 (2,2) \sim_1 (3,1) \sim_1 (1,1) \succ_1 (2,1) \sim_1 (3,2) \succ_1 (1,2) \sim_1 (2,3) \succ_1 (3,3)$$

and define $\succcurlyeq_2$ by exchanging $(1,1)$ and $(3,3)$ in the above chain, i.e.,

$$(1,3) \sim_2 (2,2) \sim_2 (3,1) \sim_2 (3,3) \succ_2 (2,1) \sim_2 (3,2) \succ_2 (1,2) \sim_2 (2,3) \succ_2 (1,1).$$

Define f by letting $f(B)$ be the set of Nash equilibria of $(B, \succcurlyeq_1, \succcurlyeq_2)$ for each $B \in \mathcal{A}$. Check that f is nonempty-valued but not deterministic. By its very construction, it is noncooperatively rationalizable. But observe that any single preference $\succcurlyeq$ that would team-rationalize it would have to satisfy

$$(1,1) \succ (2,1) \succ (2,3) \succ (3,3) \succ (3,2) \succ (1,2) \succ (1,1)$$

and therefore lack transitivity.

(3) If $f(B)$ is interpreted as the *actual* joint choice from B (and not as a set of recommended or plausible joint actions) and if the actions available to an agent are mutually exclusive (as can be assumed without loss of generality), the assumption that f is deterministic is almost tautological. One might argue that the notions of rationalizability given in Definition 2 are improperly strong. To be sure, weak versions could be defined by requiring that $f(B)$ be included in, rather than equal to, the set of Nash equilibria or the Pareto set of $(B, \succcurlyeq_1, \succcurlyeq_2)$ for every B. But *every* joint choice function would be noncooperatively and cooperatively weakly rationalizable in this sense since we may always assume that the agents are completely indifferent between all joint actions.

3. Proofs

Let us begin with the straightforward proof of Lemma 1.

Proof of Lemma 1. The "if" part is obvious. Conversely, suppose that the complete and transitive preferences $\succcurlyeq_1, \succcurlyeq_2$ rationalize the deterministic joint choice function f cooperatively. Let $\succcurlyeq_{12}$ be the Pareto relation associated with these two preferences:

$$a \succcurlyeq_{12} b \Leftrightarrow a \succcurlyeq_1 b \text{ and } a \succcurlyeq_2 b.$$

Though incomplete, $\succcurlyeq_{12}$ is transitive. We can therefore construct a complete transitive relation $\succcurlyeq_{12}^*$ which is fully compatible with it in the sense that $a \succcurlyeq_{12}^* b$ whenever $a \succcurlyeq_{12} b$ and $a \succ_{12}^* b$ whenever $a \succ_{12} b$.

It is easily seen that $\succcurlyeq_{12}^*$ team-rationalizes f. Fix $B \in \mathcal{A}$ and let $f(B) = \{b^*\}$. Then b^* is the unique efficient joint action in $(B, \succcurlyeq_1, \succcurlyeq_2)$. This implies that $b^* \succ_{12} b$ for all $b \in B$, hence b^* is a maximal element of $\succcurlyeq_{12}^*$ in B. Conversely, it is just equally clear that any maximal element of $\succcurlyeq_{12}^*$ in B must belong to $f(B)$. ∎

Theorem 1bis will follow easily from Lemma 2 below. Fix two complete and transitive preference relations $\succcurlyeq_1, \succcurlyeq_2$ on A. Define the binary relation $\succcurlyeq$ on A as follows:

$$a \succcurlyeq b \Leftrightarrow [a_2 = b_2 \text{ and } a \succcurlyeq_1 b] \text{ or } [a_1 = b_1 \text{ and } a \succcurlyeq_2 b]. \tag{3.1}$$

This relation is incomplete: it only compares joint actions belonging to a same row or a same column. Define the strict component $\succ$ of $\succcurlyeq$ in the usual way and say that a *threatens* b if $a \succ b$. The relation $\succ$ is *acyclic* if for every integer t and every sequence $a^1, a^2, ..., a^t$ in A,

$$[a^1 \succ a^2 \succ ... \succ a^t] \Rightarrow [\text{not } a^t \succ a^1].$$

In the statement and proof of Lemma 2, $\succcurlyeq_1, \succcurlyeq_2$, hence $\succcurlyeq$, are fixed. With a slight abuse of language we call any $B \in \mathcal{A}$ a subgame of A: it is understood that the agents' preferences are $\succcurlyeq_1, \succcurlyeq_2$.

Lemma 2. *If every subgame of A has a unique Nash equilibrium, then the relation $\succ$ is acyclic.*

Proof. *Step 1. Preliminaries.*

Let the assumption of Lemma 2 be satisfied. Denote by $\succ_1, \succ_2$ the strict components of $\succcurlyeq_1, \succcurlyeq_2$ defined in the usual way and call $\succcurlyeq_1, \succcurlyeq_2$ *essentially strict* if no two distinct joint actions in a row of A are indifferent to agent 2 and no two distinct joint actions in a column are indifferent to agent 1: if $a, b \in A$ and $a_1 \neq b_1, a_2 \neq b_2$, then 1) either $a \succ_1 (b_1, a_2)$ or $a \prec_1 (b_1, a_2)$ and 2) either $a \succ_2 (a_1, b_2)$ or $a \prec_2 (a_1, b_2)$. Since the assumption of a unique Nash equilibrium applies to the subgames where two actions are available to an agent and one action is available to the other, the preferences $\succcurlyeq_1, \succcurlyeq_2$ must be essentially strict.

Contrary to the claim, suppose now that $\succ$ is not acyclic. Let $C = \{a^1, ..., a^t\}$ be a cycle:

$$a^1 \succ a^2 \succ ... \succ a^t \succ a^1.$$

For future use we note that $\succ$ obviously satisfies a *restricted transitivity* property: if a, b, c are three joint actions in a same row or column, $a \succ b \succ c$ implies $a \succ c$. It is then clear that the *length* t of the cycle C must be even. Moreover, to ensure the existence of a Nash equilibrium in every 2×2 subgame, $t > 4$. We claim that a shorter cycle $b^1 \succ b^2 \succ ... \succ b^q \succ b^1, q < t$, necessarily exists. Once this fact is established, repeated application of it eventually produces a cycle of length 4, which is the desired contradiction.

To prove the fact, assume by way of contradiction that no cycle shorter than C exists. We may then safely assume that all elements of C are distinct. We may further assume that $t \leq 2m$, where $m = \min\{m_1, m_2\}$. Otherwise there must exist three distinct joint actions in C belonging to a same row or column, say, $(a_1, a_2) \succ (a_1, b_2) \succ (a_1, c_2)$, and C can obviously be shortened. There is no loss in assuming that C has in fact *maximal length* in A, i.e., $t = 2m$: otherwise we may replace A with some $A' \in \mathcal{A}$ in which C does have maximal length. Finally, renumbering actions if needed, we may write the cycle C as follows:

$$\begin{aligned}(1,1) &\succ (2,1) \succ (2,3) \succ ... \succ (m-1, m-2) \succ (m-1, m) \succ (m, m) \\ &\succ (m, m-1) \succ (m-2, m-1) \succ ... \succ (3,2) \succ (1,2) \succ (1,1).\end{aligned}$$

This is illustrated on Figure 1.[1] Define $A'_1 = A'_2 = \{1, 2, ..., m\}$. The rest of the proof focuses on $A' := A'_1 \times A'_2$.

Two definitions will be useful. The four *projections* (on C) of a joint action $a = (a_1, a_2) \in A' \backslash C$ are the four joint actions $(\alpha_1, a_2), (\alpha'_1, a_2), (a_1, \alpha_2), (a_1, \alpha'_2) \in C$ such that $\alpha_1 < \alpha'_1$ and $\alpha_2 < \alpha'_2$. The four *conjugates* of a are $(\alpha_1, \alpha_2), (\alpha_1, \alpha'_2), (\alpha'_1, \alpha_2), (\alpha'_1, \alpha'_2)$. These definitions are illustrated on Figure 2. Note that $\alpha_i = \alpha'_i - 2$ if $1 < a_j < m$ and $\alpha_i = \alpha'_i - 1$ otherwise (for two distinct agents i, j). Observe also that one (but no more) of a's conjugates might belong to C.

Step 2. Each joint action in $A' \backslash C$ either threatens all its four projections (and we call it dominant*) or else is threatened by all of them (and we call it* dominated*).*

To prove this claim, let $a \in A' \backslash C$. We give a proof for the case when a lies southwest of the cycle, i.e., $a_1 > a_2$. The arguments are easily adapted to the cases $a_1 = a_2$ and $a_1 < a_2$. Let us distinguish four cases corresponding to the four positions for a numbered (1), (2), (3), (4) on Figure 3.

Case 1: $(\alpha'_1, a_2) \prec (\alpha_1, a_2)$ and $(a_1, \alpha_2) \prec (a_1, \alpha'_2)$.

[1] On all figures, joint actions correspond to vertices of the grid, not to cells.

Suppose $a \prec (\alpha_1', a_2)$ (and therefore, by restricted transitivity of $\succ$, $a \prec (\alpha_1, a_2)$). Then $a \prec (a_1, \alpha_2)$ for otherwise we could shorten the cycle C by replacing the sequence

$$(a_1, \alpha_2) \prec (a_1, \alpha_2') \prec ... \prec (\alpha_1', a_2) \prec (\alpha_1, a_2) \tag{3.2}$$

contained in it with the shorter sequence

$$(a_1, \alpha_2) \prec a \prec (\alpha_1, a_2). \tag{3.3}$$

Using restricted transitivity once more, we conclude that a is dominated.

Next, suppose $a \succ (\alpha_1', a_2)$. Then $a \succ (a_1, \alpha_2')$ for otherwise we could replace the sequence

$$(a_1, \alpha_2') \succ (a_1, \alpha_2) \succ ... \succ (\alpha_1, a_2) \succ (\alpha_1', a_2)$$

with the shorter

$$(a_1, \alpha_2') \succ a \succ (\alpha_1', a_2).$$

By restricted transitivity, $a \succ (a_1, \alpha_2)$. This in turn implies $a \succ (\alpha_1, a_2)$ or else we could again replace sequence (3.2) with sequence (3.3). Thus a is dominant.

Case 2: $(\alpha_1', a_2) \succ (\alpha_1, a_2)$ and $(a_1, \alpha_2) \succ (a_1, \alpha_2')$.

The argument of Case 1 remains valid word for word if we interchange the symbols "$\succ$" and "$\prec$" and the words "dominated" and "dominant".

Case 3: $(\alpha_1', a_2) \succ (\alpha_1, a_2)$ and $(a_1, \alpha_2) \prec (a_1, \alpha_2')$.

Suppose $a \succ (\alpha_1', a_2)$ (hence, $a \succ (\alpha_1, a_2)$).Then $a \succ (a_1, \alpha_2')$, otherwise we could replace the sequence

$$(a_1, \alpha_2') \succ (a_1, \alpha_2) \succ ... \succ (\alpha_1', a_2) \succ (\alpha_1, a_2) \tag{3.4}$$

in C with the shorter sequence

$$(a_1, \alpha_2') \succ a \succ (\alpha_1, a_2). \tag{3.5}$$

It follows that a is dominant.

Next, suppose $a \prec (\alpha_1', a_2)$. Then $a \prec (a_1, \alpha_2)$, otherwise we could replace the sequence

$$(\alpha_1', a_2) \succ (\alpha_1, a_2) \succ ... \succ (a_1, \alpha_2') \succ (a_1, \alpha_2)$$

with the shorter

$$(\alpha_1', a_2) \succ a \succ (a_1, \alpha_2).$$

By restricted transitivity, it follows that $a \prec (a_1, \alpha_2')$, which in turn implies $a \prec (\alpha_1, a_2)$ for otherwise we could again replace (3.4) with (3.5). Thus a is dominated.

Case 4: $(\alpha_1', a_2) \prec (\alpha_1, a_2)$ and $(a_1, \alpha_2) \succ (a_1, \alpha_2')$.

The argument of Case 3 carries over provided we interchange "$\succ$" and "$\prec$" and "dominated" and "dominant". This completes Step 2.

Step 3. If a joint action a in $A' \backslash C$ is dominant, then all its conjugates which belong to $A' \backslash C$ are dominated. If a is dominated, then all its conjugates in $A' \backslash C$ are dominant.

To prove the first part of the above claim, let a be a dominant joint action in $A' \backslash C$. Suppose, contrary to the claim, that one of a's conjugates in $A' \backslash C$, say $\alpha = (\alpha_1, \alpha_2)$, is not dominated. By Step 2, α is dominant. By definition, $\alpha_1 \neq a_1$ and $\alpha_2 \neq a_2$, so that the set

$$B^*(a; \alpha) := \{a_1, \alpha_1\} \times \{a_2, \alpha_2\} = \{a, (a_1, \alpha_2), (\alpha_1, a_2), \alpha\}$$

is a 2×2 subgame of A'. Moreover, (a_1, α_2) and (α_1, a_2) are two projections on C of *both* a and α. Since a and α are both dominant,

$$a \succ (a_1, \alpha_2), a \succ (\alpha_1, a_2), \alpha \succ (a_1, \alpha_2), \alpha \succ (\alpha_1, a_2).$$

This means that a and α are two Nash equilibria of $B^*(a; \alpha)$, a contradiction.

To prove the second part of the claim, suppose now that a is a dominated joint action in $A' \backslash C$ and, by way of contradiction, that α is not dominant. By Step 2 again, α is dominated. Therefore,

$$a \prec (a_1, \alpha_2), a \prec (\alpha_1, a_2), \alpha \prec (a_1, \alpha_2), \alpha \prec (\alpha_1, a_2),$$

which means that (a_1, α_2) and (α_1, a_2) are now two Nash equilibria of $B^*(a; \alpha)$. This contradiction completes the proof of Step 3.

Step 4. End of the proof.

By Step 2, $(2, 2)$ is either dominant or dominated. Suppose it is dominated. By Step 3, its conjugates $(1, 3)$ and $(3, 1)$ are both dominant. But this contradicts Step 3 because $(1, 3)$ and $(3, 1)$ are also conjugates of each other. Similarly, if $(2, 2)$ is dominant, then $(1, 3)$ and $(3, 1)$ are both dominated, which again contradicts Step 3. ■

Proof of Theorem 1bis. Let f be a deterministic joint choice function and let $\succcurlyeq_1, \succcurlyeq_2$ rationalize f noncooperatively. For each $B \in \mathcal{A}$, $(B, \succcurlyeq_1, \succcurlyeq_2)$ has a unique

Nash equilibrium: call it $a^*(B)$. Define $\succcurlyeq$ according to (3.1) and let $\succ$ be its strict component. By Lemma 2, $\succ$ is acyclic. This guarantees that for each $a \in A$ there is a sequence

$$a = a^1 \prec a^2 \prec \ldots \prec a^t = a^*(A).$$

Moreover, at least one such sequence has maximal length. Call this maximal length $t(a)$ and define $\succcurlyeq^*$ as follows: for any $a, b \in A$,

$$a \succcurlyeq^* b \Leftrightarrow t(a) \leq t(b).$$

This is obviously a complete and transitive relation. We claim that its set of maximal elements in B coincides with $f(B)$ for every $B \in \mathcal{A}$. If $b^* \notin f(B)$, then b^* is not a Nash equilibrium of $(B, \succcurlyeq_1, \succcurlyeq_2)$: there exists, say, $(b_1, b_2^*) \in B$ such that $(b_1, b_2^*) \succ_1 b^*$, hence $(b_1, b_2^*) \succ b^*$. But then $t(b^*) > t(b_1, b_2^*)$, and b^* is not a maximal element of $\succcurlyeq^*$ in B. Conversely, if $b^* \in f(B)$, then b^* is the unique Nash equilibrium of $(B, \succcurlyeq_1, \succcurlyeq_2)$. For any $b \in B \backslash \{b^*\}$ there is a sequence $b \prec \ldots \prec b^*$ in B. This implies that $t(b) > t(b^*)$, hence b^* is the unique maximal element of $\succcurlyeq^*$ in B. ■

Theorem 1 follows from Theorem 1bis and Lemma 1.

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FIGURE 1

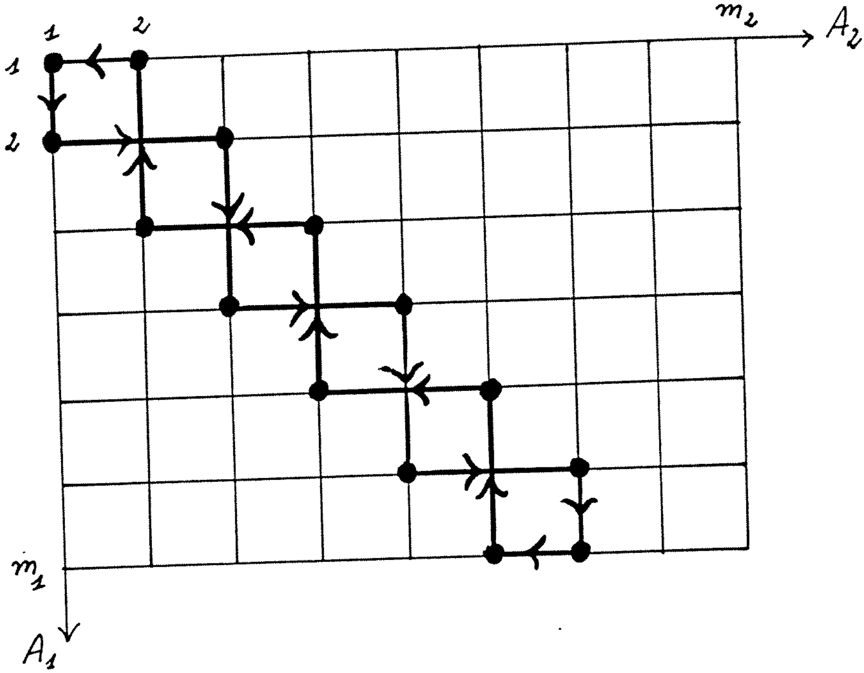

FIGURE 2

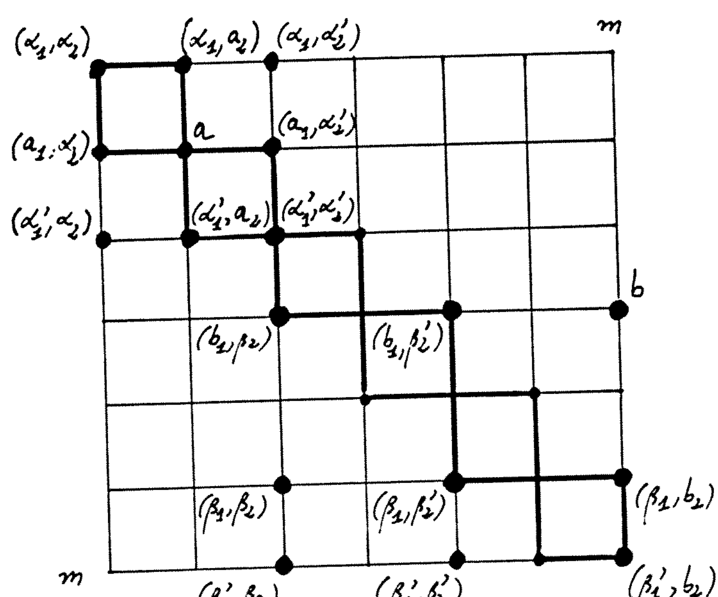

FIGURE 3